A Duality-Theoretic Perspective on Formal Languages and Recognition

by

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Abstract

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A connection between recognisable languages and profinite identities is established through the composition of two famous theorems: Eilenberg's theorem and Reiterman's theorem. In this work, we present a detailed account of the duality-theoretic approach by Gehrke et al. that has been shown to bridge the gap and demonstrate that Eilenberg's varieties and profinite theories are directly linked: they are at opposite ends of an extended Stone-type duality, instantiating a Galois correspondence between subobjects and quotients and resulting in an equational theory of recognisable languages. We give an in-depth overview of relevant components of algebraic language theory and the profinite equational theory of pseudovarieties in order to show how they are tied together by the duality-theoretic developments. Furthermore, we provide independent proofs of the key Galois connections at the heart of these bridging results.

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Notation

Much of the notation used in this work will be introduced in context or is included in the preliminaries in the next section. However, we make a few general comments about conventions that apply throughout:

- We use square brackets to indicate that we are applying a function pointwise to a subset of the domain (for example, $f[A]$ and $f^{-1}[B]$). When there are multiple set-theoretic levels, the context should make it clear whether we are applying an image/preimage map to a subset or a collection of subsets.
- We often shorten the algebraic signature of certain algebraic structures. For example, a Boolean residuation algebra $(B, \wedge, \vee, \neg, 0, 1, /, \backslash)$ may be written as $(B, /, \backslash)$ if we have already mentioned that it is a Boolean algebra.
- The kernel of a homomorphism f will be denoted $\sim_f$.
- We often switch from talking about topological algebras to the underlying algebras or even to the underlying sets. This is expected to be clear from the context, so we omit explicit reference to forgetful functors in diagrams and proofs.
- Stone spaces may be seen as Priestley spaces with a discrete ordering, and we freely treat them as such without making it explicit every time. In the same way, finite algebras may be seen as finite topological algebras with a discrete topology and we do not always include the topology in the signature explicitly.
- Set theoretic complements are written as a unary operation $(\_)^C$, to avoid overuse and ambiguity of the $\backslash$ symbol. However, we do still use $/$ to symbolize quotienting by a congruence, and this distinction should be clear from context as we choose to denote congruences in such a way that they may not be confused with elements in the domain of a residuation operation.

- Usually, the same notation will be used for a function and its restriction to an underlying set, unless we choose to make this explicit for clarity.

Chapter 1

Introduction

Formal language theory is a rich field at the intersection of computer science, mathematics and linguistics. Formal languages are purely syntactic constructions; in the traditional case they are collections of strings made up of characters from some finite alphabet. A language may be specified by various structures such as automata, formal grammars, regular expressions or sentences in formal logics. Given a specified language, such structures let us test whether a word or string belong to that language. A standard application is the parsing of computer code, where the compiler has to recognise if a string of code belongs to the programming language it is built to recognise.

The notion of *recognition* is central to this study. An algebraic approach to language theory headed by Eilenberg added monoids to the pool of structures which capture language recognition. The languages recognised by finite monoids are exactly the regular (or "recognisable") languages, those which can be specified by a finite state automaton, regular grammar or regular expression. Although all of these structures recognise the same languages, the algebraic tools from monoid theory have made a significant contribution. A famous example is Schutzenburger's theorem [27], which gives a condition which allows one to compute whether a language is "star-free", i.e. can be recognised by a regular expression which does not feature the Kleene star. This previously unsolved problem now boils down to checking whether a finite monoid is aprediodic.

Many other such instances were studied, but the phenomenon was captured more generally by Eilenberg's variety theorem ([10]) which demonstrates that there is a one-to-one correspondence between certain subclasses of regular languages and certain classes of finite monoids, which sometimes have a finite decidable characterization in terms of equations. The relevant collections of finite monoids are pseudovarieties, which have their own rich theory akin to Birkhoff's varieties in universal algebras. The study of pseudovarieties of finite monoids has an important role in finding the right characteristics of the monoids which are of interest, such as the condition of aperiodicity in the

Schutzenburger case. A famous result by Reiterman [25] (although the presentation in this work is due to more recent treatments such as [21]) gives a kind of "equational" theory for pseudovarieties of monoids via pseudoidentities. Pseudoidenties are pairs of "profinite words" which live in the pro-completion of the monoid of words on our fixed alphabet. The sets of pseudoidentities that correspond to pseudovarieties are known as profinite theories.

This two-fold connection between Eilenberg classes of regular languages and profinite theories has been identified as an instance of Stone duality, a well-known dual equivalence of categories between Boolean algebras and Stone spaces. It was shown by Gehrke and Pin (although it should be mentioned that the first investigation of the stone dual of the Boolean algebra of regular languages on a finite alphabet was done by Pippenger) that the Eilenberg varieties of languages and profinite theories are dual to each other with respect to an extended version of Stone duality, and this gives us a direct insight into the connection between recognisable languages and pseudoidentities which bypasses the reliance on recognition by monoids.

In this work, we present some background on algebraic language theory, the theory of pseudovarieties and extended Stone duality so as to provide a more self-contained account of the work relevant to the contributions of Gehrke et al. in [14], [13] and [11], so far as is relevant to the case of traditional recognisable languages. We delve into particular detail and provide independent proofs of the Galois connections which have been shown to form the bridging connection between Eilenberg's language varieties and profinite theories.

The paper is organised as follows: In the first chapter, we provide a selected background on the algebraic approach to formal language theory which leads up to Eilenberg's variety theorem. In the second chapter, we showcase the connection between pseudovarieties and profinite algebras, culminating in Reiterman's characterisation of pseudovarieties in terms of pseudoidentities. In the third chapter, we sketch the relevant parts of the duality theories that have led to a direct bridge between Eilenberg's language varieties and profinite theories. We give independent proofs for the crucial Galois correspondences underlying this bridging observation, and provide an exposition on Gehrke, Gregorieff and Pin's ([14],[13],[12]) ensuing work on the equational theory of regular languages [14].

Chapter 2

Algebraic Language Theory

2.1 Algebraic Formulation of Recognition

Traditionally, recognition in language theory is defined in terms of finite state automata or regular expressions. We will work with the definition of recognition in terms of finite monoids (see [20, Chapter IV] for a proof that the definitions are equivalent). Either way, recognisable languages are essentially those that have some form of finite representation, be it through a finite state machine, a regular expression or a monadic second order sentence [7]. The algebraic approach grew out of the study of *syntactic monoids*. This was spearheaded by Schutzenburger, Myhill, Rabin and Scott [24]. Many developments arose from the algebraic approach, especially pertaining to connections between membership of certain classes of recognisable languages and corresponding algebraic properties of their syntactic monoids. A famous result is that of Schutzenburger [27], which observed a correspondence between *star-free* languages and aperiodic monoids. Several such correspondences were studied before Eilenberg famously characterised these connections and captured them in greater generality, and his theorem is the key result of this chapter.

For a finite set A of symbols, which we will henceforth refer to as an *alphabet*, any finite sequence of symbols is called a *word*. For the purpose of our discussion, throughout this work we arbitrarily fix a countable set of symbols $\mathcal{A} = \{a_1, a_2, \ldots, a_n, \ldots\}$. Without loss of generality, every time we refer to any alphabet A, we default to A being a subset of $\mathcal{A}$ of equal size.

For a fixed alphabet A, the free monoid $(A^*, \cdot)$ may be seen as the *monoid of words*, with the product operation being concatenation of words. The empty word, which we will denote by 1, is the identity of the concatenation operation.

Definition 2.1.1. A *language* for an alphabet A is any subset $L \subseteq A^*$.

Definition 2.1.2. A language $L \subseteq A^*$ is said to be *recognised* by a finite monoid M if there exists a homomorphism $\sigma : A^* \to M$ and a subset $P \subseteq M$

such that $L = \sigma^{-1}[P]$. If there exists a finite monoid that recognises some language L, then L is said to be *recognisable*.

Remark 2.1.3. When L is recognised by a monoid M, we often also say that it is recognised by the homomorphism σ.

Lemma 2.1.4. *A language L is recognisable if and only if it can be recognised by a surjective monoid homomorphism $\sigma : A^* \to M$*

Proof. If L is recognisable, the image of the existing monoid homomorphism $\sigma : A^* \to M$ is again a finite monoid, and it recognises L because $L = \sigma^{-1}[P] = \sigma^{-1}[P \cap \sigma[A^*]]$ and $P \cap \sigma[L] \subseteq \sigma[A^*]$. □

Remark 2.1.5. In this case, we say that L is *fully* recognised by σ.

2.2 Properties of Recognition

For the rest of this subsection, we fix an arbitrary alphabet A.

Definition 2.2.1. The set of all recognisable languages of A^* is called $Rec(A^*)$.

Lemma 2.2.2. *The subset $Rec(A^*)$ contains $\emptyset$ and A^*, and is closed under finite intersections, finite unions and complementation, so that it is a Boolean subalgebra of $\mathcal{P}(A^*)$.*

Proof. The sets $\emptyset$ and A^* are recognised by the trivial monoid, so both are in $Rec(A^*)$. Let L_1, L_2 be recognisable languages. Thus there exist monoids M_1, M_2, monoid homomorphisms $\sigma_1 : A^* \to M_1$ and $\sigma_2 =: A^* \to M_2$ and subsets $P_1 \subseteq M_1, P_2 \subseteq M_2$ for which $L_1 = \sigma^{-1}[P_1]$ and $L_2 = \sigma^{-1}[P_2]$.
The language $L_1 \cap L_2$ is recognised by $\sigma : A^* \to M_1 \times M_2$, the monoid homomorphism induced by the universal property of products, because

$$L = \sigma^{-1}[P_1 \times P_2].$$

The language $L_1 \cup L_2$ is recognised by σ as well, because

$$L = \sigma^{-1}[(M_1 \times P_2) \cup (P_1 \times M_2)].$$

The language $(L_1)^C$ is recognised by σ_1, since

$$(L_1)^C = \sigma^{-1}[(P_1)^C].$$

Thus, each of these languages is recognisable and $Rec(A^*)$ is closed under the Boolean operations. □

The binary operation on any monoid $(M, \cdot)$ (including the concatenation operation on $(A^*, \cdot)$) lifts to a binary operation on $\mathcal{P}(M)$ given by

$$K \cdot L = \{v \cdot w \mid v \in K, w \in L\}$$

for $K, L \subseteq M$. Furthermore, there exist binary operations $/, \backslash : \mathcal{P}(M) \times \mathcal{P}(M) \to \mathcal{P}(M)$ such that

$$H \cdot K \subseteq L \iff K \subseteq H \backslash L \iff H \subseteq L/K$$

for $H, K, L \subseteq M$. These operations $/$ and $\backslash$ are called the right and left *residuals* of the lifted binary operation: we discuss residual operations in more detail in chapter 4. In this context, they are given, for $K, L \subseteq A^*$, by:

$$K \backslash L = \{w \in A^* \mid \forall v \in K(v \cdot w \in L)\}$$
$$L/K = \{w \in A^* \mid \forall v \in K(w \cdot v \in L)\}.$$

The Boolean algebra $Rec(A^*)$ happens to be closed under these operations on $\mathcal{P}(A^*)$. In fact, it is closed with respect to these operations with respect to residuation by arbitrary subsets of A^*. The proofs of the next two results are from [11].

Lemma 2.2.3. *The Boolean algebra $Rec(A^*)$ is closed under the left and right residuals of the lifted concatenation operation on $\mathcal{P}(A^*)$ with respect to arbitrary subsets of A^*. Namely, for any $L \in Rec(A^*)$ and any $K \subseteq A^*$, we have $K \backslash L \in Rec(A^*)$ and $L/K \in Rec(A^*)$.*

Proof. Suppose $L \in Rec(A^*)$, so that it is recogised by a surjective monoid homomorphism $\sigma : A^* \to M$. For any $K \subseteq A^*$, σ also recognises $K \backslash L$ and L/K, since $\sigma^{-1}[\sigma[K] \backslash P] = K \backslash \sigma^{-1}[P] = K \backslash L$ and $\sigma^{-1}[P/\sigma[K]] = \sigma^{-1}[P]/K = L/K$. □

Lemma 2.2.4. *A sublattice $C \subseteq Rec(A^*)$ is closed under the left and right residuals of the lifted concatenation operation on $\mathcal{P}(A^*)$ with respect to arbitrary subsets of A^* if and only if it is closed under the left and right residuals with respect to singletons, namely for any $w \in A^*$, $\{w\} \backslash L \in C$ and $L/\{w\} \in C$.*

Proof. Assuming the same set-up as in the previous proof, note that $K \backslash L = \sigma^{-1}[\sigma[K] \backslash P]$ and $\sigma[K] \backslash P = \bigcap_{v \in K} \sigma(v) \backslash P = \bigcap_{v \in K'} \sigma(v) \backslash P$ for some finite subset $K' \subseteq K$ (since $\sigma[K] \subseteq M$ is finite). Hence,

$$\begin{aligned} K \backslash L &= \sigma^{-1}[\bigcap_{v \in K'} \sigma(v) \backslash P] \\ &= \bigcap_{v \in K'} \sigma^{-1}[\sigma(v) \backslash P] \\ &= \bigcap_{v \in K'} v \backslash L. \end{aligned}$$

The result follows. □

Lemma 2.2.5. *Let A, B be two alphabets. If $h : A^* \to B^*$ is a homomorphism, then $L \in Rec(B^*)$ implies that $h^{-1}[L] \in Rec(A^*)$. In other words, $Rec(\_^*)$ is* closed under inverses of homomorphisms.

Proof. Suppose that $\sigma : B^* \to M$ recognises L, so that $L = \sigma^{-1}[P]$ for some $P \subseteq M$. Then the language $h^{-1}[L] \subseteq A^*$ is recognised by the the composite $\sigma \circ h : A^* \to M$, since $h^{-1}[L] = h^{-1}[\sigma^{-1}[P]]$. □

2.3 Syntactic Monoids and Pseudovarieties

Definition 2.3.1. Let $L \in Rec(A^*)$. We may define a congruence $\sim_L$ on A^* by

$$v \sim_L w \text{ if and only if } \forall_{x,y \in A^*}(x \cdot v \cdot y \in L \iff x \cdot w \cdot y \in L).$$

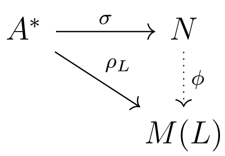
Then $M(L) := A^*/\sim_L$ is called the *syntactic monoid* of L., and $\sim_L$ is called the *syntactic congruence* of L. The quotient homomorphism $\rho_L : A^* \to M(L)$ is called the *syntactic morphism*.

This is easily seen to be a congruence, and the syntactic monoid plays a crucial role in recognition.

Lemma 2.3.2. *For a language $L \in Rec(A^*)$, the syntactic congruence* saturates *L, in the sense that L is equal to a union of congruence classes. As a consequence, if the syntactic monoid is finite, it recognises L.*

Proof. For each $w \in L$, if $v \sim_L w$ for any $v \in A^*$, then $1 \cdot v \cdot 1 \in L \iff 1 \cdot w \cdot 1 \in L$. Hence, $L = \bigcup_{w \in L} \rho_L(w)$ and $L = \rho_L^{-1}[\rho_L[L]]$. It follows that if $M(L)$ is finite, then L is recognised by it. □

Lemma 2.3.3. *Let $L \in Rec(A^*)$ and let $\sigma : A^* \to N$ be a surjective monoid homomorphism with N being a finite monoid. Then σ recognises L if and only if there exists a surjective homomorphism $\phi : N \to M(L)$ such that the following diagram commutes:*



Proof. Suppose that σ recognises L. We show that $ker(\sigma) \subseteq ker(\rho_L)$: to this end, let $(v, w) \in ker(\sigma)$. Recall that $L = \sigma^{-1}[P]$ for some $P \subseteq N$. For any $x, y \in A^*$, $x \cdot v \cdot y \in \sigma^{-1}[P] \iff x \cdot w \cdot y \in \sigma^{-1}[P]$, since $\sigma(v) = \sigma(w)$. As a consequence, we get a surjective homomorphism ϕ as desired.

On the other hand, assume that such a ϕ already exists. Then L is recognised by N, since $L = \rho_L^{-1}[P]$ for some $P \subseteq M(L)$ and thus $L = \sigma^{-1}[\phi^{-1}[P]]$. □

Corollary 2.3.4. *A language $L \subseteq A^*$ is recognisable if and only if its syntactic monoid is finite.*

Definition 2.3.5. We say that a monoid M *divides* a monoid N (denoted $M \prec N$) if and only if M is the quotient of a submonoid of N.

Corollary 2.3.6. *The syntactic monoid divides every finite monoid which recognises L.*

Proof. This is a consequence of lemma 2.3.3 and the observation in the proof of lemma 2.1.4, where it is observed that any monoid recognising a language L has a submonoid which fully recognises it. □

Syntactic monoids provide a canonical way of associating a finite monoid to a recognisable language, and they are a key component of Eilenbergs theorem. This theorem connects certain collections of recognisable languages with classes of monoids called *pseudovarieties*, which we now define. These are analogous to varieties in unviversal algebra; the latter being closed under arbitrary products ensures that no variety contains only finite algebras. Restricting this condition to closure under *finite* products yields the notion of pseudovarieties.

Definition 2.3.7. A *pseudovariety of finite algebras* of a given signature is a class of finite algebras closed under subalgebras, homomorphic images and finite (including empty) products.

We restrict our attention to monoids for the purposes of language theory. Much more will be said on the subject of pseudovarieties in 3, but we include certain results here which are required for the Eilenberg theorem.

Definition 2.3.8. Let $\mathcal{M}$ be a set of finite monoids. The pseudovariety *generated* by $\mathcal{M}$, denoted $\langle\mathcal{M}\rangle$, is the smallest pseudovariety containing each monoid in $\mathcal{M}$.

Lemma 2.3.9. *[20, XI, Proposition 1.1] A monoid belongs to the pseudovariety $\langle\mathcal{M}\rangle$ if and only if it divides a finite product of monoids from $\mathcal{M}$.*

2.4 Varieties of Languages and Eilenberg's Theorem

The closure properties we have observed for $Rec(A^*)$ are exactly the conditions required to describe the collections of recognisable languages that correspond to pseudovarieties of finite monoids. These are Eilenberg's *varieties of languages*.

Definition 2.4.1. A mapping $A \mapsto \mathcal{C}(A)$ where $C(A) \subseteq Rec(A^*)$ for finite alphabets A is called a *variety of languages* if and only if

1. $\mathcal{C}(A)$ is a Boolean subalgebra of $Rec(A^*)$,
2. $\mathcal{C}(A)$ is closed under the operations $\{w\}\backslash L$ and $L/\{w\}$ for every $L \in \mathcal{C}(A)$ and $w \in A^*$, and
3. $\mathcal{C}$ is closed under inverses of homomorphisms, in the sense that if A, B are alphabets and $h : A^* \to B^*$ is a homomorphism, then $L \in \mathcal{C}(B)$ implies that $h^{-1}[L] \in \mathcal{C}(A)$.

We include one technical lemma on varieties of languages that is relevant to Eilenberg's theorem:

Lemma 2.4.2. *[20, Lemma XIII.4.11] Let $\mathcal{C}$ be a variety of languages and let $L \in Rec(A^*)$ for some alphabet A. Then for every $m \in M(L)$, $\rho_L^{-1}(m)$ belongs to $\mathcal{C}(A)$.*

This version of the proof of Eilenberg's theorem (which essentially comprises the remainder of this section) is from [20], although the original source for the theorem is [10].

We begin by describing two mappings which will be shown to be mutually inverse bijections. Given any pseudovariety $\mathbf{V}$ of finite monoids,

$$\mathbf{V} \mapsto \mathcal{C}_{\mathbf{V}}$$

assigns to it a variety of languages of follows: For each alphabet A, $\mathcal{C}_{\mathbf{V}}(A)$ is defined to be the set of all languages $L \subseteq Rec(A^*)$ for which $M(L) \in \mathbf{V}$. On the other hand, given any variety of languages $\mathcal{C}$,

$$\mathcal{C} \mapsto \mathbf{V}_{\mathcal{C}}$$

assigns to it the pseudovariety generated by the collection of syntactic monoids: that is to say, all monoids $M(L)$ so that $L \in C(A)$ for some finite alphabet A.

Lemma 2.4.3. *For a pseudovariety of monoids $\mathbf{V}$, $\mathcal{C}_{\mathbf{V}}$ is a variety of languages.*

Proof. For any fixed alphabet A, let $L_1, L_2, L \in \mathcal{C}_{\mathbf{V}}(A)$. By inspection of the proof of lemma 2.2.2, we see that $L_1 \cap L_2, L_1 \cup L_2$ will be recognised by products of the syntactic monoids $M(L_1)$ and $M(L_2)$, and L^C is recognised by the syntactic monoid $M(L)$. Because $\mathbf{V}$ is a pseudovariety, each of these recognising monoids is contained in $\mathbf{V}$.

Now let $w \in A^*$. Then $\{w\}\backslash L$ and $L/\{w\}$ are recognised by the syntactic monoid $M(L)$, by the proof of lemma 2.2.3. A monoid recognising a language is in a pseudovariety $\mathbf{V}$ if and only if its syntactic monoid is in $\mathbf{V}$ (as a consequence of theorem 2.3.6), so the syntactic monoids of each of these languages will be contained in $\mathbf{V}$. It follows that $L_1 \cap L_2, L_1 \cup L_2$ and L^C will be in $\mathcal{C}_{\mathbf{V}}(A)$.

Lastly, suppose that $h : A^* \to B^*$ and $L \in \mathcal{C}_{\mathbf{V}}(B)$. Then syntactic monoid $M(L)$ is in $\mathbf{V}$, and we have seen in lemma 2.2.5 that the same monoid recognises $h^{-1}[L]$. Once again, this implies that $M(h^{-1}[L]) \prec M(L)$, so that $M(h^{-1}[L]) \in \mathbf{V}$ and thus $h^{-1}[L] \in \mathcal{C}_{\mathbf{V}}(A)$.

□

Theorem 2.4.4. *The mapping $\mathbf{V} \mapsto \mathcal{C}_{\mathbf{V}}$ is injective.*

Proof. Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of finite monoids. We show that $\mathbf{V} \subseteq \mathbf{W}$ if and only if $\mathcal{C}_{\mathbf{V}}(A) \subseteq \mathcal{C}_{\mathbf{W}}(A)$ for every alphabet A.

The forward implication easily follows from the definitions; if for $L \in Rec(A^*)$, $M(L) \in \mathbf{V}$, then clearly $M(L) \in \mathbf{W}$. For the converse, by proposition $XIII.4.8$ in [20], there exists a finite alphabet A and a finite set of languages $L_1, \ldots, L_n \in \mathcal{C}_{\mathbf{V}}(A)$ such that M is (up to isomorphism) a submonoid of $M(L_1) \times \ldots \times M(L_n)$. Then $L_1, \ldots, L_n \in \mathcal{C}_{\mathbf{W}}(A)$ as well, so that $M \in \mathbf{W}$.

As a consequence, $\mathbf{V} = \mathbf{W} \iff \mathcal{C}_{\mathbf{V}}(A) = \mathcal{C}_{\mathbf{W}}(A)$, so that the given map is injective. □

Theorem 2.4.5. *The mappings $\mathbf{V} \mapsto \mathcal{C}_{\mathbf{V}}$ and $\mathcal{C} \mapsto \mathbf{V}_{\mathcal{C}}$ give a mutually inverse bijective correspondence between pseudovarieties of finite monoids and varieties of languages.*

Proof. From theorem 2.4.4, we already know that the map $\mathbf{V} \mapsto \mathcal{C}_{\mathbf{V}}$ is injective. To show that it is also surjective, let $\mathcal{C}$ be a variety of languages. Let $\mathcal{D} := \mathcal{C}_{\mathbf{V}_{\mathcal{C}}}$. We will prove that $\mathcal{C} = \mathcal{D}$, so that $\mathcal{C}$ is in the image of the aforementioned map. For every alphabet A, $\mathcal{C}(A) \subseteq \mathcal{D}(A)$. Given $L \in \mathcal{C}(A)$, $M(L)$ belongs to $\mathbf{V}_{\mathcal{C}}$ by definition. Again by definition, L therefore belongs to $\mathcal{D}$.

It remains to show that $\mathcal{D}(A) \subseteq \mathcal{C}(A)$. To this end, let $L \in \mathcal{D}(A)$. From the way $\mathcal{D}$ was defined, $M(L) \in \mathbf{V}_{\mathcal{C}}$. Now since $\mathbf{V}_{\mathcal{C}}$ is generated by the collection of all the syntactic monoids of $\mathcal{C}$, there must exists alphabets $A_1, \ldots, A_n$ and n languages $L_i \subseteq A_i$ such that $M(L)$ divides the monoid $M := M(L_i) \times \ldots \times M(L_n)$. Therefore there exists submonoid T of M of which $M(L)$ is a quotient. By lemma 2.3.3, T recognises L, so there exists a surjective monoid morphism $\sigma : A^* \to T$ and a subset $P \subseteq T$ such that $L = \sigma^{-1}(T)$. Note that each $s \in M$ is equal to $(s_1, \ldots, s_n)$ with $s_i \in M(L_i)$. Let $\sigma_i = \pi_i \circ \sigma$ and let ρ_i denote the syntactic morphism $\rho_i : A_i^* \to M(L_i)$. Since each ρ_i is surjective, we may define a monoid homomorphism $\psi_i : A^* \to A_i^*$ by mapping $w \in A^*$ to any element in the non-empty set $\rho_i^{-1}(\sigma_i(w))$. Then $\rho_i \circ \psi_i = \sigma_i$. In summary, we have the following commutative diagram:

$$\begin{array}{ccc} A^* & \xrightarrow{\psi_i} & A_i^* \\ \downarrow \sigma & \searrow^{\sigma_i} & \downarrow \rho_i \\ T \subseteq M & \xrightarrow{\pi_i} & M(L_i) \end{array}$$

If we show that $\sigma^{-1}(s) \in \mathcal{C}(A)$ for every $s \in P$, then because $\mathcal{C}(A)$ is a lattice and T is finite, this will show that

$$L = \sigma^{-1}[P] = \bigcup_{s \in P} \sigma^{-1}[P]$$

is also in $\mathcal{C}(A)$. Fix any $s \in P$, and recall that $(s_1, \ldots, s_n)$ for some $s_i \in M(L_i)$, and so $\{s\} = \bigcap_{1 \leq i \leq n} \pi_i^{-1}(s_i)$. By the commutativity of the above diagram,

$$\begin{aligned} \sigma^{-1}(s) &= \bigcap_{1 \leq i \leq n} \sigma^{-1}[\pi_i^{-1}(s_i)] \\ &= \bigcap_{1 \leq i \leq n} \sigma_i^{-1}(s_i). \end{aligned}$$

Once again, closure under finite intersections means that we need only show that for every $1 \leq i \leq n$, we have $\sigma_i^{-1}(s_i) \in \mathcal{C}(A)$.

Using the definition of σ_i, we know that $\sigma^{-1}(s) = \psi_i^{-1}(\rho^{-1}(s_i))$. Since $\mathcal{C}$ is closed under inverse morphisms, it will suffice to show that $\rho_i^{-1}(s_i) \in \mathcal{C}$ for each $1 \leq i \leq n$. This follows from lemma 2.4.2. $\square$

Chapter 3

Profinite Algebras and Reiterman's Theorem

3.1 Motivation and Construction of Profinite Algebras

By Birkhoff's theorem, varieties of algebras have characterisations in terms of sets of identities given by pairs of elements of a term algebra with respect to a fixed finitary algebraic type [4]. There is no immediate analogue for Birkhoff's theorem for pseudovarieties of finite algebras, as identities are no longer enough to distinguish them. It is straightforward that all the finite members of a variety constitute a pseudovariety (which is thus an equational class of finite algebras), but the converse is not necessarily true [3].

It is possible to construct an object which plays the role of a "free algebra" with respect to pseudovarieties. Its elements are generalized terms, pairs of which are referred to as "pseudoidentities". The project of this chapter is to present these constructions and the ensuing Birkhoff-like theorem (due to Reiterman in [25]) for pseudovarieties.

The appropriate construction is that of a certain cofiltered limit, which will exhibit exactly the behaviour we would expect of such a free object. Pseudovarieties of finite algebras are not in general closed under cofiltered limits, and therefore it is necessary to introduce the pro-completion of a category.

Definition 3.1.1. Let **C** be a category. A *pro-object* in **C** is a cofiltered limit on **C**. The category **Pro-C**) of all the cofiltered limits of **C** is called the pro-completion of **C**.

The reader is referred to [17] for a proof of the existence of the pro-completion of a category, as well as for the proofs of two upcoming results which imply that we may work in the setting of topological algebras when talking about objects in the pro-completion of a pseudovariety.

Note. For the up-coming discussion, we will fix an algebraic type $\mathcal{F}$ and a variety of $\mathcal{F}$-algebras $\mathcal{V}$. Henceforth, by an "algebra" we mean a member of $\mathcal{V}$. We denote by $\mathcal{V}_f$ the category of its finite members.

Definition 3.1.2. A *topological algebra* is a topological space (X, Ω) such that X is also an algebra whose operations $f : X^n \to X$ are continuous with respect to the product topology on X^n. A topological algebra morphism is a continuous homomorphism between topological algebras. A substructure of a topological algebra is a subspace that is also a subalgebra.

Note. It is sufficient for us to work in the setting of *Hausdorff* topological algebras. For brevity of notation, we will henceforth assume all the topological algebras are Hausdorff.

A Stone-topological algebra is a topological algebra whose underlying topological space is a Stone space. The following two results establish an interesting connection between objects in the pro-completion of a category of finite algebras and Stone-topological algebras.

Proposition 3.1.3. *[17] The pro-completion of the category **Set**$_f$ of finite sets (denoted **Pro-Set**$_f$) is the category **Stone** of stone spaces.*

In light of the previous result, the next should be unsurprising:

Proposition 3.1.4. *[17] The category **Pro-**$\mathcal{V}_f$ is equivalent to a full subcategory of **Stone-**$\mathcal{V}$.*

Remark 3.1.5. For certain algebraic varieties such as monoids, groups and lattices, it so happens that **Pro-**$\mathcal{V}_f$ is exactly equivalent to **Stone-**$\mathcal{V}$, but this is not in general true. There are various characterisations of algebras that satisfy this property, such as [9, section 8].

We will thus treat **Pro-**$\mathcal{V}_f$ as a category of topological algebras, which allows for the following definition:

Definition 3.1.6. A topological algebra is said to be *profinite* if it belongs to **Pro-**$\mathcal{V}_f$, which is to say that it is a cofiltered limit of objects in $\mathcal{V}_f$ considered as topological algebras with the discrete topology.

Note. In the ensuing discussion on profinite structures, "finite algebra" will often be considred shorthand for a finite topological algebra with the discrete topology.

Cofiltered limits of topological algebras can be constructed as substructures of products, so we first recall the product construction.

Proposition 3.1.7. *Let $(X_i)_{i\in I}$ denote an indexed family of topological algebras of type $\mathcal{F}$. We can define operations for each $f \in \mathcal{F}$ on the topological space $X = \prod_{i\in I} X_i$ such that X is also their product as a topological algebra.*

Proof. For an n-ary operational symbol f, we define $f^X : X^n \to X$ as follows. Firstly, consider the diagram below and for each $i \in I$ denote by k_i the unique continuous map that makes the square commute (arising from the universal property of the product X_i^n).

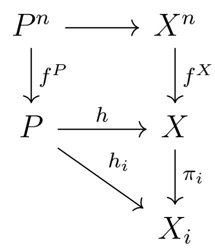
$$\begin{array}{ccc} X^n & \xrightarrow{k_i} & X_i^n \\ \big\downarrow{\scriptstyle \pi_j^{(X^n)}} & & \big\downarrow{\scriptstyle \pi_j^{(X_i^n)}} \\ X & \xrightarrow{\pi_i} & X_i \end{array} \qquad i \in I,\ j \in \{0, 1, ..., n\}.$$

Likewise, define f^X to be the unique continuous map that makes the subsequent diagram commute, arising from the universal property satisfied by X. Commutativity of this diagram also ensures that each of the projections π_i is a homomorphism.

$$\begin{array}{ccc} X^n & \xrightarrow{k_i} & X_i^n \\ \big\downarrow{\scriptstyle f^X} & & \big\downarrow{\scriptstyle f^{X_i}} \\ X & \xrightarrow{\pi_i} & X_i \end{array} \qquad i \in I.$$

Lastly, given a topological algebra P and continuous homomorphisms $h_i : P \to X_i$, we need to show that the continuous map $h : P \to X$ induced by the universal property of X as a product of topologial spaces is an $\mathcal{F}$-algebra homomorphism. This amounts to showing that the upper square in the diagram below commutes, which is to say that for $(a_1, ..., a_n) \in P^n$ we have that $h(f^P(a_1, ..., a_n)) = f^X(h(a_1), ..., h(a_n))$. This follows by composing each side with the projection π_i for each $i \in I$, observing that the bottom triangle commutes and that both π_i and h_i are homomorphisms.

□



The literature on profinite topological algebras usually refers to *inverse* (or *projective*) limits rather than cofiltered limits. It has been shown (for example in [1]) that these terms can be used interchangeably, since each inverse/projective system is a cofiltered diagram and every cofiltered diagram can be composed with a functor from a co-directed (also known as down-directed) poset with the same limit in the codomain category.

Definition 3.1.8. An *inverse system* in a category $\mathbf{C}$ is a contravariant diagram $D : \mathbf{I} \to \mathbf{C}$ such that $\mathbf{I}$ is a directed poset, considered as a category. Equivalently, it is a covariant diagram $D : I \to C$ where I is a co-directed poset. An *inverse limit* is the limit of an inverse system.

Note. For simplicity, we often conflate the diagram with the image of the diagram, denoting an inverse system or directed system as a collection of objects and leaving the morphisms implied.

The dual concept is of use later in this work and we provide the definition here. Note that a *direct limit* is in fact a colimit categorically speaking, but we stick with this nomenclature to keep in step with the algebraic literature.

Definition 3.1.9. A *direct system* in a category C is an inverse system in C^{op}. Likewise, a *direct limit* in a category C is an inverse limit in C^{op}, which is a colimit of a direct system in C.

The following construction of inverse limits of topological algebras gives in particular a concrete description of the objects in $\mathbf{Pro}\text{-}\mathcal{V}_f$ as topological algebras.

Proposition 3.1.10. *Let I be a directed poset and $(X_i)_{i \in I}$ an inverse system of topological algebras. The inverse limit can be constructed as the following subalgebra of the topological algebra product:*

$$\varprojlim (X_i)_{i \in I} = \{(x_i)_{i \in I} \in X \mid \forall i \geq j \quad \phi_{i,j}(x_i) = x_j\},$$

where $X = \prod_{i \in I} X_i$.

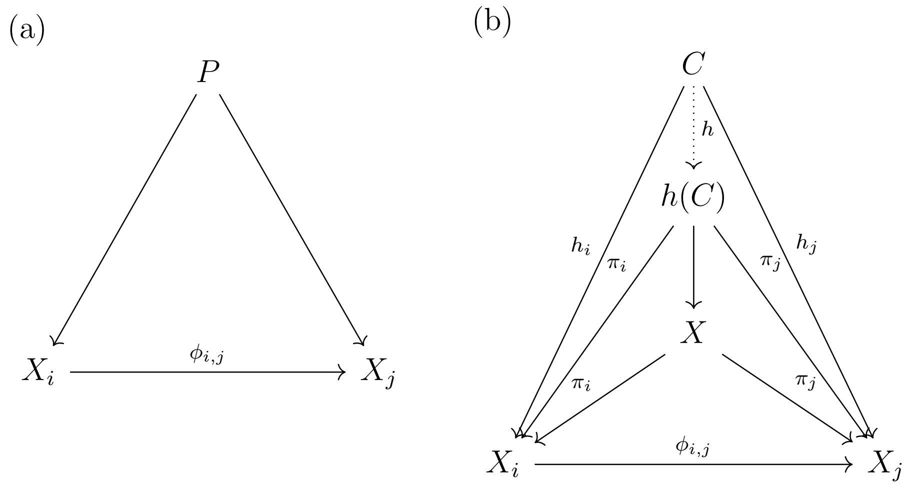
Proof. Let P denote the set above, equipped with the subspace topology. To show that P is also a subalgebra, let $f \in \mathcal{F}$ with arity n and let

$$(x_i^{(1)})_{i\in I}, ..., (x_i^{(n)})_{i\in I}$$

be elements of P. By definition,

$$f^X((x_i^{(1)})_{i\in I}, ..., (x_i^{(n)})_{i\in I}) = (f^{X_i}(x_i^{(1)}, ..., x_i^{(n)}))_{i\in I}.$$

Since for each $i \geq j$ we have that $\phi_{i,j}$ is a homomorphism and $\phi_{i,j}(x_i^{(k)}) = x_j^{(k)}$, it follows readily that $\phi_{i,j}(f^{X_i}(x_i^{(1)}, ..., x_i^{(n)})) = f^{X_j}(x_j^{(1)}, ..., x_j^{(n)})$. Hence, P is a subalgebra. From the definition it follows that P together with the restricted projection morphisms (easily shown to be continuous homomorphisms) give a cone over the inverse system in the sense that for every $i \geq j$, diagram (a)



commutes. To show that this construction is universal, consider any other such cone $(h_i : C \to X_i)_{i\in I}$. Denote by $h : C \to X$ the unique continuous homomorphism into X arising from its universal property as a product. The image of h lies entirely in P, as can be shown by chasing elements of $h(C)$ around diagram (b). Since $\pi_i|_P$ agrees with π_i on elements of $h(C)$, we get the commutativity of the diagram required to assert that P is indeed the limit, considering h as a morphism from C to P.

□

Definition 3.1.11. Inverse limits of an inverse system of finite topological algebras are called *profinite* topological algebras or just profinite algebras. If we fix a pseudovariety $\mathbf{V}$ of finite algebras, the limits of inverse systems of its members are called *pro*-$\mathbf{V}$ algebras.

Remark 3.1.12. From the preceding proof it can be deduced that P is also the limit of the inverse system acquired by taking the projections $(\pi_i(X_i))_{i\in I}$. Hence, we may assume without loss of generality that the projection mappings from a given profinite algebra are surjective.

Remark 3.1.13. The underlying algebra of any profinite algebra belongs to the same variety as the members of the inverse system used to construct it, because it is a subalgebra of a product. For example, we are therefore justified in calling an inverse limit of finite monoids a *profinite monoid*, as Birkhoff's variety theorem ensures that it will satisfy all of the same identities.

3.2 Properties of Profinite Algebras

We now mention a few useful properties of profinite algeras. For this section, we fix a profinite algebra P which is the limit of an inverse system $(X_i)_{i\in I}$ of finite algebras, with $\pi_i : P \to X_i$ denoting the projection morphisms.

Lemma 3.2.1. *The topology of P has a basis consisting of sets of the form $\pi_i^{-1}(V)$ for all open $V \subseteq X_i$.*

Proof. Recall that if we were not to restrict the projection morphisms to P, this collection would form a subbase for $X = \prod_{i\in I} X_i$. It follows that the above-mentioned sets are a subbase for P. The fact that I is directed implies that for every open $U \subseteq P$ and $\bigcap_{i\in F} \pi_i^{-1}(V_i) \subseteq U$ (where $F \subseteq I$ is finite), we can find X_k with $i \leq k$ from which we can construct $V = \bigcap_{i\in F} \phi_{i,k}^{-1}(V_i)$. so that $\pi_k^{-1}(V) = \bigcap_{i\in F} \pi_i^{-1}(V_i) \subseteq U$. □

Proposition 3.2.2. *Profinite algebras are Stone spaces.*

Proof. The inverse limit P is always a closed substructure of $X = \prod_{i\in I} X_i$. To show this, note that for every $x = (x_i)_{i\in I} \notin P$ there exist $i \geq j$ such that $\phi_{i,j}(x_i) \neq (x_j)$. This allows for the construction of a neighborhood of x disjoint from P, namely

$$V_x = \pi_i^{-1}(X_i \setminus \phi_{i,j}^{-1}(\{x_j\})).$$

Profinite algebras are inverse limits where all of the X_is are finite. By Tychonoff's theorem, their product is compact and thus the corresponding subspace is also. As a subspace of a product of Hausdorff spaces, P is again Hausdorff. That P has a basis of clopen sets follows from the previous lemma, as each π_i is a continuous map to a finite topological algebra with the discrete topology, so that every basic open set is clopen. □

Remark 3.2.3. It follows that a pro-$\mathbf{V}$ algebra P is also the limit of the same inverse system in the smaller category of Stone-topological algebras from our fixed algebraic variety.

Lemma 3.2.4. *Let $g : A \to B$ and $h : A \to C$ be topological algebra homomorphisms such that h is surjective and A is compact. If $\sim_h \subseteq \sim_g$, then there exists a continuous homomorphism $k : C \to B$ such that $k \circ h = g$.*

Proof. A homomorphism exists by an analogous result in universal algebra. We show that it is also continuous. To this end, consider a closed subset $X \subseteq B$. From the surjectivity of h, it can be checked that $k^{-1}(X) = h(g^{-1}(X))$. By the compactness of A, $g^{-1}(X)$ is compact because it is closed. By continuity of h, we see that $h(g^{-1}(X))$ is a compact subset of the Hausdorff space C and hence closed. □

Lemma 3.2.5. *Let P be a profinite topological algebra. Every continuous homomorphism $g : P \to T$ to finite algebra T factors through one of the $\pi_i : P \to X_i$.*

Proof. [21] Let $\sim_g$ and $\sim_i$ denote the congruences on P corresponding to g and each π_i. Because we have assumed topological algebras to be Hausdorff, each is a closed subalgebra of $P \times P$. Furthermore, it can be shown that $C := P \times P \setminus \sim_g$ is closed. This is a consequence of the fact that singletons are open in the discrete space T, so that for $(x, y) \in P \times P$ we have that $g^{-1}(g(x)) \times g^{-1}(g(x))$ is an open neighborhood of (x, y) contained in $\sim_g$. Now consider the closed set $\bigcap_{i \in I}(\sim_i)$, which is easily seen to be exactly the diagonal on $P \times P$ so that $C \cap \bigcap_{i \in I}(\sim_i) = \emptyset$. By the compactness of P, we can find a finite subset $F \subseteq I$ such that $C \cap \bigcap_{i \in F}(\sim_i) = \emptyset$. Since I is directed, there exists $j \in I$ which is an upper bound to every $i \in F$. This in turn implies that $\sim_j \subseteq \sim_i$ for every $i \in F$ following from the definition of P, and finally $\sim_j \subseteq \bigcap_{i \in F}(\sim_i) \subseteq \sim_g$. By lemma 3.2.4, we get a continuous homomorphism $g_j : X_i \to T$ such that $g_j \circ \pi_j = g$. □

3.3 Free Pro-V Algebras

We turn now to the construction of the *free profinite algebra* over a finite set (which we sometimes refer to as an alphabet to maintain the connection with formal language theory). More generally, we construct a free Pro-**V** algebra for some pseudovariety **V** of finite $\mathcal{V}$-algebras, which we now fix. Finite sets are all that is needed to present the upcoming Birkhoff-like theorem, but note that this can be generalised to free profinite algebras over *profinite* sets ([21]).

Definition 3.3.1. Let A be a finite set. Consider the category of all set morphisms $\sigma : A \to X_\sigma$ from A to A-generated members of **V**, so that the subalgebra generated by $\sigma(A)$ is exactly X_σ. Let the morphisms of this category be the algebra homomorphisms $\phi_{\sigma,\tau} : X_\sigma \to X_\tau$ for which $\tau = \phi_{\sigma,\tau} \circ \sigma$. Any two homomorphisms for which this holds must be equal, for they will agree on the generators of X_σ. Hence, this category is a preorder. It is also co-directed, with pairwise lower bounds given as follows: For any pair $\sigma : A \to X_\sigma$ and $\tau : A \to X_\tau$, there is a set map $\theta : A \to X_\sigma \times X_\tau$ given by the restriction to A of the homomorphism $\overline{\theta} : F_A(\mathcal{V}) \to X_\sigma \times X_\tau$ which arises from the universal property of the product construction. The image of this map is the required A-generated finite algebra.

By taking the subcategory of only those X_σ which are exactly the distinct finite quotients by a congruence of $F_A(\mathcal{V})$ (which we can always get by quotienting by the kernel of $\overline{\sigma}$), we obtain a skeletal subcategory which we denote $\mathbf{V}_A$ and thus we effectively have a co-directed poset.

The diagram sending $\sigma : A \to X_\sigma$ to X_σ and $\phi_{\sigma,\tau}$ to itself is therefore an inverse system, roughly illustrated below:

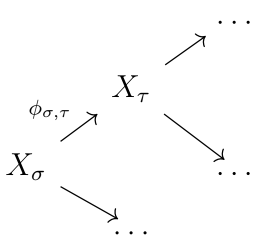
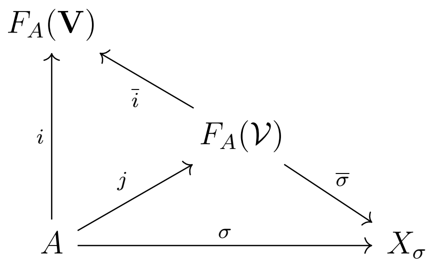


Figure 3.1: The constructed inverse system of finite topological monoids.

We define $\hat{F}_A(\mathbf{V})$ to be the limit of this diagram in the category of topological algebras, and refer to it as the free pro-$\mathbf{V}$ algebra over A. When $\mathbf{V} = \mathcal{V}_f$, we call it the free profinite algebra over A. We denote by ρ_σ the respective projections of this limit.

Proposition 3.3.2. *The free algebra $F_A(\mathcal{V})$ is isomorphic to a dense subalgebra of the free pro-$\mathbf{V}$ algebra $\hat{F}_A(\mathbf{V})$.*

Proof. Let $i : A \to \hat{F}_A(\mathbf{V})$ denote the map defined by $a \mapsto (\sigma(a))_\sigma$. If $\mathbf{V}$ is non-trivial, there exists an algebra of size greater than that of A. Hence it is possible to find an injective map to an algebra in $\mathbf{V}$, and the map from A to the subalgebra generated by this map is (up to isomorphism) in $\mathbf{V}_A$. Injectivity of this map implies that i is also injective. By the universal property of the free algebra $F_A(\mathcal{V})$, there exists a homomorphism $\overline{i} : F_A(\mathcal{V}) \to \hat{F}_A(\mathbf{V})$ which uniquely extends i in the manner that $\overline{i} \circ j = i$, where j is the usual inclusion map from A into $F_A(\mathcal{V})$. It can be seen that $\overline{i}$ is a dense embedding. That it is injective follows because both i and j are, and because $\overline{i}$ is a homomorphism. To show that it is dense, consider $x \in \hat{F}_A(\mathbf{V})$ and any open U containing x. From the description of the basis in 3.2.1, for some $\sigma : A \to X_\sigma$ there is a basic open set $\rho_\sigma^{-1}(V)$ for some open $V \subseteq X_\sigma$, with $x \in \rho_\sigma^{-1}(V)$. For this σ, we get the usual unique homomorphic extension $\overline{\sigma} : F_A(\mathcal{V}) \to X_\sigma$. We can see that



$\rho_\sigma \circ \bar{i} = \overline{\sigma}$ by the commutativity of the rest of the below diagram (considered in **Set**), as the two homomorphisms agree on the generators of $F_A(\mathcal{V})$.

Since X_σ is A-generated, $\overline{\sigma}$ is surjective and thus there exists $a \in F_A(\mathcal{V})$ such that $\overline{\sigma}(a) = \pi_\sigma(x)$. From this it follows that $\rho_\sigma \circ \bar{i}(a) = \overline{\sigma}(a) = \rho_\sigma(x)$, so that $\bar{i}(a) \in \rho_\sigma^{-1}(\pi_\sigma(x))$. Taking note that $\rho_\sigma^{-1}(\rho_\sigma(x)) \subseteq \rho_\sigma^{-1}(V)$, this completes the proof that $\bar{i}(F_A(\mathcal{V})) \cap \rho_\sigma^{-1}(V) \neq \emptyset$, so that $\bar{i}(F_A(\mathcal{V})$ is dense in $\hat{F}_A(\mathbf{V})$. □

Lemma 3.3.3. *The projection morphisms $\rho_\sigma : \hat{F}_A(\mathbf{V}) \to X_\sigma$ are always surjective.*

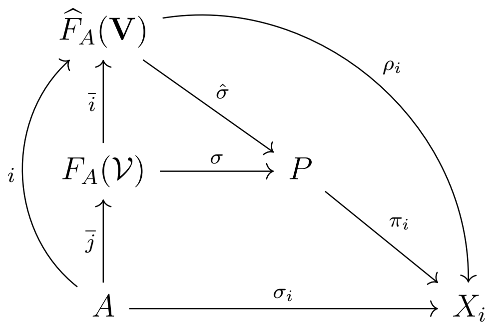
Proof. For $\sigma : A \to X_\sigma$ in $\mathbf{V}_A$, by the definition of i as above it is clear that $\rho_\sigma \circ i = \hat{\sigma}$. The homomorphism $\overline{\sigma} = \rho_\sigma \circ \bar{i}$ as they agree on generators. Then ρ_σ is surjective because $\rho_\sigma \circ \bar{i}$ is. □

The free pro-**V** algebra on a set satisfies the following property, which states that it is universal among A-generated pro-**V** algebras. Note first how the definition of being generated by a finite set differs for topological algebras.

Definition 3.3.4. A topological algebra X is said to be *generated* by a finite set A if there exists a homomorphism $\sigma : F_A(\mathcal{V}) \to X$ such that the image of σ is topologically dense in X.

Proposition 3.3.5. *Let $\sigma : F_A(\mathcal{V}) \to P$ be a homomorphism generating a pro-**V** algebra P. Then there exists a unique surjective continuous homomorphism $\hat{\sigma} : \hat{F}_A(\mathbf{V}) \to P$ such that such that $\hat{\sigma} \circ \bar{i} = \sigma$.*

Proof. The following diagram is included for clarity, but its commutativity will be the result of this proof.



By definition, P is the limit of some inverse system $(X_i)_{i\in I}$ of finite algebras in $\mathbf{V}$. Let $\sigma_i : F_A(\mathcal{V}) \to X_i$ denote $\pi_i \circ \sigma$. This map is surjective, as can be seen by recalling that we may assume π_i to be surjective and observing that for each $x \in P$, we have that $\pi_i^{-1}(\pi_i(x)) \cap \sigma(F_A(\mathcal{V})) \neq \emptyset$ by the density of the image of σ in P. Hence, each X_i is an A-generated algbera so that $\sigma_i : A \to X_i$ is (up to isomorphism) in $\mathbf{V}_A$, as described in the above construction of $\hat{F}_A(\mathbf{V})$. We therefore have access to the canonical projections ρ_{σ_i} (which we denote here by ρ_i). These projections constitute a cone over $(X_i)_{i\in I}$, because the latter is necessarily (again up to isomorphism) a subdiagram of the inverse system of which $\hat{F}_A(\mathbf{V})$ is the limit. The universal property of P thus gives us a topological algebra morphism which we call $\hat{\sigma} : \hat{F}_A(\mathbf{V}) \to P$. It remains to show that $\hat{\sigma}$ extends σ in the sense that $\hat{\sigma} \circ \bar{i}$ and is unique in doing so. The latter is immediate because P is Hausdorff and any second continuous homomorphism satisfying the same equation will agree with $\hat{\sigma}$ on the dense subset $\bar{i}(F_A(\mathcal{V}))$. To show that it extends σ, it suffices to show that $\pi_i \circ \hat{\sigma} \circ \bar{i} = \pi_i \circ \sigma$, equivalently that $\rho_i \circ \bar{i} = \sigma_i$. It is enough to show that both sides agree on the generators of $F_A(\mathcal{V})$, and we may observe that $\rho_i \circ \bar{i} \circ j = \sigma_i \circ j$ by the definition of $i = \bar{i} \circ j$ in the preceding result.

□

Corollary 3.3.6. *Any homomorphism $\sigma : F_A(\mathcal{V}) \to T$ to a finite algebra $T \in \mathbf{V}$ extends uniquely to a topological algebra morphism $\hat{\sigma} : \hat{F}_A(\mathbf{V}) \to T$ such that $i \circ \hat{\sigma} = \sigma$.*

Proof. The subalgebra of T generated by σ is trivially an A-generated profinite algebra (when considered as a topological algebra). □

This can be considered an intermediate step to showing that $\hat{F}_A(\mathbf{V})$ is the free pro-$\mathbf{V}$ algebra over A, which we now deduce:

Theorem 3.3.7. *Any homomorphism $\sigma : F_A(\mathcal{V}) \to S$ to a pro-$\mathbf{V}$ algebra S extends uniquely to a topological algebra morphism $\hat{\sigma} : \hat{F}_A(\mathbf{V}) \to S$ such that $i \circ \hat{\sigma} = \sigma$.*

Proof. To show that this follows from proposition 3.3.5, we demonstrate that S has an A-generated topological subalgebra which is itself pro-$\mathbf{V}$. Let $(X_i)_{i\in I}$ denote the inverse system of finite algebras in V of which S is the limit. Let $P_i = \pi_i \circ \sigma(F_A(\mathcal{V}))$ for each $i \in I$. The homomorphisms $\phi_{i,j|_{P_i}}$ have their range in P_j, so that the $(P_i)_{i\in I}$ with the $\phi_{i,j|_{P_i}}$ constitute a new inverse system of algebras in $\mathbf{V}$ indexed by I. Let P denote its limit. We may treat elements of P as elements of the product of the $(X_i)_{i\in I}$. Projecting these elements, considering them included in each X_i and then chasing them along the $\phi_{i,j}$ we see that they satisfy the required condition to lie in S.

The topology of P as a profinite algebra coincides with the subspace topology because each open $U \subseteq P_i$ is equal to $V \cap P_i$ for some open $V \subseteq X_i$ and thus $\pi_i^{-1}(V) \cap P_i = \pi_i^{-1}(U)$, so that the profinite and subspace topology have the same basis of open sets. Lastly, we show that σ is dense in P, so that P is an A-generated pro-$\mathbf{V}$ algebra.

Firstly, note that the image of σ in fact lies in P since each $\sigma(w)$ for $w \in F_A(\mathcal{V})$ is of the form $(\pi_i \circ \sigma(w))_{i\in I}$ and we already have that $\pi_i \circ \phi_{i,j|_{P_i}} \circ \sigma(w) = \pi_j \circ \sigma(w)$.

Next, recall that that each $x \in P$ lies in some basic open set $\pi_i^{-1}(U)$ for some open $U \subseteq P_i$. It can be shown that $\pi_i^{-1}(U)$ meets $\sigma(F_A(\mathcal{V}))$ by observing that $\pi_i(x) = \sigma(w)$ for some $w \in \sigma(F_A(\mathcal{V}))$ and that $\pi_i^{-1}(U)$ will contain this $\sigma(w)$. The result follows by composing the $\hat{\sigma}$ we obtain from proposition 3.3.5 with the inclusion of P into S. The remaining details follow easily from the analogous properties of $\hat{\sigma} : \hat{F}_A(\mathbf{V}) \to P$. □

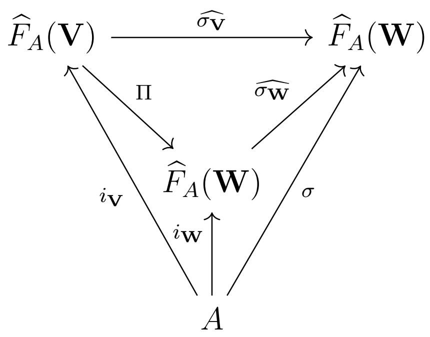
This universal property and its consequences are the cornerstones of Reiterman's theorem. The following corollary clarifies how the free pro-$\mathbf{V}$ algebras behave with respect to subpseudovarieties and substitution of variables. We also obtain a lemma which characterises members of $\mathbf{V}$ as quotients of free pro-$\mathbf{V}$ algebras. These results are reminiscent of the machinery required for Birkhoff's variety theorem.

Corollary 3.3.8. *Let $g : A \to B$ be a map between finite sets, and $\mathbf{W} \subseteq \mathbf{V}$ a pseudovariety. Denote by i_A and i_B the respective natural inclusions of A and B into the free pro-$\mathbf{V}$ and pro-$\mathbf{W}$ algebras and let $\sigma = i_B \circ g$. Then there exists a topological algebra morphism $\hat{\sigma} : \hat{F}_A(\mathbf{V}) \to \hat{F}_B(\mathbf{W})$ for which $i_B \circ g = \widehat{\sigma} \circ i_A$.*

Proof. All we need is to observe that $\hat{F}_B(\mathbf{W})$ is pro-$\mathbf{V}$ and apply proposition 3.3.7. □

Note. Of particular interest is the case where g is the identity from A to itself. We then use the notation Π in place of $\hat{\sigma}$ and refer to this as the *canonical projection* of $\hat{F}_A(\mathbf{V})$ onto $\hat{F}_A(\mathbf{W})$. Because the latter is an A-generated topological algebra, Π is surjective so that we may see $\hat{F}_A(\mathbf{W})$ as a topological algebra quotient of $\hat{F}_A(\mathbf{V})$.

Lemma 3.3.9. *Let $\sigma : A \to \hat{F}_A(\mathbf{W})$ be a map, where $\mathbf{W} \subseteq \mathbf{V}$ is a pseudovariety. Then the two extensions $\hat{\sigma}_{\mathbf{V}} : \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{W})$ and $\hat{\sigma}_{\mathbf{W}} : \hat{F}_A(\mathbf{W}) \to \hat{F}_A(\mathbf{W})$ make the following diagram commute:*



Proof. The result follows because $\hat{\sigma}_{\mathbf{W}} \circ \Pi \circ i_{\mathbf{V}} = \hat{\sigma}_{\mathbf{W}} \circ i_{\mathbf{W}} = \sigma$, so that $\hat{\sigma}_{\mathbf{W}} \circ \Pi = \hat{\sigma}_{\mathbf{V}}$ by the uniqueness of the latter extension. □

Lemma 3.3.10. *A finite algebra $T \in \mathcal{V}$ lies in $\mathbf{V}$ if and only if it is the continuous homomorphic image of a free pro-$\mathbf{V}$ algebra on some finite set.*

Proof. Assume that T is a member of $\mathbf{V}$. Because it is finite, it is generated by some $\sigma : A \to T$ and the forward direction follows immediately by corollary 3.3.6. To show the converse, let $g : \hat{F}_A(\mathbf{V}) \to T$ be a surjective continuous homomorphism. By lemma 3.2.5, g factors through some A-generated member of V, say X_σ. Namely, there exists a homomorphism $g_\sigma : X_\sigma \to T$ so that $g_\sigma \circ \rho_\sigma = g$. Now g_σ is surjective because both g and ρ_σ are, so the result follows by closure of $\mathbf{V}$ under homomorphic images. □

3.4 Profinite Terms, Pseudoidentities and Reiterman's Thereom

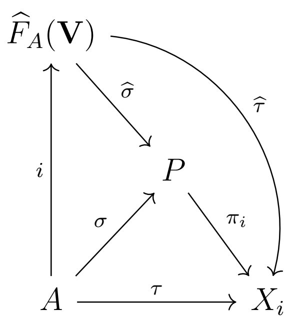
The universal property of $\hat{F}_A(\mathbf{V})$ with respect to finite algebras in $\mathbf{V}$ gives us a way to "interpret" elements of $\hat{F}_A(\mathbf{V})$ for every assignment of the variables in A to any algebra in $\mathbf{V}$. Recall from universal algebra that each element of $F_A(\mathcal{V})$ corresponds to a natural transformation which essentially gives an operation $u : M^n \to M$ for each $M \in \mathcal{V}$, where $n = |A|$. Not all such operations arise in this manner, only those obtained by some finite application of operations $f \in \mathcal{F}$. These, which correspond one-to-one with elements of $F_A(\mathcal{V})$, are called explicit operations. The larger collection is that of *implicit operations*, and it so happens that for each set of variables A these are in bijection with with elements of $\hat{F}_A(\mathbf{V})$ via our schema for interpreting its elements in $\mathbf{V}$-algebras. This means we can treat free profinite algebras as generalised term algebras, and its elements generalised terms.

Definition 3.4.1. A pro-$\mathbf{V}$ term over a finite set A is an element u of $\hat{F}_A(\mathbf{V})$. The *interpretation* of u in any pro-$\mathbf{V}$ algebra P given an interpretation $\sigma : A \to P$ of the variables is the element $\hat{\sigma}(u)$.

Definition 3.4.2. A pseudoidentity on A is a pair (u, v) where $u, v \in \hat{F}_A(\mathbf{V})$. An algebra $T \in \mathbf{V}$ *satisfies* this pseudoidentity if $\hat{\sigma}(u) = \hat{\sigma}(v)$ for every $\sigma : A \to T$, in which case we say that $T \models u = v$. For a pseudovariety $\mathbf{W} \subseteq \mathbf{V}$, we say that $\mathbf{W} \models u = v$ if $T \models u = v$ for every $T \in \mathbf{W}$. If Σ is a set of pseudoidentites over various alphabets, we say that $T \models \Sigma$ iff T models each of those identities arising from $\hat{F}_A(\mathbf{V})$ for every finite set A.

Proposition 3.4.3. *Let P be a limit of an inverse system $(X_i)_{i \in I}$ of finite members of $\mathbf{V}$ and let (u, v) be a pseudoidentity. Then $P \models u = v$ if and only if each $X_i \models u = v$.*

Proof. Suppose that $P \models u = v$ and let $\tau : A \to X_i$. Define $\sigma : A \to P$ by choosing $\sigma(a)$ to be any element of $\pi_i^{-1}(\tau(a))$, so that $\tau = \pi_i \circ \sigma$. Then the diagram below commutes by uniqueness of the map extending τ. From this it follows that $\hat{\tau}(u) = \widehat{\tau(v)}$.



To show the converse, suppose each $X_i \models u = v$ and let $\sigma : A \to P$ be a map. Then $\pi_i \circ \sigma$ is an interpretation of the elements of A in X_i, and the extension $\widehat{\pi_i \circ \sigma}$ is equal to $\hat{\sigma} \circ \pi_i$. The result follows since each $\hat{\sigma}(u)$ is of the form $(\pi_i \circ \hat{\sigma}(u))_{i \in I}$. □

We now fix a pseudovariety $\mathbf{W} \subseteq \mathbf{V}$. Let Σ denote the set of all the pseudoidentities (u, v) on $\mathbf{V}$ such that $\mathbf{W} \models u = v$. We refer as Σ_A to the subset of Σ consisting only of those pseudoidentities which are pairs of elements of $\hat{F}_A(\mathbf{V})$ for a given set A of variables.

Proposition 3.4.4. *Let (u, v) be a pseudoidentity on A. Then $(u, v) \in \Sigma_A$ if and only if $\Pi_A(u) = \Pi_A(v)$, where $\Pi_A : \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{W})$ is the canonical projection as described in the note following corollary 3.3.8.*

Proof. Suppose that $(u, v) \in \Sigma_A$. By the previous result, $\hat{F}_A(\mathbf{W}) \models u = v$ as it is a pro-**V** algebra which is also pro-**W**. Recall that Π_A is the extension of $i : A \to \hat{F}_A(\mathbf{W})$, hence $\Pi_A(u) = \Pi_A(v)$. On the other hand, if we assume that $\Pi_A(u) = \Pi_A(v)$ and let $\sigma : A \to \hat{F}_A(\mathbf{W})$ be any map. By lemma 3.3.9, the extension $\hat{\sigma}_{\mathbf{V}} = \hat{\sigma}_{\mathbf{W}} \circ \Pi_A$, so we may conclude that $\hat{F}_A(\mathbf{W}) \models u = v$. □

Corollary 3.4.5. *Each $\Sigma_A = \sim_{\Pi_A}$, so that Σ_A is a congruence on $\hat{F}_A(\mathbf{V})$.*

Theorem 3.4.6 (Reiterman)**.** *A collection of finite algebras $\mathbf{W} \subseteq \mathbf{V}$ is a subpseudovariety if and only if it is equal to the collection of all models for some set of pseudoidentities on $\mathbf{V}$.*

Proof. Suppose that **W** is a pseudovariety. Let Σ denote all pseudoidentities on **V** modeled by every algebra in **W**. Denote by $[[\Sigma]]$ the set of all models of Σ. Clearly $W \subseteq [[\Sigma]]$. To show equality, let $T \in \mathbf{V}$ be such that $T \models \Sigma$. Since $T \in \mathbf{V}$, by lemma 3.3.10 it is a continuous homomorphic quotient of $\hat{F}_A(\mathbf{V})$ and we let $\hat{\sigma} : \hat{F}_A(\mathbf{V}) \to T$ denote this quotient map. Since **W** is a pseudovariety, it is known that $\sim_{\Pi_A} = \Sigma_A$ by proposition 3.4.4, and we may deduce that $\sim_{\Pi_A} \subseteq \sim_{\hat{\sigma}}$ since we took T to be in $[[\Sigma]]$. By lemma 3.2.4, this means we have an induced continuous homomorphism $q : \hat{F}_A(\mathbf{W}) \to T$ such that $q \circ \Pi_A = \hat{\sigma}$ which is necessarily surjective because $\hat{\sigma}$ and Π_A are. The converse is straightforward to prove. □

Definition 3.4.7. A collection Σ of pseudoidentities is called a *profinite theory* if it is the set of pseudoidentities modelled by some pseudovariety.

Profinite theories have been characterised, for example in [2], as described in the theorem below (which we state in the case of monoids). We omit the proof, which takes a lengthy detour through several of the alternative characterisations of pseudoidentities via nets and filter congruences.

Theorem 3.4.8. *A collection Σ of pseudoidentities is a profinite theory if and only if:*

1. *Each Σ_A is a congruence on $\widehat{F}_A(\mathbf{V})$.*

2. *For each $(u, v) \notin \Sigma_A$ there exists a clopen set U which is a union of congruence classes such that $u \in U$ and $v \notin U$.*

3. *Σ is* closed under substitution*, in the sense that for any two alphabets A, B and any monoid homomorphism $h : A^* \to B^*$ (which, when composed with the inclusion of B^* into $\widehat{F}_B(\mathbf{V})$, yields a continuous homomorphism $\widehat{h} : \widehat{F}_A(\mathbf{V}) \to \widehat{F}_B(\mathbf{V})$ as in lemma 3.3.8), if $(u, v) \in \Sigma_A$ then $(\widehat{h}(u), \widehat{h}(v)) \in \Sigma_B$.*

Chapter 4

A Duality-Theoretic Perspective

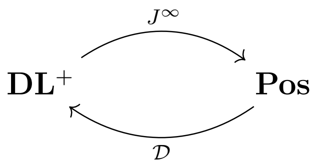
The composition of Eilenberg and Reitermans' theories yields an equational characterisation of languages via their recognising monoids. In light of the observed duality-theoretic connection that follows, the reliance on syntactic structures can be bypassed and we may see Eilenberg's language varieties as model classes for systems of profinite equations in a more direct way. This is the subject of this chapter and based on work done by Gehrke, Grigorieff and Pin in [14] and [13]. It relies on a technical excursion into duality theory, to which we devote a large portion of this chapter. The duality theories we discuss feature Galois correspondences between certain subobjects and quotients. These are central to this work and we give independent proofs for these. Utilizing these connections and the result in [13] that the residuated Boolean algebra of recognisable languages for a given alphabet is dual to its free profinite monoid, we demonstrate the key idea of [13] that that language varieties and profinite theories are at opposite ends of a dual equivalence of categories. Their correspondence is an instance of translation between subobjects and quotients in dually equivalent categories, and can be understood as a correspondence between profinite equational theories and their language-based models.

4.1 Discrete Dualities

There are several famous categorical dual equivalences for various classes of lattices and Boolean algebras. In some cases, the dual categories are topological in nature. In simpler cases, the duals are non-topological categories such as **Set** and **Pos**; we refer to these as *discrete* dualities, because they can be seen as a restriction of the topological dualities to the cases where the duals have the discrete topology. It is not our mission here to provide an exhaustive catalogue of lattice dualities, but there is much benefit to introducing duality-theoretic concepts in the discrete context, since many important intuitions could be missed by jumping into the complexity of the topological case. Thus we spend some time developing discrete analogs of the theory we will use in the topological case.

The starting point of discrete duality is two famous and well-known observations: firstly, the fact that every finite Boolean algebra is isomorphic to the power set of its atoms. Secondly, Birkhoff's representation theorem, which observes that every finite distributive lattice can be represented by a finite poset; every element is equal to the join of the join-irreducible elements below it, and thus the lattice can be recovered from the poset of its join-irreducible elements. We choose to give a summary of a more general result, namely the dual equivalence between the category $\mathbf{DL}^+$ (whose members are complete and completely distributive lattices whose completely join-irreducible elements are completely join dense) and the category **Pos** of partially ordered sets or *posets*. DL^+ lattices are in a sense those lattices that behave rather like finite lattices, at least enough so that it is possible to establish an analogous discrete duality.

Theorem 4.1.1. *The category* $\mathbf{DL}^+$ *is dually equivalent to the category* **Pos**, *with the relevent functors and natural isomorphisms given as follows:*



J^∞	$C \mapsto (J^\infty(C), \leq_C)$	*(Completely join-irreducibles)*
$\mathcal{D}$	$X \mapsto \mathcal{D}(X)$	*(Down-closed sub-sets)*
$J^\infty(C \xrightarrow{h} D)$	$J^\infty(D) \xrightarrow{h^\flat} J^\infty(C)$	*(Lower adjoint of h)*
$\mathcal{D}(X \xrightarrow{g} Y)$	$\mathcal{D}(Y) \xrightarrow{g^{-1}} \mathcal{D}(X)$	*(Preimage)*
$\eta_A : A \to \mathcal{D}(J^\infty(A))$	$a \mapsto J^\infty(A) \cap \downarrow a$	
$\eta_A^{-1} : \mathcal{D}(J^\infty(A)) \to A$	$K \mapsto \bigvee(K)$	
$\mu_X : X \to J^\infty(\mathcal{D}(X))$	$x \mapsto \downarrow x$	
$\mu_X^{-1} : J^\infty(\mathcal{D}(X)) \to X$	$M \mapsto \bigvee M$	

Although we will not provide proofs of the various components of this duality, we will include some technical results which are useful later. In particular, we prove the fact that the dual of a complete injective lattice homomorphism $h : C \to D$ between two DL^+ lattices, namely the lower adjoint $h^\flat$, turns out to be a surjective poset morphism $h^\flat : J^\infty(D) \to J^\infty(C)$.

Lemma 4.1.2. *Let C and D be complete and completely distributive lattices. For a complete lattice homomorphism $h : C \to D$ with upper and lower adjoints $h^\flat$ and $h^\sharp$, the lower adjoint $h^\flat$ maps completely join-irreducible elements of D to completely join-irreducible elements of C.*

Proof. Because h has an both upper and lower adjoints, it preserves arbitrary joins and meets. If $d \in D$ is completely join-irreducible, we wish to show that $h^\flat(d)$ is completely join-irreducible. To this end, suppose $h^\flat(d) = \bigvee S$ for a subset $S \subseteq C$. Then

$$hh^\flat(d) = h(\bigvee(S)) = \bigvee(h(S)).$$

Now $d \leq hh^\flat(d))$ by lemma A.1.7, so that $d \leq \bigvee(h(S)$. Since d was assumed to be completely join-irreducible, $d \leq h(s)$ for some $s \in S$. By the Galois property, $h^\flat(d) \leq s$. Since being completely join prime is equivalent to being completely join-irreducible in completely distributive lattices, this completes the proof. □

Lemma 4.1.3. *Let $i : C \to D$ be an inclusion morphism in* $\mathbf{DL}^+$*. The dual morphism $i^\flat : J^\infty(D) \to J^\infty(C)$ is surjective.* [1]

Proof. Let $a \in J^\infty(C)$. Then $a \in D$ and $a = \bigvee K$ for some $K \subseteq J^\infty(D)$. Recall that for $x \in D$, $i^\flat(x) = \bigwedge i^{-1}(\uparrow x)$ (the least element of C that is above x). Hence it must be that $i^\flat(a) = a$. Because $i^\flat$ has an upper adjoint, it is completely join-preserving. Hence,

$$\begin{aligned} a &= i^\flat(a) \\ &= i^\flat(\bigvee(K)) \\ &= \bigvee(i^\flat(K)). \end{aligned}$$

Given that a is a completely join-irreducible element of C, $a = i^\flat(k)$ for some $k \in K$. □

This allows us to prove another useful technical result which we apply liberally from this point forward:

Lemma 4.1.4. *Let D be a DL^+ and let C be a complete sublattice of D. Then C is itself a DL^+.*

[1] This is easily deduced if we note that complete lattice embeddings are the monomorphisms in $\mathbf{DL}^+$ and surjective poset morphisms are the epimorphisms in **Pos**, given that dual equivalences send monomorphisms to epimorphisms and vice versa. However, to avoid routinely characterising epimorphisms and monomorphisms in every category we work with, we will sometimes opt to prove these kind of properties explicitly for dual maps.

Proof. By discrete duality for DL^+ lattices, D is isomorphic to $\mathcal{D}(X)$, where $X = J^\infty(D)$. Without loss of generality, we may consider C a sublattice of $\mathcal{D}(X) = D$. Let $i : C \to \mathcal{D}(X)$ denote the inclusion map. This is a complete lattice morphism, so that i has both a lower and an upper adjoint. Further more, by lemmas 4.1.2 and 4.1.3, it follows that its lower adjoint is a surjective poset morphism $i^\flat : J^\infty(D) \to J^\infty(C)$ (notice that $J^\infty(C)$ cannot be empty because $J^\infty(D)$ is not). We aim to show that any $A \in C$ is a join of completely join-irreducible elements of C; all other relevant DL^+ properties are inherited from D. It turns out that $A = \bigcup\{i^\flat(\downarrow x) \mid x \in A\}$. Firstly, $x \in A$ implies $\downarrow x \subseteq A$, and we know that $i^\flat(\downarrow x)$ is the least element of C containing $\downarrow x$. Hence $i^\flat(\downarrow x) \subseteq A$ for every $x \in A$. If some $B \in C$ satisfies $i^\flat(\downarrow x) \subseteq B$ for every $x \in A$, then $\downarrow x \subseteq B$ for all such x and thus $A \subseteq B$, since $A = \bigcup\{\downarrow x \mid x \in A\}$ (as is true for every element of $\mathcal{D}(X)$). □

4.1.1 Other Birkhoff-Style Discrete Dualities

Complete and completely atomic Boolean algebras are exactly whose who are isomorphic to the power set of some set. The duality between CABAs and sets can be recovered from the duality for DL^+ lattices by observing that every CABA can be seen as a DL^+ which happens to have a complement for every element, and that DL^+ preserve complements. The completely join-irreducible elements of a CABA are exactly its atoms, and the ordering on atoms is discrete so that the dual poset is in fact just a set. The lattice of down-closed sets for a poset with the discrete ordering is just the power set of the underlying set. Hence, the duality between sets and CABAs, with its functors respectively collecting atoms and taking the power set (and acting the same way on morphisms), can be seen as a restriction of the DL^+ duality to CABAs and sets. We note also that finite lattices and finite Boolean algebras are DL^+ lattices and CABAs respectively, so Birkhoff's representation theorem and the duality for finite Boolean algebras fall within the scope of theorem 4.1.1 as well.

4.1.2 Galois Correspondence

Given any set, there is a lattice isomorphism between the complete lattice of equivalence relations on that set and the complete lattice of complete Boolean subalgebras of the power set. This elegant connection can be seen as the correspondence between the lattices of Galois closed sets of a Galois connection at opposite ends of the categorical duality between sets and CABAs. Gehrke [13] shows that an analogous connection exists in the case of extended Stone duality. We will end up presenting this result in detail, as it is central to the duality-theoretic approach resulting in the profinite equational theory for varieties of recognisable languages.

We first demonstrate this correspondence for the case of DL$^+$ lattices and note that CABAs and equivalence relations are a restricted case. We show in particular that we obtain a polarity whose respective Galois-closed sets are complete sublattices and poset quotients. It is useful to first note that poset quotients correspond one-to-one with preorders on the domain which extend the partial order, as the Galois connection we now discuss is given in terms of preorders on posets rather than poset quotients. However, the translation to quotients allows for the easy application of discrete duality theory when proving the following results.

Lemma 4.1.5. *Any poset quotient $q : (X, \leq) \to (Y, \leq)$ gives rise to a preorder Δ on X which extends the partial order on X, and vice versa. These may be recovered from each other in a one-to-one fashion.*

Proof. Given a preorder Δ on $(X, \leq)$, we may construct an equivalence relation on X out of the preorder Δ by setting $x \sim y \iff ((x, y) \in \Delta$ and $(y, x) \in \Delta)$. Let q_Δ denote the quotient map and let $(X/\Delta, \preceq)$ denote the quotient set with the partial order given by $q_\Delta(x) \preceq q_\Delta(y) \iff x\Delta y$. It is immediate from the definition that this is well-defined and that the quotient map q_Δ is a poset morphism.

On the other hand, given an order-preserving quotient map $q : X \to Y$, we may define a preorder Δ_q on X by setting $(x, y) \in \Delta_q$ if and only if $q(x) \leq q(y)$ in Y. This clearly extends the partial order on X, since q is order-preserving. It is reflexive and transitive because the partial order on Y is, but antisymmetry is not inherited in this way.

Lastly, for any $x, y \in X$:

$$(x, y) \in \Delta_{q_\Delta} \iff q_\Delta(x) \preceq q_\Delta(y) \iff (x, y) \in \Delta$$

and

$$q_{\Delta_q}(x) \preceq q_{\Delta_q}(y) \iff (x, y) \in \Delta_q \iff q(x) \leq q(y).$$

It follows that these quotients and preorders are in a one-to-one correspondence and that X/Δ is order-isomorphic to Y. $\square$

Let $(X, \leq)$ be a poset and $\mathcal{D}(X)$ the DL$^+$ of its down-closed subsets.

With this in mind, we construct a polarity as follows: Let R be the relation on $\mathcal{D}(X) \times (X \times X)$ given by $A\,R\,(x, x') \iff (x' \in A \Rightarrow x \in A)$. The polarity $(\mathcal{D}(X), X \times X, R)$ yields the standard antitone Galois connection

$$\mathcal{E} : \mathcal{P}(\mathcal{D}(X)) \rightleftarrows \mathcal{P}(X \times X) : \mathcal{S}.$$

Recall that for any $U \subseteq \mathcal{D}(X)$ and $V \subseteq X \times X$,

$$\mathcal{S}(V) = \{A \in \mathcal{D}(X) \mid \forall_{(x,x') \in V}(x' \in A \Rightarrow x \in A)\}$$

and

$$\mathcal{E}(U) = \{(x, x') \in X \times X \mid \forall_{A \in U}(x' \in A \Rightarrow x \in A)\}.$$

The following properties are useful for categorising the galois-closed sets as complete sublattices and preorders extending the order of the relevant poset.

Lemma 4.1.6. *For any $U \subseteq \mathcal{D}(X)$, $\mathcal{E}(U)$ is a preorder on X which extends its partial order. For any $V \subseteq X \times X$, $\mathcal{S}(V)$ is a complete sublattice of $\mathcal{D}(X)$.*

Lemma 4.1.7. *Let Δ be a preorder on X which extends its partial order, and let $q : X \to X/\Delta$ denote the corresponding quotient map as given in 4.1.5. The range of the dual (preimage) map $q^{-1} : \mathcal{D}(X/\Delta) \to \mathcal{D}(X)$ is exactly the sublattice $\mathcal{S}(\Delta)$.*

Proof. Any $A = q^{-1}[M]$ for some $M \in \mathcal{D}(X/\Delta)$ belongs to $S(\Delta)$: for any $(y, y') \in \Delta$, if it is given that $y' \in A$, then $q(y') \in M$. By the definition of the partial order on X/Δ, $q(y) \preceq q(y')$ and thus $q(y) \in M$ by the down-closure of M, so that $y \in A$. On the other hand, any $A \in S(\Delta)$ satisfies $A = q^{-1}[q[A]]$: any $q(y) \in q(A)$ is equal to $q(y')$ for some $y' \in A$, giving a pair (y, y') such that $(y, y') \in \Delta$ and $(y', y) \in \Delta$. Following the definition of $S(\Delta)$, it must be that $y \in A$ as well. □

Lemma 4.1.8. *Let C be a complete sublattice of $\mathcal{D}(X)$. Let $i : C \to \mathcal{D}(X)$ denote the inclusion map. Its dual quotient, namely $i^\flat : J^\infty(\mathcal{D}(X)) \to J^\infty(C)$, gives rise to a preorder on $J^\infty(\mathcal{D}(X))$ which extends its partial order. This poset is isomorphic to X, so this allows us to construct a preorder Δ on X which extends its own partial order. Then $\Delta = \mathcal{E}(C)$.*

Proof. We let $\Delta \subseteq X \times X$ be defined by setting $(x, x') \in \Delta$ if and only if $i^\flat(\downarrow x) \leq i^\flat(\downarrow x')$ in $J^\infty(C)$, keeping in mind that the completely join-irreducible elements of $\mathcal{D}(X)$ are sets of the form $\downarrow x$ for $x \in X$. This essentially amounts to constructing the standard preorder on $J^\infty(\mathcal{D}(X))$ arising from $i^\flat$ and defining the "same" preorder on X, up to isomorphism.

We may now prove that any $(x, x') \in \mathcal{E}(C)$ lies in Δ: It follows from the basic properties of adjoint maps (A.1.7) that $x' \in i^\flat(\downarrow x')$, since $i^\flat(\downarrow x')$ is the least $M \in C$ such that $\downarrow x' \subseteq i(M)$ in $\mathcal{D}(X)$. By the assumption that $(x, x') \in \mathcal{E}(C)$, we may observe that $\downarrow x' \subseteq i^\flat(\downarrow x')$ implies $\downarrow x \subseteq i^\flat(\downarrow x')$, so that $x \in i^\flat(\downarrow x')$. The down-closure of $i^\flat(\downarrow x')$ ensures that $\downarrow x \subseteq i(i^\flat(\downarrow x'))$, and by the Galois property we may conclude that $i^\flat(\downarrow x) \leq i^\flat(\downarrow x')$. We thus see that $(x, x') \in \Delta$.

On the other hand, suppose we begin with some $(x, x') \in \Delta$. To the end of showing that $(x, x') \in \mathcal{E}(C)$, let $M \in C$ and assume $x' \in M$. Since $M \in C$ and C is a DL$^+$, there is a set $\mathcal{K} \subseteq J^\infty(C)$ such that $M = \bigcup \mathcal{K}$. Then $\downarrow x' \subseteq \bigcup \mathcal{K}$, and because $\downarrow x'$ is completely join-irreducible in $\mathcal{D}(X)$ it follows that $\downarrow x' \subseteq K$ for some $K \in \mathcal{K}$. Applying the Galois property after noticing that $K = i(K)$

(which we can say because $K \in C$), we see that $i^\flat(\downarrow x') \leq K$ in $J^\infty(C)$. From the assumption that $(x, x') \in \Delta$, $i^\flat(\downarrow x) \leq i^\flat(\downarrow x')$, and hence $i^\flat(\downarrow x) \leq K$. Applying the Galois property here as before, we notice that $\downarrow x \subseteq i(K) \subseteq M$, so that $x \in M$. We may thus conclude that $\Delta \subseteq \mathcal{E}(C)$, as required.

□

Theorem 4.1.9. *The Galois closed sets of the above given Galois connection are the complete sublattices of $\mathcal{D}(X)$ and preorders on $X \times X$ which extend the partial order of X. More specifically:*

1. *Any $\Delta \subseteq X \times X$ is a preorder extending the partial order on X if and only if $\mathcal{ES}(\Delta) = \Delta$.*
2. *Any $C \subseteq \mathcal{D}(X)$ is a complete sublattice of $\mathcal{D}(X)$ if and only if $\mathcal{SE}(C) = C$.*

Proof.

1. If $\mathcal{ES}(\Delta) = \Delta$, the relevant deduction follows from lemma 4.1.6.

On the other hand, suppose that Δ is a preorder on X extending its partial order. It is already known that $\Delta \subseteq \mathcal{ES}(\Delta)$, because $\mathcal{ES}$ is a closure operator as given in the preliminaries on Galois connections. Let $q : X \to X/\Delta$ denote the corresponding quotient as defined in lemma 4.1.5 and let $(x, x') \in \mathcal{ES}(\Delta)$. Since $S(\Delta) = q^{-1}[\mathcal{D}(X/\Delta)]$, it follows that for every $q^{-1}[M] \in S(\Delta)$ with $M \in \mathcal{D}(X/\Delta)$, if $x' \in q^{-1}[M]$ then $x \in q^{-1}[M]$. In particular, for $M =\downarrow q(x') \subseteq X/\Delta$, since $x' \in q^{-1}[\downarrow q(x')]$ we may conclude that $x \in q^{-1}[\downarrow q(x')]$. This implies that $q(x) \preceq q(x')$ in X/Δ, which means that $(x, x') \in \Delta$. Hence, $\mathcal{ES}(\Delta) \subseteq \Delta$.

2. If $\mathcal{SE}(C) = C$, the relevant deduction follows from lemma 4.1.6.

Conversely, if C is a complete sublattice of $\mathcal{D}(X)$, let q denote the quotient map to the poset $X/\mathcal{E}(C)$, arising from the preorder $\mathcal{E}(C)$. It is already known that $C \subseteq \mathcal{SE}(C)$, so we wish to show that $\mathcal{SE}(C) \subseteq C$. By lemma 4.1.7, $\mathcal{SE}(C) = q^{-1}[\mathcal{D}(X/\mathcal{E}(C))]$. To the end of showing $A \in C$, suppose that $A \in q^{-1}[\mathcal{D}(X/\mathcal{E}(C))]$. This assumption implies that $A = q^{-1}[M]$ for some $M \subseteq X/\mathcal{E}(C)$. Because q is surjective, $q[A] = qq^{-1}[M] = M$. We have previously noted the connection between the preorder on X associated with the quotient map $q : X \to X/\mathcal{E}(C)$ and the preorder on $J^\infty(\mathcal{D}(X))$ (which is isomorphic to X) associated with the quotient map $i^\flat : J^\infty(\mathcal{D}(X)) \to J^\infty(C)$; in summary, $q(x) \leq q(x') \iff i^\flat(\downarrow x) \leq i^\flat(\downarrow x')$. This implies that there is a well-defined order isomorphism between $X/\mathcal{E}(C)$ and $J^\infty(C)$ given by $q(x) \mapsto i^\flat(\downarrow x)$. Let K denote the isomorphic copy of M in $J^\infty(C)$. Since $M = \{q(x) \mid x \in A\}$, we see that $K = \{i^\flat(\downarrow x) \mid x \in A\}$. Because M is down-closed, so is K, and thus $K \in \mathcal{D}(J^\infty(C))$. The discrete dual of $i^\flat$ is the preimage map $i^{\flat^{-1}} : \mathcal{D}(J^\infty(C)) \to \mathcal{D}(J^\infty(\mathcal{D}(X)))$. By discrete duality and the preliminaries on categorical equivalences, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}(J^\infty(C)) & \xrightarrow{\eta_C^{-1}} & C \\ \Big\downarrow {\scriptstyle i^{\flat^{-1}}} & & \Big\downarrow {\scriptstyle i} \\ \mathcal{D}(J^\infty(\mathcal{D}(X))) & \xrightarrow{\eta_{\mathcal{D}(X)}^{-1}} & \mathcal{D}(X) \end{array}$$

Chasing K around the diagram from the top left, we get that

$$\begin{aligned} i(\eta_C^{-1}(K)) &= \eta_{\mathcal{D}(X)}^{-1}(i^{\flat^{-1}}(K)) \\ \bigvee K &= \bigvee\{\downarrow x \mid i^\flat(\downarrow x) \in K\} \\ &= \bigvee\{\downarrow x \mid x \in A\} \\ &= A. \end{aligned}$$

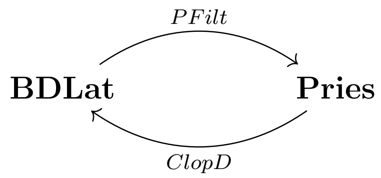
Now that A has been shown to be equal to a join of completely join irreducible elements of C, we may conclude that $A \in C$. □

Remark 4.1.10. The same result holds for any C in $\mathbf{DL}^+$ and its dual poset $J^\infty(C)$ if we set the relation R to be $a\ R\ (x, x') \iff (x' \leq a \Rightarrow x \leq a)$, as can easily be seen by recalling that by duality, $C \cong \mathcal{D}(X)$ if we define $X = J^\infty(C)$. The Galois-closed sets are complete sublattices of C and preorders on $J^\infty(C)$ extending its partial order.

4.2 Stone-Type Dualities

A topological analog for the above discrete dualities was developed by Stone [28], by observing the connection between prime filters and join-irreducible elements. In the absence of a representative set of join-irreducible elements (or atoms in the Boolean case), a Boolean algebra may be successfully represented and retrieved from a certain topological space constructed out of its set of prime filters. The category of Boolean algebras turns out to be equivalent to the category of *Stone spaces*. This was later generalised for bounded distributive lattices by Priestley [23], who noticed that the incorporation of the inclusion ordering on the prime filter space gave rise to Priestley's duality theorem, giving a dual equivalence between the category of Bounded distributive lattices and *Priestley spaces*.

Theorem 4.2.1. *The category* **BDLat** *of bounded, distributive lattices is dually equivalent to the category* **Pries**, *with the relevant functors and natural isomorphisms given as follows:*



$PFilt$	$A \mapsto (X_A, \leq, \Omega)^*$	*(Prime filters, reverse inclusion, Priestley topology)*
$ClopD$	$X \mapsto ClopD(X)$	*(Clopen down-closed subsets)*
$PFilt(A \stackrel{h}{\to} B)$	$X_B \stackrel{h^{-1}}{\to} X_A$	*(Preimage)*
$ClopD(X \stackrel{g}{\to} Y)$	$ClopD(Y) \stackrel{g^{-1}}{\to} ClopD(X)$	*(Preimage)*
η_A	$a \mapsto \{F \in X_A \mid a \in F\}$	
μ_X	$x \mapsto \{U \in ClopD(X) \mid x \in U\}$	

* The Priestley topology Ω is generated by the set

$$\{\eta_A(a) \mid a \in A\} \cup \{\eta_A(a)^C \mid a \in A\}.$$

Remark 4.2.2. Keeping in mind that Priestley spaces are duals of bounded distributive lattices, we shall henceforth often treat a Priestley space $(X, \Omega, \leq)$ as a prime filter space $(X_B, \Omega, \leq)$ of a corresponding bounded distributive lattice B. To bypass having to constantly reference the isomorphism

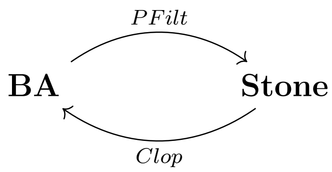
$$(X, \Omega, \leq) \cong (PFilt(ClopD(X)), \tau_{ClopD(X)}, \leq),$$

we refer to any element of $x \in X$ by F_x, since we may consider it to be a prime filter of B.

Remark 4.2.3. Note that the injectivity of η_A has always been proved using some form of the axiom of choice (in particular, the prime ideal theorem), so we do not have a concrete construction of η_A^{-1}, only the assertion of its existence.

Boolean algebras are special kinds of bounded distributive lattices (as described in the order-theoretic preliminaries). A bounded distributive lattice that happens to be a Boolean algebra always has the discrete ordering on its set of prime filters, and so the resulting dual space is just a Stone space. In the other direction, the lattice of clopen down-sets for a Priestley space with the discrete ordering is merely the Boolean algebra of clopen sets. We may thus see Stone duality as a specialisation of Priestley duality, but we will state it in the table below nevertheless.

Theorem 4.2.4. *The category* **BA** *of Boolean algebras is dually equivalent to the category* **Stone** *of Stone spaces, with the relevant functors and natural isomorphisms given as follows:*



$PFilt$	$A \mapsto (X_A, \Omega)^*$	*(Prime filters, Stone topology)*
$Clop$	$X \mapsto Clop(X)$	*(Clopen subsets)*
$PFilt(A \xrightarrow{h} B)$	$X_B \xrightarrow{h^{-1}} X_A$	*(Preimage)*
$Clop(X \xrightarrow{g} Y)$	$Clop(Y) \xrightarrow{g^{-1}} Clop(X)$	*(Preimage)*
η_A	$a \mapsto \{F \in X_A \mid a \in F\}$	
μ_X	$x \mapsto \{U \in Clop(X) \mid x \in U\}$	

* The Stone topology Ω is generated by the set $\{\eta_A(a) \mid a \in A\}$.

4.2.1 Galois Correspondence

Much like in theorem 4.1.9, there is an analagous Galois correspondence connecting substructures and quotients across Priestley duality. We present a different proof to the one in [13], to match the structure of our proof for the discrete case and to highlight the observation that $\mathcal{S}(\Delta)$ and $\mathcal{E}(A)$ essentially give the Priestley duals if we start with bounded lattice morphisms or Priestley quotient morphisms. This gives some insight as to why the bounded sublattices and Priestley quotients are the exactly Galois closed objects of this connection. We first give the result analagous to lemma 4.1.5, which shows that we may talk interchangeably about quotients of Priestley spaces and *compatible* preorders on the domain space in the following sense:

Definition 4.2.5. Let $(X_B, \Omega_{X_B}, \leq)$ be a Priestley space dual to a bounded distributive lattice B. A preorder Δ on X_B is said to be *Priestley-compatible* if it satisfies

$$\forall_{F_x, F_y \in X_B}((F_x, F_y) \notin \Delta \Rightarrow$$
$$\exists a \in B(a \in F_y \text{ and } a \notin F_x \text{ and } \eta_B(a) \text{ is a } \Delta\text{-down-set})).$$

Lemma 4.2.6. *Any Priestley quotient $q : (X_B, \Omega_{X_B}, \leq) \to (Y, \Omega_Y, \preceq)$ gives rise to a Priestley-compatible preorder Δ on X_B which extends the partial order on X_B, and vice versa. These may be recovered from each other in a one-to-one fashion.*

Proof. Given a quotient map as stipulated, by lemma 4.1.5 we obtain a preorder Δ_q on X_B which extends its partial order. It remains to be shown that this preorder is Priestley-compatible. To this end, suppose that for some $F_x, F_y \in X_B$, $(F_x, F_y) \notin \Delta_q$. Thus, $q(F_x) \not\preceq q(F_y)$, and since Y is a Priestley space we obtain a clopen upset $U \subseteq Y$ such that $q(F_x) \in U$ and $q(F_y) \notin U$. Let V denote the clopen down-set $Y \backslash U$ satisfying $q(F_x) \notin V$ and $q(F_y) \in V$. Since q is a topological quotient map, $q^{-1}[V]$ is a clopen subset of X_B with $F_y \in q^{-1}[V]$ and $F_x \notin q^{-1}[V]$. It is easy to verify that $q^{-1}[V]$ is a $\leq$-down-set using the fact that q is order-preserving. Thus $q^{-1}[V] = \eta_B(a)$ for some $a \in B$, given the characterisation of the clopen down-sets of X_B given by Priestley duality. To show that $\eta_B(a)$ is a Δ_q-down-set, suppose that $(F_x, F_y) \in \Delta_q$ and $F_y \in q^{-1}[V]$. From the former assumption we may deduce that $q(F_x) \leq q(F_y)$. Now because $q(F_y) \in V$ and V is down-closed, $q(F_x) \in V$ so that $F_x \in q^{-1}[V] = \eta_B(a)$.

On the other hand, suppose Δ is a Priestley-compatible preorder on X_B. As in 4.1.5, Δ gives rise to a poset quotient $q : X_B \to X_B/\Delta$. If we choose to equip X_B/Δ with the quotient topology, q is also continuous. The resulting space $(X_B/\Delta, \Omega_q \preceq)$ is a Priestley space; it is compact because the topological quotient map q will preserve the compactness of X_B. Towards showing that the Priestley axiom holds, fix any $q(F_x), q(F_y) \in X_B/\Delta$ such that $q(F_x) \not\preceq q(F_y)$. The fact that $(q(F_x), q(F_y)) \notin \Delta$ is equivalent. By the compatibility of Δ, there exists some $a \in B$ for which $\eta_B(a)$ is Δ-down-closed and satisfies $F_x \notin \eta_B(a)$ and $F_y \in \eta_B(a)$. Because $\eta_B(a)$ is Δ-down-closed, we can show that $\eta_B(a) = q^{-1}q[\eta_B(a)]$: given any $F_{y'} \in X_B$ such that $q(F_{y'}) \in q[\eta_B(a)]$, it follows that $q(F_{y'}) = q(F_{x'})$ for some $F_{x'} \in \eta_B(a)$. From the definition of q, we know that $(F_{x'}, F_{y'}) \in \Delta$ and $(F_{y'}, F_{x'}) \in \Delta$. Hence, $F_{y'} \in \eta_B(a)$, and we see that $q^{-1}q[\eta_B(a)] \subseteq \eta_B(a)$. The reverse inclusion is a basic set-theoretic property.

Now $q[\eta_B(a)]$ is clopen since $q^{-1}q[\eta_B(a)]$ is clopen, by virtue of its equality to $\eta_B(a)$. The equivalence $(F_x, F_y) \in \Delta \iff q(F_x) \preceq q(F_y)$ ensures that $q[\eta_B(a)]$ is $\preceq$-down-closed because $\eta_B(a)$ is Δ-down-closed. Let U denote the complement of $q^{-1}q[\eta_B(a)]$ in X/Δ, which is a clopen up-set satisfying $F_y \in U$ and $F_x \notin U$. We may therefore conclude that the Priestley separation axiom is satisfied, which completes the proof that the ordered topological space arising as described from the Priestley-compatible preorder Δ is in fact a Priestley space.

□

Remark 4.2.7. Since Stone spaces are just Priestley spaces with a discrete ordering, they fall within the scope of this theorem and we may make the following observations

Definition 4.2.8. Let (X_B, Ω_{X_B}) be a Stone space dual to a Boolean algebra B. A preorder Δ on X_B is said to be *Stone-compatible* if it is an equivalence relation and it is Priestley-compatible with respect to the discrete ordering, namely it satisfies

$$\forall_{F_x,F_y\in X_B}((F_x,F_y)\notin\Delta\Rightarrow \exists a\in B(a\in F_y \text{ and } a\notin F_x \text{ and } \eta_B(a) \text{ is a } \Delta\text{-down-set})).$$

Remark 4.2.9. A Priestley-compatible preorder on a Stone space (considered as a Priestley space with the discrete topology) yields a quotient space which is a Priestley space. If, as described above, this preorder is an equivalence relation, then the resulting Priestley space has the discrete topology (because of the symmetry of Δ), and therefore is a stone space.

In a similar vein to the discrete context, we set up a polarity as follows: Let B be a bounded distributive lattice and X_B its dual Priestley space. Let R be the relation on $B\times(X_B\times X_B)$ given by

$$a\,R\,(F_x,F_y) \iff (a\in F_y\Rightarrow a\in F_x).$$

The polarity $(B, X_B\times X_B, R)$ yields the standard antitone Galois connection

$$\mathcal{E}:\mathcal{P}(B)\rightleftarrows\mathcal{P}(X_B\times X_B):\mathcal{S}$$

where for any $U\subseteq B$ and $V\subseteq X_B\times X_B$:

$$\mathcal{S}(V)=\{b\in B\mid \forall_{(F_x,F_y)\in V}(b\in F_y\Rightarrow b\in F_x)\}$$

and

$$\mathcal{E}(U)=\{(F_x,F_y)\in X_B\times X_B\mid \forall_{b\in U}(b\in F_y\Rightarrow b\in F_x)\}.$$

In the subsequent results and proofs, any mention of a lattice is assumed to mean a bounded lattice.

Lemma 4.2.10. *For any $U\subseteq B$ and $V\subseteq X_B\times X_B$, $\mathcal{S}(V)$ is a sublattice of B and $\mathcal{E}(U)$ is a Priestley-compatible preorder on X_B which extends its partial order.*

Proof. That $S(V)$ is a bounded sublattice of B follows from the defining characteristics of prime filters, as well as the consequence that every prime filter contains the top element but never the bottom element. Now, $\mathcal{E}(U)$ extends the partial order on X_B because

$$\begin{aligned} F_x\le F_y &\iff F_y\subseteq F_x \\ &\Rightarrow F_y\cap U\subseteq F_x\cap U \\ &\Rightarrow (F_x,F_y)\in\mathcal{E}(V). \end{aligned}$$

That $\mathcal{E}(U)$ is reflexive and transitive is easily seen. To show that $\mathcal{E}(U)$ is Priestley-compatible, fix any $(F_x, F_y) \in X_B \times X_B$ with $(F_x, F_y) \notin \mathcal{E}(U)$. Negating the defining property of $\mathcal{E}(U)$, we get that there exists some $b \in U$ with $b \in F_y$ and $b \notin F_x$. Further more, $\eta_B(b)$ is $\mathcal{E}(U)$-down-closed, which follows immediately from the definition of $\mathcal{E}(U)$ and the fact that $b \in U$. □

The next two results tell us that if we start with a bounded sublattice of B or a Priestley quotient of X_B (where we understand Priestley quotients to be surjective, continuous, order-preserving maps between Priestley spaces), the operations $\mathcal{S}$ and $\mathcal{E}$ on either end of the Galois connection end up reproducing the *dual map* of the inclusion and quotient morphisms respectively. Of course, $\mathcal{E}$ yields the Priestley-compatible preorder which gives rise to the correct morphism, but this is not a problem. In both cases this is of course only up to isomorphism, but we give a concrete characterisation as subsets in terms of the "actual" maps dual to the starting inclusion and quotient.

Lemma 4.2.11. *Let Δ be a Priestley-compatible preorder on X_B which extends its partial order, and let $q : X_B \to X_B/\Delta$ denote the corresponding quotient map as given in 4.2.6. The range of of the dual (preimage) map $q^{-1} : ClopD(X_B/\Delta) \to ClopD(X_B)$ is equal to the isomorphic copy of $S(\Delta)$ in $ClopD(X_B)$ under the canonical Priestley duality isomorphism $\eta_B : B \to ClopD(X_B)$.*

Proof. To show that $q^{-1}[ClopD(X_B/\Delta)] = \eta_B(\mathcal{S}(\Delta))$, first suppose that $b \in \mathcal{S}(\Delta)$. It may be shown that $\eta_B(b) = q^{-1}q[\eta_B(b)]$: Any F_x will be contained in $q^{-1}q[\eta_B(b)]$ if and only if $q(F_x) = q(F_y)$ for some F_y with $b \in F_y$. Because $b \in S(\Delta)$ and $(F_x, F_y) \in \Delta$, it follows that $b \in F_x$ as well (by the assumption that $b \in \mathcal{S}(\Delta)$. This tells us that $F_x \in \eta_B(b)$. The reverse inclusion is a straightforward set-theoretic result.

Now because q is a toplogical quotient map, the fact that $q^{-1}q[\eta_B(b)]$ is equal to the clopen set $\eta_B(b)$ ensures that $q[\eta_B(b)]$ is clopen. It is also down-closed, for if $q(F_y) \in q[\eta_B(b)]$ for some F_y and $q(F_x) \leq q(F_y)$ for some $q(F_x) \in X_B/\Delta$, we know that $q(F_y) = q(F_{y'})$ with $F_{y'} \in \eta_B(b)$. Then $(F_x, F_{y'}) \in \Delta$ because $q(F_x) \leq q(F_{y'})$. This ensures that $F_x \in \eta_B(b)$ as well, again because $b \in S(\Delta)$. We have shown that $q[\eta_B(b)] \in ClopD(X_B/\Delta)$ and may therefore conclude that $\eta_B(b) \in q^{-1}q[ClopD(X_B/\Delta)]$ and therefore that $\eta_B[S(\Delta)] \subseteq q^{-1}[ClopD(X_B/\Delta)]$.

On the other hand, assume $A \in q^{-1}[ClopD(X_B/\Delta)]$. We aim to prove that $A = \eta_B(b)$ for some $b \in S(\Delta)$. It follows from the assumption that $A = q^{-1}[M]$ for some $M \in ClopD(X_B/\Delta)$. From Priestley duality we know that q^{-1} (which is dual to q) maps clopen down-sets to clopen down-sets, so that $A \in ClopD(X_B)$. Thus we may characterise A as being equal to $\eta_B(b)$ for some $b \in X_B$. Furthermore, by the surjectivity of q it is known that $q[\eta_B(b)] = qq^{-1}[M] = M$, so that $\eta_B(b) = q^{-1}q[M] = q^{-1}q[\eta_B(b)]$.

It follows that $b \in \mathcal{S}(\Delta)$, as we now show. Let $(F_x, F_y) \in \Delta$, which tells us that $q(F_x) \leq q(F_y)$. If $b \in F_y$, it follows that $q(F_y) \in q[\eta_B(b)]$. Since the latter set is known to be down-closed we may deduce that $q(F_x) \in q[\eta_B(b)]$, and so $F_x \in q^{-1}q[\eta_B(b)] = \eta_B(b)$. Thus, $A \in \eta_B[\mathcal{S}(\Delta)]$.

We now see that $S(\Delta)$ is isomorphic as a lattice to the image of $ClopD(X_B/\Delta)$ in $ClopD(X_B)$. Let $k : S(\Delta) \to ClopD(X_B/\Delta)$ denote the map defined by $a \mapsto M_a$, where M_a is the unique element of $ClopD(X_B/\Delta)$ such that $q^{-1}[M_a] = \eta_B(a)$. It is known to be unique because of the injectivity of the preimage map. From the preceding facts, it easily follows that k is bijective, and we may verify that k preserves the lattice bounds and operations: Firstly,

$$\begin{aligned} q^{-1}[M_{a\wedge b}] &= \eta_B(a \wedge b) \\ &= \eta_B(a) \cap \eta_B(b) \\ &= q^{-1}[M_a] \cap q^{-1}[M_b] \\ &= q^{-1}[M_a \cap M_b], \end{aligned}$$

which allows us to conclude that $M_{a\wedge b} = M_a \cap M_b$. Lastly, $q^{-1}[\emptyset] = \emptyset = \eta_B(0)$ and $q^{-1}[X_B/\Delta] = X_B = \eta_B(1)$. Thus k is an isomorphism of lattices, and we may conclude with the diagram below: □

$$\begin{array}{ccccc} X_B/\Delta & & ClopD(X_B/\Delta) & \overset{k}{\cong} & S(\Delta) \\ {\scriptstyle q}\uparrow & & \downarrow{\scriptstyle q^{-1}} & & \downarrow{\scriptstyle i} \\ X_B & & ClopD(X_B) & \overset{\eta_B}{\cong} & B \end{array}$$

We may see the inclusion of $S(\Delta)$ into B as dual to q, up to isomorphism.

Lemma 4.2.12. *Let A be a bounded sublattice of B, and let $i : A \to B$ denote the inclusion map. Let $q : X_B \to X_B/\mathcal{E}(A)$ denote the Priestley quotient map associated with the Priestley-compatible preorder $\mathcal{E}(A)$ as defined in lemma 4.2.6. Then the two Priestley quotient maps q and the preimage map i^{-1} dual to i yield the same Priestley-compatible preorder on X_B, and their codomains are isomorphic as Priestley spaces.*

Proof. Note that for any $F_x, F_y \in X_B$

$$\begin{aligned} q(F_x) \leq q(F_y) &\iff (F_x, F_y) \in \mathcal{E}(A) \\ &\iff \forall_{a \in A}(a \in F_y \Rightarrow a \in F_x) \\ &\iff F_y \cap A \subseteq F_x \cap A \\ &\iff F_x \cap A \leq F_y \cap A \text{ (In } X_A) \\ &\iff i^{-1}[F_x] \leq i^{-1}[F_y]. \end{aligned}$$

The fourth equivalence holds because the intersection of a prime filter of B with the sublattice A yields a prime filter of A, and we recall that the prime filters in X_A are ordered by *reverse* inclusion. We can see that q and i^{-1} give rise the same Priestley-compatible preorder on X_B. Furthermore, the equivalence $q(F_x) \leq q(F_y) \iff i^{-1}[F_x] \leq i^{-1}[F_y]$ can be seen to imply that there is a well-defined order isomorphism $j : X_A \to X_B/\mathcal{E}(A)$ given by $i^{-1}[F_x] \mapsto q(F_x)$. It is also continuous: to show this, let U be an open subset of $X_B/\mathcal{E}(A)$. Then

$$\begin{aligned} j^{-1}[U] &= \{i^{-1}[F_x] \in X_A \mid j(i^{-1}[F_x]) \in U\} \\ &= \{i^{-1}[F_x] \in X_A \mid q(F_x) \in U\} \\ &= \{i^{-1}[F_x] \in X_A \mid F_x \in q^{-1}[U]\} \\ &= i^{-1}[q^{-1}[U]], \end{aligned}$$

so that U is open in X_A because $q^{-1}[U]$ is, since q is a topological quotient map and i^{-1} is continuous. This is enough to show that j is an isomorphism of Priestley spaces, since they are compact and Hausdorff and we may refer to preliminary fact A.2.2 to see that this continuous bijection will be a homeomorphism. The Priestley quotient map q corresponding to $\mathcal{E}(\Delta)$ can therefore be seen as the dual of the inclusion map $i : A \to B$, up to isomorphism. □

$$\begin{array}{ccccc} A & \quad & X_A & \overset{j}{\cong} & X_B/\mathcal{E}(A) \\ \big\downarrow{\scriptstyle i} & & {\scriptstyle i^{-1}}\big\uparrow & & {\scriptstyle q}\big\uparrow \\ B & & X_B & = & X_B \end{array}$$

The commutative square on the right shows us that q is dual to the inclusion map.

Theorem 4.2.13. *The Galois closed sets of the Galois connection given on page 36 are the bounded sublattices of B and Priesley-compatible preorders on $X_B \times X_B$ which extend the partial order of X_B. More specifically:*

1. *Any $\Delta \subseteq X_B \times X_B$ is a Priestley-compatible preorder extending the partial order on X_B if and only if $\mathcal{ES}(\Delta) = \Delta$.*

2. *Any $A \subseteq B$ is a bounded sublattice of B if and only if $\mathcal{SE}(A) = A$.*

Proof.
1. If $\mathcal{ES}(\Delta) = \Delta$, the relevant consequence follows from lemma 4.2.10. On the other hand, let $(F_x, F_y) \in \mathcal{ES}(\Delta)$.

Let i denote the inclusion of $S(\Delta)$ into B, and let $r : X_B \to X_B/\mathcal{ES}(\Delta)$ denote the quotient map corresponding to the Priestley-compatible preorder $\mathcal{ES}(\Delta)$. Referring to lemma 4.2.12 and the corresponding figure with A set to be $\mathcal{S}(\Delta)$, we may observe that $r(F_x) \leq r(F_y)$ in $X_B/\mathcal{S}(\Delta)$ if and only if $i^{-1}[F_x] \leq i^{-1}[F_y]$ in $X_{\mathcal{S}(\Delta)}$.

By lemma 4.2.11, we may obtain the images under the isomorphism k of $i^{-1}[F_x] \subseteq \mathcal{S}(\Delta)$ and $i^{-1}[F_y] \subseteq \mathcal{S}(\Delta)$, which are in turn subsets of $ClopD(X_B/\Delta)$. We denote these images by $\mathcal{M}_x$ and $\mathcal{M}_y$ respectively, and take note that they are prime filters in $ClopD(X_B/\Delta)$ because k is a lattice isomorphism, and because $i^{-1}[F_x] = F_x \cap \mathcal{S}(\Delta)$ (likewise for F_y). Recall that

$$\mathcal{M}_x = k[i^{-1}[F_x]] = \{M \in ClopD(X_B/\Delta) \mid q^{-1}[M] = \eta_B(a) \text{ and } a \in F_x \cap \mathcal{S}(\Delta)\},$$

and an analogous equation holds for $\mathcal{M}_y$.

Furthermore,

$$\begin{aligned} i^{-1}[F_x] \leq i^{-1}[F_y] &\Rightarrow i^{-1}[F_y] \subseteq i^{-1}[F_x] \\ &\Rightarrow \mathcal{M}_y \subseteq \mathcal{M}_x. \end{aligned}$$

At this stage, we need to keep in mind the following diagram, which is known to be commutative by Priestley duality:

$$\begin{array}{ccc} X_B/\Delta & \overset{\mu_{X_{B/\Delta}}}{\cong} & PFilt(ClopD(X_B/\Delta)) \\ {\scriptstyle q}\uparrow & & \uparrow{\scriptstyle (q^{-1})^{-1}} \\ X_B & \overset{\mu_{X_B}}{\cong} & PFilt(ClopD(X_B)) \end{array}$$

Note that

$$\begin{aligned} \mu_{X_B}(F_x) &= \{U \in ClopD(X_B) \mid F_x \in U\} \\ &= \{\eta_B(a) \in ClopD(X_B) \mid F_x \in \eta_B(a)\} \\ &= \{\eta_B(a) \in ClopD(X_B) \mid a \in F_x\}. \end{aligned}$$

We may also observe that

$$
\begin{aligned}
(q^{-1})^{-1}[\mu_{X_B}(F_x)] &= \{M \in ClopD(X_B/\Delta) \mid q^{-1}[M] = \eta_B(a) \text{ and } a \in F_x\} \\
&= \{M \in ClopD(X_B/\Delta) \mid q^{-1}[M] = \eta_B(a) \text{ and } a \in F_x \cap \mathcal{S}(\Delta)\} \\
&= \mathcal{M}_x,
\end{aligned}
$$

where the next to last equality holds because $q^{-1}[ClopD(X_B/\Delta)] = \eta_B[\mathcal{S}(\Delta)]$. Chasing F_x and F_y respectively around the diagram in figure on the previous page, we find that $\mathcal{M}_x = \mu_{X_B/\Delta}(q(F_x))$ and $\mathcal{M}_y = \mu_{X_B/\Delta}(q(F_y))$. Since $\mu_{X_B/\Delta}$ is an order isomorphism (as it is an isomorphism of Priestley spaces), it is known that

$$
\begin{aligned}
\mathcal{M}_y \subseteq \mathcal{M}_x &\iff \mathcal{M}_x \leq \mathcal{M}_y \\
&\iff q(F_x) \leq q(F_y).
\end{aligned}
$$

We may conclude that the last fact holds, from which it follows that $(F_x, F_y) \in \Delta$. Thus, $\mathcal{E}S(\Delta) \subseteq \Delta$.

2. If $\mathcal{SE}(A) = A$, the relevant consequence follows from 4.2.10. Conversely, If A is a bounded sublattice, then we have shown that $\mathcal{E}(A)$ is a Priestley-compatible preorder on X_B. Let $q : X_B \to X_B/\mathcal{E}(A)$ denote the corresponding Priestley quotient and let $i : A \to B$ denote the inclusion bounded lattice homomorphism. Applying lemma 4.2.11, recall that $\eta_B[\mathcal{SE}(A)] = q^{-1}[ClopD(X_B/\mathcal{E}(A)]$, so that for any $b \in \mathcal{SE}(A)$ we know that $\eta_B(b) = q^{-1}[M]$ for some $M \in ClopD(X_B/\mathcal{E}(A))$. Recall also that $X_A \cong X_B/\mathcal{E}(A)$, and let K denote the isomorphic copy of M in X_A under the isomorphism j as given in lemma 4.2.12. Then

$$K = j[M] = \{i^{-1}[F_x] \mid q(F_x) \in M\}.$$

Because $q(F_x) \in M$ if and only if $i^{-1}[F_x] \in K$ by definition, it follows that $q^{-1}[M] = (i^{-1})^{-1}[K]$ in X_B. Chasing K around the following diagram (which commutes by the definition of dual equivalence of categories A.3.4),

$$
\begin{array}{ccc}
ClopD(X_A) & \cong & A \\
\big\downarrow {\scriptstyle (i^{-1})^{-1}} & & \big\downarrow {\scriptstyle i} \\
ClopD(X_B) & \cong & B
\end{array}
$$

we find that $K = \eta_A(a)$ for some $a \in A$ and that $(i^{-1})^{-1}[K] = q^{-1}[N] = \eta_B(b)$. Thus we may conclude that since $\eta_B(b) = \eta_B(a)$, then $b \in A$. Hence, $\mathcal{SE}(A) \subseteq A$.

$\square$

We give the analogous theorem for the Boolean case, which requires some adjustment:

Theorem 4.2.14. *Let B be a Boolean algebra and X_B its dual Stone space. Let R be the relation on $B \times (X_B \times X_B)$ given by*

$$a\,R\,(F_x, F_y) \iff (a \in F_y \iff a \in F_x).$$

The polarity $(B, X_B \times X_B, R)$ yields the standard antitone Galois connection

$$\mathcal{E}' : \mathcal{P}(B) \rightleftarrows \mathcal{P}(X_B \times X_B) : \mathcal{S}'$$

where for any $U \subseteq B$ and $V \subseteq X_B \times X_B$:

$$\mathcal{S}'(V) = \{b \in B \mid \forall_{(F_x,F_y)\in V}(b \in F_y \iff b \in F_x)\}$$

and

$$\mathcal{E}'(U) = \{(F_x, F_y) \in X_B \times X_B \mid \forall_{b\in U}(b \in F_y \iff b \in F_x)\}.$$

The Galois-closed sets of this Galois connection are the Boolean subalgebras of B and the Stone-compatible equivalence relations on X_B.

Proof. First we show that each $\mathcal{S}'(\Delta)$ is a Boolean subalgebra: that it is a bounded sublattice follows much the same reasoning as in theorem 4.2.10. To see that it is also closed under complements, observe that

$$\begin{aligned}
b \in \mathcal{S}'(V) &\iff \forall_{(F_x,F_y)\in V}(b \in F_y \iff b \in F_x)\\
&\iff \forall_{(F_x,F_y)\in V}(F_y \in \eta_B(b) \iff F_x \in \eta_B(b))\\
&\iff \forall_{(F_x,F_y)\in V}(F_y \notin \eta_B(b) \iff F_x \notin \eta_B(b))\\
&\iff \forall_{(F_x,F_y)\in V}(F_y \in (\eta_B(b))^C \iff F_x \in (\eta_B(b))^C)\\
&\iff \forall_{(F_x,F_y)\in V}(F_y \in \eta_B(\neg b) \iff F_x \in \eta_B(\neg b))\\
&\iff \forall_{(F_x,F_y)\in V}(\neg b \in F_y \iff \neg b \in F_x)\\
&\iff \neg b \in \mathcal{S}'(V).
\end{aligned}$$

Each $\mathcal{E}'(U)$ is an equivalence relation: again, that it is a preorder follows much the same reasoning as theorem 4.2.10. To see that it is symmetric follows by inspection, by the symmetry of the condition of membership.

A crucial observation is that if a preorder Δ on X_B is also an equivalence relation, then symmetry implies that

$$\begin{aligned}
\mathcal{S}'(\Delta) &= \{b \in B \mid \forall_{(F_x,F_y)\in V}(b \in F_y \Rightarrow b \in F_x)\}\\
&= \mathcal{S}(\Delta).
\end{aligned}$$

Similiarly, closure of a Boolean subalgebra $A \subseteq B$ under complements allows us to reason that

$$\begin{aligned}\mathcal{E}'(A) &= \{(F_x, F_y) \in X_B \times X_B \mid \forall_{a \in A}(a \in F_y \Rightarrow b \in F_x)\} \\ &= \mathcal{E}(A).\end{aligned}$$

These two observations allow us to refer to results in the previous theorem.

Now, $\mathcal{E}'\mathcal{S}'(\Delta) = \Delta$ if and only if Δ is an equivalence relation: The forward implication follows from the first observation in this proof. The the converse is implied by the fact that Δ is an equivalence relation, $\mathcal{S}'(\Delta)$ is a Boolean algebra and therefore $\mathcal{E}'\mathcal{S}'(\Delta) = \mathcal{E}\mathcal{S}(\Delta) = \Delta$, by theorem 4.2.13.

By a similar argument, $\mathcal{S}'\mathcal{E}'(A) = A$ if and only if A is a Boolean subalgebra of B. □

4.3 Extended Discrete and Stone-Type Dualities

It was observed by Pippenger ([22]) that for any finite alphabet A, the Boolean algebra $Rec(A^*)$ happens to be dual to the Stone space underlying $\hat{F}_A(\mathbf{Mon_f})$, the free profinite monoid over A (or equivalently, the profinite completion of A^*). This is in itself interesting and led to topological characterisations of notions in language theory, but it also hints at stronger and deeper result. Recall that the Boolean algebra $Rec(A^*)$ was also closed under certain *cancellation* operations. Moreover, varieties of languages are systems of substructures of $(Rec(A^*), /, \backslash)$ which interact with these operations in a way that is critical to the Eilenberg theorem.

Extended versions of Stone duality allow us to take these operations into account: we will introduce some background theory on additional operations on lattices and summarise the extended duality theories which show that Boolean algebras and distributive lattices with certain kinds of additional operations may be seen as dual to Stone spaces and Priestley spaces with additional relational structure. Gehrke [13] has shown that under this extended version of Stone duality, the entire structure $(Rec(A^*), /, \backslash)$ is dual to the Stone-topological monoid $\hat{F}_A(\mathbf{Mon_f})$. This is the result we are working towards with the background theory in this section.

As before, we ease into the relevant concepts by first illustrating them in the case of discrete duality.

Let C be a DL^+. An operation $f : C^n \to C$ only falls within the scope of the duality in theorem 4.1.1 if it is a homomorphism of complete lattices, but for many operations of interest this is not the case. However, we will see that certain operations can be captured by incorporating additional structure on the dual poset, from which the original operation may be recovered on the dual of the dual.

Definition 4.3.1. Let C be any lattice. An operation $f : C^n \to C$ is said to *preserve joins in the i-th coordinate* if and only if for every $(a_1, ..., a_n) \in C^n$ and b_i in C:

$$f(a_1, \ldots, a_i \vee b_i, \ldots, a_n) = f(a_1, \ldots, a_i, \ldots, a_n) \vee f(a_1, \ldots, b_i, \ldots, a_n).$$

An operation $f : C^n \to C$ is said to *preserve arbitrary joins in the i-th coordinate* if and only if for every $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in C^{n-1}$ and $\{a_i^{(j)} \mid j \in J\} \subseteq C$:

$$f(a_1, \ldots, \bigvee_{j \in J} a_i^{(j)}, \ldots, a_n) = \bigvee_{j \in J} f(a_1, \ldots, a_i^{(j)}, \ldots, a_n)$$

Definition 4.3.2. Let C be a DL$^+$. An operation $f : C^n \to C$ is called a *complete operator* if and only if it preserves arbitrary joins in every coordinate.

Since we are working in the DL$^+$ setting, the action of any complete operator is completely determined by its value at completely join-irreducible elements. Let $X = J^\infty(C)$, the poset dual to C. For any $\overline{a} \in C^n$,

$$f(\overline{a}) = \bigvee \{f(\overline{x}) \mid \overline{x} \in X^n \text{ and } \overline{x} \leq \overline{a}\}.$$

From this, we may define a relation $R_f \subseteq X^n \times X$ given by

$$\overline{x} R_f x \iff x \leq f(\overline{x}).$$

This definition allows one to retrieve the operation f, since

$$f(\overline{x}) = \bigvee \{x \mid \overline{x} R_f x\},$$

and we have already seen that this is sufficient to fully recover f.

The complete lattice isomorphism $\eta_C : C \to \mathcal{D}(X)$ allows us to define a function f_{R_f} on $\mathcal{D}(X)$ so that the following diagram commutes:

$$\begin{array}{ccc} C^n & \overset{\eta_C \times \ldots \times \eta_C}{\cong} & \mathcal{D}(X)^n \\ \downarrow f & & \downarrow f_{R_f} \\ C & \overset{\eta_C}{\cong} & \mathcal{D}(X) \end{array}$$

Those relations $R \subseteq X^n \times X$ which arise from complete operators have a the following characterisation:

Definition 4.3.3. A relation $R \subseteq X^n \times X$ for a poset $(X, \leq)$ is said to be *order compatible* if and only if for every $\overline{x}, \overline{x'} \in X^n$ and $x, x' \in X$, if $\overline{x'} \geq \overline{x}$ and $\overline{x}Rx$ and $x \geq x'$, then $\overline{x'}Rx'$.

In the case of a ternary relation, this is just the condition that

$$(\geq \times \geq) \circ R \circ \geq = R.$$

It is straightforward to show that R_f is order compatible, and this property also results in the fact that

$$\begin{aligned} f_{R_f}(U_1, \ldots, U_2) &= f_{R_f}(\eta_C(a_1), \ldots, \eta_C(a_n)) \\ &= \eta_C(f(\overline{a})) \\ &= \{x \in X \mid \exists_{\overline{x}} \, [\overline{x}Rx \text{ and } \overline{x} \leq \overline{a}]\} \\ &= \{x \in X \mid \exists_{\overline{x}} \, [\overline{x}Rx \text{ and } \overline{x} \in U_1 \times \ldots \times U_n]\} \end{aligned}$$

where $\overline{a} := (a_1, \ldots, a_n) \in C^n$ such that $U_i = \eta_C(a_i)$, and with order compatibility implying the third equality. This suggests a definition for f_R for any $R \subseteq X^n \times X$ and $U_1, \ldots, U_n \in \downarrow(X)$:

$$\begin{aligned} f_R(U_1, \ldots, U_n) &= R[U_1, \ldots, U_n, \_\_] \\ &= \{x \in X \mid \exists_{\overline{x}} \, [\overline{x}Rx \text{ and } \overline{x} \in U_1 \times \ldots \times U_n]\} \end{aligned}$$

As a result we have the following theorem:

Theorem 4.3.4. *([13]) Let C be a DL^+ and X its dual poset. The functions $f \mapsto R_f$ and $R \mapsto f_R$ constitute a one-to-one correspondence between complete n-ary operators on C and order compatible relations on $X^n \times X$.*

This does extend to a full categorical duality, but we will rather state it in the topological context and for a single binary operation. As usual, this result subsumes the analagous result for complete and atomic Boolean algebras and sets, since this is just the case where the ordering on X is discrete. A full account of this discrete duality in the Boolean case, encompassing the duality for morphisms, may be found in [15].

More can be said for operations with other lattice-theoretic preservation properties, but of particular interest to us are so called *residual* maps and residuated families of maps. For each $a_1, \ldots, a_i, \ldots, a_n \in C$, a complete operator $f : C^n \to C$ yields a completely join-preserving function

$$f(a_1, \ldots, \_\_, \ldots, a_n) : C \to C$$

by fixing all but the i-th coordinate. Complete operators are necessarily monotone, so by lemma A.1.7 we see that these maps have an upper adjoint which we shall denote $(f(a_1, \ldots, \_\_, \ldots, a_n))^\sharp$.

Definition 4.3.5. Let $f : C^n \to C$ be a complete operator. For each domain coordinate i, the i-th *residual* of f is the map

$$f_i^\sharp : C^n \to C$$

given by

$$f_i^\sharp(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, a) = (f(a_1, \ldots, \_\_, \ldots, a_n))^\sharp(a)$$

for every $a_1, \ldots, a_n, a \in C$. We refer to f and its n residuals as a *residuated family* of maps.

Lemma 4.3.6. *A complete operator $f : C^n \to C$ and each of its n residuals $f_i^\sharp$ satisfy the following Galois property for every $a_1, \ldots, a_n, a \in C$:*

$$f(a_1, \ldots, a_n) \leq a \iff a_i \leq f_i^\sharp(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, a)$$

Proof. This is immediate from the definition of $f_i^\sharp$ and the Galois property of adjoint maps. □

Lemma 4.3.7. *For any complete operator $f : C^n \to C$, each of its n residuals preserves arbitrary meets in the last coordinate and sends arbitrary joins to meets in every other coordinate.*

Proof. By lemma A.1.7, the fact that each residual preserves meets in the last coordinate is a consequence of $(f(a_1, \ldots, \_\_, \ldots, a_n))^\sharp$ being an upper adjoint. Without loss of generality, we show that $f_i^\sharp$ sends arbitrary joins in the first coordinate to meets in the codomain. We therefore aim to show that

$$f_i^\sharp(\bigvee T, \ldots, a_{i-1}, a_i, \ldots, a_n, a) = \bigwedge_{t \in T} f_i^\sharp(t, \ldots, a_{i-1}, a_i, \ldots, a_n, a).$$

Recall from lemma A.1.7 that (for every choice of i as the blank coordinate)

$$(f(\bigvee T, \ldots, \_\_, \ldots, a_n))^\sharp(a) = \bigvee\{s \mid f(\bigvee T, \ldots, s, \ldots, a_n) \leq a\}$$

and for every $t \in T$,

$$(f(t, \ldots, \_\_, \ldots, a_n))^\sharp(a) = \bigvee\{s \mid f(t, \ldots, s, \ldots, a_n) \leq a\}.$$

Now, note that for any $s \in C$, $f(t, \ldots, s, \ldots, a_n) \leq f(\bigvee T, \ldots, s, \ldots, a_n)$, so that $\{s \mid f(\bigvee T, \ldots, a_n) \leq a\} \subseteq \{s \mid f(t, \ldots, a_n) \leq a\}$ for every $t \in T$. Since taking joins preserves order and by the previous two equations, $(f(\bigvee T, \ldots, \_\_, \ldots, a_n))^\sharp(a) \leq (f(t, \ldots, \_\_, \ldots, a_n))^\sharp(a)$ for every $t \in T$.

Furthermore, if $b \in C$ is any upper bound for $\{(f(t, \ldots, \_\_, \ldots, a_n))^\sharp(a) \mid t \in T\}$, then by the galois property $f(t, \ldots, b, \ldots, a_n) \leq a$ for every $t \in T$, from which it follows that $\bigvee_{t\in T} f(t,\ldots,b,\ldots,a_n) \leq a$. By the fact that f preserves arbitrary joins in each coordinate, $f(\bigvee T,\ldots,b,\ldots,a_n) \leq a$ and thus $b \leq (f(\bigvee T,\ldots,\_\_,\ldots,a_n))^\sharp(a)$. We conclude that

$$(f(\bigvee T,\ldots,\_\_,\ldots,a_n))^\sharp(a) = \bigwedge_{t\in T}(f(t,\ldots,\_\_,\ldots,a_n))^\sharp(a),$$

which confirms the desired result by applying the definition of residuals. □

Remark 4.3.8. The residuals of f_R may be retrieved from the same relation R via the construction

$$(U_1,\ldots,U_n) \mapsto (R[U_1,\ldots,U_{i-1},\_\_,U_{i+1},\ldots,U_n,U^C])^C.$$

Although the above results and upcoming analogues in the topological setting apply for n-ary operations and their n residuals, for the benefit of readability and for the purpose of applications to language theory we will henceforth stick to binary operations. When a complete operator is binary, we adapt an infix notation for the two residuals: for an operation $f : C^2 \to C$ written as $a \cdot b = f(a,b)$, for every $a,b,c \in C$ we denote the the residuals by

$$c/b := f_1^\sharp(b,c)$$
$$a\backslash c := f_2^\sharp(a,c).$$

The corresponding Galois property now becomes

$$a \cdot b \leq c \iff a \leq c/b \iff b \leq a\backslash c.$$

These are often refered to as the *left residual* and the *right residual* respectively. Note that in this case, the operation $\backslash$ turns arbitrary joins into meets in the first coordinate and preserves arbitrary meets in the second, while $/$ turns arbitrary joins into meets in the second coordinate and preserves arbitrary meets in the first coordinate.

All n-ary complete operators have n residuals, but we can extend the notion of residuation to operations that are not necessarily complete operators.

Definition 4.3.9. Let X be a poset. A binary operation $\_\cdot\_ : X^2 \to X$ is said to be *residuated* and has a left and a right residual if there exist binary operations $\_/\_ : X^2 \to X$ and $\_\backslash\_ : X^2 \to X$ such that for every $a,b,c \in X$

$$a \cdot b \leq c \iff a \leq c/b \iff b \leq a\backslash c.$$

We refer to $(\cdot,/,\backslash)$ as a residuated family.

Lemma 4.3.10. *A residuated family $(\cdot,/,\backslash)$ of binary operators on a bounded distributive lattice B satisfy the following lattice-theoretic preservation properties:*

OP[·]	The operation "·" preserves finite joins in both coordinates.	$(a \vee b) \cdot c = (a \cdot c) \vee (b \cdot c)$, $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$
OP[\]	The operation "\" preserves finite meets in the second coordinate and sends finite joins to meets in the first coordinate.	$(a \vee b)\backslash c = (a\backslash c) \wedge (b\backslash c)$, $a\backslash(b \wedge c) = (a\backslash b) \wedge (a\backslash c)$
OP[/]	The operation "/" preserves finite meets in the first coordinate and sends finite joins to meets in the second coordinate.	$(a \wedge b)/c = (a\backslash c) \wedge (b\backslash c)$, $a/(b \vee c) = (a \cdot b) \wedge (a\backslash c)$

It is rudimentary to check these properties, but proofs may be found in [5]. We may describe one-to-one correspondences between binary operations on bounded distributive lattices satisfying each of the above properties respectively and ternary order-compatible relations on the dual Priestley space X. The order-topological properties corresponding to each of the above preservation properties are listed below:

REL[·]	1. For each $x \in X$, the set $R[\_, \_, x]$ is closed. 2. For all U, V clopen down-sets of X the set $R[U, V, \_]$ is clopen.
REL[\]	1. For each $x \in X$, the set $R[\_, x, \_]$ is closed. 2. For all U clopen down-set of X and V clopen up-set of X, the set $R[U, \_, V]$ is clopen.
REL[/]	1. For each $x \in X$, the set $R[x, \_, \_]$ is closed. 2. For all U clopen down-set of X and V clopen up-set of X, the set $R[\_, U, V]$ is clopen.

The remaining theorems and intermittent remarks are specific cases of the theory developed by Goldblatt in [16], although the form in which they appear here is largely based on Gehrke's summary in [13]. The study of additional operations and corresponding relations on the dual structure goes back to the work of Jónsson and Tarski [18] on Boolean algebras with operators. We refer the reader to the above sources for full details, as we have only extracted those components which are useful in our application to formal language theory.

Theorem 4.3.11. *Let B be a bounded distributive lattice and $(X, \leq, \Omega)$ its dual Priestley space. There is a one-to-one correspondence between operations on D satisfying the preservation properties* OP[·] *(respectively* OP[/] *and* OP[\]*) and order compatible relations on X satisfyng* REL[·] *(respectively* REL[/], REL[\]*). These correspondences are given as follows:*

1. *Given a binary operation $\_ \cdot \_ : B^2 \to B$ satisfying* OP[·]*, we first define a lifted version of this operation $\_ \cdot \_ : Filt(B)^2 \to Filt(B)$ on the lattice $Filt(B)$ of filters of B by*

$$F \cdot G = \langle \{a \cdot b \mid a \in F, b \in G\} \rangle_{Filt}$$

for every F, G in $Filt(B)$.

We may define a ternary relation $R.$ on the dual prime filter space X_B by

$$\begin{aligned} R. &= \{(F_x, F_y, F_z) \in (X_B)^3 \mid F_x \cdot F_y \geq F_z\} \\ &= \{(F_x, F_y, F_z) \in (X_B)^3 \mid F_x \cdot F_y \subseteq F_z\} \end{aligned}$$

On the other hand, from a ternary order compatible relation $R \subseteq (X_B)^3$ satisfying REL[·]*, define a binary operation*

$$\_ \cdot_R \_ : ClopD(X_B)^2 \to ClopD(X_B)$$

given by

$$\begin{aligned} U \cdot_R V &= R[U, V, \_] \\ &= \{F_z \mid \exists_{F_x, F_y} [F_x \in U \text{ and } F_y \in V \text{ and } R(F_x, F_y, F_z)]\}. \end{aligned}$$

These constructions give a one-to-one correspondence between binary operations on B satisfying OP[·] *and order-compatible ternary relations on X_B satisfying* REL[·].

2. *Given a binary operation $\_/\_ : B^2 \to B$ satisfying* OP[/]*, we first define a multisorted operation $\_/\_ : Idl(B) \times Filt(B) \to Idl(B)$ by*

$$I/F = \langle \{a/b \mid a \in I, b \in F\} \rangle_{Idl}$$

for every $F \in Filt(B)$ and $I \in Idl(B)$.

We may define a ternary relation $R_/$ *on the dual prime filter space* X_B *by*

$$R_/ = \{(F_x, F_y, F_z) \in (X_B)^3 \mid I_z/F_y \subseteq I_x\}$$

where I_x *is the prime ideal* $(F_x)^C$.

On the other hand, from a ternary order compatible relation $R \subseteq (X_B)^3$ *satisfying* REL[/], *define a binary operation*

$$\_/_R\_ : ClopD(X_B)^2 \to ClopD(X_B)$$

given by

$$\begin{aligned} V/_RU &= (R[\_, U, V^C])^C \\ &= \{F_x \mid \forall_{F_y,F_z}[(F_y \in U \text{ and } R(F_x, F_y, F_z)) \Rightarrow z \in V]\}. \end{aligned}$$

These constructions give a one-to-one correspondence between binary operations on B *satisfying* OP[\] *and order-compatible ternary relations on* X_B *satisfying* REL[\].

3. *Given a binary operation* $\_\backslash\_ : B^2 \to B$ *satisfying* OP[\], *we first define a multisorted operation* $\_\backslash\_ : Filt(B) \times Idl(B) \to Idl(B)$ *by*

$$F\backslash I = \langle\{a\backslash b \mid a \in F, b \in I\}\rangle_{Idl}$$

for every $F \in Filt(B)$ *and* $I \in Idl(B)$.

We may define a ternary relation $R_\backslash$ *on the dual prime filter space* X_B *by*

$$R_\backslash = \{(F_x, F_y, F_z) \in (X_B)^3 \mid F_x\backslash I_z \subseteq I_y\}$$

where I_x *is the prime ideal* $(F_x)^C$.

On the other hand, from a ternary order compatible relation $R \subseteq (X_B)^3$ *satisfying* REL[\], *define a binary operation*

$$\_\backslash_R\_ : ClopD(X_B)^2 \to ClopD(X_B)$$

given by

$$\begin{aligned} U\backslash_RV &= (R[U, \_, V^C])^C \\ &= \{F_y \mid \forall_{F_x,F_z}[(F_x \in U \text{ and } R(F_x, F_y, F_z)) \Rightarrow z \in V]\} \end{aligned}$$

These constructions give a one-to-one correspondence between binary operations on B *satisfying* OP[\] *and order-compatible ternary relations on* X_B *satisfying* REL[\].

Remark 4.3.12. More can be said if we start with a residuated family $(\cdot, /, \backslash)$ of binary operators on B: because it may be shown that

$$F_x \cdot F_y \geq F_z \iff F_x \backslash I_z \leq I_y \iff I_z / F_y \leq I_x,$$

it follows that

$$R_\cdot = R_\backslash = R_/.$$

Furthermore, given an order-compatible ternary relation R on X_B, the family of binary operations $(\cdot_R, \backslash_R, /_R)$ on the DL$^+$ of down-sets of X_B is a residuated family, as mentioned in remark 4.3.8. If they satisfy any one or any combination of REL[$\cdot$], REL[$\backslash$] or REL[/], then the corresponding operations restrict to $ClopD(X_B)$ and R is dual to each of those for which the condition is satisfied [13].

The structure $(Rec(A^*), /, \backslash)$ is not closed under the binary operation "$\cdot$" for which / and \ are residuals, but the above results are leading us towards a characterisation of a Boolean space (Priestley space with a discrete ordering) which is dual to it. We first present a formalisation of such structures with a definition of so-called *residuation algebras* (as given in[13]), and then proceed to give the relevant notion of morphisms which are dual to morphisms preserving both the lattice and the residuation structure.

Definition 4.3.13. A bounded distributive lattice with two additional binary operations $(B, /, \backslash)$ is called a *residuation algebra* if the operations respectively satisfy OP[/] and OP[$\backslash$], as well as the Galois property

$$\forall a, b, c \in B \quad b \leq a \backslash c \iff a \leq c/b.$$

Although residuation algebras do not have a binary operation for which / and \ are residuals, one may define a binary operation on the corresponding filter lattice for which a similar relationship to that in remark 4.3.12 holds.

Lemma 4.3.14. *[13, Proposition 3.14] Let $(B, /, \backslash)$ be a residuation algebra and let $(X_B, \leq, \Omega, R)$ be the dual Priestley space with the corresponding relation R as given in theorem 4.3.11. One may define an operation $\_ \cdot \_ : Filt(B) \times Filt(B) \to Filt(B)$ for every $F, G \in Filt(B)$ by*

$$F \cdot G = \{c \in B \mid \exists_{a \in F}(a \backslash c \in G)\}.$$

Then for every $F_x, F_y, F_z \in X_B$:

$$R(F_x, F_y, F_z) \iff F_x \cdot F_y \geq F_z \iff F_x \backslash I_z \leq I_y \iff I_z / F_y \leq I_x.$$

The following notion of *bounded morphisms* included below may be credited to Goldblatt [16].

Definition 4.3.15. Let (X, R) and (Y, S) be Priestley spaces with ternary order-compatible relations. Let $\phi : X \to Y$ be a continuous, order-preserving function.

If R and S satisfy REL[$\cdot$], then ϕ is said to be a *bounded morphism for these relations with respect to the last coordinate* if for any $x_1, x_2, x_3 \in X$ and $y_1, y_2 \in Y$, it satisfies

BM[$\cdot$]	1. $R(x_1, x_2, x_3) \Rightarrow S(\phi(x_1), \phi(x_2), \phi(x_3))$ 2. $(S(y_1, y_2, \phi(x_3))) \Rightarrow$ $\exists x_1, x_2[(y_1, y_2) \geq (\phi(x_1), \phi(x_2))$ and $R(x_1, x_2, x_3)]$.

If R and S satisfy REL[$\backslash$], then ϕ is said to be a *bounded morphism for these relations with respect to the second coordinate* if $x_1, x_2, x_3 \in X$ and $y_1, y_3 \in Y$ it satisfies

BM[$\backslash$]	1. $R(x_1, x_2, x_3) \Rightarrow S(\phi(x_1), \phi(x_2), \phi(x_3))$ 2. $(S(y_1, \phi(x_2), y_3)) \Rightarrow$ $\exists x_1, x_3[y_1 \geq \phi(x_1), \phi(x_3) \geq y_3$ and $R(x_1, x_2, x_3)]$.

If R and S satisfy REL[/], then ϕ is said to be a *bounded morphism for these relations with respect to the first coordinate* if for every $x_1, x_2, x_3 \in X$ and $y_2, y_3 \in Y$ it satisfies

BM[/]	1. $R(x_1, x_2, x_3) \Rightarrow S(\phi(x_1), \phi(x_2), \phi(x_3))$ 2. $(S(\phi(x_1), y_2, y_3)) \Rightarrow$ $\exists x_2, x_3[y_2 \geq \phi(x_2), \phi(x_3) \geq y_3$ and $R(x_1, x_2, x_3)]$.

The goal of this chapter has been to present enough background on extended duality theory to state a full categorical duality theorem for residuation algebras:

Theorem 4.3.16. *The category of residuation algebras together with morphisms preserving both* $\backslash$ *and* / *(as well as the bounded lattice structure) is dual to the category of Priestley spaces with relations satisfying* REL[$\backslash$] *and* REL[/] *together with morphisms satisfying* BM[$\backslash$] *and* BM[/].

The functors are given exactly as for Priestley duality, together with the above correspondence for additional operations and relations on the Priestley dual. The full proof (in the more general case) may be found in [16].

Remark 4.3.17. We could also have introduced the concept of residuation for all n-ary operations, and at every step along the way defined preservation properties OP[i] for each of its n residuals, n Priestley compatibility properties REL[i] for $n+1$-ary relations, as well as definitions BM[i] for bounded morphisms with respect to the i-th coordinate as output; these definitions follow the same pattern as for the binary case. These definitions result in duality theorems for Priestley spaces with relations satisfying some subset of the REL[i] properties with morphisms satisfying the corresponding subset of BM[i] properties and bounded distributive lattices with a corresponding number of n-ary residuation operations [13, 16].

4.3.1 Residuation Ideals

We show that all Priestley topological algebras (including Boolean-topological algebras) are extended Priestley duals of a residuation algebra. We have already seen the correspondence between Priestley quotients and bounded sublattices, and one may inquire as to what property on the corresponding bounded sublattice corresponds to a Priestley quotient map being a morphism of Priestley-topological algebras. This is answered in [13], and in this subsection we present their solution to this question.

Theorem 4.3.18. *Let $(X, \leq, \Omega, \cdot)$ be a Priestley-topological algebra with one binary operation, and let R denote the graph of "$\cdot$". Then R satisfies* REL[\] *and* REL[/]*, so that $(X, \leq, \Omega, \cdot)$ is the extended Priestley dual of a residuation algebra.*

Proof. We show that R satisfies REL[/]; the proof that R satisfies REL[\] proceeds in much the same vein. Notice that for every $x \in X$,

$$R[x, \_, \_] = \{(x', y) \mid R(x, x', y)\} = G_{x\cdot}$$

where $G_{x\cdot}$ is the graph of the continuous function $x \cdot \_ : X \to X$. The graph of a continuous function is closed ([30]), and thus so is $R[x, \_, \_]$.

Now, let U be a clopen down-set of X and V a clopen up-set. The preimage of V, namely $(\_\cdot\_)^{-1}[V]$ is clopen by the continuity of "$\cdot$". Let $\pi_1 : X \times X \to X$ denote the continuous projection map for the first coordinate. We may observe that

$$\begin{aligned} R[\_, U, V] &= \{x \in X \mid \exists_{x' \in U, y \in V} R(x, x', y)\} \\ &= \pi_i((\_ \cdot \_)^{-1}[V] \cap (X \times U)). \end{aligned}$$

Because X is compact and Hausdorff, the projection of a clopen set is clopen. Now $(X \times U) = \pi_i^{-1}[U]$, which is also clopen. Since a finite intersection of clopen sets is clopen, these facts allow us to conclude that $R[\_, U, V]$ is clopen, so that R satisfies REL[/]. □

Remark 4.3.19. Although every Priestley-topological algebra is the dual of a residuation algebra (with n residuation operations for each n-ary operation on the topological algebra), the extended duals of residuation algebras and Boolean residuation algebras are not always functional. In the Boolean case, when the dual relation is functional, the resulting operation is necessarily continuous, so that Boolean topological algebras are exactly the extended dual spaces of Boolean residuation algebras whose dual relations are functional. This result and characterisations of those (Boolean) residuation algebras whose duals are topological algebras may be found in [13]. In summary, a residuation algebra has a functional dual if the operation given in 4.3.14 sends prime filters to prime filters. This property is refered to as the residuation algebra being "join-preserving at primes". In the Boolean case, this ensures that the dual space is a topological algebra.

Definition 4.3.20. Let $(B, /, \backslash)$ be a residuation algebra. A subset $A \subseteq B$ is a *residuation ideal* if A is a bounded sublattice of B and for every $a \in A$ and $b \in B$, $a/b \in A$ and $b\backslash \in A$.

Definition 4.3.21. Let $(X, \leq, \Omega, R)$ be an Priestley space with an additional $(n+1)$-ary relation R, and let Δ be a compatible preorder on X. Then Δ is called a *relational congruence* if for every $x_1, \dots, x_n, x'_1, \dots, x'_n, z \in X$ we have

$$\begin{aligned}&[(x'_1, x_1) \in \Delta, \dots, (x'_n, x_n) \in \Delta \text{ and } R(x_1, \dots, x_n, z)] \\ &\Rightarrow \exists z' \in X[R(x'_1, \dots, x'_n, z') \text{ and } (z.z') \in \Delta].\end{aligned}$$

Theorem 4.3.22. *Let $(B, /, \backslash)$ be a residuation algebra and $(X_B, \leq, \Omega, R)$ its dual extended Priestley space. A bounded sublattice $C \subseteq B$ is a residuation ideal if and only if the corresponding compatible preorder Δ on X_B is a relational congruence on X.*

Proof. Let $C \subseteq B$ be a residuation ideal. The preorder Δ is equal to $\mathcal{E}(C)$, as given in the definitions preceding theorem 4.2.13. Let $F_x, F_{x'}, F_y, F_{y'}, F_z \in X_B$ with $(F_x, F_{x'}) \in \Delta$, $(F_y, F_{y'}) \in \Delta$ and $R(F_x, F_y, F_z)$. Recall that

$$\Delta = \{(F_x, F_y) \in X_B \times X_B \mid \forall_{c \in C}[c \in F_y \Rightarrow c \in F_x]\},$$

and we may see that $F_{x'} \cap C \subseteq F_x$, $F_{y'} \cap C \subseteq F_y$, $I_x \cap C \subseteq I_{x'}$ and $I_y \cap C \subseteq I_{y'}$. Let $F = F_{x'} \cdot F_{y'}$ (where "$\cdot$" is given by the construction in lemma 4.3.14) and let $I = {\downarrow_B} (I_z \cap C)$; the latter is an ideal because $I_z \cap C$ is, and it is easy to verify that its down-closure will still be closed under joins. Towards showing that F and I are disjoint, assume $a \in F$ and $c \in C$ with $a \leq c$. By the definition of F, there exists some $b \in F_{x'}$ with $b\backslash a \in F_{y'}$. Because $b\backslash\_$ is a monotone map (as a consequence of OP[$\backslash$]), $a \leq c$ implies that $b\backslash a \leq b\backslash c$, so that $b\backslash c \in F_{y'}$ as well. Moreover, because the maps $c/\_$ and $\_\backslash c$ form an antitone Galois connection, the composite $(c/\_)\backslash c$ is a closure operator and thus

$$b\backslash c = (c/(b\backslash c))\backslash c \text{ and } b \leq (c/b)\backslash c.$$

Now $b\backslash c \in F_{y'}$ implies that $b\backslash c \in F_y$, using the fact that $F_{y'} \cap C \subseteq F_y$ and that C is a residuation ideal. It follows that $(c/(b\backslash c))\backslash c \in F_y$, and by similar reasoning we obtain $(c/b)\backslash c \in F_x$, since F_x is up-closed. Hence, c satisfies the condition for being in $F_x \cdot F_y$. Now because $R(F_x, F_y, F_z)$, by lemma 4.3.14 it is know that $F_x \cdot F_y \geq F_z$, so that $F_x \cdot F_y \subseteq F_z$ and thus $c \in F_z$. Therefore, $c \notin I_z$ and so $a \notin\downarrow_B (I_z \cap C)$, and we may conclude that F and I are disjoint. By the prime filter theorem, it follows that there exists a prime filter $F_{z'}$ with $F \subseteq F_{z'}$ and $F_{z'} \cap I = \emptyset$. The latter equation implies that $I_z \cap C \subseteq I_{z'}$. This in turn may be seen to imply that $F_{z'} \cap C \subseteq F_z$, so that $(F_z, F_{z'}) \in \Delta$. Finally, because $F_{x'} \cdot F_{y'} = F \subseteq F_{z'}$, we may conclude that $R(F_{x'}, F_{y'}, F_{z'})$. On the other hand, suppose that Δ is a relational congruence on X and let $C = \mathcal{S}(\Delta)$, the corresponding bounded sublattice of B. We aim to show that C is a residuation ideal of B. Let $c \in C$ and $b \in B$. Note that $b\backslash c \in C$ if for every $(F_y, F_{y'}) \in \Delta$, $b\backslash c F_{y'} \Rightarrow b\backslash c \in F_y$, which holds if and only if $b\backslash c \in I_y \Rightarrow b\backslash c \in I_{y'}$. To show that this holds, fix $(F_y, F_{y'}) \in \Delta$ and let $b\backslash c \in I_y$. Recall the construction of $\backslash_R$ in theorem 4.3.11 and note that

$$\begin{aligned}
b\backslash c \in I_y &\iff b\backslash c \notin F_y \\
&\iff F_y \notin \eta_B(b\backslash c) \\
&\iff F_y \notin \eta_B(b\backslash c) \\
&\iff F_y \notin (\eta_B(b))\backslash_R(\eta_B(c)) \\
&\iff F_y \notin (R[\eta_B(b),,(\eta_B(c))^C])^C \\
&\iff F_y \in R[\eta_B(b),,(\eta_B(c))^C)] \\
&\iff \exists_{F_x,F_z \in X_B}[R(F_x, F_y, F_z) \text{ and } b \in F_x \text{ and } c \in I_z].
\end{aligned}$$

We may therefore fix some F_x, F_z for which the latter property holds. Now since $(F_y, F_{y'}) \in \Delta$ and (by reflexivity) $(F_x, F_x) \in \Delta$, the fact that Δ is a relational congruence implies that there exists $F_{z'} \in X_B$ such that $R(F_x, F_{y'}, F_{z'})$ and $(F_z, F_{z'}) \in \Delta$. From this last fact it follows that $I_z \cap C \subseteq I_{z'}$, and thus since $c \in I_z$ we may conclude that $c \in I_{z'}$. Recall that $R(F_x, F_{y'}, F_{z'}) \iff F_x\backslash I_{z'} \subseteq I_{y'}$, as per the definition in theorem 4.3.11. It follows that $b\backslash c \in I_{y'}$, which is what we were required to show.

□

Remark 4.3.23. In the case that $(X_B, \leq, \Omega, R)$ is has the discrete topology and is a Stone-topological algebra (so that B is a Boolean algebra and R is functional and continuous and denoted by "$\cdot$"), note that Δ is an equivalence relation and being a relational congruence implies that it is a congruence for the algebraic operation "$\cdot$". We state the special case of this theorem for Stone spaces:

Theorem 4.3.24. *Let $(B, /, \backslash)$ be a Boolean residuation algebra such that its dual extended Stone space $(X_B, \Omega, \cdot)$ is a binary topological algebra. A Boolean subalgebra $C \subseteq B$ is a residuation ideal if and only if the corresponding compatible equivalence relation Δ on X_B is a congruence on X_B.*

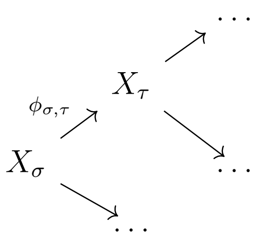
Proof. This follows from the previous remark and theorem 4.3.22 if we recall that in the case that B is a Boolean algebra, C is Boolean subalgebra if and only if its corresponding preorder Δ is an equivalence relation. □

4.4 Application to Formal Language Theory

As topological algebras, profinite algebras are necessarily Stone duals of a Boolean residuation algebra with residuation operations corresponding to their algebraic type.

Consider the diagram given in definition 3.3.1, which sets up the inverse system of finite topological algebras represented in figure 3.1. In particular, we illustrate this for the case of $\mathcal{V}$ is the variety of monoids so that $F_{\mathcal{V}}(A) = A^*$, and $\mathbf{V}$ is the pseudovariety $\mathbf{Mon}_f$ of all finite monoids. Note that now $\mathbf{V}_A$ consists of all finite quotients of A^*, but in keeping with previous notation we denote them by X_σ, where $\sigma : A \to X_\sigma$ is the unique morphism from A for which the unique extension $\overline{\sigma} : F_A(\mathcal{V}) \to X_\sigma$ is the quotient morphism. We denote $\hat{F}_{\mathbf{Mon}f}(A)$ by $\widehat{A^*}$.

This gives us an inverse system of finite topological monoids. Remark 3.1.12 implies that each $\phi_{\sigma,\Omega}$ is a *surjective* continuous homomorphism between the Stone-topological monoids.

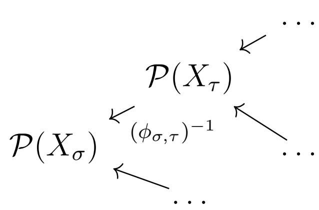


Taking the Stone dual of this inverse system, we get a direct system of Boolean algebras.

Furthermore, the fact that these continuous homomorphisms is surjective allows us to use the following theorem from [13] which ensures that the duals of *surjective* Priestley-topological algebra morphisms (including Stone-topological algebra morphisms) will preserve the residuation operations dual to an additional operation on the Priestley spaces.

Theorem 4.4.1. *[13] Let* $(X, \leq, \Omega_X, \cdot)$ *and* $(Y, \leq, \Omega_Y, \cdot)$ *be Priestley topological algebras with a single binary operation. If* $\phi : X \to Y$ *is a surjective homomorphism, then the dual bounded lattice homomorphism* $h : C \to D$ *between residuation algebras* $(C, /, \backslash)$ *and* $(D, /, \backslash)$ *is a residuation algebra homomorphism.*

The morphisms dual to each $\phi_{\sigma,\tau}$ are thus homomorphisms of Boolean residuation algebras, and we have an direct system in this category. Because these finite topological monoids have the discrete topology, the set of clopens is just the power-set, and thus we may roughly represent this direct system as follows:



A direct system of algebras of a fixed signature is generally given by a quotient of the disjoint union with appropriately defined algebraic operations; we direct the reader to Bourbaki [6] or Ribes and Zalesskii [26] for further details. In this particular case, where the system of Boolean residuation algebras of which we are taking the direct limit of may be embedded into the residuation algebra $\mathcal{P}(A^*)$ in the following way, as can be seen by taking the *discrete* dual of the above inverse system and the quotient morphisms from A^*:

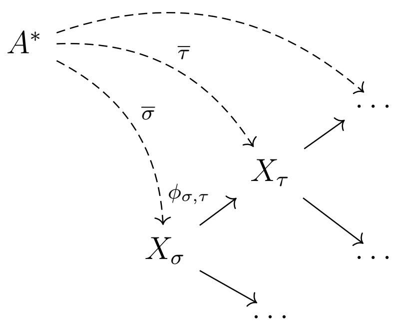


Figure 4.1: The inverse system $(X_\sigma)_{\sigma\in\mathbf{V}_A}$ consists of finite quotients of A^* and homomorphisms commuting with these quotient morphisms, as described in definition 3.3.1.

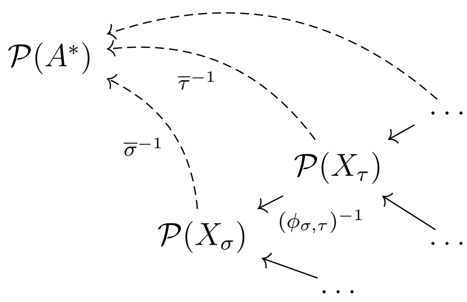


Figure 4.2: Discrete dual of the data on the left, applying discrete duality between sets and CABAs.

Viewing $(A^*, \cdot)$ and each $(X_\sigma, \cdot)$ as posets with the discrete order and thus an order-compatible relation R which is the graph of "$\cdot$", we may apply lemma 4.3.4 and the ensuing comments to see each is dual to the power-set with a residuated family of binary operations. In particular, $(\mathcal{P}(A^*), \cdot, /, \backslash)$ is equipped with the lifted concatenation and its residuals. Note that this is exactly the structure on $\mathcal{P}(A^*)$ that featured in the chapter on language theory. However, since we are interested only in Boolean and residuation operations, we ignore the "$\cdot$" operation on each of these power-sets.

The direct limit turns out to be the following Boolean residuation subalgebra of $(\mathcal{P}(A^*), /, \backslash)$:

$$\begin{aligned}\varinjlim(\mathcal{P}(X_\sigma))_{\sigma\in\mathbf{V}_A} &= \bigcup\{\sigma^{-1}[\mathcal{P}(X_\sigma)] \mid \sigma : A \to X_\sigma \in \mathbf{V}_A\} \\ &= \bigcup\{\sigma^{-1}[P] \mid P \in \mathcal{P}(X_\sigma) \text{ and } \sigma : A \to X_\sigma \in \mathbf{V}_A\} \\ &= \mathit{Rec}(A^*).\end{aligned}$$

We already know $\mathit{Rec}(A^*)$ to be closed under the relevant operations (by lemma 2.2.2). Now we also know that it is in fact the colimit of a direct system dual to the inverse system of which $\widehat{A^*}$ is the limit.

Since inverse limits are dual to direct limits (which, it is important to recall, is a colimit), it would seem we have shown that which was claimed: that $\mathit{Rec}(A^*)$ is dual to $\widehat{A^*}$. However, there is one important problem: the inverse limit resulting in $\widehat{A^*}$ is taken in the category of topological algebras, while the direct limit $\mathit{Rec}(A^*)$ is taken in the category of Boolean residuation algebras. These do not exactly match as far as our knowledge of which categories are dual to each other goes. Gehrke proved ([13]) that $\widehat{A^*}$ is still the inverse limit of the same system in the category of extended Stone spaces which is dual to the category of Boolean residuation algebras, which contains the category of Stone-topological algebras. An alternative approach using canonical extensions is given in [12], where it is shown that $Rec(A^*)$ is in the category dual to Stone-topological algebras, namely that it is join-preserving at primes (see remark 4.3.19).

The discussion in this section so far leads to the following result:

Theorem 4.4.2. *[13] Let A be a finite set. Under extended Stone duality, the Boolean residuation algebra $(Rec(A^*), /, \backslash)$ is the dual of the free profinite monoid $(\widehat{A^*}, \cdot)$.*

4.4.1 Equational Theory

In the chapter on Reiterman's theorem, we explored the connection between profinite identities and pseudovarieities. The chapter on formal language theory conlcuded with Eilenberg's theorem, detailing the connection between varieties of languages and pseudovarieties of monoids. Now that we know that $(Rec(A^*), /, \backslash)$ is the extended Stone dual of $\widehat{A^*}$, we are now equipped to describe a direct connection between various collections of languages and corresponding sets of profinite monoid identities.

We already have a system in place connecting sublattices, Boolean subalgebras, residuation ideals and Boolean residuation ideals of $Rec(A^*)$ with certain preorders and equivalence relations on $\widehat{A}$.

We summarise the known one-to-one correspondences in the table in figure 4.3.

We already know from chapter 3 that elements of $\widehat{A^*}$ are profinite terms which have an interpretation in every finite monoid, and that pairs in $\widehat{A^*} \times \widehat{A^*}$ can be interpreted as profinite identities that may or may not hold in a given monoid. Now, we use the correspondences above to develop a similar interpretation of profinite identities with respect to recognizable languages based on the relation in the discussion leading up to theorem 4.2.13.

Being the dual of the Boolean Algebra $Rec(A^*)$, we may note that $\widehat{A^*} \cong (PrF(Rec(A^*), \Omega, \cdot)$, the extended dual prime filter space. Thus, we may associate each $u \in \widehat{A^*}$ with a prime filter $F_u \in PrF(Rec(A^*))$, and this allows us to give the following definitions:

Definition 4.4.3. Let L be recognisable language and let $(u, v) \in \widehat{A^*} \times \widehat{A^*}$.

1. We call (u, v) a *profinite lattice identity* when it is denoted $u \to v$, and we say that $L \models u \to v$ (read "L models $u \to v$") if and only if $L \in F_v \Rightarrow L \in F_u$.

Substructures of $Rec(A^*)$	**Subsets of** $\widehat{A^*} \times \widehat{A^*}$	**Quotients of** $\widehat{(A^*)}$	**Theorem**
Bounded Sublattices	Priestley-Compatible Preorders	Quotients of Priestley Spaces	4.2.13
Boolean Subalgebras	Stone-Compatible Equivalence Relations	Quotients of Stone Spaces	4.2.14
Residuation Ideals	Priestley-Compatible Preorders which are Relational Congruences	Priestley-Topological Monoid Quotients	4.3.22
Boolean Residuation Ideals	Stone-Compatible Congruences	Stone-Topological Monoid Quotients	4.3.24

Figure 4.3

2. We call (u, v) a *profinite symmetric lattice identity* when it is denoted $u \leftrightarrow v$ and we say that $L \models u \leftrightarrow v$ if and only if $L \models u \to v$ and $L \models v \to u$.

3. We call (u, v) a *profinite monoid inequality* when it is denoted $u \leq v$, and we say that $L \models u \leq v$ if and only if for every $x, y \in \widehat{A^*}$, $L \models x \cdot u \cdot y \to x \cdot u \cdot y$.

4. We call (u, v) a *profinite monoid identity* or a *pseudoidenity* when it is denoted $u = v$, and we say that $L \models u = v$ if and only if $L \models u \leq v$ and $L \models u \leq v$.

Definition 4.4.4. Let $C \subseteq Rec(A^*)$. We say that $C \models u \to v$ (respectively $u \leftrightarrow v, u \leq v$ and $u = v$) if and only if $L \models u \to v$ for every $L \in C$.

With these definitions in mind, the one-to-one correspondences in theorems 4.2.13 and 4.2.14 becomes a Galois connection between theories and models:

Corollary 4.4.5. *Let* $R_{\to}$ *be the relation on* $Rec(A^*) \times (\widehat{A^*} \times \widehat{A^*})$ *given by* $L\, R_{\to}\, (u, v) \iff (L \models u \to v)$. *The polarity* $(Rec(A^*), \widehat{A^*} \times \widehat{A^*}, R_{\to})$ *yields the standard antitone Galois connection*

$$Eqn_{\rightarrow} : \mathcal{P}(Rec(A^*)) \rightleftarrows \mathcal{P}(\widehat{A^*} \times \widehat{A^*}) : Mod_{\rightarrow}$$

where for any $U \subseteq Rec(A^*)$ *and* $V \subseteq \widehat{A^*} \times \widehat{A^*}$*:*

$$Mod_{\rightarrow}(V) = \{L \in Rec(A^*) \mid \forall_{(u,v) \in V}(L \models u \rightarrow v)\}$$

and

$$Eq_{\rightarrow}(U) = \{(u,v) \in \widehat{A^*} \times \widehat{A^*} \mid \forall_{L \in U}(L \models u \rightarrow v)\}.$$

The Galois-closed sets of this Galois connection are the bounded sublattices of $Rec(A^*)$ *and the Priestley-compatible preorders on* $\widehat{A^*}$

Corollary 4.4.6. *Let* $R_{\leftrightarrow}$ *be the relation on* $Rec(A^*) \times (\widehat{A^*} \times \widehat{A^*})$ *given by* $L\, R_{\leftrightarrow}\, (u,v) \iff (L \models u \leftrightarrow v)$. *The polarity* $(Rec(A^*, \widehat{A^*} \times \widehat{A^*}, R_{\leftrightarrow})$ *yields the standard antitone Galois connection*

$$Eqn_{\leftrightarrow} : \mathcal{P}(Rec(A^*)) \rightleftarrows \mathcal{P}(\widehat{A^*} \times \widehat{A^*}) : Mod_{\leftrightarrow}$$

where for any $U \subseteq Rec(A^*)$ *and* $V \subseteq \widehat{A^*} \times \widehat{A^*}$*:*

$$Mod_{\leftrightarrow}(V) = \{L \in Rec(A^*) \mid \forall_{(u,v) \in V}(L \models u \leftrightarrow v)\}$$

and

$$Eq_{\leftrightarrow}(U) = \{(u,v) \in \widehat{A^*} \times \widehat{A^*} \mid \forall_{L \in U}(L \models u \leftrightarrow v)\}.$$

The Galois-closed sets of this Galois connection are the Boolean subalgebras of $Rec(A^*)$ *and the Stone-compatible equivalence relations on* $\widehat{A^*}$

To keep in step with this notation, we define $Mod_{\leq}$ and $Mod_{=}$ accordingly.

For a fixed alphabet A, we now have a one-to-one correspondence between bounded lattices of recognisable languages and sets of profinite lattice identities which are Priestley-compatible preorders, as well as for Boolean algebras of recognisable languages and sets of profinite symmetric lattice identities which are Stone-compatible equivalence relations. We do not have an analogous Galois connection for residuation ideals, but as we will see below there is still connection between residuation ideals and sets of profinite monoid inequalities. The following proof was presented for a more general case in [13].

Lemma 4.4.7. *Let* $C \subseteq (Rec(A^*), /, \backslash)$ *be a bounded sublattice. Then* C *is a residuation ideal if and only if it is the set of models for some set of profinite monoid inequalities.*

Proof. Firstly, suppose that C is a residuation ideal. By theorem 4.3.22, C corresponds to a Priestley-compatible preorder Δ on $(\widehat{A^*}, \cdot)$ which is a relational congruence for "$\cdot$". Recall that by theorem 4.4.5, $C = Mod_{\rightarrow}(\Delta)$.

Now, Δ being a relational congruence implies that for every $(x_1, x_2) \in \Delta$ and $(x_1', x_2') \in \Delta$, we have $(x_1 \cdot x_2, x_1' \cdot x_2') \in \Delta$. We will show that $C = Mod_{\leq}(\Delta)$. It is trivial to see that $Mod_{\leq}(\Delta) \subseteq Mod_{\rightarrow}(\Delta) = C$ by choosing x and y to be the identity, so now it remains to fix some $L \in C$ and show that for every $(u, v) \in \Delta$, $L \models u \leq v$. To this end, let $x, y \in \widehat{A^*}$. By reflexivity, $(x, x) \in \Delta$ and $(y, y) \in \Delta$. Because $(u, v) \in \Delta$ and Δ is a relational congruence for the graph of "$\cdot$", it follows that $(x \cdot u \cdot y, x \cdot v \cdot y) \in \Delta$. Thus, $L \models x \cdot u \cdot y \rightarrow x \cdot v \cdot y$, and because x and y were arbitrary, we may conclude that $L \models u \leq v$. Hence, $C = Mod_{\leq}(\Delta)$.

To show the converse, suppose that $C = Mod_{\leq}(\Delta)$ for some $\Delta \subseteq \widehat{A^*} \times \widehat{A^*}$. Let $L \in C$ and $(u, v) \in \Delta$. We aim to show that $L \models u \leq v$ implies that for any $K \in Rec(A^*)$, it must be that $K \backslash L \models u \leq v$, so that $K \backslash L \in C$. To this end, assume $L \models u \leq v$, let $x, y \in \widehat{A^*}$ and suppose that $K \backslash L \in F_{x \cdot v \cdot y}$. By Stone duality, this means $F_{x \cdot v \cdot y} \in \eta_{Rec(A^*)}(K \backslash L)$. Recall that this is the Stone duality isomorphism

$$\eta_{Rec(A^*)} : Rec(A^*) \rightarrow Clop(\widehat{A^*}),$$

which we will refer to as η for brevity throughout this proof. Let $R \subseteq (\widehat{A^*})^3$ denote the graph of "$\cdot$". Recall that by remark 4.3.12, R gives rise to a residuated family $(\cdot_R, /_R, \backslash_R)$ on $\mathcal{P}(\widehat{A^*})$, where $\cdot_R$ is just the lifting of $\cdot$ to $\mathcal{P}(\widehat{A^*})$ and $/_R, \backslash_R$ restrict to $Clop(\widehat{A^*})$ and are exactly the duals of "$\cdot$", so that

$$(Rec(A^*), /, \backslash) \cong (Clop(\widehat{A^*}), /_R, \backslash_R)).$$

By lemma 4.3.14,

$$\begin{aligned} \eta(K \backslash L) &= (\eta(K)) \backslash_R (\eta(L)) \\ &= (R[\eta(K), , (\eta(L))^C])^C. \end{aligned}$$

Using the fact that $\cdot_R$, $/_R$, and $\backslash_R$ are a residuated family, we may observe that

$$\{F_{x \cdot v \cdot y}\} \subseteq (\eta(K)) \backslash_R (\eta(L)) \iff \eta(K) \cdot_R \{F_{x \cdot v \cdot y}\} \subseteq \eta(L).$$

Th latter property holds if and only if for every $F_w \in \eta(K)$, it follows that $F_{w \cdot (x \cdot v \cdot y)} \in \eta(L)$. Equivalently, $L \in F_{w \cdot (x \cdot v \cdot y)}$. Now because $L \models u \leq v$, for every $F_w \in \eta(K)$, since $L \in F_{w \cdot (x \cdot v \cdot y)}$, it will also be true that $L \in F_{w \cdot (x \cdot u \cdot y)}$. Following the same string of equivalences backwards, we may deduce that $\eta(K) \cdot_R \{F_{x \cdot u \cdot y}\} \subseteq \eta(L)$ and thus $F_{x \cdot u \cdot v} \in \eta(K \backslash L)$. Hence, $K \backslash L \models x \cdot u \cdot y \rightarrow x \cdot v \cdot y$ for every $x, y \in \widehat{A^*}$ and so $K \backslash L \models u \leq v$. The proof that $K / L \models u \leq v$ is similar, and we see that C is a residuation ideal. □

Theorem 4.4.8. *Let $C \subseteq (Rec(A^*), /, \backslash)$ be a Boolean subalgebra. Then C is a Boolean residuation ideal if and only if it is the set of models for some set of profinite monoid equations.*

Proof. Suppose C is a Boolean residuation ideal. Then by theorem 4.4.7, $C = Mod_{\leq}(\Delta)$ for the corresponding Priestley-compatible preorder Δ that is also a relational congruence. Because C is a Boolean subalgebra, by the proof of 4.2.13 we know that this Δ is a Stone-compatible congruence. The symmetry of Δ will imply that $Mod_{\leq}(\Delta) = Mod_{=}(\Delta)$.

On the other hand, if $C = Mod_{=}(A)$ for some $A \subseteq \widehat{A^*} \times \widehat{A^*}$, closure of C under complements ensures that for every $(u, v) \in A$, $L \models u \leq v$ if and only if $L \models v \leq u$. Hence, $C = Mod_{\leq}(A)$ as well and theorem 4.4.7 allows us to conclude that C is a residuation ideal. □

We summarise the above connections between subsets of $Rec(A^*)$ and pairs of profinite monoid terms in the following theorem:

Theorem 4.4.9. *A subset* $C \subseteq (Rec(A^*), /, \backslash)$ *is a:*

1. *bounded sublattice if and only if it is the set of models for a set of profinite lattice equations.*

2. *Boolean subalgebra if and only if it is the set of models for a set of profinite symmetric lattice equations.*

3. *residuation ideal if and only if it is the set of models for a set of profinite algebra equations.*

4. *Boolean residuation ideal if and only if it is the set of models for a set of profinite monoid identities or pseudoidentities.*

As a consequence of the above results and one-to-one the correspondences between

1. Bounded sublattices which are residuation ideals and Priestley-compatible preorders which are relational congruences, and

2. Boolean subalgebras which are residuation ideals and Stone-compatible congruences,

we essentially have a characterisation of those subsets of $\widehat{A^*} \times \widehat{A^*}$ which give rise to residuation ideals and Boolean residuation ideals of recognisable languages.

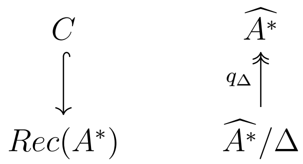
Lemma 4.4.10. *There exists a one-to-one correspondence between Priestley-compatible preorders on* $(\widehat{A^*}, \cdot)$ *which are relational congruences for "·" and subsets of* $Rec(A^*)$ *defined by a set of profinite monoid inequalities.*

Proof. The bijection is given by mapping any Priestley-compatible preorder Δ which is a relational congruence to the subset $C = Mod_{\rightarrow}(\Delta)$. It is known by theorem 4.3.22 that this will be a residuation ideal, and by theorem 4.4.7 $C = Mod_{\rightarrow}(\Delta) = Mod_{\leq}(\Delta)$. This map is injective because we already know that the correspondence $\Delta \mapsto C = Mod_{\rightarrow}(\Delta)$ is a bijection onto the set of bounded sublattices of $Rec(A^*)$. It is also surjective, because any $C = Mod_{\leq}(A)$ for some $A \subseteq \widehat{A^*} \times \widehat{A^*}$ will be a residuation ideal by theorem 4.4.7 and thus equal to $Mod_{\rightarrow}(\Delta)$ for a Priestley-compatible preorder Δ which is a relational congruence. $\square$

Lemma 4.4.11. *There exists a one-to-one correspondence between Stone-compatible congruences on $(\widehat{A^*}, \cdot)$ and subsets of $Rec(A^*)$ defined by a set of pseudoidentities.*

Proof. The proof of lemma 4.4.10 can be adapted for this context, referencing theorem 4.3.24 instead of theorem 4.3.22 and theorem 4.4.8 instead of theorem 4.4.7. $\square$

We have thus completed the description of Gehrke, Grigorieff and Pin's characterisation [13, 14, 12] of various substructures of $Rec(A^*)$ as sets models for different types of "equations" and identified the corresponding substructures of $\widehat{A^*} \times \widehat{A^*}$ which give rise to them. This is summarised in the image below and the table on the next page. For any bounded sublattice C, there is a corresponding Priestley-compatible preorder Δ on $\widehat{A^*}$. This preorder corresponds to a quotient which is dual to the inclusion morphism of C into $Rec(A^*)$.



Now Δ is a set of profinite words, and the table below summarises the corresondences we have shown in this chapter. The bounded sublattice C is a substructure of a given type if and only if Δ is a preorder or congruence of the corresponding type. In this case, C is exactly the set of models of Δ for the given equation type.

Substructures of $Rec(A^*)$	**Types of Equations**	**Subsets of** $\widehat{A^*} \times \widehat{A^*}$
Bounded Sublattices	$u \to v$	Priestley-Compatible Preorders
Boolean Subalgebras	$u \leftrightarrow v$	Stone-Compatible Equivalence Relations
Residuation Ideals	$u \leq v$	Priestley-Compatible Preorders which are Relational Congruences
Boolean Residuation Ideals	$u = v$	Stone-Compatible Congruences

Remark 4.4.12. Upon careful inspection, one can see that the definition a Stone-compatible congruence corresponds exactly to the conditions on each Σ_A in theorem 3.4.8, which gives a characterisation for Reiterman's profinite theories. We are therefore very close to a result which captures the composition of Eilenberg and Reitermans' theorems, but for one last component: behaviour with respect to different alphabets. On the lattice side, we require closure on the inverse morphisms, and on the profinite identities side, closure under substitution.

Theorem 4.4.13. *[13] Let $\mathcal{C}$ denote a mapping $A \mapsto \mathcal{C}(A)$, where each $\mathcal{C}(A)$ is a bounded sublattice of $Rec(A^*)$. Let Σ denote the mapping $A \mapsto \Sigma_A$ where Σ_A is the Priestley-compatible preorder on $\widehat{A^*}$ corresponding to $\mathcal{C}_A$. p Then $\mathcal{C}$ is closed under inverse morphisms if and only if the equational class $\Sigma = \bigcup_{A \subseteq \mathcal{A}} \Sigma_A$ is closed under substitution. The latter property is that for any monoid homomorphism $h : A^* \to B^*$ which extends uniquely to $\widehat{h} : \widehat{A^*} \to \widehat{B^*}$ (by theorem 3.3.7, because we may consider it a homomorphism into $\widehat{B^*}$, if $(u, v) \in \Sigma_A$ then $(\widehat{h}(u), \widehat{h}(v)) \in \Sigma_B$.*

Proof. Let $h : A^* \to B^*$ me a monoid homomorphism. Its preimage may be restricted to $h^{-1} : Rec(B^*) \to Rec(A^*)$ by lemma 2.2.5. The Stone dual of this morphism is $(h^{-1})^{-1} : \widehat{A^*} \to \widehat{B^*}$, and because it agrees with h on the elements of A^*, by theorem 3.3.5 it is equal to the unique extension $\widehat{h}$. Implicitly applying an isomorphism and treating the elements of $\widehat{A^*}$ and $\widehat{B^*}$ as prime filters of $Rec(A^*)$ and $Rec(B^*)$ respectively (theorem 4.4.2), we have the following observation for every $L \in Rec(B^*)$ and $u \in \widehat{A^*}$:

$$\begin{aligned} L \in F_{\widehat{h}(u)} &\iff \hat{h}(u) \in \eta_{Rec(B^*)}(L) \\ &\iff u \in \hat{h}^{-1}[\eta_{Rec(B^*)}(L)] = \eta_{Rec(B^*)}(h^{-1}(L)) \\ &\iff h^{-1}(L) \in F_u. \end{aligned}$$

Now, suppose $\mathcal{C}$ is closed under inverse homomorphisms. Fix two alphabets A and B, a homomorphism $h : A^* \to B^*$ and let $(u, v) \in \widehat{A^*}$. Let $L \in C(B)$, so that $h^{-1}[L] \in C(A)$ and thus $h^{-1}[L] \models u \to v$. We aim to show that $L \models \hat{h}(u) \to \hat{(h)}(v)$. For this purpose, assume that $\hat{h}(v) \in \Sigma_A$. By the above string of equivalences, this implies that $h^{-1}(L) \in F_v$. Because $h^{-1}[L] \models u \to v$, we know that $h^{-1}(L) \in F_u$, from which we may conclude $L \in F_{\hat{h}(u)}$ as needed. On the other hand, suppose Σ is closed under substitution and let $L \in C(B)$. We are required to prove that $h^{-1}[L] \in C(A)$, which is equivalent to saying $h^{-1}[L] \models u \to v$ for every $(u, v) \in \Sigma_A$. Fixing any $(u, v) \in \Sigma_A$, we know that $(\hat{h}(u), \hat{h}(v)) \in \Sigma_B$ so that $L \models \hat{h}(u) \to \hat{h}(v)$. Now, assume that $h^{-1}[L] \in F_{\hat{h}(v)}$. Then $L \in F_{\hat{h}(v)}$, and by the next-to-last observation it follows that $L \in F_{\hat{h}(u)}$, so that $h^{-1}[L] \in F_u$, which is what we wished to prove. □

Thus, mappings of the form $A \to \mathcal{C}(A)$ where each $\mathcal{C}(A)$ is a Boolean residuation ideal of $Rec(A^*)$ correspond one-to-one to mappings $A \mapsto \Sigma_A$ where each $\Sigma_A \subseteq \widehat{A^*} \times \widehat{A^*}$ is a Stone-compatible congruence on the profinite monoid $\widehat{A^*}$. Because this assignment is closed under substitution if and only if $\mathcal{C}$ is closed under inverse morphisms, Σ is a profinite theory exactly when $\mathcal{C}$ is a variety of recognisable languages.

Conclusion and Further Work

Eilenberg's theorem gave a one-to-one correspondence between varieties of languages and pseudovarieties of monoids. Reiterman's theorem gave a one-to-one correspondence between pseudovarieties of monoids and profinite theories, or systems of Stone-compatible congruences on free profinite monoids closed under substitution. The duality-theoretic observation that for each alphabet A, the residuated Boolean algebra $(Rec(A^*), /, \backslash)$ is the extended stone dual of the free profinite monoid $\widehat{A^*}$ and the application of existing correspondences between certain subobjects of $Rec(A^*)$ and certain quotients of $\widehat{A^*}$ yields a short-cut linking varieties of languages and profinite theories. When combined with Reiterman's theorem, this may be seen as an alternative proof for Eilenberg's theorem.

Furthermore, the work in [14, 13] has yielded a correspondence for collections of recognisable languages that are not varieties of languages - they have other closure properties such as not being closed under complements or under the residuation operations (these correspond to the bounded sublattice and boolean subalgebra cases). The closure conditions for varieties of languages correspond exactly to Boolean residuation ideals, which are the substructures of $Rec(A^*)$ that are exactly the duals of Stone-topological monoid quotients (equivalently, the model classes for Stone-compatible congruences of pseudoidentities). The classes of monoids corresponding to these alternative classes of languages have not, to our knowledge, been characterised, and this could potentially be an interesting direction for further work.

It is worth noting theorem 4.4.2 chapter 3 for the case of free monoids was first shown in a weaker form by Pippinger [22]; that the underlying topological space of the profinite completion of the free monoid was dual under stone duality to the Boolean algebra of recongisable languages over that free monoid. Incorporating the additional operations into the duality-theoretic understanding of the connection between $Rec(A^*)$ and $\widehat{A^*}$ allowed for the characterisation of those Boolean subalgebras dual to stone-compatible congruences, and these are exactly the residuation ideals.

A fruitful consequence of the duality-theoretic perspective is that it has served as a springboard for generalisation; the work in [13] and [11] applies to algebras with any finite number of finitary operation and corresponding residuation algebras with as many residuation operations as there are input coordinates to each of the corresponding algebraic operations. These ideas have also been extended to a more category-theoretic approach, where the focus is on T-algebras, monads and appropriately adapted definitions of recognition [29], and the existence of such a dual equivalence is a key part of their constructions.

Appendices

Appendix A

Preliminaries

- We assume knowledge of the basics of universal algebra such as signatures, homomorphisms, subalgebras, products, varieties of algebras, free algebras and Birkhoff's theorem. A standard reference is [8]. We generally denote varieties by calligraphic script such as $\mathcal{V}$, and the free algebra over a set A is denoted as $F_A(\mathcal{V})$.

- We assume basic knowledge of topological spaces such as separation properties, bases and subases, product constructions, and properties of continuous maps. A standard reference is [30]. We include some results and definitions which are pertinent to chapters 3 and 4.

- We assume knowledge of basic category theoretic concepts such as categories, functors, natural transformations, diagrams and limits. A standard reference is [19]. We will generally denote categories in bold-face script, such as **Set** or **C**.

- We assume basic knowledge of partially ordered sets (posets), lattices and Boolean algebras. A standard reference is [23].

We include some definitions and lemmas which are pertinent to this work for ease of reference, especially concepts from order theory and topology which are relevant to the duality-theoretic work explored in chapter 3.

A.1 Ordered Structures

Definition A.1.1. Let L be a lattice. We call L

1. *bounded* if it has a greatest element 1 and a least element 0

2. *distributive* if for every $a, b, c \in L$,we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

3. *complete* if $\bigvee S$ and $\bigwedge S$ exist for every $S \subseteq L$

4. *completely distributive* if arbitrary meets distribute over arbitrary joins, and vice versa.

Definition A.1.2. Let L be a lattice and let $a, b, x \in L$ and $S \subseteq L$. An element x is called

1. *join irreducible* if $a \vee b = x \Rightarrow x = a$ or $x = b$
2. *meet irreducible* if $a \wedge b = x \Rightarrow x = a$ or $x = b$
3. *join prime* if $x \leq a \vee b \Rightarrow x \leq a$ or $x \leq b$
4. *meet prime* if $a \wedge b \leq x \Rightarrow a \leq x$ or $b \leq x$
5. *completely join irreducible* if $x = \bigvee S \Rightarrow x \in S$
6. *completely meet irreducible* if $x = \bigwedge S \Rightarrow x \in S$.

Definition A.1.3. Let L be a lattice and $S \subseteq L$. We say that S is

1. *join dense* in L if every element of L is equal to the join of some finite $S' \subseteq S$
2. *meet dense* in L if every element of L is equal to the meet of some finite $S' \subseteq S$
3. *completely join dense* in L if every element of L is equal to the join of some $S' \subseteq S$
4. *completely meet dense* in L if every element of L is equal to the meet of some $S' \subseteq S$
5. a *filter* if S is up-closed and closed with respect to finite meets
6. a *prime* filter if S is a filter and $a \vee b \in S$ imply that $a \in S$ or $b \in S$
7. an *ideal* if S is down-closed and closed with respect to finite joins
8. a *prime* ideal if S is an ideal and $a \wedge b \in S$ imply that $a \in S$ or $b \in S$.

Definition A.1.4. A lattice L is called a DL^+ if it is complete, completely distributive and if the completely join-irreducible elements are completely join dense in L and the completely meet irreducible elements are completely meet dense in L.

Definition A.1.5. A bounded distributive lattice $(B, 0, 1, \wedge, \vee)$ is a *Boolean algebra* if for every $a \in B$ there exists a unique element $\neg a$ such that

$$a \wedge \neg a = 0$$
$$a \vee \neg a = 1.$$

It is complete if the underlying lattice is complete, and is a *CABA* (complete and atomic Boolean algebra) if the underlying lattice is a DL^+.

Definition A.1.6. Let $(X, \leq)$ and $(Y, \leq)$ be posets with monotone maps

$$f : X \rightleftarrows Y : g.$$

If for each $x \in X, y \in Y$ we have f and g satisfying the property

$$f(x) \leq y \iff x \leq g(y)$$

then we say that f is the *lower adjoint* of g and g is the *upper adjoint* of f. We refer to this as a *monotone galois connection* or just a *galois connection.*

Lemma A.1.7. *Properties of Adjoints*

1. *Adjoints, when they exists, are unique.*

2. *A monotone map $f : X \to Y$ between posets with an upper adjoint $f^\sharp$ satisfies*
$$f^\sharp(y) = \bigvee f^{-1}(\downarrow y)$$
for every $y \in Y$. Moreover, $f^\sharp(y)$ is the largest element $x \in X$ such that $f(x) \leq y$.

3. *A monotone map $f : X \to Y$ between posets with a lower adjoint $f^\flat$ satisfies*
$$f^\flat(y) = \bigwedge f^{-1}(\uparrow y)$$
for every $y \in Y$. Moreover, $f^\flat(y)$ is the least element $x \in X$ such that $y \leq f(x)$.

4. *For every $x \in X$ and $y \in Y$,*
$$ff^\sharp(y) \leq y \text{ and } x \leq ff^\flat(x).$$

5. *A monotone map between complete lattices has a upper adjoint iff it preserves arbitrary joins.*

6. *A monotone map between complete lattices has a lower adjoint iff it preserves arbitrary meets.*

Definition A.1.8. Let $(X, \leq)$ and $(Y, \leq)$ be posets with antitone maps

$$f : X \rightleftarrows Y : g.$$

If for each $x \in X, y \in Y$ we have f and g satisfying the property

$$y \leq f(x) \iff x \leq g(y)$$

We refer to this as an *antitone galois connection.*

Definition A.1.9. A unary operator $f : X \to X$ on a poset is know as a *closure operator* if for every $x \in X$ it satisfies

1. $x \leq x' \Rightarrow f(x) \leq f(x')$
2. $f(x) = x$
3. $x \leq f(x)$.

Lemma A.1.10. *If $f : X \rightleftarrows Y : g$ form an antitone Galois connection, then both $f \circ g$ and $g \circ f$ are closure operators.*

Definition A.1.11. Let X,Y be sets and $R \subseteq X \times Y$ a relation between them. Then the triple (X, Y, R) is a *polarity.*

Definition A.1.12. Let $P = (X, Y, R)$ be a polarity. This gives rise to an antitone Galois-connection

$$\mathcal{E} : P(X) \rightleftarrows P(Y) : \mathcal{S}$$

Where for $A \subseteq X$, $B \subseteq Y$

$$\mathcal{E}(A) = \{y \in Y \mid \forall a \in A : aRy\}$$
$$\mathcal{S}(B) = \{x \in X \mid \forall b \in B : xRb\}.$$

The lattices of *Galois-closed* sets are

$$(\mathcal{P}(X))^G = \{A \subseteq X \mid \mathcal{ES}(A) = A\}$$

and

$$(\mathcal{P}(Y))^G = \{B \subseteq Y \mid \mathcal{SE}(B) = B\}.$$

In the former, the meet operation is given by intersection and for $A, A' \in (\mathcal{P}(X))^G$ we have $A \vee A' = \mathcal{ES}(A \cup A')$. The complete lattice operations on $(\mathcal{P}(Y))^G$ are defined accordingly. It is useful to note that as a consequence of forming an antitone Galois connection, the composites $\mathcal{ES}$ and $\mathcal{SE}$ are closure operators.

A.2 General Topology

Definition A.2.1. Let (X, Ω_X) be a topological space. Let $\sim$ an equivalence relation on X and $q : X \to X/\sim$ denote the corresponding quotient map. The *quotient* topology on $(X/\sim)$ is defined as $\Omega_{X/\sim} = \{U \subseteq X/\sim \mid q^{-1}[U] \in \Omega_X\}$. We call $(X/\sim, \Omega_{X/\sim})$ the *quotient space*.

Lemma A.2.2. *A bijective continuous map between compact Hausdorff spaces is a homeomorphism.*

Proposition A.2.3. *If $\mathcal{B}$ is a base for a topological space X and $A \subseteq X$ is a subspace, then the collection of $B \cap A$ for $B \in \mathcal{B}$ is a base for A.*

Definition A.2.4. A topological space (X, Ω) is called *zero-dimensional* if and only if it has a base of clopen sets.

Definition A.2.5. A topological space (X, Ω) is called a *Stone Space* if it is compact, Hausdorff and zero-dimensional.

Definition A.2.6. An ordered topological space $(X, \Omega, \leq)$ is *totally order-disconnected* if and only if for every $x, y \in X$ with $x \not\leq y$ there exists a clopen up-set U such that $x \in U$ but $y \notin U$.

Definition A.2.7. An ordered topological space $(X, \Omega, \leq)$ is called a *Priestley Space* if it is compact and totally order-disconnected.

Theorem A.2.8. *The following is true of Priestley spaces:*

1. *Every Priestley space is Hausdorff and zero-dimensional, and thus the underlying topological space is a Stone space.*

2. *The product of Priestley spaces is again a Priestley space.*

A.3 Category Theory

Definition A.3.1. A category **C** is *filtered* if every finite diagram to it has a cocone. The dual notion is that of a *cofiltered* category.

Remark A.3.2. This is essentially the categorical version of a directed pre-order.

Definition A.3.3. A diagram $F : \mathbf{D} \to \mathbf{C}$ is *filtered* if **D** is a filtered category. A *filtered limit* is the limit of a filtered diagram, likewise a *cofiltered limit* is the limit of a cofiltered diagram.

We include the definition of equivalence of catogories, as it is central to duality theory and we will be consistent in our use of the below notation.

Definition A.3.4. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A pair of covariant functors

$$\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$$

is said to give an *equivalence of categories* if and only if there is exist natural isomorphisms $\eta : GF \to 1_{\mathbf{C}}$ and $\mu : 1_{\mathbf{D}} \to FG$ such that for every $f : A \to A'$ in $\mathbf{C}$ and $g : B \to B'$ in $\mathbf{D}$ the following diagrams commute:

$$\begin{array}{ccc} GF(A) & \overset{\eta_A}{\cong} & A \\ \downarrow{\scriptstyle GF(f)} & & \downarrow{\scriptstyle f} \\ GF(A') & \overset{\eta_{A'}}{\cong} & A' \end{array} \qquad\qquad \begin{array}{ccc} B & \cong & FG(B) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle i} \\ B' & \cong & FG(B') \end{array}$$

If these functors are instead contravariant, this is called a *dual equivalence* of categories, and these are the connections studied in duality theory.

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