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HOPF AND LIE ALGEBRAS IN SEMI-ADDITIVE VARIETIES

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To Jiří Adámek for his 70^{th} birthday

Abstract. We study Hopf and Lie algebras in entropic semi-additive varieties with an emphasis on adjunctions related to the enveloping monoid functor and the primitive element functor. These investigations are in part based on the concept of the abelian core of a semi-additive variety and its monoidal structure in case the variety is entropic.

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Introduction

The class of entropic varieties, that is, varieties whose algebraic theory is commutative, provides the canonical setting for generalizing classical Hopf algebra theory. By this the following is meant: In every entropic variety one not only can (as in any symmetric monoidal category) express the fundamental concepts of Hopf algebra theory, but also prove a substantial part of the theory of Hopf algebras. This is shown in [15], a paper based on combining results and methods from the theories of varieties, locally presentable categories, and coalgebra, of which I had the pleasure of learning a lot in my collaboration with Jiří Adámek (see e.g. [2], [3]and [4]).

But, clearly, there are aspects of this theory which cannot be dealt with in an arbitrary entropic variety $\mathcal{V}$. For example, the concepts of primitive element or Lie algebra require that every $\mathcal{V}$-algebra A is an internal monoid in $\mathcal{V}$, while the familiar equivalence of the various descriptions of the Sweedler dual of a k-algebra depends on the fact that the varieties of k-vector spaces are semi-additive. Since these conditions turn out to be equivalent for an entropic variety, it is natural to pay special attention to Hopf monoids in these categories.

Noting that entropic semi-additive varieties can be viewed as entropic Jónsson-Tarski varieties or categories of semimodules over commutative semirings, respectively (see Proposition 1.1 below), one observes that this has to some extent been done before in the recent paper [16]. However, this paper does not deal with the following questions which arise naturally in its context:

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(1) Can one generalize the underlying Lie algebra of an algebra and the enveloping algebra of a Lie algebra?
(2) Can one generalize the adjunction between bialgebras and Lie algebras, determined by the primitive element functor?

In this note we therefore consider these questions.

In spite of the title of this note we avoid talking about Hopf algebras and bialgebras in varieties (except when these are module categories) and prefer the terms Hopf monoid and bimonoid, respectively, in order to avoid possible confusion, since the objects of the monoidal categories under consideration in this note already are called algebras.

The paper is organized as follows:

In Section 1, after briefly recalling some fundamentals about entropic varieties, we characterize entropic semi-additive varieties as entropic Jónsson-Tarski varieties, or, equivalently, as categories of semimodules over commutative semirings. Moreover, we define the abelian core of an entropic Jónsson-Tarski variety and analyze its monoidal structure. This preliminary section is complemented by a couple of results, which will be used later.

Section 2 starts with the introduction of tensor bimonoids and Lie algebras in entropic Jónsson-Tarski varieties. We then define a generalization of the familiar underlying Lie algebra of an algebra and show that the respective functor has a left adjoint, as in the case of modules.

Section 3 starts with a discussion of primitive elements in a more conceptual way than this is done in [16]; for example, the algebra of primitive elements of a bialgebra in an entropic Jónsson-Tarski variety $\mathcal{V}$ is characterized as an equalizer in the variety $\mathcal{V}$. We then construct a generalization of the familiar primitive element functor and analyze its adjunction properties.

1. Preliminaries

1.1. **Terminology.** By a variety $\mathcal{V}$ we mean a finitary one-sorted variety, considered as a concrete category over **Set**, the category of sets. Up to concrete equivalence, $\mathcal{V}$ this is the same as the category of product preserving functors $A\colon \mathcal{T} \to \mathbf{Set}$, where $\mathcal{T}$ is an algebraic theory (see e.g. [5],[8]). An n-ary term, thus, can be thought of as a $\mathcal{T}$-morphism $t\colon n \to 1$ and its interpretation in an algebra A is $t^A := A(t)$. Recall that $\mathcal{T}$ is equivalent to the dual of the full subcategory of $\mathcal{V}$ spanned by all finitely generated free $\mathcal{V}$-algebras Fn; thus, one may think of an n-ary term as a $\mathcal{V}$-homomorphism $F1 \to Fn$. Every variety is a locally finitely presentable category.

Throughout we make use of the following convention: Given an element x of a $\mathcal{V}$-algebra A, the $\mathcal{V}$-homomorphism $F1 \to A$ with $1 \mapsto x$ will be denoted by x as well.

1.2. **Entropic varieties.** It is well known (see [8] or [10]), that every variety whose theory is commutative, is a symmetric monoidal closed category; following [10] we call any such symmetric monoidal closed category $\mathcal{V}$ an *entropic variety*. The tensor product of $\mathcal{V}$, called the *entropic tensor product*, is given by universal bimorphisms in the sense of [7]. In more details, for algebras A and B in $\mathcal{V}$ their tensor product $A \otimes B$ is characterized by the fact, that there is a bimorphism $A \times B \xrightarrow{-\otimes-} A \otimes B$ over which each bimorphism $A \times B \to C$ factors uniquely as $f = g \circ (- \otimes -)$ with a homomorphism $g\colon A \otimes B \to C$. The internal hom-functor of $\mathcal{V}$ is given by the $\mathcal{V}$-algebra $[A, B]$ of all $\mathcal{V}$-homomorphisms from A to B, considered as a subalgebra of B^A. In an entropic variety, all operations are homomorphisms. The variety ${}_c\mathbf{Monoids}$ of commutative monoids is a paradigmatic example of an entropic variety.

We will make use of the following constructions with respect to an entropic variety $\mathcal{V}$.

(1) $\mathsf{Mon}\mathcal{V}$, the category of $\mathcal{V}$-monoids $\mathsf{A} = (A, A \otimes A \xrightarrow{m} A, F1 \xrightarrow{e} A)$ in $\mathcal{V}$. $\mathsf{Mon}\mathcal{V}$ contains the category ${}_c\mathsf{Mon}\mathcal{V}$ of commutative $\mathcal{V}$-monoids as a full reflective subcategory and the latter is an entropic variety.
(2) $\mathsf{Sg}\mathcal{V}$, the category of $\mathcal{V}$-semigroups[1] $\mathsf{A} = (A, A \otimes A \xrightarrow{m} A)$ in $\mathcal{V}$. and ${}_c\mathsf{Sg}\mathcal{V}$, the category of commutative $\mathcal{V}$-semigroups.
(3) $\mathsf{Comon}\mathcal{V}$, the category of $\mathcal{V}$-comonoids $\mathsf{C} = (C, C \xrightarrow{\mu} C \otimes C, C \xrightarrow{\epsilon} F1)$.
(4) $\mathsf{Bimon}\mathcal{V}$, the category of $\mathcal{V}$-bimonoids $\mathsf{B} = (B, m, e, \mu, \epsilon)$.
(5) $\mathsf{Hopf}\mathcal{V}$, the category of $\mathcal{V}$-Hopf monoids $\mathsf{H} = (H, m, e, \mu, \epsilon, S)$.
(6) Mod_A, the category of right A-modules $(M, M \otimes A \xrightarrow{l} M)$ in $\mathcal{V}$, for any $\mathcal{V}$-monoid A; this again is an entropic variety, if the monoid A is commutative.

1.3. **Entropic semi-additive varieties.** A variety $\mathcal{V}$ is called *semi-additive* or *linear* if it is enriched over the monoidal closed category ${}_c\mathbf{Monoids}$. Alternatively, these varieties can be characterized as being pointed (that is, they have a zero object) and having binary biproducts. These are precisely the Jónsson-Tarski varieties whose binary Jónsson-Tarski operation $+$ satisfies the axiom

$$t(x_1, \ldots, x_n) + t(y_1, \ldots, y_n) = t(x_1 + y_1, \ldots, x_n + y_n) \tag{1.1}$$

for each n-ary operation t (see [9, 1.10.8]). In particular, $+$ is a homomorphism. An entropic semi-additive variety $\mathcal{V}$ has a unique nullary operation 0, every $\mathcal{V}$-algebra A contains $\{0\}$ as a one-element subalgebra and the constant maps with value 0 are homomorphisms.

Every such variety is equivalent to a category of S-semimodules over some semiring[2] S. In fact, thinking of an n-ary operation symbol as a $\mathcal{V}$-homomorphism $F1 \xrightarrow{\omega} Fn$ one concludes that $\omega = f_1 + \cdots + f_n$ with endomorphisms $f_1, \ldots, f_n \in S$, the endomorphism monoid of the free $\mathcal{V}$-algebra $F1$, since $\mathcal{V}$ has biproducts (see [9, 1.10.8]). S also is a commutative monoid, with addition defined pointwise, by enrichment of $\mathcal{V}$ over ${}_c\mathbf{Monoids}$ (that is, by the Jónsson-Tarski operation $+$). Since every endomorphism preserves $+$, S is a semiring. Now S acts on the underlying set $|A|$ of a $\mathcal{V}$-algebra by $s \cdot a := s^A(a)$ and this makes A an S-semimodule, by Equation (1.1). Consequently, the following holds, since every category $\mathbf{SMod}_S$ of S-semimodules over some commutative semiring S is well known to be entropic (see e.g. [11]).

Proposition 1.1. *The following are equivalent for a variety $\mathcal{V}$.*

(1) *$\mathcal{V}$ is an entropic semi-additive variety.*
(2) *$\mathcal{V}$ is an entropic Jónsson-Tarski variety.*
(3) *$\mathcal{V}$ is equivalent to the variety of S-semimodules over some commutative semiring S.*

Examples 1.2. The following varieties are entropic semi-additive varieties.

(1) $\mathbf{Ab}$, the category of abelian groups and, more generally, $\mathbf{Mod}_R$, for any commutative unital ring R.

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[1] By this we mean the obvious generalization of $\mathcal{V}$-monoids (see [13]).

[2] In this note by a *semiring* is meant what also (and more appropriately) is called a *rig*, meaning, a ring without negatives. We only use the traditional notion since it still seems to be more common.

(2) $\mathbf{SMod}_S = \mathsf{Mod}_S$, where S is a commutative monoid in the entropic variety ${}_c\mathbf{Monoids}$; this is the category of all S-semimodules, for a commutative unital semiring S.
(3) $\mathbf{SLat_0}$, the variety of lower bounded semilattices. $\mathbf{SLat}$, the variety of (join) semilattices, is entropic but not semi-additive.
(4) $\mathbf{DLat_0} = {}_c\mathsf{Sg}\mathbf{SLat_0}$, the variety of lower bounded distributive lattices[3].

Fact 1.3. Let A and B be algebras in an entropic Jónsson-Tarski variety $\mathcal{V}$ and $b \in B$. Then there are homomorphisms

(1) $b_r\colon A \simeq A \otimes F1 \xrightarrow{A\otimes b} A \otimes B$ and $b_l\colon A \simeq F1 \otimes A \xrightarrow{b\otimes A} B \otimes A$,
for each $a \in A$ one has $b_r(a) = a \otimes b$ and $b_r(a) = b \otimes a$;
(2) $\bar{b}\colon A \xrightarrow{\langle b_r, b_l\rangle} (A \otimes B) \times (B \otimes A)$ with coordinates b_r and b_l,
for each $a \in A$ one has $\bar{b}(a) = (a \otimes b, b \otimes a)$;
(3) $\pi_A\colon A \xrightarrow{\bar{e}} (A \otimes A) \times (A \otimes A) \xrightarrow{+} A \otimes A$, if (A, m, e) is a $\mathcal{V}$-monoid;
for each $a \in A$ one has $\pi_A(a) = a \otimes 1 + 1 \otimes a$.[4]

Lemma 1.4. *Let A be an algebra in an entropic Jónsson-Tarski variety $\mathcal{V}$. Then the following hold.*

(1) *The triple $(A, A \times A \xrightarrow{+^A} A, 0)$ is a commutative internal monoid in $\mathcal{V}$ and, thus, a (commutative) monoid. This is the only internal monoid on A by the Eckmann-Hilton argument. By this construction $\mathcal{V}$ is isomorphic to the category of commutative internal monoids in $\mathcal{V}$ (see* [9, 1.10.5]*).*
(2) *An n-fold sum $x+\cdots+x$ in A can be written as $n\cdot x$ with n the n-fold sum $1+\cdots+1 \in F1$; that is, $x+\cdots+x$ is the value of $1+\cdots+1$ under the canonical isomorphism $F1\otimes A \simeq A$.*
(3) *The variety $\mathcal{V}$ is isomorphic to the category $\mathsf{Mod_{F1}}$ of modules of the commutative monoid $\mathsf{F1}$ (see* [10]*).*

An element x of an algebra A in a Jónsson-Tarski variety is called *invertible*, if it is invertible in the monoid $(A, A \times A \xrightarrow{+^A} A, 0)$, that is, if there is an element $y \in A$ with $x + y = 0$ $(= y + x)$; such an element is uniquely determined and will be denoted by $-x$. $Inv(A)$ denotes the set of invertible elements of $(A, +^A, 0)$.

Lemma 1.5. *Let $\mathcal{V}$ be an entropic Jónsson-Tarski variety. Then, for every $\mathcal{V}$-algebra A, the following holds.*

(1) *$Inv(A)$ is a $\mathcal{V}$-subalgebra of A with an embedding υ_A and, hence, an internal submonoid of $(A, +^A, 0)$.*
(2) *Every $\mathcal{V}$-homomorphism $A \xrightarrow{f} B$ is, by restriction and corestriction, a $\mathcal{V}$-homomorphism $Inv(A) \to Inv(B)$.*
In particular, for every $b \in B$, the homomorphism $b_r\colon A \to A \otimes B$ restricts to a homomorphism $Inv(A) \to Inv(A \otimes B)$ and, hence, the homomorphism $\upsilon_A \otimes \upsilon_B$ factors as $Inv(A) \otimes Inv(B) \xrightarrow{\upsilon_{A,B}} Inv(A \otimes B) \hookrightarrow A \otimes B$; moreover one has $m(a \otimes b) \in Inv(A)$, for every monoid (A, m, e) in $\mathcal{V}$ and for all $a, b \in Inv(A)$.
(3) *For every $\mathcal{V}$-homomorphism $A \xrightarrow{f} B$ one has $f(-x) = -f(x)$, for each $x \in Inv(A)$.*
(4) *The map $Inv(A) \xrightarrow{i} Inv(A)$ with $x \mapsto -x$ is a $\mathcal{V}$-homomorphism. Consequently, $Inv(A)$ is a (commutative) internal group and, in fact the largest such contained in A.*

[3]The equations used here follow from [10].
[4]We here use the notation $e(1) = 1$ in view of the usual notation in the context of primitive elements.

Proof. For every k-ary term t and invertible elements $m_1, \dots, m_k \in A$ the element $t^A(m_1, \dots, m_k) \in A$ is invertible, since

$$t^A(m_1, \dots, m_k) + t^A(-m_1, \dots, -m_k) = t^A(m_1 + (-m_1), \dots, m_k + (-m_k)) = t^A(0, \dots, 0) = 0.$$

This calculation shows, moreover, that the map $Inv(A) \xrightarrow{i} Inv(A)$ with $x \mapsto -x$ is a $\mathcal{V}$-homomorphism. Thus, $Inv(A)$ is an internal subgroup in $\mathcal{V}$ and, when considered as an internal monoid, it is an internal submonoid of A.

Obviously $f(-x) = -f(x)$ for every homomorphism $A \xrightarrow{f} B$ and every $x \in Inv(A)$, which proves items 2 and 4.

The rest is trivial. □

We denote by $\mathcal{V}_{\mathsf{Ab}}$ the full subcategory of $\mathcal{V}$ spanned by all $\mathcal{V}$-algebras A with $Inv(A) = A$. If $\mathcal{V} = \mathbf{Alg}(\Omega, \mathcal{E})$, then $\mathcal{V}_{\mathsf{Ab}}$ is the variety $\mathbf{Alg}(\Omega', \mathcal{E}')$ with Ω' obtained from Ω by adding a unary operation $-$, and $\mathcal{E}'$ obtained from $\mathcal{E}$ by adding the equations

(1) $x + (-x) = 0$,

(2) $\omega(-x_1, \dots, -x_n) = -\omega(x_1, \dots, x_n)$ for all n-ary operations $\omega \in \Omega$, for all $n \in \mathbb{N}$.

Obviously, $\mathcal{V}_{\mathsf{Ab}}$ is an entropic Jónsson-Tarski variety and coincides with the category of all internal groups in $\mathcal{V}$. Consequently, $\mathcal{V}_{\mathsf{Ab}}$ is an additive category (see [9, 1.10.13]) and, in fact, the largest additive subvariety of $\mathcal{V}$. Being exact as a variety, $\mathcal{V}_{\mathsf{Ab}}$ even is an abelian category (see [8, 2.6.11]). In accordance with [12] we call $\mathcal{V}_{\mathsf{Ab}}$ the *abelian core* of $\mathcal{V}$.

Proposition 1.6. *For every entropic Jónsson-Tarski variety $\mathcal{V}$ the following hold.*

(1) *$\mathcal{V}_{\mathsf{Ab}}$ is a full isomorphism-closed reflective subcategory of $\mathcal{V}$.*

(2) *The assignment $A \mapsto Inv(A)$ defines a functor $\mathcal{V} \xrightarrow{Inv} \mathcal{V}_{\mathsf{Ab}}$ and this is right adjoint to the embedding $\mathcal{V}_{\mathsf{Ab}} \hookrightarrow \mathcal{V}$. Moreover, $\mathcal{V}_{\mathsf{Ab}} \hookrightarrow \mathcal{V} \xrightarrow{Inv} \mathcal{V}_{\mathsf{Ab}} = Id$.*

(3) *$\mathcal{V}_{\mathsf{Ab}}$ is closed under the entropic tensor product $- \otimes_{\mathcal{V}} -$ of $\mathcal{V}$; consequently, the entropic monoidal structure of $\mathcal{V}_{\mathsf{Ab}}$ is given by $- \otimes_{\mathcal{V}} -$ and the reflection $RF1$ of $F1$ into $\mathcal{V}_{\mathsf{Ab}}$ as the unit object.*

(4) *The embedding $\mathcal{V}_{\mathsf{Ab}} \hookrightarrow \mathcal{V}$ is a symmetric monoidal functor.*

Proof. $\mathcal{V}_{\mathsf{Ab}}$ is a full isomorphism-closed subcategory of $\mathcal{V}$ by the preceding lemma. Since its embedding into $\mathcal{V}$ commutes with the forgetful functors, it is an algebraic functor and, thus, has a left adjoint. Obviously every morphism of internal groups $f\colon G \to H$ factors over the embedding $Inv(A) \hookrightarrow A$, which shows that Inv is a coreflection. This proves items (1) and (2).

Denoting the entropic tensor product of $\mathcal{V}_{\mathsf{Ab}}$ by $- \otimes -$ and that of $\mathcal{V}$ by $- \otimes_{\mathcal{V}} -$, we first deduce from Lemma 1.5 that, for internal groups G and H, all elements $g \otimes_{\mathcal{V}} h \in G \otimes_{\mathcal{V}} H$ are invertible since the map $g \otimes_{\mathcal{V}} -$ is a homomorphism. Thus, $G \otimes_{\mathcal{V}} H$ is an internal group. We then have, for every triple G, H, K of internal groups in $\mathcal{V}$, $\mathcal{V}_{\mathsf{Ab}}(G \otimes H, K) \simeq \mathcal{V}_{\mathsf{Ab}}(G, [H, K]) = \mathcal{V}(G, [H, K]) \simeq \mathcal{V}(G \otimes_{\mathcal{V}} H, K) \simeq \mathcal{V}_{\mathsf{Ab}}(G \otimes_{\mathcal{V}} H, K)$, since the internal hom-functors of $\mathcal{V}$ and $\mathcal{V}_{\mathsf{Ab}}$ coincide.

To complete the proof of items (3) and (4) it remains to show that the following diagram commutes for every internal group G, where $F1 \xrightarrow{r} RF1$ is the reflection map.

$$\begin{tikzcd} F1 \otimes_{\mathcal{V}} G \arrow[r, "r\otimes_{\mathcal{V}} id"] \arrow[d, "can_{\mathcal{V}}"'] & RF1 \otimes_{\mathcal{V}} G \arrow[d, equal] \\ G & RF1 \otimes G \arrow[l, "can"] \end{tikzcd} \tag{1.2}$$

But this is clear, since $F1 \otimes_{\mathcal{V}} G$ is generated by the elements $1 \otimes g$, $g \in G$, and r maps the free generator of $F1$ to the free generator of the free $\mathcal{V}_{\mathsf{Ab}}$-algebra $RF1$. □

Examples 1.7.

(1) $({}_c\mathbf{Monoids})_{\mathsf{Ab}} = \mathbf{Ab}$.

(2) ${}_c\mathsf{Mon}\mathcal{V}_{\mathsf{Ab}}$ is isomorphic to $({}_c\mathsf{Mon}\mathcal{V})_{\mathsf{Ab}}$.
In fact, $({}_c\mathsf{Mon}\mathcal{V})_{\mathsf{Ab}}$ is the full subcategory of ${}_c\mathsf{Mon}\mathcal{V}$, consisting of all commutative monoids (M, m, e) with $Inv(M) = M$ and, since the embedding $\mathcal{V}_{\mathsf{Ab}} \xrightarrow{E} \mathcal{V}$ is monoidal, it embeds $\mathsf{Mon}\mathcal{V}_{\mathsf{Ab}}$ into this category. Conversely, if $(M, m, e) \in {}_c\mathsf{Mon}\mathcal{V}$ satisfies $Inv(M) = M$ and $F1 \xrightarrow{r} RF1$ is the reflection of $F1$, denote by $RF1 \xrightarrow{e'} M$ the unique homomorphism with $e' \circ r = e$; then $(M, m, e') \in \mathsf{Mon}\mathcal{V}_{\mathsf{Ab}}$ (use Lemma 1.5 and Diagram (1.2)) and $(M, m, e) = E(M, m, e')$.

(3) $(\mathbf{Mod}_R)_{\mathsf{Ab}} = \mathbf{Mod}_R$, for every commutative ring R and, more generally,

(4) $(\mathbf{SMod}_S)_{\mathsf{Ab}} = \mathbf{Mod}_{RS}$, for every commutative semiring S, where RS is the reflection of S into the category of commutative rings[5].
This is easily seen when recalling the fact that, in every entropic variety $\mathcal{V}$, the left A-modules $(M, A \otimes M \xrightarrow{l} M)$ of a $\mathcal{V}$-monoid A are in one-to-one correspondence with $\mathcal{V}$-monoid morphisms $\mathsf{A} \xrightarrow{\phi} [M, M]$, where ϕ corresponds to l by the adjunction $- \otimes M \dashv [M, -]$ (see e.g. [15]). Thus, an S-semimodule M with $M = Inv(M)$ is a semiring homomorphism $S \xrightarrow{\phi} [M, M]$, where $[M, M]$ is the endomorphism monoid of M in ${}_c\mathbf{Monoids}$. This is a monoid in $\mathbf{Ab}$ by item (1) and, thus, ϕ corresponds to a unique ring homomorphism $RS \xrightarrow{\phi} [M, M]$, that is, to an RS-module.

(5) $(\mathbf{SLat_0})_{\mathsf{Ab}} = \{0\}$.

2. The universal envelope functor

2.1. **Tensor bimonoids and Lie algebras.** Let $\mathcal{V}$ be an entropic variety. By the standard construction of free monoids in monoidal closed categories (see [18]) the free monoid TA in $\mathsf{Mon}\mathcal{V}$ over a $\mathcal{V}$-algebra A has $TA = \coprod_{n\in\mathbb{N}} A^{\otimes n}$ as its underlying $\mathcal{V}$-algebra[6] with unit $F1 \xrightarrow{\iota_0} TA$ and multiplication given by "concatination". The coproduct injection $\iota_1 \colon A \to TA$ is its universal morphism.

[5]The embedding of the category of commutative rings into the category of commutative semirings is an algebraic functor and therefore has a left adjoint R.

[6]$A^{\otimes n}$ is to be understood in the obvious way: $A^{\otimes 0} = F1$ and $A^{\otimes n+1} = A \otimes A^{\otimes n}$

If $\mathcal{V}$ is an entropic Jónsson-Tarski variety, this $\mathcal{V}$-monoid becomes a $\mathcal{V}$-bimonoid (TA, μ, ϵ), called the *$\mathcal{V}$-tensor bimonoid*, as follows, where π_{TA} is the homomorphism defined in Fact 1.3.

(1) $\mu\colon TA \to TA \otimes TA$ is the homomorphic extension of the $\mathcal{V}$-homomorphism $A \xrightarrow{\iota_1} TA \xrightarrow{\pi_{TA}} TA \otimes TA$ to a morphism of $\mathcal{V}$-monoids.

(2) $\epsilon\colon TA \to F1$ is the homomorphic extension the $\mathcal{V}$-homomorphism $A \xrightarrow{0} F1$ to a morphism of $\mathcal{V}$-monoids.

That (TA, μ, ϵ) this way becomes a $\mathcal{V}$-comonoid can be shown literally the same way is in the case of modules. Since μ and ϵ are $\mathcal{V}$-monoid morphisms by definition, TA is a $\mathcal{V}$-bimonoid.

For every $A \in \mathcal{V}_{\mathsf{Ab}}$ the tensor bimonoid (TA, μ, ϵ) even becomes a Hopf monoid in $\mathcal{V}_{\mathsf{Ab}}$ and, thus, in $\mathcal{V}$. The required antipode acts as $a_1 \otimes \cdots a_n \mapsto (-1)^n a_n \otimes \cdots \otimes a_1$[7]. The proof again is literally the same as in the case of modules.

Every Hopf algebra H, has an underlying Lie algebra, obtained as the underlying Lie algebra of the underlying algebra of H. This construction is not possible over an arbitrary entropic variety $\mathcal{V}$, since neither can the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \tag{2.1}$$

be expressed, nor can the underlying Lie algebra of a monoid be defined.

Definition 2.1. Let $\mathcal{V}$ be an entropic Jónsson-Tarski variety. A *$\mathcal{V}$-Lie algebra* is a pair $(A, [-,-])$ consisting of a $\mathcal{V}$-algebra A and a $\mathcal{V}$-bimorphism $[-,-]\colon A \times A \to A$ satisfying the identity (2.1) and, in addition, the identity $[x,x] = 0$.

A *$\mathcal{V}$-Lie morphism* $(A, [-,-]) \xrightarrow{f} (B, [-,-])$ is a $\mathcal{V}$-homomorphism $A \xrightarrow{f} B$ making the following diagram commute.

$$\begin{tikzcd}
A \times A \arrow[r, "f\times f"] \arrow[d, "{[-,-]}"'] & B \times B \arrow[d, "{[-,-]}"] \\
A \arrow[r, "f"'] & B
\end{tikzcd} \tag{2.2}$$

This defines the category $\mathsf{Lie}\mathcal{V}$ of $\mathcal{V}$-Lie algebras. There is a forgetful functor $\mathsf{Lie}\mathcal{V} \xrightarrow{V} \mathcal{V}$.

Denoting by $[-]\colon A \otimes A \to A$ the $\mathcal{V}$-homomorphism corresponding to the bimorphism $[-,-]$ one may say, equivalently, that $(A, [-])$ is a $\mathcal{V}$-Lie algebra, if $[-]$ satisfies the axioms

$$[x,[y\otimes z]] + [y,[z\otimes x]] + [z,[x\otimes y]] = 0 \text{ and } \ [x\otimes x] = 0.$$

In this language the Lie homomorphism axiom obviously is commutativity of Diagram (2.2) with $\times$ replaced by $\otimes$ and the bimorphism $[-,-]$ replaced by $[-]$.

Obviously, $\mathsf{Lie}\mathcal{V}$ is a Jónsson-Tarski variety and $(\mathsf{Lie}\mathcal{V})_{\mathsf{Ab}} = \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}}$.

2.2. **The enveloping (bi)monoid of a Lie algebra.** Since there is no underlying functor from $\mathcal{V}$-monoids to $\mathcal{V}$-Lie algebras for an arbitrary entropic Jónsson-Tarski variety, unless the theory of $\mathcal{V}$ admits an inverse of the Jónsson-Tarski operation $+$, that is, if $\mathcal{V} \simeq \mathbf{Mod}_R$, the standard categorical argument for the existence of the universal enveloping algebra UL of a Lie-algebra L does not apply[8]. We show next that one can in spite of that generalize

[7]Note that the notation $(-1)^n x$ is symbolic and short for $(-(-\cdots(-x)\cdots)$.

[8]An underlying functor would be algebraic and, thus, have a left adjoint.

the standard construction, where the resulting $\mathcal{V}$-monoid even carries the structure of a $\mathcal{V}$-bimonoid, as in the case of $\mathcal{V} = \mathbf{Mod}_R$.

Theorem 2.2. *There exist functors* $Lie\colon \mathsf{Mon}\mathcal{V} \to \mathsf{Lie}\mathcal{V}$ *and* $U\colon \mathsf{Lie}\mathcal{V} \to \mathsf{Mon}\mathcal{V}$, *where we call* $Lie\mathsf{A}$ *the* underlying Lie algebra of the monoid A, *and* $U\mathsf{L}$ *the* enveloping monoid of the Lie algebra L, *such that*

(1) *the diagram commutes, that is, Lie takes its values in* $\mathsf{Lie}\mathcal{V}_{\mathsf{Ab}}$

$$\begin{tikzcd} \mathsf{Mon}\mathcal{V} \arrow[r, "Lie"] \arrow[d] & \mathsf{Lie}\mathcal{V} \arrow[d] \\ \mathcal{V} \arrow[r, "Inv"] & \mathcal{V} \end{tikzcd}$$

(2) *there is a natural quotient* $q\colon T \circ V \Rightarrow U$

$$\begin{tikzcd} \mathsf{Lie}\mathcal{V} \arrow[rr, "U"] \arrow[dr, "V"'] & & \mathsf{Mon}\mathcal{V} \\ & \mathcal{V} \arrow[ur, "T"'] \arrow[u, phantom, "\Uparrow q"] & \end{tikzcd}$$

(3) *The restriction* $U_{\mathsf{Ab}} : \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}} \hookrightarrow \mathsf{Lie}\mathcal{V} \xrightarrow{U} \mathsf{Mon}\mathcal{V}$ *of* U *is left adjoint to the corestriction* $Lie_{\mathsf{Ab}}\colon \mathsf{Mon}\mathcal{V} \to \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}}$ *of Lie. Consequently, Lie is right adjoint to* $U_{\mathsf{Ab}} \circ R$, *where* R *is the reflection of* $\mathsf{Lie}\mathcal{V}$ *into* $\mathsf{Ab}(\mathsf{Lie}\mathcal{V}) = \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}}$.

(4) U *factors as* $\mathsf{Lie}\mathcal{V} \xrightarrow{U_{Bi}} \mathsf{Bimon}\mathcal{V} \xrightarrow{|-|} \mathsf{Mon}\mathcal{V}$, *that is, the enveloping monoid of a Lie algebra* L *carries a bimonoid structure and this construction is functorial.*

(5) U_{Ab} *factors as* $\mathsf{Lie}\mathcal{V}_{\mathsf{Ab}} \xrightarrow{U_H} \mathsf{Hopf}\mathcal{V} \to \mathsf{Mon}\mathcal{V}$, *the enveloping monoid of a Lie algebra* L *carries a Hopf monoid structure in a functorial way, provided that* $InvL = L$.

Proof. Consider, for a $\mathcal{V}$-monoid $\mathsf{A} = (A, m, e)$, the map $[-]\colon Inv(A) \otimes Inv(A) \to Inv(A)$ given by $[a \otimes b] := m(a \otimes b) - m(b \otimes a)$. This is a $\mathcal{V}$-homomorphism by item (2) of Lemma 1.5. $Lie(A, m) := (Inv(A), [-])$ then is a Lie algebra in $\mathcal{V}$ (in fact in $\mathcal{V}_{\mathsf{Ab}}$) and, for every $\mathsf{Mon}\mathcal{V}$-morphism $f\colon (A, m, e) \to (B, n, u)$ its restriction $Lie(f)$ to a homomorphism $Inv(A) \to Inv(B)$ satisfies $f([a \otimes b] = [(f \otimes f)(a \otimes b)]$, for all $a, b \in A$. This defines the functor Lie.

Given a $\mathcal{V}$-Lie algebra $(L, [-])$, form the free $\mathcal{V}$-monoid (TL, m, e) over L as in Section 2.1. Let $\rho\colon L \otimes L \to TL \times TL$ be the $\mathcal{V}$-homomorphism with $\pi_1 \circ \rho = m \circ \sigma \circ (\iota_1 \otimes \iota_1)$ and $\pi_2 \circ \rho = \iota_1 \circ [-]$. Then the family $\mathcal{S}_L$ of all $\mathcal{V}$-monoid morphisms $TL \xrightarrow{f_i} \mathsf{A}_i$ with $f_i \circ m \circ (\iota_1 \otimes \iota_1) = f_i \circ (+^L \circ \rho)$ has a (regular epi, monosource)-factorization $TL \xrightarrow{f_i} A_i = TL \xrightarrow{q_L} U\mathsf{L} \xrightarrow{m_i} A_i$ in $\mathsf{Mon}\mathcal{V}$, since $\mathsf{Mon}\mathcal{V}$ is a variety. If $l\colon \mathsf{L} \to \mathsf{L}'$ is a Lie-morphism, the $\mathcal{V}$-homomorphism $Tl\colon TL \to TL'$ is a monoid homomorphism and, for each $f_i' \in \mathcal{S}_{L'}$, $f_i' \circ Tl \in \mathcal{S}_L$. Thus, there exists a unique monoid morphism $Ul\colon UL \to UL'$ with $Ul \circ q_L = q_{L'} \circ TL$. This defines a functor U as well as a natural transformation $q\colon T \circ |-| \Rightarrow U$ being pointwise a quotient. We call the monoid $U\mathsf{L}$ the *enveloping monoid of* L.

Denote, for $\mathsf{L} \in \mathsf{Lie}(\mathcal{V}_{\mathsf{Ab}})$, by $\eta\colon L \to LieUL$ the $\mathcal{V}$-homomorphism $q_L \circ \iota_1$ ($L \in \mathcal{V}_{\mathsf{Ab}}$ implies $q_L \circ \iota_1(c) \in Inv(UL)$, for each $x \in L$). Since $q_L(m(\iota_1 x \otimes \iota_1 y)) = q_L(m(\iota_1 y \otimes \iota_1 x) + \iota_1[x \otimes y]) = q_L m(\iota_1 y \otimes \iota_1 x) + q_L \iota_1[x \otimes y]$ by definition of q_L, $[q_L \iota_1 x \otimes q_L \iota_1 y] = m(q_L \iota_1 x \otimes q_L \iota_1 y) - m(q_L \iota_1 x \otimes q_L \iota_1 x)$ by definition of Lie, and since q_L is a monoid morphism, one concludes that η is a $\mathsf{Lie}\mathcal{V}$-morphism.

For any Lie-morphism $\mathsf{L} \xrightarrow{f} Lie\mathsf{A}$ with $\mathsf{A} = (A, m, e) \in \mathsf{Mon}\mathcal{V}$ let $TL \xrightarrow{f^\sharp} A$ be its extension to a monoid morphism. By the definition of U this morphism factors as $f^\sharp = TL \xrightarrow{q_L} UL \xrightarrow{\tilde{f}} \mathsf{A}$ with a monoid morphism $UL \xrightarrow{\tilde{f}} \mathsf{A}$, which is the unique such morphism with $Lie\tilde{f} \circ \eta = f$. This proves the first statement of item 3. The second statement follows by composing this adjunction with the adjunction given by the embedding $(\mathsf{Lie}\mathcal{V})_{\mathsf{Ab}} \hookrightarrow \mathsf{Lie}\mathcal{V}$ and its left adjoint (see Proposition 1.6).

The monoid $U\mathsf{L}$ can be supplied with a bimonoid structure as follows. Let $u\colon L \to UL \otimes UL$ be map with $x \mapsto (q \circ \iota_1)x \otimes 1 + 1 \otimes (q \circ \iota_1)x$. In other words, with notation as in Section 2.1, u is the $\mathcal{V}$-homomorphism

$$L \xrightarrow{\iota_1} TL \xrightarrow{\mu} TL \otimes TL \xrightarrow{q\otimes q} UL \otimes UL = L \xrightarrow{\pi} L \otimes L \xrightarrow{\iota_1 \otimes \iota_1} TL \otimes TL \xrightarrow{q\otimes q} UL \otimes UL,$$

which has $\nu := TL \xrightarrow{\mu} TL \otimes TL \xrightarrow{q\otimes q} UL \otimes UL$ as its unique extension to a $\mathcal{V}$-monoid morphism. Now the straightforward calculation

$$\begin{aligned}
u([x \otimes y] + m(y \otimes x)) &= (q \otimes q) \circ (\iota_1 \otimes \iota_1)\big(([x \otimes y] + m(y \otimes x)) \otimes 1 + 1 \otimes ([x \otimes y] + m(y \otimes x))\big) \\
&= \big(q \circ \iota_1([x \otimes y] + m(y \otimes x))\big) \otimes 1 + 1 \otimes \big(q \circ \iota_1([x \otimes y] + m(y \otimes x))\big) \\
&= \big(q \circ \iota_1(m(x \otimes y))\big) \otimes 1 + 1 \otimes \big(q \circ \iota_1(m(x \otimes y))\big) \\
&= (q \otimes q) \circ (\iota_1 \otimes \iota_1)\big(m(x \otimes y)\big) \\
&= u(m(x \otimes y))
\end{aligned}$$

with $x \otimes y \in L \otimes L$ proves the equation

$$u \circ m = \nu \circ \iota_1 \circ m = \nu \circ \iota_1 \circ (+ \circ \rho) = u \circ (+ \circ \rho)$$

which shows that ν belongs to $\mathcal{S}_L$. Consequently there exists a morphism of $\mathcal{V}$-monoids $UL \xrightarrow{\delta} UL \otimes UL$, such that the following diagram commutes

$$\begin{tikzcd}
TL \arrow[r, "q"] \arrow[d, "\mu"'] & UL \arrow[d, "\delta"] \\
TL \otimes TL \arrow[r, "q\otimes q"'] & UL \otimes UL
\end{tikzcd} \tag{2.3}$$

Since the extension $\epsilon\colon TL \to F1$ of the 0-homomorphism $L \to F1$ belongs to the family $(f_i)_i$ as well, as is easily seen, ϵ factors in $\mathsf{Mon}\mathcal{V}$ as $TL \xrightarrow{q} UL \xrightarrow{\upsilon} F1$. It now follows trivially that $(TL, \mu, \epsilon) \xrightarrow{q} (UL, \delta, \upsilon)$ is a morphism of $\mathcal{V}$-bimonoids, since q is surjective. This construction is functorial: If $f\colon L \to L'$ is a Lie-morphism and Tf the corresponding monoid-morphism $TL \to TL'$, then $q_{L'} \circ Tf \in \mathcal{S}_L$, such that there is a unique monoid-morphism $Uf\colon UL \to UL'$ with $Uf \circ q_L = q_{L'} \circ Tf$. Uf is a comonoid morphism as well; it is compatible with the comonoid structures just defined, as is easily seen. This proves item 4.

Item 5 now follows literally as in the case of modules: the required antipode S is the extension of the $\mathcal{V}$-homomorphism $L \to L$ given by $x \mapsto -x$ and this is preserved by the bimonoid morphisms Uf just defined. □

3. Primitive element functors

Recall that in the classical case, where $\mathcal{V} = \mathbf{Mod}_R$, there exists a so-called the *primitive element functor* $P\colon \mathbf{Bialg}_R \to \mathbf{Lie}_R$, which is right adjoint to $U_{Bi}\colon \mathbf{Lie}_R \to \mathbf{Bialg}_R$. This cannot be generalized to arbitrary entropic varieties, since here one cannot even define so-called *primitive elements*. This problem is partly addressed in [16] for Jónsson-Tarski varieties. The problem whether the functor in this theorem gives rise to an adjunction is not considered. We here deal with this problem as follows.

3.1. **Primitive elements.**

Definition 3.1. Let $\mathcal{V}$ be an entropic Jónsson-Tarski variety. An element p of a $\mathcal{V}$-bimonoid B with comultiplication μ is called *primitive*, provided that $\mu(p) = \pi_B(a) = p \otimes 1 + 1 \otimes p$ and $\epsilon(p) = 0$.[9]

Note that, by definition of the comultiplication of the tensor bimonoid TA, the elements of A (more precisely, the elements $\iota_1(a)$ for $a \in A$) are primitive elements in TA.

We next provide a conceptual description of primitive elements in bimonoids.

Proposition 3.2. *Let $\mathsf{B} = (B, m, e, \mu, \epsilon)$ be a bimonoid in an entropic Jónsson-Tarski variety $\mathcal{V}$.*

Let E_1 be the equalizer of the homomorphisms $B \xrightarrow{\pi_B} B \otimes B$ and $B \xrightarrow{\mu} B \otimes B$ and E_2 the equalizer of the homomorphisms $B \xrightarrow{\epsilon} F1$ and $B \xrightarrow{0} F1$.

Then the underlying set of the $\mathcal{V}$-algebra $E_1 \cap E_2$ is the set of all primitive elements. In particular, the primitive elements of B form a $\mathcal{V}$-subalgebra $Prim(\mathsf{B})$ of B.

The assignment $\mathsf{B} \mapsto Prim(\mathsf{B})$ defines a faithful functor $Prim\colon \mathsf{Bimon}\mathcal{V} \to \mathcal{V}$.

Proof. Only the last statement requires an argument. If $f\colon \mathsf{B} \to \mathsf{B}'$ is a morphism in $\mathsf{Bimon}\mathcal{V}$ and p is a primitive element in B, then $\mu'(fp) = (f \otimes f) \circ \mu(p) = (f \otimes f)(p \otimes 1 + 1 \otimes p) = fp \otimes 1 = 1 \otimes fp$ by the morphism properties of f; hence f yields by restriction and corestriction a $\mathcal{V}$-homomorphism $Prim(f)\colon Prim(\mathsf{B}) \to Prim(\mathsf{B}')$. This proves functoriality of the construction $Prim$. □

Lemma 3.3. *Let (H, S) be a Hopf monoid in an entropic Jónsson-Tarski variety $\mathcal{V}$. Then*

(1) *S can be restricted to a $\mathcal{V}$-homomorphisms $Inv(H) \to Inv(H)$ and $Prim(\mathsf{H}) \to Prim(\mathsf{H})$.*
(2) *For each $x \in Inv(H)$ one has $Sx = -x$.*
(3) *$Inv(H)$ contains $Prim(\mathsf{H})$ as a $\mathcal{V}$-subalgebra.*
(4) *$Prim(\mathsf{H})$ is an internal group in $\mathcal{V}$.*

The assignment $\mathsf{H} \mapsto Prim(\mathsf{H})$ defines a faithful functor $Prim\colon \mathsf{Hopf}\mathcal{V} \to \mathcal{V}_{\mathsf{Ab}}$.

Proof. S, being a homomorphism, preserves inverses by Lemma 1.5. Since the antipode S of a Hopf monoid (H, S) is a bimonoid morphism $\mathsf{H} \to \mathsf{H}^{\mathsf{op},\mathsf{cop}}$ and the antipode equation is satisfied, Sp is primitive for each primitive element p, by the simple calculation $\mu(Sp) = S \circ \sigma(\mu p) = S(1 \otimes p + p \otimes 1) = 1 \otimes Sp + Sp \otimes 1$ (recall that $+$ is commutative).

It remains to show that Sp is an additive inverse of p, for each primitive element p. In fact, $Sp + p = m(Sp \otimes 1) + m(1 \otimes p) = m(Sp \otimes 1 + 1 \otimes p) = m \circ (S \otimes id)(p \otimes 1 + 1 \otimes p) = m \circ (S \otimes id) \circ \mu(p) = e \circ \epsilon(p) = 0$. □

[9] Here 1 is short for $e(1)$, the image of 1 under the unit $F1 \xrightarrow{e} \mathsf{B}$.

3.2. **An adjunction between $\mathsf{Hopf}\mathcal{V}$ and $\mathsf{Lie}\mathcal{V}$.** The following result generalizes part of the famous Milner-Moore Theorem.

Theorem 3.4. *Let $\mathcal{V}$ be an entropic Jónsson-Tarski variety. Then there exists a faithful functor $\bar{P}\colon \mathsf{Hopf}\mathcal{V} \to \mathsf{Lie}\mathcal{V}$, such that the following diagram commutes.*

$$\begin{tikzcd}
\mathsf{Hopf}\mathcal{V} \arrow[r, "\bar{P}"] \arrow[d, "Prim"'] & \mathsf{Lie}\mathcal{V} \arrow[d, "|-|"] \\
\mathcal{V}_{\mathsf{Ab}} \arrow[r, hook] & \mathcal{V}
\end{tikzcd}$$

$\mathsf{Hopf}\mathcal{V} \xrightarrow{P_H} \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}}$, the corestriction of $\bar{P}$, is right adjoint to $\mathsf{Lie}\mathcal{V}_{\mathsf{Ab}} \xrightarrow{U_H} \mathsf{Hopf}\mathcal{V}$. Consequently, $\bar{P}$ is right adjoint to $\mathsf{Lie}\mathcal{V} \xrightarrow{R} \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}} \xrightarrow{U_H} \mathsf{Hopf}\mathcal{V}$, where R is the reflection functor.

Proof. Given a $\mathcal{V}$-Hopf monoid H, one can define $\mathcal{V}$-homomorphisms $[-,-]_{\mathsf{H}}$ and ${}_{\mathsf{H}}[-,-]$ $Prim(\mathsf{H}) \otimes Prim(\mathsf{H}) \to Prim(\mathsf{H})$ by

$$[x,y]_{\mathsf{H}} = m(x \otimes y) + m(y \otimes Sx) \quad \text{and} \quad {}_{\mathsf{H}}[x,y] = m(x \otimes y) + m(Sy \otimes x)$$

In fact, since homomorphisms preserve invertible elements and inverses and since S and $m(x \otimes -)$ are homomorphisms, by item (2) of Lemma 1.5 the elements $m(x \otimes y)$, $m(y \otimes Sx)$ and $m(Sy \otimes x)$ are invertible, provided that x and y are primitive, hence invertible (see item (3) of Lemma 3.3). Consequently one has for primitive elements x and y

$$\begin{aligned}
[x,y]_{\mathsf{H}} &= m(x \otimes y) + m(y \otimes Sx) = m(x \otimes y) + m(y \otimes (-x)) \\
&= m(x \otimes y) + (-m(y \otimes x)) = m(x \otimes y) + Sm(y \otimes x) \\
&= {}_{\mathsf{H}}[x,y]
\end{aligned}$$

Now, using the equation $[x,y]_{\mathsf{H}} = m(x \otimes y) + (-m(y \otimes x))$ just shown to hold, one proves literally the same way as in the case of R-modules

(1) If $x, y \in H$ are primitive, so is $[x,y]_{\mathsf{H}}$.
(2) $[-,-]_{\mathsf{H}}$ satisfies Equation (2.1) as well as $[x,x]_{\mathsf{H}} = 0$.

This shows that $\bar{P}\mathsf{H} := (Prim(\mathsf{H}), [-]_{\mathsf{H}})$ is a Lie algebra in $\mathcal{V}_{\mathsf{Ab}}$. Obviously this construction is functorial.

Recall from the construction of the tensor bimonoid that all elements of the form $\iota_1(x)$ are primitive in TL. Since $TH \xrightarrow{q} UH$ is a bimonoid morphism by commutativity of Diagram (2.3), it follows from Proposition 3.2 that, for each $L \in \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}}$, the $\mathcal{V}$-morphism $\eta = L \xrightarrow{\iota_1} TL \xrightarrow{q} LieUL = InvUL$ factors as

$$L \xrightarrow{\iota_1'} PrimTL \xrightarrow{q'} PrimUL = \bar{P}U_H(L) \hookrightarrow Inv(UL) = LieUL$$

and that $\eta' := q' \circ \iota_1'$ is a Lie-morphism.

If now, for some Hopf monoid H, $\mathsf{L} \xrightarrow{f} \bar{P}H$ is a Lie-morphism, so is $f' = L \xrightarrow{f} \bar{P}H \hookrightarrow Lie_{\mathsf{Ab}}H$, such that by item 3 of Proposition 2.2 there exists a unique monoid morphism $\tilde{f}\colon U_{\mathsf{Ab}}L \to H$ with $f' = Lie_{\mathsf{Ab}}(\tilde{f}) \circ q \circ \iota_1$.

From $\operatorname{Im} f' \subset Prim(\mathsf{H})$ one concludes $\operatorname{Im} \tilde{f} \subset Prim(\mathsf{H})$, and this implies that the outer frame of the following diagram commutes.

$$\begin{tikzcd}
L \arrow[r, "\iota_1"] \arrow[d, "\pi"'] & TL \arrow[r, "q_L"] \arrow[d, "\mu"] & UL \arrow[r, "\tilde{f}"] \arrow[d, "\delta"] & H \arrow[d, "\mu_H"] \\
L \otimes L \arrow[r, "\iota_1 \otimes \iota_1"'] & TL \otimes TL \arrow[r, "q_L \otimes q_L"'] & UL \otimes UL \arrow[r, "\tilde{f} \otimes \tilde{f}"'] & H \otimes H
\end{tikzcd}$$

Since the left and middle cells commute by definition of μ and commutativity of Diagram (2.3), respectively, $\tilde{f}$ is compatible with the comultiplications. Similarly, $\tilde{f}$ preserves counits and so is a morphism in $\mathsf{Hopf}\mathcal{V}$. This shows that U_H is left adjoint to $\bar{P}$. □

Remarks 3.5. It follows from the theorem above that, as in the classical case, we also obtain a functor $\mathsf{Bimon}\mathcal{V} \to \mathsf{Lie}\mathcal{V}$ by forming the composition $I \circ P_H \circ C$, where $C\colon \mathsf{Bimon}\mathcal{V} \to \mathsf{Hopf}\mathcal{V}$ is the coreflection functor, available in every locally presentable monoidal closed category (see [14]), hence in every entropic variety, and I is the embedding $\mathsf{Lie}\mathcal{V}_{\mathsf{Ab}} \hookrightarrow \mathsf{Lie}\mathcal{V}$. This can be considered a substitute for P, since $I \circ P_H \circ C \vdash E \circ U_H \circ R = U_{Bi} \circ I \circ R$ by composition of adjunctions and since, by construction, $U_{Bi} \circ I = E \circ U_H$. Hence, for $\mathcal{V} = \mathbf{Mod}_R$, where $I = id$, one has $I \circ P_H \circ C = P_H \circ C \vdash U_{Bi}$ and, thus, $I \circ P_H \circ C \simeq P$.

Finally, denoting $\mathit{Cof}\colon \mathsf{Mon}\mathcal{V} \to \mathsf{Bimon}\mathcal{V}$ the cofree bimonoid functor, that is the right adjoint of the forgetful functor $|-|\colon \mathsf{Bimon}\mathcal{V} \to \mathsf{Mon}\mathcal{V}$, available in every locally presentable monoidal closed category as well (see [14]), one obtains $U = (|-| \circ U_{Bi}) \dashv (P \circ \mathit{Cof})$, such that $\mathit{Lie} \simeq P \circ \mathit{Cof}$ follows. We note that we could not find the last equation in the literature. It says in particular that, for any R-algebra A, the R-module A is isomorphic to the R-module $\mathit{Prim}(\mathit{Cof}\,\mathsf{A})$ of primitive elements in the cofree bialgebra over A.

For the convenience of the reader we visualize the various functors as follows.

$$\begin{tikzcd}[column sep=large]
\mathsf{Lie}\mathcal{V} \arrow[r, shift left, "R"] \arrow[rrrr, bend left=30, "U"] \arrow[rrr, bend left=20, "U_{Bi}"] & \mathsf{Lie}\mathcal{V}_{\mathsf{Ab}} \arrow[l, shift left, "I"] \arrow[r, shift left, "U_H"] & \mathsf{Hopf}\mathcal{V} \arrow[l, shift left, "P_H"] \arrow[r, shift left, "E"] & \mathsf{Bimon}\mathcal{V} \arrow[l, shift left, "C"] \arrow[r, shift left, "|-|"] \arrow[lll, dotted, bend left=20, "P"] & \mathsf{Mon}\mathcal{V} \arrow[l, shift left, "\mathit{Cof}"] \arrow[llll, bend left=30, "\mathit{Lie}"]
\end{tikzcd}$$

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