On a generalisation of k-Dyck paths

by

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Prof. Helmut Prodinger

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Date: December 2019

Abstract

On a generalisation of k-Dyck paths

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We consider a family of non-negative lattice paths consisting of the step set $\{(1,1),(1,-k)\}$ called k-Dyck paths, which are enumerated by the generalised Catalan numbers $\frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n}$. By removing the non-negativity condition but restricting the path to stay above the line $y=-t$ we obtain a family of lattice paths called k_t-Dyck paths which are enumerated by 'generalised' generalised Catalan numbers

$$\frac{t+1}{(k+1)n+t+1}\binom{(k+1)n+t+1}{n}.$$

We provide proofs of the enumeration of these paths by means of a bijection, the kernel method, the cycle lemma, and the symbolic method. Analysis of parameters associated with the paths is also performed using symbolic equations – particularly the number of peaks, the number of valleys, and the number of returns.

These k_t-Dyck paths find application in enumerating a family of walks in the quarter plane ($\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$) with step set $\{(1,1),(1,-k+1),(-k,0)\}$. Such walks can be decomposed into ordered pairs of k_t-Dyck paths and thus their enumeration can be proved via a simple bijection. Through this bijection some parameters in k_t-Dyck paths are preserved.

Finally, we discuss two different families of lattice paths, S-Motzkin and T-Motzkin paths, which are related to k_t-Dyck paths when $k=2$ along with $t=0$ and $t=1$. We provide bijections between these paths and other combinatorial objects, and perform analysis of parameters in these paths.

Uittreksel

Oor 'n veralgemening van k-Dyck paaie

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Ons beskou 'n familie van nie-negatiewe roosterpaaie wat bestaan uit die stapversameling $\{(1,1),(1,-k)\}$ genoem k-Dyck paaie, wat deur die veralgemeende Catalan getalle, $\frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n}$, getel word. Deur die nie-negatiwiteitsvereiste te verwyder, maar die pad tot bokant die lyn $y=-t$ te beperk kry ons 'n familie roosterpaaie genaamd k_t-Dyck paaie wat getel word deur 'veralgemeende' veralgemeende Catalan getalle

$$\frac{t+1}{(k+1)n+t+1}\binom{(k+1)n+t+1}{n}.$$

Ons lewer 'n bewys van die aftelling van hierdie paaie deur middel van 'n bijeksie, die kernmetode, die sikluslemma en die simboliese metode. Analise van parameters wat met die paaie geassosieer word, word ook uitgevoer met behulp van simboliese vergelykings – veral die aantal pieke, die aantal valleie en die aantal terugkomste.

Hierdie k_t-Dyck paaie vind 'n toepassing in 'n familie van wandelinge in die kwartvlak ($\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}$) met stapversameling $\{(1,1),(1,-k+1),(-k,0)\}$. Sulke wandelinge kan ontleed word in geordende pare k_t-Dyck paaie en dus kan hul aftelling deur middel van 'n eenvoudige bijeksie bewys word. Deur hierdie bijeksie word 'n paar parameters in k_t-Dyck paaie bewaar.

Laastens bespreek ons twee verskillende families van roosterpaaie, S-Motzkin en T-Motzkin paaie, wat verband hou met k_t-Dyck paaie wanneer $k=2$ saam met $t=0$ en $t=1$. Ons bied bijeksies tussen hierdie paaie en ander kombinatoriese voorwerpe, en doen 'n analise van parameters op hierdie paaie.

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Contents

Chapter 1

Introduction

To introduce lattice paths we begin with the classic ballot problem [3]: "When counting the election outcome in a two candidate election with a total of n votes, suppose that candidate A wins with α votes and candidate B has $\beta = n - \alpha$ votes. What is the probability that at each count of a vote, A is always ahead of or tied with B?"

This is equivalent to asking how many possible ways n votes can be counted such that at each vote count the number of votes A has received (total votes α) subtracted by the number of votes B has received (total votes β) is greater than or equal to 0. From this the probability can easily be obtained.

For example, consider a 10 vote election with 7 votes for candidate A and 3 votes for candidate B. A possible way of counting these votes which would satisfy the conditions is:

Vote (in order)	1	2	3	4	5	6	7	8	9	10
Candidate A	✓		✓	✓		✓	✓	✓		✓
Candidate B		✓			✓				✓	

Table 1.1: Each count of the vote for candidates A and B

This can also be represented in a visual manner, which allows for easy interpretation. For a vote for candidate A we will draw a vector $(1, 1)$, and a vote for candidate B will be represented as a vector $(1, -1)$. Starting at a coordinate $(0, 0)$ and drawing these vectors head to tail, we obtain the picture in Figure 1.1.

With this interpretation of votes, a vote count which satisfies the conditions is represented by a set of vectors which form a path from $(0, 0)$ to $(10, 4)$ and never goes below the line $y = 0$. Such a path is called a *lattice path* with *step set* $\{(1, 1), (1, -1)\}$ and the condition that the path never goes below the line $y = 0$ is called *non-negativity*.

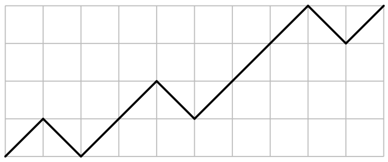


Figure 1.1: A visual representation of the vote count in Table 1.1

Using this terminology, the ballot problem can be formulated as: "Let n be a positive integer and β be a non-negative integer such that $0 \leq \beta \leq n/2$. What is the probability that a lattice path from $(0,0)$ to $(n, n-2\beta)$ is non-negative?"

A lattice path of length n that ends at height $n-2\beta$ has β steps of the form $(1,-1)$ and $n-\beta$ steps of the form $(1,1)$. The total number of lattice paths from $(0,0)$ to $(n, n-2\beta)$ is then given by $\binom{n}{\beta}$, since there are β positions for a $(1,-1)$ step to occur.

To calculate the number of non-negative paths we note that any negative path will reach the line $y=-1$. Considering just the section of the path from the first time the path reaches $y=-1$ to the end of the path, we reflect this about the line $y=-1$. Note that this reflection can be uniquely reversed. Altogether, we now have a path from $(0,0)$ to $(n, -n+2\beta-2)$ (see Figure 1.2).

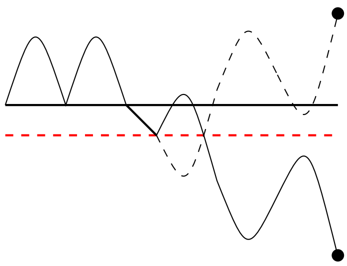


Figure 1.2: Reflection about $y=-1$, a dashed line represents the original path.

Therefore the number of negative paths is equal to the number of paths from $(0,0)$ to $(n, -n+2\beta-2)$ and thus given by $\binom{n}{\beta-1}$ (such a path would consist of $\beta-1$ $(1,1)$ steps and $n-\beta+1$ $(1,-1)$ steps). To work out how many non-negative paths there are we take the difference:

$$\binom{n}{\beta} - \binom{n}{\beta-1} = \frac{n!}{(\beta-1)!(n-\beta)!}\left(\frac{1}{\beta} - \frac{1}{n-\beta+1}\right) = \frac{n-2\beta+1}{n-\beta+1}\binom{n}{\beta}.$$

Finally, the probability is this quantity divided by total number of paths, and thus equal to $\frac{n-2\beta+1}{n-\beta+1}$.

In this way the ballot problem can be solved by a simple argument involving the enumeration of two families of lattice paths. In fact, lattice paths can be used to model complex problems in statistics, economics, and physics. Apart from applications, lattice paths have also been studied for their own merits – mathematically, lattice paths are connected to a number of other mathematical objects including trees.

Lattice path enumeration and analysis has been an active area of research since the first recorded usage of lattice paths in the late 19th century, and a full history of lattice path enumeration by Humphreys can be found in [18]. Here it is said that the number of papers related to lattice path enumeration has more than doubled each decade since 1960.

Some of the methods involved in the enumeration of lattice paths are bijections, generating functions, reflections, the cycle lemma, transfer matrix methods, the kernel method, orthogonal polynomials, and continued fractions. These methods (and more) are discussed in a comprehensive overview of lattice path enumeration by Krattenthaler [21].

We would not be able to discuss the analysis of lattice paths without mentioning the seminal paper "Basic analytic combinatorics of directed lattice paths" by Banderier and Flajolet [1], or the book *Analytic Combinatorics* by Flajolet and Sedgewick [12]. For further background, either of these can be consulted.

In Chapter 2 we define a family of lattice paths which have already been studied in the literature called k-Dyck paths [16, 17] and introduce a slight generalisation of k-Dyck paths called k_t-Dyck paths. The theme throughout the remaining chapters will be the enumeration, analysis, and application of k_t-Dyck paths.

In Chapters 2, 4, 5, and 6 the enumeration of k_t-Dyck paths is given using a bijection, the kernel method, the cycle lemma, and symbolic method respectively. It is shown that the number of k_t-Dyck paths is equal to

$$\frac{t+1}{(k+1)n+t+1}\binom{(k+1)n+t+1}{n}.$$

This is also equal to the n-th term of a generalised binomial series, and Chapter 3 briefly discusses generalised binomial series and some of their properties. This chapter also provides all roots of the equation

$$a - bx + cx^n = 0$$

where n is a positive integer in terms of generalised binomial series. This is a result of Lagrange, which pre-dates modern notation. Explicitly determining these roots in terms of generalised binomial series has applications in the kernel method as well as obtaining a formula for generalised Fibonacci numbers.

We also perform analysis on k_t-Dyck paths to determine statistics such as the average number of peaks, valleys, and returns in k_t-Dyck paths, as well as the variance in the number of peaks, valleys, and returns in k_t-Dyck paths of a given length. This analysis uses the symbolic decomposition of k_t-Dyck paths along with bivariate generating functions, and can be found in Chapter 6.

In Chapter 7 we briefly discuss walks in the quarter plane, a very active and young area of research related to lattice path enumeration. A family of walks in the quarter plane provide the motivation for considering k_t-Dyck paths, as this family of walks can be decomposed into ordered pairs of k_t-Dyck paths in a natural way. Using this we provide the enumeration of these walks from a (slightly) restricted starting point to an arbitrary $(i, j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Some statistics related to k_t-Dyck paths transfer through to the walks and are discussed.

Finally, in Chapter 8 we discuss two families of paths which are closely related to 2_0-Dyck and 2_1-Dyck paths called S-Motzkin and T-Motzkin paths. Here we perform a similar analysis of parameters to that done in Chapter 6, but for this slightly different lattice path. During the analysis of parameters in Chapters 6 and 8 some simplifications occur when calculating some of the variances. There are sums of binomial coefficients which unexpectedly simplify to single binomial coefficients. We briefly discuss why these simplifications occur, as well as some identities which can be obtained from the simplifications.

Chapter 2

k-Dyck Paths and a Generalisation

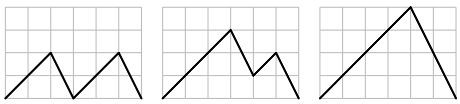
Throughout this thesis concepts will be applied to a family of lattice paths called k_t-Dyck paths, which are a generalisation of a family of lattice paths called k-Dyck paths. In this chapter we provide definitions and examples of these paths, as well as bijective proofs of their enumeration.

2.1 Definitions and examples

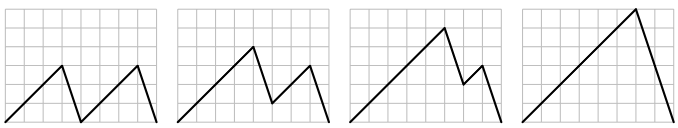
Definition 2.1.1. *A* k-Dyck path *is a non-negative lattice path with a length of* $(k+1)n$ *consisting of steps from the step set* $\{(1,1),(1,-k)\}$ *from* $(0,0)$ *to* $((k+1)n,0)$.

Note that 1-Dyck paths are what are commonly known as Dyck paths. Some examples of k-Dyck paths are given.

2-Dyck paths of length $3n$ for $n=2$:



3-Dyck paths of length $4n$ for $n=2$:

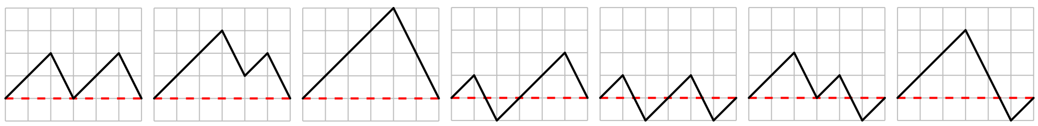


Definition 2.1.2. *Let* $t \in \mathbb{Z}_{\geq 0}$. *A* k_t-Dyck path *of length* $(k+1)n$ *is a lattice path consisting of steps from the step set* $\{(1,1),(1,-k)\}$ *from* $(0,0)$ *to* $((k+1)n,0)$ *with the condition that the path remains on or above the line* $y=-t$.

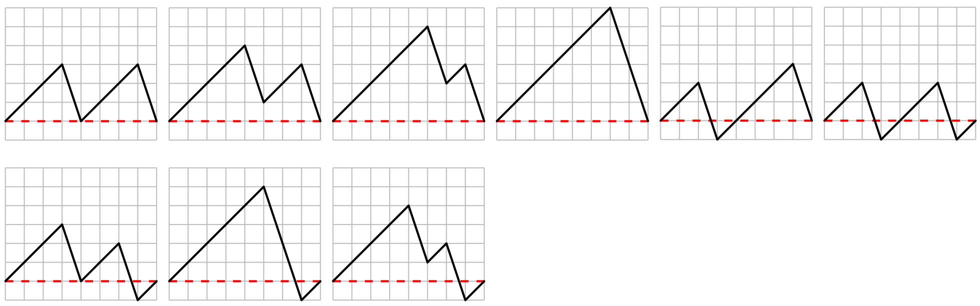
Here, note that k_0-Dyck paths are the same as k-Dyck paths. Also note that all k_{t-1}-Dyck paths of length $(k+1)n$ are also k_t-Dyck paths of length $(k+1)n$. In Section 2.2.2 we will show that the number of k_t-Dyck paths is $\frac{t+1}{(k+1)n+t+1}\binom{(k+1)n+t+1}{n}$ for $0 \le t \le k-1$. Therefore the number of k_t-Dyck paths that are not also k_{t-1}-Dyck paths is

$$\begin{aligned}
&\frac{t+1}{(k+1)n+t+1}\binom{(k+1)n+t+1}{n} - \frac{t}{(k+1)n+t}\binom{(k+1)n+t}{n}\\
&= \binom{(k+1)n+t}{n}\left[\frac{t+1}{kn+t+1} - \frac{t}{(k+1)n+t}\right]\\
&= \frac{(t+k+1)n}{(kn+t+1)((k+1)n+t)}\binom{(k+1)n+t}{n}.
\end{aligned}$$

2_1-Dyck paths of length $3n$ for $n = 2$:



3_1-Dyck paths of length $4n$ for $n = 2$:



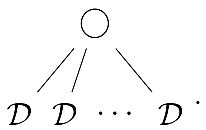
2.2 Bijections

2.2.1 Enumeration of k-Dyck paths

To enumerate k-Dyck paths we will first discuss the enumeration of d-ary trees.

Definition 2.2.1. *A d-ary tree is a tree where each internal node has exactly d children.*

Symbolically, each d-ary tree is either empty (ε) or an internal node ($\bigcirc$) with d subtrees. If we let $\mathcal{D}$ be the class of d-ary trees, then

$\mathcal{D} = \varepsilon +$  . (2.1)

Let d_n be the number of d-ary trees with n internal nodes. Then the generating function for d-ary trees is

$$D(x) = \sum_{n \geq 0} d_n x^n.$$

Therefore from (2.1) we see that

$$D(x) = 1 + xD(x)^d. \tag{2.2}$$

To determine a solution to this equation we will use the Lagrange Inversion Theorem as given in *Analytic Combinatorics* [12, Theorem A.2].

Theorem 2.2.1 (Lagrange Inversion Theorem)**.** *If a functional equation can be expressed as $A(z) = z\phi(A(z))$, then the coefficients of $A(z)$ are determined by*

$$[z^n]A(z) = \frac{1}{n}[w^{n-1}]\phi(w)^n \qquad \text{and} \qquad [z^n]A(z)^k = \frac{k}{n}[w^{n-k}]\phi(w)^n.$$

Applying this theorem to equation (2.2) with $A(x) = D(x) - 1$ we get

$$A(x) = x(1 + A(x))^d,$$

so $\phi(w) = (1 + w)^d$. Altogether,

$$[x^n]D(x) = \frac{1}{n}[w^{n-1}](1+w)^{dn} = \frac{1}{n}\binom{dn}{n-1} = \frac{1}{(d-1)n+1}\binom{dn}{n}.$$

Therefore the number of d-ary trees with n nodes is equal to the generalised Catalan number

$$\frac{1}{(d-1)n+1}\binom{dn}{n}. \tag{2.3}$$

We will prove the enumeration of k-Dyck paths through a bijection with $(k+1)$-ary trees.

Proposition 2.2.1. *The number of k-Dyck paths of length $(k+1)n$ is given by*

$$\frac{1}{kn+1}\binom{(k+1)n}{n}.$$

Proof. First we provide the mapping from a $(k+1)$-ary tree to a k-Dyck path. Consider an arbitrary $(k+1)$-ary tree with a total of $(k+1)n+1$ internal and external nodes. We traverse this tree in postorder (recursively visiting the root followed by the subtrees from right to left) and number the internal and external nodes in the order that they are visited (descending from $(k+1)n$ to 0). Then looking at the nodes numbered from 1 to $(k+1)n$ in numerical order, if a node is an internal node it becomes a $(1, -k)$ step, and if it is an external node it becomes a $(1, 1)$ step. It can be shown inductively that this gives a valid k-Dyck path: assume that all of the $k + 1$ subtrees of the root are valid k-Dyck paths with an additional $\diagup$ step at the beginning of the path. Appending these paths in order from left to right and removing the leftmost external node (initial $\diagup$ step) results in a non-negative path that ends at a height of k. The root (always numbered $(k + 1)n$) can then be added to the path to return to $y = 0$.

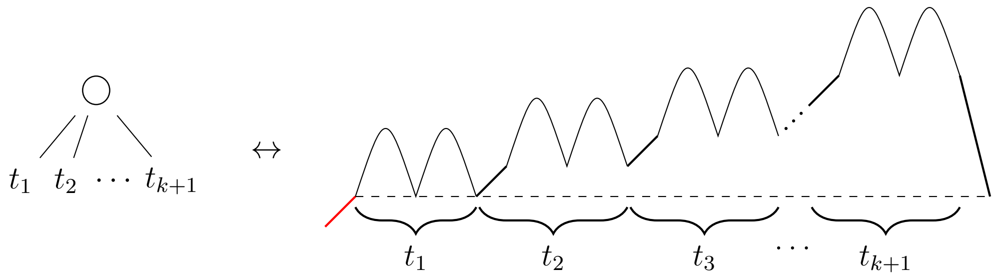


Figure 2.1: Representation of the path as pieces according to subtrees

For the mapping from a k-Dyck path to a $(k + 1)$-ary tree, consider an arbitrary k-Dyck path. Reading the steps of the path from right to left each step is of the form $(1, y)$ where $y = 1$ or $y = -k$. The final step of the path (or first step considering steps from right to left) is a $(1, -k)$ step, so we draw the root (internal) node in the resulting tree. Then, for the rest of the steps from right to left, draw a node in the rightmost available position in the resulting tree. If $y = 1$ draw an external node, and if $y = -k$ draw an internal node. The resulting tree will have one external node missing, so append the final external node (in the reverse mapping this is the node numbered 0 which is ignored). $\square$

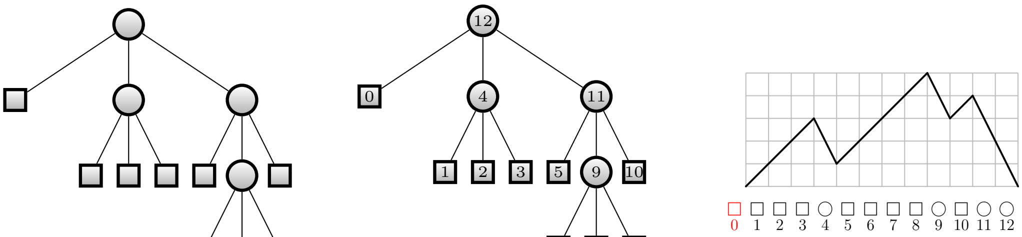
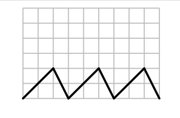
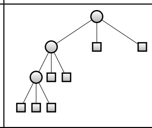
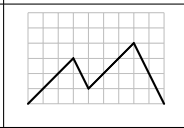
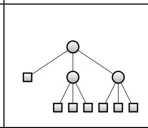
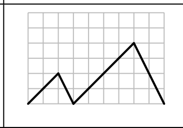
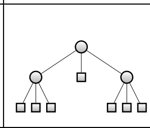
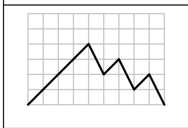
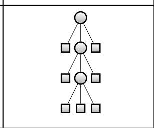
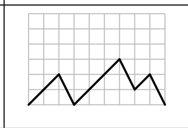
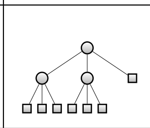
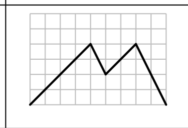
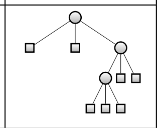
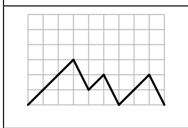
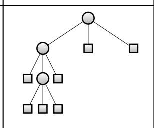
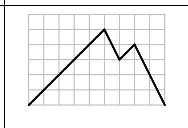
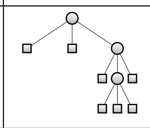
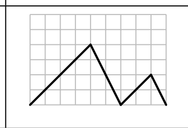
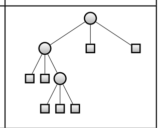
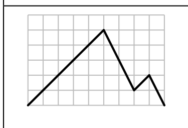
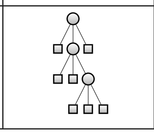
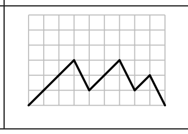
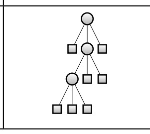
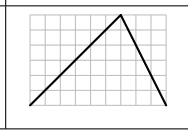
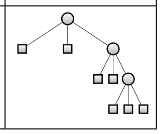


Figure 2.2: The mapping from a 3-ary (ternary) tree to a 2-Dyck path

The bijection is demonstrated in Figure 2.2. Since the reverse mapping of the one demonstrated is similar, it can be seen by looking at the figure from right to left. A small table of the bijection between $(k+1)$-ary trees with $(k+1)n$ nodes and k-Dyck paths of length $(k+1)n$ for $k=2$ and $n=3$ is given in Table 2.1.

Table 2.1: Bijection for 2-Dyck paths and 3-ary trees for $n=3$.

2-Dyck path	3-ary tree	2-Dyck path	3-ary tree	2-Dyck path	3-ary tree
					
					
					
					

2.2.2 Enumeration of k_t-Dyck paths

Proposition 2.2.2. *Let $0 \leq t \leq k$. The number of k_t-Dyck paths of length $(k+1)n$ is given by*

$$\frac{t+1}{(k+1)n+t+1}\binom{(k+1)n+t+1}{n}. \tag{2.4}$$

When $t=0$ we have k_0-Dyck paths, and equation (2.4) simplifies to the case for k-Dyck paths as shown in the calculation below:

$$\begin{aligned}\frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n} &= \frac{((k+1)n+1)!}{((k+1)n+1)n!(kn+1)!} = \frac{((k+1)n)!}{(kn+1)n!(kn)!}\\ &= \frac{1}{kn+1}\binom{(k+1)n}{n}.\end{aligned}$$

Gu, Prodinger, and Wagner [15] give an equivalent formulation of these k_t-Dyck paths and provide a bijection between them and k-plane trees to prove a result equivalent to Proposition 2.2.2. Here we will prove this via a bijection between k_t-Dyck paths and ordered tuples of k-Dyck paths.

Note that equation (2.4) is the n-th coefficient of the generalised binomial series $(\mathcal{B}_{k+1}(x))^{t+1}$ as can be found in *Concrete Mathematics* [13, Section 5.4] or discussed in Chapter 3. Since k-Dyck paths have generating function $\mathcal{B}_{k+1}(x)$, to prove Proposition 2.2.2 we will demonstrate a bijection between k_t-Dyck paths of length $(k+1)n$ and ordered tuples of $t+1$ k_0-Dyck paths whose total (summed) length is $(k+1)n$.

Proof of Proposition 2.2.2. We first give a mapping from ordered tuples of $t+1$ k_0-Dyck paths whose total (summed) length is $(k+1)n$ to k_t-Dyck paths of length $(k+1)n$. Let $(p_0, p_1, \dots, p_t)$ be an ordered tuple of $t+1$ k_0-Dyck paths. We form a new tuple $(p'_0, p'_1, \dots, p'_t)$ where p'_i is obtained by shifting each $(1, -k)$ step in the path p_i a total of $t-i$ positions to the left. By joining the end of p'_i and the start of p'_{i+1} for $0 \le i \le t-1$ we obtain a k_t-Dyck path.

For the inverse mapping, consider an arbitrary k_t-Dyck path of length $(k+1)n$ and we again form a $(t+1)$-tuple $(m_0, m_1, \dots, m_t)$. Let

$$\mathcal{H} : \{0, 1, \dots, (k+1)n\} \to \{-t, -t+1, \dots, kn\}$$

be a function from the x-coordinates of the path to the corresponding y-value which represents the height of the path. Now, let

$$m_0 = \mathsf{max}(\{\ell : \mathcal{H}(\ell) = -t \text{ and } 0 \le \ell \le (k+1)n\} \cup \{-t\})$$

and for $0 \le i \le t-1$ let

$$m_{i+1} = \mathsf{max}(\{\ell : \mathcal{H}(\ell) = -t+i+1 \text{ and } m_i + t - i \le \ell \le (k+1)n\} \cup \{m_i + 1\}).$$

Let q_0 be the subpath of the original k_t-Dyck path from x-coordinates $x = 0$ to $x = m_0 + t$. For $1 \le i \le t-1$, let q_i be the subpath of the k_t-Dyck path from $x = m_{i-1} + t - i + 1$ to $x = m_i + t - i$. Finally, let q_t be the subpath from $x = m_{t-1} + 1$ to $x = m_t = (k+1)n$ respectively. In this way, the subpath q_i is a k_{t-i}-Dyck path. To 'shift' each of these paths into k_0-Dyck paths, let q'_i be the path q_i with each $(1, -k)$ step shifted $t-i$ positions to the right. From these we obtain the desired $(t+1)$-tuple of k_0-Dyck paths, $(q'_0, q'_1, \dots, q'_t)$. □

To demonstrate this mapping, we provide a concrete example for a 4_3-Dyck path, as well as a table for 2_2-Dyck paths. Consider the 4_3-Dyck path given in Figure 2.3.

Here the largest x-value at which $\mathcal{H}(x) = -3$ is $m_0 = 7$, and it can be verified that $m_1 = \mathsf{max}(\{\ell : \mathcal{H}(\ell) = -2 \text{ and } 10 \le \ell \le 15\} \cup \{8\}) = 8$ and $m_2 = \mathsf{max}(\{\ell : \mathcal{H}(\ell) = -1 \text{ and } 10 \le \ell \le 15\} \cup \{9\}) = 9$. Finally, we have that $m_3 = \mathsf{max}(\{\ell : \mathcal{H}(\ell) = 0 \text{ and } 10 \le \ell \le 15\} \cup \{10\}) = 15$. To determine q_0, q_1, q_2, q_3 we consider the subpaths from 0 to $7+3 = 10$, 10 to $8+3-1 = 10$

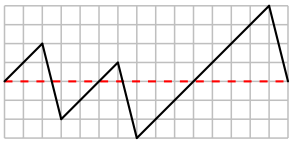
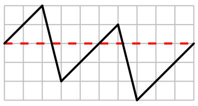
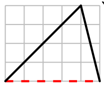
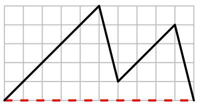
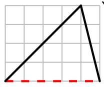


Figure 2.3: A 4_3-Dyck path of length 15

(empty), 10 to $9 + 3 - 2 = 10$ (empty), and 10 to 15. Therefore the 4-tuple (q_0, q_1, q_2, q_3) is given by

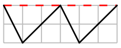
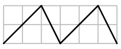
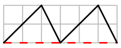
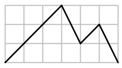
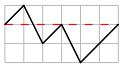
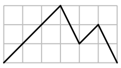
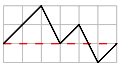
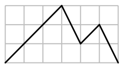
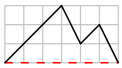
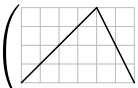
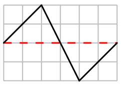
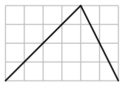
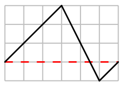
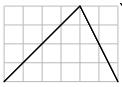
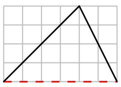
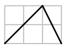
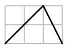
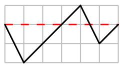
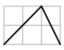
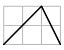
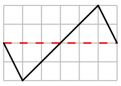
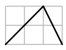
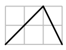
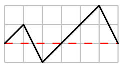
( , ε , ε , ),

where ε represents an empty path. After performing the appropriate shifts on the $(1, -4)$ steps we obtain the 4-tuple of 4_0-Dyck paths

( , ε , ε , ).

The inverse mapping can be seen by reading the example in reverse.

Table 2.2: Bijection for 2_2-Dyck paths of length 6

Triple of 2_0-Dyck paths	2_2-Dyck path	Triple of 2_0-Dyck paths	2_2-Dyck path
( , ε , ε)		(ε ,  , ε)	
(ε , ε , )		( , ε , ε)	
(ε ,  , ε)		(ε , ε , )	
( , ε , ε)		(ε ,  , ε)	
(ε , ε , )		( ,  , ε)	
( , ε , )		(ε ,  , )	

Chapter 3

Generalised Binomial Series

Generalised binomial series were introduced by Lambert in [23], and definitions and identities related to generalised binomial series can be found in *Concrete Mathematics* [13, Section 5.4]. In this chapter we will discuss some of the theory and applications of generalised binomial series, as well as discuss a result of Lagrange [22] relating to the roots of the equation $a - bx + cx^n = 0$ in terms of generalised binomial series.

3.1 Definitions and examples

Definition 3.1.1. *The* generalised binomial series $\mathcal{B}_m(x)$ *is given by*

$$\mathcal{B}_m(x) = \sum_{n\geq 0} \binom{mn+1}{n} \frac{1}{mn+1} x^n.$$

These generalised binomial series satisfy some useful identities. In particular, we will frequently use the following identity

$$(\mathcal{B}_m(x))^r = \sum_{n\geq 0} \binom{mn+r}{n} \frac{r}{mn+r} x^n.$$

To prove this, we will use a clever substitution as well as *Cauchy's integral formula*, [12, Theorem IV.4].

Theorem 3.1.1 (Cauchy's integral formula)**.** *Let $f(x)$ be analytic in a region Ω containing 0. Then, the coefficient $[x^n]f(x)$ admits the integral representation*

$$[x^n]f(x) = \frac{1}{2\pi i} \oint \frac{f(x)}{x^{n+1}}\, dx.$$

Proof. Let $x = z(1-z)^{m-1}$. We will show that $(1-z)^{-1} = \mathcal{B}_m(x)$. Applying Cauchy's integral formula to obtain the n-th coefficient of $(1-z)^{-1}$ we have

$dx = (1-z)^{m-2}(1-mz)\,dz$ and thus

$$\begin{aligned}
[x^n]\frac{1}{1-z} &= \frac{1}{2\pi i}\oint \frac{\frac{1}{1-z}}{x^{n+1}}\,dx = \frac{1}{2\pi i}\oint \frac{(1-z)^{m-2}(1-mz)}{z^{n+1}(1-z)^{(m-1)n+(m-1)+1}}\,dz \\
&= \frac{1}{2\pi i}\oint \frac{1-mz}{z^{n+1}(1-z)^{(m-1)n+2}}\,dz = [z^n]\frac{1-mz}{(1-z)^{(m-1)n+2}} \\
&= [z^n]\frac{1}{(1-z)^{(m-1)n+2}} - m[z^{n-1}]\frac{1}{(1-z)^{(m-1)n+2}} \\
&= \binom{mn+1}{n} - m\binom{mn}{n-1} = \binom{mn}{n-1}\Big(\frac{mn+1}{n} - m\Big) \\
&= \binom{mn+1}{n}\frac{1}{mn+1}.
\end{aligned}$$

Therefore $(1-z)^{-1} = \mathcal{B}_m(x)$. Now, we would like to find $\big(\mathcal{B}_m(x)\big)^r$. Using Cauchy's integral formula again, we can easily compute coefficients of $\big(\mathcal{B}_m(x)\big)^r = (1-z)^{-r}$:

$$\begin{aligned}
[x^n]\frac{1}{(1-z)^r} &= \frac{1}{2\pi i}\oint \frac{\frac{1}{(1-z)^r}}{x^{n+1}}\,dx = \frac{1}{2\pi i}\oint \frac{1-mz}{z^{n+1}(1-z)^{(m-1)n+r+1}}\,dz \\
&= [z^n]\frac{1-mz}{(1-z)^{(m-1)n+r+1}} = \binom{mn+r}{n} - m\binom{mn+r-1}{n-1} \\
&= \binom{mn+r-1}{n-1}\Big(\frac{mn+r}{n} - m\Big) = \frac{r(mn+r-1)!}{n!((m-1)n+r)!} \\
&= \binom{mn+r}{n}\frac{r}{mn+r}.
\end{aligned}$$

Therefore

$$\big(\mathcal{B}_m(x)\big)^r = \sum_{n\geq 0}\binom{mn+r}{n}\frac{r}{mn+r}x^n.$$

□

Lemma 3.1.1. *The following identities hold*

1. $\mathcal{B}_m(x)^{1-m} - \mathcal{B}_m(x)^{-m} = x.$
2. *Let ζ be a primitive k-th root of unity. Then*

$$\mathcal{B}_{k+1}(x) = \prod_{j=1}^{k}\mathcal{B}_{(k+1)/k}(\zeta^{-j}x^{\frac{1}{k}})^{\frac{1}{k}}.$$

Since we have shown that the generating function $\mathcal{B}_m(x)$ satisfies equation (2.2), we know that

$$\mathcal{B}_m(x) = 1 + x\big(\mathcal{B}_m(x)\big)^m.$$

Dividing this expression by $\mathcal{B}_m(x)^m$ gives the first identity in the above Lemma. The second identity follows easily from a theorem discussed later in this chapter.

3.2 Applications

3.2.1 The roots of the functional equation for k-ary trees

Here we explicitly determine all of the roots of the functional equation involving the generating function for k-ary trees. Combinatorially, we usually only consider the root which provides the generating function which enumerates k-ary trees. However, the other roots are useful in applications such as the kernel method outlined in Chapter 4.

Theorem 3.2.1. *Let $k \geq 2$. The equation $T = 1 + xT^k$ has k roots given by $\mathcal{B}_k(x)$ and for $1 \leq j \leq k-1$ where ζ is a fixed primitive $(k-1)$-th root of unity*

$$\zeta^j x^{-\frac{1}{k-1}} \mathcal{B}_{\frac{k}{k-1}}(\zeta^{-j} x^{\frac{1}{k-1}})^{-\frac{1}{k-1}}. \tag{3.1}$$

Proof. Firstly, we show that $y^{-1}\mathcal{B}_{\frac{k}{k-1}}(y)^{-\frac{1}{k-1}}$ is a root of the original equation $T = 1 + xT^k$, where $y = \zeta^{-j} x^{\frac{1}{k-1}}$. Substituting the root into the equation,

$$y^{-1}\mathcal{B}_{\frac{k}{k-1}}(y)^{-\frac{1}{k-1}} = 1 + x\big(y^{-1}\mathcal{B}_{\frac{k}{k-1}}(y)^{-\frac{1}{k-1}}\big)^k.$$

From this, we can manipulate it into the form

$$\mathcal{B}_{\frac{k}{k-1}}(y)^{1-\frac{k}{k-1}} - xy^{-k+1}\mathcal{B}_{\frac{k}{k-1}}(y)^{-\frac{k}{k-1}} = y.$$

This satisfies identity 1 of Lemma 3.1.1, provided that $y^{1-k} = x^{-1}$. Since $y^{k-1} = (\zeta^{-j}x^{\frac{1}{k-1}})^{k-1} = x$, this holds. Therefore $\zeta^j x^{-\frac{1}{k-1}}\mathcal{B}_{\frac{k}{k-1}}(\zeta^{-j}x^{\frac{1}{k-1}})^{-\frac{1}{k-1}}$ is a root of the equation for $1 \leq j \leq k-1$.

The fact that $\mathcal{B}_k(x)$ is a root is well known, and follows from the Lagrange Inversion Theorem (Theorem 2.2.1). □

With the above theorem we can prove the second identity in Lemma 3.1.1. In fact, with the roots of the equation $T = 1 + xT^k$ we can prove a number of identities by comparing coefficients in a similar manner to the proof that follows.

Proof of the second identity in Lemma 3.1.1. From Theorem 3.2.1 we know that the k roots of the equation $T = 1 + xT^k$ are given by

$$T_k = \mathcal{B}_k(x) \quad \text{and} \quad T_j = \zeta^j x^{-\frac{1}{k-1}}\mathcal{B}_{\frac{k}{k-1}}(\zeta^{-j}x^{\frac{1}{k-1}})^{-\frac{1}{k-1}} \quad \text{for} \quad 1 \leq j \leq k-1.$$

Now, the original equation $T = 1 + xT^k$ can be expressed in terms of its roots $x(T-T_1)(T-T_2)\cdots(T-T_k) = 0$. From $[x^0](xT^k - T + 1) = 1$ we know that

$$1 = [x^0]x(T-T_1)(T-T_2)\cdots(T-T_k) = (-1)^k xT_1T_2\cdots T_k.$$

Therefore

$$1 = (-1)^k x\mathcal{B}_k(x)\prod_{j=1}^{k-1}\zeta^j x^{-\frac{1}{k-1}}\mathcal{B}_{\frac{k}{k-1}}(\zeta^{-j}x^{\frac{1}{k-1}})^{-\frac{1}{k-1}},$$

or equivalently,

$$\mathcal{B}_k(x) = (-1)^k\prod_{j=1}^{k-1}\zeta^{-j}\mathcal{B}_{\frac{k}{k-1}}(\zeta^{-j}x^{\frac{1}{k-1}})^{\frac{1}{k-1}}. \tag{3.2}$$

Note that

$$\prod_{j=1}^{k-1}\zeta^{-j} = \zeta^{-\sum_{j=1}^{k-1} j} = \zeta^{-\frac{k(k-1)}{2}} = \begin{cases} 1 & \text{if } k \text{ is even,} \\ -1 & \text{if } k \text{ is odd.}\end{cases}$$

It follows that $(-1)^k\prod_{j=1}^{k-1}\zeta^{-j} = 1$ and equation (3.2) simplifies to the second identity in Lemma 3.1.1. □

3.2.2 Generalised Fibonacci numbers

This section is based on an excerpt of a paper by Prodinger and Selkirk [32].

These generalised binomial series can also be used to determine an explicit formula for *generalised Fibonacci numbers*. That is, the h-Fibonacci numbers are a sequence of numbers f_n defined by the recursion $f_n = \sum_{i=1}^{h} f_{n-i}$ where $f_0 = 0$, $f_1 = 1$, and $f_\ell = 0$ for $\ell < 0$. This sequence can also be defined by the generating function

$$\sum_{n\geq 0} f_n z^n = \frac{z}{1 - z - z^2 - \cdots - z^h}. \tag{3.3}$$

Note that the denominator of equation (3.3) can be rewritten as

$$1 - z - z^2 - \cdots - z^h = 1 - z(1 + z + \cdots + z^{h-1}) = 1 - z\cdot\frac{1-z^h}{1-z} = \frac{1 - 2z + z^{h+1}}{1-z}.$$

Hence the generating function in equation (3.3) can be expressed as

$$\frac{z(1-z)}{1 - 2z + z^{h+1}}. \tag{3.4}$$

The dominant root of the rational function in equation (3.4) already occurs in [30]. However, using generalised binomial series we can describe all the roots of the denominator, and use this to obtain a Binet-type formula.

Given the expression $1 - \frac{z}{u} + z^{h+1}$, let ζ be a primitive h-th root of unity. Then the $h+1$ roots can be expressed in terms of these generalised binomial series as

$$u\mathcal{B}_{h+1}(u^{h+1}) \qquad \text{and} \qquad \zeta^{-j}u^{-\frac{1}{h}}\mathcal{B}_{(h+1)/h}\big(\zeta^j u^{\frac{h+1}{h}}\big)^{-\frac{1}{h}} \quad \text{for } 0 \le j \le h-1.$$

In equation (3.4), we have the special case where $u = \frac{1}{2}$. It is easy to verify (and the calculation for $u = \frac{1}{2}$ has appeared in [30]) that

$$1 - u\mathcal{B}_{h+1}(u^{h+1}) + u^{h+1}\mathcal{B}_{h+1}(u^{h+1})^{h+1} = 0.$$

The other roots can be checked by letting $y = \zeta^j u^{\frac{h+1}{h}}$, and

$$1 - u^{-1}(uy^{-1})\mathcal{B}_{(h+1)/h}\big(y\big)^{-\frac{1}{h}} + \big((uy^{-1})\mathcal{B}_{(h+1)/h}\big(y\big)^{-\frac{1}{h}}\big)^{h+1} = 0$$

can be rearranged into

$$y = \mathcal{B}_{\frac{h+1}{h}}(y)^{-\frac{1}{h}} - u^{h+1}y^{-h}\mathcal{B}_{\frac{h+1}{h}}(y)^{-\frac{h+1}{h}}.$$

Since $u^{h+1}y^{-h} = u^{h+1}(\zeta^j u^{\frac{h+1}{h}})^{-h} = \zeta^{-hj} = 1$, this follows from identity 1 in Lemma 3.1.1, with $m = \frac{h+1}{h}$.

From this, we can explicitly compute the coefficients of

$$\frac{1}{1 - 2z + z^{h+1}}.$$

For ease of notation, let $r_h = \frac{1}{2}\mathcal{B}_{h+1}\big(\frac{1}{2^{h+1}}\big)$, and for $0 \le j \le h-1$ let

$$r_j = \zeta^{-j}2^{\frac{1}{h}}\mathcal{B}_{(h+1)/h}\left(\zeta^j\left(\frac{1}{2}\right)^{\frac{h+1}{h}}\right)^{-\frac{1}{h}}.$$

Then using partial fractions and these r_i values where $0 \le i \le h$, we can compute that $[z^n]\frac{1}{1-2z+z^{h+1}}$ is equal to

$$\begin{aligned}
[z^n]\frac{1}{(z-r_0)(z-r_1)\cdots(z-r_h)} &= [z^n]\sum_{i=0}^{h}\frac{1}{(z-r_i)}\prod_{\substack{j=0\\ j\ne i}}^{h}(r_i - r_j)^{-1} \\
&= -\sum_{i=0}^{h}\frac{1}{r_i^{n+1}}\prod_{\substack{j=0\\ j\ne i}}^{h}(r_i - r_j)^{-1}.
\end{aligned}$$

Therefore we have obtained a Binet-type formula for generalised Fibonacci numbers.

To provide a concrete example of how one would use these roots to compute generalised Fibonacci numbers, we provide the calculation of the formula for the case with Tetranacci numbers. Tetranacci numbers correspond to $h = 4$, so the five roots are (as calculated by a computer, rounding non-integers):

$$\begin{aligned}
r_4 &= \frac{1}{2}\mathcal{B}_5\Big(\frac{1}{2^5}\Big) = 0.518790063675884,\\
r_0 &= 2^{\frac{1}{4}}\mathcal{B}_{5/4}\Big(\Big(\frac{1}{2}\Big)^{\frac{5}{4}}\Big)^{-\frac{1}{4}} = 1,\\
r_1 &= -i2^{\frac{1}{4}}\mathcal{B}_{5/4}\Big(i\Big(\frac{1}{2}\Big)^{\frac{5}{4}}\Big)^{-\frac{1}{4}} = -0.114070631164587 - 1.21674600397435i,\\
r_2 &= -2^{\frac{1}{4}}\mathcal{B}_{5/4}\Big(-\Big(\frac{1}{2}\Big)^{\frac{5}{4}}\Big)^{-\frac{1}{4}} = -1.29064880134671,\\
r_3 &= i2^{\frac{1}{4}}\mathcal{B}_{5/4}\Big(-i\Big(\frac{1}{2}\Big)^{\frac{5}{4}}\Big)^{-\frac{1}{4}} = -0.114070631164587 + 1.21674600397435i.
\end{aligned}$$

Using these roots and the initial values, we can determine the values of A, B, C, D, and E in the expression

$$u_n = A \cdot r_4^{-n} + B \cdot r_0^{-n} + C \cdot r_1^{-n} + D \cdot r_2^{-n} + E \cdot r_3^{-n}.$$

Again using a computer, we find that these are given by

$$\begin{aligned}
A &= 0.293813062773642\\
B &= 0\\
C &= -0.0504502052166080 - 0.169681902881564i\\
D &= -0.192912652340427\\
E &= -0.0504502052166080 + 0.169681902881564i
\end{aligned}$$

Therefore the n-th Tetranacci number can be calculated via the formula:

$$\begin{aligned}
u_n = {}& \frac{0.293813062773642}{(0.518790063675884)^n} - \frac{0.0504502052166080 + 0.169681902881564i}{(-0.114070631164587 - 1.21674600397435i)^n}\\
& - \frac{0.192912652340427}{(-1.29064880134671)^n} + \frac{-0.0504502052166080 + 0.169681902881564i}{(-0.114070631164587 + 1.21674600397435i)^n}.
\end{aligned}$$

By using this formula and a computer, we obtain for example that when $n = 42$, $u_{42} = 274423830033$. Analogous computations provide similar formulas for other generalised Fibonacci numbers.

3.2.3 A result of Lagrange

In 1770 Lagrange found all roots of the equation

$$a - bx + cx^n = 0,$$

where n is a positive integer. The paper in which he presented a solution, [22], is in French and pre-dates modern mathematical notation such as summations, factorials, and binomial coefficients. Here we will present and prove Lagrange's result in terms of generalised binomial series.

Theorem 3.2.2. *The n solutions $x_1, x_2, \dots, x_n$ to the equation $a - bx + cx^n = 0$ are given by*

$$x_n = \frac{a}{b}\sum_{k\geq 0}\binom{nk+1}{k}\frac{1}{nk+1}\left(\frac{a^{n-1}c}{b^n}\right)^k = \frac{a}{b}\mathcal{B}_n\left(\frac{a^{n-1}c}{b^n}\right),$$

and

$$\begin{aligned} x_j &= \zeta^j\left(\frac{b}{c}\right)^{\frac{1}{n-1}}\sum_{k\geq 0}\binom{\frac{nk}{n-1}-\frac{1}{n-1}}{k}\frac{-\frac{1}{n-1}}{\frac{nk}{n-1}-\frac{1}{n-1}}\left(\frac{a}{b}\cdot\zeta^{-j}\cdot\left(\frac{c}{b}\right)^{\frac{1}{n-1}}\right)^k \\ &= \zeta^j\left(\frac{b}{c}\right)^{\frac{1}{n-1}}\mathcal{B}_{\frac{n}{n-1}}\left(\frac{a}{b}\cdot\zeta^{-j}\cdot\left(\frac{c}{b}\right)^{\frac{1}{n-1}}\right)^{-\frac{1}{n-1}}, \end{aligned}$$

where $1 \leq j \leq n-1$, $n^n|a^{n-1}c| < (n-1)^{n-1}|b^n|$, and ζ is a primitive $(n-1)$-th root of unity.

Proof. The solution $x = x_n$ follows from the first identity in Lemma 3.1.1, $\mathcal{B}_m(x)^m = x^{-1}(\mathcal{B}_m(x) - 1)$. Here

$$\begin{aligned} a - bx_n + cx_n^n &= a - b\cdot\frac{a}{b}\mathcal{B}_n\left(\frac{a^{n-1}c}{b^n}\right) + c\left(\frac{a}{b}\mathcal{B}_n\left(\frac{a^{n-1}c}{b^n}\right)\right)^n \\ &= a - a\mathcal{B}_n\left(\frac{a^{n-1}c}{b^n}\right) + \frac{a^nc}{b^n}\cdot\frac{b^n}{a^{n-1}c}\left(\mathcal{B}_n\left(\frac{a^{n-1}c}{b^n}\right) - 1\right) \\ &= a - a\mathcal{B}_n\left(\frac{a^{n-1}c}{b^n}\right) + a\left(\mathcal{B}_n\left(\frac{a^{n-1}c}{b^n}\right) - 1\right) = 0. \end{aligned}$$

For the conditions on a, b, and c, by the ratio test for convergence, we know that

$$\lim_{k\to\infty}\left|\frac{\binom{n(k+1)+1}{k+1}\frac{1}{n(k+1)+1}}{\binom{nk+1}{k}\frac{1}{nk+1}}\right|\cdot\left|\frac{a^{n-1}c}{b^n}\right| < 1.$$

We can calculate the limit explicitly:

$$\begin{aligned} \lim_{k\to\infty}\left|\frac{\binom{n(k+1)+1}{k+1}\frac{1}{n(k+1)+1}}{\binom{nk+1}{k}\frac{1}{nk+1}}\right| &= \lim_{k\to\infty}\frac{(n(k+1))!((n-1)k+1)!}{(k+1)(nk)!((n-1)(k+1)+1)!} \\ &= \lim_{k\to\infty}\frac{(nk+n)\cdots(nk+1)}{(k+1)((n-1)k+n)\cdots((n-1)k+2)} \\ &= \frac{n^n}{(n-1)^{n-1}}. \end{aligned}$$

Therefore $\frac{n^n}{(n-1)^{n-1}}\left|\frac{a^{n-1}c}{b^n}\right| < 1$ and thus $n^n|a^{n-1}c| < (n-1)^{n-1}|b^n|$.

For the other solutions $x_1, x_2, \ldots, x_{n-1}$ let $y = \zeta^{-j}\left(\frac{c}{b}\right)^{\frac{1}{n-1}}$ and $1 \le j \le n-1$:

$$\begin{aligned}
&a - bx_j + cx_j^n \\
&= a - by^{-1}\mathcal{B}_{\frac{n}{n-1}}\left(\frac{a}{b}\cdot y\right)^{1-\frac{n}{n-1}} + cy^{-n}\mathcal{B}_{\frac{n}{n-1}}\left(\frac{a}{b}\cdot y\right)^{-\frac{n}{n-1}} \\
&= a - by^{-1}\mathcal{B}_{\frac{n}{n-1}}\left(\frac{a}{b}\cdot y\right)^{1-\frac{n}{n-1}} + cy^{-n}\left(\mathcal{B}_{\frac{n}{n-1}}\left(\frac{a}{b}\cdot y\right)^{1-\frac{n}{n-1}} - \frac{a}{b}\cdot y\right) \\
&= a - (by^{-1} - cy^{-n})\mathcal{B}_{\frac{n}{n-1}}\left(\frac{a}{b}\cdot y\right)^{1-\frac{n}{n-1}} - \frac{ac}{b}\cdot y^{1-n}.
\end{aligned}$$

The step from the second to third line follows from Lemma 3.1.1, identity 1. We calculate that $y^{1-n} = \left(\zeta^{-j}\left(\frac{c}{b}\right)^{\frac{1}{n-1}}\right)^{-(n-1)} = (\zeta^{n-1})^j\left(\frac{b}{c}\right) = \frac{b}{c}$, and since $\zeta^n = \zeta^{(n-1)+1} = 1\cdot\zeta^1$, we have that

$$by^{-1} - cy^{-n} = \zeta^j\left(b\left(\frac{c}{b}\right)^{-\frac{1}{n-1}} - c\left(\frac{c}{b}\right)^{-\frac{n}{n-1}}\right) = \zeta^j\left(b^{1+\frac{1}{n-1}}c^{-\frac{1}{n-1}} - b^{\frac{n}{n-1}}c^{1-\frac{n}{n-1}}\right) = 0.$$

Therefore we can continue the calculation:

$$a - (by^{-1} - cy^{-n})\mathcal{B}_{\frac{n}{n-1}}\left(\frac{a}{b}\cdot y\right)^{1-\frac{n}{n-1}} - \frac{ac}{b}\cdot y^{1-n} = a - 0 - \frac{ac}{b}\cdot\frac{b}{c} = 0.$$

Again, we check convergence of the series using the ratio test

$$\lim_{k\to\infty}\left|\frac{\binom{\frac{n(k+1)-1}{n-1}}{k+1}\frac{-1}{n(k+1)-1}}{\binom{\frac{nk-1}{n-1}}{k}\frac{-1}{nk-1}}\right|\cdot\left|\left(\frac{a}{b}\cdot\zeta^{-j}\cdot\left(\frac{c}{b}\right)^{\frac{1}{n-1}}\right)\right| < 1.$$

Now, we can calculate the limit:

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{\binom{\frac{n(k+1)-1}{n-1}}{k+1}\frac{-1}{n(k+1)-1}}{\binom{\frac{nk-1}{n-1}}{k}\frac{-1}{nk-1}}\right| &= \lim_{k\to\infty}\frac{(nk-1)\left(\frac{n(k+1)-1}{n-1}\right)\left(\frac{nk}{n-1}\right)\cdots\left(\frac{k+n-1}{n-1}\right)}{(n(k+1)-1)(k+1)\left(\frac{nk-1}{n-1}\right)\left(\frac{nk-n}{n-1}\right)\cdots\left(\frac{k+n-2}{n-1}\right)} \\
&= \lim_{k\to\infty}\frac{(nk)\cdots(k+n-1)}{(n-1)(k+1)(nk-n)\cdots(k+n-2)} \\
&= \frac{n}{n-1}.
\end{aligned}$$

Therefore $n|ac^{\frac{1}{n-1}}| < |b^{\frac{n}{n-1}}|(n-1)$ and thus $n^{n-1}|a^{n-1}c| < |b^n|(n-1)^{n-1}$. But this follows since for the root x_n we require that $n^n|a^{n-1}c| < (n-1)^{n-1}|b^n|$. $\square$

By reviewing the applications found in Subsections 3.2.1 and 3.2.2, it can be seen that the results follow directly from Theorem 3.2.2.

Chapter 4

The Kernel Method

The kernel method is a powerful enumerative tool made popular by Knuth [19, Exercise 2.2.1.4] and can be used to obtain various enumerative results [2, 6, 28]. In the section that follows we will discuss the theory as set out by Banderier and Flajolet [1], and Flajolet and Sedgewick in *Analytic Combinatorics* [12, Section VII.8.1]. Thereafter we will apply it to enumerate partial k-Dyck paths and k_t-Dyck paths ending at arbitrary height, from which the enumeration of k-Dyck paths and k_t-Dyck paths follows as a corollary.

4.1 The kernel method

Definition 4.1.1. *Let* $\mathcal{S} = \{(1, y_1), (1, y_2), \ldots, (1, y_n)\}$ *be the step set of a lattice path. Then the* step polynomial *of* $\mathcal{S}$ *is given by*

$$S(z) = z^{y_1} + z^{y_2} + \cdots + z^{y_n}.$$

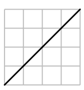
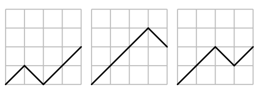
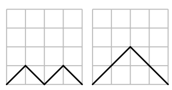
For example, the step polynomial of Dyck paths (non-negative lattice paths with step set $\{(1, 1), (1, -1)\}$) is $S(z) = z + z^{-1}$.

Definition 4.1.2. *The generating function for non-negative lattice paths with fixed step set is given by*

$$F(x, u) = \sum_{n \geq 0} f_n(u) x^n,$$

where $f_n(u)$ *is such that* $[x^n u^k]F(x, u)$ *is the number of partial paths of length* n *that end at height* k.

Continuing with the Dyck path example, the generating function for Dyck paths is $F(x, u) = 1 + xu + x^2(u^2 + 1) + x^3(u^3 + 2u) + x^4(u^4 + 3u^2 + 2) + \cdots$. Looking at $[x^4]F(x, u)$ (or the partial paths of length 4) we have $u^4 + 3u^2 + 2$. The partial paths represented by this polynomial in u are:

u^4 :  $3u^2$:  $2u^0$: 

Proposition 4.1.1. *Let $S(u)$ be the step polynomial of a non-negative lattice path where $[u^k]f_n(u)$ is the number of partial paths of length n that end at height k. Then*

$$f_{n+1}(u) = S(u)f_n(u) - r_n(u), \tag{4.1}$$

where $r_n(u)$ is the polynomial representing all possible steps ending in the region $y < 0$. So if $S(u) = a_{-c}u^{-c} + a_{-c+1}u^{-c+1} + \cdots + a_d u^d$, then

$$r_n(u) = \sum_{j=-c}^{-1} u^j([u^j]S(u)f_n(u)).$$

Let $f_n(u) = b_0 + b_1 u + b_2 u^2 + \cdots$, then for $-c \le m \le -1$,

$$[u^m]S(u)f_n(u) = [u^m](a_{-c}u^{-c} + \cdots + a_d u^d)(b_0 + b_1 u + \cdots) = \sum_{i=-c}^{m} a_i b_{m-i}.$$

Note that $b_j = \frac{f_n^{(j)}(0)}{j!}$, with this we have

$$[u^m]S(u)f_n(u) = \sum_{i=-c}^{m} \frac{a_i}{(m-i)!} f_n^{(m-i)}(0).$$

Therefore equation (4.1) becomes

$$f_{n+1}(u) = S(u)f_n(u) - \sum_{j=-c}^{-1} u^j \sum_{i=-c}^{j} \frac{a_i}{(j-i)!} f_n^{(j-i)}(0).$$

In the Dyck path example, with $S(u) = u + u^{-1}$, we have that $c = 1$ and thus

$$r_n = \sum_{j=-1}^{-1} u^j \sum_{i=-1}^{j} \frac{a_i}{(j-i)!} f_n^{(j-i)}(0) = u^{-1} \cdot \frac{1 \cdot f_n^0(0)}{0!} = u^{-1} f_n(0). \tag{4.2}$$

It follows from Proposition 4.1.1 that $f_{n+1}(u) = (u + u^{-1})f_n(u) - u^{-1}f_n(0)$. Multiplying by x^{n+1} and summing over n yields

$$\sum_{n\ge 0} f_{n+1}(u)x^{n+1} = (u + u^{-1}) \sum_{n\ge 0} f_n(u)x^{n+1} - u^{-1} \sum_{n\ge 0} f_n(0)x^{n+1},$$

or equivalently in terms of the bivariate generating function $F(x, u)$,

$$F(x, u) - f_0(u) = x(u + u^{-1})F(x, u) - xu^{-1}F(x, 0).$$

Solving for $F(x,u)$ and noting that $f_0(u)=1$, we obtain the expression

$$F(x,u)=\frac{1-xu^{-1}F(x,0)}{1-x(u+u^{-1})}. \tag{4.3}$$

How to further deal with the expression in equation (4.3) is discussed later in this chapter. For now, we return to how to use Proposition 4.1.1 in a general setting.

In a similar manner to that of the Dyck path example, we multiply equation (4.1) by x^{n+1} and sum over all $n \geq 0$ to find that

$$F(x,u)=1+xS(u)F(x,u)-x\sum_{j=-c}^{-1}u^j([u^j]S(u)F(x,u)).$$

By collecting $F(x,u)$ terms we obtain

$$F(x,u)=\frac{1-x\sum_{j=-c}^{-1}u^j([u^j]S(u)F(x,u))}{1-xS(u)}. \tag{4.4}$$

Further simplification of the bivariate generating function will be discussed later in this section after the necessary theory is established.

Definition 4.1.3. *The equation* $1-xS(u)=0$ *is called the* kernel equation.

As shown in equation (4.4), the left hand side of the kernel equation is always in the denominator of the bivariate generating function $F(x,u)$, and solutions to the kernel equation need to be 'cancelled' in order to compute the generating function. Hence the name "kernel method".

Proposition 4.1.2. *Let* $\mathcal{S}=\{(1,y_1),(1,y_2),\ldots,(1,y_n)\}$ *be the step set of a non-negative lattice path. Let* $c=\mathsf{min}\{y_1,y_2,\ldots,y_n\}$ *and* $d=\mathsf{max}\{y_1,y_2,\ldots,y_n\}$*. Then the kernel equation has* $c+d$ *roots which are branches of an algebraic function. Furthermore,* c *branches tend to* 0 *as* $x\to 0$ *and* d *branches tend to infinity as* $x \to 0$*.*

Proof. The kernel equation, $1-xS(u)$, multiplied by u^c is a polynomial of degree $c+d$, which is known to be algebraic and have $c+d$ roots in u. We show that $xu^{-c}\sim 1$ or $xu^d \sim 1$. Let $m>0$, $a_m\neq 0$, and

$$\gamma=\sum_{n\geq m}a_nx^n$$

be a solution to $1-xS(u)=0$. Then

$$\begin{aligned}
x^{-1}=S(\gamma)&=\gamma^{-c}+\cdots+\gamma^d\\
&=(a_mx^m+a_{m+1}x^{m+1}+\cdots)^{-c}+\cdots+(a_mx^m+a_{m+1}x^{m+1}+\cdots)^d\\
&=a_m^{-c}x^{-cm}(1+\tfrac{a_{m+1}}{a_m}x+\cdots)^{-c}+\cdots+(a_mx^m+a_{m+1}x^{m+1}+\cdots)^d.
\end{aligned}$$

Since $a_m x^{-cm}$ is the term of $S(\gamma)$ with the smallest exponent, we have that $a_m x^{-cm} = x^{-1}$. Therefore $a_m = \zeta$ where ζ is a c-th root of unity, and $m = 1/c$. Now, we have that

$$\lim_{x\to 0} x\gamma^{-c} = \lim_{x\to 0} x(\zeta x^{1/c} + \cdots)^{-c} = \lim_{x\to 0} x\zeta^{-c}x^{-1}(1+\cdots)^{-c} = 1.$$

Since there are c possible c-th roots of unity, there are c such γ solutions. To obtain the other solutions we set $m > 0$, $a_{-m} \neq 0$, and set

$$\gamma = \sum_{n\geq -m} a_n x^n,$$

where $1 - xS(\gamma) = 0$. Then

$$\begin{aligned} x^{-1} = S(\gamma) &= \gamma^d + \cdots + \gamma^{-c} \\ &= (a_{-m}x^{-m} + \cdots)^d + \cdots + (a_{-m}x^{-m} + \cdots)^{-c} \\ &= (a_{-m})^d x^{-md}(1+\cdots)^d + \cdots + (a_{-m}x^{-m} + \cdots)^{-c}. \end{aligned}$$

Again, since the term with the smallest exponent is $(a_{-m})^d x^{-md}$, we have that $(a_{-m})^d x^{-md} = x^{-1}$ and thus $a_{-m} = \xi$ where ξ is a d-th root of unity and $m = 1/d$. Altogether,

$$\lim_{x\to 0} x\gamma^{d} = \lim_{x\to 0} x(\xi x^{-1/d} + \cdots)^{d} = \lim_{x\to 0} x\xi^{d}x^{-1}(1+\cdots)^{d} = 1.$$

There are d possible d-th roots of unity, and so there are d possible γ solutions.

Therefore there are c branches of the kernel equation such that $xu^{-c} \sim 1$ (and thus $u \to 0$ as $x \to 0$), and there are d branches of the kernel equation such that $xu^d \sim 1$ (and thus $u \to \infty$ as $x \to 0$). □

To complete the Dyck path example, the kernel equation for Dyck paths $1 - x(u + u^{-1}) = 0$ has roots $u_1 = x^{-1}\mathcal{B}_2(x^2)^{-1}$ and $u_2 = x\mathcal{B}_2(x^2)$. This can be checked by the computation

$$1 - x(u_1 + u_1^{-1}) = 1 - \mathcal{B}_2(x^2)^{-1} - x^2\mathcal{B}_2(x^2) = \frac{\mathcal{B}_2(x^2) - 1 - x^2\mathcal{B}_2(x^2)^2}{\mathcal{B}_2(x^2)}.$$

Similarly for u_2, the expression $1 - x(u_2 + u_2^{-1})$ simplifies to the u_1 case since $u_1^{-1} = u_2$.

Here $u_1 \to \infty$ as $x \to 0$ and $u_2 \to 0$ as $x \to 0$, as predicted by Proposition 4.1.2. Because $F(x,u)$ has an expansion around $(0,0)$, the factor of the kernel equation $(u - u_2)$ must be a factor of the numerator of equation (4.3). Hence

$$1 - xu_2^{-1}F(x,0) = 0, \qquad \text{and thus} \qquad F(x,0) = \frac{1}{xu_2^{-1}} = \mathcal{B}_2(x^2).$$

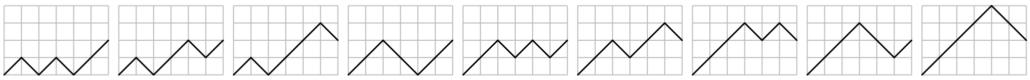
Since $u(1 - x(u + u^{-1})) = -x(u - u_1)(u - u_2)$, equation (4.3) becomes

$$\begin{aligned} F(x,u) &= \frac{u - x\mathcal{B}_2(x^2)}{-x(u - x^{-1}\mathcal{B}_2(x^2)^{-1})(u - x\mathcal{B}_2(x^2))} = \frac{1}{-x(u - x^{-1}\mathcal{B}_2(x^2)^{-1})} \\ &= \frac{1}{\mathcal{B}_2(x^2)^{-1}(1 - ux\mathcal{B}_2(x^2))} = \mathcal{B}_2(x^2)\sum_{t\geq 0}(ux\mathcal{B}_2(x^2))^t \\ &= \sum_{t\geq 0} u^t x^t \mathcal{B}_2(x^2)^{t+1} = \sum_{t\geq 0} u^t \sum_{n\geq 0} \binom{2n+t+1}{n}\frac{t+1}{2n+t+1}x^{2n+t}. \end{aligned}$$

From this, it is clear that the number of partial Dyck paths of length $2n + t$ and final height t is equal to

$$\binom{2n+t+1}{n}\frac{t+1}{2n+t+1}.$$

So to calculate the number of partial Dyck paths of length 6 and final height 2 we have $t = 2$, $n = 2$, and $\binom{7}{2}\frac{3}{7} = 9$ paths:



Theorem 4.1.1. *The bivariate generating function of non-negative walks with u marking final altitude is bivariate algebraic and given by*

$$F(x,u) = \frac{1}{u^c(1 - xS(u))}\prod_{\ell=0}^{c-1}(u - u_\ell(x)),$$

where $u_0(x)$, $u_1(x)$, $\ldots$, $u_{c-1}(x)$ are the c roots which are branches that tend to 0 as $x \to 0$.

Proof. Note that equation (4.4) can be expressed as

$$F(x,u) = \frac{u^c - u^c x \sum_{j=-c}^{-1} u^j [u^j](S(u)F(x,u))}{u^c - u^c xS(u)}.$$

With some manipulation and comparing coefficients, it can be shown that

$$\sum_{j=-c}^{-1} u^j [u^j](S(u)F(x,u)) = \sum_{j=0}^{c-1} \frac{1}{j!} \sum_{m=-c}^{-1} u^m [u^m](u^j S(u)) \left[\frac{\partial^j}{\partial u^j} F(x,u)\right]_{u=0}.$$

Owing to the combinatorial origin of the bivariate generating function $F(x,u)$ it must be bivariate analytic at $(x,u) = (0,0)$. Therefore the c branches of the kernel equation that tend to 0 as $x \to 0$ (as in Proposition 4.1.2) must be factors of the numerator as well as the denominator.

Considering just the numerator of $F(x,u)$, set

$$M(u) = u^c - u^c x \sum_{j=0}^{c-1} \frac{1}{j!} \sum_{m=-c}^{-1} u^m [u^m](u^j S(u)) \Big[\frac{\partial^j}{\partial u^j} F(x,u)\Big]_{u=0}.$$

Clearly, $M(u)$ is a monic polynomial of degree c, and the c branches $u_0(x)$, $u_1(x)$, …, $u_{c-1}(x)$ must satisfy $M(u_\ell) = 0$ for $0 \le \ell \le c-1$. Therefore

$$M(u) = \prod_{\ell=0}^{c-1} (u - u_\ell(x)).$$

It follows that

$$F(x,u) = \frac{1}{u^c(1 - xS(u))} \prod_{\ell=0}^{c-1} (u - u_\ell(x)).$$

□

4.2 Applications

4.2.1 k-Dyck paths

We begin with k-Dyck paths, since these are non-negative lattice paths and the theory in the previous section can be applied immediately. In Lemma 4.2.1 and Theorem 4.2.1 that follow, the partial Dyck paths that we discussed in the previous section are the special case where $k = 2$.

Lemma 4.2.1. *The kernel equation for a path with step set* $\{(1,1),(1,-k+1)\}$ *(which corresponds to a* $(k-1)$*-Dyck path) with* $k \ge 2$ *is given by the expression* $0 = 1 - x(u + u^{-k+1})$. *This equation has the roots*

$$x^{-1}\mathcal{B}_k(x^k)^{-1} \qquad \text{and} \qquad \zeta^j x^{\frac{1}{k-1}} \mathcal{B}_{k/(k-1)}\big(\zeta^j x^{\frac{k}{k-1}}\big)^{\frac{1}{k-1}} \qquad \text{for } 1 \le j \le k-1,$$

where ζ *is a primitive* $(k-1)$*-th root of unity.*

Proof. For the roots of the form $\zeta^j x^{\frac{1}{k-1}} \mathcal{B}_{k/(k-1)}\big(\zeta^j x^{\frac{k}{k-1}}\big)^{\frac{1}{k-1}}$, let $y = \zeta^j x^{\frac{k}{k-1}}$. Then it suffices to show that

$$\begin{aligned} 0 &= 1 - x(yx^{-1})\mathcal{B}_{k/(k-1)}(y)^{\frac{1}{k-1}} - x((yx^{-1})\mathcal{B}_{k/(k-1)}(y)^{\frac{1}{k-1}})^{1-k} \\ &= 1 - y\mathcal{B}_{k/(k-1)}(y)^{\frac{1}{k-1}} - x^k y^{1-k} \mathcal{B}_{k/(k-1)}(y)^{-1}, \end{aligned}$$

or equivalently,

$$y = \mathcal{B}_{k/(k-1)}(y)^{-\frac{1}{k-1}} - x^k y^{1-k} \mathcal{B}_{k/(k-1)}(y)^{-1-\frac{1}{k-1}}.$$

Since $x^k y^{1-k} = x^k(\zeta^j x^{\frac{k}{k-1}})^{1-k} = 1$ and $\frac{1}{k-1} = \frac{k}{k-1} - 1$, this is the first identity in Lemma 3.1.1 with $m = \frac{k}{k-1}$.

The final root can be obtained either via the Lagrange Inversion Theorem (Theorem 2.2.1), or by noting that

$$1 - x(x^{-1}\mathcal{B}_k(x^k)^{-1} + x^{k-1}\mathcal{B}_k(x^k)^{k-1}) = \frac{\mathcal{B}_k(x^k) - 1 - x^k\mathcal{B}_k(x^k)^k}{\mathcal{B}_k(x^k)} = 0.$$

This is again as a result of identity 1 in Lemma 3.1.1, since we have that $\mathcal{B}_k(x^k) - 1 = x^k\mathcal{B}_k(x^k)^k$. □

The root $x^{-1}\mathcal{B}_k(x^k)^{-1}$ tends to ∞ as $x \to 0$ and for $1 \le j \le k-1$ the roots $\zeta^j x^{\frac{1}{k-1}}\mathcal{B}_{k/(k-1)}\left(\zeta^j x^{\frac{k}{k-1}}\right)^{\frac{1}{k-1}}$ tend to 0 as $x \to 0$.

Theorem 4.2.1. *The bivariate generating function of non-negative walks with step set $\{(1,1),(1,-k+1)\}$ ($(k-1)$-Dyck paths) is given by*

$$F(x,u) = \sum_{t\ge 0} u^t \sum_{n\ge 0} \binom{kn+t+1}{n} \frac{t+1}{kn+t+1} x^{kn+t}.$$

Proof. The expression $u^c(1 - xS(u))$ factors as $-x(u-u_1)(u-u_2)\cdots(u-u_k)$ where $\{u_1, u_2, \dots, u_k\}$ is the set of roots as described in Lemma 4.2.1 with $u_k = x^{-1}\mathcal{B}_k(x^k)^{-1}$. By Theorem 4.1.1,

$$\begin{aligned}
F(x,u) &= \frac{1}{-x(u-u_1)\cdots(u-u_k)} \prod_{\ell=0}^{k-1}(u-u_\ell) = \frac{1}{-x(u-u_k)} \\
&= \frac{1}{-x(u - x^{-1}\mathcal{B}_k(x^k)^{-1})} = \frac{1}{\mathcal{B}_k(x^k)^{-1}(1 - ux\mathcal{B}_k(x^k))} \\
&= \mathcal{B}_k(x^k)\sum_{t\ge 0} u^t x^t \mathcal{B}_k(x^k)^t = \sum_{t\ge 0} u^t x^t \mathcal{B}_k(x^k)^{t+1} \\
&= \sum_{t\ge 0} u^t \sum_{n\ge 0} \binom{kn+t+1}{n} \frac{t+1}{kn+t+1} x^{kn+t}.
\end{aligned}$$

□

Lemma 4.2.2. *The kernel equation for a path with step set $\{(1,k-1),(1,-1)\}$ with $k \ge 2$ (reverse $(k-1)$-Dyck paths) is $0 = 1 - x(u^{-1} + u^{k-1})$, which has the roots*

$$x\mathcal{B}_k(x^k) \qquad \textit{and} \qquad \zeta^{-j}x^{-\frac{1}{k-1}}\mathcal{B}_{k/(k-1)}\left(\zeta^j x^{\frac{k}{k-1}}\right)^{-\frac{1}{k-1}} \qquad \textit{for} \qquad 1 \le j \le k-1,$$

where ζ is a primitive $(k-1)$-th root of unity.

This follows by noting that with the substitution $v = u^{-1}$ we have the kernel equation given in Lemma 4.2.1. In this case the root $x\mathcal{B}_k(x^k)$ tends to 0 as $x \to 0$ and for $1 \le j \le k-1$ the roots $\zeta^{-j}x^{-\frac{1}{k-1}}\mathcal{B}_{k/(k-1)}\left(\zeta^j x^{\frac{k}{k-1}}\right)^{-\frac{1}{k-1}}$ tend to ∞ as $x \to 0$.

Theorem 4.2.2. *The bivariate generating function of non-negative walks with step set $\{(1,-1),(1,k-1)\}$ (reverse $(k-1)$-Dyck paths) is given by*

$$F(x,u) = \sum_{\ell \ge 0} u^\ell \sum_{i=1}^{k-1} \frac{1}{x u_i^{\ell+1} \prod_{\substack{t=1 \\ t \ne i}}^{k-1} (u_i - u_t)},$$

where $u_j = \zeta^{-j}x^{-\frac{1}{k-1}}\mathcal{B}_{k/(k-1)}\left(\zeta^j x^{\frac{k}{k-1}}\right)^{-\frac{1}{k-1}}$ for $1 \le j \le k-1$.

Proof. The expression $u^c(1 - xS(u))$ factors as $-x(u-u_1)(u-u_2)\cdots(u-u_k)$ where $\{u_1, u_2, \ldots, u_k\}$ is the set of roots as described in Lemma 4.2.2 with $u_k = x\mathcal{B}_k(x^k)$. By Theorem 4.1.1,

$$\begin{aligned} F(x,u) &= \frac{1}{-x(u-u_1)\cdots(u-u_k)}(u-u_k) = \frac{1}{-x(u-u_1)\cdots(u-u_{k-1})} \\ &= \sum_{i=1}^{k-1} \frac{-1}{x(u-u_i)\prod_{\substack{t=1 \\ t \ne i}}^{k-1}(u_i-u_t)} = \sum_{\ell \ge 0} u^\ell \sum_{i=1}^{k-1} \frac{1}{x u_i^{\ell+1} \prod_{\substack{t=1 \\ t \ne i}}^{k-1}(u_i - u_t)}. \end{aligned}$$

□

Although the above expression for reverse paths is not easily simplified, in cases such as $k = 2$ it simplifies nicely, which allows us to calculate statistics related to 2-Dyck paths such as the area under paths. This can be done by appending a normal path of arbitrary height t to a reverse path which is also of height t.

4.2.2 k_t-Dyck paths

Let $g_m(x)$ be the generating function for the length of partial k_t-Dyck paths which end at a height of m. Then $G(x,u) = \sum_{m \ge -t} g_m(x)u^m$.

We can set up a recurrence for $g_m(x)$, since a path ending at height m is either a path ending at height $m-1$ with an additional up step $(xg_{m-1}(x))$ or a path ending at height $m+k$ with an additional down step $(xg_{m+k}(x))$. Note that the two exceptions occur at $m = 0$ (a path may be empty) and $m = -t$ $(xg_{m-1}(x)$ is not allowed). Therefore for $m \ge 1$ or $-t < m \le -1$ we have

$$g_m(x) = xg_{m-1}(x) + xg_{m+k}(x), \tag{4.5}$$

and for the other two cases,

$$g_0(x) = 1 + xg_{-1}(x) + xg_k(x),$$
$$g_{-t}(x) = xg_{-t+k}(x).$$

Multiplying $g_m(x)$ by u^m and summing over $m \geq -t$ gives

$$\sum_{m\geq -t} g_m(x)u^m = g_{-t}(x)u^{-t} + \sum_{m=-t+1}^{-1} g_m(x)u^m + g_0(x) + \sum_{m\geq 1} g_m(x)u^m.$$

We can then use the recursions (as in equation (4.5)) to obtain

$$\begin{aligned}
G(x,u) &= xg_{-t+k}(x)u^{-t} + \sum_{m=-t+1}^{-1} (xg_{m-1}(x) + xg_{m+k}(x))u^m + 1 + xg_{-1}(x) \\
&\quad + xg_k(x) + \sum_{m\geq 1}(xg_{m-1}(x) + xg_{m+k}(x))u^m \\
&= x\Big(\sum_{m=-t+1}^{-1} g_{m-1}(x)u^m + g_{-1}(x) + \sum_{m\geq 1} g_{m-1}(x)u^m \Big) \\
&\quad + \sum_{m=-t}^{-1} xg_{m+k}(x)u^m + \sum_{m\geq 0} xg_{m+k}(x)u^m + 1 \\
&= 1 + uxG(x,u) + xu^{-k} \sum_{m\geq -t} g_{m+k}(x)u^{m+k} \\
&= 1 + uxG(x,u) + xu^{-k}\Big(G(x,u) - \sum_{m=-t}^{-t+k-1} g_m(x)u^m \Big).
\end{aligned}$$

By making $G(x,u)$ the subject of the formula we obtain

$$G(x,u) = \frac{1 - \frac{x}{u^k}\sum_{m=-t}^{-t+k-1} g_m(x)u^m}{1 - xu - xu^{-k}}. \tag{4.6}$$

Note that the denominator is again of the form $1 - xS(u)$ and the numerator is also similar to the non-negative case. We can manipulate the denominator into $u(-x + (u^{-1}) - x(u^{-1})^{k+1})$ and thus apply Lagrange's solution to the equation $a - bx + cx^n = 0$ given in Theorem 3.2.2 to find the roots in terms of u^{-1}. Here $a = -x$, $b = -1$, $c = -x$, and $n = k+1$. Therefore the roots are $(u_{k+1})^{-1} = x\mathcal{B}_{k+1}(x^{k+1})$, and

$$(u_j)^{-1} = \zeta^j x^{-\frac{1}{k}} \mathcal{B}_{\frac{k+1}{k}}\big(\zeta^{-j} x^{\frac{k+1}{k}}\big)^{-\frac{1}{k}},$$

where $1 \leq j \leq k$ and ζ is a primitive k-th root of unity.

Reverting our attention back to equation (4.6), we can multiply the numerator and denominator by u^k to obtain

$$G(x,u) = \frac{u^k - x\sum_{m=-t}^{-t+k-1} g_m(x)u^m}{u^k - xu^{k+1} - x} = \frac{u^k - x\sum_{m=-t}^{-t+k-1} g_m(x)u^m}{-x(u-u_1)\cdots(u-u_k)(u-u_{k+1})}.$$

Here the 'bad' factors, $(u-u_i)$ for $1 \leq i \leq k$, cancel and thus $G(x,u)$ must simplify to something of the form

$$\frac{P_t(u)}{u^t(u-u_{k+1})}$$

where $P_t(u)$ is a polynomial of degree t. A factor of u^{-t} can be taken out of the numerator, which accounts for the u^t in the denominator of the above expression. Since

$$\frac{P_t(u)}{u^t(u-u_{k+1})} = \frac{u^k - x\sum_{m=-t}^{-t+k-1} g_m(x)u^m}{u^k - xu^{k+1} - x},$$

we can multiply both sides by the denominators to obtain the equality

$$(u^k - xu^{k+1} - x)P_t(u) = u^t(u-u_{k+1})\Big(u^k - x\sum_{m=-t}^{-t+k-1} g_m(x)u^m\Big). \tag{4.7}$$

Since $P_t(u)$ is a polynomial of degree t we can express it as

$$P_t(u) = c_t u^t + c_{t-1}u^{t-1} + \cdots + c_1 u + c_0.$$

Therefore equation (4.7) can be written as

$$\begin{aligned}
&u^{k+t+1} - x\sum_{m=-t}^{-t+k-1} g_m(x)u^{m+t+1} - u^{k+t}u_{k+1} + x\sum_{m=-t}^{-t+k-1} g_m(x)u^{m+t}u_{k+1} \\
&= (u^k - xu^{k+1} - x)(c_t u^t + c_{t-1}u^{t-1} + \cdots + c_0) \\
&= -xc_t u^{k+t+1} + (c_t - xc_{t-1})u^{k+t} + \sum_{i=1}^{t-1}(c_{t-i} - xc_{t-i-1})u^{k+t-i} + c_0 u^k - xP_t(u).
\end{aligned}$$

By comparing coefficients we see that $1 = -xc_t$ and thus $c_t = -\frac{1}{x}$ as well as $c_t - xc_{t-1} = -u_{k+1}$ and thus $c_{t-1} = \frac{-1+u_{k+1}x}{x^2}$. Since the coefficient of u^i is 0 for $1 \leq i \leq t-1$ on the left hand side of the equation, we have that $c_{t-i} - xc_{t-i-1} = 0$ and thus $c_{t-i-1} = x^{-i-1}(-\frac{1}{x} + u_{k+1})$ for $0 \leq i \leq t-1$.

It follows that

$$P_t(u) = -\frac{1}{x}u^t + \sum_{i=1}^{t} x^{-i}(-x^{-1}+u_{k+1})u^{t-i} = -\frac{1}{x}u^t + u^t(-x^{-1}+u_{k+1})\sum_{i=1}^{t} x^{-i}u^{-i}.$$

Therefore

$$\begin{aligned}
G(x,u) &= \frac{-\frac{1}{x}u^t + u^t(-x^{-1}+u_{k+1})\sum_{i=1}^t x^{-i}u^{-i}}{u^t(u-u_{k+1})} \\
&= \frac{-\frac{1}{x} + (-x^{-1}+u_{k+1})\sum_{i=1}^t x^{-i}u^{-i}}{-u_{k+1}(1-u(u_{k+1})^{-1})},
\end{aligned}$$

and thus we can find the coefficient of u^ℓ:

$$\begin{aligned}
&[u^\ell]G(x,u) \\
&\quad= [u^\ell]\left(\frac{1}{xu_{k+1}}\sum_{r\geq 0}(u_{k+1})^{-r}u^r + \left(\frac{1}{xu_{k+1}}-1\right)\sum_{i=1}^t (xu)^{-i}\sum_{r\geq 0}u^r(u_{k+1})^{-r}\right) \\
&\quad= \frac{1}{x(u_{k+1})^{\ell+1}} + \left(\frac{1}{xu_{k+1}}-1\right)\sum_{i=1}^t x^{-i}(u_{k+1})^{-i-\ell} \\
&\quad= \frac{1}{x(u_{k+1})^{\ell+1}} + \left(\frac{1-xu_{k+1}}{xu_{k+1}}\right)\cdot\frac{1}{(u_{k+1})^\ell}\cdot\frac{1}{(xu_{k+1})^t}\left(\frac{1-x^t(u_{k+1})^t}{1-xu_{k+1}}\right) \\
&\quad= \frac{1}{x(u_{k+1})^{\ell+1}} + \frac{1-x^t(u_{k+1})^t}{x^{t+1}(u_{k+1})^{t+\ell+1}} \\
&\quad= x^{-t-1}(u_{k+1})^{-t-\ell-1}.
\end{aligned}$$

Since $u_{k+1} = x^{-1}\mathcal{B}_{k+1}(x^{k+1})^{-1}$, we have that

$$[u^\ell]G(x,u) = x^{-t-1}(x^{-1}\mathcal{B}_{k+1}(x^{k+1})^{-1})^{-t-\ell-1} = x^\ell\mathcal{B}_{k+1}(x^{k+1})^{t+\ell+1}.$$

Now, we compute the coefficient of x^n in $[u^\ell]G(x,u)$:

$$\begin{aligned}
[x^nu^\ell]G(x,u) &= [x^n]x^\ell\mathcal{B}_{k+1}(x^{k+1})^{t+\ell+1} \\
&= [x^{n-\ell}]\sum_{m\geq 0}\binom{(k+1)m+t+\ell+1}{m}\frac{t+\ell+1}{(k+1)m+t+\ell+1}x^{(k+1)m}.
\end{aligned}$$

This is only non-zero if $n-\ell \equiv 0 \mod k+1$. In this case we have

$$[x^nu^\ell]G(x,u) = \binom{n+t+1}{\frac{n-\ell}{k+1}}\frac{t+\ell+1}{n+t+1}.$$

Therefore we have the following proposition.

Proposition 4.2.1. *The number of partial k_t-Dyck paths of length n which end at a height of ℓ is equal to*

$$\binom{n+t+1}{\frac{n-\ell}{k+1}}\frac{t+\ell+1}{n+t+1},$$

where $n-\ell \equiv 0 \mod k+1$.

We can obtain the enumeration of k_t-Dyck paths as a corollary of this, where $\ell = 0$ and $n := (k+1)n$. Then the number of k_t-Dyck paths of length $(k+1)n$ is equal to

$$\frac{t+1}{(k+1)n+t+1}\binom{(k+1)n+t+1}{n}.$$

To demonstrate Proposition 4.2.1 for partial 3_2-Dyck paths of length 7 ending at height -1 we have that $k=3$, $t=2$, $n=7$, and $\ell=-1$ and thus the 9 paths are:

Chapter 5

The Cycle Lemma

In this chapter we will discuss another useful tool for enumeration. It was first introduced by Dvoretzky and Motzkin [11], and after being rediscovered several times these approaches were consolidated and presented in an appealing manner by Dershowitz and Zaks [7]. This chapter is largely based on the paper by Dershowitz and Zaks.

5.1 The cycle lemma

Definition 5.1.1. *A sequence $p_1p_2\cdots p_\ell$ of boxes and circles is called k-*dominating *(for positive integer k) if for every position i, $1 \leq i \leq \ell$, the number of boxes in $p_1p_2\cdots p_i$ is more than k times the number of circles.*

For example, the sequence □ □ ◯ □ □ ◯ ◯ □ □□ is 1-dominating but not 2-dominating, since at the 6th position two ◯ have occurred while four □ have occurred previously ($4 \not< 2 \cdot 2$).

Lemma 5.1.1 (The cycle lemma)**.** *For any sequence $p_1p_2\cdots p_{m+n}$ of m boxes and n circles, $m \geq kn$, there exist exactly $m-kn$ (out of $m+n$) cyclic permutations $p_jp_{j+1}\cdots p_{m+n}p_1\cdots p_{j-1}$, $1 \leq j \leq m+n$, that are k-dominating.*

Proof. Arrange a sequence of m boxes and n circles in a loop (the first item in the sequence is adjacent to the last item in the sequence). In order for a cyclic permutation of this sequence to be k-dominating, there must be a subsequence with at least k boxes followed by a circle. This is because a k-dominating sequence will start with at least k boxes. Remove a subsequence with exactly k boxes followed by 1 circle. By the pigeonhole principle, if $m \geq kn$ there must always be at least one subsequence consisting of k boxes followed by one circle. Hence we can keep removing such subsequences until only boxes remain. The remaining boxes are then starting points for k-dominating sequences in the original sequence. This is because with one of the remaining boxes as a starting point, the rest of the sequence is made up of some combination of

$\square^k\bigcirc$ or $\square$, both of which keep the condition that the number of boxes is greater than k times the number of circles. $\square$

Sticking to the sequence discussed previously, there are 10 cyclic permutations of the original sequence (labelled 0):

0	$\square\square\bigcirc\square\square\bigcirc\bigcirc\square\square\square$	5	$\bigcirc\bigcirc\square\square\square\square\square\bigcirc\square\square$
1	$\square\bigcirc\square\square\bigcirc\bigcirc\square\square\square\square$	6	$\bigcirc\square\square\square\square\square\bigcirc\square\square\bigcirc$
2	$\bigcirc\square\square\bigcirc\bigcirc\square\square\square\square\square$	7	$\square\square\square\square\square\bigcirc\square\square\bigcirc\bigcirc$
3	$\square\square\bigcirc\bigcirc\square\square\square\square\square\bigcirc$	8	$\square\square\square\square\bigcirc\square\square\bigcirc\bigcirc\square$
4	$\square\bigcirc\bigcirc\square\square\square\square\square\bigcirc\square$	9	$\square\square\square\bigcirc\square\square\bigcirc\bigcirc\square\square$

By the cycle lemma (Lemma 5.1.1), we know that the number of 1-dominating sequences is $7 - 1 \cdot 3 = 4$ (sequences 0, 7, 8, and 9) and the number of 2-dominating sequences is $7 - 2 \cdot 3 = 1$ (sequence 7).

We can also find all 1-dominating cyclic permutations by using the method outlined in the proof of Lemma 5.1.1. Starting from the original sequence we repeatedly remove the subsequence "$\square\bigcirc$":

$$\begin{array}{cccccccccc}
\square & \square & \bigcirc & \square & \square & \bigcirc & \bigcirc & \square & \square & \square \\
\square & \square & \bigcirc & \square & & & \bigcirc & \square & \square & \square \\
\square & & & \square & & & \bigcirc & \square & \square & \square \\
\square & & & & & & & \square & \square & \square
\end{array}$$

Therefore the 4 possible starting positions for cyclic permutations which are 1-dominating are the first, eighth, ninth, and tenth.

5.2 Applications

The cycle lemma can be used to enumerate many combinatorial objects. In this section we will establish some enumerative properties of the family of k-Dyck paths. These results will be used in further sections.

Proposition 5.2.1. *The number of $(k-1)$-Dyck paths of length kn is given by*

$$\frac{1}{(k-1)n+1}\binom{kn}{n}.$$

Proof. A path of length kn with step set $\{(1,1),(1,-k+1)\}$ consists of $(k-1)n$ steps of type $(1,1)$ and n steps of type $(1,-k+1)$. Furthermore, to satisfy the non-negativity condition there must be at least $k-1$ steps of type $(1,1)$ before each step of type $(1,-k+1)$. As a result, this sequence of steps can be considered $(k-1)$-dominating if we insert an additional $(1,1)$ step at the start. We can then represent a step of type $(1,1)$ as a $\square$ and the step $(1,-k+1)$ as a $\bigcirc$. By Lemma 5.1.1 there is exactly one $(k-1)$-dominating sequence in all possible cyclic permutations of any sequence of $(k-1)n+1$ $\square$'s and n $\bigcirc$'s.

There are $\binom{(k-1)n+1+n}{n}$ such sequences of $\square$'s and $\bigcirc$'s, and one out of every $kn+1$ cyclic permutations is $(k-1)$-dominating. Therefore the number of non-negative lattice paths is given by

$$\frac{1}{kn+1}\binom{kn+1}{n} = \frac{1}{(k-1)n+1}\binom{kn}{n}.$$

$\square$

Proposition 5.2.2. *The number of k_t-Dyck paths of length $(k+1)n$ is given by*

$$\frac{t+1}{(k+1)n+t+1}\binom{(k+1)n+t+1}{n}.$$

This result is a slight extension of Proposition 5.2.1, that is, when $t = 0$ we have Proposition 5.2.1.

Proof. Here we claim that a valid k_t-Dyck path is given by a k-dominating sequence consisting of $kn+t+1$ steps of type $(1,1)$ and n steps of type $(1,-k)$. By removing the first $t+1$ steps of type $(1,1)$ of a k-dominating sequence we obtain a valid k_t-Dyck path and vice versa. By Lemma 5.1.1, in each sequence of $kn+t+1$ steps of type $(1,1)$'s and n steps of type $(1,-k)$'s there are $t+1$ possible starting points for a k-dominating cyclic permutation of the sequence. However, these starting points might not give unique sequences. For example, the sequence $\square\square\bigcirc\square\square\bigcirc$ has two possible 1-dominating cyclic permutations but due to periodicity both 1-dominating sequences are the same sequence. It is clear that all non-unique k-dominating cyclic permutations are periodic with a period that divides the length of the sequence. Therefore proportionally, there are $\frac{t+1}{(k+1)n+t+1}$ (unique) k-dominating cyclic permutations per sequence. The number of possible sequences is given by $\binom{(k+1)n+t+1}{n}$, and thus the number of k_t-Dyck paths is

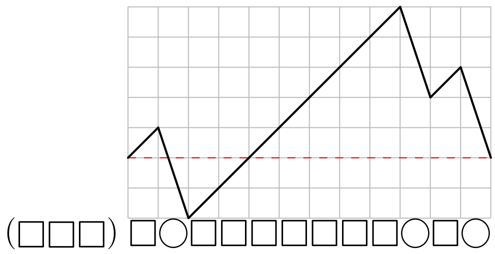
$$\frac{t+1}{(k+1)n+t+1}\binom{(k+1)n+t+1}{n}.$$

$\square$

To demonstrate how we convert from a k-dominating sequence of $kn+t+1$ steps of type $(1,1)$ (or $\square$'s) and n steps of type $(1,-k)$ (or $\bigcirc$'s), consider the 3-dominating sequence with $n=3$ and $t=2$:

$$\square\square\square\square\bigcirc\square\square\square\square\square\square\square\square\bigcirc\square\bigcirc.$$

We remove the first $t+1$ $\square$'s, and then replace each $\square$ with $(1,1)$ and $\bigcirc$ with $(1,-k)$:



Proposition 5.2.3. *The number of k_t-Dyck paths of length n that end at height i is given by*

$$\frac{t+i+1}{n+t+1}\binom{n+t+1}{\frac{n-i}{k+1}}.$$

Proof. Suppose that we have a k-dominating sequence of $t+1+r$ steps of type $(1,1)$ and $n-r$ steps of type $(1,-k)$. The total length of the sequence is then $n+t+1$ and it represents a k_t-Dyck path of length n with a 'buffer' of $t+1$ steps of type $(1,1)$. Then the final height of the associated k_t-Dyck path is $r-k(n-r)=(k+1)r-nk=i$. Thus $r=\frac{kn+i}{k+1}$. By Lemma 5.1.1, the number of k-dominating sequences with $t+1+r$ steps of type $(1,1)$ and $n-r$ steps of type $(1,-k)$ is

$$t+1+r-k(n-r)=t+1-kn+(k+1)\frac{kn+i}{k+1}=t+i+1.$$

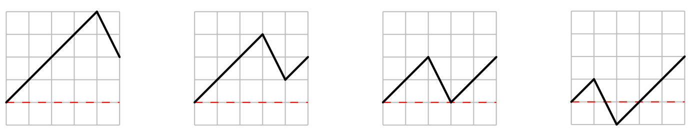
From the total $n+t+1$ steps there are $\binom{n+t+1}{n-r}$ ways to arrange a sequence of $t+1+r$ steps of type $(1,1)$ and $n-r$ steps of type $(1,-k)$. Furthermore, $t+i+1$ out of the $n+t+1$ possible cyclic permutations are k-dominating. Therefore the total number of k_t-Dyck paths of length n and end height i is

$$\frac{t+i+1}{n+t+1}\binom{n+t+1}{n-\frac{kn+i}{k+1}}=\frac{t+i+1}{n+t+1}\binom{n+t+1}{\frac{n-i}{k+1}}.$$

$\square$

Note that when $i=0$ we obtain Proposition 5.2.2 from Proposition 5.2.3.

As an example, let us consider partial 2_1-Dyck paths of length 5 that end at height 2. Here $k=2$, $t=1$, $n=5$, and $i=2$. Therefore the number of such paths is $\frac{4}{7}\binom{7}{\frac{3}{3}}=4$. The four possible partial 2_1-Dyck paths of length 5 with end height 2 are:



Chapter 6

Analysis of k_t-Dyck Paths

In previous chapters we have established the enumeration of k_t-Dyck paths using different methods. Bijectively (Proposition 2.2.2), via the kernel method (Proposition 4.2.1), and via the cycle lemma (Proposition 5.2.2).

In this chapter we will provide a symbolic decomposition for k_t-Dyck paths and give a fourth (and final) proof of the enumeration of k_t-Dyck paths, as well as study some statistics related to k_t-Dyck paths. Here we will look at the number of returns, number of peaks, number of valleys, and number of returns to the line $y = -t$. Similar parameters have been studied in the case of Dyck paths [8] and some parameters involving k-Dyck paths have been studied in [16, 31], but not for the k_t-Dyck paths we introduced in Chapter 2.

6.1 Functional equations for k_t-Dyck paths

Prodinger, Selkirk, and Wagner [33] introduced S-Motzkin and T-Motzkin paths (Definitions 8.1.1 and 8.1.2), which are bijective to 2_0-Dyck paths and 2_1-Dyck paths respectively, as shown in Subsection 8.2.1. Analysis of parameters in S-Motzkin and T-Motzkin paths can be found in Chapter 8. Here we will adapt and generalise the symbolic decomposition given for S-Motzkin and T-Motzkin paths from 2_t-Dyck paths to general k_t-Dyck paths.

Theorem 6.1.1. *Let $k \geq 2$ and $0 \leq t \leq k$. The symbolic decomposition of k_t-Dyck paths is given by*

$$\mathcal{K}_t = \varepsilon + \sum_{i=0}^{t} \big(\{\diagup\}^{k-i} \times \mathcal{K}_{k-1-i} \times \{\diagdown\} \times \{\diagup\}^{i} \times \mathcal{K}_t\big),$$

where we define $\mathcal{K}_{-1}$ to be ε and $\diagup$ represents a $(1,1)$ step and $\diagdown$ represents a $(1,-k)$ step. This decomposition is shown graphically in Figure 6.1

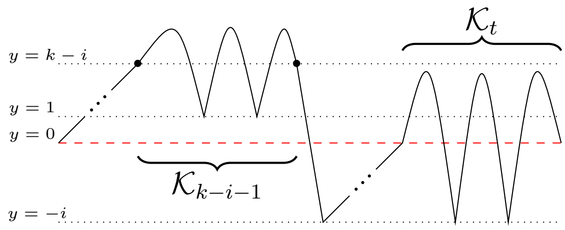


Figure 6.1: The symbolic decomposition of a k_t-Dyck path for a fixed i

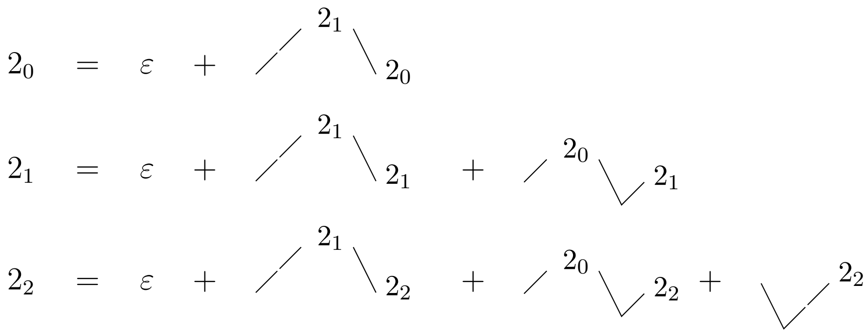
Proof. This decomposition is based on the first return of the path to the x-axis. By return we mean a step whose end point is on the line $y = 0$.

Since a k_t-Dyck path must stay above the line $y = -t$, there can be a maximal chain of at most t ⁄ steps ending at the first return, and at least none. So if i denotes the length of the maximal chain of ⁄ steps before the first return, then $0 \leq i \leq t$. Preceding this maximal chain must then be a $(1, -k)$ step (\).

Each path begins with j initial ⁄ steps, where $0 \leq j \leq k$, followed by a k_{j-1}-Dyck path. Since a k_{j-1}-Dyck path must remain above $j - 1$ positions below its starting point, it does not introduce any returns of the path to $y = 0$.

Applying this to a path whose maximal chain of ⁄ steps before the first return is equal to i, we see that there must be $k - i$ ⁄ steps, followed by a k_{k-i-1}-Dyck path, followed by a \ step, followed by the i ⁄ steps until the return. After the first return, the remaining path is then a k_t-Dyck path. □

As an example, the symbolic decomposition for 2_t-Dyck paths with $0 \leq t \leq 2$ is:



From Theorem 6.1.1, we obtain the system of functional equations for $0 \leq t \leq k$ where $K_t(x)$ is the generating function for k_t-Dyck paths of length $(k+1)n$:

$$K_t(x) = 1 + x^{k+1} K_t(x) \sum_{i=0}^{t} K_{k-i-1}(x). \tag{6.1}$$

Lemma 6.1.1. *With the substitution $x^{k+1} = z(1-z)^k$, for every $0 \le t \le k$,*

$$K_t(x) = \mathcal{B}_{k+1}(x^{k+1})^t = \frac{1}{(1-z)^{t+1}}$$

is a solution to the system of equations given in equation (6.1).

Proof. From $K_t(x) = 1 + x^{k+1} K_t(x) \sum_{i=0}^{t} K_{k-i-1}(x)$ it follows that

$$\frac{1}{K_t(x)} = 1 - x^{k+1} \sum_{i=0}^{t} K_{k-i-1}(x).$$

By subtracting consecutive t-values, we obtain

$$\frac{1}{K_t(x)} - \frac{1}{K_{t+1}(x)} = x^{k+1} K_{k-t-2}(x). \tag{6.2}$$

By definition, $K_{-1}(x) = 1$, and so for $t = -1$ we have

$$1 - \frac{1}{K_0(x)} = x^{k+1} K_{k-1}(x) \quad \text{or} \quad K_0(x) = \frac{1}{1 - x^{k+1} K_{k-1}(x)}.$$

Similarly, for $t = k-1$ we have that

$$\frac{1}{K_{k-1}(x)} - \frac{1}{K_k(x)} = x^{k+1} K_{-1}(x) = x^{k+1},$$

which simplifies to

$$K_k(x) = K_{k-1}(x)(1 - x^{k+1} K_{k-1}(x))^{-1}.$$

Now, we prove by induction that for $0 \le j \le k-1$

$$K_j(x) = (1 - x^{k+1} K_{k-1}(x))^{-j-1},$$
$$K_{k-j}(x) = K_{k-1}(x)(1 - x^{k+1} K_{k-1}(x))^{j-1}.$$

In both cases we have already proved the base case ($j = 0$) to be true. Assume that $K_j(x) = (1 - x^{k+1} K_{k-1}(x))^{-j-1}$ is true for all $0 \le j \le \ell - 2$ and $K_{k-j}(x) = K_{k-1}(x)(1 - x^{k+1} K_{k-1}(x))^{j-1}$ is true for all $1 \le j \le \ell - 1$. Now,

$$\frac{1}{K_{k-\ell}(x)} - \frac{1}{K_{k-\ell+1}(x)} = x^{k+1} K_{k-(k-\ell)-2}(x) = x^{k+1} K_{\ell-2}(x).$$

Therefore

$$\begin{aligned}
(K_{k-\ell}(x))^{-1} &= \frac{1}{K_{k-\ell+1}(x)} + x^{k+1} K_{\ell-2}(x) = \frac{1 + x^{k+1} K_{\ell-2}(x) K_{k-\ell+1}(x)}{K_{k-\ell+1}(x)} \\
&= \frac{1 + x^{k+1}(1 - x^{k+1} K_{k-1}(x))^{-\ell+1} K_{k-1}(x)(1 - x^{k+1} K_{k-1}(x))^{\ell-2}}{K_{k-1}(x)(1 - x^{k+1} K_{k-1}(x))^{\ell-2}} \\
&= \frac{1}{K_{k-1}(x)(1 - x^{k+1} K_{k-1}(x))^{\ell-1}}.
\end{aligned}$$

Now, we assume that $K_{k-j}(x) = K_{k-1}(x)(1 - x^{k+1}K_{k-1}(x))^{j-1}$ is true for all $0 \le j \le \ell + 2$ and $K_j(x) = (1 - x^{k+1}K_{k-1}(x))^{-j-1}$ is true for all $0 \le j \le \ell$. From equation (6.2) we have

$$\frac{1}{K_\ell(x)} - \frac{1}{K_{\ell+1}(x)} = x^{k+1}K_{k-\ell-2}(x).$$

Then with substitutions

$$\begin{aligned}\frac{1}{K_{\ell+1}(x)} &= \frac{1}{K_\ell(x)} - x^{k+1}K_{k-(\ell+2)}(x)\\ &= (1 - x^{k+1}K_{k-1}(x))^{\ell+1} - x^{k+1}K_{k-1}(x)(1 - x^{k+1}K_{k-1}(x))^{\ell+1}\\ &= (1 - x^{k+1}K_{k-1}(x))^{\ell+2}.\end{aligned}$$

Therefore $K_{\ell+1}(x) = (1 - x^{k+1}K_{k-1}(x))^{-(\ell+1)-1}$.

Each induction proof uses the other identity in the proof. This is allowed, because we can prove the expression by alternating: $k-2$, 1, $k-3$, 2, $k-4$, $3, \ldots$. We have that

$$K_j(x) = (1 - x^{k+1}K_{k-1}(x))^{-j-1},$$

and with $j = k-1$ it becomes

$$K_{k-1}(x) = (1 - x^{k+1}K_{k-1}(x))^{-k}. \tag{6.3}$$

Let $z = x^{k+1}K_{k-1}(x)$, then equation (6.3) becomes $z(1-z)^k = x^{k+1}$ and it follows that $K_j(x) = (1-z)^{-j-1}$. Cauchy's integral formula can be used to check that

$$K_j(x) = (1-z)^{-j-1} = \mathcal{B}_{k+1}(x^{k+1})^j.$$

□

6.2 The analysis of parameters associated with k_t-Dyck paths

Here we will analyse some parameters associated with k_t-Dyck paths, by means of the symbolic equations that were provided in the previous section. In particular, we will look at the number of peaks, the number of valleys, the number of returns, and the number of returns to $y = -t$ of k_t-Dyck paths.

In order to do this, we will need to calculate the average number of occurrences of a certain parameter of interest (represented by the variable u), as well as the variance in the number of occurrences of the parameter. To calculate the average we need to determine

$$K_{\text{ave}} = [x^{n(k+1)}]\frac{\partial K_t(x,u)}{\partial u}\bigg|_{u=1} \bigg/ [x^{n(k+1)}]K_t(x,1).$$

Similarly, for the variance we must calculate

$$K_{\text{var}} = [x^{n(k+1)}]\frac{\partial^2}{(\partial u)^2}K_t(x,u)\Big|_{u=1} \Big/ [x^{n(k+1)}]K_t(x,1) + K_{\text{ave}} - \big(K_{\text{ave}}\big)^2.$$

In the sections that follow we will need to compute coefficients of a certain form, and thus we will perform the general calculation for each of the three types here.

Lemma 6.2.1. *Given a function $f(x)$ of a particular form, the coefficients $[x^{(k+1)n}]f(x)$ for $p,q,r \in \mathbb{Z}$ are given by*

Type	$f(x)$	$[x^{(k+1)n}]f(x)$
1	$\frac{z^p}{(1-z)^{qt+r}}$	$\frac{pk+qt+r}{n-p}\binom{(k+1)n-p+qt+r-1}{n-p-1}$
2	$\frac{z^p}{(1-z)^{t+r}(1-(k+1)z)}$	$\binom{(k+1)n-p+t+r}{n-p}$
3	$\frac{z^p}{(1-z)^{t+r}(1-(k+1)z)^3}$	$\sum_{m=0}^{n-p}(m+1)(k+1)^m\binom{(k+1)n-p-m+t+r+1}{n-p-m}$

Proof. In each case we apply Cauchy's integral formula to $f(x)$ along with the substitution $x^{k+1} = w = z(1-z)^k$. With this substitution we have that $dw = (1-z)^{k-1}(1-(k+1)z)\,dz$. For a function of type 1 we have

$$\begin{aligned}
[w^n]\frac{z^p}{(1-z)^{qt+r}} &= \frac{1}{2\pi i}\oint \frac{\frac{z^p}{(1-z)^{qt+r}}}{w^{n+1}}\,dw = \frac{1}{2\pi i}\oint \frac{z^p(1-z)^{k-1}(1-(k+1)z)}{z^{n+1}(1-z)^{kn+k+qt+r}}\,dz \\
&= \frac{1}{2\pi i}\oint \frac{1-(k+1)z}{z^{n+1-p}(1-z)^{kn+qt+r+1}}\,dz = [z^{n-p}]\frac{1-(k+1)z}{(1-z)^{kn+qt+r+1}} \\
&= [z^{n-p}]\frac{1}{(1-z)^{kn+qt+r+1}} - (k+1)[z^{n-p-1}]\frac{1}{(1-z)^{kn+qt+r+1}} \\
&= \binom{(k+1)n-p+qt+r}{n-p} - (k+1)\binom{(k+1)n-p+qt+r-1}{n-p-1} \\
&= \frac{pk+qt+r}{n-p}\binom{(k+1)n-p+qt+r-1}{n-p-1}.
\end{aligned}$$

Similarly, for a function of type 2 we have (some steps done in the type 1 case are left out for brevity),

$$\begin{aligned}
&[w^n]\frac{z^p}{(1-z)^{t+r}(1-(k+1)z)} \\
&= \frac{1}{2\pi i}\oint \frac{z^p(1-z)^{k-1}(1-(k+1)z)}{z^{n+1}(1-z)^{kn+k}(1-z)^{t+r}(1-(k+1)z)}\,dz \\
&= \frac{1}{2\pi i}\oint \frac{1}{z^{n+1-p}(1-z)^{kn+t+r+1}}\,dz \\
&= [z^{n-p}]\frac{1}{(1-z)^{kn+t+r+1}} \\
&= \binom{(k+1)n-p+t+r}{n-p}.
\end{aligned}$$

Finally, a function of type 3 has coefficients

$$\begin{aligned}
&[w^n]\frac{z^p}{(1-z)^{t+r}(1-(k+1)z)^3}\\
&= \frac{1}{2\pi i}\oint \frac{z^p(1-z)^{k-1}(1-(k+1)z)}{z^{n+1}(1-z)^{kn+k}(1-z)^{t+r}(1-(k+1)z)^3}\,dz\\
&= \frac{1}{2\pi i}\oint \frac{1}{z^{n+1-p}(1-z)^{kn+1+t+r}(1-(k+1)z)^2}\,dz\\
&= [z^{n-p}]\frac{1}{(1-z)^{kn+t+r+1}(1-(k+1)z)^2}\\
&= [z^{n-p}]\sum_{m\geq 0}(m+1)(k+1)^m\frac{z^m}{(1-z)^{kn+t+r+1}}\\
&= [z^{n-p}]\sum_{m\geq 0}(m+1)(k+1)^m z^m\sum_{\ell\geq 0}\binom{\ell+kn+t+r+1}{\ell}z^\ell\\
&= \sum_{m=0}^{n-p}(m+1)(k+1)^m\binom{(k+1)n-p-m+t+r+1}{n-p-m}.
\end{aligned}$$

□

6.2.1 Returns to $y = -t$

Let u be a variable which counts returns of the path to the line $y = -t$. Although it is easy to calculate the number of paths that reach the line $y = -t$ (the number of k_{t-1}-Dyck paths subtracted from the number of k_t-Dyck paths), here we are calculating the number of times the line $y = -t$ is reached which can happen multiple times in a k_t-Dyck path if at all. For $0 \leq t \leq k$ we can adapt the system of functional equations to account for returns to $y = -t$, which yields

$$K_t(x,u) = 1 + x^{k+1}K_t(x,u)\sum_{i=0}^{t-1}K_{k-i-1}(x,1) + ux^{k+1}K_t(x,u)K_{k-t-1}(x,1).$$

By differentiating this system with respect to u we obtain the following result.

Proposition 6.2.1. *The total number of returns to $y = -t$ in k_t-Dyck paths of length $(k+1)n$ is equal to*

$$\frac{k+t+2}{n-1}\binom{(k+1)n+t}{n-2},$$

and the average number of returns to $y = -t$ in k_t-Dyck paths of length $(k+1)n$ is

$$\frac{n(k+t+2)}{(t+1)(kn+t+2)} = \frac{k+t+2}{k(t+1)} + \mathcal{O}\Big(\frac{1}{n}\Big).$$

Proof. For $0 \leq t \leq k$, differentiating the bivariate generating functions with respect to u gives

$$\frac{\partial K_t(x,u)}{\partial u} = \frac{x^{k+1}K_t(x,u)K_{k-t-1}(x,1)}{1 - x^{k+1}\sum_{i=0}^{t-1} K_{k-i-1}(x,1) - ux^{k+1}K_{k-t-1}(x,1)},$$

and evaluating at $u = 1$ results in

$$\left.\frac{\partial K_t(x,u)}{\partial u}\right|_{u=1} = \frac{x^{k+1}K_{k-t-1}(x,1)}{(1 - x^{k+1}\sum_{i=0}^{t} K_{k-i-1}(x,1))^2}.$$

With the substitutions $x^{k+1} = z(1-z)^k$ and $K_i(x,u) = \frac{1}{(1-z)^{i+1}}$ we obtain

$$\left.\frac{\partial K_t(x,u)}{\partial u}\right|_{u=1} = \frac{z(1-z)^t}{(z(1-z)^t - (1-z)^t)^2} = \frac{z(1-z)^t}{(1-z)^{2(t+1)}} = \frac{z}{(1-z)^{t+2}}.$$

Using this expression and Lemma 6.2.1 for functions of type 1 with $p = 1$, $q = 1$, and $r = 2$, we find that the total number of times a k_t-Dyck path of length $(k+1)n$ returns to the line $y = -t$ is

$$\frac{k+t+2}{n-1}\binom{(k+1)n+t}{n-2}.$$

From this we see that the average number of returns of the path to $y = -t$ is given by dividing by the total number of k_t-Dyck paths of length $(k+1)n$, which is $\frac{t+1}{(k+1)n+t+1}\binom{(k+1)n+t+1}{n}$. Hence the average number of returns to the line $y = -t$ is equal to

$$\frac{n((k+1)n+t+1)}{(t+1)(kn+t+2)} - (k+1)\frac{n(n-1)}{(t+1)(kn+t+2)} = \frac{n(k+t+2)}{(t+1)(kn+t+2)}.$$

This average can be expressed as a sum,

$$\frac{n(k+t+2)}{(t+1)(kn+t+2)} = \frac{k+t+2}{k(t+1)(1+\frac{t+2}{kn})} = \frac{k+t+2}{k(t+1)}\sum_{m\geq 0}(-1)^m\left(\frac{t+2}{k}\right)^m\frac{1}{n^m},$$

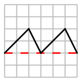
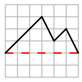
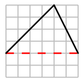
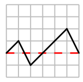
from which it is easily seen that the average tends to $\frac{k+t+2}{k(t+1)}$ as n becomes large. □

We demonstrate this result by means of an example. From Table 6.1 we see that when $t = 1$, $k = 2$, and $n = 2$ the number of returns to $y = -t$ equals 5. This is consistent with

$$\frac{1+2+2}{2-1}\binom{(2+1)2+1}{2-2} = 5.$$

To calculate the variance we first compute the second derivative of $K_t(x,u)$ with respect to u. Again, this gives a general formula in terms of z and t from which we can compute coefficients.

Table 6.1: Number of returns to $y = -1$ for 2_1-Dyck paths of length 6

Path							
Returns	2	0	1	0	1	0	1

Proposition 6.2.2. *The $(k+1)n$-th coefficient of the second derivative of $K_t(x,u)$ is given by*

$$[x^{(k+1)n}]K_t(x,u) = \frac{2(2k+t+3)}{n-2}\binom{(k+1)n+t}{n-3},$$

and the variance in the number of returns to the line $y=-t$ in k_t-Dyck paths of length $(k+1)n$ is

$$\frac{((k+t)(knt+t+t^2+6)-2k(4-n(2t+1))+4t^2)((k+1)n+t+1)n}{(kn+t+3)(kn+t+2)^2(t+1)^2}$$
$$= \frac{(k+1)(t^2+(k+4)t+2)}{k^2(t+1)^2} + \mathcal{O}\Big(\frac{1}{n}\Big).$$

Proof. We differentiate the expression $\frac{\partial K_t(x,u)}{\partial u}$ as given in the proof of Proposition 6.2.1 with respect to u to obtain

$$\frac{\partial^2 K_t(x,u)}{\partial u^2} = \frac{2x^{2(k+1)}K_{k-t-1}(x,1)^2}{(1-x^{k+1}\sum_{i=0}^{t-1}K_{k-i-1}(x,1)-ux^{k+1}K_{k-t-1}(x,1))^3}.$$

With the substitutions $u=1$, $x^{k+1}=z(1-z)^k$, and $K_i(x,1)=(1-z)^{-(i+1)}$, we can simplify these expressions to

$$\frac{\partial^2 K_t(x,u)}{\partial u^2}\bigg|_{u=1} = \frac{2z^2(1-z)^{2t}}{(1-z)^{3t+3}} = \frac{2z^2}{(1-z)^{t+3}}.$$

Using this expression and Lemma 6.2.1 of type 1 with $p=2$, $q=1$, and $r=3$ we calculate that

$$[x^{(k+1)n}]\frac{\partial^2 K_t(x,u)}{\partial u^2}\bigg|_{u=1} = 2\Big[\frac{2k+t+3}{n-2}\binom{(k+1)n+t}{n-3}\Big].$$

Furthermore, this quantity divided by $[x^{(k+1)n}]K_t(x,1)$ is equal to

$$\frac{2n(n-1)(2k+t+3)}{(kn+t+3)(kn+t+2)(t+1)},$$

and the variance is thus

$$\frac{2n(n-1)(2k+t+3)}{(kn+t+3)(kn+t+2)(t+1)} + \frac{n(k+t+2)}{(t+1)(kn+t+2)} - \left(\frac{n(k+t+2)}{(t+1)(kn+t+2)}\right)^2$$
$$= \frac{((k+t)(knt+t+t^2+6) - 2k(4-n(2t+1)) + 4t^2)((k+1)n+t+1)n}{(kn+t+3)(kn+t+2)^2(t+1)^2}$$
$$= \frac{(k+1)(t^2+(k+4)t+2)}{k^2(t+1)^2} + \mathcal{O}\Big(\frac{1}{n}\Big).$$

□

6.2.2 Peaks

A peak is an $\diagup$ step followed by a $\diagdown$ step in the path. For peaks we will use a modified symbolic decomposition of k_t-Dyck paths. This decomposition is

$$\mathcal{K}_t = \varepsilon + \sum_{i=0}^{t} \left(\mathcal{K}_t \times \{\diagup\}^{k-i} \times \mathcal{K}_{k-1-i} \times \{\diagdown\} \times \{\diagup\}^i\right),$$

and can be viewed as a last return decomposition of k_t-Dyck paths, with explanation similar to that given in the proof of Theorem 6.1.1.

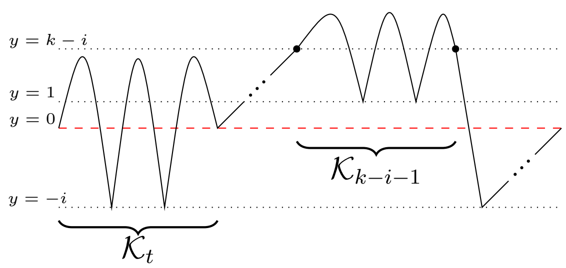


Figure 6.2: Adapted (last return) symbolic decomposition of a k_t-Dyck path

The reason that we use this slightly different decomposition is that with k_t-Dyck paths a path can start or end with either an $\diagup$ step or a $\diagdown$ step. For the decomposition in Figure 6.1 to count peaks we would have had to take into account $\mathcal{K}_{k-i-1}$ ending in an $\diagup$ step and $\mathcal{K}_t$ starting with a $\diagdown$ step (only possible when $t = k$). The decomposition in Figure 6.2 allows us to only count peaks when $\mathcal{K}_{k-i-1}$ ends in an $\diagup$ step.

Let $\mathcal{K}_t^{(2)}$ be the class of k_t-Dyck paths that are empty or end in an $\diagup$ step, and $\mathcal{K}_t^{(1)}$ be the class of k_t-Dyck paths that end in a $\backslash$ step. Then symbolically,

$$\begin{aligned}
\mathcal{K}_t &= \mathcal{K}_t^{(1)} + \mathcal{K}_t^{(2)},\\
\mathcal{K}_t^{(1)} &= \mathcal{K}_t \times \{\diagup\}^k \times \mathcal{K}_{k-1} \times \{\backslash\},\\
\mathcal{K}_t^{(2)} &= \varepsilon + \sum_{i=1}^{t} \big(\mathcal{K}_t \times \{\diagup\}^{k-i} \times \mathcal{K}_{k-i-1} \times \{\backslash\} \times \{\diagup\}^i\big).
\end{aligned}$$

In terms of functional equations, with the substitutions $x^{k+1} = z(1-z)^k$ and $K_t(x) = (1-z)^{-(t+1)}$, we find that

$$\begin{aligned}
K_t^{(1)}(x) &= \frac{z(1-z)^k}{(1-z)^{t+1}} \cdot \frac{1}{(1-z)^k} = \frac{z}{(1-z)^{t+1}},\\
K_t^{(2)}(x) &= 1 + \frac{z(1-z)^k}{(1-z)^{t+1}} \sum_{i=1}^{t} \frac{(1-z)^i}{(1-z)^k} = 1 + \frac{z((1-z)-(1-z)^{t+1})}{z(1-z)^{t+1}} = \frac{1}{(1-z)^t}.
\end{aligned}$$

Note that if we add $K_t^{(1)}(x)$ and $K_t^{(2)}(x)$ we get

$$K_t^{(1)}(x) + K_t^{(2)}(x) = \frac{z}{(1-z)^{t+1}} + \frac{1}{(1-z)^t} = \frac{z+1-z}{(1-z)^{t+1}} = \frac{1}{(1-z)^{t+1}},$$

as expected.

Proposition 6.2.3. *The number of k_t-Dyck paths that end in a $\backslash$ step is equal to*

$$\frac{t+k+1}{n-1}\binom{(k+1)n+t-1}{n-2},$$

and the number of k_t-Dyck paths that end in an $\diagup$ step is given by

$$\frac{t}{(k+1)n+t}\binom{(k+1)n+t}{n}.$$

This can be seen by applying Cauchy's integral formula to the expressions for $K_t^{(1)}(x)$ and $K_t^{(2)}(x)$ given previously. These expressions are functions of type 1 in Lemma 6.2.1 and with $p=1$, $q=1$, $r=1$, and $p=0$, $q=1$, $r=0$ respectively.

Proposition 6.2.4. *The total number of peaks in k_t-Dyck paths of length $(k+1)n$ where $0 \le t \le k-1$ is equal to*

$$(t+1)\binom{(k+1)n+t-1}{n-1},$$

and thus the average number of peaks in k_t-Dyck paths of length $(k+1)n$ is

$$\frac{n(kn+t+1)}{(k+1)n+t} = \frac{k}{k+1}n + \frac{k+t+1}{(k+1)^2} + \mathcal{O}\Big(\frac{1}{n}\Big).$$

Proof. From the symbolic equations and letting u count peaks, if a path ends in a $\backslash$ step then $i = 0$ in the symbolic decomposition. Here $\mathcal{K}_{k-1}$ introduces a peak if it ends in an $\diagup$ step, $uK_{k-1}^{(2)}(x, u)$ and does not introduce a peak if it ends in a $\backslash$ step, $K_{k-1}^{(1)}(x, u)$. Similarly, if a path ends in an up step $i \neq 0$, and again $\mathcal{K}_{k-i-1}$ introduces a peak if it ends in an $\diagup$ step, $uK_{k-i-1}^{(2)}(x, u)$, and does not introduce a peak if it ends in a $\backslash$ step, $K_{k-i-1}^{(1)}(x, u)$. Therefore we obtain the bivariate functional equations for the number of peaks as

$$\begin{aligned}
K_t(x,u) &= K_t^{(1)}(x,u) + K_t^{(2)}(x,u),\\
K_t^{(1)}(x,u) &= x^{k+1}K_t(x,u)\Big(uK_{k-1}^{(2)}(x,u) + K_{k-1}^{(1)}(x,u)\Big),\\
K_t^{(2)}(x,u) &= 1 + x^{k+1}K_t(x,u)\Big(u\sum_{i=1}^{t} K_{k-i-1}^{(2)}(x,u) + \sum_{i=1}^{t} K_{k-i-1}^{(1)}(x,u)\Big).
\end{aligned}$$

We then differentiate this system of functional equations with respect to u:

$$\begin{aligned}
\frac{\partial K_t(x,u)}{\partial u} &= \frac{\partial K_t^{(1)}(x,u)}{\partial u} + \frac{\partial K_t^{(2)}(x,u)}{\partial u},\\
\frac{\partial K_t^{(1)}(x,u)}{\partial u} &= x^{k+1}\frac{\partial K_t(x,u)}{\partial u}\Big(uK_{k-1}^{(2)}(x,u) + K_{k-1}^{(1)}(x,u)\Big)\\
&\quad + x^{k+1}K_t(x,u)\Big(u\frac{\partial K_{k-1}^{(2)}(x,u)}{\partial u} + K_{k-1}^{(2)}(x,u) + \frac{\partial K_{k-1}^{(1)}(x,u)}{\partial u}\Big),\\
\frac{\partial K_t^{(2)}(x,u)}{\partial u} &= x^{k+1}\frac{\partial K_t(x,u)}{\partial u}\Big(u\sum_{i=1}^{t} K_{k-i-1}^{(2)}(x,u) + \sum_{i=1}^{t} K_{k-i-1}^{(1)}(x,u)\Big)\\
&\quad + x^{k+1}K_t(x,u)\Big(u\sum_{i=1}^{t} \frac{\partial K_{k-i-1}^{(2)}(x,u)}{\partial u} + \sum_{i=1}^{t} K_{k-i-1}^{(2)}(x,u)\Big)\\
&\quad + x^{k+1}K_t(x,u)\sum_{i=1}^{t} \frac{\partial K_{k-i-1}^{(1)}(x,u)}{\partial u}.
\end{aligned}$$

Let $u = 1$ and substitute $K_t^{(1)}(x,1) = \frac{z}{(1-z)^{t+1}}$ and $K_t^{(2)}(x,1) = \frac{1}{(1-z)^t}$ to obtain (this can be checked by substituting these values into the differentiated system):

$$\begin{aligned}
\frac{\partial K_t^{(1)}(x,u)}{\partial u}\Big|_{u=1} &= \frac{tz^2 + z}{(1-z)^t(1-(k+1)z)},\\
\frac{\partial K_t^{(2)}(x,u)}{\partial u}\Big|_{u=1} &= \frac{tz}{(1-z)^{t-1}(1-(k+1)z)}.
\end{aligned}$$

Therefore the total number of peaks is equal to

$$[x^{(k+1)n}]\frac{\partial K_t(x,u)}{\partial u}\Big|_{u=1} = [x^{(k+1)n}]\frac{(t+1)z}{(1-z)^t(1-(k+1)z)}.$$

Using this and Lemma 6.2.1 for a function of type 2 with $p = 1$, $r = 0$, we find that

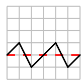
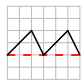
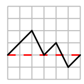
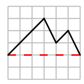
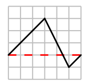
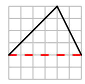
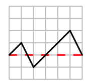
$$[x^{(k+1)n}]\frac{\partial K_t(x,u)}{\partial u}\Big|_{u=1} = (t+1)\binom{(k+1)n+t-1}{n-1}.$$

Therefore the average number of peaks in paths of length $(k+1)n$ is

$$\frac{n(kn+t+1)}{(k+1)n+t} = \frac{kn}{k+1} + \frac{k+t+1}{(k+1)^2} + \mathcal{O}\Big(\frac{1}{n}\Big).$$

□

Table 6.2: Number of peaks for 2_1-Dyck paths of length 6

Path							
Peaks	2	2	2	2	1	1	2

From Table 6.2 we can see that the total number of peaks in 2_1-Dyck paths is 12. Using the formula with $k = 2$, $t = 1$, and $n = 2$ we have

$$(1+1)\binom{(2+1)\cdot 2+1-1}{2-1} = 2\binom{6}{1} = 12,$$

as expected.

Proposition 6.2.5. *The second derivatives of the bivariate generating functions $K_t^{(1)}(x,u)$ and $K_t^{(2)}(x,u)$ at $u = 1$ where $0 \le t \le k-1$ are equal to*

$$\frac{\partial^2 K_t^{(1)}(x,u)}{(\partial u)^2}\Big|_{u=1} = \frac{z^2[2(t+k)+(t(t-3)-k(k+1))z - t(k+1)(t+k-1)z^2]}{(1-(k+1)z)^3(1-z)^{t-1}}$$

$$\frac{\partial^2 K_t^{(2)}(x,u)}{(\partial u)^2}\Big|_{u=1} = \frac{z^2[(t^2+(2k-1)t) - t(k+1)(t+k-1)z]}{(1-(k+1)z)^3(1-z)^{t-2}}$$

and therefore the variance for the number of peaks in k_t-Dyck paths of length $(k+1)n$ is asymptotically

$$\frac{k^2}{(k+1)^3}n - \frac{k(k+t)(k-1)-k(t+1)}{(k+1)^4} + \mathcal{O}\Big(\frac{1}{n}\Big).$$

By differentiating the system involving $\frac{\partial K_t(x,u)}{\partial u}$, $\frac{\partial K_t^{(1)}(x,u)}{\partial u}$, and $\frac{\partial K_t^{(2)}(x,u)}{\partial u}$ with respect to u again and substituting $u = 1$ we obtain the expressions for $\frac{\partial^2 K_t^{(1)}(x,u)}{(\partial u)^2}\Big|_{u=1}$ and $\frac{\partial^2 K_t^{(2)}(x,u)}{(\partial u)^2}\Big|_{u=1}$ given in the above proposition. Owing to the complexity of this calculation and the use of computer algebra to determine the solution, the calculation will not be given here. Adding these two expressions gives

$$\frac{\partial^2 K_t(x,u)}{(\partial u)^2}\Big|_{u=1} = \frac{z^2[(2k+t)(t+1)-(k+1)(t+1)(k+t)z]}{(1-(k+1)z)^3(1-z)^{t-1}}.$$

We can then extract coefficients by using Lemma 6.2.1 and splitting the right-hand side of the above expression into two functions of type 3 with $p=2$, $r=-2$ and $p=3$, $r=-2$ gives

$$\begin{aligned}(2k+t)(t+1)\sum_{m=0}^{n-2}\binom{(k+1)n+t-m-3}{n-m-2}(m+1)(k+1)^m\\ -(k+1)(t+1)(k+t)\sum_{m=0}^{n-3}\binom{(k+1)n+t-m-4}{n-m-3}(m+1)(k+1)^m.\end{aligned}$$

Surprisingly, this can be simplified to

$$(t+1)(kn+t)\binom{(k+1)n+t-2}{n-2}$$

as follows. The expression for the second derivative at $u=1$ can be represented in terms of derivatives as

$$\begin{aligned}
w\frac{d}{dw}\frac{z(t+k)-1}{(t-1)(1-(k+1)z)(1-z)^{t-1}} &= w\frac{dz}{dw}\frac{d}{dz}\frac{z(k+t)-1}{(t-1)(1-(k+1)z)(1-z)^{t-1}}\\
&= \frac{z(1-z)^k}{(1-(k+1)z)(1-z)^{k-1}}\frac{d}{dz}\frac{z(k+t)-1}{(t-1)(1-(k+1)z)(1-z)^{t-1}}\\
&= z\cdot\frac{(1-(k+1)z)(1-z)(k+t)+(z(k+t)-1)[(k+t)-t(k+1)z]}{(t-1)(1-(k+1)z)^3(1-z)^{t-1}}\\
&= z\cdot\frac{(k+t)[(1-(k+1)z)(1-z)+(z(k+t)-1)]-t(z(k+t)-1)(k+1)z}{(t-1)(1-(k+1)z)^3(1-z)^{t-1}}\\
&= z\cdot\frac{z(2k+t)(t-1)+z^2(k+t)(k+1)(1-t)}{(t-1)(1-(k+1)z)^3(1-z)^{t-1}}\\
&= \frac{z^2(2k+t)-z^3(k+t)(k+1)}{(1-(k+1)z)^3(1-z)^{t-1}} = \frac{1}{t+1}\cdot\frac{\partial^2 K_t(x,u)}{(\partial u)^2}\Big|_{u=1}.
\end{aligned}$$

Therefore

$$\frac{\partial^2 K_t(x,u)}{(\partial u)^2}\Big|_{u=1} = (t+1)w\frac{d}{dw}\frac{z(t+k)-1}{(t-1)(1-(k+1)z)(1-z)^{t-1}}. \tag{6.4}$$

Applying Lemma 6.2.1 for functions of type 2 we find that

$$\begin{aligned}
&[w^n]\frac{z(t+k)-1}{(t-1)(1-(k+1)z)(1-z)^{t-1}}\\
&= \frac{k+t}{t-1}\binom{(k+1)n+t-2}{n-1} - \frac{1}{t-1}\binom{(k+1)n+t-1}{n}\\
&= \frac{1}{t-1}\binom{(k+1)n+t-2}{n-1}\left[(k+t) - \frac{(k+1)n+t-1}{n}\right]\\
&= \binom{(k+1)n+t-2}{n-1}\frac{n-1}{n} = \binom{(k+1)n+t-2}{n-2}\frac{kn+t}{n}.
\end{aligned}$$

As a result, the generating function for the second derivative can be expressed as

$$\begin{aligned}
\frac{\partial^2 K_t(x,u)}{(\partial u)^2}\bigg|_{u=1} &= (t+1)w\frac{d}{dw}\sum_{n\geq 0}\binom{(k+1)n+t-2}{n-2}\frac{kn+t}{n}w^n\\
&= (t+1)\sum_{n\geq 1}\binom{(k+1)n+t-2}{n-2}(kn+t)w^n.
\end{aligned}$$

Therefore the variance is given by

$$\frac{n(n-1)(kn+t+1)(kn+t)}{((k+1)n+t)((k+1)n+t-1)} + \frac{n(kn+t+1)}{(k+1)n+t} - \left(\frac{n(kn+t+1)}{(k+1)n+t}\right)^2.$$

Asymptotically, this is

$$\frac{k^2}{(k+1)^3}n - \frac{k(k+t)(k-1)-k(t+1)}{(k+1)^4} + \mathcal{O}\Big(\frac{1}{n}\Big).$$

6.2.3 Valleys

Let u be a variable which counts the number of valleys. That is, a $\diagdown$ step followed by an $\diagup$ step. Then for $0 \leq t \leq k-1$, adapting the symbolic equations in Theorem 6.1.1 to account for valleys, for $1 \leq i \leq t$ the $\diagdown$ step followed by an $\diagup$ step introduces a valley. When $i = 0$ no valley is introduced. Therefore we obtain the bivariate generating functions

$$K_t(x,u) = 1 + ux^{k+1}K_t(x,u)\sum_{i=0}^{t}K_{k-i-1}(x,u) + (1-u)x^{k+1}K_{k-1}(x,u). \quad (6.5)$$

Proposition 6.2.6. *The total number of valleys in a k_t-Dyck path of length $(k+1)n$ is equal to*

$$\frac{(kn+t)(t+1)-k}{n-1}\binom{(k+1)n+t-1}{n-2},$$

and the average for the number of valleys in a k_t-Dyck path of length $(k+1)n$ is

$$\frac{n((kn+t)(t+1)-k)}{(t+1)((k+1)n+t)} = \frac{k}{k+1}n - \frac{(k-t)(k+t+1)}{(1+t)(1+k)^2} + \mathcal{O}\Big(\frac{1}{n}\Big).$$

Proof. We first differentiate the expression in equation (6.5) to obtain

$$\begin{aligned}\frac{\partial K_t(x,u)}{\partial u} &= x^{k+1}K_t(x,u)\sum_{i=0}^{t} K_{k-i-1}(x,u) + ux^{k+1}\frac{\partial K_t(x,u)}{\partial u}\sum_{i=0}^{t} K_{k-i-1}(x,u)\\ &\quad + ux^{k+1}K_t(x,u)\sum_{i=0}^{t}\frac{\partial K_{k-i-1}(x,u)}{\partial u} + (1-u)x^{k+1}\frac{\partial K_{k-1}(x,u)}{\partial u}\\ &\quad - x^{k+1}K_{k-1}(x,u),\end{aligned}$$

which with the substitutions $u=1$, $x^{k+1} = z(1-z)^k$, and $K_t(x,1) = (1-z)^{-t-1}$ gives

$$\begin{aligned}\frac{\partial K_t(x,u)}{\partial u}\Big|_{u=1}\Big(1 - z\sum_{i=0}^{t}(1-z)^i\Big) &= z(1-z)^{-t-1}\sum_{i=0}^{t}(1-z)^i\\ &\quad + z(1-z)^{k-t-1}\sum_{i=0}^{t}\frac{\partial K_{k-i-1}(x,u)}{\partial u}\Big|_{u=1} - z.\end{aligned}$$

Note that the simplification

$$\sum_{i=0}^{t}(1-z)^i = \frac{1-(1-z)^{t+1}}{1-(1-z)} = \frac{1-(1-z)^{t+1}}{z}$$

occurs, which leads to

$$\begin{aligned}\frac{\partial K_t(x,u)}{\partial u}\Big|_{u=1} &= \frac{1-(1-z)^{t+1}}{(1-z)^{2t+2}} + z(1-z)^{k-2t-2}\sum_{i=0}^{t}\frac{\partial K_{k-i-1}(x,u)}{\partial u}\Big|_{u=1}\\ &\quad - z(1-z)^{-t-1}.\end{aligned}$$

We check that $\frac{\partial K_t(x,u)}{\partial u}\Big|_{u=1} = \frac{(k-t)z^2+tz}{(1-(k+1)z)(1-z)^{t+1}}$ is a solution for $0 \le t \le k-1$. Note that

$$\sum_{i=0}^{t}\frac{\partial K_{k-i-1}(x,u)}{\partial u}\Big|_{u=1} = \frac{z(t+1)(1-z)^{t+2} - (1-(k+1)z)(1-(1-z)^{t+1})}{z(1-(k+1)z)(1-z)^k}.$$

Considering just the right-hand side of the equation of interest, we find that it simplifies to

$$\begin{aligned}
&\frac{1-(1-z)^{t+1}}{(1-z)^{2t+2}} + z(1-z)^{k-2t-2}\sum_{i=0}^{t}\frac{\partial K_{k-i-1}(x,u)}{\partial u}\Big|_{u=1} - \frac{z}{(1-z)^{t+1}} \\
&= \frac{(1-z)^{-t-1}-1-z}{(1-z)^{t+1}} + \frac{z(t+1)(1-z)-(1-(k+1)z)((1-z)^{-t-1}-1)}{(1-(k+1)z)(1-z)^{t+1}} \\
&= \frac{-z(1-(k+1)z)}{(1-(k+1)z)(1-z)^{t+1}} + \frac{z(t+1)(1-z)}{(1-(k+1)z)(1-z)^{t+1}} \\
&= \frac{-z[1-(k+1)z-(t+1)(1-z)]}{(1-(k+1)z)(1-z)^{t+1}} = \frac{-z[1-kz-z-t+tz-1+z]}{(1-(k+1)z)(1-z)^{t+1}} \\
&= \frac{-z[-kz-t+tz]}{(1-(k+1)z)(1-z)^{t+1}} = \frac{(k-t)z^2+tz}{(1-(k+1)z)(1-z)^{t+1}} = \frac{\partial K_t(x,u)}{\partial u}\Big|_{u=1}.
\end{aligned}$$

Expressing $\frac{(k-t)z^2+tz}{(1-(k+1)z)(1-z)^{t+1}}$ as $\frac{(k-t)z^2}{(1-(k+1)z)(1-z)^{t+1}} + \frac{tz}{(1-(k+1)z)(1-z)^{t+1}}$ we can apply Lemma 6.2.1 for a formula of type 2 with $p=2$, $r=1$ and $p=1$, $r=1$ we find that

$$[x^{(k+1)n}]\frac{\partial K_t(x,u)}{\partial u}\Big|_{u=1} = (k-t)\binom{(k+1)n+t-1}{n-2} + t\binom{(k+1)n+t}{n-1}.$$

Therefore the total number of valleys in k_t-Dyck paths of length $(k+1)n$ is

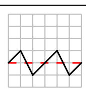
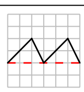
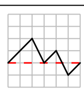
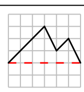
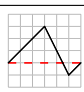
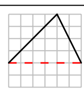
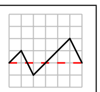
$$\begin{aligned}
&t\binom{(k+1)n+t}{n-1} + (k-t)\binom{(k+1)n+t-1}{n-2} \\
&= \frac{(kn+t)(t+1)-k}{n-1}\binom{(k+1)n+t-1}{n-2}.
\end{aligned}$$

From this we can divide by the total number of k_t-Dyck paths of length $(k+1)n$ to obtain the average number of valleys,

$$\frac{n((kn+t)(t+1)-k)}{(t+1)((k+1)n+t)} = \frac{k}{k+1}n - \frac{(k-t)(k+t+1)}{(1+t)(1+k)^2} + \mathcal{O}\Big(\frac{1}{n}\Big).$$

□

Table 6.3: Number of valleys for 2_1-Dyck paths of length 6

Path							
Valleys	2	1	2	1	1	0	1

So the total number of valleys in k_t-Dyck paths with $k = 2$, $t = 1$, and $n = 2$ is equal to 8, which corresponds to the formula:

$$\frac{(2\cdot 2+1)(1+1)-2}{2-1}\binom{(2+1)2+1-1}{2-2} = 8\cdot 1 = 8.$$

Proposition 6.2.7. *The second derivative of the bivariate generating function $K_t(x,u)$ with respect to u at $u = 1$ is equal to*

$$\begin{aligned}\frac{\partial^2 K_t(x,u)}{(\partial u)^2}\Big|_{u=1} &= \frac{t(t+2k-1)z^2-(t(k+3)(t+k-1)-k(3k-1))z^3}{(1-(k+1)z)^3(1-z)^{t+1}}\\ &+\frac{(t(2k+3)(t-1)-2k(k^2+k-1))z^4}{(1-(k+1)z)^3(1-z)^{t+1}}\\ &-\frac{(k+1)(t-1)(t-k)z^5}{(1-(k+1)z)^3(1-z)^{t+1}},\end{aligned}$$

and the variance for the number of valleys in k_t-Dyck paths of length $(k+1)n$ is asymptotically given by

$$\frac{k^2}{(k+1)^3}n-\frac{k((t^2+k)(k-1)(t+1)-k(k+1)^2t-(t+1)^3)}{(k+1)^4(1+t)^2}+\mathcal{O}\Big(\frac{1}{n}\Big).$$

To obtain the expression in the above proposition we differentiate the system of functional equations once again with respect to u, and substitute $u = 1$. Owing to the complexity of this calculation, we omit the proof. It can be easily be checked using computer algebra. To simplify this expression for extraction of coefficients, let

$$\begin{aligned}a &= t(t+2k-1),\\ b &= t(k+3)(t+k-1)-k(3k-1),\\ c &= t(2k+3)(t-1)-2k(k^2+k-1),\\ d &= (k+1)(t-1)(t-k).\end{aligned}$$

Using this and Lemma 6.2.1 for functions of type 3 with $p = 2$ and $r = 1$, $p = 3$ and $r = 1$, $p = 4$ and $r = 1$, and $p = 5$ and $r = 1$ respectively we have that

$$\begin{aligned}[x^{(k+1)n}]\frac{\partial^2 K_t(x,u)}{(\partial u)^2}\Big|_{u=1} &= a\sum_{m=0}^{n-2}(m+1)(k+1)^m\binom{(k+1)n+t-m-1}{n-m-2}\\ &- b\sum_{m=0}^{n-3}(m+1)(k+1)^m\binom{(k+1)n+t-m-2}{n-m-3}\\ &+ c\sum_{m=0}^{n-4}(m+1)(k+1)^m\binom{(k+1)n+t-m-3}{n-m-4}\\ &- d\sum_{m=0}^{n-5}(m+1)(k+1)^m\binom{(k+1)n+t-m-4}{n-m-5}.\end{aligned}$$

This sum simplifies to the binomial coefficient

$$\binom{(k+1)n+t-2}{n-2}\frac{(kn+t-1)((t+1)(kn+t)-2k)}{kn+t+1}.$$

As done in the case for peaks, we find that

$$\begin{aligned}
&\frac{az^2-bz^3+cz^4-dz^5}{(1-(k+1)z)^3(1-z)^{t+1}}\\
&\quad= w\frac{d}{dw}\Big[\frac{t(k-t)z^2+(t-k)(2k+t+1)z-kz-t+\frac{2k((tk-1)z+1)}{(t+1)(1-z)}}{t(1-(k+1)z)(1-z)^t}\Big]\\
&\quad= w\frac{d}{dw}\sum_{n\geq 0}\binom{(k+1)n+t-2}{n-2}\frac{(kn+t-1)((t+1)(kn+t)-2k)}{n(kn+t+1)}w^n\\
&\quad= \sum_{n\geq 1}\binom{(k+1)n+t-2}{n-2}\frac{(kn+t-1)((t+1)(kn+t)-2k)}{(kn+t+1)}w^n.
\end{aligned}$$

Therefore

$$[x^{(k+1)n}]\frac{\partial^2 K_t(x,u)}{(\partial u)^2}\Big|_{u=1} = \binom{(k+1)n+t-2}{n-2}\frac{(kn+t-1)((t+1)(kn+t)-2k)}{(kn+t+1)}. \tag{6.6}$$

Then $[x^{(k+1)n}]\frac{\partial^2 K_t(x,u)}{(\partial u)^2}\Big|_{u=1}\Big/[x^{(k+1)n}]K_t(x,1)$ is given by

$$\frac{n(n-1)(kn+t-1)((t+1)(kn+t)-2k)}{(t+1)((k+1)n+t)((k+1)n+t-1)},$$

and the variance is equal to

$$\begin{aligned}
&\frac{n(n-1)(kn+t-1)((t+1)(kn+t)-2k)}{(t+1)((k+1)n+t)((k+1)n+t-1)}+\frac{n((kn+t)(t+1)-k)}{(t+1)((k+1)n+t)}\\
&-\Big(\frac{n((kn+t)(t+1)-k)}{(t+1)((k+1)n+t)}\Big)^2\\
&=\frac{k^2}{(k+1)^3}n-\frac{k((t^2+k)(k-1)(t+1)-k(k+1)^2t-(t+1)^3)}{(k+1)^4(1+t)^2}+\mathcal{O}\Big(\frac{1}{n}\Big).
\end{aligned}$$

6.2.4 Returns

Now let u count the number of returns to $y=0$. A return is a step whose endpoint is on the line $y=0$. Using the symbolic decomposition in Theorem 6.1.1 we obtain the following bivariate generating functions for $0\leq t\leq k$,

$$K_t(x,u)=1+ux^{k+1}K_t(x,u)\sum_{i=0}^{t}K_{k-i-1}(x,1).$$

Differentiating this system with respect to u yields the following result.

Proposition 6.2.8. *The total number of returns to $y = 0$ in k_t-Dyck paths of length $(k+1)n$ is equal to*

$$\frac{2(t+1)}{n}\binom{(k+1)n+2t+1}{n-1} - \frac{t+1}{n}\binom{(k+1)n+t}{n-1},$$

and asymptotically the average number of returns to $y = 0$ in k_t-Dyck paths of length $(k+1)n$ is given by

$$\frac{2\cdot((k+1)n+2t+1)\cdots((k+1)n+t+1)}{(kn+2t+2)\cdots(kn+t+2)} - 1 = \frac{2(k+1)^{t+1}-k^{t+1}}{k^{t+1}} + \mathcal{O}\Big(\frac{1}{n}\Big).$$

Proof. The derivative with respect to u of the system of functional equations is:

$$\frac{\partial K_t(x,u)}{\partial u} = \frac{x^{k+1}\sum_{i=0}^{t} K_{k-i-1}(x,1)}{(1-ux^{k+1}\sum_{i=0}^{t} K_{k-i-1}(x,1))^2},$$

which with the substitutions $u = 1$, $x^{k+1} = z(1-z)^k$, and $K_{i-1}(x,1) = \frac{1}{(1-z)^i}$, gives

$$\frac{\partial K_t(x,u)}{\partial u}\Big|_{u=1} = \frac{1-(1-z)^{t+1}}{(1-z)^{2(t+1)}} = \frac{1}{(1-z)^{2(t+1)}} - \frac{1}{(1-z)^{t+1}}.$$

Using the above expression and Lemma 6.2.1 for functions of type 1 with $p = 0$, $q = 2$, $r = 2$ and $p = 0$, $q = 1$, $r = 1$ we find that the number of returns of a k_t-Dyck path of length $(k+1)n$ to the line $y = 0$ is

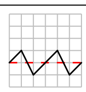
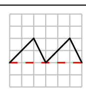
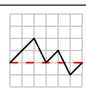
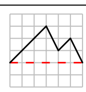
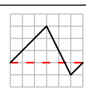
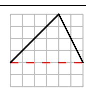
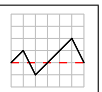
$$\frac{2(t+1)}{n}\binom{(k+1)n+2t+1}{n-1} - \frac{t+1}{n}\binom{(k+1)n+t}{n-1}.$$

The average number of returns to the line $y = 0$ is then

$$\frac{2\cdot((k+1)n+2t+1)\cdots((k+1)n+t+1)}{(kn+2t+2)\cdots(kn+t+2)} - 1 = \frac{2(k+1)^{t+1}-k^{t+1}}{k^{t+1}} + \mathcal{O}\Big(\frac{1}{n}\Big).$$

□

Table 6.4: Number of returns for 2_1-Dyck paths of length 6

Path							
Valleys	2	2	2	1	1	1	2

From Table 6.4 we see that the total number of returns in k_t-Dyck paths with $k = 2$, $t = 1$, and $n = 2$ is equal to 11, which corresponds to the formula:

$$\frac{2(1+1)}{2}\binom{(2+1)2+2+1}{2-1} - \frac{1+1}{2}\binom{(2+1)2+1}{2-1} = 2\cdot 9 - 7 = 11.$$

We will now calculate the variance. To do this, we first calculate the second derivative of $K_t(x,u)$.

Proposition 6.2.9. *The second derivative of $K_t(x,u)$ with respect to u, where u counts the number of returns to the line $y = 0$ of a k_t-Dyck path is given by*

$$\frac{\partial^2 K_t(x,u)}{\partial u^2}\Big|_{u=1} = \frac{2(1-(1-z)^{t+1})^2}{(1-z)^{3t+3}} = \frac{2}{(1-z)^{3t+3}} - \frac{4}{(1-z)^{2t+2}} + \frac{2}{(1-z)^{t+1}},$$

and the variance for the number of returns in k_t-Dyck paths of length $(k+1)n$ is asymptotically

$$2\Big(\frac{(k+1)^{2t+2} - k^{t+1}(k+1)^{t+1}}{k^{2t+2}}\Big) + \mathcal{O}\Big(\frac{1}{n}\Big).$$

Proof. The first derivative of $K_t(x,u)$ can be found in the proof of Proposition 6.2.8. We now use this to find the second derivative with respect to u:

$$\frac{\partial^2 K_t(x,u)}{\partial u^2} = \frac{2x^{2(k+1)}\big(\sum_{i=0}^{t} K_{k-i-1}(x,1)\big)^2}{(1-ux^{k+1}\sum_{i=0}^{t} K_{k-i-1}(x,1))^3}.$$

Using these along with the substitutions $u = 1$, $x^{k+1} = w = z(1-z)^k$, and $K_{i-1}(x,1) = (1-z)^{-i}$ gives

$$\begin{aligned}\frac{\partial^2 K_t(x,u)}{\partial u^2}\Big|_{u=1} &= \frac{2(z(1-z)^t - (1-z)^t + 1)^2}{((1-z)^t - z(1-z)^t)^3} = \frac{2(1-(1-z)^{t+1})^2}{(1-z)^{3t+3}}\\ &= \frac{2}{(1-z)^{3t+3}} - \frac{4}{(1-z)^{2t+2}} + \frac{2}{(1-z)^{t+1}}.\end{aligned}$$

Using the above Lemma as well as Lemma 6.2.1 for formulas of type 1 with $p = 0$ and $q = r = 3$, $p = 0$ and $q = r = 2$, $p = 0$ and $q = r = 1$ then yields

$$\begin{aligned}&\frac{t+1}{n}\left[6\binom{(k+1)n+3t+2}{n-1} - 8\binom{(k+1)n+2t+1}{n-1} + 2\binom{(k+1)n+t}{n-1}\right]\\ &= \frac{2(t+1)}{n}\binom{(k+1)n+t}{n-1}\left[\frac{3((k+1)n+3t+2)\cdots((k+1)n+t+1)}{(kn+3t+3)\cdots(kn+t+2)}\right.\\ &\qquad\left. - \frac{4((k+1)n+2t+1)\cdots((k+1)n+t+1)}{(kn+2t+2)\cdots(kn+t+2)} + 1\right].\end{aligned}$$

This quantity divided by the total number of k_t-Dyck paths of length $(k+1)n$ is equal to

$$2 + \frac{6((k+1)n+3t+2)\cdots((k+1)n+t+1)}{(kn+3t+3)\cdots(kn+t+2)} - \frac{8((k+1)n+2t+1)\cdots((k+1)n+t+1)}{(kn+2t+2)\cdots(kn+t+2)}.$$

Finally, we compute the variance, which when simplified is equal to

$$\begin{aligned}
&6\frac{((k+1)n+3t+2)\cdots((k+1)n+t+1)}{(kn+3t+3)\cdots(kn+t+2)} \\
&- 2\frac{((k+1)n+2t+1)\cdots((k+1)n+t+1)}{(kn+2t+2)\cdots(kn+t+2)} \\
&- 4\Big(\frac{((k+1)n+2t+1)\cdots((k+1)n+t+1)}{(kn+2t+2)\cdots(kn+t+2)}\Big)^2 \\
&= \frac{6(k+1)^{2t+2}}{k^{2t+2}} - \frac{2(k+1)^{t+1}}{k^{t+1}} - \frac{4(k+1)^{2t+2}}{k^{2t+2}} + \mathcal{O}\Big(\frac{1}{n}\Big) \\
&= 2\Big(\frac{(k+1)^{2t+2} - k^{t+1}(k+1)^{t+1}}{k^{2t+2}}\Big) + \mathcal{O}\Big(\frac{1}{n}\Big).
\end{aligned}$$

□

6.2.5 Table of results

For easy reference, below is a table summarising the results for k_t-Dyck paths of length $(k+1)n$ with parameter as indicated.

Table 6.5: Results for parameters associated with k_t-Dyck paths

Valleys	
Average	$\frac{k}{k+1}n - \frac{kt^2+k^2-kt-2t+k}{(t+1)(k+1)^2} + \mathcal{O}\left(\frac{1}{n}\right)$
Variance	$\frac{k^2}{(k+1)^3}n - \frac{k((t^2+k)(k-1)(t+1)-k(k+1)^2t-(t+1)^3)}{(k+1)^4(1+t)^2} + \mathcal{O}\left(\frac{1}{n}\right)$
Returns to $y=-t$	
Average	$\frac{k+t+2}{k(t+1)} + \mathcal{O}\left(\frac{1}{n}\right)$
Variance	$\frac{(k+1)(t^2+(k+4)t+2)}{k^2(t+1)^2} + \mathcal{O}\left(\frac{1}{n}\right)$
Returns to $y=0$	
Average	$\frac{2(k+1)^{t+1}-k^{t+1}}{k^{t+1}} + \mathcal{O}\left(\frac{1}{n}\right)$
Variance	$2\left(\frac{(k+1)^{2t+2}-k^{t+1}(k+1)^{t+1}}{k^{2t+2}}\right) + \mathcal{O}\left(\frac{1}{n}\right)$
Peaks	
Average	$\frac{k}{k+1}n + \frac{k+t+1}{(k+1)^2} + \mathcal{O}\left(\frac{1}{n}\right)$
Variance	$\frac{k^2}{(k+1)^3}n - \frac{k(k+t)(k-1)-k(t+1)}{(k+1)^4} + \mathcal{O}\left(\frac{1}{n}\right)$

Chapter 7

Walks in the Quarter Plane

7.1 General background

A walk in the quarter plane is a lattice path starting at the origin which is restricted to $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. More formally, it is a sequence of ordered pairs (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) such that for every $1 \leq k \leq n$,

$$\sum_{i=1}^{k} x_i \geq 0 \qquad \text{and} \qquad \sum_{i=1}^{k} y_i \geq 0.$$

The subject of walks in the quarter plane has become a very active area of research over the last two decades, with the main focus of research being determining which walks of length n with a fixed step set $\mathcal{S}$ that start at $(0, 0)$ and end at arbitrary (i, j) have holonomic generating functions [4, 5, 25]. In 2010 Mishna and Bousquet-Mélou [5] were able to classify all 79 unique (up to symmetries) walks with steps from the step set $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \backslash \{(0, 0)\}$. Only recently has some theory to deal with step sets containing steps with x- or y-coordinate greater that one been developed by Bostan, Bousquet-Mélou, and Melczer [4].

We firstly deal with walks in the quarter plane that start and end at the origin, with step set $\mathcal{S}_k = \{(1, 1), (1, -k + 1), (-k, 0)\}$ and $k \geq 2$. As a consequence of a result in [4, Proposition 21], these have a beautiful enumeration,

$$\frac{1}{(k-1)n+1}\binom{kn}{n} \cdot \frac{1}{kn+1}\binom{(k+1)n}{n},$$

which is the product of two consecutive generalised Catalan numbers. Here we extend this result further to walks with step set $\mathcal{S}_k$ which start at some (ℓ, r) with $0 \leq r < k$ and $0 \leq \ell \leq k$ and end at arbitrary (i, j).

To assist in doing this, we provide some terminology and results. In Definition 4.1.1 we defined the *step polynomial* for directed walks (the x-coordinate was constantly 1). Here we give the definition of a step polynomial for two-dimensional steps.

Definition 7.1.1. *Let $\mathcal{S} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ be the step set of a walk in the quarter plane. Then the step polynomial of this step set is given by*

$$S(x, y) = \sum_{i=1}^{n} x^{x_i} y^{y_i}.$$

This step polynomial can be used to determine whether or not a step set is considered '*Hadamard*' in the following sense.

Definition 7.1.2. *A step set $\mathcal{S}$ is called* Hadamard *if the step polynomial can be expressed as $S(x, y) = U(x) + V(x)T(y)$ for some Laurent polynomials U, V, and T.*

As a consequence of [4, Proposition 21], walks with a Hadamard step set have a pleasant and manageable enumeration (which is not the case for many other walks in the quarter plane). While it has been shown that Hadamard walks in the quarter plane from $(0, 0)$ to $(0, 0)$ have nice enumeration, specific cases have not yet been studied. Particularly, Hadamard walks starting and ending at arbitrary points have not yet been enumerated. Here we will discuss one such family of walks.

7.2 The enumeration of a family of walks in the quarter plane

Let us consider a family of walks in the quarter plane ($\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$) with step set $\mathcal{S}_k = \{(1, 1), (1, -k + 1), (-k, 0)\}$, $k \geq 2$. These walks have step polynomial

$$S(x, y) = xy + xy^{1-k} + x^{-k} = x^{-k} + x(y + y^{1-k}).$$

and thus $\mathcal{S}_k$ is a Hadamard step set. Unless stated otherwise, for the rest of the chapter by walk we refer to a walk in the quarter plane with step set $\mathcal{S}_k$.

7.2.1 The enumeration of walks starting and ending at the same point

At first we will only consider walks that start and end at the origin, and so these walks must be of length $(k+1)n$ with $n(k-1)$ steps of type $(1, 1)$, n steps of type $(1, -k + 1)$, and n steps of type $(-k, 0)$. As an example, Figure 7.1 has a walk of length 9 when $k = 2$. This walk consists of steps (in order) $(1, 1), (1, 1), (-2, 0), (1, -1), (1, -1), (1, 1), (1, -1), (-2, 0), (-2, 0)$.

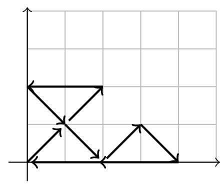


Figure 7.1: An example of a walk with step set $\mathcal{S}_2$

These walks with step set $\mathcal{S}_k$ are enumerated by very beautiful numbers involving the product of two adjacent numbers in the generalised Catalan family.

Proposition 7.2.1. *Let $\mathcal{S}_k = \{(1,1),(1,-k+1),(-k,0)\}$ and $k \geq 2$. The number of walks of length $(k+1)n$ with steps from the step set $\mathcal{S}_k$ in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ that start and end at the origin is given by*

$$\frac{1}{(k-1)n+1}\binom{kn}{n} \cdot \frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n}.$$

This proposition is in fact a corollary of Theorem 7.2.1 which involves the enumeration of a much larger family of walks existing in an *extended quarter plane*, $\mathbb{Z}_{\geq -\ell} \times \mathbb{Z}_{\geq -r}$.

Theorem 7.2.1. *Let $\mathcal{S}_k = \{(1,1),(1,-k+1),(-k,0)\}$ and $k \geq 2$. For $0 \leq \ell \leq k$, $0 \leq r < k$, the number of walks of length $(k+1)n$ with steps from the step set $\mathcal{S}_k$ in $\mathbb{Z}_{\geq -\ell} \times \mathbb{Z}_{\geq -r}$ that start and end at the origin is given by*

$$\frac{r+1}{kn+r+1}\binom{kn+r+1}{n} \cdot \frac{\ell+1}{(k+1)n+\ell+1}\binom{(k+1)n+\ell+1}{n}.$$

In Theorem 7.2.1 we are in an extended quarter plane, but an equivalent formulation would involve walks of length $(k+1)n$ in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ that start and end at (ℓ, r). To prove Theorem 7.2.1 we first require the following lemmas.

Lemma 7.2.1. *Let $[(x,y)]_m$ denote the number of steps of type (x,y) that occur before the m-th step in a walk. Then for walks in $\mathbb{Z}_{\geq -\ell} \times \mathbb{Z}_{\geq -r}$ with step set $\mathcal{S}_k$,*

$$[(1,1)]_m + [(1,-k+1)]_m \geq k[(-k,0)]_m - \ell$$

and

$$[(1,1)]_m \geq (k-1)[(1,-k+1)]_m - r.$$

Proof. Consider an arbitrary walk consisting of steps (in order) $(x_1,y_1),(x_2,y_2),\ldots,(x_N,y_N)$ with $(x_i,y_i) \in \mathcal{S}_k$ in $\mathbb{Z}_{\geq -\ell} \times \mathbb{Z}_{\geq -r}$. Then for $1 \leq m \leq N$

$$\sum_{i=1}^{m} x_i \geq -\ell \qquad \text{and} \qquad \sum_{i=1}^{m} y_i \geq -r.$$

From this we can conclude that

$$[(1,1)]_m + [(1,-k+1)]_m - k[(-k,0)]_m \geq -\ell$$

and

$$[(1,1)]_m - (k-1)[(1,-k+1)]_m \geq -r.$$

The result follows. □

Lemma 7.2.2. *In an arbitrary k_t-Dyck path, the minimum number of $(1,1)$ steps that occur before the j-th $(1,-k)$ step in the path (from left to right) is given by $kj-t$.*

Proof. Clearly the first $(1,-k)$ step must occur after at least $k-t$ steps of type $(1,1)$, and thereafter the minimum number of $(1,1)$ steps that need to occur before another $(1,-k)$ step is k. Hence $k-t+k(j-1)=kj-t$ is the minimum number of $(1,1)$ steps required. □

Using these results we can then prove Theorem 7.2.1 by means of a bijection.

Proof of Theorem 7.2.1. We prove this by providing a bijection between walks of length $(k+1)n$ in $\mathbb{Z}_{\geq -\ell} \times \mathbb{Z}_{\geq -r}$ with step set $\mathcal{S}_k$ and ordered pairs consisting of $(k-1)_r$-Dyck paths of length kn and k_ℓ-Dyck paths of length $(k+1)n$.

First we consider the mapping from walks of length $(k+1)n$ in $\mathbb{Z}_{\geq -\ell} \times \mathbb{Z}_{\geq -r}$ to ordered pairs. Let $W = w_1 w_2 \cdots w_{(k+1)n}$ be a word representing an arbitrary walk of length $(k+1)n$ in $\mathbb{Z}_{\geq -\ell} \times \mathbb{Z}_{\geq -r}$. Here $w_i \in \mathcal{S}_k$, with $(k-1)n$ steps of type $(1,1)$, and n of each of the steps $(1,-k+1)$, $(-k,0)$. We can construct two words, U and V, from the word W as follows. Initially, U and V are empty words. To form U, read W from left to right, and if $w_i = (1,1)$ append $(1,1)$ to the word U, if $w_i = (1,-k+1)$ append $(1,-k+1)$ to U. If $w_i = (-k,0)$ append nothing to U. Similarly, to form V, read W from left to right and if $w_i = (1,1)$ or $w_i = (1,-k+1)$ append $(1,1)$ to V, otherwise append $(1,-k)$ to V. From applying Lemma 7.2.1 to the mapping we can conclude that the resulting words U and V represent a $(k-1)_r$-Dyck path and a k_ℓ-Dyck path respectively.

For the inverse mapping, suppose that we have an arbitrary $(k-1)_r$-Dyck path of length kn. We write it as a word $U = u_1 u_2 \cdots u_{kn}$ where $u_i \in \{1, \overline{1}\}$, 1 represents a $(1,1)$ step and $\overline{1}$ represents a $(1,-k+1)$ step. Similarly, given an arbitrary k_ℓ-Dyck path we write it as a word $V = v_1 v_2 \cdots v_{(k+1)n}$ with $v_i \in \{1, \overline{2}\}$ where 1 represents a $(1,1)$ step, and $\overline{2}$ represents a $(1,-k)$ step.

Let $W = w_1 w_2 \cdots w_{(k+1)n}$ and for $1 \leq i \leq (k+1)n$ set $w_i = 0$. Now for all $1 \leq i \leq (k+1)n$, if $v_i = \overline{2}$ set $w_i = v_i$. Let $s = 1$ and let $1 \leq m \leq (k+1)n$ be the smallest value such that $w_m = 0$, set $w_m = u_s$ and increase s by 1. Again, let $1 \leq m \leq (k+1)n$ be the smallest value such that $w_m = 0$, set $w_m = u_s$

and increase s by 1. Repeat this process until there is no m-value such that $w_m = 0$.

From the word W we can now construct a walk of length $(k+1)n$ in $\mathbb{Z}_{\geq -\ell} \times \mathbb{Z}_{\geq -r}$ that starts and ends at the origin. Here 1 represents the step $(1,1)$, $\overline{1}$ represents $(1,-k+1)$, and $\overline{2}$ represents $(-k,0)$. We then have a walk with $(k-1)n$ steps of type $(1,1)$, n steps of type $(1,-k+1)$, and n steps of type $(-k,0)$. With Lemma 7.2.2 applied to V, we know that the minimum number of steps in W with x-component 1 (either $(1,1)$ or $(1,-k+1)$) that must occur before a step with x-component $-k$ can occur is $kj-\ell$. Similarly, Lemma 7.2.2 applied to U implies that the minimum number of $(1,1)$ steps that must occur in W before the j-th $(1,-k+1)$ step is $(k-1)j-r$. Therefore if $w_i = (x_i, y_i)$ and $1 \leq N \leq (k+1)n$,

$$\sum_{i=0}^{N} x_i \geq -\ell \qquad \text{and} \qquad \sum_{i=0}^{N} y_i \geq -r.$$

Hence W is a valid walk of length $(k+1)n$ in $\mathbb{Z}_{\geq -\ell} \times \mathbb{Z}_{\geq -r}$. $\square$

As an example, consider the walk with step set $\mathcal{S}_3 = \{(1,1),(1,-2),(-3,0)\}$ and $\ell = 2$, $r = 0$ given in Figure 7.2.

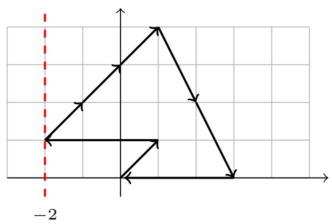


Figure 7.2: A walk of length 8 with step set $\mathcal{S}_3$ and $\ell = 2$, $r = 0$.

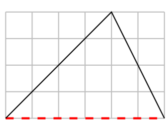
This walk has word $W = (1,1)(-3,0)(1,1)(1,1)(1,1)(1,-2)(1,-2)(-3,0)$. Therefore

$$U = (1,1)(1,1)(1,1)(1,1)(1,-2)(1,-2)$$

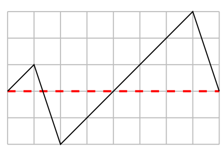
and

$$V = (1,1)(1,-3)(1,1)(1,1)(1,1)(1,1)(1,1)(1,-3)$$

and the 2_0-Dyck path and 3_2-Dyck path are given by



and



respectively. For the inverse mapping $U = 1111\overline{1}\overline{1}$ and $V = 1\overline{2}11111\overline{2}$, from which we find $W = 00000000 \to W = 0\overline{2}00000\overline{2} \to W = 1\overline{2}111\overline{1}\overline{1}\overline{2}$. The word W then translates into the walk given in Figure 7.2.

7.2.2 The enumeration of walks starting and ending at arbitrary points

Theorem 7.2.2. *Let $\mathcal{S}_k = \{(1,1),(1,-k+1),(-k,0)\}$ be the step set of a walk in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ starting at (ℓ, r), where $0 \leq \ell \leq k$, $0 \leq r < k$. Then the number of walks of length n that end at $(i+\ell, j+r)$ is*

$$\frac{\ell+i+1}{n+\ell+1}\binom{n+\ell+1}{\frac{n-i}{k+1}} \cdot \frac{r+j+1}{\frac{kn+i}{k+1}+r+1}\binom{\frac{kn+i}{k+1}+r+1}{\frac{k(n-j)+i-j}{k(k+1)}}.$$

Here we require that $n \equiv i \mod k+1$ and $i \equiv j \mod k$.

Proof. The above theorem is equivalent to the number of walks from $(0,0)$ to (i,j) in the extended quarter plane $\mathbb{Z}_{\geq -\ell} \times \mathbb{Z}_{\geq -r}$. The decomposition into ordered pairs of k_ℓ-Dyck paths and $(k-1)_r$-Dyck paths is given in the proof of Theorem 7.2.1. A walk of length n that ends at an x-coordinate i corresponds to a k_ℓ-Dyck path that ends at a height of i. By Proposition 5.2.3 the number of such paths is

$$\frac{\ell+i+1}{n+\ell+1}\binom{n+\ell+1}{\frac{n-i}{k+1}}.$$

Furthermore, if there are $q \nearrow$ steps in a k_ℓ-Dyck path of length n that ends at a height i, then $q - k(n-q) = i$ and thus $q = \frac{kn+i}{k+1}$. From the decomposition in the proof of Theorem 7.2.1 we know that the number of $\nearrow$ steps in the partial k_ℓ-Dyck path must equal the length of the corresponding $(k-1)_r$-Dyck path. Hence we need to calculate the number of partial $(k-1)_r$-Dyck paths of length q which end at height j. Again from Proposition 5.2.3 the number of paths is

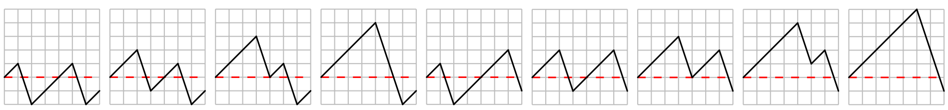
$$\frac{r+j+1}{q+r+1}\binom{q+r+1}{\frac{q-j}{k}} = \frac{r+j+1}{\frac{kn+i}{k+1}+r+1}\binom{\frac{kn+i}{k+1}+r+1}{\frac{k(n-j)+i-j}{k(k+1)}}.$$

Finally, the number of ordered pairs of k_ℓ-Dyck paths of length n which end at height i and $(k-1)_r$-Dyck paths of length $\frac{kn+i}{k+1}$ which end at height j is equal to

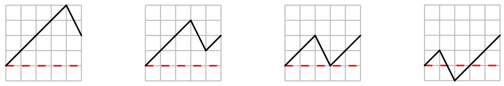
$$\frac{\ell+i+1}{n+\ell+1}\binom{n+\ell+1}{\frac{n-i}{k+1}} \cdot \frac{r+j+1}{\frac{kn+i}{k+1}+r+1}\binom{\frac{kn+i}{k+1}+r+1}{\frac{k(n-j)+i-j}{k(k+1)}}.$$

The conditions on n, i, and j follow from divisibility arguments. □

As an example, let us consider the walks of length 7 from $(2,1)$ to $(1,3)$ with step set $\mathcal{S}_3 = \{(1,1),(1,-2),(-3,0)\}$. Here $k=3$, $i=-1$, $j=2$, $\ell=2$, and $r=1$. The nine partial k_ℓ-Dyck paths of length 7 that end at height -1 are:

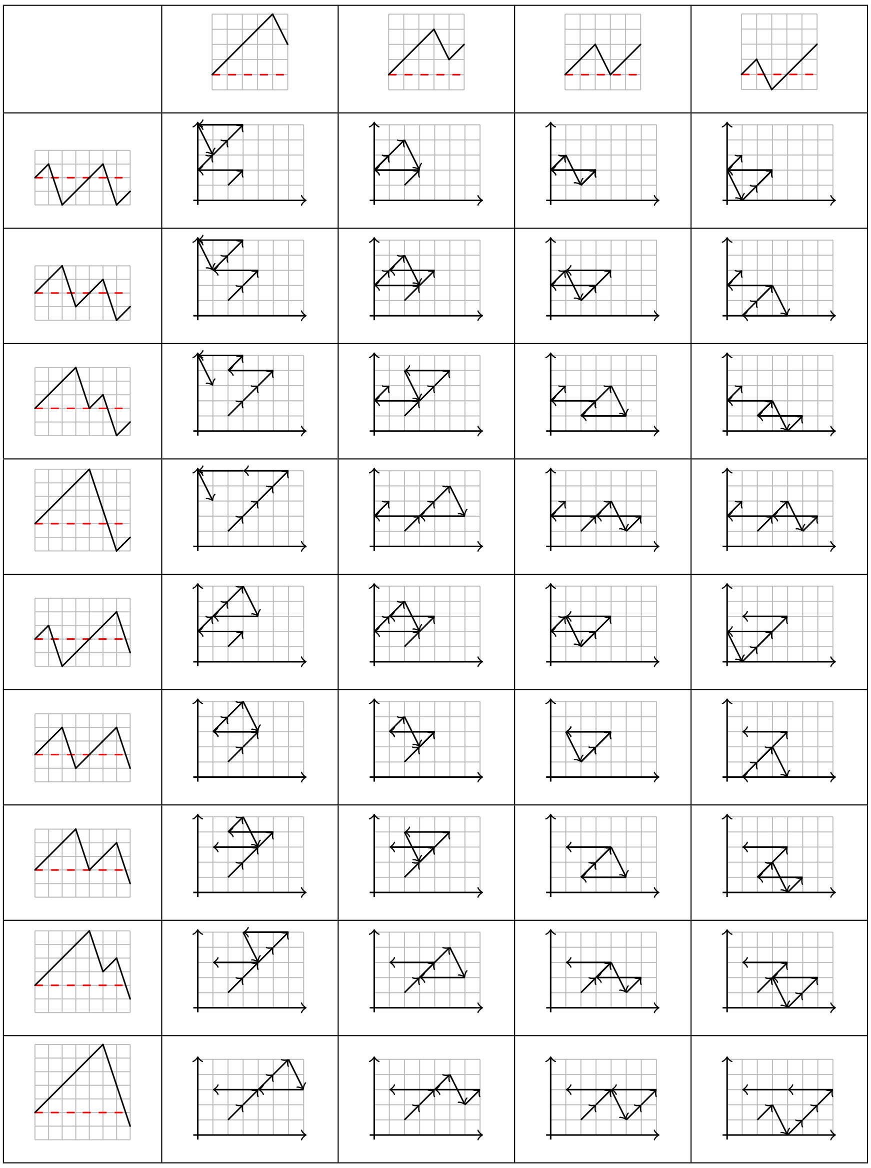


The four partial $(k-1)_r$-Dyck paths of length $\frac{kn+i}{k+1} = \frac{20}{4} = 5$ which end at height 2 are:



The resulting 36 walks in the quarter plane with step set $\mathcal{S}_3$ from $(2,1)$ to $(1,3)$ are given in Table 7.1. Note that some representations of two different walks might have the same diagram as order of steps taken cannot be easily indicated.

Table 7.1: Combining partial 3_2- and 2_1-Dyck paths to obtain a $\mathcal{S}_3$ walk in the quarter plane of length 7 from $(2,1)$ to $(1,3)$



7.3 Statistics related to walks with step set $\mathcal{S}_k$

Using the decomposition given in the proof of Theorem 7.2.1 as well as the analysis of parameters in k_t-Dyck paths given in Section 6.2, we can calculate statistics related to walks in the quarter plane with step set $\mathcal{S}_k$ which start and end at the same position. As an example, we calculate the number of returns of a walk to the boundary, but other statistics can be calculated in a similar manner.

Let us consider an arbitrary walk in the quarter plane of length $(k+1)n$ with step set $\mathcal{S}_k$ and starting point (ℓ, r) where $0 \le \ell, r \le k-1$. With the decomposition as given in Theorem 7.2.1 we obtain a k_ℓ-Dyck path $\mathcal{P}$ and a $(k-1)_r$-Dyck path $\mathcal{Q}$.

A return to the boundary is a point on the line $y=0$ or $x=0$ that a walk reaches. From the decomposition, it is clear that a return to the boundary along $y=0$ is given by a return of $\mathcal{P}$ to the line $y=-\ell$. From Subsection 6.2.1 we know that the average number of returns to $y=-\ell$ is given by

$$\frac{n(k+\ell+2)}{(\ell+1)(kn+\ell+2)}.$$

Similarly, a return to the boundary along the line $x=0$ is given by a return of $\mathcal{Q}$ to the line $y=-r$, and thus the average number of returns to $y=-r$ is equal to

$$\frac{n(k+r+1)}{(r+1)((k-1)n+r+2)}.$$

Altogether, the average number of returns to the boundary in a walk of length $(k+1)n$ with steps in $\mathcal{S}_k$ is

$$\begin{aligned}&\frac{n(k+\ell+2)}{(\ell+1)(kn+\ell+2)}+\frac{n(k+r+1)}{(r+1)((k-1)n+r+2)}\\ &\quad=\frac{(k+\ell+2)(r+1)(k-1)+(k+r+1)(\ell+1)k}{k(k-1)(\ell+1)(r+1)}+\mathcal{O}\Big(\frac{1}{n}\Big).\end{aligned}$$

Analogously we can calculate the average number of times the walk returns to the lines $x=\ell$ or $y=r$ using the number of returns to the line $y=0$ as discussed in Subsection 6.2.4. This works out to be

$$\begin{aligned}&\Big(\frac{2\cdot((k+1)n+2\ell+1)\cdots((k+1)n+\ell+1)}{(kn+2\ell+2)\cdots(kn+\ell+2)}-1\Big)\\ &\quad+\Big(\frac{2\cdot(kn+2r+1)\cdots(kn+r+1)}{((k-1)n+2r+2)\cdots((k-1)n+r+2)}-1\Big)\\ &\quad=\frac{2(k+1)^{\ell+1}}{k^{\ell+1}}+\frac{2k^{r+1}}{(k-1)^{r+1}}-2+\mathcal{O}\Big(\frac{1}{n}\Big).\end{aligned}$$

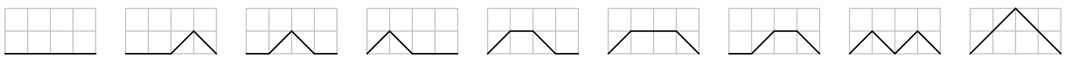
Chapter 8

Analysis of S-Motzkin and T-Motzkin Paths

This chapter is based on a paper by Prodinger, Selkirk, and Wagner [33]. It provides the analysis of S-Motzkin and T-Motzkin paths, which are bijective to 2_0-Dyck paths and 2_1-Dyck paths respectively.

8.1 Definitions

A Motzkin path is a non-negative lattice path with steps from the step set $\{\_, \diagup, \diagdown\}$ such that the path starts and ends on the x-axis. For example, the 9 Motzkin paths of length 4 are:

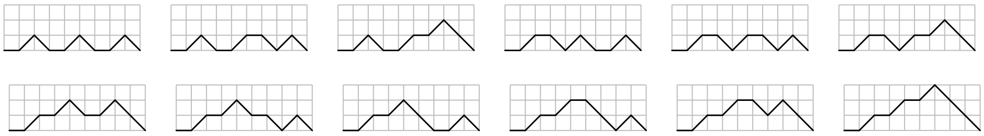


By placing further restrictions on Motzkin paths we obtain an interesting subclass.

Definition 8.1.1. *An* S-Motzkin path *is a Motzkin path of length* $3n$ *with* n *of each type of step such that the following conditions hold*

1. *The initial step must be* $\_$*, and*
2. $\_$ *and* $\diagup$ *steps alternate.*

This definition was inspired by a question at the 2018 International Mathematics Competition [27] involving restricted three-dimensional walks which can be translated into the two-dimensional S-Motzkin paths. These paths are enumerated by the generalised Catalan number, $\frac{1}{2n+1}\binom{3n}{n}$, and thus are bijective to ternary trees and non-crossing trees, as well as many other combinatorial objects [9, 14, 26, 29, 35]. As an example, the 12 S-Motzkin paths of length 9 are given below.

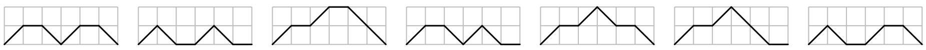


We define another subclass of Motzkin paths which is related to both S-Motzkin paths and ternary trees.

Definition 8.1.2. *A* T-Motzkin path *is a Motzkin path of length* $3n$ *with* n *of each type of step such that*

1. *The initial step is* $\diagup$, *and*
2. $\diagup$ *and* $\_$ *steps alternate.*

As an example, all T-Motzkin paths of length 6 are:



Note that although similar in definition, the class of T-Motzkin paths is larger than the class of S-Motzkin paths. Interchanging the $\_$ and $\diagup$ steps in an arbitrary S-Motzkin path provides a T-Motzkin path, but the converse is not true. T-Motzkin paths of length $3n$ are enumerated by $\frac{1}{n+1}\binom{3n+1}{n}$ and thus bijective to the class of ordered pairs of ternary trees introduced by Knuth [20]. There are several other equinumerous objects which can be found on the Online Encyclopedia of Integer Sequences A006013 [35].

Introducing another type of path is necessary for finding the symbolic decomposition of S-Motzkin and T-Motzkin paths, from which we can obtain functional equations. Therefore we define a *U-path* to be an S-Motzkin path without the initial $\_$ step. Symbolic equations for T-Motzkin paths and U-paths can be obtained in terms of each other by making use of a decomposition based on the first return of the path. Since S-Motzkin paths and U-paths are 'almost' the same, the generating function for S-Motzkin paths can be easily obtained from that of U-paths.

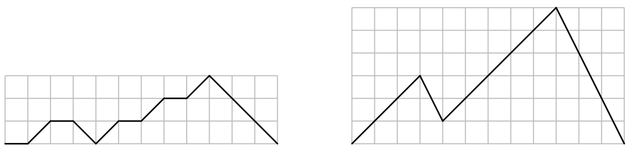
8.2 Bijections

8.2.1 S-Motzkin paths to 2_2-Dyck paths and T-Motzkin paths to 2_1-Dyck paths

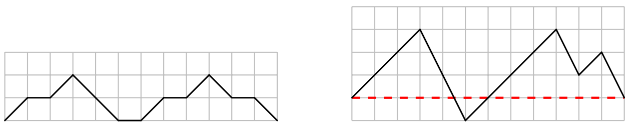
The mapping from both S-Motzkin and T-Motzkin paths to 2_0-Dyck paths and 2_1-Dyck paths respectively is very simple. All $\_$ and $\diagup$ steps become $(1,1)$ steps, and the $\diagdown$ steps become $(1,-2)$ steps.

The mapping from 2_0-Dyck paths and 2_1-Dyck paths to S-Motzkin and T-Motzkin paths respectively requires us to change every $(1,-2)$ step to a $\diagdown$ step, and the $(1,1)$ steps are changed to either $\_$ or $\diagup$ steps depending on the position of the $(1,1)$ step (in terms of the other $(1,1)$ steps) modulo 2. For S-Motzkin paths if the position of the $(1,1)$ step is even then it is an $\diagup$ step and if the position is odd is is a $\_$ step. For T-Motzkin paths if the position is even the $(1,1)$ step is mapped to a $\_$ step and if it is odd it is mapped to an $\diagup$ step.

For example, we have the following mapping from a S-Motzkin path of length 12 to a 2_0-Dyck path of length 12:



Similarly, we have the following mapping from a T-Motzkin path of length 12 to a 2_1-Dyck path of length 12:



8.2.2 S-Motzkin paths and ternary trees

A bijection between S-Motzkin paths of length $3n$ and ternary trees with n nodes is provided. This bijection can be found in [34].

S-Motzkin paths to ternary trees

We define $\varnothing$ to be the empty path. For an arbitrary S-Motzkin path $\mathcal{M}$, the canonical decomposition is given by

$$\Phi(\mathcal{M}) = \big(\mathcal{A},\, \mathcal{B},\, \mathcal{C}\big),$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ represent the S-Motzkin paths associated with the left, middle, and right subtrees respectively. Furthermore,

- $\mathcal{C}$ is the path from the penultimate to the final return of $\mathcal{M}$, with the initial and final step removed,
- $\mathcal{A}$ is the path from y to x (not including x), where x is the first $\_$ to the left of $\mathcal{C}$, y is a $\_$ step, and the path from y to x is a Motzkin path of maximal length, and
- $\mathcal{B}$ is what remains of $\mathcal{M}$ after removing the path from the penultimate to the final return of $\mathcal{M}$, as well as the path from y to x (including x).

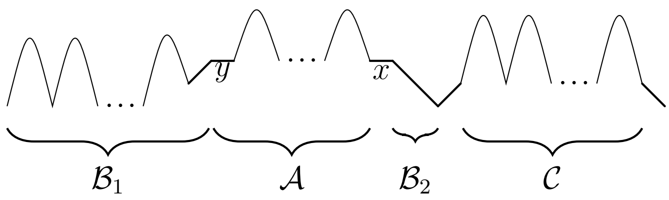


Figure 8.1: Canonical decomposition of an arbitrary S-Motzkin path

This process is performed recursively and terminates at an empty path. Note that each application of Φ adds one node and removes one of each type of step. This proves inductively that an S-Motzkin path of length $3n$ maps to a ternary tree with n (internal) nodes.

Ternary trees to S-Motzkin paths

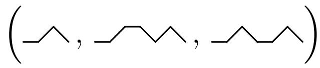
The inverse mapping is performed recursively on the end nodes as follows. Each node of a ternary tree has three (possibly empty) subtrees. Call the paths associated with the left, middle, and right subtrees $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ respectively.

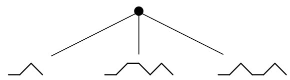
Starting at the end nodes, replace each node with $\mathcal{B}_1\,\mathcal{A}\,\_\,\mathcal{B}_2 \diagup \mathcal{C} \diagdown$, where $\mathcal{B}_1$ is the path from the start of $\mathcal{B}$ to the final $\diagup$ step of $\mathcal{B}$. The path $\mathcal{B}_2$ is what remains of $\mathcal{B}$ after removing $\mathcal{B}_1$. This process is continued recursively on each set of end nodes and terminates at the root to produce an S-Motzkin path. Note that for each node that is removed one of each type of step is added, and thus a ternary tree with n nodes produces an S-Motzkin path of length $3n$.

Example

As an example, we map the following S-Motzkin path into a ternary tree. Since the steps are reversible, the inverse mapping can be seen by reading the example in reverse. Let $\mathcal{M}$ be the S-Motzkin path

.

The canonical decomposition of $\mathcal{M}$ is then $\Phi(\mathcal{M}) =$ .
Hence



Continuing recursively:

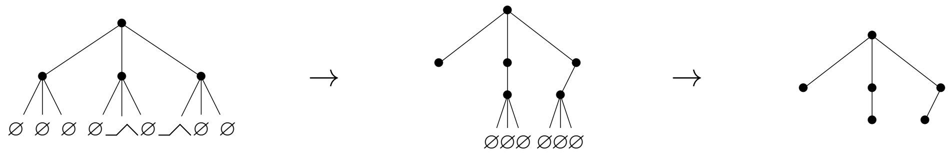
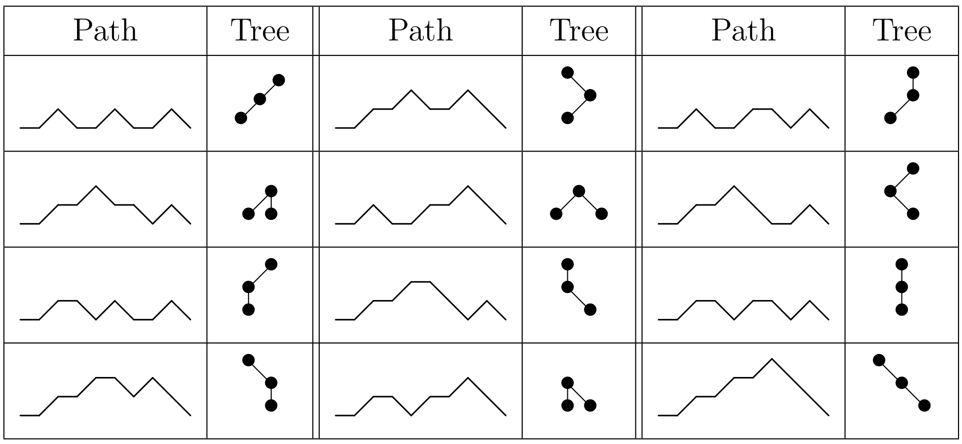


Table 8.1: Mapping from S-Motzkin paths to ternary trees for $n = 3$



8.2.3 T-Motzkin paths and pairs of ternary trees

T-Motzkin paths to pairs of ternary trees

Since a bijection between S-Motzkin paths and ternary trees is already provided, we show that every T-Motzkin path can be decomposed uniquely into an ordered pair of S-Motzkin paths (possibly including an empty path).

Given an arbitrary T-Motzkin path $\mathcal{N}$, we perform a canonical decomposition $\Omega(\mathcal{N}) = (\mathcal{A}, \mathcal{B})$ where

- $\mathcal{B}$ is the path from y to x (not including x) where x is the rightmost $\_$ step of $\mathcal{N}$, y is a $\_$ step, and the path from y to x is a Motzkin path of maximal length, and
- $\mathcal{A}$ is what remains of $\mathcal{N}$ after removing the path from y to x (including x), with an additional $\_$ step at the start of the path. In Figure 8.2 this is the path $\_\mathcal{A}_1\mathcal{A}_2$.

Note that both $\mathcal{A}$ and $\mathcal{B}$ are S-Motzkin paths.

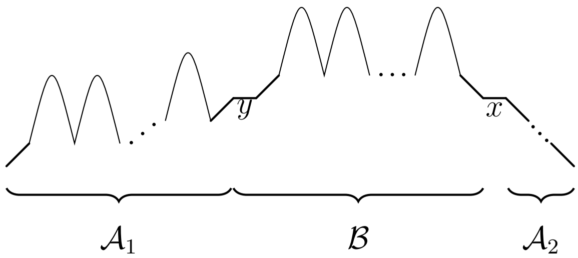


Figure 8.2: Canonical decomposition of an arbitrary T-Motzkin path

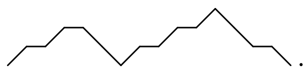
Pairs of ternary trees to T-Motzkin paths

Given an arbitrary pair of ternary trees, we can use the bijection given in Section 8.2.2 to obtain an ordered pair of S-Motzkin paths, $(\mathcal{A}, \mathcal{B})$. All S-Motzkin paths start with a $\_$ step and end in an $\diagup$ step followed by a series of $\diagdown$ steps. To obtain a T-Motzkin path from $(\mathcal{A}, \mathcal{B})$ we

- remove the initial $\_$ step from $\mathcal{A}$, and
- insert the path $\mathcal{B}\_$ immediately after the final $\diagup$ step of $\mathcal{A}$.

Example

We provide an example of the mapping from T-Motzkin paths to ternary trees. The inverse mapping can be seen by reading this example in reverse. Let $\mathcal{N}$ be



Then $\Omega(\mathcal{N})$ is given by

8.3 Analysis of parameters

8.3.1 Generating functions and related paths

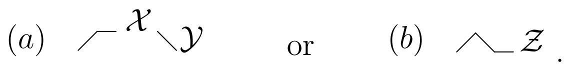
The symbolic decomposition for S-Motzkin and T-Motzkin paths given in this subsection have been included in [34].

Let $\mathcal{T}$ be the class of T-Motzkin paths, $\mathcal{U}$ be the class of U-paths, and

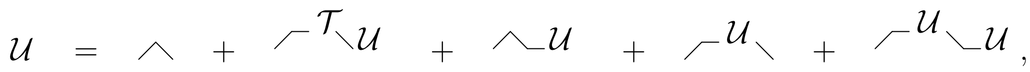
$$T(z) = \sum_{n\geq 0} t_n z^n \qquad \text{and} \qquad U(z) = \sum_{n\geq 0} u_n z^n$$

be their respective generating functions, where t_n and u_n represent the number of paths of length n in the given class.

We derive symbolic equations for the two types of paths based on a first return decomposition. Note that the only U-path of length less than five is given by $\wedge$. Taking into account the first return of a U-path, it is clear that a U-path of length five or more can be decomposed as either



In (a), $\mathcal{X}$ can either be a U-path or a T-Motzkin path. If $\mathcal{X}$ is a U-path, then $\mathcal{Y}$ is either empty or $\mathcal{Y}$ is a $\_$ step followed by a U-path. If $\mathcal{X}$ is a T-Motzkin path, then $\mathcal{Y}$ is a U-path. In (b), $\mathcal{Z}$ has to be a U-path. This then results in the symbolic equation



from which we obtain the equation

$$U(z) = z^2 + z^3T(z)U(z) + 2z^3U(z) + z^4U(z)^2. \tag{8.1}$$

Again, any T-Motzkin path of length $3n$ is either empty or, considering the first return of the path, of the form (a) or (b) as given in the U-path case.

In (a), $\mathcal{X}$ can either be a U-path or a T-Motzkin path. If $\mathcal{X}$ is a U-path, then an 'extra' $\_$ step needs to appear in $\mathcal{Y}$, and thus $\mathcal{Y}$ is given by a $\_$ step followed by a T-Motzkin path. If $\mathcal{X}$ is a T-Motzkin path, then $\mathcal{Y}$ is also a T-Motzkin path. With analogous reasoning we can see that for (b) the only possibility for $\mathcal{Z}$ is a T-Motzkin path. Using this we obtain the symbolic equation

which results in the equation

$$T(z) = 1 + z^3T(z)^2 + z^3T(z) + z^4U(z)T(z). \tag{8.2}$$

Solving the system of equations given in (8.1) and (8.2) yields

$$\begin{aligned} T(z) &= 1 + 2z^3T(z)^2 - z^6T(z)^3, \\ U(z) &= z^2 + 3z^3U(z) + 3z^4U(z)^2 + z^5U(z)^3 \end{aligned}$$

which, with substitutions, is amenable to application of the Lagrange Inversion Theorem (Theorem 2.2.1). To demonstrate this, consider the equation

$$T(z) = 1 + 2z^3T(z)^2 - z^6T(z)^3.$$

This can be factorised as $T(z)(1-z^3T(z))^2 = 1$, and with substitutions $R = z^3T(z)$ and $x = z^3$ we find that $x = R(1-R)^2$. Therefore

$$[z^{3n}]T(z) = [x^{n+1}]R = \frac{1}{n+1}[w^n]\frac{1}{(1-w)^{2n+2}} = \frac{1}{n+1}\binom{3n+1}{n},$$

which results in

$$T(z) = \sum_{n\geq 0}\frac{1}{n+1}\binom{3n+1}{n}z^{3n} \text{ and similarly } U(z) = \sum_{n\geq 1}\frac{1}{2n+1}\binom{3n}{n}z^{3n-1}.$$

Since U-paths are S-Motzkin paths without the initial horizontal step, the generating function for S-Motzkin paths is given by $S(z) = \sum_{n\geq 1}\frac{1}{2n+1}\binom{3n}{n}z^{3n}$. Note that $(1+S(z))^2 = T(z)$, which was pointed out by Knuth in his 2014 Christmas lecture [20]. We have also proved this by means of the bijection provided in Section 8.2.3.

8.3.2 Analysis of various parameters

In this section the analysis of the number of returns is done in detail, and results for the number of peaks, the number of valleys, and the number of valleys on the x-axis can be done similarly. The study of these parameters in Dyck paths can be found in [9, 24].

The number of returns

From the generating functions for U-paths and T-Motzkin paths along with the substitutions $x = z^3$ and $x = t(1-t)^2$, we obtain

$$T(z) = \frac{1}{(1-t)^2} \qquad \text{and} \qquad S(z) = \frac{t}{1-t}.$$

We introduce the variable u to count the number of returns, and also count the right end of a $\_$ step on the x-axis as a return in this context. From the symbolic equations for U-paths and T-Motzkin paths we obtain the bivariate generating functions, where $T(z,1) = T(z)$ and $S(z,1) = S(z)$:

$$\begin{aligned} S(z,u) &= u^2z^3 + uz^3S(z,u)T(z,1) + u^2z^3S(z,u) + u^2z^3S(z,1) \\ &\quad + u^2z^3S(z,1)S(z,u), \\ T(z,u) &= 1 + uz^3T(z,1)T(z,u) + u^2z^3T(z,u) + u^2z^3S(z,1)T(z,u). \end{aligned}$$

Solving this system of equations we find that

$$S(z,u) = \frac{(1-t)tu^2}{1-tu-tu^2+t^2u^2} \qquad \text{and} \qquad T(z,u) = \frac{1}{1-tu-tu^2+t^2u^2}.$$

In the sections that follow some simplifications occur when calculating variances. These are discussed in more detail in Section 8.3.3.

To determine the average number of returns we calculate the derivative of $S(z,u)$ and $T(z,u)$ with respect to u,

$$\frac{\partial}{\partial u}S(z,u)\Big|_{u=1} = \frac{(2-t)t}{(1-t)^3} \qquad \text{and} \qquad \frac{\partial}{\partial u}T(z,u)\Big|_{u=1} = \frac{t(3-2t)}{(1-t)^4}.$$

The total number of returns in all paths of length $3n$ is then obtained by extracting the coefficients of these expressions by means of Cauchy's integral formula. For S-Motzkin paths this results in

$$\begin{aligned}
[x^n]\frac{(2-t)t}{(1-t)^3} &= \frac{1}{2\pi i}\oint \frac{1}{(t(1-t)^2)^{n+1}} \cdot \frac{t(2-t)}{(1-t)^3} \cdot (1-t)(1-3t)\,dt \\
&= \frac{1}{2\pi i}\oint \frac{1}{t^n} \cdot \frac{2-7t+3t^2}{(1-t)^{2n+4}}\,dt = [t^{n-1}]\frac{2-7t+3t^2}{(1-t)^{2n+4}} \\
&= 2\binom{3n+2}{n-1} - 7\binom{3n+1}{n-2} + 3\binom{3n}{n-3},
\end{aligned}$$

and for T-Motzkin paths we obtain

$$[x^n]\frac{t(3-2t)}{(1-t)^4} = 3\binom{3n+3}{n-1} - 11\binom{3n+2}{n-2} + 6\binom{3n+1}{n-3}.$$

Therefore in S-Motzkin paths the average number of returns for paths of length $3n$ is

$$\frac{2\binom{3n+2}{n-1} - 7\binom{3n+1}{n-2} + 3\binom{3n}{n-3}}{\frac{1}{2n+1}\binom{3n}{n}} = \frac{n(23n+17)}{2(2n+3)(n+1)} = \frac{23}{4} - \frac{81}{8n} + \mathcal{O}\Big(\frac{1}{n^2}\Big)$$

and for T-Motzkin paths the average number of returns is

$$\frac{3\binom{3n+3}{n-1} - 11\binom{3n+2}{n-2} + 6\binom{3n+1}{n-3}}{\frac{1}{n+1}\binom{3n+1}{n}} = \frac{(19n+26)n}{2(2n+3)(n+2)} = \frac{19}{4} - \frac{81}{8n} + \mathcal{O}\Big(\frac{1}{n^2}\Big).$$

To calculate the variance in the number of returns for paths of length $3n$, we find the second derivatives of $S(z,u)$ and $T(z,u)$ with respect to u:

$$\frac{\partial^2}{(\partial u)^2}S(z,u)\Big|_{u=1} = \frac{2t(1+3t-4t^2+t^3)}{(1-t)^5}$$

and

$$\frac{\partial^2}{(\partial u)^2}T(z,u)\Big|_{u=1} = \frac{2t(1+6t-9t^2+3t^3)}{(1-t)^6}.$$

We again determine the coefficients using Cauchy's integral formula,

$$[x^n]\frac{2t(1+3t-4t^2+t^3)}{(1-t)^5} = 2\binom{3n+4}{n-1} - 26\binom{3n+2}{n-3} + 26\binom{3n+1}{n-4} - 6\binom{3n}{n-5}$$

and

$$[x^n]\frac{2t(1+6t-9t^2+3t^3)}{(1-t)^6} = 2\binom{3n+5}{n-1} + 6\binom{3n+4}{n-2} - 54\binom{3n+3}{n-3} + 60\binom{3n+2}{n-4} - 18\binom{3n+1}{n-5},$$

with which we find that the variance for the number of returns for S-Motzkin paths of length $3n$ is

$$\frac{2(313n^3+652n^2+53n-178)n}{(2n+5)(2n+4)(2n+3)(2n+2)} + \frac{n(23n+17)}{2(2n+3)(n+1)} - \Big(\frac{n(23n+17)}{2(2n+3)(n+1)}\Big)^2 = \frac{3(14n^2+31n+8)(3n+2)(3n+1)(n-1)n}{4(2n+5)(2n+3)^2(n+2)(n+1)^2}.$$

Similarly, the variance for the number of returns for T-Motzkin paths of length $3n$ is given by

$$\frac{3(79n^3+252n^2+91n-142)n}{2(2n+5)(2n+3)(n+3)(n+2)} + \frac{(19n+26)n}{2(2n+3)(n+2)} - \Big(\frac{(19n+26)n}{2(2n+3)(n+2)}\Big)^2 = \frac{3(14n^3+45n^2+19n-18)(3n+4)(3n+2)n}{4(2n+5)(2n+3)^2(n+3)(n+2)^2}.$$

Limiting distributions

We have defined t implicitly by $t(1-t)^2 = x$. It is well known that this type of implicit equation leads to a square root singularity [12, Section VII.4]. In this particular case, the singularity occurs at $x = \frac{4}{27}$ with $t = \frac{1}{3}$, where $\frac{d}{dt}t(1-t^2) = (1-t)(1-3t) = 0$. At this point, the singular expansion of t with respect to x is

$$t = \frac{1}{3} - \frac{2}{3\sqrt{3}}\Big(1-\frac{27x}{4}\Big)^{1/2} + \mathcal{O}\Big(1-\frac{27x}{4}\Big).$$

The generating function for the number of returns in S-Motzkin paths is given by

$$S(z,u) = \frac{(1-t)tu^2}{1-tu-tu^2+t^2u^2}.$$

Note that for $|x| \leq \frac{4}{27}$ and $|u| \leq 1$, we have $|t| \leq \frac{1}{3}$ and thus

$$|1 - tu - tu^2 + t^2u^2| \geq 1 - |t||u| - |t||u|^2 - |t|^2|u|^2 \geq 1 - \frac{1}{3} - \frac{1}{3} - \frac{1}{9} = \frac{2}{9} > 0,$$

so the denominator is non-zero and the singularity of t remains the dominant singularity. This generating function has the Taylor expansion (with substitution of the singular expansion of t):

$$\begin{aligned} S(z,u) &= \frac{2u^2}{9 - 3u - 2u^2} + \frac{9u^2}{27 - 27u + 4u^3}\Big(t - \frac{1}{3}\Big) + \mathcal{O}\Big(\Big(t - \frac{1}{3}\Big)^2\Big) \\ &= \frac{2u^2}{9 - 3u - 2u^2} - \frac{2\sqrt{3}\,u^2}{27 - 27u + 4u^3}\Big(1 - \frac{27x}{4}\Big)^{\frac{1}{2}} + \mathcal{O}\Big(1 - \frac{27x}{4}\Big). \end{aligned}$$

Applying singularity analysis [12, Section VI], we obtain

$$[x^n]S(z,u) \sim \frac{2\sqrt{3}\,u^2}{27 - 27u + 4u^3} \cdot \frac{1}{2\sqrt{\pi}} \cdot n^{-3/2}\Big(\frac{27}{4}\Big)^n.$$

Therefore, the probability generating function for the number of returns in S-Motzkin paths of length $3n$, which is given by $[x^n]S(x,u)/[x^n]S(x,1)$, converges to

$$\frac{4u^2}{27 - 27u + 4u^3} = \frac{4u^2}{(2u-3)^2(u+3)} = \frac{4}{27}u^2 + \frac{4}{27}u^3 + \frac{4}{27}u^4 + \frac{92}{729}u^5 + \cdots.$$

By [12, Theorem IX.1], the distribution of the number of returns in S-Motzkin paths converges to the discrete distribution given by this probability generating function. The probability that the number of returns is precisely k converges to

$$[u^k]\frac{4u^2}{(2u-3)^2(u+3)} = \frac{4}{3^{k+3}}(3k \cdot 2^{k-1} - 2^k + (-1)^k).$$

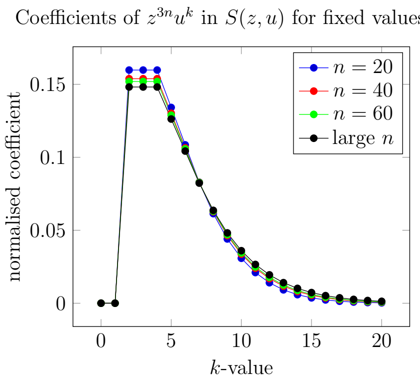
In a similar manner, we find that the limiting probability generating function for the number of returns in T-Motzkin paths of length $3n$ is given by

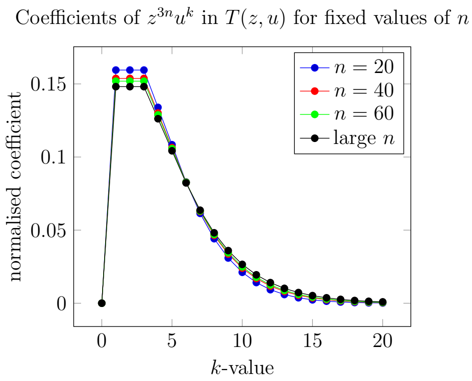
$$\frac{4u}{(2u-3)^2(u+3)} = \frac{4}{27}u + \frac{4}{27}u^2 + \frac{4}{27}u^3 + \frac{92}{729}u^4 + \frac{76}{729}u^5 + \cdots,$$

and the probability that the number of returns is precisely k converges to

$$[u^k]\frac{4u}{(2u-3)^2(u+3)} = \frac{4}{3^{k+4}}(3k \cdot 2^k + 2^k - (-1)^k).$$

The convergence in both cases is demonstrated in the figures below.





The number of peaks

There are two possible types of peaks:

$$(1) \quad \wedge \qquad \text{and} \qquad (2) \quad \diagup\!\!\overline{}\!\!\diagdown$$

We first consider peaks of type (1) and again use the variable u to count them. Then from the symbolic equations

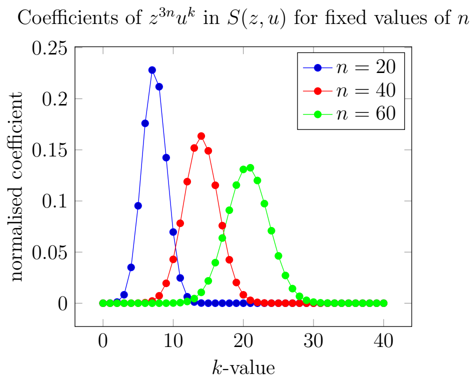
$$S(z,u) = uz^3 + z^3T(z,u)S(z,u) + uz^3S(z,u) + z^3S(z,u) + z^3S(z,u)^2,$$
$$T(z,u) = 1 + z^3T(z,u)^2 + uz^3T(z,u) + z^3S(z,u)T(z,u)$$

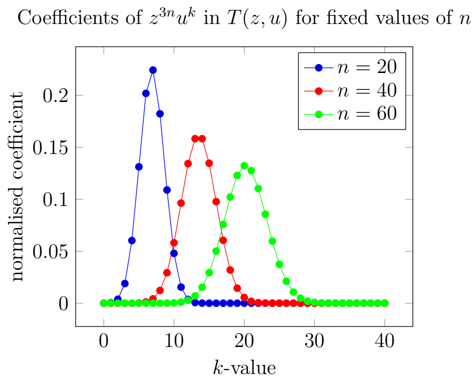
we obtain the results in Table 8.2.

Table 8.2: Results for peaks of type (1)

$K(z,u)$	$S(z,u)$	$T(z,u)$
$\frac{\partial}{\partial u}K(z,u)\Big\vert_{u=1}$	$\frac{t(1-2t)}{(1-3t)(1-t)}$	$\frac{t}{(1-3t)(1-t)}$
$[x^n]\frac{\partial}{\partial u}K(z,u)\Big\vert_{u=1}$	$\binom{3n}{n-1} - 2\binom{3n-1}{n-2}$	$\binom{3n}{n-1}$
Mean	$\frac{n}{3} + \frac{2}{3}$	$\frac{n(n+1)}{3n+1}$
$\frac{\partial^2}{(\partial u)^2}K(z,u)\Big\vert_{u=1}$	$\frac{2t^2(1-5t+8t^2-3t^3)}{(1-3t)^3(1-t)}$	$\frac{2t^2(1-2t)}{(1-3t)^3(1-t)}$
$[x^n]\frac{\partial^2}{(\partial u)^2}K(z,u)\Big\vert_{u=1}$	$\binom{3n-2}{n-3}\frac{n(n+3)}{(n-2)}$	$\binom{3n-1}{n-2}n$
Variance	$\frac{2(2n+1)(n-1)}{9(3n-1)} \sim \frac{4}{27}n$	$\frac{2(2n+1)(n+1)n}{3(3n+1)^2} \sim \frac{4}{27}n$

The system of equations for $S(z,u)$ and $T(z,u)$ satisfies the technical conditions of [10], where it is shown that we have convergence to a normal law in a rather general setting. By the main result of [10], the number of peaks (of both types) asymptotically follows a Gaussian distribution.





We now consider peaks of type (2), and again use the variable u to count them. From the symbolic equations we obtain

$$\begin{aligned}
S(z,u) &= z^3 + z^3T(z,u)S(z,u) + uz^3S(z,u) + z^3S(z,u) + z^3S(z,u)^2,\\
T(z,u) &= 1 + z^3T(z,u)^2 + uz^3T(z,u) + z^3S(z,u)T(z,u).
\end{aligned}$$

Note here that $\mathcal{T}$ contains an empty path. As a result, for paths of the form

$$\diagup^{\mathcal{T}}\diagdown\mathcal{U} \qquad \text{and} \qquad \diagup^{\mathcal{T}}\diagdown\mathcal{T}\ ,$$

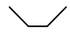
we obtain $uz^3S(z,u)$ and $uz^3T(z,u)$ respectively. If the path is not empty we obtain $z^3(T(z,u)-1)S(z,u)$ and $z^3(T(z,u)-1)T(z,u)$. Using the generating function equations we obtain the results in Table 8.3.

Table 8.3: Results for peaks of type (2)

$K(z,u)$	$S(z,u)$	$T(z,u)$
$\frac{\partial}{\partial u}K(z,u)\Big\vert_{u=1}$	$\frac{t^2}{1-3t}$	$\frac{t}{1-3t}$
$[x^n]\frac{\partial}{\partial u}K(z,u)\Big\vert_{u=1}$	$\binom{3n-2}{n-2}$	$\binom{3n-1}{n-1}$
Mean	$\frac{(2n+1)(n-1)}{3(3n-1)} \sim \frac{2}{9}n$	$\frac{(2n+1)(n+1)}{3(3n+1)} \sim \frac{2}{9}n$
$\frac{\partial^2}{(\partial u)^2}K(z,u)\Big\vert_{u=1}$	$\frac{2(1-2t)(1-t)t^3}{(1-3t)^3}$	$\frac{2(1-3t+3t^2)(1-t)t^2}{(1-3t)^3}$
$[x^n]\frac{\partial^2}{(\partial u)^2}K(z,u)\Big\vert_{u=1}$	$\binom{3n-3}{n-3}\frac{2n}{3}$	$\binom{3n-3}{n-2}n$
Variance	$\frac{2(10n^2-11n+2)(2n+1)(n-1)}{9(3n-1)^2(3n-2)}$ $\sim \frac{40}{243}n$	$\frac{2(30n^3-23n^2-3n+2)(2n+1)(n+1)}{9(3n+1)^2(3n-1)(3n-2)}$ $\sim \frac{40}{243}n$

Valleys

There are two possible types of valleys:

(1)  and (2) 

For valleys of type (1), using the variable u to count them we obtain generating function equations

$$S(z,u) = z^3 + uz^3T(z,u)S(z,u) + z^3S(z,u) + z^3S(z,u) + z^3S(z,u)^2,$$
$$T(z,u) = 1 + uz^3T(z,u)(T(z,u)-1) + 2z^3T(z,u) + z^3S(z,u)T(z,u).$$

This is taking into account that the empty T-Motzkin path does not contribute a valley of type (1). From the generating function equations we obtain the results in Table 8.4.

Table 8.4: Results for valleys of type (1)

$K(z,u)$	$S(z,u)$	$T(z,u)$
$\frac{\partial}{\partial u}K(z,u)\Big\vert_{u=1}$	$\frac{t^2}{(1-3t)(1-t)}$	$\frac{2t^2}{(1-3t)(1-t)^2}$
$[x^n]\frac{\partial}{\partial u}K(z,u)\Big\vert_{u=1}$	$\binom{3n-1}{n-2}$	$2\binom{3n}{n-2}$
Mean	$\frac{n}{3}-\frac{1}{3}$	$\frac{n(n-1)}{3n+1}$
$\frac{\partial^2}{(\partial u)^2}K(z,u)\Big\vert_{u=1}$	$\frac{2(1-t-3t^2)t^3}{(1-3t)^3(1-t)}$	$\frac{2(2-t-9t^2)t^3}{(1-3t)^3(1-t)^2}$
$[x^n]\frac{\partial^2}{(\partial u)^2}K(z,u)\Big\vert_{u=1}$	$(n-1)\binom{3n-2}{n-3}$	$\frac{(n-1)(n-2)}{(n+1)}\binom{3n-1}{n-2}$
Variance	$\frac{2(2n+1)(n-1)}{9(3n-1)} \sim \frac{4}{27}n$	$\frac{2(2n+1)(n+1)(n-1)}{3(3n+1)^2} \sim \frac{4}{27}n$

As in our analysis of peaks in the previous subsection, we can apply the main result of [10] to prove that the number of valleys (of both types) asymptotically follows a Gaussian distribution.

The generating function equations for valleys of type (2), again using u to count the number of valleys, are given by

$$\begin{aligned} S(z,u) &= z^3 + z^3T(z,u)S(z,u) + uz^3S(z,u) + z^3S(z,u) + uz^3S(z,u)^2, \\ T(z,u) &= 1 + z^3T(z,u)^2 + uz^3(T(z,u)-1) + z^3 + uz^3S(z,u)(T(z,u)-1) \\ &\quad + z^3S(z,u). \end{aligned}$$

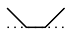
Again, we take into account the absence of a valley in the case of an empty T-Motzkin path. The equations yield the results in Table 8.5.

Table 8.5: Results for valleys of type (2)

$K(z,u)$	$S(z,u)$	$T(z,u)$
$\frac{\partial}{\partial u}K(z,u)\Big\vert_{u=1}$	$\frac{t^2}{(1-3t)}$	$\frac{2t^2}{(1-3t)(1-t)}$
$[x^n]\frac{\partial}{\partial u}K(z,u)\Big\vert_{u=1}$	$\binom{3n-2}{n-2}$	$2\binom{3n-1}{n-2}$
Mean	$\frac{(n-1)(2n+1)}{3(3n-1)} \sim \frac{2}{9}n$	$\frac{2(n+1)(n-1)}{3(3n+1)} \sim \frac{2}{9}n$
$\frac{\partial^2}{(\partial u)^2}K(z,u)\Big\vert_{u=1}$	$\frac{2(1-2t)(1-t)t^3}{(1-3t)^3}$	$\frac{2(2-3t-3t^2)t^3}{(1-3t)^3}$
$[x^n]\frac{\partial^2}{(\partial u)^2}K(z,u)\Big\vert_{u=1}$	$\frac{2n}{3}\binom{3n-3}{n-3}$	$2(n-1)\binom{3n-3}{n-3}$
Variance	$\frac{2(10n^2-11n+2)(2n+1)(n-1)}{9(3n-1)^2(3n-2)}$ $\sim \frac{40}{243}n$	$\frac{4(15n^2-19n+8)(2n+1)(n+1)(n-1)}{9(3n+1)^2(3n-1)(3n-2)}$ $\sim \frac{40}{243}n$

Valleys on the x-axis

We now consider valleys that lie on the x-axis. Keeping the two types of valleys discussed in the previous subsection, a valley of type (1) contributes one return, and a valley of type (2) contributes two returns.

(1)  and (2) 

For valleys on the x-axis of type (1), using the variable u to count them we obtain generating function equations

$$S(z,u) = z^3 + uz^3T(z,1)S(z,u) + z^3S(z,u) + z^3S(z,1) + z^3S(z,u)S(z,1),$$
$$T(z,u) = 1 + uz^3T(z,1)(T(z,u)-1) + z^3T(z,1) + z^3T(z,u) + z^3S(z,1)T(z,u).$$

Let $v_1 = 30n^3 + 43n^2 + 154n + 288$ and

$$v_2 = 778n^6 + 3953n^5 + 11212n^4 + 24373n^3 + 30064n^2 + 16260n + 2160,$$

then we obtain the results in Table 8.6.

In a similar manner to that of Section 8.3.2, we find that the limiting probability generating function for valleys of type (1) on the x-axis are given by $\frac{4(u+3)}{(7-3u)^2}$ for S-Motzkin paths, and $\frac{4(u+11)}{3(7-3u)}$ for T-Motzkin paths (both of length $3n$). Therefore the probability that the number of valleys of type (1) on the x-axis in S-Motzkin paths of length $3n$ is exactly k is

$$[u^k]\frac{4(u+3)}{(7-3u)^2} = \frac{4}{7^2}k\left(\frac{3}{7}\right)^{k-1} + \frac{12}{7^2}(k+1)\left(\frac{3}{7}\right)^k = \left(\frac{3}{7}\right)^k\left(\frac{112}{147}k + \frac{12}{49}\right),$$

and the probability that the number of valleys of type (1) on the x-axis in T-Motzkin paths of length $3n$ is exactly k is

$$[u^k]\frac{4(u+11)}{3(7-3u)} = \frac{4}{21}\left(\frac{3}{7}\right)^{k-1} + \frac{44}{21}\left(\frac{3}{7}\right)^k = \frac{160}{63}\left(\frac{3}{7}\right)^k.$$

Table 8.6: Results for valleys on the x-axis of type (1)

$K(z,u)$	$S(z,u)$	$T(z,u)$
$\frac{\partial}{\partial u}K(z,u)\Big\vert_{u=1}$	$\frac{t^2}{(1-t)^3}$	$\frac{(2-t)t^2}{(1-t)^4}$
$[x^n]\frac{\partial}{\partial u}K(z,u)\Big\vert_{u=1}$	$\binom{3n+1}{n-2}-3\binom{3n}{n-3}$	$2\binom{3n+2}{n-2}-7\binom{3n+1}{n-3}+3\binom{3n}{n-4}$
Mean	$\frac{7(n-1)n}{2(2n+3)(n+1)}\sim\frac{7}{4}$	$\frac{(19n+18)(n-1)n}{2(3n+1)(2n+3)(n+2)}\sim\frac{19}{12}$
$\frac{\partial^2}{(\partial u)^2}K(z,u)\Big\vert_{u=1}$	$\frac{2t^3}{(1-t)^5}$	$\frac{2(2-t)t^3}{(1-t)^6}$
$[x^n]\frac{\partial^2}{(\partial u)^2}K(z,u)\Big\vert_{u=1}$	$2\binom{3n+2}{n-3}-6\binom{3n+1}{n-4}$	$4\binom{3n+3}{n-3}-14\binom{3n+2}{n-4}+6\binom{3n+1}{n-5}$
Variance	$\frac{v_1(3n+1)(n-1)n}{4(2n+5)(2n+3)^2(n+2)(n+1)^2}$ $\sim\frac{45}{16}$	$\frac{v_2(n-1)n}{4(3n+1)^2(2n+5)(2n+3)^2(n+3)(n+2)^2}$ $\sim\frac{389}{144}$

For valleys on the x-axis of type (2), again using u to count the number of valleys, we obtain the generating function equations

$$
\begin{aligned}
S(z,u) &= z^3 + z^3T(z,1)S(z,u) + uz^3S(z,u) + z^3S(z,1) + uz^3S(z,u)S(z,1),\\
T(z,u) &= 1 + z^3T(z,u)T(z,1) + z^3 + uz^3(T(z,u)-1) + uz^3S(z,1)(T(z,u)-1)\\
&\qquad + z^3S(z,1).
\end{aligned}
$$

From these the results in Table 8.7 are obtained.

Table 8.7: Results for valleys on the x-axis of type (2)

$K(z,u)$	$S(z,u)$	$T(z,u)$
$\frac{\partial}{\partial u}K(z,u)\Big\vert_{u=1}$	$\frac{t^2}{(1-t)^2}$	$\frac{(2-t)t^2}{(1-t)^3}$
$[x^n]\frac{\partial}{\partial u}K(z,u)\Big\vert_{u=1}$	$\binom{3n}{n-2}-3\binom{3n-1}{n-3}$	$2\binom{3n+1}{n-2}-7\binom{3n}{n-3}+3\binom{3n-1}{n-4}$
Mean	$\frac{n-1}{n+1}\sim 1$	$\frac{(11n+6)(n-1)}{2(3n+1)(2n+3)}\sim\frac{11}{12}$
$\frac{\partial^2}{(\partial u)^2}K(z,u)\Big\vert_{u=1}$	$\frac{2t^3}{(1-t)^3}$	$\frac{2(2-t)t^3}{(1-t)^4}$
$[x^n]\frac{\partial^2}{(\partial u)^2}K(z,u)\Big\vert_{u=1}$	$2\binom{3n}{n-3}-6\binom{3n-1}{n-4}$	$4\binom{3n+1}{n-3}-14\binom{3n}{n-4}+6\binom{3n-1}{n-5}$
Variance	$\frac{(3n+1)(n-1)n}{(2n+3)(n+1)^2}\sim\frac{3}{2}$	$\frac{(203n^3+437n^2+268n+12)(n-1)n}{4(3n+1)^2(2n+3)^2(n+2)}\sim\frac{203}{144}$

The limiting probability generating functions for the number of valleys of type (2) on the x-axis is given by $\frac{4}{(3-u)^2}$ for S-Motzkin paths and $\frac{13-u}{3(3-u)^2}$ for T-Motzkin paths (both of length $3n$). Therefore the probability that the number of valleys of type (2) on the x-axis in S-Motzkin paths is exactly k is equal to

$$[u^k]\frac{4}{(3-u)^2} = (k+1)\frac{4}{3^{k+2}},$$

and the probability that the number of valleys of type (2) on the x-axis in T-Motzkin paths is exactly k is

$$[u^k]\frac{13-u}{3(3-u)^2} = \frac{13}{27}(k+1)\frac{1}{3^k} - \frac{k}{27}\frac{1}{3^{k-1}} = \frac{10k+13}{3^{k+3}}.$$

8.3.3 Identities

Although not shown, in Sections 8.3.2 and 8.3.2 the coefficients of the generating functions used to find the variance were greatly simplified by using derivatives (compared to extracting coefficients using Cauchy's integral formula). This is similar to the simplifications that occur in equations (6.4) and (6.6). An example of the simplifications in Sections 8.3.2 and 8.3.2 is given: Using Lemma 6.2.1 for functions of type 3 with $k = 2$ we obtain the coefficients

$$[x^n]\frac{2t^2(1-2t)}{(1-3t)^3(1-t)} = 2\sum_{k\geq 0}(k+1)3^k\left[\binom{3n-k-1}{n-k-2} - 2\binom{3n-k-2}{n-k-3}\right].$$

On the other hand, the generating function can be expressed as a derivative, and by using the formula $\frac{t^3}{1-3t} = \sum_{n\geq 3}\binom{3n-3}{n-3}x^n$ we find that

$$[x^n]\frac{2t^2(1-2t)}{(1-3t)^3(1-t)} = [x^n]\frac{2}{3}\cdot\frac{d}{dx}\frac{t^3}{1-3t} = \binom{3n-1}{n-2}n.$$

It follows that

$$2\sum_{k\geq 0}(k+1)3^k\left[\binom{3n-k-1}{n-k-2} - 2\binom{3n-k-2}{n-k-3}\right] = \binom{3n-1}{n-2}n,$$

which is a special case of the more general identity

$$2\sum_{k\geq j}3^k(k+1)\left[\binom{3n-k-1}{n-k-2} - 2\binom{3n-k-2}{n-k-3}\right] = \binom{3n-j-1}{n-j-2}(n+j)3^j.$$

This and other beautiful identities such as

$$2\sum_{k\geq 0}3^k(k+2i)\binom{3n-k+i-4}{n-k-i-1} = \binom{3n+i-3}{n-i}(n-i)$$

and

$$2\sum_{k\geq 0}3^k(k+2i+1)\binom{3n-k+i-2}{n-k-i-1} = \binom{3n+i-1}{n-i}(n-i)$$

can be proved directly by induction.

Generally speaking, the coefficients of higher moments of bivariate generating functions used in the analysis of parameters of other combinatorial objects do not usually simplify from sums of binomial coefficients to single binomial coefficients. For example, the simplification in Section 6.2.2 of

$$\begin{aligned}
&(2k+t)\sum_{m=0}^{n-2}\binom{(k+1)n+t-m-3}{n-m-2}(m+1)(k+1)^m \\
&-(k+1)(k+t)\sum_{m=0}^{n-3}\binom{(k+1)n+t-m-4}{n-m-3}(m+1)(k+1)^m \\
&=(kn+t)\binom{(k+1)n+t-2}{n-2},
\end{aligned}$$

is unexpected and is unique to that exact linear combination of summations. That is, a minor change such as the change of the factor $2k+t$ to $2k+t+1$ would make the expression unable to be simplified. In the case that we have dealt with, it can be shown that higher derivatives (with respect to u and at $u=1$) of any bivariate generating function $K_t(x,u)$ where x represents the length of a k_t-Dyck path and u represents some parameter of interest will simplify to a single binomial coefficient. An explanation and algorithm for simplifying these will be discussed in further work.

Conclusion and Future Work

We have determined the enumeration of k_t-Dyck paths and studied parameters related to these paths. These k_t-Dyck paths and their parameters can be used to study a family of walks in the quarter plane in a very simple way in comparison to the usual complexity of the study of walks in the quarter plane. A number of natural research questions related to the study of k_t-Dyck paths, walks in the quarter plane, or other content of this thesis arise:

- In Chapter 7 we considered walks in the quarter plane with the step set $\{(1, 1), (1, -k + 1), (-k, 0)\}$. It seems as if this can be generalised to $\{(1, 1), (1, -a), (-b, 0)\}$, which would give a decomposition into a_t-Dyck and b_t-Dyck paths. These details need to be worked out.
- What further statistics related to the walks in the quarter plane discussed in Chapter 7 can be calculated by means of the bijection or the kernel method?
- How do ordered pairs of other types of lattice paths with a fixed step set 'compose' into walks in the quarter plane? Can anything be said about these paths or statistics related to these paths generally? The general case where the starting and ending point is the origin has been done already, but can this be done for an arbitrary starting and ending point?
- The identities for S-Motzkin and T-Motzkin (or 2_0-Dyck and 2_1-Dyck) paths discussed in Subsection 8.3.3 can be generalised to k_t-Dyck paths. These details need to be worked out, as well as a general explanation and algorithm for the simplifications which occur.
- In Chapter 3 we were able to determine all roots of polynomials of a particular form in terms of generalised binomial series. Are there any other families of polynomials whose roots can be determined in terms of generalised binomial series, or a similar family of series?

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