COMPUTING VARIOUS TYPES OF LATTICES FREELY GENERATED BY POSETS

Marcel Wild

1) Introduction

Let V be a vectorspace and let $S_p \subseteq V$ $(p \in P)$ be finitely many subspaces. How big is the modular lattice $\mathcal{L}(S_p \,|\, p \in P)$ generated by these subspaces, ie. how many subspaces does one obtain by taking arbitrary intersections and sums of S_p's such as $[(S_2 \cap (S_1 + S_3)) + S_4] \cap (S_1 + S_5)$? The question is eg. relevant in connection with Gross's lattice method [G], [KKW] whereby V is a quadratic space, but no further reference to quadratic spaces will be made in this article. The answer is as follows. Consider the partially ordered set (poset) $P = (P, \leq)$ defined by $p \leq q :\Leftrightarrow S_p \subseteq S_q$. Then $|\mathcal{L}(S_p \,|\, p \in P)| \leq |\mathcal{M}(P)|$ where $\mathcal{M}(P)$ is the so called *modular lattice freely generated by the poset* P. One usually has $|\mathcal{M}(P)| = \infty$ but those P with $|\mathcal{M}(P)| < \infty$ can be neatly characterized (Thm.5). If $\mathcal{M}(P)$ happens to be finite then it is a subdirect product of 2-element lattices $\mathcal{D}_2$ and 5-element lattices $\mathcal{M}_3$, which eg. implies that the upper bound $|\mathcal{M}(P)|$ (finite or infinite) can always be realized by $|\mathcal{L}(S_p \,|\, p \in P)|$ for appropriate spaces V and $S_p \subseteq V$. We shall exhibit in detail an algorithm for computing $|\mathcal{M}(P)|$ for any finite poset P (section 6 and 7).

Before that we look at the related but simpler problem of describing the distributive lattice $\mathcal{D}(P)$ freely generated by P (section 5). Its cardinality is an upper bound to the number of *sets* one can obtain by taking arbitrary intersections and unions of fixed subsets S_p of a *set* V (whereby $S_p \subseteq S_q \Leftrightarrow p \leq q$). The even easier problems concerning the freely generated semilattice $\mathcal{S}(P)$, respectively the freely generated Boolean lattice $\mathcal{B}(P)$ are dealt with in sections 3 and 4. All of these lattices can be viewed as closure systems given by a family Σ of implications. In section 2 we outline an algorithm for calculating the cardinality of any closure system given by such a Σ.

2) An enumerative principle of exclusion

For a finite set C_0 and a "property" P applying to elements of C_0 denote by $N(P)$ the number of $x \in C_0$ with $P(x)$ (ie. sharing property P). If $P_1, \ldots, P_n$ are several properties, then the well known principle of "inclusion-exclusion" [S, p.65] states that

$$(1) \quad N(P_1 \wedge \ldots \wedge P_n) = \sum_{1 \leq i \leq n} N(P_i) - \sum_{1 \leq i < j \leq n} N(P_i \vee P_j) + \cdots \pm N(P_1 \vee \cdots \vee P_n)$$

Unfortunately $2^n - 1$ many terms $N(P_i \vee P_j \vee \cdots \vee P_k)$ need to be added and subtracted on the righthand side of (1). Moreover, sometimes there is no easy way to compute the terms themselves. We propose a principle of "exclusion" whose basic idea is very simple. Namely, starting with C_0, *exclude* iteratively all elements which fail to have property $P_1, P_2, \ldots, P_n$:

(2) $\mathbf{C}_0 \supseteq \mathbf{C}_1 := \{x \in \mathbf{C}_0 : P_1(x)\} \supseteq \mathbf{C}_2 := \{x \in \mathbf{C}_1 : P_2(x)\} \supseteq \mathbf{C}_3 \supseteq \cdots \supseteq \mathbf{C}_n$

Obviously the searched for value $N(P_1 \wedge \cdots \wedge P_n)$ is just $|\mathbf{C}_n|$. Observe that this is a n step procedure as opposed to 2^n! This sounds too good to be true. Indeed there are some tradeoffs that will be discussed elsewhere (essentially an exponential *time* problem is converted in an exponential *space* problem, but in a more benign one). Suffice it to say that the principle of exclusion outperformed the principle of inclusion-exclusion on many occasions. So far I have developed the approach under the restriction that the groundset $\mathbf{C}_0$ is a *powerset*, ie. $\mathbf{C}_0 = \mathcal{P}(M)$. Of particular interest*) is the case where each P_i is the property of an associated "implication" $A_i \rightarrow B_i$ being satisfied. Hereby an *implication* on a set M is a pair of subsets (A, B), written as $A \rightarrow B$. Let

(3) $\sum := \{A_1 \rightarrow B_1,\ A_2 \rightarrow B_2, \cdots, A_n \rightarrow B_n\}$

be a family of implications. A subset $X \subseteq M$ is Σ-*closed* if

(4) $(\forall 1 \leq i \leq n) \quad (A_i \subseteq X \Rightarrow B_i \subseteq X)$

In other words, for each i one must have either $A_i \not\subseteq X$ or $B_i \subseteq X$ (or both). It is easy to see that the family $\mathbf{C}(\Sigma)$ of all Σ-closed sets X is closed under intersections, ie. is a *closure system*. The connection between $\mathbf{C}(\Sigma)$ and $\mathbf{C}_n$ in (2) is straightforward. Let $\Sigma := \{A_1 \rightarrow B_1, \ldots, A_n \rightarrow B_n\}$ be a family of implications on M. Put $\mathbf{C}_0 := \mathcal{P}(M)$ and say that $X \in \mathbf{C}_0$ has property P_i if either $A_i \not\subseteq X$ or $B_i \subseteq X$. Then $|\mathbf{C}(\Sigma)| = N(P_1 \wedge \cdots \wedge P_n) = |\mathbf{C}_n|$.

Before we discuss in more detail how $|\mathbf{C}(\Sigma)|$ can be computed efficiently, let us indicate that this framework fits a lot of applications [W3], [W4]. For instance, let M be a universal algebra, say a semigroup with multiplication $(x, y) \mapsto x \circ y$. Then $\Sigma := \{\{x, y\} \rightarrow \{x \circ y, y \circ x\} : x, y \in M\}$ yields the closure system $\mathbf{C}(\Sigma)$ of all subsemigroups. Or, let M be a matroid with family of circuits $\mathbf{F} \subseteq \mathcal{P}(M)$. Then the family Σ of all implications $(A - \{a\}) \rightarrow \{a\} \quad (A \in \mathbf{F}, a \in A)$ yields the closure system $\mathbf{C}(\Sigma)$ of all flats of M. Or, let Σ be a family of implications $A_i \rightarrow B_i$ where $A_i \subseteq M$ is a set of "attributes" which implies**) another set of attributes $B_i \subseteq M$. The resulting lattice $\mathbf{C}(\Sigma)$ of Σ-closed sets is the so called *formal concept lattice* [GW]. In section 3, 5 and 6 of this article we shall see that also finitely presented semilattices, distributive lattices, or modular lattices, can advantageously be conceived and computed as closure systems $\mathbf{C}(\Sigma)$.

Now let us point out some computational details of our principle of exclusion (some more follow in section 7). We shall fix a linear order on our groundset M, say $M := \{p_1, p_2, \ldots, p_t\}$ with $p_1 < \cdots < p_t$. Subsets $X \subseteq M$ will be identified with their charact-

*) Yet I am also experimenting with other types of properties P_i, eg. yielding an enumeration of all linear extensions of a poset, or an enumeration of other sorts of combinatorial objects. How well the principle of exclusion competes with existing algorithms in these cases remains to be seen.

**) More precisely, this means the following. Given a *context* (G, M, I) with *object* set, G, *attribute* set M, and relation $I \subseteq G \times M$, a subset $A \subseteq M$ is said to *imply* a subset $B \subseteq M$ if

$\{g \in G | (\forall m \in A) g\, \mathrm{I} m\} \subseteq \{g \in G | (\forall m \in B) g\, \mathrm{I}\, m\}.$

eristic functions. Namely, X corresponds to a *2-valued row* $\mathbf{r} = (r_1, \ldots, r_t)$ with $r_i = 1$ $(p_i \in X)$ and $r_i = 0$ $(p_i \notin X)$. More generally, consider rows with entries from $\{0, 1, 2\}$. By definition, such a *3-valued row* $\mathbf{r} = (r_1, \ldots, r_t)$ represents the *family* of all subsets $X \subseteq M$ which satisfy

$$(\forall 1 \leq i \leq t) \quad ((r_i = 0 \Rightarrow p_i \notin X) \quad \textit{and} \quad (r_i = 1 \Rightarrow p_i \in X))$$

When $r_i = 2$ then both $p_i \in X$ or $p_i \notin X$ is allowed. For $k \in \{0, 1, 2\}$ define the *k-set* of $\mathbf{r}$ as $\mathbf{r}\{k\} := \{p_i \in M : r_i = k\}$. Identifying $\mathbf{r}$ with the family of X's it represents, we see that $\mathbf{r}$ is the Boolean interval sublattice $\{X \in \mathcal{P}(M) : \mathbf{r}\{1\} \subseteq X \subseteq \mathbf{r}\{1\} \cup \mathbf{r}\{2\}\}$ of $\mathcal{P}(M)$.

Example 1: Let $M := \{v, w, x, y, z\}$ and

$$\Sigma := \{\{v, w, x\} \to \{y, z\},\ \{v, z\} \to \{x, y\}, \{w\} \to \{x, y\}\}.$$

What is the cardinality of the closure system $\mathbf{C} = \mathbf{C}(\Sigma)$? We iteratively compute $\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3 = \mathbf{C}$. The closure system determined by $\Sigma_0 = \emptyset$ is $\mathbf{C}_0 = \mathcal{P}(M)$, ie. the 3-valued row $(2, 2, 2, 2, 2)$. The implication $\{v, w, x\} \to \{y, z\}$ holds in $X \in (2, 2, 2, 2, 2)$ iff either $\{v, w, x\} \subseteq X$ and (whence) $\{y, z\} \subseteq X$ or $\{v, w, x\} \not\subseteq X$. The latter case splits into the subcases $v \notin X$, or $w \notin X$, or $x \notin X$. Therefore $\mathbf{C}_1$ is the union of the four 3-valued row shown in Figure 1. Note that eg. the intersection of the second and the third 3-valued row is $(0, 0, 2, 2, 2)$. But in order to get the cardinalities $|\mathbf{C}_k|$ more easily, it will be necessary to have *disjoint* 3-valued rows. This can be achieved as in Figure 2, which represents $\mathbf{C}_1$ as a disjoint union of 3-valued row (check that). Generally 3-valued rows $\mathbf{r}$ and $\mathbf{s}$ are disjoint iff there is an i with ($r_i = 0$ and $s_i = 1$) or ($r_i = 1$ and $s_i = 0$). By "inforcing" the implication $A_2 \to B_2$, ie. $\{v, z\} \to \{x, y\}$, upon the 3-valued rows of Figure 2, we shall now derive a decomposition of $\mathbf{C}_2$ as disjoint union of 3-valued rows (Figure 3). The implication $\{v, z\} \to \{x, y\}$ clearly holds in the set M represented by the first row of Figure 2. It also holds in all $X \in (0, 2, 2, 2, 2)$ since $v \notin X$ for all these X. Therefore the first two rows of Figure 2 carry over to Figure 3. The third row of Fig. 2 is more cumbersome, because $\{v, z\} \to \{x, y\}$ only holds for *some* $X \in (1, 0, 2, 2, 2)$. In order to resolve that problem we split $(1, 0, 2, 2, 2)$ "with respect to" the elements $z \in A_2$. Thus $(1, 0, 2, 2, 2)$ is the disjoint union of $(1, 0, 2, 2, 0)$ and $(1, 0, 2, 2, 1)$. Now $\{v, z\} \to \{x, y\}$ holds in all of $(1, 0, 2, 2, 0)$ (why?), and $(1, 0, 2, 2, 1)$ can be dealt with easily. Namely, because $\{v, z\} \subseteq X$ for all $X \in (1, 0, 2, 2, 1)$, only those X with $\{x, y\} \subseteq X$ will survive. So $(1, 0, 2, 2, 1)$ becomes $(1, 0, 1, 1, 1)$. Summarizing, the third row of Fig. 2 is replaced by the third and fourth row of Fig.3. Finally, look at the fourth row of Fig. 2. Because $\{x, y\} \not\subseteq X$ for all $X \in (1, 1, 0, 2, 2)$, the implication $\{v, z\} \to \{x, y\}$ can only hold for the X's with $\{v, z\} \not\subseteq X$. Hence $(1, 1, 0, 2, 2)$ becomes $(1, 1, 0, 2, 0)$ in Fig. 3. We leave it to the reader to verify that enforcing the last implication $\{w\} \to \{x, y\}$ results in the 3-valued rows shown in Fig.4. The cardinality of a 3-valued row $\mathbf{r}$ is obviously 2^m where $m := |\mathbf{r}\{2\}|$. Hence $|\mathbf{C}| = 2^0 + 2^1 + 2^3 + 2^2 + 2^0 = 16$.

(1,1,1,1,1)	(1,1,1,1,1)	(1,1,1,1,1)	(1,1,1,1,1)
(0,2,2,2,2)	(0,2,2,2,2)	(0,2,2,2,2)	(0,1,1,1,2)
(2,0,2,2,2)	(1,0,2,2,2)	(1,0,2,2,0)	(0,0,2,2,2)
(2,2,0,2,2)	(1,1,0,2,2)	(1,0,1,1,1)	(1,0,2,2,0)
		(1,1,0,2,0)	(1,0,1,1,1)
Figure 1	Figure 2	Figure 3	Figure 4

A diagram of this closure system $\mathbf{C} = \mathbf{C}(\Sigma)$ of Σ-closed sets is given below.

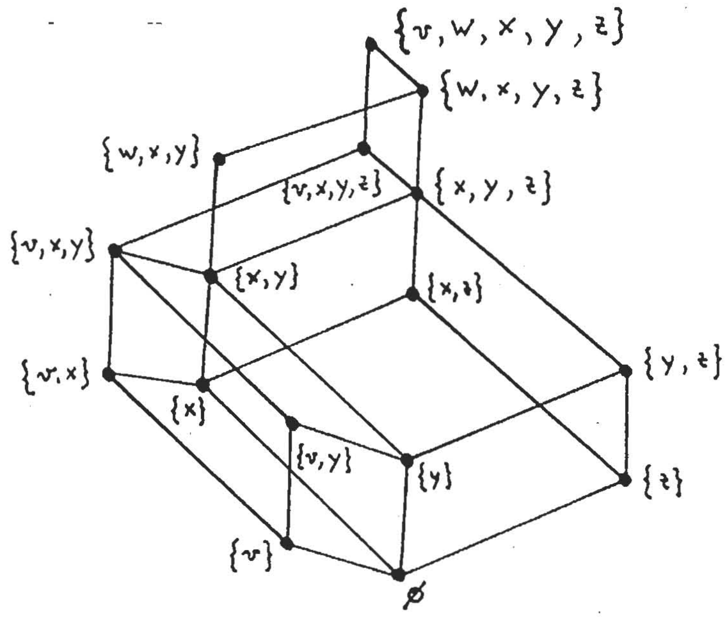


Fig. 5

3) Semilattices and lattices freely generated by posets

All of section 3, 4 and 5 is folklore, yet may not readily be found in the form presented here. Let us first consider semilattices, say join semilattices $\mathcal{S} = (\mathcal{S}, \vee)$. If $\mathcal{S}$ is generated by m elements $s_1, \cdots, s_m$ then trivially

$$\mathcal{S} = \left\{ \bigvee_{i \in I} s_i \,\middle|\, I \subseteq \{1, \cdots, m\},\ I \neq \emptyset \right\},$$

whence $|\mathcal{S}| \leq 2^m - 1$ is finite. Thus the "word problem" for finitely presented semilattices $\mathcal{S}$ is trivial. It is however still interesting to see how to compute $|\mathcal{S}|$ *efficiently*. Assume that Σ is a family of join semilattice relations $\bigvee A_i \geq \bigvee B_i$ on a set M of symbols ($A_i, B_i \subseteq M$, $1 \leq i \leq n$). Thinking of Σ as a set of implications $A_i \to B_i$ the following holds.

Theorem 1: The join semilattice $\mathcal{S}(\Sigma)$ freely generated by M under the set of relations Σ is up to isomorphism given by $(\mathbf{C}(\Sigma) - \{\emptyset\}, \vee)$.

As always for closure systems, the supremum of $X_1, X_2 \in \mathbf{C}(\Sigma)$ is given by $X_1 \vee X_2 = \overline{X_1 \cup X_2}$. Thus finitely presented semilattices can conveniently be computed with the algorithm of section 2. The precise definition of $\mathcal{S}(\Sigma)$ (as an algebra *finitely presented* by generators and relations), the easy proof of Theorem 1, and how Theorem 1 ties in with the more general finitely presented commutative semi-groups is discussed in [W5, 1.4.4]. See also [W1, Thm.5].

Example 2: How big is the $\vee$-semilattice $\mathcal{S}(\Sigma)$ freely generated by v, w, x, y, z subject to the relations $v \vee w \vee x \geq y \vee z$ and $v \vee z \geq x \vee y$ and $w \geq x \vee y$? According to Example 1 and Theorem 1 one has $|\mathcal{S}(\Sigma)| = |\mathbf{C}(\Sigma)| - 1 = 15$. One obtains a diagram of $\mathcal{S}(\Sigma)$ by dropping ϕ in Fig.5 and by eg. writing $v \vee x \vee y \vee z$ for $\{v, x, y, z\}$, etc. Of course eg. $v \vee x \vee y \vee z = v \vee z$ because of $v \vee z \geq x \vee y$. Generally we chose a minimal representation for all elements of $\mathcal{S}(\Sigma)$ in Fig.6 below.

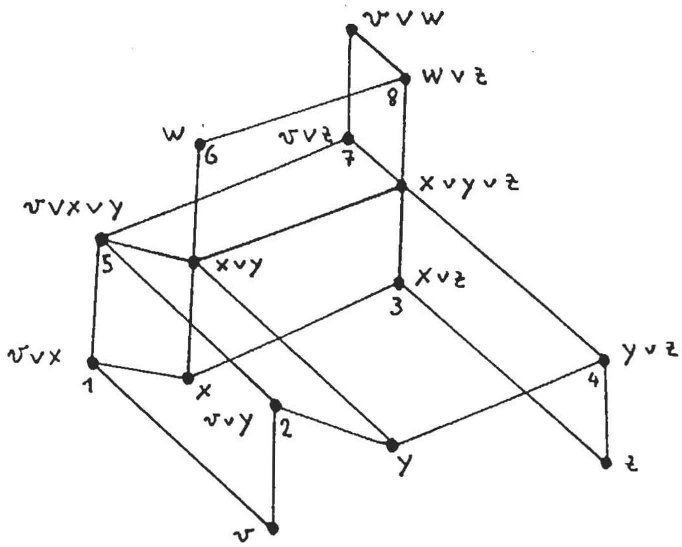


Fig. 6

It is well known that each join semilattice $\mathcal{S}$ is isomorphic to a subsemilattice of $(\mathcal{P}(M), \cup)$ where $\mathcal{P}(M)$ is the powerset of some appropriate set M (*set representation* of $\mathcal{S}$). In the finite case, taking M as the set of meet*) irreducibles of $\mathcal{S}$, an injective homomorphism $\mathcal{S} \to (\mathcal{P}(X), \cup)$ is given by $a \mapsto \{m \in M | m \not\geq a\}$:

*) Meaning "meet irreducible" in the *lattice* $\mathcal{S} \cup \{0\}$.

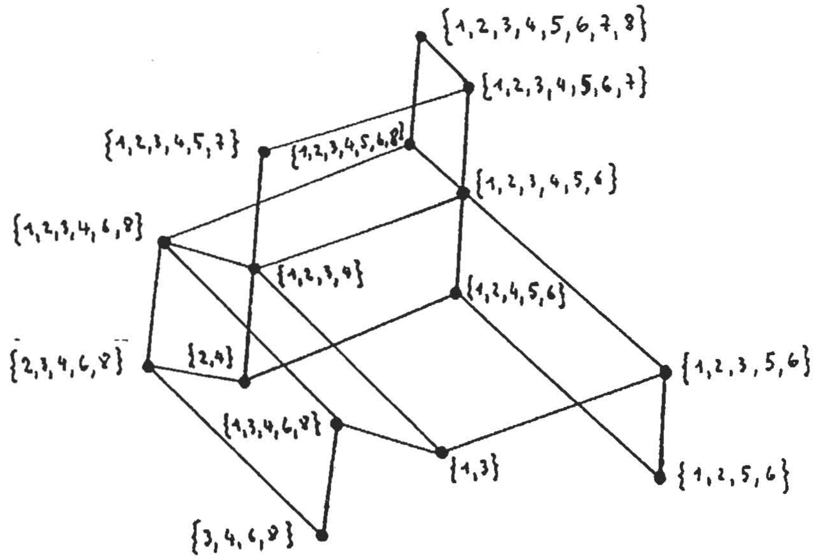


Fig. 7

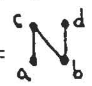
For instance, the sets corresponding to v, z, and $v \vee z$ are $\{3,4,6,8\}, \{1,2,5,6\}$, and $\{1,2,3,4,5,6,8\}$ (their union). Existing infima need not correspond to intersections of sets (example ?). One can show [W1, Thm.9] that the above M has smallest possible cardinality. □

The definition of $\mathcal{S}(\Sigma)$ is similar to the definition of the *(join) semilattice* $\mathcal{S}(P)$ *freely generated by the poset* P. The latter is the unique (up to isomorphism) semilattice $\mathcal{S}$ with the following properties.

(i) There is a generating set $P' \subseteq \mathcal{S}$ of $\mathcal{S}$ such that P' endowed with the induced order of $(\mathcal{S}, \leq)$ is isomorphic to $(P, \leq)$.

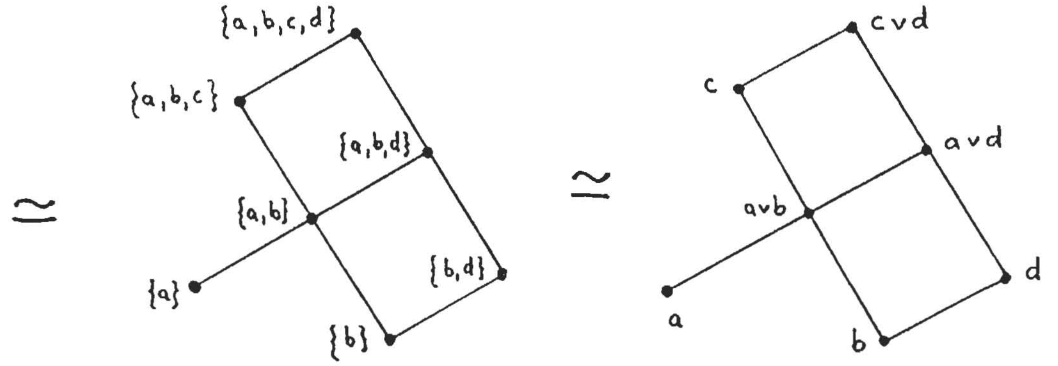
(ii) Assume $\mathcal{S}_0$ is a semilattice generated by $Q' \subseteq \mathcal{S}_0$ and $\phi : P' \to Q'$ is a surjective *order preserving* map, ie. $x \leq y \Rightarrow \phi(x) \leq \phi(y)$ for all $x, y \in P'$. Then ϕ extends to a surjective homomorphism $\Phi : \mathcal{S} \to \mathcal{S}_0$ of semilattices (ie. $\Phi \restriction P' = \phi$).

Example 3: What is the join semilattice $\mathcal{S}(P)$ freely generated by the poset

$(P, \leq) :=$  ?

Setting $\Sigma := \{\{c\} \to \{a,b\}, \{d\} \to \{b\}\}$ it comes as no surprise that

$$\mathcal{S}(P) \;\cong\; \mathcal{S}(\Sigma) \;\cong\; (\mathrm{C}(\Sigma) - \{\emptyset\}, \vee)$$



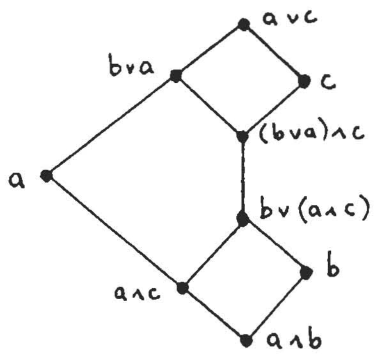
□

Let us say a few words about the *lattice* $\mathcal{L}(P)$ *freely generated by the poset* P. It is also defined by (i) and (ii) (substitute everywhere "semilattice" by "lattice"). One can show [Wi3] that $\mathcal{L}(P)$ is finite iff P does *not* contain a subposet*) Q which is isomorphic to either

1 + 1 + 1 (• • •) or **2 + 2** ( ) or **1 + 4** ( )

It is an exercise to show that there are exactly ? posets P satisfying these conditions. For instance $P := \mathbf{1} + \mathbf{2} =$ a c b is one of them and

$\mathcal{L}(P) =$



4) Boolean lattices freely generated by posets

The *Boolean lattice* $\mathcal{B}(P)$ *freely generated by the poset* P is defined analogous to $\mathcal{S}(P)$ and $\mathcal{L}(P)$. Recall that each finite Boolean lattice $\mathcal{B}$ is isomorphic to a powerset lattice $\mathcal{P}(X)$. If $|X| = t$, then $\mathcal{B}$ has t atoms and cardinality 2^t. In order to describe $\mathcal{B}(P)$ for finite P first observe that the Boolean sublattice $\mathcal{B}(A_1, \cdots, A_s)$ of $\mathcal{P}(X)$ which is generated by the subsets $A_1, \cdots, A_s \in \mathcal{P}(X)$ by distributivity and De Morgan's laws equals

$$(5) \quad \mathcal{B}(A_1, \cdots, A_s) = \left\{ \bigcup_{\delta \in I} A_1^{\delta_1} \cap \cdots \cap A_s^{\delta_s} \,\middle|\, I \subseteq \{0,1\}^s \right\}$$

*) Analogous to (i), any subset $Q \subseteq P$ of a poset $P = (P, \leq)$ yields a *subposet* $(Q, \leq')$ by just restricting the order (ie. $\leq' = \leq \cap (Q \times Q)$).

Hereby $\delta = (\delta_1, \cdots, \delta_s)$ is a 0,1-vector and $A_i^1 := A_i$, $A_i^0 := \overline{A_i}$. From (5) follows at once that the $t \leq 2^s$ many atoms of $\mathcal{B}(A_1, \cdots, A_s)$ are exactly the *nonempty* subsets of type $A_1^{\delta_1} \cap \cdots \cap A_s^{\delta_s}$.

Example 4: Let $X := \{a, b, \cdot\cdot, i\}$. How many atoms has the Boolean lattice $\mathcal{B}(A_1, A_2, A_3, A_4) \subseteq \mathcal{P}(X)$ generated by $A_1 := \{a, c, d, f\}$, $A_2 := \{b, e, f, i\}$, $A_3 := \{a, b, c, d, e, g, i\}$ and $A_4 := \{e, g, h\}$?

	a	b	c	d	e	f	g	h	i
$A_1 =$	1	0	1	1	0	1	0	0	0
$A_2 =$	0	1	0	0	1	1	0	0	1
$A_3 =$	1	1	1	1	1	0	1	0	1
$A_4 =$	0	0	0	0	1	0	1	1	0

Here the characteristic vectors of the subsets $A_i \subseteq S$ are listed as the rows of a 4×9 matrix. Then the number t of equivalence classes of equal columns obviously is the number of $(\delta_1, \cdot\cdot, \delta_s)$ with $A_1^{\delta_1} \cap \cdot \cdot \cap A^{\delta_1} \neq \emptyset$. For instance, the three equal columns labelled by a, c, d yield $(\delta_1, \delta_2, \delta_3, \delta_4) = (1, 0, 1, 0)$ and $A_1^1 \cap A_2^0 \cap A_3^1 \cap A_4^0 = \{a, c, d\} \neq \emptyset$. In our case $t = 6$ whence $|\mathcal{B}(A_1, A_2, A_3, A_4)| = 2^6 = 64$. □

From the above one immediately derives

> **Corollary 2:** Let $|X| = r$ be fixed. Pick s (not necessarily distinct) sets $A_i \in \mathcal{P}(X)$. The probability that $\mathcal{B}(A_1, \cdot\cdot, A_s) = \mathcal{P}(X)$ equals $\frac{2^s - 1}{2^s} \cdot \frac{2^s - 2}{2^s} \cdots \frac{2^s - (r-1)}{2^s}$.

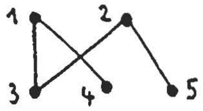
For instance, the probability that $s = 5$ random subsets $A_i \subseteq X := \{1, 2, 3, 4, 5\}$ generate the whole powerset is 0.72. When all $A_i \subseteq \{1, 2, \cdot\cdot, 10\}$, the probability is 0.21. When $r = 2^s$, the probability for $|\mathcal{B}(A_1, \cdot\cdot, A_s)| = |\mathcal{P}(X)| = 2^{(2^s)}$ is still > 0. Thus the inequality $t \leq 2^s$ mentioned after (5) is sharp. The case $t = 2^s$ can also be considered as the special case "$(P, \leq)$ = antichain" in the Theorem below. Recall that an *order ideal* of a poset $(P, \leq)$ is a set $I \subseteq P$ such that $x \in I$ implies $y \in I$ for all $y \leq x$. Dually an *order filter* is a set $F \subseteq P$ such that $x \in F$ implies $y \in F$ for all $y \geq x$.

> **Theorem 3:** Let $(P, \leq)$ be a finite poset. Then the Boolean lattice $\mathcal{B}(P)$ freely generated by $(P, \leq)$ has exactly t atoms, where t is the number of order filters $\emptyset \subseteq F \subseteq P$.

Proof: Assume w.l.o.g. $(P, \leq)$ is $(\{1, 2, \cdots, s\}, \leq)$ and that $(\{A_1, \cdots, A_s\}, \subseteq)$ is any fixed set system such that $i \leq j \Leftrightarrow A_i \subseteq A_j$ for all $i, j \in P$. Obviously $A_1^{\delta_1} \cap \cdot \cdot \cap A_s^{\delta_s}$ can be nonempty only if "$\delta_i = 1 \Rightarrow \delta_j = 1$" whenever $i \leq j$. Hence the number of $\delta = (\delta_1, \cdots, \delta_s)$ with $A_1^{\delta_1} \cap \cdots \cap A_s^{\delta_s} \neq \emptyset$ is *at most* the number t of order filters of P. On the other hand, choosing $(\{A_1, \cdots, A_s\}, \subseteq)$ appropriately (as in Example 5 below), one can indeed obtain

t nonempty sets $A_1^{\delta_1} \cap \cdot \cdot \cap A^{\delta_s}$. ■

Example 5: What is the size of the Boolean lattice $\mathcal{B}(P)$ freely generated by

$(P, \leq) :=$ 

?

As opposed to Example 4 here the 0, 1-matrix is generated by concatenating the *columns*. Namely, if we list the characteristic vectors of the $t = 13$ order filters of $(P, \leq)$ as the columns of a 5×13 matrix, then the sets $A_1, A_2, A_3, A_4, A_5 \subseteq \{a, b, \cdots, m\}$ corresponding to the rows yield a set system isomorphic to $(P, \leq)$ (check) and all intersections $A_1^{\delta_1} \cap A_2^{\delta_2} \cap A_3^{\delta_3} \cap A_4^{\delta_4} \cap A_5^{\delta_5}$, where δ is a column of the matrix, are nonempty (as in Example 4). Thus $|\mathcal{B}(P)| = 2^{13} = 8192$.

	a	b	c	d	e	f	g	h	i	j	k	l	m
$A_1 =$	0	1	0	1	1	1	1	1	0	1	1	1	1
$A_2 =$	0	0	1	1	0	1	1	1	1	1	1	1	1
$A_3 =$	0	0	0	1	0	0	0	1	0	1	0	0	1
$A_4 =$	0	0	0	0	1	0	1	1	0	0	0	1	1
$A_5 =$	0	0	0	0	0	0	0	0	1	1	1	1	1

□

5) Distributive lattices freely generated by posets

Let $(P, \leq)$ be a poset. The *distributive lattice freely generated by P* is the unique distributive lattice $\mathcal{D}(P)$ with the two obvious properties analogous to (i) and (ii) in section 3. The distributive lattice of all order ideals I of $(P, \leq)$ (with $I_1 \vee I_2 = I_1 \cup I_2$ and $I_1 \wedge I_2 = I_1 \cap I_2$) is denoted by $Id(P, \leq)$. Let $J = J(\mathcal{D})$ be the set of nonzero join irreducibles of the finite distributive lattice $\mathcal{D}$. Then $(J, \leq)$ with the partial order inherited from $\mathcal{D}$ is a poset and $\mathcal{D} \simeq Id(J, \leq)$ (Birkhoff's Theorem). The isomorphism is given by $a \mapsto J(a) := \{p \in J | p \leq a\}$. Dually the family of all order filters F of $(P, \leq)$ is denoted by $Fil(P)$. The family $Fil^*(P) := Fil(P) - \{\emptyset, P\}$ of *proper* order filters, ordered by inclusion, is generally not a lattice, just a poset. Compare the role of the order filters in Theorem 3 with the role in Theorem 4 below. This gives a feeling of why $\mathcal{B}(P)$ is much bigger than $\mathcal{D}(P)$.

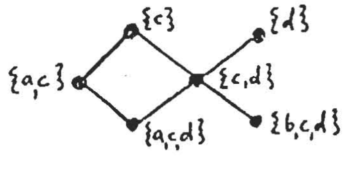
Theorem 4: Let $(P, \leq)$ be a finite poset. Then the distributive lattice $\mathcal{D}(P)$ freely generated by P is isomorphic to $Id(Fil^*(P), \supseteq)$.

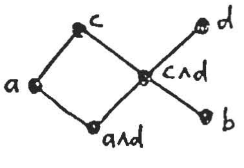
Sketch of proof: We omit to verify the (by the freeness of $\mathcal{D}(P)$ plausible) fact that $\bigwedge S \in \mathcal{D}(P)$ is join irreducible for each proper nonempty subset $S \subset P$ (note that

$\bigwedge S = 0$ and $\bigwedge \emptyset = 1$). By distributivity each $x \in \mathcal{D}(P)$ is a finite join of such $\bigwedge S$. So $J := \{\bigwedge S : S \subseteq P \textit{ antichain}, \emptyset \neq S \neq P\} \subseteq \mathcal{D}(P)$ is the poset of nonzero join irreducibles, whence $\mathcal{D}(P) \simeq Id(J, \leq)$. Consider the partial order on $J' :=$
$\{S : S \subseteq P \textit{ antichain}, \emptyset \neq S \neq P\}$ defined by $S \leq' T :\Leftrightarrow (\forall t \in T)(\exists s \in S)\ s \leq t$. Obviously $J' \to J : S \mapsto \bigwedge S$ is order preserving. Assume that $S \not\leq' T$ in J'. Then there is a $t_0 \in T$ with $s \not\leq t_0$ for all $s \in S$. Let $\mathcal{D}_2 := \{0, 1\}$ be the two element lattice. The surjective order preserving map $\phi : P \to \mathcal{D}_2$, $\phi(x) := 1$ $(x \geq s$ for some $s \in S)$, $\phi(x) := 0$ otherwise, has by (ii) in section 3 a homomorphic extension $\Phi : \mathcal{D}(P) \to \mathcal{D}_2$ with $\Phi(\bigwedge S) = 1$, $\Phi(\bigwedge T) = 0$. Hence $\bigwedge S \not\leq \bigwedge T$ in $(J, \leq)$. So $(J, \leq)$ is order isomorphic to $(J', \leq')$. But $(J', \leq')$ is clearly order isomorphic to $(Fil^*(P), \supseteq)$. ■

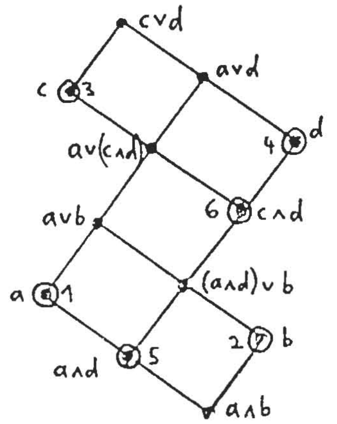
Example 6: Let $(P, \leq) :=$ 

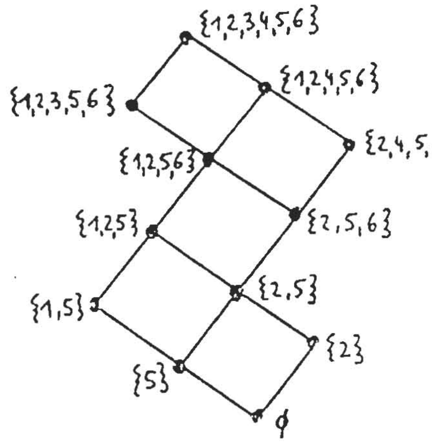
. How does $\mathcal{D}(P)$ look like? One has

$(Fil^*(P), \supseteq)$ $\simeq$ 

$\simeq$ 

Therefore $\mathcal{D}(P)$ $\simeq$ $Id(Fil^*(P), \supseteq)$ $\simeq$



$\simeq$


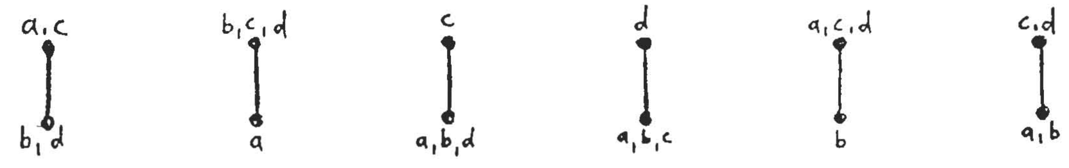
The join irreducibles (0 included) of $\mathcal{D}(P)$ are exactly the infima of the generators a, b, c, d, the generators themselves being doubly irreducible. Observe that eg. $(a \wedge d) \vee b$ can also be written as $(a \vee b) \wedge d = (a \wedge d) \vee (b \wedge d) = (a \wedge d) \vee b$ because $b < d$. Not surprisingly the freely generated semilattice $\mathcal{S}(P)$ of Example 3 is a join subsemilattice of $\mathcal{D}(P)$. Like every finite distributive lattice $\mathcal{D}(P)$ allows for a *set representation* which is obtained by labelling the nonzero join irreducibles p with say $1, 2, \cdots$ and mapping each $x \in \mathcal{D}(P)$ to $J(x)$ (rightmost diagram). As in Example 2 this is a *minimal* set representation. □

Let $\mathcal{D}$ be any finite distributive lattice. Recall that $\mathcal{L}^{(1)} := \mathcal{D}_2 = \{0, 1\}$ is the only subdirectly irreducible factor of $\mathcal{D}$. More precisely, the meet irreducible congruence relations $\theta_i \subseteq \mathcal{D} \times \mathcal{D}$ $(i \leq s)$ all have a factor lattice $\mathcal{D}/\theta_i \simeq \mathcal{D}_2$. These $\theta_i = \theta_p$ are in bijection with the join irreducibles $p \in J(\mathcal{D})$. The two congruence classes of θ_p are $\{x \in \mathcal{D} | x \geq p\}$ which is mapped to 1 under the canonical epimorphism $\Phi_i : \mathcal{D} \to \mathcal{D}_2$, and $\{x \in \mathcal{D} | x \not\geq p\}$

which is mapped to 0 under Φ_i. Because $\mathcal{D} \simeq \mathcal{D}/\theta_1 \times \cdots \times \mathcal{D}/\theta_s$ the structure of $\mathcal{D}$ is determined as soon as we know the Φ_i-images of some generators of $\mathcal{D}$.

Example 7: Consider $\mathcal{D} = \mathcal{D}(P)$ of Example 6 with generators a, b, c, d. The canonical epimorphisms $\Phi_i := \Phi_i^{(1)} : \mathcal{D}(P) \to \mathcal{M}_i^{(1)}$ (corresponding to $p = a, b, c, d, a \wedge d, c \wedge d$) map the generators as follows (eg. a on top in $\mathcal{M}_1^{(1)}$ means $\Phi_1^{(1)}(a) = 1$, and b below means $\Phi_1^{(1)}(b) = 0$):

(6)



$\mathcal{M}_1^{(1)}$ $\mathcal{M}_2^{(1)}$ $\mathcal{M}_3^{(1)}$ $\mathcal{M}_4^{(1)}$ $\mathcal{M}_5^{(1)}$ $\mathcal{M}_6^{(1)}$

Suppose only the $\Phi_i^{(1)}$-images (6) of the generators a, b, c, d of the (unknown) distributive lattice $\mathcal{D}$ were known. Because $\mathcal{D} \subseteq \mathcal{M}_1^{(1)} \times \mathcal{M}_2^{(1)} \times \cdots \times \mathcal{M}_6^{(1)}$ is a subdirect product it follows that

(7)

		1	2	3	4	5	6
a	$\mapsto$	(1,	0,	0,	0,	1,	0)
b	$\mapsto$	(0,	1,	0,	0,	0,	0)
c	$\mapsto$	(1,	1,	1,	0,	1,	1)
d	$\mapsto$	(0,	1,	0,	1,	1,	1)
$a \wedge d$	$\mapsto$	(0,	0,	0,	0,	1,	0)
$c \wedge d$	$\mapsto$	(0,	1,	0,	0,	1,	1)

Check that the order among the 0, 1-vectors is isomorphic to the order of the poset $\{a, b, c, d, a\wedge d, c\wedge d\}$ given in Example 6. In fact the 0, 1-vectors are just the characteristic vectors of some sets in the set representation of $\mathcal{D}$. Generally, the set K of 0, 1-vectors corresponding to the generators, and *all* infima of the latter *comprises* the set $J(\mathcal{D}) \subseteq \mathcal{M}_1^{(1)} \times \cdots \times \mathcal{M}_s^{(1)}$, whence allows to compute $\mathcal{D}$ as the ideal lattice $Id(K, \leq)$.

If $\mathcal{D} = \mathcal{D}(P)$ is *freely generated by a poset* $(P, \leq)$ then another method works which will be generalized in section 6. For $1 \leq k, h \leq s$ let $\beta_{k,h} : \mathcal{M}_k^{(1)} \to \mathcal{M}_h^{(1)}$ be the biggest among the order preserving maps which "map labels below labels". For instance $\beta : \mathcal{M}_1^{(1)} \to \mathcal{M}_5^{(1)} : 1 \mapsto 0,\ 0 \mapsto 0$ maps labels below labels: $\beta(a) = \beta(1) = 0 \leq a$, $\beta(b) \leq b$, $\beta(c) \leq c$, $\beta(d) \leq d$. However β is (componentwise) smaller than $\beta_{1,5} : \mathcal{M}_1^{(1)} \to \mathcal{M}_5^{(1)} : 1 \mapsto 1$, $0 \mapsto 0$, which obviously is the biggest such map. Consider

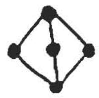
$$
\begin{array}{lllllll}
p_1 & := & (\beta_{1,1}(1), & \beta_{1,2}(1), & \cdots & ,\beta_{1,6}(1)\,) & = & (1,0,0,0,1,0) \\
p_2 & := & (\beta_{2,1}(1), & \beta_{2,2}(1), & \cdots & ,\beta_{2,6}(1)\,) & = & (0,1,0,0,0,0) \\
p_3 & := & (\beta_{3,1}(1), & \beta_{3,2}(1), & \cdots & ,\beta_{3,6}(1)\,) & = & (1,1,1,0,0,1) \\
p_4 & := & (\beta_{4,1}(1), & \beta_{4,2}(1), & \cdots & ,\beta_{4,6}(1)\,) & = & (0,1,0,1,1,1) \\
p_5 & := & (\beta_{5,1}(1), & \beta_{5,2}(1), & \cdots & ,\beta_{5,6}(1)\,) & = & (0,0,0,0,1,0) \\
p_6 & := & (\beta_{6,1}(1), & \beta_{6,2}(1), & \cdots & ,\beta_{6,6}(1)\,) & = & (0,1,0,0,1,1)
\end{array}
$$

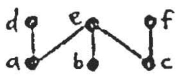
Thus one obtains the same result as in (7). □

6) Modular lattices freely generated by posets

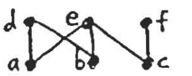
All of the following can be found in [HW], [Wi1], [Wi2] but proofs being replaced by telling examples our section 6 may better serve as a first introduction into the topic. Let $(P, \leq)$ be a poset. The *modular lattice freely generated by P* is the unique modular lattice $\mathcal{M}(P)$ with the two obvious properties analogous to (i) and (ii) in section 3. Usually $\mathcal{M}(P)$ is infinite even for finite P, but at least one can say precisely when it *is* finite:

Theorem 5 [Wi2]: The modular lattice $\mathcal{M}(P)$ freely generated by the finite poset $(P, \leq)$ is finite iff P does *not* contain a subposet isomorphic to either $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$ or $\mathbf{1}+\mathbf{2}+\mathbf{2}$. In the latter case all subdirectly irreducible factors of $\mathcal{M}(P)$ are isomorphic to

$\mathcal{D}_2 =$  or $\mathcal{M}_3 =$  .

Example 8: For $(P, \leq) :=$ 

the poset induced by $\{a, b, c, d, f\}$ is isomorphic to $\mathbf{1}+\mathbf{2}+\mathbf{2}$. Hence $\mathcal{M}(P)$ is infinite. On the other hand,

$(P', \leq) :=$ 

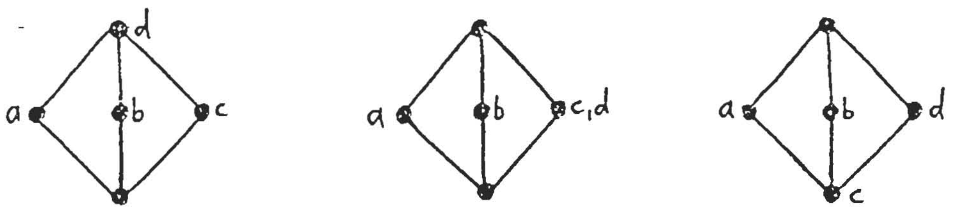
does neither contain a subposet $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$ nor $\mathbf{1}+\mathbf{2}+\mathbf{2}$.

Hence $\mathcal{M}(P')$ must be finite. □

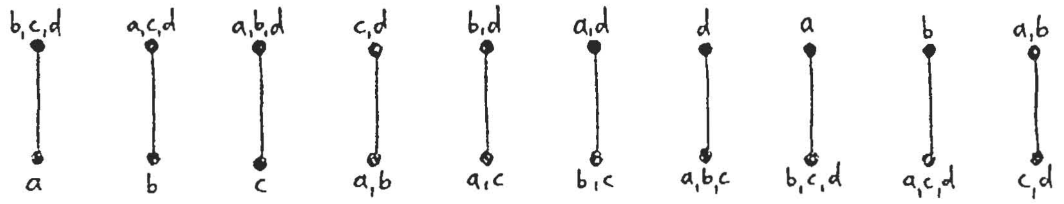
Our aim is to describe the structure and the computation of $\mathcal{M}(P)$ in case $|\mathcal{M}(P)| < \infty$. This will be done in two steps.

First step. Let $\mathcal{V}$ be a fixed *variety* [BS] of modular lattices and let $(P, \leq)$ be a finite poset. The *modular lattice freely generated by P within* $\mathcal{V}$ is the unique modular lattice $\mathcal{M}(P, \mathcal{V})$ with the obvious properties (the modular lattices $\mathcal{M}_0$ in (ii) are restricted to come from $\mathcal{V}$). For instance, if $\mathcal{V}$ is the variety of *all* modular lattices then $\mathcal{M}(P, \mathcal{V}) = \mathcal{M}(P)$, and if $\mathcal{V}$ is the variety of all distributive lattices then $\mathcal{M}(P, \mathcal{V}) = \mathcal{D}(P)$. A variety $\mathcal{V}$ is *locally finite* [BS, p.69] iff $|\mathcal{M}(P, \mathcal{V})| < \infty$ for all finite posets $(P, \leq)$. This eg. happens, and we shall only consider that case, when $\mathcal{V}$ contains only finitely many subdirectly irreducible members $\mathcal{L}^{(i)} (i \leq t)$ which all are finite. In particular, denote by $\mathcal{V}_3$ the variety generated by $\mathcal{L}^{(1)} := \mathcal{D}_2$ and $\mathcal{L}^{(2)} := \mathcal{M}_3$, ie. $\mathcal{V}_3$ is the class of all subdirect products with factors isomorphic to $\mathcal{L}^{(1)}$ or $\mathcal{L}^{(2)}$. By Theorem 5 one has $\mathcal{M}(P) \simeq \mathcal{M}(P, \mathcal{V}_3)$ whenever $\mathcal{M}(P)$ is finite. Hence we know the structure of all finite $\mathcal{M}(P)$ if we know the structure of all $\mathcal{M}(P, \mathcal{V}_3)$.

Example 9: Let $P :=$  . How big is $\mathcal{M}(P)$ and what are vector spaces $A, B \subseteq V$, $C \subseteq D \subseteq V$ such that $\mathcal{L}(V|A, B, C, D) \simeq \mathcal{M}(P)$? By Thm.5 $\mathcal{M}(P)$ is a finite subdirect product of factors $\mathcal{D}_2$ and $\mathcal{M}_3$. Applying the algorithm discussed below one obtains the following 13 subdirectly irreducible factors (the images of the generators $a, b, c, d \in \mathcal{M}(P)$ under the canonical epimorphisms are indicated):



$\mathcal{M}_1^{(1)}$ $\mathcal{M}_2^{(1)}$ $\mathcal{M}_3^{(1)}$



$\mathcal{M}_1^{(2)}$ $\mathcal{M}_2^{(2)}$ $\mathcal{M}_3^{(2)}$ $\mathcal{M}_4^{(2)}$ $\mathcal{M}_5^{(2)}$ $\mathcal{M}_6^{(2)}$ $\mathcal{M}_7^{(2)}$ $\mathcal{M}_8^{(2)}$ $\mathcal{M}_9^{(2)}$ $\mathcal{M}_{10}^{(2)}$

From this one can compute $|\mathcal{M}(P)| = 138$ (as discussed later). Furthermore, let k be any field and let V be a k-vectorspace with basis $e_1, e_2, \cdots, e_{16}$. Say $\langle e_1 \rangle, \langle e_2 \rangle, \langle e_1 + e_2 \rangle, \langle e_1, e_2 \rangle$ correspond to a, b, c, d in $\mathcal{M}_1^{(1)}$, and $\langle e_3 \rangle, \langle e_4 \rangle, \langle e_3 + e_4 \rangle$ correspond to $a, b, c = d$ in $\mathcal{M}_2^{(1)}$, etc. Setting

$$A := \langle e_1, e_3, e_5, e_8, e_9, e_{12}, e_{14}, e_{16} \rangle,$$

$$B := \langle e_2, e_4, e_6, e_7, e_9, e_{11}, e_{15}, e_{16} \rangle,$$

$$C := \langle e_1 + e_2, e_3 + e_4, e_7, e_8, e_{10} \rangle,$$

$$D := \langle e_1, e_2, e_3 + e_4, e_5 + e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13} \rangle,$$

it is clear that $\mathcal{L}(V|A, B, C, D) \simeq \mathcal{M}(P)$, ie. $\mathcal{L}(V|A, B, C, D)$ is a *linear representation* of $\mathcal{M}(P)$. In fact $V = \langle e_1, e_2 \rangle \oplus \langle e_3, e_4 \rangle \oplus \langle e_5, e_6 \rangle \oplus \langle e_7 \rangle \oplus \cdots \oplus \langle e_{16} \rangle$ is a decomposition of V which "induces" a subdirect decomposition of $\mathcal{L}(V|A, B, C, D)$ in the sense that eg. $\{X \cap \langle e_1, e_2\rangle | X \in \mathcal{L}(V|A, B, C, D)\} \simeq \mathcal{M}_3$ (more precisely $\simeq \mathcal{M}_1^{(1)}$).

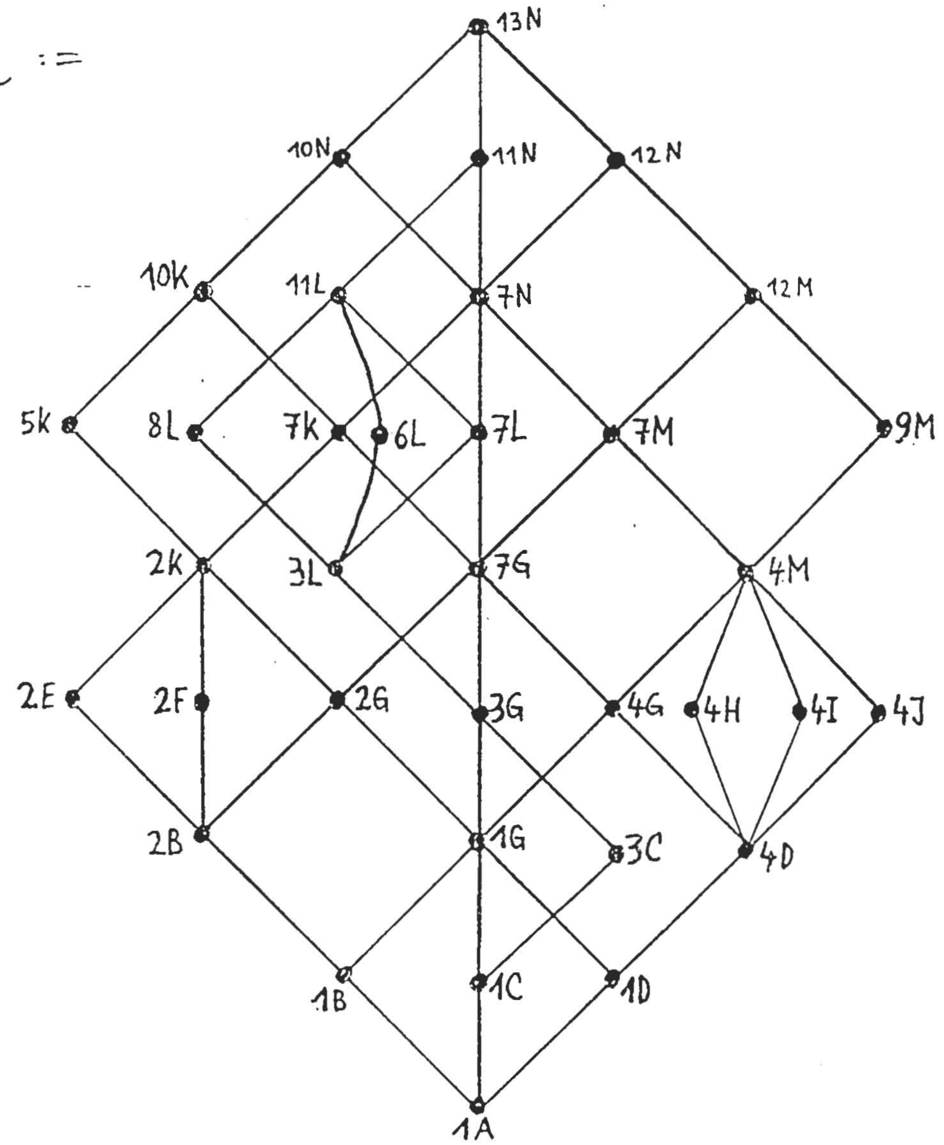
More challenging than the above is the *converse* problem. Namely say $V' := \langle e_1, e_2, \cdot\cdot, e_7\rangle$, $A' := \langle e_1, e_2, e_5, e_7\rangle$, $B' := \langle e_3, e_4, e_6\rangle$, $C' := \langle e_1+e_3, e_2+e_4, e_5+e_6, e_7\rangle$, $D' := \langle e_1, e_2, e_3, e_4, e_5+e_6, e_7\rangle$. Because $\{A', B', C', D'\}$ ordered by $\subseteq$ is isomorphic to $\mathbf{1}+\mathbf{1}+\mathbf{2}$, the lattice $\mathcal{L} := \mathcal{L}(V'|A', B', C', D')$ must be an epimorphic image of the 138-element lattice $\mathcal{M}(\mathbf{1}+\mathbf{1}+\mathbf{2})$. Because each of the subdirectly irreducible factors of $\mathcal{L}$ is thus *acyclic* (namely $\simeq \mathcal{D}_2$ or $\simeq \mathcal{M}_3$ in our case) it follows from [HW, Thm.7.3] that the subdirect decomposition of $\mathcal{L}$ must be *linearly induced* by some decomposition $V = V_1 \oplus V_2 \oplus \cdot\cdot \oplus V_s$. If the *whole* lattice $\mathcal{L}$ is given, the subspaces V_i can be found according to [HW]. It would be interesting to know how to proceed if only the *generators* (partially ordered by $\subseteq$) of some finite lattice of subspaces are known! For instance, in our case it is impossible that any nonzero $V_i \subseteq V$ induces a subdirectly irreducible factor $\simeq \mathcal{M}_{10}^{(2)}$ because $A' \cap B' = \{0\}$ implies that $V_i \cap A' = V_i \cap B' = V_i$ cannot occur. Whether or not any nonzero $V_i \subseteq V$ can induce a subdirectly irreducible factor $\simeq \mathcal{M}_2^{(1)}$ is a priori not so clear. The reader may verify that $V = \langle e_1, e_2, e_3, e_4\rangle \oplus \langle e_5, e_6\rangle \oplus \langle e_7\rangle$ does in fact induce the subdirect decomposition of $\mathcal{L}$ into its factors $\mathcal{M}_1^{(1)}, \mathcal{M}_3^{(1)}$, and $\mathcal{M}_2^{(2)}$. □

Second step. More generally than needed in the first step we shall clarify the structure of $\mathcal{M}(P, \mathcal{V})$ for *all* locally finite varieties $\mathcal{V}$ (of the mentioned type). To do so we first need to review some structure theory [HW], [W2] of *arbitrary* finite modular lattices $\mathcal{M}$.

Other than in the distributive case the poset $(J(\mathcal{M}), \leq)$ does no longer determine $\mathcal{M}$ uniquely. Generalizing $\mathcal{M}_3$ in Theorem 5 we denote by $\mathcal{M}_n$ the length two modular lattice with n atoms. An element $x \in \mathcal{M}$ is a *$\mathcal{M}_n$-element* if there is a length two interval $[x_0, x] \simeq \mathcal{M}_n$ $(n \geq 3)$ which contains *all* lower covers $x_i \prec x$ $(1 \leq i \leq n)$. A *line* corresponding to a fixed $\mathcal{M}_n$-element $x \in \mathcal{M}$ is a n-element subset $\ell = \ell_x = \{p_1, \cdot\cdot, p_n\} \subseteq J(\mathcal{M})$ such that $\ell \cap (J(x_i) - J(x_0)) = \{p_i\}$ for all $1 \leq i \leq n$. A *base of lines* of a finite modular lattice $\mathcal{M}$ is a family $\bigwedge$ of lines ℓ_x with exactly one line corresponding to each $\mathcal{M}_n$-element $x \in \mathcal{M}$. Note that $\bigwedge$ is empty iff $\mathcal{M} = \mathcal{D}$ is distributive. Generally $\mathcal{M}$ admits *several* bases of lines $\bigwedge$ but each fixed $\bigwedge$, together with the partial order on $J(\mathcal{M})$, determines $\mathcal{M}$ uniquely in the following way. An order ideal I of $(J(\mathcal{M}), \leq)$ is called *$\bigwedge$-closed* if for all $\ell \in \bigwedge$ one either has $|\ell \cap J(\mathcal{M})| \leq 1$ or $\ell \subseteq J(\mathcal{M})$. It follows at once that the family $\mathrm{C}(J(\mathcal{M}, \bigwedge)$ of all $\bigwedge$-closed order ideal of $J(\mathcal{M})$ is a closure system. According to [HW, Thm.2.5] $\mathrm{C}(J(\mathcal{M}, \bigwedge)$ as a lattice is isomorphic to $\mathcal{M}$. Let us illustrate this and further general facts by way of a concrete example.

Example 10: Consider the modular lattice below.

$\mathcal{M}$:=



There are six $\mathcal{M}_3$-elements, namely $1G, 2K, 7G, 11L, 7N, 13N$, and one $\mathcal{M}_4$-element $4M$. One possible base of lines $\bigwedge$ consists of the seven lines

$$\ell_{1G} := \{1B, 1C, 1D\}$$

$$\ell_{2K} := \{2E, 2F, 1D\}$$

$$\ell_{7G} := \{2B, 3C, 4D\}$$

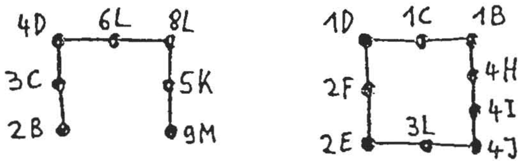
(8) $$\ell_{11L} := \{8L, 6L, 4D\}$$

$$\ell_{7N} := \{2E, 3L, 4J\}$$

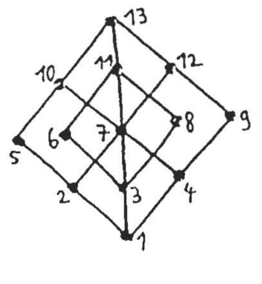
$$\ell_{13N} := \{5K, 8L, 9M\}$$

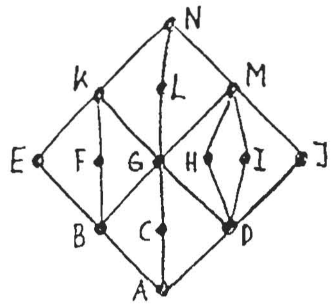
$$\ell_{4M} := \{1B, 4H, 4I, 4J\}$$

For instance for $x := 7N$ the interval $[x_0, x] = \{x_0, x_1, x_2, x_3, x\} \simeq \mathcal{M}_3$ is $\{7G, 7K, 7L, 7M, 7N\}$. One has $2E \leq 7K$ but $2E \not\leq 7G$, $3L \leq 7L$ but $3L \not\leq 7G$, $4J \leq 7M$ but $4J \not\leq 7G$. Another admissible line corresponding to $7N$ would have been $\ell'_{7N} := \{2F, 3L, 4H\}$. The partial linear space produced by $\bigwedge$ is

(9) $(J(\mathcal{M}), \bigwedge) =$ 

Let I be an order ideal of $(J(\mathcal{M}), \leq)$ of the special type $I = J(a) = \{p \in J(\mathcal{M}) | p \leq a\}$ for some $a \in \mathcal{M}$ (instead of $\mathcal{M}$ one may as well imagine *any* finite modular lattice). It is clear that *such* an order ideal is $\bigwedge$-closed: Suppose $\ell = \{p_1, p_2, \cdot\cdot, p_i \cdot\cdot\}$ is a line from $\bigwedge$, say corresponding to the $\mathcal{M}_n$-element x. If $|\ell \cap I| \geq 2$, say $p_1, p_2 \in I$, then trivially $p_1 \vee p_2 \leq a$ but also*) $p_1 \vee p_2 = x$. Hence $p_i \leq x \leq a$ and $\ell \subseteq I$. The nontrivial part of [HW, Thm.2.5] consists in showing that *every* $\bigwedge$-closed ideal $I \subseteq J(\mathcal{M})$ must be of type $I = J(a)$ for some $a \in \mathcal{M}$. According to [HW, p.20] the s subdirectly irreducible factors $\mathcal{M}^{(i)}$ of any finite modular $\mathcal{M}$ correspond bijectively to the connected components (definition clear) of any base of lines $\bigwedge$. Our $\mathcal{M}$ happens to have $s = 2$ subdirectly irreducible factors

(10) $\mathcal{M}^{(1)} :=$ 

and $\mathcal{M}^{(2)} :=$ 

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*) This is an easy exercise. Moreover from $p_1 \vee p_2 = x$ follows that $|\ell \cap \ell'| \leq 1$ for all $\ell \neq \ell'$. Indeed, if ℓ corresponds to x and ℓ' to y then $p_1, p_2 \in \ell \cap \ell'$ would yield the contradiction $x = p_1 \vee p_2 = y$. Structures $(J, \bigwedge)$ consisting of a set J of "points" and a set $\bigwedge$ of "lines" $\ell \subseteq J$ such that $\ell \neq \ell' \Rightarrow |\ell \cap \ell'| \leq 1$ are sometimes called *partial linear spaces* [M].

For instance $3 \vee 9 = 12$ in $\mathcal{M}^{(1)}$, $L \vee M = N$ in $\mathcal{M}^{(2)}$, and indeed $3L \vee 9M = 12N$ in $\mathcal{M}$. Consider the canonical epimorphisms $\Phi^{(1)} : \mathcal{M} \to \mathcal{M}^{(1)} : 1A \mapsto 1, \cdot\cdot, 13N \mapsto 13$, and $\Phi^{(2)} : \mathcal{M} \to \mathcal{M}^{(2)} : 1A \mapsto A, \cdot\cdot, 13N \mapsto N$. Fix $x \in \mathcal{M}^{(1)}$. The preimage of x under $\Phi^{(1)}$ is a sublattice of $\mathcal{M}$, in particular $\{y \in \mathcal{M} | \Phi^{(1)}(y) = x\}$ has a *smallest element* $\phi^{(1)}(x)$. For instance $\{y \in \mathcal{M} | \Phi^{(1)}(y) = 3\} = \{3C, 3G, 3L\}$ and $\phi^{(1)}(3) = 3C$. One can show that generally $\phi^{(i)} : \mathcal{M}^{(i)} \to \mathcal{M}$ $(1 \le i \le s)$ is a $\vee$-homomorphism and that $J(\mathcal{M})$ is partitioned as

$$(11) \quad J(\mathcal{M}) = \biguplus_{1 \le i \le s} \phi^{(i)}(J(\mathcal{M}^{(i)}))$$

Furthermore, let $\bigwedge^{(i)}$ be a base of lines for $\mathcal{M}^{(i)}$ $(1 \le i \le s)$. Then a base of lines $\bigwedge$ for $\mathcal{M}$ can be built as

$$(12) \quad \bigwedge := \{\phi^{(i)}(\ell) |\ 1 \le i \le s,\ \ell \in \bigwedge^{(i)}\}$$

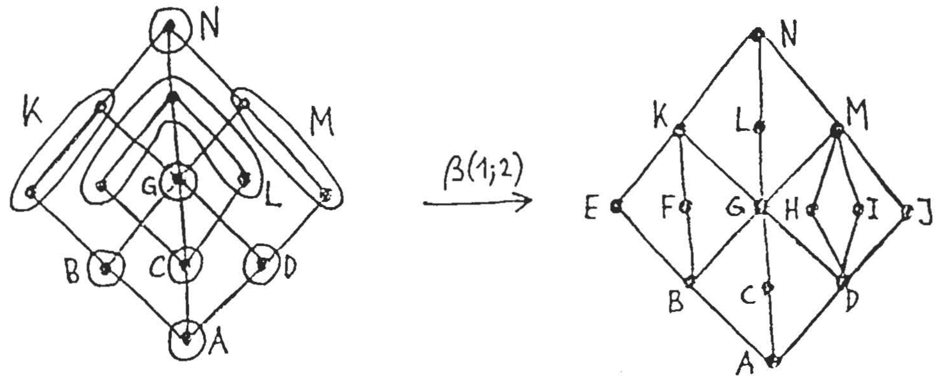
Here eg. $\bigwedge^{(1)} := \{\{2,3,4\}, \{4,6,8\}, \{5,8,9\}\}$ and $\bigwedge^{(2)} := \{\{B,C,D\}, \{D,E,F\}, \{B,H,I,J\}, \{E,J,L\}\}$ are bases of lines of $\mathcal{M}^{(1)}$ respectively $\mathcal{M}^{(2)}$ (see (10)) and the via (12) derived $\bigwedge$ happens to coincide with the base of lines in (9).

There is more to subdirect products $\mathcal{M} \subseteq \mathcal{M}^{(1)} \times \cdot\cdot \times \mathcal{M}^{(s)}$ than meets the eye [Wil]. For all $1 \le i, j \le s$ set $\beta(i;j) := \Phi^{(j)} \circ \phi^{(i)} : \mathcal{M}^{(i)} \to \mathcal{M}^{(j)}$. Trivially all $\beta(i;j)$ are $\vee$-homomorphisms and it is an exercise to verify that

$$(13) \quad (\forall\, 1 \le i, j, k \le s) \quad \beta(i;k) \ge \beta(j;k) \circ \beta(i;j)$$

(and that $\beta(i;i)$ is the identity map on $\mathcal{M}^{(i)}$ for all $1 \le i \le s$). Conversely, given any family $\beta(i;j) : \mathcal{M}^{(i)} \to \mathcal{M}^{(j)}$ $(1 \le i, j \le s)$ which satisfies (13), one can construct*) a subdirect product $\mathcal{M} \subseteq \mathcal{M}^{(1)} \times \cdot\cdot \times \mathcal{M}^{(s)}$ such that $\Phi^{(j)} \circ \phi^{(i)} = \beta(i;j)$. For our $\mathcal{M}$ one has

(14)



Thus eg. $\beta(1;2)(5) = \beta(1;2)(10) = K$. In the other direction

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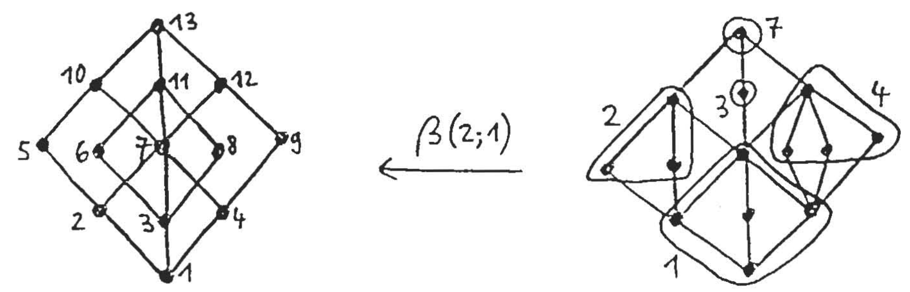
*) A $\vee$-generating set of $\mathcal{M}$ is given by the tuples $(\beta(i;1)(a), \cdot\cdot, \beta(i;i)(a), \cdots, \beta(i;s)(a))$ where $1 \le i \le s$ and $a \in \mathcal{M}^{(i)}$.

(15)



For example $(\beta(2;1) \circ \beta(1;2))(11) = \beta(2;1)(L) = 3 \leq 11$. Generally $\beta(2;1) \circ \beta(1;2) \leq \beta(1;1) = id$ and $\beta(1;2) \circ \beta(2;1) \leq \beta(2;2)$. □

Let $\bigwedge$ be a base of lines of a finite modular lattice $\mathcal{M}$. One can immediately write down a family $\Sigma = \Sigma(\mathcal{M})$ of implications such that the $\bigwedge$-closed order ideals of $J(\mathcal{M})$ coincide with the Σ-closed subsets of $J(\mathcal{M})$. Just take Σ as the family of all implications $\{p\} \to J(p)$ $(p \in J(\mathcal{M}))$ and $\{p,q\} \to \ell$ $(\ell \in \bigwedge,\ p \neq q \in \ell)$. Then trivially $\mathbf{C}(\Sigma) = \mathbf{C}(J(\mathcal{M}, \bigwedge) \simeq \mathcal{M}$, whence $|\mathcal{M}| = |\mathbf{C}(\Sigma)|$ can be computed with the algorithm of section 2. (As to an "optimal" family Σ see section 7.)

Suppose now $\mathcal{M} := \mathcal{M}(P, \mathcal{V})$ but only $(P, \leq)$ and the subdirectly irreducible members $\mathcal{L}^{(1)}, \cdots, \mathcal{L}^{(t)}$ of $\mathcal{V}$ are known! How can one from this information obtain a family $\Sigma = \Sigma(P, \mathcal{L}^{(1)}, \cdot\cdot, \mathcal{L}^{(t)})$ with $\mathbf{C}(\Sigma) \simeq \mathcal{M}$? The recipe consists of the following four steps.

1. A *labelling* of $\mathcal{L}^{(i)}$ with the elements of P is an order preserving map $\lambda : P \to \mathcal{L}^{(i)}$ such that $\lambda(P) \subseteq \mathcal{L}^{(i)}$ generates $\mathcal{L}^{(i)}$. Two labellings λ and λ' of $\mathcal{L}^{(i)}$ are *equivalent* if there is a lattice automorphism $\alpha : \mathcal{L}^{(i)} \to \mathcal{L}^{(i)}$ such that $\lambda' = \alpha \circ \lambda$. Now for each $1 \leq i \leq t$ determine a maximal family

   (16) $\quad \mathcal{M}_1^{(i)} := (\mathcal{L}^{(i)}, \lambda_1^{(i)}),\ \mathcal{M}_2^{(i)} := (\mathcal{L}^{(i)}, \lambda_2^{(i)}), \quad \cdots \quad, \mathcal{M}_{s_i}^{(i)} := (\mathcal{L}^{(i)}, \lambda_{s_i}^{(i)})$

   of non-equivalent labellings of $\mathcal{L}^{(i)}$. The $s := s_1 + \cdots + s_t$ lattices underlying the $\mathcal{M}_k^{(i)}$ will turn out to be the subdirectly irreducible factors of $\mathcal{M}(P, \mathcal{V})$. It remains to find the maps $\beta(\cdot\cdot)$ in (13) which determine the fine structure of $\mathcal{M}(P, \mathcal{V})$.

2. Call β a *morphism* between $\mathcal{M}_k^{(i)}$ and $\mathcal{M}_h^{(j)}$ if it is a $\vee$-homomorphism $\beta : \mathcal{L}^{(i)} \to \mathcal{L}^{(j)}$ such that

   (17) $\quad (\forall a \in P) \quad \beta(\lambda_k^{(i)}(a)) \leq \lambda_h^{(j)}(a)$

   One verifies easily that there is a *biggest* morphism $\beta(i,k;j,h)$ between $\mathcal{M}_k^{(i)}$ and $\mathcal{M}_h^{(j)}$ (meaning that $\beta(i,k;j,h)(x) \geq \beta(x)$ for all morphisms $\beta : \mathcal{M}_k^{(i)} \to \mathcal{M}_h^{(j)}$ and all $x \in \mathcal{L}^{(i)}$). Determine $\beta(i,k;j,h)$ for all $1 \leq i,j \leq t$ and all $1 \leq k \leq s_i$, $1 \leq h \leq s_j$.

3. For each fixed $i \in \{1, 2, \cdots t\}$ and $k \in \{1, 2, \cdots s_i\}$ let $p, q \cdots$ be the nonzero join irreducibles of the lattice underlying $\mathcal{M}_k^{(i)}$ and write down

$$\phi_k^{(i)}(p) = (\beta(i,k;1,1)(p),\cdots,\beta(i,k;1,s_1)(p),\cdots,\beta(i,k;t,s_t)(p))$$

$$(18) \quad \phi_k^{(i)}(q) = (\beta(i,k;1,1)(q),\cdots,\beta(i,k;1,s_1)(q),\cdots,\beta(i,k;t,s_t)(q))$$

$$\vdots$$

One can show that the s-tuples ($s = s_1 + \cdots + s_t$) in (18) are indeed the images under $\phi_k^{(i)} : \mathcal{M}_k^{(i)} \to \mathcal{M}(P,\mathcal{V})$ of $p, q \cdots$. Recall that $\phi_k^{(i)}$ yields the smallest preimages of the canonical epimorphism $\Phi_k^{(i)} : \mathcal{M}(P,\mathcal{V}) \to \mathcal{M}_k^{(i)}$. In view of (11) the s-tuples in (18) constitute the set $J(\mathcal{M}(P,\mathcal{V}))$.

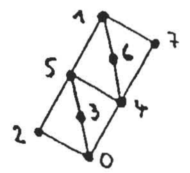
4. Because of (18) and because the partial order of each $\mathcal{L}^{(i)}$ is known, we know the partial order of $J(\mathcal{M}(P,\mathcal{V}))$. Assume $\mathcal{M}_k^{(i)}$ has a base of lines $\bigwedge_k^{(i)}$ $(1 \le i \le t)$. We know (see (12)) that a base of lines $\bigwedge$ for $\mathcal{M}(P,\mathcal{V})$ is then given by

$$\bigwedge := \{\phi_k^{(i)}(\ell) \mid 1 \le i \le t,\ 1 \le k \le s_i,\ \ell \in \bigwedge_k^{(i)}\}$$

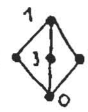
As explained before, the knowledge of $\bigwedge$ and the partial order on $J(\mathcal{M}(P,\mathcal{V}))$ yields immediately a family Σ of implications such that $C(\Sigma) \simeq \mathcal{M}(P,\mathcal{V})$.

If $\mathcal{V}$ is the variety of all distributive lattices then $\mathcal{M}(\mathcal{V},P) = \mathcal{D}(P)$. The only subdirectly irreducible member of $\mathcal{V}$ is $\mathcal{L}^{(1)} = \mathcal{D}_2$ which via the labelling process **1** yields the copies $\mathcal{M}_1^{(1)},\cdots,\mathcal{M}_s^{(1)}$ (see (6) of Example 7). As to step **3**, we wrote $\beta_{h,k}$ for $\beta(1,h;1k)$ at the end of Example 7. The following example illustrates the steps **1, 2, 3, 4** for more general locally finite varieties $\mathcal{V}$.

Example 11: Let $\mathcal{V}$ be the variety generated by

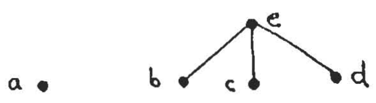
$\mathcal{L}^{(1)} := \mathcal{M}_{3,3} :=$ 

The two further subdirectly irreducible members of $\mathcal{V}$ are

$\mathcal{L}^{(2)} := \mathcal{M}_3 =$ 

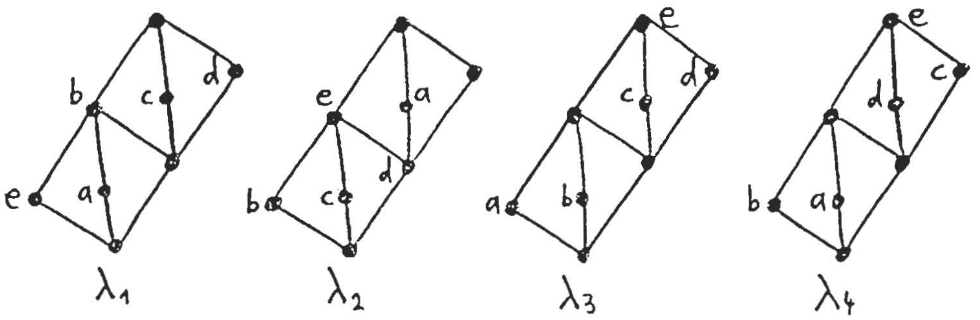
and $\mathcal{L}^{(3)} := \mathcal{D}_2 =$ 

Consider the poset

$(P, \le) :=$ 

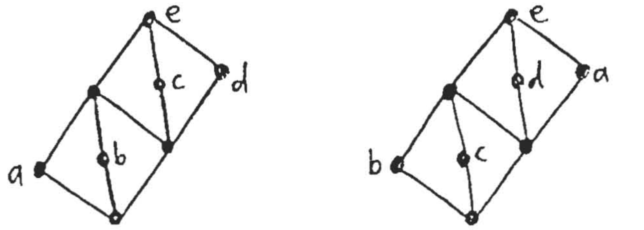
How does the modular lattice $\mathcal{M}(P,\mathcal{V})$ look like? As to step **1**, consider the four functions $\lambda_k : P \to \mathcal{L}^{(1)}$ below (eg. $\lambda_1(a) = 3$, $\lambda_1(b) = 5$) :

(19)

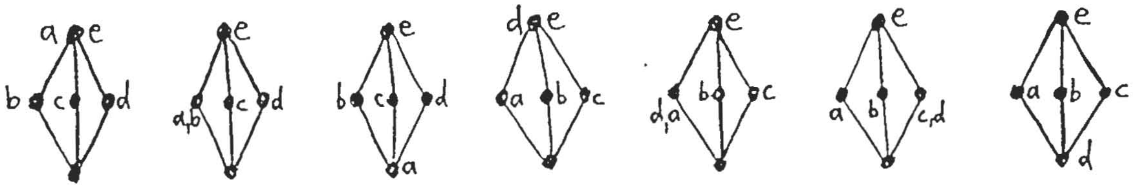


Because of $b \leq e$ and $\lambda_1(b) \not\leq \lambda_1(e)$ the function λ_1 is not order preserving. The λ_2 is order preserving but $\lambda_2(P) \not\ni 7$ does not generate $\mathcal{L}^{(1)}$ (generally the *doubly irreducibles* must always be contained in $\lambda(P)$). The λ_3 is both order preserving and $\lambda_3(P)$ generates $\mathcal{L}^{(1)}$. Hence λ_3, and similarly λ_4, qualify as labellings of $\mathcal{L}^{(1)}$. The function $\alpha : \mathcal{L}^{(1)} \to \mathcal{L}^{(1)}$ which switches 2 and 3, switches 6 and 7, and leaves the other elements fixed, is an automorphism of $\mathcal{L}^{(1)}$ and $\lambda_4 = \alpha \circ \lambda_3$. Hence λ_3 and λ_4 are equivalent. The reader may verify that the two labellings $\mathcal{M}_1^{(1)}, \mathcal{M}_2^{(1)}$ of $\mathcal{L}^{(1)}$ exhibited in (20) are non-equivalent, and that each labelling of $\mathcal{L}^{(1)}$ is equivalent to one of them. The same holds for the 7 labellings $\mathcal{M}_k^{(2)}$ of $\mathcal{L}^{(2)}$ and the 16 labellings $\mathcal{M}_k^{(3)}$ of $\mathcal{L}^{(3)}$ shown in (20).

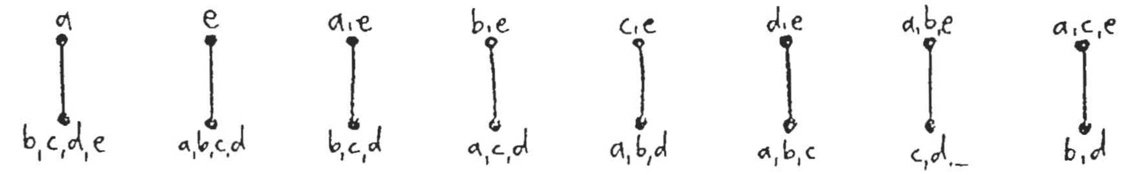
(20)



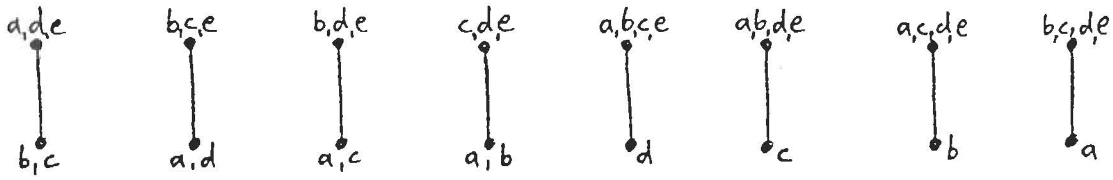
$\mathcal{M}_1^{(1)}$ $\mathcal{M}_2^{(1)}$



$\mathcal{M}_1^{(2)}$ $\mathcal{M}_2^{(2)}$ $\mathcal{M}_3^{(2)}$ $\mathcal{M}_4^{(2)}$ $\mathcal{M}_5^{(2)}$ $\mathcal{M}_6^{(2)}$ $\mathcal{M}_7^{(2)}$



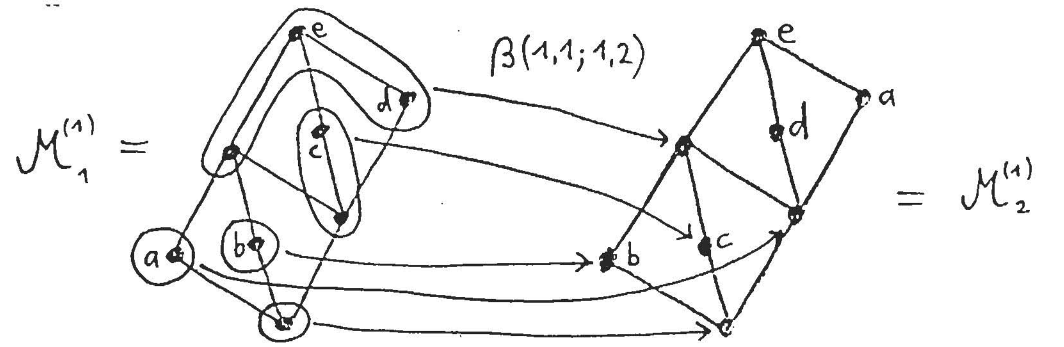
$\mathcal{M}_1^{(3)}$ $\mathcal{M}_2^{(3)}$ $\mathcal{M}_3^{(3)}$ $\mathcal{M}_4^{(3)}$ $\mathcal{M}_5^{(3)}$ $\mathcal{M}_6^{(3)}$ $\mathcal{M}_7^{(3)}$ $\mathcal{M}_8^{(3)}$



$\mathcal{M}_9^{(3)}$ $\mathcal{M}_{10}^{(3)}$ $\mathcal{M}_{11}^{(3)}$ $\mathcal{M}_{12}^{(3)}$ $\mathcal{M}_{13}^{(3)}$ $\mathcal{M}_{14}^{(3)}$ $\mathcal{M}_{15}^{(3)}$ $\mathcal{M}_{16}^{(3)}$

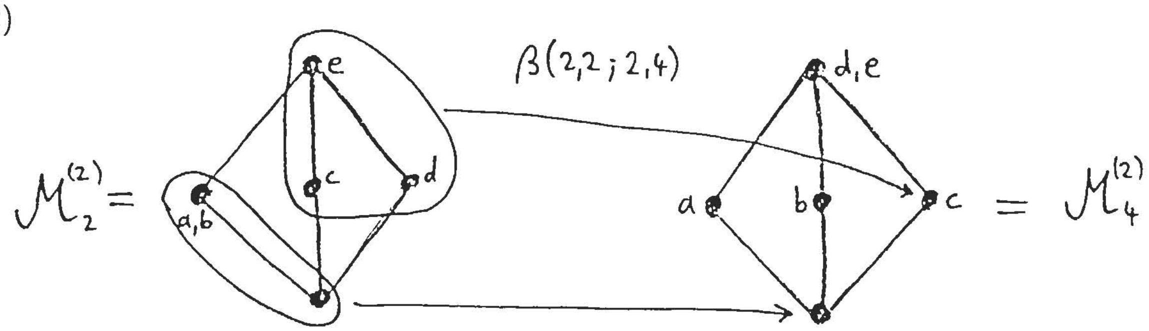
As to step 2, what is the biggest morphism $\overline{\beta} := \beta(1,1;1,2)$ between $\mathcal{M}_1^{(1)}$ and $\mathcal{M}_2^{(1)}$? We claim that $\overline{\beta}(0) = 0$, $\overline{\beta}(2) = 4$, $\overline{\beta}(3) = 2$, $\overline{\beta}(4) = \overline{\beta}(6) = 3$, $\overline{\beta}(1) = \overline{\beta}(5) = \overline{\beta}(7) = 5$:

(21)

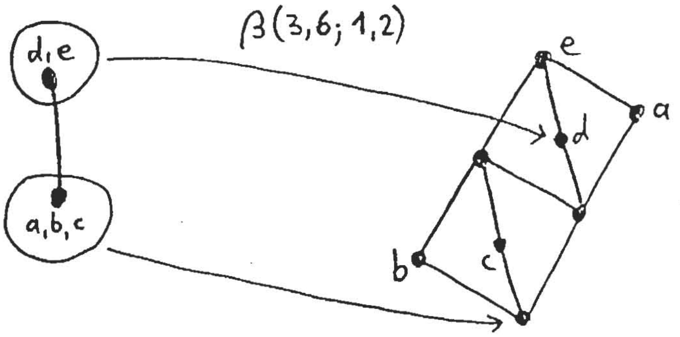


Indeed, no morphism β can do better*) than $\overline{\beta}(b) = b$, and $\overline{\beta}(c) = c$, and $\overline{\beta}(4) = c$. Also no β can do better than $\overline{\beta}(e) = 5$ (since $\beta(e) = \beta(b \vee c) = \beta(b) \vee \beta(c) \leq b \vee c = 5$), than $\overline{\beta}(5) = \overline{\beta}(7) = 5$ (since eg. $7 \leq e \Rightarrow \beta(7) \leq \beta(e) \leq 5$), than $\overline{\beta}(a) = 4$ (since $\beta(a) \leq a \wedge \beta(e) \leq a \wedge 5 = 4$), and than $\overline{\beta}(0) = 0$ (since $\beta(0) \leq \beta(a) \wedge \beta(b) \leq 4 \wedge b = 0$). It remains to check that the map $\overline{\beta}$ *itself* is a $\vee$-homomorphism. This amounts to check that the $\overline{\beta}$-preimages (the five "bubbles" in (21)) are the congruence classes of a $\vee$-congruence relation; in particular each bubble must be a $\vee$-subsemilattice. In similar fashion one easily verifies that eg. $\beta(2,2;2,4)$ is the map shown in (22), and $\beta(3,6;1,2)$ is the map shown in (23):

(22)



*) We write $\overline{\beta}(b) = b$ rather than the correct but clumsy $\overline{\beta}(\lambda_1^{(1)}(b)) = \lambda_2^{(1)}(b)$ of (17). Because every morphism β must satisfy $\beta(b) \leq b$ there can be no β with $\beta(b) > \overline{\beta}(b)$.

(23) $\mathcal{M}_6^{(3)} =$  $= \mathcal{M}_2^{(1)}$

As to step 3, let p_1, p_2, p_3, p_4, p_5 be the join irreducibles 2, 3, 4, 6, 7 of $\mathcal{M}_1^{(1)}$, let $p_6, p_7, p_8, p_9, p_{10}$ be the join irreducibles 2, 3, 4, 6, 7 of $\mathcal{M}_2^{(1)}$, let p_{11}, p_{12}, p_{13} be the join irreducibles 2, 3, 4 of $\mathcal{M}_1^{(2)}$, and so forth. Finally p_{46} is the join irreducible 1 of $\mathcal{M}_{15}^{(3)}$ and p_{47} is the join irreducible 1 of $\mathcal{M}_{16}^{(3)}$. In our case $t = 3$ and $s = s_1 + s_2 + s_3 = 2 + 7 + 16 = 25$. Fix eg. $i := 2 \in \{1, 2, t\}$ and $k := 2 \in \{1, 2, \cdot\cdot, s_2\}$. Then $\mathcal{M}_k^{(i)} = \mathcal{M}_2^{(2)}$ has the join irreducibles p_{14}, p_{15}, p_{16} and one can show that (18) consists of the following s-tuples:

$$\phi_2^{(2)}(p_{14}) = (0,0,2,2,0,\mathbf{0},0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0)$$

$$\phi_2^{(2)}(p_{15}) = (4,0,3,3,0,\mathbf{4},0,4,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,1,1)$$

$$\phi_2^{(2)}(p_{16}) = (4,0,4,4,0,\mathbf{4},0,4,0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,1,1)$$

For instance the boldface $6th$ components are the values $\beta(2,2;2,4)(p_{14}) = 0$, $\beta(2,2,;2,4)$ $(p_{15}) = 4$, $\beta(2,2;2,4)(p_{16}) = 4$ according to (22). If we eg. take $i := 3 \in \{1, 2, t\}$ and $k := 6 \in \{1, 2, \cdot\cdot, s_3\}$ then $\mathcal{M}_k^{(i)} = \mathcal{M}_6^{(3)}$ has only one join irreducible p_{37} and

$$\phi_6^{(3)}(p_{37}) = (7,\mathbf{6},4,4,4,1,2,4,0,0,0,0,0,0,1,0,0,1,0,1,1,0,1,1,1)$$

For instance the boldface second component is the value $\beta(3,\mathbf{6};\mathbf{1},2)(p_{37}) = 6$ according to (23).

As to step 4, the elements $\phi_1^{(1)}(p_1), \phi_1^{(1)}(p_2), \cdot\cdot, \phi_{16}^{(3)}(p_{47})$ are exactly the join irreducibles of $\mathcal{M}(P, \mathcal{V}) \subseteq \mathcal{M}_1^{(1)} \times \mathcal{M}_2^{(1)} \times \cdot\cdot \times \mathcal{M}_{16}^{(3)}$. Note that eg. $\phi_2^{(2)}(p_{16}) < \phi_6^{(3)}(p_{37})$ by componentwise comparison. Possible bases of lines for $\mathcal{M}_1^{(1)}$ respectively $\mathcal{M}_2^{(1)}$ are $\Lambda_1^{(1)} := \{\{p_1, p_2, p_3\}, \{p_1, p_4, p_5\}\}$ and $\Lambda_2^{(1)} := \{\{p_6, p_7, p_8\}, \{p_6, p_9, p_{10}\}\}$, and (unique) bases of lines for $\mathcal{M}_1^{(2)}, \cdot\cdot, \mathcal{M}_7^{(2)}$ are $\Lambda_1^{(2)} := \{\{p_{11}, p_{12}, p_{13}\}\}, \quad \cdots, \Lambda_7^{(2)} := \{\{p_{29}, p_{30}, p_{31}\}\}$. Hence

$$\Lambda := \{\{\phi_1^{(1)}(p_1), \phi_1^{(1)}(p_2), \phi_1^{(1)}(p_3)\}, \cdots, \{\phi_7^{(2)}(p_{29}), \phi_7^{(2)}(p_{30}), \phi_7^{(2)}(p_{31})\}\}$$

is a base of lines for $\mathcal{M}(P, \mathcal{V})$ with 11 lines and 9 connected components. □

In [HPR] an *inherent* characterisation of those partial linear spaces $(J, \bigwedge)$ that occur as bases of lines of modular lattices is given. Generating these $(J, \bigwedge)$ systematically, and applying the principle of exclusion to the corresponding $\Sigma(\bigwedge)$, this opens the possibility of generating *all* modular lattices of small cardinality.

7) Some computational details

Let $\mathcal{M}$ be a finite modular lattice with a base of lines $\bigwedge$. Let $\Sigma = \{A_i \to B_i | 1 \leq i \leq n\}$ be the family of all implications

(24) $\{p\} \to J(p) \quad (p \in J(\mathcal{M})), \quad \{p, q\} \to \ell \quad (\ell \in \bigwedge, p, q \in \ell,\ p \neq q)$

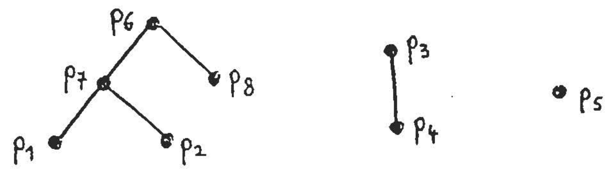
We know that $\mathcal{M}$ is isomorphic to the closure system $\mathrm{C}(\Sigma)$ of al Σ-closed subsets $X \subseteq J(\mathcal{M})$. Are there "smaller" Σ' such that still $\mathrm{C}(\Sigma') \simeq \mathcal{M}$? The answer is "yes" and "no". No, the *number* n of implications in Σ essentially cannot be reduced. Merely those m implications $\{p\} \to J(p)$ where p is minimal in $(J(\mathcal{M}), \leq)$ boil down to $\{p\} \to \{p\}$ and can thus be dropped from Σ. Let Σ_1 be the new family of implications with $|\Sigma_1| = n' := n - m$. One can show [W6] that there is *no* Σ' with $|\Sigma'| < n'$ that still satisfies $C(\Sigma') \simeq \mathcal{M}$. But yes, the *size* $|A_i| + |B_i|$ of each individual implication in Σ_1 can be reduced substantially. Namely, instead of $\{p\} \to J(p)$ it obviously suffices to take $\{p\} \to \{q, q', \cdots\}$ where $q, q', \cdots$ are the lower covers of p *in* $(J(\mathcal{M}), \leq)$. Furthermore, each implication $\{p, q\} \to \ell$ can be replaced by a suitable implication $\{p, q\} \to \{r\}$. Let Σ_2 be this new family of implications. One can show [W6] that for no Σ' with $C(\Sigma') \simeq \mathcal{M}$ is the sum $s(\Sigma')$ of the cardinalities of all premises and conclusions smaller than $s(\Sigma_2)$.

Let us look in particular at lattices $\mathcal{M} = \mathcal{M}(P, \mathcal{V}_3)$ where $\mathcal{V}_3$ is the variety generated by $\mathcal{M}_3$. They are of special interest because they are exactly the *finite* ones among the lattices $\mathcal{M}(P)$. Let Σ_2 as above be such that $\mathrm{C}(\Sigma_2) \simeq \mathcal{M}$. Rather than applying the principle of exclusion in the manner of Example 1 upon Σ_2 in order to obtain $|\mathrm{C}(\Sigma_2)| = |\mathcal{M}|$, one can add a couple of tricks that reduce the number of 3-valued rows r.

First, say $J(\mathcal{M}) = \{p_1, p_2, \cdots, p_t\}$ and $\ell = \{p_i, p_j, p_k\}$ is a line of a base of lines $\bigwedge$ of $\mathcal{M}$. It gives rise to the three implications $\{p_i, p_j\} \to \{p_k\}, \{p_i, p_k\} \to \{p_j\}, \{p_j, p_k\} \to \{p_i\}$ of Σ_2. Obviously the subsets of $\{p_i, p_j, p_k\}$ which satisfy all three implications are $\phi, \{p_i\}, \{p_j\}, \{p_k\}$, and $\{p_i, p_j, p_k\}$. Instead of rows r with components $0, 1$, or 2, we now consider "many-valued" rows r of length t. Namely, for $\ell = \{p_i, p_j, p_k\}$ we shall write the "minor" $(3, 3, 3)$ in the i-th, j-th and k-th component of r to indicate that the $0, 1$-vectors (subsets of $J(\mathcal{M})$) represented by r when projected to the i-th, j-th and k-th component must be one of $(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,1)$. For all other lines of $\bigwedge$ we use minors $(3', 3', 3')$, etc. As will be clear in a moment, it is convenient to use two other types of minors $(4, 4)$ and $(5, 5)$. Hereby $(4, 4)$ (or $(4', 4')$ etc.) stands for $\{(0,0), (1,0), (0,1)\}$ and $(5, 5)$ (or $(5', 5')$ etc.) stands for $\{(0,0), (1,1)\}$.

Example 12: Suppose $\mathcal{M}$ is a modular lattice*) with partial order

$(J(\mathcal{M}), \leq) =$



and a base of lines $\bigwedge = \{\{p_1, p_2, p_3\}, \{p_4, p_5, p_6\}\}$. In order to compute the cardinality of $\mathcal{M}$ we start out with the many-valued line in Fig. 8.1 which takes care of $\bigwedge$. (Generally the implications arising from $\bigwedge$ are advantageously handled at the *beginning* by using just *one* many-valued row.) Next we have to enforce the implications $\{p_6\} \to \{p_7, p_8\}$, $\{p_3\} \to \{p_4\}$, and $\{p_7\} \to \{p_1, p_2\}$ which take care of $(J(\mathcal{M}), \leq)$. (Generally all these implications have *singleton* premises which, as will be seen, entails that enforcing one such implication to a many- valued row can produce *at most one new* many-valued row.) To prepare the enforcement of $\{p_6\} \to \{p_7, p_8\}$ write the row in Fig. 8.1 as a disjoint union of a row with "$p_6 = 0$" and a row with "$p_6 = 1$" (see Fig. 8.2). For instance, forcing the last component of $(3', 3', 3') = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$ to be 0 yields $\{(0,0,0), (1,0,0), (0,1,0)\} = (4,4,0)$. It is clear that enforcing $\{p_6\} \to \{p_7, p_8\}$ upon the two rows in Fig. 8.2 yields the two rows in Fig. 8.3. To prepare the enforcement of $\{p_3\} \to \{p_4\}$ each row in Fig. 8.3 is split in a row with $p_3 = 0$ and a row with $p_3 = 1$ (see Fig. 8.4). Enforcing $\{p_3\} \to \{p_4\}$ upon the rows in Fig. 8.4 yields the rows in Fig. 8.5. By "preparing rows" in one's head the enforcement of $\{p_7\} \to \{p_1, p_2\}$ upon the rows in Fig. 8.5 yields the rows in Fig. 8.6 (the first and fourth row locally change, the second splits, the third is cancelled). The first row in Fig. 8.6 represents $3 \cdot 3 \cdot 2 = 18$ subsets of $\{p_1, \cdot\cdot, p_8\}$. Taking into account all rows of Fig. 8.6. we derive $|\mathcal{M}| = 18 + 4 + 2 + 1 = 25$.

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
3	3	3	3′	3′	3′	2	2

Fig. 8.1

3	3	3	4	4	0	2	2
3	3	3	5	5	1	2	2

Fig. 8.2

3	3	3	4	4	0	2	2
3	3	3	5	5	1	1	1

Fig. 8.3

4′	4′	0	4	4	0	2	2
5	5	1	4	4	0	2	2
4	4	0	5	5	1	1	1
5′	5′	1	5	5	1	1	1

Fig. 8.4

*) For the purpose of illustrating the enhanced prinicple of exclusion we ignore whether such a $\mathcal{M}$ can actually exist.

4'	4'	0	4	4	0	2	2
5	5	1	1	0	0	2	2
4	4	0	5	5	1	1	1
5'	5'	1	1	1	1	1	1

Fig. 8.5

4'	4'	0	4	4	0	0	2
5	5	1	1	0	0	0	2
1	1	1	1	0	0	1	2
1	1	1	1	1	1	1	1

Fig. 8.6

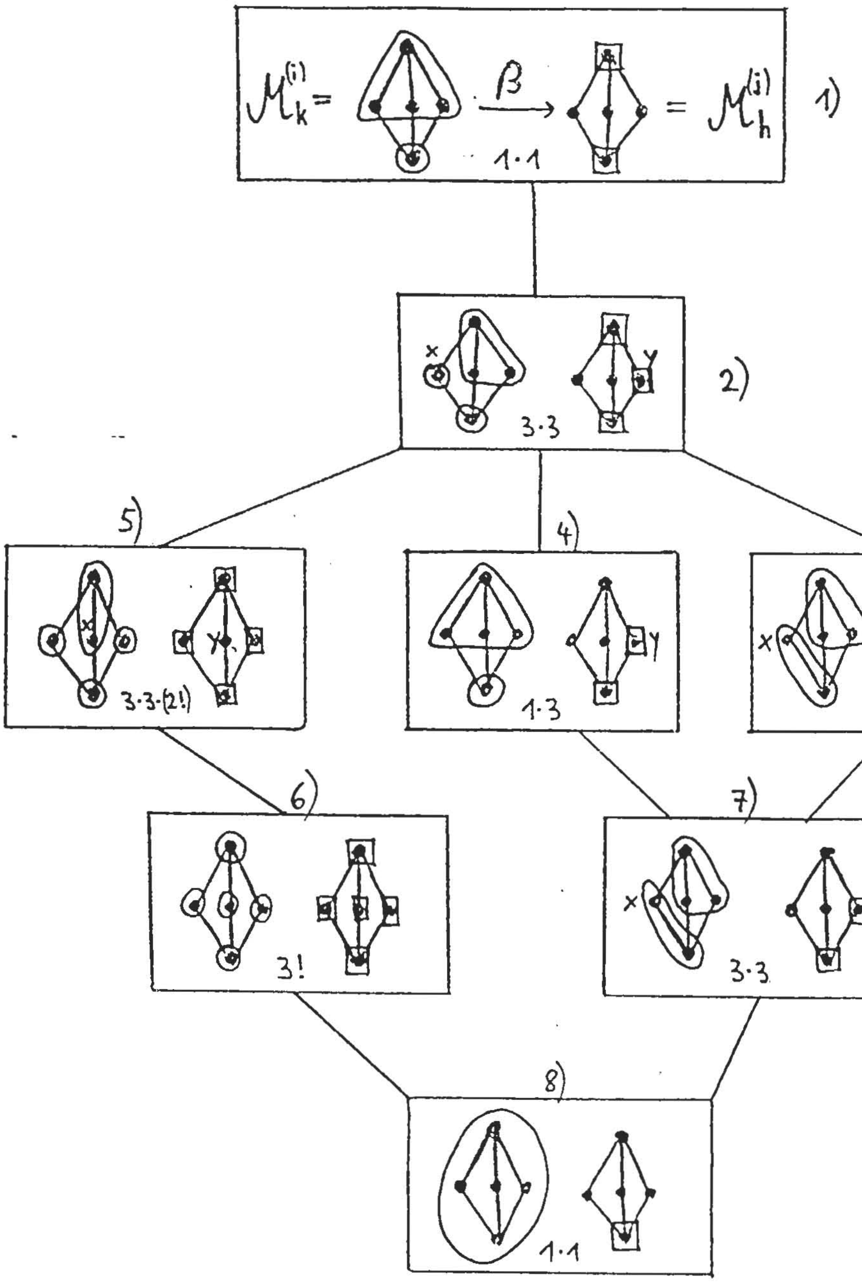
□

Let us mention that other types of minors further reduce the number of many-valued rows. For instance, let the m-tuple $(6,6,\cdot\cdot,6)$ represent the subsets of $\{p_1,\cdot\cdot,p_m\}$ which satisfy *all* of the implications $\{p_1\} \to \{p_2\}, \{p_2\} \to \{p_3\}, \cdots, \{p_{m-1}\} \to \{p_m\}$. One eg. verifes at once that the set system represented by the first row below is the disjoint union of the set systems represented by the other two rows:

$$(6,6,6,6,6,6,6,6)$$
$$(0,0,0,0,6,6,6,6)$$
$$(6,6,6,1,1,1,1,1)$$

More generally, tree-like conglomerations of the implications (with singleton premises but not necessarily singleton conclusions) can be abbreviated in a similar fashion.

So much about the computation of $C(\Sigma_2)$ once Σ_2 is *known*. But how to get Σ_2 or, what amounts to the same, how to compute the poset $(J(\mathcal{M}), \leq)$ and a base of lines $\bigwedge$ of $\mathcal{M}$? Recall from step 1 in section 6 that first one has to generate all non-equivalent labellings of $\mathcal{L}^{(1)} = \mathcal{D}_2$ and $\mathcal{L}^{(2)} = \mathcal{M}_3$ with the elements $a \in P$. The labellings of $\mathcal{D}_2$ are easily obtained (corresponding to the proper order filters), but more interesting is it to write a programme for labelling $\mathcal{M}_3$ in all possible ways with the elements of an arbitrary finite poset. We omit the details. Let us rather investigate more closely step 2 where all biggest morphisms $\beta(i,k;j,h)$ between two labellings of $\mathcal{D}_2$, or a labelling of $\mathcal{D}_2$ and $\mathcal{M}_3$, or two labellings of $\mathcal{M}_3$ have to be generated. The third case is the most interesting one and can be mechanised as follows:



If the V-homomorphism β in 1) happens to be a morphism (ie. maps labels below labels) then $\beta(i,k;\bar{j},h)=\beta$. Otherwise check whether one the 9 (there are $3\cdot3$ choices for x and y) V-homomorphisms in 2) happens to be a morphism. If yes, there can be only one such β (why?) and $\beta(i,k;\bar{j},h)=\beta$. Otherwise check the 3 V-homomorphisms β in 3), the 3 β's in 4), the $3\cdot3\cdot2=18$ β's in 5), the 6 β's in 6), the 9 β's in 7) in *that order*. If none of these β's maps labels below labels than $\beta(i,k;\bar{j},h)$ must be the constant map $x\mapsto 0$ in 8). We leave it to the reader to verify that this algorithm yields the correct morphism $\beta(i,k;\bar{j},h)$. For instance, why isn't it possible to come across some β in 6) *and* some β'

in 7) which both map labels below labels? As an exercise one may apply the algorithm to come up with the morphism $\beta(2,2;2,4)$ in (22).

The author plans to use the algorithms discussed in this article to compute the cardinalities of all $\mathcal{M}(P)$ for say $|P| \leq 5$. Of course, for bigger posets P or varieties $\mathcal{V}$ with higher number t of subdirectly irreducibles, the family of implications Σ_2 with $C(\Sigma_2) \simeq \mathcal{M}(P,\mathcal{V})$ will be too big to be handled by the principle of exclusion. But it is clear that at least $J(\mathcal{M}(P,\mathcal{V}))$ and its partial order, the number s of sudirectly irreducible factors of $\mathcal{M}(P,\mathcal{V})$, and the length of $\mathcal{M}(P,\mathcal{V})$ (which is just the sum of the lengths of these factors) can be determined for much bigger P and t.

References

[BS] S. Burris, H.P. Sankappanavar, A course in universal algebra, Graduate texts in Mathematics, Springer 1981.

[GW] B. Ganter and R. Wille, Formal Concept Analysis - Mathematical foundations, Springer Verlag 1999.

[G] H. Gross, Lattices and infinite dimensional forms. "The Lattice Method", Order 4 (1987) 233-256.

[HPR] C. Herrmann, D. Pickering and M. Roddy, A geometric description of modular lattices, Algebra Universalis 31 (1994) 365-396.

[HW] C. Herrmann and M. Wild, Acyclic modular lattices and their representations, Journal of Algebra 136 (1991) 17-36.

[KKW] H.A. Keller, U.M. Künzi, M. Wild (editors), Orthogonal geometry in infinte dimensional vector spaces, Bayreuther Mathematische Schriften 53 (1998).

[M] Metsch, Linear spaces with few lines, Springer Lecture Notes in Mathematics 1490 (1991).

[S] R.P. Stanley, Enumerative Combinatorics, Vol.1, Wadsworth 1986.

[W1] M. Wild, Implicational bases for finite closure systems, Technische Hochschule Darmstadt, Preprint No. 1210 (1989).

[W2] M. Wild, Modular lattices of finite length, unpublished manuscript, 28 pages, MIT 1990.

[W3] M. Wild, A theory of finite closure spaces based on implications, Advances in Mathematics 108 (1994) 118-139.

[W4] M. Wild, Computations with finite closure systems and implications, Springer Lecture Notes in Computer Science 959 (1995) 111-120.

[W5] M. Wild, Halbgruppen und formale Sprachen, typed lecture notes 1996, Library of the Mathematisches Institut der Universität Zürich, X00-147.

[W6] M. Wild, Optimal implicational bases for finite modular lattices, to appear in Quaestiones Mathematicae.

[Wi1] R. Wille, Subdirekte Produkte vollständiger Verbände, Journal für die reine und angewandte Mathematik 283/284 (1976) 53-70.

[Wi2] R. Wille, Über modulare Verbände, die von einer endlichen halbgeordneten Menge frei erzeugt werden, Mathematische Zeitschrift 131 (1973) 241-249.

[Wi3] R. Wille, On lattices freely generated by finite partially ordered sets, Contributions to Universal Algebra, Coll. Math. Janos Bolyai 17 (B. Csakany, E.T. Schmidt eds), Budapest, Amsterdam 1977.

Marcel Wild
Department of Mathematics
University of Stellenbosch
Private Bag X1
7602 Matieland
South Africa

mwild@land.sun.ac.za
http://www.sun.ac.za/maths/MW/