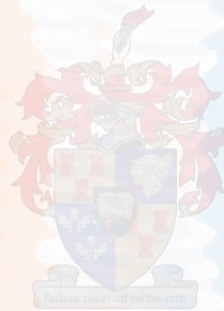

Thermalisation

OF A TWO-SPECIES CONDENSATE COUPLED TO A BOSONIC BATH

Author:
Jan Cillié Louw

Supervisor:
Prof. Michael Kastner
Co-Supervisor:
Dr. Johannes Kriel

April 2019



Thesis presented in partial fulfillment of the requirements for the degree of Master of Science in
Theoretical Physics in the Faculty of Science at Stellenbosch University

Declaration

By submitting this thesis electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the sole author thereof (save to the extent explicitly otherwise stated), that reproduction and publication thereof by Stellenbosch University will not infringe any third party rights and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Date April 2019

Abstract

Motivated by recent experiments, we study the time evolution of a two-species Bose-Einstein condensate which is coupled to a bosonic bath. For the particular condensate, unconventional thermodynamics have recently been predicted. To study these thermal properties we find the conditions under which this *open* quantum system thermalises—equilibrates to the Gibbs state describing the canonical ensemble.

The condensate is mapped from its bosonic representation, describing N interacting bosons, onto a Schwinger spin representation, with spin angular momentum $S = 2N$. The corresponding Hamiltonian takes the form of a Lipkin-Meshkov-Glick (LMG) model. We find that the total system-bath Hamiltonian is too difficult to solve. Fortunately, in the case where the LMG model is only weakly coupled to a near-memoryless bath, we may derive an approximate differential equation describing the LMG model's evolution. Upon further approximations this equation describes a quantum dynamical semigroup—preserving all the rules of quantum mechanics. The generator of this semigroup, known as the Lindbladian, is our main object of interest. Its nullspace is populated by the stationary state(s), which we wish to compare to the canonical ensemble. If the only state in this nullspace is the Gibbs state, then the system will thermalise to that state.

For extensive bath temperatures we show that the Gibbs state is stationary in the thermodynamic limit $N = S/2 \rightarrow \infty$. Numerically we find that this is the only stationary state, meaning the system thermalises.

To prove thermalisation for the intensive temperature case we diagonalise the Lindbladian. To do this we first perform a Holstein-Primakoff (HP) mapping from spins onto bosons. Such a mapping may be approximated in the region of interest—close to the stationary state(s)—given one has knowledge of that region's expectation values, such as the magnetisation $\langle S_z \rangle_{\text{eq}}$. The new approximated Lindbladian may then be diagonalised via a Bogoliubov transform. To find these expectation values we study the observables' time evolution in the Heisenberg picture. We find that energy dissipation happens on a time scale purely characterised by the weak coupling. This allows us to set up an approximate differential equation for the energy which we solve, hence finding the equilibrium energy. Using this energy together with spin coherent states we are then able to find $\langle S_z \rangle_{\text{eq}}$ and $\langle S_x^2 \rangle_{\text{eq}}$.

Performing the HP mapping in the region of interest then leaves a quadratic, in bosonic ladder operators, Lindbladian which we subsequently diagonalise with a Bogoliubov transform. The stationary states are found to be approximate Gibbs states. In the non-degenerate phase of the LMG model we find a single stationary state, thus proving thermalisation. In contrast, we find two (orthogonal) near stationary states in degenerate phase. Numerically we find, for finite system size, that these two states form an equal fraction of the true stationary state which is a Gibbs state. However the numerics further indicates that the spectral gap between this Gibbs state and its closest state tends to zero in the thermodynamic limit. As such we expect two (orthogonal) stationary states in the degenerate phase as $S \rightarrow \infty$.

Opsomming

Gemotiveer deur onlangse eksperimente bestudeer ons die tydontwikkeling van 'n twee-spesie Bose-Einstein kondensaat wat gekoppel is aan 'n bosoniese bad. Vir die spesifieke kondensaat was onkonvensionele termodinamika onlangs voorspel. Om hierdie termiese eienskappe te bestudeer vind die kondisies waaronder hierdie *oop* kwantumstelsel ekwilibreer na die Gibbs-toestand van die kanoniese ensemble.

Die kondensaat word eers afgebeeld vanaf sy oorspronklike bosoniese voorstelling, wat N -wisselende bosone beskryf, na 'n Schwinger-spinvoorstelling, met 'n hoekmomentum van $S = 2N$. Die ooreenstemmende Hamiltoniaan het dan die vorm van 'n Lipkin-Meshkov-Glick (LMG) model. Die totale stelsel-bad Hamiltoniaan is te ingewikkeld om eksak te hanteer. Gelukkig, in die geval waar die LMG-model swak aan die bad gekoppel is, kan ons 'n benaderde differensiaalvergelyking aflei wat die LMG-model se evolusie beskryf. Na verdere benaderings word hierdie vergelyking 'n kwantum-dinamiese semigroep en gehoorsaam dit al die reëls van kwantumeganika. Die generator van hierdie semigroep, bekend as die Lindblad operator, is van sentrale belang. Die nulruimte van hierdie operator bevat die stationêre toestande. Indien hierdie nulruimte slegs die Gibbs-toestand bevat, sal die stelsel hierna ekwilibreer.

Vir ekstensiewe bad temperature toon ons dat die Gibbs-toestand stationêr is in die termodinamiese limiet waar $N = S/2 \rightarrow \infty$. Numeries vind ons dat hierdie die enigste stationêre toestand is, wat aandui dat die sisteem wel termiese ewewig bereik.

Om intensiewe temperature te ondersoek diagonaliseer ons die Lindblad operator. Hiervoor voer ons eers 'n Holstein-Primakoff (HP) afbeelding uit. Hierdie afbeelding word verder benader deur gebruik te maak van kennis omtrent die verwagtingswaardes van waarneembares, soos die magnetisasie $\langle S_z \rangle_{\text{eq}}$, relatief tot die stasionêre toestand. Die benaderde Lindblad operator kan dan deur 'n Bogoliubov-transformasie gediagonaliseer word. Om die nodige verwagtingswaardes te bepaal, bestudeer ons die waarneembares se tydsontwikkeling in die Heisenberg-beeld. Ons vind dat energie-dissipasie op 'n tydskaal plaasvind wat deur die koppelingskonstante gekarakteriseer word. Dit stel ons in staat om 'n benaderde differensiaalvergelyking op te stel vir die energie-verwagtingswaarde waaruit ons die ewewig waarde kan bepaal. Deur hierdie resultaat te gebruik te same met spin-koherente toestande kan ons die waardes van $\langle S_z \rangle_{\text{eq}}$ en $\langle S_x^2 \rangle_{\text{eq}}$ vind wat met die grondtoestand ooreenstem.

Die uitvoer van die HP-afbeelding produseer dan 'n kwadratiese, in die bosoniese leeroperatore, Lindblad operator wat ons met 'n Bogoliubov-transformasie diagonaliseer. Ons toon dat die stationêre toestand benaderd 'n Gibbs-toestand is. In die nie-ontaarde fase van die LMG-model vind ons 'n enkele stationêre toestand, wat toon dat termiese ewewig bereik word. In teenstelling hiermee, vind ons twee (ortogonale) stationêre toestande in die ontaarde fase.

Numeries vind ons, vir eindige stelselgrootte, dat hierdie twee state 'n gelyke fraksie van die ware stationêre toestand vorm, wat 'n Gibbs-toestand is. Die numeriese resultate dui egter verder aan dat die spektrale gaping tussen hierdie Gibbs-toestand en sy naaste toestand neig tot nul in die termodinamiese limiet. As sodanig verwag ons twee (ortogonale) stationêre toestande in die ontaarde fase soos $S \rightarrow \infty$.

Acknowledgements

I would like to express my sincere gratitude to my supervisors, Michael Kastner and Hannes Kriel, for captivating meetings and guidance throughout this project. Without their physical insight and mathematical expertise I would not have been able to finish this thesis in the year (plus 3 months) that I had.

I would also like to thank NITheP, the NRF and Stellenbosch University for their financial support during my studies. Further I would like to thank my friends at NITheP for their help on this thesis. In particular Penn Cassidy who aided in the creation of some of the figures in this thesis, Philip Uhrich for his insight into the experimental setup, and Hugo Touchette for his help on the fixed point analysis part and his style advice on this thesis.

I would also like to thank Ilya Sinayskiy for sharing his insight into the more rigorous parts of open quantum systems.

I would like to thank my parents and brothers for their emotional and financial support throughout this thesis. Lastly I would like to thank my grandfather Jan Cillié Malan for his proof reading of this thesis.

Contents

Abstract	I
Opsomming	I
Acknowledgements	V
List of Figures	IX
1 Introduction	1
1.1 Quantum thermalisation	1
1.1.1 Open quantum systems	2
1.2 The system-bath dynamics	3
1.3 Thesis outline	4
2 The Models	5
2.1 The Lipkin-Meshkov-Glick model	5
2.1.1 Parity conservation	6
2.2 Coupling to a thermal bosonic bath	6
2.2.1 Thermalisation and conserved quantities	7
3 The Master Equation	9
3.1 The Born (weak coupling) approximation	10
3.2 Bath correlations	11
3.2.1 Continuous field limit	13
3.3 The Markov (short memory) approximation	15
3.3.1 Explicit form of the coupling operator	16
3.4 The Lindbladian	17
3.5 The stationary states	18
3.5.1 Pure dephasing	18
3.5.2 Extensive temperature	19
4 Evolution of Observables	22
4.1 The adjoint master equation	22
4.2 Dissipation for intensive temperature	24
4.2.1 The range of the energy spectrum	27
4.3 The equilibrium energy for intensive temperature	29
4.3.1 Energy dissipation rate	29
4.4 Canonical thermodynamics	30
5 Diagonalisation of the Lindbladian	33
5.1 Gap of the Lindbladian	33
5.2 Bosonisation of spins	36
5.2.1 Bosonisation in the symmetric phase	36
5.2.2 The Bogoliubov transform	36
5.3 Lindbladian in the symmetric phase	37
5.3.1 Fixed particle number difference subspace	38
5.3.2 The stationary state in the symmetric phase	41
5.4 Bosonisation in the symmetry broken phase	42
5.4.1 Restriction to a well	42
5.5 Lindbladian in the symmetry broken phase	44
5.5.1 Comparing to numerics	45

6	Conclusion	47
6.1	Outlook	47
6.1.1	Studying the phase transition	47
6.1.2	Bounding the gap	47
Appendix A	Identities	49
A.1	Operator algebra identities and proofs	49
A.1.1	Gibbs free energy	52
A.2	Lindbladian properties	53
A.2.1	The polaron transform	55
A.3	Bogoliubov transforms	55
A.3.1	Bosonic Dissipater	57
A.4	Sokhotski-Plemelj theorem	57
A.5	Lorenz-Drude cutoff	57
A.6	Jacobi elliptical functions	59
A.6.1	Differential equation constant values	59
A.6.2	Extensive temperature thermodynamics	60
A.7	Spin Coherent States	61
7	Bibliography	63

List of Figures

1.1	BEC interaction illustration	1
1.2	Diagram of thermalisation	2
1.3	System in bath illustration	3
2.1	Fixed N states	5
2.2	Open quantum system	7
2.3	System-bath coupling	7
3.1	Lorentz-Drude spectral density	14
3.2	Plots of the decoherence kernel	14
3.3	Plot of ν_1 integral	16
3.4	Lindbladian gap for intensive temperature	21
4.1	Jacobi elliptic function	25
4.2	Jacobi elliptic periods	26
4.3	Spin coherent state illustration	27
4.4	Symmetric phase energy symbol	28
4.5	Symmetry broken phase energy symbol	28
4.6	Energy symbol differential equation plot 1	29
4.7	Energy symbol differential equation plot 2	29
4.8	Energy dissipation plot	30
4.9	Symmetry broken phase Gibbs free energy integrand	32
5.1	Double well of symmetry broken ground state manifold	43
5.2	Fidelities of Gibbs state and stationary state	45
5.4	Hinton diagrams of the stationary states	46
5.3	Lindbladian gaps in the symmetry broken phase	46
6.1	Gershgorin discs	48
6.2	CUT procedure	48

1 | Introduction

In this thesis we study the thermal properties of a two-species Bose-Einstein condensate (BEC) [41]. This is a state of matter whereby n_1 (n_2) atoms lose their individuality, acting as one quantum object. Recently *unconventional* thermal properties have been predicted for this model [70], linked to the linear growth of its fixed particle Hilbert space. For instance it was predicted that for *any* intensive temperature there exists a system size large enough such that the system's thermal expectation values become temperature independent. In our analysis we study the conditions under which this unconventional thermodynamics may be observed.

The specific BEC of interest may be realized experimentally by having N rubidium atoms coherently distributed between two hyperfine states¹ of the ground state (GS) manifold [57, 71]. These two states are coupled via resonant two-photon transition with Rabi frequency \hbar . By photon emission or absorption the rubidium atoms will transition between the two available hyperfine states at a rate given by the Rabi frequency \hbar . Pictorially one can think of this as moving one atom from n_2 to n_1 and vice versa, as shown in Fig. 1.1. The atoms also interact non-linearly, at a strength J , via atom-atom s-wave scattering. This interaction will, like a biased scale, either balance or imbalance the populations of the two bosonic species, depending on the sign of J . For dominating atom-atom interaction $J \geq \hbar$, the GS becomes two-fold degenerate in the thermodynamic limit² $N \rightarrow \infty$ [17]. Above this second order phase transition at $J = \hbar^-$, this degenerate phase is characterised by a non-zero mean population imbalance $\langle n_2 - n_1 \rangle \neq 0$. Further for $J > 2\hbar$ the relative phase between the two species diverges, which is analogous to the AC Josephson effect in superconductors [34].

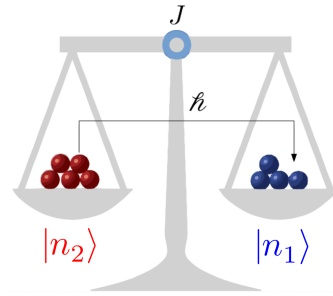


Figure 1.1: *The various interactions of the two species BEC. The two-photon transition is responsible for atoms transitioning from one state to another. The coupling strength J serves to balance or imbalance n_1 and n_2 like a biased scale.*

1.1 Quantum thermalisation

To study the thermal properties of the two-species BEC we must first find the conditions under which this system thermalises. This is studied in the field of quantum thermalisation, which addresses the question of how classical thermodynamics can arise from a fundamentally quantum world [40, 47, 51]. In particular why is it that a quantum system, which evolves unitarily, can be effectively described by equilibrium statistical mechanics?

In the quantum description of closed systems one typically considers the evolution of a *pure* state $|\psi\rangle\langle\psi|$, where the matrix elements encode inherent quantum uncertainty. In contrast to this, classical thermodynamics considers *mixed* states, a sum of pure states $\varrho = \sum_n \varrho_n |n\rangle\langle n|$, where the matrix elements ϱ_n encode uncertainty due to lack of information. These are states such as the canonical ensemble, which is energy diagonal. As such we may reformulate the question: Why is it that a *pure* state can be effectively described by a *mixed* state?

In the setting where the system remains isolated it will evolve unitarily, remaining pure. In an experiment, however, one does not have access to all the information in $|\psi\rangle\langle\psi|$. As such one typically describes a state by its observable quantities, for instance a population imbalance $n_2 - n_1$ or a relative phase. Thus a pure state would appear mixed if all accessible

¹In particular there are n_1 atoms in the state $|F = 1, m_F = 1\rangle$ and n_2 atoms in the state $|F = 2, m_F = -1\rangle$. Here m_F is the projection of the total atomic angular momentum \mathbf{F} with magnitude squared $F(F + 1)$. The specific experiment $N = n_1 + n_2 = 500$ atoms, however this state has been studied for up to $N = \mathcal{O}(10^7)$ [60].

²The gap between the GS and the first excited state takes the form $\sim e^{-\text{const.}N}$ [48]

observables would yield similar results. This is the idea behind the eigenstate thermalisation hypothesis (ETH). It states that, for every single energy eigenstate, the expectation value of a local observable, coincides with its microcanonical average plus some small fluctuations [64]. Whether the system thermalises in this sense is dependent on the system's integrability [40]. For this thesis integrability can be thought of as a measure of the system's local conserved quantities, that is, local operators that commute with the system's Hamiltonian \mathcal{H}_S . If a specific observable is conserved, then its time average does not change, and cannot tend to an ensemble average i.e. it cannot thermalise.

In current experiments ultra-cold atomic gases are usually very well isolated from their environment and unitary dynamics can be observed. If not integrable, then such closed systems may thermalise in the sense of the ETH.

The particular BEC we consider may be mapped onto a Lipkin-Meshkov-Glick [44] (LMG) type Hamiltonian \mathcal{H}_S , which was originally used as a toy model for two shell nuclear interactions [38]. This is done via a Schwinger pseudo-spin mapping [29, 53], which identifies the particle number N with twice the spin angular momentum $2S$. This LMG model is, however, integrable in the sense that it is solvable by Bethe ansatz [50]. This integrability leads to oscillatory motion with a short period which prevents thermalisation [62].

1.1.1 Open quantum systems

Despite the integrability of the LMG model, the system may still relax to a canonical ensemble³ by considering an open quantum system, rather than a closed system. This is because the act of coupling to an environment typically breaks the symmetries responsible for the invariance of particular observables, leading to fewer conserved quantities.

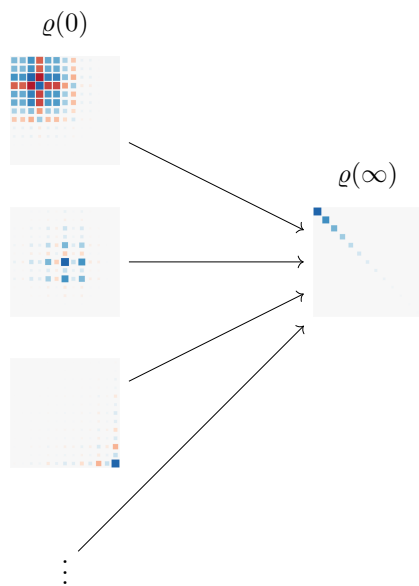


Figure 1.2: *Multiple initial states $\rho(0)$ thermalising to a Gibbs states $\rho(\infty)$.*

The total system is then described by the sum of Hamiltonians $\mathcal{H}_{S\mathcal{E}} = \mathcal{H}_S + \mathcal{H}_{\mathcal{E}} + \mathcal{V}$, where $\mathcal{H}_{\mathcal{E}}$ and \mathcal{V} correspond to the environment and its interaction with the target system (LMG model) respectively. The target is then said to be *open*, and will typically evolve non-unitarily, thereby becoming mixed. Our particular environment will be a thermal state, meaning a bath with inverse temperature β .

In an open system two processes play a role. The first is decoherence, which is responsible for the off-diagonal (in the interaction Hamiltonian's eigenbasis) terms of the density matrix to decay. Since the off-diagonal terms encode quantum coherence, this process is seen as the state becoming classical. The other process is dissipation associated with energy transfer, which typically happens on a slower time scale compared to decoherence. These two processes then allow for the possibility of a different version of *thermalisation* to occur—the evolution of any density matrix towards the canonical ensemble $e^{-\beta\mathcal{H}_S}$, also known as a Gibbs state. This process is illustrated in Fig. 1.2, where all initial states evolve towards the energy diagonal Gibbs state.

As the initial state may be pure, while the Gibbs state is mixed, such evolution will only be possible for an open system. The total system, target plus bath, is still closed, with unitary evolution. Thus if the total Hilbert space has finite dimension, then the system has a finite recurrence time as a result of Poincaré's recurrence theorem [61]. To ensure that the state remains a Gibbs state an infinite recurrence time is required. In hopes of seeing equilibration to a Gibbs state we couple the LMG model to an

³Another way to induce thermalisation is in the LMG model, in the sense of ETH, is via a quench [47].

environment described by a, possibly infinite, set of harmonic oscillators in thermal equilibrium. As a model for quantum Brownian motion [60] and blackbody radiation [24], this particular environment is prototypical choice for estimating decoherence and dissipative effects, especially on macroscopic scales. The system-bath coupling is between the position operators of the oscillators and population imbalance $n_2 - n_1$ of the system. Such coupling is a model for continuous environmental monitoring of $n_2 - n_1$. If this coupling is weak it will act as a weak measurement leading to decoherence of the system in the $n_2 - n_1$ basis.

Whether these conditions are enough to ensure thermalisation is, again, deeply dependent on the target's conserved quantities [40]. For example if the target system's energy \mathcal{H}_S is conserved $[\mathcal{H}_S, \mathcal{H}_{SE}] = 0$, then the system cannot evolve to a Gibbs state which has energy proportional to $\text{Tr}\{\mathcal{H}_S e^{-\beta \mathcal{H}_S}\}$. This is an explicit reason why closed systems cannot thermalise in our sense, as $[\mathcal{H}_S, \mathcal{H}_S] = 0$.

This total system reduces to various models often studied in the literature. For instance considering spin-1/2 particles, this model reduces to the *spin-boson model* [34]. This is an important paradigm for studying dissipative quantum systems, as it models multiple two state processes such as spontaneous emission of a photon [49]. This system has been shown to thermalise [5]. Further the spin-boson model may be mapped onto a Kondo model [37], important for modeling correlated electrons, via fermionization techniques [36].

Setting the atom-atom coupling to zero $J = 0$ yields the multi-modal Dicke Hamiltonian, which has been studied extensively in the zero temperature regime [18, 19]. For the single-model case it has been shown that there exists a critical temperature above which super-radiance can be observed: spontaneous emission of light with intensity proportional to N^2 rather than the expected N [67–69].

1.2 The system-bath dynamics

We want to study thermalisation of the LMG model, and as such we are uninterested in the bath's evolution. Hence we eliminate, by tracing out, the bath degrees of freedom⁴. This, however, yields a non-local (in time) differential equation for the target which is intractable. Fortunately, if the bath is near-memoryless and only weakly coupled to the system, as illustrated in Fig. 1.3, we may derive a simpler approximate differential equation. This is because these properties allow us to make various standard approximations [12], which drastically simplify the equation of motion. The weak coupling implies that the system-bath correlations build up over

a long time and may be ignored for short times. The size of the bath provides a large recurrence time for the entire system, aiding in the desired memoryless condition. Lastly, if the bath is energetic, then its correlations decay fast relative to the typical frequency of the LMG model. We may then average the bath over these fast time scales, while keeping the system constant. This will be the condition we require for a near-memoryless bath and thus a time-local differential equation, known as the *Born-Markov master equation*. Upon a further approximation one may derive from this equation a quantum-dynamical semi-group, which preserves all the rules of quantum mechanics. The Lindbladian, which is the generator of this group, is the main object of interest. This is because it takes the role of the Hamiltonian for open quantum systems. As such we may find the stationary states by diagonalising the Lindbladian and compare them to the Gibbs state. If we find that the Gibbs state is the only stationary state, then we have proven that the system thermalises.

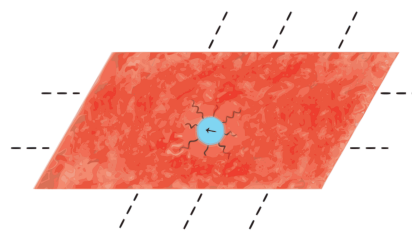


Figure 1.3: *System (blue sphere) weakly coupled to large almost memoryless bath.*

⁴This is effectively done experimentally by focusing on a subsystem, due to the inability to track the evolution of such a large environment.

1.3 Thesis outline

We start Chap. 2 by introducing the LMG model and its spin representation. This system is then coupled to a bath of harmonic oscillators as described in Sect. 2.2. Here we also briefly discuss some of the conserved quantities of this total system and their relevance to thermalisation.

In Chap. 3 we discuss the dynamics of this open system. As the full problem is intractable, we use the standard approximations [12] to set up a *Born-Markov master equation*. This equation describes the dynamics of the LMG model in the regime of weak coupling to an energetic and large bath. This master equation is time local and trace preserving, but does not yet preserve positivity. Upon one final approximation we ensure that all the rules of quantum mechanics are preserved, hence the differential equation characterises a quantum dynamical semi-group. We then study the Lindbladian, which is the generator of this group, and thus the generator of time evolution. In the case where $\mathcal{R} = 0$ we find a pure decoherence process. As such there is no dissipation and the system does not thermalise. We conclude this chapter by proving that the Gibbs state is stationary for an extensively large temperature. Numerically, for finite S , we find that thermalisation occurs due to the Gibbs state being the only stationary state.

To find the stationary state in the non-extensive temperature case we require a different approach, known as a Holstein-Primakoff mapping [30]. This mapping includes a square root of operators which may be expanded in the region of interest. In our case we are interested in the equilibrium region. Hence to perform this expansion we require the equilibrium expectation values of the LMG model. We study these expectation values in Chap. 4 by considering the time evolution of the relevant observables. Here we will find that the Hamiltonian evolves on a slow time scale. Averaging over another fast time scale, using the two-timing method [65], we obtain a differential equation purely in terms of the Hamiltonian. This equation allows us to find the energy dissipation rate which upper bounds the equilibration rate. Further we may find the equilibrium energy. We then use variational methods together with spin coherent states to find the equilibrium expectation values, which turn out to be temperature independent. We compare these to the thermal expectation values, found via spin coherent states, and find that both match those of the ground state, which coincides with the predictions of [70].

Having derived the equilibrium expectation values, we perform a Holstein-Primakoff expansion in Chap. 5. We use degenerate perturbation theory, in the weak coupling, to find a simplified Lindbladian. We subsequently diagonalise this Lindbladian via a four-dimensional Bogoliubov transform. We find that the stationary state indeed corresponds to the thermal state to a high degree of accuracy. Further in the non-degenerate phase of the LMG model there is only one stationary state, thus the system thermalises. For the degenerate case we find two different vacuum states. Numerically it seems that these states only become stationary in the thermodynamic limit. In the finite S regime they each form half of the true stationary state, which is a Gibbs state.

We will use natural units: $\hbar = k_B = c = 1$ throughout this thesis. However to link to experiments and everyday experience we will sometimes write expressions in standard units. Further all numerical data presented in this thesis was computed in python, in particular making use of the *QuTiP* library [33].

2 | The Models

2.1 The Lipkin-Meshkov-Glick model

The target system of interest is a two-species Bose-Einstein condensate (BEC) [15, 53] modeled by the Hamiltonian

$$\mathcal{H}_S = -\frac{J}{N} \left[\frac{a_1^\dagger a_1 - a_2^\dagger a_2}{2} \right]^2 - \frac{\hbar}{2} [a_1^\dagger a_2 + a_2^\dagger a_1], \quad (2.1)$$

where N is the (conserved) total number of bosonic particles. The prefactor of $1/N$ ensures extensivity, leading to a finite energy per particle in the thermodynamic limit. Here a_q^\dagger and a_q are bosonic ladder operators corresponding to the creation and annihilation of a q -species boson respectively:

$$a_q^\dagger |n_q\rangle = \sqrt{n_q + 1} |n_q + 1\rangle, \quad a_q |n_q\rangle = \sqrt{n_q} |n_q - 1\rangle. \quad (2.2)$$

where n_q is the number of q -species bosons. In this number basis it becomes clear that the fixed N particle Hilbert space \mathcal{H}_S is populated by $N + 1$ states, as shown in Fig. 2.1. As bosonic operators, a_q^\dagger and a_q satisfy the following commutation relations

$$[a_q, a_p^\dagger] = \delta_{qp}, \quad [a_q, a_p] = [a_q^\dagger, a_p^\dagger] = 0. \quad (2.3)$$

From (2.2) we can see how the two-photon transition, the second operator in (2.1), is responsible for changing one species to another, as illustrated by the arrows in Fig. 2.1. Using (2.2) one may show that the number operator $a_q^\dagger a_q$ tallies the q -species particles: $a_q^\dagger a_q |n_q\rangle = n_q |n_q\rangle$. As such the first operator in (2.1), corresponding to J , acts as a biased scale, as was illustrated in Fig. 1.1. For negative J it reaches a minimum when the populations are balanced $n_1 = n_2$. For positive J this number difference operator lowers the energy the more unbalanced the populations become: $n_1 \gg n_2$ or $n_1 \ll n_2$. This ambiguity with regards to maximising n_1 or n_2 is responsible for a two-fold ground state degeneracy at $J = \hbar$, for $J > 0$. Hoping to study the physics on either side of this phase transition, we will focus on the parameter regime $J \geq 0$ throughout this thesis.

We bring the Hamiltonian into a simpler looking form by performing a Schwinger pseudo-spin mapping [29, 53]. This procedure identifies the bosons with spins

$$S_x \equiv \frac{a_1^\dagger a_1 - a_2^\dagger a_2}{2}, \quad S_y \equiv \frac{a_2^\dagger a_1 - a_1^\dagger a_2}{2i}, \quad S_z \equiv \frac{a_1^\dagger a_2 + a_2^\dagger a_1}{2}, \quad (2.4)$$

where the symbol \equiv is used to mean ‘‘equality by definition’’. As spin operators they satisfy the $\mathfrak{su}(2)$ commutation relations

$$[S_j, S_k] = i\varepsilon_{jkl} S_l, \quad (2.5)$$

where ε_{jkl} is the *Levi-Civita* symbol, identifying (x, y, z) with $(1, 2, 3)$. This yields the Lipkin-Meshkov-Glick (LMG) [44] type Hamiltonian

$$\mathcal{H}_S = -\frac{J}{2S} S_x^2 - \hbar S_z. \quad (2.6)$$

The newly defined spin angular momentum (quantum number) S is half the particle number $N = 2S$, hence it is also conserved $[\hat{S}^2, \mathcal{H}_S] = 0$. Here $\hat{S}^2 \equiv S_x^2 + S_y^2 + S_z^2$ is the total spin

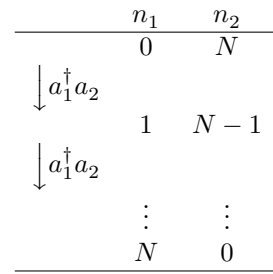


Figure 2.1: Allowed states in the fixed N Hilbert space and the ladder operators connecting said states.

angular momentum operator with eigenvalues $S(S+1)$, which is proportional to the identity $\mathbb{1}$ when restricted to a particular angular momentum subspace. This relation between spin and particle number allows us to relate the large S -limit to the thermodynamic limit $N \rightarrow \infty$, for which the LMG dynamics becomes *quasi-solvable* [56]. In other words this model may be solved in the thermodynamic limit via a semi-classical expansion, where $1/S$ plays the role of \hbar [20].

2.1.1 Parity conservation

In the Schwinger pseudo-spin representation we may expand any operator in the convenient $\mathfrak{su}(2)$ Dicke basis $|S, m\rangle$, where $m \in \{-S, -S+1, \dots, S-1, S\}$ is the eigenvalue of the S_z operator: $S_z|S, m\rangle = m|S, m\rangle$. The range of m is symmetric around zero which leads to an energy spectrum independent of $\text{sgn}(\hbar)$. As such we will take \hbar to be positive without loss of generality. The state $|S, m\rangle$ may be raised or lowered by the spin ladder operators S_+ and S_- in the following way:

$$S_{\pm}|S, m\rangle = \sqrt{(S \mp m)(S \pm m + 1)}|S, m \pm 1\rangle. \quad (2.7)$$

In terms of these ladder operator the typical spin operators become

$$S_y \equiv \frac{S_+ - S_-}{2i} \quad S_x \equiv \frac{S_+ + S_-}{2}. \quad (2.8)$$

This structure of the S_x operator, given by (2.8), implies a further conserved quantity: the parity $e^{i\pi S_z}$. Its (time) invariance is rooted in the S_x^2 operator in \mathcal{H}_S (2.6) taking the form

$$4S_x^2 = S_+^2 + S_-^2 + S_-S_+ + S_+S_-. \quad (2.9)$$

As such \mathcal{H}_S always changing m by zero or two (always even numbers), in which case the parity remains unchanged: $e^{i\pi m} \rightarrow e^{i\pi(m+2)} = e^{i\pi(m+0)} = e^{i\pi m}$. Hence \mathcal{H}_S never mixes odd and even m sectors and may be separated as

$$\mathcal{H}_S = \sum_{n, m \text{ odd}} c_{nm} |n\rangle\langle m| + \sum_{p, q \text{ even}} c_{pq} |p\rangle\langle q|, \quad (2.10)$$

where c_{ij} are scalars.

2.2 Coupling to a thermal bosonic bath

Since our aim is to study non-reversible equilibration we consider another system—an *environment*—which will exchange energy and phase coherence with the LMG model. The (target) system¹ is coupled to this environment via an interaction term $\mathcal{V} = cS_x \otimes \mathcal{B}$ with c characterising the coupling strength, as depicted in Fig. 2.2. The particular environment is described by the quantum analog of the many-body harmonic oscillator, which models simple harmonic motion, such as the oscillations of a spring or electromagnetic fields. Its Hamiltonian takes the form

$$\mathcal{H}_{\mathcal{E}} = \sum_q \frac{p_q^2}{2m_q} + \frac{m_q \omega_q^2}{2} x_q^2 + \text{const.}, \quad (2.11)$$

where the q th oscillator/particle has mass m_q and frequency ω_q . Upon defining the q th particle's position and momentum operators, x_q and p_q respectively, in terms of bosonic ladder operators:

$$x_q = \sqrt{\frac{1}{2m_q \omega_q}} (a_q^\dagger + a_q), \quad p_q = \sqrt{\frac{m_q \omega_q}{2}} i (a_q^\dagger - a_q), \quad (2.12)$$

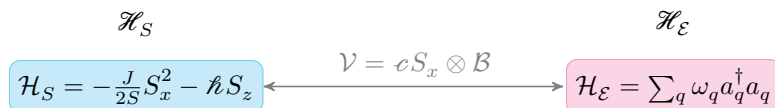


Figure 2.2: Diagram of the open quantum system, with the arrow indicating their interaction.

we may map this Hamiltonian to a sum of number operators² $\mathcal{H}_E \rightarrow \sum_q \omega_q a_q^\dagger a_q$.

The system-bath Hamiltonian may then be written as $\mathcal{H}_{S\mathcal{E}} = \mathcal{H}_0 + \mathcal{V}$, where the unperturbed part is given by $\mathcal{H}_0 = (\mathcal{H}_S + \mathcal{H}_c) \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{H}_E$. Here we have added a *counter-term* \mathcal{H}_c , which will ensure that the system's unitary dynamics remains unperturbed for large bath frequencies [12]. We do this as we are uninterested in how the coupling changes the system's unitary evolution³. Hence such contributions will merely act as to complicate our calculations. Given our particular bath and coupling, the counter-term we will require is⁴

$$\mathcal{H}_c = c^2 S_x^2 \sum_q \frac{\gamma_q^2}{\omega_q}. \quad (2.13)$$

For specific parameters $\mathcal{H}_{S\mathcal{E}}$ reduces to various models often studied in the literature. For instance setting $S = 1/2$ one recovers the *spin-boson model* [34]. While setting $J = 0$ yields the multi-modal Dicke Hamiltonian [18]. For the single-mode case it has been shown that there exists a critical temperature T_c for $c > \sqrt{2\hbar\omega_1/(S\gamma_1)}$ [19].

We define \mathcal{B} as a sum of weighted position operators

$$\mathcal{B} = \sum_q \gamma_q (a_q^\dagger + a_q), \quad (2.14)$$

which generates a change in momentum for all the coupled harmonic oscillators. The sign and magnitude of this momentum change will depend on the system's magnetisation in the x direction, as illustrated in Fig. 2.3. Since the S_x operator is the generator of rotation, coupling to the bath will inevitably lead to more complicated rotations in the system.

As the system and bath interact they exchange information and become entangled. This buildup of correlations with the many degrees of freedom (DOF) of the bath will manifest itself, once the bath's DOF are traced out, as decoherence and dissipation.

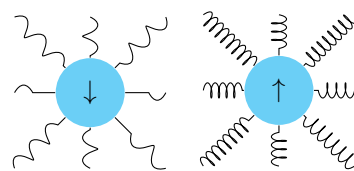


Figure 2.3: A pictorial representation of system-bath coupling. A particular oscillator's momentum is represented by the frequency/compression of a particular spring. The up and down arrows indicate a positive or negative S_x magnetisation of the system respectively.

2.2.1 Thermalisation and conserved quantities

In this thesis we define thermalisation in the following way:

Def 1 (Thermalisation) For any initial state $\rho(0) \in \mathbb{S}(\mathcal{H}_S)$ the system tends towards a Gibbs state $\rho(t \rightarrow \infty) \propto e^{-\beta \mathcal{H}_S}$, where β is the inverse temperature of the bath and \mathcal{H}_S is the system's Hamiltonian. Here $\mathbb{S}(\mathcal{H}_S)$ is the state space, the space of all bounded positive semi-definite Hermitian operators with unit trace acting on \mathcal{H}_S .

¹From henceforth we will merely refer to the target system as *the system*.

²Any states $|n_q\rangle$ will from henceforth refer to the number states of the bath, unless otherwise stated.

³Recall that we wish to study thermalisation which is a dissipative process.

⁴This will be shown in Sect. 3.3.1

Thermalisation in the sense of def 1 means that the system becomes exclusively dependent on the bath, in which case the system has lost all memory of its initial conditions. The goal of this thesis is to find the conditions under which this happens for our particular model. Let us start this analysis by considering two simple cases.

For $\hbar = 0$ we note that all operators in $\mathcal{H}_{S\mathcal{E}}$ acting on \mathcal{H}_S are powers of S_x . Since $\mathcal{H}_S \propto S_x^2$, for $\hbar = 0$, this leads to energy conservation $[\mathcal{H}_S, \mathcal{H}_{S\mathcal{E}}] = 0$, preventing thermalisation in the sense of def 1. This leads to pure dephasing—decoherence where the diagonal of ρ in the coupling's eigenbasis is unchanged—which we will explore in Sect. 3.5.1. This implies that \hbar sets a time scale for energy dissipation to occur.

One quantity which is always conserved is the parity operator $\exp(i\pi \sum_q a_q^\dagger a_q + i\pi S_z)$, which follows from the same reasoning as Sect. 2.1.1⁵. Since this parity is conserved even with respect to the coupling \mathcal{V} there would always remain some dependence on the initial states for the total system. However the LMG parity $e^{i\pi S_z}$ is no longer conserved, $[e^{i\pi S_z}, \mathcal{V}] \neq 0$, and will change upon coupling to the bath. This is one example of where coupling to a bath breaks a symmetry associated with a system, as was discussed in Sect. 1.1, leading to a less integrable system.

⁵It commutation with \mathcal{V} is due to $S_x(a_q + a_q^\dagger)$ always changing $n_q + m$ by an even number.

3 | The Master Equation

We may systematically study thermalisation by analysing the time evolution of the total density matrix $\varrho_{S\mathcal{E}}$. Since the total system plus bath is closed the density matrix $\varrho_{S\mathcal{E}}$ undergoes unitary evolution described by the von Neumann equation

$$\dot{\varrho}_{S\mathcal{E}} = i[\varrho_{S\mathcal{E}}, \mathcal{H}_{S\mathcal{E}}]. \quad (3.1)$$

One may check, by differentiation, that it has the formal solution $\varrho_{S\mathcal{E}}(t) = e^{-it\mathcal{H}_{S\mathcal{E}}} \varrho_{S\mathcal{E}}(0) e^{it\mathcal{H}_{S\mathcal{E}}}$. We are, however, only interested in the (target) system's time evolution, which we may isolate by tracing out the bath

$$\varrho(t) = \text{Tr}_{\mathcal{E}} \left\{ e^{-it\mathcal{H}_{S\mathcal{E}}} \varrho_{S\mathcal{E}}(0) e^{it\mathcal{H}_{S\mathcal{E}}} \right\}. \quad (3.2)$$

One way to evaluate (3.2) is via the Hadamard lemma—also known as the Baker-Campbell-Hausdorff (BCH) formula [58]

$$e^{-A} \bullet e^A = \bullet + [\bullet, A] + \frac{1}{2!} [[\bullet, A], A] + \frac{1}{3!} [[[\bullet, A], A], A] + \dots, \quad (3.3)$$

where the bullet sign \bullet is a place holder for some operator throughout this text. In general, this approach will not yield a closed form solution, which we require. In the simpler case of no coupling ($c = 0$) it has been shown that there exists a solution via the Bethe ansatz [50]. This, unfortunately, requires the solution to a set of non-linear algebraic equations with no general closed form solution either, making it impractical for our use. Since this simpler case is beyond us, there seems little hope to solve the problem when coupling to possibly infinitely many harmonic oscillators. We might still be able to capture the most important parts of the dynamics with an approximate differential equation of the form

$$\dot{\varrho}(t) = \lim_{\delta \rightarrow 0} \frac{\varrho(t + \delta) - \varrho(t)}{\delta} \equiv \mathcal{L}\varrho(t), \quad (3.4)$$

called a *master equation*, with the generator of time evolution \mathcal{L} being some *superoperator*. The term superoperator implies that it acts on a vector space of linear operators, namely the density matrices $\varrho \in \mathbb{S}(\mathcal{H}_S)$. This generator \mathcal{L} must preserve all the rules of quantum mechanics, meaning any initial density matrix $\varrho(0)$ must remain a density matrix $\varrho(t) \in \mathbb{S}(\mathcal{H}_S)$ for all time t . A negative expectation value for the density matrix would imply a negative probability, while a unit trace $\text{Tr}\varrho(t) = 1$ is required for probability conservation. Lastly a density matrix must remain hermitian to yield real expectation values of observables. As such the following conditions must be satisfied:

- (i) Trace-preservation: $\text{Tr}\{\mathcal{L}\varrho(t)\} = 0$
- (ii) Positivity-preservation: $\langle \psi | e^{t\mathcal{L}} \varrho(0) | \psi \rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}_S$
- (iii) Hermiticity-preservation: $(e^{t\mathcal{L}} \varrho(0))^\dagger = e^{t\mathcal{L}} \varrho(0)$.

Further since (3.4) is a time local differential equation it already satisfies the semi-group property $e^{(t+s)\mathcal{L}} = e^{t\mathcal{L}} e^{s\mathcal{L}}$. In this case \mathcal{L} is said to be the generator of a quantum dynamical semi-group [43].

Whether such an equation as (3.4) should exist is not obvious. Especially since this time local differential equation (3.4) implies that the state of $\varrho(t + dt)$ is completely determined by its prior state $\varrho(t)$ —the system is *Markovian*. To see why this might not hold consider the following example: at time t_1 some energy flows from the system to the bath, then it returns at time $t_1 + t_2$. This would imply that the state $\varrho(t_1 + t_2)$ is dependent on $\varrho(t_1)$ [52], which breaks Markovianity.

However, if the system-bath coupling \mathcal{V} was weak, then the exchange of energy and correlation build up would happen on a slow time scale, given by the inverse coupling strength. We may

then ignore these non-Markovian system-bath correlations for early times. Further if the bath has rapid fluctuations, encoded in a fast decay of its autocorrelation functions, while the system is slow to change, we may perform a separation of time scales. This amounts to averaging over the bath, while keeping the system constant. These two assumptions, weak coupling and separation of timescales, combined will indeed yield a master equation. We may then use this equation to find the stationary state(s) of the system and compare them to the Gibbs state.

3.1 The Born (weak coupling) approximation

Our derivation of the master equation (3.4) starts by transforming into the interaction picture¹ $\mathcal{V}(t) = U(t)\mathcal{V}U^\dagger(t)$, where $U(t) = \exp(i\mathcal{H}_0 t)$. In this picture the von Neumann equation (3.1) becomes

$$\dot{\varrho}_{S\mathcal{E}}^{(I)}(t) = i[\varrho_{S\mathcal{E}}^{(I)}(t), \mathcal{V}(t)]. \quad (3.5)$$

We assume that the system and bath are initially in a product state and that the interaction between them is switched on at time $t = 0$. Formally integrating both sides of (3.5) with respect to t and substituting the result back into (3.5) yields

$$\varrho_{S\mathcal{E}}^{(I)}(t) = i[\varrho^{(I)}(0) \otimes \varrho_{\mathcal{E}}^{(I)}(0), \mathcal{V}(t)] + \int_0^t ds [\mathcal{V}(t), [\mathcal{V}(s), \varrho_{S\mathcal{E}}^{(I)}(s)]]. \quad (3.6)$$

Where the bath is characterised by $\varrho_{\mathcal{E}}^{(I)}(0) \propto e^{-\beta\mathcal{H}_{\mathcal{E}}}$ with temperature β^{-1} , meaning that $\varrho_{\mathcal{E}}^{(I)}(0) = \varrho_{\mathcal{E}}(0)$, as it commutes with \mathcal{H}_0 . Writing the total density matrix as a sum of its factored part plus correlations²

$$\varrho_{S\mathcal{E}}^{(I)}(s) = \varrho^{(I)}(s) \otimes \varrho_{\mathcal{E}}^{(I)}(s) + g_{S\mathcal{E}}(s) \quad (3.7)$$

and tracing over the bath degrees of freedom (DOF), in (3.6), leads to

$$\dot{\varrho}^{(I)}(t) = i\mathcal{C}\text{Tr}\{\varrho_{\mathcal{E}}(0)\mathcal{B}(t)\} [\varrho^{(I)}(0), S_x(t)] + \int_0^t ds \text{Tr}_{\mathcal{E}}[\mathcal{V}(t), [\mathcal{V}(s), \varrho^{(I)}(s) \otimes \varrho_{\mathcal{E}}(s) + g_{S\mathcal{E}}(s)]]. \quad (3.8)$$

The first term vanishes since \mathcal{B} , as defined in (2.14), is not particle number conserving, leading to a zero mean at time zero (indicated by the subscript) $\langle \mathcal{B}(t) \rangle_0 \equiv \text{Tr}\{\varrho_{\mathcal{E}}(0)\mathcal{B}(t)\} = 0$. Using the relation $[A, [B, C]] = [A, BC] + \text{H.C.}$, we split the double commutator into a commutator and its Hermitian conjugate (H.C.):

$$\dot{\varrho}^{(I)}(t) = \int_0^t ds \text{Tr}_{\mathcal{E}}[\mathcal{V}(t), \mathcal{V}(s)(\varrho^{(I)}(s) \otimes \varrho_{\mathcal{E}}^{(I)}(s) + g_{S\mathcal{E}}(s))] + \text{H.C.} \quad (3.9)$$

Next the trace's invariance under cyclic permutations allows the bath correlations to factorize:

$$\dot{\varrho}^{(I)}(t) = \mathcal{C}^2 \int_0^t ds \langle \mathcal{B}(t)\mathcal{B}(s) \rangle_s [S_x(t), S_x(s)\varrho^{(I)}(s)] + [S_x(t), S_x(s)\text{Tr}_{\mathcal{E}}\{\mathcal{B}(t)\mathcal{B}(s)g_{S\mathcal{E}}(s)\}] + \text{H.C.} \quad (3.10)$$

If there were no coupling $\mathcal{V} = 0$, then the bath will remain stationary. As such no system-bath correlations will form $g_{S\mathcal{E}}(s) = 0$. Together we may deduce that [14]

$$\varrho_{\mathcal{E}}^{(I)}(s) = \varrho_{\mathcal{E}}(0) + \mathcal{O}(\mathcal{V}) \quad \text{and} \quad g_{S\mathcal{E}}(s) = \mathcal{O}(\mathcal{V}). \quad (3.11)$$

¹We take the convention that any operator (other than the density matrix) written with explicit time dependence is in the interaction picture.

²These correlation operators are a study in their own right, and are used in another approximation scheme called the cluster expansion [10]. Studying their differential equations: $\dot{g}_{S\mathcal{E}}(s) = i[\varrho_{S\mathcal{E}}^{(I)}(s), \mathcal{V}(s)] - i\text{Tr}_{\mathcal{E}}[\varrho_{S\mathcal{E}}^{(I)}(s), \mathcal{V}(s)]\varrho_{\mathcal{E}}$, one can make stronger statements about the validity of the approximations to come.

This leads us to the *Born approximation*—neglecting terms of cubic or higher order in the coupling, in (3.10), which yields

$$\dot{\rho}^{(I)}(t) = c^2 \int_0^t ds \langle \mathcal{B}(t)\mathcal{B}(s) \rangle_0 [S_x(t), S_x(s)\rho^{(I)}(s)] + \text{H.C.}, \quad (3.12)$$

which is valid for *weak coupling*³.

Next we perform a change of variables to the time difference $\tau = t - s$ to obtain

$$\dot{\rho}^{(I)}(t) = c^2 \int_0^t d\tau \langle \mathcal{B}(t)\mathcal{B}(t-\tau) \rangle_0 [S_x(t-\tau)\rho^{(I)}(t-\tau), S_x(t)] + \text{H.C.}, \quad (3.13)$$

where the minus sign from flipping the order of integration has been absorbed into the commutator. This is still a nonlocal (in time) differential equation. However, if it were the case that $\langle \mathcal{B}(t)\mathcal{B}(t-\tau) \rangle_0$ is strongly decaying around $\tau = 0$, then we may consider a separation in time scales of system and bath. This would then allow us to treat the system as approximately constant around the point $\tau = 0$ leading to a time local master equation. To investigate whether or not such a separation is valid we study the bath correlation functions.

3.2 Bath correlations

To further simplify the problem we study the time intervals over which bath correlations buildup and decay. Using the cyclic invariance of the trace we may write the two-time correlation functions purely in terms of τ :

$$\begin{aligned} \langle \mathcal{B}(t)\mathcal{B}(t-\tau) \rangle_0 &= \text{Tr} \left\{ e^{i\mathcal{H}_\varepsilon t} \mathcal{B} e^{-i\mathcal{H}_\varepsilon t} e^{i\mathcal{H}_\varepsilon(t-\tau)} \mathcal{B} e^{-i\mathcal{H}_\varepsilon(t-\tau)} e^{-\beta\mathcal{H}_\varepsilon} \right\} \\ &= \text{Tr} \left\{ e^{i\mathcal{H}_\varepsilon \tau} \mathcal{B} e^{-i\mathcal{H}_\varepsilon \tau} \mathcal{B} e^{-\beta\mathcal{H}_\varepsilon} \right\} \\ &= \langle \mathcal{B}(\tau)\mathcal{B} \rangle. \end{aligned}$$

To evaluate $\langle \mathcal{B}(\tau)\mathcal{B} \rangle$ we first require the time evolved ladder operators, since \mathcal{B} is a sum of such operators (2.14). Using the BCH formula (3.3) we find

$$a_q(\tau) \equiv e^{i\mathcal{H}_\varepsilon \tau} a_q e^{-i\mathcal{H}_\varepsilon \tau} = a_q - i\tau[a_q, \mathcal{H}_\varepsilon] + \frac{(-i\tau)^2}{2!} [[a_q, \mathcal{H}_\varepsilon], \mathcal{H}_\varepsilon] + \frac{(-i\tau)^3}{3!} [[[a_q, \mathcal{H}_\varepsilon], \mathcal{H}_\varepsilon], \mathcal{H}_\varepsilon] + \dots$$

where $[a, aa^\dagger] = a[a, a^\dagger] = a$, hence any iterations of such commutators yields the same result, leaving

$$a_q(\tau) = a_q - i\tau\omega_q a_q + \frac{(-i\tau\omega_q)^2}{2!} a_q + \frac{(-i\tau\omega_q)^3}{3!} a_q + \dots = a_q e^{-i\omega_q \tau}, \quad (3.14)$$

while $a_q^\dagger(\tau) = (a_q(\tau))^\dagger$. The only non-zero contribution to the bath correlations comes from number conserving terms such as $\langle a_q^\dagger a_q \rangle$, all of which couple to only a single mode, leaving only a single sum

$$\langle \mathcal{B}(\tau)\mathcal{B} \rangle = \sum_q \gamma_q^2 \langle (a_q e^{-i\omega_q \tau} + a_q^\dagger e^{i\omega_q \tau})(a_q + a_q^\dagger) \rangle. \quad (3.15)$$

The reason for this is that any other combination, such as $a_1 a_2$, changes an arbitrary state $|n_1 n_2 \dots n_m\rangle$, leading to zero overlap in the trace. This is also the reason for $\langle \mathcal{B}(t) \rangle_0 = 0$. The only non-zero terms in (3.15) are $\langle a_q^\dagger a_q \rangle$ and $\langle a_q a_q^\dagger \rangle = 1 + \langle a_q^\dagger a_q \rangle$ and as such

$$\langle \mathcal{B}(\tau)\mathcal{B} \rangle = \sum_q \gamma_q^2 \left(e^{-i\omega_q \tau} (1 + \langle a_q^\dagger a_q \rangle) + e^{i\omega_q \tau} \langle a_q^\dagger a_q \rangle \right)$$

³There exists another procedure which allows one to derive a master equation for arbitrarily strong coupling using the polaron transform [67–69]. This, however, introduces different limitations which we do not desire. See Sect. A.2.1 for a discussion on this.

$$= \sum_q \gamma_q^2 \left((1 + 2\langle a_q^\dagger a_q \rangle) \cos(\omega_q \tau) - i \sin(\omega_q \tau) \right), \quad (3.16)$$

where we have used $e^{\pm i\theta} = \cos \theta \pm i \sin \theta$. In order to calculate these expectation values $\langle a_q^\dagger a_q \rangle$ it is useful to introduce the (generating) partition function Z : the normalisation factor in $\rho_{\mathcal{E}} = Z^{-1} e^{-\beta \mathcal{H}_{\mathcal{E}}}$. Tracing both sides and using the normalisation condition $\text{Tr} \rho_{\mathcal{E}} = 1$ leads to $Z = \text{Tr} e^{-\beta \mathcal{H}_{\mathcal{E}}}$. We evaluate this trace over the number basis

$$\begin{aligned} Z &= \sum_{n_1 n_2 \dots n_m} \langle n_1 n_2 \dots n_m | \prod_q e^{-\beta \omega_q a_q^\dagger a_q} | n_1 n_2 \dots n_m \rangle \\ &= \sum_{n_1 n_2 \dots n_m} \prod_q e^{-\beta \omega_q n_q} \\ &= \sum_{n_1} \sum_{n_2} \dots \sum_{n_m} e^{-\beta \omega_1 n_1} e^{-\beta \omega_2 n_2} \dots e^{-\beta \omega_m n_m}, \end{aligned} \quad (3.17)$$

where we have used the fact that all ladder operators commute except for a_i^\dagger and a_i , to factorise the exponential. Summing over each individual n_q yields a product of geometric series

$$Z = \prod_q \frac{1}{1 - e^{-\beta \omega_q}}. \quad (3.18)$$

We may now use this expression to calculate the expected particle number of the q th mode as $\langle a_q^\dagger a_q \rangle = -\partial_{\beta \omega_q} \ln Z$, since the derivative brings an extra factor n_q into the sum (3.17). This evaluates to the Bose-Einstein distribution [58]

$$\langle a_q^\dagger a_q \rangle = \frac{1}{e^{\beta \omega_q} - 1}. \quad (3.19)$$

The real and imaginary parts of $\langle \mathcal{B}(\tau) \mathcal{B} \rangle$ are named the decoherence and dissipation/noise kernels, respectively. We write them as separate functions $\langle \mathcal{B}(\tau) \mathcal{B} \rangle = \nu(\tau) - i\eta(\tau)$. Substituting (3.19) into (3.16) yields the following expressions:

$$\nu(\tau) = \sum_q \gamma_q^2 \coth\left(\frac{\beta \omega_q}{2}\right) \cos(\omega_q \tau) \quad \text{and} \quad \eta(\tau) = \sum_q \gamma_q^2 \sin(\omega_q \tau), \quad (3.20)$$

where $1 + 2\langle a_q^\dagger a_q \rangle$ simplified⁴ to $\coth(\beta \omega_q / 2)$. Interestingly, this form (3.20) allows one to prove the Kubo-Martin-Schwinger (KMS) condition [3] via ordinary trigonometric identities, which in our case takes the form $\langle \mathcal{B}(\tau) \mathcal{B} \rangle = \langle \mathcal{B}(-\tau - i\beta) \mathcal{B} \rangle$.

Different baths

If we had instead considered a *fermionic bath*, we would have the Fermi-Dirac distribution [58]

$$\langle a_q^\dagger a_q \rangle = \frac{1}{e^{\beta \omega_q} + 1} \quad (3.21)$$

yielding the kernels [60]

$$\nu(\tau) = \sum_q \gamma_q^2 \cos(\omega_q \tau) \quad \text{and} \quad \eta(\tau) = \sum_q \gamma_q^2 \tanh(\beta \omega_q / 2) \sin(\omega_q \tau). \quad (3.22)$$

This also matches the solution for a *spin-bath*

$$\mathcal{H}_{\mathcal{E}} = - \sum_q \frac{\omega_q}{2} \sigma_q^z, \quad \mathcal{B} = \sum_q \gamma_q \sigma_q^x.$$

⁴The hyperbolic cotangent is defined as $\coth(x/2) \equiv \frac{e^x + 1}{e^x - 1} = 1 + \frac{2}{e^x - 1}$.

where σ_i^μ are the Pauli matrices acting on the i th spin

$$\sigma_i^\mu \equiv \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2 \otimes \sigma^\mu \otimes \mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2, \quad \mu \in \{x, y, z\}. \quad (3.23)$$

Comparing the zero temperature case of bosons and fermions, the thermal states tend to vacuum states $|n_1 n_2 \cdots n_m\rangle = |00 \cdots 0\rangle$, where differences between Bose and Fermi statistics disappear. This manifests itself in matching kernels, since $\tanh(\infty) = \coth(\infty) = 1$. As such these three problems coincide at zero temperature. For high temperature, however, things seem to change drastically.

3.2.1 Continuous field limit

For any finite number of harmonic oscillators in the bath we cannot have true, indefinite, equilibration⁵. This is due to Poincaré's recurrence theorem:

Thm 1 *Given a Hamiltonian with discrete eigenvalues there exists a time for which any initial state $|\psi(t_0)\rangle$ evolves to an arbitrarily close state [61]. Mathematically, for any $\epsilon > 0$ and time t_0 there exists a time $t > t_0$ such that*

$$\| |\psi(t)\rangle - |\psi(t_0)\rangle \|^2 < \epsilon,$$

where $\| |\psi\rangle \| = \sqrt{\langle \psi | \psi \rangle}$ is the vector norm.

So, in order for true non-reversible dynamics to occur we need to consider the continuum limit from the tuple of frequencies ω_q to a continuous variable ω . This would then correspond to a bosonic field which has continuous spectra for which thm 1 no longer applies. The γ_q terms in (2.14) then become functions of ω : $\gamma_q \rightarrow \gamma(\omega)$, and the sums in (3.20) map onto integrals $\sum_q \gamma_q^2 \rightarrow \int d\omega J_{SD}(\omega)$, where $J_{SD}(\omega)$ is the spectral density of the bath modes [16]. Before taking the limit we define the spectral density as

$$J_{SD}(\omega) = \sum_q \gamma_q^2 \delta(\omega - \omega_q), \quad (3.24)$$

such that the integral over continuous frequencies reproduces the sum over the bath's frequencies $\int_0^\infty d\omega J_{SD}(\omega) f(\omega) = \sum_q \gamma_q^2 f(\omega_q)$. From this we may formally write

$$\nu(\tau) = \int_0^\infty d\omega J_{SD}(\omega) \coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau), \quad \eta(\tau) = \int_0^\infty d\omega J_{SD}(\omega) \sin(\omega\tau), \quad (3.25)$$

after which we can safely take the limit.

We will now restrict ourselves to the general class of *Ohmic* dissipation $J_{SD}(\omega) = \omega j_{SD}(\omega)$, which is characterized by a linear, in ω , spectral density for low frequencies. This linear term is required to counteract the simple pole of the coth function in (3.25) at $\omega = 0$. The *cutoff* function $j_{SD}(\omega)$ exists only to regularise the integrals in (3.25). To see why such a regularisation is necessary, let us consider the ν integral in (3.25) with $j_{SD}(\omega) = 1$. The integrand is then

$$\omega \coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) = \omega \cos(\omega\tau) + \frac{2\omega \cos(\omega\tau)}{e^{\beta\omega} - 1}, \quad (3.26)$$

where the second term⁶, which is Planck's law for blackbody radiation [24] (without the cos function), decays. The first term in (3.26), which would disappear if we had first normal ordered⁷ $\mathcal{B}(\tau)\mathcal{B}$, has linearly growing oscillations associated with vacuum (zero-point) fluctuations.

⁵This is assuming that the system is finite.

⁶It evaluates to $T^2(\psi_1(1 - iT\tau) + \psi_1(1 + iT\tau))$, where ψ_1 is the trigamma function [14, page 32].

⁷Normal ordering, with respect to the vacuum, means that we consider $aa^\dagger = a^\dagger a$, instead of $aa^\dagger = a^\dagger a + 1$, hence ignoring the 1 (c-number) which is responsible for the divergence.

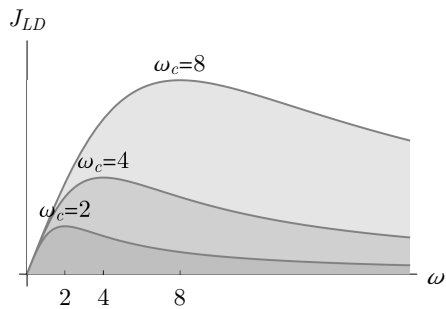


Figure 3.1: Plots of the Ohmic spectral densities with a Lorentz-Drude cutoff for different cutoff frequencies ω_c .

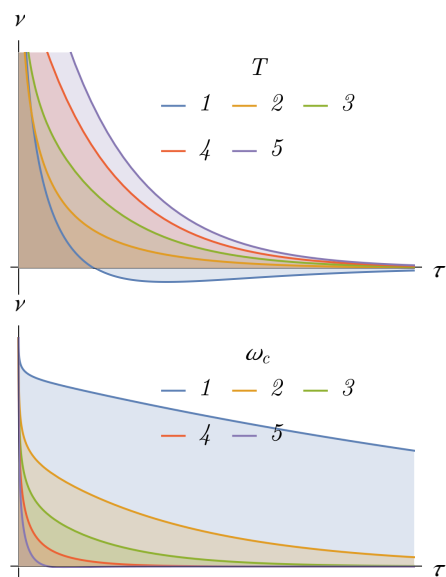


Figure 3.2: Plots of the decoherence kernel as a function of time. The top figure is for $\omega_c = 10$ with variable temperature T , while the bottom is for $T = 10$ with variable cutoff.

The integral over the linear term in (3.26) can be regularized by using its Fourier (cos) transform⁸

$$\int_0^\infty d\omega \omega \cos(\omega\tau) = -\tau^{-2}, \quad (3.27)$$

but then we encounter a second order pole at $\tau = 0$, which will only lead to divergences over the time integral in (3.13). Thus we require a cutoff.

We will use the *Lorentz-Drude* cutoff [16] described by

$$j_{LD}(\omega) = \frac{1}{\pi} \frac{\omega_c^2}{\omega_c^2 + \omega^2}, \quad (3.28)$$

where ω_c is the cutoff frequency, in the sense that $J_{LD}(\omega)$ has a single maximum at $\omega = \omega_c$, after which it decays to zero as seen in Fig. 3.1. The cut-off frequency is typically chosen such that the open system dynamics becomes independent of ω_c [16]. This cutoff leads to the following counter term (2.13)

$$\begin{aligned} \mathcal{H}_c &= e^2 S_x^2 \sum_q \frac{\gamma_q^2}{\omega_q} \\ &\rightarrow e^2 S_x^2 \int_0^\infty d\omega \frac{J_{LD}(\omega)}{\omega} \\ &= e^2 S_x^2 \int_0^\infty d\omega j_{LD}(\omega) \\ &= e^2 S_x^2 \frac{\omega_c}{2}. \end{aligned} \quad (3.29)$$

We may now test the concentration, in τ , of the correlations by evaluating the kernels (3.25). Starting with the noise kernel

$$\begin{aligned} \eta(\tau) &= \int_0^\infty d\omega \frac{1}{\pi} \omega \frac{\omega_c^2}{\omega_c^2 + \omega^2} \sin(\omega\tau) \\ &= \omega_c^2 e^{-\omega_c \tau} / 2, \end{aligned} \quad (3.30)$$

we find exponential decay around $\tau = 0$ at a rate given by the cutoff frequency ω_c . The decoherence kernel is calculated via the residue theorem [12], leaving

$$\nu(\tau) = \frac{\omega_c^2}{2} e^{-\omega_c \tau} \cot\left(\frac{\omega_c}{2T}\right) - \frac{\omega_c^2}{\pi} \sum_{n=1}^{\infty} \frac{|n| e^{-2\pi T \tau |n|}}{(\beta \omega_c / (2\pi))^2 - n^2}, \quad (3.31)$$

which is the sum of exponentially decaying functions. From this expansion we can deduce that the largest correlation time, corresponding to the slowest exponential decay, is $\tau_E = \text{Max}\{1/\omega_c, \beta/(2\pi)\}$ [12]. As seen from Fig. 3.2 the decoherence kernel also decays around $\tau = 0$. For low enough temperature it has a second smaller revival at $\tau = \mathcal{O}(\omega_c^{-1})$.

Choosing $J(\omega) = J_{LD}(\omega) \coth(\beta\omega/2) = J_{LD}(\omega) / \tanh(\beta\omega/2)$ for the spin/fermionic bath case (3.22) yields the same kernels as the bosonic case (3.25). As such the freedom of choosing the spectral density allows the two baths to act on the system in the same way.

⁸Integrals which are done without proof or reference were done in *Mathematica*. Particularly we have made extensive use of the Fourier sine and cosine routines.

3.3 The Markov (short memory) approximation

We have seen in Sect. 3.2 that the bath correlation function $\langle \mathcal{B}(\tau)\mathcal{B} \rangle_0$ concentrates around $\tau = 0$. We now assume that the system has a longer time scale, such that it may be considered constant over this clustered region. Then the integrand in (3.13)

$$\dot{\varrho}^{(I)}(t) = c^2 \int_0^t d\tau \langle \mathcal{B}(\tau)\mathcal{B} \rangle_0 [S_x(t-\tau)\varrho^{(I)}(t-\tau), S_x(t)] + \text{H.C.}, \quad (3.32)$$

will also concentrate around $\tau = 0$. From (3.31) and (3.30) we know that the concentration time is proportional to $\tau_{\mathcal{E}} = \text{Max}\{1/\omega_c, \beta/(2\pi)\}$. Hence, whatever the system time scale τ_S is we may pick a sufficiently large⁹ cutoff frequency or temperature, such that the system may be considered stationary over the contributing region of $\langle \mathcal{B}(\tau)\mathcal{B} \rangle$ [21]. In this case we may then make the replacement $\varrho^{(I)}(t-\tau) \rightarrow \varrho^{(I)}(t)$ in the integrand of (3.32), which is known as the *Markov approximation*. In truth this approximation limits us to a coarse grained time $t \in \tau_{\mathcal{E}}\mathbb{Z}$ at which we may “look” at the system [52].

Since the integrand (3.32) decays strongly for $\tau > 0$, we may push the upper bound of the integral to infinity and beyond

$$\dot{\varrho}^{(I)}(t) = c^2 \int_0^\infty d\tau \langle \mathcal{B}(\tau)\mathcal{B} \rangle_0 [S_x(t-\tau)\varrho^{(I)}(t), S_x(t)] + \text{H.C.}, \quad (3.33)$$

yielding the time-local *Born-Markov* master equation. Transforming (3.33) back into the Schrödinger picture we obtain

$$\dot{\varrho}(t) = i[\varrho(t), \mathcal{H}_S + c^2 S_x^2 \frac{\omega_c}{2}] + [S_{\mathcal{B}}\varrho(t), S_x] + \text{H.C.}, \quad (3.34)$$

where we have inserted the evaluated counter term \mathcal{H}_c (3.29) and defined the operator

$$S_{\mathcal{B}} \equiv c^2 \int_0^\infty d\tau \langle \mathcal{B}(\tau)\mathcal{B} \rangle_0 S_x(-\tau). \quad (3.35)$$

In order to assess whether (3.34) is a quantum dynamical semi-group it will serve us to separate its unitary contribution from the rest. That is, we wish to write it in “Lindblad” form $\dot{\varrho}(t) = [\mathcal{U} + \mathcal{D}_{\text{BM}}]\varrho(t)$. The unitary dynamics is driven by the *unitor*¹⁰ $\mathcal{U}\varrho = i[\varrho, \mathcal{H}_S + \mathcal{H}_\gamma]$, where

$$\mathcal{H}_\gamma \equiv \frac{1}{2i} [S_x S_{\mathcal{B}} - S_{\mathcal{B}}^\dagger S_x] + c^2 \frac{\omega_c}{2} S_x^2 \quad (3.36)$$

is the coupling contribution to said unitary dynamics. The dissipator is defined as

$$\mathcal{D}_{\text{BM}}\varrho \equiv S_{\mathcal{B}}\varrho S_x - \frac{1}{2}\{S_x S_{\mathcal{B}}, \varrho\} + \text{H.C.}, \quad (3.37)$$

where $\{A, B\} \equiv AB + BA$ is the anticommutator. This term refers to dissipation/contraction of the state space [22]: $|e^{\mathcal{D}t}\mathbb{S}(\mathcal{H}_S)| < |\mathbb{S}(\mathcal{H}_S)|$, rather than dissipation of energy. In other words, states evolve to a subset of the original state space $\mathbb{S}(\mathcal{H}_S)$. This holds for any positive definite dissipator \mathcal{D} . For the dissipator to be extensive, like the Hamiltonian, we require a coupling of

$$c^2 = \gamma S^{-1},$$

where γ is the system-size independent coupling strength.

To study whether a system thermalises it is imperative that one does not make the secular or rotating wave approximation, which ignores the fast oscillating frequencies. This is because it always leads to the Gibbs distribution being stationary [12, 59, 63].

⁹Due to this approximation we are unable to study the zero temperature limit. This does not mean that such a master equation cannot be derived in this limit [7]. Seeing as we will find that our intensive temperature master equation predicts expectation values matching those for zero temperature the insight such an equation would yield is likely minimal.

¹⁰Technically this part can be called the Liouvillian, but this term is often used interchangeably with another object we will encounter called the Lindbladian. Thus to avoid confusion and to match with the sound of dissipator we will use the term of unitor for the unitary part.

3.3.1 Explicit form of the coupling operator

To further simplify the master equation we still have to calculate $S_{\mathcal{B}}$ by evaluating the integral (3.35), which includes the integrand $S_x(-\tau)$. Unfortunately, as stated before, solving for the time evolution of S_x by use of the BCH formula (3.3) or via the Bethe ansatz, does not provide an answer in closed form. However, under the condition that $\tau_S \gg \tau_{\mathcal{E}}$ we may approximate $S_x(-\tau)$ under the integral by the first few terms of the BCH solution [12, ch. 3.6]

$$\begin{aligned} S_x(-\tau) &= S_x - i[\mathcal{H}_S + \mathcal{O}(c^2), S_x]\tau + \mathcal{O}(\tau^2) \\ &= S_x - i[-\hbar S_z, S_x]\tau + \mathcal{O}(\tau^2) + \mathcal{O}(c^2) \\ &= S_x - \tau\hbar S_y + \mathcal{O}(\tau^2) + \mathcal{O}(c^2). \end{aligned} \quad (3.38)$$

Using this series expansion (3.38) one may show that (3.35) simplifies to¹¹

$$S_{\mathcal{B}} \approx c^2 [(T - i\omega_c/2)S_x - \hbar(\nu_1 - i/2)S_y], \quad (3.39)$$

where we have ignored order $\mathcal{O}(\tau_{\mathcal{E}}^2)$ and $\mathcal{O}(c^2)$ terms.

Further we have defined

$$\nu_1 \equiv \int_0^{\infty} d\tau \nu(\tau)\tau, \quad q \equiv \frac{\beta\omega_c}{2\pi} \quad (3.40)$$

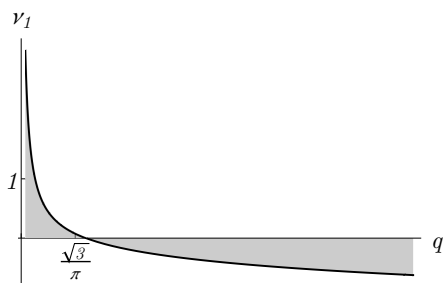


Figure 3.3: Plot of $\nu_1(q)$.

which is plotted in Fig. 3.3. In our case (3.40) will typically contribute at a lower order in S than the other terms in (3.39). Thus the explicit expression of this integral, given in (A.70), will be of little importance to us. From Fig. 3.3 we note that ν_1 has a unique zero at $q \approx \sqrt{3}/\pi$.

Using (3.39) we may now calculate \mathcal{H}_γ (3.36). Here the (anti)Hermitian parts of $S_{\mathcal{B}}$ will form (anti)commutators, thus the TS_x term cancels and we are left with

$$\begin{aligned} \mathcal{H}_\gamma/c^2 &= \frac{1}{2i} \frac{i\hbar}{2} \{S_x, S_y\} + \frac{1}{2i} \hbar \nu_1 [S_x, S_y] - \frac{\omega_c i}{2} \{S_x, S_x\} + \frac{\omega_c}{2} S_x^2 \\ &= \frac{1}{2} \frac{\hbar}{2} \{S_x, S_y\} + \frac{1}{2} \hbar \nu_1 S_z - \frac{\omega_c}{2} S_x^2 + \frac{\omega_c}{2} S_x^2 \\ &= \frac{\hbar}{4} \{S_x, S_y\} + \frac{\hbar \nu_1}{2} S_z, \end{aligned} \quad (3.41)$$

where the linear in ω_c dependent term has canceled with the counter term. The ν_1 contribution to unitor is only of order $\mathcal{O}(S^0)$, since $c^2 = \mathcal{O}(S^{-1})$. In comparing this to the $\mathcal{O}(S^1)$ contribution of \mathcal{H}_S , we find that the unitor is effectively independent of ω_c in the thermodynamic limit. As such the counter term has ensured that system's unitary dynamics remains unperturbed by large bath frequency. Further when inserting $S_{\mathcal{B}}$ (3.39) into the dissipator (3.37)

$$\mathcal{D}_{\text{BM}}\varrho \equiv S_{\mathcal{B}}\varrho S_x - \frac{1}{2}\{S_x S_{\mathcal{B}}, \varrho\} + \text{H.C.}, \quad (3.42)$$

we note that the $i\omega_c$ contribution cancels:

$$i\omega_c S_x \varrho S_x - \frac{1}{2}\{S_x i\omega_c S_x, \varrho\} + \text{H.C.} = 0. \quad (3.43)$$

¹¹Some of these integrals are evaluated in [12, ch. 3.6], see Appendix A.5 for further details on these calculations.

Since the linear in ω_c contribution to S_B cancels, we are left with an effective operator S'_B

$$S'_B = c^2 T S_x + c^2 \frac{i\hbar}{2} S_y + c^2 \hbar \nu_1 S_y. \quad (3.44)$$

We may now check whether the Born-Markov master equation (3.34) preserves the rules of quantum mechanics.

3.4 The Lindbladian

It was shown by Lindblad [43] in 1976 that the most general time-local differential equation $\dot{\rho} = \mathcal{L}\rho = (\mathcal{D} + \mathcal{U})\rho$, which is also trace, positivity and hermiticity preserving, is the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation, where

$$\mathcal{U}\rho = i[\rho, \mathcal{H}], \quad \mathcal{D}\rho = \sum_{k,l} K_{kl} \left[L_k \rho L_l^\dagger - \frac{1}{2} \{L_l^\dagger L_k, \rho\} \right]. \quad (3.45)$$

Here the terms like $L_k \rho L_l^\dagger$, are responsible for *quantum jumps*, transitions from one state to another, earning L_k the name of *jump operator*, while the anticommutators ensure normalisation in case of no jumps¹². The generator of this quantum dynamical semi-group \mathcal{L} is the *superoperator* known as the *Lindbladian*. For (3.45) to describe a GKSL equation the following conditions must be satisfied [12]:

- (i) The Hamiltonian \mathcal{H} must be Hermitian.
- (ii) The jump operators L_k must be traceless and mutually orthonormal with respect to the Hilbert-Schmidt inner product $\langle L_k | L_l \rangle = \text{Tr}\{L_k^\dagger L_l\} = \delta_{kl}$.
- (iii) The coefficient matrix K must be Hermitian and positive semi-definite, meaning all eigenvalues are non-negative.

With respect to our choice of jump operators, the convenient option satisfying ii is S_x and S_y , which need only be normalized. Next, since the Hamiltonian

$$\mathcal{H} \equiv \mathcal{H}_S + \mathcal{H}_\gamma = \mathcal{H}_S + \frac{\hbar\gamma}{4S} \{S_x, S_y\} + \frac{\hbar\nu_1\gamma}{2S} S_z \quad (3.46)$$

is already Hermitian, we only have to prove condition iii. We find the coefficient matrix K by rearranging the Born-Markov dissipator (3.37)

$$\begin{aligned} \mathcal{D}_{\text{BM}}\rho &= S'_B \rho(t) S_x - \frac{1}{2} \{S_x S'_B, \rho(t)\} + \text{H.C.} \\ &= \left(\frac{\kappa_{xx}}{2} S_x + \kappa_{yx} S_y \right) \rho(t) S_x - \frac{1}{2} \left\{ S_x \left(\frac{\kappa_{xx}}{2} S_x + \kappa_{xy} S_y \right), \rho(t) \right\} + \text{H.C.} \\ &= \sum_{k,l \in \{x,y\}} \kappa_{kl} \left[S_k \rho S_l - \frac{1}{2} \{S_l S_k, \rho\} \right]. \end{aligned} \quad (3.47)$$

Inserting the coefficients from (3.44) we have

$$K \equiv \begin{bmatrix} \kappa_{xx} & \kappa_{xy} \\ \kappa_{yx} & \kappa_{yy} \end{bmatrix} = \frac{\gamma}{2S} \begin{bmatrix} 4T & \hbar(2\nu_1 - i) \\ \hbar(2\nu_1 + i) & 0 \end{bmatrix} \quad (3.48)$$

which is Hermitian, but not positive semidefinite. This is because K has a negative determinant

$$\det(K) = \kappa_{xx}\kappa_{yy} - |\kappa_{xy}|^2 = -|\kappa_{xy}|^2 < 0, \quad (3.49)$$

¹²The validity of these statements are more apparent in the Kraus operator formalism [52, ch. 3.5]

which implies a negative eigenvalue. To restore positivity we find the *minimally invasive* change to K such that $\det(K) = 0$ [12, ch. 3.6], implying one zero and one positive eigenvalue. From (3.49) the solution is $\kappa_{yy} = |\kappa_{xy}|^2 / \kappa_{xx}$. Now writing K in its spectral decomposition we note that the zero eigenvalue term comes with a prefactor of zero, and we only have to consider the non-zero eigenvalue contribution¹³

$$L \equiv \left[\sqrt{\kappa_{xx}} S_x + \frac{\kappa_{yx}}{\sqrt{\kappa_{xx}}} S_y \right] = \sqrt{\kappa_{xx}} \left[S_x + \frac{\kappa_{yx}}{\kappa_{xx}} S_y \right] = \sqrt{\frac{2\gamma T}{S}} \left[S_x + \frac{\hbar\beta(i + 2\nu_1)}{4} \right]. \quad (3.50)$$

The dissipator then simplifies to

$$\mathcal{D}\varrho = L\varrho L^\dagger - \frac{1}{2}\{L^\dagger L, \varrho\}. \quad (3.51)$$

Since this approximation's validity is dependent on κ_{yy} being small, it is also dependent on ν_1 being small. In Chap. 5 we will choose $\omega_c \approx 2\sqrt{3}T$ in which case $\nu_1 = 0$, as seen in Fig. 3.3.

With some tedious manipulation and cancellations between the unitor and dissipator the full Lindbladian may be written as

$$\begin{aligned} \mathcal{L}\varrho = & i[\varrho, \mathcal{H}_S] - \frac{\kappa_{xx}}{2}[S_x, [S_x, \varrho]] - i\Im\kappa_{yx}[S_x, \{S_y, \varrho\}] \\ & - \frac{|\kappa_{xy}|^2}{\kappa_{xx}}[S_y, [S_y, \varrho]] + 2\Re\kappa_{xy}[S_y, [S_x, \varrho]], \end{aligned} \quad (3.52)$$

as shown in Appendix A.2. This is the most general form of our quantum dynamical semi-group (3.52). It has been written in a form such that we may use known commutation relations to prove that certain states are stationary.

3.5 The stationary states

Now that we have derived the quantum dynamical semi-group (3.52), we can turn our attention to the main question of this study: does the system equilibrate and if so to what stationary state? One immediate observation is that, as opposed to unitary dynamics, the anticommutator term in (3.52) prevents $\varrho \propto \mathbb{1}$ from being a stationary solution. This reflects the fact that, in general, the stationary state is not of infinite temperature $\lim_{\beta \rightarrow 0} e^{-\beta\mathcal{H}_S} = \mathbb{1}$. To further study this question we consider (3.52) in various parameter regimes.

3.5.1 Pure dephasing

The simplest parameter regime of our semi-group is for zero magnetic field $\hbar = 0$. In this case the Lindbladian (3.52)

$$\mathcal{L}\varrho = -\frac{iJ}{2S}[\varrho, S_x^2] - \frac{\gamma T}{S}[S_x, [S_x, \varrho]], \quad (3.53)$$

is already diagonal, with eigenoperators¹⁴ $|S_x = p\rangle\langle S_x = q|$. This equation (3.53) indicates that there exists many stationary states, such as $|S_x = p\rangle\langle S_x = p|$. As such $\varrho(\infty)$ does not necessarily correspond to the Gibbs distribution, meaning that in the case of $\hbar = 0$ the system does *not* thermalise in the sense of def 1. To obtain the time evolved state $\varrho(t)$ we expand it in the S_x basis $\varrho(t) = \sum_{pq} \varrho_{pq} |S_x = p\rangle\langle S_x = q|$, which we substitute into (3.53) yielding

$$\dot{\varrho}_{pq} = \left[-\frac{iJ}{2S}(q^2 - p^2) - \frac{\gamma T}{S}(p - q)^2 \right] \varrho_{pq}, \quad (3.54)$$

¹³As we only have one jump operator we may leave L in its unnormalized form without causing confusion.

¹⁴An eigenoperator A of a matrix B satisfies $[B, A] = \lambda A$, where λ is the eigenvalue. [24]. If we label the eigenstates of B as $|B_n\rangle$, then its eigenoperators are merely the projection operators $|B_n\rangle\langle B_m|$.

which is solved by

$$\varrho_{pq}(t) = \exp\left(-\frac{iJt}{2S}(q^2 - p^2) - \frac{\gamma Tt}{S}(p - q)^2\right) \varrho_{pq}(0). \quad (3.55)$$

This solution (3.55) indicates that all off-diagonal terms $p \neq q$ decay exponentially to zero, at a rate determined by the temperature T . As such we are left with a stationary state

$$\varrho(\infty) = \sum_p \varrho_{pp} |S_x = p\rangle \langle S_x = p|,$$

which is energy diagonal, but more importantly diagonal in the coupling (S_x) operator's basis. As such this is a pure dephasing process, where the act of interacting with a system via a particular operator, in this case S_x , acts as a (weak) measurement. After the state has decohered, become diagonal, it has lost all quantum interference terms. The weights ϱ_{pp} may then be interpreted as classical probabilities encoding ignorance (lack of information) rather than quantum uncertainty. These weights are purely determined by the initial conditions. Remembering that the T contribution came from ν it becomes clear that it earns the name of decoherence kernel.

3.5.2 Extensive temperature

Hoping to study thermalisation in the sense of def 1, which is an energy dissipative process, we will not set $\hbar = 0$. Due to this its appearance will unnecessarily complicate the expressions to come, and we will rescale time by \hbar for now. We now attempt to answer the main question of this study: is the Gibbs distribution $e^{-\beta\mathcal{H}_S}$ ¹⁵ a stationary state of the Lindbladian:

$$\hbar^{-1} \mathcal{L} e^{-\beta\mathcal{H}_S} = 0?$$

To answer this, one may insert $e^{-\beta\mathcal{H}_S}$ into (3.52). To use the spin commutation relations we need to consider the terms in the Taylor series

$$e^{-\beta\mathcal{H}_S} = \mathbb{1} - \beta\mathcal{H}_S + \frac{\beta^2}{2!} \mathcal{H}_S^2 + \mathcal{O}(S^3) \dots \quad (3.56)$$

individually. Recall from Sect. 2.1 that the spin angular momentum S is half the particle number $N = 2S$. This means that the terms in this series (3.56) diverge in the thermodynamic limit $N \rightarrow \infty$. This changes with an extensive temperature $T = (S\hbar)\tilde{T}$, with $\tilde{T} = \mathcal{O}(S^0)$, in which case $\beta = 1/(S\hbar\tilde{T}) = \mathcal{O}(S^{-1})$ and the series (3.56) is of order $\mathcal{O}(S^0)$. We may then study (3.56) term by term in the large S limit.

To take the large-spin limit, in a well-controlled way, we need to consider intensive quantities such as the rescaled spin operators¹⁶ $\hat{\mathbf{s}} = (S_x, S_y, S_z)/S = (\hat{x}, \hat{y}, \hat{z})$ and the rescaled Hamiltonian

$$\hat{h} \equiv \frac{\mathcal{H}_S}{S\hbar} = -\frac{\Lambda}{2} \hat{x}^2 - \hat{z}, \quad \Lambda \equiv \frac{J}{\hbar}. \quad (3.57)$$

The series (3.56) may then be written as

$$e^{-\tilde{\beta}\hat{h}} = \sum_{n=0}^{\infty} \frac{(-\tilde{\beta})^n}{n!} \hat{h}^n = \mathbb{1} - \tilde{\beta}\hat{h} + \frac{(-\tilde{\beta})^2}{2} \hat{h}^2 + \dots, \quad (3.58)$$

where $\tilde{\beta} = \tilde{T}^{-1}$. For an extensive temperature we have $\kappa_{xx} = \mathcal{O}(S^0)$ while $\kappa_{xy} = \mathcal{O}(S^{-1})$ ¹⁷, which allows us to write the Lindbladian (3.52) as

$$\hbar^{-1} \mathcal{L} \varrho = i[\varrho, \mathcal{H}_S/\hbar] - \gamma\tilde{T}[S_x, [S_x, \varrho]] - i\gamma/2 [S_x, \{S_y, \varrho\}]/S$$

¹⁵Here we have not normalised the Gibbs state, however this plays no role in the calculations to come.

¹⁶The reason for this notational choice of the hats will become clear later on. *Their intensive fear of melanoma.*

¹⁷Here we have considered $\nu_1 = \mathcal{O}(S^0)$, which in the case of extensive temperatures means considering an extensive cut-off as well, since in the large temperature regime ν_1 goes like T/ω_c .

$$- \mathcal{O}(S^{-1})[S_y, [\hat{y}, \varrho]] + \mathcal{O}(S^0)[S_y, [\hat{x}, \varrho]]. \quad (3.59)$$

Only the coupling's contribution in (3.59) remain, as $[e^{-\beta\mathcal{H}_S}, \mathcal{H}_S] = 0$. The identity term, in the Taylor series (3.58) commutes with all terms in (3.59), leaving only an anticommutator contribution. Inserting (3.58) into (3.59) and grouping the same orders in $\tilde{\beta}$ together then yields

$$\begin{aligned} (\gamma\hbar)^{-1}\mathcal{L}e^{-\tilde{\beta}\hat{h}} = & - \left[S_x, \sum_{n=0}^{\infty} \frac{(-\tilde{\beta})^{n+1}}{(n+1)!} \frac{1}{\tilde{\beta}} [S_x, \hat{h}^{n+1}] + \frac{(-\tilde{\beta})^n}{n!} \frac{i}{2} \{y, \hat{h}^n\} \right] \\ & + \sum_{n=0}^{\infty} \frac{(-\tilde{\beta})^{n+1}}{(n+1)!} \left[\mathcal{O}(S^0)[S_y, [\hat{x}, \hat{h}^{n+1}]] - \mathcal{O}(S^{-1})[S_y, [\hat{y}, \hat{h}^{n+1}]] \right]. \end{aligned} \quad (3.60)$$

We wish to show that the Gibbs state is stationary in the thermodynamic limit, meaning that (3.60) tends to zero as $S \rightarrow \infty$. To do this we use the fact that the rescaled spin operators commute in the large S limit:

$$[\hat{\mu}, \hat{\nu}] = \frac{1}{S^2} [S_\mu, S_\nu] = \epsilon_{\mu\nu\zeta} \frac{S_\zeta}{S^2} = \epsilon_{\mu\nu\zeta} \frac{\hat{\zeta}}{S} \rightarrow 0, \quad (3.61)$$

which allows one to prove the following bound (A.7)

$$\| [S_\mu, [\hat{\nu}, \hat{\zeta}^{n+1}]] \| \leq n^4 \mathcal{O}(S^{-1}) \| \hat{\zeta} \|^n. \quad (3.62)$$

The inequality (3.62) shows that the double commutators contribute to a lower order, for intensive powers of rescaled spin operators, than the rest of \mathcal{L} in (3.60).

Back to the problem at hand, we use (3.62) to bound the second sum in (3.60)

$$\sum_{n=0}^{\infty} \frac{|(-\tilde{\beta})^{n+1}|}{(n+1)!} \left[\mathcal{O}(S^0) \| [S_y, [\hat{x}, \hat{h}^{n+1}]] \| + \mathcal{O}(S^{-1}) \| [S_y, [\hat{y}, \hat{h}^{n+1}]] \| \right] \quad (3.63)$$

$$\leq \sum_{n=0}^{\infty} \frac{(\tilde{\beta})^{n+1} n^4 \| \hat{h} \|^n}{(n+1)!} (\mathcal{O}(S^{-1}) + \mathcal{O}(S^{-2}))$$

$$\leq \mathcal{O}(S^{-1}) \sum_{n=0}^{\infty} \frac{(\tilde{\beta})^{n+1} n^4 \| \hat{h} \|^n}{(n+1)!}$$

$$= \mathcal{O}(S^{-1}) \frac{e^{\tilde{\beta} \| \hat{h} \|} (\tilde{\beta} \| \hat{h} \| (\tilde{\beta} \| \hat{h} \| (\tilde{\beta} \| \hat{h} \| + 1)^2 - 1) + 1) - 1}{\| \hat{h} \|} \quad (3.64)$$

$$= \mathcal{O}(S^{-1}). \quad (3.65)$$

The first commutator in (3.63) came from the real part of κ_{xy} , corresponding to ν_1 . The second term in (3.63) $\mathcal{O}(S^{-1})[S_y, [\hat{y}, \hat{h}^{n+1}]]$ was the term we were forced to add for positivity preservation. Thus (3.60) reduces to

$$(\gamma\hbar)^{-1}\mathcal{L}e^{-\tilde{\beta}\hat{h}} = \sum_{n=0}^{\infty} \frac{(-\tilde{\beta})^n}{(n+1)!} \left[S_x, [S_x, \hat{h}^{n+1}] - \frac{n+1}{2} \{i\hat{y}, \hat{h}^n\} \right] + \mathcal{O}(S^{-1}). \quad (3.66)$$

To evaluate the commutators in (3.66) we make use of

$$[A, B^{n+1}] = \frac{n+1}{2} \{B^n, [A, B]\} + \frac{n-1}{2} \sum_{j=0}^{n-2} J_j + \sum_{k=0}^{n-3} k(n-k-1) J_k \quad (3.67)$$

with $J_q \equiv -B^{n-2-q} [[A, B], B], B] B^q$, which is proven in (A.1). The first nested commutator $[S_x, \hat{h}^{n+1}]$ in (3.66) may then be written as

$$[S_x, \hat{h}^{n+1}] = \frac{n+1}{2} \{ \hat{h}^n, [S_x, \hat{h}] \} + \frac{n-1}{2} \sum_{j=0}^{n-2} J_j + \sum_{k=0}^{n-3} k(n-k-1) J_k$$

$$= \frac{n+1}{2} \{i\hat{y}, \hat{h}^n\} + \frac{n-1}{2} \sum_{j=0}^{n-2} J_j + \sum_{k=0}^{n-3} k(n-k-1)J_k. \quad (3.68)$$

The anticommutator term in (3.68) cancels with the nested anticommutator term in (3.66), leaving

$$(\gamma\hbar)^{-1} \mathcal{L}e^{-\tilde{\beta}\hat{h}} = \sum_{n=0}^{\infty} \frac{(-\tilde{\beta})^n}{(n+1)!} \left[S_x, \frac{n-1}{2} \sum_{j=0}^{n-2} J_j + \sum_{k=0}^{n-3} k(n-k-1)J_k \right] + \mathcal{O}(S^{-1}). \quad (3.69)$$

Lastly, noting that $J_q \equiv -\hat{h}^{n-2-q}[[[S_x, \hat{h}], \hat{h}], \hat{h}] \hat{h}^q = \hat{h}^{n-2-q} \mathcal{O}(S^{-2}) \hat{h}^q$, leaves only

$$(\gamma\hbar)^{-1} \mathcal{L}e^{-\tilde{\beta}\hat{h}} = \sum_{n=0}^{\infty} \frac{(-\tilde{\beta})^n}{(n+1)!} \mathcal{O}(S^{-1}). \quad (3.70)$$

Hidden in this $\mathcal{O}(S^{-1})$, in (3.69), are powers and polynomials in n . The factorial in (3.69) ensure that this evaluates to a finite sum similar in appearance to (3.64). As this sum is finite we have $\mathcal{L}e^{-\tilde{\beta}\hat{h}} = \mathcal{O}(S^{-1})$, in other words, we have that the Gibbs distribution is stationary for large S and extensive temperature.

To motivate thermalisation we calculate the real part of the gap between the stationary state, with eigenvalue ℓ_0 and its closest state numerically¹⁸, plotted in Fig. 3.4. The reason for this notation, and the meaning of the gap, will become clearer in Chap. 5. For now it is only important to note that the stationary eigenvalue is zero $\ell_0 = 0$ and as such the gap is merely $|\ell_1|$. If this gap is non-zero then the system will equilibrate to this unique stationary state. As we can see from Fig. 3.4, the numerics would indicate that the gap is indeed non-zero for an extensive temperature. As we know that the stationary state corresponds to the Gibbs state this together implies thermalisation.

Lastly we mention that this proof cannot be repeated for non-extensive temperatures as we have made use of the intensivity of all the terms in the series expansion (3.56). For intensive temperatures we will require a more refined approach.

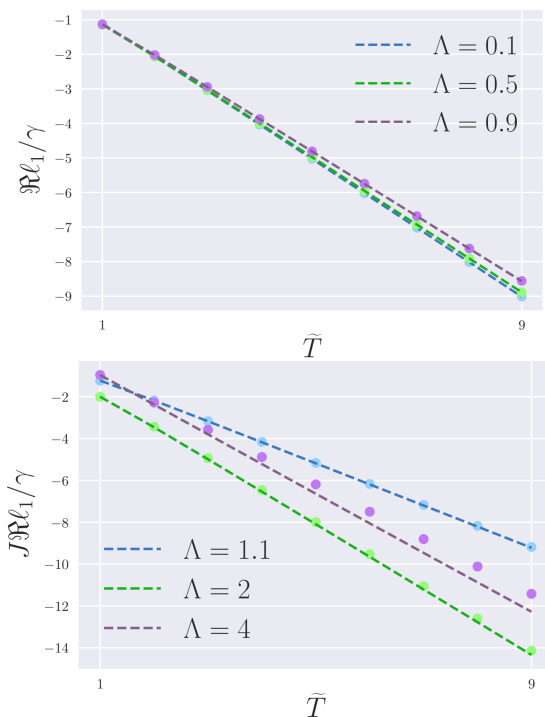


Figure 3.4: Real part of Lindbladian gap $\Re\ell_1$ for extensive temperatures. The circle plots correspond to $S = 10$, while dotted lines are for $S = 40$, plotted together to motivate convergence in the thermodynamic limit.

¹⁸All numerical work in this thesis is done in *Python*. In particular we make extensive use of the exact diagonalisation routines in the *QuTiP* library [33].

4 | Evolution of Observables

In the previous chapter we derived an approximate differential equation, known as a GKSL master equation (3.52), which describes the time evolution of the density matrix ϱ . It turns out, however, that, except for simple cases such as $\hbar = 0$, studying the time evolution of ϱ can be quite involved. Since our eventual goal is to prove equilibration to the thermal state, we are only interested in finding the stationary state(s) and those close to it. This means that for our purposes we only need to diagonalise the system in its “semi-stationary subspace” $\mathcal{L}\varrho \approx 0$. One way to restrict the analysis to such a region is via another spin-boson (bosonisation) mapping known as the Holstein-Primakoff mapping [30]

$$S_+ \rightarrow \sqrt{S}\sqrt{1 + \hat{z}} a, \quad (4.1)$$

which we will fully introduce in Chap. 5. If we are able to reduce the Lindbladian to being quadratic in these bosons, then we may diagonalize it via a Bogoliubov transform [2], introduced in Sect. 5.2.2. In hopes of obtaining such a quadratic operator one expands the square root in (4.1) around the region of interest, which in our case is the equilibrium region $\hat{z} \rightarrow \langle \hat{z} \rangle_{\text{eq}}$. For example, expanding around $\hat{z} = 1$ yields an operator of the form

$$S_+ \rightarrow \sqrt{2S}a + \mathcal{O}(S^{-1/2}). \quad (4.2)$$

This expansion will become exact, in the region of the expansion, in the thermodynamic limit $N = 2S \rightarrow \infty$.

To perform this expansion (4.2) one requires the equilibrium expectation values, in particular $\langle \hat{z} \rangle_{\text{eq}}$. To find $\langle \hat{z} \rangle_{\text{eq}}$ we turn our attention to the Heisenberg picture, where we study the time evolution of an observable instead of the density matrix.

4.1 The adjoint master equation

To find the equation of motion (EOM) for some observable \bullet , we consider its time evolved expectation value $\langle \bullet \rangle(t) = \text{Tr}\{\bullet e^{\mathcal{L}t}\varrho\}$. Seeing as any observable is Hermitian, we may write

$$\langle \bullet \rangle(t) = \text{Tr}\left\{(e^{\mathcal{L}^\dagger t}\bullet)^\dagger \varrho\right\} = \text{Tr}\{\bullet(t)\varrho\},$$

which implies that the generator of time evolution in the Heisenberg picture is the adjoint of the Lindbladian. The corresponding EOM is $\partial_t \bullet = \mathcal{L}^\dagger \bullet$, earning it the name of *adjoint master equation* [12]. In the general Lindblad form (3.45) $\mathcal{L}^\dagger \bullet$ is given by

$$\mathcal{L}^\dagger \bullet = i[\mathcal{H}, \bullet] + L^\dagger \bullet L - \frac{1}{2}\{L^\dagger L, \bullet\}. \quad (4.3)$$

which satisfies $\mathcal{L}^\dagger \mathbb{1} = 0$, as opposed to the ordinary master equation (3.52) where $\mathcal{L}\mathbb{1} \neq 0$. This reflects probability conservation, as $\langle \mathbb{1} \rangle = \text{Tr}\varrho$. Considering \mathcal{L} in (3.59), the adjoint master equation takes the form

$$\hbar^{-1}\mathcal{L}^\dagger \bullet = i[\mathcal{H}_S/\hbar, \bullet] - \gamma\tilde{T}[S_x, [S_x, \bullet]] + i\gamma/2\{\hat{y}, [S_x, \bullet]\} + [S_y, \mathcal{O}(S^{-1})[\hat{y}, \bullet] + \mathcal{O}(S^0)[\hat{x}, \bullet]]. \quad (4.4)$$

By studying (4.4) we may calculate the required expectation values. The $\mathfrak{su}(2)$ commutation relations (2.5)¹ would allow us to simplify (4.4) for some observable \bullet . However since \mathcal{L}^\dagger is a contracting operator [12] $\|\bullet(t)\|_1 \leq \|\bullet\|_1$, where $\|\bullet\|_1 \equiv \text{Tr}\sqrt{\bullet\bullet^\dagger}$, the commutation relations need not hold at later times. For instance, we may have $S_x(\infty) = 0$, in which case $[S_x(\infty), S_y(\infty)] = 0$,

¹This is something we could not do before when studying a general initial state $\varrho(0)$ with (3.59), and the reason why studying (4.4) is typically easier.

which breaks the $\mathfrak{su}(2)$ relations unless $iS_z(\infty) = 0$. For a closed system the $\mathfrak{su}(2)$ relations hold at all times due to the unitor \mathcal{U} being a *derivation* which satisfies the factorisation rule

$$e^{\mathcal{U}^\dagger t} AB = [e^{\mathcal{U}^\dagger t} A][e^{\mathcal{U}^\dagger t} B] = A(t)B(t).$$

This is because the (unitary) time evolution operator $U(t)$ allows one to write

$$[A(t), B(t)] = [U(t)AU^\dagger(t), U(t)BU^\dagger(t)] = U(t)ABU^\dagger(t) - U(t)BAU^\dagger(t) = U(t)[A, B]U^\dagger(t).$$

In general a Lindbladian is not a derivation, but a weaker property still holds due to:

$$\mathcal{L}^\dagger \bullet(t) = \mathcal{L}^\dagger e^{\mathcal{L}^\dagger t} \bullet = e^{\mathcal{L}^\dagger t} \mathcal{L}^\dagger \bullet.$$

As such we can evaluate $\mathcal{L}^\dagger \bullet$ at time zero, where the spin operators are unchanged from their initial $\mathfrak{su}(2)$ form², and only then do we evolve them. Using this for some rescaled spin operator $\hat{\mu}$, (4.4) reduces to

$$\begin{aligned} \hbar^{-1} \mathcal{L}^\dagger \hat{\mu} &= i[\mathcal{H}_S/\hbar, \hat{\mu}] - \gamma \tilde{T}[S_x, [S_x, \hat{\mu}]] + i\gamma \hat{y}[S_x, \hat{\mu}] + [S_y, \mathcal{O}(S^{-1})[\hat{y}, \hat{\mu}] + \mathcal{O}(S^0)[\hat{x}, \hat{\mu}]] \\ &= i[\mathcal{H}_S/\hbar, \hat{\mu}] - \gamma \tilde{T}[S_x, [S_x, \hat{\mu}]] + i\gamma \hat{y}[S_x, \hat{\mu}] + \mathcal{O}(S^{-1}) \\ &\sim i[\mathcal{H}_S/\hbar, \hat{\mu}] - \gamma \tilde{T}[S_x, [S_x, \hat{\mu}]] + i\gamma \hat{y}[S_x, \hat{\mu}], \end{aligned} \quad (4.5)$$

where we have used (3.62) to bound the double commutators. Here we have used the symbol \sim to mean asymptotic similarity, defined as

$$f(S) \sim g(S) \Leftrightarrow \lim_{S \rightarrow \infty} \frac{\|f(S)\|}{\|g(S)\|} = 1. \quad (4.6)$$

With this simplified equation (4.5) it is easy to check whether \mathcal{L} is a *derivation*. One rule which a derivation must satisfy is the product rule $\partial_t(\hat{\mu}\hat{\nu}) = \partial_t(\hat{\mu})\hat{\nu} + \hat{\mu}\partial_t(\hat{\nu})$. Using the identity $[A, BC] = B[A, C] + [A, B]C$, we find that the double commutator in (4.5) breaks this rule

$$[S_x, [S_x, \hat{\mu}\hat{\nu}]] = [S_x, \hat{\mu}[S_x, \hat{\nu}]] + [S_x, [S_x, \hat{\mu}]\hat{\nu}] = \hat{\mu}[S_x, [S_x, \hat{\nu}]] + [S_x, [S_x, \hat{\mu}]]\hat{\nu} + [S_x, \hat{\mu}][S_x, \hat{\nu}], \quad (4.7)$$

whilst the single commutators do not. Using (4.7) in (4.5) highlights the parameter responsible

$$\mathcal{L}^\dagger(\hat{\mu}\hat{\nu}) \sim \hat{\mu}\mathcal{L}^\dagger(\hat{\nu}) + \mathcal{L}^\dagger(\hat{\mu})\hat{\nu} - \gamma \tilde{T} \hbar [S_x, \hat{\mu}][S_x, \hat{\nu}] \approx \mathcal{L}^\dagger(\hat{\mu})\hat{\nu} + \hat{\mu}\mathcal{L}^\dagger(\hat{\nu}), \quad (4.8)$$

as the extensive temperature $T = S\hbar\tilde{T}$. This product rule is required for the factorisation rule $e^{\mathcal{L}^\dagger t} \hat{\mu}\hat{\nu} \sim [e^{\mathcal{L}^\dagger t} \hat{\mu}][e^{\mathcal{L}^\dagger t} \hat{\nu}]$ to hold, see thm 8. Without the ability to factorise the time evolution in this manner identities such as $[\hat{y}^2](t) = \hat{y}(t)\hat{y}(t)$, as well as the $\mathfrak{su}(2)$ commutation relations for the time evolved operators *no longer hold* and the various EOMs become difficult to treat. However, for intensive temperatures, the double commutator in (4.5) is of order $\mathcal{O}(S^{-1})$, for rescaled operators, and the product rule is preserved in the thermodynamic limit. The adjoint equation (4.5) may then be written as

$$\hbar^{-1} \mathcal{L}^\dagger \hat{\mu} \sim i[\mathcal{H}_S/\hbar, \hat{\mu}] + \gamma i \hat{y}[S_x, \hat{\mu}]. \quad (4.9)$$

Since we have already proven that the Gibbs distribution is stationary for extensive temperatures in Sect. 3.5.2, we will now limit our scope to *intensive temperatures* $T = \mathcal{O}(S^0)$, where the product rule holds.

²The reason why they originally satisfy the commutation relations is because the system is originally uncoupled. Having only undergone unitary evolution the operators have yet to be contracted before coupling.

4.2 Dissipation for intensive temperature

Hoping to find $\langle \hat{z} \rangle_{eq}$, we consider the time evolution of $\hat{\mathbf{s}} = (\hat{x}, \hat{y}, \hat{z})$. Using (4.9) this is given component-wise by

$$\frac{d}{d(\hbar t)} \hat{x}(t) \equiv \dot{\hat{x}}(t) \sim ie^{\mathcal{L}^\dagger t} [\mathcal{H}_S / \hbar, \hat{x}] + \gamma ie^{\mathcal{L}^\dagger t} \hat{y} [S_x, \hat{x}] \sim ie^{\mathcal{L}^\dagger t} [-S_z, \hat{x}] \sim \hat{y}(t) \quad (4.10)$$

$$\frac{d}{d(\hbar t)} \hat{y}(t) \equiv \dot{\hat{y}}(t) \sim ie^{\mathcal{L}^\dagger t} [\mathcal{H}_S / \hbar, \hat{y}] + \gamma ie^{\mathcal{L}^\dagger t} \hat{y} [S_x, \hat{y}] \sim (\Lambda \hat{z}(t) - 1) \hat{x}(t) - \gamma \hat{z}(t) \hat{y}(t) \quad (4.11)$$

$$\frac{d}{d(\hbar t)} \hat{z}(t) \equiv \dot{\hat{z}}(t) \sim -i \frac{\Lambda}{2S} e^{\mathcal{L}^\dagger t} [S_x^2, \hat{z}] + \gamma ie^{\mathcal{L}^\dagger t} \hat{y} [S_x, \hat{z}] \sim -\Lambda \hat{x}(t) \hat{y}(t) + \gamma \hat{y}^2(t). \quad (4.12)$$

From hence forth all considered operators are time evolved, as such for less cluttered notation we will suppress all time dependence. Using (4.10) to (4.12) we find the stationary solutions satisfying $\dot{\hat{\mathbf{s}}} \sim 0$, to be $\hat{y} \sim 0$ and either $\hat{z} \sim \Lambda^{-1}$ or $\hat{x} \sim 0$. To avoid this ambiguity we will instead study the time evolution of these operators³.

We start by rearranging (4.12) to find the energy dissipation

$$\frac{d}{d(\hbar t)} \left[-\frac{\Lambda}{2} \hat{x}^2 - \hat{z} \right] \equiv \dot{\hat{h}} \sim -\gamma \hat{y}^2 \sim -\gamma \hat{x}^2, \quad (4.13)$$

where we have used (4.10) to rewrite $\hat{x}\hat{y} = \hat{x}\dot{\hat{x}}$. Note that the dissipation happens on a slow $\mathcal{O}(\gamma)$ timescale, while $\hat{x} \sim \hat{y}$, is not explicitly dependent on γ . Squaring (4.10) we obtain

$$\dot{\hat{x}}^2 \sim \dot{\hat{y}}^2 \sim \mathbb{1} - \hat{z}^2 - \hat{x}^2, \quad (4.14)$$

where we have used the restriction to the unit-sphere $\mathbb{1} = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$. This holds because the spin angular momentum is a conserved quantity. Writing \hat{z} in terms of the Hamiltonian we have

$$\dot{\hat{x}}^2 = \mathbb{1} - \left[\frac{\Lambda}{2} \hat{x}^2 + \hat{h} \right]^2 - \hat{x}^2. \quad (4.15)$$

Equations (4.13) and (4.15) then translates into

$$\dot{\hat{x}} \sim \pm \sqrt{\mathbb{1} - \left[\frac{\Lambda}{2} \hat{x}^2 + \hat{h} \right]^2 - \hat{x}^2} \quad (4.16)$$

$$\dot{\hat{h}} \sim -\gamma \left[\mathbb{1} - \left[\frac{\Lambda}{2} \hat{x}^2 + \hat{h} \right]^2 - \hat{x}^2 \right], \quad (4.17)$$

which is a set of non-linear coupled differential equations and thus too difficult to solve exactly. For a time independent energy \hat{h} , the differential equation for \hat{x} (4.15) would be solved by a Jacobi elliptic function [55]. In this case \hat{h} is time dependent, yet it has a slow dissipation due to the weak coupling as seen in (4.13).

This slow time scale of \hat{h} and fast time scale of \hat{x} is the ideal application for the *two-timing method*⁴ [65]. To use this method we start by defining a *fast time* $\tau = \hbar t$ and a *slow time* $s = \gamma \hbar t$, treating them as independent of one another. We then make the ansatz that the relevant functions may be separated as

$$\hat{h}(t, \gamma) = \hat{h}^{(0)}(\tau, s) + \gamma \hat{h}^{(1)}(\tau, s) + \mathcal{O}(\gamma)^2, \quad \hat{x}(t, \gamma) = \hat{x}^{(0)}(\tau, s) + \gamma \hat{x}^{(1)}(\tau, s) + \mathcal{O}(\gamma)^2. \quad (4.18)$$

³Another alternative is to solve for the stationary solutions of some hierarchy of equations which are products of spin operators. We have tried this with little success.

⁴Interestingly enough in our case this method yields the same result as Krylov-Bogolyubov method of averaging which considers an averaged differential system over the period of oscillation.

The time derivative of $\hat{h}(t, \gamma)$ in (4.18) is evaluated with the chain rule

$$\dot{\hat{h}} = \partial_\tau \hat{h} + \gamma \partial_s \hat{h} = \partial_\tau [\hat{h}^{(0)} + \gamma \hat{h}^{(1)}] + \gamma \partial_s \hat{h}^{(0)} + \mathcal{O}(\gamma)^2, \quad (4.19)$$

which, upon ignoring all quadratic and higher orders in γ and using (4.13), becomes

$$\begin{aligned} \partial_\tau \hat{h}^{(0)} + \gamma \partial_\tau \hat{h}^{(1)} + \gamma \partial_s \hat{h}^{(0)} &\sim -\gamma \dot{\hat{x}}^2 \\ &\sim -\gamma (\partial_\tau \hat{x}^{(0)} + \gamma \partial_\tau \hat{x}^{(1)} + \gamma \partial_s \hat{x}^{(0)})^2 \\ &\sim -\gamma (\partial_\tau \hat{x}^{(0)})^2. \end{aligned} \quad (4.20)$$

Collecting the different orders in γ , (4.20) may be written as two equations

$$\partial_\tau \hat{h}^{(0)} = 0, \quad \partial_\tau \hat{h}^{(1)} + \partial_s \hat{h}^{(0)} \sim -(\partial_\tau \hat{x}^{(0)})^2. \quad (4.21)$$

The first equation in (4.21) reflects that the energy is conserved to zeroth order in γ , meaning $\hat{h}^{(0)} \equiv \hat{h}^{(0)}(\tau, s) = \hat{h}^{(0)}(s)$. We will, however, suppress the slow time dependence for now as all operators are dependent on s . The second equation in (4.21) is solved by integrating over τ

$$\hat{h}^{(1)}(\tau_f) + \text{const.} + \partial_s \hat{h}^{(0)} \tau_f = - \int_0^{\tau_f} d\tau (\partial_\tau \hat{x}^{(0)}(\tau))^2. \quad (4.22)$$

The crucial step in the two-timing method is the restriction that (4.22) may not have secular (growing without bound) in τ contributions, as they lead to an unbounded operator $\hat{h}^{(1)}$. Therefore we require that all secular terms in

$$\partial_s \hat{h}^{(0)} \tau_f + \int_0^{\tau_f} d\tau (\partial_\tau \hat{x}^{(0)})^2$$

cancel. To enforce this restriction we first need to solve for $\hat{x}^{(0)}$, such that we may evaluate the integral in (4.22). Separating over the two time scales in (4.15), we obtain

$$[\partial_\tau \hat{x}^{(0)}]^2 + \left[\frac{\Lambda}{2} (\hat{x}^{(0)})^2 + h^{(0)} \right]^2 = 1 - (\hat{x}^{(0)})^2. \quad (4.23)$$

This is the same differential equation as (4.15), with one important difference: $\hat{h}^{(0)}$ is τ independent. As such $\hat{x}^{(0)}$ is solved by the Jacobi elliptic function

$$\hat{x}^{(0)}(\tau) = C \text{cn}[\pm a\tau + u_0, m], \quad C \equiv \frac{2}{\Lambda} a \sqrt{m}. \quad (4.24)$$

Here the constants, in τ , are given by

$$a(\hat{h}^{(0)})^2 \equiv \sqrt{\Lambda^2 + 2\Lambda \hat{h}^{(0)} + 1}, \quad (4.25)$$

$$m(\hat{h}^{(0)}) \equiv \frac{1}{2} - \frac{\Lambda \hat{h}^{(0)} + 1}{2a(\hat{h}^{(0)})^2} \quad (4.26)$$

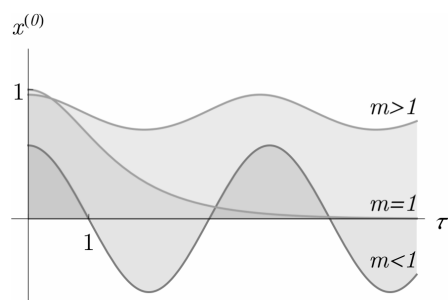


Figure 4.1: Time evolved $x^{(0)}$ for different Λ .

which are derived in Sect. A.6.1. For $m = 1$ (4.24) takes the form of a sech function as seen in Fig. 4.1. For the so-called *Rabi* regime $m < 1$, $\hat{x}^{(0)}$ oscillates around zero [71]. For the *Josephson* regime⁵ $m > 1$, the oscillations are around a non-zero value, determined by $\text{sgn}(\hat{x}^{(0)}(0))$ and $\hat{h}^{(0)}$. The constant operator u_0 in (4.24) encodes the initial condition of $\hat{x}^{(0)}(0, 0) = \hat{x}$.

⁵Note that for some numerical algorithms $\text{cn}(u, m)$ will become discontinuous for $m > 1$, in this case one is required to make use of the following identity $\text{cn}(u, m) = \text{dn}(\sqrt{m}u, 1/m)$, where dn is another Jacobi elliptical function.

Due to the periodicity of the Jacobi function, we may isolate the secular terms in (4.22) by averaging over j periods P and then taking j to infinity

$$\lim_{j \rightarrow \infty} \frac{1}{jP} \hat{h}^{(1)}(jP) + \frac{1}{jP} \text{const.} + \frac{1}{jP} \partial_s \hat{h}^{(0)} jP = -A(\hat{h}^{(0)}), \quad (4.27)$$

where we have defined the time average as

$$A(\hat{h}^{(0)}) \equiv \lim_{j \rightarrow \infty} \frac{1}{jP} \int_0^{jP} d\tau (\partial_\tau \hat{x}^{(0)})^2. \quad (4.28)$$

But with the period P being a function of an operator $\hat{h}^{(0)}$, hence itself an operator, how does one interpret equations (4.27) and (4.28)? One way would be to expand (4.24) in terms of $\hat{h}^{(0)}$'s eigenoperators. Then with the period is purely dependent on $\hat{h}^{(0)}$'s eigenvalues and we would obtain a sum of different integrals over different periods. There is, however, a caveat and that is the operator u_0 , which describes the initial conditions of (4.24). The problem is that this operator will not necessarily commute with $\hat{h}^{(0)}$, thus preventing our diagonal expansion. Fortunately this operator, like a phase in an ordinary trigonometric function, will not contribute in an infinite time average. As such, for our purposes, we may safely ignore u_0 and consider

$$\hat{x}^{(0)}(\tau) = C \text{cn}[\pm \omega \tau, m]. \quad (4.29)$$

Now both operators (4.29) and m , relevant to (4.28), are diagonal in the eigenbasis of $\hat{h}^{(0)}$. To interpret (4.28), we expand m in this basis

$$m(\hat{h}^{(0)}) = \sum_n m_n |h_n^{(0)}\rangle \langle h_n^{(0)}|, \quad m_n \equiv \langle h_n^{(0)} | m_n | h_n^{(0)} \rangle. \quad (4.30)$$

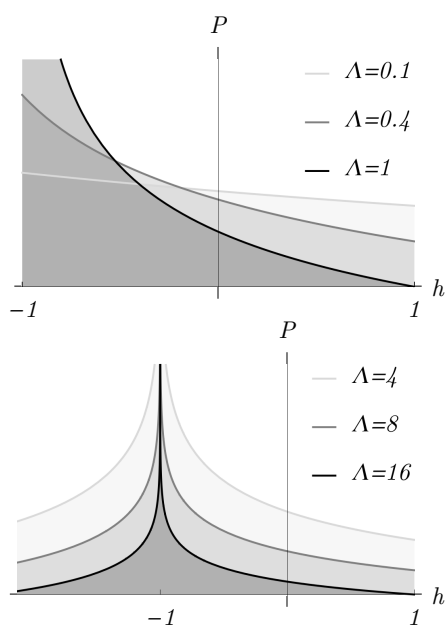


Figure 4.2: The periods of $\hat{x}^{(0)}$ plotted as a function of energy. The first figure is in the symmetric phase $\Lambda < 1$, while the second is in the symmetry broken phase.

This expression (4.30) allows us to more accurately discuss the Jacobi function's periods. Firstly (4.29) has three periods depending on the value of m_n [13]. For $m_n < 1$, (4.29) is in the Rabi regime with a period

$$P(m_n < 1) = \frac{4K(m_n)}{\omega_n}, \quad (4.31)$$

where the complete elliptic integral is defined as

$$K(u) \equiv \int_0^{\pi/2} d\varphi \frac{1}{\sqrt{1 + u \sin^2(\varphi)}}. \quad (4.32)$$

For the so-called *symmetric phase* $\Lambda < 1$, (4.29) is always restricted to the Rabi regime, as seen in Fig. 4.2.

At $m_n = 1$ the period tends to infinity, seen in the cusp of the second figure in Fig. 4.2. However, since the sech function yields a bounded integral, we set $A(h_n^{(0)}) = 0$ in this case as the integral in (4.22) has no singular contributions. In the symmetry broken phase (4.29) may cross into the Josephson regime for energies smaller than -1 , corresponding to $m_n > 1$. The Josephson regime has a period given by

$$P(m_n > 1) = \frac{K(m_n^{-1})}{\omega_n \sqrt{m_n}}. \quad (4.33)$$

We may now write the time average (4.28) more explicitly as

$$A(\hat{h}^{(0)}) \equiv \sum_n \lim_{j \rightarrow \infty} \frac{1}{jP(m_n)} \int_0^{jP(m_n)} d\tau \langle h_n^{(0)} | (\partial_\tau \hat{x}^{(0)})^2 | h_n^{(0)} \rangle | h_n^{(0)} \rangle \langle h_n^{(0)} |, \quad (4.34)$$

which evaluates to⁶

$$A(h_n^{(0)}) = \frac{4}{3\Lambda^2} \alpha_n^4 \begin{cases} 1 - m_n + (2m_n - 1) \frac{E(m_n)}{K(m_n)} & \text{for } m_n < 1 \\ m_n \left[2(1 - m_n) + (2m_n - 1) \frac{E(m_n^{-1})}{K(m_n^{-1})} \right] & \text{for } m_n > 1 \end{cases} \quad (4.35)$$

where the complete elliptic integral of the second kind is defined by

$$E(u) = \int_0^{\pi/2} d\varphi \sqrt{1 - u \sin^2(\varphi)}. \quad (4.36)$$

The non-secular terms in (4.27) vanish as $j \rightarrow \infty$, leaving $\partial_s \hat{h}^{(0)} = -A(\hat{h}^{(0)})$. This equation implies that $\hat{h}^{(0)}$ will remain diagonal in the same basis, hence may focus on the eigenvalues

$$\partial_s h_n^{(0)} = -A(h_n^{(0)}). \quad (4.37)$$

To find the stationary states of (4.37) we first need to find the energy spectrum, in particular the range of allowed eigenvalues $[h_g, h_e]$, where h_g and h_e are the ground and highest excited state energies respectively. Even though the eigenvalues of $\hat{h}^{(0)}$ are changing, the range of its eigenvalues will always remain within $[h_g, h_e]$, as shown in thm 9. It is by studying (4.37) as a continuous function of $h \in [h_g, h_e]$ that we will find $\langle \hat{z} \rangle_{\text{eq}}$.

4.2.1 The range of the energy spectrum

To find the spectral range we will use variational methods. The idea is to choose some general *variational state* parametrized by some variable(s) $|\alpha\rangle$ as a trial GS or highest excited state. One then finds the corresponding energy by maximising or minimising $\langle \alpha | \hat{h} | \alpha \rangle = \langle \hat{h} \rangle_\alpha$ with respect to α .

We use the spin coherent (variational) state [54]

$$|\alpha\rangle = (1 + |\alpha|^2)^{-S} e^{\alpha S_-} |S, S\rangle, \quad \alpha \equiv \tan \frac{\theta}{2} e^{i\phi}, \quad (4.38)$$

where the angles have ranges $(\theta, \phi) \in [0, \pi] \times [0, 2\pi]$. The overlap of two such states $\langle \alpha | \alpha' \rangle$ becomes orthonormal in the limit $S \rightarrow \infty$, proven in (A.95). One of the reasons why we use spin coherent states is that they form an over-complete basis for our Hilbert space \mathcal{H}_S . This, together with their orthogonality, allows us to “search” the entire Hilbert space, in the thermodynamic limit, to find the true GS and highest excited state energies [25].

We now introduce the key concept of an operator $\hat{\mu}$'s (lower) *symbol* $\mu \equiv \langle \hat{\mu} \rangle_\alpha = \langle \alpha | \hat{\mu} | \alpha \rangle$, note that we have dropped the hat. Expanding (4.38) yields something similar to a binomial series in magnetisation⁷ m , as shown in (A.97). Using this series one may find the following symbols [26]:

$$\langle \hat{\mathbf{s}} \rangle_\alpha \equiv \hat{\mathbf{s}} = (x, y, z) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = (\sqrt{1 - z^2} \cos \phi, \sqrt{1 - z^2} \sin \phi, z). \quad (4.39)$$

⁶We have used *Mathematica*, together with the identity $\text{cn}(u, m) = \text{dn}(\sqrt{m}u, 1/m)$ for $m > 1$, to evaluate the time average A .

⁷Recall that this is the S_z eigenvalue, thus this is a series in the Dicke basis, which was discussed in Sect. 2.1.

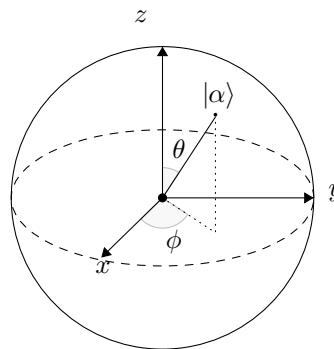


Figure 4.3: *The Majorana sphere, a Bloch sphere with $S > 1/2$, representation of spin coherent state.*

This vector (4.39) parametrizes the unit sphere in Fig. 4.3. Using the Cauchy-Schwartz inequality we then find that $\langle \hat{\mathbf{s}}^2 \rangle_\alpha \geq \langle \hat{\mathbf{s}} \rangle_\alpha^2 = 1$. Further we have

$$\langle \hat{\mathbf{s}}^2 \rangle_\alpha = \langle (S(S+1))/S^2 \rangle_\alpha \sim 1 \geq \langle \hat{\mathbf{s}} \rangle_\alpha^2 = 1,$$

implying that the inequality is saturated, meaning “ \geq ” is replaced with “ $=$ ”. This is because the symbols satisfy: $\langle \hat{\mu}^2 \rangle \sim \mu^2$, which may be proven by use of the star product⁸ [1], for instance

$$\langle \hat{x}^2 \rangle_\alpha = \langle \hat{x} \rangle_\alpha^2 + \frac{1}{2S} = x^2 + \mathcal{O}(S^{-1}). \quad (4.40)$$

As we know that $\mathbf{s}^2 \equiv \hat{x}^2 + \hat{y}^2 + \hat{z}^2 \sim 1$ is a conserved quantity, it follows from this property (4.40) that

$$x^2 + y^2 + z^2 \sim \langle \hat{x}^2 \rangle + \langle \hat{y}^2 \rangle + \langle \hat{z}^2 \rangle = \langle \mathbf{s}^2 \rangle_\alpha \sim 1,$$

is also conserved. Hence the symbols are restricted to the unit sphere in Fig. 4.3.

With (4.39) and (4.40) we calculate the rescaled Hamiltonian’s symbol $h \equiv \langle \hat{h} \rangle_\alpha$

$$\begin{aligned} h &= -\frac{\Lambda}{2}x^2 - z + \mathcal{O}(S^{-1}) \\ &\sim -\frac{\Lambda}{2}\sin^2\theta\cos^2\phi - \cos\theta. \end{aligned} \quad (4.41)$$

The minima of h will coincide with the classical ground state(s). When $\Lambda < 1$, h (4.41) has a minimum at $\theta = 0$, seen in Fig. 4.4, corresponding to $\alpha = 0$, meaning the unique GS $|S\rangle$.

For $\Lambda > 1$ we define $\Delta \equiv \hbar/J = \Lambda^{-1}$ to simplify the expressions to come. In this case we have minima at $\phi \in \{0, \pi\}$ as reflected in Fig. 4.5. Substituting this into (4.41) yields

$$h \sim -\frac{\Delta^{-1}}{2}\sin^2\theta - \cos\theta. \quad (4.42)$$

Then solving for $\partial_\theta h = 0$, we obtain

$$(1 - \Delta^{-1}\cos\theta)\sin\theta = 0 \Rightarrow \cos\theta = \Delta, \quad (4.43)$$

Figure 4.4: Plot of the energy symbol $h(\theta, \phi)$, in the symmetric phase, with $\Lambda = 1/2$.

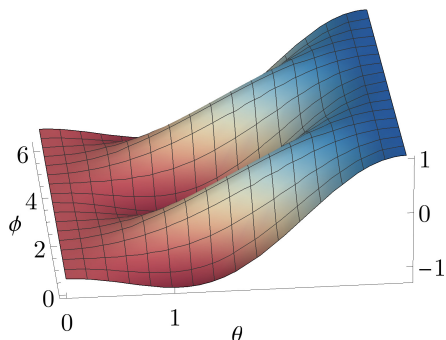


Figure 4.5: The energy symbol h in the symmetry broken phase with $\Lambda = 2$.

meaning $\mathbf{s} = (\pm\sqrt{1 - \Delta^2}, 0, \Delta)$, with a GS energy $\sim -JS(1 + \Delta^2)/2$. The two signs in the x symbol are responsible for a two-fold degeneracy, earning it the title of *symmetry broken* phase.

In summary we have the following GS energies and expectation values

$$h_g = \begin{cases} -1 & \Lambda < 1 \\ -\frac{1}{2}[\Lambda^{-1} + \Lambda] & \Lambda > 1 \end{cases} \quad \mathbf{s} = \begin{cases} (0, 0, 1) & \Lambda < 1 \\ (\pm\sqrt{1 - \Delta^2}, 0, \Delta) & \Lambda > 1 \end{cases}. \quad (4.44)$$

Lastly from both plots in Fig. 4.5 we can see that the highest excited states always correspond to $\theta = \pi$, meaning the highest excited energy is always $h_e = 1$. Thus we have the desired range $h \in [h_g, h_e]$.

⁸See Appendix A.7 for a brief introduction to the star product.

4.3 The equilibrium energy for intensive temperature

Now that we have obtained the range of h_n , we wish to find the stationary (equilibrium) energies. This is done via what is known as a fixed point analysis [65] on its differential equation (4.37). One starts this analysis by thinking of $\partial_s h$ as a velocity and h as a position. A fixed point h^* then corresponds to where the velocity is zero $\partial_s h|_{h=h^*} = 0$. If the velocity to the left of h^* is positive, then there is motion towards h^* from the left. Similarly, if the velocity to the right of h^* is negative then there is motion towards h^* from the right. This means that when the change in velocity with position, goes from positive to negative

$$\frac{(\partial_s h)|_{h=h^*+\delta} - (\partial_s h)|_{h=h^*-\delta}}{\delta} \rightarrow \partial_h(\partial_s h)|_{h=h^*} < 0,$$

then h^* corresponds to a stable fixed point (sink/attractor), indicated by a black circle in Fig. 4.6 and 4.7. By the same reasoning the reverse situation corresponds to an unstable fixed point (source/repeller), indicated by white circle. Sources can be thought of as a ball on top of a very sharp hill, where even an infinitesimal perturbation leads to the ball rolling down the hill into a valley (sink).

With this background knowledge we see from Fig. 4.6 and 4.7 that in both phases the “change of velocity with position” (gradient of plot) is negative at the GS energy, meaning that there is “motion” to the left most point: $h = h_g$. Since there exists no lower energy than the GS energy this is the most “left” point and corresponds to a sink. In the broken phase there exist a half-stable (saddle) point at the cusp ($h^{(0)} = -1$), seen in Fig. 4.7 as a half white half black circle, while the highest excited state is an unstable fixed point. One may check that this saddle point has infinite instability to the left. By this we mean that its gradient tends to $+\infty$ to the left of the fixed point. Hence any fluctuations from finite S or higher orders in γ will lead to this point becoming unstable. As such we may safely treat it as an unstable fixed point, which will be reflected in the numerics. Thus we are left with only one stable fixed point—the ground state energy. This is our equilibrium energy and looking back at (4.44) it corresponds to the following expectation values

$$\langle \hat{z} \rangle_{eq} = \min\{\Delta, 1\}, \quad \langle \hat{x}^2 \rangle_{eq} = \max\{0, 1 - \Delta^2\}. \quad (4.45)$$

Thus we have found the expectation values we require for our bosonisation (HP mapping) (4.1).

4.3.1 Energy dissipation rate

We are now in a position to calculate the energy dissipation rate close to the stationary energy h_g . We may then compare this rate to the gaps which we will derive in Chap. 5. To do this rigorously we will study the expectation value $\langle \hat{h}^{(0)} \rangle$ close to this energy, which is described by

$$\partial_s \langle \hat{h}^{(0)} \rangle = - \langle A(\hat{h}^{(0)}) \rangle. \quad (4.46)$$

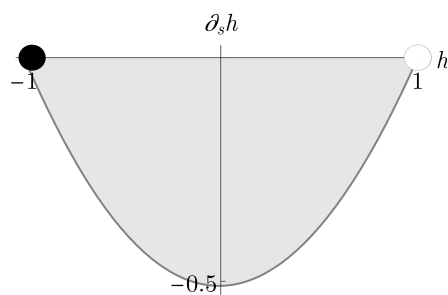


Figure 4.6: *The energy's slow derivative (4.37) $\partial_s h^{(0)}$ as a function of $h^{(0)}$, over the entire spectrum's range, in the symmetric phase, with $\Lambda = 1/2$.*

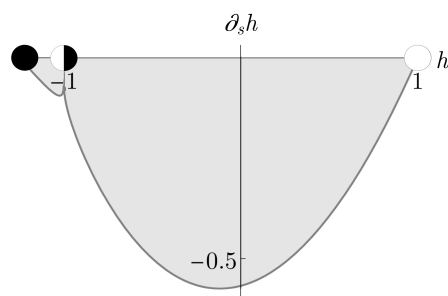


Figure 4.7: *Plot of $\partial_s h^{(0)}$ as a function of h in the symmetry broken phase with $\Lambda = 2$.*

To simplify this expression we assume that all eigenvalues have evolved such that the range is now limited to the convex region of the GS, which for the symmetry broken region is $h_n \in [h_g, -1]$. We may then use Jensen's inequality [32] to obtain

$$-A(\langle \hat{h}^{(0)} \rangle) \leq -\langle A(\hat{h}^{(0)}) \rangle \leq 0, \quad (4.47)$$

where the upper bound of 0 is merely due to the range of A . Defining $h^{(0)} \equiv \langle \hat{h}^{(0)} \rangle$, this inequality is saturated as $h^{(0)} \rightarrow h_g$. As such (4.46) may be approximated as

$$\partial_s h^{(0)} = -\langle A(\hat{h}^{(0)}) \rangle \approx -A(h^{(0)}), \quad (4.48)$$

close to h_g . We study the decay rate of (4.48) by expanding around the sink h_g :

$$A(h) = \begin{cases} (h^{(0)} - h_g) + \mathcal{O}\left(\frac{(h^{(0)} - h_g)^2}{(\Lambda - 1)^2}\right) & m < 1, \Lambda < 1 \\ \frac{4}{3}(h^{(0)} - h_g) - \mathcal{O}((h^{(0)} - h_g)^{3/2}) & m = 1, \Lambda = 1 \\ \frac{h^{(0)} - h_g}{\Lambda} + \mathcal{O}\left(\frac{(h^{(0)} - h_g)^2}{\Lambda^2 - 1}\right) & m > 1, \Lambda > 1. \end{cases} \quad (4.49)$$

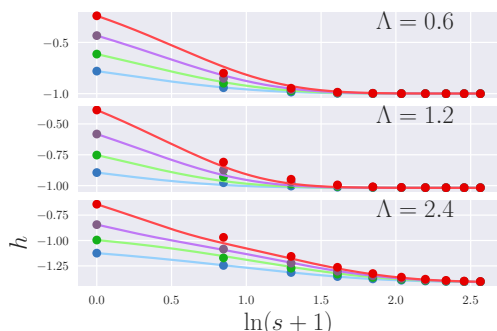


Figure 4.8: Energy dissipation plotted for different initial energies. The circles are the analytical solutions (4.52), while the solid lines are numerical solutions for $S = 40$. For the numerics $\hbar = 1$, $T = 4$ and $\gamma = 10^{-3}$.

predicted exponential behaviour, as well as strong overlap at finite system size. Further we see from the final two plots in Fig. 4.8 that the saddle point at $h = -1$ is essentially ignored by the numerics.

4.4 Canonical thermodynamics

In the previous section we found that the stationary expectation values (4.45) are, for an intensive temperature $T = \mathcal{O}(S^0)$, independent of T . Hence, at first sight, we may expect that the system does not thermalise to a Gibbs state $e^{-\beta \mathcal{H}_S}$, which has explicit temperature dependence¹⁰. To examine this further we need to calculate the exact thermal expectation values, meaning those of the state $e^{-\beta \mathcal{H}_S}$, and compare them to (4.45). To do this it convenient

⁹The nontrivial finite-size corrections to h , for a closed system, at $\Lambda = 1$ ($m = 1$) are proportional to $N^{-4/3}$, i.e. a critical exponent of $4/3$ [17]. Comparing this to the exponential decay in (4.52) at $\Lambda = 1$, there might be some link.

¹⁰For such a state to have temperature independent expectation values would not match our intuition of a critical Curie temperature existing for which the magnetic field $\langle z \rangle_{eq}$ tends to zero.

Keeping only the first terms in (4.49) and writing the differential equation (4.48) in terms of t , where $s = \gamma \hbar t$, we have

$$\partial_t h^{(0)} = -\omega_h (h^{(0)} - h_g), \quad (4.50)$$

with⁹

$$\omega_h = \gamma \hbar \begin{cases} 1 & \Lambda < 1 \\ \frac{4}{3} & \Lambda = 1 \\ \frac{1}{\Lambda} & \Lambda > 1 \end{cases}. \quad (4.51)$$

Equation (4.50) is solved by

$$h^{(0)}(t) = h_g + (h^{(0)}(0) - h_g) e^{-\omega_h t}, \quad (4.52)$$

which decays exponentially at a rate ω_h . We compare this (4.52) to finite system size numerics in Fig. 4.8. This plot shows the predicted exponential behaviour, as well as strong overlap at finite system size.

to use the partition function $Z = \text{Tr} e^{-\beta \mathcal{H}_S}$. The previous partition function we encountered was for the bosonic bath in Sect. 3.2, where the Hamiltonian \mathcal{H}_E was already diagonal. Our current Hamiltonian \mathcal{H}_S is not diagonalisable (in a practical way), thus we cannot calculate Z in the same way. We may, however, use the spin coherent states to evaluate this trace in the thermodynamic limit.

To see how we may evaluate Z using coherent states we first require a completeness relation. For the *usual* case of a complete basis this is just the sum of the eigenoperators spanning the space $\mathbb{1} = \sum_n |n\rangle\langle n|$, but the spin coherent states are an *over-complete* basis. Despite this they still have a completeness relation¹¹ [1]

$$\mathbb{1} = \int d\mu(\alpha, \bar{\alpha}) |\alpha\rangle\langle\alpha| = \frac{2S+1}{4\pi} \int_0^{2\pi} d\phi \int_{-1}^1 d\cos(\theta) |\alpha\rangle\langle\alpha|, \quad (4.53)$$

with one apparent difference to the typical resolutions of identity—the *measure*

$$d\mu(\alpha, \bar{\alpha}) = d^2\alpha \frac{2S+1}{\pi(1+|\alpha|^2)}, \quad (4.54)$$

which is required to normalise this integral with respect to the over-complete basis. Here $\bar{\alpha}$ is the complex conjugate of α . Using (4.53) the trace of $e^{-\beta \mathcal{H}_S}$ over the coherent states $|\alpha\rangle$ yields the partition function [1]

$$Z = \text{Tr} e^{-\beta \mathcal{H}_S} = \int d\mu(\alpha, \bar{\alpha}) \langle e^{-\beta \mathcal{H}_S} \rangle_\alpha. \quad (4.55)$$

This expression may then be bounded as follows¹² [42]:

Thm 2 For a spin type Hamiltonian \mathcal{H}_S the partition function is bounded as follows

$$\int d\mu(\alpha, \bar{\alpha}) e^{-\beta \langle \mathcal{H}_S \rangle_\alpha} \leq \text{Tr} e^{-\beta \mathcal{H}_S} \leq \int d\mu(\alpha, \bar{\alpha}) e^{-\beta P(\mathcal{H}_S)}, \quad (4.56)$$

with the upper symbol (*P-representative*) $P(\bullet)$ determined by [28, 39]

$$\bullet = \int d\mu(\alpha, \bar{\alpha}) P(\bullet) |\alpha\rangle\langle\alpha|. \quad (4.57)$$

For our specific Hamiltonian the upper symbol¹³ is merely $P(\mathcal{H}_S) \sim \hbar S h + \mathcal{O}(S^0)$, where h is the lower symbol of \hat{h} (4.41). Using thm 2 yields the following bounds (4.56)

$$\int d\mu(\alpha, \bar{\alpha}) \exp(-S \hbar \beta h) \leq \text{Tr} e^{-\beta \mathcal{H}_S} \leq \int d\mu(\alpha, \bar{\alpha}) \exp(-S \hbar \beta h + \mathcal{O}(S^0)). \quad (4.58)$$

Note that both the upper and lower bounds in (4.58) have a large S parameter in the exponential. This extensive exponential behaviour is the ideal situation for employing Laplace's method [6]

$$\int d\mu(\alpha, \bar{\alpha}) e^{-S \hbar \beta h + \mathcal{O}(S^0)} \sim \frac{2S+1}{4\pi} \sum_{\alpha^*} \frac{2\pi}{S \sqrt{\beta \hbar |\det(\mathbb{H}(\alpha^*))|}} e^{-S \hbar \beta h(\alpha^*) + \mathcal{O}(S^0)}, \quad (4.59)$$

¹¹Recall that $\alpha \equiv \tan \frac{\theta}{2} e^{i\phi}$, as was defined in (4.38)

¹²One proves the lower and upper bounds using the Golden-Thomson inequality $\text{Tr} e^{A+B} \leq \text{Tr}\{e^A e^B\}$ and path integrals respectively. [42]

¹³These upper symbols are more difficult to calculate hence we omit the proof. See [39] for a general formula on how to calculate these symbols.

where the sum runs over all minima of h . Here $\mathbb{H}(\alpha^*)$ is the Hessian matrix of h evaluated at said minima, its explicit expression will play no role in this analysis.

In the symmetry broken phase h has two minima (4.44), both with the same energy h_g . This means that the leading order, in S , exponential behaviour of (4.59) is $e^{-S\beta\hbar h_g}$. Since both bounds in (4.58) have the same leading order exponential behaviour so must the bounded object Z . Mathematically one writes this as $Z \asymp e^{-S\beta\hbar h_g}$, which is explicitly defined as

$$a(S) \asymp b(S) \Leftrightarrow \lim_{S \rightarrow \infty} \frac{\ln a(S)}{S} = \lim_{S \rightarrow \infty} \frac{\ln b(S)}{S}. \quad (4.60)$$

Using $Z \asymp e^{-S\beta\hbar h_g}$ one may calculate the leading order Gibbs free energy, as we have shown in (A.13), from which we may calculate the desired expectation values.

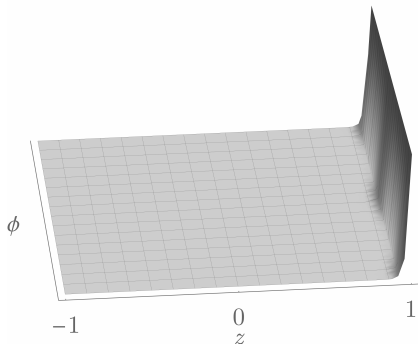


Figure 4.9: Plot of the integrands in (4.58) for $S\hbar\beta = 100$ and $\Lambda = 1/2$.

If h is in the symmetric phase, then the minimum h_g is independent of ϕ , as seen in Fig. 4.4. The bounds in (4.58) then effectively become ϕ independent, which can be seen in Fig. 4.9. We may then treat the integrands as ϕ -independent. Using $z = \cos \theta$ from (4.39), we obtain

$$\begin{aligned} & \int d\mu(\alpha, \bar{\alpha}) e^{-S\hbar\beta h - \hbar\beta \mathcal{O}(S^0)} \\ & \sim \frac{2S+1}{4\pi} \int_{-1}^1 dz e^{-S\hbar\beta h - \hbar\beta \mathcal{O}(S^0)} \Big|_{\phi=\phi_0} \int_0^{2\pi} d\phi \\ & \sim \frac{2S+1}{2} \int_{-1}^1 dz e^{-S\hbar\beta h - \hbar\beta \mathcal{O}(S^0)} \Big|_{\phi=\phi_0} \end{aligned} \quad (4.61)$$

evaluated at some arbitrary ϕ_0 . Now using Laplace's method over z we obtain

$$\int d\mu(\alpha, \bar{\alpha}) e^{-S\hbar\beta h - \hbar\beta \mathcal{O}(S^0)} \sim e^{-S\hbar\beta h_g} \mathcal{O}(S^0), \quad (4.62)$$

where the exponent is ϕ_0 independent, since h_g is ϕ independent. Again, we only have to consider the exponential behaviour (4.62) in order to obtain the expectation values. Thus in both cases we may write the Gibbs free energy as

$$g(\beta) \equiv -\frac{1}{S\beta} \ln Z \sim \hbar h_g = -\frac{J}{2} \langle \hat{x}^2 \rangle_{\text{eq}} - \hbar \langle \hat{z} \rangle_{\text{eq}} = g(\infty), \quad (4.63)$$

which is temperature *independent* to leading order in S . The thermal expectation values are evaluated¹⁴ as

$$\langle \hat{x}^2 \rangle = -2\partial_J g(\infty) = \langle \hat{x}^2 \rangle_{\text{eq}} \quad \text{and} \quad \langle \hat{z} \rangle = -\partial_{\hbar} g(\infty) = \langle \hat{z} \rangle_{\text{eq}},$$

which match the GS expectation values (4.45). In other words *all* GS and thermal expectation values calculable from $g(\beta)$ are equivalent and T -independent in the thermodynamic limit. This is a consequence of the dimension of the Hilbert space growing linearly¹⁵ [70]. In Sect. A.6.2 we show that a transition from the ferromagnetic phase to a paramagnetic phase only occurs at high enough extensive temperatures. In which case

$$\langle \hat{x}^2 \rangle_{\text{eq}} = \frac{1}{3} + \frac{2\Lambda\tilde{\beta}}{45} + \mathcal{O}(\tilde{\beta}^2, S^{-1}) \quad \langle \hat{z} \rangle_{\text{eq}} = \frac{\tilde{\beta}}{3} + \mathcal{O}(\tilde{\beta}^2, S^{-1}), \quad (4.64)$$

which reflects that for large extensive temperatures $g(\infty) \propto \mathbb{1} + \mathcal{O}(\tilde{\beta})$, as was shown in Sect. 3.5.2. Without the constraint to a specific spin angular momentum sub-sector the Hilbert space would grow exponentially. In that case \mathcal{H}_S has a phase transition, for $\Lambda > 1$, at some critical temperature $T_c = \mathcal{O}(S^0)$ from a ferromagnetic phase at $T < T_c$ to a paramagnetic phase at $T > T_c$ [35].

¹⁴More details on the Gibbs free energy and its relation to expectation values can be found in Sect. A.6.2

¹⁵The linear growth was shown in Sect. 2.1.

5 | Diagonalising the Lindbladian

In the previous chapter we found the stationary expectation values of \hat{z} for intensive temperatures, which we require for the HP expansion (4.1). Recall that we will use this expansion to limit our scope to the region $\mathcal{L}\varrho \approx 0$.

Further, we found that the stationary expectation values of \hat{x}^2 and \hat{z} match those of the thermal state for intensive temperatures. This, however, does not guarantee that the system equilibrates to a Gibbs distribution. To see why this is recall that the thermal state and GS have the same \hat{x}^2 and \hat{z} expectation values in the thermodynamic limit. Thus the question of whether ϱ equilibrates to a Gibbs distribution is still open.

In order to systematically study the stationary state(s) and relaxation effects of the master equation (3.52) we need to diagonalise the Lindbladian \mathcal{L} and find its ‘‘eigenvectors’’. To bring the operator \mathcal{L} into the familiar form of a matrix we introduce the method of *vectorisation* [45]. This is a mapping of ϱ onto a vector, and with \mathcal{L} acting on this vector it now maps onto a matrix. Formally this is a unitary transformation from $\mathcal{H}_S \otimes \mathcal{H}_S^*$, where \mathcal{H}_S^* is the dual space of \mathcal{H}_S , equipped with the Hilbert-Schmidt inner product, to an isomorphic vector space, such that the relation between inner products is preserved $\text{Tr}\{A^\dagger B\} = \langle A|B\rangle$. Notationally we write $\varrho : \varrho \rightarrow |\varrho\rangle$, which is defined as

$$\varrho = \sum_{ij} \varrho_{ij} |i\rangle\langle j| \rightarrow |\varrho\rangle = \sum_{ij} \varrho_{ij} |i\rangle \otimes \langle j|^T. \quad (5.1)$$

Left and right multiplication of matrices, such as $A\varrho B$, then maps to $A \otimes B^T |\varrho\rangle$ [46]. With this we may vectorise the dissipator $\mathcal{D}\bullet = L \bullet L^\dagger - 1/2\{L^\dagger L, \bullet\}$:

$$\mathcal{D} \rightarrow L \otimes L^* - \frac{1}{2}(L^\dagger L \otimes \mathbb{1} + \mathbb{1} \otimes L^T L^*), \quad (5.2)$$

where L^* is the complex conjugate of L . The unitor $\mathcal{U}\bullet = i[\bullet, \mathcal{H}]$ then maps onto

$$\mathcal{U} \rightarrow i\mathbb{1} \otimes \mathcal{H}^T - i\mathcal{H} \otimes \mathbb{1}. \quad (5.3)$$

Considering that the adjoint master equation (4.3), $\mathcal{L}^\dagger \bullet = i[\mathcal{H}, \bullet] + L^\dagger \bullet L - 1/2\{L^\dagger L, \bullet\}$ maps to

$$\mathcal{L}^\dagger = i\mathcal{H} \otimes \mathbb{1} - i\mathbb{1} \otimes \mathcal{H}^T + L^\dagger \otimes L^T - \frac{1}{2}(L^\dagger L \otimes \mathbb{1} + \mathbb{1} \otimes L^T L^*), \quad (5.4)$$

it is even more apparent that the two operators \mathcal{L} and \mathcal{L}^\dagger are adjoints of one another.

In Chap. 4, we mentioned that we wish to diagonalize this operator \mathcal{L} in the ‘‘semi-stationary subspace’’¹ by mapping onto bosons (4.1). With two tensor producted operators each acting on its own Hilbert space this would lead, once again, to two different species of bosons interacting only via the first term in (5.2).

5.1 Gap of the Lindbladian

In terms of this vectorized Lindbladian $\mathcal{L} = \mathcal{D} + \mathcal{U}$ (5.2) and (5.3) the formal time evolution of some density matrix maps onto iterative matrix multiplication by \mathcal{L} :

$$|\varrho(t)\rangle = e^{t\mathcal{L}} |\varrho(0)\rangle = \sum_k \frac{(t\mathcal{L})^k}{k!} |\varrho(0)\rangle. \quad (5.5)$$

¹By this we do not mean that the bosonisation restricts the Hilbert space to this subspace, but rather that the mapping is only valid for small excitations away from stationarity. Exactly how small these excitations need to be will be shown in Sect. 5.2.

Expanding $\rho(0)$ in terms of \mathcal{L} 's eigenvectors $|\ell_n\rangle$, corresponding to eigenvalues ℓ_n , (5.5) may be written as

$$e^{t\mathcal{L}}|\rho(0)\rangle = \sum_k c_k e^{t\ell_k} |\ell_k\rangle = \sum_k c_k e^{t\ell_k} |\ell_k\rangle. \quad (5.6)$$

From (5.6) we note that any positive $\Re\ell_k$ leads to certain eigenstates' "weights" $c_k e^{t\ell_k}$ growing without bound. This is in violation of trace and positivity preservation. As such all eigenvalues must satisfy $\Re\ell_n \leq 0$, as \mathcal{L} preserves the rules of quantum mechanics. We order these eigenvalues by their real part, meaning $\Re\ell_{n+1} \leq \Re\ell_n$.

Considering (5.6), the stationary state(s) correspond to the zero eigenvalue(s), as those states would not evolve. In order to see thermalisation, in the sense of def 1, one requires that the equilibrium state be independent of initial conditions. If we had two stationary states, then equilibration to the one or the other would have to depend on the initial conditions. Hence thermalisation requires a unique stationary state, meaning a unique zero eigenvalue $\ell_0 = 0$.

Assuming for now that the stationary state is unique, we can see that the relaxation time scale is set by the gap between the zero eigenvalue and the closest non-zero eigenvalue ℓ_1 , i.e. $\tau_{eq} = |\Re\ell_1|^{-1}$. This is because $|\ell_1\rangle$, has the slowest (exponential) decay out of all the states. We mentioned in Sect. 4.3.1 that the energy's dissipation rate would give us an idea of the gap. As it is currently the slowest exponential decay we are aware of, we may already conjecture that the gap $|\Re\ell_1|$ will be smaller than the exponential decay of the energy (4.52) from (4.50):

$$|\Re\ell_1| \leq \omega_h, \quad \omega_h = \gamma \hbar \begin{cases} 1 & \Lambda < 1 \\ \frac{4}{3} & \Lambda = 1 \\ \frac{1}{\Lambda} & \Lambda > 1, \end{cases} \quad (5.7)$$

If we had a positive real Lindbladian we would already know from the Perron Frobenius theorem [6] that \mathcal{L} has a unique stationary state, thus a non-zero gap $|\Re\ell_1|$, given by $\rho_{eq} = \lim_{t \rightarrow \infty} e^{t\mathcal{L}}\rho(0)$. However our Lindbladian \mathcal{L} is definitely not real due to the unitor (5.3) having imaginary contributions.

One can still prove a unique stationary state by lower bounding the gap. This may be done by use of the *Gershgorin* theorem [27]. This method, however, requires a diagonally dominant Lindbladian which we do not have².

There exists one method for estimating the gap that is particularly well suited to our problem of an extensive Lindbladian. Such a Lindbladian has a time evolution operator $e^{t\mathcal{L}} = e^{t\mathcal{O}(S)}$ which is dominated, in the thermodynamic limit, by only the slowest oscillations and exponential decay $e^{t\omega_S}$. This is the idea behind the Wentzel-Kramers-Brillouin (WKB) approximation [6]. With the zeroth eigenvalue being zero one can motivate that the spectral gap tends to this dominant frequency ω_S . This is then the frequency of the spin observables $\hat{\mathbf{s}} = (\hat{x}, \hat{y}, \hat{z})$ [11].

We may use this correspondence between frequency and the gap, together with the random phase approximation (RPA)—replacing the operators with their classical (stationary) values [9], to estimate $|\Re\ell_1|$. From (4.10) to (4.12) we have³ $\dot{\hat{x}} \sim \hat{y}$,

$$\dot{\hat{y}} \sim (\Lambda \hat{z} - 1)\hat{x} - \gamma \hat{z} \hat{y}, \quad (5.8)$$

$$\dot{\hat{z}} \sim -\Lambda \hat{x} \hat{y} + \gamma \hat{y}^2. \quad (5.9)$$

By applying the RPA we are replacing one of the operators by its stationary expectation value, hence it will no longer evolve. We consider the case where \hat{z} is stationary, which is at $\langle \hat{z} \rangle_{eq} = \min\{\Delta, 1\}$ from (4.45). This allows us to set up a *linear* matrix equation for \hat{x} and \hat{y} using (4.10) and (5.8)

$$\frac{d}{dt} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = A \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}, \quad A \equiv \hbar \begin{bmatrix} 0 & 1 \\ \Lambda \langle \hat{z} \rangle_{eq} - 1 & -\gamma \langle \hat{z} \rangle_{eq} \end{bmatrix}, \quad (5.10)$$

²See Sect. 6.1.2 for a deeper discussion.

³Recall that $\dot{\mu} \equiv \partial_{\hbar t} \mu$.

where we have used the RPA $\hat{z} \rightarrow \langle \hat{z} \rangle_{eq} = \min\{\Delta, 1\}$.

We are interested in the purely decaying/oscillating solutions $\mathbf{v} = e^{\omega_S t} \mathbf{v}$, which are dominant in the thermodynamic limit. With \hat{z} assumed stationary, we only have to consider the differential equation

$$\omega_S e^{\omega_S t} \mathbf{v} = A e^{\omega_S t} \mathbf{v}, \quad (5.11)$$

which is an eigenvalue equation. As such the “frequencies” are the eigenvalues of A given by

$$\omega_S = -\frac{\gamma \hbar}{2} \langle z \rangle_{eq} \pm \hbar \sqrt{\left[\frac{\gamma \langle z \rangle_{eq}}{2} \right]^2 + \Lambda \langle z \rangle_{eq} - 1}. \quad (5.12)$$

These eigenvalues (5.12) are conjectured to be the gaps of the Lindbladian. The real and imaginary parts of ω_S characterise the exponential decay and unitary oscillations respectively. For the symmetric phase we have $\langle z \rangle_{eq} = 1$, and (5.12) may be expanded around $\gamma = 0$, yielding

$$\omega_S = \pm i \hbar \sqrt{1 - \Lambda} - \frac{\gamma \hbar}{2} + \mathcal{O}(\gamma)^2, \quad (5.13)$$

which indeed has a real part that is upper bounded by (5.7). Note that the real part of the conjectured gap is proportional to the coupling, which reflects that the dissipation is purely due to the coupling. Further setting $\hbar = 0$ yields a zero conjectured gap, implying the case of no thermalisation as was seen in Sect. 3.5.1.

In the symmetry broken phase $z \sim \Delta$ and we have the two solutions

$$\omega_S \in \{0, -\gamma \hbar \Delta\}. \quad (5.14)$$

The fact that this gap reduces to zero for vanishing coupling reflects the degenerate GS in the symmetry broken phase. Which of these two conjectured gaps is the correct one, if either, is unknown at the moment. If the gap is zero, then the system would not thermalise in the sense of def 1. Again both gaps have real parts which are upper bounded by (5.7).

If we further enforced that the dynamics remains on the unit sphere with $|\mathbf{s}|^2 = 1$, then any small change $\delta \mathbf{s}$ away from the stationary point $\mathbf{s}_{eq} = (\pm \sqrt{1 - \Delta}, 0, \Delta)$, from (4.44), must be tangential to the surface. In other words we require the following orthogonality⁴ to hold

$$\mathbf{s}_{eq} \cdot \delta \mathbf{s} = \pm \sqrt{1 - \Delta} \delta x + \Delta \delta z = 0,$$

which implies that $\delta z = \pm \sqrt{\Lambda^2 - 1} \delta x$. Linearising [65] the EOM around the equilibrium points $\mathbf{s} = \mathbf{s}_{eq} + \delta \mathbf{s}$ and noting that $\dot{\mathbf{s}}_{eq} = 0$, we find $\delta \dot{x} = \delta y$ and

$$\delta \dot{y} = (\Lambda(\Delta + \delta z) - 1)(\pm \sqrt{1 - \Delta} + \delta x) - \gamma(\Delta + \delta z)\delta y = \pm \sqrt{1 - \Delta} \delta z - \gamma \Delta \delta y. \quad (5.15)$$

Upon writing δz in terms of δx , we are left with the matrix equation

$$\frac{d}{dt} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = B \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}, \quad B \equiv \hbar \begin{bmatrix} 0 & 1 \\ 1 - \Lambda^2 & -\gamma \Delta \end{bmatrix}, \quad (5.16)$$

which does not match (5.10) for the symmetry broken phase⁵. The eigenvalues of this matrix are both non-zero

$$\omega_B = \pm i J \sqrt{1 - \Delta^2} - \frac{\gamma \hbar \Delta}{2} + \mathcal{O}(\gamma^2). \quad (5.17)$$

This frequency cannot coincide with the real gap seeing as

$$\omega_B |_{\gamma=0} \neq 0 = |_{\gamma=0} \Re \ell_1.$$

As such it will most likely indicate the decay rate to a particular degenerate solution, i.e. the dispersion relation.

⁴This is by the same reasoning as the method of Lagrange multipliers [4]

⁵One can check that if the procedure were repeated for the symmetric phase, then one would get $A = B$, meaning $\omega_S = \omega_B$.

5.2 Bosonisation of spins

Now that we have a better understanding of the gaps we can work towards diagonalizing \mathcal{L} . After this we may compare the gaps to the RPA conjectured gaps (5.13), (5.14) and (5.17). We start by (re)introducing the Holstein-Primakoff (HP) mapping [30], which is defined by

$$S_+ = \sqrt{2S - a^\dagger a} a. \quad (5.18)$$

The relation $[S_+, S_-] = 2S_z$ allows us to derive $S_z = S - a^\dagger a$ from (5.18). This is the reason we may write $S_+ = \sqrt{S}\sqrt{1 - \hat{z}} a$, as in Chap. 4. Recall that we only care about the eigenstates of \mathcal{L} close to the thermal state, as such we expand around the equilibrium expectation values

$$\langle \hat{z} \rangle_{eq} = \min\{1, \Delta\}, \quad \langle \hat{x}^2 \rangle_{eq} = \max\{0, 1 - \Delta^2\}. \quad (5.19)$$

This expansion will be valid for the equilibrium state and small fluctuations around it.

5.2.1 Bosonisation in the symmetric phase

We start this bosonisation analysis by diagonalizing the Hamiltonian in the symmetric phase $\Lambda < 1$. This will serve as a simple example of this mapping, but will also be required for later. We wish to expand (5.18) around its equilibrium value $\langle S_z \rangle_{eq} = S$. Remembering that $S_z = S - a^\dagger a$, this translates into expanding around $a^\dagger a = 0$, which yields

$$\frac{S_+}{\sqrt{S}} = \sqrt{2} \sqrt{1 - \frac{a^\dagger a}{2S}} a = \sqrt{2} \sum_{k=0}^{\infty} \binom{1/2}{k} \left(\frac{-a^\dagger a}{2S} \right)^k a = \sqrt{2} a - \frac{a^\dagger a a}{\sqrt{8S}} - \mathcal{O}(S^{-2}). \quad (5.20)$$

Here the limitation to small excitations around a specific region has become apparent with the first ‘‘correction’’ term $a^\dagger a a/S$. To study the allowed number of excitations we consider how this correction term acts on a state $|n\rangle$ for which we have

$$\frac{(a^\dagger a) a}{S} |n\rangle = \frac{a^\dagger a \sqrt{n}}{S} |n-1\rangle = \frac{\sqrt{n}(n-1)}{S} |n-1\rangle = \frac{n^{3/2} - n}{S} |n-1\rangle. \quad (5.21)$$

Hence the correction term (5.21) will only decay with S for a sub-extensive number of excitations $n = \mathcal{O}(S^\alpha)$ with $\alpha < 2/3$.

Writing the Hamiltonian in terms of spin ladder operators we obtain

$$\mathcal{H}_S/\hbar = -\frac{\Lambda}{2S} S_x^2 - S_z = -\frac{\Lambda}{2} \left(\frac{S_+ + S_-}{2\sqrt{S}} \right)^2 - S_z. \quad (5.22)$$

Substituting the series expansion (5.20) into (5.22) we find

$$\mathcal{H}_S/\hbar \sim -\frac{\Lambda}{2} \left(\frac{a + a^\dagger}{\sqrt{2}} \right)^2 - (S - a^\dagger a) + \mathcal{O}(S^{-1}) \sim \left(-\frac{\Lambda}{2} + 1 \right) a^\dagger a - \frac{\Lambda}{4} (a^{\dagger 2} + a^2) - S, \quad (5.23)$$

which is *quadratic* in the ladder operators. The constant in (5.23) indicates the energy of the region we are expanding around. In our case this corresponds to the symmetric phase GS energy $Sh_g = -S$, derived (using coherent states) in (4.44).

5.2.2 The Bogoliubov transform

A Hamiltonian which is of the general quadratic form $\mathcal{H} \sim 2\epsilon a^\dagger a + (\varphi a^2 + \text{H.C.})$, such as (5.23), may be diagonalized⁶ by a Bogoliubov transform [2] for $\epsilon \neq |\varphi|^2$ [23]. The essence

⁶For the case where $|\varphi|^2 > \epsilon$ the Hamiltonian’s spectrum becomes complex, continuous and unbounded, and has corresponding Glauber coherent eigenstates [23].

of the Bogoliubov transform is to define a new ladder operator $a = ub + vb^\dagger$, such that the Hamiltonian is, up to a constant, proportional to a number operator $\mathcal{H}_S/\hbar \rightarrow \omega b^\dagger b + \text{const.}$ This transformation from one set of ladder operators to the other take the forms:

$$\begin{bmatrix} a \\ a^\dagger \end{bmatrix} = T_{b \rightarrow a} \begin{bmatrix} b \\ b^\dagger \end{bmatrix} = \begin{bmatrix} u & v \\ v^* & u \end{bmatrix} \begin{bmatrix} b \\ b^\dagger \end{bmatrix}, \quad \begin{bmatrix} b \\ b^\dagger \end{bmatrix} = T_{b \rightarrow a}^{-1} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} = \begin{bmatrix} u & -v \\ -v^* & u \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix}, \quad (5.24)$$

where we require $|u|^2 - |v|^2 = 1$ to preserve the bosonic commutation relations (2.3). This restricts the functional form (up to a global phase) of the Bogoliubov coefficients to $u = \cosh(r)$ and $v = e^{i\theta} \sinh(r)$, i.e. a rotation together with a boost. As shown in (A.52), the Bogoliubov transform of a Hamiltonian such as (5.23):

$$\mathcal{H}_S/\hbar \sim \omega_\uparrow b^\dagger b - S. \quad (5.25)$$

has a dispersion relation and coefficients

$$\omega_\uparrow = \sqrt{1 - \Lambda} \quad \text{and} \quad u_\uparrow \pm v_\uparrow = \sqrt{\omega_\uparrow^{\mp 1}} \quad (5.26)$$

respectively. We have used the up arrows \uparrow to distinguish between the LMG phases, we will use a down arrow \downarrow for the symmetry broken phase.

Note that the gap ω_\uparrow predicted by the HP mapping is the same as the RPA conjectured gap (5.13) for zero coupling, as expected for pure unitary evolution.

Now that we have diagonalised the Hamiltonian we are ready to apply the same methods to the Lindbladian.

5.3 Lindbladian in the symmetric phase

To start our study of the Lindbladian we wish to express the jump operator L in terms of the bosons b . As stated in Sect. 3.4 we are uninterested in the bath's frequency contribution to the LMG model and as such want to eliminate all ω_c dependence. Therefore we set $\omega_c \approx 2\sqrt{3}T$, such that $\nu_1 = 0$, in which case the jump operator (3.50) may be written as

$$L = \sqrt{\frac{2\gamma T}{S}} \left[S_x + \frac{\hbar\beta}{4} iS_y \right] \quad (5.27)$$

To write (5.27) in terms of bosons we perform this mapping for S_x and S_y individually. Rewriting S_x in terms of the spin ladder operators and expanding in terms of bosons (5.20) yields

$$\frac{S_x}{\sqrt{S}} = \frac{S_+ + S_-}{2\sqrt{S}} \sim \frac{a + a^\dagger}{\sqrt{2}}. \quad (5.28)$$

Using the Bogoliubov transform (5.24), this may be written in terms of the b bosons as⁷

$$\frac{S_x}{\sqrt{S}} \sim \frac{(u_\uparrow + v_\uparrow)b + (v_\uparrow + u_\uparrow)b^\dagger}{\sqrt{2}} = (u_\uparrow + v_\uparrow) \frac{b + b^\dagger}{\sqrt{2}} = \frac{b + b^\dagger}{\sqrt{2\omega_\uparrow}}, \quad (5.29)$$

where in the final step we have used the expression (5.26). In the same way we may obtain

$$\frac{S_y}{\sqrt{S}} \sim i \frac{u_\uparrow - v_\uparrow}{\sqrt{2}} (b^\dagger - b) = -i \sqrt{\frac{\omega_\uparrow}{2}} (b - b^\dagger) \quad (5.30)$$

Note that these forms of the S_x/\sqrt{S} and S_y/\sqrt{S} operators in (5.29) and (5.30) are exactly those of a quantum harmonic oscillator's position and momentum operators (2.12) respectively.

⁷With the following linear in b bosons S_x operator one may exactly evolve S_x in this regime. Then redoing the integrals required to find S_B in Sect. 3.3.1 one may verify the validity of the BCH solution Fig. 3.3 to a high degree of accuracy.

As such they satisfy the canonical commutation relation $[S_x, S_y]/S = i$. Comparing this to the spin commutation relation $[S_x, S_y]/S = i\hat{z}$, we find that they indeed match for the low energy state in the symmetric phase where $\langle \hat{z} \rangle_{\text{eq}} \sim 1$.

Using (5.29) and (5.30) the jump-operator (5.27) may be expressed as

$$\begin{aligned} L &= \sqrt{2\gamma T} \left[\frac{S_x}{S} + \frac{\hbar\beta}{4} i \frac{S_y}{S} \right] \\ &\sim \sqrt{2\gamma T} \left[\frac{b + b^\dagger}{\sqrt{2\omega_\uparrow}} + \frac{\hbar\beta}{4} \sqrt{\frac{\omega_\uparrow}{2}} (b - b^\dagger) \right] \\ &\sim \sqrt{\frac{\gamma\hbar}{2}} \left[\sqrt{\frac{2}{\hbar\beta\omega_\uparrow}} (b + b^\dagger) + \frac{1}{2} \sqrt{\frac{\beta\hbar\omega_\uparrow}{2}} (b - b^\dagger) \right] \\ &\sim \sqrt{\frac{\gamma\hbar}{2}} [B_+ b + B_- b^\dagger], \end{aligned} \quad (5.31)$$

where we have defined

$$B_\pm \equiv \sqrt{\frac{2}{\beta\hbar\omega_\uparrow}} \pm \frac{1}{2} \sqrt{\frac{\beta\hbar\omega_\uparrow}{2}} \quad (5.32)$$

as to lighten the notation. We are now in a position to consider the full Lindbladian.

As stated before we treat the Lindbladian as a superoperator and associate the left and right acting superoperators to two different species of bosons b_1 and b_2 . From (5.2) we have a dissipator taking the form

$$\mathcal{D} = L \otimes L^* - \frac{1}{2} (L^\dagger L \otimes \mathbb{1} + \mathbb{1} \otimes L^T L^*). \quad (5.33)$$

The left and right acting operators have translated into the operators to the left and right factors of the tensor product. Since (5.31) is a real operator, as the ladder operators are real as well as the coefficients, (5.33) may be simplified to

$$\mathcal{D} = L \otimes L - \frac{1}{2} (L^\dagger L \otimes \mathbb{1} + \mathbb{1} \otimes L^\dagger L). \quad (5.34)$$

Next, with the left and right operators acting on different Hilbert spaces, we employ the standard bosonic subscript notation L_i to indicate which Hilbert space the operator acts on, leaving

$$\mathcal{D} \sim L_1 L_2 - \frac{1}{2} (L_1^\dagger L_1 + L_2^\dagger L_2), \quad L_i \equiv \sqrt{\frac{\gamma\hbar}{2}} [B_+ b_i + B_- b_i^\dagger]. \quad (5.35)$$

When performing such a mapping onto two different sets of bosons one must take care to preserve all the correct bosonic commutation relations. The commutator $[b, b^\dagger] = 1$ can be checked as follows

$$\varrho = [b, b^\dagger] \varrho [b, b^\dagger] \rightarrow [b_1, b_1^\dagger] [b_2, b_2^\dagger]^T |\varrho\rangle = \mathbb{1} \mathbb{1}^T |\varrho\rangle = |\varrho\rangle.$$

Next $[b_i, b_j^\dagger] = \delta_{ij}$ follows from associativity

$$0 = (b_\varrho) b - b (b_\varrho) \rightarrow (b_2)^T (b_1 |\varrho\rangle) - b_1 ((b_2)^T |\varrho\rangle) = [b_1, b_2^\dagger] |\varrho\rangle,$$

where we have, again, used the fact that the ladder operators are real, meaning $b^\dagger = b^T$.

5.3.1 Fixed particle number difference subspace

Since each L_i is linear in the ladder operators, the dissipator (5.35) is quadratic in the ladder operators. In principle such an operator may be diagonalised by a four dimensional Bogoliubov

transform⁸ [8]

$$\begin{bmatrix} c_1 \\ c_2 \\ \bar{c}_1^\dagger \\ \bar{c}_2^\dagger \end{bmatrix} = \begin{bmatrix} U & V \\ V^* & U^* \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_1^\dagger \\ b_2^\dagger \end{bmatrix}, \quad (5.36)$$

however finding the exact solution is neither easy⁹ nor elegant in this case. The solutions to the coefficients will typically be complicated functions and would not yield any further insight in the regime of validity—small γ .

In hopes of finding simple solutions, which one can analyze with greater ease, we consider the problem of diagonalizing the Lindbladian $\mathcal{L}\bullet = i[\bullet, \mathcal{H}_S] + \mathcal{O}(\gamma)$ from a perturbative point of view. We know from (5.25) that for $\mathcal{L}_Q \approx 0$, meaning also close to the GS manifold, that the eigenoperators of $i[\mathcal{H}_S, \bullet]$ are of the form $|n_1\rangle\langle n_2|$. The corresponding eigenspaces, with eigenvalues $i\omega_\uparrow(n_2 - n_1)$, are highly degenerate, with each eigenvalue only depending on the particle number difference $\Delta_n \equiv n_2 - n_1$. Let us now consider \mathcal{L} as a collection of block matrices which conserve the particle difference Δ_n together with some $\mathcal{O}(\gamma)$ terms which do not:

$$\mathcal{L} = \begin{pmatrix} \ddots & & & & & & \mathcal{O}(\gamma) \\ & \boxed{\Delta_n = -1} & & & & & \\ & & \boxed{\Delta_n = 0} & & & & \\ & & & \boxed{\Delta_n = 1} & & & \\ & \mathcal{O}(\gamma) & & & & & \\ & & & & & \ddots & \end{pmatrix}.$$

Following the degenerate perturbative procedure we diagonalise the blocks, corresponding to the degenerate subspaces, individually, ignoring the small $\mathcal{O}(\gamma)$ terms which do not conserve Δ_n [58]. On the second quantised (bosonic operators) level this corresponds to ignoring all non-number-difference conserving terms in \mathcal{L} . With this in mind, we consider the dissipator (5.35) term by term, starting with

$$\begin{aligned} L_1 L_2 &\sim \frac{\gamma\hbar}{2} [B_+^2 b_1 b_2 + B_-^2 b_1^\dagger b_2^\dagger + \text{non-}\Delta_n \text{ conserving terms}] \\ &\sim \frac{\gamma\hbar}{2} [B_+^2 b_1 b_2 + B_-^2 b_1^\dagger b_2^\dagger] \\ &\sim \frac{\gamma\hbar}{2} [(\Upsilon + 1)b_1 b_2 + (\Upsilon - 1)b_1^\dagger b_2^\dagger], \quad \Upsilon \equiv B_\pm^2 \mp 1 = \frac{2}{\beta\hbar\omega_\uparrow} + \frac{\beta\hbar\omega_\uparrow}{8}, \end{aligned} \quad (5.37)$$

where we have used \sim to indicate that we are in the Δ_n conserving subspace. The next operator in (5.35) reduces to

$$\begin{aligned} L_1^\dagger L_1 &\sim \frac{\gamma\hbar}{2} [B_+ b_1^\dagger + B_- b_1] [B_+ b_1 + B_- b_1^\dagger] \\ &\sim \frac{\gamma\hbar}{2} [B_+^2 b_1^\dagger b_1 + B_-^2 b_1 b_1^\dagger] \\ &\sim \frac{\gamma\hbar}{2} [2\Upsilon b_1^\dagger b_1 + 1], \end{aligned} \quad (5.38)$$

where we have normal ordered: $b_1 b_1^\dagger = b_1^\dagger b_1 + 1$. The final operator $L_2^\dagger L_2$ takes the same form as (5.38). Substituting (5.38) and (5.37) into (5.35) we are left with

$$2D/(\gamma\hbar) \sim (\Upsilon + 1)b_1 b_2 + (\Upsilon - 1)b_1^\dagger b_2^\dagger - \frac{1}{2} (2\Upsilon b_1^\dagger b_1 + 1 + 2\Upsilon b_2 b_2^\dagger + 1)$$

⁸Note that is not required that $(c_i)^\dagger = \bar{c}_i^\dagger$ but only that $[c_i, \bar{c}_i^\dagger] = 1$.

⁹The general form of such a four dimensional Bogoliubov transform can be found in [8]. In essence it is just a four dimensional boost and rotation parametrized by ten variables. However, solving for these variables amounts to solving a set of non-linear coupled equations, which spanned multiple pages in our case.

$$\sim |(\Upsilon + 1)b_1b_2 + (\Upsilon - 1)b_1^\dagger b_2^\dagger - \Upsilon(b_1^\dagger b_1 + b_2^\dagger b_2) - \Upsilon + 1, \quad (5.39)$$

The dissipator (5.39) is still quadratic and we still require a four dimensional Bogoliubov transform. As such one might question whether we have reduced the complexity of the problem by use of degenerate perturbation theory. The answer is not obvious at all, but in this case the dissipator (5.39) is simple enough such that it can be diagonalised by an informed guess¹⁰:

$$\begin{bmatrix} c_1 \\ c_2 \\ \bar{c}_1^\dagger \\ \bar{c}_2^\dagger \end{bmatrix} = \begin{bmatrix} U & V \\ \bar{V}^* & \bar{U}^* \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_1^\dagger \\ b_2^\dagger \end{bmatrix} = u \begin{bmatrix} b_1 - v/ub_2^\dagger \\ b_2 - v/ub_1^\dagger \\ b_1^\dagger - b_2 \\ b_2^\dagger - b_1 \end{bmatrix}, \quad U = \bar{U}^* = -\bar{V}^* = u\mathbb{1}, \quad V = -v \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (5.40)$$

with the coefficients simply given by $u = \sqrt{(\Upsilon + 1)/2}$, $v = u - u^{-1}$. We may now check that the new bosons satisfy the commutation relation $[c_i, \bar{c}_i^\dagger] = 1$, which we do for the first boson:

$$[c_1, \bar{c}_1^\dagger] = [ub_1 - vb_2^\dagger, ub_1^\dagger - ub_2] = u^2[b_1, b_1^\dagger] + uv[b_2^\dagger, b_2] = u^2 - vu = u^2 - (u - u^{-1})u = 1,$$

where we have used the standard bosonic commutation relations satisfied by b_i . The proof for the second boson c_2 follows in the same way. Lastly, the two bosons should commute with another. To see this we prove it for a single case:

$$[c_1, c_2] = [ub_1 - vb_2^\dagger, ub_2 - vb_1^\dagger] = [ub_1, -vb_1^\dagger] + [-vb_2^\dagger, ub_2] = uv(-[b_1, b_1^\dagger] - [b_2^\dagger, b_2]) = uv(1 - 1) = 0.$$

One may check that indeed these new bosonic operators reduce the dissipator to

$$\mathcal{D} \sim -\frac{\gamma\hbar}{2} [\bar{c}_1^\dagger c_1 + \bar{c}_2^\dagger c_2]. \quad (5.41)$$

The easiest way to see this is by working in reverse. We start with $\bar{c}_1^\dagger c_1 + \bar{c}_2^\dagger c_2$ and insert their b boson representations (5.40)

$$\begin{aligned} \bar{c}_1^\dagger c_1 + \bar{c}_2^\dagger c_2 &= u(b_1^\dagger - b_2)(ub_1 - vb_2^\dagger) + u(b_2^\dagger - b_1)(ub_2 - vb_1^\dagger) \\ &= u^2 b_1^\dagger b_1 + uvb_2 b_2^\dagger + u^2 b_2^\dagger b_2 + uvb_1 b_1^\dagger - 2u^2 b_2 b_1 - 2uvb_1^\dagger b_2^\dagger. \end{aligned} \quad (5.42)$$

Upon normal ordering $b_i b_i^\dagger = b_i^\dagger b_i + 1$, (5.42) reduces to

$$\bar{c}_1^\dagger c_1 + \bar{c}_2^\dagger c_2 = (u^2 + uv)(b_1^\dagger b_1 + b_2^\dagger b_2) + 2uv - 2u^2 b_2 b_1 - 2uvb_1^\dagger b_2^\dagger. \quad (5.43)$$

Inserting $u = \sqrt{(\Upsilon + 1)/2}$ and $v = u - u^{-1}$, we obtain

$$\begin{aligned} \bar{c}_1^\dagger c_1 + \bar{c}_2^\dagger c_2 &= (2u^2 - 1)(b_1^\dagger b_1 + b_2^\dagger b_2) + 2(u^2 - 1) - 2u^2 b_2 b_1 - 2(u^2 - 1)b_1^\dagger b_2^\dagger \\ &= (\Upsilon + 1 - 1)(b_1^\dagger b_1 + b_2^\dagger b_2) + (\Upsilon + 1 - 2) - (\Upsilon + 1)b_2 b_1 - (\Upsilon + 1 - 2)b_1^\dagger b_2^\dagger \\ &= -(\Upsilon + 1)b_1 b_2 - (\Upsilon - 1)b_1^\dagger b_2^\dagger + \Upsilon(b_1^\dagger b_1 + b_2^\dagger b_2) + \Upsilon - 1, \end{aligned} \quad (5.44)$$

which is indeed proportional to (5.39). Now what does this mean for the unitor? Using (5.29) and (5.30) the full Hamiltonian (3.46) may be written as

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_S + \frac{\hbar\gamma}{4S} \{S_x, S_y\} + \text{const.} \\ &\sim \omega_\uparrow b^\dagger b + \frac{\hbar\gamma}{4} \left\{ \frac{b + b^\dagger}{\sqrt{2\omega_\uparrow}}, -i\sqrt{\frac{\omega_\uparrow}{2}}(b - b^\dagger) \right\} + \text{const.} \\ &\sim | \omega_\uparrow b^\dagger b - i\frac{\hbar\gamma}{8} [b^\dagger, b] + \text{const.} \end{aligned}$$

¹⁰This does of course require some luck. Luckily I have an Irish friend who has lent me just enough luck.

$$\sim |\omega_\uparrow b^\dagger b + \text{const.}| \quad (5.45)$$

Using (5.45) the unitor $\mathcal{U} = i[\bullet, \mathcal{H}]$ is given by (5.3) to be

$$\mathcal{U} = i\mathbb{1} \otimes \mathcal{H}^T - i\mathcal{H} \otimes \mathbb{1} \sim i\omega_\uparrow (b_2^\dagger b_2 - b_1^\dagger b_1). \quad (5.46)$$

One may now check that this is proportional to the new particle number difference $\bar{c}_1^\dagger c_1 - \bar{c}_2^\dagger c_2$. Using (5.40) we express $\bar{c}_1^\dagger c_1 - \bar{c}_2^\dagger c_2$ in terms of the b bosons

$$\begin{aligned} \bar{c}_1^\dagger c_1 - \bar{c}_2^\dagger c_2 &= u(b_1^\dagger - b_2)(ub_1 - vb_2^\dagger) - u(b_2^\dagger - b_1)(ub_2 - vb_1^\dagger) \\ &= u^2 b_1^\dagger b_1 + uvb_2 b_2^\dagger - u^2 b_2^\dagger b_2 - uvb_1 b_1^\dagger, \end{aligned} \quad (5.47)$$

where the non-number-conserving terms have canceled. If we now normal order (5.47), then the generated c -numbers (constants) cancel. This leaves

$$\bar{c}_1^\dagger c_1 - \bar{c}_2^\dagger c_2 = (u^2 - uv)(b_1^\dagger b_1 - b_2^\dagger b_2), \quad (5.48)$$

where one may check, using $v = u - u^{-1}$, that $u^2 - uv = -1$. This together with the diagonalised dissipator in (5.41) leads to the following Lindbladian

$$\mathcal{L} \sim i\hbar\omega_\uparrow [\bar{c}_1^\dagger c_1 - \bar{c}_2^\dagger c_2] - \frac{\gamma\hbar}{2} [\bar{c}_1^\dagger c_1 + \bar{c}_2^\dagger c_2]. \quad (5.49)$$

The gap is then the difference between the vacuum state (with zero eigenvalue) and the first excited state (of either c boson):

$$\pm i\hbar\omega_\uparrow - \frac{\gamma\hbar}{2}. \quad (5.50)$$

Seeing as $\omega_\uparrow = \sqrt{1 - \Lambda}$ from (5.26), we note that this gap matches the conjectured gap (5.13)

$$\omega_S = \pm i\hbar\sqrt{1 - \Lambda} - \frac{\gamma\hbar}{2} + \mathcal{O}(\gamma)^2 \quad (5.51)$$

up to linear order in γ . Thus we have shown that our stationary state is indeed unique in the symmetric phase. As such to prove thermalisation we only have to show that vacuum state is the Gibbs state $e^{-\beta\hbar\omega_\uparrow b^\dagger b}$.

5.3.2 The stationary state in the symmetric phase

The stationary state corresponds to the zero eigenvalue of (5.49). Since we are dealing with bosons, this is the vacuum state $c_i|\emptyset_c\rangle = 0$. In terms of the b bosons we have [9]:

$$|\emptyset_c\rangle \propto \exp\left(-\frac{1}{2} \sum_{i,j} (U^{-1}V)_{ij} b_i^\dagger b_j^\dagger\right) |0_{b_1}, 0_{b_2}\rangle, \quad (5.52)$$

where $|0_{b_1}, 0_{b_2}\rangle$ is the vacuum state of both b bosons. From (5.40) we note that U is proportional to the identity, $U_{ij} = u\delta_{ij}$, and V is off-diagonal, $V_{ij} = -v(1 - \delta_{ij})$, meaning (5.52) reduces to

$$|\emptyset_c\rangle \propto \exp\left(-\frac{1}{2} \sum_{i,j} -\frac{1}{u}v(1 - \delta_{ij})b_i^\dagger b_j^\dagger\right) |0_{b_1}, 0_{b_2}\rangle = \exp\left(\frac{v}{u}b_1^\dagger b_2^\dagger\right) |0_{b_1}, 0_{b_2}\rangle. \quad (5.53)$$

Expanding this exponential in (5.53) and using the formula $(b_1^\dagger b_2^\dagger)^n |0_{b_1}, 0_{b_2}\rangle = n! |n, n\rangle$, which is proven by iterative use of (2.2), we obtain

$$|\emptyset_c\rangle \propto \sum_{n=0}^{\infty} \left(\frac{v}{u}\right)^n \frac{(b_1^\dagger b_2^\dagger)^n}{n!} |0_{b_1}, 0_{b_2}\rangle = \sum_{n=0}^{\infty} \left(\frac{v}{u}\right)^n |n, n\rangle. \quad (5.54)$$

We may now map the vectorised object back to a density matrix as $|n, n\rangle \rightarrow |n\rangle\langle n|$, upon which (5.54) becomes

$$\varrho_{eq} \propto \sum_{n=0}^{\infty} \left(\frac{v}{u}\right)^n |n\rangle\langle n| = \sum_n \exp\left(n \ln\left(\frac{v}{u}\right)\right) |n\rangle\langle n| = \exp\left(\ln\left(\frac{v}{u}\right) b^\dagger b\right), \quad (5.55)$$

where the last equality is due to $b^\dagger b = \sum_n n |n\rangle\langle n|$. The logarithm in (5.55) may be simplified as

$$\ln\left(\frac{v}{u}\right) = \ln(1 - u^{-2}) = \ln\left(1 - \frac{2}{\Upsilon + 1}\right) = 2 \ln\left|\frac{\beta\hbar\omega_\uparrow - 4}{\beta\hbar\omega_\uparrow + 4}\right|, \quad (5.56)$$

where we have used the definition (5.37) of Υ . Expanding this logarithm (5.56) around $\beta\hbar\omega_\uparrow = 0$ yields

$$2 \ln\left|\frac{\beta\hbar\omega_\uparrow - 4}{\beta\hbar\omega_\uparrow + 4}\right| = -\beta\hbar\omega_\uparrow - \frac{(\beta\hbar\omega_\uparrow)^3}{48} - \frac{(\beta\hbar\omega_\uparrow)^5}{1280} + \dots = -\beta\hbar\omega_\uparrow + \mathcal{O}\left((\beta\hbar\omega_\uparrow)^3\right). \quad (5.57)$$

With $\mathcal{H}_S = \hbar\omega_\uparrow b^\dagger b + \text{const.}$ we may finally write the stationary state (5.55) as

$$\varrho_{eq} \propto \exp\left(-\beta\mathcal{H}_S + \mathcal{O}\left((\beta\hbar\omega_\uparrow)^3\right) b^\dagger b\right), \quad (5.58)$$

which is the Gibbs distribution with some small corrections. Remember that we derived the master equation under the assumption that the system frequency, hence $\hbar\omega_\uparrow$, is small in comparison to the bath's frequencies, which scaled with the temperature. As such the master equation's region of validity—small $\hbar\omega_\uparrow/T$ —coincides with the region in which the stationary state is a thermal state.

To see the (small) size of these corrections in (5.58) more explicitly, consider that in standard units

$$\beta\hbar\omega_\uparrow = \hbar \frac{\hbar\omega_\uparrow}{k_B T} \approx 10^{-11} \hbar \frac{\omega_\uparrow}{T} [\text{J} \cdot \text{s} \cdot \text{K}].$$

Thus we have proven that the system has a unique stationary state (non-zero gap) and that this stationary state is very close to a Gibbs distribution—together we have proven that the system *thermalises* in the symmetric phase. Yaaaaaaayyyyy! So happy!

5.4 Bosonisation in the symmetry broken phase

Having proven that the system thermalises in the symmetric phase, we move on to the degenerate symmetry broken phase with $\Delta \equiv \hbar/J = \Lambda^{-1} < 1$. Its equilibrium value $\langle S_z \rangle_{\text{eq}} = \langle S - a^\dagger a \rangle_{\text{eq}} = S\Delta$ would imply that we expand the HP mapped S_+ operator (5.18) around $\langle a^\dagger a \rangle_{\text{eq}} = S(1 - \Delta)$. This point of expansion is no longer zero, as opposed to the symmetric phase where we had expanded around $a^\dagger a = 0$, and no longer intensive. This leads to an expansion

$$\frac{S_+}{\sqrt{S}} \rightarrow \sqrt{2} \sqrt{1 - \frac{a^\dagger a}{2S}} a = \sqrt{1 + \Delta} a + \mathcal{O}(a^\dagger a - S(1 - \Delta)) a,$$

which is no longer damped in S . The damping is crucial to ensure the accuracy of the HP expansion in the thermodynamic limit, and as such we require an adjusted strategy.

5.4.1 Restriction to a well

To perform an *effective* HP expansion we rotate the spin operators $\mathbf{S} = R\bar{\mathbf{S}}$ such that the new spin- z operator \bar{S}_z may again be considered $\sim S$. In other words, we rotate the spin operator

such that \bar{S}_z is aligned with the new (semi-classical) magnetic field orientations [17]. The corresponding rotations are

$$\begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix} = R_{\pm} \begin{bmatrix} \bar{S}_x \\ \bar{S}_y \\ \bar{S}_z \end{bmatrix}, \quad R_{\pm} \equiv \begin{bmatrix} \Delta & 0 & \pm\sqrt{1-\Delta^2} \\ 0 & 1 & 0 \\ \mp\sqrt{1-\Delta^2} & 0 & \Delta \end{bmatrix}, \quad (5.59)$$

which may be verified by using the symmetry broken expectation values (5.19) to obtain

$$\langle \bar{\hat{z}} \rangle_{\text{eq}} = \pm\sqrt{1-\Delta^2} \langle \hat{x} \rangle_{\text{eq}} + \Delta \langle \hat{z} \rangle_{\text{eq}} = (\pm\sqrt{1-\Delta^2})(\pm\sqrt{1-\Delta^2}) + \Delta\Delta = 1 - \Delta^2 + \Delta^2 = 1.$$

Here we have used the orthogonality of a rotation matrix, $R_{\mp}R_{\pm} = 1$, to find $\bar{S}_z/S \equiv \bar{\hat{z}}$ in terms of S_x and S_z . Further, restriction to the unit sphere forces $\langle \bar{\hat{x}} \rangle \sim 0$. In total $\langle \bar{\hat{s}} \rangle$ matches $\langle \hat{s} \rangle$ in the symmetric phase (5.19).

These two rotations have the effect of focusing the analysis to a particular *well* (subspace) of \mathcal{H}_S depicted in Fig. 5.1. The bottoms of these wells correspond to the low energy states, which have either extensively positive or negative S_x expectation values. For instance the ground states have $\langle S_x \rangle_{\pm} = \pm S\sqrt{1-\Delta^2}$. Since the sign of \hat{x} is not a conserved quantity¹¹, these two wells will interact with one another. However only the sign of the *bridging states* $|m_x \in \{-1, 0, 1\}\rangle$, with $S_x|m_x\rangle = m_x|m_x\rangle$, are changed when acted upon by S_z or S_y . As for the other states, with $m_x \notin \{-1, 0, 1\}$, we have

$$[\text{sgn}(S_x), S_z]|m_x\rangle = [\text{sgn}(S_x), S_y]|m_x\rangle = 0$$

Hence $\text{sgn}(S_x)$ is conserved as long as the states are restricted to these wells. In this case we may think of $\text{sgn}(S_x) \equiv \pm = -\mp$. Further these bridging states form an ever smaller fraction of the Hilbert space as S increases, meaning that low energy states—the bottom of the wells—become extensively separated. As such we may hypothesise that the *tunneling* (transitions) from one well to the other happens over a slow time scale. It is important to note that this method will fail close to the phase transition where the two wells become closer as $\lim_{\Delta \rightarrow 1} \langle S_x \rangle_{\pm} = \pm 0$.

With $\langle \bar{\hat{z}} \rangle \sim 1$ we may use the same expansion on \bar{S}_+ as we did with (5.18)

$$S_x = \Delta \bar{S}_x \pm \sqrt{1-\Delta^2} \bar{S}_z \sim \Delta \sqrt{S/2}(a + a^\dagger) \pm \sqrt{1-\Delta^2}(S - a^\dagger a), \quad (5.60)$$

where we may think of this operator acting on a state $|n, \pm\rangle$ which is in either the left or right well¹². Following the same reasoning as (5.21) this expansion (5.60) is only valid for sub-extensive number of excitations $n = \mathcal{O}(S^\alpha)$ with $\alpha < 2/3$. As such the expansion in (5.60) is only valid if the operator acts on the low lying, well-restricted states. Substituting (5.60) and a similar expression for S_z into the Hamiltonian yields

$$\mathcal{H}_S/J = -\frac{1}{2S}(\Delta \bar{S}_x \pm \sqrt{1-\Delta^2} \bar{S}_z)^2 - \Delta(\mp\sqrt{1-\Delta^2} \bar{S}_x + \Delta \bar{S}_z)$$

¹¹In terms of the S_x eigenstates $|m_x\rangle$, $S_y \equiv \frac{S_{x,+} + S_{x,-}}{2}$ and $S_z \equiv \frac{S_{x,+} - S_{x,-}}{2i}$, which raise and lower $|m_x\rangle$ in the same way as (2.7). As such the S_z operator in the Hamiltonian may change this sign by raising or lowering $|S_x = 0\rangle$.

¹²These states are only orthogonal to another in the low-energy subspace, in the thermodynamic limit. As such introducing them is purely for heuristic reasons and should not be taken as rigorous.

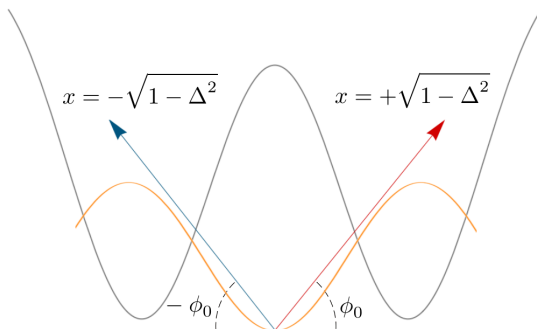


Figure 5.1: *Rotation of the spin operators by angles $\phi = \pm\phi_0$ onto the semi-classical magnetic fields, and thus two separate wells. Their states act almost independently, indicated by the orange curve decaying in the connecting part of the wells.*

$$\sim \left(1 - \frac{\Delta^2}{2}\right) a^\dagger a - \frac{\Delta^2}{4} (a^2 + a^{\dagger 2}) - S \frac{\Delta^2 + 1}{2} - \frac{\Delta^2}{4} \quad (5.61)$$

which is quadratic: $2\epsilon a^\dagger a + \varphi(a^2 + \text{H.C.}) + \text{const.}$, meaning diagonalisable by a Bogoliubov transform. Assuringly we see that (5.61) predicts the same GS energy as (4.44). The Hamiltonian (5.61) has a dispersion relation and coefficients which are given by

$$\omega_\downarrow = \sqrt{1 - \Delta^2}, \quad u_\downarrow \pm v_\downarrow = \sqrt{\omega_\downarrow^{\mp 1}}, \quad (5.62)$$

respectively. In terms of the new basis given by the Bogoliubov transform (5.61) becomes

$$\mathcal{H}_S/J \sim \omega_\downarrow b^\dagger b + S\Delta h_g. \quad (5.63)$$

The fact that \mathcal{H}_S is independent of \pm , up to correction terms, implies the two fold degeneracy over low lying energy states in the symmetry broken phase, in the thermodynamic limit. The sign of S_x will however play a role in the correction terms and close to the bridging states. This means that our expansion cannot fully describe tunneling between these two wells.

It is of importance to note that $J\omega_\downarrow$ is the gap between states in the particular wells Fig. 5.1, so should not be compared with the RPA conjectured gap ω_S (5.14), which is zero for an uncoupled system. Lastly ω_B (5.17) matches $J\omega_\downarrow$ for $\gamma = 0$, which motivates the interpretation that ω_B is the ‘‘decay’’ rate in a particular well.

5.5 Lindbladian in the symmetry broken phase

Now that we have diagonalised the Hamiltonian we will proceed with the same steps as with the symmetric phase Sect. 5.3. We find, in a similar way as for (5.29), that

$$\frac{S_x}{\sqrt{S}} \sim \frac{\Delta}{\sqrt{2\omega_\downarrow}} (b + b^\dagger) \pm \sqrt{S}\omega_\downarrow + \mathcal{O}\left(\frac{1}{\sqrt{S}}\right) b^\dagger b + \mathcal{O}\left(\frac{1}{\sqrt{S}}\right) (b^2 + b^{\dagger 2}), \quad \frac{S_y}{\sqrt{S}} \sim i\sqrt{\frac{\omega_\downarrow}{2}} (b^\dagger - b), \quad (5.64)$$

With these equations (5.64) we find the jump operator

$$\begin{aligned} L &= \sqrt{2\gamma T} \left[\frac{S_x}{\sqrt{S}} + \frac{\hbar\beta}{4} i \frac{S_y}{\sqrt{S}} \right] \\ &\sim \sqrt{2\gamma T} \left[\frac{\Delta}{\sqrt{2\omega_\downarrow}} (b + b^\dagger) - \frac{\hbar\beta}{4} \sqrt{\frac{\omega_\downarrow}{2}} (b^\dagger - b) + \mathcal{O}(\sqrt{S}) + \mathcal{O}(S^{-1/2}) b^\dagger b + \mathcal{O}(S^{-1/2}) (b^2 + b^{\dagger 2}) \right]. \end{aligned} \quad (5.65)$$

To simplify (5.65) we again use degenerate perturbation theory, restricting ourselves to the fixed (particle) number difference subspace. In this way we obtain a dissipator of the form (5.35)

$$\mathcal{D} \sim L_1 L_2 - \frac{1}{2} (L_1^\dagger L_1 + L_2^\dagger L_2), \quad (5.66)$$

where L_i acts on the i th b boson. Note that in the fixed particle difference subspace the terms like $\mathcal{O}(S^{-1/2})b^2$ in (5.65) will only contribute when combining with their counterpart $\mathcal{O}(S^{-1/2})b^{\dagger 2}$, due to being the only terms raising or lowering twice. But in that case their contribution would be of the order $\mathcal{O}(S^{-\alpha})$, with $\alpha > 0$. For this reason we may ignore them in the thermodynamic limit and write (5.65) as

$$L \sim \sqrt{2\gamma T} \left[\frac{\Delta}{\sqrt{2\omega_\downarrow}} (b + b^\dagger) - \frac{\hbar\beta}{4} \sqrt{\frac{\omega_\downarrow}{2}} (b^\dagger - b) + \mathcal{O}(\sqrt{S}) + \mathcal{O}\left(\frac{1}{\sqrt{S}}\right) b^\dagger b \right]. \quad (5.67)$$

Now let us consider the contribution of the term

$$\mathcal{O}(\sqrt{S}) + \mathcal{O}(1/\sqrt{S}) b^\dagger b \propto q + \hat{n}_i \equiv l_i$$

from (5.67) in dissipator (5.66). Here q is some constant and $\hat{n}_i \equiv b_i^\dagger b_i$. First notice that as the only number conserving term in (5.67) it will only contribute in (5.66), in the fixed particle difference subspace, when it multiplies with itself. Thus its only contribution in (5.66) is

$$\begin{aligned} l_1 l_2 - \frac{1}{2}(l_1^\dagger l_1 + l_2^\dagger l_2) &= (q + n_1)(q + n_2) - \frac{1}{2}((q + n_1)(q + n_1) + (q + n_2)(q + n_2)) \\ &= q(n_1 + n_2) - \frac{1}{2}(q(2n_1 + 2n_2)) \\ &= 0 \end{aligned} \quad (5.68)$$

where the remaining terms have canceled. As such the dissipator, like the Hamiltonian, is also $\text{sgn}(S_x)$ independent up to correction terms and (5.67) reduces to the form of (5.31)

$$\begin{aligned} L &\sim \sqrt{2\gamma T} \left[\frac{\Delta}{\sqrt{2\omega_\downarrow}}(b + b^\dagger) - \frac{\hbar\beta}{4} \sqrt{\frac{\omega_\downarrow}{2}}(b^\dagger - b) \right] \\ &\sim \hbar \sqrt{\frac{\gamma}{2J}} [B_+ b + B_- b^\dagger], \quad B_\pm = \sqrt{\frac{2}{\beta J \omega_\uparrow}} \pm \frac{1}{2} \sqrt{\frac{\beta J \omega_\uparrow}{2}}, \end{aligned} \quad (5.69)$$

albeit with slightly different constants. Hence we merely have to substitute the different constants into (5.49) to yield the diagonalised Lindbladian

$$\mathcal{L} \sim iJ\omega_\downarrow [\bar{c}_1^\dagger c_1 - \bar{c}_2^\dagger c_2] - \frac{\gamma\hbar\Delta}{2} [\bar{c}_1^\dagger c_1 + \bar{c}_2^\dagger c_2]. \quad (5.70)$$

Since we have $\omega_\downarrow = \sqrt{1 - \Delta^2}$ from (5.62), we can see that the gap matches the conjectured gap (5.17), up to linear order in γ :

$$\omega_B = \pm iJ\sqrt{1 - \Delta^2} - \frac{\gamma\hbar\Delta}{2} + \mathcal{O}(\gamma^2). \quad (5.71)$$

Substituting the constants of (5.69) into the symmetric phase vacuum state (5.58) we obtain the rotated symmetry broken vacuum

$$\varrho_{eq} \propto \exp(-\beta J\omega_\downarrow b^\dagger b + \mathcal{O}(\beta J\omega_\downarrow)^3) \propto \exp(-\beta\mathcal{H}_S + \mathcal{O}((\beta\hbar\omega_\uparrow)^3) b^\dagger b). \quad (5.72)$$

Again, (5.72) matches the Gibbs state to a high degree of accuracy, in this case however it is restricted to a particular well. Since these two wells interact at higher energies we do not expect the vacuum states in (5.72) to be stationary states for finite S .

5.5.1 Comparing to numerics

To measure the ‘‘closeness’’ of two states ϱ_1, ϱ_2 numerically we use the fidelity [31]

$$F_{12} = F(\varrho_1, \varrho_2) = \left[\text{Tr} \sqrt{\sqrt{\varrho_1} \varrho_2 \sqrt{\varrho_1}} \right]^2,$$

which is symmetric $F_{12} = F_{21}$. The fidelity can be thought of as an overlap of two states, and as such has a range $F_{12} \in [0, 1]$ with $F_{11} = 1$. The closer to 1 the fidelity is the closer the states are to one another.

Numerically we find that $|\ell_1\rangle\langle\ell_1|$ is not a valid density matrix. This indicates that for finite S , there is only one stationary state: $|\ell_0\rangle\langle\ell_0|$. Considering the plot in Fig. 5.2 we find fidelities which are very close to 1

between the Gibbs state and the stationary state of the Lindbladian. Further as we would expect, based on the correction terms in (5.72), the fidelity is closer to 1 the higher the temperature T .

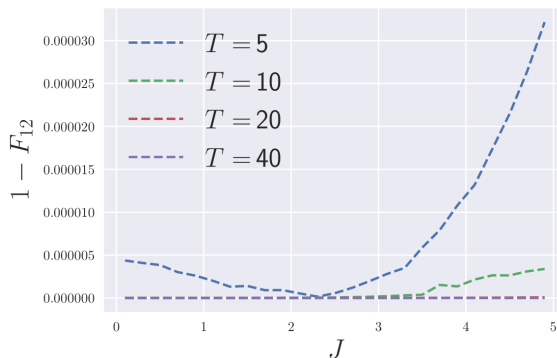


Figure 5.2: Plot of $1 - F_{12}$, with $\varrho_1 = |\ell_0\rangle\langle\ell_0|$ and $\varrho_2 \propto e^{-\beta\mathcal{H}_S}$ as a function of J , for various temperatures. Here $S = 40$, $\hbar = 1$ and $\gamma = 10^{-3}$.

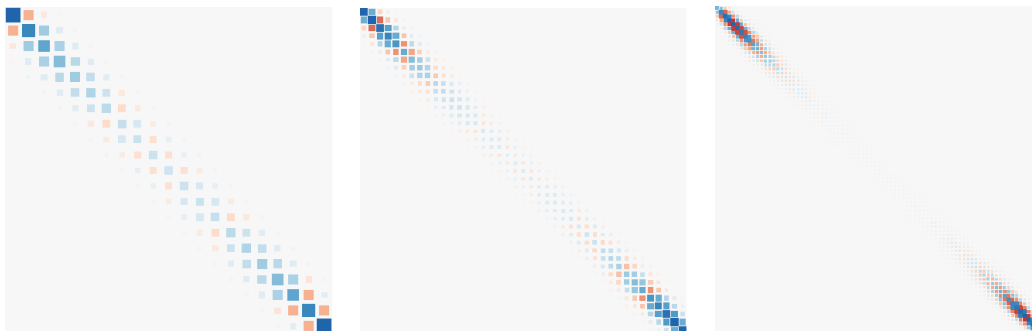


Figure 5.4: Hinton diagrams of $|\ell_0\rangle\langle\ell_0|$ in the S_x basis for $S \in \{10, 20, 40\}$. As Hinton diagrams, the size of the square indicates the magnitude of the matrix element. Blue and red indicate positive and non-positive matrix elements respectively. Here $J = 2$, $\hbar = 1$, $T = 4$ and $\gamma = 10^{-3}$.

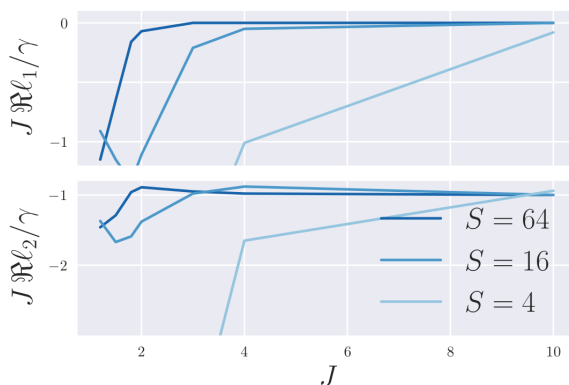


Figure 5.3: Plots of $\ell_1(J)$ and $\ell_2(J)$, for $T = 4$, $\hbar = 1$ and $\gamma = 10^{-3}$.

By exact diagonalisation we find that the stationary state $|\ell_0\rangle\langle\ell_0|$ is an equal mixture of the two well-restricted vacuum states, as seen in Fig. 5.4. Further we see that as S increases, the matrix elements corresponding to the different wells become extensively separated.

Numerically¹³ it appears that the gap $|\Re\ell_1|$ tends to zero as the system size increases, as plotted in Fig. 5.3. This would indicate two (orthogonal) stationary states in the thermodynamic limit, rather than one. This is also hinted at by the \pm terms only appearing in the correction terms of the Lindbladian (5.70). Together these two

pieces of evidence motivate that the tunneling between the wells happens on a long time scale which scales with S . Then in the thermodynamic limit the tunneling time diverges and the wells “separate”. The well-restricted vacuum states (5.72) then correspond to prethermalised states, as they are stable over times scales which scale with S .

The second plot in Fig. 5.3 seems to indicate $\Re\ell_2 \rightarrow -\gamma J$, which matches one of the conjectured gaps $\omega_S = -\gamma\hbar\Delta$ from (5.14) for $\hbar = 1$.

¹³In the intensive temperature case the parameter regimes of the numerics are severely limited by the system size. This is because the difference in intensive temperature and extensive temperature only really arises for large S , while for our Lindbladian the numerics for $S = 80$ took two days to finish. As such we needed to use a small enough temperature for the numerics, such that it is in the “intensive regime”.

6 | Conclusion

In this thesis we have studied the thermalisation process of the integrable LMG model coupled to a near-memoryless bosonic bath. For a zero magnetic field $\mathcal{H} = 0$ the LMG model's energy is conserved preventing thermalisation. This leads to a pure decoherence process. For extensive temperatures we proved that the Gibbs state is stationary, in the thermodynamic limit. We studied the gap numerically, by exact diagonalisation, which implied a non-zero gap. This implies, all together, that the system thermalises in the extensive temperature case. We further found that even in the extensive temperature case, that the thermal state yields GS expectation values for low enough temperatures.

In the intensive temperature case we considered the two different LMG phases separately. For the symmetric phase we found, using an HP mapping, a unique stationary state matching the Gibbs state up to a high degree of accuracy. This Gibbs state indeed has the GS expectation values predicted in [70]. For the symmetry broken phase we found two different vacuum states corresponding to the different wells Fig. 5.1. These wells interact with one another over a certain timescale, and as such these vacuum states are not necessarily stationary. For a finite system size we found, using exact diagonalisation routines, that the true stationary state is an equal mixture of these two states. As such the numerics indicates that for finite S the system indeed thermalises to a unique Gibbs state in the symmetry broken phase. Numerically it appears as though the Lindbladian's gap tends to zero as the system size increases. This implies that there exist two (orthogonal) stationary states in the thermodynamic limit. This is further hinted at by the leading order HP expansion being independent of the particular well. As such we would expect that the two vacuum states only become true stationary states as $S \rightarrow \infty$. Then a well restricted initial state would equilibrate to the corresponding vacuum state (5.72). For a finite system we would then expect tunneling, between the two wells, to occur on time scales which scale with S , meaning that (5.72) correspond to prethermalised states. Seeing as any combination of these two stationary states would also be stationary, we note that the equilibrium state would be dependent on its initial conditions, in this case.

6.1 Outlook

6.1.1 Studying the phase transition

The methods used in this thesis become unstable close to the phase transition $\Lambda = 1$. For instance the rotation method, introduced in Sect. 5.4.1, will fail close to the phase transition as the wells' separation goes to zero $\lim_{\Delta \rightarrow 1} \langle S_x \rangle_{\pm} = \pm 0$. One method which we have not included in this thesis does yield promising results close to $\Lambda = 1$. The procedure starts by separating the Hamiltonian over its conserved parity $e^{i\pi S_z}$ as was done in (2.10). One then performs an adjusted HP mapping onto bosons with spin-1/2 degrees of freedom. This method's predictions compares well with numerics, *but only close to* $\Lambda = 1$. Without knowing the reason for its failure away from the phase transition, its validity is brought into question. If however we could understand this failure in terms of some limitation of the mapping, then we may use it to study the phase transition. This method further gives insight into the expected tunneling times between the two prethermalised state.

6.1.2 Bounding the gap

As discussed throughout this thesis, for thermalisation to occur one requires a unique stationary state. For the extensive temperature case, as well as the symmetry broken phase in the intensive temperature case, we have only given finite system size numerics to indicate that thermalisation occurs. However, as mentioned in Sect. 5.1, given a diagonally dominant operator, one may lower bound the gap $|\Re \ell_1|$ via the Gershgorin theorem [27]:

Thm 3 For any n by n complex matrix \mathcal{L} , the Gershgorin disc D_k is centered at \mathcal{L}_{kk} in the complex plane, with radius $R_k = \sum_{j \neq k} |\mathcal{L}_{kj}|$. If the union of k discs is disjoint from that of $n - k$ discs then the former union contains exactly k and the latter $n - k$ eigenvalues of \mathcal{L} .

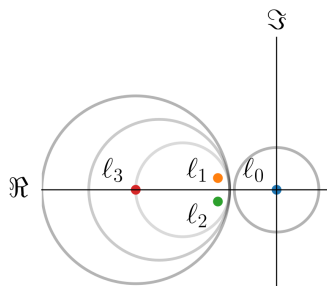


Figure 6.1: Gershgorin disc for some diagonally dominant Lindbladian, with eigenvalues ℓ_i .

Take for instance the Gershgorin discs in Fig. 6.1 for some Lindbladian. Due to the disc around zero not intersecting with any other discs, we know from thm 3 that there can only be one eigenvalue at the origin¹, i.e. one zero eigenvalue. But we also know that the closest eigenvalue describing the gap must be in another disc. As such we may lower bound the gap by the distance between the origin and the closest disc.

In order to obtain the tightest bounds, we would desire a matrix that is diagonally dominant as to minimise the radius in which the eigenvalues fall. As it turns out our Lindbladian (3.52) \mathcal{L} is not diagonal enough in either the S_x, S_y or S_z basis as to give us any useful information about the bound. Using continuous unitary transformations, one may however evolve an operator such that it becomes ever more diagonal. This is done via the flow equations [36]². The general structure of the flow is depicted in Fig. 6.2.

After flowing the Lindbladian one may then apply the Gershgorin theorem on the now approximately diagonal matrix to achieve a lower bound on the gap which is of arbitrary precision.

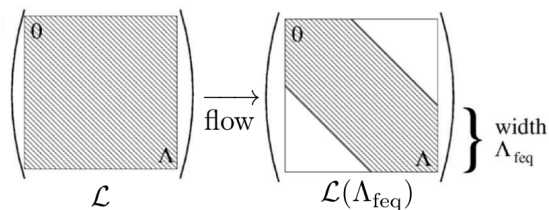


Figure 6.2: Left is the original Lindbladian, while on the right it has been evolved according to the flow equations which has made it band-diagonal with an effective band width Λ_{feq} . This figure was adapted from [36].

¹Remember that a Lindbladian must have at least one zero eigenvalue to preserve the rules of quantum mechanics.

²This method is typically used to perform time evolution, where one requires the inverse unitary transformations. These inverses introduce a difficulty which we will not encounter using the flow equations to bound the gap as we do not require them.

A | Identities

A.1 Operator algebra identities and proofs

Thm 4 Defining $J_q \equiv -B^{n-2-q}[[[A, B], B], B]B^q$ we have

$$[A, B^{n+1}] = \frac{(n+1)}{2} \{B^n, [A, B]\} + \frac{n-1}{2} \sum_{j=0}^{n-2} J_j + \sum_{k=0}^{n-3} k(n-k-1)J_k. \quad (\text{A.1})$$

Proof: Iterative use of the following identity $[A, BC] = B[A, C] + [A, B]C$ yields

$$[A, B^{n+1}] = \sum_{k=0}^n B^{n-k} [A, B] B^k, \quad n \in \mathbb{N}. \quad (\text{A.2})$$

Now defining $iC \equiv [A, B]$ and rearranging terms in (A.2), we obtain

$$\begin{aligned} [A, B^{n+1}] &= \frac{i}{2} \sum_{k=0}^n B^n C + B^{n-k} [C, B^k] + C B^n + [B^{n-k}, C] B^k \\ &= \frac{i}{2} \sum_{k=0}^n \{B^n, C\} + B^{n-k} [C, B^k] + [B^{n-k}, C] B^k \\ &= \frac{i(n+1)}{2} \{B^n, C\} + \frac{i}{2} \sum_{k=0}^n B^{n-k} [C, B^k] + \text{H.C.}, \end{aligned} \quad (\text{A.3})$$

where the fact that the last two terms sum to hermitian conjugates of another is due to the symmetry of the sum. Since the first term in the sum (A.3) is zero, as $B^0 = \mathbb{1}$, we shift $k \rightarrow k+1$ leading to the sum term

$$\frac{i}{2} \sum_{k=0}^n B^{n-k} [C, B^k] + \text{H.C.} = \frac{i}{2} \sum_{k=0}^{n-1} B^{n-1-k} [C, B^{k+1}] + \text{H.C.}$$

Defining $iD \equiv [C, B]$ and using (A.2) yet again leads to

$$\begin{aligned} &= \frac{i}{2} \sum_{k=0}^{n-1} \sum_{p=0}^k B^{n-1-k} B^{k-p} [C, B] B^p + \text{H.C.} \\ &= \frac{i}{2} \sum_{k=0}^{n-1} \sum_{p=0}^k B^{n-1-k} B^{k-p} i D B^p + \text{H.C.} \\ &= -\frac{1}{2} \sum_{k=0}^{n-1} \sum_{p=0}^k B^{n-1-p} D B^p - \text{H.C.} \\ &= -\frac{1}{2} \sum_{p=0}^{n-1} (n-p) B^{n-1-p} D B^p - \text{H.C.}, \end{aligned} \quad (\text{A.4})$$

where we have summed over k . Due to the subtraction of the Hermitian conjugate (A.4) has an asymmetry that causes the p -independent terms to cancel:

$$nB^{n-1-p} i D B^p + nB^p i D B^{n-1-p} + \text{H.C.}$$

$$\begin{aligned}
&= nB^{n-1-p}iDB^p - nB^{n-1-p}iDB^p + nB^p iDB^{n-1-p} - nB^p iDB^{n-1-p} \\
&= 0.
\end{aligned}$$

This leaves only p -dependent terms in (A.4)

$$\begin{aligned}
&\frac{1}{2} \sum_{p=0}^{n-1} pB^{n-1-p}DB^p - \text{H.C.} \\
&= \frac{1}{2} \sum_{p=0}^{n-1} p(B^{n-1-p}DB^p - B^pDB^{n-1-p}) \\
&= \frac{1}{2} \sum_{p=0}^{n-1} p(B^{n-1-p}(B^pD + [D, B^p]) - B^p(B^{n-1-p}D + [D, B^{n-1-p}])) \\
&= \frac{1}{2} \sum_{p=0}^{n-1} p(B^{n-1-p}[D, B^p] - B^p[D, B^{n-1-p}]) \\
&= \frac{1}{2} \sum_{p=0}^{n-1} p \left(\sum_{k=0}^{p-1} J_k - \sum_{q=0}^{n-2-p} J_q \right), \tag{A.5}
\end{aligned}$$

where we have again used (A.2) and defined $J_q \equiv B^{n-2-q}[D, B]B^q$. Summing over all of p then finally yields

$$\frac{n-1}{2} \sum_{j=0}^{n-2} J_j + \sum_{k=0}^{n-3} k(n-k-1)J_k \tag{A.6}$$

or written explicitly with $J_q \equiv -B^{n-2-q}[[A, B], B]B^q$. Substituting this back into (A.3) the identity follows.

Thm 5 For any rescaled spin operators we have the following bounds on the double commutator

$$\| [S_\mu, [\hat{\nu}, \hat{\zeta}^{n+1}]] \| \leq n^4 \mathcal{O}(S)^{-1} \| \hat{\zeta} \|^n. \tag{A.7}$$

Proof: For rescaled spin operators $\hat{\nu}$ and $\hat{\zeta}$ we have

$$[\hat{\nu}, \hat{\zeta}^{n+1}] = \frac{(n+1)}{2} \{ \hat{\zeta}^n, [\hat{\nu}, \hat{\zeta}] \} + \frac{n-1}{2} \sum_{j=0}^{n-2} J_j + \sum_{k=0}^{n-3} k(n-k-1)J_k, \tag{A.8}$$

with $J_q \equiv -\hat{\zeta}^{n-2-q}[[\hat{\nu}, \hat{\zeta}], \hat{\zeta}] \hat{\zeta}^q = -\hat{\zeta}^{n-2-q} \mathcal{O}(S)^{-3} \hat{\zeta}^q$. The double commutator is then

$$\begin{aligned}
\| [S_\mu, [\hat{\nu}, \hat{\zeta}^{n+1}]] \| &= \frac{n+1}{2} \| [S_\mu, \{ \hat{\zeta}^n, [\hat{\nu}, \hat{\zeta}] \}] \| + \frac{n-1}{2} \sum_{j=0}^{n-2} \| [S_\mu, J_j] \| + \sum_{k=0}^{n-3} k(n-k-1) \| [S_\mu, J_k] \| \\
&\leq \frac{n+1}{2} \| [S_\mu, \{ \hat{\zeta}^n, \mathcal{O}(S)^{-1} \}] \| + \frac{n-1}{2} \sum_{j=0}^{n-2} \mathcal{O}(S)^{-2} \| \hat{\zeta} \|^{n-2} \\
&\quad + \sum_{k=0}^{n-3} k(n-k-1) \mathcal{O}(S)^{-2} \| \hat{\zeta} \|^{n-2} \\
&\leq \frac{n+1}{2} \| \{ [S_\mu, \hat{\zeta}^n], \mathcal{O}(S)^{-1} \} \| + \| \{ [S_\mu, \mathcal{O}(S)^{-1}], \hat{\zeta}^n \} \|
\end{aligned}$$

$$\begin{aligned}
& + \mathcal{O}(S)^{-2} \|\hat{\zeta}\|^{n-2} \left[\frac{(n-1)(n-2)}{2} + \sum_{k=0}^{n-3} k(n-k-1) \right] \\
& \leq \frac{n+1}{2} \mathcal{O}(S)^{-1} \|\{[S_\mu, \hat{\zeta}^n]\|\| + \frac{n+1}{2} \mathcal{O}(S)^{-1} \|\hat{\zeta}\|^n \\
& \quad + \mathcal{O}(S)^{-2} \|\hat{\zeta}\|^{n-2} \left[\frac{(n-1)(n-2)}{2} + \frac{(n-3)(n^2-4)}{6} \Theta(n-3) \right] \\
& \leq \frac{n+1}{2} \mathcal{O}(S)^{-1} \|\{[S_\mu, \hat{\zeta}^n]\|\| + \frac{n+1}{2} \mathcal{O}(S)^{-1} \|\hat{\zeta}\|^n \\
& \quad + \mathcal{O}(S)^{-2} \|\hat{\zeta}\|^{n-2} n(n-1)(n-2), \tag{A.9}
\end{aligned}$$

where $\Theta(n-3)$ is the heavy-side theta function, which is 0 when the argument is negative and 1 for positive arguments. We have bounded these two sums by a falling factorial $n(n-1)(n-2)$. Now we focus on bounding the first term in (A.9); using the same formula (A.8) we have

$$\|\{[S_\mu, \hat{\zeta}^n]\|\| = \frac{n}{2} \|\{\hat{\zeta}^{n-1}, [S_\mu, \hat{\zeta}]\|\| + \frac{n-2}{2} \sum_{j=0}^{n-3} \|J'_j\| + \sum_{k=0}^{n-4} k(n-k-1) \|J'_k\|$$

with $J'_q \equiv -\hat{\zeta}^{n-3-q} [[S_\mu, \hat{\zeta}], \hat{\zeta}], \hat{\zeta}] \hat{\zeta}^q = -\hat{\zeta}^{n-2-q} \mathcal{O}(S)^{-2} \hat{\zeta}^q$

$$\begin{aligned}
\|\{[S_\mu, \hat{\zeta}^n]\|\| & \leq \frac{n}{2} \mathcal{O}(S)^0 \|\hat{\zeta}\|^{n-1} + \mathcal{O}(S)^{-2} \|\hat{\zeta}\|^{n-3} \left[\frac{(n-2)(n-3)}{2} \Theta(n-3) \right. \\
& \quad \left. + \frac{(n^2-16)(n-3)}{6} \Theta(n-4) \right] \\
& \leq \frac{n}{2} \mathcal{O}(S)^0 \|\hat{\zeta}\|^{n-1} + \mathcal{O}(S)^{-2} \|\hat{\zeta}\|^{n-3} (n-1)(n-2)(n-3)
\end{aligned}$$

Again we have bounded the two sums by a falling factorial. Now using this bound in (A.9)

$$\begin{aligned}
\|\{[S_\mu, [\hat{\nu}, \hat{\zeta}^{n+1}]]\|\| & \leq \frac{n+1}{2} \mathcal{O}(S)^{-1} \left[\frac{n}{2} \mathcal{O}(S)^0 \|\hat{\zeta}\|^{n-1} + \mathcal{O}(S)^{-2} \|\hat{\zeta}\|^{n-3} (n-1)(n-2)(n-3) \right] \\
& \quad + \frac{n+1}{2} \mathcal{O}(S)^{-1} \|\hat{\zeta}\|^n + \mathcal{O}(S)^{-2} \|\hat{\zeta}\|^{n-2} n(n-1)(n-2) \\
& \leq \frac{n}{2} \frac{n+1}{2} \mathcal{O}(S)^{-1} \|\hat{\zeta}\|^{n-1} + \mathcal{O}(S)^{-3} \|\hat{\zeta}\|^{n-3} \frac{n+1}{2} (n-1)(n-2)(n-3) \\
& \quad + \frac{n+1}{2} \mathcal{O}(S)^{-1} \|\hat{\zeta}\|^n + \mathcal{O}(S)^{-2} \|\hat{\zeta}\|^{n-2} n(n-1)(n-2), \tag{A.10}
\end{aligned}$$

Taking the largest power of n and smallest decay in S we have the generous upper bound

$$\|\{[S_\mu, [\hat{\nu}, \hat{\zeta}^{n+1}]]\|\| \leq n^4 \mathcal{O}(S)^{-1} \|\hat{\zeta}\|^n. \tag{A.11}$$

A.1.1 Gibbs free energy

Thm 6 Consider a Hamiltonian of the form $\mathcal{H} = -S \sum_i x_i \mathcal{O}_i$, where every \mathcal{O}_i is an intensive observable. We write the partition function as

$$Z = \text{Tr} \left\{ e^{-S\beta \sum_i x_i \mathcal{O}_i} \right\} = e^{-S\beta \tilde{g}(\mathbf{x})} \left[f_0(\mathbf{x}) + \sum_j f_j(\mathbf{x}) e^{-S\beta g_j(\mathbf{x})} \right], \quad \mathbf{x} = (x_1, x_2, \dots) \quad (\text{A.12})$$

where every $g_j(\mathbf{x})$ is some positive function. Then one needs only to consider the leading exponential behaviour in Z

$$Z \asymp e^{-S\beta \tilde{g}(\mathbf{x})}. \quad (\text{A.13})$$

to evaluate all related equilibrium expectation values $\langle \mathcal{O}_i \rangle_{\text{eq}} \equiv \text{Tr} \{ \mathcal{O}_i e^{-\beta \mathcal{H}} / Z \}$.

Using (A.12) the Gibbs free energy

$$g \equiv -\frac{1}{S\beta} \ln Z$$

may then be evaluated as

$$g = \tilde{g}(\mathbf{x}) - \frac{1}{S\beta} \ln \left[f_0(\mathbf{x}) + \sum_j f_j(\mathbf{x}) e^{-S\beta g_j(\mathbf{x})} \right]. \quad (\text{A.14})$$

The Gibbs free energy has the property that

$$\frac{\partial}{\partial x_i} g \Big|_{\mathbf{x}=0} \equiv \partial_i g \Big|_{\mathbf{x}=0} = \text{Tr} \{ \mathcal{O}_i e^{-\beta \mathcal{H}} \} = \text{Tr} \{ \mathcal{O}_i \varrho_{\text{eq}} \} = \langle \mathcal{O}_i \rangle_{\text{eq}}.$$

Taking the derivative of (A.14) with respect to some parameter x_i

$$\partial_i g = \partial_i \tilde{g} + \frac{\sum_i f_i \partial_i g_i e^{-S\beta g_i}}{f_0 + \sum_i f_i e^{-S\beta g_i}} - \frac{\partial_i f_0 + \sum_i \partial_i f_i e^{-S\beta g_i}}{S\beta [f_0 + \sum_i f_i e^{-S\beta g_i}]}. \quad (\text{A.15})$$

Due to the exponential decay we have $\partial_i g \sim \partial_i \tilde{g}$, meaning $\partial_i \tilde{g} \Big|_{\mathbf{x}=0} \sim \langle \mathcal{O}_i \rangle_{\text{eq}}$. Here the asymptotic similarity can be with respect to S or β .

Now we consider the continuous case, where we may use Laplace's method

$$Z = \int d^d r f(\mathbf{r}) e^{-S\beta g(\mathbf{r})} \quad (\text{A.16})$$

$$= \left[\frac{2\pi}{S\beta} \right]^{d/2} \sum_i |\det(\mathbb{H}(f))|^{-1/2} f(\mathbf{r}_i) e^{-S\beta g(\mathbf{r}_i)} + \mathcal{O}((S\beta)^{-1/5}) \quad (\text{A.17})$$

$$= e^{-S\beta \tilde{g}} \left[\frac{2\pi}{S\beta} \right]^{d/2} \sum_i |\det(\mathbb{H}(f(\mathbf{r}_i)))|^{-1/2} f(\mathbf{r}_i) e^{-S\beta g_i} + \mathcal{O}((S\beta)^{-1/5}), \quad (\text{A.18})$$

where we have defined the Hessian matrix $\mathbb{H}_{i,j}(f) = \partial_i \partial_j f$. This is of the same form as (A.12), hence we again only have to consider the leading order exponential behaviour. In the language of large deviation theory [66] one would simply write

$$Z \asymp e^{-S\beta \tilde{g}(x)}. \quad (\text{A.19})$$

This explicitly means

$$a(S) \asymp b(S) \Leftrightarrow \lim_{S \rightarrow \infty} \frac{\ln a(S)}{S} = \lim_{S \rightarrow \infty} \frac{\ln b(S)}{S}. \quad (\text{A.20})$$

A.2 Lindbladian properties

Thm 7 For $L = H + A$, with $(H, A)^\dagger = (H, -A)$, in particular where

$$H = \sqrt{\kappa_{xx}}S_x + \frac{\Re\kappa_{xy}}{\sqrt{\kappa_{xx}}}S_y, \quad A = \frac{i\Im\kappa_{xy}}{\sqrt{\kappa_{xx}}}S_y,$$

we have that

$$\begin{aligned} \mathcal{L}\varrho = & i[\varrho, \mathcal{H}_S] - \frac{\kappa_{xx}}{2}[S_x, [S_x, \varrho]] - i\Im\kappa_{yx}[S_x, \{S_y, \varrho\}] \\ & - \frac{|\kappa_{xy}|^2}{\kappa_{xx}}[S_y, [S_y, \varrho]] + 2\Re\kappa_{xy}[S_y, [S_x, \varrho]]. \end{aligned} \quad (\text{A.21})$$

The dissipater may be written as

$$\begin{aligned} \mathcal{D}\varrho = & L\varrho L^\dagger - \frac{1}{2}\{L^\dagger L, \varrho\} \\ = & -\frac{1}{2}[H, [H, \varrho]] + \frac{1}{2}[A, [A, \varrho]] + A\varrho H - H\varrho A - \frac{1}{2}\{\{H, A\}, \varrho\} \\ = & -\frac{1}{2}[H, [H, \varrho]] + \frac{1}{2}[A, [A, \varrho]] + \frac{1}{2}[A, \{H, \varrho\}] - \frac{1}{2}[H, \{A, \varrho\}] \\ = & -\frac{1}{2}[H, [H, \varrho] + \{A, \varrho\}] + \frac{1}{2}[A, [A, \varrho] + \{H, \varrho\}] \\ = & -\frac{1}{2}[H, [H, \varrho]] + \frac{1}{2}[A, [A, \varrho]] + \frac{1}{2}\{[A, \varrho], H\} - \frac{1}{2}\{\{H, \varrho\}, A\} - \{\{H, A\}, \varrho\}. \end{aligned} \quad (\text{A.22})$$

We consider the Lindblad operator with Hermitian and anti-Hermitian parts

$$H = \sqrt{\kappa_{xx}}S_x + \frac{\Re\kappa_{xy}}{\sqrt{\kappa_{xx}}}S_y, \quad A = \frac{i\Im\kappa_{xy}}{\sqrt{\kappa_{xx}}}S_y,$$

The first double commutator in (A.22) is then

$$[H, [H, \varrho]] = \left[\sqrt{\kappa_{xx}}S_x + \frac{\Re\kappa_{xy}}{\sqrt{\kappa_{xx}}}S_y, \left[\sqrt{\kappa_{xx}}S_x + \frac{\Re\kappa_{xy}}{\sqrt{\kappa_{xx}}}S_y, \varrho \right] \right] \quad (\text{A.23})$$

$$= \kappa_{xx}[S_x, [S_x, \varrho]] + \frac{(\Re\kappa_{xy})^2}{\kappa_{xx}}[S_y, [S_y, \varrho]] + \Re\kappa_{xy}([S_y, [S_x, \varrho]] + [S_x, [S_y, \varrho]]), \quad (\text{A.24})$$

using the Jacobi identity $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$, we may write

$$\begin{aligned} [H, [H, \varrho]] = & \kappa_{xx}[S_x, [S_x, \varrho]] + \frac{(\Re\kappa_{xy})^2}{\kappa_{xx}}[S_y, [S_y, \varrho]] + \Re\kappa_{xy}(2[S_y, [S_x, \varrho]] + [\varrho, [S_x, S_y]]), \\ = & \kappa_{xx}[S_x, [S_x, \varrho]] + \frac{(\Re\kappa_{xy})^2}{\kappa_{xx}}[S_y, [S_y, \varrho]] + 2\Re\kappa_{xy}[S_y, [S_x, \varrho]] + i\Re\kappa_{xy}[\varrho, S_z]. \end{aligned} \quad (\text{A.25})$$

The last term in (A.25) cancels with the $\Re\kappa_{xy}$ dependent term in the unitary part

$$\mathcal{U}\varrho = i[\varrho, \mathcal{H}_S + \frac{\Re\kappa_{xy}}{2}S_z - \frac{\Im\kappa_{xy}}{2}\{S_x, S_y\}] \quad (\text{A.26})$$

$$= i[\varrho, \mathcal{H}_S] - \frac{i\Im\kappa_{xy}}{2}\{[S_y, \varrho], S_x\} + \{[S_x, \varrho], S_y\} + i\frac{\Re\kappa_{xy}}{2}[\varrho, S_z], \quad (\text{A.27})$$

where we have used $[A, \{B, C\}] = \{[A, B], C\} + \{[A, C], B\}$. Next we consider the following two terms in (A.22)

$$\frac{1}{2}\{[A, \varrho], H\} - \frac{1}{2}\{\{H, \varrho\}, A\} = \frac{i}{2}\frac{\Im\kappa_{xy}}{\sqrt{\kappa_{xx}}}\left[\{[S_y, \varrho], \sqrt{\kappa_{xx}}S_x + \frac{\Re\kappa_{xy}}{\sqrt{\kappa_{xx}}}S_y\}\right]$$

$$- \left\{ \left[\sqrt{\kappa_{xx}} S_x + \frac{\Re \kappa_{xy}}{\sqrt{\kappa_{xx}}} S_y, \varrho \right], S_y \right\}. \quad (\text{A.28})$$

The terms containing two S_y operators cancel, leaving

$$\frac{1}{2} \{[A, \varrho], H\} - \frac{1}{2} \{[H, \varrho], A\} = \frac{i \Im \kappa_{xy}}{2} \{[S_y, \varrho], S_x\} - \{[S_x, \varrho], S_y\}, \quad (\text{A.29})$$

where the first term cancels with the unitary part, while the second adds. Lastly, since S_y commutes with itself it is clear that the final term in (A.22) merely yields

$$\{[H, A], \varrho\} = i \Im \kappa_{xy} \{[S_x, S_y], \varrho\} = -i \Im \kappa_{xy} \{S_z, \varrho\} \quad (\text{A.30})$$

and in total we have

$$\begin{aligned} \mathcal{L}\varrho &= i[\varrho, \mathcal{H}_S] - \frac{\kappa_{xx}}{2} [S_x, [S_x, \varrho]] + \Im \kappa_{yx} (\{S_z, \varrho\} - i\{[S_x, \varrho], S_y\}) \\ &\quad - \frac{(\Re \kappa_{xy})^2 + \Im \kappa_{xy})^2}{\kappa_{xx}} [S_y, [S_y, \varrho]] + 2 \Re \kappa_{xy} [S_y, [S_x, \varrho]] \\ &= i[\varrho, \mathcal{H}_S] - \frac{\kappa_{xx}}{2} [S_x, [S_x, \varrho]] - i \Im \kappa_{yx} [S_x, \{S_y, \varrho\}] \\ &\quad - \frac{|\kappa_{xy}|^2}{\kappa_{xx}} [S_y, [S_y, \varrho]] + 2 \Re \kappa_{xy} [S_y, [S_x, \varrho]] \end{aligned} \quad (\text{A.31})$$

Thm 8 *The product rule $\mathcal{L}^\dagger(\hat{\mu}\hat{\nu}) \sim \mathcal{L}^\dagger(\hat{\mu})\hat{\nu} + \hat{\mu}\mathcal{L}^\dagger(\hat{\nu})$ implies the factorization property $\mathcal{G}_t \hat{\mu}\hat{\nu} = [\mathcal{G}_t \hat{\mu}][\mathcal{G}_t \hat{\nu}]$*

Proof: Using the general Leibniz formula we may prove

$$\mathcal{G}_t \hat{\mu}\hat{\nu} \sim \sum_n \frac{t^n \mathcal{L}^{\dagger n}}{n!} \hat{\mu}\hat{\nu} \quad (\text{A.32})$$

$$\sim \sum_n \sum_{r=0}^n \binom{n}{r} \frac{t^n}{n!} \mathcal{L}^{\dagger n-r}(\hat{\mu}) \mathcal{L}^{\dagger r}(\hat{\nu}) \quad (\text{A.33})$$

$$\sim \sum_n \sum_{r=0}^n \frac{t^{n-r} t^r}{(n-r)! n!} \mathcal{L}^{\dagger n-r}(\hat{\mu}) \mathcal{L}^{\dagger r}(\hat{\nu}) \quad (\text{A.34})$$

$$\sim \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{t^{p+q}}{p! q!} \mathcal{L}^{\dagger p}(\hat{\mu}) \mathcal{L}^{\dagger q}(\hat{\nu}) \quad (\text{A.35})$$

$$\sim [\mathcal{G}_t \hat{\mu}][\mathcal{G}_t \hat{\nu}], \quad (\text{A.36})$$

where we have used the Cauchy product to separate the two sums.

Thm 9 *For any observable \bullet , the expectation value $\langle \bullet \rangle$ will always remain within the range of its spectrum, for any quantum dynamic semi-group evolution.*

Proof: For any trace and positivity preserving evolution we have

$$\langle \bullet(s) \rangle = \text{Tr}\{\bullet \varrho(s)\} = \sum_n \bullet_n \varrho_n(s), \quad \varrho_n(s) \equiv \langle \bullet_n | \varrho(s) | \bullet_n \rangle. \quad (\text{A.37})$$

where the trace has been taken with respect to the eigenstates of \bullet in the Schrödinger picture. The eigenvalues have been ordered as $\bullet_n < \bullet_{n+1}$. Since $\varrho_n(s) > 0$ and $\sum_n \varrho_n(s) = 1$, we know that the minimum of (A.37) is when $\varrho_0(s) = 1$, and corresponds to the smallest eigenvalue of \bullet . Similarly the maximum corresponds to the largest eigenvalue of \bullet . The expectation value can never evolve to a value outside of the operator's original spectral range.

A.2.1 The polaron transform

In the large coupling c regime one may also derive a master equation derivation. This makes use of the polaron transform [67–69]

$$\mathcal{H}_{S\mathcal{E}} \rightarrow e^{\sigma S_x} \mathcal{H}_{S\mathcal{E}} e^{-\sigma S_x}, \quad \sigma \equiv c \sum_q \frac{\gamma_q}{\omega_q} (a_q^\dagger - a_q). \quad (\text{A.38})$$

To evaluate (A.38) we use the spin rotation and bosonic shift identities

$$e^{i\theta S_\mu} S_\nu e^{-i\theta S_\mu} = S_\nu \cos \theta - \epsilon_{\mu\nu\rho} S_\rho \sin \theta, \quad e^{\alpha a^\dagger - \bar{\alpha} a} a e^{-\alpha a^\dagger + \bar{\alpha} a} = a - \alpha, \quad (\text{A.39})$$

which may be proven by use of the BCH formula (3.3). Here $\bar{\alpha}$ is the complex conjugate of α . This in turn yields¹

$$e^{\sigma S_x} \mathcal{H}_{S\mathcal{E}} e^{-\sigma S_x} = -\frac{J}{2S} S_x^2 + \mathcal{H}_{\mathcal{E}} - \hbar [S_y \cosh \sigma + i S_z \sinh \sigma], \quad (\text{A.40})$$

where the key observation is that we have substituted one weak coupling for another $c \rightleftharpoons \hbar$, thus the derivation would follow in the same manner. This limits one to the parameter regime $\hbar \ll J$, which is very far away from our point of interest $\hbar = J$ at which a phase transition occurs, whilst also being decoherence dominated, hence we will *not* use this transform.

A.3 Bogoliubov transforms

The Bogoliubov transform [2] serves to diagonalize Hamiltonians of the form

$$\mathcal{H} = 2\epsilon a^\dagger a + [\varphi a^2 + \text{H.C.}] \rightarrow \omega b^\dagger b.$$

We are required to find new bosonic operators for which the Hamiltonian “factorizes”. The transformation must be of the form

$$\begin{bmatrix} a \\ a^\dagger \end{bmatrix} = T_{b \rightarrow a} \begin{bmatrix} b \\ b^\dagger \end{bmatrix} = \begin{bmatrix} u & v \\ v^* & u \end{bmatrix} \begin{bmatrix} b \\ b^\dagger \end{bmatrix}, \quad \begin{bmatrix} b \\ b^\dagger \end{bmatrix} = T_{b \rightarrow a}^{-1} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} = \begin{bmatrix} u & -v \\ -v^* & u \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix}, \quad (\text{A.41})$$

where we require $|u|^2 - |v|^2 = 1$ to preserve the bosonic commutation relations. This restricts the functional form (up to a global phase) of u, v to

$$u = \cosh(r) \quad v = e^{i\theta} \sinh(r) \Rightarrow u^2 + |v|^2 = \cosh(2r), \quad uv = \sinh(2r)e^{i\theta}/2.$$

¹It is interesting to note that the polaron transform also removes the counter term.

Thm 10 Any Hamiltonian that is quadratic in the bosonic operators

$$\mathcal{H} = 2\epsilon a^\dagger a - \left[\phi a^{\dagger 2} + H.C. \right], \quad (\text{A.42})$$

with $|\phi| < |\epsilon|$ is diagonalizable by a transformation

$$\begin{bmatrix} a \\ a^\dagger \end{bmatrix} = T_{b \rightarrow a} \begin{bmatrix} b \\ b^\dagger \end{bmatrix} = \begin{bmatrix} u & v \\ v^* & u \end{bmatrix} \begin{bmatrix} b \\ b^\dagger \end{bmatrix}$$

with dispersion relation and coefficients

$$\omega = 2\text{sgn}(\epsilon)\sqrt{\epsilon^2 - |\phi|^2} \quad u = \sqrt{\frac{\epsilon}{\omega} + \frac{1}{2}} \quad v = \frac{\phi}{\omega u}. \quad (\text{A.43})$$

Consider the diagonal Hamiltonian $\mathcal{H} = \omega b^\dagger b$. Reversing the normal ordering we obtain

$$\mathcal{H} = \frac{\omega}{2}(b^\dagger b + b b^\dagger - 1) \quad (\text{A.44})$$

and using (A.41) this may be written as

$$\mathcal{H} = \frac{\omega}{2} \begin{bmatrix} b^\dagger & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b \\ b^\dagger \end{bmatrix} - \frac{\omega}{2} \quad (\text{A.45})$$

$$= \frac{\omega}{2} \begin{bmatrix} a^\dagger & a \end{bmatrix} \begin{bmatrix} u & -v \\ -v^* & u \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u & -v \\ -v^* & u \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} - \frac{\omega}{2} \quad (\text{A.46})$$

$$= \frac{\omega}{2} \begin{bmatrix} a^\dagger & a \end{bmatrix} \begin{bmatrix} u^2 + |v|^2 & -2uv \\ -2uv^* & u^2 + |v|^2 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} - \frac{\omega}{2} \quad (\text{A.47})$$

$$= \frac{\omega}{2} [(u^2 + |v|^2) \underbrace{(a^\dagger a + a a^\dagger - 1)}_{a^\dagger a + a a^\dagger - a a^\dagger + a^\dagger a} - 2u(v(a^\dagger)^2 + v^* a^2)] \quad (\text{A.48})$$

$$= \underbrace{\omega(u^2 + |v|^2)}_{2\epsilon} a^\dagger a - \underbrace{\omega uv}_{\phi} a^{\dagger 2} - \omega v v^* a^2 \quad (\text{A.49})$$

$$= 2\epsilon a^\dagger a - \left[\phi a^{\dagger 2} + H.C. \right], \quad (\text{A.50})$$

meaning that $e^{i\theta} = \phi/|\phi|$. Further one can show the following

$$\omega = 2\text{sgn}(\epsilon)\sqrt{\epsilon^2 - |\phi|^2} \quad u = \sqrt{\frac{\epsilon}{\omega} + \frac{1}{2}} \quad v = \frac{\phi}{\omega u}, \quad (\text{A.51})$$

where $\omega \in \mathbb{R}$ only when $|\phi| < |\epsilon|$.

Thm 11 For the special case where the coefficients in (A.42) satisfy $\epsilon = 1/2 - \phi$ and $\phi \in \mathbb{R}$, the Bogoliubov coefficients satisfy

$$|u \pm v|^2 = \frac{-2\phi \pm 2\phi + 1}{\omega} = \omega^{\mp 1}, \quad (\text{A.52})$$

We may write

$$|u \pm v|^2 = (u^2 + |v|^2) \pm u(v + v^*) = \frac{2\epsilon \pm (\phi + \phi^*)}{\omega}$$

where the last equality is only true if $\epsilon = 1/2 - \phi$ and $\phi \in \mathbb{R}$ then $\omega = \text{sgn}(\epsilon)\sqrt{1-4\phi}$

$$|u \pm v|^2 = \frac{-2\phi \pm 2\phi + 1}{\omega} = \omega^{\mp 1}. \quad (\text{A.53})$$

A.3.1 Bosonic Dissipater

Consider the dissipater $\mathcal{D} = \ell_1 \ell_2^* - (\ell_1^\dagger \ell_1 + \ell_2^\dagger \ell_2)$, with

$$\ell_i = B_+ b_i + B_- b_i^\dagger, \quad B_\pm = \frac{1}{\sqrt{q}} \pm \frac{e^{i\theta}}{2} \sqrt{q}, \quad \Upsilon = \frac{1}{q \cos \theta} + \frac{q}{4 \cos \theta}, \quad \frac{|B_\pm|^2}{\cos \theta} = \Upsilon \pm 1 \quad (\text{A.54})$$

Then keeping only the number difference conserving terms we have

$$\ell_1 \ell_2^* = (B_+ b_1 + B_- b_1^\dagger)(B_+^* b_2 + B_-^* b_2^\dagger) \quad (\text{A.55})$$

$$= |B_+|^2 b_1 b_2 + |B_-|^2 b_1^\dagger b_2^\dagger, \quad (\text{A.56})$$

$$\ell_1^\dagger \ell_1 = (B_+^* b_1^\dagger + B_-^* b_1)(B_+ b_1 + B_- b_1^\dagger) \quad (\text{A.57})$$

$$= |B_+|^2 b_1^\dagger b_1 + |B_-|^2 b_1 b_1^\dagger. \quad (\text{A.58})$$

Then we are left with

$$\frac{\mathcal{D}}{\cos \theta} = -\Upsilon(n_1 + n_2) + (\Upsilon - 1)b_1^\dagger b_2^\dagger + (\Upsilon + 1)b_1 b_2 - \Upsilon + 1. \quad (\text{A.59})$$

A.4 Sokhotski-Plemelj theorem

We will now use the generalized Sokhotski-Plemelj theorem

$$\int_0^\infty d\tau e^{i\omega\tau} \tau^n = \frac{\pi}{i^n} \delta^{(n)}(\omega) + n! \mathcal{P} \left(\frac{i}{\omega} \right)^{n+1}, \quad (\text{A.60})$$

where $\delta(\omega)$ is the Dirac delta function and \mathcal{P} the Cauchy principle value. The principle value is defined by

$$\mathcal{P} \int dx \frac{f(x)}{x} = \lim_{\epsilon \rightarrow 0} \int dx \frac{x}{x^2 + \epsilon^2} f(x). \quad (\text{A.61})$$

This formula allows one to derive the formulae like

$$\int_0^\infty d\tau \sin(\omega\tau) = \mathcal{P} \left(\frac{1}{\omega} \right), \quad \int_0^\infty d\tau \tau \sin(\omega\tau) = -\pi \delta'(\omega) \quad (\text{A.62})$$

$$\int_0^\infty d\tau \cos(\omega\tau) = \pi \delta(\omega) \quad \int_0^\infty d\tau \tau \cos(\omega\tau) = -\mathcal{P} \left(\frac{1}{\omega^2} \right). \quad (\text{A.63})$$

A.5 Lorenz-Drude cutoff

For an ohmic spectral density $J_{SD}(\omega) = \omega j_{SD}(\omega)$ we have the kernels

$$\nu(\tau) = \int_0^\infty d\omega j_{SD}(\omega) \omega \coth \left(\frac{\beta\omega}{2} \right) \cos(\omega\tau) \quad \text{and} \quad \eta(\tau) = \int_0^\infty d\omega j_{SD}(\omega) \omega \sin(\omega\tau). \quad (\text{A.64})$$

We will consider a Lorenz-Drude cutoff

$$j_{LD}(\omega) = \frac{1}{\pi} \frac{\omega_c^2}{\omega_c^2 + \omega^2}. \quad (\text{A.65})$$

One may calculate $\nu(\tau)$ using the residue theorem around its poles as [12, 22]

$$\begin{aligned}
\nu(\tau) &= \int_0^\infty d\omega J_{SD}(\omega) \coth\left(\frac{\beta\omega}{2}\right) \cos(\omega\tau) \\
&= T\omega_c^2 \sum_{n=-\infty}^{\infty} \frac{\omega_c e^{-\omega_c\tau} - 2\pi T|n|e^{-2\pi T\tau|n|}}{\omega_c^2 - (2\pi Tn)^2} \\
&= \omega_c^2 e^{-\omega_c\tau} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{q}{q^2 - n^2} - \frac{\omega_c^2}{2\pi} \sum_{n=-\infty}^{\infty} \frac{|n|e^{-2\pi T\tau|n|}}{q^2 - n^2}, \quad q \equiv \frac{\omega_c}{2\pi T} \\
&= \frac{\omega_c^2}{2} e^{-\omega_c\tau} \cot(\pi q) - \frac{\omega_c^2}{2\pi} \sum_{n=-\infty}^{\infty} \frac{|n|e^{-2\pi T\tau|n|}}{q^2 - n^2} \\
&= \frac{\omega_c^2}{2} e^{-\omega_c\tau} \cot\left(\frac{\omega_c}{2T}\right) - \frac{\omega_c^2}{2\pi} \sum_{n=-\infty}^{\infty} \frac{|n|e^{-2\pi T\tau|n|}}{(\beta\omega_c/(2\pi))^2 - n^2}
\end{aligned} \tag{A.66}$$

where we have used the series expansion

$$\cot(\pi q) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{q}{q^2 - n^2} = \frac{2}{\pi q} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{q}{q^2 - n^2}.$$

Now since the exponentials are absolutely convergent we may integrate this series term by term

$$\begin{aligned}
\int_0^\infty d\tau \nu(\tau) &= \omega_c^2 \frac{1}{2\omega_c} \cot(\pi q) - 2 \frac{\omega_c^2}{2\pi} \sum_{n=1}^{\infty} \frac{n}{q^2 - n^2} \frac{1}{2\pi Tn} \\
&= \omega_c/2 \cot(\pi q) - \frac{\omega_c^2}{\pi} \frac{1}{2\pi T} \sum_{n=1}^{\infty} \frac{1}{q^2 - n^2} \\
&= \omega_c/2 \cot(\pi q) - \frac{\omega_c^2}{\pi} \frac{-1 + \pi q \cot(\pi q)}{2q^2} \\
&= \omega_c/2 \cot(\pi q) + T - \omega_c \cot(\pi q)/2 \\
&= T.
\end{aligned} \tag{A.67}$$

We may compare this to the solution

$$\begin{aligned}
\int_0^\infty d\tau \nu(\tau) &= \int_0^\infty d\tau \int_0^\infty d\omega J_{LD}(\omega) \coth\left(\frac{\omega}{2T}\right) \cos(\omega\tau) \\
&= \int_0^\infty d\omega J_{LD}(\omega) \coth\left(\frac{\omega}{2T}\right) \underbrace{\int_0^\infty d\tau \cos(\omega\tau)}_{\pi\delta(\omega)} \\
&= \frac{1}{2} \lim_{\omega \rightarrow 0} \omega \coth\left(\frac{\omega}{2T}\right) \\
&= T,
\end{aligned} \tag{A.68}$$

where the Dirac delta function was obtained via use of (A.63).

This may be extended in general to the following integral

$$\int_0^\infty d\tau \tau^m \nu(\tau) = \frac{m!}{2\omega_c^{m-1}} \cot(\pi q) - \frac{\omega_c^2}{\pi} \frac{2m!}{(2\pi T)^{m+1}} \sum_{n=1}^{\infty} \frac{1}{n^m (q^2 - n^2)} \tag{A.69}$$

Which for $m = 1$ simplifies to

$$\nu_1 = \int_0^\infty d\tau \nu(\tau) \tau = \frac{1}{2\pi q} - \frac{1}{\pi} [\psi(q+1) - \psi(1)], \quad q \equiv \frac{\beta\omega_c}{2\pi} \tag{A.70}$$

where $\psi(q) = \partial_q \ln(\Gamma(q))$ is the *digamma* function. For small or large q we may use the following expressions

$$\nu_1 = \begin{cases} \frac{1}{2\pi q} - \frac{\pi q}{6} + \mathcal{O}(q^2) & q < 1 \\ -\ln(q)/\pi + \mathcal{O}(q^{-1}) & q \gg 1, \end{cases}$$

meaning that in the high temperature case this integral is well approximated by $2T/\omega_c$, but we also have $\nu_1 \sim \log \omega_c$ as $\omega_c \rightarrow \infty$. The expression (A.70) has a root at $q \approx \sqrt{3}/\pi$ meaning when $\omega_c \approx 2\sqrt{3}T$.

A.6 Jacobi elliptical functions

A.6.1 Differential equation constant values

In this section we prove that the following differential equation

$$[\partial_t x]^2 + \left[\frac{\Lambda}{2} x^2 + h \right]^2 = 1 - x^2. \quad (\text{A.71})$$

is solved by $x(t) = C \text{cn}[\pm at - u_0, m]$ and derive the form of the constants.

We start by consider the following identities [13]

$$\text{sn}^2[u, m] = 1 - \text{cn}^2[u, m], \quad \text{dn}^2[u, m] = 1 - m \text{sn}^2[u, m], \quad \partial_t \text{cn}[u, m] = -\text{asn}[u, m] \text{dn}[u, m] \quad (\text{A.72})$$

for $u = at + u_0$. Using these identities we get

$$\begin{aligned} \dot{x}^2 &= a^2 C^2 \text{sn}^2 \text{dn}^2 \\ &= a^2 C^2 \left[(1 - \text{cn}^2) - m(1 - \text{cn}^2)^2 \right] \\ &= a^2 C^2 \left[(1 - m) + (2m - 1)\text{cn}^2 - m \text{cn}^4 \right] \\ &= (1 - m)a^2 C^2 + a^2(2m - 1)x^2 - \frac{m a^2}{C^2} x^4, \end{aligned} \quad (\text{A.73})$$

where we have substituted $\text{cn} = C^{-1}x$. Rearranging (A.73), we obtain

$$\dot{x}^2 - (1 - m)a^2 C^2 + 1 - (a^2(2m - 1) + 1)x^2 + \frac{m a^2}{C^2} x^4 = 1 - x^2. \quad (\text{A.74})$$

Comparing the coefficients of x^4 in (A.74) to those in (A.71) we find $C^{-2} m a^2 = (\Lambda/2)^2$ implying that x has the following amplitude (squared)

$$C^2 = m a^2 \left(\frac{2}{\Lambda} \right)^2. \quad (\text{A.75})$$

Substituting this into (A.74), we obtain

$$\dot{x}(t)^2 + (m - 1)m a^4 \left(\frac{2}{\Lambda} \right)^2 + 1 + (a^2(1 - 2m) - 1)x^2 + \left(\frac{\Lambda}{2} \right)^2 x^4 = 1 - x^2$$

which we again compare to (A.71)

$$4m(1 - m)a^4 = \Lambda^2(1 - h^2) \quad (\text{A.76})$$

$$(2m - 1)a^2 = -(\Lambda h + 1). \quad (\text{A.77})$$

When $h = h_g$, then the only solution is $\mathfrak{a} = 0$, meaning we have a constant function. When $h = -1$ then $m = \Theta(\Lambda - 1)$. Here Θ is the Heaviside step function, with $\Theta(0) = 1/2$. Lastly when $h = 1$, then $m = 0$. Otherwise the solution is

$$m^2 - m + \frac{\delta}{4\delta + 1} = 0, \quad \delta \equiv \frac{\Lambda^2}{4} \frac{1 - h^2}{(\Lambda h + 1)^2}, \quad (\text{A.78})$$

which is solved by

$$m = \frac{1}{2} \left(1 \pm \sqrt{\frac{(\Lambda h + 1)^2}{\Lambda^2 + 2\Lambda h + 1}} \right). \quad (\text{A.79})$$

Then from $(2m - 1)\mathfrak{a}^2 = -\Lambda h - 1$ we get

$$\mathfrak{a}^2 = \mp \text{sgn}(\Lambda h + 1) \sqrt{\Lambda^2 + 2\Lambda h + 1} = \sqrt{\Lambda^2 + 2\Lambda h + 1}, \quad (\text{A.80})$$

and since \mathfrak{a}^2 must be positive we have that $\pm = -\text{sgn}(\Lambda h + 1)$ and

$$m = \frac{1}{2} \left(1 - \frac{\text{sgn}(\Lambda h + 1)|\Lambda h + 1|}{\sqrt{\Lambda^2 + 2\Lambda h + 1}} \right) = \frac{1}{2} \left(1 - \frac{\Lambda h + 1}{\sqrt{\Lambda^2 + 2\Lambda h + 1}} \right). \quad (\text{A.81})$$

So we have

$$x(t) = C \text{cn}[\pm \mathfrak{a} t - u_0, m]. \quad (\text{A.82})$$

A.6.2 Extensive temperature thermodynamics

From [56] we have that the integrated density of states for the LMG model takes the form

$$\mathcal{N}(h) = 1 + \frac{2}{\pi \Lambda \hbar^2 \mathfrak{a}(h)} K(m(h)) \mathbb{1}(h \in (-1, 1]). \quad (\text{A.83})$$

Here $\mathbb{1}(h \in (-1, 1])$ being the indicator or window function, which is 1 if the variable is within the specified interval and 0 outside. The complete elliptic integral of the first kind $K(m)$ is defined in (4.32). The functions $\mathfrak{a}(h)$ and $m(h)$ are given in terms of the rescaled Hamiltonian's symbol as defined in (A.80) and (A.81) respectively. Defining $\tilde{\beta} = (T\hbar)^{-1} S = \tilde{T}^{-1} = \mathcal{O}(S^0)$ the partition function may be written as

$$\begin{aligned} Z(\tilde{\beta}) &= \hbar \int dh \mathcal{N}(h) e^{-\tilde{\beta} h} \\ &= \hbar \int_{h_0}^{-1} dh e^{-\tilde{\beta} h} + \hbar \int_{-1}^1 dh \frac{2}{\pi \Lambda \hbar^2 \mathfrak{a}} K(m) e^{-\tilde{\beta} h} \\ &= \frac{\hbar}{\tilde{\beta}} e^{-\tilde{\beta} h_0} \left[1 - \exp\left(-\tilde{\beta} \frac{(\Lambda - 1)^2}{2\Lambda}\right) \right] + \int_{-1}^1 dh \frac{2}{\pi \Lambda \hbar \mathfrak{a}} K(m) e^{-\tilde{\beta} h}. \end{aligned} \quad (\text{A.84})$$

For a low extensive temperature $\tilde{T} \ll 1$, we have large $\tilde{\beta}$ and expanding around the main contribution at $h = -1$ we have

$$\frac{K(m)}{\mathfrak{a}} = \sum_{n=0}^{\infty} (c_0^n \ln(1+h) + c_1^n) (1+h)^n, \quad c_0^0 = -\frac{1}{2\sqrt{\Lambda-1}} \quad (\text{A.85})$$

together yielding

$$\frac{2}{\pi \Lambda \hbar} \int_{-1}^1 dh e^{-\tilde{\beta} h} \frac{K(m)}{\mathfrak{a}} = \frac{2}{\pi \Lambda \hbar} \frac{e^{\tilde{\beta}}}{\tilde{\beta}} (c_1^0 - (\gamma_E + \ln(\tilde{\beta}) - \ln \Lambda) c_0^0 + \mathcal{O}(\tilde{\beta}^{-1}))$$

$$= \frac{e^{\tilde{\beta}}}{\tilde{\beta}\pi\Lambda\hbar\sqrt{\Lambda-1}} \left(\ln(\tilde{\beta}) + \mathcal{O}(\tilde{\beta}^0) \right). \quad (\text{A.86})$$

Leaving

$$Z = \frac{\hbar}{\tilde{\beta}} e^{-\tilde{\beta}h_0} \left[1 + \exp\left(-\tilde{\beta}\frac{(\Lambda-1)^2}{2\Lambda}\right) \left[\frac{1}{\pi\hbar^2} \frac{\ln(\tilde{\beta})}{\Lambda\sqrt{\Lambda-1}} + \mathcal{O}(\tilde{\beta}^0) \right] \right] \quad (\text{A.87})$$

which again means that $Z \asymp e^{-\tilde{\beta}\hbar h_0}$, hence in the small temperature region we would expect the GS expectation values with *exponentially small corrections*.

For a high extensive temperature $T \gg 1$ we turn back to the ordinary partition function and expand around $\tilde{\beta}h = 0$

$$Z = \text{Tr}\{1\} - \tilde{\beta}\text{Tr}\{h\} + \tilde{\beta}^2\text{Tr}\{h^2\} + \mathcal{O}(\tilde{\beta})^3 \quad (\text{A.88})$$

$$= 2S \left[1 + \frac{\tilde{\beta}\Lambda}{4S}\text{Tr}\{x^2\} + \frac{\tilde{\beta}^2}{4S} \left[\frac{\Lambda^2}{4}\text{Tr}\{x^4\} + \Lambda\text{Tr}\{x^2z\} + \text{Tr}\{z^2\} \right] \right] + \mathcal{O}(\tilde{\beta}^2, S^0) \quad (\text{A.89})$$

$$= 2S \left[1 + \frac{\tilde{\beta}\Lambda}{6} + \frac{\tilde{\beta}^2}{4} \left[\frac{\Lambda^2}{4} \frac{2}{5} + 0 + \frac{2}{3} \right] \right] + \mathcal{O}(\tilde{\beta}^2, S^0) \quad (\text{A.90})$$

$$= 2S \left[1 + \frac{\tilde{\beta}\Lambda}{6} + \frac{\tilde{\beta}^2}{2} \left[\frac{\Lambda^2}{4} \frac{1}{5} + \frac{1}{3} \right] \right] + \mathcal{O}(\tilde{\beta}^2, S^0). \quad (\text{A.91})$$

Then using our standard Gibbs free energy we get

$$\langle \hat{x}^2 \rangle_{\text{eq}} = \frac{1}{3} + \frac{2\Lambda\tilde{\beta}}{45} + \mathcal{O}(\tilde{\beta}^2, S^{-1}) \quad \langle z \rangle_{\text{eq}} = \frac{\tilde{\beta}}{3} + \mathcal{O}(\tilde{\beta}^2, S^{-1}), \quad (\text{A.92})$$

which is exactly what one would expect for an infinite temperature Gibbs distribution $\varrho \propto \mathbb{1}$.

As such the system changes from a ferromagnetic phase to a paramagnetic phase for high enough extensive temperatures.

A.7 Spin Coherent States

Introduced in 1970 by Radcliffe [54], the spin coherent states

$$|\alpha\rangle = (1 + |\alpha|^2)^{-S} e^{\alpha S_-} |S, S\rangle, \quad \alpha \equiv \tan \frac{\theta}{2} e^{i\phi}, \quad (\theta, \phi) \in [0, \pi] \times [0, 2\pi] \quad (\text{A.93})$$

are flexible trial states for variational methods; they allow one to not only find the ground state energy, but to first order in S also the ground state (GS) of a spin system [25]. Like all standard coherent states they are maximally classical in the sense of saturating (equality of an inequality) an uncertainty relation $\Delta S_x \Delta S_y \geq |\langle S_z \rangle|/2$, with ΔS_x being a standard deviation.

The overlap of two such states $\langle \alpha | \alpha' \rangle$ becomes orthonormal in the limit $S \rightarrow \infty$. To see this let us first consider the following inequality

$$(1 + \bar{\alpha}\alpha')^2 \leq (1 + |\alpha|^2)(1 + |\alpha'|^2), \quad (\text{A.94})$$

with equality if and only if $\alpha = \alpha'$. As such the overlap

$$\langle \alpha | \alpha' \rangle = \left(\frac{(1 + \bar{\alpha}\alpha')^2}{(1 + |\alpha|^2)(1 + |\alpha'|^2)} \right)^S \quad (\text{A.95})$$

will be less than one if $\alpha \neq \alpha'$, implying the limit $\langle \alpha | \alpha' \rangle \rightarrow \delta_{\alpha, \alpha'}$ as $S \rightarrow \infty$.

Being the maximally projected states along some direction $(\theta, \phi) \in [0, \pi] \times [0, 2\pi]$, the spin coherent states are eigenstates of the rotated S_z operator $R_{\theta, \phi} S_z R_{\theta, \phi}^{-1}$, with

$$R_{\theta, \phi} = \exp\left(\frac{\theta}{2} e^{i\phi} S_- - \frac{\theta}{2} e^{-i\phi} S_+\right), \quad (\text{A.96})$$

and given explicitly by $R_{\theta, \phi}|S, S\rangle$. Using the Zassenhaus formula this exponential may be factorized as [26]

$$\begin{aligned} |\alpha\rangle &= \exp(\alpha S_-) \exp(-\ln(1 + |\alpha|^2) S_z) \exp(-\alpha^* S_+) |S, S\rangle, \quad \alpha \equiv \tan \frac{\theta}{2} e^{i\phi} \\ &= (1 + |\alpha|^2)^{-S} e^{\alpha S_-} |S, S\rangle \\ &= \frac{1}{(1 + |\alpha|^2)^S} \sum_{m=-S}^S \alpha^{S-m} \sqrt{\binom{2S}{S+m}} |S, m\rangle \end{aligned} \quad (\text{A.97})$$

where the final line follows from

$$S_-^q |S, S\rangle = q! \sqrt{\binom{2S}{q}} |S, S-q\rangle. \quad (\text{A.98})$$

Using $S_- |S, m\rangle = \sqrt{S(S+1) - m(m+1)} |S, m-1\rangle$ we may prove (A.98) as

$$S_-^q |S, S\rangle = \prod_{k=1}^q \sqrt{k(2S-k+1)} |S, S-q\rangle = \sqrt{\frac{q!(2S)!}{(2S-q)!}} |S, S-q\rangle = q! \sqrt{\binom{2S}{q}} |S, S-q\rangle.$$

As a function of m

$$\langle m|\alpha\rangle \sim \frac{2^S}{(\pi S)^{1/4} (1 + |\alpha|^2)^S} \alpha^{S-m} e^{-m^2/(2S)}, \quad (\text{A.99})$$

has a Gaussian form with a maximum at $m = \lceil S \cos \theta \rceil$. Note how the symbol z corresponds to the maximum of the spin coherent state.

We now introduce the key concepts of an operator $\hat{\sigma}$'s (lower) *symbol* $\sigma = \langle \hat{\sigma} \rangle_\alpha = \langle \alpha | \hat{\sigma} | \alpha \rangle$. We have dropped the hat here to indicate that we are dealing with a symbol. Using (A.97) one can find the corresponding symbols [54]

$$\langle \mathbf{s} \rangle_\alpha = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = (\sqrt{1-z^2} \cos \phi, \sqrt{1-z^2} \sin \phi, z). \quad (\text{A.100})$$

The restriction to the unit sphere is reflected in the identity $x^2 + y^2 + z^2 = 1$, which, again, is due to conservation of angular momentum on S . The symbol for $\langle \mathcal{H}_S \rangle_\alpha / S$.

The simplest way to calculate symbols of multiple operators, for instance $\hat{\sigma} \hat{\rho}$, is done via the star product [1] $\langle \alpha | \hat{\sigma} \hat{\rho} | \alpha \rangle = \sigma \star \rho$ defined as

$$\star = 1 + \overleftarrow{\partial}_\alpha \frac{(1+r^2)^2}{2S} \overrightarrow{\partial}_{\bar{\alpha}} + \mathcal{O}(S)^{-2}, \quad (\text{A.101})$$

where the arrows indicate acting in the left or right direction. Here we have defined the Wirtinger derivatives over the complex plane $\partial_\alpha = (\partial_{\Re\alpha} - i\partial_{\Im\alpha})/2$. Since we are considering large S , we will ignore the terms $\mathcal{O}(S)^{-2}$.

7 | Bibliography

- [1] G. Alexanian, A. Pinzul, and A. Stern. Generalized coherent state approach to star products and applications to the fuzzy sphere. *Nucl. Phys. B*, 600(3):531–547, 2001.
- [2] A. Altland and B. Simons. *Condensed Matter Field Theory*. Cambridge University Press, 2006.
- [3] H. Araki. Multiple time analyticity of a quantum statistical state satisfying the KMS boundary condition. *Publ. Res. Inst. Math. Sci.*, 4(2):361–371, 1968.
- [4] G. B. Arfken, H. J. Weber, and F. E. Harris. *Mathematical Methods for Physicists, Seventh Edition: A Comprehensive Guide*. 2012.
- [5] S. Attal and A. Joye. The Langevin equation for a quantum heat bath. *J. Funct. Anal.*, 247(2):253–288, jun 2007.
- [6] C. M. Bender and S. A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers I*. Springer New York, 1999.
- [7] C. Benedetti, F. S. Sehdaran, M. H. Zandi, and M. G. A. Paris. Quantum probes for the cutoff frequency of Ohmic environments. 2017.
- [8] R. F. Bishop and A. Vourdas. General two-mode squeezed states. *Zeitschrift für Phys. B Condens. Matter*, 71(4):527–529, 1988.
- [9] J. P. Blaizot and G. Ripka. *Quantum Theory of Finite Systems*. MIT Press, 1986.
- [10] M. Bonitz. *Quantum Kinetic Theory*. Springer International Switzerland, 1998.
- [11] R. Botet and R. Jullien. Large-size critical behavior of infinitely coordinated systems. *Phys. Rev. B*, 28(7):3955–3967, 1983.
- [12] H. P. Breuer and F. Petruccione. *The Theory of Open Quantum Systems*. Oxford University Press, 2007.
- [13] P. F. Byrd and M. D. Friedman. Handbook of Elliptic Integrals for Engineers and Scientists. *Math. Comput.*, 26(119):804, 1972.
- [14] H. Carmichael. *Statistical Methods in Quantum Optics 1: Master Equations and Fokker-Planck Equations*. Springer New York, 1999.
- [15] J. I. Cirac, M. Lewenstein, K. Mo, and P. Zoller. Quantum superposition states of Bose-Einstein condensates. Technical report, 1998.
- [16] I. de Vega and D. Alonso. Dynamics of non-Markovian open quantum systems. *Rev. Mod. Phys.*, 89:015001, 2017.
- [17] S. Dusuel and J. Vidal. Continuous unitary transformations and finite-size scaling exponents in the Lipkin-Meshkov-Glick model. *Phys. Rev. B - Condens. Mater Phys.*, 71(22), 2005.
- [18] C. Emary and T. Brandes. Quantum Chaos Triggered by Precursors of a Quantum Phase Transition: The Dicke Model. *Phys. Rev. Lett.*, 90(4):4, 2003.
- [19] C. Emary and T. Brandes. Chaos and the quantum phase transition in the Dicke model. *Phys. Rev. E*, 67(6):066203, 2003.
- [20] J. S. Ferreira and P. Ribeiro. The Lipkin-Meshkov-Glick model with Markovian dissipation -A description of a collective spin on a metallic surface. 2018.

-
- [21] C. H. Fleming and N. I. Cummings. Accuracy of perturbative master equations. *Phys. Rev. E - Statistical, Nonlinear, and Soft Matter Physics*, 83(3), 2011.
- [22] C. H. Fleming and B. L. Hu. Non-Markovian dynamics of open quantum systems: Stochastic equations and their perturbative solutions. *Ann. Phys. (N. Y.)*, 327:1238–1276, 2012.
- [23] J. Garcia and R. Rossignoli. Spectrum and normal modes of non-Hermitian quadratic boson operators. *Phys. Rev. A*, 96(6):1–10, 2017.
- [24] C. W. Gardiner and P. Zoller. *Quantum noise: a handbook of Markovian and non-Markovian quantum stochastic methods with applications to quantum optics*. Springer, 2004.
- [25] A. Garg and M. Stone. Bohr-Sommerfeld Quantization of Spin Hamiltonians. *Phys. Rev. Lett.*, 92(1):4, 2004.
- [26] J. P. Gazeau. *Coherent states in quantum physics*. Wiley-VCH, 2009.
- [27] S. Gerschgorin. Über die Abgrenzung der Eigenwerte einer Matrix. *Cl. des Sci. mathématiques*, (6):749–754, 1931.
- [28] R. Gilmore. Q and P representatives for spherical tensors. *J. Phys. A Gen. Phys.*, 9(7):L65–L66, 1976.
- [29] R. Gilmore and D. H. Feng. Studies of the ground state properties of the LMG model via the atomic coherent states. *Phys. Lett. B*, 76(1):26–28, 1978.
- [30] T. Holstein and H. Primakoff. Field Dependence of the Intrinsic Domain Magnetization of a Ferromagnet. *Phys. Rev.*, 58(12):1098–1113, 1940.
- [31] K. Jacobs. *Chapter 1. Quantum measurement theory*. Cambridge University Press, 2014.
- [32] J. L. W. V. Jensen. Sur les fonctions convexes et les inegalites entre les valeurs moyennes. *Acta Math.*, 30(1):175–193, 1906.
- [33] J. R. Johansson, P. D. Nation, and F. Nori. QuTiP 2: A Python framework for the dynamics of open quantum systems. *Comput. Phys. Commun.*, 184(4):1234–1240, 2013.
- [34] B.D. Josephson. Possible new effects in superconductive tunnelling. *Phys. Lett.*, 1(7):251–253, 1962.
- [35] M. Kastner. Nonequivalence of ensembles in the Curie-Weiss anisotropic quantum Heisenberg model. *J. Stat. Mech. Theory Exp.*, 2010.
- [36] S. Kehrein. *The flow equation approach to many-particle systems*. Springer, 2006.
- [37] J. Kondo. Resistance Minimum in Dilute Magnetic Alloys. Technical Report 1, 1964.
- [38] J. N. Kriel, A. Y. Morozov, and F. G. Scholtz. Non-perturbative flow equations from continuous unitary transformations. *J. Phys. A. Math. Gen.*, 38(1):205–226, 2005.
- [39] J. Kutzner. Eine Phasenraumdarstellung fuer Spinsysteme. *Zeitschrift fuer Physik A Hadrons and nuclei*, 188(2):177–188, 1973.
- [40] J. Larson. Integrability versus quantum thermalization. *J. Phys. B At. Mol. Opt. Phys.*, 46(22):224016, 2013.
- [41] A. J. Leggett. Bose-Einstein condensation in the alkali gases: Some fundamental concepts. *Rev. Mod. Phys.*, 73(2):307–356, 2001.

- [42] E. H. Lieb. The classical limit of quantum spin systems. *Commun. Math. Phys.*, 31(4):327–340, 1973.
- [43] G. Lindblad. On the generators of quantum dynamical semigroups. *Commun. Math. Phys.*, 48(2):119–130, 1976.
- [44] H. J. Lipkin, N. Meshkov, and A. J. Glick. Validity of many-body approximation methods for a solvable model. (I). Exact solutions and perturbation theory. *Nucl. Phys.*, 62(2):188–198, 1965.
- [45] H. D. Macedo and J. N. Oliveira. Typing linear algebra: A biproduct-oriented approach. *Sci. Comput. Program.*, 78(11):2160–2191, 2013.
- [46] J. R. Magnus and H. Neudecker. *Matrix Differential Calculus with Applications in Statistics and Econometrics*, volume 31. John Wiley, 1989.
- [47] T. Mori. Classical ergodicity and quantum eigenstate thermalization: Analysis in fully connected Ising ferromagnets. Technical Report 1, 2017.
- [48] C. M. Newman and L. S. Schulman. Metastability and the analytic continuation of eigenvalues. *J. Math. Phys.*, 18(1):23–30, 1977.
- [49] A. Pálffy. Spontaneous Emission: Weisskopf-Wigner Theory. Technical report.
- [50] F. Pan and J. P. Draayer. Analytical solutions for the LMG model. *Phys. Lett. Sect. B Nucl. Elem. Part. High-Energy Phys.*, 451(1-2):1–10, 1999.
- [51] S. Popescu, A. J. Short, and A. Winter. Entanglement and the foundations of statistical mechanics. *Nat. Phys.*, 2(11):754–758, 2006.
- [52] J. Preskill. Lecture Notes for Ph219/CS219: Quantum Information Chapter 3. 2015.
- [53] G. Queraltó and S. Isach. *MSc in Photonics Master in Photonics Quantum phase transitions of a Bose-Einstein condensate*. PhD thesis, 2015.
- [54] J. M. Radcliffe. Some properties of coherent spin states. *J. Phys. A Gen. Phys.*, 4(3):313–323, 1971.
- [55] S. Raghavan, A. Smerzi, S. Fantoni, and S. R. Shenoy. Coherent oscillations between two weakly coupled Bose-Einstein condensates: Josephson effects, oscillations, and macroscopic quantum self-trapping. *Phys. Rev. A*, 59(1):620–633, 1999.
- [56] P. Ribeiro, J. Vidal, and R. Mosseri. Exact spectrum of the Lipkin-Meshkov-Glick model in the thermodynamic limit and finite-size corrections. *Phys. Rev. E - Statistical, Nonlinear, and Soft Matter Physics*, 78(2):1–13, 2008.
- [57] T. F. Rønnow, Z. Wang, J. Job, S. Boixo, S. V. Isakov, D. Wecker, J. M. Martinis, D. A. Lidar, and M. Troyer. Defining and detecting quantum speedup. *Science (80-.)*, 345(6195):420–424, 2014.
- [58] J. J. Sakurai and J. Napolitano. *Modern Quantum Mechanics*. Cambridge University Press, 2 edition, 2017.
- [59] G. Schaller. *Non-Equilibrium Master Equations*. 2015.
- [60] M. Schlosshauer. *Decoherence and the quantum-to-classical transition*. 2007.
- [61] L. S. Schulman. Note on the quantum recurrence theorem. *Phys. Rev. A*, 18:2379–2380, 1978.

- [62] B. Sciolla and G. Biroli. Dynamical transitions and quantum quenches in mean-field models. *J. Stat. Mech. Theory Exp.*, 2011(11):P11003, nov 2011.
- [63] H. Spohn and J. L. Lebowitz. Irreversible thermodynamics for quantum systems weakly coupled to thermal reservoirs. *Adv. Chem. Phys. Ilya Prigogine, Vol. 38*, x:109–142, 2007.
- [64] M. Srednicki. *Chaos and Quantum Thermalization*. 1994.
- [65] S. H. Strogatz. *Nonlinear dynamics and chaos : with applications to physics, biology, chemistry, and engineering*. Levant Books, 2007.
- [66] H. Touchette. *The large deviation approach to statistical mechanics*, 2009.
- [67] T. Vorrath and T. Brandes. Dynamics of a large spin with strong dissipation. *Phys. Rev. Lett.*, 95(7):1–4, 2005.
- [68] T. Vorrath, T. Brandes, and B. Kramer. Dynamics of a large spin with weak dissipation. *Chemical Physics*, 296(2-3):295–300, 2004.
- [69] T. Vorrath, T. Brandes, and B. Kramer. Dynamics of a large-spin-boson system in the strong coupling regime. (1):1–4, 2007.
- [70] J. R. Webster and M. Kastner. Subexponentially Growing Hilbert Space and Nonconcentrating Distributions in a Constrained Spin Model. *J. Stat. Phys.*, 171(3):449–461, 2018.
- [71] T. Zibold, E. Nicklas, C. Gross, and M. K. Oberthaler. Classical bifurcation at the transition from rabi to Josephson dynamics. *Phys. Rev. Lett.*, 105(20):1–4, 2010.