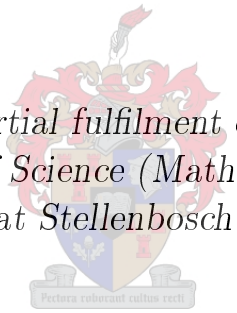
An Age Structured Model for Substance Abuse Dynamics in the Western Cape Province of South Africa.

by

Filister Chinake



Thesis presented in partial fulfilment of the requirements for the degree of Master of Science (Mathematics) in the Faculty of Science at Stellenbosch University

Department of Mathematical Sciences,
University of Stellenbosch,
Private Bag X1, Matieland 7602, South Africa.

Supervisor: Prof. F. Nyabadza

March 2017

Declaration

By submitting this thesis electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the sole author thereof (save to the extent explicitly otherwise stated), that reproduction and publication thereof by Stellenbosch University will not infringe any third party rights and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Date: ..... March 2017 .....

Abstract

An Age Structured Model for Substance Abuse Dynamics in the Western Cape Province of South Africa.

F. Chinake

Department of Mathematical Sciences,
University of Stellenbosch,
Private Bag X1, Matieland 7602, South Africa.

Thesis: MSci(Math)

January 2017

The substance abuse problem has escalated in the Western Cape province of South Africa. This has resulted in high rates of gangsterism and other problems associated with substance abuse. The problem has evolved as gangsters have aggressively extended their turf by recruiting school learners to sell drugs within school premises. The effect is that more age groups are being recruited into abusing substance. In order to reverse the current trends of substance abuse it is imperative that the dynamics of the problem are fully understood. More insight can be gained if age structure was incorporated into the substance abuse models as the processes like initiation, escalation into problematic substance abuse and quitting are influenced by age. Thus we propose an age structured model of substance abuse. A form of the reproduction number R_0 is calculated and the model is shown to be well posed. A suitable finite difference scheme is discussed for the numerical solution of our partial differential equations. Sensitivity analysis is undertaken using the Latin Hypercube Sampling and Partial Rank Correlation Coefficient. Parameters for the model are obtained by fitting the model to the age structured data for individuals in the rehabilitation centres. The dynamics of the model are described by the results from the numerical simulations. The model is used to predict the dynamics of substance abuse until the year 2020. Substance abuse is predicted to increase with time and higher incidence of substance abuse expected for the older age groups.

Uittreksel

'n Ouderdom gestruktureerde model vir die dinamika vir substansie misbruik in die Wes-Kaap Provinsie van Suid Afrika.

("An Age Structured Model for Substance Abuse Dynamics in the Western Cape Province of South Africa.")

F. Chinake

Departement Wiskundige Wetenskappe,
Universiteit van Stellenbosch,
Privaatsak X1, Matieland 7602, Suid Afrika.

Tesis: MScie (Wiskunde)

Januarie 2017

In die Wes-Kaapse Provinsie van Suid-Afrika het die misbruik van dwelms vererger. Dit het groot skaalse probleme en rampokkery wat geassosieer word met dwelms veroorsaak. Die probleem het gergroei toe rampokkers met aggressie hul turf vergroot het deur skoolkinders te rekruut om dwelms op die skoolgrand te verkoop. Die effek is dat meer ouderdomsgroepe epgelei word om dwelms te misbruik. Om die huidige koers van dwelm misbruik om te keer, is dit noodsaaklik om die dinamika van die probleem volledig te verstaan. Meer insig kan herwin word as 'n ouderdomsstruktuur opgerig word in dwelm misbruik-modelle sodat die prosesse soos inisiasies,die escaleer in problematiese dwelmmisbruik en opgee gein vloed word deur ouderdom. Dus stel ons voor in ouderdomstruktuur model van dwelmimisbruik vorm van die reproduksie nommer R_0 is bereken en die model is goed gestel. 'n verkose finiet verskil skema is bespreek vir die numerieke oplossing van ons partydige differensiale gelyke. Sensitiewe analise is onderneem deur die Latin Hypercube Sampling en Partial Rank Correlation Coefficient te gebruik. Die paramaters vir die model is verwerf deur pas die model by ouderdomstruktuur data vir individuele in die rehabilitasie sentrums aan te pas. Die dinamika vir die kompartemente van die model is beskryf deur die resultate van die numerieke simulasies. Die

model word gebruik om die dinamika van dwelmmisbruik tot in die jaar 2020 te voorspel. Dit word voorspel dat dwelmmisbruik met tyd gaan verhoog en hoer insidente van dwelmmisbruik verwag word in ouer ouderdomsgroepe.

Acknowledgements

I would like to thank the Almighty God for taking me this far. My profound gratitude goes to my supervisor Professor Farai Nyabadza for his invaluable support and patience. The financial assistance of the University of Stellenbosch and the African Institute for Mathematical Sciences (AIMS) who jointly funded this project towards this research is hereby acknowledged.

Dedications

To my husband Noah Chinake and my son Jubilee Victor Chinake.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Substance Abuse

Substance abuse is a complex problem which has widespread consequences. Various research findings have consistently pointed to the enormity of the problem worldwide. Research findings estimate that around 246 million individuals were reported to have used an illicit drug in 2013, with 27 million reported to have progressed to become high risk drug users [2]. High risk drug users is a new term referring to the group of people previously defined as the problem drug users. High risk drug use is defined as 'injecting drug use or long duration or regular use of opioids, cocaine and/or amphetamines' [3].

The number one drug for which people seek treatment in Africa is cannabis. Substance abuse in Africa is mainly fuelled by the continent's role in illicit drug trafficking. The African continent is a transit route for drugs transported across the globe with South Africa considered the regional hub [2]. Although the drugs that enter Africa are destined for Europe and North America, the transit countries have a habit of becoming user countries [4]. These countries become vulnerable to drug abuse along with crime related to drugs. Usually there is increase in organized crime further propagating the economic influence of drug traffickers threatening the security, health and development of the continent.

Alcohol is reported as the most abused substance in South Africa [5]. The Western Cape province was reported to have the second highest prevalence of harmful drinking amongst expecting mothers [6] . High numbers of Foetal Alcohol Spectrum Disorders (FASD) signify the extend of alcohol abuse among women in the province. The rate of babies born with (FASD) in the province is reported to be among the highest in the country. Substance abuse is defined as the hazardous or harmful use of alcohol and illicit drugs. Usually people do not regard alcohol abuse as a real problem for which they should seek treatment. The Ministry of Social Development is mainly responsible for programmes that

seek to address substance abuse issues in the Western Cape province. In order to accomplish the task it is imperative that at risk individuals are timeously identified in order for them to be given necessary help before they progress into dependence.

Drug abuse in South Africa is mainly driven by increased supply of drugs. Illicit drugs are easily available because the country is the largest transit zone in Africa [2]. Increase in poverty escalates street level dealing of some of the illicit drugs in the province. The Cape flats in the city of Cape Town experience high levels of crimes related to drugs as a result of higher levels of unemployment and poverty [7]. Other factors responsible for substance abuse amongst the youth population are family dysfunction and the phenomenon of absent parents. As the drug problem escalates the province is faced with even higher rates of gangsterism. Learners are recruited as merchandisers who facilitate the sale of drugs in school premises as gangs seek to extend their turf. Schools become dangerous places where gang violence and robbery takes place and school activities are disrupted. In the Western Cape province alone results from a survey of 133 schools revealed that 61.6% of these schools had experienced gang related disturbances with 2 out of every 5 schools confirming the presence of drug merchants and peddlers within their schools [8].

In some communities drug dealing is so intricately interwoven into the community micro economics. This usually breeds powerful gang structures with leadership in place exercising power through a patronage system. In the case where a community is impoverished and experiences high unemployment levels the culture of drug abuse and peddling easily takes root as the gangs become the much needed source of employment. Most of the communities begin to rely on the income from drug peddling. When that happens the problem becomes so entrenched into the community thus posing a bigger challenge in addressing the supply of drug. Ultimately there is a criminal network extremely difficult to break [9].

Drug abuse is a multifaceted and relapsing chronic health condition which is difficult to control because it also usually considered a criminal offence [2]. Data cannot be collected using conventional data collection methods and thus epidemiological indicators of substance abuse are usually estimated from incomplete data. This hinders efforts to monitor and control the spread of substance abuse [10]. Information on substance abuse is obtained from general population and school surveys, estimates of problem drug use, data collected from treatment centres, information in relation to drug related deaths and also drug related infections such as HIV.

In the Western Cape province of South Africa one of the challenges is that there is no adequate information on the nature and extent of substance abuse [11]. A related challenge is that of identifying the population subgroups that have relatively high unmet needs. Ability to predict the substance abuse trends

in the province among the various subgroups of the population assists in the the formulation of intervention measures that are targeted at the specific population. The current comprehensive data on the level of substance abuse is collected routinely by the South African Community Epidemiology Network on Drug Use (SACENDU) from the treatment centres. We thus use the data in predicting the trends in the community.

Many studies usually focus on distinguishing people vulnerable to substance abuse by gender where in most cases substance abuse has been indicated to be more prevalent among males than females. Demand for treatment services amongst the youth addicted to methamphetamine has escalated in the Western Cape province[11]. Consequently the older alcohol depended adults found it difficult to access treatment. In order to ensure that resources are optimally allocated we need to understand how substance abuse evolves with time and age. Youth initiation to a drug is expected to rise in an environment where the risk perception is declining and availability is increasing [12]. Hence we can clearly see that factors that sustains a culture of abusing substances are different across ages. Prevention and treatment efforts are more targeted prior to a proper assessment of the current distribution of services measured against the actual need for the service.

1.1.1 Vulnerability to Substance Abuse.

Development of drug use disorders is a result of complex multi-factorial interaction between repeated exposure to drugs, biological and environmental factors [2]. There is no single cause of substance abuse but results from the interplay between various factors such as age, social class, occupation, school status, gender and geographical location.

Problematic substance abuse usually start at the adolescence stage. People from different age groups are recruited into abusing substances via different mechanisms. In most cases young people are influenced by peer pressure to start experimenting with drugs. The culture of communal drinking particularly promotes alcohol abuse among adults [5]. Some factors that are influential in making individuals susceptible to substance abuse are chemical dependence on alcohol, poor social conditions and boredom. On the other hand some people abuse substances because they seek to alter their mood states, while for some it is a mechanism for coping with stressful situations as well as a way of enjoyment. This is exacerbated by lack of social mechanisms that are put in place to deal with people abusing alcohol and illicit drugs.

1.1.2 Impact of Drug Abuse

The abuse of stimulants such as methamphetamine in Cape Town is also linked to other health related problems. Usually this leads to a rise in risky sexual

behaviour [6]. There is also the escalation of social problems associated with abusing substances. Substance abuse leads to lack of motivation in youths as well as interfere with cognitive processes which ultimately results in educational failure. Usually young people experience debilitating mood disorders and increases the risk of accidents that can injure or kill them.

Substance abuse can have devastating effects to individuals, families and the community at large. Failure to deal with the problem effectively is costly. High costs in health are inevitable because substance abuse results in other health related problems. Abusing substances is a catalyst for sexual risky behaviour likely to result in widespread transmission of HIV. Other health complications emanating from abusing substance are mental health problems, overdosing as well as injuries due to violence and accidence. There are also costs associated with treating individuals with drug and alcohol abuse issues.

Abusing substances is usually associated with crime and offending behaviour as well as risky sexual behaviour. People who abuse substances are more likely to commit crime [13]. The most common crimes committed by individuals abusing substances are housebreaking, robbery, domestic violence and theft. Information quantifying number of crimes related to substance abuse gives an indication of the level of substance abuse in a community.

1.1.3 Substances Abused In Cape Town

Cape Town experiences high levels of substance abuse with most people reporting methamphetamine as their primary substance of abuse. There has been an increase in demand of drug abusing treatment [14] with methamphetamine accounting for 35% of patient admission. Other common drugs abused are alcohol and cannabis.

Fighting the substance abuse issue is more complex because there are so many substances available for an addict to experiment with. Alcohol is the number one substance of abuse in South Africa. Alcohol consumption is more common among farm workers around South Africa which is a result of the 'dop system' of the apartheid era. Under the 'dop system', farm employees were given alcohol as the benefit for employment [15]. Methamphetamine is the most popular substance abused in Cape Town. 98% of tik addicts who seek help in South Africa are from the Western Cape [9]. According to research by SACENDU, alcohol is the second substance of abuse in the Western Cape Province and cannabis is the third most abused substances.

1.1.3.1 Methamphetamine

Methamphetamine is a powerful highly addictive stimulant which affects the central nervous system. The street name of the drug in Cape Town is called Tik. Worldwide methamphetamine goes by the name meth, chalk, ice and

crystal. It is a white, colorless, bitter-tasting crystalline powder that dissolves easily in water or alcohol. It is a drug that is medically used for patients who suffers from attention deficit hyperactivity disorder. It is also used as a short component of weight loss treatment. To the user the drug results in a pleasurable sense of well being or euphoria. Greater amounts of the drug gets into the brain and will cause longer lasting harmful effects to the central nervous system. This drug has a high potential for widespread abuse [16].

1.1.3.2 Alcohol

Alcohol Use Disorder (AUD) is a diagnosed medical condition for problematic alcohol drinking that becomes severe [17]. Alcohol addiction is distinguished from alcohol abuse with addiction defined to be the psychological and physical dependence on alcohol while alcohol abusers are usually heavy drinkers, not necessarily addicted, who will perpetuate their drinking despite the consequences [18]. Abusing alcohol has effects on the functioning of the body thus affecting the mood and behaviour of a person. Usually a person has difficulties thinking and making movement co-ordinations [19].

1.1.3.3 Cannabis

Cannabis is commonly known as marijuana [20]. Recreational users of cannabis perceive it as harmless. It is obtained from the plant Cannabis sativa and its subspecies. Cannabis contains Δ^9 tetrahydrocannabinol (THC) responsible for marijuana intoxication resulting in its use for recreational purposes [21]. It can be smoked in hand rolled pipes, water pipes and also in blunts. Some people prefer ingesting marijuana after mixing it with food [22]. The use of cannabis can result in mood altering effects such as euphoria. This ability to produce a high often results in wide spread and often chronic recreational use. Fatuous laughter and talkativeness often results if the substance is taken in a social gathering setting. In naive users the most common side effects are anxiety, panic reactions, increased risk of accident as well as increased risk of psychotic symptoms [20; 21]. The long term use of cannabis increases the risk of respiratory cancer as well as acute and chronic bronchitis. Smoking in pregnancy is indicated for increased risk in birth defects such as ventricular septal defect and low birth weight [23].

1.2 Project Motivation and Objectives of the Study

1.2.1 Motivation

Africa is no longer just the transit route for illicit drugs, but has also become a consumer. Consequently the substance abuse problem has evolved into a more complex problem. The devastating effects to communities affected by substance abuse are too enormous to miss. In some suburbs of Cape Town reports of violence and crime related to drug trafficking activities have increasingly become common. There is also a big challenge of providing support for the ever increasing number of drug users a direct result of drug trafficking activities in the city. One of the aims of the African Union as reported in their "Action plan on drug control (2013-2017)" is to increase monitoring of changing and emerging trends of drug use as well as the implementation of evidence based responses [24]. Thus it is imperative that in order to gain an understanding of the changing and emerging trends we need to consider important characteristics that are influential in susceptibility to substance abuse. Age is one such factor that needs to be monitored and help detect the changing trends in relation to the substance abusing problem.

1.2.2 Objectives of the Study

The main objective of the study is to understand the dynamics of substance abuse. In particular we want to incorporate age structure into our model.

The specific objectives are to;

1. Formulate an age structured model of substance abuse.
2. Carry out mathematical analysis of the model formulated
3. Carry out numerical simulations of the age structured model of substance abuse.
4. Fit the model to the Cape Town Data for substance abusing people in rehabilitation and estimate the parameters of the model.
5. Predict the trend of the age related substance abuse for the Cape Town population.

1.3 Mathematical Preliminaries

1.3.1 The McKendrick-von Forster Partial Differential Equation

The McKendrick equation gives a description of the time evolution of a population that is structured in age. It is one of the ways of modelling the evolution of an age structured population and it takes the form of the following partial differential equation:

$$\frac{\partial P(a,t)}{\partial t} + \frac{\partial P(a,t)}{\partial a} = -\mu(a)P(a,t). \tag{1.3.1}$$

$P(a,t)$ is the density of the population age a at time t and $\mu(a)$ is the instantaneous death rate. A standard way of solving equation (1.3.1) is by using a method of characteristics.

1.3.1.1 Method Of Characteristics

There are curves in the a-t plane called characteristic curves. Along these curves the solution is constant and equal to its initial value.

In order to find the characteristic curves we introduce the following parametric curves

$$a = a(s) \quad \textit{and} \quad t = t(s)$$

and we have the following

$$K(s) = P(a(s), t(s))$$

taking

$$\frac{dK}{ds} = \frac{dP(a(s),t(s))}{ds} = \underbrace{\frac{\partial P}{dt}\frac{dt}{ds} + \frac{\partial P}{da}\frac{da}{ds}}_{\text{using the chain rule}}. \tag{1.3.2}$$

If we chose

$$\frac{da}{ds} = 1 \tag{1.3.3}$$

$$\frac{dt}{ds} = 1 \tag{1.3.4}$$

we have

$$\frac{dK}{ds} = \frac{\partial P}{dt} + \frac{\partial P}{da}. \tag{1.3.5}$$

Eventually we get

$$\frac{dK}{ds} = -\mu(a(s))K \tag{1.3.6}$$

integrating equations (1.3.3) and (??) we obtain the following characteristic curves

$$t = t_0 + s \quad \text{and} \quad a = a_0 + s$$

Integrating equation (1.3.6) we get

$$\begin{aligned} \int_0^s \frac{dK(\alpha)}{K(\alpha)} &= -\int_0^s \mu(a(\alpha))d\alpha \\ \ln\left[\frac{K(s)}{K(0)}\right] &= -\int_{a_0}^{a_0+s} \mu(\theta)d\theta \\ K(s) &= K_0 \frac{e^{-\int_0^{a_0+s} \mu(\theta)d\theta}}{e^{-\int_0^{a_0} \mu(\theta)d\theta}} \end{aligned} \tag{1.3.7}$$

Considering the initial condition $P(a, 0)$ where $K(s) = P(a(s), t(s))$

Equating $P(a, 0) = P(a(s), t(s))$ we get

$$a(s) = a_0 + s = a$$

$$t(s) = t_0 + s = 0.$$

Taking

$$t_0 = 0 \Longrightarrow t = s \quad \text{and} \quad a_0 = a - t$$

But we know that

$$K(s) = P(a_0 + s, s)$$

and

$$K(0) = P(a_0, 0) = P(a - t, 0),$$

hence

$$P(a, t) = P(a - t, 0)e^{\int_{a-t}^{a} \mu(\theta)d\theta} \quad \text{for} \quad a > t. \tag{1.3.8}$$

If we consider the initial condition $P(0, t)$ where $K(s) = P(a(s), t(s))$,

equating $P(0, t) = P(a(s), t(s))$ we get

$$a(s) = a_0 + s = 0$$

$$t(s) = t_0 + s = t.$$

Taking

$$a_0 = 0 \Longrightarrow a = s \quad \text{and} \quad t_0 = t - a$$

But we know that

$$K(s) = P(s, t_0 + s)$$

and

$$K(0) = P(0, t - a)$$

Hence

$$P(a,t) = P(0,t-a)e^{\int_0^a \mu(\theta)d\theta} \quad \text{for} \quad t > a. \tag{1.3.9}$$

Thus

$$P(a,t) = \begin{cases} P(0,t-a)e^{-\int_0^a \mu(\theta)\mathrm{d}\theta} & \text{if } a < t, \\ P(a-t,0)e^{-\int_{a-t}^a \mu(\theta)\mathrm{d}\theta} & \text{if } a > t. \end{cases}$$

1.3.2 The Reproduction Number

The reproduction number denoted R_0 is defined as the average number of secondary cases generated by a primary case [25]. In the context of substance abuse R_0 is defined as the number of substance abusers produced when a single substance abusing individual is introduced into a population of susceptible individuals.

To obtain an expression for the basic reproduction number we make use of the Euler Lotka characteristic equation.

1.3.2.1 The Euler Lotka Characteristic Equation

The Euler Lotka characteristic equation is given as

$$\int_\alpha^\beta e^{-ra} m(a) l(a) da = 1 \tag{1.3.10}$$

where $l(a)$ is defined as the survival rate and $m(a)$ is defined as the maternity function depicting the birth rate per capita for mothers of age a.

Equation (1.3.10) has a unique solution r_0 which denotes the intrinsic rate of natural increase of a population. This is computed as the dominant real root of the Euler Lotka characteristic equation.

1.3.3 Existence and Uniqueness of Solutions

One of the issues when dealing with a model that is given as a coupled set of partial differential equations is that it is not easy to derive an analytical solution to the system. As a result we appeal to a numerical solution that best approximates the solution. Before resorting to a numerical implementation there are some issues that must first be verified. We need to determine if our system of equations is well posed. The main question to be ascertained is that do there exist a solution to our system? If that is satisfied we want the solution to be unique and to continuously depend upon the given initial

data. In order to establish the above conditions we formulate our system as an abstract Cauchy problem of the form

$$\frac{dx}{dt} = Ax(t) + f(t, x) \tag{1.3.11}$$

The solution of the above equation can be established using the semi group approach as follows

1.3.3.1 The Semi Group Approach

If we consider the following differential equation

$$\frac{dx}{dt} = Ax(t), \quad t \geq 0, \quad x(0) = f \quad \text{on a Banach space X} \tag{1.3.12}$$

where A is a given linear operator with domain $D(A)$ and $f \in X$. We obtain the generator A of $T(.)$ which is given by setting

$$D(A) = \{f | \lim_{t \to 0^+} \frac{1}{t}(T(t)f - f) \quad \text{exists}\}$$

and is such that

$$Af = \lim_{t \to 0^+} \frac{1}{t}(T(t)f - f) \quad \text{for} \quad f \in D(A)$$

Since the generator defined above is a generator of a strongly continuous semi group $T(t)_{t \geq 0}$ we have that for every $f \in X$ the orbit map

$$x : t \longrightarrow x(t) = T(t)f$$

is the unique mild solution of the abstract Cauchy equation (1.3.12)

1.4 Outline of the Thesis

In Chapter 1 we introduce and discuss the substance abusing problem. Chapter 2 is dedicated to reviewing age structured models and models of substance abuse. We introduce the age structured model of substance abuse in Chapter 3 and carry out the mathematical analysis of this model, formulate the numerical scheme as well as fitting of the model to the data for Cape Town is undertaken. Chapter 4 is dedicated to discussing our findings and the limitations of our model.

Chapter 2

Literature Review

2.1 Mathematical Models

Mathematical models are useful tools that can be employed to describe the dynamics of a disease within a population. Even the simplest of models does provide useful insights vital for understanding complex processes [26]. A mathematical model allows for conceptual experiments that would be physically difficult or impossible to do [27]. Models enable us to extrapolate from available information and predict if there will be a disease outbreak and how this outbreak will evolve. This ability to predict is a vital tool that facilitates that measures for curbing disease spread are put in place. Mathematical models can also be applied to assess the impact of intervention measures by policy makers [28].

2.2 Deterministic Models of Substance Abuse

Epidemiological models are used to study the dynamics involved in the initiation and use of drugs because substance abuse is naturally contagious [29]. In the case of infectious diseases there is need for an agent that is transmitted via physical contact while substance abuse is spread as an innovative socially acceptable practice to those susceptible to substance abuse [30]. Studying the evolution of substance abuse is more complex because we must not only consider the susceptible individual's immediate contacts as the only forces behind possible initiation. Another very influential force can be the overall perception of drugs in the susceptible person's society as sometimes portrayed in movies and news [29].

The SIR model by Kermack and McKendrick is a simple compartmental model that divides the population into 3 disease classes namely the susceptibles, infectives and the recovereds. Assumptions are made with regard to the nature and the rates of movement between the disease states. Compartmental mod-

els are a useful tool for predicting the dynamics of a disease within a population. Usually the independent variable is time and the dynamics of the compartments are given as coupled differential equations. Substance abuse is so similar to infectious diseases because it can be passed on from person to person. Many compartmental models of substance abuse have already been proposed by many authors. The models seek to understand the current dynamics of substance abuse, predict the prevalence of drug use as well as assess the effectiveness of the intervention measures.

Earlier models of substance abuse were formulated in [31; 32; 33]. In [33] the model seeks to understand the influence of substance abuse on HIV. The population is classified according to where they belong in the drug using career. Thus the population is divided into four compartments that consist of individuals susceptible to illicit drug use in one compartment. The other 3 compartments are for individuals that are using illicit drugs. Users of illicit drugs are categorised into individuals injecting drugs, individuals who are crack-cocaine users and individuals that are users of both crack -cocaine users and injecting drug users. All the individuals are considered as susceptible to HIV. Results from the analysis of the model showed that HIV/AIDS was expected to increase with the increase in the use of drugs and specifically where the mode of intake is injecting. Alternatively in the open version of the model it was established that positive correlation between between HIV/AIDS and the use of drugs only occured when the death rate due to abusing drugs was above a certain threshold value.

White and Comisky in [31] alludes to the fact that many authors focused on the impact of opiate to the individual and the society. This is not adequate in light of the reality that opiate/heroine addiction is a global phenomenon. In [31] the authors provided an initial framework in the mathematical epidemiology context within which some characteristics of the opiate using career could be identified. Similar to other infectious disease model an interpretation of R_0 was given. A sensitivity analysis on R_0 shows that it increases with increase in the transmission parameter. $R_0 > 1$ means that on average a drug user introduces at least one new drug user during their drug using career.

The stability of the positive equilibrium of the model in [31] was later investigated Mulone *et al.* in [34]. Mulone *et al.* established that an unstable endemic equilibrium signals an epidemic in heroine use. Of interest is some of the ideas for further consideration in [34] that could be explored to further understand the dynamics of substance abuse. The authors alluded in [34] to the fact that models could include a delay effect while accounting for relapse from rehabilitation thus using delay differential equations instead of the ordinary differential equations. Another possible consideration is to account for the male and females in the substance abuse model since they exhibit different dynamics. More insight could be gained by accounting for spatial movement

of individuals in our models as well as study the effects of influences like peer pressure.

The model formulated by Fabio Sanchez *et al.* in [32] is a typical SIR model. The population is divided into three compartments namely susceptibles, problem drinkers and recovereds. Drinking alcohol is usually a socially acceptable activity unless it becomes problematic. Thus in the model individuals who are casual and moderate drinkers are considered as susceptibles that could potentially advance into problematic drinking. The drinking free equilibrium is defined as the state where a drinking culture does not exist. The reproduction number is thus defined as the measure of resilience of the drinking free equilibrium to the invasion of problem drinkers. It gives the ratio of the average number of secondary cases generated by a typical drinker where 2 forms of reproduction numbers are defined. A reproduction number $R_0 = \frac{\beta}{\mu}$ is defined in the absence of treatment and there is no class of those recovered from drug use yet. The assumption is that recovery or quitting only occurs after undergoing treatment. The other formulation of the reproduction number $R_\phi = \frac{\beta}{\mu+\phi}$ caters for the existence of recovered people after undergoing treatment. Clearly R_0 is greater than R_ϕ whenever ϕ is greater than 0. The implications of these different reproduction numbers is that $R_0 < 1$ guarantees that the culture of drinking will not be established as long as the initial number of drinkers is low . On the other hand $R_0 > 1$ ensures that just the introduction of a single problem drinker will result in the eruption of a culture of drinking. $R_\phi < 1$ does not guarantee that a culture of drinking will not be established because the R class can produce more people who are susceptible to alcohol abuse. The possibility of relapsing after recovering makes it imperative for treatment to be effective to avoid the blowing out of the alcohol abuse even with high rates of uptake into rehabilitation and quitting after treatment.

Recently a number of mathematical models have been studied on specific substances for the Western Cape province recently by [7; 30]. In [30] a model for methamphetamine abuse is considered while in [7] a model studying the dynamics of amphetamine use in the Western Cape Province is formulated. The model by Kalula and Nyabadza [30] is a modification of the model in [7]. The total population is divided into core and non core groups. Both models includes 2 classes of drug users namely light drug users and hard drug users. The model in [7] considers only in-patient rehabilitation hence there is no relapse while in rehabilitation. The type of rehabilitation for the model in [30] is outpatient and thus there is a possibility to relapse even while in rehabilitation.

A common assumption in recent models of substance abuse is that before a susceptible individual progress into the hard drug use state they are first recruited as light drug users [7; 29; 30]. The supporting assumption is that light drug users provide a positive feedback that has more potential of initiating susceptible persons into substance abusing. On the other hand hard drug users

are perceived negatively and alert people to the danger of substance abuse [7; 29; 30].

The model in [7] seeks to predict the prevalence of drug use in the Western Cape province. Data for amphetamine abusers in treatment for the period 1996 to 2008 was used for fitting. An assumption of the model is that those who quit while in treatment move to the compartment of quitters who can relapse after they have first recovered. Another very realistic assumption is that a quitter who relapses will most likely move straight into the hard drug user state because of the previous familiarity with abusing drugs. The incidence function includes an exponential function so as to cater for behaviour change in the model. Behaviour change is likely to result if a susceptible individual is exposed to the adverse effects of drugs such as death as they occur to a person already abusing substances. The model reproduction number is defined as the number of new initiates generated by one index case in a population that is entirely susceptible. It is a threshold number that determines the persistence of amphetamine abuse. Reduction in reproduction number results in the reduction in the number of drug users. Practical ways as informed by the model is to reduce the contact rate between the susceptible and those on drugs, encourage increase in the behaviour change as well as reduce the time that those on drugs spend in the light drug use state where they have the greatest potential to recruit more susceptible people.

Kalula and Nyabadza in [30] divides the population into core and non core groups. Through some interaction with individuals in the core group the people in the non core group are recruited into the core group as susceptibles. Once there are in the core group they can advance into the compartments such as light drug use and hard drug use. This differs with the other models of substance abuse where everyone else not currently involved in substance abuse is considered as susceptible. The model is shown to be well posed by establishing a feasible region that is positively invariant where the state variables remained non negative for positive initial conditions. R_0 is defined to represent the average number of secondary cases that one drug user can generate in a population of potential drug users. R_0 is calculated using the next generation method. Numerical simulations to verify the theorem on stability of the drug free equilibrium are carried out using the Runge Kutta scheme in Matlab. A critical value of R_0 is established below which no drug persistent equilibria exists. It is not enough for $R_0 < 1$ but for an effective drug abuse control the reproduction number must be brought below the critical reproduction number. The role of key parameters on the value of the reproduction number was investigated and it was established that reducing R_0 can be achieved by accelerating the rate of transference into the hard drug use compartment and the rate into the quitters compartment and into the rehabilitation compartment. The model proposes that in order to effectively fight the substance abusing problem it is imperative to limit the time spend in the light substance abus-

ing compartment since these individuals are assumed to be more capable of recruiting more people than hard drug users. The data applied for the model was that of methamphetamine users in South Africa that sought treatment for the ten year period up to 2009.

The model in [7] was modified in [35]. The disease compartments are the same as in [7] but in the case of [35] initiation into drug use occurs via two processes. In the case of other models initiation occurs as a result of contact with drug users [7; 29; 30]. Another process of initiation captured in the model by [35] is a result of the influence of drug supply chains. As a result this model has two contact rates namely person to person contact and the supply chain to person contact. The other modification is that the model does incorporate the density of drugs in the community and includes some aspect of policing into the model.

The substance abuse models reviewed in this chapter assume that individuals in the same disease class are homogenous. In reality there are so many differences amongst individuals in the same disease compartment. Some of the characteristics can be useful in helping us gain more useful insight into the dynamics of a disease. Various factors are known to contribute when considering why individuals decide to abuse substances. One of these factors is age. Substance abuse models that fail to capture age structure are not likely to capture the important influence of age thus failing to adequately understand the dynamics of the substance abuse problem in a community.

However, preliminary work has been done by some authors addressing the problem. A non compartmental model which incorporated age as an explanatory variable was proposed by Gfroerer *et al.* in [36]. A regression model that incorporated the known predictor variables of substance abuse was considered for the population of the United States Of America. The predictor variables were mainly assigned to the model equations pertaining to individuals who are at high risk of abusing substances in old age. These predictor variables are namely age, gender, race and the history of cigarette, alcohol and marijuana use. The dependent variable was whether the respondent had a need for substance abuse treatment within the past 12 months. The studies established that the there is going to be a large demand for treatment among older adults in the US. It was anticipated that as the baby boom aged, there is going to be a notable increase in the number of people around age 50 experiencing substance abusing problems . This is contrary to the usual expectation where younger people are the ones dominating in terms of having higher numbers involved in substance abusing activities. The implication of the finding is that there will be a need for a different focus in treatment methods. Understanding and ability of treatment programs to adequately cater for the special needs of the older population of substance abuse will ensure the success of these treatment programs. It is assumed that youths who are initiated into illicit drug and

alcohol abuse are most likely to experience problematic substance abuse issues as adults.

Another model that considered age was also proposed by Almeder *et al.* [29]. In [29] a model was formulated where the total population is divided into two groups mainly non- users and users. Death and migration factors are ignored to simplify the model. They also assumed a constant birth cohort size. Movement is only one directional from the non user group to the user group. User population does not just include the current users but includes everyone who has ever consumed or partaken of drugs. The model describing the dynamics of non-users is given as a form of the McKendrick equation. This is given as

$$P_a + P_t = -\mu(a,t)P(a,t) \tag{2.2.1}$$

where in this case $\mu(a,t)$ represents the initiation rate. This initiation rate $\mu(a,t)$ is assumed to be the product of three different factors namely a basic age specific initiation rate, the influence of the reputation of the drug as well as a prevention factor which includes the effects of age specific prevention programs.

The findings from both models [29; 36] indicate that more valuable insight into the dynamics of substance abuse can be gained if age structure is incorporated in the models. We want to explore an age structured compartmental model that includes more disease classes than those considered in [29; 36]. In the next section we explore different models of various diseases that included age structure to gain more understanding into the age structured models.

2.3 Models with Age Structure

Even though most compartmental models assume homogeneity within a compartment in reality, there is still some heterogeneity between individuals in that same disease class. Structured population models seek to account for these differences. Characteristics that distinguish individuals are geographical location, size and age. These factors do influence the population dynamics. Vital population measures such as birth rates, growth rates and death rates are known to differ between age groups [37].

The age structured models capture the effects of demographic behaviour of individuals [38]. Age structured models are the most appropriate for understanding certain diseases. According to [39] the age of an individual can account for the risk of contracting cholera as well as the efficacy of vaccines. Age may also have some influence on reproduction, survival rates and behaviours of individuals. Behavioural changes are the major focus in the control and prevention of many infectious diseases [40].

Disease models with age structure have been used for studying various diseases. Shim *et al.* [41] included age structure in modelling the transmission of rotaries infection. Carlos Castillo Chavez and Wenzhang Huang [42] studied the influence of age structure on the dynamics of sexually transmitted diseases. Both models assume that the contact rate between susceptibles and infectives is age related. In both these models there is an infective agent that can be passed from one infected individual to a susceptible individual.

An age structured model that seek to understand the dynamics of cholera was done by Alexanderian *et al.* [39]. The model studies the dynamics of humans as well as the dynamics of the cholera. The dynamics of the humans are given as a system of partial differential equations and that of cholera is given as a system of ordinary differential equation. For most infectious diseases transmission is a result of close contact between the infected and the susceptible individuals. Cholera is a water-borne disease hence transmission can occur even without physical contact between the infective and the susceptible. Two control strategies are considered in this model namely hydration and treatment by antibiotics. A method of characteristics was employed to prove the existence of solutions by obtaining a representation of the solution. The Banach Contraction Mapping Principle is further employed to build a map that will be used in fixed point argument for existence and uniqueness.

A similar model to that Alexanderian *et al.* of was studied by Agusto *et al.* in [43]. The model consist of 3 classes for humans and the classes for mosquitoes. The ages are classified into 3 age groups namely juvenile, adult and senior. The model equations describing the dynamics of Chikungunya virus are presented as non linear differential equations where each disease class for each age class evolves with time. There are 15 equations that are presented. A sensitivity analysis is undertaken using Partial Correlation Coefficient where the response variable of interest is R_0 .

Some of the age structured models we have explored assume partial immunity to the disease at certain ages. This partial immunity is a result of either vaccination or the presence of maternal antibodies for infants currently being breast fed [39; 41]. Although maternal antibodies have been known to offer immunity to babies some diseases like HIV can be passed on from mother to child. In such cases usually an assumption of vertical transmission is made [44; 45]. In the case of substance abuse there is no partial immunity as a result of vaccination or the presence of the mothers' antibodies. However, there are interventions for prevention that do not necessarily offer immunity which we are not including in our model for simplicity. We are also not taking into account the possible vertical transmission that could occur between parents who abuse substances and their children.

In most disease models the parameters of the model are assumed to be constant while individuals in each disease compartment change with time. In the case

of age structured models most parameters change with age and compartments change with time and age. The transmission parameter is unlike other parameters in most cases. In [45; 46] the age dependent transmission parameter is given as $\beta(a,b)$ which depicts the probability that an infectious person of age b meets a susceptible individual aged a and infects them. In order to prove the existence and uniqueness of solutions condition the problem is considered as an abstract Cauchy problem on the Banach space. The abstract Cauchy problem is shown to have a unique positive global (mild) solution that continuously depend on initial conditions. This is accomplished by employing the semi group approach.

The reproduction number is an important threshold number in disease modelling. This gives insight into the conditions necessary for the disease to either spread or die out eventually. In the case of a disease model consisting of ordinary differential equations detailed explanation on the calculation of R_0 are given in [47]. We compute the basic reproduction number in a similar manner to [41] who also computed the vaccination dependent reproduction number

The models reviewed in this work were not fitted to data. In our case we have available data that we want to fit to the model. A model by Nyabadza and Dieter [48] and Dieter in [1] was fitted to available data. Thus we are going to discuss this model in detail since we are going to explore a model very similar to it.

In [48; 1] an age structured model for the Cape Town Metropole was studied. The model proposed is an initial boundary value problem that seeks to understand the effect of age structure on the dynamics of TB in Cape Town and thus shows that the TB in Cape Town is driven by HIV. The compartments for the TB disease are given as the susceptibles, those who have been vaccinated, the latently infected, individuals with active TB and those recovered from TB.

Their model can be adequately described by the coupled differential equations (2.3.1), corresponding initial and boundary conditions as well as the model diagram in Figure 2.1.

$$\begin{aligned}
\frac{\partial S(a,t)}{\partial t}+\frac{\partial S(a,t)}{\partial a} &= -[\lambda(a,t)+\mu(a)+\psi(a)]S(a,t),\\
\frac{\partial V(a,t)}{\partial t}+\frac{\partial V(a,t)}{\partial a} &= \psi(a)S(a,t)-[\theta\lambda(a,t)+\mu(a)]V(a,t),\\
\frac{\partial L(a,t)}{\partial t}+\frac{\partial L(a,t)}{\partial a} &= \lambda(a,t)[pS(a,t)+\theta V(a,t)+r_2R(a,t)]\\
&\quad -[r_1\lambda(a,t)+\sigma+\mu(a)]L(a,t), \qquad (2.3.1)\\
\frac{\partial I(a,t)}{\partial t}+\frac{\partial I(a,t)}{\partial a} &= \lambda(a,t)(1-p)S(a,t)+[\lambda(a,t)r_1+\sigma]L(a,t)+\phi R(a,t)\\
&\quad -[\mu(a)+\delta+\rho]I(a,t),\\
\frac{\partial R(a,t)}{\partial t}+\frac{\partial R(a,t)}{\partial a} &= \rho I(a,t)-[r_2\lambda(a,t)+\phi+\mu(a)]R(a,t).
\end{aligned}$$

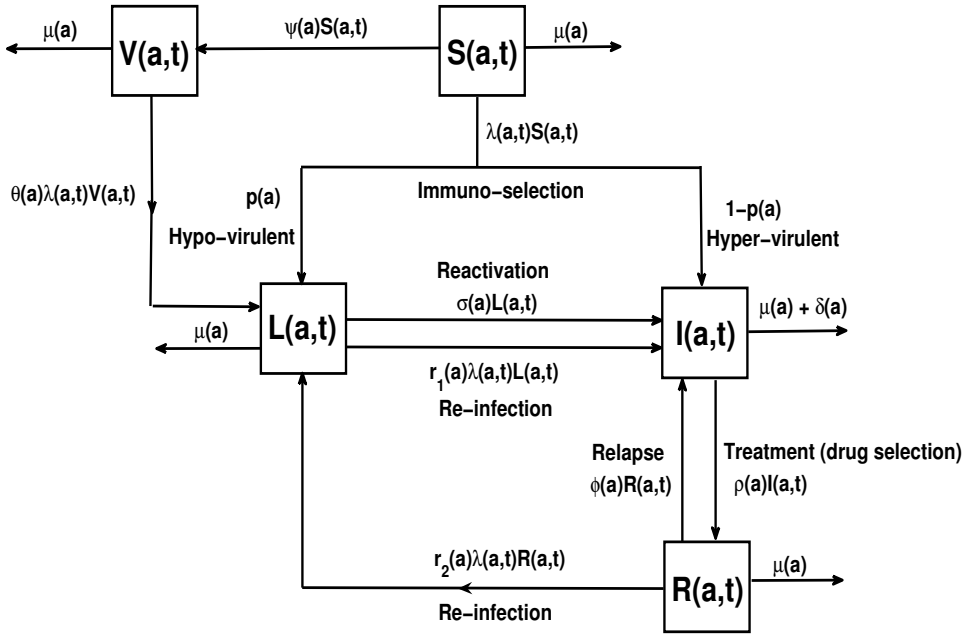


Figure 2.1: SVLIR-Model which incorporates re-infection and reactivation of TB [1].

An analytic solution for the model (2.3.1) is not possible and thus a numerical scheme is applied. The system of equations is discretized by a finite difference scheme similar to the one in [41]. A rectangular domain is considered where $a \in [0, A]$ and $t \in [0, T]$ with A and T denoting the maximum age and maximum time respectively. The scheme in [41] approximate a partial differential equation as follows:

$$\begin{aligned} \frac{\partial X(a,t)}{\partial a} + \frac{\partial X(a,t)}{\partial t} &\approx \frac{X(a_i,t_j) - X(a_{i-1},t_j) + X(a_{i-1},t_j) - X(a_{i-1},t_{j-1})}{h}, \\ &= \frac{X(a_i,t_j) - X(a_{i-1},t_{j-1})}{h}, \\ &= \frac{X_i^j - X_{i-1}^{j-1}}{h}. \end{aligned} \tag{2.3.2}$$

The partial differential equation of model (2.3.1) are approximated with the difference equations as in equation (2.3.2). The resulting discretised system of equations is implemented in Matlab and fitted to the available data for TB cases in Taiwan and Cape Town. For both Taiwan and Cape Town the resulting incidence output is an upper triangular matrix. This is not suitable since the numerical solution for the whole rectangular domain is being sought. The finite difference scheme in [41] is modified to become

$$\begin{aligned} \frac{\partial X(a,t)}{\partial a} + \frac{\partial X(a,t)}{\partial t} &\approx \frac{X(a_i,t_j) - X(a_{i-1},t_j) + X(a_i,t_j) - X(a_i,t_{j-1})}{h}, \\ &= \frac{2X(a_i,t_j) - X(a_{i-1},t_{j-1}) + X(a_i,t_{j-1})}{h}, \\ &= \frac{2X_i^j - X_{i-1}^j + X_i^{j-1}}{h}. \end{aligned} \tag{2.3.3}$$

In most models with age structure the transmission parameter is given as a function of the age of the susceptible individual and the age of the infectious individual [45; 46] with the force of infection given as an integral function. In the model in [48], the transmission is given as a function of age and time. The integral form of the force of infection is discretised using the trapezium rule. However the discretised integral form is not implemented because the resulting force of infection will only be in terms of time and not accounting for age. In the end the discretised form for the force of infection is given as:

$$\lambda_i^j = \beta_i^j I_i^j$$

The discretised TB model is fitted to the data for reported TB cases for Taiwan and Cape Town. Initial and boundary conditions are formulated by making suitable assumptions. Sensitivity analysis was undertaken using Latin Hypercube Sampling and Partial Correlation Coefficient to determine the most important parameters for the TB model. Eventually numerical simulations are implemented in Matlab and the results revealed some big differences in age distribution of TB between Taiwan and Cape Town. Thus the results show that TB in Cape Town is mainly driven by HIV.

2.4 Summary

In this Chapter we have considered two different types of models that have a relationship with the model that we wish to study. The reason we had to explore these models separately is because even though there is some models of substance abuse none of them has really included age structure. Thus we wish to extend and build on the current work and include age structure in our model in a way very similar to [48].

Chapter 3

Model of Substance Abuse with Age Structure.

3.1 Introduction

The concept of epidemiological models was extended to drug abuse by White and Comiskey [31]. A further study of the model by White and Comiskey was undertaken in [34] and possible considerations to the model were outlined. Accounting for the sex and spatial movement of the people in each substance abuse compartment could be possible modifications to the future substance abuse models. To date other models of substance abuse have been formulated with different assumptions. Usually before a drug user progresses to hard drug use there are first initiated as light drug users, thus recent models have separated these individuals into 2 different compartments [7; 30]. The more recent model by Nyabadza [35] also accounts for the influence of drug supply chains and incorporates the density of drugs in the community and includes some aspect of policing into the model. Even with all this work that has been undertaken on substance abuse there is still more that needs to be done in order to better understand the processes involved in the spread of substance abuse. Incorporating age structure into the models of substance abuse will bring more enlightenment on the current and future distribution of the population of substance abuse. The information gained from the model that incorporates age structure will help in the formulation of control strategies that are tailor made to the predicted dynamics of the substance abusing phenomenon. Thus we propose a model similar to [7; 30] but do not make a distinction between light drug users and hard drug users. We extend the models to include age structure motivated by the model in [48].

3.2 Model Formulation

Here we formulate a model that monitors four populations in relation to substance abuse. The first compartment consist of individuals susceptible to drug use denoted $S(a,t)$. Uptake into the susceptibles occurs through birth by assuming that the majority of children born survive to a drug using age. The second compartment consist of individuals involved in abusing drugs denoted by $D(a,t)$. The third compartment consists of individuals who are in rehabilitation denoted $R(a,t)$ recruited from $D(a,t)$. The last compartment $Q(a,t)$ consists of individuals who have stopped using drugs and called quitters. The total population is thus given by

$$N = N(a,t) = S(a,t) + D(a,t) + R(a,t) + Q(a,t).$$

Our model assumptions are as follows:

We assume that the total population is constant. We also assume that that there is a homogeneous mixing among all individuals from different compartments. The class responsible for initiation is $D(a,t)$ which consists of drug users not in treatment. Drug users who are undergoing treatment in compartment $R(a,t)$ are assumed not to initiate new cases. We also assume that the susceptible class $S(a,t)$ consists of people who have never been involved in abusing substances before. Those who quit do not revert back to $S(a,t)$ but there is a possibility that they can still relapse and move straight back into $D(a,t)$.

In formulating the age structured model of substance abuse we introduce the following parameters :

Table 3.1: Model parameters and their description.

Parameter	Description
$\lambda(a,t)$	The force of initiation for susceptibles of age a at time t.
$\mu(a)$	The age specific natural death rate.
$\sigma(a)$	The rate of movement into rehabilitation as a function of age.
$\gamma(a)$	The rate of relapsing while in rehabilitation as a function of age.
$\rho(a)$	The recovery rate as a function of age.
$\omega(a)$	The relapse rate of the recovereds as a function of age.
$\psi(a)$	The drug induced death rate as a function of age.

The dynamics of all the compartments is fully described by Figure 3.1.

The model diagram and assumptions lead to the following system of differential equations,

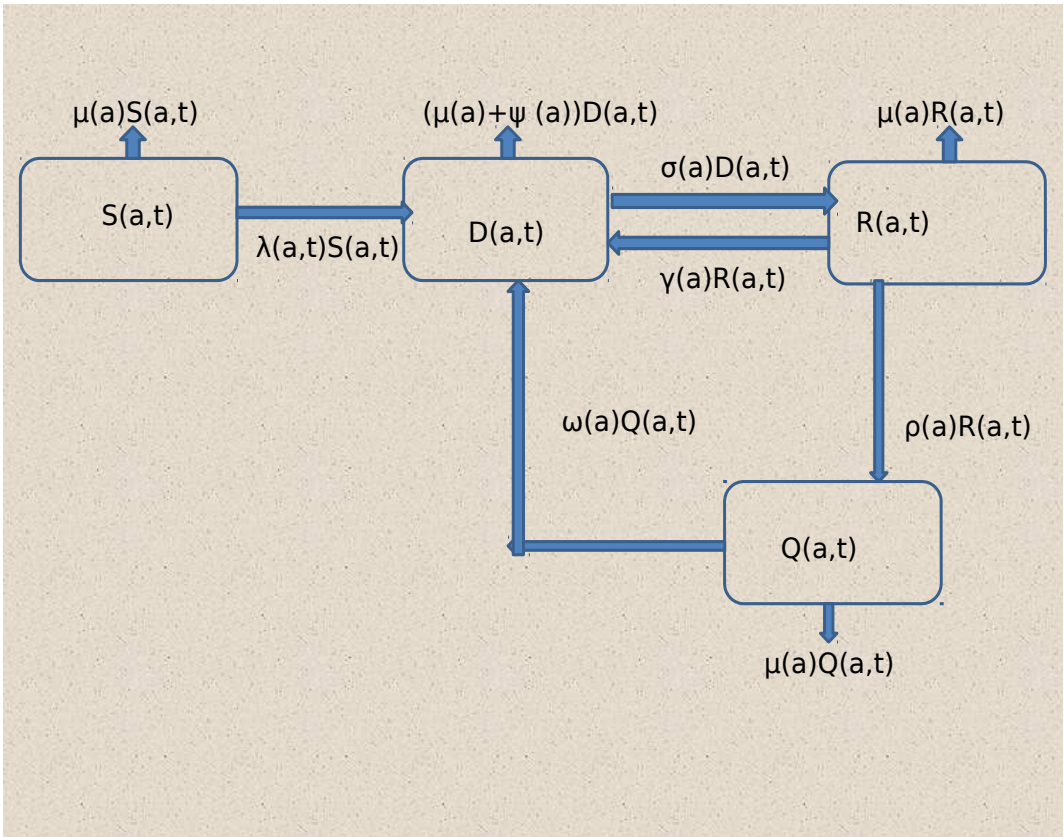


Figure 3.1: The diagrammatic representation of the substance abuse model with the compartments densities $S(a,t), D(a,t), R(a,t), Q(a,t)$ and $\mu(a), \gamma(a), \sigma(a), \omega(a), \lambda(a), \rho(a), \psi(a)$ denoting the transition rates which are ≥ 0.

$$\begin{aligned}
\frac{\partial S(a,t)}{\partial t} + \frac{\partial S(a,t)}{\partial a} &= -\mu(a)S(a,t) - \lambda(a,t)S(a,t), \\
\frac{\partial D(a,t)}{\partial t} + \frac{\partial D(a,t)}{\partial a} &= \lambda(a,t)S(a,t) + \gamma(a)R(a,t) + \omega(a)Q(a,t) \\
&\quad -(\mu(a) + \sigma(a) + \psi(a))D(a,t), \\
\frac{\partial R(a,t)}{\partial t} + \frac{\partial R(a,t)}{\partial a} &= \sigma(a)D(a,t) \\
&\quad -(\mu(a) + \rho(a) + \gamma(a))R(a,t), \\
\frac{\partial Q(a,t)}{\partial t} + \frac{\partial Q(a,t)}{\partial a} &= \rho(a)R(a,t) - (\mu(a) + \omega(a))Q(a,t).
\end{aligned} \tag{3.2.1}$$

We assume that the number of births is equal to the number infants aged zero so that $N(0,t) = S(0,t)$.

As a result, we have the following boundary conditions:

$$S(0,t) = \theta(t) \quad \text{and} \quad D(0,t) = R(0,t) = Q(0,t) = 0.$$

Adding the boundary conditions we get:

$$\begin{aligned} N(0,t) &= S(0,t) + D(0,t) + R(0,t) + Q(0,t), \\ &= \theta(t). \end{aligned} \tag{3.2.2}$$

Adding up equations (3.2.1)) we get

$$\frac{\partial N(a,t)}{\partial t} + \frac{\partial N(a,t)}{\partial a} = -\mu N(a,t) - \psi(a)D(a,t).$$

Thus we have

$$\frac{\partial N(a,t)}{\partial t} + \frac{\partial N(a,t)}{\partial a} \leq -\mu N(a,t). \tag{3.2.3}$$

We can see that equation (3.2.3) is a von McKendrick form with a solution of the form

$$N(a,t) \leq \begin{cases} \theta(t)e^{-\int_0^a \mu(\epsilon)\mathrm{d}\epsilon} & \text{if } a \leq t, \\ N(a-t,0)e^{-\int_{a-t}^a \mu(\epsilon)\mathrm{d}\epsilon} & \text{if } a \geq t. \end{cases}$$

The above result thus gives us the upper bound of our total population $N(a,t)$. The population is thus bounded as follows:

$$\begin{aligned} 0 \leq N(a,t) &\leq \theta(t)e^{-\int_0^a \mu(\epsilon)\mathrm{d}\epsilon} && \text{if a} \leq t, \\ 0 \leq N(a,t) &\leq N(a-t,0)e^{-\int_{a-t}^a \mu(\epsilon)\mathrm{d}\epsilon} && \text{if a} \geq t. \end{aligned} \tag{3.2.4}$$

3.3 Rescaling the system.

We simplify our model by non-dimensionalising equation (3.2.1). To accomplish this we introduce the scaled state variables:

$$s(a,t) = \frac{S(a,t)}{N(a,t)}, \quad d(a,t) = \frac{D(a,t)}{N(a,t)}, \quad r(a,t) = \frac{R(a,t)}{N(a,t)} \quad \text{and} \quad q(a,t) = \frac{Q(a,t)}{N(a,t)}.$$

To obtain the following rescaled version of equation (3.2.1)

$$\begin{aligned}
\frac{\partial s(a,t)}{\partial t}+\frac{\partial s(a,t)}{\partial a} &= -\lambda(a,t)s(a,t),\\
\frac{\partial d(a,t)}{\partial t}+\frac{\partial d(a,t)}{\partial a} &= \lambda(a,t)s(a,t)+\gamma(a)r(a,t)+\\
&\quad \omega(a)q(a,t)-(\sigma(a)+\psi(a))d(a,t),\\
\frac{\partial r(a,t)}{\partial t}+\frac{\partial r(a,t)}{\partial a} &= \sigma(a)d(a,t)-(\rho(a)+\gamma(a))r(a,t),\\
\frac{\partial q(a,t)}{\partial t}+\frac{\partial q(a,t)}{\partial a} &= \rho(a)r(a,t)-\omega(a)q(a,t), \qquad (3.3.1)
\end{aligned}$$

with the following boundary conditions

$s(0,t)=1$ and $d(0,t)=r(0,t)=q(0,t)=0.$

and initial conditions;

$s(a,0)=\varphi_s(a),\quad d(a,0)=\varphi_d(a),\quad r(a,0)=\varphi_r(a),\quad \text{and}\quad q(a,0)=\varphi_q(a).$

All parameters and state variables of system (3.3.1) are assumed to be non negative for $t\geq 0$.

3.4 Existence and Uniqueness of Solution

In order to establish if the system (3.3.1) is well posed we consider the system in an abstract form. In order to do that we introduce $X=L^1(0,a)^4$ with the norm $\|\varphi\|_X=\sum_{i=1}^4\|\varphi_i\|_{l^1}$ where $\varphi=((\varphi_1,\varphi_2,\varphi_3,\varphi_4)\in X)$. $(X,\|.\|_X)$ is a Banach Space for our system, see also [45; 49; 50; 51].

If we consider the operator A
defined by

$$A\varphi=\left(\frac{-d\varphi_1}{da},\frac{-d\varphi_2}{da},\frac{-d\varphi_3}{da},\frac{-d\varphi_4}{da}\right)^T$$

$$\text{where } D(A)=\left\{\varphi=(\varphi_1,\varphi_2,\varphi_3,\varphi_4)\in X;\varphi_i\in W^{1,1}(0,a)\quad\text{and}\begin{pmatrix}\varphi_1(0)\\ \varphi_2(0)\\ \varphi_3(0)\\ \varphi_4(0)\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\}$$

and the function

$$F : \overline{D(A)} \longrightarrow X \text{ defined by } F\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{pmatrix} = \begin{pmatrix}-\lambda(.,\upsilon)\varphi_1\\ -\lambda(.,\upsilon)\varphi_1 + \gamma\varphi_3 + \omega\varphi_4 - (\sigma+\psi)\varphi_2\\ \sigma\varphi_2 - (\rho+\gamma)\varphi_3\\ \rho\varphi_3 - \omega\varphi_4\end{pmatrix}$$

where $\overline{D(A)}$ is the closure of $D(A)$. The non linear operator F is defined on the whole space X where $\lambda(.,\upsilon) \in L^1(0,a)$ such that

$$\lambda(a,\upsilon) = \int_0^A \beta(a,\vartheta)N(\vartheta)\varphi_2(\vartheta)d\vartheta.$$

where $\beta(a,\upsilon) \in L^\infty((0,a)\times(0,a))$.

Let $u(t) = (s(.,t), d(.,t), r(.,t), q(.,t))^T \in X$. We can rewrite the initial boundary value problem (3.3.1) as the abstract semi linear problem in X.

$$\frac{du(t)}{dt} = Au(t) + F(u(t)) \quad u(0) = u_0 \in X \tag{3.4.1}$$

where $u_0(a) = (s_0(a), d_0(a), r_0(a), q_0(a))^T$.

Thus A is the infinitesimal generator of a C_0 semi-group $T(t), t \geq 0$ and F is continuous and locally Lipschitz. Then for each $u_0 \in X$ there exists a maximal interval of existence $[0,t_0]$ and a unique continuous (mild) solution $t \longrightarrow u(t;u_0)$ from $[0,t_0]$ to X. We use the variation of constants formula to obtain the solution as follows:

We rewrite equation (3.4.1) as follows

$$\frac{du(t)}{dt} - Au(t) = F(u(t)) \tag{3.4.2}$$

Thus we have e^{-At} as the integrating factor.

Making use of the integrating factor equation (3.4.2) can be expressed as

$$(e^{-At}u(t))' = e^{-At}F(u(t).) \tag{3.4.3}$$

Integrating both sides of equation (3.4.3) we get

$$e^{-At}u(t) = \int_0^t e^{-As}F(u(s))ds. \tag{3.4.4}$$

Taking the limits of $e^{-At}u(t)|_0^t = e^{-At}u(t) - u(0)$.

Substituting into equation (3.4.4) we get

$$\begin{aligned} e^{-At}u(t) - u(0) &= \int_0^t e^{-As}F(u(s))ds, \\ e^{-At}u(t) &= u(0) + \int_0^t e^{-As}F(u(s))ds, \\ \Rightarrow u(t) &= e^{At}u(0) + \int_0^t e^{A(t-s)}F(u(s))ds. \end{aligned} \tag{3.4.5}$$

Thus we finally have the following unique continuous (mild) solution

$$u(t;u_0) = T(t)u_0 + \int_0^t T(t-s)F(u(s;u_0))ds, \tag{3.4.6}$$

for all $t \in [0, t_0]$ and either $t_0 = \infty$ or $\lim_{t\uparrow t_0}\|u(t;u_0)\| = \infty$. $\lim_{t\uparrow t_0}||u(t;u_0)|| = \infty$

3.4.1 The Invariant Region

We will analyse the model system (3.3.1) in a biologically feasible region Ω.

The region $\Omega \in R^4$ is assumed positively invariant and attracting with respect to model system (3.3.1).

Let $(s(a,t),\ d(a,t),\ r(a,t),\ q(a,t))$ be the solution of model (3.3.1) with non negative initial conditions.

The first equation of system (3.3.1) is a von McKendrick form with a solution of the form

$$s(a,t) = \begin{cases} e^{-\int_0^a \lambda(\alpha,t-a+\alpha)\mathrm{d}\alpha} & \text{if } a \le t, \\ \varphi_s(a)e^{-\int_0^t \lambda(a-t+\alpha,\alpha)\mathrm{d}\alpha} & \text{if } a \ge t. \end{cases}$$

Thus $s(a,t) \ge 0$ always.

Similarly for $r(a,t)$ and $q(a,t)$ we obtain

$$r(a,t) = \begin{cases} \sigma\int_0^a e^{-(\rho+\gamma)(a-\alpha)}d(\alpha,t-a+\alpha)\mathrm{d}\alpha & \text{if } a \le t, \\ r_0(a-t)e^{-(\rho+\gamma)t} + \sigma\int_0^t e^{-(\rho+\gamma)(t-\alpha)}d(a-t+\alpha,\alpha)\mathrm{d}\alpha & \text{if } a \ge t. \end{cases}$$

$$q(a,t) = \begin{cases} \rho\int_0^a e^{-(\omega)(a-\alpha)}r(\alpha,t-a+\alpha)\mathrm{d}\alpha & \text{if } a \le t, \\ q_0(a-t)e^{-(\omega)t} + \rho\int_0^t e^{-(\omega)(t-\alpha)}r(a-t+\alpha,\alpha)\mathrm{d}\alpha & \text{if } a \ge t. \end{cases}$$

We note that $q(a,t)$ can also be written in terms of $d(a,t)$.

$d(a,t)$ is positive as also shown in [45; 49; 50; 51] thus we have that $r(a,t)$ and $q(a,t)$ are also positive.

We also know that the total population has an upper bound, yielding the following result

$$N(a,t) \leq \begin{cases} \theta(t)e^{-\int_0^a \mu(\epsilon)\mathrm{d}\epsilon} & \text{if } a \leq t, \\ N(a-t,0)e^{-\int_{a-t}^a \mu(\epsilon)\mathrm{d}\epsilon} & \text{if } a \geq t. \end{cases}$$

Thus we know that our state variables remain positive given positive initial conditions as also shown in [45; 49; 50; 51],

3.5 The Reproduction Number

The drug free equilibrium of our normalised model is given by

$$E_0 = (1,0,0,0).$$

Below we linearise the $d(a,t)$ about the drug free equilibrium and make the assumption that the solutions initially change exponentially to obtain the characteristic equation.

The characteristic equation will thus be analysed to obtain the formula for the reproductive number.

If we assume the following solutions

$$d(a,t) = \bar{d}(a)e^{\kappa t} \text{ and } \lambda(a,t) = \lambda_0 e^{\kappa t} + 0(e^{2\kappa t})$$

where

$$\lambda_0 = \int_0^\infty \bar{d}(a)B_\infty(a)da \tag{3.5.1}$$

and B_∞ is the effective contact rate when the system is in equilibrium. Linearising $d(a,t)$ yields

$$\kappa e^{\kappa t}\bar{d}(a) + e^{\kappa t}\frac{d\bar{d}(a)}{da} = \lambda_0 e^{\kappa t} - (\sigma+\psi)\bar{d}(a)e^{\kappa t}.$$

Dividing by $e^{\kappa t}$ and considering the linear part of the equation we get

$$\kappa\bar{d}(a) + \frac{d\bar{d}(a)}{da} = \lambda_0 - (\sigma+\psi)\bar{d}(a). \tag{3.5.2}$$

Equation (3.5.2) is integrated as follows:

$$(\kappa+\sigma+\psi)\bar{d}(a)+\frac{d\bar{d}(a)}{da}=\lambda_0. \tag{3.5.3}$$

Using the integrating factor we obtain the following

$$\bar{d}(a)=e^{-(\sigma+\psi+\kappa)a}\int_0^a e^{(\sigma+\psi(+\kappa)\alpha}\lambda_0 d\alpha. \tag{3.5.4}$$

We can rewrite equation (3.5.4) as follows

$$\bar{d}(a)=\lambda_0 e^{-(\sigma+\psi+\kappa)a}\int_0^a e^{(\sigma+\psi+\kappa)\alpha}d\alpha. \tag{3.5.5}$$

Substituting equation (3.5.5) into equation (3.5.1) we get

$$\lambda_0=\lambda_0\int_0^\infty\int_0^a e^{(\sigma+\psi+\kappa)(\alpha-a)}B_\infty(a)d\alpha da. \tag{3.5.6}$$

Here we change our variables as follows

$$a-\alpha=\zeta,$$

and we change the order of integration as follows

$$\lambda_0=\lambda_0\int_0^\infty\int_0^\infty e^{(\sigma+\psi+\kappa)(-\zeta)}B_\infty(\zeta+\alpha)d\zeta d\alpha. \tag{3.5.7}$$

Dividing both sides by λ_0 we obtain the following equation

$$1=\int_0^\infty\int_0^\infty e^{-(\sigma+\psi+\kappa)\zeta}B_\infty(\zeta+\alpha)d\zeta d\alpha \tag{3.5.8}$$

Equation (3.5.8) is a form of the characteristic equation of Euler and Lotka from which we can obtain the intrinsic growth rate if we substitute $\kappa=0$ into equation (3.5.8). The basic reproduction number is the unique solution of our characteristic equation obtained by computing the dominant real root of equation (3.5.8).

3.6 Numerical Solution

We hereby consider a numerical simulation for the substance abuse model. For this purpose we make use of the finite difference method. We define a suitable rectangular grid in the (a,t) plane where $a \in [0,A]$ and $t \in [0,T]$. Each point in this plane is characterised by

$$\begin{aligned} a_i &= a_0 + i\Delta a \qquad i = 0,1,...,N, \\ t_j &= t_0 + j\Delta t \qquad j = 0,1,...,M, \\ X_i^j &= X(a_i,t_j). \end{aligned}$$

We consider taking the same age and time steps as follows

$$\Delta a = \Delta t = h.$$

Thus we have $\frac{A}{N} = \frac{T}{M} = h$. An approximate solution to each partial differential equation is obtained by considering a finite difference scheme that is backward in both time and age as follows

$$\begin{aligned} \frac{\partial X(a,t)}{\partial a} + \frac{\partial X(a,t)}{\partial t} &\approx \frac{X(a_i,t_j) - X(a_{i-1},t_j) + X(a_{i-1},t_j) - X(a_{i-1},t_{j-1})}{h} \\ &= \frac{X(a_i,t_j) - X(a_{i-1},t_{j-1})}{h} \\ &= \frac{X_i^j - X_{i-1}^{j-1}}{h}. \end{aligned} \tag{3.6.1}$$

After using equation (3.6.1) our discretized system is given as

$$\begin{aligned} \frac{S_i^j - S_{i-1}^{j-1}}{h} &= -(\mu_i + \lambda_i^j)S_i^j, \\ \frac{D_i^j - D_{i-1}^{j-1}}{h} &= \lambda_i^j S_i^j + \gamma_i R_i^j + \omega_i Q_i^j - (\mu_i + \sigma_i + \psi_i)D_i^j, \\ \frac{R_i^j - R_{i-1}^{j-1}}{h} &= \sigma_i D_i^j - (\mu_i + \rho_i + \gamma_i)R_i^j, \\ \frac{Q_i^j - Q_{i-1}^{j-1}}{h} &= \rho_i R_i^j - (\mu_i - \omega_i)Q_i^j. \end{aligned} \tag{3.6.2}$$

Solving equations (3.6.2) we get

$$\begin{aligned} S_i^j &= \frac{S_{i-1}^{j-1}}{1+h(\mu_i+\lambda_i^j)}, \\ D_j^j &= \frac{D_i^j + h(\lambda_i^j S_i^j + \gamma_i R_i^j + \omega_i Q_i^j)}{1+h(\mu_i+\sigma_i+\psi_i)} \\ R_i^j &= \frac{R_{i-1}^{j-1} + h\sigma_i D_i^j}{1+h(\mu_i+\rho_i+\gamma_i)}, \\ Q_i^j &= \frac{Q_{i-1}^{j-1} + h\rho_i R_i^j}{1+h(\mu_i+\omega_i)}. \end{aligned} \tag{3.6.3}$$

The finite difference scheme of equation (3.6.3) yielded a lower triangular incidence matrix. Since we are operating in a rectangular domain we need a finite difference scheme that will result in a rectangular incidence matrix. We therefore modify the above finite difference scheme to have

$$\begin{aligned} \frac{\partial X(a,t)}{\partial a} + \frac{\partial X(a,t)}{\partial t} &\approx \frac{X(a_i,t_j) - X(a_{i-1},t_j) + X(a_i,t_j) - X(a_i,t_{j-1})}{h}, \\ &= \frac{2X(a_i,t_j) - X(a_{i-1},t_{j-1}) + X(a_i,t_{j-1})}{h}, \\ &= \frac{2X_i^j - X_{i-1}^j + X_i^{j-1}}{h}. \end{aligned} \tag{3.6.4}$$

Using the above scheme our system of equations become :

$$\begin{aligned} \frac{2S_i^j - S_{i-1}^j + S_i^{j-1}}{h} &= -(\mu_i+\lambda_i^j)S_i^j, \\ 2S_i^j + h(\mu_i+\lambda_i^j)S_i^j &= S_{i-1}^j + S_i^{j-1}, \\ S_i^j &= \frac{S_{i-1}^j + S_i^{j-1}}{2+h(\mu_i+\lambda_i^j)}. \end{aligned} \tag{3.6.5}$$

Thus the system of equations that we will implement in our numerical simu-

lations is given by

$$\begin{aligned} S_i^j &= \frac{S_{i-1}^j + S_i^{j-1}}{2 + h(\mu_i + \lambda_i^j)}, \\ D_j^j &= \frac{D_{i-1}^j + D_i^{j-1} + h(\lambda_i^j S_i^j + \gamma_i R_i^j + \omega_i Q_i^j)}{2 + h(\mu_i + \sigma_i + \psi_i)}, \\ R_i^j &= \frac{R_{i-1}^j + R_i^{j-1} + h\sigma_i D_i^j}{2 + h(\mu_i + \rho_i + \gamma_i)}, \\ Q_i^j &= \frac{Q_{i-1}^j + Q_i^{j-1} + h\rho_i R_i^j}{2 + h(\mu_i + \omega_i)}. \end{aligned} \tag{3.6.6}$$

Boundary conditions are set as

$$S_0^j = \theta_j \text{ and } D_0^j = R_0^j = Q_0^j = 0$$

and initial conditions are also set as

$$S_i^0 = S_0(a_i),\ D_i^0 = D_0(a_i),\ R_i^0 = R_0(a_i),\ Q_i^0 = Q_0(a_i).$$

If we substitute the correct expressions for S_i^j, R_i^j and Q_i^j in the equation for D_i^j we see that we need to approximate D_j^j by D_i^{j-1} in order to directly solve for D_i^j and subsequently for R_i^j and Q_i^j.

We approximate the error by setting

$$\begin{aligned} S(a_i, t_j) - S_i^j &= \varrho_i^j, \\ D(a_i, t_j) - D_i^j &= \xi_i^j, \\ R(a_i, t_j) - R_i^j &= \zeta_i^j, \\ Q(a_i, t_j) - Q_i^j &= \alpha_i^j. \end{aligned}$$

Approximating the derivative at (a_i, t_j) using equation (3.6.4) and subtracting corresponding terms from from equation (3.6.6) we obtain the following error estimates

$$\begin{aligned} \frac{2\varrho_i^j - \varrho_{i-1}^j - \varrho_i^{j-1}}{h} &= -(\mu_i + \lambda_i^j)\varrho_i^j + O(h), \\ \frac{2\xi_i^j - \xi_{i-1}^j - \xi_i^{j-1}}{h} &= \lambda_i^j \varrho_i^j + \gamma_i \zeta_i^j + \omega_i \alpha_i^j - (\mu_i + \sigma_i + \psi_i)\xi_i^j + O(h), \\ \frac{2\zeta_i^j - \zeta_{i-1}^j - \zeta_i^{j-1}}{h} &= \sigma_i \xi_i^j - (\mu_i + \rho_i + \gamma_i)\zeta_i^j + O(h), \\ \frac{2\alpha_i^j - \alpha_{i-1}^j - \alpha_i^{j-1}}{h} &= \rho_i \zeta_i^j - (\mu_i + \omega_i)\alpha_i^j + O(h). \end{aligned} \tag{3.6.7}$$

The initial and boundary conditions of the above system are all equal to zero. Thus we have the following boundary conditions

$$\varrho_0^j = 0, \qquad \xi_0^j = 0, \qquad \zeta_0^j = 0, \qquad \alpha_0^j = 0$$

and initial conditions

$$\varrho_i^0 = 0, \qquad \xi_j^0 = 0, \qquad \zeta_j^0 = 0, \qquad \alpha_j^0 = 0.$$

Solving for $\varrho_i^j, \xi_i^j, \zeta_i^j$ and α_i^j we get

$$\begin{aligned}
\varrho_i^j &= \frac{\varrho_{i-1}^j + \varrho_i^{j-1}}{2 + h(\mu_i + \lambda_i^j)} + O(h^2), \\
\xi_j^j &= \frac{\xi_{i-1}^j + \xi_i^{j-1} + h(\lambda_i^j \varrho_i^j + \gamma_i \zeta_i^j + \omega_i \alpha_i^j)}{2 + h(\mu_i + \sigma_i + \psi_i)} + O(h^2), \\
\zeta_i^j &= \frac{\zeta_{i-1}^j + \zeta_i^{j-1} + h\sigma_i \xi_i^j}{2 + h(\mu_i + \rho_i + \gamma_i)} + O(h^2), \\
\alpha_i^j &= \frac{\alpha_{i-1}^j + \alpha_i^{j-1} + h\rho_i \zeta_i^j}{2 + h(\mu_i + \omega_i)} + O(h^2).
\end{aligned} \tag{3.6.8}$$

By using a suitable norm on the errors in a way similar to [41] the convergence of our numerical scheme can be established.

3.7 Fitting model to Cape Town data

The data that we are using was obtained from various reports from the South African Community Epidemiology Network on Drug Use (SACENDU). Data on substance abuse is routinely collected from six sites in South Africa. Every month data on illicit drug use is collected from treatment centres in the Western Cape, KwaZulu Natal, Eastern Cape, Gauteng, Limpopo and from the Central Region site. These sites make up the South African Community Epidemiology Network on Drug Use.

SACENDU has made available information with regard to substance on their website. A comprehensive report with detailed data on substance abuse titled, "Monitoring alcohol and drug abuse treatment admissions in South Africa" is published every six months. Even though data is collected monthly the data in the report is the total figure for six months . For the purpose of our studies we extracted data from a number of reports for which the data spanned the period July 2005 up till December 2014. According to the June 2015 report the data was collected from 33 specialist centres in Cape Town.

Table 3.2: Data on the people in rehabilitation in some centres of Western Cape for 13 age groups from 2005 to 2014

Year	AGE GROUPS											
	10-14	15-19	20-24	25-29	30-34	35-39	40-44	45-49	50-54	55-59	60-64	65+
2005b	98	576	505	257	200	159	137	89	61	27	10	8
2006a	77	647	579	340	284	239	174	134	80	51	27	19
2006b	108	652	683	342	279	241	182	143	75	47	25	18
2007a	112	688	661	357	253	230	226	149	85	56	25	13
2007b	107	704	708	437	276	254	201	157	84	76	25	21
2008a	71	553	582	435	248	214	172	150	97	55	36	16
2008b	85	569	691	463	278	193	187	149	91	53	23	14
2009a	128	764	848	633	358	324	237	157	103	62	25	14
2009b	61	545	607	488	283	215	179	120	60	42	21	9
2010a	98	595	676	627	339	258	216	114	92	41	28	11
2010b	119	691	676	576	334	245	193	144	92	41	28	11
2011a	116	496	604	616	322	274	192	140	83	45	18	11
2011b	76	353	694	605	334	205	166	137	79	44	16	8
2012a	170	696	886	845	466	276	196	151	100	56	25	13
2012b	123	531	629	674	433	255	166	155	90	53	28	12
2013a	125	617	751	825	489	308	223	146	104	62	24	19
2013b	187	701	574	755	459	264	195	140	85	62	19	21
2014a	167	635	561	796	504	255	219	159	96	65	18	11
2014b	185	597	561	725	501	302	201	154	92	53	28	19

The data in Table 3.2 is categorised according to age and thus it is suitable to validate our model. The first entry in the Table 3.2 is 98 which gives the total number of individuals age $10-14$ who were admitted in the treatment centres around the city of Cape Town for the period July 2005 - December 2005. In order to be able to fit this data to our model we need the data to be for specific ages and not for age intervals. Thus we modify the data in Table 3.2 to obtain data for these specific ages. This is essential for us to carry out the numerical

Table 3.3: Modified data on the people in rehabilitation in some centres of Western Cape for 13 age groups from 2005 to 2014

Year	AGE GROUPS											
	12	**17**	**22**	**27**	**32**	**37**	**42**	**47**	**52**	**57**	**62**	**67**
2005b	19.6	115.2	101	51.4	40	31.8	27.4	17.8	12.2	5.4	2	1.6
2006a	15.4	129.4	115.8	68	56.8	47.8	34.8	26.8	16	10.2	5.4	3.8
2006b	21.6	130.4	13.6	68.4	55.8	48.2	36.4	28.6	15	9.45	3.6	3.6
2007a	22.4	137.6	132.2	71.4	50.6	46	45.2	45.2	17	11.2	5	2.6
2007b	21.4	140.8	141.6	87.4	55.2	50.8	40.2	31.4	16.8	15.2	5	4.2
2008a	14.2	110.6	116.4	87	49.6	42.8	34.4	30	19.4	11	7.2	3.2
2008b	17	113.8	138.2	92.6	55.6	38.6	37.4	29.8	18.2	10.6	4.6	2.8
2009a	25.6	152.8	169.6	126.6	71.6	64.8	47.4	31.4	20.6	12.4	5	2.8
2009b	12.2	109	121.4	97.6	56.6	43	35.8	24	12	8.4	4.2	1.9
2010a	19.6	119	135.2	125.4	67.8	51.6	43.2	22.8	18.4	8.2	5.6	2.2
2010b	23.8	138.2	135.2	115.2	66.8	49	38.6	28.8	18.4	8.2	5.6	2.2
2011a	23.2	99.2	120.8	123.2	64.4	54.8	38.4	28	16.6	9	3.6	2.2
2011b	15.2	70.6	138.8	121	66.8	41	33.2	27.4	15.8	8.8	3.2	1.6
2012a	34	139.2	177.2	169	93.2	55.2	39.2	30.2	20	11.2	5	2.6
2012b	25.6	106.2	125.8	134.8	86.6	51	33.2	31	18	10.6	5.6	2.4
2013a	25	123.4	150.2	165	97.8	61.6	44.6	29.2	20.8	12.4	4.8	3.8
2013b	37.4	140.2	114.8	151	91.8	52.8	39	28	17	12.4	3.8	4.2
2014a	33.4	127	112.2	159.2	100.8	51	43.8	31.8	19.2	13	3.6	2.2
2014b	37	119.4	112.2	145	100.2	60.4	40.2	30.8	18.4	10.6	5.6	3.8

simulations . To obtain the entries of Table 3.3 the median age of each age group is considered as well as the average number per age from each given age interval. For instance the median age for the $10-14$ age interval is age 12 and the number of individuals aged 12 is $\frac{98}{5} = 19.6$. The remaining age intervals and number of individuals are calculated in a similar manner.

3.8 Boundary Conditions

The boundary conditions for our model is formulated in the same manner as in [48]. The starting population is calculated by taking estimated population for 2005 for the $0-4$ age group and divide it by 5 to get the population for the zero age group. The growth rate for the population in Cape Town for the period $2001-2007$ is estimated at 20.91%. We thus obtain an annual growth rate of 3.49% [52]. Thus we have the boundary condition is given as

$$S(0,t) = 90180\exp[0.03485(t-2005.5)].$$

The initial modelling time is 2005.5 the time from which we have data available for the people in rehabilitation. The number of people in the rehabilitation compartment per age group are the first column of Table 3.3.

The initial condition for the drug using compartment (D) was estimated using the number of people apprehended for committing crimes related to drugs and alcohol in the Western Cape province for the year 2005. The actual number of individuals in the drug abusing compartment are definitely more than those who are prosecuted for possession or using substances. The main challenge as been pointed out earlier remains that of knowing the actual substance abuse going on in the communities. The figure for crimes related to drugs and alcohol is reported as 30432 [53]. To apportion this figure to different age groups we use the proportions of people in rehabilitation for the same period. The total number of people per age group were obtained from the estimated figures given by the SA Statistics [54]. We assumed that the rate of quitting was 20% and thus obtained the initial condition for compartment Q to be 20% of those in R. Finally, to get the initial condition for the sucseptibles we subtracted the sum of $D(a,0), R(a,0)$ and $Q(a,0)$ from $N(a,0)$ where $N(a,0)$ is the total number of people of age a during the year 2005. The data for the initial conditions of the four disease compartments are shown in Table 3.4.

Table 3.4: Table of initial values for each age and disease class.

Condition	AGE GROUPS											
	12	**17**	**22**	**27**	**32**	**37**	**42**	**47**	**52**	**57**	**62**	**67**
$S(a,0)$	81774	74860	82733	83481	79270	61986	55327	48645	38552	32347	28814	21383
$D(a,0)$	1402	8241	7225	3677	2861	2274	1960	1273	872	386	143	114
$R(a,0)$	19.6	115.2	101	51.4	40	31.8	27.4	17.8	12.2	5.4	2	1.6
$Q(a,0)$	3.92	23.04	20.2	10.28	8	6.36	5.48	3.56	2.44	1.08	0.4	0.32

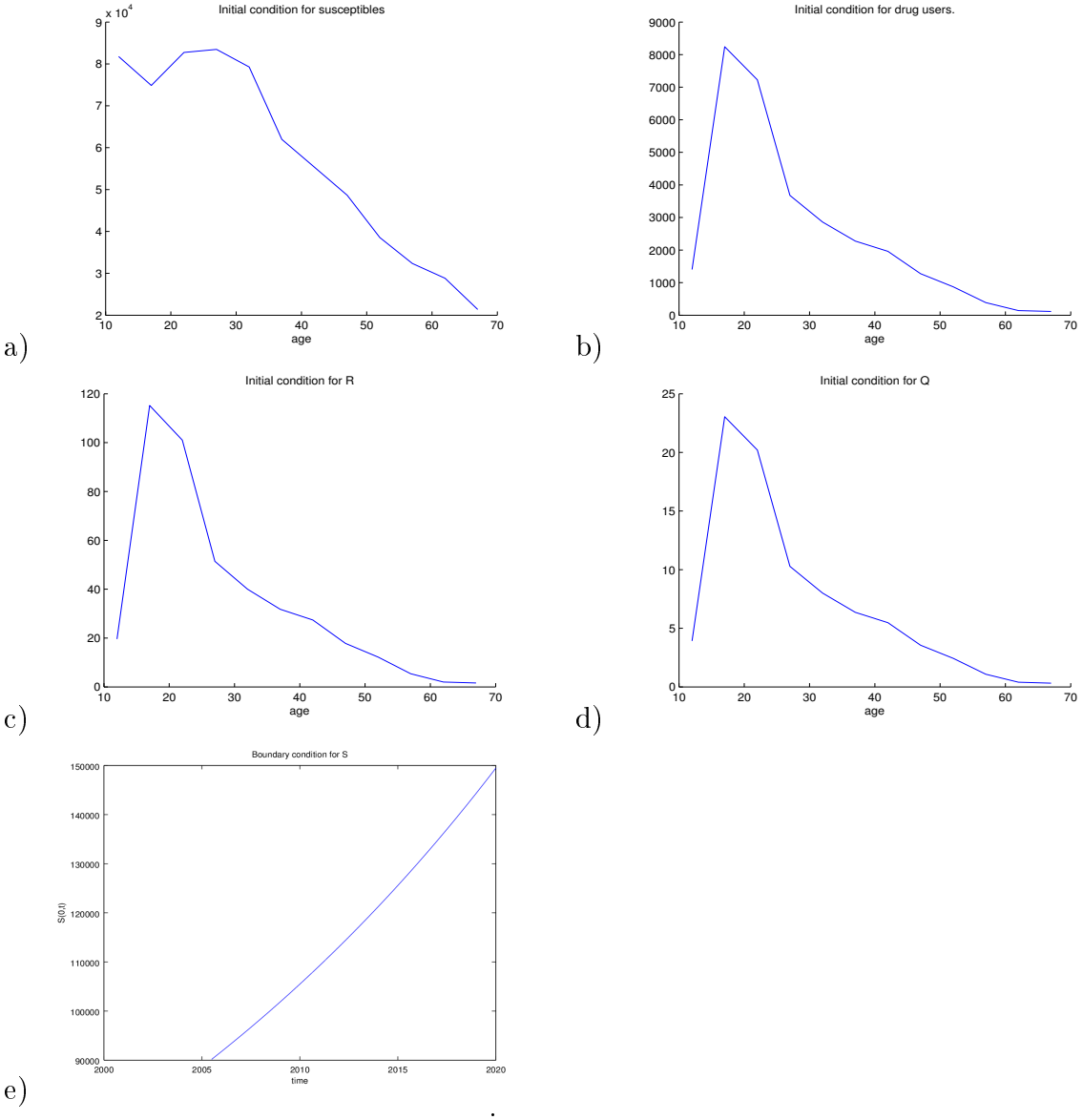


Figure 3.2: Figures a) to d) are plots of the initial conditions for each disease compartment and e) is the plot of the boundary condition $S(0,t)$.

The parameters in Table 3.5 were estimated from the data as fitted to the model.

Table 3.5: Parameter values for different ages. All the parameter values are estimated from fitting the model to the data.

Age	Parameters					
	μ	γ	σ	ω	ρ	ψ
0	0.123	0	0	0	0	0
12	$3.57e^{-09}$	0.0486	0.0208	0.1058	0.01895	0.222
17	$1.14e^{-08}$	0.5	0.021658	0.4344	0.42995	0.1075
22	$3.81e^{-08}$	0.4571	0.022308	0.022308	0.2202	0.0714
27	$6.06e^{-08}$	0.2276	0.0066092	0.3613	0.2196	0.0532
32	$1.48e^{-07}$	0.32927	0.00819	0.2182	0.495	0.0229
37	$3.02e^{-07}$	0.5	0.011505	0.1562	0.5	0.0147
42	$3.93e^{-05}$	0.1021	0.01118	0.1055	0.4822	0.0089
47	0.0011	0.0798	0.01404	0.1254	0.4949	0.0085
52	0.0037	0.2905	0.00884	0.1119	0.31575	0.0069
57	0.00487	0.7962	0.008866	0.1178	0.4838	0.0042
62	0.0091	0.5	0.007722	0.0898	0.5	0.00070021
67	0.0615	0.495	0.004862	0.1209	0.49255	$2.31e^{-14}$

3.9 Sensitivity Analysis

Mathematical models are usually characterised by uncertainties emanating mainly from lack of information. The main uncertainty is associated with parameters. Due to the unavailability of information, assumptions have to be made both in the model formulation and the choice of parameters. Thus it is imperative that we gain an understanding into the extend to which our parameters actually influence our outcomes of interest.

In order to accomplish this task we employ a method called Latin Hypercube Sampling and Partial Rank Correlation Coefficient. We undertake a sensitivity analysis for us to identify the parameters for which their uncertainty will mostly contribute to imprecision in the predictions from the model [55].

The sensitivity analysis is performed on the individuals in the quitters compartment for all age groups. Here we present the results obtained for the ages $12, 37$ and 62 and the results are a good representative of the results obtained for other age classes. As shown in Figure 3.3 parameters having the same influence on the quitters of all ages are $\omega, \rho, \gamma, \sigma$. We also note that μ the natural mortality parameter has great influence on the quitters in the older ages starting from age 52 onwards and a very small influence on the younger age groups. The other parameter is ψ the drug induced death rate which has more influence on the younger ages from $12 - 47$ and small influence on the older ages. Quitters are depicted to be largely positively influenced by σ the rate of uptake into the rehabilitation compartment and ρ the rate at which people in rehab quit abusing drugs. The parameters having a negative influ-

ence on the number of people in the quitters compartment are ω the rate of relapse from the quitters back to the drug users compartment,γ the rate of relapsing while in rehab. The scatter plots presented in Figure 3.4 are for the age 37 individuals in the Q compartment plotted against the most influential parameters on the dynamics of the Quitters.

3.10 Simulation Results

The finite difference scheme from equation (3.6.6) are implemented in Matlab to obtain results for each of the four compartments making up our model. All the parameters of the model are obtained by fitting the results of our simulation to the data in Table 3.3. The main purpose for our numerical solution is so that we can gain a qualitative understanding of how substance abuse will evolve. As as a result, effective interventions can be incorporated into models so as to impact and influence policy in the fight against drug abuse. The only available data we have is that from treatment centres. In our model we have a compartment R consisting of people in treatment. We want the simulated results for the R compartment to be as close as possible to the actual observed values of the people in the treatment centres as given in Table 3.3.

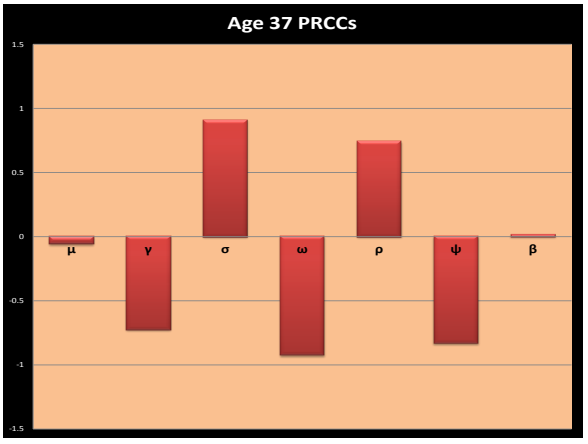
Thus we firstly present the results from the simulation of R plotted together with the observed values from treatment centres. The mesh plot of the simulation results of R alongside the actual observed age structured numbers of people in rehabilitation centres is shown in Figure 3.5. Most of observed data points lie above the simulated results. Very few data points lie below the simulated data namely the earlier years for age 12 individuals and latter years for ages $42, 47$ and 52 individuals. This does not really tell us much in terms of how well our simulation output compares to the observed data. In order to answer this we look at the output from Figure 3.5 from various viewpoints.

Figure 3.6 gives the different view points of Figure 3.5 and we can see that the results of our simulation coincides with the data points of Table 3.3. Few data points can be seen lying above and below our simulation results. Thus we can say the simulations are a good fit and we can make predictions for the R compartment as well as other compartments of our model.

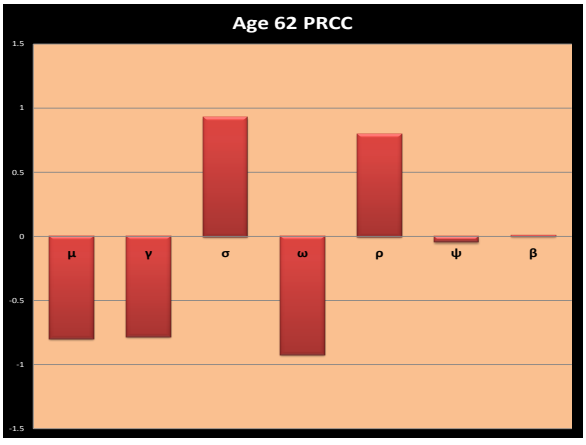
The dynamics of the people in compartment R are depicted by Figure 3.7. We can see that the highest number of people in treatment centres for the whole simulation time was experienced in the period $2007 - 2012$ within the $17 - 22$ age groups. Around 2010 individuals around the age 17 seem to be the majority in the treatment centres but not for very long. With time there is a clear shift away from the younger ages to the older ages. At first we see those in the 20s and the shift continues until the dominant age group is around age 30. We also note that the age range for the highest number in rehabilitation centres seems to be getting narrower with time. The trend continues into the future where the majority of patients in the rehabilitation centres is predicted to be individuals in the $37 - 42$ age groups as we approach the year 2020. In order to clearly see how the numbers of people are predicted to evolve we present different views of Figure 3.7. From Figure 3.8 we can see how the numbers

of people in compartment R will evolve with age. Figure 3.9 on the other helps us see how the total numbers will evolve with time. The overall outlook from Figures 3.7 and 3.9 show that the total numbers of people visiting the treatment centres are on a downward trend with time. Figure 3.8 shows two age groups for which there are peaks characterised by higher numbers around the 20s and 30s age groups.

a)



b)



c)

Figure 3.3: Figures a) to c) are are bar graphs for the Partial Correlation Coefficients of each of our model parameters for ages $12, 37$ and 62 respectively.

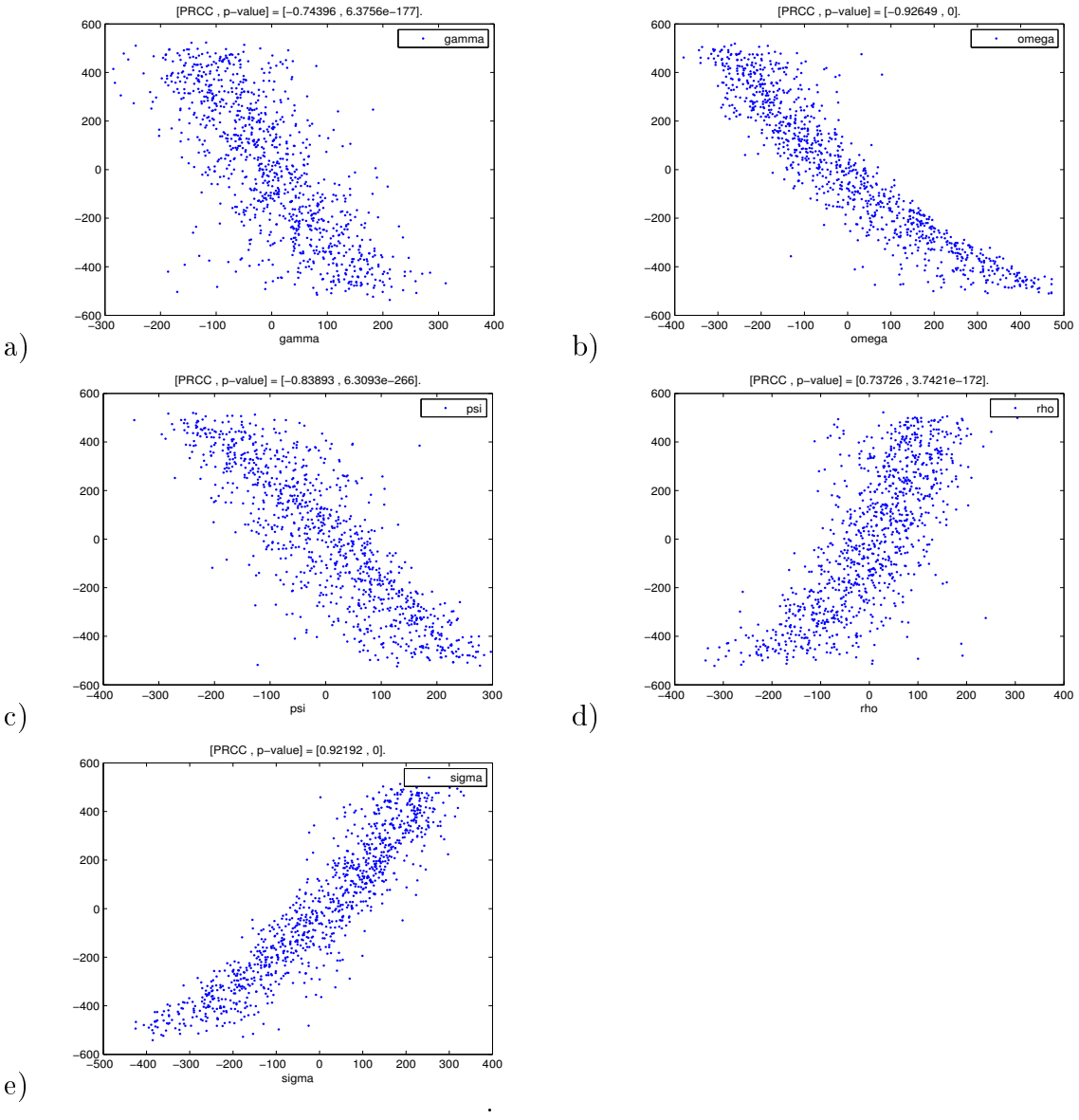


Figure 3.4: PRCC scatter plots at time 600 years of the most significant parameters to the dynamics of individuals aged 37 in compartment Q. The paramaters are (a) γ, (b) ω, (c) ψ, (d) ρ and (e) σ

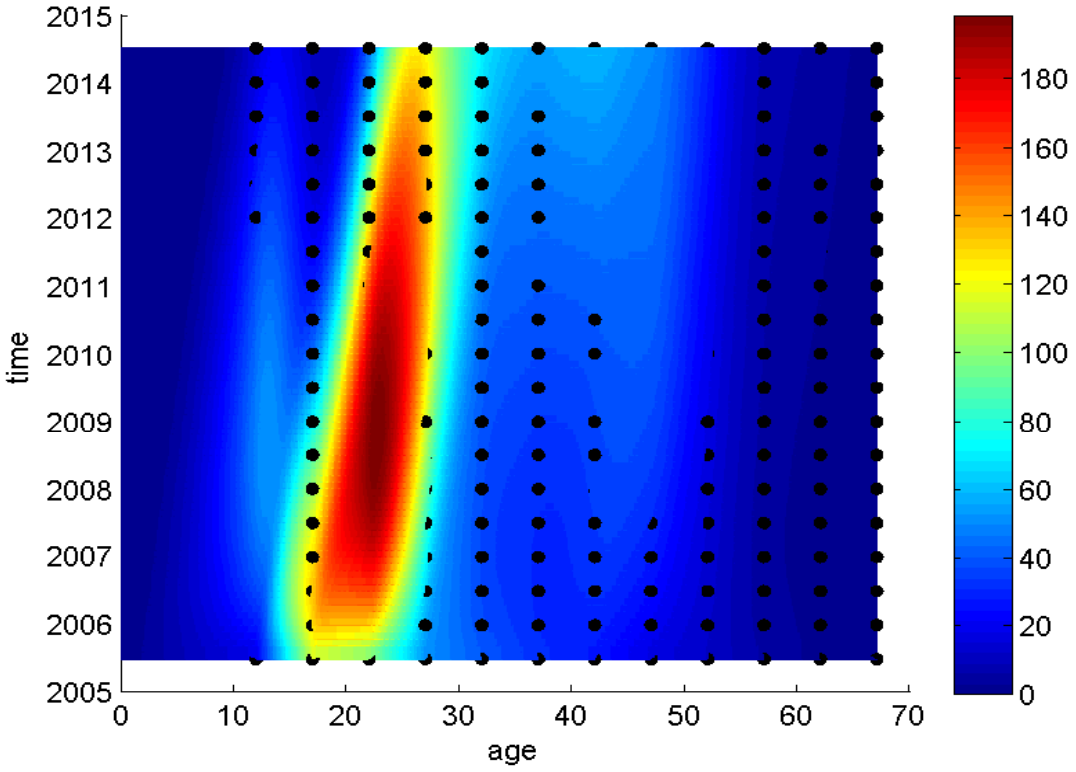


Figure 3.5: A mesh colour plot for the simulated results and observed data for the R compartment.

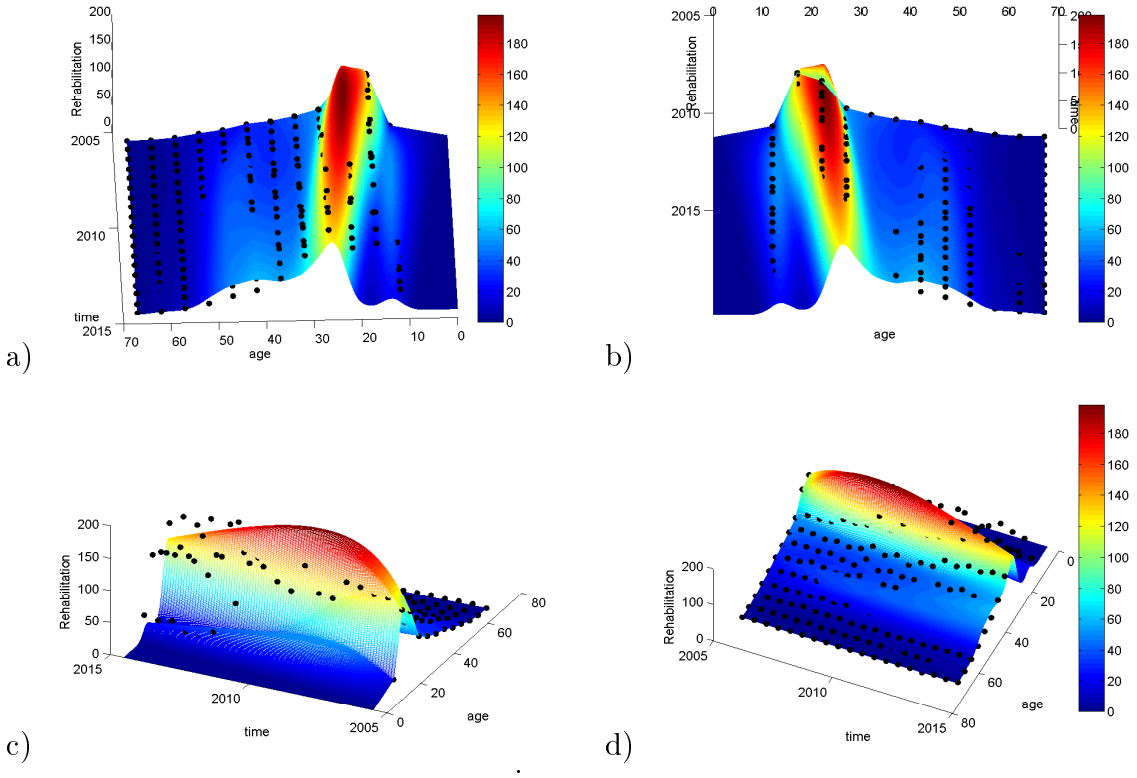


Figure 3.6: Different viewpoints of Figure 3.5 (a) top view, (b) bottom view, (c) side view showing mainly the younger age groups, and (d) side view showing mostly the older age groups

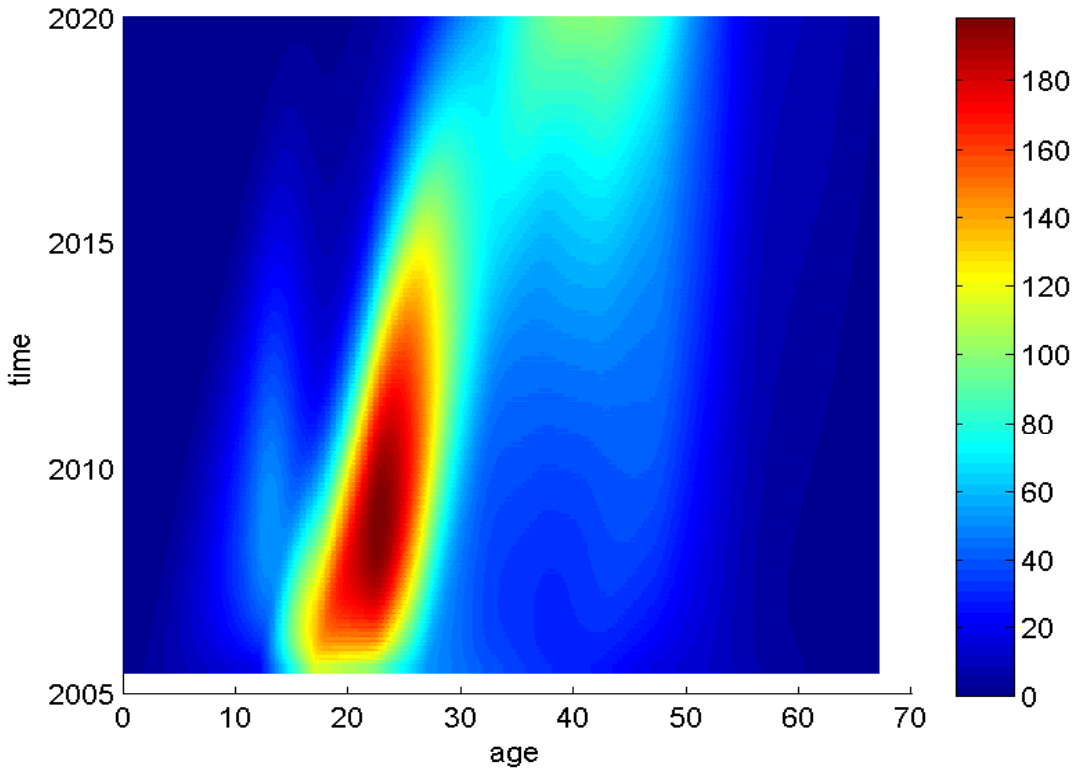


Figure 3.7: A colour mesh showing the predicted dynamics of the people in rehabilitation according to age and time.

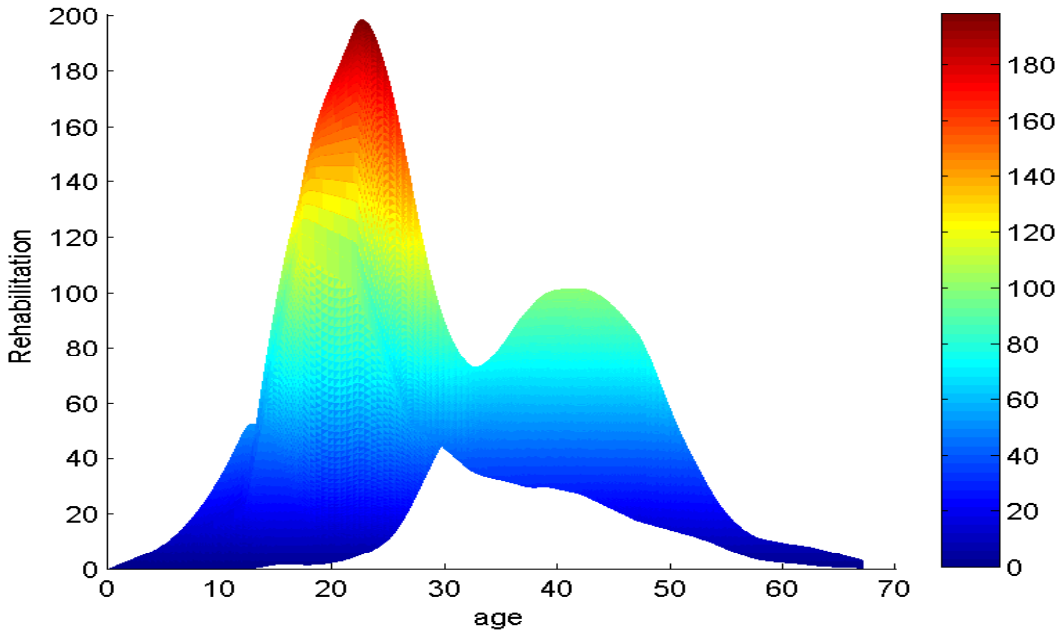


Figure 3.8: A colour mesh showing the predicted dynamics of the people in rehabilitation according to age.

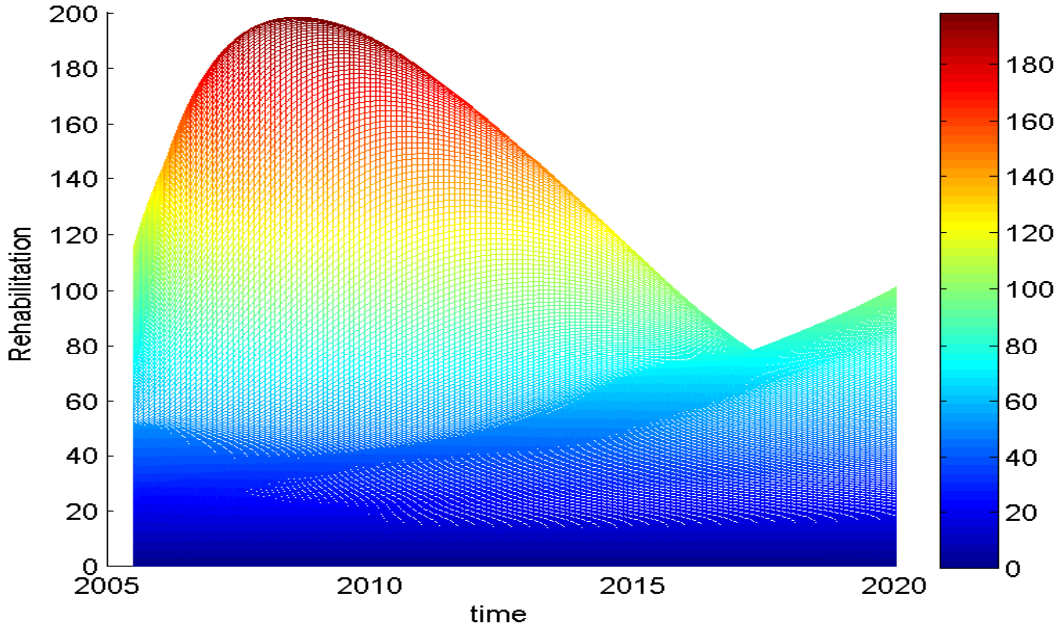


Figure 3.9: A colour mesh showing the predicted dynamics of the people in rehabilitation according to time.

In order to come up with intervention measures that are effective we need to understand how incidence of substance abuse evolves with time. This is a measure that describes the occurrence of new drug users. The formula for calculating the incidence for our model is given as

$$\text{Incidence} = \lambda(a,t)S(a,t).$$

We thus present firstly a mesh colour plot that describes how incidence of substance abuse will evolve with time and age in Figure 3.10. Figures 3.11 and 3.12 show us how incidence evolves with age and time respectively.

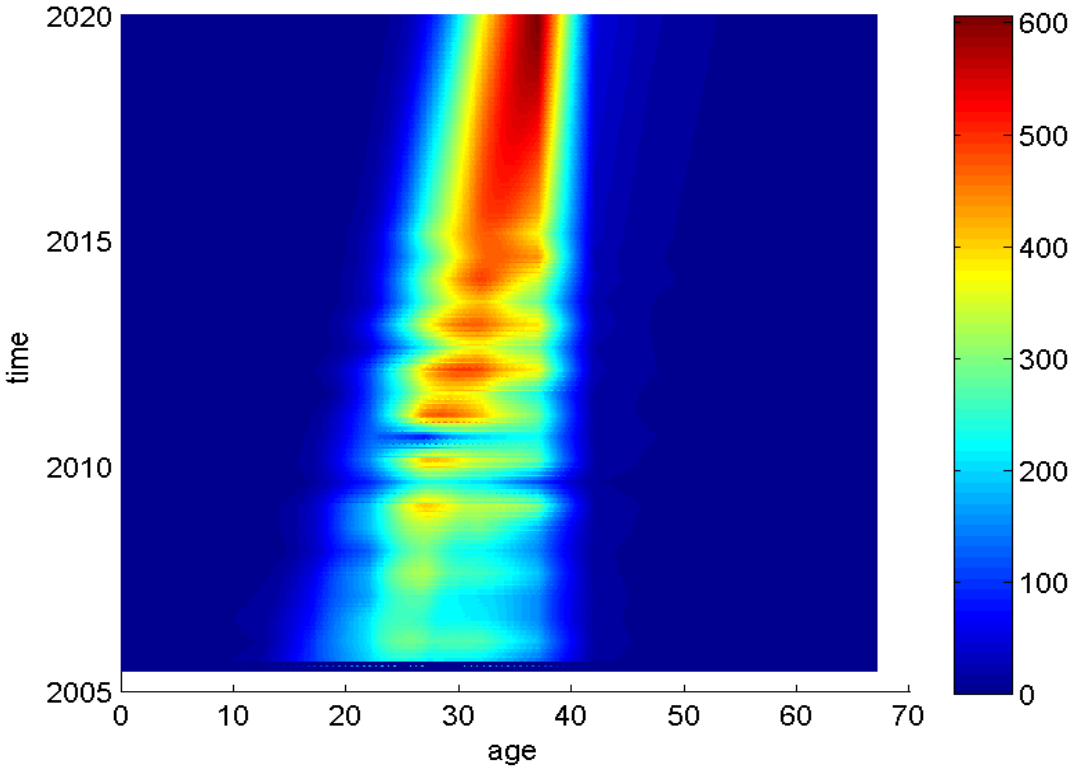


Figure 3.10: A colour mesh showing the predicted dynamics of the incidence of substance abuse according to age and time.

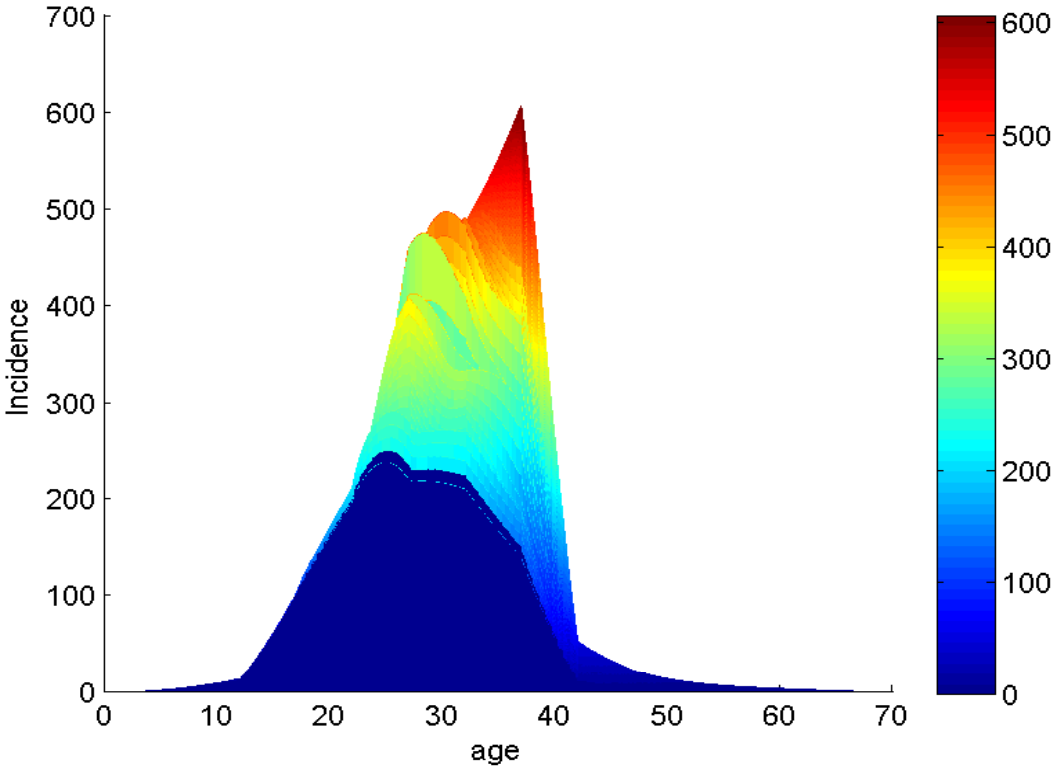


Figure 3.11: Dynamics of the substance abuse incidence with age.

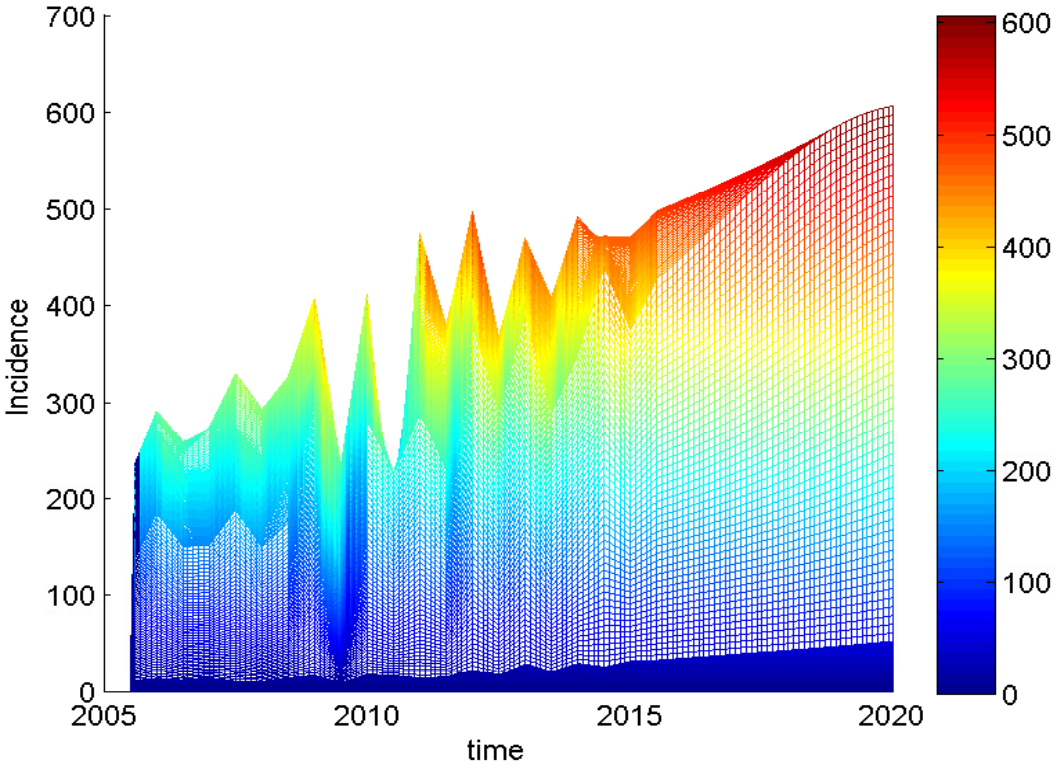


Figure 3.12: Dynamics of the substance abuse incidence with time.

A darker shade of blue for the younger ages between $0-10$ and older ages $50+$ in Figure 3.10 and Figure 3.11 represents a very low incidence of substance abuse. Figure 3.10 shows the dark blue area becoming wider with time extending beyond age 10 while for older age groups this is narrowing with time. Thus we see that over time low incidence of substance abuse spans the age group $0-20$. Initially slightly high incidence of substance abuse are seen for the age range $10-50$ yet this age range becomes thinner as other age groups are lost to the very low incidence category. Even higher incidence are depicted but for a very narrow age range. We also see increase in incidence of substance abuse within the ages $20-37$. Figure 3.10 and Figure 3.12 show that the there is an upward trend in the total incidence of substance abuse with time. There is narrowing of the age range experiencing higher incidence of substance abuse with time and generally shifting to the right towards the older age groups. Most of the new drug users are going to be coming from a slightly older population aged around 40 where we observe the peak in incidence of substance abuse.

We also present the mesh plot for the dynamics of the individuals in the drug using compartment. Figure 3.13 shows how the number of people within this compartment will evolve with time and age. We see two peaks with the highest numbers in the D compartment occurring in 2005 comprising of individuals in the early 20 age group and between the period $2018 - 2020$ but comprising of people in the late 30 age category. The D compartment also seems to be showing a widening in the age range with time thus showing that a wider population demographic is predicted to be dominating in the D compartment compared the narrower one in the R compartment. Like the dynamics of the R compartment higher numbers in the D compartment are predicted to lean towards the older age groups with time.

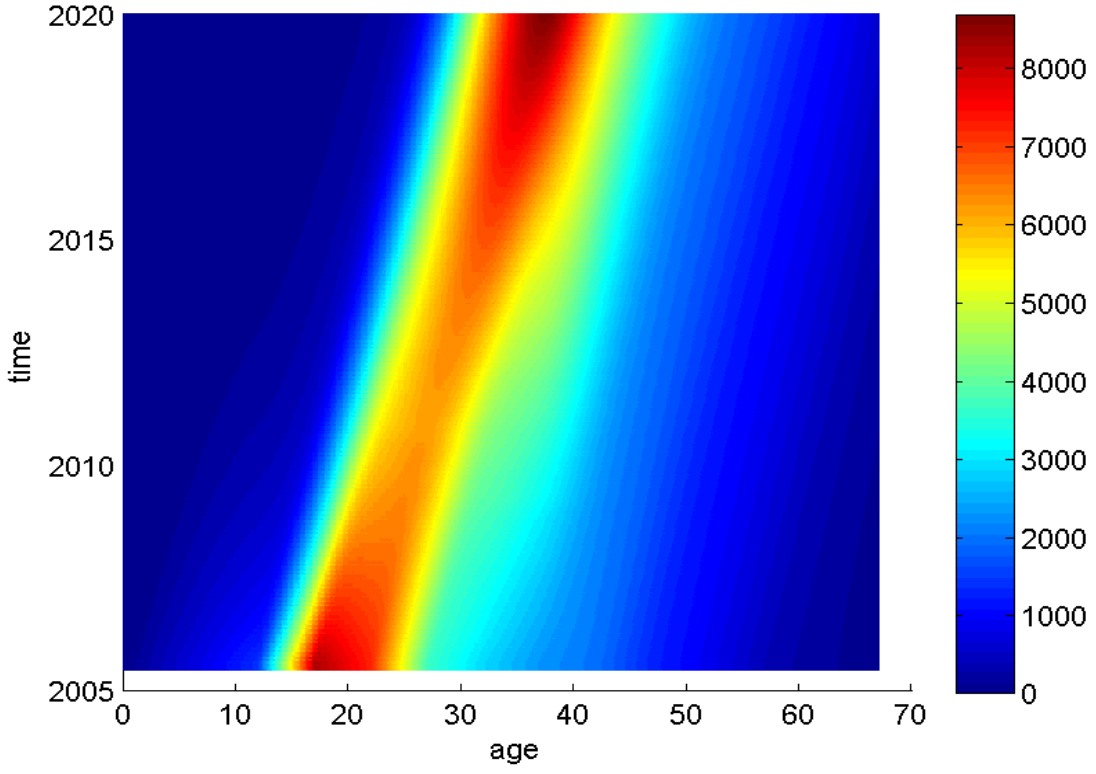


Figure 3.13: Mesh plot depicting the dynamics of the individuals in the compartment D.

The implications of the results obtained is that the people who are involved in substance abuse currently will not move out completely. Typically there is a clear cycle with individuals who are involved in abusing drugs relapsing often and as they grow they continue to be involved in abusing substances. The effect of the density comes into play where we observe higher incidences in the

older age groups. The older adults who have been in the cycle of relapsing for too long can recruit new initiates among their age mates thus we end up with the scenarios described in our simulation results. The result supports the findings by [36] who established that the there is going to be a large demand for treatment among older adults in the US. According to the studies by [36] it is anticipated that as the baby boom aged, a shift would be seen in the numbers of individuals with substance abuse problems increasing around age 50.

3.11 Summary

In this chapter we first introduced the finite difference scheme for the numerical solution of our age structured model of substance abuse introduced in the previous Chapter. The first scheme proposed is a backward scheme in both time and age. From this scheme we obtained an upper triangular incidence matrix not very useful to us as we require our solution to be on the rectangular domain. We modified the scheme and carried out simulations from which we obtained a numerical solution presented in the form of three dimensional mesh colour plots. We also carried out sensitivity analysis on the parameters using the Latin Hypercube Sampling and Partial Rank Correlation Coefficient Coefficient method.

Chapter 4

Discussion

Understanding substance abuse dynamics is often challenging due to the problems associated with obtaining accurate and up to date information with regard to this phenomenon. Usually data that is available is of the people that have been admitted into a rehabilitation facility. This information does not accurately tell us what is really happening in the communities. Estimates are often made by making use of data obtained from secondary sources such as statistics of crimes committed under the influence of illegal substances as well as records by health centres of people admitted after experiencing a health problem associated with substance abuse.

We made use of the data from rehabilitation centres around Cape Town. The data we particularly used was giving the number of individuals in rehabilitation according to the ages and time. We used this data to estimate the parameters for our age structured model of substance abuse. A numerical simulation of the model was undertaken to understand the dynamics of each of the compartments of substance abuse. We were able to simulate the incidence of substance and predict how this incidence will evolve with time.

Results from our simulation clearly show that substance abuse does evolve with both age and time. It is thus imperative that these dynamics be further explored to facilitate the finding of appropriate and effective intervention measures to the substance abusing problem.

The main finding of this study is the apparent shift in substance abuse incidence from younger ages to older ages. This can potentially be a result of the change in major factors driving substance abuse. Usually substance abuse in younger people is mainly driven by peer pressure while for adults it is usually other factors such as the general socio-economic atmosphere within a community. On the other hand the shift could also be a result of the past intervention measures having been targeting more of the younger ages in the past. We could alternatively conclude that the shift is indicative of the different mechanisms which are now influencing the use of substance abuse in Cape Town. Issues

like increased access to illegal drugs as well as the advent of new drugs are all possible reasons for this shift.

Usually initiation to substance abuse occurs at younger age groups. In the case where the rate of relapse is high usually the people who initiate at younger ages continue to be involved in substance abuse into adulthood as they go through cycles of treatment, quitting and relapsing. The impact of the density of substance abuse means that in future we might see more older people being initiated into substance abuse if these older age groups become more dominant. This means that treatment centres need to be better prepared to cater for the changing demographics of people requiring treatment services. Treatment centres also need to ensure the chances of relapsing after treatment are remarkably reduced in order to eradicate substance abuse.

Substance abuse incidence is predicted to increase in the future while the number of people in rehabilitation will decrease. This anomaly can be potentially accounted for if a further study is undertaken to determine the dynamics of particular substances of abuse. For instance most people do not consider alcohol abuse as an addictive problem for which they should seek treatment. A scenario where there is more individuals being initiated into heavy drinking will most definitely increase the incidence of substance abuse and not have the same effect on the number of people being admitted into the rehabilitation centres.

4.1 Limitations

Our model offers valuable insight into how substance abuse will evolve with time and age in the Western Cape province. However we depended so much on the data from the rehabilitation centres to obtain the parameters of our model. Obtaining information on how much substance abuse is going on in the community would greatly improve the accuracy of our model and then we could obtain not only qualitative but quantitative insight on the dynamics of substance abuse. Possible future work would be to extend this work to mapping of substance abuse in the Western Cape province.

List of References

[1] Winkler, D.: *Modelling the effect of HIV on age-specific incidence of active TB disease: a comparison between Taiwan and Cape Town Metropole*. Master's thesis, Mathematics, University of Stellenbosch, Stellenbosch, South Africa, 2012.

[2] UNODC: World drug report 2015. Tech. Rep., United Nations Publication, 2015.

[3] High risk drug use:an overview of the methods and definitions used [Online]. Available at:http://www.emcdda.europa.eu/data/2014/methods-hrdu,[2016 April 14], 2014.

[4] Segell, G.: *Stability and Intervention in Sub-Sahara Africa*. G. Segell, 1998. ISBN 9781901414134. Accessed on [2016, April 04].
Available at: https://books.google.co.za/books?id=02VRPIg62rsC

[5] Parry, C.D.: *Substance abuse in South Africa: country report focussing on young persons*. Medical Research Council, 1998.

[6] Harker, N., Kader, R., Myers, B., Fakier, N., Parry, C., Flisher, A., Peltzer, K., Ramlagan, S. and Davids, A.: Substance abuse trends in the western cape: A review of studies conducted since 2000. *Parow: Medical Research Council*, 2008.

[7] Nyabadza, F. and Hove-Musekwa, S.D.: Substance abuse in the western cape province of south africa: Insights through mathematical modelling. *The quarterly update on epidemiology from the South African Centre for Epidemiological Modelling and Analysis*, 2010.

[8] Substance abuse programme concept paper 10 year plan [Online]. Available at:https://www.westerncape.gov.za/other/2009/10/concept_paper_-_programme_substance_abuse.pdf,[2016 April 04], 2008.

[9] Workstream on the prevention and treatment of harmful alcohol and drug use [Online]. Available at:https://docplayer.net/1422614_Workstream_on_the_prevention_and_treatment_of_harmful_alcohol_and_drug_use.html,[2016 April 04], 2010.

[10] Rossi, C.: The role of dynamic modelling in drug abuse epidemiology. *Bulletin on Narcotics*, pp. 33–44, 2003.

[11] Harker, N., Kader, R., Myers, B., Fakier, N., MRC, C.P., Peltzer, K., Ramlagan, S. and Davids, A.: Substance abuse trends in the western cape. *Unpublished Report, Medical Research Council, Cape Town*, pp. 1–54, 2001.

[12] UNODC: World drug report 2014. Tech. Rep., United Nations Publication, 2014.

[13] Håkansson, A. and Berglund, M.: Risk factors for criminal recidivism–a prospective follow-up study in prisoners with substance abuse. *BMC psychiatry*, vol. 12, no. 1, p. 1, 2012.

[14] Kalula, A.S. and Nyabadza, F.: A theoretical model for substance abuse in the presence of treatment. *South African Journal of Science*, vol. 108, no. 3-4, pp. 96–107, 2012.

[15] London, L.: Thedop'system, alcohol abuse and social control amongst farm workers in south africa: a public health challenge. *Social science & medicine*, vol. 48, no. 10, pp. 1407–1414, 1999.

[16] The Science of Drug Abuse and Addiction [Online]. Available at:https://www.drugabuse.gov/publications/research-reports/methamphetamine/what-methamphetamine, [2016, April 04], 2016.

[17] Alcohol Use Disorder [Online]. Available at:http://www.niaaa.nih.gov/alcohol-health/overview-alcohol-consumption/alcohol-use-disorders/,[2016 May 17], 2016.

[18] Alcohol Abuse [Online]. Available at:http://drugabuse.com/library/alcohol-abuse/,[2016 May 17], 2016.

[19] Alcohol's effects on the body [Online]: Available at:http://www.niaaa.nih.gov/alcohol_health/alcohols_effects_body/,[2016 May 17], 2016.

[20] Hall, W. and Solowij, N.: Adverse effects of cannabis. *The Lancet*, vol. 352, no. 9140, pp. 1611–1616, 1998.

[21] Ashton, C.H.: Adverse effects of cannabis and cannabinoids. *British Journal of anaesthesia*, vol. 83, no. 4, pp. 637–649, 1999.

[22] Marijuana [Online]. Available at:https://www.drugabuse.gov/publications/drugfacts/marijuana,[2016 May 04], 2016.

[23] The Adverse Effects of Marijuana [Online]. Available at:http://www.csam-asam.org/adverse_effects_marijuana_healthcare_professionals,[2016 May 17], 2016.

[24] Fifth Session Of The African Union Conference Of Ministers Of Drug Control(camd5)[Online]. Available at:http://sa.au.int/en/content/5th_session_au_conference_ministers_drug_control_camdc5_addis_ababa_ethiopia/,[2016 May 17], 2012.

[25] Chowell, G., Nishiura, H. and Bettencourt, L.M.: Comparative estimation of the reproduction number for pandemic influenza from daily case notification data. vol. 4, no. 12, pp. 155–166, 2007. ISSN 1742-5689. Accessed On:[2016 May 17], {http://rsif.royalsocietypublishing.org/content/4/12/155.full.pdf},journal={JournalofTheRoyalSocietyInterface}. Available at: http://rsif.royalsocietypublishing.org/content/4/12/155

[26] Brauer, F., Castillo-Chavez, C. and Castillo-Chavez, C.: *Mathematical models in population biology and epidemiology*, vol. 40. Springer, 2001.

[27] Aron, J.L.: Mathematical modeling: The dynamics of infection. *Infectious Disease Epidemiology: Theory and Practice, eds. Nelson K., Williams C. Masters*, pp. 181–212, 2007.

[28] Keeling, M.J. and Danon, L.: Mathematical modelling of infectious diseases. *British Medical Bulletin*, vol. 92, no. 1, pp. 33–42, 2009.

[29] Almeder, C., Caulkins, J.P., Feichtinger, G. and Tragler, G.: An age-structured single-state drug initiation modelâcycles of drug epidemics and optimal prevention programs. *Socio-Economic Planning Sciences*, vol. 38, no. 1, pp. 91–109, 2004.

[30] Kalula, A.S. and Nyabadza, F.: A theoretical model for substance abuse in the presence of treatment. *South African Journal of Science*, vol. 108, no. 3-4, pp. 96–107, 2012.

[31] White, E. and Comistitle, C.: Heroin epidemics, treatment and ode modelling. *Mathematical biosciences*, vol. 208, no. 1, pp. 312–324, 2007.

[32] Sánchez, F., Wang, X., Castillo-Chávez, C., Gorman, D.M. and Gruenewald, P.J.: Drinking as an epidemicâa simple mathematical model with recovery and relapse. *Therapistâs guide to evidence-based relapse prevention*, p. 353, 2007.

[33] Burattini, M., Massad, E., Coutinho, F., Azevedo-Neto, R., Menezes, R. and Lopes, L.: A mathematical model of the impact of crack-cocaine use on the prevalence of hiv/aids among drug users. *Mathematical and computer modelling*, vol. 28, no. 3, pp. 21–29, 1998.

[34] Mulone, G. and Straughan, B.: A note on heroin epidemics. *Mathematical biosciences*, vol. 218, no. 2, pp. 138–141, 2009.

[35] Nyabadza, F., Njagarah, J.B. and Smith, R.J.: Modelling the dynamics of crystal meth (âtikâ) abuse in the presence of drug-supply chains in south africa. *Bulletin of mathematical biology*, vol. 75, no. 1, pp. 24–48, 2013.

[36] Gfroerer, J., Penne, M., Pemberton, M. and Folsom, R.: Substance abuse treatment need among older adults in 2020: the impact of the aging baby-boom cohort. *Drug and alcohol dependence*, vol. 69, no. 2, pp. 127–135, 2003.

[37] Auger, P., Magal, P. and Ruan, S.: *Structured population models in biology and epidemiology*, vol. 1936. Springer, 2008.

[38] Liu, H.-L., Yu, J.-Y. and Zhu, G.-T.: Stability results for an age-structured sis epidemic model with vector population. *Journal of Applied Mathematics*, vol. 2015, 2015.

[39] Alexanderian, A., Gobbert, M.K., Fister, K.R., Gaff, H., Lenhart, S. and Schaefer, E.: An age-structured model for the spread of epidemic cholera: Analysis and simulation. *Nonlinear Analysis: Real World Applications*, vol. 12, no. 6, pp. 3483–3498, 2011.

[40] Li, J. and Brauer, F.: Continuous-time age-structured models in population dynamics and epidemiology. In: *Mathematical Epidemiology*, pp. 205–227. Springer, 2008.

[41] Shim, E., Feng, Z., Martcheva, M. and Castillo-Chavez, C.: An age-structured epidemic model of rotavirus with vaccination. *Journal of mathematical biology*, vol. 53, no. 4, pp. 719–746, 2006.

[42] Castillo-Chavez, C. and Huang, W.: Age-structured core group model and its impact on std dynamics. In: *Mathematical Approaches for Emerging and Reemerging Infectious Diseases: Models, Methods, and Theory*, pp. 261–273. Springer, 2002.

[43] Agusto, F.B., Easley, S., Freeman, K. and Thomas, M.: Mathematical model of three age-structured transmission dynamics of chikungunya virus. *Computational and mathematical methods in medicine*, vol. 2016, 2016.

[44] El-Doma, M.: Analysis of an age-dependent sis epidemic model with vertical transmission and proportionate mixing assumption. *Mathematical and computer modelling*, vol. 29, no. 7, pp. 31–43, 1999.

[45] Inaba, H.: Mathematical analysis of an age-structured sir epidemic model with vertical transmission. *Discrete and Continuous Dynamical Systems Series B*, vol. 6, no. 1, p. 69, 2006.

[46] Inaba, H.: Threshold and stability results for an age-structured epidemic model. *Journal of mathematical biology*, vol. 28, no. 4, pp. 411–434, 1990.

[47] Van den Driessche, P. and Watmough, J.: Further notes on the basic reproduction number. In: *Mathematical Epidemiology*, pp. 159–178. Springer, 2008.

[48] Nyabadza, F. and Winkler, D.: A simulation age-specific tuberculosis model for the cape town metropole. *South African Journal of Science*, vol. 109, no. 9-10, pp. 01–07, 2013.

[49] Safi, M.A., Gumel, A.B. and Elbasha, E.H.: Qualitative analysis of an age-structured seir epidemic model with treatment. *Applied Mathematics and Computation*, vol. 219, no. 22, pp. 10627–10642, 2013.

[50] Djidjou, R.D., Tewa, J.J. and Bowong, S.: Analysis of an age-structured sil model. *Arima*, vol. 17, pp. 23–52, 2014.

[51] Li, X.-Z., Gupur, G. and Zhu, G.-T.: Threshold and stability results for an age-structured seir epidemic model. *Computers & Mathematics with Applications*, vol. 42, no. 6, pp. 883–907, 2001.

[52] City of cape town discussion paper[Online]. Available at:https://studylib.net/doc/10456184/city_of_cape_town_discussion_paper_demographics_scenario,[2016,May 04], 2010.

[53] Western cape crime statistics[Online]. Available at:http://www.crimestatssa.com/province.php?ShowProvince=Western+Cape,[2016,October 20], 2005.

[54] statistics [Online]: Available at:http://www.statssa.gov.za/publications/SAStatistics/Statistics2004.pdf,[2016,October 20], 2004.

[55] Gomero, B.: Latin hypercube sampling and partial rank correlation coefficient analysis applied to an optimal control problem. Accessed on: [2016, April 04], 2012.
Available at: http://trace.tennessee.edu/utk_gradthes/1278