# Spectral Theory in Commutatively Ordered Banach Algebras

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# Declaration

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## Abstract

Let $A$ be a complex Banach algebra with unit 1. A subset $C$ of $A$ is called a *cone* if $C+C \subseteq C$ and $\lambda C \subseteq C$ for all scalars $\lambda \geq 0$. If $C$ satisfies the additional properties that $C.C \subseteq C$ and $1 \in C$, then it is called an *algebra cone*. The elements of $C$ are called *positive*. A Banach algebra $A$ with an algebra cone $C$ can be ordered by $C$ in the following way: If $a, b \in A$, then $a \leq b$ if and only if $b - a \in C$. A Banach algebra ordered by an algebra cone is called an *ordered Banach algebra* (OBA).

A non-commutative $C^*$-algebra cannot be ordered by an algebra cone, since in a $C^*$-algebra, the product of positive elements is positive only if the elements commute. Every non-commutative $C^*$-algebra however, then does have the property that the product of commuting positive elements is positive. In this work we define a subset $C$ of a Banach algebra as an *algebra c-cone* if $C$ satisfies the following properties: $C+C \subseteq C$, $\lambda C \subseteq C$ for all scalars $\lambda \geq 0$, $1 \in C$, $ab \in C$ whenever $a, b \in C$ and $ab = ba$. Every algebra cone is an algebra $c$-cone. Every Banach algebra (including a non-commutative $C^*$-algebra) with an algebra $c$-cone $C$ can be ordered by $C$ in the usual way and a Banach algebra ordered by an algebra $c$-cone is called a commutatively ordered Banach algebra (COBA). Since every algebra cone is an algebra $c$-cone, every OBA is a COBA.

Spectral theory in OBAs has been studied for around twenty years. In this work, we generalize many of the results in OBAs to the more general setting of COBAs, and then obtain new results in COBAs and OBAs.

# Opsomming

Laat $A$ 'n komplekse Banach-algebra met eenheidselement 1 wees. 'n Deelversameling $C$ van $A$ word 'n *keël* genoem indien $C+C \subseteq C$ en $\lambda C \subseteq C$ vir alle skalare $\lambda \geq 0$. As $C$ die addisionele eienskappe het dat $C.C \subseteq C$ en $1 \in C$, dan word dit 'n *algebra-keël* genoem. Die elemente van $C$ word *positief* genoem. 'n Banach-algebra $A$ met 'n algebra-keël $C$ kan soos volg m.b.v. $C$ georden word: As $a, b \in A$, dan is $a \leq b$ as en slegs as $b - a \in C$. 'n Banach-algebra wat deur 'n algebra-keël georden is, word 'n *geordende Banach-algebra* (GBA) genoem.

'n Nie-kommutatiewe $C^*$-algebra kan nie m.b.v. 'n algebra-keël georden word nie, omdat die produk van positiewe elemente in 'n $C^*$-algebra slegs positief is indien die elemente kommuteer. Elke nie-kommutatiewe $C^*$-algebra het egter dan wel die eienskap dat die produk van kommuterende positiewe elemente positief is. In hierdie werk definieer ons 'n deelversameling $C$ van 'n Banach-algebra as 'n *algebra-c-keël* indien $C$ die volgende eienskappe bevredig: $C + C \subseteq C$, $\lambda C \subseteq C$ vir alle skalare $\lambda \geq 0$, $1 \in C$, $ab \in C$ wanneer $a, b \in C$ en $ab = ba$. Elke algebra-keël is 'n algebra-$c$-keël. Elke Banach-algebra (insluitende 'n nie-kommutatiewe $C^*$-algebra) wat 'n algebra-$c$-keël $C$ het, kan op die gewone manier m.b.v. $C$ georden word en 'n Banach-algebra wat deur 'n algebra-$c$-keël georden is word 'n *kommutatief-geordende Banach-algebra* (KGBA) genoem. Omdat elke algebra-keël 'n algebra-$c$-keël is, is elke GBA 'n KGBA.

Spektraalteorie in GBAs word al vir ongeveer twintig jaar bestudeer. In hierdie werk veralgemeen ons baie GBA-resultate na die meer algemene konteks van KGBAs, en verkry dan ook nuwe resultate in KGBAs en GBAs.

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My gratitude also go to my fellow postgraduate students, especially those that I shared the office with, for their support. Finally, I would like to convey special thanks to my family for their encouragement, patience and support at all times.

# Dedication

I dedicate this work to my parents; my father on the occasion of his sixtieth birthday and my mother on the occasion of her fifty sixth birthday.

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# Introduction

A subset $C$ of a complex unital Banach algebra $A$ is called an *algebra cone* if $C$ contains the unit and is closed under addition, multiplication and positive scalar multiplication. The elements of $C$ are called *positive*. Every Banach algebra with an algebra cone can be ordered by the algebra cone. Then it is called an *ordered Banach algebra* (OBA). Spectral theory in ordered Banach algebras has been investigated in the papers of S. Mouton and H. Raubenheimer; H. du T. Mouton and S. Mouton; D. Behrendt and H. Raubenheimer; G. Braatvedt, R. Brits and H. Raubenheimer; and S. Mouton. However, a non-commutative $C^*$-algebra cannot be ordered by an algebra cone since, in a non-commutative $C^*$-algebra, the product of positive elements is positive only if the elements commute. Any $C^*$-algebra, however, then does have the property that the product of commuting positive elements is positive.

R. Harte mentioned some years ago that certain known results in OBAs possibly still hold true if the assumption that the algebra cone is closed under multiplication is weakened to closedness only under multiplication of commuting positive elements. Such results would of course hold in OBAs, and would also be applicable to non-commutative $C^*$-algebras. He briefly followed up this idea in the paper [31], where he defined a *partially ordered Banach algebra* as a Banach algebra ordered by a cone that contains the unit and is closed under addition, positive scalar multiplication and multiplication by commuting positive elements. Every OBA is a partially ordered Banach algebra. R. Harte generalized two well known results in OBAs to the more general setting of partially ordered Banach algebras (see [31] theorem 1, theorem 2).

Instead of using R. Harte's terminology of 'partially ordered Banach algebra', we will adopt the more descriptive term *'commutatively ordered Banach algebra'* (COBA), which is still defined as a complex unital Banach algebra containing a subset $C$, called an *algebra c-cone*, such that $C$ contains the unit and is closed under addition, positive scalar multiplication and multiplication by commuting positive elements. Every OBA is a COBA. In this work we will give a full account of which known results in OBAs can be generalized to COBAs. Generally, OBA results that can be generalized to COBAs fall into three categories: those that involve multiplication of positive elements but the product is not required to be positive; those for which the only multiplication of positive elements involved is taking powers of positive elements; those that involve multiplication of different positive elements and require the product to be positive, but will still hold to the full or a lesser extent if we only require that the product of positive elements is positive if the elements commute. The OBA results in the first two categories are extended to COBAs with minor or no adjustments to the proofs. Those in the third category are extended to COBAs by including additional assumptions and making appropriate modifications to the proofs. After generalizing OBA results to COBAs, we will proceed to obtain new results in COBAs and OBAs. This will be done by studying a number of research papers regarding spectral theory of positive operators on Banach lattices, including the papers of V. Caselles; F. Rábiger and M. Wolff, and then obtaining analogous results in the setting of COBAs and OBAs.

Research in the broad area of spectral theory in ordered structures dates back to around 1900, when O. Perron and G. Frobenius discovered that the spectrum of a positive matrix had certain special features. Since then spectral theory in the more general context of positive operators on ordered Banach spaces or Banach lattices has been investigated intensively and several authors have made contributions, including leading mathematicians like H.H. Schaefer and A.C. Zaanen. Spectral theory in the even more general context of OBAs has only been studied in the last approximately twenty years.

This thesis is organized into five chapters. Chapter 1 contains all the preliminary material that will be needed in the rest of the document. Proofs of well known results that can be easily obtained from standard literature will generally be omitted. Only those proofs for which the reader may not easily access the appropriate literature will be included.

In Chapter 2 we introduce COBAs and obtain results giving their fundamental properties, following the development in [51]. This chapter begins with Section 2.1, where we define COBAs and the associated basic properties, and also give examples of COBAs. As will be seen in this section, there are several non-trivial examples of COBAs, some of which were suggested in [31] and others which are due to the author. Section 2.2 gives properties of algebra $c$-cones in different Banach algebras, following the corresponding development in [51] for algebra cones. The results in this section show that these properties come naturally in a COBA. Section 2.3 extends the notion of COBA to direct sums of Banach algebras, analogous to direct sums of OBAs. This section includes example 2.3.4, which shows that the direct sum of an OBA and a COBA is a COBA. This is an important result because since there are plenty of examples of OBAs, it shows that there are plenty of examples of COBAs. In Section 2.4 we consider quotient COBAs. It is established in [51] that if $(A, C)$ is an OBA and $F$ is a closed ideal of $A$, then $(A/F, \pi C)$ is also an OBA, where $\pi : A \to A/F$ is the canonical homomorphism. If $(A, C)$ is a COBA, $(A/F, \pi C)$ is in general not a COBA. This is because if $C$ is only an algebra $c$-cone, the multiplication property required for $\pi C$ to be an algebra $c$-cone is not satisfied. This means that care should be taken when dealing with the ordering in the structure $(A/F, \pi C)$. This represents a major difference between OBAs and COBAs. To handle this issue, we will introduce the concept of a *maximal positive commutative subset* (MPCS for short) of a COBA. These are subsets of COBAs with special properties that will allow us to deal with COBA results that involve quotient algebras. The results in this section are attributed to the author.

One of the most important properties of COBAs is monotonicity of the spectral radius. Chapters 3, 4 and 5 are organized around this concept. Chapter 3 contains COBA results that do not rely on this property, while nearly all the results in Chapter 4 involve the property. Some of the results in Chapter 5 do not use monotonicity of the spectral radius while others do.

We discuss the results in Chapter 3. Let $a$ be a non-zero element of a Banach algebra $A$. A point $\lambda$ in the spectrum of $a$ is called an *eigenvalue* of $a$ is there exists a non-zero element $u$ in $A$ such that $au = \lambda u$ or $ua = \lambda u$. The element $u$ is called the *eigenvector* corresponding to $\lambda$. In Section 3.1 we obtain the important Krein-Rutman theorem in the setting of COBAs. The OBA version of the Krein-Rutman theorem was proved in [47]. This theorem gives conditions under which the spectral radius of a positive element is an eigenvalue of that element, with positive corresponding eigenvector. The Krein-Rutman theorem readily extends to COBAs, with the same proof as that for OBAs. This is because, in the proof, the only multiplication of positive elements involved is taking powers.

In Section 3.2 we discuss positive elements and analytic functions. The problem is as follows: if $a$ is a positive element and $f$ an analytic function, when is $f(a)$ positive? This problem was considered for OBAs in [42] and in the proofs of most of the results, the only multiplication of positive elements involved is taking powers. With the same proofs, which we include because they are generally short, we will extend these results to COBAs.

Section 3.3 is about the problem of unit spectrum. In this problem we seek to determine when a positive element with spectrum consisting of 1 only is necessarily the unit of the Banach algebra. In [16], this problem was studied for OBAs. Since in the proofs of the results taking powers is the only multiplication involving positive elements, corresponding results in COBAs can be established with the same proofs. The results will therefore be stated without the proofs given. The problem of unit spectrum was also considered in [34] in the operator theoretic setting. With similar proofs, some of the results can be obtained in general Banach algebras. However, by restricting ourselves to positive elements in COBAs and OBAs, we will establish the results with much simpler proofs, although in certain cases the results become weaker. The results here are due to the author.

Section 3.4 is basically an extension of the problem of unit spectrum of Section 3.3. The problem is to determine when a positive element with spectrum consisting of 1 only will be greater than or equal to 1. This problem was studied in [42] for OBAs and in the proofs of the results, either there is no multiplication of positive elements or the only multiplication involving positive elements is taking powers. Therefore the same proofs can be used to establish the corresponding COBA results. We will state the results without the proofs.

We discuss the results in Chapter 4, which generally rely on monotonicity of the spectral radius. In Section 4.1 we obtain some fundamental results about monotonicity of the spectral radius in COBAs, following the development in [51]. Some of the results play a crucial role in further development of a large part of spectral theory in COBAs. The most important one is probably theorem 4.1.6, whose proof is the same as that of its OBA counterpart in ([51], theorem 5.2), since the only multiplication of positive elements involved is taking powers.

In Section 4.2 we consider a special class of algebra $c$-cones called *algebra $c'$-cones*. It was mentioned previously that if $(A, C)$ is COBA and $F$ a closed ideal of $A$, then $(A/F, \pi C)$ is in general not a COBA, where $\pi : A \to A/F$ is the canonical homomorphism. It was also said that to deal with this issue the concept of an MPCS would be introduced in Chapter 2. In Section 4.2 we will show that if $C$ is an algebra $c'$-cone, then $\pi C$ is an algebra $c'$-cone. With the resulting structure $(A/F, \pi C)$, which we will call a $C'$OBA, COBA results involving quotient algebras can be stated and proved. Alongside MPCSs, $C'$OBAs will be useful in the development of the parts of the theory that involve quotient algebras.

In Section 4.3 we discuss the boundary spectrum and continuity of the spectral radius function. Properties of the boundary spectrum were investigated in [45] and a number of results involve OBAs. We will obtain the corresponding results in COBAs. An important problem in Banach algebra theory is that of determining when the spectral radius function is continuous.

This problem was studied for OBAs in [44] and [46], and the results in [44] require that the product of positive elements is positive, whether the elements commute or not. So in order to extend these results to COBAs, the commutativity assumption is required, but this then means that the ordering is not necessary as we obtain known results in general Banach algebras. Therefore the results in [44] cannot be meaningfully extended to COBAs. On the other hand, the situation in [44] does not arise in [46] and so we will obtain COBA counterparts of results in [46]. Since the only multiplication of positive elements involved in the OBA results in [45] and [46] is taking powers, we will obtain their COBA analogues with the same proofs, which are included to illustrate how some COBA results obtained earlier are used. The material on the boundary spectrum and on continuity of the spectral radius has been put together in this section because some of the results about the boundary spectrum in [45] are used in [46] in connection with continuity of the spectral radius.

In Section 4.4 we consider results involving Riesz elements, Riesz points and quasi-inessential elements. The results obtained are COBA generalizations of OBA results in [42],[43] and [47]. Proofs that are similar to the corresponding OBA ones will be generally omitted. A number of results here involve quotient algebras and so MPCSs and $C'$OBAs are used extensively.

In Section 4.5 we consider properties of the peripheral spectrum of a positive element under perturbation by a positive Riesz element in a COBA. Similar properties were studied in [21] for certain classes of operators. The results in this section, due to the author, extend the theory of COBAs and OBAs. One of the main ones is corollary 4.5.8, which shows that if the peripheral spectrum of a positive element in a COBA has no Riesz points, then one of two very dissimilar properties holds.

Section 4.6 considers the following problem: if $(a_n)$ is a sequence of positive elements converging to an element $a$ in a COBA, which properties does the limit $a$ inherit from $(a_n)$? The problem was studied for OBAs in [43] and we obtain COBA counterparts of the results. One of them is corollary 4.6.6, which is stronger than its OBA counterpart ([43], corollary 4.10) and is established using $C'$OBAs while the proof of its OBA analogue uses different means. It is results such as ([43], corollary 4.10) that give motivation for the introduction of $C'$OBAs. In this section proofs similar to the corresponding OBA ones will in general be omitted.

Chapter 4 ends with Section 4.7, which is about the trace of a positive element. The problem here is finding conditions under which the trace of a positive element is necessarily positive. While results for this problem are obtained trivially for positive elements in a $C^*$-algebra, it is not the case in a general COBA or OBA. The results in this section extend the theory of COBAs and OBAs and are due to the author.

Finally we discuss Chapter 5, which is about domination properties for positive elements in COBAs. The general problem in this chapter is as follows: if $a$ and $b$ are positive elements in a COBA such that $0 \leq a \leq b$ and if $b$ has a certain property, under what conditions does $a$ inherit the property? In the case of positive operators on Banach lattices, this is a classical problem and has been investigated for various properties by several authors and results abound (cf. [1], [2], [3], [21], [50], [55], [60]). The vast majority of results in Chapter 5 involve quotient algebras. As such, MPCSs and $C'$OBAs are used extensively in this chapter.

Section 5.1 is on domination by positive elements and the radical. The problem here is to investigate when a dominated positive element lies in the radical, given that the dominating element is in the radical. This problem was studied in [41] for positive elements in OBAs. We generalize the results in this paper to COBAs and will generally omit proofs if they are similar to the OBA ones. While a number of results are obtainable in COBAs by using MPCSs, such results are considerable weaker than their OBA counterparts because of restrictive conditions that come with MPCSs. In fact for some of the results, the restrictions are so severe that the results become trivial.

Section 5.2 is on domination by Riesz elements in COBAs. Here we will generalize an OBA result in ([51], theorem 6.2) to COBAs. An examination of instances of ([51], theorem 6.2) given in ([51], theorem 6.5, corollary 6.6) reveals that these results are actually formulated and proved in a COBA sense, although COBAs are not introduced in this paper. It is such results that give motivation for generalization from OBAs to COBAs.

Section 5.3 is about domination by quasi inessential elements in COBAs. We obtain COBA counterparts of an OBA result in ([47], corollary 5.4). As we prove in this section, some of the assumptions in ([47], corollary 5.4) can be dropped. In Section 5.4 we deal with the domination problem for inessential elements. We give COBA generalizations of the OBA results in [13]. Some of the results are obtained by using MPCSs, although some then become trivial because of restricting conditions that come with MPCSs. Section 5.5 is about the domination problem for rank one and finite rank elements. The results here are also COBA generalizations of the OBA results in [13], and like in Section 5.4 some of the results become trivial because of the use of MPCSs.

Let $(f_n)$ be the sequence of complex valued functions defined by $f_n(\lambda) = \sum_{k=0}^{n-1} \frac{\lambda^k}{n}$. If $a$ is an element of a Banach algebra, the sequence $(f_n(a))$ is called the sequence of *ergodic sums* of $a$. We say that $a$ is *ergodic* if its sequence of ergodic sums converges. In Section 5.6 we consider the domination problem for ergodic elements in a COBA. The corresponding problem in the operator theoretic setting was studied in [50], and the result obtained relies on a theorem of N. Dunford (see [25], theorem 3.16). By obtaining a Banach algebra version of ([25], theorem 3.16), we will prove our domination results for ergodic elements under conditions similar to those for the corresponding result in ([50], theorem 4.5). In the process of obtaining the Banach algebra analogue of ([25], theorem 3.16), we will establish several Banach algebra results that are of interest in themselves. The results in Section 5.6, due to the author, extend the theory of COBAs and OBAs.

For a Banach algebra element $a$, the set of all eigenvalues of $a$ which lie in the peripheral spectrum of $a$ is called the *peripheral point spectrum* of $a$. In Section 5.7 we consider two problems. The first is that of determining when the spectral radius of a dominated positive element is an eigenvalue of the element, with positive corresponding eigenvector, given that the spectral radius of the dominating element is an eigenvalue of the element and has positive corresponding eigenvector. The second problem is that of determining when the peripheral point spectrum of a dominated positive element is contained in the peripheral point spectrum of the dominating element. A slightly more general problem was studied in [50] for positive operators on Banach lattices and the results obtained are typically operator theoretic. We will obtain our results under different conditions and by purely algebraic means. The results in Section 5.7 extend the theory of COBAs and OBAs and are due to the author.

Definitions, theorems and other results are numbered successively. By theorem 3.2.1 we mean theorem 1 of Section 2 of Chapter 3. The symbol $\Box$ indicates the end of a proof.

# Chapter 1

# Preliminaries

In this chapter we collect the pre-requisite material and establish the notation that will be used in the rest of the chapters.

## 1.1 Spectral theory

Throughout $A$ (or B) will be a complex Banach algebra with unit 1. A linear map $\pi : A \to B$ between Banach algebras is said to be a *homomorphism* if $\pi(ab) = \pi a \pi b$ for all $a, b \in A$ and $\pi 1 = 1$. The *spectrum* of $a$ in $A$ is the set $\sigma(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a$ is not invertible in $A\}$. The *spectral radius* of $a$ in $A$ is the number $r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$. When it becomes necessary to emphasize the particular Banach algebra, we will write $\sigma(a, A)$ and $r(a, A)$ to indicate the spectrum and spectral radius respectively, of $a$ in $A$. We will denote by $\delta(a)$ the *distance* $d(0, \sigma(a))$ from 0 to $\sigma(a)$. The *peripheral spectrum* of $a$ is the set $\operatorname{psp}(a) = \sigma(a) \cap \{\lambda \in \mathbb{C} : |\lambda| = r(a)\}$. It is a nonempty closed subset of $\sigma(a)$.

Let $E$ be a subset of $A$. Define $E^c = \{b \in A : ba = ab$ for all $a \in E\}$ and $E^{cc} = \{b \in A : ba = ab$ for all $a \in E^c\}$. The set $E^c$ is called the *commutant* of $E$ and $E^{cc}$ is called the *second commutant* of $E$.

**Proposition 1.1.1.** *([15], proposition 2, p.75) Let $E$ be a subset of a Banach algebra $A$. Then the following statements hold:*

*(i) $E^c$ is a closed subalgebra of $A$ containing 1.*
*(ii) $E$ is a commutative subset if and only if $E \subseteq E^c$.*
*(iii) $F^c \subseteq E^c$ if $E \subseteq F$.*
*(iv) If $E$ is a commutative subset, then $E^{cc}$ is a commutative subalgebra of $A$ and $E \subseteq E^{cc} \subseteq E^c$.*
*(v) The centre $A^c$ of $A$ is a commutative subalgebra of $A$.*

To prove some results in later chapters, the following results about the spectrum will be useful.

**Theorem 1.1.2.** *([9], theorem 3.2.8) Let $A$ be a Banach algebra and $a \in A$. Then*

*(i)* $\lambda \mapsto (\lambda 1 - a)^{-1}$ *is analytic on* $\mathbb{C} \setminus \sigma(a)$ *and goes to* $0$ *at infinity,*
*(ii)* $\sigma(a)$ *is compact and non-empty,*
*(iii)* $r(a) = \lim_{n\to\infty} ||a^n||^{\frac{1}{n}}$.

**Theorem 1.1.3.** *([9], theorem 3.3.5) Let* $A$ *be a Banach algebra. Suppose that* $a \in A$ *and that* $\alpha \notin \sigma(a)$*. Then we have* $d(\alpha, \sigma(a)) = \frac{1}{r((\alpha 1 - a)^{-1})}$.

**Proposition 1.1.4.** *Let* $A$ *be a Banach algebra and* $B$ *a closed subalgebra of* $A$ *containing* $1$*. Then* $\sigma(a, A) \subseteq \sigma(a, B)$ *for all* $a \in B$*.*

**Proposition 1.1.5.** *([9], corollary 3.2.10) Let* $A$ *be a Banach algebra and let* $a, b \in A$*. If* $ab = ba$*, then* $\sigma(ab) \subseteq \sigma(a)\sigma(b)$*.*

Let $f$ be a complex valued function which is analytic on a neighbourhood $\Omega$ of the spectrum of $a$. Let $\Gamma$ be a contour in $\mathbb{C} \setminus \sigma(a)$ surrounding $\sigma(a)$. The integral $\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda 1 - a)^{-1} d\lambda$ is well defined since the map $\lambda \mapsto (\lambda 1 - a)^{-1}$ is defined and continuous on $\Gamma$. Furthermore, this integral is independent of the contour $\Gamma$ surrounding $\sigma(a)$. To see why this true, suppose that $\Gamma_1, \Gamma_2$ are two contours in $\mathbb{C} \setminus \sigma(a)$ surrounding $\sigma(a)$ such that $a_1 = \frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)(\lambda 1 - a)^{-1} d\lambda \neq \frac{1}{2\pi i}\int_{\Gamma_2} f(\lambda)(\lambda 1 - a)^{-1} d\lambda = a_2$. Then by the Hahn-Banach theorem, there exists a linear functional $\phi$ on $A$ such that $\phi(a_1) \neq \phi(a_2)$. Now $\phi(a_1) = \frac{1}{2\pi i}\int_{\Gamma_1} h(\lambda) d\lambda$ and $\phi(a_2) = \frac{1}{2\pi i}\int_{\Gamma_2} h(\lambda) d\lambda$, where $h(\lambda) = f(\lambda)\phi((\lambda 1 - a)^{-1}))$. Since $h$ is analytic on $\mathbb{C} \setminus \sigma(a)$ by theorem 1.1.2, it follows from Cauchy's theorem that $\phi(a_1) = 0 = \phi(a_2)$, which is a contradiction. Hence the integral is independent of the contour. Therefore we can define without problems the element $f(a) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda 1 - a)^{-1} d\lambda$ in $A$ ([9], p.43).

**Lemma 1.1.6.** *([9], lemma 3.3.1) Let* $a$ *be an element of a Banach algebra* $A$ *and let* $\Gamma$ *be a smooth contour surrounding* $\sigma(a)$*. If* $s(\lambda)$ *is a rational function having no poles surrounded by* $\Gamma$*, then* $s(a) = \frac{1}{2\pi i}\int_\Gamma s(\lambda)(\lambda 1 - a)^{-1} d\lambda$.

**Theorem 1.1.7.** *([9], theorem 3.3.3) Let* $A$ *be a Banach algebra and let* $a \in A$*. Suppose that* $\Omega$ *is an open set containing* $\sigma(a)$ *and* $\Gamma$ *a smooth contour included in* $\Omega$ *and surrounding* $\sigma(a)$*. Then the mapping* $f \mapsto f(a) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda 1 - a)^{-1} d\lambda$ *from* $H(\Omega)$ *into* $A$ *has the following properties:*

*(i)* $(f_1 + f_2)(a) = f_1(a) + f_2(a)$,
*(ii)* $(f_1 \cdot f_2)(a) = f_1(a) \cdot f_2(a) = f_2(a) \cdot f_1(a)$,
*(iii)* $1(a) = 1$ *and* $I(a) = a$ *(where* $I(\lambda) = \lambda$*),*
*(iv) if* $(f_n)$ *converges to* $f$ *uniformly on compact subsets of* $\Omega$*, then* $f(a) = \lim_{n\to\infty} f_n(a)$,
*(v)* $\sigma(f(a)) = f(\sigma(a))$.

**Theorem 1.1.8.** *([9], theorem 3.3.4) Let* $A$ *be a Banach algebra. Suppose that* $a \in A$ *has a disconnected spectrum. Let* $U_0$ *and* $U_1$ *be two open disjoint sets such that* $\sigma(a) \subseteq U_0 \cup U_1$*,* $\sigma(a) \cap U_0 \neq \emptyset$ *and* $\sigma(a) \cap U_1 \neq \emptyset$*. Then there exists a non-trivial projection* $p$ *commuting with* $a$ *such that* $\sigma(pa) = (\sigma(a) \cap U_1) \cup \{0\}$ *and* $\sigma(a - pa) = (\sigma(a) \cap U_0) \cup \{0\}$*.*

**Remark 1.1.9.** *The idempotent* $p$ *in theorem 1.1.8 is called the spectral idempotent associated with* $a$ *and* $\sigma(a) \cap U_1$*, and is of particular importance in the case where* $\sigma(a) \cap U_1 = \{\alpha\}$*. In* *this case, $p$ is said to be the spectral projection associated with $a$ and $\alpha$, and $p$ takes the form $p = p(a,\alpha) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma (\lambda 1 - a)^{-1} d\lambda$, where $\Gamma$ is a contour around $\alpha$, separating $\alpha$ from the remainder of the spectrum of $a$.*

If $X$ is a metric space and $E \subseteq X$, we write $\partial_X E$ (or simply $\partial E$ if it is clear which metric space is being used) to denote the (topological) boundary of $E$ in $X$. Let $A$ be a Banach algebra and let $a \in A$. The *resolvent set* of $a$ is the set $\mathbb{C} \setminus \sigma(a)$. The boundary of the unbounded connected component of the resolvent set of $a$ will be denoted by $\partial_\infty \sigma(a)$. The *resolvent* of $a$ is the function $R(\lambda, a) = (\lambda 1 - a)^{-1}$, where $\lambda \in \mathbb{C} \setminus \sigma(a)$. A point $z \in \sigma(a)$ is called a *pole* of order $k$ of the resolvent of $a$ if $z$ is an isolated point in $\sigma(a)$ and $k$ is the smallest positive integer such that $(z1 - a)^k p(a, z) = 0$, where $p(a, z)$ is the spectral projection corresponding to $a$ and $z$. The resolvent of $a$ may be represented by a power series $R(\lambda, a) = \frac{1}{\lambda}\displaystyle\sum_{k=0}^{\infty}\left(\frac{a}{\lambda}\right)^k$ on the set $\{\lambda \in \mathbb{C} : |\lambda| > r(a)\}$ ([9], theorem 3.2.1). This is called the *Neumann series* for $R(\lambda, a)$. The following results about the resolvent function will be useful in proving important results in Chapter 4.

**Theorem 1.1.10.** *([43], theorem 5.2) Let $A$ be a Banach algebra and $(a_n)$ a sequence in $A$ such that $a_n \to a \in A$. Suppose that $(\alpha_n)$ is a sequence in $\mathbb{C}$ such that, for each $n \in \mathbb{N}$, $\alpha_n$ is a pole of the resolvent of $a_n$ of order $k_n$, and $\alpha_n \to \alpha \in \mathbb{C}$ where $\alpha$ is a pole of the resolvent of $a$ of order $k$. If*

$$(\lambda 1 - a)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - \alpha)^j b_j, \ (b_{-j} = 0 \text{ for all } j > k)$$

*and*

$$(\lambda 1 - a_n)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - \alpha_n)^j b_{n,j}, \ (b_{n,-j} = 0 \text{ for all } j > k_n)$$

*are the Laurent series of the resolvents of $a$ and $a_n$, then $b_{n,j} \to b_j$ as $n \to \infty$, for all $j \in \mathbb{Z}$.*

**Corollary 1.1.11.** *([43], corollary 5.3) Let $A$ be a Banach algebra and $(a_n)$ a sequence in $A$ such that $a_n \to a \in A$ as $n \to \infty$. Suppose that $(\alpha_n)$ is a sequence in $\mathbb{C}$ such that, for each $n \in \mathbb{N}$, $\alpha_n$ is a pole of the resolvent of $a_n$, and $\alpha_n \to \alpha \in \mathbb{C}$ where $\alpha$ is a pole of the resolvent of $a$. If $p = p(a, \alpha)$ and $p_n = p(a_n, \alpha_n)$, then $p_n \to p$ as $n \to \infty$.*

**Corollary 1.1.12.** *([43], corollary 5.4) Let $A$ be a Banach algebra and $(a_n)$ a sequence in $A$ such that $a_n \to a \in A$. Suppose that $(\alpha_n)$ is a sequence in $\mathbb{C}$ such that, for each $n \in \mathbb{N}$, $\alpha_n$ is a pole of the resolvent of $a_n$ of order $k_n$, and $\alpha_n \to \alpha \in \mathbb{C}$ where $\alpha$ is a pole of the resolvent of $a$ of order $k$. Let the Laurent series of the resolvents of $a$ and $a_n$ be as in theorem 1.1.10, and $u = b_{-k}$, $u_n = b_{n,-k_n}$ (where $au = \alpha u = ua$ and $a_n u_n = \alpha_n u_n = u_n a_n$). If there exists an $N \in \mathbb{N}$ such that $k_n \le k$ for all $n \ge N$, then $u_n \to u$ as $n \to \infty$.*

Let $A$ be a Banach algebra and $S$ the set of all non-invertible elements of $A$. For any $a \in A$ we define the set $S_\partial(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \in \partial S\}$ (or $S_\partial(a, A)$ if we need to emphasize the particular Banach algebra) in the complex plane. The set $S_\partial(a)$ is called the *boundary spectrum* of $a$ in $A$. It is known that $S_\partial(a)$ is compact and nonempty, and $S_\partial(a) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq r(a)\}$ ([46], p.1779). We also define $r_1(a) = \sup\{|\lambda| : \lambda \in \partial\sigma(a)\}$ and $r_2(a) = \sup\{|\lambda| : \lambda \in S_\partial(a)\}$.

The following results about the boundary spectrum will be useful in chapter 4.

**Proposition 1.1.13.** *([45], proposition 2.1) Let $A$ be a Banach algebra and $a \in A$. Then $\partial\sigma(a) \subseteq S_\partial(a) \subseteq \sigma(a)$ and therefore $r_1(a) = r_2(a) = r(a)$.*

**Proposition 1.1.14.** *([45], proposition 2.7) Let $a$ be an invertible element of a Banach algebra $A$. Then $S_\partial(a^{-1}) = (S_\partial(a))^{-1}$.*

Let $A$ be a Banach algebra and $a \in A$. Using $\partial S$ we define the set $T(a)$ as follows: $T(a) = \{\lambda \in \mathbb{C} : |\lambda| 1 - a \in \partial S\}$. It is known that $T(a) = \{\lambda \in \mathbb{C} : |\lambda| \in S_\partial(a)\} = \{\lambda \in \mathbb{C} : |\lambda| \in S_\partial(a) \cap \mathbb{R}^+\}$. Therefore $T(a)$ is compact and $T(a) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq r(a)\}$. Also if $\lambda_0 \in \mathbb{R}^+$, then $S_\partial(a) \cap \mathbb{R}^+ = \{\lambda_0\}$ if and only if $T(a) = \{\lambda \in \mathbb{C} : |\lambda| = \lambda_0\}$. If $T(a) \neq \emptyset$ and $\gamma(a) = \sup\{|\lambda| : \lambda \in T(a)\}$, then $\gamma(a) \in T(a)$ for all $a \in A$ ([46], 1779-1780).

Let $K(\mathbb{C})$ denote the set of compact subsets of $\mathbb{C}$. The following theorem about the set $T(a)$ will be used in chapter 4.

**Theorem 1.1.15.** *([46], theorem 4.5) Let $A$ be a Banach algebra. Then the function $a \mapsto T(a)$ from $A$ into $K(\mathbb{C})$ is upper semicontinuous on $A$.*

## 1.2 Ideals and inessential elements

In this section and the rest of the document, every ideal in a Banach algebra will be assumed to be two-sided. Also, if $S$ is a subset of a normed space $X$, we write $cl_X(S)$ to denote the closure of $S$ in $X$. If it is clear which normed space is involved, we omit $X$ and just write $cl(S)$.

A Banach algebra $A$ is said to be *semiprime* if $aAa = \{0\}$ implies $a = 0$, where $a \in A$. An element $a$ in $A$ is said to be *quasi nilpotent* if $r(a) = 0$. The set of quasi nilpotent elements of $A$ is denoted by $\mathrm{QN}(A)$. The *radical* of $A$ is the set $\mathrm{Rad}(A) = \{a \in A : aA \subseteq QN(A)\}$. If $\mathrm{Rad}(A) = \{0\}$, then $A$ is said to be *semisimple.* Every semisimple Banach algebra is semiprime.

Let $A$ be a Banach algebra and $a \in A$. If $J$ is an ideal of $A$, an isolated point $\lambda \in \sigma(a)$ is called a *Riesz point* of $\sigma(a)$ relative to $J$ if the corresponding spectral projection $p(a, \lambda)$ belongs to $J$. An ideal $I$ of $A$ is said to be *inessential* if the spectrum of every element in $I$ is either finite or a sequence converging to zero. We define the set $\mathrm{kh}(A, I) = \{a \in A : a + cl_A(I) \in \mathrm{Rad}(A/cl_A(I))\}$ of *inessential elements* of $A$. It is well known that $\mathrm{kh}(A, I)$ is a closed ideal of $A$ and $I \subseteq cl_A(I) \subseteq \mathrm{kh}(A, I)$ ([9], p.106). An element $a$ in $A$ is said to be *Riesz* relative to a closed ideal $F$ of $A$ if the spectrum of the element $a + F$ in the quotient algebra

$A/F$ consists of zero. We will use $\mathcal{R}(A,F)$ to denote the set of all Riesz elements of $A$ relative to $F$. For a closed inessential ideal $I$ of $A$, the inclusion $\mathrm{kh}(A,I) \subseteq \mathcal{R}(A,I)$ is well known.

We define the set $D_I(a,A)$ as follows:

$$\lambda \notin D_I(a,A) \Leftrightarrow \begin{cases} \lambda \notin \sigma(a) \\ \text{or} \\ \lambda \text{ is a Riesz point of } \sigma(a) \text{ relative to } I. \end{cases}$$

The following results will be used to prove some important results in Chapter 3 and Chapter 4.

**Theorem 1.2.1.** *([47], theorem 3.11) Let $A$ be a semisimple Banach algebra, $a \in A$ and $I$ a closed inessential ideal of $A$ such that $a$ is Riesz w.r.t. $I$. If $0 \neq \alpha \in \sigma(a)$ then $\alpha$ is a pole of the resolvent of $a$.*

**Lemma 1.2.2.** *([43], lemma 4.1) Let $A$ be a Banach algebra and $J$ an ideal of $A$. Suppose that $(a_n)$ is a sequence in $A$ such that $a_n \to a \in A$.*

*(1) If $(\alpha_n)$ is a sequence such that $\alpha_n \in \sigma(a_n)$ for all $n \in \mathbb{N}$ and $\alpha_n \to \alpha$, then $\alpha \in \sigma(a)$.*

*(2) If $\mathrm{psp}(a)$ consists of Riesz points of $\sigma(a)$ relative to $J$, then the following properties hold:*
*(a) There are finitely many (Riesz) points in $\mathrm{psp}(a)$.*
*(b) $r(a_n) \to r(a)$ as $n \to \infty$.*
*(c) If $(\alpha_n)$ is a sequence such that $\alpha_n \in \mathrm{psp}(a_n)$ for all $n \in \mathbb{N}$ and $\alpha_n \to \alpha$, then $\alpha \in \mathrm{psp}(a)$.*

**Lemma 1.2.3.** *([43], lemma 2.1) Let $A$ be a semisimple Banach algebra, $I$ an inessential ideal of $A$, and $a \in A$. Then a point $\alpha$ in $\sigma(a)$ is a Riesz point of $\sigma(a)$ relative to $I$ if and only if $\alpha$ is a pole of the resolvent of $a$ and $p = p(a,\alpha) \in I$.*

In later chapters, the following spectral result will be applied in several proofs.

**Proposition 1.2.4.** *Let $A$ be a Banach algebra and $F$ a closed ideal in $A$. If $a \in A$ then $\sigma(a+F, A/F) \subseteq \sigma(a,A)$. Therefore $r(a+F, A/F) \leq r(a,A)$.*

Let $I$ be a closed inessential ideal in a Banach algebra $A$. An element $a$ in $A$ is called *quasi inessential* (relative to $I$) if there exist $k \in I$ and $n \in \mathbb{N}$ such that $||a^n - k|| < 1$. The set of quasi inessential elements relative to $I$ will be denoted by $\mathrm{qkh}(A,I)$. It well known that $\mathrm{kh}(A,I) \subseteq \mathcal{R}(A,I) \subseteq \mathrm{qkh}(A,I)$.

**Proposition 1.2.5.** *([47], proposition 5.1) Let $I$ be a closed inessential ideal in a Banach algebra $A$. An element $a$ in $A$ is quasi inessential relative to $I$ if and only if $r(a+I, A/I) < 1$.*

**Proof.** Suppose that $a$ is quasi inessential relative to $I$. Then there exists a $k \in I$ and $n \in \mathbb{N}$ such that $||a^n - k|| < 1$. Therefore $r(a^n + I, A/I) \leq ||a^n + I|| = \inf_{b \in I} ||a^n - b|| \leq ||a^n - k|| < 1$. From the spectral mapping theorem, it follows that $(r(a+I, A/I))^n = r(a^n + I) < 1$. This implies that $r(a+I, A/I) < 1$. Conversely suppose that $r(a+I, A/I) < 1$. Then

$r(a + I, A/I) = \lim_{n\to\infty} ||a^n + I||^{\frac{1}{n}} < 1$. This implies that there exist $m \in \mathbb{N}$ such that $||a^m + I||^{\frac{1}{m}} < 1$, so that $||a^m + I|| < 1$. Since $||a^m + I|| = \inf_{b\in I} ||a^m - b||$, it follows that $||a^m - b|| < 1$ for some $b \in I$. Hence $a$ is quasi inessential. □

Proposition 1.2.5 is established in [47] with a rather subtle proof, different from ours.

Let $K$ be a compact subset of $\mathbb{C}$. A *hole* of $K$ is a bounded component of $\mathbb{C} \setminus K$. We denote by $\eta K$ the union of $K$ and all the holes of $K$. The following theorem will be required in proving several results in later chapters.

**Theorem 1.2.6.** *([9], theorem 5.7.4) Let $I$ be an inessential ideal of a Banach algebra $A$. For $a \in A$ and $b \in I$ we have the following properties:*

*(i) if $G$ is a connected component of $\mathbb{C}\setminus D_I(a, A)$ intesecting $\mathbb{C}\setminus\sigma(a+b)$ then it is a component of $\mathbb{C} \setminus D_I(a + b)$,*
*(ii) the unbounded connected components of $\mathbb{C} \setminus D_I(a, A)$ and $\mathbb{C} \setminus D_I(a + b, A)$ coincide, in particular $D_I(a, A)$ and $D_I(a + b, A)$ have the same external boundaries,*
*(iii) $\sigma(a + I, A/cl_A(I)) \subseteq D_I(a, A)$ and $\eta D_I(a, A) = \eta\sigma(a + I, A/cl_A(I))$.*

**Theorem 1.2.7.** *([48], theorem 5.5) Let $I$ be a closed inessential ideal of a Banach algebra $A$ and let $a \in A$. If $b$ is a Riesz element of $A$ relative to $I$ and $ab = ba$, then $D_I(a, A) = D_I(a + b, A)$.*

## 1.3 Rank one and finite rank elements

Following [49], we call a non-zero element $x$ of a semisimple Banach algebra $A$ a *rank one element* if there exists a (unique) linear functional $f_x$ on $A$ such that $xax = f_x(a)x$ for all $a \in A$. The set of all rank one elements of $A$ will be denoted by $\mathcal{F}_1(A)$. If $x \in \mathcal{F}_1(A)$, the *trace* of $x$, denoted by $\mathrm{tr}(x)$, is the number defined by $x^2 = \mathrm{tr}(x)x$. If $f_x$ is the linear functional such that $xax = f_x(a)x$ for all $a \in A$, then we see that $\mathrm{tr}(x) = f_x(1)$. With the spectral mapping theorem, it can be shown that $\sigma(x) \subseteq \{0, \mathrm{tr}(x)\}$.

In Chapter 4, we will refer to the following examples of rank one elements.

**Example 1.3.1.** *([12], example 3.4.3) The Banach algebra $A = M_2(\mathbb{C})$ is semisimple and the rank one elements of $A$ are the non-invertible matrices.*

**Example 1.3.2.** *Let $A = \ell^\infty$ be the Banach algebra of all bounded sequences of complex numbers. Then the rank one elements of $A$ are the sequences with one non-zero term and zeroes elsewhere, and the non-zero term equals the trace of the element.*

**Proof.** It is well known that $\ell^\infty$ is a semisimple Banach algebra. Let $x = (0, 0, ..., \mathrm{tr}(x), 0, 0, ...)$ be a sequence in $A$ with the $i^{th}$ term $\mathrm{tr}(x)$ and zeroes elsewhere. We show that $x \in \mathcal{F}_1(A)$. Let $a = (\alpha_1, \alpha_2, ..., \alpha_i, ...)$ be an arbitrary element in $A$. Then $xax = (0, 0, ..., \alpha_i\mathrm{tr}(x)^2, 0, 0, ...) = \alpha_i\mathrm{tr}(x)x$. Let $f_x : A \to \mathbb{C}$ be the map defined by $f_x(a) = \alpha_i\mathrm{tr}(x)$. Clearly, $f_x$ is a linear functional on $A$. Therefore $x \in \mathcal{F}_1(A)$.

Conversely, suppose that $x = (\xi_1, \xi_2, ..., \xi_i, ...) \in \mathcal{F}_1(A)$. Then $x \neq 0$. Without loss of generality we assume for fixed $i$ that $\xi_i \neq 0$ but the other terms may or may not be equal to zero. Since $x \in \mathcal{F}_1(A)$, there is a linear functional $f_x : A \to \mathbb{C}$ such that $xax = f_x(a)x$ for all $a \in A$. Take $a = (0, 0, ..., 1, 0, 0, ...)$, a sequence with 1 in the $i^{th}$ position and zeroes elsewhere. Then $xax = (0, 0, ..., \xi_i^2, 0, 0, ...)$, which has one non-zero term $\xi_i^2$ in the $i^{th}$ position and zeros elsewhere. But $xax = f_x(a)x = (f_x(a)\xi_1, f_x(a)\xi_2, ..., f_x(a)\xi_i, ...)$. Suppose that $x$ has more than one non-zero term. If $f_x(a) = 0$, then $f_x(a)x = xax = 0$, which contradicts the fact that $xax$ has only one non-zero term $\xi_i^2$. If $f_x(a) \neq 0$, then $f_x(a)x = xax$ has more than one non-zero term, which contradicts the fact that $xax$ has only one no-zero term $\xi_i^2$. Therefore $x$ must have one non-zero term and zeroes elsewhere. To prove the second part, let $x = (0, 0, ..., \xi_i, 0, 0, ...) \in \mathcal{F}_1(A)$, $\xi_i \neq 0$. Then $x^2 = (0, 0, ..., \xi_i^2, 0, 0, ...) = \xi_i x$. Since $x^2 = \operatorname{tr}(x)x$, it follows that $\operatorname{tr}(x)x = \xi_i x$. Hence $\operatorname{tr}(x) = \xi_i$. □

**Example 1.3.3.** *([49], p. 658) Let $K$ be a completely regular Hausdorff space and $C_b(K)$ be the Banach algebra of all complex valued bounded continuous functions on $K$ with the supremum norm. Then the rank one elements of $C_b(K)$ are elements of the form*

$$f_s(t) = \begin{cases} \beta & if\ t = s \\ 0 & if\ t \neq s \end{cases}$$

*where $\beta \in \mathbb{C}$ is fixed and $s$ is an isolated point of $K$.*

Let $A$ be a semisimple Banach algebra. An element $x$ in $A$ is said to be a *finite rank element* if $x = \sum_{i=1}^n x_i$, where $x_i \in \mathcal{F}_1(A)$. The set of all finite rank elements of $A$ is denoted by $\mathcal{F}(A)$. By convention, $0 \in \mathcal{F}(A)$. It is well known that $\mathcal{F}(A)$ is an ideal of $A$ and if $\operatorname{Soc}(A)$ exists, then $\mathcal{F}(A) = \operatorname{Soc}(A)$ (cf. [49], p.659). The trace of $x \in \mathcal{F}(A)$ is defined by $\operatorname{tr}(x) = \sum_{i=1}^n \operatorname{tr}(x_i)$.

The following theorem gives some of the basic properties of the trace.

**Theorem 1.3.4.** *([49], theorem 4.5) Let $A$ be a semiprime Banach algebra. Then the trace satisfies the following properties:*

*(i) The trace is a linear functional on $\mathcal{F}$.*
*(ii) If $x \in \mathcal{F}(A)$ and $x$ is nilpotent, then $\operatorname{tr}(x) = 0$.*

In [10], Aupetit and Mouton defined the trace in Banach algebras in terms of the spectrum of an element. After the following discussion, we will give this definition of the trace.

If $S$ is any set, we denote by $\#S$ the number of elements in $S$. Let $A$ be a Banach algebra and $a$ an element of $A$. The *spectral rank* of $a$ is the number $\operatorname{rank}(a) = \sup_{x \in A} \#(\sigma(xa) \setminus \{0\}) \leq \infty$ ([10], p. 117). If $\operatorname{rank}(a)$ is finite, then $a$ is said to be a *spectrally finite rank element*. Therefore if $a$ is spectrally of finite rank, the set $E(a) = \{x \in A : \#(\sigma(xa) \setminus \{0\}) = \operatorname{rank}(a)\}$ is non-empty. It is well known that in a semisimple Banach algebra, the set of spectrally finite rank elements coincides with the socle ([10], corollary 2.9).

Let $A$ be a semisimple Banach algebra and $a \in \mathrm{Soc}(A)$. Let $\Gamma$ be an oriented smooth contour not intersecting $\sigma(a)$ and denote by $\Delta_0$ the interior of $\Gamma$. Then there exists a ball $U$ in $A$ centered at 1, such that $\sigma(xa) \cap \Gamma = \emptyset$ for $x \in U$, and for $x, y \in U \cap E(a)$ we have that $\#(\sigma(xa) \cap \Delta_0) = \#(\sigma(ya) \cap \Delta_0)$ ([10], theorem 2.4). The number $\#(\sigma(xa) \cap \Delta_0)$ is denoted by $m(\Gamma, a)$. For any isolated point $\alpha$ in $\sigma(a)$, the number $m(\alpha, a)$ is defined to be $m(\Gamma, a)$, where $\Gamma$ is taken to be a small circle centered at $\alpha$ and separating $\alpha$ from the rest of $\sigma(a)$. The number $m(\alpha, a)$ satifies $m(\alpha, a) \geq 1$ and turns out to be algebraic multiplicity of $\alpha$ for $a$ ([10], p.120). For an element $a$ in $\mathrm{Soc}(A)$, we define the trace ([10], p. 130) of $a$ by

$$\mathrm{tr}(a) = \textstyle\sum_{\lambda \in \sigma(a)} \lambda m(\lambda, a).$$

The trace is a linear functional on $\mathrm{Soc}(A)$ ([10], theorem 3.3). In chapter 4, we will need the following result about the trace.

**Theorem 1.3.5.** *([10], theorem 3.5) Let $A$ be a semisimple Banach algebra and $a \in \mathrm{Soc}(A)$. Then $r(a) = \overline{\lim}_{k\to\infty} |\mathrm{tr}(a^k)|^{\frac{1}{k}}$. Moreover, if $r(a) \neq 0$, the number of spectral values of $a$ with modulus $r(a)$ is given by $\overline{\lim}_{k\to\infty} \frac{|\mathrm{tr}(a^k)|}{r(a)^k}$.*

## 1.4 Operator theory

Let $T : H_1 \to H_2$ be a bounded linear operator between Hilbert spaces. Then the *adjoint operator* $T^*$ of $T$ is the operator $T^* : H_2 \to H_1$ such that for all $x \in H_1$ and $y \in H_2$, $\langle Tx, y\rangle = \langle x, T^*y\rangle$. If $T = T^*$, then $T$ is said to be *self-adjoint* . A bounded self-adjoint linear operator $T$ on a Hilbert space $H$ is said to be *positive,* written $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

**Theorem 1.4.1.** *([37], theorem 9.3-1, p.470) If $S$ and $T$ are positive bounded linear operators on a Hilbert space and if they commute, then their product is positive.*

**Proposition 1.4.2.** *([22], proposition 2.13, p.34) If $H$ is a Hilbert space and $T$ a bounded self-adjoint linear operator on $H$, then $||T|| = \sup\{|\langle Tx, x\rangle| : ||x|| = 1\}$.*

**Theorem 1.4.3.** *([54], theorem 12.32) If $H$ is a Hilbert space and $T \in \mathcal{L}(H)$, then the following statements are equivalent:*

*(i) $\langle Tx, x\rangle \geq 0$ for every $x \in H$,*
*(ii) $T^* = T$ and $\sigma(T) \subseteq [0, \infty)$.*

Let $E$ be a Banach lattice and $x \in E$. We say that $x \geq 0$ if $x = |x|$, and then we define $E^+ = \{x \in E : x = |x|\}$. A linear operator $T : E \to E$ is positive if $T(E^+) \subseteq E^+$. An operator $T : E \to E$ is called *regular* if it can be written as a linear combination over $\mathbb{C}$ of positive operators. The set of all regular operators on $E$ is denoted by $\mathcal{L}^r(E)$ and it is a subspace of $\mathcal{L}(E)$. If we define the $r$-norm

$$||T||_r = \inf\{||S|| : S \in \mathcal{L}(E), |Tx| \leq S|x| \text{ for all } x \in E\}$$

on $\mathcal{L}^r(E)$, then $\mathcal{L}^r(E)$ is a Banach algebra which contains the unit of $\mathcal{L}(E)$ (cf. [56], [7]). For more on positive operators on Banach lattices (cf. [55]).

An operator $T$ on a Hilbert space $H$ is said to be of *finite rank* if the range of $T$ is finite dimensional. The set of all finite rank operators on $H$ is denoted by $\mathcal{F}(H)$. An operator $T$ on $H$ is called *compact* if the closure of the image of the closed unit ball in $H$ is compact. The set of all compact operators of $H$ will be denoted by $\mathcal{K}(H)$. Clearly, $\mathcal{F}(H) \subseteq \mathcal{K}(H)$.

Let $T$ be bounded linear operator on a normed space $X$. A point $\lambda \in \mathbb{C}$ is called an eigenvalue of $T$ if there an $x \in X$, with $x \neq 0$, such that $Tx = \lambda x$. The following proposition will be useful in Chapter 5.

**Proposition 1.4.4.** *If $T$ is a bounded linear operator on a normed space $X$, then every pole of the resolvent of $T$ is an eigenvalue of $T$.*

**Proof.** Suppose that $\lambda_0$ is a pole of the resolvent of $T$. Then $(\lambda_0 I - T)^{-1}$ does not exist, where $I$ is the identity operator on $X$. This means that $\lambda_0 I - T$ is not injective, so that $\mathcal{N}(\lambda_0 I - T) \neq \{0\}$, where $\mathcal{N}(\lambda_0 I - T)$ is the null space of $\lambda_0 I - T$. Thus there is an $0 \neq x \in X$ such that $(\lambda_0 I - T)x = 0$. $\square$

## 1.5 $C^*$-algebras

In this section we present some results on the theory of $C^*$-algebras that will be used in later chapters.

**Example 1.5.1.** *([22], example 1.2, p.238) If $H$ is a Hilbert space, then $A = \mathcal{L}(H)$ is a $C^*$-algebra, where for each $T \in A$, $T^*$ is the adjoint of $T$.*

**Proposition 1.5.2.** *([22], proposition 1.11 (e), p.239) If $A$ is a $C^*$-algebra and if $a \in A$ is such that $a^* = a$, then $r(a) = ||a||$.*

**Proposition 1.5.3.** *([22], proposition 3.7, p.247) Let $A$ be a $C^*$-algebra and $C = \{a \in A : a = a^* \text{ and } \sigma(a) \subseteq [0, \infty)\}$. Then $C$ is a closed subset of $A$ and $\lambda a, a + b \in C$ for all $a, b \in C$ and for all $\lambda \geq 0$.*

**Theorem 1.5.4.** *([22], theorem 4.6, p.252) Let $A$ be a $C^*$-algebra and $F$ a closed ideal in $A$. If for each $a + F$ in $A/F$, we define $(a + F)^* = a^* + F$, then $A/F$ with its quotient norm is a $C^*$-algebra.*

**Proposition 1.5.5.** *([23], proposition 12.8) Let $A$ be a $C^*$-algebra and $C = \{a \in A : a = a^* \text{ and } \sigma(a) \subseteq [0, \infty)\}$. If $a, b \in C$ and $b - a \in C$ then $||a|| \leq ||b||$.*

## 1.6 Analysis

Let $K$ be a subset of the complex plane $\mathbb{C}$. We use **acc** $K$ to denote the set of accumulation points of $K$ and **iso** $K$ to denote the set of isolated points of $K$.

In Chapter 4, we will make use of corollary 1.6.3 to prove an important result. To prove corollary 1.6.3, we need lemma 1.6.1 and theorem 1.6.2.

**Lemma 1.6.1.** *([52], lemma 1.9.1) Let $(a_n), (b_n)$ and $(c_n)$ be sequences of positive real numbers such that $a_n < b_n$ for all $n \in \mathbb{N}$, $\frac{b_n}{a_n} \to \infty$ and $\frac{b_n}{c_n} \to 1$ as $n \to \infty$. Then $a_n < c_n$ for $n$ large enough.*

**Proof.** Since $\frac{b_n}{a_n} \to \infty$ and $\frac{b_n}{c_n} \to 1$ as $n \to \infty$, we have that $\frac{a_n}{c_n} = \frac{b_n/c_n}{b_n/a_n} \to 0$ as $n \to \infty$. This implies that $\frac{a_n}{c_n} < 1$ for $n$ large enough. Therefore $a_n < c_n$ for $n$ large enough. $\square$

**Theorem 1.6.2.** *(**Stirling's formula**) ([59], p.331) If $n$ is a large positive integer, then $n! \approx \sqrt{2\pi n}n^n e^{-n}$.*

**Corollary 1.6.3.** *([52], lemma 1.9.2) If $0 < \alpha < 1$, then $e^n < \dfrac{n^n}{\alpha^n n!}$ for $n$ large enough.*

**Proof.** If $\alpha < 1$ then $\frac{1}{\alpha^n} \to \infty$ as $n \to \infty$. From the fact that $\frac{1}{\alpha^n}$ grows faster than $\sqrt{2\pi n}$, it follows that $\frac{1}{\alpha^n\sqrt{2\pi n}} \to \infty$ as $n \to \infty$. Let $M > 1$. Then there exists an $N \in \mathbb{N}$ such that $\frac{1}{\alpha^n\sqrt{2\pi n}} > M$ for all $n > N$. Therefore $\frac{e^n}{\alpha^n\sqrt{2\pi n}} > Me^n > e^n$ for all $n > N$.
Let $a_n = e^n$, $b_n = \frac{e^n}{\alpha^n\sqrt{2\pi n}}$ and $c_n = \frac{n^n}{\alpha^n n!}$. Then, by Stirling's formula, $\frac{b_n}{c_n} = \frac{e^n n!}{\sqrt{2\pi n}n^n} \to 1$ as $n \to \infty$. Since $\frac{b_n}{a_n} = \frac{1}{\alpha^n\sqrt{2\pi n}} \to \infty$ as $n \to \infty$, by the previous lemma, $a_n < c_n$ for $n$ large enough. Therefore $e^n < \frac{n^n}{\alpha^n n!}$ for $n$ large enough. $\square$

The following results about real and complex numbers will be useful in the next chapter.

**Lemma 1.6.4.** *If $(x_n)$ is a sequence of non-negative real numbers and if $x \in \mathbb{R}$ is such that $x_n \to x$ as $n \to \infty$, then $x$ is non-negative.*

**Corollary 1.6.5.** *Let $\mathbb{C}^n$ be the normed space of all ordered $n$-tuples of complex numbers. If $(z_n)$ is a sequence in $\mathbb{C}^n$ with all the components real and non-negative and if $z \in \mathbb{C}^n$ is such that $z_n \to z$ as $n \to \infty$, then all the components of $z$ are real and non-negative.*

**Corollary 1.6.6.** *Let $M_n(\mathbb{C})$ denote the normed space of all $n$ by $n$ complex matrices. If $(A_n)$ is a sequence in $M_n(\mathbb{C})$ with all entries real and non-negative, and if $A \in M_n(\mathbb{C})$ is such that $A_n \to A$ as $n \to \infty$, then $A$ has only non-negative, real entries.*

**Corollary 1.6.7.** *Let $\ell^\infty$ be the normed space of all bounded sequences of complex numbers. If $(x_n)$ is a sequence in $\ell^\infty$ such that all the components of all the terms $x_n$ are real and non-negative, and if $x \in S$ is such that $x_n \to x$ as $n \to \infty$, then all the components of $x$ are real and non-negative.*

# Chapter 2

# Commutatively ordered Banach algebras

In this chapter commutatively ordered Banach algebras will be introduced and discussed. In [51] basic properties of ordered Banach algebras are given. We will develop corresponding properties compatible with the structure of commutatively ordered Banach algebras. Following the development in [51] and partly in [42], further properties involving different Banach algebras, direct sums of Banach algebras and quotient algebras will be presented.

## 2.1 Definitions and examples

In this section, commutatively ordered Banach algebras and associated basic properties are defined. As we will see, there are several non-trivial examples of commutatively ordered Banach algebras.

**Definition 2.1.1.** *A nonempty subset $C$ of a Banach algebra $A$ is called a cone if $C$ satisfies the following:*

*(i)* $C + C \subseteq C$,
*(ii)* $\lambda C \subseteq C$ *for all* $\lambda \geq 0$.

If $C$ also satisfies the property $C \cap -C = \{0\}$, then it is called a *proper cone.* We say that $C$ is closed if it a closed subset of $A$.

Let $X$ be a nonempty set. A relation $\leq$ is said to be an *ordering* on $X$ if one or more of the following conditions are satisfied:

(i) $x \leq x$ for every $x \in X$ (reflexivity),
(ii) $x \leq y$ and $y \leq x$ imply that $x = y$ (antisymmetry),
(iii) $x \leq y$ and $y \leq z$ imply that $x \leq z$ (transitivity).

The set $X$ with an ordering $\leq$ is called an *ordered set.* If the relation $\leq$ satisfies only (i) and (iii), then $\leq$ is said to be a *partial order* on $X$.

Every cone $C$ in a Banach algebra $A$ induces an ordering $\leq$ defined by $a \leq b$ if and only if $b - a \in C$, for $a, b \in A$. It can easily be verified that:

**Proposition 2.1.2.** *If $\leq$ is an ordering induced by a cone $C$ in a Banach algebra $A$, then the ordering $\leq$ is reflexive and transitive. Therefore it is a partial order on $A$. Moreover, $C$ is a proper cone if and only if $\leq$ is antisymmetric on $A$.*

In view of the fact that $C$ induces a partial order on $A$, we find that $C = \{a \in A : a \geq 0\}$. Therefore the elements of $C$ are called *positive.*

We define a special class of cones that in effect forms the basis of our work.

**Definition 2.1.3.** *A cone $C$ in a Banach algebra $A$ is called an algebra c-cone if it satisfies the following:*

*(i) $ab \in C$ for all $a, b \in C$ such that $ab = ba$,*
*(ii) $1 \in C$, where $1$ is the unit of $A$.*

Following [51], we call a cone $C$ that satisfies the stronger conditions $ab \in C$ for all $a, b \in C$ and $1 \in C$ an *algebra cone.* Obviously, every algebra cone is an algebra $c$-cone.

**Definition 2.1.4.** *A Banach algebra $A$ is called a commutatively ordered Banach algebra (COBA) if $A$ is ordered by a relation $\leq$ in such a way that for $a, b \in A$ and $\lambda \in \mathbb{C}$*

*(i) $a, b \geq 0 \Rightarrow a + b \geq 0$,*
*(ii) $a \geq 0, \lambda \geq 0 \Rightarrow \lambda a \geq 0$,*
*(iii) $a, b \geq 0$ and $ab = ba \Rightarrow ab \geq 0$,*
*(iv) $1 \geq 0$.*

Considering that any algebra $c$-cone $C$ in $A$ defines a partial ordering satisfying these properties, we find that $(A, C)$ is a COBA. If $C$ is an algebra cone, then following [51], we say that $(A, C)$ is an *ordered Banach algebra* (OBA). Clearly, every OBA is a COBA. Note that what we have defined as a COBA is what R. Harte in ([31], definition 1) defined as a *partially ordered Banach algebra.*

The properties of algebra $c$-cones we have discussed so far reconcile the algebraic structure of the Banach algebra with the order structure. We define a property of algebra $c$-cones that reconciles the topological structure of the Banach algebra with the order structure.

**Definition 2.1.5.** *An algebra c-cone $C$ in a Banach algebra $A$ is said to be c-normal if there exists a constant $\alpha > 0$ such that $0 \leq a \leq b$ and $ab = ba$ in $A$ imply that $||a|| \leq \alpha||b||$.*

If the algebra $c$-cone $C$ has the stronger property that there exists an $\alpha > 0$ such that $0 \leq a \leq b$ w.r.t. $C$ implies that $||a|| \leq \alpha||b||$ then we say that $C$ is *normal.* Therefore every normal algebra $c$-cone is $c$-normal. It is easy to establish that a $c$-normal algebra $c$-cone of a Banach algebra satisfies the following:

**Proposition 2.1.6.** *Every c-normal algebra c-cone in a Banach algebra $A$ is proper.*

From the definition of an algebra $c$-cone, it is clear that every subalgebra $B$ of a Banach algebra $A$ is an algebra $c$-cone, so that $(A, B)$ is a COBA. Since $B \cap -B = B$, we get that $B$ is not a proper (and hence non-$c$-normal) algebra $c$-cone of $A$.

An important concept in the theory of OBAs is that of monotonicity of the spectral radius (function) (see [51]). Following is a definition of monotonicity of the spectral radius compatible with the COBA stucture.

**Definition 2.1.7.** *Let $(A, C)$ be a COBA and let $a, b \in A$. If $0 \leq a \leq b$ w.r.t. $C$ and $ab = ba$ imply that $r(a) \leq r(b)$, then we say that the spectral radius (function) is c-monotone w.r.t. $C$.*

When we have the stronger property $r(a) \leq r(b)$ whenever $0 \leq a \leq b$ w.r.t. $C$, then following [51], we say that the spectral radius is *monotone* w.r.t. $C$. Therefore monotonicity implies $c$-monotonicity.

Another property that an algebra $c$-cone may possess is that of inverse-closedness, which we define as follows:

**Definition 2.1.8.** *If an algebra c-cone $C$ in a Banach algebra has the property that if $a \in C$ and $a$ is invertible, then $a^{-1} \in C$, then $C$ is said to be inverse-closed.*

For properties of inverse-closed algebra cones we refer to, for instance, [58].

Consider $A$ as a Banach space and let $C \subseteq A$ be a cone in $A$. Then $A$ can be ordered by $C$ in the usual way, and $(A, C)$ is called an *ordered Banach space* (OBS). The terms closed, proper and normal apply to $C$ in the usual sense. We will refer to ordered Banach spaces from time to time. Clearly, every COBA is an OBS. The following is an example of an OBS.

**Example 2.1.9.** *([52], example 2.2.5) Consider the Banach space $\ell^p$ $(1 \leq p < \infty)$, with norm defined by $||(x_i)_{i \in \mathbb{N}}|| = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$, where $(x_i)_{i \in \mathbb{N}} \in \ell^p$. The set*

$$C = \{(x_i)_{i \in \mathbb{N}} \in \ell^p : x_i \geq 0 \text{ for all } i \in \mathbb{N}\}$$

*is a closed, normal cone of the Banach space $\ell^p$. Therefore $(\ell^p, C)$ is an OBS.*

If we take $p = \infty$ and consider the Banach space $\ell^\infty$, with norm defined by $||(a_i)_{i \in \mathbb{N}}|| = \sup\{|x_i| : i \in \mathbb{N}\}$, then it can easily be shown that the set $C$ as defined in the above example, is a closed, normal cone in $\ell^\infty$. This means that $(\ell^\infty, C)$ is an OBS.

The following proposition shows how some of the most common OBAs and COBAs arise.

**Proposition 2.1.10.** *Let $X$ be an OBS with a closed cone $C'$ and $A = \mathcal{L}(X)$ the Banach algebra of all bounded linear operators on $X$, with norm $||T|| = \sup\{||Tx|| : ||x|| = 1\}$. Let $C = \{T \in \mathcal{L}(X) : Tx \geq 0$ for all $x \in X$ such that $x \geq 0\}$. Then $(A, C)$ is an OBA with $C$ closed.*

**Proof.** Clearly, $C$ is an algebra cone in $A$. To show that $C$ is closed, let $T \in cl(C)$. Then there is a sequence $(T_n)$ in $C$ such that $T_n \to T$ as $n \to \infty$. Therefore $T_n x \to Tx$ for $x \in X$. If $x \geq 0$, then $T_n x \in C'$ for all $n \in \mathbb{N}$. Since $C'$ is closed, $Tx \geq 0$. Thus $T \in C$, and $C$ is closed. $\square$

In view of example 2.1.9, an immediate application of proposition 2.1.10 is the following:

**Corollary 2.1.11.** *([52], example 2.2.8) Let $1 \leq p < \infty$ and consider the Banach algebra $\mathcal{L}(\ell^p)$. The set $C = \{T \in \mathcal{L}(\ell^p) : Tx \geq 0$ for all $x \in \ell^p$ such that $x \geq 0\}$ is a closed algebra cone of $A$. Therefore $(\mathcal{L}(\ell^p), C)$ is an OBA.*

Another application of proposition 2.1.10, together with other facts, is the following:

**Corollary 2.1.12.** *([51], p.499) Let $E$ be a Banach lattice and let $C = \{x \in E : x \geq 0\}$. If $K = \{T \in \mathcal{L}(E) : TC \subseteq C\}$, then $K$ is a closed, normal algebra cone of both $\mathcal{L}(E)$ and $\mathcal{L}^r(E)$. Therefore $(\mathcal{L}(E), K)$ and $(\mathcal{L}^r(E), K)$ are both OBAs.*

We give more examples of OBAs as well as examples of COBAs that are not OBAs.

**Example 2.1.13.** *Let $\mathbb{C}$ be the Banach algebra of all complex numbers with norm $||z|| = |z|$ and $\mathbb{R}^+$ the subset of $\mathbb{C}$ of all non-negative real numbers. Then $\mathbb{R}^+$ is a closed, normal algebra cone of $\mathbb{C}$. So $(\mathbb{C}, \mathbb{R}^+)$ is an OBA.*

Example 2.1.13 is easily proved considering properties of real numbers. The next example is a generalization of example 2.1.13.

**Example 2.1.14.** *Consider the Banach algebra $\mathbb{C}^n$ with norm $||a|| = \max\{|a_i| : i = 1, ..., n\}$. Define $C = \{a = (a_1, ..., a_n) \in \mathbb{C}^n : a_i \geq 0$ for $i = 1, ..., n\}$. Then $C$ is a closed, normal algebra cone of $\mathbb{C}^n$. Therefore $(\mathbb{C}^n, C)$ is an OBA.*

**Proof.** Since the operations on $C^n$ are componentwise addition, scalar multiplication and multiplication, it is clear that $C$ is an algebra cone of $\mathbb{C}^n$. Closedness of $C$ follows from corollary 1.6.5. By direct computation, we can easily establish normality of $C$. $\square$

**Example 2.1.15.** *Let $A = M_n(\mathbb{C})$ be the Banach algebra of all $n \times n$ complex matrices with norm*

$$||A|| = \max_j \sum_{k=1}^{n} |\alpha_{jk}|, \text{ where } A = (\alpha_{jk}) \in M_n(\mathbb{C}).$$

*Let $C$ be the subset of $M_n(\mathbb{C})$ consisting of matrices with only non-negative, real entries. Then $C$ is a closed, normal algebra cone of $M_n(\mathbb{C})$. Therefore $(M_n(\mathbb{C}), C)$ is an OBA.*

**Proof.** That $C$ is an algebra cone in $A$ follows from properties of matrices. Closedness of $C$ follows from corollary 1.6.6. Normality of $C$ can be established by direct computation. $\square$

Since the inverse of a positive real number is also positive, the algebra cones in examples 2.1.13 and 2.1.14 are also inverse-closed. If $C$ in example 2.1.15 is the set all diagonal matrices with only non-negative real entries, then $C$ is a closed, normal algebra cone of $M_n(\mathbb{C})$. Moreover, $C$ is inverse-closed.

**Example 2.1.16.** *Consider the Banach algebra $A = \ell^\infty$ with norm $||a|| = \sup\{|a_i| : i \in \mathbb{N}\}$, where $a = (a_1, a_2, ...) \in \ell^\infty$. Define $C = \{(a_1, a_2, ...) \in \ell^\infty : a_i \geq 0$ for $i \in \mathbb{N}\}$. Then $(A, C)$ is an OBA with $C$ closed and normal.*

**Proof.** Since the operations on $A$ are componentwise addition, scalar multiplication and multiplication, it is clear that $C$ is an algebra cone in $A$. Closedness of $C$ follows from corollary 1.6.7. By routine computation, normality of $C$ is easily established. □

The next example demonstrates how a simple algebra cone arises in an arbitrary Banach algebra.

**Example 2.1.17.** *([52], example 2.2.3) Let $A$ be a Banach algebra and $C = \{\alpha 1 : \alpha \geq 0\}$. Then $C$ is a closed, normal algebra cone of $A$. Therefore $(A, C)$ is an OBA.*

In an OBA, normality of the algebra cone implies monotonicity of the spectral radius (see [51], theorem 4.1). Therefore the spectral radius is monotone in the OBAs in corollary 2.1.12 through example 2.1.17. Also, considering how the cones were defined and the definition of multiplication on the Banach algebra, we find that the cones in example 2.1.16 and example 2.1.17 are inverse-closed.

The examples we have given so far are those of COBAs that are OBAs. The next four are examples of COBAs that are not OBAs.

**Example 2.1.18.** *Let $H$ be a Hilbert space and $A = \mathcal{L}(H)$. The subset $C = \{T \in A : T \geq 0\}$ of $A$ is a closed, normal algebra c-cone of $A$ and the spectral radius in $(A, C)$ is monotone. Hence $(A, C)$ is a COBA.*

**Proof.** Using theorem 1.4.1, it is clear that $C$ is an algebra $c$-cone in $A$. Closedness of $C$ follows from continuity of the inner product. Let $0 \leq S \leq T$. Then $\langle Tx, x\rangle \geq \langle Sx, x\rangle$ for all $x \in H$. It follows from proposition 1.4.2 that $||S|| = \sup\{|\langle Sx, x\rangle| : ||x|| = 1\} \leq \sup\{|\langle Tx, x\rangle| : ||x|| = 1\} = ||T||$, so that $C$ is normal. Since $0 \leq S \leq T$, we have that $S = S^*$ and $T = T^*$ by theorem 1.4.3, so that $r(S) = ||S||$ and $r(T) = ||T||$ by proposition 1.5.2. Monotonicity then follows from normality of $C$. □

**Example 2.1.19.** *Let $A$ be a $C^*$-algebra and let $C = \{a \in A : a = a^*$ and $\sigma(a) \subseteq [0, \infty)\}$. Then $C$ is a closed, normal algebra c-cone of $A$ and the spectral radius in $(A, C)$ is monotone. Hence $(A, C)$ is a COBA.*

**Proof.** By proposition 1.5.3, $C$ is closed under addition and positive scalar multiplication. Clearly, $1 \in C$. If $a, b \in C$ such that $ab = ba$, then $(ab)^* = b^*a^* = ab$ and $\sigma(ab) \subseteq [0, \infty)$ by proposition 1.1.5, so that $ab \in C$. This proves that $C$ is an algebra $c$-cone of $A$. By proposition 1.5.3, $C$ is closed. That $C$ is normal follows from proposition 1.5.5. To show that the spectral radius in $(A, C)$ is monotone, suppose that $a, b \in C$ such that $0 \leq a \leq b$ w.r.t. $C$. Then $r(a) = ||a||$ and $r(b) = ||b||$ by proposition 1.5.2, monotonicity follows from normality of $C$. □

Example 2.1.19 is a prototype of a COBA. If $A$ is not commutative, then $(A, C)$ is not an OBA. If $T$ is an operator on a Hilbert space $H$, then $\langle Tx, x\rangle \geq 0$ if and only if $T = T^*$ and $\sigma(T) \subseteq [0, \infty)$ by theorem 1.4.3. Consequently, example 2.1.18 is a special case of example 2.1.19.

While example 2.1.19 (and consequently example 2.1.18) was suggested by Harte in [31], the next two examples are attributed to the author.

**Example 2.1.20.** *Let* $A = M_2(\mathbb{C})$ *and* $C = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in A : a, b, c \geq 0 \right\}$. *Then* $C$ *is a closed, normal algebra c-cone of* $A$ *and the spectral radius in* $(A, C)$ *is monotone. Hence* $(A, C)$ *is a COBA.*

**Proof.** Clearly, $C$ is a closed algebra $c$-cone of $A$. From direct calculation we get that $||a|| \leq ||b||$ whenever $a, b \in A$ and $0 \leq a \leq b$ w.r.t. $C$. Thus $C$ is normal. To show that the spectral radius in $(A, C)$ is monotone, we first note that $C \subseteq \{a \in A : a = a^*\}$, where $a^*$ denotes the complex conjugate transpose of $a$. Since $a \mapsto a^*$ is an involution on $A$, it follows from proposition 1.5.2 that $r(a) = ||a||$ for all $a \in C$, so that by normality of $C$, the spectral radius in $(A, C)$ is monotone. □

In example 2.1.20 we noted that $C \subseteq \{a \in A : a = a^*\}$. However, the elements of $C$ in general satisfy $\sigma(a) \subseteq \mathbb{R}$, rather than $\sigma(a) \subseteq [0, \infty)$. So example 2.1.20 is different from example 2.1.19. Also, since in general, $a, b \in C$ implies $ab \in C$ only if $ab = ba$, $(A, C)$ is not an OBA.

We generalize example 2.1.20 to obtain an infinite dimensional Banach algebra, as follows:

**Example 2.1.21.** *Let* $A$ *and* $C$ *be as in example 2.1.20. Let* $\ell^\infty(A) = \{(a_1, a_2, ...) : a_i \in A$ *for all* $i \in \mathbb{N}\}$ *and* $\ell^\infty(C) = \{(c_1, c_2, ...) : c_i \in C$ *for all* $i \in \mathbb{N}\} \subseteq \ell^\infty(A)$. *Then* $\ell^\infty(A)$ *is a Banach algebra and* $\ell^\infty(C)$ *is a closed, normal algebra c-cone of* $\ell^\infty(A)$ *and the spectral radius in* $(\ell^\infty(A), \ell^\infty(C))$ *is monotone. Hence* $(\ell^\infty(A), \ell^\infty(C))$ *is a COBA.*

**Proof.** If we define addition, scalar multiplication and vector multiplication componentwise, and norm $||(a_1, a_2, ...)| = \sup_{i\in\mathbb{N}} ||a_i||$ (where $||a_i||$ is the norm of the matrix $a_i$ in $A$), it can be shown that $\ell^\infty(A)$ is a normed algebra with unit $(I_2, I_2, ...)$ ($I_2$ is the identity matrix on $A$). Completeness can be shown as for $\ell^\infty$. Using example 2.1.20, it can be shown by direct calculation that $\ell^\infty(C)$ is a closed algebra $c$-cone of $\ell^\infty(A)$ and that $C$ is normal in $A$, with $||a|| \leq ||b||$ whenever $a, b \in \ell^\infty(A)$ and $0 \leq a \leq b$ w.r.t. $\ell^\infty(C)$. We show that the spectral radius in $(\ell^\infty(A), \ell^\infty(C))$ is monotone. Let $a, b \in \ell^\infty(C)$ such that $0 \leq a \leq b$ w.r.t. $\ell^\infty(C)$. For any $x = (x_1, x_2, ...) \in \ell^\infty(A)$, we define $x^* = (x_1^*, x_2^*, ...)$, where $x_i^*$ is the complex conjugate transpose of $x_i$ in $A$. It can easily be verified that $x \mapsto x^*$ is an involution on $\ell^\infty(A)$. If $x \in \ell^\infty(C)$, we see from example 2.1.20 that $x = x^*$. Thus $a = a^*$ and $b = b^*$, so that $r(a) = ||a||$ and $r(b) = ||b||$ by proposition 1.5.2. Since $||a|| \leq ||b||$, we have that $r(a) \leq r(b)$. □

More examples of COBAs will be given in Section 2.3.

Positive elements in a COBA satisfy the following:

**Proposition 2.1.22.** *Let $(A, C)$ be a COBA and let $a, b \in A$. If $0 \leq a \leq b$ w.r.t. $C$ and $ab = ba$, then $0 \leq a^n \leq b^n$ for any $n \in \mathbb{N}$.*

**Proof.** We prove the result by mathematical induction. For $n = 1$, the inequality obviously holds. Suppose that $0 \leq a^k \leq b^k$ for $k > 1$. If $ab = ba$ then from $0 \leq a \leq b$ and the induction hypothesis, we have that $a^{k+1} = aa^k \leq ab^k \leq bb^k = b^{k+1}$, and the result follows. □

Since $ab = ba$ in proposition 2.1.22 is required only to get the product $ab$ to be positive, in an OBA the condition can be dropped. The next counter example shows however that in a COBA which is not an OBA, it cannot in general be dropped.

**Example 2.1.23.** *Let $A = M_2(\mathbb{C})$ and $C = \{a \in A : a = a^* \text{ and } \sigma(a) \subseteq [0, \infty)\}$, where $a^*$ denotes the complex conjugate transpose of $a$. Then $C$ is a closed, normal algebra $c$-cone of $A$. If $a = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \in A$ then $0 \leq a \leq b$ but $a^2 \leq b^2$ does not hold.*

**Proof.** That $C$ is a closed, normal algebra $c$-cone of $A$ follows from example 2.1.19. We see that $a = a^*$, $b = b^*$, $\sigma(a) = \{0, 2\}$, $\sigma(b) = \{\frac{3}{2} - \frac{\sqrt{5}}{2}, \frac{3}{2} + \frac{\sqrt{5}}{2}\}$ and $b - a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, so that $0 \leq a \leq b$. We have that $b^2 = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$ and $a^2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, so that $b^2 - a^2 = \begin{pmatrix} 3 & -1 \\ -1 & 0 \end{pmatrix}$. It follows that $\sigma(b^2 - a^2) = \{\frac{3}{2} - \frac{\sqrt{13}}{2}, \frac{3}{2} + \frac{\sqrt{13}}{2}\}$, so that $\sigma(b^2 - a^2) \subseteq [0, \infty)$ is not satisfied and $a^2 \leq b^2$ does not hold. We note that $ab = \begin{pmatrix} 3 & -2 \\ -3 & 2 \end{pmatrix}$ and $ba = \begin{pmatrix} 3 & -3 \\ -2 & 2 \end{pmatrix}$, so that $ab \neq ba$. □

## 2.2 Cones in different Banach algebras

In the previous section we established that an algebra $c$-cone in a Banach algebra may have the properties of being closed, inverse-closed, proper, $c$-normal (or normal) or that the spectral radius in the COBA may be $c$-monotone (or monotone). In this section we discuss algebra $c$-cones in relation to different Banach algebras and the properties of algebra $c$-cones that are preserved under homomorphisms.

If $A$ and $B$ are Banach algebras such that $1 \in B \subseteq A$ and $C$ is an algebra cone of $A$, then $C \cap B$ is an algebra cone of $B$ and under suitable conditions, properties of $C$ are inherited by $C \cap B$ ([51], p.492). For COBAs, we have the following corresponding result, which can be established by routine computation:

**Theorem 2.2.1.** *Let $A$ and $B$ be Banach algebras such that $1 \in B \subseteq A$. If $C$ is an algebra $c$-cone in $A$ then we have the following:*

*(i) $C \cap B$ is an algebra $c$-cone in $B$,*
*(ii) if $C$ is a proper algebra $c$-cone in $A$, then $C \cap B$ is a proper algebra $c$-cone in $B$,*

*(iii) if $B$ has finer norm than $A$ and if $C$ is closed in $A$, then $C \cap B$ is closed in $B$,*
*(iv) if $B$ is a closed subalgebra of $A$, then c-normality (normality) of $C$ in $A$ implies c-normality (normality) of $C \cap B$ in $B$.*

In ([42], p.134) it is remarked that if $\pi : A \to B$ is a homomorphism and $C$ is an algebra cone of $A$, then $\pi C$ is an algebra cone of $B$, and under certain conditions, properties of $C$ carry over to $\pi C$. For COBAs we have

**Proposition 2.2.2.** *Let $A$ and $B$ be Banach algebras and $\pi : A \to B$ a homomorphism. Then we have the following:*

*(i) if $\pi$ is injective and if $C$ is an algebra c-cone of $A$, then $\pi C$ is an algebra c-cone of $B$,*
*(ii) if $\pi$ is injective and $C$ is proper, then $\pi C$ is proper,*
*(iii) if $\pi$ is continuous and bijective, and if $C$ is closed, then $\pi C$ is closed.*

**Proof.** (i) Let $a_1, b_1 \in \pi C$ and let $\lambda \geq 0$. Then there exist $a, b \in C$ such that $\pi a = a_1$ and $\pi b = b_1$. Since $C$ is an algebra $c$-cone, $a + b, \lambda a \in C$ and if $ab = ba$ then $ab \in C$. Therefore $a_1 + b_1 = \pi a + \pi b = \pi(a + b) \in \pi C$ and $\lambda a_1 = \lambda \pi a = \pi(\lambda a) \in \pi C$. Suppose that $a_1 b_1 = b_1 a_1$. Then $\pi a \pi b = \pi b \pi a$, so that $\pi(ab) = \pi(ba)$. Since $\pi$ is injective, $ab = ba$. Therefore, $ab \in C$, so that $a_1 b_1 = \pi a \pi b = \pi(ab) \in \pi C$. Furthermore, since $\pi 1 = 1$, we have that $1 \in \pi C$. Hence $\pi C$ is an algebra $c$-cone of $B$.

(ii) If $C$ is proper then $C \cap -C = \{0\}$. Since $\pi$ is injective, $\pi C \cap -\pi C = \pi C \cap \pi(-C) = \pi(C \cap -C) = \pi(\{0\}) = \{0\}$.

(iii) If $\pi$ is continuous and bijective, then it is a closed map. Therefore if $C$ is closed, then $\pi C$ is closed. $\square$

In (i) above, since injectivity of $\pi$ is used only to guarantee the commutativity $ab = ba$, we find that if $C$ is an algebra cone of $A$ then injectivity of $\pi$ is not necessary for $\pi C$ to be an algebra cone of $B$.

## 2.3 Direct sums

Given any finite number of algebras, we can define their direct sum in the usual way. If these algebras are normed, then using their norms, we can define a norm on the direct sum. The direct sum then becomes a normed algebra. If the algebras are normed and ordered by cones, the cones on the algebras give rise to a cone on the direct sum. Then the direct sum becomes an ordered normed algebra. In this section we extend this idea to COBAs.

The following lemma will be required in proving results in this section.

**Lemma 2.3.1.** *Let $n \in \mathbb{N}$ and $A_i$ a Banach algebra, for $i = 1, ..., n$. Let $A = A_1 \oplus \cdots \oplus A_n$ and let $a = (a_1, ..., a_n) \in A$. Then $\sigma(a) = \bigcup_{i=1}^{n} \sigma(a_i)$.*

**Proof.** Let $\lambda \in \mathbb{C}$. Then $\lambda 1 - a$ is invertible in $A$ if and only if $\lambda 1_i - a_i$ in invertible in $A_i$, for all $i = 1, ..., n$, where $1_i$ is the unit element of $A_i$. This implies that $\lambda \notin \sigma(a)$ if and only if $\lambda \notin \sigma(a_i)$ for all $i = 1, ..., n$. Therefore $\sigma(a) = \bigcup_{i=1}^{n} \sigma(a_i)$. $\square$

**Proposition 2.3.2.** *Let $n \in \mathbb{N}$ and $A_i$ a COBA with an algebra c-cone $C_i$, for $i = 1, ..., n$. Let $A = A_1 \oplus \cdots \oplus A_n$ and $C = \{(c_1, ..., c_n) \in A : c_i \in C_i$ for $i = 1, ..., n\}$. Then $(A, C)$ is a COBA and if $C_i$ is closed (inverse-closed, c-normal or normal) for all $i = 1, ..., n$ then $C$ is closed (inverse-closed, c-normal or normal). If the spectral radius is c-monotone (or monotone) in $(A_i, C_i)$ for all $i = 1, ..., n$, then the spectral radius in $(A, C)$ is c-monotone (or monotone).*

**Proof.** By direct computation, it can easily be shown that $C$ is an algebra $c$-cone of $A$ and if $C_i$ is closed (inverse-closed) for $i = 1, ..., n$, then $C$ is closed (inverse-closed). To prove $c$-normality of $C$, suppose that $C_i$ is $c$-normal for all $i = 1, ..., n$. Let $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in A$ with $ab = ba$ and $0 \leq a \leq b$ w.r.t. $C$. Then $0 \leq a_i \leq b_i$ and $a_i b_i = b_i a_i$ for all $i = 1, ..., n$. If $C_i$ is $c$-normal for all $i = 1, ..., n$, we have that $||a_i||_i \leq \alpha_i ||b_i||_i$, where $\alpha_i > 0$ for all $i = 1, ..., n$. Let $||a|| = \max\{||a_1||_1, ..., ||a_n||_n\}$ and $||b|| = \max\{||b_1||_1, ..., ||b_n||_n\}$ and let $\alpha = \max\{\alpha_1, ..., \alpha_n\}$, where $|| \cdot ||_i$ is the norm on $A_i$. Then $\alpha > 0$ and $||a|| \leq \alpha ||b||$, so that $C$ is $c$-normal. In a similar way, we can show that if $C_i$ is normal in $A_i$ for all $i = 1, ..., n$, then $C$ is normal in $A$.

To prove the last part, let $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in A$ be such that $0 \leq a \leq b$ w.r.t. $C$ and $ab = ba$. Then $0 \leq a_i \leq b_i$ w.r.t. $C_i$ and $a_i b_i = b_i a_i$ for $i = 1, ..., n$. If the spectral radius is $c$-monotone in $(A_i, C_i)$ for $i = 1, ...., n$, we have that $r(a_i) \leq r(b_i)$ for $i = 1, ..., n$. It follows from lemma 2.3.1 that $r(a) \leq r(b)$. We apply a similar argument for monotonicity. $\square$

In view of example 2.1.15 and the OBA version of proposition 2.3.2 ([42], example 3.6), we get the following:

**Example 2.3.3.** *([42], example 3.7) Let $n, k_1, ..., k_n \in \mathbb{N}$ and $A = M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_n}(\mathbb{C})$. Let $C = \{(a_1, ..., a_n) \in A : a_i$ is a $k_i \times k_i$ matrix with only non-negative entries, for all $i = 1, ..., n\}$ and $C' = \{(a_1, ..., a_n) \in A : a_i$ is a diagonal $k_i \times k_i$ matrix with only non-negative entries, for all $i = 1, ..., n\}$. Then both $C$ and $C'$ are closed, normal algebra cones of $A$. Therefore $(A, C)$ and $(A, C')$ are OBAs.*

The following is an important example: it shows that the direct sum of any COBA and any OBA is a COBA. Since there are many examples of OBAs, some of which were given in Section 2.1, the result shows that there are plenty of examples of COBAs.

**Example 2.3.4.** *Let $A_1$ be a COBA with an algebra c-cone $C_1$ and let $A_2$ be an OBA with an algebra cone $C_2$. If $A = A_1 \oplus A_2$ and $C = \{(c_1, c_2) \in A : c_1 \in C_1, c_2 \in C_2\}$, then $(A, C)$ is a COBA. If $C_1$ is closed (c-normal or normal) and $C_2$ is closed (normal), then $C$ is closed (c-normal or normal). If the spectral radius in $(A_1, C_1)$ is c-monotone (or monotone) and the spectral radius in $(A_2, C_2)$ is monotone, then the spectral radius in $(A, C)$ is c-monotone (or monotone).*

**Proof**. By routine computations, we get that $C$ is an algebra $c$-cone of $A$ and if $C_1$ and $C_2$ are closed, then $C$ is closed. To show that $C$ is $c$-normal in $A$ if $C_1$ is $c$-normal in $A_1$ and $C_2$ is normal in $A_2$, suppose that $0 \leq a \leq b$ w.r.t $C$ and $ab = ba$. We have that $||a|| = ||(a_1, a_2)|| = \max\{||a_1||, ||a_2||\}$ and $||b|| = ||(b_1, b_2)|| = \max\{||b_1||, ||b_2||\}$. Since $ab = ba$, $a_1b_1 = b_1a_1$. It follows from $c$-normality of $C_1$ that there exists an $\alpha > 0$ such that $||a_1|| \leq \alpha||b_1||$. Since $C_2$ normal, there exists a $\beta > 0$ such that $||a_2|| \leq \beta||b_2||$. Therefore $||a|| \leq \gamma \max\{||b_1||, ||b_2||\} = \gamma||b||$, where $\gamma = \max\{\alpha, \beta\}$. Thus $C$ is $c$-normal. In a similar way we can prove that if $C_1$ is normal in $A_1$ and $C_2$ is normal in $A_2$ then $C$ is normal in $A$.

To show the last part, let $a = (a_1, a_2), b = (b_1, b_2) \in A$ such that $0 \leq a \leq b$ and $ab = ba$. Then $a_1b_1 = b_1a_1$ and $a_2b_2 = b_2a_2$. If the spectral radius in $(A_1, C_1)$ is $c$-monotone and in $(A_2, C_2)$ is monotone, we have that $r(a_1) \leq r(b_1)$ and $r(a_2) \leq r(b_2)$. It follows from lemma 2.3.1 that $r(a) \leq r(b)$. Similarly, if the spectral radius in $(A_1, C_1)$ and in $(A_2, C_2)$ is monotone, then the spectral radius in $(A, C)$ is monotone. $\square$

If in example 2.3.4 the spectral radius in the OBA $(A_2, C_2)$ is not monotone, then the spectral radius in the COBA $(A, C)$ is not monotone. We verify this in the next proposition.

**Proposition 2.3.5.** *If $(A_1, C_1)$ is a COBA and $(A_2, C_2)$ is an OBA in which the spectral radius is not monotone, then the spectral radius in the COBA $(A, C) = (A_1 \oplus A_2, C_1 \oplus C_2)$ is not monotone.*

**Proof.** By example 2.3.4, $(A, C)$ is a COBA. If $a_2, b_2 \in A_2$ such that $0 \leq a_2 \leq b_2$ w.r.t. $C_2$ and $r(a_2) > r(b_2)$, then the result follows by considering $(a_1, a_2)$ and $(b_1, b_2)$, where $a_1 = r(b_2)1$, $b_1 = \frac{1}{2}(r(a_2) + r(b_2))1 \in A_1$. $\square$

The next example illustrates proposition 2.3.5. It gives us an example of a COBA (which is not an OBA) where the spectral radius is not monotone.

**Example 2.3.6.** *Consider a non-commutative $C^*$-algebra $A_1$ ordered by the algebra c-cone $C_1 = \{a_1 \in A_1 : a_1 = a_1^* \text{ and } \sigma(a_1) \subseteq [0, \infty)\}$. Let $\mathcal{L}(\ell^2)$ denote the real Banach algebra of bounded linear operators on $\ell^2$ and $\mathbf{N} = \{0, 1, 2, ...\}$. Take $C = \{(x_k)_{k\in\mathbf{N}} \in \ell^2 : \sum_{k=0}^{n} x_k \geq 0$ for all $n \in \mathbf{N}\}$ and $K = \{T \in \mathcal{L}(\ell^2) : TC \subseteq C\}$. Let $A_2$ be the complex Banach algebra $\mathcal{L}(\ell^2)_{\mathbb{C}}$ and $C_2 = K$. Then $(A_1, C_1)$ is COBA and $(A_2, C_2)$ is an OBA in which the spectral radius in not monotone. Therefore the spectral radius in the COBA $(A, C) = (A_1 \oplus A_2, C_1 \oplus C_2)$ is not monotone.*

**Proof.** That $(A_1, C_1)$ is a COBA and $(A_2, C_2)$ is an OBA in which the spectral radius is not monotone follow from example 2.1.19 and ([51], example 4.3) respectively. By example 2.3.4, $(A, C)$ is a COBA. We show that the spectral radius in this COBA is not monotone. Let $e_0 = (1, 0, 0, ...), e_1 = (0, 1, 0, ...), ...$ and $S, T \in \mathcal{L}(\ell^2)$ be defined as follows: $S(x_k)_{k\in\mathbf{N}} = \sum_{n=0}^{\infty} x_n e_{2^n}$ and $T(x_k)_{k\in\mathbf{N}} = x_0e_1 + \sum_{n=2}^{\infty}(1/2^{j(n)})(x_{j(n)} + x_{j(n)+1})e_n$, where $j(n) \in \mathbf{N}$ is such that $2^{j(n)} + 1 \leq n \leq 2^{j(n)+1}$ for $n \geq 2$. Then by ([51], example 4.3), $0 \leq S \leq T$ in $(A_2, C_2)$, $r(S) > 0$ and $r(T) = 0$. It follows that $0 \leq (r(T)I, S) \leq (\frac{1}{2}r(S)I, T)$ in $(A, C)$, and

$$r((r(T)I, S) = \max\{r(T), r(S)\} = r(S) > \tfrac{1}{2}r(S) = \max\{\tfrac{1}{2}r(S), r(T)\} = r((\tfrac{1}{2}r(S)I, T)). \ \square$$

In the following proposition we prove that if $C_2$ in example 2.3.4 is not normal, then $C_1 \oplus C_2$ in the COBA $A_1 \oplus A_2$ is not normal.

**Proposition 2.3.7.** *If $(A_1, C_1)$ is a COBA and $(A_2, C_2)$ an OBA such that $C_2$ is not normal (not proper) in $A_2$ then $C_1 \oplus C_2$ is not normal (not proper) in the COBA $A_1 \oplus A_2$.*

**Proof.** Let $\alpha > 0$ and $a_2, b_2 \in A_2$ such that $0 \leq a_2 \leq b_2$ w.r.t. $C_2$, but $||a_2|| > \alpha||b_2||$. Take $r = \min\{r(a_2), r(b_2)\}$. Then it follows that $C_1 \oplus C_2$ is not normal in $A_1 \oplus A_2$ by considering $(a_1, a_2)$ and $(b_1, b_2)$, where $a_1 = r1 = b_1$. It is trivial to show that if $C_2$ is not proper in $A_2$, then $C_1 \oplus C_2$ is not proper in $A_1 \oplus A_2$. □

The next example illustrates the previous proposition. It gives a COBA (which is not an OBA) in which the algebra $c$-cone is not normal.

**Example 2.3.8.** *Consider a non-commutative $C^*$-algebra $A_1$ ordered by the algebra $c$-cone $C_1 = \{a_1 \in A_1 : a_1 = a_1^* \text{ and } \sigma(a_1) \subseteq [0, \infty)\}$. Let $E$ be the Banach lattice $\ell^1 \oplus L^2[0,1] \oplus \ell^\infty$, $C$ be the positive cone in $E$ and let $K = \{T \in \mathcal{L}^r(E) : TC \subseteq C\}$. Also let $\pi : \mathcal{L}^r(E) \to \mathcal{L}^r(E)/\mathcal{K}^r(E)$ be the canonical homomorphism. Then $(A_1, C_1)$ is a COBA and $(A_2, C_2)$ is an OBA such that $C_2$ is not normal in $A_2$, where $A_2 = \mathcal{L}^r(E)/\mathcal{K}^r(E)$ and $C_2 = \pi K$. Therefore $(A_1 \oplus A_2, C_1 \oplus C_2)$ is COBA such that $C_1 \oplus C_2$ is not normal in $A_1 \oplus A_2$.*

**Proof.** By example 2.1.19, $(A_1, C_1)$ is a COBA and by ([51], example 4.2), $(A_2, C_2)$ is an OBA such that $C_2$ is not normal in $A_2$. That $(A_1 \oplus A_2, C_1 \oplus C_2)$ is a COBA follows from example 2.3.4. In ([51], example 4.2), it is shown that $C_2$ is not even proper in $A_2$. This implies that $C_1 \oplus C_2$ is not proper in $A_1 \oplus A_2$, and so it not normal. □

## 2.4 Quotient algebras

Let $A$ and $B$ be Banach algebras and $\pi : A \to B$ a homomorphism. If $C$ is an algebra cone of $A$, then $\pi C = \{\pi c : c \in C\}$ is an algebra cone in $B$. In particular, if $F$ is a closed ideal in the OBA $(A, C)$ and if $\pi : A \to A/F$ is the canonical homomorphism, then $\pi C$ is an algebra cone of $A/F$, although normality or closedness of $\pi C$ cannot be deduced from the corresponding properties of $C$ ([51], p.492). In our current setting, if $C$ is an algebra $c$-cone of $A$, then $\pi C$ is in general not an algebra $c$-cone of $A$ (see proposition 2.2.2), since $\pi$ is not injective and so the multiplication property for an algebra $c$-cone is not satisfied.

In order to work with quotient algebras for COBAs, we will need the following definition and results, which are due to the author.

**Definition 2.4.1.** *Let $(A, C)$ be a COBA. A subset $M$ of $A$ is called a maximal positive commutative set (MPCS) in $A$ if $M$ is a commutative subset of $C$ and it is not a proper subset of another commutative subset of $C$.*

**Theorem 2.4.2.** *Let $A$ be a COBA with an algebra $c$-cone $C$ and let $M$ be an MPCS in $A$. Suppose that $F$ is a closed ideal of $A$ and $\pi : A \to A/F$ is the canonical homomorphism. Then we have the following:*

*(i) $M$ is an algebra cone of $A$,*
*(ii) $M$ is contained in a closed commutative subalgebra of $A$ containing the unit of $A$,*
*(iii) if $C$ is a closed (proper, c-normal or normal) cone of $A$, then $M$ is a closed (proper, normal) cone of $A$,*
*(iv) $\pi M$ is an algebra cone of $A/F$.*

**Proof.** (i) Let $a, b \in M$ and $\lambda \geq 0$. Then for any $c \in M$, we have that $c(a+b) = (a+b)c$, $c(\lambda a) = (\lambda a)c$ and $c(ab) = (ab)c$. Now since $M \subseteq C$ and $C$ is an algebra $c$-cone, $a+b, \lambda a, ab \in C$. Since also 1 commutes with every element of $A$, all the elements of the set $\{a+b, \lambda a, ab, 1\}$ commute with one another and with all the elements of $M$. Therefore $M \cup \{a+b, \lambda a, ab, 1\}$ is a commutative subset of $C$. Maximality of $M$ then implies that $M = M \cup \{a+b, \lambda a, ab, 1\}$, so that $a+b, \lambda a, ab, 1 \in M$. Hence $M$ is an algebra cone of $A$.

(ii) Follows from proposition 1.1.1.

(iii) Let $(a_n)$ be a sequence in $M$ and $a \in A$ such that $a_n \to a$ as $n \to \infty$. Since $M \subseteq C$ and $C$ is closed, $a \in C$. Now for any $b \in M$, we have that $ba_n = a_n b$ for all $n \in \mathbb{N}$. This implies that $ab = ba$, so that $M \cup \{a\}$ is a commutative subset of $C$. Maximality of $M$ then implies that $M = M \cup \{a\}$, so that $a \in M$. Hence $M$ is closed. If $C$ is proper, it follows from $M \subseteq C$ and $-M \subseteq -C$ that $M \cap -M \subseteq C \cap -C = \{0\}$, so that $M$ is proper. If $C$ is $c$-normal (or normal), then normality of $M$ follows from the fact that the norm of an element of $C$ does not change if the element is considered as a member of $M$.

(iv) Using the fact that $1 \in M$ and if $a+F, b+F \in \pi M$, then there exist $a_1, b_1 \in M$ such that $a+F = a_1+F$ and $b+F = b_1+F$, it can easily be shown that $\pi M$ is an algebra cone of $A/F$. $\square$

Closedness or $c$-normality of $\pi M$ can not be deduced from the corresponding properties of $M$.

The next result shows that every positive element in an algebra $c$-cone is contained in an MPCS.

**Proposition 2.4.3.** *Let $A$ be a COBA with an algebra c-cone $C$ and let $a \in C$. Then there exists an MPCS $M$ in $A$ such that $a \in M$.*

**Proof.** Let $a \in C$. We show that $a$ is contained in a commutative subset of $C$. Since 0 and 1 are in $C$ and they commute with every element of $A$, we have that $\{0, 1, a\}$ is a commutative subset of $C$ containing $a$. Let $\mathcal{P} = \{M : M \text{ is a commutative subset of } C \text{ containing } a\}$. Then $\mathcal{P}$ is nonempty. We show that $\mathcal{P}$ contains a maximal element. Consider the inclusion relation $\subseteq$ on $\mathcal{P}$. By definition, the relation $\subseteq$ has the reflexive, antisymmetry and transitive properties on $\mathcal{P}$. Now let $\mathcal{P}_0 = \{M_\alpha \in \mathcal{P} : \alpha \in I\}$ be a chain in $\mathcal{P}$, where $I$ is an indexing set. We show that $\mathcal{P}_0$ has an upper bound. Define $M^* = \bigcup_{\alpha \in I} M_\alpha$. Then $M_\alpha \subseteq M^*$ for all $\alpha \in I$. Therefore $M^*$ is an upper bound for $\mathcal{P}_0$. By Zorn's lemma, $\mathcal{P}$ has a maximal element. $\square$

We end this chapter with the following observation: When we defined COBAs, it appeared that $c$-normality and $c$-monotonicity are the natural COBA analogues of normality and monotonicity of OBAs. However, in all the examples known to us so far where the algebra $c$-cone is $c$-normal (or the spectral radius $c$-monotone), it is in fact normal (or the spectral radius is monotone). As we will see in the rest of the document, most results are still obtainable under the weaker assumptions of $c$-normality and $c$-monotonicity, although in certain cases we have to make a corresponding commutativity assumption. It is for this reason that in the remaining chapters, in COBAs, we will generally assume $c$-normality or $c$-monotonicity, rather than normality or monotonicity.

# Chapter 3

# Cones and spectral theory

In the previous chapter we saw that algebra $c$-cones may have the properties of being closed, proper, $c$-normal (or normal), inverse-closed and the spectral radius $c$-monotone (or monotone). In this chapter we will give results in COBAs that do not rely on the property of the spectral radius being $c$-monotone (or monotone). As we will see, these results include the important Krein-Rutman thorem, which is discussed in the first section of the chapter.

## 3.1 Krein-Rutman theorem

Let $a$ be a non-zero element of a Banach algebra $A$. A point $\lambda$ in $\sigma(a)$ is called an *eigenvalue* of $a$ if there exists a non-zero element $u$ in $A$ such that $au = \lambda u$ or $ua = \lambda u$. The element $u$ is called the *eigenvector* corresponding to $\lambda$.

This section deals with the Krein-Rutman theorem, which generally describes conditions under which the spectral radius of a positive element will be an eigenvalue of that element, with positive corresponding eigenvector. The original result was proved in 1948 by M.G. Krein and M.A. Rutman in the operator-theoretic context (cf. [36]). For more on the Krein-Rutman theorem see [57], [60]. In [47], S. Mouton and H. Raubenheimer proved several versions of the result in the OBA setting. Two of their results use purely algebraic methods. One of the two results is the basic form from which the other one is obtained. In this section we adapt the proof and give the basic form of the Krein-Rutman theorem in the COBA setting. We will defer the other version of the Krein-Rutman theorem to Chapter 4, as it relies on monotonicity of the spectral radius.

Before proceeding we first establish some notation. Let $A$ be a Banach algebra and $a \in A$. If $\lambda_0$ is a pole of order $k$ of the resolvent $R(\lambda, a)$ of $a$, then $R(\lambda, a)$ has a Laurent series expansion in a deleted neighbourhood of $\lambda_0$. In this section, $a_{-k}$ will denote the coefficient of $(\lambda - \lambda_0)^{-k}$ in the Laurent series expansion of $R(\lambda, a)$.

The basic form of the Krein-Rutman theorem in OBAs is ([47], theorem 3.2). Before we prove its COBA version, we first note that in the proof of ([47], theorem 3.2), the ordering in the Banach algebra is used only to get the eigenvector $u$ to be positive. In the general setting of Banach algebras, we have the following result:

**Theorem 3.1.1.** *([43], theorem 3.1) Let $A$ be a Banach algebra and let $0 \neq a \in A$. If $\lambda_0$ is a pole of order $k$ of the resolvent of $a$, then $\lambda_0$ is an eigenvalue of $a$ with corresponding eigenvector $a_{-k}$. Moreover, $aa_{-k} = a_{-k}a$ and $aa_{-k}a = {\lambda_0}^2 a_{-k}$.*

With theorem 3.1.1, we prove the basic form of the Krein-Rutman theorem, which is the COBA version of ([47], theorem 3.2).

**Theorem 3.1.2.** *Let $A$ be a COBA with a closed algebra c-cone $C$ and let $0 \neq a \in C$ such that $r(a) > 0$. If $r(a)$ is a pole of the resolvent of $a$, then there exists $0 \neq u \in C$ such that $au = ua = r(a)u$ and $aua = r(a)^2 u$.*

**Proof.** By theorem 3.1.1, we have that $r(a)$ is an eigenvalue of $a$ with corresponding eigenvector $a_{-k}$ that satisfies $aa_{-k} = a_{-k}a$ and $aa_{-k}a = r(a)^2 a_{-k}$. Since $r(a)$ is a pole of order $k$ of the resolvent of $a$, we have that $R(\lambda, a) = \frac{a_{-k}}{(\lambda - r(a))^k} + \frac{a_{-k+1}}{(\lambda - r(a))^{k-1}} + \cdots + a_0 + a_1(\lambda - r(a)) + \cdots$ (where $a_i \in A$ for $i = -k, -k+1, \ldots$) in a deleted neighbourhood of $r(a)$. Multiplying both sides of the Laurent expansion by $(\lambda - r(a))^k$ and then taking limits as $\lambda \to r(a)^+$, we have that $a_{-k} = \lim_{\lambda \to r(a)^+} (\lambda - r(a))^k R(\lambda, a)$. Since $C$ is a closed algebra $c$-cone and $a \in C$, from the Neumann series $R(\lambda, a) = \sum_{j=i}^{\infty} \frac{a^j}{\lambda^{j+1}}$ $(\lambda > r(a))$ for $R(\lambda, a)$ and the fact all powers $a^j$ are in $C$, it follows that $R(\lambda, a) \in C$. Hence $a_{-k} \in C$. Taking $u = a_{-k}$, the result follows. □

Note that the proof of theorem 3.1.2 is the same as in the OBA case; the only multiplication of positive elements involved is taking powers.

From the proof of theorem 3.1.2 we can deduce the following important result, which will be used in Chapter 5 in connection with an ergodicity problem.

**Proposition 3.1.3.** *Let $A$ be a COBA with a closed algebra c-cone $C$ and let $0 \neq a \in C$ such that $r(a) > 0$. If $r(a)$ is a simple pole of the resolvent of $a$, then $p = p(r(a), a) \in C$, $ap = pa = r(a)p$ and $apa = r(a)^2 p$.*

The following result is a consequence of theorem 3.1.1. It provides a spectral characterization of the rank one idempotents. We will refer to this result in Chapter 4.

**Proposition 3.1.4.** *Let $A$ be a semiprime Banach algebra and let $x \in \mathcal{F}_1(A)$. Then $x$ is an idempotent if and only if $r(x) = 1$ and $r(x)$ is a pole of the resolvent of $x$.*

**Proof.** Since $x \in \mathcal{F}_1$, we have that $x \neq 0$. Thus if $x$ is an idempotent, then $r(x) = 1$. It is well known that $p(e, 1) = e$ for every non zero idempotent $e$. Therefore $p(x, r(x)) = x$, which implies that $r(x)$ is a simple pole of the resolvent of $x$. Conversely, suppose that $r(x) = 1$ and $r(x)$ is a pole of the resolvent of $x$. Then by theorem 3.1.1, there exists a $0 \neq u \in A$ such that $xu = ux = u$. Therefore $xux = u$. Since $x \in \mathcal{F}_1(A)$, there exists $0 \neq \lambda \in \mathbb{C}$ such that $xux = \lambda x$. From $xux = u$ it follows that $u = \lambda x$. Using $ux = u$ we obtain that $\lambda x^2 = \lambda x$, so that $x^2 = x$. □

Note: it is well known that if $x$ is a rank one element which is not quasinilpotent, then $\mathrm{tr}(x) \neq 0$. In this case it can be shown by direct calculation that $\mathrm{tr}(x)^{-1}x$ is an idempotent.

## 3.2 Positive elements and analytic functions

Let $A$ be a Banach algebra, $a \in A$ and let $f$ be a complex valued function analytic on a neighbourhood $\Omega$ of the spectrum $\sigma(a)$ of $a$. Suppose that $C$ is an algebra $c$-cone of $A$. We consider the problem of finding conditions under which $a \in C$ implies $f(a) \in C$. This problem was investigated for OBAs in [42]; we will extend the results to COBAs. The proofs of the results are similar to the proofs of the corresponding OBA results. We include them for the sake of completeness since they are short.

For polynomial and exponential functions, we have proposition 3.2.1. Note that in this result, we define $e^a = \sum_{n=0}^{\infty} \frac{1}{n!} a^n$ ([15], definition 1, p.38).

**Proposition 3.2.1.** *Let $A$ be a COBA with an algebra $c$-cone $C$ and let $a \in C$.*

*(i) If $p(\lambda) = \alpha_n \lambda^n + \cdots + \alpha_1 \lambda + \alpha_0$ with $\alpha_n, ..., \alpha_0$ real and positive, then $p(a) \in C$.*
*(ii) If $C$ is closed and if $f(\lambda) = e^{\lambda}$, then $f(a) \in C$.*

**Proof.** Since $a \in C$, the multiplicative property of $C$ implies that $a^n \in C$ for all $n \in \mathbb{N}$. Since $C$ is closed under addition and positive scalar multiplication, we obtain (i) and since in addition $C$ is a closed subset of $A$, we obtain (ii). $\square$

**Corollary 3.2.2.** *Let $(A, C)$ be a COBA with $C$ inverse-closed and let $a \in C$. Let $p(\lambda) = \alpha_n \lambda^n + \cdots + \alpha_1 \lambda + \alpha_0$ and $q(\lambda) = \beta_m \lambda^m + \cdots + \beta_1 \lambda + \beta_0$ with $\alpha_n, ..., \alpha_0, \beta_m, ..., \beta_0$ real and positive. Suppose that $q(\lambda)$ has no zeroes in $\sigma(a)$ and let $s(\lambda) = \frac{p(\lambda)}{q(\lambda)}$. Then $s(a) \in C$.*

**Proof.** From proposition 3.2.1 we obtain that $p(a), q(a) \in C$. Now, the spectral mapping theorem implies that $q(a)$ is invertible. Since $C$ is inverse-closed, this implies that $(q(a))^{-1} \in C$. By lemma 1.1.6, $s(a) = p(a)(q(a))^{-1}$. Since lemma 1.1.6 implies that $p(a)(q(a))^{-1} = (q(a))^{-1}p(a)$, we obtain that $s(a) \in C$. $\square$

For the resolvent function, we have:

**Proposition 3.2.3.** *Let $(A, C)$ be a COBA with $C$ closed and let $a \in C$. If $\lambda > r(a)$, then $(\lambda 1 - a)^{-1} \in C$.*

**Proof.** Since $\lambda > r(a)$, the resolvent of $a$ has a Neumann series representation $(\lambda 1 - a)^{-1} = \frac{1}{\lambda}\left(\sum_{n=0}^{\infty} (\frac{a}{\lambda})^n\right)$. Since $C$ is an algebra $c$-cone and $a \in C$, all the powers $a^n$ of $a$ are in $C$. Also since $\lambda > r(a) \geq 0$, we have that $(\frac{a}{\lambda})^n \in C$ for all $n \in \mathbb{N}$. Since $C$ is closed, this implies that $\frac{1}{\lambda}\left(\sum_{n=0}^{\infty} (\frac{a}{\lambda})^n\right) \in C$. Hence $(\lambda 1 - a)^{-1} \in C$. $\square$

The original OBA results corresponding to proposition 3.2.1, corollary 3.2.2 and proposition 3.2.3 are ([42], proposition 4.10, 4.20, 4.6) respectively.

Let $A$ be a Banach algebra. A subset $I$ of $A$ is called a *multiplicative ideal* of $A$ if $IA \subseteq I$ and $AI \subseteq I$. Recalling that $I^c$ denotes the commutant of the set $I$, we prove the following corollary, which is a consequence of proposition 3.2.3.

**Corollary 3.2.4.** *Let $(A, C)$ be a COBA with $C$ closed and let $I$ be a non-trivial multiplicative ideal in $A$. For every element $a$ in $A$, there exists $1 \neq c \in A$ such that $ac - a \in I$. Moreover, if $a \in C \cap I^c$ and $I \cap C \neq \{0\}$, then $ac - a \in C$.*

**Proof.** Let $0 \neq b \in I$ and $\lambda \notin \sigma(b)$. Then $(\lambda 1 - b)^{-1}$ exists. Take $c = \lambda(\lambda 1 - b)^{-1}$. Since $b \neq 0$, we have that $c \neq 1$. Now by ([13], (3), p.929), with $p = 1$, we have that $ac - a = a\lambda(\lambda 1 - b)^{-1} - a = a(1 + \frac{b}{\lambda} \cdot \lambda(\lambda 1 - b)^{-1}) - a = a(\lambda 1 - b)^{-1} b \in I$. This proves the first part.

To prove the second part, let $a \in C \cap I^c$. If $I \cap C \neq \{0\}$, take $0 \neq b \in I \cap C$ with $\lambda \notin \sigma(b)$ such that $\lambda > r(b)$. Then with $c$ defined as before, $ac - a = a(\lambda 1 - b)^{-1} b$. Since $a \in I^c$ and $b \in I$, we have that $a(\lambda 1 - b) = (\lambda 1 - b)a$ and consequently, $a(\lambda 1 - b)^{-1} = (\lambda 1 - b)^{-1} a$. From $ac - a = a(\lambda 1 - b)^{-1} b$ and proposition 3.2.3, it follows that $ac - a \in C$. □

The original OBA result corresponding to corollary 3.2.4 is ([13], theorem 2.1).

In theorem 3.2.6, we give a generalization of proposition 3.2.1. In order to prove it, the following lemma will be required.

**Lemma 3.2.5.** *Let $A$ be a Banach algebra and $a \in A$ such that $r(a)$ is a pole of order $k$ of the resolvent of $a$. Suppose that $f$ is a complex valued function, analytic at least on an open disk of the form $D(r(a), R)$. Let $g(\lambda) = f(\lambda)(\lambda 1 - a)^{-1}$ and let $a_n$ denote the coefficient of $(\lambda - r(a))^n$ in the Laurent series expansion of $g$ around $r(a)$ for all $n \in \mathbb{Z}$.*

*(i) If $f(r(a)) = 0$ and the order of $f$ at $r(a)$ is $k$, then $a_{-1} = 0$.*

*Moreover, if $(A, C)$ is a COBA with $C$ closed, $a \in C$ and $f(\lambda) > 0$ for all $\lambda$ in the real interval $(r(a), r(a) + R)$, then*
*(ii) if $f(r(a)) > 0$ then $a_{-k} \in C$,*
*(iii) if $f(r(a)) = 0$ and the order of $f$ at $r(a)$ is $k - 1$, then $a_{-1} \in C$.*

**Proof.** (i) This is ([42], theorem 4.11 (1)).

(ii) If $f(r(a)) > 0$, then the order of $g$ at $r(a)$ is $-k$. Therefore $a_{-k} = \lim_{\lambda \to r(a)} (\lambda - r(a))^k g(\lambda)$. Restricting $\lambda$ to the interval $(r(a), r(a) + R)$, we have that $a_{-k} = \lim_{\lambda \to r(a)^+} (\lambda - r(a))^k f(\lambda)(\lambda 1 - a)^{-1}$. Since $C$ is closed, it follows from proposition 3.2.3 and the assumption on $f$ that $a_{-k} \in C$.

(iii) Suppose that $f(r(a)) = 0$ and the order of $f$ at $r(a)$ is $k - 1$. Then the order of $g$ at $r(a)$ is $-1$. Therefore $a_{-1} = \lim_{\lambda \to r(a)} (\lambda - r(a)) g(\lambda) = \lim_{\lambda \to r(a)^+} (\lambda - r(a)) f(\lambda)(\lambda 1 - a)^{-1}$. Proposition 3.2.3 and the assumptions on $f$ then imply that $a_{-1} \in C$. □

**Theorem 3.2.6.** *Let $A$ be a Banach algebra and $a \in A$ such that $\sigma(a) = \{\lambda_1, ..., \lambda_m\}$ $(m \geq 1)$ where $\lambda_1 = r(a)$ and $\lambda_j$ is a pole of order $k_j$ of the resolvent of $a$ $(j = 1, ..., m)$. Let $f$ be any complex valued function, analytic at least on a neighbourhood of $\sigma(a)$, such that $f$ has a zero of order $k_j$ at $\lambda_j$ $(j = 2, ..., m)$.*

*(i) If $f(r(a)) = 0$ and the order of $f$ at $r(a)$ is $k_1$, then $f(a) = 0$.*

*In addition, suppose that $(A, C)$ is a COBA with $C$ closed, $a \in C$ and $f(\lambda) > 0$ for all $\lambda$ in a real interval of the form $(r(a), r(a) + R)$. Then*

*(ii) if $f(r(a)) > 0$ and $k_1 = 1$, then $f(a) \in C$,*
*(iii) if $f(r(a)) = 0$ and the order of $f$ at $r(a)$ is $k_1 - 1$, then $f(a) \in C$.*

**Proof.** From the proof of ([42], theorem 4.14), we have that $f(a) = a_{-1}$. From lemma 3.2.5, we obtain the results. $\square$

Lemma 3.2.5 and theorem 3.2.6 were originally proved in the OBA case in ([42], theorem 4.11, theorem 4.14) respectively.

By application of theorem 3.2.6, COBA analogues of ([42], corollary 4.16, 4.17, 4.18, 4.19) can be obtained.

We prove theorem 3.2.8 about invertible positive elements. The following lemma will be needed:

**Lemma 3.2.7.** *Let $(A, C)$ be a COBA with $a$ and $b$ invertible elements of $A$ such that $0 \leq a \leq b$, $ab = ba$ and $a^{-1}, b^{-1} \geq 0$. Then $0 \leq b^{-1} \leq a^{-1}$.*

**Proof.** Suppose that $0 \leq a \leq b$ and $ab = ba$. Then $0 \leq a^{-1}a \leq a^{-1}b$, that is, $0 \leq 1 \leq a^{-1}b$. Therefore $0 \leq b^{-1} \leq a^{-1}bb^{-1} = a^{-1}$. $\square$

In a $C^*$-algebra, lemma 3.2.7 holds without the assumptions $ab = ba$ and $a^{-1}, b^{-1} \geq 0$, if $a \in C$ (see [22], p.249).

**Theorem 3.2.8.** *Let $(A, C)$ be a COBA with $C$ closed and inverse-closed. If $a \in C$ and $a$ is invertible, then*

*(i) $a \geq \alpha 1$ for all $\alpha \geq 0$, where $\alpha < \delta(a)$,*
*(ii) $a \leq \beta 1$ for all $\beta > r(a)$.*

**Proof.** (i) If $0 < \alpha < \delta(a)$, then $\frac{1}{\delta(a)} < \frac{1}{\alpha}$. Therefore $\frac{1}{\alpha} > r(a^{-1})$. It follows from proposition 3.2.3 that $((\frac{1}{\alpha})1 - a^{-1})^{-1} \geq 0$. Since $C$ is inverse-closed, $(\frac{1}{\alpha})1 - a^{-1} \geq 0$, so that $a^{-1} \leq (\frac{1}{\alpha})1$. Lemma 3.2.7 then implies that $a \geq \alpha 1$.

(ii) Suppose that $\beta > r(a)$. Then proposition 3.2.3 implies that $(\beta 1 - a)^{-1} \geq 0$. Since $C$ is inverse-closed, $\beta 1 - a \geq 0$, so that $a \leq \beta 1$. $\square$

The last two results in the OBA case are also due to S. Mouton ([42], lemma 4.21, theorem 4.22). Note also that if $A$ in theorem 3.2.8 is a $C^*$-algebra ordered by $C = \{a \in A : a = a^*$ and $\sigma(a) \subseteq [0, \infty)\}$, then the result is trivial and holds without the assumptions that $C$ is inverse-closed and $a$ is invertible.

## 3.3 Unit spectrum I

Let $a$ be an element of a Banach algebra such that $\sigma(a) = \{1\}$. A problem that naturally arises is that of determining when $a$ is necessarily the unit element. We refer to this problem as the problem of unit spectrum. The problem of unit spectrum has been investigated in the operator-theoretic setting in for instance [28], [40] and [34]. In OBAs, this problem has been studied in [16]. In this section we will generalize these results to COBAs. We will also obtain COBA counterparts of some of the results in [34]. The proofs of all the results arising from [16] are verbatim those of the corresponding OBA results, since every COBA has the property that all powers $a^k$ of a positive element $a$ are positive. For this reason, proofs of results arising from [16] will generally be omitted.

An element of a Banach algebra $a$ is said to be *Cesáro bounded* if there exists a $D > 0$ such that $||\frac{1}{n+1}\sum_{k=0}^{n} a^k|| \leq D$ for all $k \in \mathbb{N}$. If there is a $D > 0$ such that $||(1-\theta)\sum_{k=0}^{\infty} \theta^k a^k|| \leq D$ for all $\theta \in (0, 1)$, then $a$ is said to be *Abel bounded*. If $a$ is invertible and both $a$ and $a^{-1}$ are Abel bounded, then $a$ is said to be *doubly Abel bounded*. Let $N \in \mathbb{N}$ be fixed. If there exists a $D > 0$ such that $||(1-\theta)^N \sum_{k=0}^{\infty} \theta^k a^k|| \leq D$ for all $\theta \in (0, 1)$, then $a$ is said to be *$(N)$-Abel bounded*. If $a$ is invertible and both $a$ and $a^{-1}$ are $(N)$-Abel bounded, then $a$ is said to be *doubly $(N)$-Abel bounded*.

Our first result for the problem of unit spectrum is corollary 3.3.2. To obtain this result, we need theorem 3.3.1.

**Theorem 3.3.1.** *Let $(A, C)$ be a COBA with $C$ closed and c-normal. If $a \in C$ and $a$ is Abel bounded, then $a$ is Cesáro bounded.*

Theorem 3.3.1 is a COBA version of ([16], theorem 2.1).

Using theorem 3.3.1 and ([40], theorem 2), we can establish corollary 3.3.2.

**Corollary 3.3.2.** *Let $(A, C)$ be a COBA with $C$ closed and c-normal. If $\sigma(a) = \{1\}$, $a, a^{-1} \in C$ and $a$ is doubly Abel bounded, then $a = 1$.*

Corollary 3.3.2 is the COBA version of the original OBA version ([16], corollary 2.2).

If we replace Abel boundedness of $a$ in theorem 3.3.1 with the weaker condition of $(N)$-Abel boundedness, we obtain theorem 3.3.3. Its original OBA counterpart is ([16], theorem 3.2).

**Theorem 3.3.3.** *Let $(A, C)$ be a COBA with $C$ closed and c-normal. If $a \in C$ is $(N)$-Abel bounded, then $||\frac{1}{n+1}\sum_{k=0}^{n} a^k|| = o(n^N)$ as $n \to \infty$.*

In ([24], theorem 2) it is proved that if $a$ is an element of a Banach algebra such that $\sigma(a) = \{1\}$, $||\frac{1}{n+1}\sum_{k=0}^{n} a^k|| = o(n^p)$ as $n \to \infty$ and $||\frac{1}{n+1}\sum_{k=0}^{n}(a^{-1})^k|| = o(n^q)$ as $n \to \infty$ for some $p, q \in \mathbb{N}$ then $(a-1)^s = 0$, where $s = \min(p, q)$. An immediate consequence of theorem 3.3.3 and this result is the following corollary. Its original OBA version is ([16], corollary 3.3).

**Corollary 3.3.4.** *Let $(A, C)$ be a COBA with $C$ closed and c-normal. Let $a \in A$ such that $\sigma(a) = \{1\}$. If $a, a^{-1} \in C$ and if $a$ is doubly $(N)$-Abel bounded, then $(a-1)^N = 0$.*

We present our next result regarding the problem of unit spectrum. It is the COBA version of ([16], theorem 2.7).

**Theorem 3.3.5.** *Let $(A, C)$ be a COBA with $C$ closed and proper. If $a \in A$ such that $\sigma(a) = \{1\}$, then $a = 1$ if and only if $a^L$ is Abel bounded and $a^N \geq 1$ for some $L, N \in \mathbb{N}$.*

Following along the lines of ([16], theorem 3.1), we can prove the following result, which may be seen as a generalization of theorem 3.3.5.

**Theorem 3.3.6.** *Let $(A, C)$ be a COBA with $C$ closed and proper. Let $a \in A$ such that $\sigma(a) = \{1\}$. If $a \geq 1$ and if $a$ is $(N)$-Abel bounded, then $a = 1$.*

In corollary 3.3.2 and theorem 3.3.5 we had to assume that the element $a$ is Abel bounded or doubly Abel bounded. In the next theorem, which is a COBA version of ([16], theorem 4.1), we obtain the result under conditions that do not use any of these assumptions.

**Theorem 3.3.7.** *Let $(A, C)$ be a COBA with $C$ closed, proper and inverse closed. If $a \in A$ is such that $\sigma(a) = \{1\}$ and if $a^N \in C$ for some $N \in \mathbb{N}$ then $a = 1$.*

We establish the following result, which is a COBA counterpart of ([16], proposition 4.2). It will be used in Chapter 4 in proving a result that provides a criterion for a positive element in a COBA to be Abel bounded and Chapter 5 in connection with ergodicity.

**Proposition 3.3.8.** *Let $(A, C)$ be a COBA with $C$ closed and inverse-closed. If $a \in C$, then $0 \leq a \leq r(a)1$.*

**Proof.** Let $(\lambda_n)$ be the sequence defined by $\lambda_n = r(a) + \frac{1}{n}$. The resolvent $(\lambda 1 - a)^{-1}$ of $a$ has a Neumann series representation $(\lambda 1 - a)^{-1} = \sum_{k=0}^{\infty} \frac{a^k}{\lambda^{k+1}}$ for $|\lambda| > r(a)$. Since $\lambda_n > r(a) \geq 0$ for all $n \in \mathbb{N}$ and since $C$ is an algebra $c$-cone, we have that $\frac{a^k}{\lambda_n^{k+1}} \in C$ for all $k$. Closedness of $C$ then implies that $(\lambda_n 1 - a)^{-1} \in C$ for all $n \in \mathbb{N}$. Since $C$ is inverse-closed, $\lambda_n 1 - a \in C$ for all $n \in \mathbb{N}$. From closedness of $C$ and the fact that $r(a)1 - a = \lim_{n\to\infty}(\lambda_n 1 - a)$, we get that $r(a)1 - a \in C$. □

The problem of unit spectrum was studied in [34] for operators on Banach spaces. Some of the results in this paper can be obtained in the more general setting of Banach algebras. If we restrict ourselves to positive elements in Banach algebras, we obtain corresponding results with simpler proofs, although in some cases the results are weaker. These are the next four results. The results extend the theory of COBAs and OBAs and are due to the author.

Suppose that $T$ is a bounded linear operator on a Banach space, with $\sigma(T) = \{0\}$, and that $0 < t \in \mathbb{R}$. In ([34], theorem 2.1) conditions on $||tTe^{tT}||$ are given that imply that $T = 0$. We prove the following corresponding theorem for COBAs and then apply it to obtain corollary 3.3.10.

**Theorem 3.3.9.** *Let $A$ be a COBA with a closed and c-normal algebra c-cone $C$ and let $a \in C$. If $\underline{\lim}_{t\to\infty}||tae^{ta}|| < \infty$ then $a = 0$.*

**Proof.** We have that $ae^{ta} = a + ta^2 + \frac{t^2a^3}{2!} + \cdots$. Since $C$ is a closed algebra $c$-cone, $ae^{ta} \in C$. Clearly, $a \leq ae^{ta}$. Since $C$ is $c$-normal, there exists an $\alpha > 0$ such that $||a|| \leq \alpha||ae^{ta}||$. From the hypothesis, it follows that $\underline{\lim}_{t\to\infty}||ta|| \leq \alpha\underline{\lim}_{t\to\infty}||tae^{ta}|| < \infty$. If $||a|| > 0$, then $\underline{\lim}_{t\to\infty}||ta|| = \infty$. Therefore $||a|| = 0$, so that $a = 0$. □

Note that the ordering has enabled us to relax the condition on $||tae^{ta}||$ in ([34], theorem 2.1). This yields a slightly stronger result than ([34], theorem 2.1) if the operator $T$ is assumed to be positive. Note also that the conditions in theorem 3.3.9 imply $\sigma(a) = \{0\}$. If not, then $0 < r(a) \leq ||a||$, so that $\underline{\lim}_{t\to\infty}||ta|| = \infty$, and then $c$-normality implies $\underline{\lim}_{t\to\infty}||tae^{ta}|| = \infty$.

The next corollary follows immediately from theorem 3.3.9.

**Corollary 3.3.10.** *Let $A$ be a COBA with a closed and c-normal algebra c-cone $C$ and let $a \in C$. If $a \leq 1$ or if $a \geq 1$ and if $\underline{\lim}_{t\to\infty}||t(1-a)e^{t(1-a)}|| < \infty$ (respectively $\underline{\lim}_{t\to\infty}||t(a-1)e^{t(a-1)}|| < \infty$), then $a = 1$.*

**Theorem 3.3.11.** *Let $A$ be a COBA with a c-normal algebra c-cone $C$ and let $a \in C$. Suppose that $a \geq 1$ or $a \leq 1$. If $\underline{\lim}_{n\to\infty}n||a^{n+1} - a^n|| < \infty$, then $a = 1$.*

**Proof.** Suppose that $a \geq 1$. We have that $(a^{n+1} - a^n) - (a - 1) = (a - 1)(a^n - 1)$. By proposition 2.1.22, we get that $(a - 1)(a^n - 1) \geq 0$, so that $0 \leq a - 1 \leq a^{n+1} - a^n$. Since $C$ is $c$-normal, there exists an $\alpha > 0$ such that $||a - 1|| \leq \alpha||a^{n+1} - a^n||$. By hypothesis, $\underline{\lim}_{n\to\infty}n||a - 1|| \leq \alpha\underline{\lim}_{n\to\infty}n||a^{n+1} - a^n|| < \infty$. If $||a - 1|| > 0$ then $\underline{\lim}_{n\to\infty}n||a - 1|| = \infty$. Therefore $||a - 1|| = 0$, and so $a = 1$. By making obvious adjustments, we can prove the result for the case $a \leq 1$ in a similar way. □

Theorem 3.3.11 is a COBA analogue of ([34], theorem 2.2). Note however that because of the ordering, the condition on $||a^{n+1} - a^n||$ has been relaxed. Similarly as in theorem 3.3.9, this yields a slightly stronger result than ([34], theorem 2.2) in the special case where the operator $T$ is positive and either $T \geq I$ or $T \leq I$ (where $I$ is the identity operator). Also, the conditions in theorem 3.3.11 imply $\sigma(a) = \{1\}$ (see remarks following theorem 3.3.9).

**Theorem 3.3.12.** *Let $A$ be a COBA with a c-normal algebra c-cone $C$ and let $a \in C$. If $a \leq 1$ or $a \geq 1$ and if $\lambda = 1$ is the only complex solution of the system of inequalities $|1 - \lambda^n| \leq ||1 - a^n||$, $n = 1, 2, ...$, then $a = 1$.*

**Proof.** Let $\alpha \in (0, 1)$. Since $\lambda = 1$ is the only complex solution of the system of inequalities $|1-\lambda^n| \leq ||1-a^n||$, $n = 1, 2...$, there exist $n \in \mathbb{N}$ such that $||1-a^n|| < 1-\alpha^n$. Now if $a \leq 1$, we have by proposition 2.1.22 that $1-a^{n-1} \geq 0$. Therefore $(1-a^n)-(1-a) = a(1-a^{n-1}) \geq 0$, so that $1-a \leq 1-a^n$. Since $C$ is $c$-normal, there exists a constant $\beta > 0$ such that $||1-a|| \leq \beta||1-a^n|| < \beta(1-\alpha^n)$. Suppose that $||1-a|| > 0$. Since $\lim_{\alpha\to 1^-} \beta(1-\alpha^n) = 0$, there exists an $\alpha_0 \in (0,1)$ such that $\beta(1-\alpha_0^n) \leq ||1-a||$. This contradicts the fact that $||1-a|| < \beta(1-\alpha^n)$ for all $\alpha \in (0,1)$. Thus $||1-a|| = 0$, so that $a = 1$. By making obvious adjustments, we can prove the result for the case $a \geq 1$ in a similar way. $\square$

Note that the conditions in theorem 3.3.12 imply that $\sigma(a) = \{1\}$. To see this let $\lambda \in \sigma(a)$. Then by the spectral mapping theorem, $1-\lambda^n \in \sigma(1-a^n)$ for all $n \in \mathbb{N}$. Therefore $|1-\lambda^n| \leq r(1-a^n) \leq ||1-a^n||$ for all $n \in \mathbb{N}$. Since $\lambda = 1$ is the only complex solution of the system of inequalities $|1-\lambda^n| \leq ||1-a^n||$, $n = 1, 2, ...$, it follows that $\sigma(a) = \{1\}$.

Theorem 3.3.12 is a COBA analogue of ([34], theorem 4.1). It is a weaker result than ([34], theorem 4.1), but, under the assumption that the operator $T$ is positive and either $T \geq I$ or $T \leq I$, it can be established with a simpler proof.

## 3.4 Unit spectrum II

In Section 3.3 we dealt with the problem of unit spectrum. Huijsmans and de Pagter (see [62]) asked the following more general question: when will it be true that if $T$ is a positive bounded linear operator on a complex Banach lattice with $\sigma(T) = \{1\}$, then $T \geq 1$? This problem has been investigated in the operator-theoretic setting in, for instance, [61], [62]. In [42], S. Mouton studied this problem in OBAs. The aim of this section is to generalize the results of Mouton to COBAs. The proofs of all the results in this section are the same as the proofs of the corresponding original OBA results and will therefore not be included.

In [42], S. Mouton proved that if $(M_n(\mathbb{C}), C)$ is the OBA of all $n \times n$ complex matrices with $C$ the subset of $M_n(\mathbb{C})$ consisting of all $n \times n$ complex matrices with non-negative entries, and if $a \in C$ with $\sigma(a) = \{1\}$, then $a - 1 \in C$ ([42], theorem 4.1). In the same paper a generalization of this result to direct sums of arbitrary OBAs was obtained ([42], theorem 4.2). The following is a corresponding result for direct sums of arbitrary COBAs, whose existence is guaranteed by proposition 2.3.2:

**Theorem 3.4.1.** *Let $n \in \mathbb{N}$ and $(A_i, C_i)$ a COBA for all $i = 1, ..., n$. Suppose that $(A, C)$ is the COBA with $A = A_1 \oplus \cdots \oplus A_n$ and $C = \{(c_1, ..., c_n) \in A : c_i \in C_i$ for $i = 1, ..., n\}$, and suppose that each $(A_i, C_i)$ satisfies the following property: if $a_i \in C_i$ with $\sigma(a_i) = \{1\}$ then $a_i - 1 \in C_i$. If $a \in C$ with $\sigma(a) = \{1\}$, then $a - 1 \in C$.*

In terms of different Banach algebras, we have the following result, whose original OBA counterpart is ([42], theorem 4.5).

**Theorem 3.4.2.** *Let $B$ be a COBA with a proper algebra c-cone $C_1$ and with $B$ isomorphic (as an algebra) to a COBA $A$, with a proper algebra c-cone $C$ which has the property that if $a \in C$ and $\sigma(a, A) = \{1\}$, then $a - 1 \in C$. Suppose that $C$ is the only proper algebra c-cone of $A$. If $a_1 \in C_1$ and $\sigma(a_1, B) = \{1\}$, then $a_1 - 1 \in C_1$.*

We now consider the case where the spectral radius of $a$ is a pole of the resolvent $(\lambda 1 - a)^{-1}$ of $a$, and extend the problem to the case where $\sigma(a) = \{r(a)\}$ with $r(a) \geq 1$. We have the following result, whose OBA counterpart is ([42], corollary 4.9 (2) and (4)).

**Theorem 3.4.3.** *Let $(A, C)$ be a COBA and let $a \in C$ with $\sigma(a) = \{r(a)\}$ such that $r(a) \geq 1$. Then*

*(i) if $r(a)$ is a simple pole of the resolvent of $a$, then $a - 1 \in C$.*
*(ii) If $C$ is closed and $r(a)$ is a pole of order 2 of the resolvent of $a$, then $a - 1 \in C$.*

Theorem 3.4.3 may be obtained as a corollary of theorem 3.2.6.

Recall from Chapter 1 that $\delta(a)$ denotes the distance from 0 to $\sigma(a)$. S. Mouton extended the problem under consideration to the case where the cone is inverse-closed and to the case where $\delta(a) \geq 1$, with no other restrictions on $\sigma(a)$ ([42], theorem 4.23). The following is the COBA counterpart of this result.

**Theorem 3.4.4.** *Let $(A, C)$ be a COBA with $C$ closed and inverse-closed, and let $a \in C$. Then we have the following implications:*

*(i) $\delta(a) > 1 \Rightarrow a > 1$ and $\delta(a) = 1 \Rightarrow a \geq 1$, hence $\delta(a) \geq 1 \Rightarrow a - 1 \in C$.*
*(ii) If $a$ is invertible: $r(a) < 1 \Rightarrow a < 1$ and $r(a) = 1 \Rightarrow a \leq 1$, hence $r(a) \leq 1 \Rightarrow 1 - a \in C$.*
*If in addition, $C$ is proper, then we also have*

*(iii) $\sigma(a) \subseteq C(0, 1) \Rightarrow a = 1$, where $C(0, 1)$ is the circle with centre at 0 and radius 1 in the complex plane.*
*(iv) $\sigma(a) = \{1\} \Rightarrow a = 1$.*

In the case of a $c$-normal algebra $c$-cone, the behaviour of the spectrum in (iii) above is quite restricted. We will present the result showing this fact in the next chapter, where it has been deferred as it relies on monotonicity of the spectral radius.

# Chapter 4

# Monotonicity of the spectral radius

In Chapter 3 we gave results in COBAs that do not rely on $c$-monotonicity (or monotonicity) of the spectral radius. In this chapter we will discuss those results in COBAs that do rely on $c$-monotonicity (or monotonicity) of the spectral radius.

## 4.1 Basic properties

This section deals mostly with basic results involving $c$-monotonicity or monotonicity of the spectral radius in COBAs. Some of these results are crucial for further development of spectral theory in COBAs.

If $(A, C)$ is a COBA and $M$ an MPCS in $A$ and if $F$ is a closed ideal in $A$, then $(A, M)$ and $(A/F, \pi M)$ are OBAs by theorem 2.4.2. Consequently, we refer to the spectral radius in $(A, M)$ and $(A/F, \pi M)$ as being monotone, rather than $c$-monotone.

We begin by providing the following elementary result, due to the author, about $c$-monotonicity in relation to algebra $c$-cones and MPCSs.

**Theorem 4.1.1.** *If $(A, C)$ is a COBA such that the spectral radius in $(A, C)$ is c-monotone and if $M$ is an MPCS in $A$, then the spectral radius in $(A, M)$ is monotone.*

**Proof.** The result follows from theorem 2.4.2 and the fact that the spectrum of an element of $C$ remains the same if this element is considered as a member of $M$. □

In ([51], theorem 4.1 1) H. Raubenheimer and S. Rode proved that in an OBA, normality of the algebra cone implies monotonicity of the spectral radius. Results of this type have already been proved for positive operators on ordered Banach spaces (cf. ([38], theorem 4.2), and ([19], theorem 1.1)). For COBAs, we have

**Theorem 4.1.2.** *If $A$ is a COBA with a c-normal algebra c-cone $C$, then the spectral radius in $(A, C)$ is c-monotone.*

**Proof.** Let $a, b \in A$ such that $0 \leq a \leq b$ w.r.t. $C$ and $ab = ba$. Then by proposition 2.1.22, we have that $0 \leq a^n \leq b^n$ for any $n \in \mathbb{N}$. Since $a^n$ and $b^n$ commute and $C$ is $c$-normal, it follows that $||a^n|| \leq \alpha ||b^n||$ for some $\alpha > 0$. Therefore $\lim_{n\to\infty} ||a^n||^{\frac{1}{n}} \leq \lim_{n\to\infty} \alpha^{\frac{1}{n}} ||b^n||^{\frac{1}{n}} = \lim_{n\to\infty} ||b^n||^{\frac{1}{n}}$. This implies that $r(a) \leq r(b)$. $\square$

Note that theorem 4.1.2 is essentially ([31], theorem 2), although our proof is more detailed.

In view of the fact that proposition 2.1.22 is used in the proof of theorem 4.1.2, we suspect that normality of $C$ would not necessarily imply monotonicity of the spectral radius relative to $C$ in a COBA which is not an OBA. However, in all the examples known to us where the spectral radius is $c$-monotone, it is in fact monotone. So giving an example where normality implies only $c$-monotonicity and not monotonicity of the spectral radius remains an open problem.

The next example shows that the converse of theorem 4.1.2 is not true.

**Example 4.1.3.** *Let $E$ be the Banach lattice $\ell^1 \oplus L^2[0,1] \oplus \ell^\infty$, $C$ the positive cone in $E$ and let $K = \{T \in \mathcal{L}^r(E) : TC \subseteq C\}$. If $\pi : \mathcal{L}^r(E) \to \mathcal{L}^r(E)/\mathcal{K}^r(E)$ is the canonical homomorphism, then $(A, \pi K) = (\mathcal{L}^r(E)/\mathcal{K}^r(E), \pi K)$ is an OBA, and hence a COBA. The spectral radius in $(A, \pi K)$ is monotone (and hence c-monotone) but $\pi K$ is not c-normal in $A$.*

**Proof.** By ([51], example 4.2) $(A, \pi K)$ is an OBA (and hence a COBA), and the spectral radius is monotone (and hence $c$-monotone). We show that $\pi K$ is not $c$-normal in $A$. By ([51], example 4.2), there exist positive operators $S$,$T$ on $E$ such that $0 \leq S \leq T$ w.r.t. $K$, $T \in \mathcal{K}^r(E)$ and $S \notin \mathcal{K}^r(E)$. Thus $0 \leq S + \mathcal{K}^r(E) \leq T + \mathcal{K}^r(E)$, $||S + \mathcal{K}^r(E)|| > 0$ and $||T + \mathcal{K}^r(E)|| = 0$. Since $\mathcal{K}^r(E)$ is an ideal, $ST - TS \in \mathcal{K}^r(E)$, so that $S + \mathcal{K}^r(E)$ and $T + \mathcal{K}^r(E)$ commute but $||S + \mathcal{K}^r(E)|| > \alpha ||T + \mathcal{K}^r(E)||$ for every scalar $\alpha > 0$. $\square$

The next corollary is an immediate consequence of theorems 2.4.2, 4.1.1 and 4.1.2.

**Corollary 4.1.4.** *If $A$ is a COBA with a c-normal algebra c-cone $C$ and if $M$ is an MPCS in $A$, then the spectral radius in $(A, M)$ is monotone.*

The next result is about $c$-monotonicity of the spectral radius when different Banach algebras are involved. Its original OBA counterpart is ([51], proposition 4.5).

**Theorem 4.1.5.** *Let $(A, C)$ be a COBA and $B$ a Banach algebra with finer norm than $A$ such that $1 \in B \subseteq A$. Suppose that the spectral radius in $(B, C \cap B)$ is c-monotone. If $a, b \in B$ such that $0 \leq a \leq b$ w.r.t. $C$, $ab = ba$ and $\sigma(b, B) = \sigma(b, A)$, then $r(a, A) \leq r(b, A)$.*

**Proof.** Let $a, b \in B$ with $0 \leq a \leq b$ w.r.t. $C$ and $ab = ba$. Since the spectral radius in $(B, C \cap B)$ is $c$-monotone, $r(a, B) \leq r(b, B)$. From the assumption that $B$ has finer norm than $A$, we obtain that $r(a, A) \leq r(a, B)$. The assumption $\sigma(b, B) = \sigma(b, A)$ implies that $r(b, B) = r(b, A)$. Therefore $r(a, A) \leq r(a, B) \leq r(b, B) = r(b, A)$. $\square$

H. Raubenheimer and S. Rode proved that if the spectral radius in an OBA is monotone, then the spectral radius of a positive element in the OBA belongs to the spectrum of that element ([51], theorem 5.2). This is a generalization of the original matrix theorem of O. Perron. We prove the corresponding result in COBAs. This result will play a crucial role in the further development of spectral theory in COBAs. In fact, it is because this result can be generalized from OBAs to COBAs that generalization of a large part of the theory of OBAs to COBAs is made possible.

**Theorem 4.1.6.** *Let $A$ be a COBA with a closed algebra c-cone $C$ such that the spectral radius in $(A, C)$ is c-monotone. If $a \in C$ then $r(a) \in \sigma(a)$.*

**Proof.** Let $a \in C$. Without loss of generality we assume that $r(a) = 1$. Suppose that $1 \notin \sigma(a)$. Then there exists $0 \leq \alpha < 1$ such that $\sigma(a) \subseteq \{\lambda \in \mathbb{C} : \text{Re } \lambda \leq \alpha\}$. Let $t \geq 0$ and let $f(z) = e^{tz}$. Using the spectral mapping theorem, $\sigma(e^{ta}) = e^{t\sigma(a)} \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq e^{t\alpha}\}$. Therefore $r(e^{ta}) \leq e^{t\alpha}$ for all $t \geq 0$. From the fact that $C$ is an algebra $c$-cone, $a \in C$ and $t \geq 0$, we obtain that all positive integer powers $\frac{(ta)^n}{n!}$ are in $C$. Since $C$ is closed, it follows that $e^{ta} = 1 + ta + \dfrac{(ta)^2}{2!} + \cdots \in C$. This implies that $0 \leq \dfrac{t^n}{n!}a^n \leq e^{ta}$ for all $n \in \mathbb{N}$ and for all $t \geq 0$. Since the spectral radius in $(A, C)$ is $c$-monotone and $r(a) = 1$, we have that $0 \leq r\left(\left(\dfrac{t^n}{n!}\right)a^n\right) = \dfrac{t^n}{n!} \leq e^{t\alpha}$. Taking $t = \frac{n}{\alpha}$, we get that $\dfrac{n^n}{\alpha^n n!} \leq e^n$. This contradicts corollary 1.6.3. Hence $r(a) \in \sigma(a)$. $\square$

The proof of theorem 4.1.6 is that of ([51], theorem 5.2), since the only multiplication of positive elements involved is taking powers. We include it in the interest of completeness. Note that theorem 4.1.6 is essentially ([31], theorem 3). Also, if $A$ in theorem 4.1.6 is a $C^*$-algebra ordered by $C = \{a \in A : a = a^* \text{ and } \sigma(a) \subseteq [0, \infty)\}$, the result holds without using monotonicity or the proof of theorem 4.1.6. This is so because if $a \in C$, then $\sigma(a) \subseteq [0, \infty)$. Similar remarks apply for similar results in this section.

From theorem 2.4.2 and ([51], theorem 5.2), we can obtain the following result, which will be useful in proving other results in this chapter.

**Theorem 4.1.7.** *Let $A$ be a COBA with a closed algebra c-cone $C$ and $M$ an MPCS in $A$ such that the spectral radius in $(A, M)$ is monotone. If $a \in M$ then $r(a) \in \sigma(a)$.*

In ([33], theorem 2), Herzog and Schmoeger proved the converse of ([51], theorem 5.2). Adapting this result to our present setting, we obtain the following result, which is a converse of theorem 4.1.6.

**Theorem 4.1.8.** *Let $A$ be a Banach algebra and $a \in A$. If $r(a) \in \sigma(a)$, then there exists a closed algebra c-cone $C$ in $A$ such that the spectral radius in $(A, C)$ is c-monotone and $a \in C$.*

Note that since an algebra cone is an algebra $c$-cone and since monotonicity implies $c$-monotonicity, theorem 4.1.8 can alternatively be obtained as a direct corollary of ([33], theorem 2).

In the setting of quotient algebras, H. Raubenheimer and S. Mouton obtained ([51], theorem 5.3) and ([51], corollary 5.4), which is a generalization of a result of J. Martinez and J.M. Mazón ([39], corollary 2.14). We give the corresponding results for COBAs.

**Theorem 4.1.9.** *Let $(A, C)$ be a COBA with $C$ closed and let $M$ be an MPCS in $A$. Suppose that $F$ is a closed ideal of $A$ such that the spectral radius in $(A/F, \pi M)$ is monotone. If $a \in M$ then $r(a + F) \in \sigma(a + F)$.*

**Proof.** By theorem 2.4.2, we have that $(A, M)$ is an OBA with $M$ closed. From ([51], theorem 5.3), the result follows. $\square$

Note that if $(A, C)$ is a COBA and $M$ an MPCS in $A$ and if $B$ is a Banach algebra with $1 \in B \subseteq A$, then by theorem 2.4.2 and the remarks on ([51], p. 492), $M \cap B$ is an algebra cone in $B$. Therefore we have the following corollary of theorem 4.1.9:

**Corollary 4.1.10.** *Let $(A, C)$ be a COBA and $M$ an MPCS in $A$, and let $B$ be a Banach algebra with $1 \in B \subseteq A$ such that $M \cap B$ is closed in $B$. Suppose that $I$ is an inessential ideal of both $A$ and $B$ such that the spectral radius function in the OBA $(B/cl_B(I), \pi(M \cap B))$ is monotone. If $a \in M \cap B$ is such that $\sigma(a, A) = \sigma(a, B)$ then $r(a + cl_B(I), B/cl_B(I)) \in \sigma(a + cl_B(I), B/cl_B(I))$ and $r(a + cl_A(I), A/cl_A(I)) \in \sigma(a + cl_A(I), A/cl_A(I))$.*

Since $C$ in the corollary above is used only to guarantee the existence of $M$, the proof of this corollary is similar to the proof of the corresponding original OBA result in ([51], corollary 5.4).

The next two results, due to the author, are related to what was discussed in Chapter 3. They have been placed here because they rely on monotonicity of the spectral radius.

Proposition 3.1.4 gives a spectral characterization of the rank one idempotents in a general semiprime Banach algebra. If we take a semiprime COBA in which the spectral radius is $c$-monotone, then we obtain the following:

**Proposition 4.1.11.** *Let $A$ be a semiprime COBA with a closed algebra $c$-cone $C$ such that the spectral radius is $c$-monotone and let $x \in \mathcal{F}_1(A) \cap C$. Then $x$ is an idempotent if and only if $r(x) = 1$.*

**Proof.** If $x$ is an idempotent, that $r(x) = 1$ follows from the fact that $x \in \mathcal{F}_1$ means $x \neq 0$. Conversely, suppose that $r(x) = 1$. With the spectral mapping theorem, it can be shown that $\sigma(a) = \{\mathrm{tr}(a)\}$ if dim $(A) = 1$ and $\sigma(a) = \{0, \mathrm{tr}(a)\}$ if dim $A > 1$ for every $a \in \mathcal{F}_1(A)$. By theorem 4.1.6, this implies that $r(x) = \mathrm{tr}(x) = 1$. From $x^2 = \mathrm{tr}(x)x$, the result follows. $\square$

An element $a$ in a Banach algebra is said to be *power bounded* if there exists a scalar $M > 0$ such that $||a^n|| \leq M$ for all $n \in \mathbb{N}$. It is well known that every power bounded element is Abel bounded. The following result gives a criterion for a positive element in a COBA with a closed, inverse closed and $c$-normal algebra $c$-cone to be power bounded.

**Proposition 4.1.12.** *Let $A$ be a COBA with a closed, inverse closed and $c$-normal algebra $c$-cone $C$. If $a \in C$ and $r(a) \leq 1$, then $a$ is power bounded.*

**Proof.** From proposition 3.3.8 and the spectral mapping theorem, we have that $0 \leq a^n \leq 1$ for all $n \in \mathbb{N}$. It follows from $c$-normality of $C$ that there exists a constant $\alpha > 0$ such that $||a^n|| \leq \alpha$ for all $n \in \mathbb{N}$. $\square$

## 4.2 Algebra $c'$-cones

We recall that in order to obtain the result in theorem 4.1.9, we had to work with MPCSs. This is because from the COBA $(A, C)$, we could not get $(A/F, \pi C)$ to be a COBA. If we relax the multiplication axiom in the definition of an algebra $c$-cone, we obtain another type of cone, which we shall call algebra $c'$-cone. An algebra $c'$-cone is more general than an algebra $c$-cone. In this section we show that if $C$ is an algebra $c'$-cone, then $\pi C$ is also an algebra $c'$-cone. Using this fact we prove a variant of theorem 4.1.9 in terms of $\pi C$. We start by defining algebra $c'$-cones.

**Definition 4.2.1.** *Let $A$ be a Banach algebra. A subset $C$ of $A$ is called an algebra $c'$-cone if $C$ satisfies the following:*

*(i) $C + C \subseteq C$,*
*(ii) $\lambda C \subseteq C$ for all $\lambda \geq 0$,*
*(iii) If $a \in C$, then $a^n \in C$ for any $n \in \mathbb{N}$,*
*(iv) $1 \in C$.*

Every Banach algebra can be ordered by an algebra $c'$-cone in the usual way. A Banach algebra ordered by a $c'$-cone is called a $C'$OBA. Clearly, every algebra $c$-cone is an algebra $c'$-cone. Therefore every COBA is a $C'$OBA.

In the following proposition we prove that if $C$ is an algebra $c'$-cone, then $\pi C$ is also an algebra $c'$-cone.

**Proposition 4.2.2.** *Let $(A, C)$ be a $C'$OBA and let $F$ be a closed ideal of $A$. If $\pi : A \to A/F$ is the canonical homomorphism, then $\pi C = \{c + F : c \in C\}$ is an algebra $c'$-cone in $A/F$.*

**Proof.** Let $a + F, b + F \in \pi C$. Clearly, $(a + F) + (b + F) \in \pi C$ and if $\lambda \geq 0$, then $\lambda(a + F) \in \pi C$. It is also clear that $1 + F \in \pi C$. We show that $(a + F)^n \in \pi C$ for every $n \in \mathbb{N}$. Since $a + F \in \pi C$, there exists an $a_1 \in C$ such that $a + F = a_1 + F$. Since $C$ is an algebra $c'$-cone, $a_1^n \in C$, so that $(a + F)^n \in \pi C$. $\square$

Note that $\pi C$ is not in general an algebra $c$-cone, even if $C$ is an algebra $c$-cone.

The following two are examples of $C'$OBAs.

**Example 4.2.3.** *Let $A$ be a $C^*$-algebra and $C = \{a \in A : a = a^* \text{ and } \sigma(a, A) \subseteq [0, \infty)\}$. Suppose that $F$ is a closed ideal in $A$ and for each $a + F$ in $A/F$, define $(a + F)^* = a^* + F$. If $\pi : A \to A/F$ is the canonical homomorphism, then $\pi C$ is a normal algebra $c'$-cone in $A/F$ and the spectral radius in $(A/F, \pi C)$ is monotone. Therefore $(A/F, \pi C)$ is a $C'$OBA.*

**Proof.** By example 2.1.19, $C$ is an algebra $c$-cone in $A$ and hence by proposition 4.2.2, $\pi C$ is an algebra $c'$-cone in $A/F$. Let $K = \{a+F \in A/F : (a+F)^* = a+F \text{ and } \sigma(a+F, A/F) \subseteq [0,\infty)\}$. By theorem 1.5.4, $A/F$ is a $C^*$-algebra. In view of example 2.1.19, $(A/F, K)$ is a COBA with $K$ normal and the spectral radius in $(A/F, K)$ monotone. Since $\pi C \subseteq K$, it follows that $\pi C$ is a normal algebra $c'$-cone in $A/F$ and the spectral radius in $(A/F, \pi C)$ is monotone. □

By theorem 1.4.3 the following is a special case of example 4.2.3.

**Example 4.2.4.** *Let $H$ be a Hilbert space and $A = \mathcal{L}(H)$ and $C = \{T \in A : T \geq 0\}$. Suppose that for all $T + \mathcal{K}(H)$ in $A/\mathcal{K}(H)$, we define $(T + \mathcal{K}(H))^* = T^* + \mathcal{K}(H)$. If $\pi : A \to A/\mathcal{K}(H)$ is the canonical homomorphism, then $\pi C$ is a normal algebra $c'$-cone in $A/\mathcal{K}(H)$. Therefore $(A/\mathcal{K}(H), \pi C)$ is a $C'$OBA.*

From proposition 4.2.2, if $(A, C)$ is a $C'$OBA and $F$ is a closed ideal of $A$, then $(A/F, \pi C)$ is a quotient $C'$OBA. The following result is a variant of theorem 4.1.9 for quotient $C'$OBAs.

**Theorem 4.2.5.** *Let $(A, C)$ be a COBA with $C$ closed. Suppose that $F$ is a closed ideal of $A$ such that the spectral radius in the $C'$OBA $(A/F, \pi C)$ is c-monotone. If $a \in C$ then $r(a + F) \in \sigma(a + F)$.*

Note that the $C'$OBA structure in theorem 4.2.5 is used only to allow the result to be formulated in the quotient algebra $(A/F, \pi C)$. Otherwise the result is proved in a similar way to ([51], theorem 5.3).

The following result is a $C'$OBA counterpart of ([51], corollary 5.4). Its proof is verbatim the same as that of ([51], corollary 5.4) by using theorem 2.2.1 and theorem 4.2.5.

**Corollary 4.2.6.** *Let $(A, C)$ be a COBA and $B$ a subalgebra of $A$ with $1 \in B \subseteq A$ and such that $C \cap B$ is closed in $B$. Suppose that $I$ is an inessential ideal of both $A$ and $B$ such that the spectral radius in the $C'$OBA $(B/cl_B(I), \pi(C \cap B))$ is c-monotone. If $a \in C \cap B$ is such that $\sigma(a, A) = \sigma(a, B)$, then $r(a + cl_B(I), B/cl_B(I)) \in \sigma(a + cl_B(I), B/cl_B(I))$ and $r(a + cl_A(I), A/cl_A(I)) \in \sigma(a + cl_A(I), A/cl_A(I))$.*

The next two results are COBA versions of ([51], theorem 6.1).

**Theorem 4.2.7.** *Let $(A, C)$ be a COBA and $M$ an MPCS in $A$. Suppose that $a, b \in A$ and that $F$ is a closed ideal in $A$. Then the following two conditions are equivalent:*

*(i) if $0 \leq a \leq b$ w.r.t. $M$ and $b \in F$, then $a \in F$;*
*(ii) the algebra cone $\pi M$ in the quotient algebra $A/F$ is proper.*

**Proof.** The result follows from theorem 2.4.2 and ([51], theorem 6.1). □

**Theorem 4.2.8.** *Let $(A, C)$ be a COBA with $a, b \in A$ and let $F$ be a closed ideal in $A$. Then the following two conditions are equivalent:*

*(i) if $0 \leq a \leq b$ w.r.t. $C$ and $b \in F$, then $a \in F$;*
*(ii) the algebra $c'$-cone $\pi C$ in the quotient algebra $A/F$ is proper.*

**Proof** (i) $\Rightarrow$ (ii): Let $c + F \in \pi C \cap -\pi C$. Then $c + F = c_1 + F = -c_2 + F$, where $c_1, c_2 \in C$. Therefore $c_1 + c_2 + F = F$, so that $c_1 + c_2 \in F$. Now since $c_1, c_2 \in C$, we have that $0 \leq c_1 \leq c_1 + c_2$ w.r.t. $C$. Condition (i) then implies that $c_1 \in F$. Therefore $c+F = c_1+F = F$, and since $c + F$ was arbitrary in $\pi C \cap -\pi C$, we get that $\pi C \cap -\pi C = \{F\}$. Hence $\pi C$ is a proper algebra $c'$-cone in $A/F$.

(ii) $\Rightarrow$ (i): Suppose that $0 \leq a \leq b$ and $b \in F$. Then $F \leq a + F \leq b + F = F$. Since $\pi C$ is a proper algebra $c'$-cone in $A/F$ by condition (ii), the order $\leq$ in $A/F$ is antisymmetric by theorem 2.1.2. It follows that $a + F = F$, so that $a \in F$. $\square$

The proof of theorem 4.2.8 is essentially that of ([51], theorem 6.1).

## 4.3 Boundary spectrum and spectral continuity of positive elements

In [45] S. Mouton studied the boundary spectrum of elements of Banach algebras and applied some of the results in ordered Banach algebras. This work was continued in [46], where results on the boundary spectrum were applied in proving spectral continuity of positive elements. In this section we generalize some of the OBA results in the papers cited above to the COBA setting. The proofs follow along the lines of the proofs of the corresponding original results in OBAs; we include them to illustrate how some COBA analogues of known OBA results, which have already been presented, are utilized.

Recall that the boundary spectrum $S_\partial(a)$ of an element $a$ in a Banach algebra $A$ is the set $S_\partial(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \in \partial S\}$, where $S$ is the set of all non-invertible elements of $A$.

In theorem 4.1.6 we proved that if $a$ is a positive element in a COBA, then under certain conditions, $r(a) \in \sigma(a)$. The following is an analogous result for the boundary spectrum. Its original version in OBAs is ([45], proposition 3.3).

**Proposition 4.3.1.** *Let $(A, C)$ be a COBA with $C$ closed and such that the spectral radius in $(A, C)$ is c-monotone. If $a \in C$ then $r(a) \in S_\partial(a)$.*

**Proof.** If $a \in C$, then $r(a) \in \sigma(a)$ by theorem 4.1.6. Therefore $r(a) \in \partial\sigma(a)$ and by proposition 1.1.13, we have that $r(a) \in S_\partial(a)$. $\square$

If $C$ is inverse-closed, then we obtain the following result, whose original OBA version is ([45], proposition 3.4). As set out in Chapter 1, the notation $\delta(a)$ here indicates the distance $d(0, \sigma(a))$ from 0 to $\sigma(a)$.

**Proposition 4.3.2.** *Let $(A, C)$ be a COBA with $C$ closed and inverse-closed, and such that the spectral radius in $(A, C)$ is c-monotone. If $a$ is an invertible element of $C$, then $\delta(a) \in S_\partial(a)$.*

**Proof.** Since $C$ is inverse-closed, we have that $a^{-1} \in C$. It follows from proposition 4.3.1 that $r(a^{-1}) \in S_\partial(a^{-1})$. Proposition 1.1.14 then implies that $r(a^{-1}) = \frac{1}{\lambda_0}$, for some $\lambda_0 \in S_\partial(a)$.

Since $r(a^{-1}) = \frac{1}{\delta(a)}$, it follows that $\delta(a) \in S_\partial(a)$. $\square$

Using the boundary spectrum we obtain the following result, which is a stronger version of theorem 4.1.5.

**Theorem 4.3.3.** *Let $(A, C)$ be a COBA and $B$ a Banach algebra with finer norm than $A$ and such that $1 \in B \subseteq A$.*

*(i) Suppose that the spectral radius in $(A, C)$ is c-monotone. If $a, b \in B$ with $ab = ba$ and $0 \leq a \leq b$, and either $\partial\sigma(a, B) = \partial\sigma(a, A)$ or $S_\partial(a, B) = S_\partial(a, A)$, then $r(a, B) \leq r(b, B)$.*

*(ii) Suppose that the spectral radius in $(B, C \cap B)$ is c-monotone. If $a, b \in B$ with $0 \leq a \leq b$ and $ab = ba$, and either $\partial\sigma(b, B) = \partial\sigma(b, A)$ or $S_\partial(b, B) = S_\partial(b, A)$, then $r(a, A) \leq r(b, A)$.*

**Proof.** (i) Since $B$ has finer norm than $A$, we have that $r(b, A) \leq r(b, B)$. From $c$-monotonicity of the spectral radius in $(A, C)$, we obtain that $r(a, A) \leq r(b, A)$. By assumption and proposition 1.1.13, we have that $r(a, B) = r(a, A)$. Therefore $r(a, B) = r(a, A) \leq r(b, A) \leq r(b, B)$.

(ii) From the assumptions and proposition 1.1.13, we get that $r(b, B) = r(b, A)$. The assumption that $B$ has finer norm than $A$ and that the spectral radius in $(B, C \cap B)$ is $c$-monotone lead to the inequalities $r(a, A) \leq r(a, B)$ and $r(a, B) \leq r(b, B)$. Hence $r(a, A) \leq r(a, B) \leq r(b, B) = r(b, A)$. $\square$

The original OBA result corresponding to theorem 4.3.3 was proved in ([45], theorem 3.5).

In [46], S. Mouton proved an OBA result giving conditions under which the spectral radius of a positive element is continuous (see [46], theorem 4.6). We generalize this result to the COBA setting. We start with the following lemma, whose original OBA version is also due to S. Mouton ([46], lemma 4.3). We first recall from Chapter 1 that if $a$ is an element of a Banach algebra $A$, then $T(a)$ is the set $\{\lambda \in \mathbb{C} : |\lambda|1 - a \in \partial S\}$, where as usual, $S$ is the set of all non-invertible elements of $A$. Also, $\gamma(a)$ is the number $\sup\{|\lambda| : \lambda \in T(a)\}$.

**Lemma 4.3.4.** *Let $(A, C)$ be a COBA with $C$ closed and c-normal. If $a \in C$, then $\gamma(a) = r(a)$ and $r(a) \in T(a)$.*

**Proof.** From the fact that $T(a) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq r(a)\}$, we have (if $T(a) \neq \emptyset$) that $\gamma(a) \leq r(a)$. Since $a \in C$, theorem 4.1.2 and proposition 4.3.1 imply that $r(a) \in S_\partial(a)$. Since $r(a) \in \mathbb{R}^+$, it follows that $r(a) \in T(a)$. Hence $T(a) \neq \emptyset$ and $r(a) \leq \gamma(a)$. $\square$

**Theorem 4.3.5.** *Let $(A, C)$ be a COBA with $C$ closed and c-normal, and let $a \in C$ be such that $S_\partial(a) \cap \mathbb{R}^+ = \{r(a)\}$. If $(a_n)$ is a sequence in $C$ such that $a_n \to a$ as $n \to \infty$, then $r(a_n) \to r(a)$ as $n \to \infty$.*

The proof of theorem 4.3.5 is the same as that of ([46], theorem 4.6), using lemma 4.3.4 and theorem 1.1.15.

## 4.4 Riesz elements, Riesz points and quasi inessential elements

In this section we apply $c$-monotonicity and monotonicity of the spectral radius to prove results involving Riesz elements, Riesz points and quasi inessential elements and the peripheral spectrum. The results in this section are COBA generalizations of the corresponding results in OBAs. The results involving MPCSs in COBAs will generally not be presented if they can be obtained directly from theorem 2.4.2 and the associated OBA result. It will be mentioned whenever such results occur.

In the previous chapter we gave a basic form of the Krein-Rutman theorem (theorem 3.1.2) and mentioned that the other version of the result was deferred to this chapter. We are now in a position to give the second version of the Krein-Rutman theorem, which is the next result.

**Theorem 4.4.1.** *Let $A$ be a semisimple COBA with a closed, c-normal algebra c-cone $C$ and let $0 \neq a \in C$ be such that $r(a) > 0$. If $I$ is a closed inessential ideal in $A$ such that $a$ is Riesz relative to $I$, then there exists $0 \neq u \in C$ such that $ua = au = r(a)u$ and $aua = r(a)u^2$.*

**Proof.** By theorem 4.1.2, the spectral radius in $(A, C)$ is $c$-monotone. It follows from theorem 4.1.6 that $r(a) \in \sigma(a)$. From the hypothesis and theorem 1.2.1, it follows that $r(a)$ is a pole of the resolvent of $a$. Theorem 3.1.2 then implies that there exists $0 \neq u \in C$ such that $ua = au = r(a)u$ and $aua = r(a)^2u$. $\square$

The original OBA counterpart of the previous theorem is due to S. Mouton and H. Raubenheimer (see [47], theorem 3.7).

In ([43], lemma 4.2) S. Mouton proved that under suitable conditions, the spectral radius of a positive element of an OBA is strictly greater than the spectral radius of the corresponding element in the quotient algebra. We give the corresponding result in the COBA setting.

**Theorem 4.4.2.** *Let $(A, C)$ be a COBA with $C$ closed, $M$ an MPCS in $A$ and $I$ a closed inessential ideal of $A$ such that the spectral radius in $(A/I, \pi M)$ is monotone. Let $a \in M$.*

*(i) If $r(a)$ is a Riesz point of $\sigma(a)$, then $r(a + I) < r(a)$.*
*(ii) If in addition, the spectral radius in $(A, M)$ is monotone, then $r(a)$ is a Riesz point of $\sigma(a)$ if and only if $r(a + I) < r(a)$.*

Theorem 4.4.2 follows directly from theorem 2.4.2 and ([43], lemma 4.2). It has been presented because it is used to prove other results in this section.

In $C'$OBAs, the following is the corresponding result to theorem 4.4.2. Its proof is essentially that of ([43], lemma 4.2).

**Theorem 4.4.3.** *Let $(A, C)$ be a COBA with $C$ closed and $I$ a closed inessential ideal of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. Let $a \in C$.*

*(i) If $r(a)$ is a Riesz point of $\sigma(a)$, then $r(a+I) < r(a)$.*
*(ii) If in addition the spectral radius in $(A, C)$ is c-monotone, then $r(a)$ is a Riesz point of $\sigma(a)$ if and only if $r(a+I) < r(a)$.*

**Proof.** (i) If $r(a+I) = r(a)$ then by theorem 4.2.5 we get that $r(a) \in \sigma(a+I)$. It follows from theorem 1.2.6 that $r(a) \in D_I(a, A)$, so that $r(a)$ is not a Riesz point of $\sigma(a)$.

(ii) If $r(a)$ is not a Riesz point of $\sigma(a)$, then by theorem 4.1.6, we have that $r(a) \in D_I(a, A)$. Therefore $r(a) \in \eta\sigma(a+I)$ by theorem 1.2.6. This implies that $r(a) \leq r(a+I)$. $\square$

Note that if $A$ is any Banach algebra, $I$ a closed inessential ideal in $A$ and $a \in A$ such that $\sigma(a) \subseteq [0, \infty)$, then it follows from theorem 1.2.6 that $r(a)$ is a Riesz point of $\sigma(a)$ if and only if $r(a+I) < r(a)$. So if $A$ in theorems 4.4.2 and 4.4.3 is a $C^*$-algebra ordered by $C = \{a \in A : a = a^* \text{ and } \sigma(a) \subseteq [0, \infty)\}$, then the results hold without using monotonicity.

It is well known that if $T$ is a positive operator on a Banach lattice $E$ such that the spectral radius of $T$ is a Riesz point of $\sigma(T, \mathcal{L}(E))$, then $\operatorname{psp}(T)$ consists of Riesz points ([55], theorem V.5.5). In the following theorem we give a corresponding result for a positive element of a COBA. The original result was proved for OBAs in ([43], theorem 4.3) (also see [47], theorem 4.1).

**Theorem 4.4.4.** *Let $(A, C)$ be a COBA with $C$ closed, $M$ an MPCS in $A$ and $I$ a closed inessential ideal of $A$ such that the spectral radius in $(A/I, \pi M)$ is monotone. If $a \in M$ is such that $r(a)$ is a Riesz point of $\sigma(a)$ relative to $I$, then $\operatorname{psp}(a)$ consists of Riesz points of $\sigma(a)$.*

Theorem 4.4.4 follows directly from theorem 2.4.2 and ([43], theorem 4.3). We have presented it because we refer to it later in this section.

The $C'$OBA counterpart of theorem 4.4.4 is the following. The proof mimics that of ([43], theorem 4.3) by using theorem 4.4.3.

**Theorem 4.4.5.** *Let $(A, C)$ be a COBA with $C$ closed and $I$ a closed inessential ideal of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. If $a \in C$ is such that $r(a)$ is a Riesz point of $\sigma(a)$, then $\operatorname{psp}(a)$ consists of Riesz points of $\sigma(a)$.*

If $A$ in theorems 4.4.4 and 4.4.5 is a $C^*$-algebra ordered by $C = \{a \in A : a = a^* \text{ and } \sigma(a) \subseteq [0, \infty)\}$, then the results hold trivially. This follows from the fact that if $a \in C$, then $\operatorname{psp}(a) = \{r(a)\}$.

The next theorem is a COBA version of ([43], theorem 4.4).

**Theorem 4.4.6.** *Let $(A, C)$ be a COBA with $C$ closed and $M$ an MPCS in $A$ such that the spectral radius in $(A, M)$ is monotone. Let $I$ be a closed inessential ideal of $A$ such that the spectral radius in $(A/I, \pi M)$ is monotone. Suppose that $a, b \in A$ with $0 \leq a \leq b$ relative to $M$ and $r(a) = r(b)$. If $r(b)$ is a Riesz point of $\sigma(b)$, then $\operatorname{psp}(a)$ consists of Riesz points of $\sigma(a)$.*

**Proof.** Since $r(b)$ is a Riesz point of $\sigma(b)$, it follows from theorem 4.4.2 that $r(b+I) < r(b)$. By the monotonicity of the spectral radius in $(A/I, \pi M)$, we have that $r(a+I) \leq r(b+I)$. Since $r(a) = r(b)$, it follows that $r(a+I) < r(a)$. Theorem 4.4.2 then implies that $r(a)$ is a Riesz point of $\sigma(a)$. The result now follows from theorem 4.4.4. $\square$

An immediate consequence of theorem 4.4.6 is the following, which was originally proved for OBAs in ([47], theorem 4.3).

**Corollary 4.4.7.** *Let $(A, C)$ be a COBA with $C$ closed and $M$ an MPCS in $A$ such that the spectral radius in $(A, M)$ is monotone. Let $I$ be a closed inessential ideal of $A$ such that the spectral radius in $(A/I, \pi M)$ is monotone. Suppose $a, b \in A$ with $0 \leq a \leq b$ w.r.t. $M$ and $r(a) = r(b)$. If $r(b)$ is a Riesz point of $\sigma(b)$ relative to $I$, then $r(a)$ is a Riesz point of $\sigma(a)$ relative to $I$.*

In terms of $C'$OBAs, we get the following result corresponding to theorem 4.4.6.

**Theorem 4.4.8.** *Let $(A, C)$ be a COBA with $C$ closed and such that the spectral radius in $(A, C)$ is c-monotone. Let $I$ be a closed inessential ideal of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. Suppose that $a, b \in A$ with $0 \leq a \leq b$ relative to $C$, $ab = ba$ and $r(a) = r(b)$. If $r(b)$ is a Riesz point of $\sigma(b)$, then $\mathrm{psp}(a)$ consists of Riesz points of $\sigma(a)$.*

**Proof.** Since $r(b)$ is a Riesz point of $\sigma(a)$, theorem 4.4.3 implies that $r(b+I) < r(b)$. Since the spectral radius in $(A/I, \pi C)$ is $c$-monotone, $r(a+I) \leq r(b+I)$. It follows from $r(b+I) < r(b)$ and from the assumption $r(a) = r(b)$ that $r(a+I) < r(a)$. Theorem 4.4.3 then implies that $r(a)$ is a Riesz point of $\sigma(a)$. By theorem 4.4.5, we get that $\mathrm{psp}(a)$ consists of Riesz points of $\sigma(a)$. $\square$

Note that if in theorem 4.4.8 we assume that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is monotone, then $a$ and $b$ need not commute. The next corollary follows immediately from theorem 4.4.8.

**Corollary 4.4.9.** *Let $(A, C)$ be a COBA with $C$ closed and such that the spectral radius in $(A, C)$ is c-monotone. Let $I$ be a closed inessential ideal of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. Suppose $a, b \in A$ with $0 \leq a \leq b$, $ab = ba$ and $r(a) = r(b)$. If $r(b)$ is a Riesz point of $\sigma(b)$ relative to $I$, then $r(a)$ is a Riesz point of $\sigma(a)$ relative to $I$.*

In a $C^*$-algebra, theorem 4.4.6, corollary 4.4.7, theorem 4.4.8 and corollary 4.4.9 hold trivially (see remarks following theorem 4.4.5).

Following the remarks after theorem 3.4.4, we give the following result. The original version in the OBA setting was proved in ([42], theorem 4.25).

**Theorem 4.4.10.** *Let $(A, C)$ be a COBA with $C$ closed and c-normal. If $a \in A$ and there is a $k \in \mathbb{N}$ and an $\alpha > 0$ such that $a^k \geq \alpha 1$, then*

*(i)* $\mathrm{psp}(a^k) = \{r(a)^k\}$,
*(ii)* $\#\mathrm{psp}(a) \leq k$.

The proof of the above result is the same as that of ([42], theorem 4.25) by using theorems 4.1.2 and 4.1.6.

Note that in a $C^*$-algebra $A$ ordered by $C = \{a \in A : a = a^* \text{ and } \sigma(a) \subseteq [0, \infty)\}$, theorem 4.4.10 holds trivially. This is so since if $a \in C$, $\operatorname{psp}(a) = \{r(a)\}$.

Combining theorem 3.2.8 (i) and theorem 4.4.10 (i) we obtain

**Theorem 4.4.11.** *Let $(A, C)$ be a COBA with $C$ closed, c-normal and inverse-closed. If $a \in C$ is an invertible element, then $\operatorname{psp}(a) = \{r(a)\}$.*

In the OBA setting, ([42], theorem 4.26) is the original version of the above result. Also, as in theorem 4.4.10, in a $C^*$-algebra the result holds trivially.

The following theorem gives a characterization of quasi inessential elements. Its original version in the OBA setting was proved by S. Mouton and H. Raubenheimer in ([47], theorem 5.2), but our proof is somewhat neater.

**Theorem 4.4.12.** *Let $(A, C)$ be a COBA with $C$ closed and such that the spectral radius in $(A, C)$ is c-monotone. Let $I$ be a closed inessential ideal of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. If $a \in C$ with $r(a) = 1$, then the following statements are equivalent:*

*(i) $a$ is quasi inessential relative to $I$.*
*(ii) $1 \notin \sigma(a + I)$.*
*(iii) $1$ is a Riesz point of $\sigma(a)$ relative to $I$.*

**Proof.** (i) $\Rightarrow$ (ii): By proposition 1.2.5, we have that $r(a + I) < 1$, so that $1 \notin \sigma(a + I)$.

(ii) $\Rightarrow$ (iii): We have that $r(a + I) \leq r(a) = 1$. By theorem 4.2.5, $r(a + I) \in \sigma(a + I)$. It follows that if $1 \notin \sigma(a + I)$ then $r(a + I) < 1$, so that by theorem 4.4.3, 1 is a Riesz point of $\sigma(a)$ relative to $I$.

(iii) $\Rightarrow$ (i): If 1 is a Riesz point of $\sigma(a)$ relative to $I$, then $r(a + I) < 1$ by theorem 4.4.3. It follows from proposition 1.2.5 that $a$ is quasi inessential relative to $I$. $\square$

A result corresponding to theorem 4.4.12 can be obtained in terms of maximal positive commutative sets directly from theorem 2.4.2 and ([47], theorem 5.2).

## 4.5 Peripheral spectrum: Perturbation by Riesz elements

In [21] V. Caselles studied the peripheral spectrum in relation to compact positive perturbations of certain classes of positive operators. The results obtained are ([21], theorem 5.1, corollary 5.2, theorem 5.4, corollary 5.5). We prove related results for the peripheral spectrum in relation to perturbations of positive elements by positive Riesz elements in the setting of COBAs. All these results have analogues in the OBA setting as well. The main results are corollary 4.5.8 and corollary 4.5.10. These and other results in this section extend the theory of COBAs and OBAs and are attributed to the author. We start with the following proposition.

**Proposition 4.5.1.** *Let $(A, C)$ be a COBA with $C$ closed and such that the spectral radius in $(A, C)$ is monotone, and let $I$ be a closed inessential ideal of $A$. If $a, b \geq 0$ with $b$ Riesz relative to $I$, then the following two statements are equivalent:*

*(i)* $r(a) < r(a + b)$
*(ii)* $\mathrm{psp}(a) \cap \mathrm{psp}(a + b) = \emptyset$.

**Proof.** We prove the non-trivial implication (ii) $\Rightarrow$ (i). By theorem 4.1.6, we have that $r(a) \in \sigma(a)$ and $r(a+b) \in \sigma(a+b)$, so that $r(a) \in \mathrm{psp}(a)$ and $r(a+b) \in \mathrm{psp}(a+b)$. Now since $a, b \geq 0$, monotonicity of the spectral radius implies that $r(a) \leq r(a + b)$. If $r(a) = r(a + b)$ then $\mathrm{psp}(a) \cap \mathrm{psp}(a + b) \neq \emptyset$, which yields the result. □

Note that proposition 4.5.1 remains true if we replace monotonicity with $c$-monotonicity, as long as $a$ and $b$ commute.

In a $C^*$-algebra, $\mathrm{psp}(a) = \{r(a)\}$ for every positive element $a$. Thus if in proposition 4.5.1 $A$ is a $C^*$-algebra then the result holds without using monotonicity.

Before we proceed we introduce the following notation: if $a$ is an element in a Banach algebra $A$ and $I$ a closed inessential ideal of $A$, we denote by $\mathcal{R}_p(a, I)$ the set of Riesz points of $\sigma(a)$ that are in $\mathrm{psp}(a)$.

In theorem 4.5.5 we consider the relationship between $\mathrm{psp}(a)$ and $\mathrm{psp}(a + b)$ when $\mathrm{psp}(a)$ has no Riesz points, where $a$ and $b$ are positive elements in a COBA and $b$ is Riesz relative to a closed inessential ideal. To prove the result we will need corollary 4.5.4, which is a consequence of the following theorem:

**Theorem 4.5.2.** *Let $A$ be a Banach algebra, $a \in A$ and $I$ a closed inessential ideal of $A$. If $b \in A$ is Riesz relative to $I$ and $ab = ba$, then at least one of the conditions*

$$\mathrm{psp}(a) \subseteq \mathrm{psp}(a + b) \cup \mathcal{R}_p(a, I) \text{ and } \mathrm{psp}(a + b) \subseteq \mathrm{psp}(a) \cup \mathcal{R}_p(a + b, I)$$

*holds.*

**Proof.** Case 1: $\mathrm{psp}(a)$ consists of Riesz points of $\sigma(a)$ and $\mathrm{psp}(a + b)$ consists of Riesz points of $\sigma(a + b)$. Then obviously both inclusions hold.

Case 2: $\mathrm{psp}(a)$ consists of Riesz points of $\sigma(a)$ and $\mathrm{psp}(a + b)$ does not consist of Riesz points of $\sigma(a + b)$, i.e.

$$\mathrm{psp}(a + b) \nsubseteq \mathcal{R}_p(a + b, I). \qquad (*)$$

In this case we have $r(a+b) < r(a)$. To prove this, we get from theorem 1.2.7 that $D_I(a) = D_I(a+b)$. Take $\lambda \in \operatorname{psp}(a+b) \setminus \mathcal{R}_p(a+b, I)$. Then $\lambda \in D_I(a)$ and if $|\lambda| = r(a)$, we have that $\lambda \in \operatorname{psp}(a)$, which contradicts the fact that $\operatorname{psp}(a)$ consists of Riesz points of $\sigma(a)$. Thus $|\lambda| < r(a)$, so that $r(a+b) < r(a)$. It follows that $\operatorname{psp}(a+b) \cap \operatorname{psp}(a) = \emptyset$. Together with (*) this implies that $\operatorname{psp}(a+b) \nsubseteq \operatorname{psp}(a) \cup \mathcal{R}_p(a+b, I)$, and clearly $\operatorname{psp}(a) \subseteq \operatorname{psp}(a+b) \cup \mathcal{R}_p(a, I)$.

Case 3: $\operatorname{psp}(a)$ does not consist of Riesz points of $\sigma(a)$ and $\operatorname{psp}(a+b)$ consists of Riesz points of $\sigma(a+b)$. In this case we have $r(a) < r(a+b)$, and similarly as in Case 2 we find that $\operatorname{psp}(a) \nsubseteq \operatorname{psp}(a+b) \cup \mathcal{R}_p(a, I)$ while, clearly, $\operatorname{psp}(a+b) \subseteq \operatorname{psp}(a) \cup \mathcal{R}_p(a+b, I)$.

Case 4: $\operatorname{psp}(a)$ does not consist of Riesz points of $\sigma(a)$ and $\operatorname{psp}(a+b)$ does not consist of Riesz points of $\sigma(a+b)$, i.e. $\operatorname{psp}(a) \cap D_I(a) \neq \emptyset$ and $\operatorname{psp}(a+b) \cap D_I(a+b) \neq \emptyset$. By theorem 1.2.7 $D_I(a) = D_I(a+b)$ and so $\operatorname{psp}(a) \cap D_I(a+b) \neq \emptyset$ and $\operatorname{psp}(a+b) \cap D_I(a) \neq \emptyset$, say $\lambda_1 \in \sigma(a+b)$ with $|\lambda_1| = r(a) \leq r(a+b)$ and $\lambda_2 \in \sigma(a)$ with $|\lambda_2| = r(a+b) \leq r(a)$. It follows that $r(a) = r(a+b)$.

We show that $\operatorname{psp}(a) \subseteq \operatorname{psp}(a+b) \cup \mathcal{R}_p(a, I)$. Let $\lambda \in \operatorname{psp}(a)$. If $\lambda$ is a Riesz point of $\sigma(a)$ then we are done. If $\lambda$ is not a Riesz point of $\sigma(a)$ then $\lambda \in D_I(a)$, so that $\lambda \in D_I(a+b)$ by theorem 1.2.7. Therefore $\lambda \in \sigma(a+b)$ and it follows from $r(a) = r(a+b)$ that $\lambda \in \operatorname{psp}(a+b)$. Similarly, $\operatorname{psp}(a+b) \subseteq \operatorname{psp}(a) \cup \mathcal{R}_p(a+b, I)$. $\square$

Note that if $b$ in theorem 4.5.2 is an element of $I$, the condition $ab = ba$ can be dropped. This is because then theorem 1.2.6 can be used in the place of theorem 1.2.7. The following example, however, shows that the condition $ab = ba$ cannot in general be omitted.

**Example 4.5.3.** *Let $u \in \mathcal{L}(\ell^2)$ be the linear operator $u(x_1, x_2, ...) = (0, x_1, x_2, ...)$ for every $x = (x_1, x_2, ...) \in \ell^2$. Let $T$ and $S$ in the algebra $A = \mathcal{L}(\ell^2 \oplus \ell^2)$ be defined by $T(x, y) = (uy, x)$ and $S(x, y) = (0, x)$ for every $(x, y) \in \ell^2 \oplus \ell^2$. Then $S$ is a Riesz operator relative to the closed inessential ideal of compact operators $I = \mathcal{K}(\ell^2 \oplus \ell^2)$ in $A$, $TS \neq ST$ and neither of the inclusions $\operatorname{psp}(T) \subseteq \operatorname{psp}(T+S) \cup \mathcal{R}_p(T, I)$ and $\operatorname{psp}(T+S) \subseteq \operatorname{psp}(T) \cup \mathcal{R}_p(T+S, I)$ holds.*

**Proof.** From ([30], example 1) we get that $S$ is Riesz relative to $I$ and that $ST \neq TS$. From the same result we also get that $\sigma(T) = \mathbf{B}$, $\sigma(T+S) = \sqrt{2}\mathbf{B}$, $D_I(T) = \sigma(T)$ and $D_I(T+S) = \sigma(T+S)$, where $\mathbf{B} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. This means that $\operatorname{psp}(T) \cap \operatorname{psp}(T+S) = \emptyset$, $\operatorname{psp}(T)$ has no Riesz points of $\sigma(T)$ and $\operatorname{psp}(T+S)$ has no Riesz points of $\sigma(T+S)$. Hence neither of the mentioned inclusions holds. $\square$

It follows from cases 1 and 4 in the proof of theorem 4.5.2 that:

**Corollary 4.5.4.** *Let $A$ be a Banach algebra, $a \in A$ and $I$ a closed inessential ideal of $A$. Suppose that $b \in A$ is Riesz relative to $I$, that $r(a) = r(a+b)$ and $ab = ba$. Then $\operatorname{psp}(a) \subseteq \operatorname{psp}(a+b) \cup \mathcal{R}_p(a, I)$ and $\operatorname{psp}(a+b) \subseteq \operatorname{psp}(a) \cup \mathcal{R}_p(a+b, I)$.*

Using corollary 4.5.4 we prove the following result:

**Theorem 4.5.5.** *Let $(A, C)$ be a COBA with $C$ closed and such that the spectral radius in $(A, C)$ is c-monotone, and let $I$ be a closed inessential ideal of $A$. Suppose that $a, b \geq 0$ with*

*$ab = ba$ and that $b$ is Riesz relative to $I$. If $\operatorname{psp}(a)$ has no Riesz points of $\sigma(a)$ then one of $r(a) < r(a+b)$ or $\operatorname{psp}(a+b) = \operatorname{psp}(a) \cup \mathcal{R}_p(a+b, I)$ holds.*

**Proof.** Since $a, b \geq 0$ and $ab = ba$, $c$-monotonicity of the spectral radius implies that $r(a) \leq r(a+b)$. If $r(a) < r(a+b)$ then we are done. If $r(a) = r(a+b)$ then by the previous corollary, we get the inclusions $\operatorname{psp}(a) \subseteq \operatorname{psp}(a+b) \cup \mathcal{R}_p(a, I)$ and $\operatorname{psp}(a+b) \subseteq \operatorname{psp}(a) \cup \mathcal{R}_p(a+b, I)$. Since $\operatorname{psp}(a)$ has no Riesz points, from the first inclusion we obtain that $\operatorname{psp}(a) \subseteq \operatorname{psp}(a+b)$. Together with the second inclusion, this implies that $\operatorname{psp}(a+b) = \operatorname{psp}(a) \cup \mathcal{R}_p(a+b, I)$. $\square$

The next example gives an instance of theorem 4.5.5.

**Example 4.5.6.** *Let $I, P \in A = \mathcal{L}(\ell^2)$, where $I$ is the identity operator and $P$ a projection of $\ell^2$ on a finite dimensional subspace of $\ell^2$. Then we have that $I, P \geq 0$, $IP = PI$ and $P$ is Riesz relative to $F = \mathcal{K}(\ell^2)$. Furthermore, $\operatorname{psp}(I)$ has no Riesz points of $\sigma(I)$. Also, $r(I) < r(I+P)$ but $\operatorname{psp}(I+P) \neq \operatorname{psp}(I) \cup \mathcal{R}_p(I+P, F)$, illustrating theorem 4.5.5.*

**Proof.** By example 2.1.18, $A$ is a COBA with a closed, normal algebra $c$-cone $C = \{T \in A : T \geq 0\}$. Obviously $I, P \geq 0$, $IP = PI$, $\sigma(I) = \{1\}$ and $\sigma(I+P) = \{1, 2\}$, so that $r(I) < r(I+P)$. Clearly, $P \in F$, and so it is Riesz relative to $F$. To show that $\operatorname{psp}(I)$ has no Riesz points of $\sigma(I)$, we note that $\operatorname{psp}(I) = \sigma(I) = \{1\}$. Since the spectral projection corresponding to 1 and $I$ is $I$ and $F$ is a proper ideal of $A$, it follows that 1 is not a Riesz point of $\sigma(I)$ relative to $F$. Now, from theorem 1.2.7, we get that $D_F(I+P) = D_F(I) = \{1\}$. Therefore $\mathcal{R}_p(I+P, F) = \{2\}$, and so $\operatorname{psp}(I+P) \neq \operatorname{psp}(I) \cup \mathcal{R}_p(I+P, F)$. $\square$

Note also that the converse of theorem 4.5.5 is false, as the following counter example shows.

**Example 4.5.7.** *Consider $A = M_2(\mathbb{C})$ ordered by $C = \{a \in A : a = a^* \text{ and } \sigma(a) \subseteq [0, \infty)\}$, where $a^*$ denotes the complex conjugate transpose. Let $a = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in A$. Then $b, c$ are Riesz relative to $A$, $ab = ba$, $ac = ca$ and $a, b, c \in C$. We have $r(a) < r(a+b)$ and $\operatorname{psp}(a+c) = \operatorname{psp}(a) \cup \mathcal{R}_p(a+c, A)$, but $\operatorname{psp}(a)$ consists of Riesz points of $\sigma(a)$.*

**Proof.** For the closed inessential ideal of $A$ we take the whole algebra $A$. We have that $r(a) = 2 < r(a+b) = 3$, and $\operatorname{psp}(a) = \{2\}$ consists of Riesz points of $\sigma(a)$. Also, it follows from $\operatorname{psp}(a+c) = \{2\}$ and $\mathcal{R}_p(a+c, A) = \{2\}$ that $\operatorname{psp}(a+c) = \operatorname{psp}(a) \cup \mathcal{R}_p(a+c, A)$. $\square$

Finally, combining proposition 4.5.1 and theorem 4.5.5 we obtain the following result, which shows that if $\operatorname{psp}(a)$ has no Riesz points of $\sigma(a)$, then one of two very dissimilar properties holds.

**Corollary 4.5.8.** *Let $(A, C)$ be a COBA with $C$ closed and such that the spectral radius in $(A, C)$ is $c$-monotone, and let $I$ be a closed inessential ideal of $A$. Suppose that $a, b \geq 0$ with $ab = ba$ and that $b$ is Riesz relative to $I$. If $\operatorname{psp}(a)$ has no Riesz points of $\sigma(a)$, then one of $\operatorname{psp}(a) \cap \operatorname{psp}(a+b) = \emptyset$ or $\operatorname{psp}(a+b) = \operatorname{psp}(a) \cup \mathcal{R}_p(a+b, I)$ holds.*

If $A$ in corollary 4.5.8 is a $C^*$-algebra, then the result holds without the commutativity or monotonicity assumptions, and without any assumptions on $\operatorname{psp}(a)$, since $\operatorname{psp}(a) = \{r(a)\}$ for all positive $a$ in a $C^*$-algebra.

In corollary 4.5.8, we gave the relationship between $\operatorname{psp}(a)$ and $\operatorname{psp}(a+b)$ when $\operatorname{psp}(a)$ has no Riesz points. When we only know that $r(a)$ is not a Riesz point of $\sigma(a)$, we get the following result.

**Theorem 4.5.9.** *Let $(A, C)$ be a COBA such that $C$ is closed and the spectral radius in $(A, C)$ is c-monotone, and let $I$ be a closed inessential ideal of $A$. Suppose that $a, b \geq 0$ with $b$ Riesz relative to $I$ and $ab = ba$. If $r(a)$ is not a Riesz point of $\sigma(a)$, then the following two assertions are equivalent:*

*(i)* $r(a) < r(a+b)$,
*(ii)* $\operatorname{psp}(a+b)$ *consists of Riesz points of* $\sigma(a+b)$.

**Proof.** (i) $\Rightarrow$ (ii): Suppose that $r(a) < r(a+b)$. Let $\lambda \in \operatorname{psp}(a+b)$. If $\lambda$ is not a Riesz point of $\sigma(a+b)$, then $\lambda \in D_I(a+b)$. It follows from theorem 1.2.7 that $\lambda \in D_I(a) \subseteq \sigma(a)$, so that $|\lambda| \leq r(a)$. This contradicts $r(a) < r(a+b)$. Thus $\lambda$ is a Riesz point of $\sigma(a+b)$.

(ii) $\Rightarrow$ (i): From theorem 4.1.6 we obtain that $r(a) \in \sigma(a)$ and $r(a+b) \in \sigma(a+b)$, so that $r(a) \in \operatorname{psp}(a)$ and $r(a+b) \in \operatorname{psp}(a+b)$. Since $a, b \geq 0$ and $ab = ba$, $c$-monotonicity of the spectral radius implies that $r(a) \leq r(a+b)$. Since $r(a)$ is not a Riesz point of $\sigma(a)$ by assumption, we obtain from theorem 1.2.7 that $r(a) \in D_I(a) = D_I(a+b)$. Now, since $\operatorname{psp}(a+b)$ consists of Riesz points of $\sigma(a+b)$, we have that $r(a+b) \notin D_I(a+b)$, so that $r(a) \neq r(a+b)$. Thus $r(a) < r(a+b)$. $\square$

Finally, combining proposition 4.5.1 and theorem 4.5.9 we obtain the following result.

**Corollary 4.5.10.** *Let $(A, C)$ be a COBA with $C$ closed and such that the spectral radius in $(A, C)$ is c-monotone, and let $I$ be a closed inessential ideal of $A$. Suppose that $a, b \geq 0$ with $ab = ba$ and that $b$ is Riesz relative to $I$. If $r(a)$ is not a Riesz point of $\sigma(a)$, then the following two assertions are equivalent:*

*(i)* $\operatorname{psp}(a) \cap \operatorname{psp}(a+b) = \emptyset$,
*(ii)* $\operatorname{psp}(a+b)$ *consists of Riesz points of* $\sigma(a+b)$.

A number of results in this section rely on theorem 1.2.7. (Note that ([30], theorem 2) can also be used instead.) Therefore for these results, the condition $ab = ba$ cannot be dropped even in an OBA.

## 4.6 Convergence properties

If a sequence $(a_n)$ of positive elements of an OBA converges to an element $a$ in the OBA, the problem of determining which properties of $r(a)$ are inherited by $r(a_n)$ was studied by S.

Mouton in [43]. This problem was originally studied in the context of positive operators on Banach lattices by Arándiga and Caselles in [4], [5] and [6]. In this section we will adapt the results in [43] to the more general COBA setting. The results involving MPCSs in COBAs will generally not be presented if they can be obtained directly from theorem 2.4.2 and the associated OBA result. It will be mentioned whenever such results occur.

We start with the following result, which is about a continuity property of the spectral radius. Its original OBA version is ([43], theorem 4.5).

**Theorem 4.6.1.** *Let $(A, C)$ be a COBA with $C$ closed and $I$ a closed inessential ideal of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. Suppose that $a \in A$, $a_n \in C$ for all $n \in \mathbb{N}$ and $a_n \to a$ as $n \to \infty$. If $r(a)$ is a Riesz point of $\sigma(a)$, then $r(a_n) \to r(a)$ as $n \to \infty$.*

**Proof.** Since $C$ is closed, $a \in C$. From the assumption that $r(a)$ is a Riesz point of $\sigma(a)$ and from theorem 4.4.5, it follows that $\mathrm{psp}(a)$ consists of Riesz points of $\sigma(a)$. By lemma 1.2.2, we have that $r(a_n) \to r(a)$ as $n \to \infty$. $\square$

A result corresponding to theorem 4.6.1 can be obtained in terms of MPCSs directly from theorem 2.4.2 and ([43], theorem 4.5).

The next result is a COBA analogue of ([43], theorem 4.6).

**Theorem 4.6.2.** *Let $(A, C)$ be a COBA with $C$ closed and such that the spectral radius in $(A, C)$ is c-monotone. Let $I$ be a closed inessential ideal of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. Suppose that $a \in A$, $a_n \in C$ for all $n \in \mathbb{N}$ and $a_n \to a$ as $n \to \infty$. If $r(a)$ is a Riesz point of $\sigma(a)$, then there is a natural number $N$ such that, for all $n \geq N$, $r(a_n)$ is a Riesz point of $\sigma(a_n)$.*

The proof of theorem 4.6.2 is the same as the proof of ([43], theorem 4.6) by using theorems 4.4.3 and 4.6.1. Using theorem 2.4.2 and ([43], theorem 4.6), we can obtain a result corresponding to theorem 4.6.2 in terms of MPCSs.

A COBA version of ([43], theorem 4.7) can be obtained using MPCSs directly from theorem 2.4.2 and ([43], theorem 4.7). The following is its corollary, and will be used to prove another result later in this section.

**Corollary 4.6.3.** *Let $(A, C)$ be a COBA with $C$ closed, $M$ an MPCS in $A$ and $I$ a closed inessential ideal of $A$ such that the spectral radius in $(A/I, \pi M)$ is monotone. Suppose that $a \in A$, $a_n \in M$ for all $n \in \mathbb{N}$, that $a_n \to a$ as $n \to \infty$ and that $r(a)$ is a Riesz point of $\sigma(a)$. If $\alpha \in \mathbb{C}$, $\alpha_n \in \mathrm{psp}(a_n)$ for all $n \in \mathbb{N}$ and $\alpha_n \to \alpha$ as $n \to \infty$, then there is a natural number $N$ such that for all $n \geq N$, $\alpha_n$ is a Riesz point of $\sigma(a_n)$.*

The original OBA counterpart of corollary 4.6.3 is ([43], corollary 4.8) and it can be established with a similar proof.

The following are COBA counterparts of ([43], theorem 4.7) and corollary 4.6.3 in terms of $C'$OBAs. Their proofs are the same as those of ([43], theorem 4.7, corollary 4.8) respectively.

**Theorem 4.6.4.** *Let $(A, C)$ be a COBA with $C$ closed and $I$ a closed inessential of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. Suppose that $a \in A$, $a_n \in C$ for all $n \in \mathbb{N}$, that $a_n \to a$ as $n \to \infty$, and that $r(a)$ is Riesz point of $\sigma(a)$. If $\alpha \in \operatorname{psp}(a)$, $\alpha_n \in \partial_\infty \sigma(a_n)$ for all $n \in \mathbb{N}$ and $\alpha_n \to \alpha$ as $n \to \infty$, then there is an $N \in \mathbb{N}$, such that for all $n \geq N$, $\alpha_n$ is a Riesz point of $\sigma(a_n)$.*

**Corollary 4.6.5.** *Let $(A, C)$ be a COBA with $C$ closed and $I$ a closed inessential ideal of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. Suppose that $a \in A$, $a_n \in C$ for all $n \in \mathbb{N}$, that $a_n \to a$ as $n \to \infty$ and that $r(a)$ is a Riesz point of $\sigma(a)$. If $\alpha \in \mathbb{C}$, $\alpha_n \in \operatorname{psp}(a_n)$ for all $n \in \mathbb{N}$ and $\alpha_n \to \alpha$ as $n \to \infty$, then there is an $N \in \mathbb{N}$ such that, for all $n \geq N$, $\alpha_n$ is a Riesz point of $\sigma(a_n)$.*

We prove ([43], corollary 4.10) in our current setting. We first remark that if $\mathcal{L}(X)$ is the Banach algebra of all bounded linear operators on a Banach space $X$, then the set $\mathcal{K}(X)$ of all compact operators in $\mathcal{L}(X)$ is a closed inessential ideal of $\mathcal{L}(X)$ ([9], p.106).

**Corollary 4.6.6.** *Let $H$ be a Hilbert space. The positive operators on $H$ have the following properties:*

*(i) If $T$ is a positive operator on $H$ and $r(T, \mathcal{L}(H))$ is a Riesz point of $\sigma(T, \mathcal{L}(H))$, then the peripheral spectrum of $T$ in $\mathcal{L}(H)$ consists of Riesz points of $\sigma(T, \mathcal{L}(H))$.*

*(ii) Suppose that $(T_n)$ is a sequence of positive operators converging uniformly to an operator $T$. If $r(T, \mathcal{L}(H))$ is a Riesz point of $\sigma(T, \mathcal{L}(H))$, then*
*(a) $r(T_n, \mathcal{L}(H)) \to r(T, \mathcal{L}(H))$ as $n \to \infty$ and*
*(b) for all $n$ big enough $r(T_n, \mathcal{L}(H))$ is a Riesz point of $\sigma(T_n, \mathcal{L}(H))$.*

*(iii) Suppose that $S$ and $T$ are positive operators on $H$ satisfying $0 \leq S \leq T$ and $ST = TS$. If $r(S, \mathcal{L}(H)) = r(T, \mathcal{L}(H))$ and $r(T, \mathcal{L}(H))$ is a Riesz point of $\sigma(T, \mathcal{L}(H))$, then the peripheral spectrum of $S$ in $\mathcal{L}(H)$ consists of the Riesz points of $\sigma(S, \mathcal{L}(H))$.*

*(iv) Suppose that $T_n$ is a sequence of positive operators converging uniformly to an operator $T$. Suppose that $r(T, \mathcal{L}(H))$ is a Riesz point of $\sigma(T, \mathcal{L}(H))$. If $\alpha_n \in \partial_\infty \sigma(T_n, \mathcal{L}(H))$ for all $n \in \mathbb{N}$ and $\alpha_n \to \alpha$ as $n \to \infty$ where $\alpha$ is in the peripheral spectrum of $T$ in $\mathcal{L}(H)$, then, for all $n$ big enough, $\alpha_n$ is a Riesz point of $\sigma(T_n, \mathcal{L}(H))$.*

**Proof.** Let $C = \{S \in \mathcal{L}(H) : S \geq 0\}$. Then by example 2.1.18, we have that $(\mathcal{L}(H), C)$ is a COBA with $C$ closed and normal, and the spectral radius in $(\mathcal{L}(H), C)$ is monotone. Let $I = \mathcal{K}(H)$. If $\pi : \mathcal{L}(H) \to \mathcal{L}(H)/I$ is the canonical homomorphism, then $(\mathcal{L}(H)/I, \pi C)$ is a $C'$OBA and $\pi C$ is $c$-normal; hence the spectral radius is $c$-monotone. Therefore:

(i) Follows from theorem 4.4.5.
(ii) Part (a) follows from theorem 4.6.1 and (b) follows from theorem 4.6.2.
(iii) Follows from theorem 4.4.8.
(iv) Follows from theorem 4.6.4. □

Recall that $\mathcal{L}(H)$ is a non-commutative $C^*$-algebra. Therefore the structure under consideration in ([43], corollary 4.10) is a typical COBA. Although the corresponding quotient algebra does not have the COBA structure, it does however, have the $C'$OBA structure. We have used this fact to prove corollary 4.6.6, whereas ([43], corollary 4.10) uses different means to prove the result. So ([43], corollary 4.10) is a typical result that motivates the introduction of $C'$OBAs. Also note that corollary 4.6.6 is a stronger result than ([43], corollary 4.10), since the assumption that operators in the sequence $(T_n)$ in corollary 4.6.6 (ii) and (iv) mutually commute has been dropped.

Theorems 4.6.7 and 4.6.8 are COBA analogues of ([43], theorem 5.5) and ([43], theorem 5.6) respectively. They will be followed by the $C'$OBA analogues. These results give conditions under which certain properties of the Laurent series of the resolvents of a sequence of positive elements are inherited by the Laurent series of the limit of such a sequence.

**Theorem 4.6.7.** *Let $(A, C)$ be a semisimple COBA with $C$ closed, $M$ an MPCS in $A$ and $I$ a closed inessential ideal in $A$ such that the spectral radius in $(A/I, \pi M)$ is monotone. Suppose that $a \in A$, $a_n \in M$ for all $n \in \mathbb{N}$, that $a_n \to a$ as $n \to \infty$ and that $r(a)$ is a Riesz point of $\sigma(a)$. If $\alpha_n \in \operatorname{psp}(a_n)$ such that $\alpha_n \to \alpha$, then the following hold:*

*(i) For all $n$ large enough, $\alpha_n$ is a pole, say of order $k_n$, of $(\lambda 1 - a_n)^{-1}$, and $\alpha$ is a pole, say of order $k$, of $(\lambda 1 - a)^{-1}$.*

*(ii) If $(\lambda 1 - a)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - \alpha)^j b_j$ ($b_{-j} = 0$ for all $j > k$) and if for all $n \geq N$, we have that $(\lambda 1 - a_n)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - \alpha_n)^j b_{n,j}$ ($b_{n,-j} = 0$ for all $j > k_n$), then $b_{n,j} \to b_j$ as $n \to \infty$, for all $j \in \mathbb{Z}$.*
*(iii) If $p = p(a, \alpha)$ and $p_n = p(a_n, \alpha_n)$, then $p_n \to p$ as $n \to \infty$.*
*(iv) If $k_n \leq k$ for all $n \geq N_1$, for some $N_1 \in \mathbb{N}$, and $u = b_{-k}$, $u_n = b_{n,-k_n}$ (where $au = \alpha u = ua$ and $a_n u_n = \alpha_n u_n = u_n a_n$), then $u_n \to u$ as $n \to \infty$.*

**Proof.** (i) From theorem 4.4.4, we have that $\operatorname{psp}(a)$ consists of Riesz points of $\sigma(a)$. It follows from lemma 1.2.2 2(c) that $\alpha \in \operatorname{psp}(a)$, so that $\alpha$ is a Riesz point of $\sigma(a)$. By lemma 1.2.3 we have that $\alpha$ is a pole of $(\lambda 1 - a)^{-1}$. From corollary 4.6.3 we have that $\alpha_n$ is a Riesz point of $\sigma(a_n)$, and hence a pole of $(\lambda 1 - a_n)^{-1}$, for all $n$ big enough.

(ii) This result follows from (i) and theorem 1.1.10.
(iii) Follows from (i) and corollary 1.1.11.
(iv) Follows from (i) and corollary 1.1.12. □

**Theorem 4.6.8.** *Let $(A, C)$ be a semisimple COBA with $C$ closed, $M$ an MPCS in $A$ such that the spectral radius in $(A, M)$ is monotone. Suppose that $I$ is a closed inessential ideal in $A$ such that the spectral radius in $(A/I, \pi M)$ is monotone, and that $a \in A$, $a_n \in M$ for all $n \in \mathbb{N}$. If $a_n \to a$ as $n \to \infty$ and $r(a)$ is a Riesz point of $\sigma(a)$, then the following hold:*

*(i) For all $n$ large enough, $r(a_n)$ is a pole, say of order $k_n$, of $(\lambda 1 - a_n)^{-1}$, and $r(a)$ is pole, say of order $k$, of $(\lambda 1 - a)^{-1}$.*

*(ii) If* $(\lambda 1 - a)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - r(a))^j b_j$ *($b_{-j} = 0$ for all $j > k$) and if for all $n \geq N$ we have that* $(\lambda 1 - a_n)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - r(a_n)) b_{n,j}$ *($b_{n,-j} = 0$ for all $j > k_n$), then $b_{n,j} \to b_j$ as $n \to \infty$, for all $j \in \mathbb{Z}$.*
*(iii) If $p = p(a, r(a))$ and $p_n = p(a_n, r(a_n))$, then $p_n \to p$ as $n \to \infty$.*
*(iv) Let $u$ denote the positive Laurent eigenvector of the eigenvalue $r(a)$ of $a$, and $u_n$ the positive Laurent eigenvector of the eigenvalue $r(a_n)$ of $a_n$. If $k_n \leq k$ for all $n \geq N_1$, for some $N_1 \in \mathbb{N}$, then $u_n \to u$ as $n \to \infty$.*

**Proof.** By corollary 4.1.7 we have that $r(a_n) \in \sigma(a_n)$, and hence $r(a_n) \in \operatorname{psp}(a_n)$ for all $n \in \mathbb{N}$. Since $r(a)$ is a Riesz point of $\sigma(a)$, theorem 4.4.4 implies that $\operatorname{psp}(a)$ consists of Riesz points of $\sigma(a)$. It follows from lemma 1.2.2 2(b) that $r(a_n) \to r(a)$. The results (i)-(iv) then follow from theorem 4.6.7. □

If the sequence $(a_n)$ in theorem 4.6.8 has the property that $r(a_n) \in \sigma(a_n)$ for all $n$ large enough, then monotonicity of the spectral radius in $(A, M)$ is not required, since it is needed only to guarantee that $r(a_n) \in \sigma(a_n)$.

**Theorem 4.6.9.** *Let $(A, C)$ be a semisimple COBA with $C$ closed and let $I$ be a closed inessential ideal of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. Suppose that $a \in A$, $a_n \in C$ for all $n \in \mathbb{N}$, that $a_n \to a$ as $n \to \infty$ and that $r(a)$ is a Riesz point of $\sigma(a)$. If $\alpha_n \in \operatorname{psp}(a_n)$ such that $\alpha_n \to \alpha$ as $n \to \infty$, then the following hold:*

*(i) For all $n$ large enough, $\alpha_n$ is a pole, say of order $k_n$, of $(\lambda 1 - a_n)^{-1}$, and $\alpha$ is a pole, say of order $k$, of $(\lambda 1 - a)^{-1}$.*
*(ii) If* $(\lambda 1 - a)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - \alpha)^j b_j$ *($b_{-j} = 0$ for all $j > k$) and for all $n \geq N$* $(\lambda 1 - a_n)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - \alpha_n)^j b_{n,j}$ *($b_{n,-j} = 0$ for all $j > k_n$), then $b_{n,j} \to b_j$ as $n \to \infty$, for all $j \in \mathbb{Z}$.*
*(iii) If $p = p(a, \alpha)$ and $p_n = p(a_n, \alpha_n)$, then $p_n \to p$ as $n \to \infty$.*
*(iv) If $k_n \leq k$ for all $n \geq N_1$, for some $N_1 \in \mathbb{N}$, and $u = b_{-k}$, $u_n = b_{n,-k_n}$ (where $au = \alpha u = ua$ and $a_n u_n = \alpha_n u_n = u_n a_n$, then $u_n \to u$ as $n \to \infty$.*

**Proof.** (i) By theorem 4.4.5, $\operatorname{psp}(a)$ consists of Riesz points of $\sigma(a)$. It follows from lemma 1.2.2 (2c) that $\alpha \in \operatorname{psp}(a)$, so that $\alpha$ is a Riesz point of $\sigma(a)$. By lemma 1.2.3, we have that $\alpha$ is a pole of $(\lambda 1 - a)^{-1}$. It follows from corollary 4.6.5 that $\alpha_n$ is a Riesz point of $\sigma(a_n)$ and hence a pole of $(\lambda 1 - a_n)^{-1}$, for $n$ large enough.

(ii) Follows from (i) and theorem 1.1.10.
(iii) Follows from (i) and corollary 1.1.11.
(iv) Follows from (i) and corollary 1.1.12. □

**Theorem 4.6.10.** *Let $(A, C)$ be a semisimple COBA with $C$ closed and such that the spectral radius in $(A, C)$ is c-monotone. Let $I$ be a closed inessential ideal of $A$ such that the spectral* *radius in the $C'OBA$ $(A/I, \pi C)$ is c-monotone. Suppose that $a \in A$, $a_n \in C$ for all $n \in \mathbb{N}$, that $a_n \to a$ as $n \to \infty$ and that $r(a)$ is a Riesz point of $\sigma(a)$. Then the following hold:*

*(i) For all $n$ large enough, $r(a_n)$ is a pole, say of order $k_n$, of $(\lambda 1 - a_n)^{-1}$, and $r(a)$ is a pole, say of order $k$, of $(\lambda 1 - a)^{-1}$.*
*(ii) If $(\lambda 1 - a)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - r(a))^j b_j$ ($b_{-j} = 0$ for all $j > k$) and for all $n \geq N$ $(\lambda 1 - a_n)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - r(a_n))^j b_{n,j}$ ($b_{n,-j} = 0$ for all $j > k_n$), then $b_{n,j} \to b_j$ as $n \to \infty$, for all $j \in \mathbb{Z}$.*
*(iii) If $p = p(a, r(a))$ and $p_n = p(a_n, r(a_n))$, then $p_n \to p$ as $n \to \infty$.*
*(iv) Let $u$ denote the positive Laurent eigenvector of the eigenvalue $r(a)$ of $a$, and $u_n$ the positive Laurent eigenvector of the eigenvalue $r(a_n)$ of $a_n$. If $k_n \leq k$ for all $n \geq N_1$, for some $N_1 \in \mathbb{N}$, then $u_n \to u$ as $n \to \infty$.*

**Proof.** Since the spectral radius in $(A, C)$ is $c$-monotone, $r(a_n) \in \sigma(a_n)$ by theorem 4.1.6. Therefore $r(a_n) \in \operatorname{psp}(a_n)$. By theorem 4.4.5, $\operatorname{psp}(a)$ consists of Riesz points of $\sigma(a)$. It follows from lemma 1.2.2 (2b) that $r(a_n) \to r(a)$. From theorem 4.6.9, the results (i)-(iv) follow. □

## 4.7 Trace

The trace of elements of Banach algebras has been studied by several authors including [49] and [10]. In the context of ordered Banach algebras, a problem that naturally arises is that of determining when a positive element in a Banach algebra has a positive trace. In this section we will investigate this problem and give some results. Our main result is theorem 4.7.6. The results in this section extend the theory of COBAs and OBAs and are due to the author.

We will start with the following remark.

**Remark 4.7.1.** *Let $A$ be a semisimple Banach algebra and $a \in \operatorname{Soc}(A)$ such that $\sigma(a) \subseteq [0, \infty)$. Then in terms of the spectral definition of the trace, it is clear that the element $a$ has a positive trace. In view of this, if $A$ is a $C^*$-algebra ordered by the algebra c-cone $C = \{a \in A : a^* = a$ and $\sigma(a) \subseteq [0, \infty)\}$, then every element in $\operatorname{Soc}(A) \cap C$ has a positive trace.*

The following proposition, which gives conditions under which a positive rank one element has a positive trace, will be required in the proof of our main result.

**Proposition 4.7.2.** *Let $A$ be a semisimple COBA with a closed algebra c-cone $C$ such that the spectral radius in $(A, C)$ is c-monotone. If $x \in \mathcal{F}_1(A) \cap C$ then $\operatorname{tr}(x) \geq 0$.*

**Proof.** We have that $x^2 = \operatorname{tr}(x)x$. Using the spectral mapping theorem, it can easily be shown that if $\dim(A) = 1$ then $\sigma(x) = \{\operatorname{tr}(x)\}$ and if $\dim(A) > 1$, then $\sigma(x) = \{0, \operatorname{tr}(x)\}$. Now by theorem 4.1.6, we have that $r(x) \in \sigma(x)$. This implies that $\operatorname{tr}(x) \geq 0$. □

Let $A$ be a semisimple COBA with an algebra $c$-cone $C$. We define the set

$$\mathcal{F}_+ = \{0\} \cup \{x \in \mathcal{F}(A) : x = \sum_{i=1}^{n} x_i, \text{ with } x_i \in \mathcal{F}_1(A) \cap C \text{ for } i = 1, ..., n\}.$$

From the definition of $\mathcal{F}_+$, it is clear that $\mathcal{F}_+ \subseteq \mathcal{F}(A) \cap C$. In some cases, the sets $\mathcal{F}_+$ and $\mathcal{F}(A) \cap C$ actually coincide. We illustrate this with the following examples.

**Example 4.7.3.** *Let $A = M_2(\mathbb{C})$ and $C = \{(\alpha_{ij}) \in A : \alpha_{ij} \geq 0, i, j = 1, 2\}$. Then $C$ is a closed, normal algebra cone in $A$ and $\mathcal{F}_+ = \mathcal{F}(A) \cap C$.*

**Proof.** The first part follows from example 2.1.15. Since $A$ is semisimple (example 1.3.1) and finite dimensional, $\mathcal{F}(A) = \text{Soc}(A) = A$, so that $\mathcal{F}(A) \cap C = C$. So to show that $\mathcal{F}_+ = \mathcal{F}(A) \cap C$, we show that $\mathcal{F}_+ = C$. For the non-trivial inclusion, let $M = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in C$. Then the non-invertible matrices $M_1 = \begin{pmatrix} \alpha_{11} & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & \alpha_{12} \\ 0 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 \\ \alpha_{21} & 0 \end{pmatrix}$ and $M_4 = \begin{pmatrix} 0 & 0 \\ 0 & \alpha_{22} \end{pmatrix}$ are all in $C$. From example 1.3.1, we have that $M_1, M_2, M_3, M_4 \in \mathcal{F}_1(A)$. Since $M = \sum_{i=1}^{4} M_i$, it follows that $M \in \mathcal{F}_+$. Hence $C \subseteq \mathcal{F}_+$. $\square$

**Example 4.7.4.** *Let $A = \ell^\infty$ and $C = \{(\alpha_i) \in A : \alpha_i \geq 0$ for all $i \in \mathbb{N}\}$. Then $C$ is a closed, normal algebra cone of $A$ and $\mathcal{F}_+ = \mathcal{F}(A) \cap C$.*

**Proof.** The first part follows from example 2.1.16. We show the non-trivial inclusion $\mathcal{F}(A) \cap C \subseteq \mathcal{F}_+$. Let $x \in \mathcal{F}(A) \cap C$. If $x = 0$ then $x \in \mathcal{F}_+$ and we are done. If $x \neq 0$ then $x = \sum_{i=1}^{n} x_i$, where $x_i \in \mathcal{F}_1(A)$, $i = 1, ..., n$. By example 1.3.2, the rank one elements of $A$ are the sequences with one non-zero term and zeroes elsewhere. Now since $x \in C$ its terms are either zeroes or positive real numbers. This implies that the non-zero term of $x_i$ is a positive real number for all $i = 1, ..., n$. Therefore $x_i \in C$. Hence $x \in \mathcal{F}_+$. $\square$

**Example 4.7.5.** *Let $K$ be a completely regular Hausdorff space and let $C_b(K)$ be the Banach algebra of all complex valued bounded continuous functions on $K$ with the supremum norm. If $C = \{f \in C_b(K) : f(k) \geq 0$ for all $k \in K\}$, then $C$ is a normal algebra cone of $C_b(K)$ and $\mathcal{F}_+ = \mathcal{F}(C_b(K)) \cap C$.*

**Proof.** By ([13], example 5.1), we have that $C$ is a normal algebra cone of $C_b(K)$. We prove the non-trivial inclusion $\mathcal{F}(C_b(K)) \cap C \subseteq \mathcal{F}_+$. Let $f \in \mathcal{F}(C_b(K)) \cap C$. Then $f(k) \geq 0$ for all $k \in K$ and $f = \sum_{i=1}^{n} f_i$, where $f_i \in \mathcal{F}_1(C_b(K))$. By example 1.3.3, for each $i = 1, ..., n$, there is an $s_i \in$ **iso** $(K)$ and an $\alpha_i \in \mathbb{C}$ such that

$$f_i(t) = \begin{cases} \alpha_i & if\ t = s_i \\ 0 & if\ t \neq s_i. \end{cases}$$

It follows from $f = \sum_{i=1}^{n} f_i$ that $f(t) = \alpha_i$ if $t = s_i$ for $i = 1, ..., n$, and $f(t) = 0$ otherwise. Since $f(k) \geq 0$ for all $k \in K$, we get that $\alpha_i \geq 0$ for $i = 1, ..., n$. Hence $f \in \mathcal{F}_+$. $\square$

We now present our main result.

**Theorem 4.7.6.** *Let $A$ be a semisimple COBA with a closed algebra c-cone $C$ such that the spectral radius in $(A, C)$ is c-monotone. If $x \in \mathcal{F}_+$, then* $\mathrm{tr}(x) \geq 0$.

**Proof.** If $x = 0$ then we are done. If $x \neq 0$, then since $x \in \mathcal{F}_+$, there exist $n \in \mathbb{N}$ and $x_i \in \mathcal{F}_1(A) \cap C$ such that $x = \sum_{i=1}^n x_i$. Therefore $\mathrm{tr}(x)$ is defined and $\mathrm{tr}(x) = \sum_{i=1}^n \mathrm{tr}(x_i)$. The result now follows from proposition 4.7.2. □

The following corollary follows immediately from theorem 4.7.6.

**Corollary 4.7.7.** *Let $A$ be a semisimple COBA with a closed algebra c-cone $C$ such that the spectral radius in $(A, C)$ is c-monotone and $\mathcal{F}(A) \cap C = \mathcal{F}_+$. If $x \in \mathcal{F}(A) \cap C$, then* $\mathrm{tr}(x) \geq 0$.

The following theorem gives conditions under which the trace of a dominated positive finite rank element is less than the trace of the dominating element.

**Theorem 4.7.8.** *Let $A$ be a semisimple COBA with a closed algebra c-cone $C$ such that the spectral radius in $(A, C)$ is c-monotone, and suppose that $\mathcal{F}(A) \cap C = \mathcal{F}_+$. If $x, y \in \mathcal{F}(A) \cap C$ such that $0 \leq x \leq y$, then* $\mathrm{tr}(x) \leq \mathrm{tr}(y)$.

**Proof.** We have that $y - x \in \mathcal{F}(A) \cap C$. From corollary 4.7.7, it follows that $\mathrm{tr}(y - x) \geq 0$. Since the trace is a linear functional on $\mathcal{F}(A)$ by theorem 1.3.4, the result follows. □

If $A$ in theorem 4.7.8 is a $C^*$-algebra, then the result holds without using monotonicity and the assumption $\mathcal{F}(A) \cap C = \mathcal{F}_+$. This follows directly from linearity of the trace and the fact that in a $C^*$-algebra, $\mathrm{tr}(a) \geq 0$ for every positive finite rank element.

Note that proposition 4.7.2 and theorem 4.7.6 through theorem 4.7.8 hold in a semisimple OBA with a closed algebra cone such that the spectral radius in the OBA is monotone.

In the rest of this section we will use corollary 4.7.7 and theorem 4.7.8 to establish Banach algebra analogues of some well known trace inequalities in matrix theory.

In [14] trace inequalities for positive semidefinite matrices were studied and one of the results is ([14], theorem 1.1). We prove theorem 4.7.9, which is an adaptation of this result in our present setting.

**Theorem 4.7.9.** *Let $A$ be a semisimple COBA with a closed, inverse-closed algebra c-cone $C$ such that the spectral radius in $(A, C)$ is c-monotone, and suppose that $\mathcal{F}(A) \cap C = \mathcal{F}_+$. Let $a, b, c, d \in \mathcal{F}(A) \cap C$, with $0 \leq a \leq b$ and $0 \leq c \leq d$. If $a, b, c, d$ mutually commute and $a^{-1}, b^{-1}, (a+c)^{-1}, (b+d)^{-1}$ exist, then*

$$\mathrm{tr}[(b - a)(a^{-1} - b^{-1}) + (d - c)((a + c)^{-1} - (b + d)^{-1})] \geq 0.$$

**Proof.** Since $C$ is inverse-closed, $a^{-1}, b^{-1}, (a + c)^{-1}, (b + d)^{-1} \in C$. Since $0 \leq a \leq b$ and $0 \leq c \leq d$, and $a, b, c, d$ mutually commute, lemma 3.2.7 implies that $b^{-1} \leq a^{-1}$ and $(b + d)^{-1} \leq (a + c)^{-1}$. It follows that $(b - a)(a^{-1} - b^{-1}) \geq 0$ and $(d - c)((a + c)^{-1} -$

$(b+d)^{-1}) \geq 0$, so that $[(b-a)(a^{-1}-b^{-1})+(d-c)((a+c)^{-1}-(b+d)^{-1})] \in C$. Clearly, $[(b-a)(a^{-1}-b^{-1})+(d-c)((a+c)^{-1}-(b+d)^{-1})] \in \mathcal{F}(A)$. The result then follows from corollary 4.7.7. $\square$

Since commutativity of the elements $a, b, c, d$ in theorem 4.7.9 is used only to get products of positive elements to be positive, in an OBA with a closed and inverse-closed algebra cone such that the spectral radius in the OBA is monotone, the result holds without this requirement. Similar remarks can be made for the rest of the results in this section.

In [26], S. Furuichi, K. Kuriyama and K. Yanagi studied trace inequalities for products of two matrices and proved ([26], theorem 1.3). The following theorem is a version of this result in our current setting.

**Theorem 4.7.10.** *Let $A$ be a semisimple COBA with a closed algebra c-cone $C$ such that the spectral radius in $(A, C)$ is c-monotone, and suppose that $\mathcal{F}(A) \cap C = \mathcal{F}_+$. If $a, b \in \mathcal{F}(A) \cap C$ such that $0 \leq a \leq \frac{b}{2}$ and $ab = ba$, then $\mathrm{tr}(ab) \leq \frac{1}{2}\mathrm{tr}(a^2+b^2)$.*

**Proof.** From the assumption $0 \leq a \leq \frac{b}{2}$, we get that $\frac{1}{2}(a^2+b^2) - ab = \frac{a^2}{2} + (\frac{b}{2} - a)b \in C$. Now since $a, b \in \mathcal{F}(A)$ and $\mathcal{F}(A)$ is an ideal of $A$, we also have that $\frac{1}{2}(a^2+b^2), ab \in \mathcal{F}(A)$. The result then follows from theorem 4.7.8. $\square$

Let $a$ be an element of a Banach algebra and let $n \in \mathbb{N}$. We define $\exp_n(a) = (1+\frac{a}{n})^n$. If $\lambda \in \mathbb{C}$, it is well known that $(1+\frac{\lambda}{n})^n \to e^\lambda$ as $n \to \infty$. If $\Omega$ is a neighbourhood of $\sigma(a)$, then clearly both functions $(1+\frac{\lambda}{n})^n$ and $e^\lambda$ are analytic on $\Omega$. Therefore if $\Gamma$ is a smooth contour surrounding $\sigma(a)$, then we have that $(1+\frac{a}{n})^n = \frac{1}{2\pi i}\int_\Gamma (1+\frac{\lambda}{n})^n(\lambda 1 - a)^{-1}d\lambda \to \frac{1}{2\pi i}\int_\Gamma e^\lambda(\lambda 1 - a)^{-1}d\lambda = e^a$. We have the following result.

**Theorem 4.7.11.** *Let $A$ be a finite-dimensional semisimple COBA with a closed algebra c-cone $C$ such that the spectral radius in $(A, C)$ is c-monotone, and suppose that $\mathcal{F}(A) \cap C = \mathcal{F}_+$. If $a, b \in C$ with $ab = ba$, then $\mathrm{tr}(\exp_n(a+b)) \leq \mathrm{tr}(\exp_n(a)\exp_n(b))$.*

**Proof.** We have that $A = \mathcal{F}(A) = \mathrm{Soc}(A)$. Also, $\exp_n(a+b) = (1+\frac{1}{n}(a+b))^n$ and $\exp_n(a)\exp_n(b) = (1+\frac{a}{n})^n(1+\frac{b}{n})^n$. Since $ab = ba$, we get that $(1+\frac{a}{n})^n(1+\frac{b}{n})^n = \big((1+\frac{a}{n})(1+\frac{b}{n})\big)^n = (1+\frac{1}{n}(a+b)+\frac{ab}{n^2})^n$. Clearly, $1+\frac{1}{n}(a+b) \leq 1+\frac{1}{n}(a+b)+\frac{ab}{n^2}$ for all $n \in \mathbb{N}$. It follows from proposition 2.1.22 that $(1+\frac{a}{n})^n(1+\frac{b}{n})^n \geq (1+\frac{1}{n}(a+b))^n \geq 0$, so that the result follows from theorem 4.7.8. $\square$

Theorem 4.7.11 is an adaptation of ([27], proposition 3.2). This result is the famous Golden-Thompson inequality. For more on the Golden-Thompson inequality (see [27]) and the references given there.

The following theorem gives conditions under which the number of points in the peripheral spectrum of a dominated positive element of the socle is less than or equal to the number of points in the peripheral spectrum of the dominating element. In this result, we apply the spectral definition of the trace.

**Theorem 4.7.12.** *Let $(A, C)$ be a semisimple COBA such that the spectral radius in $(A, C)$ is c-monotone. Suppose that $a, b \in \mathrm{Soc}(A)$ such that $0 \leq a \leq b$ and $ab = ba$. If $|\mathrm{tr}(a^k)| \leq |\mathrm{tr}(b^k)|$ for all $k \in \mathbb{N}$, then either $\mathrm{psp}(a) \cap \mathrm{psp}(b) = \emptyset$ or $\#\mathrm{psp}(a) \leq \#\mathrm{psp}(b)$.*

**Proof.** Since the spectral radius is $c$-monotone, $r(a) \leq r(b)$. If $r(a) < r(b)$, then $\mathrm{psp}(a) \cap \mathrm{psp}(b) = \emptyset$. If $r(a) = r(b)$, then by theorem 1.3.5 and the assumption $|\mathrm{tr}(a^k)| \leq |\mathrm{tr}(b^k)|$, we have that $\#\mathrm{psp}(a) = \overline{\lim}_{k\to\infty} \frac{|\mathrm{tr}(a^k)|}{r(a)^k} \leq \overline{\lim}_{k\to\infty} \frac{|\mathrm{tr}(b^k)|}{r(b)^k} = \#\mathrm{psp}(b)$. $\square$

The condition $ab = ba$ in theorem 4.7.12 is needed only because of $c$-monotonicity.

# Chapter 5

# Domination

The motivation for the results in this chapter is the following general problem (which we will refer to as the domination problem) that was studied in Banach lattices: Let $E$ be a Banach lattice and let $S$ and $T$ be positive operators such that $0 \leq S \leq T$ holds. If $T$ has certain properties, does it follow that $S$ has the same properties? This problem has been investigated if $T$ has certain topological properties, e.g. $T$ compact [3], $T$ weakly compact [2], $T$ Dunford-Pettis [1] and if $T$ has certain spectral properties, see [21]. The domination problem in OBAs has been studied and several authors have made contributions. In this chapter we start by generalizing some of the results to COBAs and then proceed to obtain new results in COBAs and OBAs. Some of the results in this chapter rely on $c$-monotonicity (or monotonicity) of the spectral radius while others do not.

Note that corollary 4.4.7 is in fact a domination result. It is however placed in a section of the previous chapter because it is closely related to the other results in that section.

## 5.1 The radical

Let $a$ and $b$ be positive elements of a COBA with $a \leq b$. This section deals with results describing conditions under which if $b$ is in the radical, then $a$ is also in the radical. Such results were obtained in [51] and [41] for OBAs. We obtain corresponding results for COBAs. The proofs of our results will follow along the lines of the proofs of corresponding results for OBAs.

We start with the following theorem, whose original OBA version is ([51], theorem 4.1).

**Theorem 5.1.1.** *Let $A$ be a COBA with a c-normal algebra c-cone $C$ and let $a, b \in A$ with $ab = ba$ and $0 \leq a \leq b$ w.r.t. $C$. Then*

*(i) if $b \in \mathrm{QN}(A)$ then $a \in \mathrm{QN}(A)$,*
*(ii) if $b \in \mathrm{Rad}(A)$ then $a \in \mathrm{QN}(A)$,*
*(iii) if $b \in \mathrm{Rad}(A)$ and $a \in A^c$ then $a \in \mathrm{Rad}(A)$.*

**Proof.** (i) If $b \in \mathrm{QN}(A)$ then $r(b) = 0$. Since $0 \leq a \leq b$ and $ab = ba$, it follows from theorem 4.1.2 that $r(a) \leq r(b)$, so that $r(a) = 0$. Hence $a \in \mathrm{QN}(A)$.

(ii) Let $b \in \mathrm{Rad}(A)$. Then $bA \subseteq \mathrm{QN}(A)$, so that $b \in \mathrm{QN}(A)$. By (i), we have that $a \in \mathrm{QN}(A)$.

(iii) Let $x \in A$. Since $a \in A^c$, it follows from proposition 1.1.5 that $r(ax) \leq r(a)r(x)$. If $b \in \mathrm{Rad}(A)$, then $a \in \mathrm{QN}(A)$ by (ii), so that $r(a) = 0$. Therefore $r(ax) = 0$. Since $x$ was an arbitrary element of $A$, we have that $aA \subseteq \mathrm{QN}(A)$. $\square$

We prove theorem 5.1.3, which also gives conditions required for a dominated positive element to be in the radical, if the dominating element is in the radical. We will need the following lemma, whose original OBA counterpart is ([41], lemma 4.1).

**Lemma 5.1.2.** *Let $A$ be a COBA with a c-normal algebra c-cone $C$ and let $M$ be an MPCS in $A$. If $a, b \in M$ such that $0 \leq a \leq b$ w.r.t. $C$ and $b \in \mathrm{Rad}(A)$, then $aM \subseteq \mathrm{QN}(A)$.*

**Proof.** Let $b \in \mathrm{Rad}(A)$. Then $bA \subseteq \mathrm{QN}(A)$, so that $bM \subseteq \mathrm{QN}(A)$. From $0 \leq a \leq b$ we have that $b - a \in C$. Let $c \in M$. Then $(b-a)c = bc - ac \in C$, so that $0 \leq ac \leq bc$. Since $C$ is $c$-normal, the spectral radius in $(A, C)$ is $c$-monotone by theorem 4.1.2. This implies that $r(ac) \leq r(bc)$. Since $bM \subseteq \mathrm{QN}(A)$, we have that $aM \subseteq \mathrm{QN}(A)$. $\square$

Note that if $A$ in theorem 5.1.1 and lemma 5.1.2 is a $C^*$-algebra, these results are trivial since a $C^*$-algebra is semisimple. This also applies to some of the other results in this section.

**Theorem 5.1.3.** *Let $A$ be a COBA with a c-normal algebra c-cone $C$ and $M$ any MPCS in $A$. Suppose that for every $x$ in $A$ there is a $0 \neq \lambda \in \mathbb{C}$ such that $\lambda x \in M$. If $a, b \in M$ with $0 \leq a \leq b$ relative to $C$ and $b \in \mathrm{Rad}(A)$, then $a \in \mathrm{Rad}(A)$.*

**Proof.** Suppose that $a, b \in M$ with $0 \leq a \leq b$ relative to $C$ and $b \in \mathrm{Rad}(A)$. Then $aM \subseteq \mathrm{QN}(A)$ by lemma 5.1.2. We must show that $aA \subseteq \mathrm{QN}(A)$. If $x \in A$ then by assumption, there is a $0 \neq \lambda \in \mathbb{C}$ such that $\lambda x \in M$. Therefore $a(\lambda x) \in aM \subseteq \mathrm{QN}(A)$. This implies that $r(a(\lambda x)) = |\lambda| r(ax) = 0$. Since $\lambda \neq 0$, we have that $r(ax) = 0$. This means that $ax \in \mathrm{QN}(A)$. Since $x$ was arbitrary, $aA \subseteq \mathrm{QN}(A)$. $\square$

The original OBA result corresponding to theorem 5.1.3 is ([41], theorem 4.2).

The following result is an immediate consequence of theorem 5.1.3. Its OBA counterpart is ([41], corollary 4.3).

**Corollary 5.1.4.** *Let $A$ be a COBA with a c-normal algebra c-cone $C$ and $M$ an MPCS in $A$. Suppose that for every $x$ in $A$ there is a line segment $L$ in $\mathbb{C}$ such that $\lambda x \in M$ for all $\lambda \in L$. If $a, b \in M$ such that $0 \leq a \leq b$ w.r.t. $C$ and $b \in \mathrm{Rad}(A)$, then $a \in \mathrm{Rad}(A)$.*

Since the proof of lemma 5.1.2 involves multiplication of positive elements, commutativity is required to establish this result, and consequently the same holds for theorem 5.1.3 and corollary 5.1.4. To prove the corresponding results for OBAs, commutativity is not necessary ([41], lemma 4.1, theorem 4.2, corollary 4.3).

The next result is a COBA version of ([41], lemma 4.4) and can be established with the same proof.

**Proposition 5.1.5.** *Let $A$ be a COBA with a c-normal algebra c-cone $C$. If $aC \subseteq \mathrm{QN}(A)$, then $a$(span $C$) $\subseteq \mathrm{QN}(A)$.*

If $(A, C)$ is a COBA where $C$ is only an algebra $c$-cone, we do not know if the analogue of ([41], theorem 4.6) holds. It cannot be proved in the same way as ([41], theorem 4.6), because although we have proposition 5.1.5, there is no suitable COBA analogue of ([41], lemma 4.1). However, if we take an MPCS $M$ in $A$ the result follows immediately from theorem 5.1.1 (i) under the assumption $A = \text{span } M$ because then $A$ is commutative and this implies $\mathrm{Rad}(A) = \mathrm{QN}(A)$. This also applies to ([41], theorem 4.9) since if $A$ is commutative, then continuity of the spectral radius function is automatically implied and does not play a role.

With the assumptions of the foregoing results, we obtain the following result, which gives characterizations of the radical in terms of the cone of the Banach algebra. Its OBA counterpart is ([41], theorem 4.17).

**Theorem 5.1.6.** *Let $A$ be a COBA with a c-normal algebra c-cone $C$ and let $M$ be an MPCS in $A$. Suppose that at least one of the following conditions holds:*

*(i) For every $x \in A$ there is a $0 \neq \lambda \in \mathbb{C}$ such that $\lambda x \in M$.*
*(ii) For every $x \in A$ there is a line segment $L$ in $\mathbb{C}$ such that $\lambda x \in M$ for all $\lambda \in L$.*
*Then* $\mathrm{Rad}(A) = \{a \in A : aM \subseteq \mathrm{QN}(A)\}$.

**Proof.** Let $a \in \mathrm{Rad}(A)$. Then $aA \subseteq \mathrm{QN}(A)$. Since $aM \subseteq aA$, we have that $aM \subseteq \mathrm{QN}(A)$. Therefore $\mathrm{Rad}(A) \subseteq \{a \in A : aM \subseteq \mathrm{QN}(A)\}$. So this inclusion holds for both cases. For both cases, the inclusion $\{a \in A : aM \subseteq \mathrm{QN}(A)\} \subseteq \mathrm{Rad}(A)$ is proved in a similar way to ([41], theorem 4.17). □

The COBA results corresponding to ([41], theorem 4.17 (3) and (4)) hold trivially under the assumptions $A = \text{span } M$ and $A = cl(\text{span } M)$ (see remarks following proposition 5.1.5).

Note that since elements in an MPCS have the restrictive requirement of being positive and commutative, the COBA results mentioned in the preceding discussion will in general be considerably weaker than their OBA counterparts.

In [13] D. Behrendt and H. Raubenheimer proved a result giving conditions, in terms of polynomials, under which an element in an OBA dominated by a positive element lies in the radical (see [13], theorem 3.3, corollary 3.4). With similar proofs we establish the corresponding results for COBAs.

**Theorem 5.1.7.** *Let $(A, C)$ be a COBA such that the spectral radius in $(A, C)$ is c-monotone and let $a, b \in A$ such that $0 \leq a \leq b$, $ab = ba$ and $b \in \mathrm{QN}(A)$. If $g(a) = \lambda_k a^k + \lambda_{k+1}a^{k+1} + \cdots + \lambda_n a^n$ $(\lambda_k, \lambda_{k+1}, ..., \lambda_n \in \mathbb{C}, \lambda_k \neq 0)$ is a polynomial in $a$ and if $g(a) \in \mathrm{Rad}(A)$, then $a^k \in \mathrm{Rad}(A)$.*

**Proof.** Since the spectral radius in $(A, C)$ is $c$-monotone, $r(a) \leq r(b)$. From the assumption that $b \in \mathrm{QN}(A)$, it follows that $r(a) = 0$, so that $a \in \mathrm{QN}(A)$. By the spectral mapping theorem, this implies that $\sigma(\lambda_k + \lambda_{k+1}a + \cdots + \lambda_n a^{n-k}) = \{\lambda_k\}$. Therefore $\lambda_k + \lambda_{k+1}a + \cdots + \lambda_n a^{n-k}$ is invertible. Since $g(a) = \lambda_k a^k + \lambda_{k+1}a^{k+1} + \cdots + \lambda_n a^n = a^k(\lambda_k + \lambda_{k+1}a + \cdots \lambda_n a^{n-k})$, it follows from $g(a) \in \mathrm{Rad}(A)$ that $a^k = g(a)(\lambda_k + \lambda_{k+1}a + \cdots + \lambda_n a^{n-k})^{-1} \in \mathrm{Rad}(A)$. □

**Corollary 5.1.8.** *Let $(A, C)$ be a COBA such that the spectral radius in $(A, C)$ is c-monotone and let $a, b \in A$ such that $0 \leq a \leq b$, $ab = ba$ and $b \in \mathrm{QN}(A)$. If $a + a^2 \in \mathrm{Rad}(A)$ then $a \in \mathrm{Rad}(A)$.*

Note that in the proof of theorem 5.1.7, the multiplication involved does not require the products to be positive. Therefore we can use the same proof to establish the corresponding result in the $C'$OBA setting. This is the following theorem. We will use it to prove another result in Section 5.4.

**Theorem 5.1.9.** *Let $(A, C)$ be a $C'$OBA such that the spectral radius in $(A, C)$ is c-monotone and let $a, b \in A$ such that $0 \leq a \leq b$, $ab = ba$ and $b \in \mathrm{QN}(A)$. If $g(a) = \lambda_k a^k + \lambda_{k+1}a^{k+1} + \cdots + \lambda_n a^n$ $(\lambda_k, \lambda_{k+1}, ..., \lambda_n \in \mathbb{C}, \lambda_k \neq 0)$ is a polynomial in $a$ and if $g(a) \in \mathrm{Rad}(A)$, then $a^k \in \mathrm{Rad}(A)$.*

**Corollary 5.1.10.** *Let $(A, C)$ be a $C'$OBA such that the spectral radius in $(A, C)$ is c-monotone and let $a, b \in A$ such that $0 \leq a \leq b$, $ab = ba$ and $b \in \mathrm{QN}(A)$. If $a + a^2 \in \mathrm{Rad}(A)$ then $a \in \mathrm{Rad}(A)$.*

In the last four results, the commutativity condition is required only because of $c$-monotonicity.

## 5.2 Riesz elements

Here we prove domination results that involve Riesz elements. We start with the following theorem, whose original OBA version is due to H. Raubenheimer and S. Rode ([51], theorem 6.2).

**Theorem 5.2.1.** *Let $(A, C)$ be a COBA and $M$ an MPCS in $A$. Suppose that $F$ is a closed ideal of $A$ such that the spectral radius in $(A/F, \pi M)$ is monotone. If $a, b \in A$ such that $0 \leq a \leq b$ w.r.t. $M$ and $b$ is Riesz relative to $F$, then $a$ is Riesz relative to $F$.*

**Proof.** From theorem 2.4.2 and ([51], theorem 6.2) we obtain the result. □

For $C'$OBAs, we get the following counterpart of theorem 5.2.1.

**Theorem 5.2.2.** *Let $(A, C)$ be a COBA and $F$ a closed ideal of $A$ such that the spectral radius in the $C'$OBA $(A/F, \pi C)$ is c-monotone. If $a, b \in A$ such that $0 \leq a \leq b$ w.r.t. $C$, $ab = ba$ and $b$ is Riesz relative to $F$, then $a$ is Riesz relative to $F$.*

**Proof.** Since $0 \leq a \leq b$ w.r.t. $C$, we have that $F \leq a + F \leq b + F$ w.r.t. $\pi C$. From $c$-monotonicity of the spectral radius in $(A/F, \pi C)$, it follows that $r(a + F) \leq r(b + F)$. If $b$

is Riesz relative to $F$, then $r(b+F)=0$. Therefore $r(a+F)=0$, so that $a$ is Riesz relative to $F$. $\square$

Note that if $a$ and $b$ in theorem 5.2.2 satisfy $(a+F)(b+F)=(b+F)(a+F)$ rather than $ab=ba$, then theorem 5.2.2 still holds and in this case we have a stronger result.

Instances of theorem 5.2.2 are given in ([51], corollary 6.3, proposition 6.4, theorem 6.5, corollary 6.6). Note in particular that the formulation and proofs of theorem 6.5 and corollary 6.6 in [51] are in the COBA sense, although COBAs are not introduced in this paper. It is results such as these that give motivation for generalization from OBAs to COBAs.

## 5.3 Quasi inessential elements

In this section we prove results about domination by quasi inessential elements. We start with the following theorem.

**Theorem 5.3.1.** *Let $(A,C)$ be a COBA and $M$ an MPCS in $A$. Suppose that $I$ is a closed inessential ideal in $A$ such that the spectral radius in $(A/I,\pi M)$ is monotone, and $a,b\in A$ such that $0\le a\le b$ w.r.t. $M$. If $b$ is quasi inessential relative to $I$, then $a$ is quasi inessential relative to $I$.*

**Proof.** Suppose that $b$ is quasi inessential relative to $I$. Then $r(b+I)<1$ by proposition 1.2.5. Since $0\le a\le b$ w.r.t. $M$, we have that $I\le a+I\le b+I$ w.r.t. $\pi M$. From monotonicity of the spectral radius in $(A/I,\pi M)$, it follows that $r(a+I)\le r(b+I)<1$. Proposition 1.2.5 then implies that $a$ is quasi inessential relative to $I$. $\square$

This result was proved by J. Martinez and J.M. Mazón in the setting of positive operators on Banach lattices ([39], proposition 2.5). S. Mouton and H. Raubenheimer proved the result in OBAs ([47], corollary 5.4), although in view of proposition 1.2.5, closedness of the cone and monotonicity of the spectral radius in the original OBA are not necessary, and no restrictions on the spectral radii of the elements involved are required.

In terms of $C'$OBAs, we get the following version of the above theorem.

**Theorem 5.3.2.** *Let $(A,C)$ be a COBA and suppose that $I$ is a closed inessential ideal in $A$ such that the spectral radius in the $C'$OBA $(A/I,\pi C)$ is c-monotone. Let $a,b\in A$ such that $0\le a\le b$ w.r.t. $C$ and $ab=ba$. If $b$ is quasi inessential relative to $I$, then $a$ is quasi inessential relative to $I$.*

**Proof.** If $b$ is quasi inessential relative to $I$, then $r(b+I)<1$ by proposition 1.2.5. Since $0\le a\le b$ w.r.t. $C$, we have that $I\le a+I\le b+I$ w.r.t. $\pi C$. From $c$-monotonicity of the spectral radius in $(A/I,\pi C)$, it follows that $r(a+I)\le r(b+I)<1$. Proposition 1.2.5 then implies that $a$ is quasi inessential relative to $I$. $\square$

If $a$ and $b$ in the above theorem satisfy $(a+I)(b+I)=(b+I)(a+I)$ rather than $ab=ba$, the theorem is still valid and in this case it stronger than in the form given.

## 5.4 Inessential elements

In [13] the domination problem was studied for inessential elements in the OBA setting. In this section we will obtain corresponding results for COBAs. We start with the following theorem, whose original OBA counterpart is ([13], theorem 4.1).

**Theorem 5.4.1.** *Let $(A, C)$ be a COBA and $M$ an MPCS in $A$. Suppose that $F$ is a closed ideal in $A$ such that the spectral radius in $(A/F, \pi M)$ is monotone. If $a, b \in A$ such that $0 \leq a \leq b$ w.r.t. $M$ and if $b$ is inessential relative to $F$, then we have the following:*

*(i) $a$ is Riesz relative to $F$,*
*(ii) If $a$ is in the center $A^c$ of $A$, then $a$ is inessential relative to $F$.*

**Proof.** (i) Since $b$ is inessential relative to $F$, we have that $b + F \in \mathrm{Rad}(A/F)$. Since the spectral radius in $(A/F, \pi M)$ is monotone, it follows that $r(a + F, A/F) = 0$, so that $a + F \in \mathrm{QN}(A/F)$. Hence $a$ is Riesz relative to $F$.

(ii) Let $x + F \in A/F$. If $a \in A^c$, then $(a + F)(x + F) = (x + F)(a + F)$, so that $r((a + F)(x + F)) \leq r(a + F)r(x + F) = 0$ by (i). Therefore $(a + F)A/F \subseteq \mathrm{QN}(A/F)$. Hence $a + F \in \mathrm{Rad}(A/F)$ and $a$ is inessential relative to $F$. □

We do not know if ([13], theorem 4.1 (iii)) holds if $C$ is only an algebra $c$-cone. It cannot be established like ([13], theorem 4.1 (iii)) because there is no suitable COBA version of ([41], theorem 4.6). However, if we replace $C$ by $M$, the result holds trivially. This is because the condition $A = \mathrm{span}\ M$ implies $A$ is commutative, so that $\mathrm{Rad}(A) = \mathrm{QN}(A)$, and then the Riesz and inessential elements of $A$ coincide. We then get the conclusion directly from theorem 5.2.1.

The following is the $C'$OBA version of theorem 5.4.1 and can be established with a similar proof.

**Theorem 5.4.2.** *Let $(A, C)$ be a COBA and suppose that $F$ is a closed ideal in $A$ such that the spectral radius in the $C'$OBA $(A/F, \pi C)$ is c-monotone. If $a, b \in A$ with $0 \leq a \leq b$ w.r.t. $C$, $ab = ba$ and if $b$ is inessential relative to $F$, then we have the following:*

*(i) $a$ is Riesz relative to $F$,*
*(ii) If $a$ is in the center $A^c$ of $A$, then $a$ is inessential relative to $F$.*

We obtain the following analogue of theorem 5.1.7 in terms of inessential elements. We follow this from ([13], theorem 4.6).

**Theorem 5.4.3.** *Let $(A, C)$ be a COBA, $M$ an MPCS in $A$ and $F$ a closed ideal in $A$ such that the spectral radius in $(A/F, \pi M)$ is monotone. Suppose that $a, b \in A$ such that $0 \leq a \leq b$ w.r.t. $M$ and $b$ is Riesz relative to $F$. If $g(a) = \lambda_k a^k + \lambda_{k+1} a^{k+1} + \cdots + \lambda_n a^n$ ($\lambda_k, \lambda_{k+1}, ..., \lambda_n \in \mathbb{C}, \lambda_k \neq 0$) is a polynomial in $a$ and if $g(a) \in \mathrm{kh}(A, F)$, then $a^k \in \mathrm{kh}(A, F)$.*

**Proof.** Since $b$ is Riesz relative to $F$, we have that $b + F \in \mathrm{QN}(A/F)$ and since $g(a) \in \mathrm{kh}(A, F)$, it follows that $g(a) + F \in \mathrm{Rad}(A/F)$. With the assumption that the spectral radius in $(A/F, \pi M)$ is monotone, we apply ([13], theorem 3.3) in the quotient OBA $(A/F, \pi M)$ to obtain that $a^k + F \in \mathrm{Rad}(A/F)$. Hence $a^k \in \mathrm{kh}(A, F)$. $\square$

The following theorem is the $C'$OBA version of theorem 5.4.3:

**Theorem 5.4.4.** *Let $(A, C)$ be a COBA and $F$ a closed ideal in $A$ such that the spectral radius in the $C'$OBA $(A/F, \pi C)$ is c-monotone. Suppose that $a, b \in A$ such that $0 \leq a \leq b$ w.r.t. $C$, $ab = ba$ and $b$ is Riesz relative to $F$. If $g(a) = \lambda_k a^k + \lambda_{k+1} a^{k+1} + \cdots + \lambda_n a^n$ $(\lambda_k, \lambda_{k+1}, ..., \lambda_n \in \mathbb{C}, \lambda_k \neq 0)$ is a polynomial in $a$ and if $g(a) \in \mathrm{kh}(A, F)$, then $a^k \in \mathrm{kh}(A, F)$.*

**Proof.** From $0 \leq a \leq b$ and $ab = ba$, we obtain that $F \leq a + F \leq b + F$ and $(a+F)(b+F) = (b+F)(a+F)$. Now since $b$ is Riesz relative to $F$, we have that $b + F \in \mathrm{QN}(A/F)$ and since $g(a) \in \mathrm{kh}(A, F)$, it follows that $g(a) + F \in \mathrm{Rad}(A/F)$. Applying theorem 5.1.9 in the $C'$OBA $(A/F, \pi C)$, we get that $a^k + F \in \mathrm{Rad}(A/F)$. Hence $a^k \in \mathrm{kh}(A, F)$. $\square$

If $A$ in theorems 5.4.1 and 5.4.2 is a $C^*$-algebra, then results are trivial. This is because a $C^*$-algebra is semisimple. Also, in theorems 5.4.2 and 5.4.4, the condition $ab = ba$ is required only because of $c$-monotonicity.

## 5.5 Rank one and finite rank elements

In this section we deal with the domination problem in relation to rank one and finite rank elements. As ([13], examples 5.2 and 5.3) demonstrate, if $b$ is a positive finite rank element in a semiprime OBA and if $a$ is positive and dominated by $b$, then in general, $a$ is not finite rank. This scenario extends to COBAs.

The following theorem (whose original OBA version is ([13], theorem 5.4 (ii)) shows that if a positive element in a COBA is dominated by a finite rank element, then under certain natural conditions, the dominated element is at most in the closure of the set of finite rank elements.

**Theorem 5.5.1.** *Let $(A, C)$ be a COBA with $A$ semiprime, and $M$ an MPCS in $A$ such that the spectral radius in $(A/\mathrm{cl}_A(\mathcal{F}(A)), \pi M)$ is monotone. Suppose that $a, b \in A$ such that $0 \leq a \leq b$ w.r.t. $M$ and $b \in \mathcal{F}(A)$. If $a$ is in the center of $A$, then $a \in \mathrm{kh}(A, \mathcal{F}(A))$. Therefore if $A/\mathrm{cl}_A(\mathcal{F}(A))$ is semisimple, then $a \in \mathrm{cl}_A(\mathcal{F}(A))$.*

**Proof.** Since $b \in \mathcal{F}(A)$, we have that $b \in \mathrm{kh}(A, \mathcal{F}(A))$ and replacing $F$ by $\mathrm{cl}_A(\mathcal{F}(A))$ in theorem 5.4.1 (ii), the first part of the result follows. To prove the second part, if $A/\mathrm{cl}_A(\mathcal{F}(A))$ is semisimple, then $\mathrm{Rad}(A/\mathrm{cl}_A(\mathcal{F}(A))) = \{\mathrm{cl}_A(\mathcal{F}(A))\}$. Since $a + \mathrm{cl}_A(\mathcal{F}(A)) \in \mathrm{Rad}(A/\mathrm{cl}_A(\mathcal{F}(A)))$ by the first part, we have that $a + \mathrm{cl}_A(\mathcal{F}(A)) = \mathrm{cl}_A(\mathcal{F}(A))$, so that $a \in \mathrm{cl}_A(\mathcal{F}(A))$. $\square$

Similar remarks to those following theorem 5.4.1 apply to ([13], theorem 5.4 (i)). The following theorem is the $C'$OBA version of theorem 5.5.1.

**Theorem 5.5.2.** *Let $(A, C)$ be a COBA with $A$ semiprime and such that the spectral radius in the $C'OBA$ $(A/\mathrm{cl}_A(\mathcal{F}(A)), \pi C)$ is c-monotone. Suppose that $a, b \in A$ such that $0 \le a \le b$ w.r.t. $C$ and $b \in \mathcal{F}(A)$. If $a$ is in the center of $A$, then $a \in \mathrm{kh}(A, \mathcal{F}(A))$.*

**Proof.** Since $b \in \mathcal{F}(A)$, we get that $b \in \mathrm{kh}(A, \mathcal{F}(A))$ and since $a$ is in the center of $A$, $ab = ba$. Replacing $F$ by $\mathrm{cl}_A(\mathcal{F}(A))$ in theorem 5.4.2 (ii), we obtain that $a \in \mathrm{kh}(A, \mathcal{F}(A))$. $\square$

If $A$ is a semiprime Banach algebra which is not semisimple, $\mathrm{cl}_A(\mathcal{F}_1(A)) \cap \mathrm{Rad}(A) = \{0\}$ and $\mathrm{cl}_A(\mathcal{F}_1(A)) \cdot \mathrm{Rad}(A) = \{0\} = \mathrm{Rad}(A) \cdot \mathrm{cl}_A(\mathcal{F}_1(A))$ hold ([17], theorem 2.10). Consequently, $\mathcal{F}(A) \cdot \mathrm{Rad}(A) = \{0\} = \mathrm{Rad}(A) \cdot \mathcal{F}(A)$. Using these facts we obtain the following results. They are COBA analogues of ([13], theorem 5.5, 5.6; corollary 5.7) respectively.

**Theorem 5.5.3.** *Let $A$ be a semiprime COBA which is not semisimple, with a proper algebra c-cone $C$. Suppose that $a, b \in A$ with $0 \le a \le b$ and $ab = ba$.*

*(i) If $b \in \mathcal{F}(A)$ and $a \in \mathrm{Rad}(A)$, then $a^2 = 0$.*
*(ii) If $b \in \mathrm{Rad}(A)$ and $a \in \mathcal{F}(A)$, then $a^2 = 0$.*

**Proof.** Since $C$ is an algebra $c$-cone, $0 \le a \le b$ and $ab = ba$, we have that $0 \le a^2 \le ab$. If $a \in \mathrm{Rad}(A)$ and $b \in \mathcal{F}(A)$, it follows from $\mathrm{Rad}(A) \cdot \mathcal{F}(A) = \{0\}$ that $ab = 0$. Since $C$ is proper, $a^2 = 0$. Part (ii) is proved similarly. $\square$

As for a COBA analogue of ([13], theorem 5.5 (iii)), see remarks after theorem 5.4.1. Note in theorem 5.5.3 that the condition $ab = ba$ is required because we need the product $ab$ to be positive.

**Theorem 5.5.4.** *Let $A$ be a semiprime COBA with a proper algebra c-cone $C$ and $M$ an MPCS in $A$. Suppose that $b \in M \cap \mathcal{F}(A)$ and $M \cap \mathrm{Rad}(A) \ne \{0\}$. Then there does not exist an invertible element $a$ such that $0 \le a \le b$ w.r.t. $M$.*

**Proof.** Let $0 \ne c \in M \cap \mathrm{Rad}(A)$. Suppose that $a$ is invertible and $0 \le a \le b$ w.r.t. $M$. From $b, c \in M$, we have that $0 \le ac \le bc$ w.r.t. $M$. Since $b \in \mathcal{F}(A)$ and $c \in \mathrm{Rad}(A)$, it follows from $\mathcal{F}(A) \cdot \mathrm{Rad}(A) = \{0\}$ that $bc = 0$. Since $C$ is proper, $M$ is proper by theorem 2.4.2, so that $ac = 0$. Since $a$ is invertible, it follows that $c = 0$, which is a contradiction. $\square$

**Corollary 5.5.5.** *Let $A$ be a semiprime COBA with a proper algebra c-cone $C$ and $M$ an MPCS in $A$. If $M \cap \mathrm{Rad}(A) \ne \{0\}$, there does not exist elements $b \in \mathcal{F}(A)$ such that $0 \le 1 \le b$ relative to $C$.*

## 5.6 Ergodic elements

Let $A$ be a Banach algebra, $a \in A$ and let $(f_n(\lambda))$ be the sequence of functions defined by $f_n(\lambda) = \sum_{k=0}^{n-1} \frac{\lambda^k}{n}$, $\lambda \in \mathbb{C}$. Note that $f_n(1) = 1$ for all $n \in \mathbb{N}$. The terms of the sequence $(f_n(a))$ are called *ergodic sums* of $a$. The element $a$ is said to be *ergodic* if its sequence of ergodic sums converges.

In this section we consider the problem of determining when a positive element dominated by an ergodic element is ergodic. This problem was considered in the operator-theoretic setting in [50] and a result was obtained (see [50], theorem 4.5). The proof of this result depends on a theorem by N. Dunford (see [25], theorem 3.16). Here we will get results corresponding to ([50], theorem 4.5) in our current setting. The results in this section extend the theory of COBAs and OBAs and are attributed to the author. The main ones are theorems 5.6.11, 5.6.12, 5.6.16 and 5.6.17. A key result that will be used to establish these theorems is theorem 5.6.10, which is a generalization of part of ([25], theorem 3.16) to Banach algebras. The results from theorem 5.6.1 through proposition 5.6.9, some of which are Banach algebra versions of results in [25] (see theorems 2.19, 2.20, 2.21, 3.6), lead to theorem 5.6.10. The proofs of these results rely heavily on operator theory, while our proofs are completely algebraic. We start with the following theorem.

**Theorem 5.6.1.** *Let $A$ be a Banach algebra and let $a \in A$. Suppose that $f$ is an analytic function on a neighbourhood $\Omega$ of $\sigma(a)$ such that*

*(i) for every pole $\lambda$ of the resolvent of $a$ of order $k$, $f^{(j)}(\lambda) = 0$ $(j = 0, 1, ..., k-1)$ and*
*(ii) $\sigma(a)$ contains at least one pole $\lambda_1$ of the resolvent of $a$, and there exists a neighbourhood $U$ of $\sigma(a) \setminus \{\lambda_1, \lambda_2, ..., \lambda_n\}$ for some $n \geq 1$, where $\lambda_1, \lambda_2, ..., \lambda_n$ are poles of the resolvent of $a$, such that $f(\lambda) = 0$ for all $\lambda \in U$.*

*Then $f(a) = 0$.*

**Proof.** Let $\Gamma$ be a smooth contour surrounding $\sigma(a)$. Then $f(a) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda 1 - a)^{-1} d\lambda$. We may take $\Gamma$ to be the union of $\Gamma_1$ and $C_1, C_2, ..., C_n$, where $\Gamma_1$ is a smooth contour contained in $U$ and surrounding $\sigma(a) \setminus \{\lambda_1, \lambda_2, ..., \lambda_n\}$, and $C_i$ is a small circle centered at the pole $\lambda_i$ of the resolvent of $a$ and separating $\lambda_i$ from the rest of $\sigma(a)$. Therefore $f(a) = \frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)(\lambda 1 - a)^{-1} d\lambda + \sum_{i=1}^{n} \frac{1}{2\pi i}\int_{C_i} f(\lambda)(\lambda 1 - a)^{-1} d\lambda$. From assumption (ii) we have that $\frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)(\lambda 1 - a)^{-1} d\lambda = 0$. Now since $f^{(j)}(\lambda_i) = 0$ for $i = 1, 2, ..., n$ and for $j = 0, 1, ..., k_i - 1$ (with $k_i$ the order of the pole $\lambda_i$) by assumption (i), there exist $g_i \in H(\Omega)$ such that $f(\lambda) = (\lambda - \lambda_i)^{k_i} g_i(\lambda)$. Therefore $\frac{1}{2\pi i}\int_{C_i} f(\lambda)(\lambda 1 - a)^{-1} d\lambda = \frac{1}{2\pi i}\int_{C_i} (\lambda - \lambda_i)^{k_i} g_i(\lambda)(\lambda 1 - a)^{-1} d\lambda$. Since $\lambda_i$ is a pole of order $k_i$ of the resolvent of $a$, the resolvent of $a$ has a Laurent series expansion $(\lambda 1 - a)^{-1} = \frac{a_{-k_i}}{(\lambda - \lambda_i)^{k_i}} + \frac{a_{-k_i+1}}{(\lambda - \lambda_i)^{k_i - 1}} + \cdots + a_0 + a_1(\lambda - \lambda_i) + \cdots$. Therefore $(\lambda - \lambda_i)^{k_i} g_i(\lambda)(\lambda 1 - a)^{-1} = g_i(\lambda)[a_{-k_i} + a_{-k_i+1}(\lambda - \lambda_i) + \cdots + a_0(\lambda - \lambda_i)^{k_i} + a_1(\lambda - \lambda_i)^{k_i+1} + \cdots]$ on a deleted neighbourhood of $\lambda_i$ which includes $C_i$. Since $g_i$ is analytic on a neighbourhood of $\sigma(a)$, it has no singularities on or inside $C_i$. This implies that $(\lambda - \lambda_i)^{k_i} g_i(\lambda)(\lambda 1 - a)^{-1}$ has no singularities on or inside $C_i$. It follows from Cauchy's theorem that $\frac{1}{2\pi i}\int_{C_i} (\lambda - \lambda_i)^{k_i} g_i(\lambda)(\lambda 1 - a)^{-1} d\lambda = 0$ for $i = 1, 2, ..., n$. Hence $f(a) = 0$. $\square$

The following result is an immediate consequence of theorem 5.6.1.

**Corollary 5.6.2.** *Let $A$ be a Banach algebra and let $a \in A$. Suppose that $f, g$ are functions analytic on a neighbourhood $\Omega$ of $\sigma(a)$ such that*

*(i) for every pole $\lambda$ of the resolvent of $a$ of order $k$, $f^{(j)}(\lambda) = g^{(j)}(\lambda)$ $(j = 0, 1, ..., k-1)$*

*and*
*(ii)* $\sigma(a)$ *contains at least one pole* $\lambda_1$ *of the resolvent of* $a$*, and there exists a neighbourhood* $U$ *of* $\sigma(a) \setminus \{\lambda_1, \lambda_2, ..., \lambda_n\}$ *for some* $n \geq 1$*, where* $\lambda_1, \lambda_2, ..., \lambda_n$ *are poles of the resolvent of* $a$*, such that* $f(\lambda) = g(\lambda)$ *for all* $\lambda \in U$*.*

*Then* $f(a) = g(a)$*.*

The following theorem is an important consequence of corollary 5.6.2.

**Theorem 5.6.3.** *Let* $A$ *be a Banach algebra and let* $a \in A$*. Suppose that* $f$ *is a complex valued function analytic on a neighbourhood* $\Omega$ *of* $\sigma(a)$*. If* $\alpha$ *is a pole of order* $k$ *of the resolvent of* $a$*, then* $f(a) = f(a)(1-p) + \sum_{n=0}^{k-1} \frac{(a-\alpha 1)^n}{n!} f^{(n)}(\alpha)p$*, where* $p = p(\alpha, a)$*.*

**Proof.** Since $\alpha$ is an isolated point in $\sigma(a)$, we can take two open sets $U_0$ and $U_1$ such that $\sigma(a) \setminus \{\alpha\} \subseteq U_0$, $\{\alpha\} \subseteq U_1$, $U_0 \cap U_1 = \emptyset$ and $f$ is analytic on $U = U_0 \cup U_1$. Let $\chi : U \to \mathbb{C}$ be the function defined by $\chi(\lambda) = 0$ if $\lambda \in U_0$ and $\chi(\lambda) = 1$ if $\lambda \in U_1$. Then $\chi$ is analytic on $U$ and from remark 1.1.9, we get that $p = p(\alpha, a) = \chi(a)$. Now let $g : U \to \mathbb{C}$ be the function defined by $g(\lambda) = f(\lambda)(1 - \chi(\lambda)) + \sum_{n=0}^{k-1} \frac{(\lambda-\alpha)^n}{n!} f^{(n)}(\alpha)\chi(\lambda)$. We show that $f$ and $g$ satisfy conditions (i) and (ii) of corollary 5.6.2. If $\lambda \in U_0$ then $\chi(\lambda) = 0$ and so $g(\lambda) = f(\lambda)$. Hence $f$ and $g$ satisfy condition (ii) of corollary 5.6.2. Since $g(\lambda) = f(\lambda)$ for all $\lambda \in U_0$, we have that $g^{(j)}(\lambda_i) = f^{(j)}(\lambda_i)$ $(j = 0, 1, ..., k_i - 1)$ for every pole $\lambda_i \in U_0$ of order $k_i$ of the resolvent of $a$. We show that $g^{(j)}(\alpha) = f^{(j)}(\alpha)$ for $j = 0, 1, ..., k-1$. We restrict the functions $f$ and $g$ to the set $U_1$. For $j = 0$, it is clear that $f(\alpha) = g(\alpha)$. For $j = 1$, we have that

$$g'(\lambda) = f'(\alpha) + \tfrac{2(\lambda-\alpha)}{2!} f''(\alpha) + \tfrac{3(\lambda-\alpha)^2}{3!} f'''(\alpha) + \cdots + \tfrac{(k-1)(\lambda-\alpha)^{k-2}}{(k-1)!} f^{(k-1)}(\alpha).$$

Therefore $g'(\alpha) = f'(\alpha)$. Next,

$$g''(\lambda) = f''(\alpha) + (\lambda-\alpha) f'''(\alpha) + \tfrac{3 \cdot 4(\lambda-\alpha)^2}{4!} f^{(4)}(\alpha) + \cdots + \tfrac{(k-2)(k-1)(\lambda-\alpha)^{k-3}}{(k-1)!} f^{(k-1)}(\alpha).$$

Hence $g''(\alpha) = f''(\alpha)$. Continuing in this way, we obtain $g^{(k-1)}(\lambda) = \frac{1 \cdot 2 \cdots (k-1)}{(k-1)!} f^{(k-1)}(\alpha) = f^{(k-1)}(\alpha)$, and so $g^{(k-1)}(\alpha) = f^{(k-1)}(\alpha)$. Therefore condition (i) of corollary 5.6.2 is satisfied and so the result follows. $\square$

The following three corollaries are consequences of theorem 5.6.3.

**Corollary 5.6.4.** *Let* $A$ *be a Banach algebra and let* $a \in A$*. Let* $(f_n)$ *be a sequence of complex valued functions analytic on a neighbourhood* $\Omega$ *of* $\sigma(a)$*. Suppose that* $\alpha \neq 0$ *is a pole of order* $k$ *of the resolvent of* $a$ *such that* $f_n(\alpha) \to 1$ *as* $n \to \infty$ *and* $f_n^{(j)}(\alpha) \to 0$ $(j = 1, 2, ..., k-1)$ *as* $n \to \infty$*. If* $(\alpha 1 - a) f_n(a) \to 0$ *as* $n \to \infty$ *then* $f_n(a) \to p$ *as* $n \to \infty$*, where* $p = p(a, \alpha)$*.*

**Proof.** Since $\alpha$ is an isolated point in $\sigma(a)$, we can take two open sets $U_0$ and $U_1$ such that $\sigma(a) \setminus \{\alpha\} \subseteq U_0$, $\{\alpha\} \subseteq U_1$, $U_0 \cap U_1 = \emptyset$ and $f_n$ is analytic on $U = U_0 \cup U_1$ for all $n \in \mathbb{N}$. By remark 1.1.9, we have that $\sigma((1-p)a) = (\sigma(a) \cap U_0) \cup \{0\}$, where $p = p(a, \alpha)$. Since $\alpha \neq 0$, it follows that $\alpha \notin \sigma((1-p)a)$, so that $\alpha 1 - (1-p)a$ is invertible. Let $b = \alpha 1 - (1-p)a$. Then,

$$\begin{aligned} f_n(a)(1-p) &= f_n(a)(1-p)bb^{-1} \\ &= f_n(a)(1-p)(\alpha 1 - (1-p)a)b^{-1} \\ &= f_n(a)(1-p)(\alpha 1 - a)b^{-1} \\ &= f_n(a)(\alpha 1 - a)(1-p)b^{-1}, \end{aligned}$$

since $(1-p)p = 0$. From the assumption $(\alpha 1 - a)f_n(a) \to 0$ as $n \to \infty$, it follows that $f_n(a)(1-p) \to 0$ as $n \to \infty$. Also, from theorem 5.6.3, we get that $f_n(a) = f_n(a)(1-p) + \sum_{j=0}^{k-1} \frac{(a-\alpha 1)^j}{j!} f_n^{(j)}(\alpha)p$. Together with the assumptions $f_n(\alpha) \to 1$ as $n \to \infty$ and $f_n^{(j)}(\alpha) \to 0$ $(j = 1, 2, ..., k-1)$ as $n \to \infty$, it follows that $f_n(a) \to p$ as $n \to \infty$. $\square$

In the case where $\alpha$ in corollary 5.6.4 is a simple pole, we get the following result.

**Corollary 5.6.5.** *Let $A$ be a Banach algebra and $a \in A$. Let $(f_n)$ be a sequence of functions analytic on a neighbourhood $\Omega$ of $\sigma(a)$. Suppose that $\alpha \neq 0$ is a simple pole of the resolvent of $a$ such that $f_n(\alpha) \to 1$ as $n \to \infty$. Then $f_n(a) \to p$ as $n \to \infty$ if and only if $(\alpha 1 - a)f_n(a) \to 0$ as $n \to \infty$, where $p = p(\alpha, a)$.*

**Proof.** Suppose that $f_n(a) \to p$ as $n \to \infty$. Then $(\alpha 1 - a)f_n(a) \to (\alpha 1 - a)p$ as $n \to \infty$. Since $\alpha$ is a simple pole of the resolvent of $a$, we have that $(\alpha 1 - a)p = 0$. Hence $(\alpha 1 - a)f_n(a) \to 0$ as $n \to \infty$. The converse follows from corollary 5.6.4. $\square$

**Corollary 5.6.6.** *Let $A$ be a Banach algebra, $a \in A$ and suppose that $\alpha$ is a pole of order at most $k \geq 1$ of the resolvent of $a$. Suppose that $(f_n)$ is a sequence of complex valued functions analytic on a neighbourhood of $\sigma(a)$. If $(a - \alpha 1)f_n(a) \to 0$ as $n \to \infty$ and $f_n(\alpha) \to 1$ as $n \to \infty$, then $\alpha$ is a simple pole of the resolvent of $a$.*

**Proof.** Let $p = p(\alpha, a)$. By theorem 5.6.3 we may write

$$f_n(a) = f_n(a)(1-p) + \sum_{j=0}^{k-1} \frac{(a-\alpha 1)^j}{j!} f_n^{(j)}(\alpha)p. \qquad (*)$$

For $k = 1$, the result is trivial. For $k = 2$ we have from (*) that $f_n(a) = f_n(a)(1-p) + f_n(\alpha)p + f_n'(\alpha)(a - \alpha 1)p$. Therefore

$$(a - \alpha 1)f_n(a) = (a - \alpha 1)f_n(a)(1-p) + (a - \alpha 1)f_n(\alpha)p + (a - \alpha 1)^2 f_n'(\alpha)p. \qquad (**)$$

Since $k = 2$, we have that $\alpha$ is pole of order at most 2 of the resolvent of $a$. Therefore $(a - \alpha 1)^2 f_n'(\alpha)p = 0$. Using the assumptions $(a - \alpha 1)f_n(a) \to 0$ as $n \to \infty$ and $f_n(\alpha) \to 1$ as $n \to \infty$, it then follows from (**) that $(a - \alpha 1)p = 0$. Hence $\alpha$ is a simple pole of the resolvent of $a$. For any $k > 2$, the general procedure is as follows: In the first step multiply both sides of (*) by $(a - \alpha 1)^{k-1}$. Since $\alpha$ is a pole of order at most $k$, this makes all but the first two terms of the expression on the right hand side of (*) zero. On the resulting equation, we take limits as $n \to \infty$ and use the assumptions $(a - \alpha 1)f_n(a) \to 0$ as $n \to \infty$

and $f_n(\alpha) \to 1$ as $n \to \infty$ to obtain that $(a - \alpha 1)^{k-1} p = 0$. In the second step multiply both sides of (*) by $(a - \alpha 1)^{k-2}$. Using arguments similar to the first step and the fact that $(a - \alpha 1)^{k-1} p = 0$, we get that $(a - \alpha 1)^{k-2} p = 0$. After $k - 1$ steps it follows that $(a - \alpha 1)p = 0$. $\square$

In the rest of this section, we will consider only sequences of analytic functions of the form $f_n(\lambda) = \sum_{k=0}^{n-1} \frac{\lambda^k}{n}$, as they are of particular interest for the problem at hand. The next proposition shows that if we take this sequence of functions and $\alpha = 1$ in corollary 5.6.5, then we obtain a stronger form of the forward implication in corollary 5.6.5. To prove this result we will use the following lemma.

**Lemma 5.6.7.** *Let $A$ be a Banach algebra and $a \in A$. Then $(1 - a) \sum_{k=0}^{n-1} \frac{a^k}{n} \to 0$ if and only if $\frac{a^n}{n} \to 0$.*

**Proof.** We have that $(1 - a) \sum_{k=0}^{n-1} \frac{a^k}{n} = \sum_{k=0}^{n-1} \frac{a^k}{n} - \sum_{k=0}^{n-1} \frac{a^{k+1}}{n} = \frac{1}{n} - \frac{a^n}{n}$. Now, $\frac{1}{n} - \frac{a^n}{n} \to 0$ if and only if $\frac{a^n}{n} \to 0$. $\square$

**Proposition 5.6.8.** *Let $A$ be a Banach algebra and $a \in A$. If $(f_n(a))$ converges, where $f_n(a) = \sum_{k=0}^{n-1} \frac{a^k}{n}$, then $(1 - a) f_n(a) \to 0$.*

**Proof.** Suppose that $(f_n(a))$ converges, say, $f_n(a) \to b$. We have that $\frac{1}{n}((n + 1) f_{n+1}(a) - n f_n(a)) = \frac{a^n}{n}$. Since $\frac{1}{n}((n + 1) f_{n+1}(a) - n f_n(a)) = \frac{n+1}{n} f_{n+1}(a) - f_n(a) \to b - b = 0$, we get that $\frac{a^n}{n} \to 0$. It follows from lemma 5.6.7 that $(1 - a) f_n(a) \to 0$. $\square$

In the following proposition, we establish that if $\sigma(a)$ contains only one element, then a stronger form of the reverse implication in corollary 5.6.5 holds. This result is not used further; it is included for the sake of interest.

**Proposition 5.6.9.** *Let $A$ be a Banach algebra and $a \in A$. Suppose that $\sigma(a) = \{\alpha\}$ and $\alpha$ is a simple pole of the resolvent of $a$. If $(f_n)$ is a sequence of functions analytic on a neighbourhood $\Omega$ of $\sigma(a)$ and if $f_n(\alpha) \to 1$, then $f_n(a) \to p(\alpha, a)$.*

**Proof.** Let $\Gamma$ be a small circle centred at $\alpha$. Then $f_n(a) = \frac{1}{2\pi i} \int_\Gamma f_n(\lambda)(\lambda 1 - a)^{-1} d\lambda$. Since $\alpha$ is a simple pole of the resolvent of $a$, we obtain the Laurent series expansion $(\lambda 1 - a)^{-1} = \frac{a_{-1}}{\lambda - \alpha} + a_0 + a_1(\lambda - \alpha) + a_2(\lambda - \alpha)^2 + \cdots$ $(a_j \in A, j = -1, 0, 1, 2, ...)$ on a deleted neighbourhood $N_0$ of $\alpha$ which contains $\Gamma$. Let $S(\lambda)$ be the sum of the power series $a_0 + a_1(\lambda - \alpha) + a_2(\lambda - \alpha)^2 + \cdots$. Then $S(\lambda)$ is analytic on $N_0$ and then $(\lambda 1 - a)^{-1} = S(\lambda) + \frac{a_{-1}}{\lambda - \alpha}$. Clearly, $f_n(\lambda)S(\lambda)$ is analytic on $N_0$. Since $\Gamma$ is contained in $N_0$, it follows that

$$\begin{aligned} f_n(a) &= \frac{1}{2\pi i} \int_\Gamma f_n(\lambda) \left( S(\lambda) + \frac{a_{-1}}{\lambda - \alpha} \right) d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma f_n(\lambda) S(\lambda) d\lambda + \frac{a_{-1}}{2\pi i} \int_\Gamma \frac{f_n(\lambda)}{\lambda - \alpha} d\lambda \\ &= f_n(\alpha) a_{-1}. \end{aligned}$$

Since $a_{-1} = p(\alpha, a)$ and $f_n(\alpha) \to 1$, we obtain that $f_n(a) \to p(\alpha, a)$. $\square$

We now give our key result corresponding to part of ([25], theorem 3.16).

**Theorem 5.6.10.** *Let $A$ be a Banach algebra and $a \in A$. Suppose that $1 \in$ **iso** $\sigma(a)$. Let $(f_n)$ be the sequence of functions $f_n(\lambda) = \sum_{k=0}^{n-1} \frac{\lambda^k}{n}$ $(\lambda \in \mathbb{C})$. Then the following statements are equivalent:*

*(i) $(f_n(a))$ converges, with $f_n(a) \to p(1, a)$.*
*(ii) $(1 - a)f_n(a) \to 0$ as $n \to \infty$ and 1 is a simple pole of the resolvent of $a$.*

**Proof.** (i) $\Rightarrow$ (ii): Suppose that $f_n(a) \to p(1, a)$. Then by proposition 5.6.8, we have that $(1-a)f_n(a) \to 0$. Since $(1-a)f_n(a) \to (1-a)p(1, a)$, by uniqueness of limits, $(1-a)p(1, a) = 0$. Therefore 1 is a simple pole of the resolvent of $a$.

(ii) $\Rightarrow$ (i): We have that $f_n(1) = \sum_{k=0}^{n-1} \frac{1^k}{n} = 1$. If $(1 - a)f_n(a) \to 0$ as $n \to \infty$ and 1 is a simple pole of the resolvent of $a$, then by corollary 5.6.5, we get that $(f_n(a))$ is convergent, with $f_n(a) \to p(1, a)$. $\square$

Recall that a Banach algebra element $a$ is ergodic if the sequence $(f_n(a))$ converges, where $f_n(\lambda) = \sum_{k=0}^{n-1} \frac{\lambda^k}{n}$, $\lambda \in \mathbb{C}$. We give our first main result regarding the domination problem for ergodic elements.

**Theorem 5.6.11.** *Let $(A, C)$ be a COBA with $C$ closed and proper, and let $a, b \in A$ such that $0 \leq 1 \leq a \leq b$ and $ab = ba$. Suppose that $r(b) = 1 \in$ **iso** $\sigma(a)$ and that 1 is a simple pole of the resolvent of $b$. If $b$ is ergodic with $f_n(b) \to p(1, b)$ and if $p(1, a) = p(1, b)$, then $a$ is ergodic, with $f_n(a) \to p(1, a)$.*

**Proof.** Since 1 is a simple pole of the resolvent of $b$, by proposition 3.1.3, we get that 1 is an eigenvalue of $b$ with positive corresponding eigenvector $p(1, b)$. From the assumption $p(1, a) = p(1, b)$ it follows that $p(1, a) \in C$. Clearly, $(a - 1)p(1, a) = p(1, a)(a - 1)$ and $(b - 1)p(1, b) = p(1, b)(b - 1)$. Since $p(1, a) = p(1, b)$ and since $C$ is an algebra $c$-cone, the assumption $0 \leq 1 \leq a \leq b$ implies that $0 \leq (a - 1)p(1, a) \leq (b - 1)p(1, b) = 0$. From the fact that $C$ is proper, it follows that $(1 - a)p(1, a) = 0$, so that 1 is a simple pole of the resolvent of $a$. Now since $b$ is ergodic, proposition 5.6.8 and lemma 5.6.7 imply that $\frac{b^n}{n} \to 0$ as $n \to \infty$. It follows from proposition 2.1.22 that $\frac{a^n}{n} \to 0$ as $n \to \infty$. Lemma 5.6.7 then implies that $(1 - a)f_n(a) \to 0$ as $n \to \infty$. It follows from theorem 5.6.10 that $a$ is ergodic, with $f_n(a) \to p(1, a)$. $\square$

In an OBA, the condition $ab = ba$ in theorem 5.6.11 can be dropped since it is used only to guarantee that $0 \leq a^n \leq b^n$ $(n \in \mathbb{N})$ by proposition 2.1.22.

The remaining results about domination by ergodic elements will be proved under conditions similar to those of ([50], theorem 4.5). We start with theorem 5.6.12, which is the basic result from which the others will be obtained.

**Theorem 5.6.12.** *Let $A$ be a COBA with a closed algebra $c$-cone $C$ such that the spectral radius in $(A, C)$ is $c$-monotone, and let $a, b \in A$ such that $0 \leq a \leq b$ and $ab = ba$. Suppose that $1 \in$ **iso** $\sigma(a)$ is a pole of the resolvent of $a$. If $b$ is ergodic, then $a$ is ergodic.*

**Proof.** From proposition 2.1.22 we have that $0 \leq \frac{a^n}{n} \leq \frac{b^n}{n}$ for all $n \in \mathbb{N}$. Since $b$ is ergodic, $\frac{b^n}{n} \to 0$ as $n \to \infty$ by proposition 5.6.8 and lemma 5.6.7. Therefore $\frac{a^n}{n} \to 0$ as $n \to \infty$. Lemma 5.6.7 then implies that $(1-a)f_n(a) \to 0$ as $n \to \infty$. Since 1 is pole of the resolvent of $a$, it follows from corollary 5.6.6 that 1 is a simple pole of the resolvent of $a$. It follows from theorem 5.6.10 that $a$ is ergodic, with $f_n(a) \to p(1,a)$ as $n \to \infty$. $\square$

Even if $c$-monotonicity is replaced with monotononicity, we do not know whether the condition $ab = ba$ in theorem 5.6.12 can be dropped, since the proof uses proposition 2.1.22. Although we showed in example 2.1.23 that the assumption is essential for proposition 2.1.22, we could not show that it is essential for theorem 5.6.12. However, in OBA in which we have monotonicity, the condition can be dropped.

To prove our next two main theorems, the following three lemmas will be required.

**Lemma 5.6.13.** *Let $A$ be a Banach algebra and $a \in A$. If $a$ is ergodic, then $r(a) \leq 1$.*

**Proof.** Since $a$ is ergodic, $(f_n(a))$ converges. It follows from proposition 5.6.8 that $(1-a)f_n(a) \to 0$ as $n \to \infty$. Lemma 5.6.7 then implies that $\frac{||a^n||}{n} \to 0$ as $n \to \infty$. Therefore there exists a constant $c > 0$ such that $\frac{||a^n||}{n} \leq c$ for all $n \in \mathbb{N}$, so that $||a^n|| \leq cn$ for all $n \in \mathbb{N}$. It follows that $r(a) = \lim_{n\to\infty} ||a^n||^{\frac{1}{n}} \leq \lim_{n\to\infty} c^{\frac{1}{n}} n^{\frac{1}{n}} = 1$. $\square$

**Lemma 5.6.14.** *Let $(A, C)$ be an OBA with $C$ closed and such that the spectral radius in $(A, C)$ is monotone, and let $a, b \in A$ such that $0 \leq a \leq b$. Let $I$ be a closed inessential ideal of $A$ such that the spectral radius in $(A/I, \pi C)$ is monotone. If $r(a) = r(b)$ and if $r(b)$ is a Riesz point of $\sigma(b)$, then $r(a)$ is a Riesz point of $\sigma(a)$.*

Lemma 5.6.14 follows immediately from ([43], theorem 4.4) and it will also be used in the next section.

**Lemma 5.6.15.** *Let $A$ be a Banach algebra and $a \in A$. If $\frac{a^n}{n} \to 0$ as $n \to \infty$ and $1 \notin \sigma(a)$, then $\sum_{k=0}^{n-1} \frac{a^k}{n}$ converges to $0$ as $n \to \infty$.*

**Proof.** If $\frac{a^n}{n} \to 0$ as $n \to \infty$, then by lemma 5.6.7 $(1-a)\sum_{k=0}^{n-1} \frac{a^k}{n} \to 0$ as $n \to \infty$. If also $1 \notin \sigma(a)$, then $1-a$ is invertible, which yields the result. $\square$

**Theorem 5.6.16.** *Let $(A, C)$ be a semisimple OBA with $C$ closed and such that the spectral radius in $(A, C)$ is monotone, and let $a, b \in A$ such that $0 \leq a \leq b$ w.r.t. $C$. Let $I$ be a closed inessential ideal of $A$ such that the spectral radius in $(A/I, \pi C)$ is monotone. If $b$ is ergodic and if $r(b)$ is a Riesz point of $\sigma(b)$, then $a$ is ergodic.*

**Proof.** Since $b$ is ergodic, lemma 5.6.13 and the fact that the spectral radius in $(A, C)$ is monotone imply that $r(a) \leq r(b) \leq 1$. Then we have four cases: $r(a) < r(b) < 1$, $r(a) < r(b) = 1$, $r(a) = r(b) < 1$ and $r(a) = r(b) = 1$. Now from the OBA version of proposition 2.1.22, we have that $0 \leq \frac{a^n}{n} \leq \frac{b^n}{n}$ for all $n \in \mathbb{N}$. Since $b$ is ergodic, $\frac{b^n}{n} \to 0$ as $n \to \infty$ by proposition 5.6.8 and lemma 5.6.7. Therefore $\frac{a^n}{n} \to 0$ as $n \to \infty$. In the first three cases, we get that $1 \notin \sigma(a)$. Therefore $\sum_{k=0}^{n-1} \frac{a^k}{n} \to 0$ as $n \to \infty$ by lemma 5.6.15, so that $a$ is ergodic. To deal with the last case suppose that $r(a) = r(b) = 1$. Since $r(b)$ is a Riesz point of $\sigma(b)$, by lemma 5.6.14, we have that $r(a)$ is a Riesz point of $\sigma(a)$. Lemma 1.2.3 then implies that $r(a)$ is a pole of the resolvent of $a$. From the OBA version of theorem 5.6.12, it follows that $a$ is ergodic. □

For $C'$OBAs, we have the following version of theorem 5.6.16.

**Theorem 5.6.17.** *Let $(A, C)$ be a semisimple COBA with $C$ closed and such that the spectral radius in $(A, C)$ is c-monotone, and let $a, b \in A$ with $0 \leq a \leq b$ and $ab = ba$. Let $I$ be a closed inessential ideal of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. If $b$ is ergodic and if $r(b)$ is a Riesz point of $\sigma(b)$, then $a$ is ergodic.*

**Proof.** Since $b$ is ergodic, lemma 5.6.13 and the fact that the spectral radius in $(A, C)$ is $c$-monotone imply that $r(a) \leq r(b) \leq 1$. The cases $r(a) < r(b) < 1$, $r(a) < r(b) = 1$, $r(a) = r(b) < 1$ are dealt with in a similar way to theorem 5.6.16. For the case $r(a) = r(b) = 1$, since $r(b)$ is a Riesz point of $\sigma(b)$, we have by corollary 4.4.9 that $r(a)$ is a Riesz point of $\sigma(a)$. Lemma 1.2.3 then implies that $r(a)$ is a pole of the resolvent of $a$. The result then follows from theorem 5.6.12. □

Note that there is a result in terms of MPCSs corresponding to theorem 5.6.16. Since it can easily be deduced, we have not presented it.

We end this section with the following observation, which we formulate as a proposition:

**Proposition 5.6.18.** *Let $A$ be a COBA with a closed and inverse-closed algebra c-cone $C$. If $a \in C$ and $a$ is ergodic, then $a \leq 1$.*

Proposition 5.6.18 follows from proposition 3.3.8 and lemma 5.6.13.

## 5.7 Peripheral point spectrum

Let $a$ be a non-zero element of a Banach algebra $A$. We denote by $P\sigma(a)$ the set of all eigenvalues of $a$, and $P\sigma(a)$ is called the *point spectrum* of $a$. The set $P\sigma(a) \cap \operatorname{psp}(a)$ is called the *peripheral point spectrum* of $a$.

In this section we consider two problems. The first problem is that of determining when the spectral radius of a dominated positive element is an eigenvalue of the element, with positive corresponding eigenvector, given that the spectral radius of the dominating element is an eigenvalue of the element and has positive corresponding eigenvector. The main results are theorems 5.7.4 and 5.7.5. The second problem is that of determining conditions under which the peripheral point spectrum of a dominated positive element is contained in the peripheral point spectrum of the dominating element. The main results are theorems 5.7.6 and 5.7.9 for OBAs and theorems 5.7.8 and 5.7.10 for COBAs. Our results are analogous to those in [50], where these problems were dealt with in the operator-theoretic context. The results in this section, due to author, extend the theory of COBAs and OBAs. Whenever we have a result involving $C'$OBAs in this section, there is a corresponding result in terms of MPCSs, which will not be presented as it can easily be deduced from the corresponding OBA result.

The first part of the following proposition is related to our first problem. It provides conditions under which an eigenvalue of a dominated positive element with positive corresponding eigenvector will be an eigenvalue of the dominating element, with positive corresponding eigenvector. A result for the corresponding problem for positive operators in Banach lattices is ([50], lemma 1.3). Since the proof of this result is typically operator theoretic, we will obtain our result under different conditions. Our proof then will be purely algebraic. Conversely, the second part of the result gives conditions under which an eigenvalue of a dominating positive element with positive corresponding eigenvector will be an eigenvalue of the dominated element, with positive corresponding eigenvector. This part will be useful in the proof of theorem 5.7.3.

**Proposition 5.7.1.** *Let $A$ be an OBA with a proper algebra cone $C$ and let $a, b \in A$. Suppose that $\lambda$ is an eigenvalue of $a$ with positive corresponding eigenvector $u$. If either $0 < a \leq b \leq \lambda 1$ or $0 < \lambda 1 \leq b \leq a$, then $\lambda$ is an eigenvalue of $b$ with corresponding eigenvector $u$.*

**Proof.** To prove the first part, if $\lambda$ is an eigenvalue of $a$ with positive corresponding eigenvector $u$, then it follows from $0 < a \leq b \leq \lambda 1$ that $0 \leq au \leq bu \leq \lambda u = au$. Since $C$ is proper, the result follows. The second part is established similarly. $\square$

If we assume in proposition 5.7.1 that $u$ commutes with both $a$ and $b$, then with the same proof, we obtain the result in the COBA setting.

The following example, which can be easily verified, shows that the condition $b \leq \lambda 1$ in the first part of proposition 5.7.1 cannot be dropped. In a similar way, it can be shown that the condition $\lambda 1 \leq b$ in the second part of the result cannot be omitted.

**Example 5.7.2.** *Let $A = M_2(\mathbb{C})$ and $M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, N = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \in A$. Then 2 is an eigenvalue of $M$ with corresponding eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and 2 is an eigenvalue of $N$ but $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is not a corresponding eigenvector.*

Recall that our first problem is that of determining when the spectral radius of a dominated positive element is an eigenvalue of the element, with positive corresponding eigenvector, given that the spectral radius of the dominating element is an eigenvalue of the element and has positive corresponding eigenvector. The following theorem is the basic result for this problem.

**Theorem 5.7.3.** *Let $A$ be an OBA with a proper, closed algebra cone $C$ and let $a, b \in A$. Suppose that $0 < r(a) = r(b)$ and that $0 < r(b)1 \leq a \leq b$. If $r(b)$ is a pole of the resolvent of $b$, then $r(b)$ is an eigenvalue of $b$ with positive corresponding eigenvector $u$, and $r(a)$ is an eigenvalue of $a$ with corresponding eigenvector $u$.*

**Proof.** Suppose that $r(b)$ is a pole of the resolvent of $b$. Then by the Krein-Rutman theorem ([47], theorem 3.2), $r(b)$ is an eigenvalue of $b$, with positive corresponding eigenvector $u$. With the assumptions $r(a) = r(b)$ and $0 < r(b)1 \leq a \leq b$, we apply proposition 5.7.1 to obtain that

$r(a)$ is an eigenvalue of $a$, with corresponding eigenvector $u$. $\square$

If $u$ in theorem 5.7.3 commutes with $a$, then using theorem 3.1.2 and the COBA version of proposition 5.7.1, the corresponding result for COBAs can be established.

Under certain natural conditions, the assumption $0 < r(b)1 \leq a \leq b$ in theorem 5.7.3 can be relaxed to $0 < a \leq b$. We prove this in the following result.

**Theorem 5.7.4.** *Let $(A, C)$ be a semisimple OBA with $C$ closed and such that the spectral radius in $(A, C)$ is monotone. Let $I$ be a closed inessential ideal of $A$ such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a, b \in A$ with $0 < a \leq b$ w.r.t. $C$ and $0 < r(a) = r(b)$. If $r(b)$ is an eigenvalue of $b$ with positive corresponding eigenvector and with $r(b)$ a Riesz point of $\sigma(b)$, then $r(a)$ is an eigenvalue of $a$ with positive corresponding eigenvector.*

**Proof.** By lemma 5.6.14, $r(a)$ is a Riesz point of $\sigma(a)$. Lemma 1.2.3 then implies that $r(a)$ is a pole of the resolvent of $a$. It follows from the Krein-Rutman theorem that $r(a)$ is an eigenvalue of $a$, with positive corresponding eigenvector. $\square$

In terms of $C'$OBAs, we get the following analogue of theorem 5.7.4.

**Theorem 5.7.5.** *Let $(A, C)$ be a semisimple COBA with $C$ closed and such that the spectral radius in $(A, C)$ is c-monotone. Let $I$ be a closed inessential ideal of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. Suppose that $a, b \in A$ such that $0 < a \leq b$ w.r.t. $C$ and that $0 < r(a) = r(b)$. If $r(b)$ is an eigenvalue of $b$ with positive corresponding eigenvector and with $r(b)$ a Riesz point of $\sigma(b)$, then $r(a)$ is an eigenvalue of $a$ with positive corresponding eigenvector.*

**Proof.** By corollary 4.4.9, $r(a)$ is a Riesz point of $\sigma(a)$. It follows from lemma 1.2.3 and theorem 3.1.2 that $r(a)$ is an eigenvalue of $a$ with positive corresponding eigenvector. $\square$

We now turn to our second problem, that of determining when the peripheral point spectrum of a dominated positive element is contained in the peripheral point spectrum of the dominating element. Results for the corresponding problem for positive operators on a Banach lattice are contained in ([50], theorem 2.2; corollaries 2.3, 2.4; theorem 2.6). Since the proofs of these results are typically operator theoretic, we will obtain our results under different conditions, so that the proofs are by purely algebraic means. We start with the following result.

**Theorem 5.7.6.** *Let $(A, C)$ be a semisimple OBA with $C$ closed and such that the spectral radius in $(A, C)$ is monotone. Let $I$ be a closed inessential ideal of $A$ such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a, b \in A$ such that $0 < a \leq b$ w.r.t. $C$ and $0 < r(a) = r(b)$. If $r(b)$ is a Riesz point of $\sigma(b)$ and $\operatorname{psp}(a) \subseteq \operatorname{psp}(b)$, then $P\sigma(a) \cap \operatorname{psp}(a) \subseteq P\sigma(b) \cap \operatorname{psp}(b)$.*

**Proof.** Suppose that $r(b)$ is a Riesz point of $\sigma(b)$. Then by ([43], theorem 4.3) and ([43], theorem 4.4), we have that $\operatorname{psp}(a)$ consists of Riesz points of $\sigma(a)$ and $\operatorname{psp}(b)$ consists of Riesz points of $\sigma(b)$. It follows from lemma 1.2.3 that $\operatorname{psp}(a)$ consists of poles of the resolvent of $a$ and $\operatorname{psp}(b)$ consists of poles of the resolvent of $b$. By theorem 3.1.1, we have that $\operatorname{psp}(a)$ consists of eigenvalues of $a$ and $\operatorname{psp}(b)$ consists of eigenvalues of $b$. Therefore $\operatorname{psp}(a) \subseteq P\sigma(a)$ and $\operatorname{psp}(b) \subseteq P\sigma(b)$. From the assumption $\operatorname{psp}(a) \subseteq \operatorname{psp}(b)$, the result follows. □

The next proposition shows that if we take positive operators on a Banach lattice, then some of the assumptions in theorem 5.7.6 can be omitted.

**Proposition 5.7.7.** *Let $E$ be a Banach lattice and $S, T$ positive operators on $E$ such that $0 \leq S \leq T$. If $r(S) = r(T)$ and if $r(T)$ is a Riesz point of $\sigma(T)$, then $P\sigma(S) \cap \operatorname{psp}(s) \subseteq P\sigma(T) \cap \operatorname{psp}(T)$.*

**Proof.** Since $r(S) = r(T)$ and $r(T)$ is a Riesz point of $\sigma(T)$, $\operatorname{psp}(S) \subseteq \operatorname{psp}(T)$ by ([50], corollary 1.6). Also, $r(S)$ is a Riesz point of $\sigma(S)$ by ([21], theorem 4.1). It follows (cf. [50], p.25) that $\operatorname{psp}(S)$ consists of Riesz points of $\sigma(S)$ and $\operatorname{psp}(T)$ consists of Riesz points of $\sigma(T)$. Since Riesz points of $\sigma(S)$ are poles of the resolvent of $S$ (cf. [50]), $\operatorname{psp}(S)$ consists of poles of the resolvent of $S$. Similarly, $\operatorname{psp}(T)$ consists of poles of the resolvent of $T$. From proposition 1.4.4, we get that $\operatorname{psp}(S)$ consists of eigenvalues of $S$ and $\operatorname{psp}(T)$ consists of eigenvalues of $T$. This implies that $P\sigma(S)\cap\operatorname{psp}(S) = \operatorname{psp}(S)$ and $P\sigma(T)\cap\mathrm{T} = \operatorname{psp}(T)$, and the result follows. □

Note that if $E$ in the proposition above is a Dedekind complete Banach lattice, then in view of ([43], example 3.2), the result can be obtained directly from ([50], corollary 1.6) and theorem 5.7.6.

In terms of $C'$OBAs, we have the following analogue of theorem 5.7.6.

**Theorem 5.7.8.** *Let $(A, C)$ be a semisimple COBA with $C$ closed and such that the spectral radius in $(A, C)$ is c-monotone. Let $I$ be a closed inessential ideal of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. Suppose that $a, b \in A$ such that $0 < a \leq b$ w.r.t. $C$, $ab = ba$ and $0 < r(a) = r(b)$. If $r(b)$ is a Riesz point of $\sigma(b)$ and if $\operatorname{psp}(a) \subseteq \operatorname{psp}(b)$, then $P\sigma(a) \cap \operatorname{psp}(a) \subseteq P\sigma(b) \cap \operatorname{psp}(b)$.*

By using lemma 1.2.3 and theorems 4.4.5, 4.4.8 and 3.1.1, theorem 5.7.8 can be proved similarly to theorem 5.7.6.

If the elements involved are Riesz elements, some of the restrictions in theorem 5.7.6 and theorem 5.7.8 can be relaxed. We prove this in the following results.

**Theorem 5.7.9.** *Let $(A, C)$ be a semisimple OBA and $I$ a closed inessential ideal of $A$ such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a, b \in A$ such that $0 < a \leq b$ w.r.t. $C$ and $r(a), r(b) > 0$. If $b$ is Riesz relative $I$ and if $\operatorname{psp}(a) \subseteq \operatorname{psp}(b)$, then $P\sigma(a) \cap \operatorname{psp}(a) \subseteq P\sigma(b) \cap \operatorname{psp}(b)$.*

**Proof.** If $b$ is Riesz relative to $I$, then $a$ is Riesz relative to $I$ by ([51], theorem 6.2). It follows from theorem 1.2.1 that every non-zero point in $\sigma(a)$ is a pole of the resolvent of $a$ and every non-zero point in $\sigma(b)$ is a pole of the resolvent of $b$. Since $r(a), r(b) > 0$, we have that $\operatorname{psp}(a)$ consists of poles of the resolvent of $a$ and $\operatorname{psp}(b)$ consists of poles of the resolvent of $b$. By theorem 3.1.1, it follows that $\operatorname{psp}(a)$ consists of eigenvalues of $a$ and $\operatorname{psp}(b)$ consists of eigenvalues of $b$. Therefore $P\sigma(a) \cap \operatorname{psp}(a) = \operatorname{psp}(a)$ and $P\sigma(b) \cap \operatorname{psp}(b) = \operatorname{psp}(b)$. From the assumption $\operatorname{psp}(a) \subseteq \operatorname{psp}(b)$, the result follows. □

**Theorem 5.7.10.** *Let $(A, C)$ be a semisimple COBA and $I$ a closed inessential ideal of $A$ such that the spectral radius in the $C'$OBA $(A/I, \pi C)$ is c-monotone. Suppose that $a, b \in A$ such that $0 < a \leq b$ w.r.t. $C$, $ab = ba$ and $r(a), r(b) > 0$. If $b$ is Riesz relative to $I$ and if $\operatorname{psp}(a) \subseteq \operatorname{psp}(b)$, then $P\sigma(a) \cap \operatorname{psp}(a) \subseteq P\sigma(b) \cap \operatorname{psp}(b)$.*

Theorem 5.7.10 can be proved in a similar way to theorem 5.7.9, by using theorems 1.2.1, 3.1.1 and 5.2.2.

Note that if $A$ is a $C^*$-algebra ordered by $C = \{a \in A : a = a^* \text{ and } \sigma(a) \subseteq [0, \infty)\}$, theorems 5.7.6, 5.7.8, 5.7.9 and 5.7.10 hold trivially under merely the assumptions $a, b \in C$ and $\operatorname{psp}(a) \subseteq \operatorname{psp}(b)$. This is because if $a, b \in C$, then $\operatorname{psp}(a) = \{r(a)\}$ and $\operatorname{psp}(b) = \{r(b)\}$.

# Bibliography

[1] C.D. Aliprantis and O. Burkinshaw: Dunford Pettis operators on Banach lattices, *Trans. Amer. Math. Soc.* **274** (1982), 227–238.

[2] C.D. Aliprantis and O. Burkinshaw: On weakly compact operators on Banach lattices, *Proc. Amer. Math. Soc.* **83** (1981), 573–587.

[3] C.D. Aliprantis and O. Burkinshaw: Positive compact operators in Banach lattices, *Math. Z.* **174** (1980), 289–298.

[4] F. Arándiga and V. Caselles: Approximations of positive operators and continuity of the spectral radius, *J. Operator Theory* **26** (1991), 53–71.

[5] F. Arándiga and V. Caselles: Approximations of positive operators and continuity of the spectral radius II, *Math. Z.* **209** (1992), 547–558.

[6] F. Arándiga and V. Caselles: Approximations of positive operators and continuity of the spectral radius III, *J. Aust. Math. Soc. Series A* **57** (1994), 330–340.

[7] W. Arendt: On the $o$-spectrum of regular operators and the spectrum of measures, *Math. Z.* **178** (1981), 271–287.

[8] N. Asmar: *Applied complex analysis with partial differential equations*, Prentice Hall, New Jersey, 2002.

[9] B. Aupetit: *A primer on spectral theory*, Springer-Verlag, New York, 1991.

[10] B. Aupetit, H. du T. Mouton: Trace and determinant in Banach algebras, *Studia Math.* **121(2)** (1996), 115–136.

[11] B.A. Barnes, G.J. Murphy, M.R.F. Smyth and T.T. West: *Research notes in Mathematics* **67**. *Riesz and Fredholm theory in Banach algebras*, Pitman, 1982.

[12] D.R. Behrendt: On inessential elements in Banach algebras, Ph.D. Thesis, University of Johannesburg, 2005.

[13] D. Behrendt and H. Raubenheimer: On domination of inessential elements in ordered Banach algebras, *Illinois J. Math.* **51(3)** (2007), 927–936.

[14] E.V. Belmega, S. Lasaulce and M. Debbah: A trace inequality for positive definite matrices, *J. Inequal. Pure Appl. Math. (JIPAM)* **10(1)** (2009), 1–4.

[15] F.F. Bonsall and J. Duncan: *Complete normed algebras*, Springer-Verlag, New York–Heidelberg–Berlin, 1973.

[16] G. Braatvedt, R. Brits and H. Raubenheimer: Gelfand-Hille type theorems in ordered Banach algebras, *Positivity* **13** (2009), 39–50.

[17] R.M. Brits, L. Lindeboom and H. Raubenheimer: On the structure of rank one elements in Banach algebras, *Extracta Math.* **18** (2003), 297–309.

[18] L. Burlando: A generalization of the uniform ergodic theorem to poles of arbitrary order, *Studia Math.* **122(1)** (1997), 75–98.

[19] L. Burlando: Monotonicity of the spectral radius for positive operators on ordered Banach spaces, *Arch. Math.* **56** (1991), 49–57.

[20] S.R. Caradus, W.E. Pfaffenberger, B. Yood: *Calkin Algebras and Algebras of Operators on Banach Spaces. Lecture Notes in Pure and Applied Mathematics* **9**, Marcel Dekker Inc., New York, 1974.

[21] V. Caselles: On the peripheral spectrum of positive operators, *Israel J. Math.* **58** (1987), 144–160.

[22] J.B. Conway: *A course in functional analysis*, Springer-Verlag, New York–Heidelberg–Tokyo, 1985.

[23] R.S. Doran and V.A. Belfi: *Characterizations of $C^*$-algebras: The Gelfand–Naimark theorems*, Marcel Dekker Inc., New York–Basel, 1986.

[24] D. Drissi, J. Zemánek: Gelfand-Hille theorems for Cesáro means, *Quaest. Math.* **23** (2000), 375–381.

[25] N. Dunford: Spectral theory. I. Convergence to projections, *Trans. Amer. Math. Soc.* **54** (1943), 185–217.

[26] S. Furuichi, K. Kuriyama and K. Yanagi: Trace inequalities for products of matrices, *Linear Algebra Appl.* **430** (2009), 2271–2276.

[27] S. Furuichi, M. Lin: A matrix trace inequality and its application, *Linear Algebra Appl.* **433** (2010), 1324–1328.

[28] J.J. Grobler and C.B. Huijsmans: Doubly Abel bounded operators with single spectrum, *Quaest. Math.* **18** (1995), 397–406.

[29] J.J. Grobler and H. Raubenheimer: Spectral properties of elements in different Banach algebras, *Glasg. Math. J.* **33** (1991), 11–20.

[30] L. Groenewald, R.E. Harte and H. Raubenheimer: Perturbation by inessential and Riesz elements, *Quaest. Math.* **12** (1989), 439–446.

[31] R. Harte: Tim Starr, Trevor West and a positive contribution to operator theory, *Filomat* **21(1)** (2007), 129–135.

[32] W.K. Hayman and P.B. Kennedy: *Subharmonic functions I*, London Math. Soc. Monogr. 9, Academic Press, London, 1976.

[33] G. Herzog and C. Schmoeger: A note on a theorem of Raubenheimer and Rode, *Proc. Amer. Math. Soc.* **131 (11)** (2003), 3507–3509.

[34] N. Kalton, S. Montgomery-Smith, K. Oleszkiewicz, Y. Tomilov: Power-bounded operators and related norm estimates, *J. Lond. Math. Soc.* **70 (2)** (2004), 463–478.

[35] K. Knopp: *Theory of functions parts I and II*, Dover, New York, 1996.

[36] M.G. Krein and M.A. Rutman: Linear operators leaving invariant a cone in a Banach space, *Amer. Math. Soc. Transl.* **10 (1)** (1962), 199–325.

[37] E. Kreyszig: *Introductory functional analysis with applications*, John Wiley & Sons, New York, 1989.

[38] I. Marek: Frobenius theory for positive operators: comparison theorems and applications, *SIAM J. Appl. Math.* **19** (1970), 607–628.

[39] J. Martinez and J.M. Mazón: Quasi-compactness of dominated positive operators and $C_0-$ semigroups, *Math. Z.* **207** (1991), 109–120.

[40] M. Mbekhta and J. Zemánek: 'Sur le théoréme ergodique uniforme et le spectre', *C.R. Acad. Sci. Paris Sér. I Math.* **317** (1993), 1155–1158.

[41] H. du T. Mouton and S. Mouton: Domination properties in ordered Banach algebras, *Studia Math.* **149 (1)** (2003), 63–73.

[42] S. Mouton: A spectral problem in ordered Banach algebras, *Bull. Austral. Math. Soc.* **67** (2003), 131–144.

[43] S. Mouton: Convergence properties of positive elements in Banach algebras, *Math. Proc. R. Ir. Acad.* **102** A (2) (2002), 149–162.

[44] S. Mouton: On spectral continuity of positive elements, *Studia Math.* **174 (1)** (2006), 75–84.

[45] S. Mouton: On the boundary spectrum in Banach algebras, *Bull. Austral. Math. Soc.* **74** (2006), 239–246.

[46] S. Mouton: A condition for spectral continuity of positive elements, *Proc. Amer. Math. Soc.* **137 (5)** (2009), 1777–1782.

[47] S. Mouton and H. Raubenheimer: More spectral theory in ordered Banach algebras, *Positivity* **1** (1997), 305–317.

[48] T. Mouton and H. Raubenheimer: More on Fredholm theory relative to a Banach algebra homomorphism, *Proc. R. Ir. Acad.* **93** A (1) (1993), 17–25.

[49] J. Puhl: The trace of finite and nuclear elements in Banach algebras, *Czechoslovak Math. J.* **28(103)** (1978), 656–676.

[50] F. Räbiger and M.P.H. Wolff: Spectral and asymptotic properties of dominated operators, *J. Austral. Math. Soc. (Series A)* **63** (1997), 16–31.

[51] H. Raubenheimer and S. Rode: Cones in Banach algebras, *Indag. Math.* **7** (1996), 489–502.

[52] S. Rode: Cones in Banach algebras, Ph.D. Thesis, University of the Orange Free State, 1994.

[53] H.L. Royden: *Real analysis*, Macmillan Publishing Company, New York, 1988.

[54] W. Rudin: *Functional analysis*, McGraw-Hill, New York, 1973.

[55] H.H. Schaefer: *Banach lattices and positive operators*, Springer-Verlag, 1974.

[56] H.H. Schaefer: On the $o$-spectrum of order bounded operators, *Math. Z.* **154** (1977), 79–84.

[57] H.H. Schaefer: *Topological vector spaces*, Springer-Verlag, New York Heidelberg Berlin, 1971.

[58] H.H. Schaefer, M. Wolff and W. Arendt: On lattice isomorphisms with positive real spectrum and groups of positive operators, *Math. Z.* **164** (1978), 115–123.

[59] M.R. Spiegel, S. Lipschutz, J.J. Schiller, D. Spellman: *Complex variables*, McGraw-Hill, New York, 2009.

[60] A.C. Zaanen: *Riesz spaces II*, North-Holland, Amsterdam, 1983.

[61] X.-D. Zhang: Some aspects of the spectral theory of positive operators, *Acta Appl. Math.* **27** (1992), 135–142.

[62] X.-D. Zhang: On spectral properties of positive operators, *Indag. Math. (N.S.)* **4** (1993), 111–127.

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