

Generalising Some Geometrical Theorems and Objects¹

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INTRODUCTION

Generalisation is an important mathematical process, one that lies at the very heart of mathematical thinking. Not only does generalisation expand knowledge, but it often contributes to increasing our understanding by revealing relationships between different mathematical concepts and topics. With this in mind it is unfortunate that most formal school systems around the world tend to focus largely on ‘routine’ problems and the ‘drill and practice’ of mathematical techniques. Little opportunity is typically left to stretch and challenge more able learners. Given the pivotal role that generalisation plays in mathematics, it seems educationally necessary to design suitable activities for engaging talented mathematical learners (as well as prospective Mathematics teachers) in the process of generalisation that goes beyond the normal curriculum, and can possibly stimulate them to explore generalisations of their own. This article presents two possible examples of appropriate generalisation activities. The first is the generalisation of a familiar theorem for cyclic quadrilaterals to cyclic polygons, while the second is the generalisation of the concept of a rectangle to a higher order polygons.

INVESTIGATION 1

Let us consider the familiar high school result that the opposite angles of a (convex) cyclic quadrilateral are supplementary. An obvious question to ask is what happens when we have a cyclic hexagon? Are the opposite angles also supplementary? Or is there another relationship? Quickly making a construction in dynamic geometry, or through deductive reasoning, students should be able to discover the conjecture for themselves, shown in Figure 1, that the two sums of the alternate angles of a (convex) cyclic hexagon are both equal to 360° , i.e. $\angle A + \angle C + \angle E = \angle B + \angle D + \angle F = 360^\circ$.

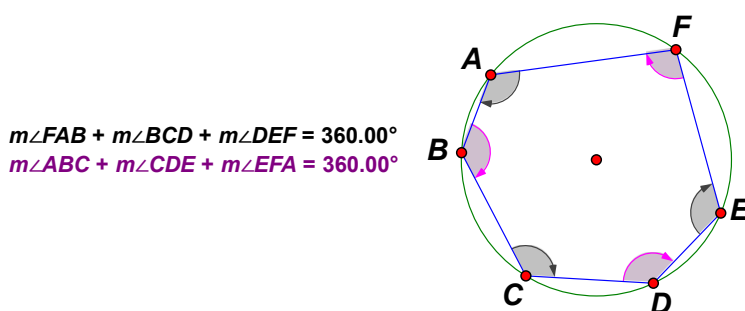


FIGURE 1: Cyclic hexagon

This leads to an alternative wording of the result for a cyclic quadrilateral, namely ‘the alternate angles of a (convex) cyclic quadrilateral are supplementary’.

¹ This article is an adaptation of a paper presented by the author at the 13th International Congress on Mathematical Education in Hamburg, July 2016.

The next question to ask is what happens in the case of a cyclic octagon? From this point students should then try to extend the idea to make the further generalisation that for any (convex) cyclic $2n$ -gon, the sum of alternate angles equals $(n - 1)180^\circ$.

But what about the converse for the above generalisation to any convex cyclic $2n$ -gon? If the sum of alternate angles of a $2n$ -gon is equal to $(n - 1)180^\circ$, does it follow that it is necessarily cyclic? It certainly does in the case of quadrilaterals, but what about hexagons? Starting with three angles adding up to 360° by arranging them around a point as shown in Figure 2, it is not difficult to construct a dynamic hexagon with three alternate angles correspondingly equal to these angles. Now consider the hexagon ABCDEF shown in Figure 2. We have $\angle B + \angle D + \angle F = 360^\circ$, i.e. the sum of the alternate angles equals 360° . However, by constructing the perpendicular bisectors for all six sides it can be seen that they are not concurrent. From this it follows that there is no equidistant point in relation to all six vertices, and it is thus not possible to draw a circle through all six vertices. The hexagon, in other words, is not cyclic!

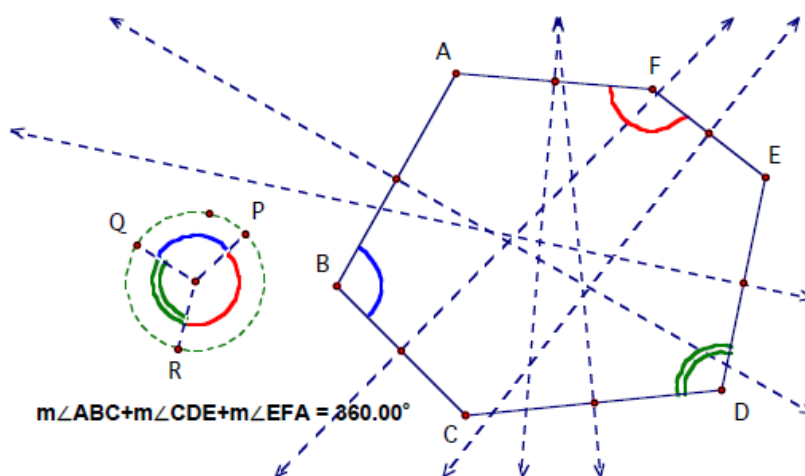


FIGURE 2: Refuting the converse case for a hexagon

Werner Olivier, the top student in my 2005 geometry class for prospective high school teachers, came up with the following elegant alternative argument. Instead of constructing a dynamic hexagon as described in Figure 2, consider any cyclic hexagon as shown in Figure 3. Respectively extend AF and CD to F' and D', and then draw lines through these points respectively parallel to FE and DE to intersect in E'. Clearly, all the angles of ABCD'E'F' are the same as that of ABCDEF, but it is obviously not cyclic, and therefore a counter-example.

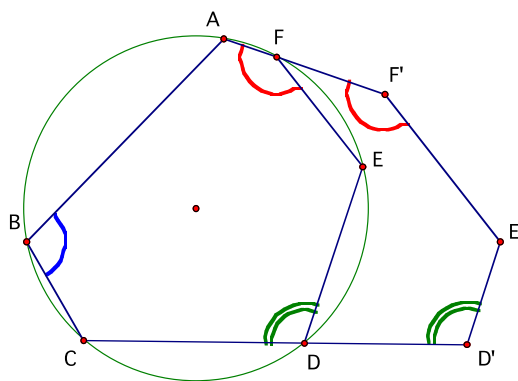


FIGURE 3: An elegant refutation of the converse case for a hexagon

One of the problems with the traditional Euclidean approach to geometry is that there are very few cases where the converses of theorems are false, and students inevitably assume, or develop the misconception, that geometric converses are always true. It is therefore a valuable strategy to go beyond dealing only with the special cases of triangles and quadrilaterals as in Euclid's Elements, and to examine analogous cases for polygons where appropriate. This often provides ample opportunities for showing the difference between a statement and its converse, and often highlights the "specialness" of triangles and quadrilaterals. Moreover, genuine mathematical research involves both proving and disproving, and both these need to be reflected in our teaching, especially for mathematically talented learners. It is not sufficient only to focus on developing students' skills in proving true statements. We also need to provide instructive opportunities for developing their ability to find counter-examples.

INVESTIGATION 2

Consider the geometrical figure of a square. How could we generalise the concept of 'square' to other polygons? What would be equivalent to the concept of 'square' in the case of a pentagon, hexagon, heptagon etc.? A natural place to start would perhaps be with the (usual) definition of a square, namely: "a square is a quadrilateral with equal sides and equal angles." Generalizing this definition leads to the familiar concept of a 'regular polygon', i.e. a polygon with all sides equal and all angles equal. One could then explore the common properties of regular polygons, finding for example that they all have equal (main) diagonals, are both cyclic and circumscribed, and have line and rotational symmetry.

Generalising the concept of 'square' to other polygons was reasonably straight forward, but what about generalising the concept of 'rectangle' to other polygons? What would be equivalent to the concept of 'rectangle' in the case of a pentagon, hexagon or heptagon? This perhaps seems like a strange question to ask, but go with it! Let us once again start with the (usual) definition of a rectangle, namely: "a rectangle is a quadrilateral with equal angles." From this definition one might be tempted to extend the idea of 'rectangle' to that of equi-angled polygons, i.e. polygons with all angles equal². But is this really a good generalisation of the concept of 'rectangle'? Let us consider the equi-angled hexagon shown in Figure 4, where all angles are equal to 120° . While the hexagon still has opposite sides parallel (and one can ask students to prove that!), it clearly doesn't have equal diagonals, nor is it cyclic. It has no sides equal, no lines of symmetry, and no rotational symmetry. A great number of properties found in a rectangle have thus been lost, so perhaps this isn't a good generalisation of the concept of 'rectangle' after all.

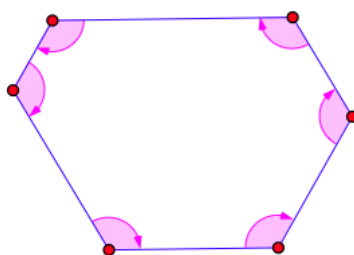


Figure 4: Equi-angled hexagon

² Equi-angular polygons have many interesting properties that could also be explored by mathematically able learners. Several of these properties are discussed and presented in Ball (2002).

Perhaps we could add another property of a rectangle to the definition so that the generalisation to other polygons retains more of the rectangle's properties? But which specific property should we choose? After a bit of reflection one might perhaps consider refining the definition to include the condition that apart from being equi-angled, it also has to be cyclic. A good idea at this point might be to construct a hexagon using dynamic geometry software, as illustrated in Figure 5.

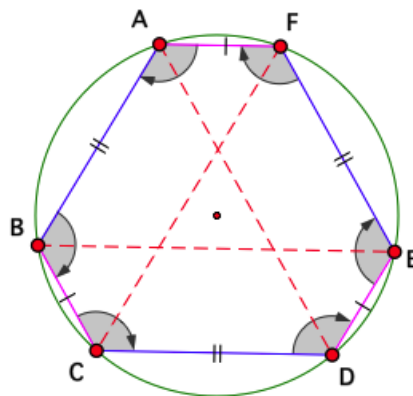


Figure 5: Cyclic equi-angled hexagon

Like a rectangle, the constructed cyclic, equi-angled hexagon has the following corresponding properties, namely: alternate sides equal; main diagonals equal; three lines of symmetry; and rotational symmetry of order 3. This now looks like a much better generalisation of the concept of ‘rectangle’. It also provides a rich context for posing several modestly challenging conjectures to students to prove, such as: “If a cyclic hexagon has all angles equal, then the two sets of alternate sides are equal”. This result can easily be proved in various ways, and is a good example of a simple task that can lead to multiple solutions. There is great educational value in this because each proof involves the application of different concepts. For example, compare the following two different proofs.

First proof

With reference to Figure 6, it is easy to immediately see that $\triangle ABF$ is congruent to $\triangle FEA$ (\angle, \angle, s), and hence that (alternate sides) $AB = FE$. Applying the same argument at other adjacent vertices, it follows that the two sets of alternate sides are equal, i.e. $AB = CD = EF$ and $BC = DE = FA$.

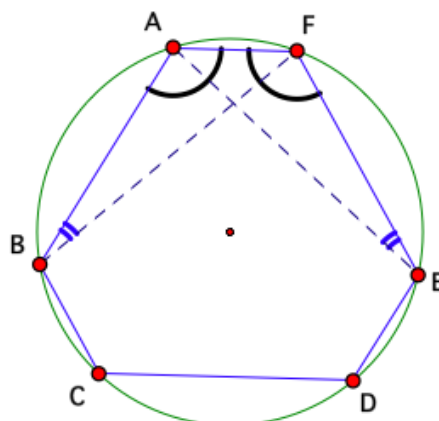


Figure 6: First proof

Second proof

With reference to Figure 7, construct an arbitrary point G on arc CD. Now, since $\angle A = \angle F$ it follows that $BF = EA$ (equal chords subtend equal angles). Using the same theorem, it follows that $\angle BGF = \angle EGA$, and subtracting the common $\angle AGF$ from both sides of this equality gives $\angle BGA = \angle EGF$. It thus follows that the chords subtended by these two angles are equal, i.e. $AB = FE$. The same argument applies to the other adjacent vertices, from which it follows that the two sets of alternate sides are equal.

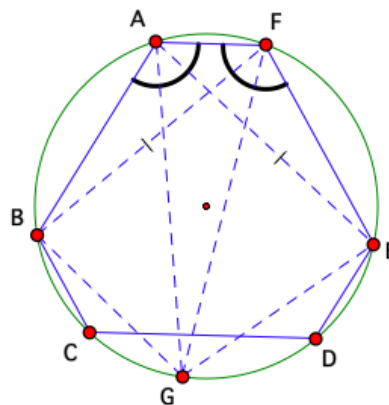


Figure 7: Second proof

The above result for a cyclic hexagon with equal angles naturally leads to the general theorem as discussed in De Villiers (2011a) that “If a cyclic $2n$ -gon has all angles equal, then the two sets of alternate sides are equal”, and where cyclic $2n$ -gons with all angles equal have been called semi-regular angle-gons. One could also fruitfully challenge students to consider the converse result, namely: if the two sets of alternate sides of a cyclic hexagon are equal, does it imply that all its angles are equal? See for example Samson (2015) which describes an examination task similar to this where Grade 12 students were able to provide multiple proofs for this converse result³.

CONCLUDING REMARKS

Following up on the preceding activity, students could next be asked to similarly explore the generalisation of the concept of a rhombus to higher polygons, or that of an isosceles trapezium, kite (e.g. see De Villiers, 2011b), or parallelogram (e.g. see De Villiers, 2009). However, there are many other suitable concepts and theorems in secondary geometry, for example Varignon’s theorem (see De Villiers, 2007), that could equally be used as stimulating starting points to engage students in the exciting process of further generalisation beyond the narrow confines of the prescribed curriculum. As argued by Polya (1954, p. 30), while quoting Schopenhauer, we understand a mathematical relationship or concept more broadly and more purely when we recognize it as the same in widely different cases and between completely heterogeneous objects.

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³ Interactive dynamic geometry Java sketches for cyclic $2n$ -gons with all angles equal, as well as circumscribed $2n$ -gons with all sides equal, are available at: <http://dynamicmathematicslearning.com/semi-regular-anglegon.html>