Weighted centrality, and a further approach to categorical commutativity

by

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Abstract

Weighted centrality, and a further approach to categorical commutativity

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We investigate weighted commutators, that is, weighted subobject commutator and weighted normal commutator, as well as commutators in the sense of Huq, Higgins, Ursini and Smith, which are all special cases of weighted commutators. One of the main aims is to establish further properties of weighted commutators, and explore new relationships among commutators. In a normal Mal'cev category $\mathbb{C}$ with finite colimits, we show that the Huq commutator of a pair of local representations (i.e. equivalence relations considered as subobjects in a category of points over a fixed object) is the local representation of the Smith commutator of the equivalence relations corresponding to the original local representations. We also show that the weighted normal commutator can be obtained as the image of the kernel functor applied to the Huq commutator of another type of morphisms in a category of points over a fixed object. In addition, the weighted normal commutator is characterized as the largest monotone ternary operation C defined on subobjects in a finitely cocomplete normal Barr-exact Mal'cev category, such that: (a) $C(X,Y,W) \leq \overline{X} \wedge Y$; (b) $C(f(X), f(Y), f(W)) = \overline{f(C(X,Y,W))}$, for subobjects $(X,x),(Y,y)$, and (W,w) of an object A, and every morphism f whose domain is A. The weighted subobject commutator is characterized in a similar way, and furthermore, known characterizations of Higgins, Huq, and Ursini commutators are recovered as special cases.

Another aim is to extend the notion of commuting morphisms to a more general context, and in particular, to a subtractive category with finite joins of subobjects, where we show that commuting morphisms are related to the notion of internal partial subtraction structures. Furthermore, we show that several results about central morphisms, commutative objects, and abelian objects, which usually require a category to be at least (strongly) unital, also hold in the context of (regular) subtractive category with finite joins of subobjects.

Uittreksel

Geweegde sentraliteit en ’n verdere benadering tot kategoriese kommutatiwiteit

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Ons ondersoek geweegde kommutator, dit wil sê geweegde subobjekkommutator en geweegde normale kommutator, sowel as kommutators in die sin van Huq, Higgins, Ursini en Smith, wat almal spesiale gevalle van geweegde kommutator is. Een van die hoofdoelwitte is om verdere eienskappe van geweegde kommutator te vestig, en om nuwe verhoudings tussen kommutator te ondersoek. In ’n normale Mal’cev-kategorie $\mathbb{C}$ met eindige kolimiete, wys ons dat die Huq-kommutator van ’n paar plaaslike voorstellings (dit wil sê ekwivalensieverhoudinge wat as subobjekte in ’n kategorie punte oor ’n vaste objek beskou word) die plaaslike voorstelling van die Smith is kommutator van die ekwivalensieverhoudinge wat ooreenstem met die oorspronklike plaaslike voorstellings. Ons wys ook dat die geweegde normale kommutator verkry kan word as die beeld van die “kernel” funktor wat op die Huq-kommutator van ’n ander soort morfismes toegepas word in ’n kategorie punte oor ’n vaste objek. Daarbenewens word die geweegde normale kommutator gekenmerk as die grootste monotone ternêre werking C gedefinieër op sub-objekte in ’n eindelik klaargemaakte normale Barr-exact Mal’cev-kategorie, sodat: (a) $C(X,Y,W) \leq \cancel{X} \wedge Y$; (b) $C(f(X), f(Y), f(W)) = \cancel{f(C(X,Y,W))}$, vir subobjekte $(X,x), (Y,y)$, en (W,w) van ’n objek A, en elke morfisme f waarvan die domein A is. Die geweegde subobjekkommutator word op ’n soortgelyke manier gekenmerk, en voorts word bekende karakterisering van Higgins, Huq en Ursini-kommutators as spesiale gevalle herwin.

’N Ander doel is om die idee van die pendel van morfismes uit te brei na ’n meer algemene konteks, en veral tot ’n subtraktiewe kategorie met ’n eindige samevoeging van sub-onderwerpe, waar ons wys dat die pendel-morfisme verband hou met die idee van interne gedeeltelike aftrek-strukture. Verder toon ons dat verskeie resultate oor sentrale morfismes, kommutatiewe objekte en abeliese objekte, wat gewoonlik vereis dat ’n kategorie ten minste (sterk) uniaal is, ook in die konteks van ’n (gereelde) aftrekkategorie met ’n beperkte samevoeging van subobjekte geld.

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Dedications

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Introduction

The classical *commutator* measures the obstruction for a pair of subgroups of a given group to *commute*. Indeed, for a pair of subgroups X and Y of a group G, their commutator is trivial precisely when every element of X commutes with every element of Y, i.e. $xy = yx$ for all $x \in X$ and $y \in Y$. There are various generalizations of the classical commutator in universal algebras, which have been even extended further to categorical contexts. Let us briefly mention the generalizations we are going to study in this thesis.

S. A. Huq [22] introduced the categorical notion of commuting pair of morphisms having the same codomain (*commuting morphisms*), which gives rise to the notion of *Huq commutator*. In the case of groups, for two subgroups X and Y of a group G, the inclusion maps $X \hookrightarrow G$ and $Y \hookrightarrow G$ commute in the sense of Huq if and only if every element of X commutes with every element of Y. In [21] P. J. Higgins introduced the *Higgins commutator*; a universal-algebraic commutator, which generalizes the classical commutator of groups to Ω-*groups*. A categorical description of Higgins commutator has been given by S. Mantovani and G. Metere [30], who have also shown that the Huq commutator is the *normal closure* of the Higgins commutator. There is another notion of commutator (*Smith commutator*), introduced by J. D. H. Smith [36] in *Mal'cev (congruence permutable)* varieties. Commutators of congruences have also been investigated in more general varieties, such as *congruence modular* varieties by J. Hagemann and C. Herrmann [20] (see also H. P. Gumm [18], and R. Freese and R. McKenzie [16]). The categorical notion of commutator of equivalence relations has been introduced by M. C. Pedicchio [33] in the context of Mal'cev category, and this has provided a first categorical approach to commutator theory of congruences. In the same paper, the notion of a pair of equivalence relations *centralizing* each other in a Mal'cev category is given. In [37] A. Ursini introduced BIT varieties (also called ideal-determined varieties in H. P. Gumm and A. Ursini [19]), and in this varietal setting, commutators of ideals (*Ursini commutators*) in the sense of Ursini [38], can be obtained from Smith commutators of congruences (see [19]). A description of Ursini commutator in a categorical context has been given by S. Mantovani [29], who also proved that the varietal relationship between the Ursini commutator and Smith commutator extends to a categorical setting. Recently, M. Gran, G. Janelidze, and A. Ursini [17] introduced categorical notions of *weighted commutators* and *weighted centrality*. The Huq, Higgins, and Ursini commutators are all special cases of the weighted commutators. Similarly, both notions of commutation of morphisms and centralization of equivalence relations are all obtained as special cases of weighted centrality. Weighted commutators comprise of two notions of commutators, namely *weighted subobject commutator* and *weighted normal commutator*, and the two are related in the sense that the weighted normal commutator is the normal closure of the weighted subobject commutator (see [17]).

J. Hagemann and C. Herrmann [20] discovered a lattice-theoretic characterization of the

Smith commutator in a congruence modular variety, as the largest monotone binary operation C defined on the congruence lattice of each algebra, satisfying the conditions (that): (a) $C(R,S)$ is always less or equal to the meet of R and S; (b) $C(R,S)$ is preserved by images under surjective homomorphisms, i.e. $f(C(R,S)) = C(f(R), f(S))$, for every pair of congruences R and S on an algebra X, and every surjective homomorphism f whose domain is X. In [35] the author characterized the Huq commutator in the context of normal unital category with finite colimits, as the largest monotone binary operation C defined for subobjects, such that: (a) $C(H,K)$ is contained in the meet of the normal closures of (H,h) and (K,k); (b) $C(f(H), f(K)) \leq \overline{f(C(H,K))}$, for every pair of subobjects (H,h) and (K,k) of X, and every morphism f whose domain is X. Furthermore, the author showed in [34] that the Higgins commutator can be characterized in a similar way in the context of ideal-determined unital category. Since weighted commutators could be thought of as ternary operations defined on all subobjects of each object, we show in Section 3.4 that weighted commutators admit a similar characterization, but as ternary operations. We also show that the characterizations of Huq, Higgins, and Ursini/Smith commutators can be recovered as special cases.

Recall that an equivalence relation on an object A in a Mal'cev category $\mathbb{C}$ can be identified with a subobject in a category of *points* over A (that is, a category of split epimorphisms over A), which, for instance, D. Bourn, N. Martins-Ferreira, and T. Van der Linden [13] called such a subobject a *local representation*. In a Mal'cev category, a pair of equivalence relations centralize each other precisely when their corresponding local representations commute in the sense of Huq (see D. Bourn [5]). In a finitely cocomplete normal Mal'cev category, the Huq commutator of a pair of subobjects of an object and the Smith commutator of a pair of equivalence relations on an object, can always be constructed. So in the present work, it will be shown that the Huq commutator of a pair of local representations is the local representation of the Smith commutator of the equivalence relations corresponding to the original local representations. More generally, since it is also possible to present weighted centrality in terms of commuting morphisms in a category of points over a fixed object, similar description of weighted normal commutators in terms of Huq commutators will be given.

An interesting class of morphisms is that of central morphisms; that is, those morphisms which commute with the identity morphisms of their codomains. The notion of central morphisms has been investigated, for instance, by D. Bourn [6], where the following have been shown: (a) In every unital category, the class $Z(X,Y)$ of central morphisms from X to Y forms a commutative monoid; (b) commutative objects (those objects whose identity morphisms are central) are exactly objects that admit internal commutative monoid structures; (c) in a strongly unital category, $Z(X,Y)$ is always an abelian group, and in particular, every commutative object is an abelian object (that is, an object which admits an internal abelian group structure).

Z. Janelidze [25] introduced the notion of a subtractive category, which is a categorical generalization of a pointed subtractive variety introduced by A. Ursini [39]. In addition, the following "categorical equation" has been observed:

$$\text{strongly unital} \;=\; \text{unital} \;+\; \text{subtractive}.$$

In this thesis, the notion of commuting morphisms is extended to a more general context, and in particular to a subtractive category with finite joins of subobjects, where we show in Section 4.2 that commuting morphisms are related to the notion of *partial subtraction structures* [12]. We will generalize several results about central morphisms, commutative objects, and abelian objects in a (regular) subtractive category with finite joins of subobjects; in particular those of [6], which were originally proven in a (strongly) unital context.

The thesis consists of the following chapters:

Chapter 1: In this chapter we recall the necessary background for the ensuing chapters. In addition to already known materials recalled in this chapter, we prove Theorem 1.3.7, which leads to useful facts about decomposition of certain regular epimorphisms in a regular strongly unital category.

Chapter 2: This chapter deals with binary commutators in the sense of Huq, Higgins, Ursini, and Smith. We recall some of their basic properties, and relationships among them. In addition, we characterize the Higgins commutator in a normal strongly unital category $\mathbb{C}$ with finite colimits; the same result has been already given (see Theorem 3.1 of [34]) in the context of ideal-determined unital category.

Chapter 3: In this chapter we explore some other properties of weighted centrality and weighted commutators, and also establish new relationships with the other commutators. In Section 3.1 we recall an equivalent formulation of weighted centrality in terms of commuting pairs of morphisms (in the sense of Huq) in a category of points over a fixed object, and this provides an alternative approach to see Smith centrality as a special case of weighted centrality. In Section 3.2 we investigate regular images of weighted commutators under arbitrary morphisms. We show (Theorem 3.2.6) that in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, the weighted subobject commutator is preserved by regular images under arbitrary morphisms. We deduce a similar result for the weighted normal commutator by applying the fact that the weighted normal commutator is the normal closure of the weighted subobject commutator. In Section 3.3 we describe Smith commutator in terms of Huq commutator of local representations. More specifically, we explain (Theorem 3.3.1) that in a normal Mal'cev category $\mathbb{C}$ with finite colimits, the local representation of the Smith commutator of a pair of equivalence relations R and S is the Huq commutator of the local representations of R and S. As a corollary, we recover an already known fact, given in [17] (see also [29]), that the $1-$weighted normal commutator (defined on normal subobjects) is the associated normal subobject (i.e. *normalisation*) of the Smith commutator. In Section 3.4, the weighted subobject commutator is characterized (Theorem 3.4.6) as the largest monotone ternary operation C defined for subobjects in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, such that: (a) $C(X,Y,W) \leq \overline{X} \wedge \overline{Y}$; (b) $C(f(X),f(Y),f(W)) = f(C(X,Y,W))$, for subobjects $(X,x),(Y,y)$, and (W,w) of an object A, and every morphism f whose domain is A. A similar characterization is given in Theorem 3.4.7 for the weighted normal commutator. As special cases, we recovered already known characterizations of Higgins, Huq, and Ursini commutators. In Section 3.5 we investigate another relationship between weighted normal commutator and Huq commutator. More precisely, we show (Theorem 3.5.1) that in a normal Mal'cev category $\mathbb{C}$ with finite colimits, the weighted normal commutator can be obtained as the image of the *kernel functor* applied to the Huq commutator of some morphisms in a category of points over a fixed object.

Chapter 4: This chapter is devoted to investigate commuting morphisms in subtractive categories. In a subtractive category there is a fundamental notion of partial subtraction structures [12], which happens to be related to commuting morphisms. In Section 4.1 we show that in a regular subtractive category the class of morphisms between X and Y which admit partial subtraction structures forms an abelian group. In contrast with the unital context, in which commutative objects are not necessarily abelian objects, in Section 4.2, we prove (Theorem 4.2.12) that in a subtractive category with finite joins of subobjects commutative objects are precisely those objects which admit internal abelian group structures, which as shown already in Corollary 2.7 of [12], are equivalent to those objects which admit (partial) subtraction structures. In Section 4.3 we prove a stronger fact (Theorem 4.3.2) that central morphisms are precisely those morphisms which admit partial subtraction structures in a regular subtractive category with finite joins of subobjects. As a consequence, we obtain that in a regular subtractive category $\mathbb{C}$ with finite joins of subobjects the class of central morphisms between X and Y admits an abelian group structure; extending the result valid for strongly unital categories, given in [6]. In addition, (thanks to Theorem 4.3.2) we extend to any regular subtractive category $\mathbb{C}$ with finite joins of subobjects and cokernels, the following universal construction, usually known to hold in a finitely cocomplete regular unital category (see e.g F. Borceux and D. Bourn [4]): For every morphism $f : X \longrightarrow Y$ there is a universal morphism $g : Y \longrightarrow Q$ which, by composition, makes f central. As a particular case, an already known construction of abelian objects in regular subtractive categories with cokernels (Theorem 4.3 of [12]) can be recovered.

Chapter 5: In this chapter we explore commuting morphisms in a more general setting. In Section 5.1 we describe commuting morphisms in several examples such as categories of pointed sets, *implication algebras*, and some other "non-unital" categories. In Section 5.2 we show that some known results about commuting morphisms can be generalized to a wider context. In particular, we show that in a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, the commutes relation is an example of a *cover relation* arising from a special kind of a monoidal structure. In addition, we observe that when $\mathbb{C}$ is a subtractive category with finite joins of subobjects, monoids in $\mathbb{C}$ equipped with the monoidal structure whose corresponding cover relation is the commutes relation, are exactly abelian objects.

Contents

Chapter 1

Preliminaries

In this chapter we recall the necessary background for the ensuing chapters. As for a detailed account of the categorical notions recalled in this chapter, we recommend [4].

Let us explain some notation we will be using. In a pointed category $\mathbb{C}$, we will write 0 to denote the null (zero) morphism between any two objects, and just 1 (instead of 1_X) to denote the identity morphism on any object X. For a category $\mathbb{C}$ with finite products and coproducts, we will write $A \times B$ for the product $(A \times B, \pi_1, \pi_2)$ of A and B, where π_1 and π_2 denote the first and second product projections respectively. Dually, we will write $A + B$ for the coproduct $(A + B, i_1, i_2)$ of A and B, where i_1 and i_2 denote the coproduct inclusions. For morphisms $f : A \longrightarrow B$ and $g : A \longrightarrow C$ in $\mathbb{C}$, $\langle f, g\rangle$ will denote the unique morphism $A \longrightarrow B \times C$ with $f = \pi_1\langle f, g\rangle$ and $g = \pi_2\langle f, g\rangle$. Dually, for morphisms $u : U \longrightarrow W$ and $v : V \longrightarrow W$ in $\mathbb{C}$, $[u, v]$ will denote the unique morphism $U + V \longrightarrow W$, with $u = [u, v]i_1$ and $v = [u, v]i_2$. For an object X in a category $\mathbb{C}$, a subobject of X is a class of isomorphic monomorphisms with a common codomain of X. We write (H, h) to denote the class of monomorphisms into X which are isomorphic to a monomorphism $h : H \rightarrowtail X$. $\mathsf{Sub}(X)$ denotes the class of all subobjects of X, and it can be preordered by a relation $\leq$ defined by:

$$(H, h) \leq (K, k) \Leftrightarrow h \text{ factors through } k, \text{ i.e. } h = k\alpha \text{ for some morphism } \alpha : H \longrightarrow K.$$

For convenience, we will write $H \leq K$ instead of $(H, h) \leq (K, k)$. When $\mathbb{C}$ has pullbacks, the meet exists for any two subobjects (H, h) and (K, k) in $\mathsf{Sub}(X)$, and is given by the pullback of $h : H \rightarrowtail X$ and $k : K \rightarrowtail X$. A category $\mathbb{C}$ has finite joins of subobjects when for every object X of $\mathbb{C}$, $\mathsf{Sub}(X)$ has finite joins.

1.1 Regular categories

In the categories $\mathsf{Set}, \mathsf{Gp}$, and Rng of (respectively) sets, groups, and rings, and more generally, in varieties of universal algebras, every morphism can be factored uniquely through its image. This is one of the key properties of *regular categories*. Before we recall the definition of a regular category, let us first recall several types of epimorphisms.

A morphism $f : A \longrightarrow B$ in a category $\mathbb{C}$ is called an epimorphism if for every pair of parallel morphisms u and v such that $uf = vf$, one has $u = v$. The dual notion of epimorphism is *monomorphism*. In Set and Gp, epimorphisms are exactly surjective maps and surjective group homomorphisms respectively. In general epimorphisms are not always surjective, for instance, we will recall below that in Rng the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism but not surjective:

Proposition 1.1.1. *In the category* Rng *of rings, the inclusion* $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ *is an epimorphism but not surjective.*

Proof. The fact that the inclusion is not surjective is clear. Let $f, g : \mathbb{Q} \longrightarrow B$ be parallel morphisms in Rng such that $fi = gi$. Writing (b/a) for an arbitrary element of $\mathbb{Q}$, $i(a) = a/1$ for all $a \in \mathbb{Z}$, and so $f(a/1) = g(a/1)$. Therefore, $f(a/1)f(1/a) = f(1/1) = g(1/1) = g(a/1)g(1/a)$ for $a \in \mathbb{Z} - \{0\}$, and moreover, $f(b/a)f(1/1) = f(1/1)f(b/a) = f(b/a)$ and $g(b/a)g(1/1) = g(1/1)g(b/a) = g(b/a)$ for all $(b/a) \in \mathbb{Q}$. We also see that $f(1/a) = f(1/a)[g(a/1)g(1/a)] = [f(1/a)g(a/1)]g(1/a) = g(1/a)$. Hence, for $(b/a) \in \mathbb{Q}$, $f(b/a) = f(b/1)f(1/a) = g(b/a)$, and this means $f = g$. □

Definition 1.1.2. *Let $f : X \longrightarrow Y$ be a morphism in a category $\mathbb{C}$.*

(i) The morphism f is an extremal epimorphism if for each factorization $f = mg$ where m is a monomorphism, m is necessarily an isomorphism.

(ii) The morphism f is a strong epimorphism if for each commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \arrow[dl, dashed, "t"'] \\ C \arrow[r, tail, "m"'] & D \end{tikzcd}$$

with m a monomorphism, there exists a morphism $t : Y \longrightarrow C$ such that $g = tf$ and $h = mt$.

Lemma 1.1.3 (see e.g [4]). *In an arbitrary category $\mathbb{C}$, every strong epimorphism $f : X \longrightarrow Y$ is an extremal epimorphism. If $\mathbb{C}$ has pullbacks, a morphism $f : X \longrightarrow Y$ is a strong epimorphism if and only if it is an extremal epimorphism.*

Proof. Suppose $f : X \longrightarrow Y$ is a strong epimorphism, and $f = mg$, where m is a monomorphism. In the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "1"] \arrow[dl, dashed, "t"'] \\ C \arrow[r, tail, "m"'] & Y \end{tikzcd}$$

since f is a strong epimorphism and m is a monomorphism, there exists a morphism $t : Y \longrightarrow C$ such that $g = tf$ and $mt = 1$. We also have that $mtm = m$, but since m is a monomorphism, $tm = 1$ and thus m is an isomorphism. It remains to prove that in a category $\mathbb{C}$ with pullbacks, every extremal epimorphism is a strong epimorphism. Given that $f : X \longrightarrow Y$ is an extremal epimorphism in a category $\mathbb{C}$ with pullbacks, consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ C \arrow[r, tail, "m"'] & D \end{tikzcd}$$

where m is a monomorphism. In the diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\ & C \times_D Y \arrow[r, tail, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "h"] \\ & C \arrow[r, tail, "m"'] & D \end{tikzcd}$$

we see that f factors through the pullback π_2 of m along h. But since π_2 is a monomorphism and f is an extremal epimorphism, π_2 is an isomorphism. Writing π_2^{-1} for the inverse of π_2, it can be easily checked that the composite $\pi_1\pi_2^{-1} : Y \longrightarrow C$ is a morphism such that $(\pi_1\pi_2^{-1})f = g$ and $m(\pi_1\pi_2^{-1}) = h$. $\square$

Remark 1.1.4. *In the previous lemma it has been implicitly shown that if m is a monomorphism and t is a morphism such that mt is the identity morphism, then m is an isomorphism. Using this fact it can be easily shown that*

$$\textit{strong epimorphism} + \textit{monomorphism} = \textit{isomorphism}.$$

Nevertheless, the analogous "equation" for extremal epimorphism, from which the above follows since strong epimorphisms are extremal epimorphisms, holds for rather obvious reason.

Proposition 1.1.5. *In an arbitrary category $\mathbb{C}$, the composite of two strong epimorphisms is a strong epimorphism.*

Proof. Let $f : X \twoheadrightarrow Y$ and $g : Y \twoheadrightarrow Z$ be strong epimorphisms. For each commutative diagram

$$\begin{tikzcd} X \arrow[r, two heads, "f"] \arrow[d, "p"'] & Y \arrow[r, two heads, "g"] \arrow[dl, dashed, "u"] & Z \arrow[d, "s"] \arrow[dll, dashed, "v"'] \\ A \arrow[rr, tail, "m"'] & & B \end{tikzcd}$$

where m is a monomorphism, since f is a strong epimorphism, there exists a morphism u such that $uf = p$ and $mu = sg$. Similarly, since g is a strong epimorphism and $mu = sg$, there exists a morphism v such that $u = vg$ and $mv = s$, and thus $p = vgf$. Therefore gf is a strong epimorphism.

□

Definition 1.1.6. *In a category $\mathbb{C}$, a pair of morphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ is*

(a) jointly epimorphic if for every pair of parallel morphisms $u, v : Z \rightrightarrows D$ such that $uf = vf$ and $ug = vg$, one has $u = v$;

(b) jointly strongly epimorphic when for each commutative diagram

$$\begin{tikzcd}
& M \arrow[d, "m"] & \\
& Q & \\
X \arrow[r, "f"'] \arrow[uur, bend left, "f'"] & Z \arrow[u, "q"'] \arrow[uu, dashed, bend left, "\varphi"] & Y \arrow[l, "g"] \arrow[uul, bend right, "g'"']
\end{tikzcd}$$

if m is a monomorphism, then there exists a morphism $\varphi : Z \longrightarrow M$ such that $m\varphi = q$;

(c) jointly extremal-epimorphic when for each commutative diagram

$$\begin{tikzcd}
& M \arrow[d, "m"] & \\
X \arrow[ur, "f'"] \arrow[r, "f"'] & Z & Y \arrow[ul, "g'"'] \arrow[l, "g"]
\end{tikzcd}$$

if m is a monomorphism, then m is an isomorphism.

Remark 1.1.7. *In an arbitrary category $\mathbb{C}$, a morphism f is an (extremal) epimorphism if and only if the pair (f, f) is jointly (extremal)-epimorphic. Similarly, f is a strong epimorphism if and only if the pair (f, f) is jointly strongly epimorphic.*

The next lemma is the analogy of Lemma 1.1.3 in the case of jointly strongly epimorphic and jointly extremal-epimorphic pairs of morphisms, and nevertheless, Lemma 1.1.3 can be easily deduced as a consequence of the previous remark. The proof is essentially the same.

Lemma 1.1.8. *In an arbitrary category $\mathbb{C}$, if a pair of morphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ is jointly strongly epimorphic, then it is also jointly extremal-epimorphic. When $\mathbb{C}$ has pullbacks, a pair $f : X \longrightarrow Z$, $g : Y \longrightarrow Z$ is jointly strongly epimorphic if and only if it is jointly extremal-epimorphic.*

Proof. Suppose a pair of morphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ is jointly strongly epimorphic. If f and g factor through a monomorphism $m : M \longrightarrow Z$ as $f = mr$ and $g = ms$, then the diagram

$$\begin{tikzcd}
 & M \arrow[d, "m"] & \\
 & Z & \\
X \arrow[uur, bend left, "r"] \arrow[r, "f"'] & Z \arrow[u, "1"'] \arrow[uu, dashed, bend left, "\varphi"] & Y \arrow[l, "g"] \arrow[uul, bend right, "s"']
\end{tikzcd}$$

commutes. Since f and g are jointly strongly epimorphic, there exists a morphism $\varphi : Z \longrightarrow M$ such that $m\varphi = 1$. Thus m is an isomorphism, since $m\varphi = 1$ and m is a monomorphism. For the second part of the lemma we only need to prove the "if " part. Suppose a category $\mathbb{C}$ has pullbacks, and a pair of morphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ is jointly extremal-epimorphic. For every commutative diagram

$$\begin{tikzcd}
 & M \arrow[d, tail, "m"] & \\
 & Q & \\
X \arrow[uur, bend left, "r"] \arrow[r, "f"'] & Z \arrow[u, "\phi"'] & Y \arrow[l, "g"] \arrow[uul, bend right, "s"']
\end{tikzcd}$$

where m is a monomorphism, since in the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, dashed] \arrow[dd, "r"'] & & Z \arrow[dd, "\phi"] & & Y \arrow[ll, "g"'] \arrow[dl, dashed] \arrow[dd, "s"] \\
 & M \times_Q Z \arrow[ur, tail, "\pi_2"] \arrow[dl, "\pi_1"] & & M \times_Q Z \arrow[ul, tail, "\pi_2"'] \arrow[dr, "\pi_1"'] & \\
M \arrow[rr, tail, "m"'] & & Q & & M \arrow[ll, tail, "m"]
\end{tikzcd}$$

the pullback π_2 of m along ϕ is a monomorphism, and the pair f, g is jointly extremal-epimorphic, π_2 is an isomorphism. Writing π_2^{-1} for the inverse of π_2, it can be easily checked that the composite $\pi_1\pi_2^{-1} : Z \longrightarrow M$ is a morphism such that $m(\pi_1\pi_2^{-1}) = \phi$. $\square$

Proposition 1.1.9. *In a category $\mathbb{C}$ with equalizers, every jointly extremal-epimorphic pair of morphisms is jointly epimorphic.*

Proof. Let $f : A \longrightarrow B$ and $g : C \longrightarrow B$ be a jointly extremal-epimorphic pair of morphisms in a category $\mathbb{C}$ with equalizers. For a pair of parallel morphisms u and v such that $uf = vf$ and $ug = vg$, both f and g factor through the equalizer $m : E \rightarrowtail B$ of u and v

$$\begin{tikzcd} & E \arrow[d, tail, "m"] & \\ A \arrow[ur, dashed, bend left, "\alpha"] \arrow[r, "f"'] & B \arrow[d, shift right, "v"'] \arrow[d, shift left, "u"] & C \arrow[ul, dashed, bend right, "\beta"'] \arrow[l, "g"] \\ & D. & \end{tikzcd}$$

Since equalizers are monomorphisms, and the pair of morphisms f and g is jointly extremal-epimorphic, m is an isomorphism, and this implies that $u = v$.

$\square$

Definition 1.1.10. *In a category $\mathbb{C}$, a morphism $f : A \longrightarrow B$ is a regular epimorphism if it is the coequalizer of a pair of morphisms.*

Remark 1.1.11. *The following can be easily observed:*

- *In every category $\mathbb{C}$ regular epimorphisms are always epimorphisms, and this follows immediately from the universal property of coequalizers.*
- *In a category $\mathbb{C}$ with equalizers, extremal epimorphisms are epimorphisms.*

Proposition 1.1.12. *In an arbitrary category $\mathbb{C}$, every regular epimorphism is a strong epimorphism.*

Proof. Let $f : X \twoheadrightarrow Y$ be a regular epimorphism, and k_1, k_2 be a pair of parallel morphisms such that f is their coequalizer. For each commutative square

$$\begin{tikzcd} U \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & X \arrow[r, two heads, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \arrow[dl, dashed, "t"'] \\ & C \arrow[r, tail, "m"'] & D \end{tikzcd}$$

where m is a monomorphism, it is not difficult to see that $gk_1 = gk_2$. But since f is the coequalizer of k_1 and k_2, there exists a morphism $t : Y \longrightarrow C$ such that $g = tf$. Furthermore, we see that $mtf = hf$, and since f is a (regular) epimorphism, the lower triangle also commutes, i.e. $h = mt$. □

Next we recall the internalized notion of relations in a category $\mathbb{C}$ with finite limits.

Definition 1.1.13. *In a finitely complete category $\mathbb{C}$, a relation from X to Y is a span*

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ X & & Y \end{tikzcd}$$

such that r_1 and r_2 are jointly monomorphic, in other words, the factorization $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$ is a monomorphism. When $X = Y$, one says R is a relation on X. We shall denote a relation from X to Y by a triple (R, r_1, r_2).

Let us also recall that in a finitely complete category $\mathbb{C}$, a relation (R, r_1, r_2) on X is

(i) **reflexive** if there is a morphism $\triangle : X \longrightarrow R$ such that $r_1\triangle = 1 = r_2\triangle$;

(ii) **symmetric** if there is a morphism $\sigma : R \longrightarrow R$ such that $r_1\sigma = r_2$ and $r_2\sigma = r_1$;

(iii) **transitive** if for the pullback

$$\begin{tikzcd} R \times_X R \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_2"'] & X \end{tikzcd}$$

there is a morphism $\tau : R \times_X R \longrightarrow R$ such that $r_1\tau = r_1\pi_1$ and $r_2\tau = r_2\pi_2$.

A relation (R, r_1, r_2) on X is an equivalence relation if it is reflexive, symmetric, and transitive.

Remark 1.1.14. *In the category of sets, equivalence relations in the above sense (i.e. internal equivalence relations) are essentially the usual ones. While in the category of groups, an equivalence relation on a group G is essentially a usual equivalence relation on the underlying set of the group G, which is also a subgroup of $G \times G$, (i.e. a congruence on G).*

Definition 1.1.15. *Let $\mathbb{C}$ be a category with pullbacks. For every morphism $f : X \longrightarrow Y$ in $\mathbb{C}$, the kernel pair of f is the pullback of f with itself in the diagram*

$$\begin{tikzcd} \underset{<f,f>}{X \times X} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Remark 1.1.16. *In a category $\mathbb{C}$ with finite limits,*

(i) for every morphism $f : X \longrightarrow Y$, the kernel pair of f

$$\underset{<f,f>}{X \times X} \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X$$

determines an equivalence relation $\left(\underset{<f,f>}{X \times X}, \pi_1, \pi_2\right)$ *on X;*

(ii) every regular epimorphism is a coequalizer of its kernel pair.

Let us recall an even stronger notion of epimorphism.

Definition 1.1.17. *A morphism $f : X \longrightarrow Y$ in a category $\mathbb{C}$ is called a split epimorphism if there exists a morphism $s : Y \longrightarrow X$, called a section of f, such that $fs = 1$.*

In an arbitrary category $\mathbb{C}$, if $f : X \longrightarrow Y$ is a split epimorphism, with a section s, then for every pair of parallel morphisms u and v such that $uf = vf$, one has $u = ufs = vfs = v$, and this implies that split epimorphisms are epimorphisms. This also follows from the fact that f is a regular epimorphism; being the coequalizer of the identity morphism of X and the composite sf. Therefore in every category $\mathbb{C}$, we have

$$\textit{split epimorphism} \Rightarrow \textit{regular epimorphism} \Rightarrow \textit{epimorphism.}$$

Definition 1.1.18 (see [2])**.** *A category $\mathbb{C}$ is regular when*

(a) $\mathbb{C}$ has finite limits;

(b) every kernel pair has a coequalizer;

(c) regular epimorphisms are pullback stable along any morphism.

Every variety of universal algebras is a regular category, and here regular epimorphisms are exactly surjective homomorphisms. The category Top of topological spaces is not regular, as regular epimorphisms (open surjective continuous functions) are not necessarily pullback stable. However, the category $\mathsf{Grp}(\mathsf{Top})$ of topological groups is regular.

For a category $\mathbb{C}$ and an object I in $\mathbb{C}$, the slice category $(\mathbb{C} \downarrow I)$ is one whose objects are denoted by pairs (X, p) where X is an object in $\mathbb{C}$ and $p : X \longrightarrow I$ is a morphism in $\mathbb{C}$. A morphism $f : (X, p) \longrightarrow (Y, q)$ in $(\mathbb{C} \downarrow I)$ is a morphism $f : X \longrightarrow Y$ in $\mathbb{C}$ such that $qf = p$. The dual notion of slice category is coslice category, and is denoted by $(I \downarrow \mathbb{C})$ for an object I in $\mathbb{C}$. We shall observe that if $\mathbb{C}$ is a regular category then for every object I in $\mathbb{C}$ the slice (resp. coslice) category $(\mathbb{C} \downarrow I)$ (resp. $(I \downarrow \mathbb{C})$) is also regular.

Proposition 1.1.19 (see e.g [4]). *For a regular category $\mathbb{C}$ and any object I in $\mathbb{C}$, the category $(\mathbb{C} \downarrow I)$ (resp. $(I \downarrow \mathbb{C})$) is regular.*

Proof. Let us first show that the category $(\mathbb{C} \downarrow I)$ has finite limits. Equalizers in $(\mathbb{C} \downarrow I)$ are computed as in $\mathbb{C}$: For a pair of parallel morphisms $u, v : (X, p) \longrightarrow (Y, r)$ in $(\mathbb{C} \downarrow I)$, the equalizer of u and v in $(\mathbb{C} \downarrow I)$ is given by the morphism $w : (A, pw) \longrightarrow (X, p)$, where $w : A \longrightarrow X$ is the equalizer of a pair $u, v : X \longrightarrow Y$. For a pair of objects (X, p) and (Y, r) in $(\mathbb{C} \downarrow I)$, their product is given by the object $(X \times_I Y, p\pi_1)$, where $X \times_I Y$ is the pullback of p along r in $\mathbb{C}$, and π_1, π_2 are pullback projections, with $p\pi_1 = r\pi_2$. Thus $(\mathbb{C} \downarrow I)$ has finite limits. Pullbacks in $(\mathbb{C} \downarrow I)$ are computed as in $\mathbb{C}$, and this implies the same for kernel pairs in $(\mathbb{C} \downarrow I)$. Therefore, for a morphism $f : (X, p) \longrightarrow (Y, r)$ its kernel pair is given by the pair

$$\pi_1, \pi_2 : \left(X \underset{<f,f>}{\times} X, p\pi_1 \right) \rightrightarrows (X, p), \text{ with } \pi_1, \pi_2 : X \underset{<f,f>}{\times} X \rightrightarrows X$$

the kernel pair of $f : X \longrightarrow Y$ in $\mathbb{C}$. Let $q : X \longrightarrow Q$ be the coequalizer of π_1 and π_2 in $\mathbb{C}$. Then since $f\pi_1 = f\pi_2$, f factors as $f = \alpha q$ through q in $\mathbb{C}$, and immediately one obtains a morphism $q : (X, p) \longrightarrow (Q, r\alpha)$, which is necessarily the coequalizer of π_1 and π_2 in $(\mathbb{C} \downarrow I)$. Note that when $f : (X, p) \longrightarrow (Y, r)$ is a regular epimorphism, it is also the coequalizer of π_1 and π_2 in $(\mathbb{C} \downarrow I)$, and so $f \cong q$. This implies that for a regular category $\mathbb{C}$, a morphism $f : (X, p) \longrightarrow (Y, r)$ is a regular epimorphism in $(\mathbb{C} \downarrow I)$ if and only if $f : X \longrightarrow Y$ is a regular epimorphism in $\mathbb{C}$. Since pullbacks in $(\mathbb{C} \downarrow I)$ are computed as in $\mathbb{C}$, it follows that regular epimorphisms are pullback stable along any morphism in $(\mathbb{C} \downarrow I)$. □

Lemma 1.1.20 (see e.g [4]). *In a regular category $\mathbb{C}$, for a regular epimorphism $f : X \longrightarrow Y$ and any morphism $g : Y \longrightarrow Z$, the induced morphism*

$$f \times_Z f : X \times_Z X \longrightarrow Y \times_Z Y$$

is an epimorphism.

Proof. Using the following facts:

(a) for each commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "p"] \arrow[d, "v"'] \arrow[dr, phantom, "(1)"] & B_1 \arrow[r, "u"] \arrow[d, "s"'] \arrow[dr, phantom, "(2)"] & C_1 \arrow[d, "q"] \\
A \arrow[r, "k"'] & B \arrow[r, "r"'] & C
\end{tikzcd}$$

if square (2) and the outer rectangle (1) + (2) are pullbacks, then square (1) is also a pullback;

(b) regular epimorphisms are pullback stable, it is easily seen that all the squares in the diagram

$$\begin{tikzcd}
X \times_Z X \arrow[r, two heads, "j"] \arrow[d, two heads, "i"'] \arrow[dr, dotted, "f \times_Z f"] & Y \times_Z X \arrow[r, "h"] \arrow[d, two heads, "e"] & X \arrow[d, two heads, "f"] \\
X \times_Z Y \arrow[r, two heads, "d"] \arrow[d, "c"'] & Y \times_Z Y \arrow[r, "b"] \arrow[d, "a"] & Y \arrow[d, "g"] \\
X \arrow[r, two heads, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

are pullbacks, and the morphisms d, e, i, and j are all regular epimorphisms. But since regular epimorphisms are epimorphisms, and the composite of two epimorphisms is an epimorphism, the morphism $f \times_Z f = di = ej$ is an epimorphism.

□

Let us now recall the fact about the existence of "*(regular epi,mono)-factorizations*" in every regular category.

Theorem 1.1.21 (see e.g [4]). *In a regular category $\mathbb{C}$, every morphism factors as a regular epimorphism followed by a monomorphism. This factorization is unique up to isomorphism.*

Proof. For a morphism $f : X \longrightarrow Y$, let π_1, π_2 be the kernel pair of f, and e be the coequalizer of π_1 and π_2. Since $f\pi_1 = f\pi_2$, there is a morphism m such that $f = me$

$$\begin{tikzcd}
\underset{<f,f>}{X \times X} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "q = e \times_Y e"'] & X \arrow[r, "f"] \arrow[d, two heads, "e"] & Y \\
\underset{<m,m>}{I \times I} \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & I. \arrow[ur, "m"'] &
\end{tikzcd}$$

We shall prove that m is a monomorphism. For that it is enough to prove that, if r_1, r_2 is the kernel pair of m then $r_1 = r_2$. Since $me\pi_1 = me\pi_2$, there exists a morphism $q = e \times_Y e$ such that $r_1 q = e\pi_1 = e\pi_2 = r_2 q$. Applying the previous lemma, the morphism $q = e \times_Y e$ is a an epimorphism, and therefore $r_1 = r_2$. To prove the second part of the theorem, let us suppose that $f = m'e'$, where e' is a regular epimorphism and m' is a monomorphism. Since the diagram

$$\begin{tikzcd}
X \arrow[r, two heads, "e"] \arrow[d, two heads, "e'"'] & I \arrow[d, tail, "m"] \arrow[dl, dashed, "\alpha"'] \\
I' \arrow[r, tail, "m'"'] & Y
\end{tikzcd}$$

commutes and e is also a strong epimorphism (being a regular epimorphism), there exists a morphism α making the upper and lower triangles commute. Clearly, α is a monomorphism, and since e' is also an extremal epimorphism, it follows that α is an isomorphism. $\square$

Remark 1.1.22. *In a regular category $\mathbb{C}$, if f is an extremal epimorphism then it is a regular epimorphism, since from the (regular epi, mono)-factorization $f = me$, m is an isomorphism. More generally, in a regular category, strong epimorphisms and regular epimorphisms coincide. As a consequence of this and Proposition 1.1.5, we obtain that the composite of two regular epimorphisms is again a regular epimorphism.*

As a well-known fact, for each morphism $g : X \longrightarrow Y$ and any object A in a category $\mathbb{C}$ with products, the two commutative squares in the diagram

$$\begin{tikzcd}
A \times X \arrow[r, "1 \times g"] \arrow[d, "\pi_2"'] & A \times Y \arrow[d, "\pi_2"] \\
X \arrow[r, "g"'] & Y \\
X \times A \arrow[r, "g \times 1"'] \arrow[u, "\pi_1"] & Y \times A \arrow[u, "\pi_1"']
\end{tikzcd}$$

are pullbacks. Thus for a regular category $\mathbb{C}$, the morphisms $1 \times g$ and $g \times 1$ are regular epimorphisms if g is a regular epimorphism.

Proposition 1.1.23 (see e.g [4]). *In a regular category $\mathbb{C}$, the product of two regular epimorphisms is a regular epimorphism.*

Proof. For regular epimorphisms f and g, the morphism $f \times g$ can be expressed as the composite $(1 \times g)(f \times 1)$ of two strong epimorphisms, and so it is a strong epimorphism and hence, a regular epimorphism.

$\square$

Next we recall the interchange property of limits (see e.g [3]), and we will briefly illustrate how certain facts about limits and colimits are obtained from this property and its dual property respectively. Given two small categories $\mathbb{D}$ and $\mathbb{D}'$, and a category $\mathbb{C}$ which admits limits of $\mathbb{D}$ and $\mathbb{D}'$, for every functor $F : \mathbb{D}' \times \mathbb{D} \longrightarrow \mathbb{C}$ and the corresponding functors

$$F_{\mathbb{D}} : \mathbb{D}' \longrightarrow \mathbb{C}^{\mathbb{D}} \quad \text{and} \quad F_{\mathbb{D}'} : \mathbb{D} \longrightarrow \mathbb{C}^{\mathbb{D}'}$$

assigning to every object D' of $\mathbb{D}'$ and every object D of $\mathbb{D}$, the functors

$$F_{\mathbb{D}}(D', -) : \mathbb{D} \longrightarrow \mathbb{C} \quad \text{and} \quad F_{\mathbb{D}'}(-, D) : \mathbb{D}' \longrightarrow \mathbb{C},$$

respectively, where $F_{\mathbb{D}'}(D', D) = F(D', D)$ and $F_{\mathbb{D}}(D', D) = F(D', D)$, and morphisms are assigned accordingly, the following hold

$$\lim F \cong \lim_{\mathbb{D}} \left(\lim_{\mathbb{D}'} F_{\mathbb{D}} \right) \cong \lim_{\mathbb{D}'} \left(\lim_{\mathbb{D}} F_{\mathbb{D}'} \right).$$

Intuitively, the above property shows that limits "commute" with limits. The interchange property of limits and its dual property can be easily used to establish facts about limits and colimits respectively. For example, for morphisms $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ in a category $\mathbb{C}$ with finite limits, the interchange property of limits implies that

$$(X \times Y) \underset{<f\times g, f\times g>}{\times} (X \times Y) \cong \left(X \underset{<f,f>}{\times} X \right) \times \left(Y \underset{<g,g>}{\times} Y \right),$$

i.e. the kernel pair of the product $f \times g$ is the product of kernel pairs of f and g. The same is true for kernels, i.e. $\ker(f) \times \ker(g) = \ker(f \times g)$. Similarly, using the dual property in a category $\mathbb{C}$ with finite colimits, if $f = \mathsf{Coeq}(r_1, r_2)$ and $g = \mathsf{Coeq}(k_1, k_2)$ are coequalizers of pairs of morphisms r_1, r_2 and k_1, k_2 respectively, then

$$\mathsf{Coeq}(r_1 + k_1, r_2 + k_2) \cong \mathsf{Coeq}(r_1, r_2) + \mathsf{Coeq}(k_1, k_2) = f + g.$$

This means that in a category $\mathbb{C}$ with finite colimits, the sum of two regular epimorphisms is again a regular epimorphism. In particular, $\mathrm{coker}(f) + \mathrm{coker}(g) = \mathrm{coker}(f + g)$.

The last notion of epimorphism that we are going to recall in this section is that of *normal epimorphism*, and it only makes sense in pointed categories.

Definition 1.1.24. *A morphism $f : X \longrightarrow Y$ in a pointed category $\mathbb{C}$ is a normal epimorphism if it is the cokernel of some morphism. The dual notion of normal epimorphism is normal monomorphism; that is, kernel of some morphism.*

In a pointed category $\mathbb{C}$, normal epimorphisms are regular epimorphisms: If f is the cokernel of u, then f is the coequalizer of u and the zero morphism. Furthermore, in a pointed category $\mathbb{C}$ with kernels, normal epimorphisms are cokernels of their kernels: If f is the cokernel of u and $\ker(f)$ is the kernel of f, then u factors through $\ker(f)$ as $u = \ker(f)\alpha$. So for any morphism s such that $s\ker(f) = 0$, one has $su = s\ker(f)\alpha = 0$, but since f is the cokernel of u, the morphism s factors as $s = \beta f$. Therefore f is the cokernel of $\ker(f)$.

A pointed regular category where every regular epimorphism is a normal epimorphism is called a *normal category* [27]. According to [27], "a pointed variety of universal algebras is normal if and only if it is a variety with ideals in the sense of K. Fichtner [8], also known in universal algebra as a 0-regular variety".

We shall later make a remark on the two notions of normal subobjects; that is, normal subobjects defined as normal monomorphisms and Bourn-normal subobjects [8]. We will specify which notion of normal subobjects we will be working with in this thesis.

1.2 Unital categories

Definition 1.2.1 (see [7])**.** *A category $\mathbb{C}$ is unital when*

(a) $\mathbb{C}$ is pointed;

(b) $\mathbb{C}$ has finite limits;

(c) for each pair of objects X and Y in $\mathbb{C}$, the pair of morphisms $\langle 1,0\rangle : X \longrightarrow X \times Y$ and $\langle 0,1\rangle : Y \longrightarrow X \times Y$ is jointly extremal-epimorphic.

For algebraic varieties, it is a well-known fact (see e.g Theorem 1.2.15 [4]) that an algebraic variety $\mathcal{V}$ is unital if and only if it is Jonsson-Tarski, i.e. its theory contains a binary term p and a unique constant 0, satisfying $p(x,0) = x = p(0,x)$. A set X together with a binary term p and a constant 0, satisfying that $p(x,0) = x = p(0,x)$ is called a unitary magma. One writes UMag for the category whose objects are unitary magmas, and the morphisms are those which preserve the binary term p and the constant 0. The categories UMag, Mon, CoM, Gp, Ab, and Rg of unitary magmas, monoids, commutative monoids, groups, abelian groups, and rings respectively, are all unital. The dual category $\mathsf{Set}_*^{\mathsf{op}}$ of pointed sets is also unital.

Remark 1.2.2. *For objects X and Y in a unital category $\mathbb{C}$, the pair of morphisms $\langle 1,0\rangle :$ $X \longrightarrow X \times Y$ and $\langle 0,1\rangle : Y \longrightarrow X \times Y$ being jointly extremal-epimorphic amounts to the identity morphism of $X \times Y$ being a minimal element for those subobjects of $X \times Y$ through which both $\langle 1,0\rangle$ and $\langle 0,1\rangle$ factor. But as a general fact, in any poset where the meet of every two elements exists, if there is a minimal element it is necessarily the minimum element. So since $\mathbb{C}$ has finite limits, $\mathsf{Sub}(X \times Y)$ has meets defined via pullbacks, and therefore the identity morphism of $X \times Y$ is the join of $\langle 1,0\rangle$ and $\langle 0,1\rangle$; being the minimum element of those subobjects of $X \times Y$ which contain both $\langle 1,0\rangle$ and $\langle 0,1\rangle$.*

Note that for objects X and Y in a unital category $\mathbb{C}$, the pair of morphisms $\langle 1,0\rangle : X \longrightarrow X \times Y$ and $\langle 0,1\rangle : Y \longrightarrow X \times Y$ is also jointly epimorphic by Proposition 1.1.9.

Proposition 1.2.3. *For a pointed finitely complete category $\mathbb{C}$, the following are equivalent:*

(a) $\mathbb{C}$ is unital;

(b) for each commutative diagram

$$\begin{tikzcd}
 & R \arrow[d, tail, "\langle r_1, r_2\rangle"] & \\
A \arrow[ur, bend left, "u"] \arrow[r, "\langle f,0\rangle"'] & X \times Y & B, \arrow[l, "\langle 0,g\rangle"] \arrow[ul, bend right, "v"']
\end{tikzcd}$$

the morphism $f \times g : A \times B \longrightarrow X \times Y$ factors through $\langle r_1, r_2\rangle$.

Proof. $(a) \Rightarrow (b)$. In the diagram

$$
\begin{tikzcd}
 & R' \arrow[rr, "t"] \arrow[dd, tail, "\langle r_1', r_2' \rangle"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & R \arrow[dd, tail, "\langle r_1, r_2 \rangle"] & \\
 & & B \arrow[dl, "\langle 0,1 \rangle"'] \arrow[rr, "1"] \arrow[ul, dashed, "v'"'] & & B \arrow[dl, "\langle 0,g \rangle"] \arrow[ul, bend right, "v"'] \\
 & A \times B \arrow[rr, "f \times g"] & & X \times Y & \\
A \arrow[ur, "\langle 1,0 \rangle"] \arrow[rr, "1"] \arrow[uuur, dashed, bend left, "u'"] & & A \arrow[ur, "\langle f,0 \rangle"'] \arrow[uuur, bend left, "u"] & &
\end{tikzcd}
$$

since $(f \times g)\langle 1, 0\rangle = \langle r_1, r_2\rangle u$ and $(f \times g)\langle 0, 1\rangle = \langle r_1, r_2\rangle v$, one obtains factorizations u' and v' of $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$, respectively, through the pullback $\langle r_1', r_2'\rangle$. Therefore $\langle r_1', r_2'\rangle$ is an isomorphism, and clearly $f \times g$ factors through $\langle r_1, r_2\rangle$ by $t\langle r_1', r_2'\rangle^{-1}$, with $\langle r_1', r_2'\rangle^{-1}$ the inverse of $\langle r_1', r_2'\rangle$.

$(b) \Rightarrow (a)$. If $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$ is a relation such that $\langle 1, 0\rangle : X \longrightarrow X \times Y$ and $\langle 0, 1\rangle : Y \longrightarrow X \times Y$ factor through it, then (b) implies that $\langle r_1, r_2\rangle$ is a split epimorphism, and hence an isomorphism.

$\square$

For objects X and Y in a pointed category $\mathbb{C}$ with finite products and coproducts, we write

$$X + Y \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} X \times Y \tag{1.1}$$

to denote the unique morphism $[\langle 1, 0\rangle, \langle 1, 0\rangle] : X+Y \longrightarrow X\times Y$ induced by $\langle 1, 0\rangle : X \longrightarrow X\times Y$ and $\langle 0, 1\rangle : Y \longrightarrow X\times Y$, which, by the universal properties of coproducts and products, is equal to the unique morphism $\langle [1, 0], [0, 1]\rangle : X + Y \longrightarrow X \times Y$ induced by $[1, 0] : X + Y \longrightarrow X$ and $[0, 1] : X + Y \longrightarrow Y$.

For objects X and Y in a regular unital category $\mathbb{C}$ with coproducts, the morphism (1.1) above is a regular epimorphism: As seen in the commutative diagram

$$
\begin{tikzcd}
 & X + Y \arrow[d, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] \arrow[dr, two heads, "e"] & \\
 & & X \star Y \arrow[dl, tail, "m"] \\
X \arrow[uur, bend left, "i_1"] \arrow[r, "\langle 1,0 \rangle"'] & X \times Y & Y, \arrow[l, "\langle 0,1 \rangle"] \arrow[uul, bend right, "i_2"']
\end{tikzcd}
$$

with me the (regular epi, mono)-factorization of the morphism (1.1), m is an isomorphism since the pair $\langle 1, 0\rangle, \langle 0, 1\rangle$ is jointly-extremal epimorphic. More generally, for every commutative diagram

$$\begin{tikzcd}
 & T \arrow[d, dotted] & \\
X \arrow[ur, bend left] \arrow[r, "u"'] & Z & Y \arrow[l, "v"] \arrow[ul, bend right]
\end{tikzcd}$$

in a regular category $\mathbb{C}$, if u and v are jointly extremal-epimorphic, then the dotted arrow is a regular epimorphism.

The kernel of the morphism (1.1) (when it exists) is denoted by a morphism $\kappa_{X,Y} : X \diamond Y \rightarrowtail X + Y$, where $X \diamond Y$ is the co-smash product of X and Y.

1.3 Subtractive categories

A pointed subtractive variety (in the sense of Ursini [39]) is one whose theory contains a binary term s and a unique constant 0, satisfying $s(x,x) = 0$ and $s(x,0) = x$. The notion of a subtractive category was subsequently introduced in [25], as a categorical generalisation of a pointed subtractive variety.

Definition 1.3.1 (see [25]). *A subtractive category is a pointed finitely complete category $\mathbb{C}$ such that every left punctual reflexive relation in $\mathbb{C}$ is right punctual, i.e. for every relation $\langle r_1, r_2 \rangle : R \rightarrowtail X \times X$, if $\langle 1, 1 \rangle : X \longrightarrow X \times X$ and $\langle 1, 0 \rangle : X \longrightarrow X \times X$ factor through $\langle r_1, r_2 \rangle$, then $\langle 0, 1 \rangle : X \longrightarrow X \times X$ factors through $\langle r_1, r_2 \rangle$ as well.*

According to [25], a variety of universal algebras is a subtractive category if and only if it is a pointed subtractive variety. The categories Gp, Ab, and Rng, of groups, abelian groups, and rings respectively, are subtractive. In addition, the category of (nonempty) implication algebras [1] (defined in Chapter 5 below) and the category of loops are subtractive categories.

The next proposition is the analogy of Proposition 1.2.3 in a subtractive category.

Proposition 1.3.2 (see [12]). *Let $\mathbb{C}$ be a pointed category with finite limits. The following are equivalent:*

(a) $\mathbb{C}$ is subtractive;

(b) for any subobject $\langle r_1, r_2 \rangle : R \rightarrowtail X \times Y$ and morphisms $f : A \longrightarrow X$ and $g : A \longrightarrow Y$, if $\langle f, g \rangle$ and $\langle f, 0 \rangle$ factor through $\langle r_1, r_2 \rangle$, then $\langle 0, g \rangle$ factors through $\langle r_1, r_2 \rangle$ as well.

Proof. The implication $(b) \Rightarrow (a)$ is obvious, so we will only prove the implication $(a) \Rightarrow (b)$: Given that $\langle f, g \rangle$ and $\langle f, 0 \rangle$ factor through $\langle r_1, r_2 \rangle$ in the diagram

$$\begin{tikzcd}
& & R' \arrow[rr, "t"] \arrow[dd, tail, "{\langle r_1', r_2' \rangle}"'] \arrow[ddrr, phantom, "\lrcorner", very near start] & & R \arrow[dd, tail, "{\langle r_1, r_2 \rangle}"] & \\
& & & A \arrow[dl, "{\langle 1,0 \rangle}"'] \arrow[rr, "1"] \arrow[ul, dashed, bend right, "v'"'] & & A, \arrow[dl, "{\langle f,0 \rangle}"] \arrow[ul, bend right, "v"'] \\
A \arrow[rr, "{\langle 0,1 \rangle}"] \arrow[uurr, dashed, bend left, "w"] & & A \times A \arrow[rr, "f \times g"] & & X \times Y & \\
& A \arrow[ur, "{\langle 1,1 \rangle}"] \arrow[rr, "1"] \arrow[uuur, dashed, bend left, "u'"] & & A \arrow[ur, "{\langle f,g \rangle}"'] \arrow[uuur, bend left, "u"] & &
\end{tikzcd}$$

we see that the morphisms $\langle 1,1\rangle$ and $\langle 1,0\rangle$ factor through the pullback $\langle r_1', r_2'\rangle$. Therefore, by subtractivity the morphism $\langle 0,1\rangle$ factors through $\langle r_1', r_2'\rangle$, which implies that $\langle 0,g\rangle = (f\times g)\langle 0,1\rangle$ factors through $\langle r_1, r_2\rangle$. $\square$

Recall that in a pointed category $\mathbb{C}$, a diagram

$$0 \longrightarrow K \overset{k}{\rightarrowtail} A \overset{g}{\twoheadrightarrow} B \longrightarrow 0$$

is a *short exact sequence* if k is the kernel of g, and g is the cokernel of k.

The following fact is usually called the upper 3×3 lemma.

Proposition 1.3.3 (see [27]). *Let $\mathbb{C}$ be a normal subtractive category. In the diagram*

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "u_1"'] \arrow[d, "f_1"'] & B_1 \arrow[r, "v_1"'] \arrow[d, "g_1"'] & C_1 \arrow[r] \arrow[d, "h_1"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "u_2"'] \arrow[d, "f_2"'] & B_2 \arrow[r, "v_2"'] \arrow[d, "g_2"'] & C_2 \arrow[r] \arrow[d, "h_2"] & 0 \\
0 \arrow[r] & A_3 \arrow[r, "u_3"'] \arrow[d] & B_3 \arrow[r, "v_3"'] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

where all the columns are short exact sequences, if the second and third rows are short exact sequences, then the first row is also a short exact sequence.

Proof. See [27], Lemma 5.1. $\square$

Many of categories from classical algebras are both unital and subtractive. For example, the categories $\mathsf{Gp}, \mathsf{Rng}$, and Ab are both unital and subtractive. Not all unital categories are subtractive and vice-versa: The categories UMag and Mon of unitary magmas and monoids respectively, are unital but not subtractive, and the category of (nonempty) implication algebras is subtractive but not unital. Nevertheless, there is an interesting characterization of pointed finitely complete categories which are both unital and subtractive, which we will recall later.

Definition 1.3.4 (see [7]). *A pointed category $\mathbb{C}$ with finite limits is strongly unital if every left punctual reflexive relation is indiscrete, in other words, for each object X, the pair of morphisms $\langle 1, 1\rangle : X \longrightarrow X \times X$ and $\langle 1, 0\rangle : X \longrightarrow X \times X$ is jointly extremal-epimorphic.*

Recall

Proposition 1.3.5. *For a pointed finitely complete category $\mathbb{C}$, the following statements are equivalent:*

(a) *$\mathbb{C}$ is strongly unital;*

(b) *for any subobject $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$ and morphisms $f : A \longrightarrow X$ and $g : A \longrightarrow Y$, if $\langle f, g\rangle$ and $\langle f, 0\rangle$ factor through $\langle r_1, r_2\rangle$, then the morphism $f \times g$ factors through $\langle r_1, r_2\rangle$ as well.*

Proof. The implication $(b) \Rightarrow (a)$ is obvious. To prove the implication $(a) \Rightarrow (b)$, let $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$ be a relation such that $\langle f, g\rangle : A \longrightarrow X \times Y$ and $\langle f, 0\rangle : A \longrightarrow X \times Y$ factor through it. In the commutative diagram

$$\begin{tikzcd}
& R' \arrow[rrr] \arrow[dd, tail, "{\langle r_1', r_2'\rangle}"] \arrow[dr, phantom, "\lrcorner", very near start] & & & R \arrow[dd, tail, "{\langle r_1, r_2\rangle}"] & \\
& & & A \arrow[ull, dashed, bend right] \arrow[dll, "{\langle 1,0\rangle}"'] \arrow[rr, "1"] & & A \arrow[ul, bend right] \arrow[dl, "{\langle f,0\rangle}"] \\
& A \times A \arrow[rrr, "f \times g"] & & & X \times Y & \\
A \arrow[ur, "{\langle 1,1\rangle}"] \arrow[rr, "1"] \arrow[uuur, dashed, bend left] & & A \arrow[urr, "{\langle f,g\rangle}"'] \arrow[uuurr, bend left] & & &
\end{tikzcd}$$

the morphisms $\langle 1, 1\rangle$ and $\langle 1, 0\rangle$ factor through the pullback $\langle r_1', r_2'\rangle$, which is then an isomorphism since $\mathbb{C}$ is strongly unital. Now it immediately follows that $f \times g$ factors through $\langle r_1, r_2\rangle$.

□

Here is the characterization of pointed finitely complete categories which are both unital and subtractive:

Proposition 1.3.6 (see [25]). *A pointed category $\mathbb{C}$ with finite limits is strongly unital if and only if it is unital and subtractive.*

Proof. The "if" part is straightforward. As for the "only if " part, let us first show that $\mathbb{C}$ is unital, by proving that for each commutative diagram

$$\begin{tikzcd}
 & R \arrow[d, tail, "\langle r_1, r_2\rangle"] & \\
X \arrow[ur, bend left, "u"] \arrow[r, "\langle 1,0\rangle"'] & X \times Y & Y \arrow[ul, bend right, "v"'] \arrow[l, "\langle 0,1\rangle"]
\end{tikzcd}$$

$\langle r_1, r_2\rangle$ is an isomorphism. Using the previous diagram, we obtain the following commutative diagram

$$\begin{tikzcd}
 & R \arrow[d, tail, "\langle r_1, r_2\rangle"] & \\
R \arrow[ur, bend left, "1"] \arrow[r, "\langle r_1,r_2\rangle"'] & X \times Y & R, \arrow[ul, bend right, "ur_1"'] \arrow[l, "\langle r_1,0\rangle"]
\end{tikzcd}$$

which, after applying the previous proposition to it, implies the morphism $r_1 \times r_2$ factors through $\langle r_1, r_2\rangle$. Therefore, the morphism $(r_1 \times r_2)(u \times v)$, which is the identity morphism of $X \times Y$, also factors through $\langle r_1, r_2\rangle$. Hence $\langle r_1, r_2\rangle$ is an isomorphism. The second part of the "only if" is rather straightforward: If for a relation $\langle r_1, r_2\rangle : R \rightarrowtail X \times X$, the morphisms $\langle 1, 1\rangle : X \longrightarrow X \times X$ and $\langle 1, 0\rangle : X \longrightarrow X \times X$ factor through it, then strongly unital forces $\langle r_1, r_2\rangle$ to be an isomorphism, through which $\langle 0, 1\rangle$ factors.

□

The following theorem is a slight modification of Lemma 1.8.18 in [4], and it will be useful in Chapter 3.

Theorem 1.3.7. *Let $\mathbb{C}$ be a strongly unital category. Consider the following commutative diagram*

$$\begin{tikzcd}
W \arrow[r, "\langle 1,w\rangle"] \arrow[dr, "h"'] \arrow[ddr, bend right, "1"'] & W \times A \arrow[d, "\varphi"] \arrow[dd, dotted, bend right=40, "\pi_1"'] & A \arrow[l, "\langle 0,1\rangle"'] \arrow[dl, "f"] \arrow[ddl, bend left, "0"] \\
 & Q \arrow[d, "g"] & \\
 & W &
\end{tikzcd} \tag{1.2}$$

with $g\varphi = \pi_1$. If (R, r_1, r_2) is the kernel pair relation of f, then $(W \times R, 1 \times r_1, 1 \times r_2)$ is the kernel pair relation of φ.

Proof. Let (K, k, k') be the kernel pair relation of φ. Writing $\langle k, k' \rangle = \langle \langle k_1, k_2 \rangle, \langle k_1', k_2' \rangle \rangle : K \rightarrowtail (W \times A) \times (W \times A)$, since $g\varphi = \pi_1$, we see that $k_1 = \pi_1 \langle k_1, k_2 \rangle = g\varphi \langle k_1, k_2 \rangle = g\varphi \langle k_1', k_2' \rangle = \pi_1 \langle k_1', k_2' \rangle = k_1'$. Now we can see in the diagram below that $\langle k_1, k_2 \rangle$ and $\langle k_1', k_2' \rangle$ factor by $\langle k_1, \langle k_2, k_2' \rangle \rangle$ through $1 \times \pi_1$ and $1 \times \pi_2$ respectively

$$\begin{tikzcd}
 & W \times (A \times A) \arrow[dr, shift left, "1 \times \pi_1"] \arrow[dr, shift right, "1 \times \pi_2"'] & \\
K \arrow[ur, tail, "\langle k_1, \langle k_2, k_2' \rangle \rangle"] \arrow[rr, shift left, "\langle k_1, k_2 \rangle"] \arrow[rr, shift right, "\langle k_1', k_2' \rangle"'] & & W \times A.
\end{tikzcd}$$

Since $fr_1 = \varphi\langle 0, 1 \rangle r_1 = \varphi\langle 0, 1 \rangle r_2 = fr_2$ and $\varphi\langle 1, 0 \rangle = \varphi\langle 1, 0 \rangle$, the morphisms $\langle \langle 0, r_1 \rangle, \langle 0, r_2 \rangle \rangle$ and $\langle \langle 1, 0 \rangle, \langle 1, 0 \rangle \rangle$ factor through $\langle \langle k_1, k_2 \rangle, \langle k_1', k_2' \rangle \rangle$, which (respectively) mean that the morphisms $\langle 0, \langle r_1, r_2 \rangle \rangle$ and $\langle 1, \langle 0, 0 \rangle \rangle$ factor through $\langle k_1, \langle k_2, k_2' \rangle \rangle$

$$\begin{tikzcd}
 & & K \arrow[d, tail, "\langle k_1, \langle k_2, k_2' \rangle \rangle"] & \\
 & W \times R \arrow[ur, dashed, "\lambda"] \arrow[dr, tail, "1 \times \langle r_1, r_2 \rangle"'] & & \\
W \arrow[uurr, dashed, bend left, "\alpha"] \arrow[rr, tail, "\langle 1, \langle 0, 0 \rangle \rangle"'] & & W \times (A \times A) & R. \arrow[l, tail, "\langle 0, \langle r_1, r_2 \rangle \rangle"] \arrow[uul, dashed, bend right, "\beta"']
\end{tikzcd}$$

Applying Proposition 1.2.3 to the previous diagram, the morphism $1 \times \langle r_1, r_2 \rangle$ factors through $\langle k_1, \langle k_2, k_2' \rangle \rangle$. It remains to show that $\langle k_1, \langle k_2, k_2' \rangle \rangle$ factors through $1 \times \langle r_1, r_2 \rangle$. But since $\varphi\langle k_1, k_2 \rangle = \varphi\langle k_1', k_2' \rangle$ and $\varphi\langle k_1, 0 \rangle = \varphi\langle k_1', 0 \rangle$ (since $k_1 = k_1'$), it follows by subtractivity that $\varphi\langle 0, k_2 \rangle = \varphi\langle 0, k_2' \rangle$. This means $fk_2 = \varphi\langle 0, 1 \rangle k_2 = \varphi\langle 0, 1 \rangle k_2' = fk_2'$, which implies that $\langle k_2, k_2' \rangle$ factors through $\langle r_1, r_2 \rangle$ via a morphism τ. Thus the diagram

$$\begin{tikzcd}
W \times R \arrow[r, tail, "1 \times \langle r_1, r_2 \rangle"] & W \times (A \times A) \\
K \arrow[u, "\langle k_1, \tau \rangle"] \arrow[ur, tail, "\langle k_1, \langle k_2, k_2' \rangle \rangle"'] &
\end{tikzcd}$$

commutes, and hence the result follows. ∎

Remark 1.3.8. *In a strongly unital category $\mathbb{C}$, for each commutative diagram*

$$
\begin{tikzcd}
A \arrow[r, "{\langle 1,1\rangle}"] \arrow[dr, "h"'] \arrow[ddr, bend right, "1"'] & A\times A \arrow[d, "\varphi"] & A \arrow[l, "{\langle 0,1\rangle}"'] \arrow[dl, "f"] \arrow[ddl, bend left, "0"] \\
& Q \arrow[d, "g"] & \\
& A &
\end{tikzcd}
\tag{1.3}
$$

since the morphisms $\langle 1,1\rangle$ and $\langle 0,1\rangle$ are jointly epimorphic, $g\varphi = \pi_1$. Using the previous theorem, if (R, r_1, r_2) is the kernel pair relation of f, then $(A\times R, 1\times r_1, 1\times r_2)$ is the kernel pair relation of φ.

As a consequence of Theorem 1.3.7, we can make the following useful observation:

Corollary 1.3.9. *Let $\mathbb{C}$ be a regular strongly unital category. In diagram (1.2) of Theorem 1.3.7, if φ is a regular epimorphism, then it is of the form $1\times\tilde{q} : W\times A \longrightarrow W\times\tilde{Q}$, where $\tilde{q} : A \longrightarrow \tilde{Q}$ is the regular epimorphism in the (regular epi, mono)-factorization of $\varphi\langle 0,1\rangle$.*

Proof. From Theorem 1.3.7, the kernel pair of φ is given by the pair $1\times r_1, 1\times r_2$, where r_1, r_2 is the kernel pair of $\varphi\langle 0,1\rangle$. Let $\tilde{q} : A \longrightarrow \tilde{Q}$ be the regular epimorphism in the (regular epi, mono)- factorization of $\varphi\langle 0,1\rangle$. Clearly, $\tilde{q}$ is the coequalizer of the pair r_1, r_2, and therefore the morphism $1\times\tilde{q}$ is the coequalizer of its kernel pair $1\times r_1, 1\times r_2$. Hence $1\times\tilde{q}$ is isomorphic to φ since they are both coequalizers of $1\times r_1$ and $1\times r_2$.

□

In the next remark, we state an equivalent formulation of Corollary 1.3.9 in a category of points over a fixed object. Before doing so, let us recall what a category of points over a fixed object is. For each object A in a category $\mathbb{C}$, we write $\mathsf{Pt}(A)$ to denote the category of points over A, whose objects are triples (X, r, s), with X an object in $\mathbb{C}$ and $r : X \longrightarrow A, s : A \longrightarrow X$ are morphisms such that $rs = 1$. A morphism $f : (X, r, s) : X \longrightarrow (Y, q, t)$ in $\mathsf{Pt}(A)$ is a morphism $f : X \longrightarrow Y$ in $\mathbb{C}$, such that $qf = r$ and $fs = t$. Note that this category is pointed; the zero object is given by $(A, 1, 1)$ and the zero morphism between any two objects (X, r, s) and (Y, q, t) is the composite $tr : (X, r, s) \longrightarrow (Y, q, t)$.

Remark 1.3.10. *Let $\mathbb{C}$ be a regular strongly unital category. Corollary 1.3.9 is equivalent to the following statement: For any object W in $\mathbb{C}$, every regular epimorphism φ in $\mathsf{Pt}(W)$ whose domain is $(W\times A, \pi_1, \langle 1, w\rangle)$ can be chosen to be of the form $1\times\tilde{q}$, with $\tilde{q}$ a regular epimorphism in $\mathbb{C}$*

$$
\begin{tikzcd}
W \times A \arrow[r, two heads, "\varphi"] \arrow[d, shift left, "\pi_1"] & Q \arrow[d, shift left, "g"] \\
W \arrow[r, "1"'] \arrow[u, shift left, "\langle 1, w\rangle"] & W. \arrow[u, shift left, "h"]
\end{tikzcd}
$$

The following fact will be useful.

Proposition 1.3.11. *Let $\mathbb{C}$ be a regular category, and $\tilde{q} : A \longrightarrow \tilde{Q}$ be a regular epimorphism in $\mathbb{C}$. If k_1, k_2 is the kernel pair of $\tilde{q}$, then the factorization $\langle k_1, k_2\rangle$ is the kernel of the morphism*

$$1 \times \tilde{q} : (A \times A, \pi_1, \langle 1, 1\rangle) \longrightarrow (A \times \tilde{Q}, \pi_1, \langle 1, \tilde{q}\rangle)$$

in $\mathsf{Pt}(A)$

$$
\begin{tikzcd}
A \underset{<\tilde{q},\tilde{q}>}{\times} A \arrow[r, tail, "\langle k_1, k_2\rangle"] \arrow[d, shift left, "k_1"] & A \times A \arrow[r, two heads, "1 \times \tilde{q}"] \arrow[d, shift left, "\pi_1"] & A \times \tilde{Q} \arrow[d, shift left, "\pi_1"] \\
A \arrow[r, "1"'] \arrow[u, shift left, "\triangle"] & A \arrow[r, "1"'] \arrow[u, shift left, "\langle 1, 1\rangle"] & A. \arrow[u, shift left, "\langle 1, \tilde{q}\rangle"]
\end{tikzcd}
$$

Proof. In the diagram

$$
\begin{tikzcd}
A \underset{<\tilde{q},\tilde{q}>}{\times} A \arrow[r, "k_1"] \arrow[d, "\langle k_1, 1\rangle"] \arrow[dd, bend right=60, "\langle k_1, k_2\rangle"'] \arrow[dr, phantom, "(1)"] & A \arrow[d, "\langle 1, 1\rangle"'] \arrow[dd, bend left=60, "\langle 1, \tilde{q}\rangle"] \\
A \times (A \underset{<\tilde{q},\tilde{q}>}{\times} A) \arrow[r, two heads, "1 \times k_1"] \arrow[d, two heads, "1 \times k_2"] \arrow[dr, phantom, "(2)"] & A \times A \arrow[d, two heads, "1 \times \tilde{q}"'] \\
A \times A \arrow[r, two heads, "1 \times \tilde{q}"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "(3)"] & A \times \tilde{Q} \arrow[d, "\pi_2"] \\
A \arrow[r, two heads, "\tilde{q}"'] & \tilde{Q}
\end{tikzcd}
$$

since diagram (3) and the outer diagram (1)+(2)+(3) are pullbacks, the outer diagram (1)+(2) is also a pullback. Recalling the computation of kernels in $\mathsf{Pt}(A)$, diagram (1) + (2) being a pullback implies $\langle k_1, k_2\rangle$ is the kernel of the morphism $1 \times \tilde{q}$ in $\mathsf{Pt}(A)$.

□

1.4 Mal'cev and Barr-exact categories

Definition 1.4.1 (see [15]). *A category $\mathbb{C}$ with finite limits is a Mal'cev category when every reflexive relation in $\mathbb{C}$ is an equivalence relation.*

For varieties, the condition asking that every reflexive relation is an equivalence relation corresponds to the existence of a ternary term p, satisfying that $p(x, y, y) = x$ and $p(x, x, y) = y$. A variety of universal algebras whose theory contains such a ternary term p is called a Mal'cev variety [36], and p is called a Mal'cev term. The category Gp of groups is Mal'cev, with a Mal'cev term p defined by $p(x, y, z) = x - y + z$. More generally, every variety containing a group operation, i.e. a variety of $\Omega-$groups, is Mal'cev. Furthermore, the categories Heyt and LCMag of Heyting algebras and left closed magmas respectively, are Mal'cev (see e.g [4]). A magma is left closed [4] when it is furnished with a second binary operation written $x \setminus y$, satisfying that $x + (x \setminus y) = y$ and $x \setminus (x + y) = y$. There are Mal'cev categories which are non-varietal; the dual category of elementary topos is such example.

Mal'cev, strongly unital, and unital categories are related in the following implications:

$$\textit{Mal'cev} \Rightarrow \textit{strongly unital} \Rightarrow \textit{unital.}$$

At varietal level, strongly unital corresponds to the existence of a ternary term p satisfying $p(x, 0, 0) = x$ and $p(x, x, y) = y$, while for the unital case p is only required to satisfy $p(x, 0, 0) = x = p(0, 0, x)$. These are clearly weaker versions of a Mal'cev terms. Categorically, the first implication (Mal'cev $\Rightarrow$ strongly unital) follows immediately from the fact that every right punctual reflexive relation is an equivalence relation, and so symmetry forces any such relation to be also left punctual, and through transitivity it can be shown that it is indiscrete.

Let us also mention that (strongly) unital and subtractive categories can be used to characterize Mal'cev categories in terms of categories of points over fixed objects. The characterization in terms of (strongly) unital categories appears in [7], and it asserts that, a category $\mathbb{C}$ with finite limits is a Mal'cev category if and only if, for each object A, $\mathsf{Pt}(A)$ is (strongly) unital. An analogy of this characterization in terms of subtractive categories is given in [12]: A category with finite limits is a Mal'cev category if and only if for every object X, the category of points over X is subtractive.

Recall that (see e.g Proposition 2.2.11 [4]) for a pullback preserving functor $U : \mathbb{C} \longrightarrow \mathbb{C}'$ between categories with finite limits, if U reflects isomorphisms and $\mathbb{C}'$ is a Mal'cev category, then $\mathbb{C}$ is a Mal'cev category as well. Let us apply this result to observe the following: For each object X in a Mal'cev category $\mathbb{C}$, the functor

$$d_0 : \mathsf{Pt}(A) \longrightarrow \mathbb{C},$$

which sends each morphism $f : (X, r, s) \longrightarrow (Y, q, p)$ (resp. object (X, r, s)) to $f : X \longrightarrow Y$ (resp. X), is a pullback preserving functor which reflects isomorphisms, and therefore the category $\mathsf{Pt}(A)$ is Mal'cev.

In [14] it has been shown that a regular category is Mal'cev if and only if composition of equivalence relations is commutative, i.e. $R \circ S = S \circ R$ for every pair of equivalence relations R and S on a given object. Let us also observe that for each object X in a normal category $\mathbb{C}$, the category $\mathsf{Pt}(X)$ is normal. Recall that $\mathsf{Pt}(X) \cong ((X, 1) \downarrow (\mathbb{C} \downarrow X))$, and since regular categories are stable under slice and co-slice categories, it follows that $\mathsf{Pt}(X)$ is a regular category whenever $\mathbb{C}$ is regular. It has been shown already that $\mathsf{Pt}(X)$ is pointed, so it remains only to show that every regular epimorphism in $\mathsf{Pt}(X)$ is a normal epimorphism: For every regular epimorphism $f : (A, r, s) \longrightarrow (B, q, p)$ in $\mathsf{Pt}(X)$, $f : A \longrightarrow B$ is a normal epimorphism in $\mathbb{C}$. Since pushouts in $\mathsf{Pt}(X)$ are computed as in $\mathbb{C}$, to show that f is a normal epimorphism in $\mathsf{Pt}(X)$ it is enough to show that the bottom square in the diagram

$$\begin{tikzcd}
K' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
K \arrow[r, two heads, "\tau"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
A \arrow[r, two heads, "f"'] & B
\end{tikzcd}$$

is a pushout. But this follows immediately since the outer diagram is a pushout and τ is a (normal) epimorphism (or the fact that the first square is also a pushout).

Definition 1.4.2 (see [2]). *A regular category $\mathbb{C}$ is called Barr-exact when every equivalence relation in $\mathbb{C}$ is a kernel pair relation.*

In a general categorical context, equivalence relations which are kernel pair relations of some morphisms are called *effective equivalence relations*. A simple example of a Barr-exact category is the category Set of sets. In Set if R is an equivalence relation on X, then for the quotient $q : X \longrightarrow X/R$, $q(x) = q(x')$ if and only if $(x, x') \in R$, and this means R is the kernel pair relation of q. Some other examples of Barr-exact categories include each variety of universal algebras and the dual of each elementary topos.

In normal Barr-exact categories the following important fact holds: For every object X there is a bijection between the class of normal monomorphisms whose codomain is X and the class of equivalence relations on X.

1.5 Ideal-determined categories

Let us recall the following classical facts: (a) Every congruence R on a group G is determined by the equivalence class of the unit since

$$(x, y) \in R \Leftrightarrow x^{-1}y \in [1]_R,$$

and (b) every normal subgroup of G is the unit class of a unique congruence on G. To explain (b) we will show that for any normal subgroup H of G, the relation R defined by

$$(x, y) \in R \Leftrightarrow x^{-1}y \in H$$

is a congruence on G whose unit class is H. Clearly, R is reflexive and thus, it is an equivalence relation on G. The normality of H will be used to show that R is a congruence. By definition, $(x, y), (a, b) \in R$ if and only if $x^{-1}y \in H$ and $a^{-1}b \in H$. But since H is a normal subgroup we also have $a^{-1}x^{-1}ya \in H$, which together with $a^{-1}b \in H$, implies $(xa)^{-1}yb = a^{-1}x^{-1}yb \in H$, i.e. $(xa, yb) \in R$. Let us show that R is closed under inversion: For $(x, y) \in R$, $x^{-1}y \in H$ implies $y^{-1}x \in H$. Using the fact that H is a normal subgroup, $xy^{-1}xx^{-1} = xy^{-1} \in H$ and hence, $(x^{-1}, y^{-1}) \in R$. Thus R is a congruence on G. Lastly, $(x, 1) \in R$ if and only if $x^{-1} \in H$, and this shows that H is precisely the unit class of R. The uniqueness of R is straightforward. These classical properties have been investigated in universal algebras, replacing normal subgroups with ideals (i.e. "zero" classes of congruences), and for that the notion of ideal-determined varieties (in the sense of [19]) was introduced; that is, those pointed varieties whose congruences are completely determined by their ideals. Ideal-determined varieties were also called BIT varieties in [37]. A categorical counterpart of an ideal-determined variety, called *ideal-determined category*, has been introduced in [24]. In modern terms an ideal-determined category can be defined as follows:

Definition 1.5.1 (see [24])**.** *A normal category $\mathbb{C}$ with finite colimits is called ideal-determined if normal monomorphisms are preserved by regular images along regular epimorphisms.*

Again going back to groups, a normal subgroup can be described in two equivalent ways: (a) as a kernel of a group homomorphism; (b) as a unit class of a congruence. However, for an arbitrary pointed category the two descriptions are not equivalent. Thus, there are two notions of normal subobjects, namely, normal subobjects defined to be kernels, and *Bourn-normal subobjects* [8], which in pointed categories are "zero" classes of equivalence relations. These two notions of normal subobjects coincide under certain assumptions, that we will see in the next section. Nevertheless, for the purpose of this project, we will only define normal subobjects to be normal monomorphisms (kernels).

1.6 Protomodular categories

Protomodular categories were introduced in [9], and sometimes called Bourn-protomodular categories. In protomodular categories, most properties of categories of "group-like" structures hold. In particular, in pointed Barr-exact protomodular categories, the two notions of normal subobjects coincide, that is, Bourn-normal subobjects are precisely normal monomorphisms (see, for instance, Proposition 3.2.20 [4]).

Definition 1.6.1 (see [9])**.** *A category $\mathbb{C}$ is protomodular when*

(a) $\mathbb{C}$ has pullbacks of split epimorphisms along any morphism;

(b) For every morphism $v : A \longrightarrow B$ in $\mathbb{C}$, the "inverse image" functor $v^\star$, that is the functor

$$v^\star : \mathsf{Pt}(B) \longrightarrow \mathsf{Pt}(A)$$

induced by pulling back along v, reflects isomorphisms.

A pointed category $\mathbb{C}$ with finite limits is protomodular when the split short five lemma holds (see e.g [4]), that is, for each commutative diagram

$$\begin{tikzcd}
X \arrow[r, tail, "v"] \arrow[d, "a"'] & Y \arrow[r, shift left, "p"] \arrow[d, "b"] & Z \arrow[l, shift left, "r"] \arrow[d, "c"] \\
X' \arrow[r, tail, "u"'] & Y' \arrow[r, shift left, "q"] & Z' \arrow[l, shift left, "s"]
\end{tikzcd}$$

with q and p split epimorphisms

$$qs = 1, \; pr = 1,$$

and $u = \ker q, v = \ker p$ the respective kernels, if a and c are isomorphisms then so is b. The short five lemma holds for groups, and in particular, the category of groups is often used as a leading and guiding example when studying protomodularity. Other classical properties such as, (a) monomorphisms are precisely those morphisms with trivial kernels, (b) any congruence is completely determined by the unit class, and (c) reflexive relations are equivalence relations, all lift to pointed protomodular categories (see e.g [9], [8], [7]). From (c) we see that protomodularity implies Mal'cev.

Theorem 1.6.2 (see [11])**.** *Let $\mathcal{V}$ be a variety of universal algebras. $\mathcal{V}$ is protomodular if and only if it has $0-$ary terms $e_1, ..., e_n$, binary terms $s_1, ..., s_n$, and $(n+1)-$ary term p such that $p(x, s_1(x, y), ..., s_n(x, y)) = y$ and $s_i(x, x) = e_i$ for each $i = 1, ..., n$.*

The categories of (abelian) groups, non-unitary rings, Lie algebras, crossed modules of groups, and more generally, every variety of $\Omega-$groups are protomodular categories.

1.7 Semi-abelian categories

Although in this thesis we are working in a strictly weaker context than *semi-abelian category* [23], we will end this chapter with a brief summary on the historical background of semi-abelian categories.

Over the years different people were working to find a right categorical framework which exhibits properties of "group-like" structures, just as abelian category allows a generalized treatment of abelian groups. What are nowadays called *old style axioms* were conditions given in different papers, which as mentioned in [23] required a "good behaviour" of normal epimorphisms and monomorphisms, to capture properties of groups, rings, and modules. The notion of semi abelian categories serves the same purpose, it incorporates the old style axioms and also acts as a bridge between the work done in the past and "modern categorical algebra".

According to [23], "a pointed, Barr-exact, and protomodular category $\mathbb{C}$ with finite colimits is called a semi-abelian category".

Let $\mathcal{V}$ be a variety of universal algebras. According to [11], $\mathcal{V}$ is semi-abelian if and only if its theory has a unique constant 0, binary terms $s_1, ..., s_n$ and a $(n+1) - ary$ term p such that $p(x, s_1(x, y), ..., s_n(x, y)) = y$ and $s_i(x, x) = 0$ for each $i = 1, ..., n$.

Chapter 2

Binary commutators

The classical notion of commutator of two subgroups has been generalized in several ways to various types of universal algebras and further to certain categorical contexts. In this chapter we study, in categorical contexts, the notions of binary commutators in the sense of Huq, Higgins, Ursini and Smith.

2.1 Higgins commutator

The Higgins commutator [21] is a universal-algebraic commutator, which generalizes the commutator of groups to Ω-groups. In [30] the Higgins commutator has been defined categorically for a pair of subobjects of an object in an ideal-determined category. The main aim of this section is to characterize the Higgins commutator as the largest binary operation defined on all subobjects of every object in a normal strongly unital category $\mathbb{C}$ with finite colimits, satisfying certain conditions. The same characterization already appears in the author's work (see [34]), but given in the context of ideal-determined unital category; in the present work we replace the condition that kernels are preserved by images under normal epimorphisms (in the definition of an ideal-determined category $\mathbb{C}$) with the condition that $\mathbb{C}$ is subtractive. So this section is essentially as in [34].

Definition 2.1.1 (see [21], [30]). *Let (H, h) and (K, k) be subobjects of X in a normal strongly unital category $\mathbb{C}$ with finite colimits. The Higgins commutator $[H, K]_{\mathsf{H}}$ is obtained as the regular image of the normal subobject $\kappa_{H,K} : H \diamond K \rightarrowtail H + K$ under the canonical morphism $[h, k] : H + K \longrightarrow X$ in the diagram*

$$\begin{tikzcd}
H \diamond K \arrow[r, two heads] \arrow[d, tail, "\kappa_{H,K}"'] & {[H,K]_{\mathsf{H}}} \arrow[d, tail] \\
H + K \arrow[r, "{[h,k]}"'] & X.
\end{tikzcd}$$

In a normal strongly unital category $\mathbb{C}$ with finite colimits, a binary operation $[-, -]$ (defined on all subobjects of every object) is said to be *monotone* when $[H', K'] \leq [H, K]$ for subobjects

(H', h'), (H, h), (K', k'), and (K, k) of X, such that $H' \leq H$ and $K' \leq K$. The Higgins commutator is monotone: For subobjects (H', h'), (H, h), (K', k'), and (K, k) of X in a normal strongly unital category $\mathbb{C}$ with finite colimits, if $H' \leq H$ and $K' \leq K$, then there exist morphisms α and β such that $h' = h\alpha$ and $k' = k\beta$. Furthermore, one obtains the following commutative diagram

$$\begin{tikzcd}
 & H' \diamond K' \arrow[r, tail, "\kappa_{H',K'}"] \arrow[d, dashed, "t"'] \arrow[ddl, bend right, two heads] & H' + K' \arrow[r, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] \arrow[d, "\alpha + \beta"] & H' \times K' \arrow[d, "\alpha \times \beta"] \\
 & H \diamond K \arrow[r, tail, "\kappa_{H,K}"'] \arrow[dd, two heads] & H + K \arrow[r, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] \arrow[dd, "{[h,k]}"] & H \times K \\
{[H', K']_{\mathsf{H}}} \arrow[drr, tail] \arrow[dr, dashed, tail] & & & \\
 & {[H, K]_{\mathsf{H}}} \arrow[r, tail] & X &
\end{tikzcd}$$

in which, by the universal property of kernels, there is a morphism t making the upper left square commute. The Higgins commutator $[H', K']_{\mathsf{H}}$ is obtained as the image of $\kappa_{H',K'}$ along $[h', k'] = [h, k](\alpha + \beta)$, but since the morphism $H' \diamond K' \twoheadrightarrow [H', K']_{\mathsf{H}}$ in the diagram is a strong epimorphism, it can be seen that $[H', K']_{\mathsf{H}} \leq [H, K]_{\mathsf{H}}$.

For a subobject (H, h) of X in any normal category with finite colimits, its normal closure, denoted by $(\overline{H}, \overline{h})$, is a subobject of X given by the kernel of the cokernel of $h : H \rightarrowtail X$.

Proposition 2.1.2 (see [34])**.** *For a pair of subobjects (H, h) and (K, k) of X in a normal strongly unital category $\mathbb{C}$ with finite colimits, one has $[H, K]_{\mathsf{H}} \leq \overline{H} \wedge \overline{K}$.*

Proof. In the commutative diagram

$$\begin{tikzcd}
H \diamond K \arrow[r, tail, "\kappa_{H,K}"] \arrow[d, two heads] & H + K \arrow[r, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] \arrow[d, "{[h,k]}"] & H \times K \arrow[d, "qk\pi_2"] \\
{[H, K]_{\mathsf{H}}} \arrow[r, tail] & X \arrow[r, two heads, "q"'] & Q
\end{tikzcd}$$

where q is the cokernel of $h : H \rightarrowtail X$, we see that $[H, K]_{\mathsf{H}} \leq \overline{H}$, with $\overline{H} \rightarrowtail X$ the kernel of q. Similarly, one can show that $[H, K]_{\mathsf{H}} \leq \overline{K}$. Hence $[H, K]_{\mathsf{H}} \leq \overline{H} \wedge \overline{K}$.

□

For a pair of objects H and K in a normal strongly unital category $\mathbb{C}$ with finite colimits, $\ker[1, 0] : H \flat K \rightarrowtail H + K$ and $\ker[0, 1] : K \flat H \rightarrowtail H + K$ denote the kernels of the induced morphisms $[1, 0] : H + K \longrightarrow H$ and $[0, 1] : H + K \longrightarrow K$ respectively. As observed in

[30], for instance, it can be seen from the diagram below that the co-smash product $H \diamond K$ coincides with the meet $H\flat K \wedge K\flat H$ in $H+K$, since they are both kernels of the morphism $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : H + K \longrightarrow H \times K$

$$\begin{tikzcd}
H\flat K \wedge K\flat H \arrow[r] \arrow[d] \arrow[dr, dotted, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & K\flat H \arrow[d, tail, "\ker[0,1]"] & \\
H\flat K \arrow[r, tail, "\ker[1,0]"'] & H+K \arrow[r, "{[0,1]}"] \arrow[d, "{[1,0]}"'] \arrow[dr, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] & K \\
 & H & H \times K. \arrow[l, "\pi_1"] \arrow[u, "\pi_2"']
\end{tikzcd}$$

Furthermore, the Higgins commutator of the coproduct inclusions $i_1 : H \longrightarrow H + K$ and $i_2 : K \longrightarrow H + K$, that we shall denote by $[i_1(H), i_2(K)]_{\mathsf{H}}$, is just the co-smash product $H \diamond K$ as seen in the diagram

$$\begin{tikzcd}
H \diamond K \arrow[r] \arrow[d, tail] & {[i_1(H), i_2(K)]_{\mathsf{H}}} \arrow[d, tail] \\
H+K \arrow[r, "{[i_1, i_2] = 1}"] \arrow[d, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] & H+K \arrow[d, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"] \\
H \times K \arrow[r, "1"] & H \times K.
\end{tikzcd}$$

Therefore, using the fact that the Higgins commutator is monotone, and, $H \leq K\flat H$ and $K \leq H\flat K$ in $H+K$, we can apply Proposition 2.1.2 to conclude that

$$H \diamond K = [i_1(H), i_2(K)]_{\mathsf{H}} \leq [K\flat H, H\flat K]_{\mathsf{H}} \leq K\flat H \wedge H\flat K = H \diamond K.$$

For a pair of subobjects (H, h) and (K, k) of X in a normal strongly unital category $\mathbb{C}$ with finite colimits, since $H \leq K\flat H$ and $K \leq H\flat K$ in $H+K$, it can be easily seen through the diagram

$$\begin{tikzcd}
 & K\flat H \arrow[r, two heads] \arrow[d, tail] & {[h,k](K\flat H)} \arrow[d, tail] \\
H \arrow[ur, dotted, tail, "\alpha"] \arrow[r, "i_1"] \arrow[rr, tail, bend right, "h"'] & H+K \arrow[r, "{[h,k]}"'] & X
\end{tikzcd}$$

that $H \leq [h,k](K\flat H)$ in X. In a similar way it can be shown that $K \leq [h,k](H\flat K)$ in X.

The following result is a special case of Lemma 5.1 of [17] but stated in a weaker context, and the proof is essentially the same.

Lemma 2.1.3. *Let $\mathbb{C}$ be a normal strongly unital category with finite colimits. If $f : X \twoheadrightarrow Y$ and $g : X' \twoheadrightarrow Y'$ are normal epimorphisms, then the induced morphism $f \diamond g : X \diamond X' \longrightarrow Y \diamond Y'$ is also a normal epimorphism.*

Proof. Consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathrm{Ker}(f+g) \arrow[r, dotted, "\langle \alpha, \beta \rangle"] \arrow[d, tail, "\ker(f+g)"'] & \mathrm{Ker}(f) \times \mathrm{Ker}(g) \arrow[d, tail, "\ker(f)\times\ker(g)"] & \\
0 \arrow[r] & X \diamond X' \arrow[r, tail] \arrow[d, "f\diamond g"'] & X + X' \arrow[r, two heads, "\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]"] \arrow[d, two heads, "f+g"'] & X \times X' \arrow[r] \arrow[d, two heads, "f\times g"] & 0 \\
0 \arrow[r] & Y \diamond Y' \arrow[r, tail] & Y + Y' \arrow[r, two heads, "\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]"'] \arrow[d] & Y \times Y' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Using the upper 3×3 lemma (Proposition 1.3.3), the morphism $f \diamond g$ is a normal epimorphism if and only if the dotted arrow is a normal epimorphism. We obtain the following commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, shift right, "s"'] \arrow[d, tail, "\ker(f)"'] & \mathrm{Ker}(f+g) \arrow[l, shift right, "r"'] \arrow[r, shift left, "r'"] \arrow[d, tail, "\ker(f+g)"] & \mathrm{Ker}(g) \arrow[l, shift left, "s'"] \arrow[d, tail, "\ker(g)"] \\
X \arrow[r, shift right, "i_1"'] \arrow[d, two heads, "f"'] & X + X' \arrow[l, shift right, "{[1,0]}"'] \arrow[r, shift left, "{[0,1]}"] \arrow[d, two heads, "f+g"] & X' \arrow[l, shift left, "i_2"] \arrow[d, two heads, "g"] \\
Y \arrow[r, shift right, "i_1"'] & Y + Y' \arrow[l, shift right, "{[1,0]}"'] \arrow[r, shift left, "{[0,1]}"] & Y' \arrow[l, shift left, "i_2"]
\end{tikzcd}$$

where r, s, r', and s' are morphisms induced by appropriate $[1,0], i_1, [0,1]$ and i_2, respectively. Clearly, $rs = 1, r's = 0, rs' = 0$, and $r's' = 1$, and since $\ker(f)r = [1,0]\ker(f+g) = \ker(f)\alpha$

and $\ker(g)r' = [0,1]\ker(f+g) = \ker(g)\beta$, we see that $r = \alpha$ and $r' = \beta$. It is now clear that the diagram

$$\begin{tikzcd}
& \mathrm{Ker}(f+g) \arrow[d, dotted, "\langle\alpha,\beta\rangle"] & \\
\mathrm{Ker}(f) \arrow[ur, bend left, "s"] \arrow[r, "\langle 1,0\rangle"'] & \mathrm{Ker}(f)\times\mathrm{Ker}(g) & \mathrm{Ker}(g) \arrow[ul, bend right, "s'"'] \arrow[l, "\langle 0,1\rangle"]
\end{tikzcd}$$

commutes, and since the pair of morphisms $\langle 1,0\rangle$ and $\langle 0,1\rangle$ is jointly extremal-epimorphic, the dotted arrow is a normal epimorphism. □

The next theorem has been already proven for ideal-determined unital categories in [35]. However, using the previous lemma, an alternative proof can be obtained in a normal strongly unital category with finite colimits.

Theorem 2.1.4. *Let $f : X \longrightarrow Y$ be a morphism in a normal strongly unital category $\mathbb{C}$ with finite colimits and (H,h), (K,k) be a pair of subobjects of X. Then, $f([H,K]_{\mathsf{H}}) = [f(H), f(K)]_{\mathsf{H}}$.*

Proof. Let $h'u$ and $k'v$ be the (regular epi, mono)-factorizations of the composites fh and fk respectively. Using Lemma 2.1.3, the induced morphism $u \diamond v$ in the diagram

$$\begin{tikzcd}
& & f(H)\diamond f(K) \arrow[rr, two heads] \arrow[dd, tail] & & f([H,K]_{\mathsf{H}}) \arrow[dd, tail] \\
H\diamond K \arrow[urr, two heads, "u\diamond v"] \arrow[rrr, two heads] \arrow[dd, tail] & & & {[H,K]_{\mathsf{H}}} \arrow[ur, two heads] \arrow[dd, tail] & \\
& & f(H)+f(K) \arrow[rr, "{[h',k']}"] & & Y \\
H+K \arrow[urr, two heads, "u+v"] \arrow[rrr, "{[h,k]}"] & & & X \arrow[ur, "f"'] &
\end{tikzcd}$$

is a normal epimorphism. Now $f([H,K]_{\mathsf{H}}) = [f(H), f(K)]_{\mathsf{H}}$ follows by the uniqueness of regular images.

□

Now we can state the main result, which has been already proven for an ideal-determined unital category in [34].

Theorem 2.1.5. *In a normal strongly unital category $\mathbb{C}$ with finite colimits, the Higgins commutator is the largest binary operation C on subobjects (defined on all subobjects of each object) satisfying the following conditions:*

(a) C is monotone;

(b) $C(H,K) \leq \overline{H} \wedge \overline{K}$ for each pair of subobjects (H,h) and (K,k) of an object X;

(c) *$C(f(H), f(K)) = f(C(H, K))$ for each pair of subobjects (H, h) and (K, k) of an object X, and every morphism f whose domain is X.*

Proof. Let us suppose there is a binary operation $[|-,-|]$ satisfying the conditions (a), (b), and (c) above. Let (H, h) and (K, k) be a pair of subobjects of X. Since $H \leq [h, k](K\flat H)$ and $K \leq [h, k](H\flat K)$ in X, we have

$$\begin{aligned} [|H, K|] &\leq [|[h, k](K\flat H), [h, k](H\flat K)|] \\ &= [h, k]([|K\flat H, H\flat K|]) \\ &\leq [h, k](K\flat H \wedge H\flat K) \qquad (2.1) \\ &= [h, k](H \diamond K) \\ &= [H, K]_{\mathsf{H}}. \end{aligned}$$

□

2.2 Huq commutator

The Huq commutator [22] is a purely categorical notion derived from the concept of commuting morphisms, also introduced by Huq [22] in a context closely related to semi-abelian. The notion of commuting morphisms has been investigated in various categorical contexts; in [5], for instance, it is shown that commuting morphisms can always be defined in a unital category. In this section we will recall some of the recent developments in the study of Huq commutator.

Definition 2.2.1 (see [22]). *Two morphisms $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ in a unital category $\mathbb{C}$ are said to commute when there exists a (necessarily unique) morphism $\varphi : X \times Z \to Y$ making the diagram*

$$\begin{tikzcd} & X \arrow[dl, "\langle 1,0\rangle"'] \arrow[dr, "f"] & \\ X \times Z \arrow[rr, dashed, "\varphi"] & & Y \\ & Z \arrow[ul, "\langle 0,1\rangle"] \arrow[ur, "g"'] & \end{tikzcd}$$

commute.

In the category of groups, for two subgroups X and Y of a group G, it can be easily shown that the inclusion maps $X \hookrightarrow G$ and $Y \hookrightarrow G$ commute in the sense of Huq if and only if every element of X commutes with every element of Y.

Definition 2.2.2 (see [22]). *For a pair of subobjects (H, h) and (K, k) of X in a normal unital category $\mathbb{C}$ with finite colimits, the Huq commutator $[H, K]_Q$ is the smallest normal subobject of X for which the composites qk and qh, where q is the cokernel of $[H, K]_Q \rightarrowtail X$, commute.*

In a normal unital category $\mathbb{C}$ with finite colimits, the Huq commutator of a pair of subobjects (H, h) and (K, k) of X always exists, and can be constructed as the kernel of q in the diagram

$$\begin{tikzcd}
& H \arrow[dl, "{\langle 1,0\rangle}"'] \arrow[d, dotted] \arrow[dr, tail, "h"] & & \\
H \times K \arrow[r, dotted, "\varphi"] & Q & X \arrow[l, two heads, dotted, "q"'] & {[H,K]_Q} \arrow[l, dotted, "\ker(q)"'] \\
& K \arrow[ul, "{\langle 0,1\rangle}"] \arrow[u, dotted] \arrow[ur, tail, "k"'] & &
\end{tikzcd}$$

where Q is the colimit of the solid arrows (see [5]). Or equivalently (see Proposition 5.5 in [30]) as the kernel of q in the following pushout

$$\begin{tikzcd}
H + K \arrow[r, "{[h,k]}"] \arrow[d, two heads, "\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]"'] & X \arrow[d, two heads, "q"] & {[H,K]_Q} \arrow[l, dotted, "\ker(q)"'] \\
H \times K \arrow[r, "\varphi"'] & Q. \arrow[ul, phantom, "\ulcorner", very near start] &
\end{tikzcd}$$

The following is a general fact.

Lemma 2.2.3. *Let $\mathbb{C}$ be a pointed category with finite colimits. For each commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, two heads, "e"'] & Y \arrow[r, two heads, "\mathrm{coker}(f)"] \arrow[d, "s"] \arrow[dr, phantom, "(1)"] & Z \arrow[d, "r"] \\
A \arrow[r, "k"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

where e is an epimorphism, diagram (1) is a pushout if and only if g is the cokernel of k.

For the next proposition we copy the proof given in [35].

Proposition 2.2.4 (see [30],[35])**.** *In a normal unital category $\mathbb{C}$ with finite colimits, the Huq commutator is the normal closure of the Higgins commutator.*

Proof. Let (H, h) and (K, k) be a pair of subobjects of X in a normal unital category $\mathbb{C}$ with finite colimits. Consider the commutative diagram

$$\begin{tikzcd}
H \diamond K \arrow[r, tail, "\kappa_{H,K}"] \arrow[d, two heads, "e"'] & H + K \arrow[r, two heads, "\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]"] \arrow[d, "{[h,k]}"'] \arrow[dr, phantom, "(1)"] & H \times K \arrow[d, dashed, "\varphi"] \\
{[H,K]_{\mathsf{H}}} \arrow[r, tail, "m"'] & X \arrow[r, two heads, "\mathsf{coker}(m)"'] & Q
\end{tikzcd}$$

in which the morphism φ is given by the universal property of cokernels. Applying Lemma 2.2.3 to the diagram above, the square (1) is a pushout. Therefore the kernel of coker(m), which is also the Huq commutator $[H, K]_Q$, is the normal closure of the Higgins commutator $[H, K]_{\mathsf{H}}$. $\square$

The next theorem describes the Huq commutator of regular images of a pair of subobjects under an arbitrary morphism.

Theorem 2.2.5 (see [35], Theorem 3.2)**.** *In a normal unital category $\mathbb{C}$ with finite colimits, for every pair of subobjects (H, h) and (K, k) of X and every morphism f whose domain is X, one has*

$$[f(H), f(K)]_Q = \overline{f([H, K]_Q)}.$$

We recall

Lemma 2.2.6 (see [35], Lemma 3.4)**.** *In a normal category $\mathbb{C}$ with finite colimits, for every subobject (H, h) of X and every morphism f whose domain is X, one has*

$$\overline{f(H)} = \overline{f(\overline{H})}.$$

In a normal strongly unital category $\mathbb{C}$ with finite colimits, Theorem 2.2.5 follows from Theorem 2.1.4 (as explained already in [34]), by applying the fact that the Huq commutator is the normal closure of the Higgins commutator, and the following application of the previous lemma:

$$\overline{f([H, K]_{\mathsf{H}})} = \overline{f(\overline{[H, K]_{\mathsf{H}}})}$$

for subobjects (H, h) and (K, k) of an object X and every morphism f whose domain is X. To make this more precise, for a morphism $f : X \longrightarrow Y$ and a pair of subobjects (H, h) and (K, k) of X in a normal strongly unital category $\mathbb{C}$ with finite colimits, we have

$$\begin{aligned}
[f(H), f(K)]_Q &= \overline{[f(H), f(K)]_{\mathsf{H}}} \\
&= \overline{f([H, K]_{\mathsf{H}})} \\
&= \overline{f(\overline{[H, K]_{\mathsf{H}}})} \\
&= \overline{f([H, K]_Q)}.
\end{aligned} \tag{2.2}$$

The Huq commutator has been characterized as follows:

Theorem 2.2.7 (see [35], Theorem 4.5). *In a normal unital category $\mathbb{C}$ with finite colimits, the Huq commutator is the largest binary operation $[-,-]$ defined on all subobjects of each object, satisfying the following conditions:*

(a) $[-,-]$ *is monotone;*

(b) $[H,K] \leq \overline{H} \wedge \overline{K}$ *for each pair of subobjects* (H,h) *and* (K,k) *of* X*;*

(c) $[f(H), f(K)] \leq \overline{f([H,K])}$ *for each pair of subobjects* (H,h) *and* (K,k) *of an object of* X *and every morphism* f *whose domain is* X.

2.3 Ursini and Smith commutators

In [38] the notion of commutator of a pair of ideals (Ursini commutator) has been introduced in the context of `BIT` varieties [37], also called ideal-determined varieties in [19]. An intrinsic description of Ursini commutator in an ideal-determined category has been given in [29] following a similar approach used in [30] to define the categorical notion of Higgins commutator. In this section we will recall the categorical definition of Ursini commutator given in [29], and how it is related to Higgins, Huq, and Smith commutators.

To begin, we recall the necessary background to define the categorical version of Ursini commutator. Let $\mathbb{C}$ be an ideal-determined category, and $(H,h),(K,k)$ be a pair of subobjects of X in $\mathbb{C}$. Observe that the morphism $[h,1,k] : H+X+K \longrightarrow X$ is a regular epimorphism since it is a split epimorphism. Following [29], we shall denote by Ω the morphism

$$\langle [h,1,0],[0,1,0],[0,1,k]\rangle = [\langle h,0,0\rangle, \langle 1,1,1\rangle, \langle 0,0,k\rangle] : H+X+K \longrightarrow X\times X\times X.$$

The morphism Ω is not a regular epimorphism in general, even for groups, so one can consider its (regular epi, mono)-factorization

$$\begin{tikzcd} H+X+K \arrow[rr, "{[\langle h,0,0\rangle, \langle 1,1,1\rangle, \langle 0,0,k\rangle]}"] \arrow[dr, two heads, "e"'] & & X\times X\times X. \\ & R \arrow[ur, tail, "m"'] & \end{tikzcd}$$

Definition 2.3.1 (see [29]). *Let* (H,h) *and* (K,k) *be subobjects of* X *in an ideal-determined category* $\mathbb{C}$. *The Ursini commutator* $[H,K]_U$ *is the regular image under* $[h,1,k] : H+X+K \longrightarrow X$ *of the kernel* $\mathrm{Ker}(\Omega)$ *of* Ω

$$\begin{tikzcd} \mathrm{Ker}(\Omega) \arrow[r, two heads] \arrow[d, tail] & {[H,K]_U} \arrow[d, tail] \\ H+X+K \arrow[r, two heads, "{[h,1,k]}"'] & X. \end{tikzcd}$$

It can be seen in the diagram below that the Ursini commutator $[H,K]_U$ can also be constructed as the kernel of the morphism q in the pushout diagram on the right

$$\begin{tikzcd}
\mathrm{Ker}(\Omega) \arrow[r, tail] \arrow[d, two heads] & H+X+K \arrow[r, two heads, "e"] \arrow[d, two heads, "{[h,1,k]}"] & R \arrow[d, two heads, "r"] \\
{[H,K]_U} \arrow[r, tail] & X \arrow[r, two heads, "q"'] & Q. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Next we compare Ursini commutator with Huq and Higgins commutators. For subobjects (H,h) and (K,k) of X in an ideal-determined unital category $\mathbb{C}$, it can be easily seen that the right hand side rectangle in the diagram

$$\begin{tikzcd}
\mathrm{Ker}(\Omega) \arrow[r, tail] & H+X+K \arrow[rr, "{[\langle h,0,0\rangle,\langle 1,1,1\rangle,\langle 0,0,k\rangle]}"] & & X\times X\times X \\
H\diamond K \arrow[u, dashed, "\beta"] \arrow[r, tail, "\kappa_{H,K}"'] & H+K \arrow[u, "{[i_1,i_3]}"'] \arrow[rr, two heads, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] & & H\times K \arrow[u, "{\langle h\pi_1,0,k\pi_2\rangle}"']
\end{tikzcd}$$

commutes, and this implies there is a factorization β through the kernel $\mathrm{Ker}(\Omega)$. Now in the diagram

$$\begin{tikzcd}
 & \mathrm{Ker}(\Omega) \arrow[rr, two heads] \arrow[dd, tail] & & {[H,K]_U} \arrow[ddd, tail] \\
H\diamond K \arrow[ur, "\beta"] \arrow[rr, two heads] \arrow[ddd, tail] & & {[H,K]_{\mathsf{H}}} \arrow[ur, dashed, two heads] \arrow[ddd, tail] & \\
 & H+X+K \arrow[ddr, two heads, "{[h,1,k]}"] & & \\
 & & & \\
H+K \arrow[uur, "{[i_1,i_3]}"] \arrow[rr, "{[h,k]}"'] & & X &
\end{tikzcd}$$

since $[h,k]=[h,1,k][i_1,i_3]$, it can be easily seen that the strong epimorphism $H\diamond K \twoheadrightarrow [H,K]_{\mathsf{H}}$ induces a factorization of the Higgins commutator $[H,K]_{\mathsf{H}} \rightarrowtail X$ through the Ursini commutator $[H,K]_U \rightarrowtail X$. But since, by definition, the Ursini commutator is always normal, and the Huq commutator is the normal closure of the Higgins commutator, it follows that

$$[H,K]_{\mathsf{H}} \leq [H,K]_Q \leq [H,K]_U \lhd X.$$

The Ursini commutator is also different from Huq and Higgins commutators in the sense that, it is invariant with respect to normal closures of subobjects, in other words, for subobjects (H,h), (K,k) of X in an ideal-determined category $\mathbb{C}$, $[H,K]_U=[\overline{H},\overline{K}]_U$ (see Proposition 4.5 in [29]). This is a property that both Huq and Higgins commutators do not satisfy even for groups; implying that the above inequality may be strict.

The Ursini commutator of two subgroups H and K of a group G is just the classical commutator of their respective normal closures. This suggests that all the three commutators coincide for normal subgroups.

Next we recall the construction of the Smith commutator.

Definition 2.3.2 (see [5],[33],[36]). *Let (R, r_1, r_2) and (S, s_1, s_2) be equivalence relations on an object X in a regular Mal'cev category $\mathbb{C}$ with finite colimits. Consider the pullback of s_1 along r_2 in the diagram*

$$\begin{tikzcd} R \times_X S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "s_1"] \\ R \arrow[r, "r_2"'] & X. \end{tikzcd}$$

The Smith commutator $[R, S]_{\mathsf{S}}$ is given by the kernel pair relation of the normal epimorphism t in the diagram

$$\begin{tikzcd} & R \arrow[dl, "{\langle 1, \triangle_S r_2 \rangle}"'] \arrow[d, dotted] \arrow[dr, "r_1"] & \\ R \times_X S \arrow[r, dotted, "\varphi"] & T & X \arrow[l, dotted, two heads, "t"'] \\ & S \arrow[ul, "{\langle \triangle_R s_1, 1 \rangle}"] \arrow[u, dotted] \arrow[ur, "s_2"'] & \end{tikzcd}$$

where T is the colimit of solid arrows. When the morphism t is an isomorphism, the Smith commutator $[R, S]_{\mathsf{S}}$ is trivial, and it precisely means that R and S centralise each other.

In a normal Barr-exact category $\mathbb{C}$, every equivalence relation is uniquely determined by its zero class, i.e. the kernel of its quotient. These corresponding normal subobjects (kernels) are called associated normal subobjects. For a normal subobject (H, h) of an object X, we write (R_H, r_1, r_2) to denote its associated equivalence relation, obtained as the kernel pair relation of the cokernel of h.

In the next theorem we recall the description of Ursini commutator in terms of Smith commutator in a categorical context, given in [29]. An independent proof has been given in [17].

Theorem 2.3.3 (see [29], Theorem 4.12). *In a normal Barr-exact Mal'cev category $\mathbb{C}$, for a pair of normal subobjects (H, h) and (K, k) of X, and their associated equivalence relations (R_H, r_1, r_2) and (R_K, r_1', r_2') respectively, the Ursini commutator $[H, K]_U$ is the associated normal subobject of the Smith commutator $[R_H, R_K]_{\mathsf{S}}$.*

In the next chapter we will prove a stronger fact from which the previous description of Ursini commutator in terms of Smith commutator can be recovered as a special case. In addition, we will obtain a characterization of the Ursini commutator as a special case of a characterization of another notion of commutator.

Chapter 3

Weighted commutators

The notion of *weighted commutators*, introduced and studied in [17], is derived from the notion of *weighted centrality*, also introduced in [17]. We shall explore further properties of weighted centrality and weighted commutators, and also investigate further relationships with the commutators discussed in the previous chapter.

3.1 Weighted centrality

Let $w : W \longrightarrow A, x : X \longrightarrow A$, and $y : Y \longrightarrow A$ be morphisms in a pointed category $\mathbb{C}$ with finite limits and colimits. A *weighted cospan* (w, x, y) in $\mathbb{C}$ is a diagram

$$\begin{tikzcd} & W \arrow[d, "w"] & \\ X \arrow[r, "x"] & A & Y \arrow[l, "y"'] \end{tikzcd}$$

whereby w plays a role of a "weight". Now consider the following pullback in $\mathbb{C}$

$$\begin{tikzcd} (W+X) \times_W (W+Y) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & W+Y \arrow[d, "{[1,0]}"] \\ W+X \arrow[r, "{[1,0]}"'] & W. \end{tikzcd}$$

Weighted centrality is then defined as follows:

Definition 3.1.1 (see [17])**.** *Let $w : W \longrightarrow A$, $x : X \longrightarrow A$, and $y : Y \longrightarrow A$ be morphisms in a pointed category $\mathbb{C}$ with finite limits and colimits. The morphisms x and y commute over w if there exists a morphism*

$$m : (W + X) \times_W (W + Y) = (W + X) \times_{([1,0],[1,0])} (W + Y) \longrightarrow A$$

making the diagram

$$\begin{tikzcd}
 & W+X \arrow[dl, "{\langle 1, i_1[1,0]\rangle}"'] \arrow[dr, "{[w,x]}"] & \\
(W+X)\times_W(W+Y) \arrow[rr, dashed, "m"] & & A \\
 & W+Y \arrow[ul, "{\langle i_1[1,0], 1\rangle}"] \arrow[ur, "{[w,y]}"'] &
\end{tikzcd}$$

commute. The morphism m is called an internal multiplication, and when such a morphism exists, one says there is an internal multiplication $X \times Y \to A$ over w.

As explained in [17], taking $W = 0$ in a unital category $\mathbb{C}$ with finite colimits, Definition 3.1.1 reduces to commuting morphisms in the sense of Huq, in other words, $x : X \longrightarrow A$ and $y : Y \longrightarrow A$ commute over the zero morphism $0 \longrightarrow A$ if and only if they commute in the sense of Huq. In the definition above one is not insisting on the uniqueness of an internal multiplication whenever it exists. However, in some categorical contexts it can be shown that an internal multiplication is necessarily unique whenever it exists. We shall observe the uniqueness of internal multiplications in a pointed finitely cocomplete Mal'cev category $\mathbb{C}$.

Let X, Y, and W be objects in a pointed finitely cocomplete Mal'cev category $\mathbb{C}$. The diagram

$$\begin{tikzcd}
W+X \arrow[dr, shift left, "{[1,0]}"] & (W+X)\times_W(W+Y) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, shift left, "{[1,0]\pi_1}"] & W+Y \arrow[dl, shift right, "{[1,0]}"'] \\
 & W \arrow[ul, shift left, "i_1"] \arrow[u, shift left, "{\langle i_1, i_1\rangle}"] \arrow[ur, shift right, "i_1"'] &
\end{tikzcd}$$

represents the product of $(W + X, [1, 0], i_1)$ and $(W + Y, [1, 0], i_1)$ in $\mathsf{Pt}(W)$. In the diagram

$$\begin{tikzcd}
W+X \arrow[r, "{[1,0]}"] \arrow[dr, shift left, "{[1,0]}"] & W \arrow[r, "i_1"] \arrow[d, shift left, "1"] & W+Y \arrow[dl, shift right, "{[1,0]}"'] \\
 & W \arrow[ul, shift left, "i_1"] \arrow[u, shift left, "1"] \arrow[ur, shift right, "i_1"'] &
\end{tikzcd}$$

we see that the composite $i_1[1, 0]$ is the zero morphism from $(W + X, [1, 0], i_1)$ to $(W + Y, [1, 0], i_1)$ in $\mathsf{Pt}(W)$. But since $\mathsf{Pt}(W)$ is unital whenever $\mathbb{C}$ is a Mal'cev category (see [7]), the pair of morphisms $\langle 1, i_1[1, 0]\rangle : (W + X, [1, 0], i_1) \longrightarrow ((W + X) \times_W (W + Y), [1, 0]\pi_1, \langle i_1, i_1\rangle)$ and $\langle i_1[1, 0], 1\rangle : (W + Y, [1, 0], i_1) \longrightarrow ((W + X) \times_W (W + Y), [1, 0]\pi_1, \langle i_1, i_1\rangle)$ is jointly extremal-epimorphic in $\mathsf{Pt}(W)$. Hence the pair of morphisms $\langle 1, i_1[1, 0]\rangle : W+X \longrightarrow (W+X)\times_W(W+Y)$ and $\langle i_1[1, 0], 1\rangle : W + Y \longrightarrow (W + X) \times_W (W + Y)$ is jointly extremal-epimorphic in $\mathbb{C}$, by applying the following general fact: In a category $\mathbb{C}$ with finite limits, for an object A, if the pair of morphisms $f : (X, r, s) \longrightarrow (Y, p, q)$ and $g : (X', r', s') \longrightarrow (Y, p, q)$ is jointly extremal-epimorphic in $\mathsf{Pt}(A)$, then the pair of morphisms $f : X \longrightarrow Y$ and $g : X' \longrightarrow Y$ is jointly extremal-epimorphic in $\mathbb{C}$. It can now be concluded that in a pointed Mal'cev category $\mathbb{C}$ with finite colimits, internal multiplications are necessarily unique. Furthermore, when $\mathbb{C}$ is a pointed regular Mal'cev category with finite colimits, from the commutativity of the diagram

$$\begin{tikzcd}
& W+X+Y \arrow[d, dotted, "{\langle [i_1,i_2,0],[i_1,0,i_2]\rangle}"] & \\
W+X \arrow[ur, bend left, "{[i_1,i_2]}"] \arrow[r, "{\langle 1, i_1[1,0]\rangle}"'] & (W+X)\times_W (W+Y) & W+Y \arrow[ul, bend right, "{[i_1,i_3]}"'] \arrow[l, "{\langle i_1[1,0],1\rangle}"]
\end{tikzcd}$$

it follows that the dotted morphism

$$[\langle 1, i_1[1,0]\rangle, \langle i_1[1,0], 1\rangle] = \langle [i_1, i_2, 0], [i_1, 0, i_2]\rangle : W + X + Y \longrightarrow (W + X) \times_W (W + Y)$$

is a regular epimorphism. Its kernel is denoted by $X \otimes^W Y \rightarrowtail W + X + Y$.

Let $w : W \longrightarrow A$, $x : X \longrightarrow A$, and $y : Y \longrightarrow A$ be morphisms in a pointed category $\mathbb{C}$ with finite limits and colimits. We write

$$\begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix} : W + X \longrightarrow W \times A \text{ and } \begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix} : W + Y \longrightarrow W \times A$$

to denote the morphisms

$$\langle [1, 0], [w, x]\rangle = [\langle 1, w\rangle, \langle 0, x\rangle] : W+X \to W\times X \text{ and } \langle [1, 0], [w, y]\rangle = [\langle 1, w\rangle, \langle 0, y\rangle] : W+Y \to W\times Y$$

respectively. Therefore, one obtains the following cospan in $\mathsf{Pt}(W)$

$$\begin{tikzcd}
W+X \arrow[r, "{\begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix}}"] \arrow[dr, shift left, "{[1,0]}"] & W\times A \arrow[d, shift left, "\pi_1"] & W+Y \arrow[l, "{\begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix}}"'] \arrow[dl, shift right, "{[1,0]}"'] \\
& W, \arrow[ul, shift left, "i_1"] \arrow[u, shift left, "{\langle 1,w\rangle}"] \arrow[ur, shift right, "i_1"'] &
\end{tikzcd} \tag{3.1}$$

and furthermore, the following diagram

$$\begin{tikzcd}
& (W+X,[1,0],i_1) \arrow[dl, "{\langle 1,i_1[1,0]\rangle}"'] \arrow[dr, "{\begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix}}"] & \\
((W+X)\times_W (W+Y),[1,0]\pi_1,\langle i_1,i_1\rangle) & & (W\times A,\pi_1,\langle 1,w\rangle) \\
& (W+Y,[1,0],i_1) \arrow[ul, "{\langle i_1[1,0],1\rangle}"] \arrow[ur, "{\begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix}}"'] &
\end{tikzcd} \tag{3.2}$$

in $\mathsf{Pt}(W)$. Therefore in a pointed Mal'cev category $\mathbb{C}$ with finite colimits, weighted centrality can be equivalently expressed by means of commuting morphisms in the sense of Huq as follow:

Proposition 3.1.2 (see e.g [32]). *Let $w : W \longrightarrow A$, $x : X \longrightarrow A$, and $y : Y \longrightarrow A$ be morphisms in a pointed Mal'cev category $\mathbb{C}$ with finite colimits. The following statements are equivalent:*
(a) the morphisms x and y commute over w;
(b) the cospan (3.1) Huq-commutes in $\mathsf{Pt}(W)$, i.e. the morphisms

$$\begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix} \text{ and } \begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix}$$

commute in the sense of Huq in $\mathsf{Pt}(W)$.

Proof. $(a) \Rightarrow (b)$. If x and y commute over w, and

$$m : (W+X)\times_W (W+Y) \longrightarrow A$$

is the internal multiplication $X\times Y \longrightarrow A$ over w, it is easy to see that the diagram

$$\begin{tikzcd}
& (W+X,[1,0],i_1) \arrow[dl, "{\langle 1,i_1[1,0]\rangle}"'] \arrow[dr, "{\begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix}}"] & \\
((W+X)\times_W (W+Y),[1,0]\pi_1,\langle i_1,i_1\rangle) \arrow[rr, "{\langle [1,0]\pi_1,m\rangle}"] & & (W\times A,\pi_1,\langle 1,w\rangle) \\
& (W+Y,[1,0],i_1) \arrow[ul, "{\langle i_1[1,0],1\rangle}"] \arrow[ur, "{\begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix}}"'] &
\end{tikzcd}$$

commutes, and this implies (b).

$(b) \Rightarrow (a)$. Assuming that (b) holds, let us use the previous diagram to picture this situation. If $\varphi = \langle \alpha, m \rangle$ is the cooperator of the cospan (3.1) in $\mathsf{Pt(W)}$, it can be easily seen after composing further with $\pi_2 : W \times A \longrightarrow A$ that the morphism $\pi_2 \varphi = m$ is an internal multiplication $X \times Y \longrightarrow A$ over w. $\square$

As mentioned before, for a normal subobject (X, x) of A in a normal Barr-exact category $\mathbb{C}$, its associated equivalence relation (R_X, r_1, r_2) is given by the kernel pair relation of the cokernel of $x : X \rightarrowtail A$. Let us observe in the next lemma that in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, the associated equivalence relation (R_X, r_1, r_2) can be computed as the join of the diagonal $\langle 1, 1 \rangle : A \longrightarrow A \times A$ and the morphism $\langle 0, x \rangle : X \longrightarrow A \times A$.

Lemma 3.1.3. *For a normal subobject (X, x) of A in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, its associated equivalence relation (R_X, r_1, r_2) is the join of the morphisms $\langle 1, 1 \rangle : A \longrightarrow A \times A$ and $\langle 0, x \rangle : X \longrightarrow A \times A$.*

Proof. In this context the join of $\langle 1, 1 \rangle : A \longrightarrow A \times A$ and $\langle 0, x \rangle : X \longrightarrow A \times A$ in $A \times A$ is just the regular image of the factorization $[\langle 1, 1 \rangle, \langle 0, x \rangle] = \langle [1, 0], [1, x] \rangle : A + X \longrightarrow A \times A$, which we shall denote by

$$\begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix} : A + X \longrightarrow A \times A.$$

Clearly, the diagonal $\langle 1, 1 \rangle$ of A factors through $\langle r_1, r_2 \rangle$, and because $\mathbb{C}$ is Mal'cev, (R_X, r_1, r_2) is an equivalence relation on A

$$\begin{tikzcd} & R_X \arrow[dr, tail, "\langle r_1, r_2 \rangle"] & \\ A + X \arrow[ur, two heads, "e"] \arrow[rr, "\begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix}"'] & & A \times A. \end{tikzcd}$$

Since $\mathbb{C}$ is Barr-exact, let $q : A \longrightarrow Q$ be the quotient of the equivalence relation (R_X, r_1, r_2). It remains to show that q is the cokernel of x, so that x is indeed the associated normal subobject of (R_X, r_1, r_2) and vice-versa. Since $qr_1 = qr_2$, and from the diagram $r_1 e = [1, 0]$ and $r_2 e = [1, x]$, one has $qx = qr_2 e i_2 = qr_1 e i_2 = q[1, 0] i_2 = 0$. Writing $\operatorname{coker}(x)$ for the cokernel of x, from $qx = 0$, we know that q factors through $\operatorname{coker}(x)$. On the other hand, since

$$\operatorname{coker}(x) r_1 e = \operatorname{coker}(x)[1, 0] = \operatorname{coker}(x)[1, x] = \operatorname{coker}(x) r_2 e$$

and e is a (regular) epimorphism, one obtains $\operatorname{coker}(x) r_1 = \operatorname{coker}(x) r_2$, which implies that $\operatorname{coker}(x)$ factors through q, since q is the coequalizer of r_1 and r_2. Hence q is the cokernel of x. $\square$

In a Mal'cev category $\mathbb{C}$, an equivalence relation (R, r_1, r_2) on an object A can be identified with the subobject $\langle r_1, r_2 \rangle : (R, r_1, \triangle_R) \rightarrowtail (A \times A, \pi_1, \langle 1, 1 \rangle)$ of $(A \times A, \pi_1, \langle 1, 1 \rangle)$ in $\mathsf{Pt(A)}$

$$\begin{tikzcd} R \arrow[r, tail, "\langle r_1, r_2 \rangle"] \arrow[d, shift left, "r_1"] & A \times A \arrow[d, shift left, "\pi_1"] \\ A \arrow[r, "1"'] \arrow[u, shift left, "\triangle_R"] & A, \arrow[u, shift left, "\langle 1,1 \rangle"] \end{tikzcd}$$

called the *local representation* of (R, r_1, r_2) (see e.g [13]). As explained in Proposition 2.3 of [5], in a Mal'cev category $\mathbb{C}$ a pair of equivalence relations (R, r_1, r_2) and (S, s_1, s_2) on an object A centralize each other, if and only if, their respective local representations $\langle r_1, r_2 \rangle : (R, r_1, \triangle_R) \rightarrowtail (A \times A, \pi_1, \langle 1, 1 \rangle)$ and $\langle s_1, s_2 \rangle : (S, s_1, \triangle_S) \rightarrowtail (A \times A, \pi_1, \langle 1, 1 \rangle)$ Huq-commute in $\mathsf{Pt}(A)$. Recall that in a regular unital category $\mathbb{C}$, a pair of composites gs and fr, where s and r are regular epimorphisms, Huq-commutes if and only if f and g Huq-commute as well (see e.g Proposition 1.6.4 in [4]). In other words, commuting in the sense of Huq allows "regular epi-cancellation". As a consequence of this, one can conclude that there is no a difference between the Huq commutator defined for a pair of morphisms f and g (having the same codomain), and the Huq commutator defined on their respective regular images. We will later establish a similar property for weighted centrality. For now let us take the weight to be the identity morphism of A in Proposition 3.1.2. Then using the previous fact ("regular epi-cancellation" for commuting morphisms in the sense of Huq), one can replace the morphisms in (b) of Proposition 3.1.2 with their respective regular images, which, as shown already, are the local representations of the equivalence relations associated to normal subobjects (X, x) and (Y, y). Using Proposition 2.3 of [5], through Proposition 3.1.2 we recover the following fact:

Remark 3.1.4. *In a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, two normal subobjects $x : X \longrightarrow A$ and $y : Y \longrightarrow A$ commute over $1 : A \longrightarrow A$, if and only if, their associated equivalence relations centralize each other (see also Remark* 1.10 *of [17] for an alternative explanation).*

We will end this section by proving a "regular epi-cancellation" property for weighted centrality, and we will do so in two parts: In the first part we consider a weighted cospan (w, xx', yy'), whereby x' and y' are regular epimorphisms, while in the second part we consider a weighted cospan (ww', x, y) where w' is a regular epimorphism.

Proposition 3.1.5. *Let $\mathbb{C}$ be a normal Mal'cev category with finite colimits. Given a weighted cospan*

$$\begin{tikzcd} & & W \arrow[d, "w"] & & \\ X' \arrow[r, two heads, "x'"] & X \arrow[r, "x"] & A & Y \arrow[l, "y"'] & Y' \arrow[l, two heads, "y'"'] \end{tikzcd}$$

where x' and y' are regular epimorphisms, the composites xx' and yy' commute over w if and only if x and y commute over w.

Proof. It can be easily seen that the following is a diagram in $\mathsf{Pt}(W)$

$$\begin{tikzcd}
(W+X',[1,0],i_1) \arrow[r, two heads, "1+x'"] \arrow[d, "{\langle 1,i_1[1,0]\rangle}"'] & (W+X,[1,0],i_1) \arrow[dr, "{\begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix}}"] & \\
((W+X')\times_W (W+Y'),[1,0]\pi_1,\langle i_1,i_1\rangle) & & (W\times A,\pi_1,\langle 1,w\rangle) \\
(W+Y',[1,0],i_1) \arrow[r, two heads, "1+y'"'] \arrow[u, "{\langle i_1[1,0],1\rangle}"] & (W+Y,[1,0],i_1). \arrow[ur, "{\begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix}}"'] &
\end{tikzcd}$$

According to Proposition 3.1.2, the morphisms xx' and yy' commute over w if and only if the morphisms

$$\begin{bmatrix} 1 & w \\ 0 & xx' \end{bmatrix} = \begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix}(1+x') \text{ and } \begin{bmatrix} 1 & w \\ 0 & yy' \end{bmatrix} = \begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix}(1+y')$$

Huq-commute in $\mathsf{Pt}(W)$. But since commuting morphisms in the sense of Huq allow "regular epi-cancellation", it follows that x and y commute over w, if and only if, the composites xx' and yy' commute over w.

□

The "second part" of the regular epi-cancellation of weighted centrality is not so straightforward, it is based on several facts that we are going to observe next.

Recall

Lemma 3.1.6. *Let $\mathbb{C}$ be a normal Barr-exact Mal'cev category with finite colimits. If $f : X' \longrightarrow X, g : Y' \longrightarrow Y$, and $h : W' \longrightarrow W$ are normal epimorphisms, then the morphism*

$$(h+f)\times_h (h+g) : (W'+X')\times_{W'} (W'+Y') \longrightarrow (W+X)\times_W (W+Y)$$

is a normal epimorphism.

Proof. Since $(h+f)\times_h(h+g) = ((h+1)\times_h(h+1))((1+f)\times_{W'}(1+g))$ and $(1+f)\times_{W'}(1+g)$ is already a normal epimorphism being the product of two normal epimorphisms in $\mathsf{Pt}(W')$, it remains only to show that $(h+1)\times_h(h+1)$ is a normal epimorphism. Consider the commutative diagram

$$
\begin{tikzcd}
(W'+X)\times_{W'}(W'+Y) \arrow[rr, two heads, "\pi_2"] \arrow[dr, "(h+1)\times_h(h+1)"] \arrow[dd, two heads, "\pi_1"'] & & W'+Y \arrow[dr, "h+1"] \arrow[dd, two heads, "{[1,0]}" near start] & & \\
& (W+X)\times_W(W+Y) \arrow[rr, two heads, "\pi_2"] \arrow[dd, two heads, "\pi_1"' near start] & & W+Y \arrow[dd, two heads, "{[1,0]}"] \arrow[dddr, dotted, bend left, "v"] & \\
W'+X \arrow[rr, two heads, "{[1,0]}"' near end] \arrow[dr, "h+1"'] & & W' \arrow[dr, two heads, "h"] \arrow[ddrr, dotted, bend right, "p"'] & & \\
& W+X \arrow[rr, two heads, "{[1,0]}"'] \arrow[drrr, dotted, bend right, "u"'] & & W \arrow[dr, dotted, "q"] & \\
& & & & T
\end{tikzcd}
$$

in which the back and front faces are pushouts, since they are pullbacks of normal epimorphisms. To show that $(h+1)\times_h(h+1)$ is a regular epimorphism, it is enough to show that the diagram

$$
\begin{tikzcd}
(W'+X)\times_{W'}(W'+Y) \arrow[r, two heads, "(h+1)\pi_2"] \arrow[d, two heads, "(h+1)\pi_1"'] & W+Y \arrow[d, two heads, "{[1,0]}"] \\
W+X \arrow[r, two heads, "{[1,0]}"'] & W
\end{tikzcd}
$$

is a pushout. For that, if u and v are morphisms such that $u(h+1)\pi_1 = v(h+1)\pi_2$, using the fact that the back and right hand faces are pushouts (in the first diagram), we obtain factorizations p and q respectively, and it can be easily seen that u and v factor through q. $\square$

For morphisms $f : X' \longrightarrow X, g : Y' \longrightarrow Y$, and $h : W' \longrightarrow W$ in a normal Mal'cev category $\mathbb{C}$ with finite colimits, we shall write $f \otimes^h g : X' \otimes^{W'} Y' \longrightarrow X \otimes^W Y$ to denote the morphism induced by the universal property of kernels in the diagram

$$
\begin{tikzcd}
X' \otimes^{W'} Y' \arrow[r, dashed, "f\otimes^h g"] \arrow[d, tail] & X \otimes^W Y \arrow[d, tail] \\
W'+X'+Y' \arrow[r, "h+f+g"] \arrow[d, two heads, "{\langle [i_1,i_2,0],[i_1,0,i_2]\rangle}"'] & W+X+Y \arrow[d, two heads, "{\langle [i_1,i_2,0],[i_1,0,i_2]\rangle}"] \\
(W'+X')\times_{W'}(W'+Y') \arrow[r, "(h+f)\times_h(h+g)"'] & (W+X)\times_W(W+Y).
\end{tikzcd}
$$

Next we recall Lemma 5.1 of [17], and we will repeat the proof since we are stating it in a weaker context.

Lemma 3.1.7. *Let $\mathbb{C}$ be a normal Barr-exact Mal'cev category with finite colimits. If $f : X' \longrightarrow X$, $g : Y' \longrightarrow Y$, and $h : W' \longrightarrow W$ are normal epimorphisms, then the induced morphism*

$$f \otimes^h g : X' \otimes^{W'} Y' \longrightarrow X \otimes^W Y$$

is also a normal epimorphism.

Proof. Consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathrm{Ker}(h+f+g) \arrow[r, dotted, "\langle r, r' \rangle"] \arrow[d, "\ker(h+f+g)"'] & \mathrm{Ker}(h+f) \times_{\mathrm{Ker}(h)} \mathrm{Ker}(h+g) \arrow[d, "\ker(h+f) \times_{\ker(h)} \ker(h+g)"] & \\
0 \arrow[r] & X' \otimes^{W'} Y' \arrow[r] \arrow[d, "f \otimes^h g"'] & W' + X' + Y' \arrow[r] \arrow[d, "h+f+g"'] & (W'+X') \times_{W'} (W'+Y') \arrow[r] \arrow[d, "(h+f) \times_h (h+g)"] & 0 \\
0 \arrow[r] & X \otimes^W Y \arrow[r] & W + X + Y \arrow[r] \arrow[d] & (W+X) \times_W (W+Y) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

in which the rows and columns are short exact sequences. Applying the upper 3×3 lemma (Proposition 1.3.3), $f \otimes^h g$ is a normal epimorphism if and only if the dotted arrow $\langle r, r' \rangle$ is a normal epimorphism. We consider the diagram

$$
\begin{tikzcd}
 & \mathrm{Ker}(h) \arrow[rr, shift right, "q'"'] \arrow[dl, shift left, "q"] \arrow[dd, tail, "\ker(h)"] & & \mathrm{Ker}(h+g) \arrow[ll, shift right, "p'"'] \arrow[dl, shift left, "s'"] \arrow[dd, tail, "\ker(h+g)"] \\
\mathrm{Ker}(h+f) \arrow[ur, shift left, "p"] \arrow[rr, shift right, "s"'] \arrow[dd, tail, "\ker(h+f)"'] & & \mathrm{Ker}(h+f+g) \arrow[ll, shift right, "r"'] \arrow[ur, shift left, "r'"] \arrow[dd, tail, "\ker(h+f+g)"] & \\
 & W' \arrow[rr, shift right, "i_1"'] \arrow[dl, shift left, "i_1"] \arrow[dd, two heads, "h"] & & W'+Y' \arrow[ll, shift right, "{[1,0]}"'] \arrow[dl, shift left, "{[i_1,i_3]}"] \arrow[dd, two heads, "h+g"] \\
W'+X' \arrow[ur, shift left, "{[1,0]}"] \arrow[rr, shift right, "{[i_1,i_2]}"'] \arrow[dd, two heads, "h+f"'] & & W'+X'+Y' \arrow[ll, shift right, "{[i_1,i_2,0]}"'] \arrow[ur, shift left, "{[i_1,0,i_2]}"] \arrow[dd, two heads, "h+f+g"] & \\
 & W \arrow[rr, shift right, "i_1"'] \arrow[dl, shift left, "i_1"] & & W+Y \arrow[ll, shift right, "{[1,0]}"'] \arrow[dl, shift left, "{[i_1,i_3]}"] \\
W+X \arrow[ur, shift left, "{[1,0]}"] \arrow[rr, shift right, "{[i_1,i_2]}"'] & & W+X+Y \arrow[ll, shift right, "{[i_1,i_2,0]}"'] \arrow[ur, shift left, "{[i_1,0,i_2]}"] &
\end{tikzcd}
$$

in which the morphisms in the top face are induced by the appropriate morphisms in the middle face, and moreover, $pq = 1$, $p'q' = 1$, $rs = 1$, and $r's' = 1$. In the diagram

$$
\begin{tikzcd}
\mathrm{Ker}(h+f) \times_{\mathrm{Ker}(h)} \mathrm{Ker}(h+g) \arrow[r, shift right, "\pi_2"'] \arrow[d, shift left, "\pi_1"] & \mathrm{Ker}(h+g) \arrow[l, shift right, "{\langle qp', 1\rangle}"'] \arrow[d, shift right, "p'"'] \\
\mathrm{Ker}(h+f) \arrow[u, shift left, "{\langle 1, q'p\rangle}"] \arrow[r, shift left, "p"] & \mathrm{Ker}(h) \arrow[l, shift left, "q"] \arrow[u, shift right, "q'"']
\end{tikzcd}
$$

the pair of morphisms $\langle 1, q'p\rangle$ and $\langle qp', 1\rangle$ is jointly extremal-epimorphic in $\mathsf{Pt}(\mathrm{Ker}(h))$, and so in $\mathbb{C}$. It now follows from the commutativity of the diagram

$$
\begin{tikzcd}
 & \mathrm{Ker}(h+f+g) & \\
\mathrm{Ker}(h+f) \arrow[ur, bend left, "s"] \arrow[r, "{\langle 1, q'p\rangle}"'] & \mathrm{Ker}(h+f) \times_{\mathrm{Ker}(h)} \mathrm{Ker}(h+g) \arrow[u, dotted, leftarrow, "{\langle r, r'\rangle}"'] & \mathrm{Ker}(h+g) \arrow[ul, bend right, "s'"'] \arrow[l, "{\langle qp', 1\rangle}"]
\end{tikzcd}
$$

that the dotted arrow is a normal epimorphism.

□

Remark 3.1.8. *In the notation of Lemma 3.1.7, since the induced morphism $f \otimes^h g$ is a (normal) epimorphism, it follows (see e.g Lemma 2.2.3) that the bottom rectangle in the diagram*

$$\begin{tikzcd}
X' \otimes^{W'} Y' \arrow[r, dashed, two heads, "f \otimes^h g"] \arrow[d, tail] & X \otimes^W Y \arrow[d, tail] \\
W' + X' + Y' \arrow[r, two heads, "h+f+g"] \arrow[d, two heads, "{\langle [i_1, i_2, 0], [i_1, 0, i_2] \rangle}"'] & W + X + Y \arrow[d, two heads, "{\langle [i_1, i_2, 0], [i_1, 0, i_2] \rangle}"] \\
(W' + X') \times_{W'} (W' + Y') \arrow[r, two heads, "(h+f) \times_h (h+g)"'] & (W + X) \times_W (W + Y)
\end{tikzcd}$$

is a pushout.

Now we are ready to state the "second part" of the regular epi-cancellation property of weighted centrality.

Proposition 3.1.9. *Let $\mathbb{C}$ be a normal Barr-exact Mal'cev category with finite colimits. For a weighted cospan*

$$\begin{tikzcd}
 & W' \arrow[d, two heads, "w'"] & \\
 & W \arrow[d, "w"] & \\
X \arrow[r, "x"] & A & Y \arrow[l, "y"']
\end{tikzcd}$$

where w' is a regular epimorphism, the morphisms x and y commute over the composite ww' if and only if they commute over w.

Proof. If x and y commute over w and $m : (W + X) \times_W (W + Y) \longrightarrow A$ is an internal multiplication $X \times Y \longrightarrow A$ over w, then clearly, the composite $m((w' + 1) \times_{w'} (w' + 1)) : (W' + X) \times_{W'} (W' + Y) \longrightarrow A$ in the diagram

$$\begin{tikzcd}
W'+X \arrow[r, "\langle 1, i_1[1,0]\rangle"] \arrow[d, two heads, "w'+1"'] & (W'+X)\times_{W'}(W'+Y) \arrow[d, two heads, "(w'+1)\times_{w'}(w'+1)"] & W'+Y \arrow[l, "\langle i_1[1,0],1\rangle"'] \arrow[d, two heads, "w'+1"] \\
W+X \arrow[r, "\langle 1, i_1[1,0]\rangle"] \arrow[dr, "{[w,x]}"'] & (W+X)\times_{W}(W+Y) \arrow[d, "m"] & W+Y \arrow[l, "\langle i_1[1,0],1\rangle"'] \arrow[dl, "{[w,y]}"] \\
& A &
\end{tikzcd}$$

is an internal multiplication $X \times Y \longrightarrow A$ over ww'. Conversely, if x and y commute over ww', and m' is an internal multiplication $X \times Y \longrightarrow A$ over ww', by pre-composing with the jointly epimorphic pair $[i_1, i_2] : W' + X \longrightarrow W' + X + Y$ and $[i_1, i_3] : W' + Y \longrightarrow W' + X + Y$ it can be seen that the outer part of the diagram

$$\begin{tikzcd}
W'+X+Y \arrow[rr, two heads, "w'+1+1"] \arrow[d, two heads, "{[\langle 1,i_1[1,0]\rangle, \langle i_1[1,0],1\rangle]}"'] & & W+X+Y \arrow[d, two heads, "{[\langle 1,i_1[1,0]\rangle, \langle i_1[1,0],1\rangle]}"'] \arrow[ddr, bend left, "{[w,x,y]}"] & \\
(W'+X)\times_{W'}(W'+Y) \arrow[rr, two heads, "(w'+1)\times_{w'}(w'+1)"'] \arrow[rrrd, bend right, "m'"'] & & (W+X)\times_{W}(W+Y) \arrow[dr, dashed, "m"] & \\
& & & A
\end{tikzcd}$$

commutes. But since the rectangle is a pushout, there is a factorization m of $[w, x, y]$ through $[\langle 1, i_1[1, 0]\rangle, \langle i_1[1, 0], 1\rangle]$, and this is equivalent to saying that x and y commute over w. $\square$

3.2 Regular images of weighted commutators under arbitrary morphisms

It has been shown already in [17] that the weighted commutators are preserved by normal-epimorphic images. In this section we investigate images of weighted commutators under arbitrary morphisms.

Definition 3.2.1 (see [17]). *Let $(X, x), (Y, y)$, and (W, w) be subobjects of an object A in a normal Mal'cev category $\mathbb{C}$ with finite colimits. The* ***weighted subobject commutator*** *$[(X, x),(Y, y)]_{(W,w)}$ is obtained as the image under $[w, x, y] : W + X + Y \longrightarrow A$ of the kernel of the morphism $[\langle 1, i_1[1, 0]\rangle, \langle i_1[1, 0], 1\rangle] : W + X + Y \longrightarrow (W + X) \times_W (W + Y)$*

$$\begin{tikzcd} X \otimes^W Y \arrow[r, two heads] \arrow[d, tail] & {[(X,x),(Y,y)]_{(W,w)}} \arrow[d, tail] \\ W+X+Y \arrow[r, "{[w,x,y]}"'] & A. \end{tikzcd}$$

Remark 3.2.2. *The* ***0-weighted subobject commutator*** *of* (X, x) *and* (Y, y)*, that is when* $W = 0$*, is denoted by* $[(X, x), (Y, y)]_0$*, and it is exactly the Higgins commutator of* (X, x) *and* (Y, y)*.*

When a pair of morphisms commute in the sense of Huq, and when two equivalence relations centralize each other in the sense of Smith, both situations correspond to the respective commutators being trivial. Let us observe next that the same thing happens between weighted centrality and the weighted subobject commutator in a normal Mal'cev category $\mathbb{C}$ with finite colimits. If (X, x) and (Y, y) commute over (W, w), and $m : (W + X) \times_W (W + Y) \to A$ is an internal multiplication $X \times Y \longrightarrow A$ over (W, w), then the bottom square in the diagram

$$\begin{tikzcd} X \otimes^W Y \arrow[r, two heads] \arrow[d, tail] & {[(X,x),(Y,y)]_{(W,w)}} \arrow[d, tail] \\ W+X+Y \arrow[r, "{[w,x,y]}"] \arrow[d, two heads, "{\langle [i_1,i_2,0],[i_1,0,i_2] \rangle}"'] & A \arrow[d, two heads, "1"] \\ (W+X)\times_W (W+Y) \arrow[r, "m"'] & A \end{tikzcd}$$

is a pushout, and this implies $[(X, x), (Y, y)]_{(W,w)} = 0$ (being inside the kernel of the identity morphism of A). On the other hand, in the previous diagram if $[(X, x), (Y, y)]_{(W,w)} = 0$, then the composite $X \otimes^W Y \rightarrowtail W + X + Y \longrightarrow A$ is the zero morphism, and thus, by definition of cokernel there is a morphism $m : (W + X) \times_W (W + Y) \to A$ making the bottom square commute. But this means m is an internal multiplication $X \times Y \longrightarrow A$ over (W, w).

Definition 3.2.3 (see [17])**.** *Let* (X, x)*,* (Y, y)*, and* (W, w) *be subobjects of an object* A *in a normal Mal'cev category* $\mathbb{C}$ *with finite colimits. The* ***weighted normal commutator*** $N[(X, x), (Y, y)]_{(W,w)}$ *is the kernel of* q *defined via the pushout*

$$\begin{tikzcd} W+X+Y \arrow[r, "{[w,x,y]}"] \arrow[d, "{\langle [i_1,i_2,0],[i_1,0,i_2] \rangle}"'] & A \arrow[d, "q"] \\ (W+X)\times_W (W+Y) \arrow[r] & Q, \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

or equivalently, the kernel of q in the diagram

$$\begin{tikzcd}
 & W+X \arrow[ld, "{\langle 1, i_1[1,0]\rangle}"'] \arrow[d, dotted] \arrow[rd, "{[w,x]}"] & \\
(W+X)\times_W (W+Y) \arrow[r, dotted, "\beta"] & \tilde{Q} & A \arrow[l, dotted, two heads, "\tilde{q}"'] \\
 & W+Y \arrow[lu, "{\langle i_1[1,0], 1\rangle}"] \arrow[u, dotted] \arrow[ru, "{[w,y]}"'] &
\end{tikzcd} \tag{3.3}$$

where Q is the colimit of the outer morphisms.

Remark 3.2.4 (see [17], Theorem 3.4)**.** *For subobjects (X,x), (Y,y), and (W,w) in a normal Mal'cev category $\mathbb{C}$ with finite colimits, applying Lemma 2.2.3, the morphism q in the diagram below is the cokernel of the weighted subobject commutator $[(X,x),(Y,y)]_{(W,w)}$. Therefore the kernel of q, which is the weighted normal commutator $N[(X,x),(Y,y)]_{(W,w)}$ by definition, is the normal closure of the weighted subobject commutator $[(X,x),(Y,y)]_{(W,w)}$*

$$\begin{tikzcd}
X\otimes^W Y \arrow[r, two heads] \arrow[d, tail] & {[(X,x),(Y,y)]_{(W,w)}} \arrow[d, tail] \\
W+X+Y \arrow[r, "{[w,x,y]}"] \arrow[d, two heads, "{\langle [i_1,i_2,0],[i_1,0,i_2]\rangle}"'] & A \arrow[d, two heads, "q"] \\
(W+X)\times_W (W+Y) \arrow[r] & Q. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Again when $W=0$ or w is the identity morphism of A in Definition 3.2.3, the respective commutators are denoted by $N[(X,x),(Y,y)]_0$ and $N[(X,x),(Y,y)]_1$, and called $0-$weighted normal commutator and $1-$weighted normal commutator respectively. Note that the $0-$weighted normal commutator is exactly the Huq commutator. We shall describe the $1-$weighted normal commutator (for normal subobjects) in the next section. For now let us observe that, taking w to be the identity morphism of A, and $\mathbb{C}$ to be an ideal-determined Mal'cev category (where normal monomorphisms are preserved by images under regular epimorphisms), in the diagram

$$\begin{tikzcd}
X \otimes^{A} Y \arrow[r, two heads] \arrow[d, tail] & {[(X,x),(Y,y)]_1} \arrow[d, tail] \\
A+X+Y \arrow[r, two heads, "{[1,x,y]}"] \arrow[d, two heads, "{\langle [i_1,i_2,0],[i_1,0,i_2] \rangle}"'] & A \arrow[d, two heads, "q"] \\
(A+X)\times_A (A+Y) \arrow[r, two heads] & Q,
\end{tikzcd}$$

the square at the bottom being a pushout amounts to q being the cokernel of its kernel

$$[(X,x),(Y,y)]_1 \rightarrowtail A.$$

Note that $[(X,x),(Y,y)]_1 \rightarrowtail A$ is a normal subobject of A, since it is the image of the normal subobject $X \otimes^A Y \rightarrowtail A+X+Y$ under the normal epimorphism $[1,x,y]$. But since $N[(X,x),(Y,y)]_1 \rightarrowtail A$ is the kernel of q by definition of $1-$weighted normal commutator, one must have $[(X,x),(Y,y)]_1 = N[(X,x),(Y,y)]_1$ in an ideal-determined Mal'cev category $\mathbb{C}$. Therefore, we have the following remark:

Remark 3.2.5 (see [17], Corollary 3.5)**.** *The $1-$weighted subobject and $1-$weighted normal commutators always coincide in an ideal-determined Mal'cev category $\mathbb{C}$.*

We shall use the following data in the next theorem: A normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, subobjects $(X,x),(Y,y)$, and (W,w) of A and their images $(X',x'),(Y',y')$, and (W',w') respectively, under a morphism $f: A \longrightarrow B$ as shown in the diagram

$$\begin{tikzcd}
 & W \arrow[rrr, two heads, "h"] \arrow[dd, tail, "w"] & & & W' \arrow[dd, tail, "w'"] & \\
X \arrow[rrr, two heads, "l"] \arrow[dr, tail, "x"] & & & X' \arrow[dr, tail, "x'"] & & \\
 & A \arrow[rrr, "f"] & & & B & \\
 & & Y \arrow[ul, tail, "y"] \arrow[rrr, two heads, "k"] & & & Y'. \arrow[ul, tail, "y'"]
\end{tikzcd}$$

Theorem 3.2.6. *For subobjects $(X,x),(Y,y)$, and (W,w) of A, and their respective images $(X',x'),(Y',y')$, and (W',w') under a morphism $f: A \longrightarrow B$ in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, the following hold:*

(a) $f([(X,x),(Y,y)]_{(W,w)}) = [(X',x'),(Y',y')]_{(W',w')}$;

(b) $\overline{f(N[(X,x),(Y,y)]_{(W,w)})} = N[(X',x'),(Y',y')]_{(W',w')}$.

Proof. (a) Using Lemma 3.1.7, the induced morphism $l \otimes^h k : X \otimes^W Y \longrightarrow X' \otimes^{W'} Y'$ is a normal epimorphism. Thus

$$f([(X,x),(Y,y)]_{(W,w)}) = [(X',x'),(Y',y')]_{(W',w')}$$

follows by the uniqueness of regular images as seen in the diagram

$$\begin{tikzcd}
& X' \otimes^{W'} Y' \arrow[rr, two heads] \arrow[dd, tail] & & f([(X,x),(Y,y)]_{(W,w)}) \arrow[dd, tail] \\
X \otimes^{W} Y \arrow[ur, two heads, "\mathfrak{l} \otimes^{\mathfrak{h}} \mathfrak{k}"] \arrow[rr, two heads] \arrow[dd, tail] & & {[(X,x),(Y,y)]_{(W,w)}} \arrow[ur, two heads] \arrow[dd, tail] & \\
& W' + X' + Y' \arrow[rr, two heads, "{[w',x',y']}"] & & B. \\
W + X + Y \arrow[ur, two heads, "\mathfrak{h}+\mathfrak{l}+\mathfrak{k}"] \arrow[rr, "{[w,x,y]}"] & & A \arrow[ur, two heads, "\mathfrak{f}"] &
\end{tikzcd}$$

(b) Since the weighted normal commutator is the normal closure of the weighted subobject commutator, we have

$$\begin{aligned}
N[(X',x'),(Y',y')]_{(W',w')} &= \overline{[(X',x'),(Y',y')]_{(W',w')}} \\
&= \overline{f([(X,x),(Y,y)]_{(W,w)})} \\
&= \overline{f(\overline{[(X,x),(Y,y)]_{(W,w)}})} \\
&= \overline{f(N[(X,x),(Y,y)]_{(W,w)})},
\end{aligned} \tag{3.4}$$

whereby Lemma 2.2.6 is applied to conclude that $\overline{f([(X,x),(Y,y)]_{(W,w)})} = \overline{f(\overline{[(X,x),(Y,y)]_{(W,w)}})}$. □

Remark 3.2.7. *In the notation of Theorem 3.2.6, taking $W = 0$ in (a) and (b), we recover Theorem 2.1.4 and Theorem 2.2.5 respectively. Taking w to be the identity morphism of A, $f([(X,x),(Y,y)]_1) = [(X',x'),(Y',y')]_1$ and $\overline{f(N[(X,x),(Y,y)]_1)} = N[(X',x'),(Y',y')]_1$ when f is a regular epimorphism and $\mathbb{C}$ is an ideal-determined Mal'cev category (see, for instance, Corollary* 5.3 *[17]).*

3.3 Huq commutator of local representations

In this section the Smith commutator of a pair of equivalence relations and the Huq commutator of the corresponding local representations are compared.

As explained before, if $\mathbb{C}$ is a normal Mal'cev category then for each object A in $\mathbb{C}$, $\mathsf{Pt}(A)$ is also a normal Mal'cev category. In addition, $\mathsf{Pt}(A)$ is Barr-exact if $\mathbb{C}$ is Barr-exact. Recall that the local representation of an equivalence relation (R, r_1, r_2) on an object A in a normal Barr-exact Mal'cev category $\mathbb{C}$ is not just an ordinary subobject in $\mathsf{Pt}(A)$, but in fact, it is the normal subobject in $\mathsf{Pt}(A)$ associated to the equivalence relation $((A \times R, \pi_1, \langle 1, \triangle_R \rangle), 1 \times r_1, 1 \times r_2)$ on $(A \times A, \pi_1, \langle 1, 1 \rangle)$

$$\begin{tikzcd}
A \times R \arrow[r, shift left, "1 \times r_1"] \arrow[r, shift right, "1 \times r_2"'] \arrow[d, shift left, "\pi_1"] & A \times A \arrow[d, shift left, "\pi_1"] \\
A \arrow[r, "1"'] \arrow[u, shift left, "\langle 1, \triangle_R \rangle"] & A. \arrow[u, shift left, "\langle 1, 1 \rangle"]
\end{tikzcd}$$

Indeed, as seen in the diagram

$$\begin{tikzcd}
R \arrow[r, "\langle r_1, 1 \rangle"] \arrow[d, "\langle r_1, r_2 \rangle"'] & A \times R \arrow[r, "1 \times r_1"] \arrow[d, "\langle 1 \times r_1, 1 \times r_2 \rangle"] & A \times A \arrow[d, "1"] \\
A \times A \arrow[r, "\langle \langle 1, 1 \rangle \pi_1, 1 \rangle"'] & (A \times A) \times (A \times A) \arrow[r, "\pi_1"'] & A \times A,
\end{tikzcd}$$

where the left hand rectangle is a pullback, the local representation of (R, r_1, r_2) is the "unit class" of the equivalence relation $((A \times R, \pi_1, \langle 1, \triangle_R \rangle), 1 \times r_1, 1 \times r_2)$. We shall investigate the relationship between the Smith commutator of equivalence relations and the Huq commutator of the corresponding local representations. It is well known that if the Smith commutator of a pair of equivalence relations is trivial then the Huq commutator of the corresponding local representations is also trivial. Before we make the above investigation, let us make some necessary observations.

Given a pair of equivalence relations (R', r'_1, r'_2) and (R, r_1, r_2) on an object A in a Mal'cev category $\mathbb{C}$, consider the pullback

$$\begin{tikzcd}
R' \times_A R \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "r_1"] \\
R' \arrow[r, "r'_2"'] & A
\end{tikzcd}$$

and the cospan

$$\begin{tikzcd}
R' \arrow[r, "\langle 1, \triangle_R r'_2 \rangle"] \arrow[d, shift left, "r'_2"] & R' \times_A R \arrow[d, shift left, "r'_2 \pi_1"] & R \arrow[l, "\langle \triangle_{R'} r_1, 1 \rangle"'] \arrow[d, shift left, "r_1"] \\
A \arrow[r, "1"'] \arrow[u, shift left, "\triangle_{R'}"] & A \arrow[u, shift left, "\langle \triangle_{R'}, \triangle_R \rangle"] & A \arrow[l, "1"] \arrow[u, shift left, "\triangle_R"]
\end{tikzcd}$$

in $\mathsf{Pt}(A)$. Note that the pair of morphisms $\langle 1, \triangle_R r_2' \rangle : R' \longrightarrow R' \times_A R$ and $\langle \triangle_{R'} r_1, 1 \rangle : R \longrightarrow R' \times_A R$ is jointly extremal-epimorphic in $\mathbb{C}$ since the cospan above is jointly extremal-epimorphic in $\mathsf{Pt}(A)$.

For morphisms $\tilde{q} : A \longrightarrow \tilde{Q}$, $q' : A \longrightarrow Q'$, and $\lambda : A \times \tilde{Q} \longrightarrow A \times Q'$ in a normal category $\mathbb{C}$ such that $\pi_1 \lambda = \pi_1$ and $\lambda(1 \times \tilde{q}) = 1 \times q'$, in the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\tilde{q}"] \arrow[dd, "\langle 0,1 \rangle"] \arrow[drr, bend right, "q'"] & \tilde{Q} \arrow[dr, "\rho"] \arrow[dd, "\langle 0,1 \rangle"] & \\
 & & Q' \arrow[dd, "\langle 0,1 \rangle"] \\
A \times A \arrow[r, "1 \times \tilde{q}"] \arrow[dd, "\pi_1"] \arrow[drr, bend right, "1 \times q'"] & A \times \tilde{Q} \arrow[dr, "\lambda"] \arrow[dd, "\pi_1"] & \\
 & & A \times Q' \arrow[dd, "\pi_1"] \\
A \arrow[r, "1"'] & A \arrow[dr, "1"'] & \\
 & & A
\end{tikzcd}$$

in which ρ is induced by λ through the universal property of kernels, it can be seen that $q' = \rho\tilde{q}$, and in addition, if $\tilde{q}$ is a regular epimorphism then $\lambda = 1 \times \rho$.

Now we can describe the Smith commutator of equivalence relations in terms of the Huq commutator of the corresponding local representations.

Theorem 3.3.1. *Let (R, r_1, r_2) and (R', r_1', r_2') be two equivalence relations on an object A in a normal Mal'cev category $\mathbb{C}$ with finite colimits. The Smith commutator $[R', R]_{\mathsf{S}}$ is the Huq commutator of the local representations of (R, r_1, r_2) and (R', r_1', r_2').*

Proof. Consider the diagram

$$\begin{tikzcd}
& (R, r_1, \triangle_R) \arrow[dl, "\langle \triangle_{R'} r_1, 1\rangle"'] \arrow[d, dotted, "u"'] \arrow[dr, tail, "\langle r_1, r_2\rangle"] & \\
(R' \times_A R, r_2'\pi_1, \langle \triangle_{R'}, \triangle_R\rangle) \arrow[r, dotted, "\varphi"] & (Q, \tau, q\langle 1,1\rangle) & (A \times A, \pi_1, \langle 1,1\rangle) \arrow[l, dotted, "q"'] \\
& (R', r_2', \triangle_{R'}) \arrow[ul, "\langle 1, \triangle_R r_2'\rangle"] \arrow[u, dotted, "v"] \arrow[ur, tail, "\langle r_2', r_1'\rangle"'] &
\end{tikzcd} \tag{3.5}$$

in $\mathsf{Pt}(A)$, where $(Q, \tau, q\langle 1,1\rangle)$ is the colimit of the solid arrows. The morphism q is the universal arrow which, by composition, makes the local representations $\langle r_1, r_2\rangle$ and $\langle r_2', r_1'\rangle$ commute in the sense of Huq (see e.g Proposition 1.9 [5]). In addition, from [5] we know that q is a regular epimorphism, and hence a normal epimorphism. So applying Corollary 1.3.9 through Remark 1.3.10, Q and q can be chosen to be of the forms $A \times \tilde{Q}$ and $1 \times \tilde{q}$ respectively, where $\tilde{q}$ is a normal epimorphism in $\mathbb{C}$. Therefore, the colimit of the outer arrows in diagram (3.5) can be chosen to be of the form $(A \times \tilde{Q}, \pi_1, \langle 1, \tilde{q}\rangle)$, and so diagram (3.5) transforms into the diagram

$$\begin{tikzcd}
& (R, r_1, \triangle_R) \arrow[dl, "\langle \triangle_{R'} r_1, 1\rangle"'] \arrow[dr, tail, "\langle r_1, r_2\rangle"] & \\
(R' \times_A R, r_2'\pi_1, \langle \triangle_{R'}, \triangle_R\rangle) \arrow[r, "\phi = \langle \alpha, \beta\rangle"] \arrow[dr, dotted, "\langle r_2'\pi_1, \beta'\rangle"] & (A \times \tilde{Q}, \pi_1, \langle 1, \tilde{q}\rangle) \arrow[d, dashed, "\lambda"'] & (A \times A, \pi_1, \langle 1,1\rangle) \arrow[l, "1 \times \tilde{q}"'] \arrow[dl, dotted, "1 \times q'"'] \\
& (A \times Q', \pi_1, \langle 1, q'\rangle) & \\
& (R', r_2', \triangle_{R'}). \arrow[uul, "\langle 1, \triangle_R r_2'\rangle"] \arrow[uur, tail, "\langle r_2', r_1'\rangle"'] &
\end{tikzcd} \tag{3.6}$$

It is not difficult to see that the diagram of solid arrows

$$
\begin{tikzcd}
& R \arrow[dl, "\langle \triangle_{R'} r_1, 1\rangle"'] \arrow[dr, "r_2"] & \\
R' \times_A R \arrow[r, "\beta"] \arrow[dr, dotted, "\beta'"] & \tilde{Q} & A \arrow[l, "\tilde{q}"'] \arrow[dl, dotted, "q'"'] \\
& Q' & \\
& R' \arrow[uul, "\langle 1, \triangle_R r'_2\rangle"] \arrow[uur, "r'_1"'] &
\end{tikzcd}
\tag{3.7}
$$

obtained from diagram (3.6) by composing further with $\pi_2 : A \times \tilde{Q} \longrightarrow \tilde{Q}$, commutes. Now if $q' : A \longrightarrow Q'$ and $\beta' : R' \times_A R \longrightarrow Q'$ are morphisms making diagram (3.7) commute, then clearly the morphisms $1 \times q'$ and $\langle r'_2\pi_1, \beta'\rangle$ also make diagram (3.6) commute, and this implies that there is a factorization λ such that $1 \times q' = \lambda(1 \times \tilde{q})$. As explained before (just prior to this result), there is a morphism $\rho : \tilde{Q} \longrightarrow Q'$ such that $1 \times \rho = \lambda$ and $\rho\tilde{q} = q'$. Thus $\tilde{Q}$ is the colimit of the outer morphisms in diagram (3.7). By definition of the Smith commutator, the kernel pair relation of $\tilde{q}$ is the Smith commutator $[R', R]_{\mathsf{S}}$, but according to Proposition 1.3.11, the kernel pair relation of $\tilde{q}$ is also the kernel of $1 \times \tilde{q}$ in $\mathsf{Pt}(A)$. □

For normal subobjects (X, x) and (Y, y) of A in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, we have already observed that their associated equivalence relations (R_X, r_1, r_2) and (R_Y, r'_1, r'_2) respectively, are given by the regular images of the morphisms

$$\begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix} \quad \text{and} \begin{bmatrix} 1 & 1 \\ 0 & y \end{bmatrix}$$

respectively. As mentioned before, there is no a difference between the Huq commutator defined for a pair of morphisms f and g (having the same codomain), and the Huq commutator defined on their respective regular images. So in the diagram

$$
\begin{tikzcd}
& (A+X, [1,0], i_1) \arrow[dl, "\langle 1, i_1[1,0]\rangle"'] \arrow[d, dotted, "u"] \arrow[dr, "\begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix}"] & \\
((A+X) \times_A (A+Y), [1,0]\pi_1, \langle i_1, i_1\rangle) \arrow[r, dotted, "\varphi"] & (Q, \tau, q\langle 1,1\rangle) & (A \times A, \pi_1, \langle 1,1\rangle) \arrow[l, dotted, "q"'] \\
& (A+Y, [1,0], i_1) \arrow[ul, "\langle i_1[1,0], 1\rangle"] \arrow[u, dotted, "v"'] \arrow[ur, "\begin{bmatrix} 1 & 1 \\ 0 & y \end{bmatrix}"'] &
\end{tikzcd}
\tag{3.8}
$$

q is the universal arrow which, by composition, makes the morphisms

$$\begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & y \end{bmatrix}$$

Huq-commute in $\mathsf{Pt}(A)$, if and only if, it is the universal arrow which, by composition, makes their respective regular images $\langle r_2, r_1\rangle$ and $\langle r_1', r_2'\rangle$ (which are also the local representations of (R_X, r_1, r_2) and (R_Y, r_1', r_2') respectively), Huq-commute in $\mathsf{Pt}(A)$. This essentially means that $(Q, \tau, q\langle 1,1\rangle)$ is the colimit of the solid arrows in diagram (3.8) if and only if $(Q, \tau, q\langle 1,1\rangle)$ is the colimit of the solid arrows in diagram (3.5), with R and R' replaced by R_X and R_Y respectively. We have seen in the previous theorem that if $(Q, \tau, q\langle 1,1\rangle)$ is the colimit of the solid arrows in diagram (3.5), then Q and q can be chosen to be of the forms $A \times \tilde{Q}$ and $1 \times \tilde{q}$ respectively. Furthermore, the kernel of $1 \times \tilde{q}$ in $\mathsf{Pt}(A)$ is the local representation of the Smith commutator of (R_X, r_1, r_2) and (R_Y, r_1', r_2').

Now we are ready to characterize the Smith commutator in terms of $1-$weighted normal commutator (for normal subobjects).

The next result has been already proven in [17] (see also Theorem 2.3.3), so we just present a different proof.

Corollary 3.3.2. *Let (X, x) and (Y, y) be normal subobjects of A and, (R_X, r_1, r_2) and (R_Y, r_1', r_2') be their associated equivalence relations respectively, in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits. The $1-$weighted normal commutator $N[(X,x),(Y,y)]_1$ is the associated normal subobject of the Smith commutator $[R_X, R_Y]_{\mathsf{S}}$.*

Proof. Given that $(A \times \tilde{Q}, \pi_1, \langle 1, \tilde{q}\rangle)$ is the colimit of the solid arrows in the diagram

$$\begin{tikzcd}
 & & (A+X,[1,0],i_1) \arrow[dll, "{\langle 1, i_1[1,0]\rangle}"'] \arrow[d, dotted, "u"] \arrow[drr, "{\begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix}}"] & & \\
((A+X)\times_A (A+Y), [1,0]\pi_1, \langle i_1, i_1\rangle) \arrow[rr, dotted, "{\varphi = \langle \alpha, \beta\rangle}"] & & (A\times \tilde{Q}, \pi_1, \langle 1, \tilde{q}\rangle) & & (A\times A, \pi_1, \langle 1,1\rangle) \arrow[ll, dotted, "1\times \tilde{q}"'] \\
 & & (A+Y,[1,0],i_1) \arrow[ull, "{\langle i_1[1,0], 1\rangle}"] \arrow[u, dotted, "v"'] \arrow[urr, "{\begin{bmatrix} 1 & 1 \\ 0 & y \end{bmatrix}}"'] & &
\end{tikzcd} \tag{3.9}$$

in $\mathsf{Pt}(A)$, in a similar way as in the previous theorem, it can be shown that $\tilde{Q}$ is the colimit of the solid arrows in the diagram

$$
\begin{tikzcd}
& A+X \arrow[dl, "{\langle 1, i_1[1,0]\rangle}"'] \arrow[d, dotted] \arrow[dr, "{[1,x]}"] & \\
(A+X)\times_A (A+Y) \arrow[r, dotted, "\beta"] & \tilde{Q} & A \arrow[l, dotted, two heads, "\tilde{q}"'] \\
& A+Y \arrow[ul, "{\langle i_1[1,0], 1\rangle}"] \arrow[u, dotted] \arrow[ur, "{[1,y]}"'] &
\end{tikzcd}
\tag{3.10}
$$

in $\mathbb{C}$. The kernel of $\tilde{q}$ is the $1-$weighted normal commutator $N[(X,x),(Y,y)]_1$, and, as explained immediately after the previous theorem, the kernel of $1 \times \tilde{q}$ in $\mathsf{Pt}(A)$ is the local representation of the Smith commutator $[R_X, R_Y]_{\mathsf{S}}$ (given by kernel pair relation of $\tilde{q}$, see Proposition 1.3.11). So $\tilde{q}$ is the quotient of the Smith commutator $[R_X, R_Y]_{\mathsf{S}}$, and hence its kernel $N[(X,x),(Y,y)]_1$ is the associated normal subobject of $[R_X, R_Y]_{\mathsf{S}}$. □

3.4 Characterization of weighted commutators

In this section we give, in the context of normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, characterization of weighted commutators. In addition, the known characterizations of Huq, Higgins, and Ursini/Smith commutators will be recovered.

Let us first show that weighted commutators, as ternary operations

$$[-,-,-] : \mathsf{Sub}(A) \times \mathsf{Sub}(A) \times \mathsf{Sub}(A) \longrightarrow \mathsf{Sub}(A),$$

defined on all subobjects of each object A, are monotone: For $(X,x), (X',x'), (Y,y), (Y',y'), (W',w')$, and (W,w) subobjects of A in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, if $W' \leq W$, $X' \leq X$, and $Y' \leq Y$, then there exist morphisms α, β, and λ such that $w' = w\alpha$, $x' = x\beta$, and $y' = y\lambda$. In addition, one obtains the following commutative diagram

$$
\begin{tikzcd}
& X \otimes^W Y \arrow[rr, two heads, "e"] \arrow[dd, tail] & & {[(X,x),(Y,y)]_{(W,w)}} \arrow[dd, tail] \\
X' \otimes^{W'} Y' \arrow[ur, "\beta \otimes^{\alpha} \lambda"] \arrow[rr, two heads, "e'"] \arrow[dd, tail] & & {[(X',x'),(Y',y')]_{(W',w')}} \arrow[ur, dashed] \arrow[dd, tail] & \\
& W+X+Y \arrow[rr, "{[w,x,y]}"] & & A \\
W'+X'+Y' \arrow[ur, "\alpha+\beta+\lambda"] \arrow[rr, "{[w',x',y']}"] & & A \arrow[ur, "1"] &
\end{tikzcd}
$$

from which, using the fact that e' is a strong epimorphism, it can be seen that

$$[(X',x'),(Y',y')]_{(W',w')} \leq [(X,x),(Y,y)]_{(W,w)}.$$

From the previous inequality we deduce the same for the weighted normal commutator, that is,

$$N[(X',x'),(Y',y')]_{(W',w')} \leq N[(X,x),(Y,y)]_{(W,w)},$$

by applying the following facts: (a) the weighted normal commutator is the normal closure of the weighted subobject commutator; (b) applying normal closure preserves the order.

Proposition 3.4.1. *Let (X,x), (Y,y), and (W,w) be subobjects of A in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits. Then one has $N[(X,x),(Y,y)]_{(W,w)} \leq \overline{X} \wedge \overline{Y}$ and $[(X,x),(Y,y)]_{(W,w)} \leq \overline{X} \wedge \overline{Y}$.*

Proof. It is not difficult to see that the outer part of the diagram

$$\begin{tikzcd}
W+X+Y \arrow[rr, "{[w,x,y]}"] \arrow[d, two heads, "{\langle [i_1,i_2,0],[i_1,0,i_2]\rangle}"'] & & A \arrow[d, two heads, "q"] \arrow[ddr, bend left, two heads, "\mathrm{coker}(x)"] & \\
(W+X)\times_W(W+Y) \arrow[rr] \arrow[dr, bend right, "\pi_2"'] & & Q \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
& W+Y \arrow[rr, bend right, "{\mathrm{coker}(x)[w,y]}"'] & & \mathrm{Coker}(x)
\end{tikzcd}$$

commutes. Therefore, by the universal property of pushouts, it immediately follows that $\mathrm{coker}(x)$ factors through q and this implies $N[(X,x),(Y,y)]_{(W,w)} \leq \overline{X}$, since $(\overline{X},\overline{x})$ is the kernel of $\mathrm{coker}(x)$ and the weighted normal commutator $N[(X,x),(Y,y)]_{(W,w)}$ is the kernel of q. Similarly, it can be shown that $N[(X,x),(Y,y)]_{(W,w)} \leq \overline{Y}$, hence $N[(X,x),(Y,y)]_{(W,w)} \leq \overline{X}\wedge\overline{Y}$. The second part of the theorem follows from this inequality:

$$[(X,x),(Y,y)]_{(W,w)} \leq N[(X,x),(Y,y)]_{(W,w)} \leq \overline{X}\wedge\overline{Y}.$$

□

Let X,Y, and W be objects in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits. We shall denote by $\ker[i_1,i_2,0] : \mathrm{Ker}[i_1,i_2,0] \rightarrowtail W+X+Y$ and $\ker[i_1,0,i_2] : \mathrm{Ker}[i_1,0,i_2] \rightarrowtail W+X+Y$, the kernels of the morphisms $[i_1,i_2,0] : W+X+Y \longrightarrow W+X$ and $[i_1,0,i_2] : W+X+Y \longrightarrow W+Y$ respectively. Let us observe the following:

Lemma 3.4.2. *For objects X,Y, and W in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, $\mathrm{Ker}[i_1,i_2,0]\wedge\mathrm{Ker}[i_1,0,i_2] = X\otimes^W Y$.*

Proof. Since $X\otimes^W Y$ is the kernel of the morphism $\langle[i_1,i_2,0],[i_1,0,i_2]\rangle$, in the diagram below it is not difficult to see that $X\otimes^W Y \leq \mathrm{Ker}[i_1,i_2,0]$ and $X\otimes^W Y \leq \mathrm{Ker}[i_1,0,i_2]$, and thus $X\otimes^W Y \leq \mathrm{Ker}[i_1,i_2,0]\wedge\mathrm{Ker}[i_1,0,i_2]$ follows

$$\begin{tikzcd}
X \otimes^W Y \arrow[r] \arrow[d] \arrow[dr, tail] & \mathrm{Ker}[i_1,i_2,0] \arrow[d, tail, "{\ker[i_1,i_2,0]}"] & \\
\mathrm{Ker}[i_1,0,i_2] \arrow[r, tail, "{\ker[i_1,0,i_2]}"'] & W+X+Y \arrow[r, "{[i_1,0,i_2]}"] \arrow[d, "{[i_1,i_2,0]}"'] \arrow[dr, "{\langle[i_1,i_2,0],[i_1,0,i_2]\rangle}"] & W+Y \\
& W+X & (W+X)\times_W(W+Y). \arrow[l, "\pi_1"] \arrow[u, "\pi_2"']
\end{tikzcd}$$

On the other hand, if one replaces $X \otimes^W Y$ with $\mathrm{Ker}[i_1,i_2,0] \wedge \mathrm{Ker}[i_1,0,i_2]$ in the diagram above, then by the universal property of products, the morphism $\mathrm{Ker}[i_1,i_2,0] \wedge \mathrm{Ker}[i_1,0,i_2] \longrightarrow W+X+Y$ equals the zero morphism after composing with $\langle[i_1,i_2,0],[i_1,0,i_2]\rangle$, and this implies $\mathrm{Ker}[i_1,i_2,0] \wedge \mathrm{Ker}[i_1,0,i_2] \leq X \otimes^W Y$. Hence the desired result follows. $\square$

For objects X, Y, and W in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, their coproduct inclusions $i_1 : W \longrightarrow W+X+Y$, $i_2 : X \longrightarrow W+X+Y$, and $i_3 : Y \longrightarrow W+X+Y$ form the following weighted cospan

$$\begin{tikzcd}
& W \arrow[d, "i_1"] & \\
X \arrow[r, "i_2"] & W+X+Y & Y. \arrow[l, "i_3"']
\end{tikzcd}$$

In the next lemma we show that the weighted subobject commutator $[(X,i_2),(Y,i_3)]_{(W,i_1)}$ and weighted normal commutator $N[(X,i_2),(Y,i_3)]_{(W,i_1)}$ coincide.

Lemma 3.4.3. *For objects X, Y, and W in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, we have*

$$[(X,i_2),(Y,i_3)]_{(W,i_1)} = N[(X,i_2),(Y,i_3)]_{(W,i_1)} = X \otimes^W Y.$$

Proof. The diagram

$$\begin{tikzcd}
W+X+Y \arrow[r, equal] \arrow[d, two heads, "{\langle[i_1,i_2,0],[i_1,0,i_2]\rangle}"'] & W+X+Y \arrow[d, two heads, "{\langle[i_1,i_2,0],[i_1,0,i_2]\rangle}"] \\
(W+X)\times_W(W+Y) \arrow[r, equal] & (W+X)\times_W(W+Y)
\end{tikzcd}$$

is clearly a pushout, and so by definition of weighted normal commutator, it follows that $N[(X,i_2),(Y,i_3)]_{(W,i_1)} = X \otimes^W Y$. Also by definition of weighted subobject commutator, it is clear that $[(X,i_2),(Y,i_3)]_{(W,i_1)} = X \otimes^W Y$.

$\square$

Again for objects X, Y, and W in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, there is also the following weighted cospan

$$\begin{tikzcd} & W \arrow[d, "i_1"] & \\ \mathrm{Ker}[i_1, i_2, 0] \arrow[r, "\mathrm{ker}[i_1,i_2,0]"] & W + X + Y & \mathrm{Ker}[i_1, 0, i_2] \arrow[l, "\mathrm{ker}[i_1,0,i_2]"'] \end{tikzcd}$$

in $\mathbb{C}$. For this weighted cospan, for simplicity, we will write $[\mathrm{Ker}[i_1, i_2, 0], \mathrm{Ker}[i_1, 0, i_2]]_{(W,i_1)}$ and $N[\mathrm{Ker}[i_1, i_2, 0], \mathrm{Ker}[i_1, 0, i_2]]_{(W,i_1)}$ to denote its weighted subobject commutator and weighted normal commutator respectively.

Proposition 3.4.4. *Let X, Y, and W be objects in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits. Then*

$$[\mathrm{Ker}[i_1, 0, i_2], \mathrm{Ker}[i_1, i_2, 0]]_{(W,i_1)} = N[\mathrm{Ker}[i_1, 0, i_2], \mathrm{Ker}[i_1, i_2, 0]]_{(W,i_1)} = X \otimes^W Y.$$

Proof. Since $X \leq \mathrm{Ker}[i_1, 0, i_2]$ and $Y \leq \mathrm{Ker}[i_1, i_2, 0]$ in $W + X + Y$, and weighted commutators are monotone, one has

$$\begin{aligned} X \otimes^W Y &= N[(X, i_2), (Y, i_3)]_{(W,i_1)} \\ &\leq N[\mathrm{Ker}[i_1, 0, i_2], \mathrm{Ker}[i_1, i_2, 0]]_{(W,i_1)} \\ &\leq \mathrm{Ker}[i_1, i_2, 0] \wedge \mathrm{Ker}[i_1, 0, i_2] \\ &= X \otimes^W Y. \end{aligned} \tag{3.11}$$

Similarly, since $X \otimes^W Y = [(X, i_2), (Y, i_3)]_{(W,i_1)}$, it can be shown that

$$[\mathrm{Ker}[i_1, 0, i_2], \mathrm{Ker}[i_1, i_2, 0]]_{(W,i_1)} = X \otimes^W Y.$$

□

Let us take (X, x), (Y, y), and (W, w) to be subobjects of A in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits. Since $X \leq \mathrm{Ker}[i_1, 0, i_2]$ and $Y \leq \mathrm{Ker}[i_1, i_2, 0]$ in $W + X + Y$, in the diagram

$$\begin{tikzcd} & \mathrm{Ker}[i_1, 0, i_2] \arrow[r, two heads] \arrow[d, tail] & {[w, x, y](\mathrm{Ker}[i_1, 0, i_2])} \arrow[d, tail] \\ X \arrow[ur] \arrow[r, "i_2"] \arrow[rr, tail, bend right, "x"'] & W + X + Y \arrow[r, "{[w,x,y]}"] & A \end{tikzcd}$$

we see that $X \leq [w, x, y](\mathrm{Ker}[i_1, 0, i_2])$. In a similar way it can be shown that $Y \leq [w, x, y](\mathrm{Ker}[i_1, i_2, 0])$. For the weighted cospan

$$\begin{tikzcd} & W \arrow[d, tail, "w"] & \\ {[w,x,y](\mathrm{Ker}[i_1,i_2,0])} \arrow[r, tail] & A & {[w,x,y](\mathrm{Ker}[i_1,0,i_2]),} \arrow[l, tail] \end{tikzcd}$$

we will denote its weighted subobject commutator and weighted normal commutator by

$$[[w,x,y](\mathrm{Ker}[i_1,0,i_2]),[w,x,y](\mathrm{Ker}[i_1,i_2,0])]_{(W,w)}$$

and

$$N[[w,x,y](\mathrm{Ker}[i_1,0,i_2]),[w,x,y](\mathrm{Ker}[i_1,i_2,0])]_{(W,w)}$$

respectively.

Then we have the following useful result.

Proposition 3.4.5. *For subobjects (X,x), (Y,y), and (W,w) of A in a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, one has*

$$[[w,x,y](\mathrm{Ker}[i_1,0,i_2]),[w,x,y](\mathrm{Ker}[i_1,i_2,0])]_{(W,w)} = [(X,x),(Y,y)]_{(W,w)}$$

and

$$N[[w,x,y](\mathrm{Ker}[i_1,0,i_2]),[w,x,y](\mathrm{Ker}[i_1,i_2,0])]_{(W,w)} = N[(X,x),(Y,y)]_{(W,w)}.$$

Proof. Applying Theorem 3.2.6 (a), we have

$$\begin{aligned}[][[w,x,y](\mathrm{Ker}[i_1,0,i_2]),[w,x,y](\mathrm{Ker}[i_1,i_2,0])]_{(W,w)} &= [w,x,y]([\mathrm{Ker}[i_1,0,i_2],\mathrm{Ker}[i_1,i_2,0]]_{(W,i_1)}) \\ &= [w,x,y](X\otimes^W Y) \\ &= [(X,x),(Y,y)]_{(W,w)}.\end{aligned} \tag{3.12}$$

We obtain the second part of the theorem by applying normal closure to the above equation. □

Now we are ready to state our main results for this section.

Theorem 3.4.6. *In a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, the weighted subobject commutator is the largest ternary operation C on subobjects (defined on all subobjects of each object) satisfying the following conditions:*

(a) C is monotone;

(b) $C((X,x),(Y,y),(W,w)) \leq \overline{X}\wedge\overline{Y}$ for subobjects $(X,x),(Y,y)$, and (W,w) of A;

(c) $f(C((X,x),(Y,y),(W,w))) = C((X',x'),(Y',y'),(W',w'))$ for subobjects $(X,x),(Y,y)$, and (W,w) of A, and their respective images $(X',x'),(Y',y')$, and (W',w'), under every morphism f whose domain is A.

Proof. Suppose there is a monotone ternary operation $[|-,-,-|]$ satisfying the conditions $(a),(b)$, and (c) above. Let $(X,x),(Y,y)$, and (W,w) be subobjects of A. We already know that $X \leq [w,x,y](\mathrm{Ker}[i_1,0,i_2]), Y \leq [w,x,y](\mathrm{Ker}[i_1,i_2,0])$, and $W = ([w,x,y](i_1(W))$, and thus

$$\begin{aligned}
[|(X,x),(Y,y),(W,w)|] &\leq [|[w,x,y](\mathrm{Ker}[i_1,0,i_2]),[w,x,y](\mathrm{Ker}[i_1,i_2,0]),[w,x,y](i_1(W))|] \\
&= [w,x,y]([|\,\mathrm{Ker}[i_1,0,i_2],\mathrm{Ker}[i_1,i_2,0],i_1(W)|]) \\
&\leq [w,x,y](\mathrm{Ker}[i_1,0,i_2] \wedge \mathrm{Ker}[i_1,i_2,0]) \\
&= [w,x,y](X \otimes^W Y) \\
&= [(X,x),(Y,y)]_{(W,w)}.
\end{aligned} \tag{3.13}$$

□

In a similar way we obtain a characterization of the weighted normal commutator below:

Theorem 3.4.7. *In a normal Barr-exact Mal'cev category $\mathbb{C}$ with finite colimits, the weighted normal commutator is the largest ternary operation C on subobjects (defined on all subobjects of each object) satisfying the following conditions:*

(a) C is monotone;

(b) $C((X,x),(Y,y),(W,w)) \leq \overline{X} \wedge \overline{Y}$ for subobjects $(X,x),(Y,y)$, and (W,w) of A;

(c) $\overline{f(C((X,x),(Y,y),(W,w)))} = C((X',x'),(Y',y'),(W',w'))$ for subobjects $(X,x),(Y,y)$, and (W,w) of A, and their respective images $(X',x'),(Y',y')$, and (W',w'), under every morphism f whose domain is A.

Proof. Suppose there is a monotone ternary operation $[|-,-,-|]$ satisfying the conditions $(a),(b)$, and (c) above. Let $(X,x),(Y,y)$, and (W,w) be subobjects of A. Then,

$$\begin{aligned}
[|(X,x),(Y,y),(W,w)|] &\leq [|[w,x,y](\mathrm{Ker}[i_1,0,i_2]),[w,x,y](\mathrm{Ker}[i_1,i_2,0]),[w,x,y](i_1(W))|] \\
&= \overline{[w,x,y]([|\,\mathrm{Ker}[i_1,0,i_2],\mathrm{Ker}[i_1,i_2,0],i_1(W)|])} \\
&\overline{\leq [w,x,y](\mathrm{Ker}[i_1,0,i_2] \wedge \mathrm{Ker}[i_1,i_2,0])} \\
&= \overline{[w,x,y](X \otimes^W Y)} \\
&= \overline{[(X,x),(Y,y)]_{(W,w)}} \\
&= N[(X,x),(Y,y)]_{(W,w)}.
\end{aligned} \tag{3.14}$$

□

Remark 3.4.8. *Taking $W = 0$ in Theorem 3.4.6 and Theorem 3.4.7, one obtains the characterizations of Higgins commutator (Theorem 2.1.5) and Huq commutator (Theorem 2.2.7) respectively.*

We state a similar characterization for the 1−weighted normal commutator in the next proposition.

Proposition 3.4.9. [1] *In an ideal-determined Barr-exact Mal'cev category $\mathbb{C}$, the $1-$weighted normal commutator is the largest binary operation C on subobjects (defined on all subobjects of each object) satisfying the following conditions:*

(a) *C is monotone;*

(b) *$C((X,x),(Y,y)) \leq \overline{X} \wedge \overline{Y}$ for subobjects (X,x) and (Y,y) of A;*

(c) *$f(C((X,x),(Y,y))) = C((X',x'),(Y',y'))$ for subobjects (X,x) and (Y,y) of A, and their respective images (X',x') and (Y',y'), under every* ***regular epimorphism*** *f whose domain is A.*

Proof. Let us suppose there is a binary operation $[|-,-|]$ defined on all subobjects of each object, satisfying the conditions $(a), (b)$, and (c) above. Now since $[1,x,y]$ is a normal epimorphism and $N[(X,x),(Y,y)]_1 = [(X,x),(Y,y)]_1$ in $\mathbb{C}$ an ideal-determined Mal'cev category, we have

$$\begin{aligned}
[|(X,x),(Y,y)|] &\leq [|[1,x,y](\mathrm{Ker}[i_1,0,i_2]), [1,x,y](\mathrm{Ker}[i_1,i_2,0])|] \\
&= [1,x,y]([|\,\mathrm{Ker}[i_1,0,i_2], \mathrm{Ker}[i_1,i_2,0]|]) \\
&\leq [1,x,y](\mathrm{Ker}[i_1,0,i_2] \wedge \mathrm{Ker}[i_1,i_2,0]) \\
&= [1,x,y](X \otimes^A Y) \\
&= [(X,x),(Y,y)]_1 \\
&= N[(X,x),(Y,y)]_1.
\end{aligned} \tag{3.15}$$

□

We will now use the previous proposition to observe that $1-$weighted normal commutator not only coincide with the Ursini commutator in the case of normal subobjects, but even for arbitrary subobjects in an ideal-determined Barr-exact Mal'cev category $\mathbb{C}$. A key tool to showing this is the fact that the Ursini commutator is invariant with respect to normal closures.

Corollary 3.4.10. *In an ideal-determined Barr-exact Mal'cev category $\mathbb{C}$, $1-$weighted normal commutator is the Ursini commutator.*

Proof. For subobjects (X,x) and (Y,y) of A in an ideal-determined Barr-exact Mal'cev category $\mathbb{C}$, $N[(X,x),(Y,y)]_1 \leq N[\overline{(X,x)},\overline{(Y,y)}]_1 = [\overline{(X,x)},\overline{(Y,y)}]_U = [(X,x),(Y,y)]_U$. To show the converse inclusion, that is $[(X,x),(Y,y)]_U \leq N[(X,x),(Y,y)]_1$, it is enough to show that the Ursini commutator satisfies the three conditions $(a), (b)$, and (c) in the previous proposition. It is not difficult to see that the Ursini commutator is monotone; since it is also constructed by taking regular image of a kernel. In addition, $[(X,x),(Y,y)]_U = [\overline{(X,x)},\overline{(Y,y)}]_U = N[\overline{(X,x)},\overline{(Y,y)}]_1 \leq \overline{X} \wedge \overline{Y}$. Furthermore, for every regular epimorphism f whose domain is A, we have $f([(X,x),(Y,y)]_U) = f([\overline{(X,x)},\overline{(Y,y)}]_U) = f(N[\overline{(X,x)},\overline{(Y,y)}]_1) = N[f(\overline{(X,x)}), f(\overline{(Y,y)})]_1$. But since f is a regular epimorphism, using Lemma 2.2.6, one has $\overline{f((X,x))} = f(\overline{(X,x)}) = f(\overline{(X,x)})$. Therefore,

$$\begin{aligned}
f([(X,x),(Y,y)]_U) &= N[f(\overline{(X,x)}), f(\overline{(Y,y)})]_1 \\
&= N[\overline{f((X,x))}, \overline{f((Y,y))}]_1 \\
&= [\overline{f((X,x))}, \overline{f((Y,y))}]_U \\
&= [f((X,x)), f((Y,y))]_U.
\end{aligned} \tag{3.16}$$

[1] Note that the $1-$ weighted subobjcet commutator and $1-$ weighted normal commutator coincide in an ideal-determined Mal'cev category $\mathbb{C}$.

Hence, by the above proposition $[(X,x),(Y,y)]_U \leq N[(X,x),(Y,y)]_1$. $\square$

Remark 3.4.11. *As a consequence of the previous corollary, Proposition 3.4.9 also gives a characterization of the Ursini commutator in an ideal-determined Barr-exact Mal'cev category* $\mathbb{C}$.

3.5 Weighted normal commutator as the Huq commutator in points.

It has been observed already in Proposition 3.1.2 that weighted centrality can be expressed in terms of Huq-commuting morphisms in a category of points over a fixed object. In this section, for subobjects (W,w), (X,x), and (Y,y) of A in a normal Mal'cev category $\mathbb{C}$ with finite colimits, we establish a relationship between the weighted normal commutator $N[(X,x),(Y,y)]_{(W,w)}$ and the Huq commutator of the pair of morphisms

$$\begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix} : (W+X,[1,0],i_1) \longrightarrow (W\times A,\pi_1,\langle 1,w\rangle)$$

and

$$\begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix} : (W+Y,[1,0],i_1) \longrightarrow (W\times A,\pi_1,\langle 1,w\rangle).$$

Recall that for an object W in a pointed category $\mathbb{C}$ with finite limits and coproducts, the *kernel functor*

$$\mathrm{Ker} : \mathsf{Pt}(W) \longrightarrow \mathbb{C}$$

assigns to every object (A,r,s) the kernel $\mathrm{Ker}(r)$ of r, and every morphism $f:(A,r,s)\longrightarrow (B,u,v)$

$$\begin{tikzcd}
\mathrm{Ker}(r) \arrow[r, dashed] \arrow[d, tail] & \mathrm{Ker}(u) \arrow[d, tail] \\
A \arrow[r, "f"] \arrow[d, shift left, "r"] & B \arrow[d, shift left, "u"] \\
W \arrow[r, "1"'] \arrow[u, shift left, "s"] & W \arrow[u, shift left, "v"]
\end{tikzcd}$$

is assigned to the induced morphism $\mathrm{Ker}(r) \dashrightarrow \mathrm{Ker}(u)$. The kernel functor has a left adjoint

$$W+(-) : \mathbb{C} \longrightarrow \mathsf{Pt}(W),$$

which assigns to every object X and every morphism $f : X \longrightarrow Y$ in $\mathbb{C}$, the object $(W+X,[1,0],i_1)$ and the morphism $1+f : (W+X,[1,0],i_1) \longrightarrow (W+Y,[1,0],i_1)$ respectively

$$
\begin{tikzcd}
W+X \arrow[r, "1+f"] \arrow[d, shift left, "{[1,0]}"] & W+Y \arrow[d, shift left, "{[1,0]}"] \\
W \arrow[u, shift left, "i_1"] \arrow[r, "1"'] & W. \arrow[u, shift left, "i_1"]
\end{tikzcd}
$$

Now we are ready to state the main result of this section.

Theorem 3.5.1. *Let $(X, x), (Y, y)$, and (W, w) be subobjects of an object A in a normal Mal'cev category $\mathbb{C}$ with finite colimits. The weighted normal commutator $N[(X, x), (Y, y)]_{(W,w)}$ is the image of the kernel functor applied to the Huq commutator of the morphisms*

$$\begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix} : (W + X, [1, 0], i_1) \longrightarrow (W \times A, \pi_1, \langle 1, w\rangle)$$

and

$$\begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix} : (W + Y, [1, 0], i_1) \longrightarrow (W \times A, \pi_1, \langle 1, w\rangle)$$

in $\mathsf{Pt}(W)$.

Proof. Consider the commutative diagram

$$
\begin{tikzcd}
& (W+X,[1,0],i_1) \arrow[dl, "{\langle 1, i_1[1,0]\rangle}"'] \arrow[d, dotted, "u"] \arrow[dr, "{\begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix}}"] & \\
((W+X)\times_W (W+Y),[1,0]\pi_1,\langle i_1,i_1\rangle) \arrow[r, dotted, "\varphi"] & (Q,\tau,q\langle 1,w\rangle) & (W\times A,\pi_1,\langle 1,w\rangle) \arrow[l, dotted, "q"'] \\
& (W+Y,[1,0],i_1) \arrow[ul, "{\langle i_1[1,0],1\rangle}"] \arrow[u, dotted, "v"'] \arrow[ur, "{\begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix}}"'] &
\end{tikzcd}
\tag{3.17}
$$

in $\mathsf{Pt(W)}$, where $(Q, \tau, q\langle 1, w\rangle)$ is the colimit of the solid arrows. As explained before, the morphism q is a normal epimorphism, and using Corollary 1.3.9 through Remark 1.3.10, Q and q can be chosen to be of the forms $W \times \tilde{Q}$ and $1 \times \tilde{q}$ respectively, where $\tilde{q}$ is a normal epimorphism in $\mathbb{C}$. So diagram (3.17) can be re-presented as follows

$$
\begin{tikzcd}
& (W+X,[1,0],i_1) \arrow[dl, "{\langle 1, i_1[1,0]\rangle}"'] \arrow[dr, "{\begin{bmatrix} 1 & w \\ 0 & x \end{bmatrix}}"] & \\
((W+X)\times_W (W+Y),[1,0]\pi_1,\langle i_1,i_1\rangle) \arrow[r, dotted, "{\varphi=\langle\alpha,\beta\rangle}"] & (W\times\tilde{Q},\pi_1,\langle 1,\tilde{q}w\rangle) & (W\times A,\pi_1,\langle 1,w\rangle) \arrow[l, dotted, "1\times\tilde{q}"'] \\
& (W+Y,[1,0],i_1) \arrow[ul, "{\langle i_1[1,0],1\rangle}"] \arrow[ur, "{\begin{bmatrix} 1 & w \\ 0 & y \end{bmatrix}}"'] &
\end{tikzcd}
\tag{3.18}
$$

Clearly, the diagram

$$
\begin{tikzcd}
& W+X \arrow[dl, "{\langle 1, i_1[1,0]\rangle}"'] \arrow[dr, "{[w,x]}"] & & \\
(W+X)\times_W (W+Y) \arrow[r, dotted, "\beta"] & \tilde{Q} & A \arrow[l, dotted, two heads, "\tilde{q}"'] \\
& W+Y \arrow[ul, "{\langle i_1[1,0],1\rangle}"] \arrow[ur, "{[w,y]}"'] &
\end{tikzcd}
$$

commutes in $\mathbb{C}$, and moreover, in the same way as in Theorem 3.3.1 and Corollary 3.3.2, $\tilde{Q}$ is the colimit of the solid arrows. The weighted normal commutator $N[(X,x),(Y,y)]_{(W,w)}$ is the kernel of $\tilde{q}$, and now the result follows immediately from the fact that the kernel functor preserves kernels: So the kernel functor sends the kernel of $1\times\tilde{q}$ (as a morphism in $\mathsf{Pt}(W)$) to the kernel of $\tilde{q}$. $\square$

Chapter 4

Centrality in subtractive categories

In a subtractive category $\mathbb{C}$, we recall the notion of partial subtraction structure introduced in [12]. In this chapter we show that there is an abelian group structure in every regular subtractive category, determined by morphisms that admit partial subtraction structures. Furthermore, we extend commuting morphisms to subtractive categories with finite joins of subobjects, and show that they can be related to the notion of partial subtraction structures.

We will write $\mathcal{S}$ to denote the variety of *subtractive algebras*, that is, the variety that has only a binary term s and a constant 0 in its theory, such that $s(x,x) = 0$ and $s(x,0) = x$.

4.1 Partial subtraction structures

Recall that (see e.g [12]) an object X in a subtractive category $\mathbb{C}$ is said to admit an internal subtraction structure when there is a morphism $s : X \times X \longrightarrow X$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\langle 1,1\rangle"] \arrow[dr, bend right, "0"'] & X \times X \arrow[d, "s"] & X \arrow[l, "\langle 1,0\rangle"'] \arrow[dl, bend left, "1"] \\ & X & \end{tikzcd}$$

commute. More generally, we have the following definition:

Definition 4.1.1. *Let $\mathbb{C}$ be a subtractive category. A morphism $f : X \longrightarrow Y$ is said to admit a subtractor along a morphism $g : Y \longrightarrow Z$, if there exists a morphism $\varphi : Y \times X \longrightarrow Z$ making the diagram*

$$\begin{tikzcd} X \arrow[r, "\langle f,1\rangle"] \arrow[dr, bend right, "0"'] & Y \times X \arrow[d, "\varphi"] & Y \arrow[l, "\langle 1,0\rangle"'] \arrow[dl, bend left, "g"] \\ & Z & \end{tikzcd}$$

commute. We will call such a morphism φ a ***subtractor*** *of f along g. When g is the identity morphism of Y, we will just call φ a subtractor of f. Note that a subtractor of the identity morphism of an object X (along the identity morphism of X) is just an internal subtraction structure s on X, usually called a* ***subtraction****.*

In the variety $\mathcal{S}$ of subtractive algebras, if $f : X \longrightarrow Y$ admits a subtractor $\varphi : Y \times X \longrightarrow Z$ along $g : Y \longrightarrow Z$, then it means for every $x \in X$ and $y \in Y$, $\varphi(y, 0) = g(y)$ and $\varphi(f(x), x) = 0$, and this implies that

$$\begin{aligned} \varphi(y, x) &= \varphi(y, x) - \varphi(f(x), x) \\ &= \varphi(y - f(x), 0) \\ &= g(y - f(x)). \end{aligned} \tag{4.1}$$

Moreover, for $x, x' \in X$ and $y, y' \in Y$ one can observe that

$$\begin{aligned} g(y - f(x)) - g(y' - f(x')) &= \varphi(y, x) - \varphi(y', x') \\ &= \varphi(y - y', x - x') \\ &= g(y - y') - g(f(x) - f(x')). \end{aligned} \tag{4.2}$$

Thus we obtain the following:

Proposition 4.1.2. *In the variety $\mathcal{S}$ of subtractive algebras, a homomorphism $f : X \longrightarrow Y$ admits a subtractor along $g : Y \longrightarrow Z$, if and only if, for every pairs $x, x' \in X$ and $y, y' \in Y$, one has*

$$g(y - f(x)) - g(y' - f(x')) = g(y - y') - g(f(x) - f(x')).$$

Proof. If f admits a subtractor φ along g, then as observed above, for every pairs $x, x' \in X$ and $y, y' \in Y$,

$$g(y - f(x)) - g(y' - f(x')) = g(y - y') - g(f(x) - f(x')).$$

Conversely, let us suppose for every pairs $x, x' \in X$ and $y, y' \in Y$, one has

$$g(y - f(x)) - g(y' - f(x')) = g(y - y') - g(f(x) - f(x')).$$

Define $\varphi : Y \times X \longrightarrow Z$ by $\varphi(y, x) = g(y - f(x))$. Clearly, $\varphi(f(x), x) = 0$ and $\varphi(y, 0) = g(y)$. Furthermore, for pairs $y, y' \in Y$ and $x, x' \in X$,

$$\begin{aligned} \varphi(y - y', x - x') &= g(y - y') - g(f(x) - f(x')) \\ &= g(y - f(x)) - g(y' - f(x')) \\ &= \varphi(y, x) - \varphi(y', x'), \end{aligned} \tag{4.3}$$

and this means φ is a homomorphism. Therefore φ is a subtractor of f.

□

Remark 4.1.3. *In the variety $\mathcal{S}$ of subtractive algebras, if a composite gf admits a subtractor then it implies f admits a subtractor along g. We shall later prove in more general that the converse is true when g is a regular epimorphism (although it is not difficult to see it for the variety $\mathcal{S}$).*

The following notion has been introduced in [12], although only a particular case was considered.

Definition 4.1.4. *A morphism $f : X \longrightarrow Y$ in a subtractive category $\mathbb{C}$ is said to admit a partial subtraction structure along g, if there is a relation $\langle r_1, r_2\rangle : R \rightarrowtail Y \times X$ and a morphism $\varphi : R \longrightarrow Z$, such that the morphisms $\langle f, 1\rangle : X \longrightarrow Y \times X$ and $\langle 1, 0\rangle : Y \longrightarrow Y \times X$ factor through $\langle r_1, r_2\rangle$, and the diagram*

$$\begin{tikzcd}
& R \arrow[d, tail, "\langle r_1, r_2\rangle"] \arrow[dd, dotted, bend right, "\varphi"'] & \\
X \arrow[ur, bend left, "u"] \arrow[r, "\langle f, 1\rangle"] \arrow[dr, bend right, "0"'] & Y \times X & Y \arrow[ul, bend right, "v"'] \arrow[l, "\langle 1, 0\rangle"'] \arrow[dl, bend left, "g"] \\
& Z &
\end{tikzcd}$$

commutes. For that we will say f admits a partial subtractor φ along g (with respect to a subobject $\langle r_1, r_2\rangle : R \rightarrowtail Y \times X$) or equivalently, f admits a partial subtraction structure along g. We will mostly require g to be identity morphism of Y, and for that we will write a triple $(f, (R, r_1, r_2), \varphi)$ to denote a partial subtraction structure on a morphism f.

Proposition 4.1.5. *In the variety $\mathcal{S}$ of subtractive algebras, a morphism admits a subtractor as soon as it admits a partial subtraction structure.*

Proof. If a homomorphism $f : X \to Y$ in $\mathcal{S}$ admits a partial subtraction structure $(f, (R, r_1, r_2), \varphi)$, then for every $x \in X$ and $y \in Y$, one has $\varphi(f(x), x) = 0$ and $\varphi(y, 0) = y$. Therefore, for every pair $(y, x) \in R$, one has

$$\begin{aligned} \varphi(y, x) &= \varphi(y, x) - \varphi(f(x), x) \\ &= \varphi(y - f(x), 0) \\ &= y - f(x). \end{aligned} \tag{4.4}$$

Furthermore, since for every $x \in X$ the pairs $(f(x), 0), (f(x), x) \in R$, it follows by subtractivity that $(0, x) \in R$. Now we can observe that

$$\begin{aligned} f(x) = \varphi(f(x), 0) &= \varphi(f(x), x) - \varphi(0, x) \\ &= 0 - \varphi(0, x) \\ &= 0 - (0 - f(x)), \end{aligned} \tag{4.5}$$

and moreover, for every pairs $x, x' \in X$ and $y, y' \in Y$, one has

$$\begin{aligned} (y - f(x)) - (y' - f(x')) =& \varphi(y, 0 - (0 - x)) - \varphi(y', 0 - (0 - x')) \\ &= \varphi(y - y', (0 - (0 - x)) - (0 - (0 - x'))) \\ &= (y - y') - ((0 - (0 - f(x))) - (0 - (0 - f(x')))) \\ &= (y - y') - (f(x) - f(x')), \end{aligned} \tag{4.6}$$

which, according to Proposition 4.1.2, implies f admits a subtractor. $\square$

We shall extend the previous proposition to a general categorical context. A particular case has already been obtained in [12], where it has been shown that in a subtractive category an identity morphism admits a subtraction as soon as it admits a partial subtractor. Proving the previous proposition in a categorical context is not so straightforward, it is based on several facts about partial subtrators that we are going to observe first. We shall begin by proving the following fact.

Proposition 4.1.6. *In a regular subtractive category* $\mathbb{C}$*, for each commutative diagram*

$$\begin{tikzcd}
 & R \arrow[d, tail, "\langle r_1, r_2\rangle"] & \\
X \arrow[ur, bend left, "u"] \arrow[r, "\langle f, 1\rangle"] \arrow[dr, bend right, "0"'] & Y \times X & Y \arrow[l, "\langle 1, 0\rangle"'] \arrow[ul, bend right, "v"'] \arrow[dl, bend left, "g"] \\
 & Z &
\end{tikzcd}$$

if there is a morphism $\varphi : R \longrightarrow Z$ *such that* $\varphi u = 0$ *and* $\varphi v = g$ *(i.e.* φ *is a partial subtractor of* f *along* g *with respect to a monomorphism* $\langle r_1, r_2\rangle$*), then* φ *is necessarily unique.*

Proof. Suppose $\varphi' : R \longrightarrow Z$ and $\varphi : R \longrightarrow Z$ are two morphisms such that $\varphi' u = 0$, $\varphi' v = g$ and $\varphi u = 0$, $\varphi v = g$. Let $\langle s_1, s_2\rangle : S \rightarrowtail R \times R$ be the joint kernel pair relation of φ and φ', i.e. the kernel pair relation of $\langle \varphi, \varphi'\rangle : R \longrightarrow Z \times Z$. Clearly, the diagonal $\langle 1, 1\rangle$ of R factors through $\langle s_1, s_2\rangle$, and since $\varphi u = 0 = \varphi' u$, the morphism $\langle 0, u\rangle$ also factors through $\langle s_1, s_2\rangle$. Now consider the relation (T, t_1, t_2) given by the regular image of $\langle s_1, s_2\rangle$ along the morphism $1 \times r_2$ in the diagram

$$\begin{tikzcd}
 & S \arrow[rrr, two heads, "e"] \arrow[dd, tail, "\langle s_1, s_2\rangle"'] & & & T \arrow[dd, tail, "\langle t_1, t_2\rangle"] & & \\
 & & & X \arrow[ull, bend right, "\beta"'] \arrow[dll, "\langle 0, u\rangle"'] \arrow[rr, "1"] & & X \arrow[ul, bend right, "e\beta"'] \arrow[dl, "\langle 0, 1\rangle"] & \\
 & R \times R \arrow[rrr, two heads, "1 \times r_2"'] & & & R \times X & & R \arrow[ll, "\langle 1, 0\rangle"] \arrow[uull, dashed, "w"'] \\
R \arrow[uuur, bend left, "\alpha"] \arrow[ur, "\langle 1, 1\rangle"] \arrow[rr, "1"] & & R \arrow[uuurr, bend left, "e\alpha"] \arrow[urr, "\langle 1, r_2\rangle"'] & & & &
\end{tikzcd}$$

It is clear the morphisms $\langle 1, r_2\rangle$ and $\langle 0, r_2\rangle = \langle 0, 1\rangle r_2$ factor through $\langle t_1, t_2\rangle$, and therefore the morphism $\langle 1, 0\rangle$ also factors through $\langle t_1, t_2\rangle$ by subtractivity. Let w in the diagram above be a morphism such that $\langle 1, 0\rangle = \langle t_1, t_2\rangle w$. Pulling back $\langle s_1, s_2\rangle$ along the morphism $1 \times v$, we see that the cube in the diagram

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\alpha"] \arrow[drrr, dotted, bend left, "h"] \arrow[dd, dotted, bend right, "l"'] \arrow[dddr, dotted, bend right, "{\langle l, r_1s_2h\rangle}"] & & & \\
& T' \arrow[rr, "k"] \arrow[dl, "t'_1"'] \arrow[dd, tail, "{\langle t'_1, t'_2\rangle}"] \arrow[dr, phantom, "\lrcorner", very near start] & & S \arrow[dl, two heads, "e"'] \arrow[dd, tail, "{\langle s_1, s_2\rangle}"] \\
R \arrow[rr, "w"] \arrow[dd, "1"'] & & T \arrow[dd, tail, "{\langle t_1, t_2\rangle}"'] & \\
& R \times Y \arrow[rr, "1 \times v"'] \arrow[dl, "\pi_1"'] & & R \times R \arrow[dl, "1 \times r_2"] \\
R \arrow[rr, "{\langle 1, 0\rangle}"'] & & R \times X &
\end{tikzcd}$$

commutes, and also $t'_1 = s_1k$ and $vt'_2 = s_2k$. But since $\varphi s_1 = \varphi s_2$, one has

$$\begin{aligned} \varphi t'_1 &= \varphi s_1 k \\ &= \varphi s_2 k \\ &= \varphi v t'_2 \\ &= g t'_2. \end{aligned} \tag{4.7}$$

In a similar way since $\varphi' s_1 = \varphi' s_2$, it can be shown that $\varphi' t'_1 = g t'_2$, and thus $\varphi t'_1 = \varphi' t'_1$. In order to conclude $\varphi = \varphi'$, we will observe that the top rectangle of the previous cube is a pullback, which will then imply t'_1 is a (regular) epimorphism: If h and l are a pair of morphisms such that $wl = eh$, then since $r_1v = 1$, $r_2v = 0$, $l = s_1h$, and the pair r_1, r_2 is jointly monomorphic, then $r_1vr_1s_2h = r_1s_2h$ and $r_2vr_1s_2h = 0 = r_2s_2h$, and this means the morphisms $\langle l, r_1s_2h\rangle$ and h make the back outer diagram commute, i.e. $(1 \times v)\langle l, r_1s_2h\rangle = \langle s_1, s_2\rangle h$. But since the back rectangle is a pullback, one obtains a factorization α through the pullback, which also turns out to be the unique morphism such that $t'_1\alpha = l$ and $k\alpha = h$. $\square$

Remark 4.1.7. *For morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ in a regular subtractive category $\mathbb{C}$, a subtractor φ of f along g in the diagram*

$$\begin{tikzcd}
X \arrow[r, "{\langle f, 1\rangle}"] \arrow[dr, bend right, "0"'] & Y \times X \arrow[d, dotted, "\varphi"'] & Y \arrow[l, "{\langle 1, 0\rangle}"'] \arrow[dl, bend left, "g"] \\
& Z &
\end{tikzcd}$$

can be seen as a partial subtractor of f along g with respect to the identity morphism on $Y \times X$. Applying the previous proposition, it follows that a morphism can only admit at most one subtractor.

We shall prove the following "cancellation" properties: In a regular subtractive category $\mathbb{C}$, for every diagram

$$\begin{tikzcd} A \arrow[r, two heads, "e"] & B \arrow[r, "f"] & C \arrow[r, tail, "m"] & D \end{tikzcd}$$

where e is a regular epimorphism and m is a monomorphism,

(a) the morphism f admits a partial subtraction structure whenever the composite fe admits a partial subtraction;

(b) the morphism f admits a partial subtraction structure whenever the composite mf admits a partial subtraction structure.

We prove (a) above in the next proposition.

Proposition 4.1.8. *Let $\mathbb{C}$ be a regular subtractive category. If a morphism $g : A \longrightarrow Y$ admits a partial subtraction structure, then for any morphism $e : X \longrightarrow A$, the composite ge also admits a partial subtraction structure. The converse is true when e is a regular epimorphism.*

Proof. Suppose $g : A \longrightarrow Y$ admits a partial subtraction structure $(g, (R, r_1, r_2), \varphi)$ as shown in the diagram

$$\begin{tikzcd}
 & R \arrow[d, tail, "\langle r_1, r_2\rangle"] \arrow[dd, dotted, bend right, "\varphi"'] & \\
A \arrow[ur, bend left, "u"] \arrow[r, "\langle g, 1\rangle"] \arrow[dr, bend right, "0"'] & Y \times A & Y \arrow[ul, bend right, "v"'] \arrow[l, "\langle 1, 0\rangle"'] \arrow[dl, bend left, "1"] \\
 & Y. &
\end{tikzcd}$$

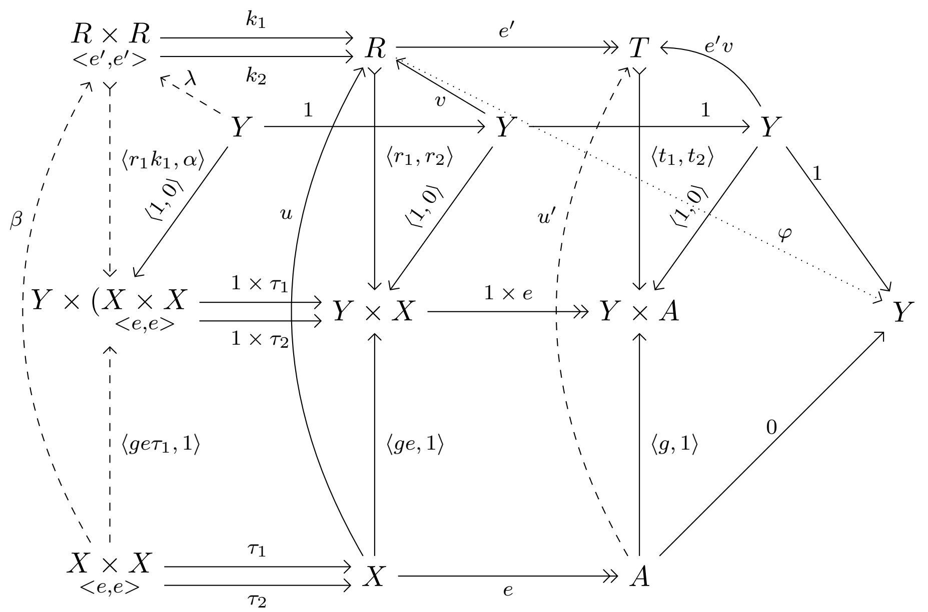
Let $\langle s_1, s_2\rangle$ be the pullback of $\langle r_1, r_2\rangle$ along $(1 \times e)$ in the diagram

$$\begin{tikzcd}
S \arrow[r, "p"] \arrow[d, tail, "\langle s_1, s_2\rangle"'] \arrow[dr, phantom, very near start, "\lrcorner"] & R \arrow[d, tail, "\langle r_1, r_2\rangle"] \\
Y \times X \arrow[r, "1 \times e"'] & Y \times A.
\end{tikzcd}$$

Since $(1 \times e)\langle ge, 1\rangle = \langle r_1, r_2\rangle ue$ and $(1 \times e)\langle 1, 0\rangle = \langle r_1, r_2\rangle v$, the morphisms $\langle ge, 1\rangle$ and $\langle 1, 0\rangle$ factor through the pullback $\langle s_1, s_2\rangle$ by morphisms u' and v' respectively. Now with $pu' = ue$ and $pv' = v$, it can be easily seen in the diagram

$$\begin{tikzcd}
 & S \arrow[d, tail, "\langle s_1, s_2\rangle"] \arrow[dd, dotted, bend right, "\varphi p"'] & \\
X \arrow[ur, bend left, "u'"] \arrow[r, "\langle ge, 1\rangle"] \arrow[dr, bend right, "0"'] & Y \times X & Y \arrow[ul, bend right, "v'"'] \arrow[l, "\langle 1, 0\rangle"'] \arrow[dl, bend left, "1"] \\
 & Y &
\end{tikzcd}$$

that $(ge, (S, s_1, s_2), \varphi p)$ is a partial subtraction structure on ge. To prove the second part of the proposition, let us suppose ge, where e is a regular epimorphism, admits a partial subtraction structure $(ge, (R, r_1, r_2), \varphi)$, and morphisms u and v are respective factorizations of $\langle ge, 1\rangle$ and $\langle 1, 0\rangle$ through $\langle r_1, r_2\rangle$. Let $\langle t_1, t_2\rangle$ be the regular image of $\langle r_1, r_2\rangle$ along the morphism $1 \times e$. Consider the diagram



where

$$(\underset{<e',e'>}{R \times R}, k_1, k_2) \text{ and } (\underset{<e,e>}{X \times X}, \tau_1, \tau_2)$$

are the kernel pair relations of e' and e respectively. The morphism λ is the unique factorization of v through k_1 and k_2, i.e. $k_1\lambda = v = k_2\lambda$. Using the commutativity of the right hand squares, and the fact that e is a strong epimorphism, there is a factorization u' of $\langle g, 1\rangle$ through $\langle t_1, t_2\rangle$. Now the rest of the dotted arrows are obtained as induced morphisms by the universal property of kernel pairs. It can be seen that

$$\varphi k_1 \beta = \varphi u \tau_1 = 0 = \varphi k_2 \beta \text{ and } \varphi k_1 \lambda = \varphi v = 1 = \varphi k_2 \lambda.$$

Now we have obtained the following commutative diagram

$$\begin{tikzcd}
 & & \underset{<e',e'>}{R\times R} \arrow[d, tail, "\langle r_1k_1,\alpha\rangle"] \arrow[dd, dotted, bend right, "\varphi k_2"'] \arrow[dd, dotted, bend left, "\varphi k_1"] & & \\
\underset{<e,e>}{X\times X} \arrow[rr, "\langle ge\tau_1,1\rangle"] \arrow[urr, bend left, "\beta"] \arrow[drr, bend right, "0"'] & & \underset{<e,e>}{Y\times(X\times X)} & & Y \arrow[ll, "\langle 1,0\rangle"'] \arrow[ull, bend right, "\lambda"'] \arrow[dll, bend left, "1"] \\
 & & Y & &
\end{tikzcd}$$

from which, by the uniqueness of partial subtractors, it follows that $\varphi k_1 = \varphi k_2$. But since e' is the coequalizer of the pair k_1, k_2, there is a unique morphism ϕ such that $\varphi = \phi e'$, and it can be easily seen that ϕ is a partial subtractor of g. $\square$

Lemma 4.1.9. *In a subtractive category $\mathbb{C}$, if a morphism $f : X \longrightarrow Y$ admits a partial subtraction structure along a composite $mg : Y \longrightarrow Z$, where $m : W \to Z$ is a monomorphism, then f also admits a partial subtraction structure along g.*

Proof. Suppose a morphism f admits a partial subtraction structure along a composite mg as shown in the diagram

$$\begin{tikzcd}
 & R \arrow[d, tail, "\langle r_1,r_2\rangle"] \arrow[dd, dotted, bend right, "\varphi"'] & \\
X \arrow[r, "\langle f,1\rangle"] \arrow[ur, bend left, "u"] \arrow[dr, bend right, "0"'] & Y\times X & Y \arrow[l, "\langle 1,0\rangle"'] \arrow[ul, bend right, "v"'] \arrow[dl, bend left, "mg"] \\
 & Z &
\end{tikzcd}$$

with m a monomorphism. Let us consider the pullback of m along φ

$$\begin{tikzcd}
T \arrow[r, "r"] \arrow[d, tail, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, tail, "m"] \\
R \arrow[r, "\varphi"'] & Z.
\end{tikzcd}$$

Since $\varphi u = 0 = m0$, there is a factorization $h : X \longrightarrow T$ through the pullback, such that $th = u$ and $rh = 0$. Similarly, for the morphisms v and g, since $\varphi v = mg$, there is $k : Y \longrightarrow T$ such that $tk = v$ and $rk = g$. These yield a partial subtraction structure on f along g as seen in the diagram

$$
\begin{tikzcd}
 & T \arrow[d, tail, "t"] \arrow[ddd, dotted, bend right, "r"'] & \\
 & R \arrow[d, tail, "\langle r_1, r_2\rangle"] & \\
X \arrow[ruu, bend left, "h"] \arrow[r, "\langle f, 1\rangle"] \arrow[rd, bend right, "0"'] & Y \times X & Y \arrow[luu, bend right, "k"'] \arrow[l, "\langle 1, 0\rangle"'] \arrow[ld, bend left, "g"] \\
 & W. &
\end{tikzcd}
$$

□

In the next lemma we show that every partial subtraction structure on a identity morphism gives rise to an internal subtraction structure. The same result is given in Theorem 2.5 of [12], hence we copy the proof.

Lemma 4.1.10. *In a subtractive category $\mathbb{C}$, for each partial subtraction structure* $(1, (R, r_1, r_2), \varphi)$

$$
\begin{tikzcd}
 & R \arrow[dd, tail, "\langle r_1, r_2\rangle"] \arrow[ddd, dotted, bend right, "\varphi"'] & \\
 & & \\
X \arrow[ruu, bend left, "u"] \arrow[r, "\langle 1, 1\rangle"] \arrow[rd, bend right, "0"'] & X \times X & X \arrow[luu, bend right, "v"'] \arrow[l, "\langle 1, 0\rangle"'] \arrow[ld, bend left, "1"] \\
 & X &
\end{tikzcd}
$$

on the identity morphism of X, the morphism $\langle r_1, r_2\rangle$ is an isomorphism, in other words, X admits a subtraction structure as soon as the identity morphism of X admits a partial subtraction structure or, equivalently, if the identity morphism of X admits a partial subtractor φ with respect to $\langle r_1, r_2\rangle$, then $\langle r_1, r_2\rangle$ is an isomorphism.

Proof. Consider the relation $\langle t_1, t_2\rangle : T \rightarrowtail X \times (X \times X)$, obtained from pulling back $\langle r_1, r_2\rangle : R \rightarrowtail X \times X$ along $1 \times s : X \times (X \times X) \longrightarrow X \times X$. Using generalized elements, T is defined as follows: for $x, y, z \in X$,

$$(x, (y, z)) \in T \Leftrightarrow (x, s(y, z)) \in R.$$

For every pair $(x, y) \in X \times X$, since $(x, (y, y)) \in T$ and $(0, (0, y)) \in T$ (for every $y \in X$, $(y, y), (y, 0) \in R$ implies $(0, y) \in R$), then by subtractivity, $(x, (y, 0)) \in T$, and this implies $(x, s(y, 0)) = (x, y) \in R$. Hence, $\langle r_1, r_2\rangle$ is an isomorphism. □

Proposition 4.1.11. *In a subtractive category $\mathbb{C}$, if a monomorphism $f : X \rightarrowtail Y$ admits a partial subtraction structure $(f, (R, r_1, r_2), \varphi)$*

$$\begin{tikzcd}
& R \arrow[d, tail, "\langle r_1, r_2\rangle"] \arrow[dd, dotted, bend right, "\varphi"'] & \\
X \arrow[ur, bend left, "u"] \arrow[r, "\langle f, 1\rangle"] \arrow[dr, bend right, "0"'] & Y \times X & Y \arrow[ul, bend right, "v"'] \arrow[l, "\langle 1, 0\rangle"'] \arrow[dl, bend left, "1"] \\
& Y &
\end{tikzcd}$$

then the following statements hold:

(a) X admits a subtraction structure;

(b) the morphism $\langle r_1, r_2\rangle$ is an isomorphism;

(c) f admits a subtractor.

Proof. (a). Consider the pullback of $\langle r_1, r_2\rangle$ along $f \times 1$ in the diagram

$$\begin{tikzcd}
S \arrow[r, tail, "r"] \arrow[d, tail, "\langle s_1, s_2\rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "\langle r_1, r_2\rangle"] \\
X \times X \arrow[r, tail, "f \times 1"'] & Y \times X.
\end{tikzcd}$$

Clearly, the pairs of morphisms $\langle 1, 1\rangle, u$ and $\langle 1, 0\rangle, vf$ factor through the pullback as $\langle s_1, s_2\rangle u' = \langle 1, 1\rangle, ru' = u$ and $rv' = vf$, $\langle s_1, s_2\rangle v' = \langle 1, 0\rangle$, respectively. These data yield the following commutative diagram

$$\begin{tikzcd}
& S \arrow[d, tail, "\langle s_1, s_2\rangle"] \arrow[dd, dotted, bend right, "\varphi r"'] & \\
X \arrow[ur, bend left, "u'"] \arrow[r, "\langle 1, 1\rangle"] \arrow[dr, bend right, "0"'] & X \times X & X \arrow[ul, bend right, "v'"'] \arrow[l, "\langle 1, 0\rangle"'] \arrow[dl, tail, bend left, "f"] \\
& Y &
\end{tikzcd}$$

which exhibits a partial subtraction structure on the identity morphism of X along f. Using Lemma 4.1.9, the identity morphism of X admits a partial subtraction structure, and hence X admits a subtraction structure, according to the previous lemma.

(b). Let $s : X \times X \longrightarrow X$ be a subtraction on X. Now consider the relation $\langle t_1, t_2 \rangle : T \rightarrowtail Y \times (X \times X)$, obtained from pulling back $\langle r_1, r_2 \rangle : R \rightarrowtail Y \times X$ along $1 \times s : Y \times (X \times X) \longrightarrow Y \times X$. Using generalized elements, T is defined as follows: for $y \in Y$ and $x, x' \in X$,

$$(y, (x, x')) \in T \Leftrightarrow (y, s(x, x')) \in R.$$

For every pair $(y, x) \in Y \times X$, since $(y, (x, x)) \in T$ and $(0, (0, x)) \in T$ (since for every $x \in X$, $(f(x), x), (f(x), 0) \in R$ imply $(0, x) \in R$), then by subtractivity, $(y, (x, 0)) \in T$, and this implies $(y, s(x, 0)) = (y, x) \in R$. Hence, $\langle r_1, r_2 \rangle$ is an isomorphism.

(c). Since the morphism $\langle r_1, r_2 \rangle$ is an isomorphism according to (b), it is not difficult to see that $\varphi \langle r_1, r_2 \rangle^{-1}$, where $\langle r_1, r_2 \rangle^{-1}$ denotes the inverse of $\langle r_1, r_2 \rangle$, is a subtractor of f. □

Corollary 4.1.12. *In a subtractive category $\mathbb{C}$, if a composite $mf : X \longrightarrow Z$, where $m : Y \longrightarrow Z$ is a monomorphism, admits a partial subtraction structure then f also admits a partial subtraction structure.*

Proof. If mf admits a partial subtraction structure $(mf, (R, r_1, r_2), \varphi)$, just as in Proposition 4.1.11 (a), it can be shown that f admits a partial subtraction structure along m; that is by showing that the composite φr, where r is the pullback of $m \times 1$ along $\langle r_1, r_2 \rangle$

$$\begin{tikzcd}
T \arrow[r, tail, "r"] \arrow[d, tail, "\langle t_1, t_2 \rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "\langle r_1, r_2 \rangle"] \\
Y \times X \arrow[r, tail, "m \times 1"'] & Z \times X
\end{tikzcd}$$

is a partial subtractor of f along m with respect to $\langle t_1, t_2 \rangle$. And using Lemma 4.1.9, it follows that f admits a partial subtraction structure.

□

Proposition 4.1.13. *In a subtractive category $\mathbb{C}$, if an object X admits a subtraction $s : X \times X \longrightarrow X$, then the following hold:*

(a) the pair $\langle 1, 1 \rangle : X \longrightarrow X \times X$, $\langle 1, 0 \rangle : X \longrightarrow X \times X$ is jointly extremal-epimorphic;

(b) the pair $\langle 1, 0 \rangle : X \longrightarrow X \times X$, $\langle 0, 1 \rangle : X \longrightarrow X \times X$ is jointly extremal-epimorphic.

Proof. (a) Let $\langle r_1, r_2 \rangle : R \rightarrowtail X \times X$ be a monomorphism such that $\langle 1, 1 \rangle$ and $\langle 1, 0 \rangle$ factor through it. In the diagram

$$\begin{tikzcd}
 & R \arrow[d, tail, "\langle r_1, r_2 \rangle"] & \\
X \arrow[ur, bend left, "u"] \arrow[r, "\langle 1, 1 \rangle"] \arrow[dr, bend right, "0"'] & X \times X \arrow[d, "s"] & X \arrow[l, "\langle 1, 0 \rangle"'] \arrow[ul, bend right, "v"'] \arrow[dl, bend left, "1"] \\
 & X &
\end{tikzcd}$$

we see that $s\langle r_1, r_2\rangle$ is a partial subtractor of the identity morphism of X with respect to $\langle r_1, r_2\rangle$. Hence, by Lemma 4.1.10, $\langle r_1, r_2\rangle$ is an isomorphism.

(b) Let $\langle r_1, r_2\rangle : R \rightarrowtail X\times X$ be a monomorphism such that $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ factor through it. Consider the relation $\langle t_1, t_2\rangle : T \rightarrowtail X\times(X\times X)$, obtained from pulling back $\langle r_1, r_2\rangle : R \rightarrowtail X\times X$ along $1 \times s : X \times (X \times X) \longrightarrow X \times X$. Using generalized elements, T is defined as follows: for $x, y, z \in X$,

$$(x,(y,z)) \in T \Leftrightarrow (x, s(y,z)) \in R.$$

Since for every $x \in X$, one has $(x, 0), (0, x) \in R$, it follows that for every pair $(x, y) \in X \times X$, $(x,(y, y)) \in T$ and $(0,(0, y)) \in T$, and hence by subtractivity, $(x,(y, 0)) \in T$, which implies $(x, s(y, 0)) = (x, y) \in R$. Therefore, $\langle r_1, r_2\rangle$ is an isomorphism.

□

Remark 4.1.14. *(see [12]) As a consequence of* (a) *of the previous proposition, it immediately follows that an object in a subtractive category can only admit at most one internal subtraction structure.*

Now we can use the "cancellation" properties, that is, Proposition 4.1.8 and Corollary 4.1.12, to prove a categorical version of Proposition 4.1.5.

Theorem 4.1.15. *Let* $\mathbb{C}$ *be a regular subtractive category. A morphism* $f : X \longrightarrow Y$ *admits a subtractor as soon as it admits a partial subtraction structure.*

Proof. Let $(f,(R, r_1, r_2), \varphi)$ be a partial subtraction structure on f, and $f = me$ be the (regular epi, mono)-factorization. Using Proposition 4.1.8, it can be concluded that the image $m : f(X) \rightarrowtail Y$ admits a partial subtraction structure. Now applying Proposition 4.1.11 (c), it follows that m admits a subtractor ϕ. Hence, as seen in the diagram

$$\begin{tikzcd}
Y \arrow[r, "1"] \arrow[d, "{\langle 1,0\rangle}"'] & Y \arrow[d, "{\langle 1,0\rangle}"] \arrow[dr, bend left, "1"] & \\
Y\times X \arrow[r, two heads, "1\times e"] & Y\times f(X) \arrow[r, dashed, "\phi"] & Y \\
X \arrow[u, "{\langle f,1\rangle}"] \arrow[r, two heads, "e"'] & f(X) \arrow[u, "{\langle m,1\rangle}"'] \arrow[ur, bend right, "0"'] &
\end{tikzcd}$$

the morphism $\phi(1 \times e)$ is a subtractor of f.

□

So far we know that when a morphism $f : X \longrightarrow Y$ in a regular subtractive category $\mathbb{C}$ admits a subtractor, it is necessarily unique. Moreover, the image $m : f(X) \rightarrowtail Y$ in the (regular epi, mono)-factorization $f = me$, also admits a subtractor. Hence, $f(X)$ admits a subtraction $s : f(X) \times f(X) \longrightarrow f(X)$ whenever f admits a subtractor. We will write φ_f and ϕ_f to denote the subtractors of f and m respectively. The three subtractors, namely, φ_f, ϕ_f, and s are related in (i) and (ii) in the remark below:

Remark 4.1.16. *Let $f : X \longrightarrow Y$ be a morphism in a regular subtractive category $\mathbb{C}$, and me be the (regular epi, mono)-factorization of f. If f admits a subtractor φ_f, then, as seen in the previous theorem, (i) $\varphi_f = \phi_f(1 \times e)$. Furthermore, for a subtraction $s : f(X) \times f(X) \longrightarrow f(X)$, it is not difficult to see that the two composites $f(X) \times f(X) \longrightarrow Y$ on the rectangle*

$$\begin{tikzcd}
f(X) \times f(X) \arrow[r, "m \times 1"] \arrow[d, "s"'] & Y \times f(X) \arrow[d, "\phi_f"] \\
f(X) \arrow[r, "m"'] & Y
\end{tikzcd}$$

are both subtractors of the identity morphism of $f(X)$ along m. Hence by the uniqueness of subtractors (see Proposition 4.1.6), (ii) $ms = \phi_f(m \times 1)$.

Let us proceed to show that there is a subtractive structure on certain classes of morphisms in a regular subtractive category.

Definition 4.1.17. *Let $f : X \longrightarrow Y$ be a morphism in a regular subtractive category $\mathbb{C}$. Given a morphism $g : X \longrightarrow Y$ in $\mathbb{C}$ which admits a subtractor φ_g, let us define $f - g : X \longrightarrow Y$ to be the morphism given by the composite $\varphi_g\langle f, 1\rangle : X \longrightarrow Y$ in the diagram*

$$\begin{tikzcd}
 & X \arrow[d, tail, "\langle f, 1\rangle"] \arrow[dd, dotted, bend right, "f - g"'] & \\
X \arrow[r, "\langle g, 1\rangle"] \arrow[dr, bend right, "0"'] & Y \times X \arrow[d, "\varphi_g"] & Y \arrow[l, "\langle 1, 0\rangle"'] \arrow[dl, bend left, "1"] \\
 & Y. &
\end{tikzcd}$$

For every pair of objects X and Y in a regular subtractive category $\mathbb{C}$, we write $Z(X, Y)$ to denote the class of all morphisms from X to Y which admit subtractors. As seen in the previous definition, it is not necessary for f to admit a subtractor in order to define $f - g$. However, for the morphism $f - g$ to also admit a subtractor, we will see that both f and g need to admit subtractors. In other words, we will see that $Z(X, Y)$ is closed under the "subtraction" in the previous definition.

Lemma 4.1.18. *Let $\mathbb{C}$ be a regular subtractive category. For $f \in Z(X, Y)$, the morphism $\langle \varphi_f, \pi_2\rangle : Y \times X \longrightarrow Y \times X$ is a monomorphism.*

Proof. To prove that the morphism $\langle \varphi_f, \pi_2\rangle$ is a monomorphism, it is enough to show that $k = k'$, where (K, k, k') is the kernel pair relation of $\langle \varphi_f, \pi_2\rangle$. Writing $\langle k, k'\rangle = \langle\langle k_1, k_2\rangle, \langle k_1', k_2'\rangle\rangle$, from $\langle \varphi_f, \pi_2\rangle\langle k_1, k_2\rangle = \langle \varphi_f, \pi_2\rangle\langle k_1', k_2'\rangle$, we see that $k_2 = k_2'$, and thus $\langle \varphi_f, \pi_2\rangle\langle 0, k_2\rangle = \langle \varphi_f, \pi_2\rangle\langle 0, k_2'\rangle$. It now follows by subtractivity that $\langle \varphi_f, \pi_2\rangle\langle k_1, 0\rangle = \langle \varphi_f, \pi_2\rangle\langle k_1', 0\rangle$. But since $\varphi_f\langle 1, 0\rangle = 1$ by definition, we have $k_1 = \varphi_f\langle 1, 0\rangle k_1 = \varphi_f\langle 1, 0\rangle k_1' = k_1'$. Hence, $k = k'$. $\square$

Proposition 4.1.19. *In a regular subtractive category $\mathbb{C}$, for every pair of objects X and Y in $\mathbb{C}$, $(Z(X,Y),-,0)$ is a subtractive algebra.*

Proof. Clearly, the zero morphism $0 : X \longrightarrow Y$ is an element of $Z(X,Y)$; its subtractor is given by $\pi_1 : Y \times X \longrightarrow Y$, and in addition, for $f \in Z(X,Y)$, one has $f - 0 = \pi_1\langle f, 1\rangle = f$ and $f - f = \varphi_f\langle f, 1\rangle = 0$. Now it remains to show that $Z(X,Y)$ is closed under "$-$". For $f, g \in Z(X,Y)$, since $\langle \varphi_g, \pi_2\rangle$ in the commutative diagram

$$\begin{tikzcd}
 & Y\times X \arrow[d, tail, "\langle\varphi_g,\pi_2\rangle"] \arrow[dd, dotted, bend right, "\varphi_f"'] & \\
X \arrow[ur, bend left, "\langle f,1\rangle"] \arrow[r, "\langle f-g,1\rangle"] \arrow[dr, bend right, "0"'] & Y\times X & Y \arrow[ul, bend right, "\langle 1,0\rangle"'] \arrow[l, "\langle 1,0\rangle"'] \arrow[dl, bend left, "1"] \\
 & Y &
\end{tikzcd}$$

is a monomorphism, we see that $f - g$ admits a partial subtractor φ_f with respect to $\langle \varphi_g, \pi_2\rangle$, and hence, it admits a subtractor, i.e. $f - g \in Z(X,Y)$. □

Having shown that $Z(X,Y)$ admits a subtraction structure for every pair of objects X and Y in a regular subtractive category $\mathbb{C}$, we will now proceed to show that this subtraction structure is part of an abelian group structure on $Z(X,Y)$. For that we are going to establish several identities on the subtractive algebra $Z(X,Y)$, sufficient to give an abelian group structure.

Lemma 4.1.20. *In a regular subtractive category $\mathbb{C}$, for $f, g, h \in Z(X,Y)$, one has $(g - f) - (h - f) = g - h$.*

Proof. The morphisms $\varphi_{h-f}\langle \varphi_f, \pi_2\rangle$ and φ_h are both partial subtractors of $h - f$ with respect to $\langle \varphi_f, \pi_2\rangle$ as seen in the diagram

$$\begin{tikzcd}
 & Y\times X \arrow[d, tail, "\langle\varphi_f,\pi_2\rangle"] \arrow[dd, dotted, bend right, "\varphi_h"'] & \\
X \arrow[ur, bend left, "\langle h,1\rangle"] \arrow[r, "\langle h-f,1\rangle"] \arrow[dr, bend right, "0"'] & Y\times X \arrow[d, "\varphi_{h-f}"] & Y \arrow[ul, bend right, "\langle 1,0\rangle"'] \arrow[l, "\langle 1,0\rangle"'] \arrow[dl, bend left, "1"] \\
 & Y. &
\end{tikzcd}$$

The uniqueness of partial subtractors forces $\varphi_{h-f}\langle \varphi_f, \pi_2\rangle$ and φ_h to be equal. Pre-composing with $\langle g, 1\rangle$ we get

$$\begin{aligned}
\varphi_{h-f}\langle \varphi_f, \pi_2\rangle\langle g, 1\rangle &= \varphi_h\langle g, 1\rangle \\
\varphi_{h-f}\langle g - f, 1\rangle &= g - h \\
(g - f) - (h - f) &= g - h.
\end{aligned} \tag{4.8}$$

□

Proposition 4.1.21. *In a regular subtractive category $\mathbb{C}$, the following identities hold:*

(a) $0-(0-g)=g$, *for every* $g \in Z(X,Y)$*;*

(b) $(f-(0-g))-g=f$, *for every pair* $f,g \in Z(X,Y)$*.*

Proof. (a). If we set f to g and h to 0 in the previous lemma, then we obtain $0-(0-g)=g$.

(b). Again we apply the previous lemma and the fact that $0-(0-g)=g$, so that $(f-(0-g))-g=(f-(0-g))-(0-(0-g))=f-0=f$.

□

Proposition 4.1.22. *Let $\mathbb{C}$ be a regular subtractive category. For $f,g \in Z(X,Y)$, one has* $\varphi_f\langle\varphi_g,\pi_2\rangle=\varphi_{f-(0-g)}$.

Proof. Since $(f-(0-g))-g=f$ according to the previous proposition, it can be seen that the diagram

$$\begin{tikzcd}
 & Y \times X \arrow[d, tail, "\langle\varphi_g,\pi_2\rangle"] \arrow[dd, dotted, bend right, "\varphi_{f-(0-g)}"'] & \\
X \arrow[ur, bend left, "\langle f-(0-g),1\rangle"] \arrow[r, "\langle f,1\rangle"] \arrow[dr, bend right, "0"'] & Y \times X \arrow[d, "\varphi_f"] & Y \arrow[ul, bend right, "\langle 1,0\rangle"'] \arrow[l, "\langle 1,0\rangle"'] \arrow[dl, bend left, "1"] \\
 & Y &
\end{tikzcd}$$

commutes. This implies that both $\varphi_f\langle\varphi_g,\pi_2\rangle$ and $\varphi_{f-(0-g)}$ are partial subtractors of f with respect to $\langle\varphi_g,\pi_2\rangle$, and now the result follows by the uniqueness of partial subtractors.

□

As a corollary, we can make the following useful observation.

Corollary 4.1.23. *For $f,g,h \in Z(X,Y)$ in a regular subtractive category $\mathbb{C}$, one has $(h-g)-f=h-(f-(0-g))$. In particular, $g-f=0-(f-g)$.*

Proof. As follows by the previous proposition, since $\varphi_f\langle\varphi_g,\pi_2\rangle=\varphi_{f-(0-g)}$, pre-composing on both side of the equation with $\langle h,1\rangle$, we obtain

$$\begin{aligned}
\varphi_f\langle\varphi_g,\pi_2\rangle\langle h,1\rangle &= \varphi_{f-(0-g)}\langle h,1\rangle \\
\varphi_f\langle h-g,1\rangle &= h-(f-(0-g)) \\
(h-g)-f &= h-(f-(0-g)).
\end{aligned} \tag{4.9}$$

Now in the equation $(h-g)-f=h-(f-(0-g))$, setting h to 0 and g to $(0-g)$, we get $(0-(0-g))-f=0-(f-(0-(0-g)))$, which simplifies to $g-f=0-(f-g)$.

□

We have the following technical lemma.

Lemma 4.1.24. *In a regular subtractive category $\mathbb{C}$, for $f \in Z(X,Y)$, let $f=me$ be the (regular epi, mono)-factorization. The following hold:*

(a) $ms\langle 0,e\rangle = 0-f$;

(b) $\varphi_{0-f} = \phi_f(1\times s\langle 0,e\rangle)$,

where $s : f(X)\times f(X) \longrightarrow f(X)$ *is the subtraction on the image* $f(X)$ *and* $\phi_f : Y\times f(X) \longrightarrow Y$ *is the subtractor of* m.

Proof. (a). In the diagram

$$\begin{tikzcd}
X \arrow[r, "{\langle 0,1\rangle}"] \arrow[d, "{\langle 0,e\rangle}"'] & Y\times X \arrow[d, "1\times e"] \arrow[dd, bend left, dotted, "\varphi_f"] \\
f(X)\times f(X) \arrow[r, "m\times 1"] \arrow[d, "s"'] & Y\times f(X) \arrow[d, "\phi_f"'] \\
f(X) \arrow[r, "m"'] & Y
\end{tikzcd}$$

the commutativity of the upper rectangle is clear. Furthermore, since the lower rectangle also commutes and $\phi_f(1\times e) = \varphi_f$ (see Remark 4.1.16), it follows that $ms\langle 0,e\rangle = \varphi_f\langle 0,1\rangle = 0-f$.

(b). Since $0-f = ms\langle 0,e\rangle$ as observed in (a), it is not difficult to see that the diagram

$$\begin{tikzcd}
Y \arrow[r, "1"] \arrow[d, "{\langle 1,0\rangle}"'] & Y \arrow[d, "{\langle 1,0\rangle}"] \arrow[dr, bend left, "1"] & \\
Y\times X \arrow[r, "{1\times s\langle 0,e\rangle}"] \arrow[rr, bend right, dotted, "\varphi_{0-f}"] & Y\times f(X) \arrow[r, "\phi_f"] & Y \\
X \arrow[u, "{\langle 0-f,1\rangle = \langle ms\langle 0,e\rangle,1\rangle}"] \arrow[r, "{s\langle 0,e\rangle}"'] & f(X) \arrow[u, "{\langle m,1\rangle}"'] \arrow[ur, bend right, "0"'] &
\end{tikzcd}$$

commutes. Now by the uniqueness of subtractors it follows that $\varphi_{0-f} = \phi_f(1\times s\langle 0,e\rangle)$. □

Proposition 4.1.25. *For* $f, g \in Z(X,Y)$ *in a regular subtractive category* $\mathbb{C}$, *we have* $(0-f)-(0-g) = g-f$.

Proof. Let me and $m'e'$ be the (regular epi, mono)-factorizations of f and g respectively. Furthermore, let us write s and s' to denote the subtractions on the images $f(X)$ and $g(X)$ respectively. Consider the diagram

$$\begin{tikzcd}
X \times X \arrow[r, two heads, "q \times q"] \arrow[d, "{\langle s(e \times e), s'(e' \times e') \rangle}"'] & (f-g)(X) \times (f-g)(X) \arrow[d, "s''"] \\
f(X) \times g(X) \arrow[d, "m \times 1"'] & (f-g)(X) \arrow[d, "p"] \\
Y \times g(X) \arrow[r, "\phi_g"'] & Y
\end{tikzcd}$$

where pq, with $q : X \twoheadrightarrow (f-g)(X)$ and $p : (f-g)(X) \rightarrowtail Y$, is the (regular epi, mono)-factorization of the morphism $f-g$, and s'' is the subtraction on the image $(f-g)(X)$. To show that the previous diagram commutes, we will observe that both the lower and upper composites $X \times X \longrightarrow Y$ on the diagram are subtractors of the identity morphism of X along $f-g$; commutativity will then follow by the uniqueness of (partial) subtractors (Proposition 4.1.6). First pre-composing with the diagonal $\langle 1, 1\rangle : X \longrightarrow X \times X$, we obtain

$$\phi_g(m \times 1)\langle s\langle e, e\rangle, s'\langle e', e'\rangle\rangle = 0 \text{ and } ps''\langle q, q\rangle = 0.$$

On the other hand, pre-composing with $\langle 1, 0\rangle : X \longrightarrow X \times X$, we obtain

$$\begin{aligned}
\phi_g\langle ms\langle e, 0\rangle, s'\langle e', 0\rangle\rangle &= \phi_g\langle me, e'\rangle \\
&= \phi_g\langle f, e'\rangle \\
&= \phi_g(1 \times e')\langle f, 1\rangle \\
&= \varphi_g\langle f, 1\rangle \quad \text{(apply Remark 4.1.16)} \\
&= f - g,
\end{aligned} \tag{4.10}$$

and $ps''\langle q, 0\rangle = pq = f - g$. Therefore the above diagram commutes. Now we obtain $(0 - f) - (0 - g) = g - f$ by pre-composing with $\langle 0, 1\rangle : X \longrightarrow X \times X$ on the two equal composites $X \times X \longrightarrow Y$; that is,

$$\begin{aligned}
\phi_g(m \times 1)\langle s(e \times e), s'(e' \times e')\rangle\langle 0, 1\rangle &= \phi_g\langle ms\langle 0, e\rangle, s'\langle 0, e'\rangle\rangle \\
&= \phi_g\langle 0 - f, s'\langle 0, e'\rangle\rangle \quad \text{(since } ms\langle 0, e\rangle = 0 - f\text{)} \\
&= \phi_g(1 \times s\langle 0, e'\rangle)\langle 0 - f, 1\rangle \\
&= \varphi_{0-g}\langle 0 - f, 1\rangle \quad \text{(since } \phi_g(1 \times s\langle 0, e'\rangle) = \varphi_{0-g}\text{)} \\
&= (0 - f) - (0 - g),
\end{aligned} \tag{4.11}$$

which is the same as $ps''(q \times q)\langle 0, 1\rangle = ps''\langle 0, q\rangle = 0 - (f - g) = g - f$ by Corollary 4.1.23. □

Now we can state our main result for this section.

Theorem 4.1.26. *In a regular subtractive category $\mathbb{C}$, for every pair of objects X and Y, $Z(X, Y)$ has an abelian group structure.*

Proof. We use the subtraction "$-$" on $Z(X, Y)$ to define a binary operation "$+$" by

$$f + g := f - (0 - g) \text{ for } f, g \in Z(X, Y).$$

Using Proposition 4.1.21 (a), for every $f \in Z(X,Y)$, $f+0 = f = 0-(0-f) = 0+f$. Furthermore, for $f \in Z(X,Y)$, there is $0-f \in Z(X,Y)$ such that

$$f + (0-f) = f - (0-(0-f)) = f - f = 0 = (0-f) - (0-f) = (0-f) + f.$$

For $f, g, h \in Z(X,Y)$, one has

$$\begin{aligned}(h+g)+f &= (h+g) - (0-f) \\ &= (h-(0-g)) - (0-f) \\ &= h - ((0-f)-g) \qquad \text{(applying Corollary 4.1.23)} \\ &= h - (0-(g-(0-f))) \quad \text{(applying Corollary 4.1.23)} \\ &= h - (0-(g+f)) \\ &= h + (g+f).\end{aligned} \tag{4.12}$$

Lastly, setting f to $0-f$ in the previous proposition, we obtain $f+g = (0-(0-f))-(0-g) = g-(0-f) = g+f$ for all $f, g \in Z(X,Y)$. □

Remark 4.1.27. *For a pair of objects X and Y in a regular subtractive category $\mathbb{C}$, the fact that $(Z(X,Y),+,0)$ is an abelian group only depends on the following identities*

(a) $f - f = 0$ *and* $f - 0 = f$*;*

(b) $(h-g) - f = h - (f-(0-g))$*;*

(c) $(0-f)-(0-g) = g - f$ *for all* $f, g, h \in Z(X,Y)$*.*

Note that $0-(0-h) = h$ is obtained from (b) by setting g to h and f to $(0-h)$. More generally, a subtractive algebra X is an abelian group, if and only if, $(x-y)-z = x-(z-(0-y))$ and $(0-x)-(0-y) = y-x$ for all $x,y,z \in X$. If a subtractive algebra X satisfies only $(x-y)-z = x-(z-(0-y))$, then it is just a group.

4.2 Commutativity in categories

In this section we extend categorical commutativity to a pointed finitely complete category with finite joins of subobjects. Our main aim is to investigate categorical commutativity in a subtractive category $\mathbb{C}$ with finite joins of subobjects. Let us first fix notation, and also give some necessary background.

For a pair of objects X and Y in a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, we write $m : X \star Y \rightarrowtail X \times Y$ to denote the join of the canonical morphisms $\langle 1,0\rangle : X \longrightarrow X \times Y$ and $\langle 0,1\rangle : Y \longrightarrow X \times Y$. The morphisms $I_1 : X \longrightarrow X \star Y$ and $I_2 : Y \longrightarrow X \star Y$ denote the respective factorizations of $\langle 1,0\rangle$ and $\langle 0,1\rangle$ through the join $m : X \star Y \longrightarrow X \times Y$

$$\begin{tikzcd} X \arrow[r, "I_1"] \arrow[dr, tail, bend right, "{\langle 1,0\rangle}"'] & X \star Y \arrow[d, tail, "m"] & Y \arrow[l, "I_2"'] \arrow[dl, tail, bend left, "{\langle 0,1\rangle}"] \\ & X \times Y. & \end{tikzcd}$$

Remark 4.2.1. *Note that for objects X and Y in a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, the morphism $m : X \star Y \rightarrowtail X \times Y$ being the join of the morphisms $\langle 1, 0\rangle : X \longrightarrow X \times Y$ and $\langle 0, 1\rangle : Y \longrightarrow X \times Y$ implies the pair of morphisms $I_1 : X \longrightarrow X \star Y$ and $I_2 : Y \longrightarrow X \star Y$ is jointly extremal-epimorphic, and hence jointly epimorphic.*

When $\mathbb{C}$ is a pointed regular category with finite coproducts, the join of a pair of subobjects (H, h) and (K, k) of X can be given by the regular image of the factorization $[h, k] : H+K \longrightarrow X$. This means for a pair of objects X and Y in a pointed regular category $\mathbb{C}$ with finite coproducts, the join of $\langle 1, 0\rangle : X \longrightarrow X \times Y$ and $\langle 0, 1\rangle : Y \longrightarrow X \times Y$ can be given by the regular image of the canonical morphism $X + Y \longrightarrow X \times Y$ in the diagram

$$\begin{tikzcd} & X \star Y \arrow[dr, tail, "m"] & \\ X + Y \arrow[ur, two heads, "e"] \arrow[rr, "\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}"'] & & X \times Y. \end{tikzcd}$$

Remark 4.2.2. *In a unital category $\mathbb{C}$, for every pair of objects X and Y the join $m : X \star Y \rightarrowtail X \times Y$ is indiscrete, i.e. $X \star Y \cong X \times Y$.*

Definition 4.2.3. *Let $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ be morphisms in a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects. The pair f, g is said to commute when there exists a morphism φ making the diagram*

$$\begin{tikzcd} X \arrow[r, "I_1"] \arrow[dr, bend right, "f"'] & X \star Y \arrow[d, dashed, "\varphi"] & Y \arrow[l, "I_2"'] \arrow[dl, bend left, "g"] \\ & Z & \end{tikzcd}$$

*commute. When such a morphism φ exists it is necessarily unique, since the pair of morphisms I_1 and I_2 is jointly epimorphic. The morphism φ is usually called the **cooperator** of f and g.*

Remark 4.2.4. *Note that when $\mathbb{C}$ is a unital category the previous definition coincides with the definition of commuting morphisms in the sense of Huq [22].*

Definition 4.2.3 allows us to explore categorical commutativity in a wide range of examples. For now we are only going to focus on subtractive categories with finite joins of subobjects. We shall describe what it means for a pair of morphisms having the same codomain to commute in the variety $\mathcal{S}$ of subtractive algebras.

Given a subtractive algebra X in $\mathcal{S}$, let us define the following recursive terms on X :

$$s_1(x_1, x_2) := s(x_1, x_2) \text{ and } s_n(x_1, x_2, ..., x_n, x_{n+1}) := s(s_{n-1}(x_1, x_2, ..., x_n), x_{n+1})$$

for $n \geq 2$ and each $x_i \in X$. Writing s_n explicitly using "$-$" instead of s,

$$s_n(x_1, x_2, ..., x_n, x_{n+1}) = ((...(((x_1 - x_2) - x_3) - x_4)..., -x_n) - x_{n+1}).$$

When necessary, we may use $-$ instead of s to simplify notation. In addition, when no confusion may arise, we can drop the subscript when writing s_n and just write $s(x_1, x_2, ..., x_n)$ instead of $s_n(x_1, x_2, ..., x_n)$.

Let us describe $X \star Y$ for any pair of subtractive algebras X and Y in $\mathcal{S}$. By definition $X \star Y$ is the subtractive algebra generated by the pairs $(x, 0)$ and $(0, y)$, with $x \in X$ and $y \in Y$. Thus, an element in $X \star Y$ is either of the form

$$s((x, 0), (0, y_1), ..., (0, y_n)) = (x, s(0, y_1, ..., y_n)) \text{ or } s((0, y), (x_1, 0), ..., (x_m, 0)) = (s(0, x_1, ..., x_m), y).$$

Thus,

$$X \star Y = \{(x, s(0, y_1, ..., y_n)) | x \in X, y_i \in Y\} \cup \{(s(0, x_1, ..., x_m), y) | x_i \in X, y \in Y\}.$$

Let us write $\mathcal{A} = \{(x, s(0, y_1, ..., y_n)) | x \in X, y_i \in Y\}$ and $\mathcal{B} = \{(s(0, x_1, ..., x_m), y) | x_i \in X, y \in Y\}$, so that $X \star Y = \mathcal{A} \cup \mathcal{B}$. The union $\mathcal{A} \cup \mathcal{B}$ is not disjoint, as for example, pairs such as $(0, 0)$ and $(0 - x, 0 - y)$ can belong to either $\mathcal{A}$ or $\mathcal{B}$. We will say two elements of $X \star Y$ are of (a) "same type" if they both belong to $\mathcal{A}$ or they both belong to $\mathcal{B}$, for example $(x, 0)$ and $(x, 0 - y)$, and, $(0 - x, y)$ and $(0, y)$; (b) "both type" if they belong to the intersection $\mathcal{A} \cap \mathcal{B}$, for example $(0, 0)$ and $(0 - x, 0 - y)$; (c) "different type" if otherwise, for example $(x, 0)$ and $(0, y)$.

Now given a pair of homomorphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ in the variety $\mathcal{S}$ of subtractive algebras, commutativity of f and g is given by the necessary and sufficient conditions for the existence of a homomorphism $\varphi : X \star Y \longrightarrow Z$ in $\mathcal{S}$, satisfying that $\varphi(x, 0) = f(x)$ and $\varphi(0, y) = g(y)$ for all $x \in X$ and $y \in Y$. Since every element $(x, s(0, y_1, ..., y_n))$ in $X \star Y$ can be expressed as $(x, s(0, y_1, ..., y_n)) = s((x, 0), (0, y_1), ..., (0, y_n))$ (similarly $(s(0, x_1, ..., x_m), y) = s((0, y), (x_1, 0), ..., (x_m, 0)))$, if φ is a homomorphism, then it is defined by

$$\varphi(x, s(0, y_1, ..., y_n)) = s(f(x), g(y_1), ..., g(y_n))$$

and

$$\varphi(s(0, x_1, ..., x_m), y) = s(g(y), f(x_1), ..., f(x_m)).$$

We can further observe below that for an element of "both type", that is an element of the form

$$(s(0, x_1, ..., x_n), s(0, y_1, ..., y_m)),$$

φ is well defined:

$$\begin{aligned} & s(s(0, f(x_1), ..., f(x_n)), g(y_1), ..., g(y_m)) \\ & = \varphi\big(s(0, x_1, ..., x_n), s(0, y_1, ..., y_m)\big) \\ & = s\big(\varphi(0, s(0, y_1, ..., y_m)), \varphi(x_1, 0), \varphi(x_2, 0), ..., \varphi(x_n, 0)\big) \quad \text{(since } \varphi \text{ is a homomorphism)} \\ & = s\big(s(0, g(y_1), ..., g(y_m)), f(x_1), ..., f(x_n)\big). \end{aligned} \tag{4.13}$$

For readability, we shall assume f and g are inclusions. Now the necessary and sufficient conditions for the existence of a homomorphism $\varphi : X \star Y \longrightarrow Z$ in $\mathcal{S}$ are given in the next theorem.

Theorem 4.2.5. *Let $f : X \hookrightarrow Z$ and $g : Y \hookrightarrow Z$ be homomorphisms in the variety $\mathcal{S}$ of subtractive algebras. The pair f, g commute if and only if the following identities hold:*

(a) $s\big(x, y_1, y_2, ..., y_n, s(y, x_1, x_2, ..., x_m)\big) = s\big(x, s(0, x_1, x_2, ..., x_m), y_1, y_2, ..., y_n, y\big)$;

(b) $s\big(x, y_1, y_2, ..., y_n, s(x', y'_1, y'_2, ..., y'_m)\big) = s\big(x, x', y_1, y_2, ..., y_n, s(0, y'_1, y'_2, ..., y'_m)\big)$;

(c) $s\big(y, x_1, x_2, ..., x_n, s(y', x'_1, x'_2, ..., x'_m)\big) = s\big(y, y', x_1, x_2, ..., x_n, s(0, x'_1, x'_2, ..., x'_m)\big)$;

(d) $s\big(y, x_1, x_2, ..., x_n, s(x, y_1, y_2, ..., y_m)\big) = s\big(y, s(0, y_1, y_2, ..., y_m), x_1, x_2, ..., x_n, x\big)$.

Proof. As observed before, the homomorphisms f and g commute when there exists a homomorphism $\varphi : X \star Y \longrightarrow Z$ in $\mathcal{S}$ such that $\varphi(x, 0) = x$ and $\varphi(0, y) = y$, for all $x \in X$ and $y \in Y$. Observe that equations (a) and (d) are obtained in a similar way; they are both obtained by subtracting two elements of "different type" and then apply the homomorphism φ to it. Similarly, equations (b) and (c) are obtained by subtracting two elements of "same type", and then apply the homomorphism φ to it. Equation (a) is obtained as follows:

$$\begin{aligned} & s\big(x, y_1, y_2, ..., y_n, s(y, x_1, x_2, ..., x_m)\big) \\ &= s\big(\varphi\big(x, s(0, y_1, ..., y_n)\big), \varphi\big(s(0, x_1, ..., x_m), y\big)\big) \\ &= \varphi\big(s(x, s(0, x_1, ..., x_m)), s(0, y_1, ..., y_n, y)\big) \\ &= s\big(x, s(0, x_1, x_2, ..., x_m), y_1, y_2, ..., y_n, y\big). \end{aligned} \tag{4.14}$$

The other three equations are obtained in a similar way. For the "if" part, let us suppose the four identities, namely, $(a), (b), (c)$, and (d) hold. We will show that the map $\varphi : X \star Y \longrightarrow Z$ defined by

$$\varphi(x, s(0, y_1, ..., y_n)) = s(x, y_1, ..., y_n)$$

and

$$\varphi(s(0, x_1, ..., x_m), y) = s(y, x_1, ..., x_m)$$

is a homomorphism in $\mathcal{S}$. Let us first show that taking

$$(s(0, x_1, ..., x_n), s(0, y_1, ..., y_m))$$

as an element of "both type", and apply φ accordingly, the two images are always the same, i.e. we need to show that

$$s\big(0, x_1, ..., x_n, y_1, ..., y_m\big) = s\big(0, y_1, ..., y_m, x_1, ..., x_n\big). \tag{4.15}$$

Note that by setting $x'_i = 0$ for $i = 1, 2, ..., m$, and $y' = y_1$ in (c), we obtain

$$s\big(y, x_1, ..., x_n, y_1\big) = s\big(y, y_1, x_1, ..., x_n\big). \tag{4.16}$$

Using the previous equation, we get

$$s\big(0, x_1, ..., x_n, y_1, ..., y_m\big) = s\big(0, y_1, x_1, ..., x_n, y_2, ..., y_m\big).$$

Applying the same process repeatedly to the term $s\big(0, x_1, ..., x_n, y_1, ..., y_m\big)$, after m steps we will obtain equation (4.15)

$$s\big(0, x_1, ..., x_n, y_1, ..., y_m\big) = s\big(0, y_1, ..., y_m, x_1, ..., x_n\big).$$

Next we show that φ is a homomorphism in $\mathcal{S}$. Since $X \star Y$ is the union of two sets, without loss of generality, we can just check that φ preserves subtraction when two elements of "same type" and when two elements of "different type" are subtracted. More precisely, for elements $(x', s(0, y'_1, ..., y'_p)), (s(0, x_1, ..., x_m), y)$, and $(x, s(0, y_1, ..., y_n))$ in $X \star Y$, we will show that

$$\varphi\big(s(x, x'), s(0, y_1, ..., y_n, s(0, y'_1, ..., y'_p))\big) = s\big(\varphi\big(x, s(0, y_1, ..., y_n)\big), \varphi\big(x', s(0, y'_1, ..., y'_p)\big)\big)$$

and

$$\varphi\big(s(x, s(0, x_1, ..., x_m)), s(0, y_1, ..., y_n, y)\big) = s\big(\varphi\big(x, s(0, y_1, ..., y_n)\big), \varphi\big(s(0, x_1, ..., x_m), y\big)\big).$$

For that, we have

$$\begin{aligned} & s\big(\varphi\big(x, s(0, y_1, ..., y_n)\big), \varphi\big(x', s(0, y'_1, ..., y'_p)\big)\big) \\ & = s\big(x, y_1, y_2, ..., y_n, s(x', y'_1, y'_2, ..., y'_p)\big) \\ & = s\big(x, x', y_1, y_2, ..., y_n, s(0, y'_1, y'_2, ..., y'_p)\big) \quad \text{(applying (b))} \\ & = \varphi\big(s(x, x'), s(0, y_1, ..., y_n, s(0, y'_1, ..., y'_p))\big), \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} & s\big(\varphi\big(x, s(0, y_1, ..., y_n)\big), \varphi\big(s(0, x_1, ..., x_m), y\big)\big) \\ & = s\big(x, y_1, y_2, ..., y_n, s(y, x_1, x_2, ..., x_m)\big) \\ & = s\big(x, s(0, x_1, x_2, ..., x_m), y_1, y_2, ..., y_n, y\big) \quad \text{(applying (a))} \\ & = \varphi\big(s(x, s(0, x_1, ..., x_m)), s(0, y_1, ..., y_n, y)\big). \end{aligned} \tag{4.18}$$

$\square$

Accordingly, commutative objects are defined as follows:

Definition 4.2.6. *In a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, an object X is said to be commutative when the identity morphism of X commute with itself, that is, when there is a morphism $\varphi : X \star X \longrightarrow X$ making the diagram*

$$\begin{tikzcd} X \arrow[r, "I_1"] \arrow[dr, bend right, "1"'] & X \star X \arrow[d, dashed, "\varphi"] & X \arrow[l, "I_2"'] \arrow[dl, bend left, "1"] \\ & X & \end{tikzcd}$$

commute.

In the previous theorem, taking both f and g to be the identity morphism of X, the equations $(a), (b), (c)$, and (d) all reduce to one equation, that is,

$$\begin{aligned} s(x, x_1, x_2, ..., x_n), s(x', x'_1, x'_2, ..., x'_m)) &= s(x, s(0, x'_1, x'_2, ..., x'_m), x_1, x_2, ..., x_n, x') \\ &= s(x, x', x_1, x_2, ..., x_n, s(0, x'_1, x'_2, ..., x'_m)). \end{aligned} \tag{4.19}$$

It can be seen that equation (4.19) is generated by the following two simple equations: (a) $(x - x_1) - x_2 = (x - x_2) - x_1$; (b) $x - (x_1 - x_2) = (x - x_1) - (0 - x_2)$ (which can also be seen as a special case of Corollary 4.3.3 below). Thus:

Remark 4.2.7. *In the variety $\mathcal{S}$ of subtractive algebras, a subtractive algebra X is commutative if and only if the following hold for all $x, y, z \in X$,*

(a) $(x - y) - z = (x - z) - y$;

(b) $x - (y - z) = (x - y) - (0 - z)$.

We can further make the following observation:

Proposition 4.2.8. *In the variety $\mathcal{S}$ of subtractive algebras, for a subtractive algebra X the following statements are equivalent:*

(a) X is commutative;

(b) X is an abelian group.

Proof. If X is commutative then $(x - y) - w = (x - w) - y$ and $x - (w - z) = (x - w) - (0 - z)$ for all $x, y, w, z \in X$. Now for $x \in X$, we see that $0 - (0 - x) = (x - x) - (0 - x) = x - (x - x) = x$. As observed in Remark 4.1.27, a subtractive algebra X has an abelian group structure if $(x - y) - z = x - (z - (0 - y))$ and $(0 - x) - (0 - y) = y - x$ for all $x, y, z \in X$. Clearly, $x - (z - (0 - y)) = (x - z) - (0 - (0 - y)) = (x - z) - y = (x - y) - z$ and $(0 - x) - (0 - y) = (0 - (0 - y)) - x = y - x$, and these give $(a) \Rightarrow (b)$. The implication $(b) \Rightarrow (a)$ is obvious. $\square$

Our next task is to generalize the previous proposition to a general subtractive category with finite joins of subobjects. We will first explain some notation and give some auxiliary facts.

For objects X and Y in a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, we will write $\pi_1^\star$ and $\pi_2^\star$ to denote the composites $\pi_1 m$ and $\pi_2 m$ respectively, in which m is the join $m : X \star Y \longrightarrow X \times Y$

$$\begin{tikzcd} X \arrow[r, "I_1"] \arrow[dr, tail, bend right, "\langle 1,0 \rangle"'] & X \star Y \arrow[d, tail, "m"] & Y \arrow[l, "I_2"'] \arrow[dl, tail, bend left, "\langle 0,1 \rangle"] \\ & X \times Y. & \end{tikzcd}$$

Since the product projections $\pi_1 : X \times Y \longrightarrow X$ and $\pi_2 : X \times Y \longrightarrow Y$ are jointly monomorphic, and m is a monomorphism, the pair $\pi_1^\star, \pi_2^\star$ is also jointly monomorphic. Given a morphism $r : R \longrightarrow X \star Y$, we will write $r_1 = \pi_1^\star r$ and $r_2 = \pi_2^\star r$. We see that $\pi_1^\star I_1 = 1$, $\pi_2^\star I_1 = 0$ and $\pi_1^\star I_2 = 0$, $\pi_2^\star I_2 = 1$.

Lemma 4.2.9. *In a subtractive category $\mathbb{C}$ with finite joins of subobjects, for a morphism $f : X \longrightarrow Y$, let $\varphi : X\star Y \longrightarrow Y$ be a morphism such that $f = \varphi I_1$ and φI_2 is the identity morphism of Y. The morphism $\langle \varphi, \pi_1^\star \rangle : X \star Y \longrightarrow Y \times X$ is a monomorphism.*

Proof. Let (K, k, k') be the kernel pair relation of $\langle \varphi, \pi_1^\star \rangle$. We will show that $k = k'$. Letting $k_1 = \pi_1^\star k, k_2 = \pi_2^\star k$ and $k'_1 = \pi_1^\star k', k'_2 = \pi_2^\star k'$, since $\langle \varphi, \pi_1^\star \rangle k = \langle \varphi, \pi_1^\star \rangle k'$, it follows that $k_1 = \pi_1^\star k = \pi_1^\star k' = k'_1$. In the diagram

$$\begin{tikzcd}
X \times (Y \times Y) \arrow[r, shift left, "1 \times \pi_1"] \arrow[r, shift right, "1 \times \pi_2"'] & X \times Y \arrow[r, "\pi_1"] & X \\
K \arrow[u, "\langle k_1, \langle k_2, k'_2 \rangle \rangle"] \arrow[r, shift left, "k"] \arrow[r, shift right, "k'"'] & X \star Y \arrow[u, tail, "m"] \arrow[r, "\langle \varphi, \pi_1^\star \rangle"'] & Y \star X \arrow[u, "\pi_2"'] \\
X \arrow[u, dashed, "\beta"'] \arrow[uu, dashed, bend left, "\langle 1, \langle 0, 0 \rangle \rangle"] \arrow[ur, bend right=20, "I_1"] \arrow[ur, bend right=40, "I_1"'] & &
\end{tikzcd}$$

in which β is a morphism such that $k\beta = I_1 = k'\beta$, we see that $\langle 1, \langle 0, 0 \rangle \rangle$ factors through $\langle k_1, \langle k_2, k'_2 \rangle \rangle$ by β. Therefore, we obtain the following commutative diagram

$$\begin{tikzcd}
& & K \arrow[dd, tail, "\langle k_1, \langle k_2, k'_2 \rangle \rangle"] & \\
& K \arrow[ur, dashed, "\lambda"] \arrow[dr, tail, "\langle 0, \langle k_2, k'_2 \rangle \rangle"'] & & \\
K \arrow[uurr, bend left, "1"] \arrow[rr, tail, "\langle k_1, \langle k_2, k'_2 \rangle \rangle"'] & & Y \times (Y \times Y) & K, \arrow[l, tail, "\langle k_1, \langle 0, 0 \rangle \rangle"] \arrow[uul, bend right, "\beta k_1"']
\end{tikzcd}$$

from which by using subtractivity, we conclude that $\langle 0, \langle k_2, k'_2 \rangle \rangle$ factors through $\langle k_1, \langle k_2, k'_2 \rangle \rangle$. Using the diagram

$$\begin{tikzcd}
& K \arrow[r, shift left, "k"] \arrow[r, shift right, "k'"'] \arrow[d, "\langle k_1, \langle k_2, k'_2 \rangle \rangle"] & X \star Y \arrow[d, tail, "m"] \\
K \arrow[ur, dashed, "\lambda"] \arrow[r, "\langle 0, \langle k_2, k'_2 \rangle \rangle"'] & X \times (Y \times Y) \arrow[r, shift left, "1 \times \pi_1"] \arrow[r, shift right, "1 \times \pi_2"'] & X \times Y
\end{tikzcd}$$

we see that

$$mk\lambda = \langle 0, k_2 \rangle = mI_2k_2 \text{ and } mk'\lambda = \langle 0, k'_2 \rangle = mI_2k'_2,$$

which imply that $k\lambda = I_2k_2$ and $k'\lambda = I_2k'_2$ respectively. Hence $k_2 = \varphi I_2k_2 = \varphi I_2k'_2 = k'_2$, and, together with $k_1 = k'_1$, we have $k = k'$.

$\square$

As a special case of the previous lemma, we obtain the following:

Remark 4.2.10. *In a subtractive category $\mathbb{C}$ with finite joins of subobjects, if X is a commutative object, with the cooperator denoted by $\varphi : X \star X \longrightarrow X$, then the morphism $\langle \varphi, \pi_1^\star \rangle : X \star X \longrightarrow X \times X$ is a monomorphism.*

We recall:

Proposition 4.2.11 (see [12], Corollary 2.6)**.** *In a subtractive category $\mathbb{C}$ if X and X' admit internal subtraction structures s and s' respectively, then every morphism $f : X \longrightarrow X'$ is a homomorphism of subtraction, i.e. the diagram*

$$\begin{tikzcd} X \times X \arrow[r, "s"] \arrow[d, "f \times f"'] & X \arrow[d, "f"] \\ X' \times X' \arrow[r, "s'"'] & X' \end{tikzcd}$$

commutes.

Proof. It is not difficult to see that $s'(f \times f)\langle 1,1\rangle = 0 = fs\langle 1,1\rangle$ and $s'(f \times f)\langle 1,0\rangle = f = fs\langle 1,0\rangle$. But since X admits a subtraction s, the pair $\langle 1,1\rangle, \langle 1,0\rangle$ is jointly epimorphic (being jointly extremal-epimorphic according to Proposition 4.1.13). Hence the diagram above commutes. $\square$

Applying the previous proposition, every subtraction $s : X \times X \longrightarrow X$ in a subtractive category is a homomorphism of subtraction, i.e. the diagram

$$\begin{tikzcd} (X \times X) \times (X \times X) \arrow[r, "s'"] \arrow[d, "s \times s"'] & X \times X \arrow[d, "s"] \\ X \times X \arrow[r, "s"'] & X, \end{tikzcd}$$

in which s' is the subtraction on $X \times X$ given by

$$\begin{tikzcd} (X \times X) \times (X \times X) \arrow[r, "i"] \arrow[rr, bend right, "s'"'] & (X \times X) \times (X \times X) \arrow[r, "s \times s"] & X \times X, \end{tikzcd}$$

where $i = \langle\langle \pi_1\pi_1, \pi_1\pi_2\rangle, \langle \pi_2\pi_1, \pi_2\pi_2\rangle\rangle$ is the "middle interchange", commutes. Explicitly, the commutativity of the previous rectangle, which represents s being a homomorphism of subtraction, means that

$$\begin{aligned} s\langle s\langle \pi_1\pi_1, \pi_1\pi_2\rangle, s\langle \pi_2\pi_1, \pi_2\pi_2\rangle\rangle &= ss' \\ &= s(s \times s) \\ &= s\langle s\pi_1, s\pi_2\rangle \\ &= s\langle s\langle \pi_1, \pi_2\rangle\pi_1, s\langle \pi_1, \pi_2\rangle\pi_2\rangle \\ &= s\langle s\langle \pi_1\pi_1, \pi_2\pi_1\rangle, s\langle \pi_1\pi_2, \pi_2\pi_2\rangle\rangle. \end{aligned} \tag{4.20}$$

And now we can state a categorical version of Proposition 4.2.8.

Theorem 4.2.12. *For an object X in a subtractive category $\mathbb{C}$ with finite joins of subobjects, the following statements are equivalent:*

(a) X is commutative;

(b) X admits an internal subtraction structure;

(c) X admits an internal abelian group structure.

Proof. $(a) \Rightarrow (b)$. Let $\varphi : X \star X \longrightarrow X$ be the cooperator of the pair of identity morphisms of X. By Remark 4.2.10 we know that $\langle \varphi, \pi_1^\star\rangle$ is a monomorphism. Since the diagram

$$\begin{tikzcd} & X \star X \arrow[d, tail, "\langle \varphi, \pi_1^\star \rangle"] \arrow[dd, dotted, bend right, "\pi_2^\star"'] & \\ X \arrow[ur, bend left, "I_1"] \arrow[r, "\langle 1,1 \rangle"] \arrow[dr, bend right, "0"'] & X \times X & X \arrow[ul, bend right, "I_2"'] \arrow[l, "\langle 1,0 \rangle"'] \arrow[dl, bend left, "1"] \\ & X & \end{tikzcd}$$

commutes, we see that $\pi_1^\star$ is a partial subtractor of the identity morphism of X with respect to $\langle \varphi, \pi_1^\star\rangle$. Now the result follows from Lemma 4.1.10.

$(b) \Rightarrow (a)$. Since $s\langle 0, s\langle 0, 1\rangle\rangle = s\langle s\langle 1, 1\rangle, s\langle 0, 1\rangle\rangle = s\langle s\langle 1, 0\rangle, s\langle 1, 1\rangle\rangle = 1$, it is not difficult to see that the diagram

$$\begin{tikzcd} X \arrow[d, "\langle 0,1 \rangle"'] \arrow[drr, bend left, "1"] & & \\ X \times X \arrow[r, "1 \times s\langle 0,1 \rangle"] & X \times X \arrow[r, "s"] & X \\ X \arrow[u, "\langle 1,0 \rangle"] \arrow[urr, bend right, "1"'] & & \end{tikzcd}$$

commutes. Hence, pre-composing with the morphism $m : X \star X \rightarrowtail X \times X$, it can be easily seen that the composite $s(1 \times s\langle 0, 1\rangle)m$ is the cooperator of the pair of identity morphisms of X.

$(b) \Leftrightarrow (c)$. According to Corollary 2.7 of [12], an internal subtraction structure on an object is always part of a unique internal abelian group structure on the same object.

□

Although we can deduce this fact from Theorem 4.1.26, we are going to explicitly extract an internal abelian group structure from a subtraction $s : X \times X \longrightarrow X$ on an object X in a subtractive category $\mathbb{C}$. Using the subtraction s, addition $p : X \times X \longrightarrow X$ is defined by the morphism $p = s(1 \times s\langle 0, 1\rangle)$

$$\begin{tikzcd} X \times X \arrow[r, "1 \times s\langle 0,1\rangle"] \arrow[rr, bend right, "p = s\langle \pi_1, s\langle 0, \pi_2\rangle\rangle"'] & X \times X \arrow[r, "s"] & X. \end{tikzcd}$$

Let us show that $s\langle 0, 1\rangle : X \longrightarrow X$ is the additive inverse of p; that is, showing that the square

$$\begin{tikzcd} X \arrow[r, "\langle 1, s\langle 0,1\rangle\rangle"] \arrow[d, "\langle s\langle 0,1\rangle, 1\rangle"'] \arrow[dr, "0"] & X \times X \arrow[d, "p"] \\ X \times X \arrow[r, "p"'] & X \end{tikzcd}$$

commutes. Since s is a homomorphism, we have

$$\begin{aligned} p\langle 1, s\langle 0,1\rangle\rangle &= s(1 \times s\langle 0,1\rangle)\langle 1, s\langle 0,1\rangle\rangle \\ &= s\langle 1, s\langle 0,1\rangle s\langle 0,1\rangle\rangle \\ &= s\langle 1, s\langle 0, s\langle 0,1\rangle\rangle\rangle \\ &= s\langle 1, 1\rangle \quad (\text{since } \ s\langle 0, s\langle 0,1\rangle\rangle = 1) \\ &= 0, \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} p\langle s\langle 0,1\rangle, 1\rangle &= s(1 \times s\langle 0,1\rangle)\langle s\langle 0,1\rangle, 1\rangle \\ &= s\langle s\langle 0,1\rangle, s\langle 0,1\rangle\rangle \\ &= s\langle 1,1\rangle s\langle 0,1\rangle \\ &= 0. \end{aligned} \tag{4.22}$$

Let us also show that $p\langle 1, 0\rangle = 1 = p\langle 0, 1\rangle$ i.e. the diagram

$$\begin{tikzcd} X \arrow[r, "\langle 1,0\rangle"] \arrow[d, "\langle 0,1\rangle"'] \arrow[dr, "1"] & X \times X \arrow[d, "p"] \\ X \times X \arrow[r, "p"'] & X \end{tikzcd}$$

commutes. Clearly, $p\langle 1,0\rangle = s(1\times s\langle 0,1\rangle)\langle 1,0\rangle = s\langle 1,0\rangle = 1$. We also see that $p\langle 0,1\rangle = s(1\times s\langle 0,1\rangle)\langle 0,1\rangle = s\langle 0, s\langle 0,1\rangle\rangle = 1$.

We prove associativity by showing that the diagram

$$\begin{tikzcd}
(X\times X)\times X \arrow[rr, "\langle s\langle \pi_1\pi_1, s\langle 0,\pi_2\pi_1\rangle\rangle, \pi_2\rangle = p\times 1"] \arrow[d, "\langle \pi_1\pi_1, \langle \pi_2\pi_1, \pi_2\rangle\rangle"'] & & X\times X \arrow[dd, "s\langle \pi_1, s\langle 0,\pi_2\rangle\rangle = p"] \\
X\times (X\times X) \arrow[d, "\langle \pi_1, s\langle \pi_1\pi_2, s\langle 0, \pi_2\pi_2\rangle\rangle\rangle = 1\times p"'] & & \\
X\times X \arrow[rr, "s\langle \pi_1, s\langle 0,\pi_2\rangle\rangle = p"'] & & X
\end{tikzcd}$$

commutes. Again we use the fact that s is a homomorphism to obtain

$$\begin{aligned}
p(1\times p)\langle \pi_1\pi_1, \langle \pi_2\pi_1, \pi_2\rangle\rangle &= s\langle \pi_1, s\langle 0,\pi_2\rangle\rangle\langle \pi_1, s\langle \pi_1\pi_2, s\langle 0,\pi_2\pi_2\rangle\rangle\rangle\langle \pi_1\pi_1, \langle \pi_2\pi_1, \pi_2\rangle\rangle \\
&= s\langle \pi_1\pi_1, s\langle 0,1\rangle s\langle \pi_2\pi_1, s\langle 0,\pi_2\rangle\rangle \\
&= s\langle \pi_1\pi_1, s\langle s\langle s\langle 0,\pi_2\rangle, \pi_2\pi_1\rangle, 0\rangle\rangle \\
&= s\langle \pi_1\pi_1, s\langle s\langle 0,\pi_2\rangle, \pi_2\pi_1\rangle\rangle \\
&= s\langle s\langle \pi_1\pi_1, 0\rangle, s\langle s\langle 0, \pi_2\pi_1\rangle, \pi_2\rangle\rangle \\
&= s\langle s\langle \pi_1\pi_1, s\langle 0, \pi_2\pi_1\rangle\rangle, s\langle 0,\pi_2\rangle\rangle \\
&= s\langle \pi_1, s\langle 0,\pi_2\rangle\rangle\langle s\langle \pi_1\pi_1, s\langle 0,\pi_2\pi_1\rangle\rangle, \pi_2\rangle \\
&= p(p\times 1).
\end{aligned} \tag{4.23}$$

Lastly, we will show that p is an abelian group operation. For that, we will show $p\langle \pi_2,\pi_1\rangle = p$; that is,

$$\begin{aligned}
p\langle \pi_2,\pi_1\rangle &= s\langle \pi_1, s\langle 0,\pi_2\rangle\rangle\langle \pi_2,\pi_1\rangle \\
&= s\langle \pi_2, s\langle 0,\pi_1\rangle\rangle \\
&= s\langle s\langle 0, s\langle 0,\pi_2\rangle\rangle, s\langle 0,\pi_1\rangle\rangle \\
&= s\langle s\langle 0, s\langle 0,\pi_1\rangle\rangle, s\langle 0,\pi_2\rangle\rangle \\
&= s\langle \pi_1, s\langle 0,\pi_2\rangle\rangle = p.
\end{aligned} \tag{4.24}$$

Remark 4.2.13. *In a subtractive category $\mathbb{C}$ with finite joins of subobjects, when X admits an internal subtraction structure $s : X\times X \longrightarrow X$, the join $m : X\star X \rightarrowtail X\times X$*

$$\begin{tikzcd}
 & X\star X \arrow[d, tail, "m"] & \\
X \arrow[ur, "I_1"] \arrow[r, "\langle 1,0\rangle"'] & X\times X & X \arrow[ul, "I_2"'] \arrow[l, "\langle 0,1\rangle"]
\end{tikzcd}$$

is an isomorphism, since the pair of morphisms $\langle 1,0\rangle$ and $\langle 0,1\rangle$ is jointly extremal-epimorphic according to Proposition 4.1.13. Therefore, since the product $X\times X$ admits a subtraction structure whenever X does, $X\times X$ is commutative, and so is $X\star X$.

4.3 Central morphisms in subtractive categories

Definition 4.3.1. *Let $\mathbb{C}$ be a pointed finitely complete category with finite joins of subobjects. A morphism $f : X \longrightarrow Y$ is said to be central if it commutes with the identity morphism of Y.*

We have the following characterization:

Theorem 4.3.2. *Let $\mathbb{C}$ be a regular subtractive category with finite joins of subobjects. A morphism $f : X \longrightarrow Y$ is central if and only if it admits a partial subtraction structure.*

Proof. Suppose a morphism $f : X \longrightarrow Y$ is central, and $\varphi : X\star Y \longrightarrow Y$ is the cooperator of f and the identity morphism of Y. According to Lemma 4.2.9, the morphism $\langle\varphi,\pi_1^\star\rangle$ is a monomorphism. Therefore, as shown in the commutative diagram

$$\begin{tikzcd}
& X\star Y \arrow[d, tail, "\langle\varphi,\pi_1^\star\rangle"] \arrow[dd, dotted, bend right, "\pi_2^\star"'] & \\
X \arrow[ur, bend left, "I_1"] \arrow[r, "\langle f,1\rangle"] \arrow[dr, bend right, "0"'] & Y\times X & Y \arrow[ul, bend right, "I_2"'] \arrow[l, "\langle 1,0\rangle"'] \arrow[dl, bend left, "1"] \\
& Y &
\end{tikzcd}$$

f admits a partial subtraction structure. Conversely, if f admits a partial subtraction structure then it also admits a subtractor. Now let re be the (regular epi, mono)-factorization of f. According to Proposition 4.1.8, $r : f(X) \longrightarrow Y$ admits a subtractor $\phi_f : Y\times f(X) \longrightarrow Y$, and so the image $f(X)$ also admits a subtraction $s : f(X)\times f(X) \longrightarrow f(X)$ by applying Proposition 4.1.9 (a). As seen before (see Remark 4.1.16), $\phi_f(r\times 1) = rs$ and this means the rectangle in the diagram

$$\begin{tikzcd}
X \arrow[r, "\langle 0,s\langle 0,e\rangle\rangle"] \arrow[dr, bend right, "e"'] & f(X)\times f(X) \arrow[r, "r\times 1"] \arrow[d, "s"] & Y\times f(X) \arrow[d, "\phi_f"] \\
& f(X) \arrow[r, "r"'] & Y
\end{tikzcd}$$

commutes. Furthermore, using the fact that s is a homomorphism of subtraction, we have

$$\begin{aligned} s\langle 0,s\langle 0,e\rangle\rangle &= s\langle s\langle 1,1\rangle, s\langle 0,1\rangle\rangle e \\ &= s\langle s\langle 1,0\rangle, s\langle 1,1\rangle\rangle e \\ &= s\langle 1,0\rangle e \\ &= e, \end{aligned} \tag{4.25}$$

which means the left triangle commutes. Using the commutativity of the previous diagram, since $f = re = \phi_f(r \times 1)\langle 0, s\langle 0, e\rangle\rangle = \phi_f\langle 0, s\langle 0, e\rangle\rangle$, we see that the diagram

$$\begin{tikzcd}
X \arrow[d, "\langle 0,1\rangle"'] \arrow[rr, bend left, "f"] & & \\
Y \times X \arrow[r, "1 \times s\langle 0,e\rangle"] & Y \times f(X) \arrow[r, "\phi_f"] & Y \\
Y \arrow[u, "\langle 1,0\rangle"] \arrow[rru, bend right, "1"'] & &
\end{tikzcd}$$

commutes. Hence, pre-composing with the morphism $m : Y \star X \rightarrowtail Y \times X$, it can be easily seen that the composite $\phi_f(1 \times s\langle 0, e\rangle)m$ is the cooperator of f and the identity morphism of Y.

□

As a consequence of the previous theorem, we can apply Proposition 4.1.2 to describe central morphisms in the variety $\mathcal{S}$ of subtractive algebras.

Corollary 4.3.3. *In the variety $\mathcal{S}$ of subtractive algebras, a homomorphism $f : X \longrightarrow Y$ is central if and only if the following identities hold for all $x \in X$ and $y, y' \in Y$:*

(a) $(y - f(x)) - y' = (y - y') - f(x)$;

(b) $y - (y' - f(x)) = (y - y') - (0 - f(x))$.

As a result of Theorem 4.3.2 and Theorem 4.1.26, we extend the following fact; already known to be always true in a strongly unital category (see [6]).

Corollary 4.3.4. *For every pair of objects X and Y in a regular subtractive category $\mathbb{C}$ with finite joins of subobjects, writing $Z(X, Y)$ for the class of central morphisms from X to Y, $Z(X, Y)$ has an abelian group structure.*

The characterization of central morphisms in terms of partial subtraction structures not only gave a simple description of central morphisms in the variety $\mathcal{S}$ of subtractive algebras, but, as we will see, it also allows to associate to every morphism $f : X \longrightarrow Y$ in a regular subtractive category $\mathbb{C}$ with finite joins of subobjects and cokernels, a universal morphism $g : Y \longrightarrow Q$ which, by composition, makes f central. More precisely, we will show that for a morphism $f : X \longrightarrow Y$ in a regular subtractive category $\mathbb{C}$ with finite joins of subobjects and cokernels, there is a universal morphism $g : Y \longrightarrow Q$ for which the following diagram

$$\begin{tikzcd}
X \arrow[r, "\langle f,1\rangle"] \arrow[dr, bend right, "0"'] & Y \times X \arrow[d, "\varphi"] & Y \arrow[l, "\langle 1,0\rangle"'] \arrow[dl, bend left, "g"] \\
 & Q &
\end{tikzcd}$$

commutes. This is an extension of Corollary 1.10 of [5].

Let us first prove the following technical lemma.

Lemma 4.3.5. *In a regular subtractive category $\mathbb{C}$ with finite joins of subobjects, if a morphism $f : X \longrightarrow Y$ admits a subtractor along a regular epimorphism $g : Y \twoheadrightarrow Z$, then the composite gf is central.*

Proof. Consider the kernel pair relation

$$(Y \underset{<g,g>}{\times} Y, k_1, k_2)$$

of g, and let $\varphi_f : Y \times X \longrightarrow Z$ be the subtractor of f along g. It is clear that there is a morphism λ such that $k_1\lambda = f = k_2\lambda$. Now consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\langle \lambda, 1\rangle"] \arrow[d, "1"'] & (Y \underset{<g,g>}{\times} Y) \times X \arrow[d, shift right, "k_1 \times 1"'] \arrow[d, shift left, "k_2 \times 1"] & Y \underset{<g,g>}{\times} Y \arrow[l, "\langle 1, 0\rangle"'] \arrow[d, shift right, "k_1"'] \arrow[d, shift left, "k_2"] \\
X \arrow[r, "\langle f, 1\rangle"] \arrow[dr, "0"'] & Y \times X \arrow[d, "\varphi_f"] & Y \arrow[l, "\langle 1, 0\rangle"'] \arrow[dl, two heads, "g"] \\
 & Z. &
\end{tikzcd}$$

It can be seen that both $\varphi_f(k_1 \times 1)$ and $\varphi_f(k_2 \times 1)$ are subtractors of the morphism λ along $gk_1 = gk_2$. Hence, by the uniqueness of subtractors (Proposition 4.1.6), it follows that $\varphi_f(k_1 \times 1) = \varphi_f(k_2 \times 1)$. Since $g \times 1 : Y \times X \twoheadrightarrow Z \times X$, is the coequalizer of the pair $(k_1 \times 1)$, $(k_2 \times 1)$, it follows that φ_f factors through $g \times 1$ by a morphism φ_{gf}, that is $\varphi_f = \varphi_{gf}(g \times 1)$. Now it can be easily seen in the diagram below that φ_{gf} is a subtractor of gf, and this means gf is central

$$\begin{tikzcd}
X \arrow[r, "\langle f, 1\rangle"] \arrow[d, "1"'] & Y \times X \arrow[d, two heads, "g \times 1"] \arrow[dd, bend right=40, dotted, "\varphi_f"'] & Y \arrow[l, "\langle 1, 0\rangle"'] \arrow[d, two heads, "g"] \\
X \arrow[r, "\langle gf, 1\rangle"] \arrow[dr, "0"'] & Z \times X \arrow[d, dashed, "\varphi_{gf}"] & Z \arrow[l, "\langle 1, 0\rangle"'] \arrow[dl, "1"] \\
 & Z. &
\end{tikzcd} \tag{4.26}$$

□

Remark 4.3.6. *In a regular subtractive category $\mathbb{C}$ with finite joins of subobjects, it can be seen that (see e.g diagram (4.26) above) if a composite gf is central then f admits a subtractor along g. As shown in the previous lemma, the converse holds when g is a regular epimorphism.*

Theorem 4.3.7. *In a regular subtractive category $\mathbb{C}$ with cokernels and finite joins of subobjects, for every morphism $f : X \longrightarrow Y$ there is a universal morphism $q : Y \longrightarrow Q$ which, by composition, makes f central. The universal morphism q is necessarily a regular epimorphism.*

Proof. Let us consider the cokernel $\varphi : Y \times X \twoheadrightarrow Q$ of the morphism $\langle f, 1\rangle$ in the diagram

$$\begin{tikzcd}
X \arrow[r, "\langle f,1\rangle"] & Y \times X \arrow[r, two heads, "\varphi"] & Q, \\
& Y \arrow[u, "\langle 1,0\rangle"'] \arrow[ur, dotted, bend right, "q"'] &
\end{tikzcd}$$

with $q = \varphi\langle 1, 0\rangle$. It can be easily seen in the diagram

$$\begin{tikzcd}
X \arrow[rr, "\langle f,1\rangle"] \arrow[drr, bend right, "0"'] & & Y \times X \arrow[d, "\varphi"] \arrow[dr, dashed, "\varphi^*"] & & Y \arrow[ll, "\langle 1,0\rangle"'] \arrow[dl, dotted, two heads, "e"'] \arrow[dll, bend left, "q"] \\
& & Q \arrow[r, dashed, "\lambda"] & q(Y) \arrow[l, dotted, bend left, "m"] &
\end{tikzcd}$$

that φ is the subtractor of f along q. In order to apply Lemma 4.3.5, we first need to show that q is a regular epimorphism. For that, let us write me for the (regular epi,mono)-factorization of q. Applying Lemma 4.1.9, f admits a subtractor φ^* along e, and this means, $\varphi^*\langle f, 1\rangle = 0$ and $\varphi^*\langle 1, 0\rangle = e$. Hence φ^* factors through the cokernel of $\langle f, 1\rangle$, that is, $\varphi^* = \lambda\varphi$. We see that $m\varphi^*$ and φ are both subtractors of f along q, therefore, $m\varphi^* = \varphi$. Now $\varphi = m\varphi^* = m\lambda\varphi$, and by the universal property of cokernels it follows that λ is a section of m, and hence m is an isomorphism. Thus, $q = me$ is a regular epimorphism. Now we can apply Lemma 4.3.5 to conclude that qf is central. If there is a morphism q' such that $q'f$ is central, as observed already in the previous remark, it means that f admits a subtractor φ' along q'. So by definition of φ', $\varphi'\langle f, 1\rangle = 0$ and $\varphi'\langle 1, 0\rangle = q'$, which implies that φ' factors as $\varphi' = \alpha\varphi$, since φ is the cokernel of $\langle f, 1\rangle$. Hence, $q' = \varphi'\langle 1, 0\rangle = \alpha\varphi\langle 1, 0\rangle = \alpha q$, and this shows that q is the universal arrow which, by composition, makes f central. □

Remark 4.3.8. *In a regular subtractive category $\mathbb{C}$ with cokernels and finite joins of subobjects, from the previous theorem it can be concluded that, for every morphism f and its associated universal morphism q for which qf is central, the kernel $[|f, 1|]$ of q measures the lack of centrality in f, in other words, f is central if and only if the kernel of q is the zero morphism.*

Remark 4.3.9. *In a regular subtractive category $\mathbb{C}$ with finite joins of subobjects and cokernels, taking f to be the identity morphism of X in the previous theorem, one recovers the following fact (see Theorem* 4.3 *of [12]); that is, the associated abelian object of X is given by $\widetilde{X}$ in the diagram*

$$X \xrightarrow{\langle 1,1 \rangle} X \times X \xrightarrow{q_X} \widetilde{X}$$

where q_X is the cokernel of the diagonal of X. But since abelian objects are exactly commutative objects in a subtractive category $\mathbb{C}$ with finite joins of subobjects, we can conclude that the full subcategory $\mathsf{Com}(\mathbb{C})$ *of commutative objects in $\mathbb{C}$ is reflective.*

Chapter 5

Monoidal sum structures

The notion of commuting morphisms gives rise to a binary relation (*commutes relation*) on the class of morphisms (defined only for those pairs of morphisms having the same codomain). As explained in [28], for a unital category $\mathbb{C}$, the commutes relation is an example of a *cover relation* [26] arising from a special type of monoidal structure on $\mathbb{C}$, called a *monoidal sum structure*. We will show that this fact extends to a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects. More specifically, we will show that in every pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, there is a monoidal sum structure on $\mathbb{C}$ whose corresponding cover relation is the commutes relation.

5.1 Commuting morphisms in several "non-unital" examples

Let us recall the following notation introduced in the second section of the previous chapter: For a pair of objects X and Y in a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, we write $m : X \star Y \rightarrowtail X \times Y$ to denote the join of the canonical morphisms $\langle 1, 0\rangle : X \longrightarrow X \times Y$ and $\langle 0, 1\rangle : Y \longrightarrow X \times Y$. The morphisms $I_1 : X \longrightarrow X \star Y$ and $I_2 : Y \longrightarrow X \star Y$ denote the respective factorizations of $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ through the join $m : X \star Y \longrightarrow X \times Y$. We write $\pi_1^\star$ and $\pi_2^\star$ to denote the composites $\pi_1 m$ and $\pi_2 m$ respectively.

Recall that, for a pair of morphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ in a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, the commutativity of f and g is defined by the existence of a cooperator $\varphi : X \star Y \longrightarrow Z$, such that $\varphi I_1 = f$ and $\varphi I_2 = g$. As explained before, this reduces to the usual definition of commuting morphisms when the category $\mathbb{C}$ is unital. In this section we will describe commuting morphisms in several "non-unital" examples.

Example 1. Consider a category $\mathcal{I}$ whose objects are sets X equipped with a binary operation "$-$" and a unique constant "0," satisfying the following:

(a) $x - 0 = x$;

(b) $x - x = 0$;

(c) $0 - x = 0$.

Morphisms in this category are maps which preserve the binary operation $-$ and the constant 0. Let us quickly recall the definition of an implication algebra [1], and later explain that an implication structure on a non-empty set gives rise to a binary operation $-$ and a unique constant

0, satisfying the previous axioms. An implication algebra is a set X together with a binary operation "$\cdot$" satisfying the following:

(I1) $(x \cdot y) \cdot x = x$;

(I2) $(x \cdot y) \cdot y = (y \cdot x) \cdot x$;

(I3) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$.

A classical implication "$x \Rightarrow y$" can be derived from the previous axioms, by setting $x \cdot x = y \cdot y = 1$ and $x \Rightarrow y := x \cdot y$ (see [1] for details). Let us show that $x \cdot x = y \cdot y$. Using (I2), $((x \cdot x) \cdot (y \cdot y)) \cdot (y \cdot y) = ((y \cdot y) \cdot (x \cdot x)) \cdot (x \cdot x)$. We will show that $((x \cdot x) \cdot (y \cdot y)) \cdot (y \cdot y) = y \cdot y$, which will also implies that $((y \cdot y) \cdot (x \cdot x)) \cdot (x \cdot x) = x \cdot x$.

$$\begin{aligned} ((x \cdot x) \cdot (y \cdot y)) \cdot (y \cdot y) &= y \cdot (((x \cdot x) \cdot (y \cdot y)) \cdot y) \quad \text{(using (I3)))} \\ &= y \cdot ((y \cdot ((x \cdot x) \cdot y)) \cdot y) \quad \text{(using (I3))} \\ &= y \cdot y \qquad \text{(since } (y \cdot ((x \cdot x) \cdot y)) \cdot y = y \text{ by (I1)).} \end{aligned} \tag{5.1}$$

Let us also show that $x \cdot 1 = 1$ and $1 \cdot x = x$: Using (I1), $1 \cdot x = (x \cdot x) \cdot x = x$, which implies that $1 = (1 \cdot x) \cdot 1 = x \cdot 1$. Now we can define a subtraction on any non-empty implication algebra by letting $0 = 1$ and

$$x - y := y \cdot x.$$

Clearly, $x - x = 0, x - 0 = 1 \cdot x = x$, and $0 - x = x \cdot 1 = 1 = 0$.

The category $\mathcal{I}$ is not unital: For objects $X = \{0, x\}$ and $Y = \{0, y\}$ in $\mathcal{I}$, the relation $R = \{(0,0),(x,0),(0,y)\}$ is a punctual relation from X to Y, but it is not indiscrete since $(x,y) \notin R$. Let us now describe commuting morphisms in this category. Given two homomorphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ in $\mathcal{I}$, the object $X \star Y$ is given by the union

$$X \star Y = \{(x,0)|x \in X\} \cup \{(0,y)|y \in Y\}.$$

The map $\varphi : X \star Y \longrightarrow Z$ defined by

$$\varphi(x,0) = f(x) \text{ and } \varphi(0,y) = g(y),$$

is a morphism in $\mathcal{I}$ if and only if

$$f(x) = f(x) - g(y) \text{ and } g(y) = g(y) - f(x)$$

for all $x \in X$ and $y \in Y$. And this is what it means for f and g to commute in $\mathcal{I}$. Let us also describe commutative objects and central morphisms in $\mathcal{I}$. Using the previous calculations, an object X is commutative if and only if for every pair $x, x' \in X$, $x = x - x'$ and $x' = x' - x$. This immediately implies that every element x of a commutative object X is 0, since $x = x - x = 0$. Hence, in $\mathcal{I}$ only the zero object $\{0\}$ is commutative. Similar argument shows that only the zero morphism is central.

Example 2. Another example of a "non-unital" category is the category $\mathbf{Set}^*$ of pointed sets. For two pointed sets X and Y, the object $X \star Y$ is given by

$$X \star Y = \{(x,0)|x \in X\} \cup \{(0,y)|y \in Y\},$$

which is exactly the coproduct of X and Y in $\mathbf{Set}^*$. Hence, any two morphisms in $\mathbf{Set}^*$ always commute.

Example 3. Consider a category $\mathcal{C}$ whose objects are semi-groups $(X, +)$ equipped with a unique constant "0", satisfying that $0 + 0 = 0$ and $x + 0 = 0 + x$. For objects X and Y in $\mathcal{S}$, one has

$$X \star Y = \{(x, 0) | x \in X\} \cup \{(0, y) | y \in Y\} \cup \{(x + 0, 0 + y) | x \in X \;\text{ and }\; y \in Y\}.$$

On the set $\mathbb{N}$ of natural numbers, define $+$ as follows:

$$x + y := \begin{cases} 0 & \text{if} \quad x = y = 0, \\ 1 & \quad \text{otherwise.} \end{cases}$$

It is not difficult to see that $(\mathbb{N}, +)$ is an object in $\mathcal{C}$. In addition, $\mathbb{N} \star \mathbb{N} = \{(1, 1), (n, 0), (0, m) |\; n, m \in \mathbb{N}\} \neq \mathbb{N} \times \mathbb{N}$, and this shows that $\mathcal{C}$ is not unital. Now given morphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ in $\mathcal{C}$, a map $\varphi : X \star Y \longrightarrow Z$ such that

$$\varphi(x, 0) = f(x) \;\text{ and }\; \varphi(0, y) = g(y)$$

is a morphism in $\mathcal{C}$ if and only if, for all $x \in X$ and $y \in Y$,

$$f(x) + g(y) = g(y) + f(x) = g(y) + f(x) + 0.$$

In other words, f and g commute if and only if, for all $x \in X$ and $y \in Y$,

$$f(x) + g(y) = g(y) + f(x) = g(y) + f(x) + 0.$$

Remark 5.1.1. *Further examples of non-unital categories can be obtained by taking the product of a unital category with any of the above examples, and commutes is obviously computed component-wise.*

5.2 Monoidal sum structures and commutes relation

For a pair of morphisms $f : A \longrightarrow C$ and $g : B \longrightarrow D$ in a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, consider the following commutative diagram

$$\begin{tikzcd}
 & A \times B \arrow[rr, "f \times g"] & & C \times D & \\
 & & B \arrow[ul, bend right, "\langle 0,1\rangle"'] \arrow[dl, "I_2"'] \arrow[rr, "g"] & & D. \arrow[ul, bend right, "\langle 0,1\rangle"'] \arrow[dl, "I_2"] \\
 & A \star B \arrow[uu, "m_1"] & & C \star D \arrow[uu, "m_2"'] & \\
A \arrow[uuur, bend left, "\langle 1,0\rangle"] \arrow[ur, "I_1"] \arrow[rr, "f"] & & C \arrow[uuur, bend left, "\langle 1,0\rangle"] \arrow[ur, "I_1"] & &
\end{tikzcd}$$

Using the commutativity of the above diagram we see that the diagram

$$\begin{tikzcd}
 & C \star D \arrow[d, "m_2"] & \\
 & C \times D & \\
A \arrow[r, "I_1"'] \arrow[ur, bend left=40, "I_1 f"] \arrow[uur, bend left=50, "I_1 f"] & A \star B \arrow[u, "(f \times g) m_1"'] \arrow[uu, dashed, bend left=30, "f \star g"] & B \arrow[l, "I_2"] \arrow[uul, bend right=50, "I_2 g"']
\end{tikzcd}$$

commutes, but since the morphisms I_1 and I_2 are jointly strongly epimorphic, there exists a unique morphism $f \star g : A \star B \longrightarrow C \star D$ such that the diagram

$$\begin{tikzcd}
A \star B \arrow[r, "f \star g"] \arrow[d, tail, "m_1"] \arrow[dd, dotted, bend right=50, "\pi_1^\star"'] & C \star D \arrow[d, tail, "m_2"'] \arrow[dd, dotted, bend left=50, "\pi_1^\star"] \\
A \times B \arrow[r, "f \times g"'] \arrow[d, dotted, "\pi_1"] & C \times D \arrow[d, dotted, "\pi_1"'] \\
A \arrow[r, dotted, "f"'] & C
\end{tikzcd}$$

commutes. It can also be seen that $(f \star g)I_1 = I_1 f$, $(f \star g)I_2 = I_2 g$, and $\pi_1^\star(f \star g) = f\pi_1^\star$. In a same way it can be shown that $\pi_2^\star(f \star g) = g\pi_2^\star$. We shall observe that for a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, $\star$ is a bifunctor

$$\star : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$$

on $\mathbb{C}$, which assigns to each pair of objects A, B and each pair of morphisms $f : A \longrightarrow C, g : B \longrightarrow D$, the object $A \star B$ and the morphism $f \star g : A \star B \longrightarrow C \star D$ respectively. For morphisms $f : X \longrightarrow Y, f' : X' \longrightarrow Y', g : Y \longrightarrow Z$, and $g' : Y' \longrightarrow Z'$ in $\mathbb{C}$, since the diagram

$$\begin{tikzcd}
X \arrow[r, "I_1"] \arrow[d, "f"'] & X \star X' \arrow[d, dashed, "f \star f'"] & X' \arrow[l, "I_2"'] \arrow[d, "f'"] \\
Y \arrow[r, "I_1"] \arrow[d, "g"'] & Y \star Y' \arrow[d, dashed, "g \star g'"] & Y' \arrow[l, "I_2"'] \arrow[d, "g'"] \\
Z \arrow[r, "I_1"'] & Z \star Z' & Z' \arrow[l, "I_2"]
\end{tikzcd}$$

commutes and the pair I_1, I_2 is jointly epimorphic, it follows that $(g \star g')(f \star f') = gf \star g'f'$, and this means $\star$ preserves compositions. In a similar way, it can be shown that $\star$ preserves identity morphisms. Hence, $\star$ is a bifunctor on $\mathbb{C}$.

Definition 5.2.1 (see [28]). *Let $\mathbb{C}$ be an arbitrary category. In the functor category $\mathbb{C}^{\mathbb{C}\times\mathbb{C}}$, a diagram*

$$\begin{tikzcd}
P_1 \arrow[r, "\iota_1"] & \bigotimes & P_2 \arrow[l, "\iota_2"']
\end{tikzcd}$$

where P_1 and P_2 are the product projections $\mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$, is called a sum structure on $\mathbb{C}$, if for every object (C_1, C_2) in $\mathbb{C} \times \mathbb{C}$, the $(C_1, C_2)-$components $\iota_{1_{C_1,C_2}}$ and $\iota_{2_{C_1,C_2}}$ are jointly epimorphic. A sum structure on $\mathbb{C}$ is usually denoted by a triple $(\bigotimes, \iota_1, \iota_2)$.

In every category $\mathbb{C}$ with coproducts, there is a sum structure $(+, i_1, i_2)$ given by the coproduct $+$ and the coproduct inclusions i_1 and i_2.

Remark 5.2.2. *In a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, $(\star, I_1, I_2)$ is a sum structure on $\mathbb{C}$, with I_1 and I_2 considered as natural transformations*

$$I_1 : P_1 \longrightarrow \star \ \text{ and } I_2 : P_2 \longrightarrow \star$$

whereby for any two objects X and Y in $\mathbb{C}$ the $(X, Y)-$components $I_{1_{X,Y}}$ and $I_{2_{X,Y}}$ are just the jointly epimorphic pair of morphisms $I_1 : X \longrightarrow X \star Y$ and $I_2 : Y \longrightarrow X \star Y$ respectively. For a morphism $(f, g) : (X, Y) \longrightarrow (X', Y')$ in $\mathbb{C} \times \mathbb{C}$, the naturalities of I_1 and I_2 can be seen in the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "I_1"] \arrow[d, "f"'] & X \star Y \arrow[d, "f \star g"] & Y \arrow[l, "I_2"'] \arrow[d, "g"] \\
X' \arrow[r, "I_1"'] & X' \star Y' & Y'. \arrow[l, "I_2"]
\end{tikzcd}$$

Next we recall what a cover relation on a category is.

Definition 5.2.3 (see [26],[28]). *A cover relation on a category $\mathbb{C}$ is a binary relation $\sqsubset$ on the class of morphisms of $\mathbb{C}$, defined only for those pairs of morphisms having the same codomain, and it has the following two properties:*

(i) if $f \sqsubset g$ and h is composable with f, then $hf \sqsubset hg$;

(ii) if $f \sqsubset g$ and e is composable with f, then $fe \sqsubset g$.

A cover relation $\sqsubset$ is called a bicover relation if its inverse relation is also a cover relation.

As explained in [28], for instance, a sum structure $(\bigotimes, \iota_1, \iota_2)$ on a category $\mathbb{C}$ induces a bicover relation $\sqsubset_{\bigotimes}$ on $\mathbb{C}$, whereby for a pair of morphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$, $f \sqsubset_{\bigotimes} g$ is defined by the existence of a (necessarily unique) morphism φ making the diagram

$$\begin{tikzcd}
X \arrow[r, "\iota_1"] \arrow[dr, bend right, "f"'] & X \otimes Y \arrow[d, dashed, "\varphi"] & Y \arrow[l, "\iota_2"'] \arrow[dl, bend left, "g"] \\
& Z &
\end{tikzcd}$$

commute. For the coproduct sum structure $(+, i_1, i_2)$ on a category $\mathbb{C}$ with coproducts, the induced bicover relation $\sqsubset_+$ is indiscrete, since for morphisms f and g having the same codomain, $f \sqsubset_+ g$ is defined by the existence of the coproduct induced morphism $[f, g]$.

Remark 5.2.4. *For the sum structure $(\star, I_1, I_2)$ on a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, the induced bicover relation $\sqsubset_\star$ on $\mathbb{C}$ is exactly the commutes relation, in other words, for morphisms f and g having the same codomain, $f \sqsubset_\star g$ if and only if f and g commute.*

Definition 5.2.5 (see [28]). *A sum structure $(\otimes, \iota_1, \iota_2)$ on a category $\mathbb{C}$ is preassociative if for every three objects X, Y, Z in $\mathbb{C}$ there is a morphism*

$$X \otimes (Y \otimes Z) \xrightarrow{\ \alpha\ } (X \otimes Y) \otimes Z$$

called the associativity morphism at X, Y, Z, such that the diagram

$$\begin{tikzcd}
X \arrow[r, "\iota_1"] \arrow[dd, "\iota_1"'] & X \otimes (Y \otimes Z) \arrow[dd, bend left, "\alpha"] & Y \otimes Z \arrow[l, "\iota_2"'] \\
& Y \arrow[ur, "\iota_1"] \arrow[dl, "\iota_2"'] & \\
X \otimes Y \arrow[r, "\iota_1"'] & (X \otimes Y) \otimes Z & Z \arrow[l, "\iota_2"] \arrow[uu, "\iota_2"']
\end{tikzcd}$$

commutes. The morphism α is uniquely determined, and it is natural in all three arguments. The resulting natural transformation is called the associativity natural transformation, and when it is a natural isomorphism, the sum structure $(\otimes, \iota_1, \iota_2)$ is said to be associative.

In a category $\mathbb{C}$ with products, for objects X, Y, and Z there is a canonical isomorphism $\alpha = \langle \pi_1\pi_1, \langle \pi_2\pi_1, \pi_2\rangle\rangle$ making the diagram

$$\begin{tikzcd}
X & X \times (Y \times Z) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[dd, dotted, bend right=60, "\langle \pi_1, \pi_1\pi_2\rangle"'] & Y \times Z \arrow[dd, "\pi_2"] \arrow[dl, "\pi_1"'] \\
 & Y & \\
X \times Y \arrow[uu, "\pi_1"] \arrow[ur, "\pi_2"] & (X \times Y) \times Z \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[uu, bend left, "\alpha"] \arrow[uu, dotted, bend right=60, "\langle \pi_2\pi_1, \pi_2\rangle"] & Z
\end{tikzcd} \tag{5.2}$$

commute. Now for objects X, Y, and Z in a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, and the morphism α in diagram (5.2), consider the diagram

$$\begin{tikzcd}
(X \star Y) \star Z \arrow[r, dashed, "\lambda"] \arrow[d, tail, "m_0"'] & X \star (Y \star Z) \arrow[d, tail, "m_1"] \\
(X \star Y) \times Z \arrow[d, tail, "m_0' \times 1"'] & X \times (Y \star Z) \arrow[d, tail, "1 \times m_1'"] \\
(X \times Y) \times Z \arrow[r, "\alpha"'] & X \times (Y \times Z)
\end{tikzcd} \tag{5.3}$$

where m_0, m_0', m_1, and m_1' denote joins of suitable pairs of $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$. Let us explain how the morphism λ in diagram (5.3) is obtained. In the diagram

$$
\begin{tikzcd}
 & X \star (Y \star Z) \arrow[d, tail, "(1 \times m_1')m_1"] & \\
X \star Y \arrow[ur, bend left, "1 \star I_1"] & X \times (Y \times Z) & Y \star Z \arrow[ul, bend right, "I_2"'] \\
X \star Y \arrow[u, "1"] \arrow[r, "I_1"'] & (X \star Y) \star Z \arrow[u, "\alpha(m_0' \times 1)m_0"'] \arrow[uu, dashed, bend left=40, "\lambda"] & Z \arrow[u, "I_2"'] \arrow[l, "I_2"]
\end{tikzcd}
\tag{5.4}
$$

we see that

$$
\begin{aligned}
\alpha(m_0' \times 1)m_0 I_2 &= \alpha\langle 0, 1\rangle \\
&= \langle \pi_1\pi_1, \langle \pi_2\pi_1, \pi_2\rangle\rangle\langle 0, 1\rangle \\
&= \langle 0, \langle 0, 1\rangle\rangle
\end{aligned}
\tag{5.5}
$$

and

$$
\begin{aligned}
(1 \times m_1')m_1 I_2 I_2 &= (1 \times m_1')\langle 0, 1\rangle I_2 \\
&= \langle 0, m_1' I_2\rangle \\
&= \langle 0, \langle 0, 1\rangle\rangle.
\end{aligned}
\tag{5.6}
$$

In a similar way it can be shown that

$$(1 \times m_1')m_1(1 \star I_1) = \langle \pi_1^\star, \langle \pi_2^\star, 0\rangle\rangle = \alpha(m_0' \times 1)m_0 I_1,$$

and thus, diagram (5.4) commutes. The morphism λ in diagram (5.3) is then obtained (in diagram (5.4)) from the fact that the pair of morphisms $I_1 : X \star Y \longrightarrow (X \star Y) \star Z$ and $I_2 : Z \longrightarrow (X \star Y) \star Z$ is jointly strongly epimorphic. As a result we obtain the following commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "I_1"] \arrow[dd, "I_1"'] & X \star (Y \star Z) \arrow[dd, dotted, bend left, "\lambda^{-1}"] & Y \star Z \arrow[l, "I_2"'] \\
 & Y \arrow[ur, "I_1"] \arrow[dl, "I_2"'] & \\
X \star Y \arrow[r, "I_1"'] & (X \star Y) \star Z \arrow[uu, bend left, "\lambda"] & Z. \arrow[l, "I_2"] \arrow[uu, "I_2"']
\end{tikzcd}
\tag{5.7}
$$

The morphism λ is an isomorphism; its inverse λ^{-1} is the morphism induced by the inverse of α in diagram (5.2). Now we can make the following remark:

Remark 5.2.6. *In a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, the sum structure $(\star, I_1, I_2)$ is associative.*

According to [28], an object I in a category $\mathbb{C}$ is a unit of a sum structure $(\otimes, \iota_1, \iota_2)$ if for every object X in $\mathbb{C}$ the morphisms $\iota_1 : X \longrightarrow X \otimes I$ and $i_2 : X \longrightarrow I \otimes X$ are isomorphisms. Let us show that the zero object is a unit of the sum structure $(\star, I_1, I_2)$ on a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects: For every object X in $\mathbb{C}$, since the pair of morphisms $\pi_1^\star$ and $\pi_2^\star$ is jointly monomorphic, in the commutative diagram

$$\begin{tikzcd}
0 & 0 \star X \arrow[l, "\pi_1^\star"'] \arrow[r, "\pi_2^\star"] & X \\
& X \arrow[u, "I_2"] \arrow[ul, bend left, "0"] \arrow[ur, bend right, "1"'] & \\
& 0 \star X \arrow[u, "\pi_2^\star"] \arrow[uul, bend left, "0"] \arrow[uur, bend right, "\pi_2^\star"'] &
\end{tikzcd}$$

we see that $I_2 : X \longrightarrow 0 \star X$ is an isomorphism (being a monomorphism and a split epimorphism). In a similar way, it can be shown that $I_1 : X \longrightarrow X \star 0$ is an isomorphism. Hence, the zero object 0 is a unit of the sum structure $(\star, I_1, I_2)$, and according to [28], this means $(\star, I_1, I_2, 0)$ is a *monoidal sum structure* on $\mathbb{C}$, that is, an associative sum structure that has a unit. But as observed in Theorem 2.6.2 of [28], for instance, a monoidal sum structure gives rise to a monoidal structure on $\mathbb{C}$. Now writing λ for the associativity natural transformation of the sum structure $(\star, I_1, I_2)$, and $\rho = I_2 : X \longrightarrow 0 \star X$ and $\beta = I_1 : X \longrightarrow X \star 0$ for the left and right units respectively, it follows that $(\star, 0, \lambda, \rho, \beta)$ is a monoidal structure on a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects. Furthermore, for objects X and Y in $\mathbb{C}$, writing $m : X \star Y \rightarrowtail X \times Y$ and $m' : Y \star X \rightarrowtail Y \times X$ for the respective joins of the pairs $\langle 1, 0\rangle : X \longrightarrow X \times Y$, $\langle 0, 1\rangle : Y \longrightarrow X \times Y$ and $\langle 1, 0\rangle : Y \longrightarrow Y \times X$, $\langle 0, 1\rangle : X \longrightarrow Y \times X$, the diagram

$$\begin{tikzcd}
 & Y \star X \arrow[d, "\mathsf{tw}m'"] & \\
 & X \times Y & \\
X \arrow[uur, bend left, "I_2"] \arrow[r, "I_1"'] & X \star Y \arrow[u, "m"'] \arrow[uu, dashed, bend left, "\mathsf{tw}^\star"] & Y \arrow[l, "I_2"] \arrow[uul, bend right, "I_1"']
\end{tikzcd}$$

in which tw is a morphism such that $\pi_1 = \pi_2\mathsf{tw}$ and $\pi_2 = \pi_1\mathsf{tw}$, commutes, and this implies that there is a morphism $\mathsf{tw}^\star$ such that $\mathsf{tw}^\star I_1 = I_2$ and $\mathsf{tw}^\star I_2 = I_1$. The morphism $\mathsf{tw}^\star$ is necessarily an isomorphism. Therefore, the monoidal structure $(\star, 0, \lambda, \rho, \beta)$ on $\mathbb{C}$ is symmetric.

Next we give a characterization of monoids in the monoidal category $(\mathbb{C}, \star)$, with $\mathbb{C}$ a pointed finitely complete category with finite joins of subobjects.

Proposition 5.2.7. *In a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, an object X is a monoid in the monoidal category $(\mathbb{C}, \star)$ if and only if X is a commutative object.*

Proof. If X is a monoid in $(\mathbb{C}, \star)$, with $a : X \star X \longrightarrow X$ the multiplication, then the diagram

$$\begin{tikzcd}
X \arrow[r, "I_1"] \arrow[dr, "1"'] & X \star X \arrow[d, "a"] & X \arrow[l, "I_2"'] \arrow[dl, "1"] \\
 & X &
\end{tikzcd}$$

commutes, and this means that a is the cooperator of the pair of identity morphisms of X, and thus X is commutative. Conversely, let us suppose X is a commutative object, and $\varphi : X \star X \longrightarrow X$ is the cooperator of the pair of identity morphisms of X. Clearly, the cooperator is a multiplication on X, with the zero map $0 : 0 \longrightarrow X$ as a unit. Furthermore, using the morphism λ in diagram (5.7), and the fact that the morphisms I_1 and I_2 are jointly epimorphic, it can be seen after pre-composing with I_1 and I_2 that the diagram

$$\begin{tikzcd}
(X \star X) \star X \arrow[r, "\varphi \star 1"] \arrow[d, "\lambda"'] & X \star X \arrow[dd, "\varphi"] \\
X \star (X \star X) \arrow[d, "1 \star \varphi"'] & \\
X \star X \arrow[r, "\varphi"'] & X
\end{tikzcd}$$

commutes, and this shows associativity. $\square$

Remark 5.2.8. *For a pointed finitely complete category $\mathbb{C}$ with finite joins of subobjects, since the monoidal category $(\mathbb{C}, \star)$ is symmetric, every monoid is automatically commutative.*

For $\mathbb{C}$ a subtractive category with finite joins of subobjects, we can observe the following:

Corollary 5.2.9. *In a subtractive category $\mathbb{C}$ with finite joins of subobjects, an object X is a monoid in the monoidal category $(\mathbb{C}, \star)$, if and only if, it has an internal abelian group structure in the monoidal category $(\mathbb{C}, \times)$ (with the monoidal structure induced by the product).*

Proof. According to Theorem 4.2.12, an object in $\mathbb{C}$ is commutative if and only if it is endowed with an internal abelian group structure in $(\mathbb{C}, \times)$. But since commutative objects are precisely monoids in the monoidal category $(\mathbb{C}, \star)$, the result follows. $\square$

References

[1] J. C Abbott, *Implication algebras*, Bull. Math. Soc. Sci. Math. R. S. Roumanie **11** (59), 1967, 3-23.

[2] M. Barr, *Exact categories,* Lecture Notes in Mathematics **236**, Springer-Verlag, 1971, 1-120.

[3] F. Borceux, *Handbook of categorical algebra 1,* Encyclopedia of Mathematics and its Applications **50**, Cambridge University Press, 1994.

[4] F. Borceux and D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories,* Mathematics and its Applications **566**, Kluwer Academic Publishers, 2004.

[5] D. Bourn, *Commutator theory in regular Mal'cev categories,* in Hopf algebras and Semi-Abelian Categories, G Janelidze, B. Pareigis, W. Tholen eds., Fields Institute Communication **43**, 2004, 61-75.

[6] D. Bourn, *Intrinsic centrality and associated classifying properties,* Journal of Algebra **256** (1), 2002, 126-145.

[7] D. Bourn, *Mal'cev categories and fibration of pointed objects,* Applied Categories Structures **4** (2-3), 1996, 307-327.

[8] D. Bourn, *Normal subobjects and abelian objects in protomodular categories,* Journal of Algebra **228** (1), 2000, 143-164.

[9] D. Bourn, *Normalization equivalence, kernel equivalence, and affine categories*, Lecture Notes in Mathematics **1488**, Springer-Verlag, 1991, 43-62.

[10] D. Bourn, and M. Gran, *Centrality and normality in protomodular categories,* Theory and Applications of Categories **9** (8), 2002, 151-165.

[11] D. Bourn and G. Janelidze, *Characterization of protomodular varieties of universal algebras,* Theory and Applications of Categories **11** (6), 2003, 143-147.

[12] D. Bourn and Z. Janelidze, *Subtractive categories and extended subtractions,* Applied Categorical Structures **17** (4), 2009, 317-343.

[13] D. Bourn, N. Martins-Ferreira, and T. Van der Linden, *A characterization of the "Smith is Huq" condition in the pointed Mal'cev setting,* Cah. Topologie Géom. Différ. Catégoriques **54** (3), 2013, 163-183.

[14] A. Carboni, G. M. Kelly, and M. C. Pedicchio, *Some remarks on Mal'cev and Goursat categories*, Applied Categorical Structures **1** (4), 1993, 385-421.

[15] A. Carboni, J. Lambek, and M. C. Pedicchio, *Diagram chasing in Mal'cev categories*, Journal of Pure and Applied Algebra **69** (3), 1991, 271-284.

[16] R. Freese and R. McKenzie, *Commutator theory for congruence varieties*, London Mathematical Society, Lecture Notes Series **125**, Cambridge University Press, 1987.

[17] M. Gran, G. Janelidze, and A. Ursini, *Weighted commutators in semi-abelian categories,* Journal of Algebra **397**, 2014, 643-665.

[18] H. P. Gumm, *Geometrical methods in congruence modular algebras*, Mem. AMS. **45** (286), 1983.

[19] H. P. Gumm and A. Ursini, *Ideals in universal algebras,* Algebra Universalis **19** (1), 1984, 45-54.

[20] J. Hagemann and C. Herrmann, *A concrete ideal multiplication for algebraic systems and its relation to congruence distributivity,* Arch. Math. **32** (1), 1979, 234-245.

[21] P. J. Higgins, *Groups with multiple operators,* Proc. Lond. Math. Soc. **6** (3), 1956, 366-416.

[22] S. A. Huq, *Commutator, nilpotency and solvability in categories,* Quart. J. Math. **19** (1), 1968, 363-389.

[23] G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories,* Journal of Pure and Applied Algebra **168** (2-3), 2002, 367-386.

[24] G. Janelidze, L. Márki, W. Tholen, and A. Ursini, *Ideal-determined categories,* Cah. Topol. Géom. Différ. Catég. **51** (2), 2010, 115-125.

[25] Z. Janelidze, *Subtractive categories,* Applied Categorical Structures **13** (4), 2005, 343-350.

[26] Z. Janelidze, *Closedness properties of internal relations V: Linear Mal'cev conditions*, Algebra Universalis **58** (1), 2008, 105-117.

[27] Z. Janelidze, *The pointed subobject functor,* 3×3 *lemmas, and subtractivity of spans,* Theory and Applications of Categories **23** (11), 2010, 221-242.

[28] Z. Janelidze, *Cover relations on categories,* Applied Categorical Structures **17** (4), 2009, 351-371.

[29] S. Mantovani, *The Ursini commutator as normalized Smith-Pedicchio commutator,* Theory and Applications of Categories **27** (8), 2012, 174-188.

[30] S. Mantovani and G. Metere, *Normalities and commutators,* Journal of Algebra **324** (9), 2010, 2568-2588.

[31] N. Martins-Ferreira and T. Van der Linden, *A decomposition formula for the weighted commutator,* Applied Categorical Structures **22** (5-6), 2014, 899-906.

[32] N. Martins-Ferreira and T. Van der Linden, *Further remarks on the "Smith is Huq" condition*, Applied Categorical Structures **23** (4), 2015, 527-541.

[33] M. C. Pedicchio, *A categorical approach to commutator theory,* Journal of Algebra **177** (3), 1995, 647-657.

[34] V. T. Shaumbwa, *A characterization of the Higgins commutator,* Applied Categorical Structures **27** (2), 2019, 159-162.

[35] V. T. Shaumbwa, *A characterization of the Huq commutator,* Theory and Applications of Categories **32** (46), 2017, 1588-1600.

[36] J. D. H. Smith, *Mal'cev varieties*, Lecture Notes in Mathematics **554**, Springer-Verlaq, 1976.

[37] A. Ursini, *Sulle varietá di algebre con una buona teoria degli ideali*, Boll. Unione Mat. Ital. **4** (6), 1972, 90-95.

[38] A. Ursini, *Ideals and their calculus I,* Rapporto Matematico n. 41, Universita di Sienå, 1981.

[39] A. Ursini, *On subtractive varieties I*, Algebra Universalis **31** (2), 1994, 204-222.