Partitioning the hypercube Q_n into n isomorphic edge-disjoint trees

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1 Introduction

The problem of finding edge-disjoint trees in a hypercube e.g. arises in the context of parallel computing [3]. Independent of applications it is of high aesthetic appeal. A hypercube of dimension n, denoted by Q_n, comprises 2^n vertices each corresponding to a distinct binary string of length n. An edge exists between two vertices if and only if their corresponding binary strings differ in exactly one position. Since each vertex of Q_n has degree n, the number of edges is $n2^{n-1}$. A variety of decomposability options derive from this fact. In the remainder of the introduction we focus on three of them. The first two have been dealt with before in the literature, the third is the topic of this article.

a) Roskind and Tarjan [7] found a polynomial-time algorithm for computing the *spanning tree packing number* $\sigma(G)$ of a connected graph G, i.e. the maximum number of edge-disjoint spanning trees of G. Since each spanning tree of Q_n has $2^n - 1$ edges, one concludes

$$\sigma(Q_n) \leq \left\lfloor \frac{n2^{n-1}}{2^n - 1} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor .$$

It turns out that $\sigma(Q_n) = \lfloor \frac{n}{2} \rfloor$. This is an instance of the fact that a variety of graphs [4] which are regular of degree n have $\sigma(G) = \lfloor \frac{n}{2} \rfloor$. For n even, an explicit construction of $m = \frac{n}{2}$ edge-disjoint spanning trees $S_1, \cdots, S_m$ of Q_n is given in [1]. These S_i's are *not* isomorphic, and because of $m(2^n - 1) = n2^{n-1} - m$, there are m edges left over (which constitute a path in Q_n). For n odd, an analogous construction remains to be found.

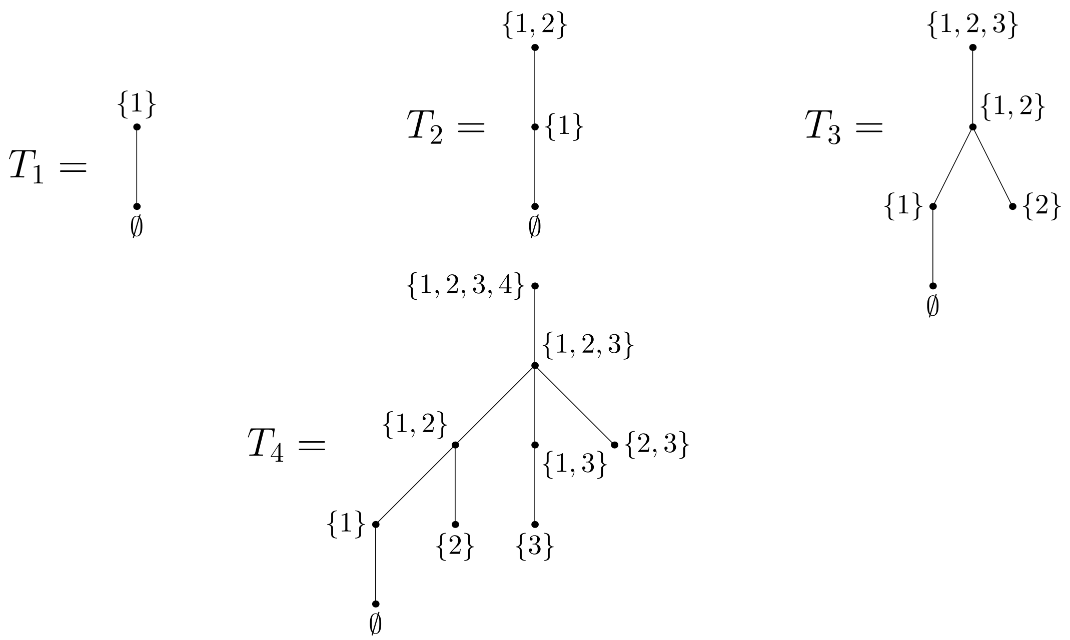
b) So partitioning the edge set of Q_n into *spanning* trees is impossible. What about a partitioning into *isomorphic* trees? Surprisingly, for *every* tree T with n edges, the edge set of Q_n can by [2] be covered by 2^{n-1} trees all of which isomorphic to T. This has been extended in [5], [6], and will be followed up in Section 4.

c) As opposed to partitioning Q_n into 2^{n-1} isomorphic trees of size n, in this article we partition Q_n into n trees of size 2^{n-1}, all of which isomorphic to some tree T_n. Our tree T_n needs to have a very specific shape.

As proposed by one referee, the authors at one time pondered to condense the whole article to Theorem 2 and its easy (once found) proof. Eventually we opted for a more leisurely approach which includes two equivalent definitions of T_n and some more historical remarks. The direct definition of T_n is useful in Theorem 2, whereas the recursive definition underlies our pictures of T_1 up to T_5.

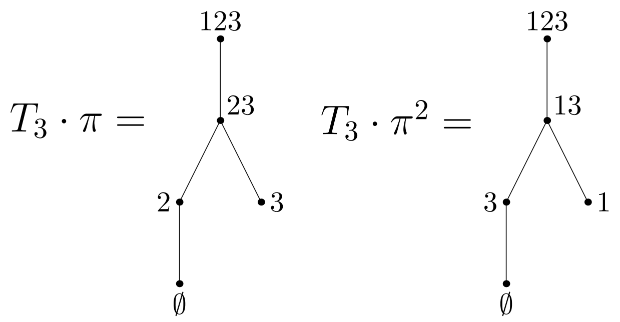
2 Construction of T_n

Rather than dealing with 0, 1-strings, it will be more convenient to let the vertex set $V(Q_n)$ be the power set $\mathcal{P}([n])$ of $[n] := \{1, 2, \cdots, n\}$, with $X, X' \in V(Q_n)$ being adjacent if and only if their symmetric difference has cardinality one. Before tackling the formal definition of T_n, let us do some small cases; of course, we must have $|V(T_n)| = 2^{n-1} + 1$ for every n.

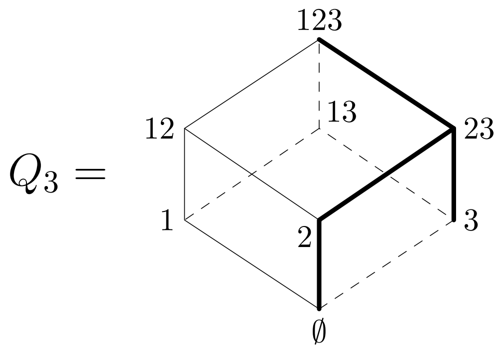


The following notation shall be employed throughout the article. Permutations π of $[n]$ will be written in cycle notation and to the *right* of the argument, separated by a dot. Thus if $\pi = (3,4)(2,5,7)$ (for some $n \geq 7$), then $3 \cdot \pi = 4$ and $7 \cdot \pi = 2$. Each permutation of $[n]$ induces, in the obvious way, a permutation of the vertex set $V(Q_n)$. Carrying over the described notation, applying the permutation π above to the vertex (say) $X = \{1,2,4\}$ of Q_n yields $X \cdot \pi := \{1,5,3\}$. Furthermore, for $\{X_1, X_2, \cdots\} \subseteq V(Q_n)$ we set $\{X_1, X_2, \cdots\} \cdot \pi := \{X_1 \cdot \pi, X_2 \cdot \pi, \cdots\}$. If T is a subgraph of Q_n and π a permutation, then $T \cdot \pi$ is defined as the subgraph of Q_n which has vertex set $V(T) \cdot \pi$ and edge set $\{\{X \cdot \pi, Y \cdot \pi\} \mid \{X,Y\} \in E(T)\}$.

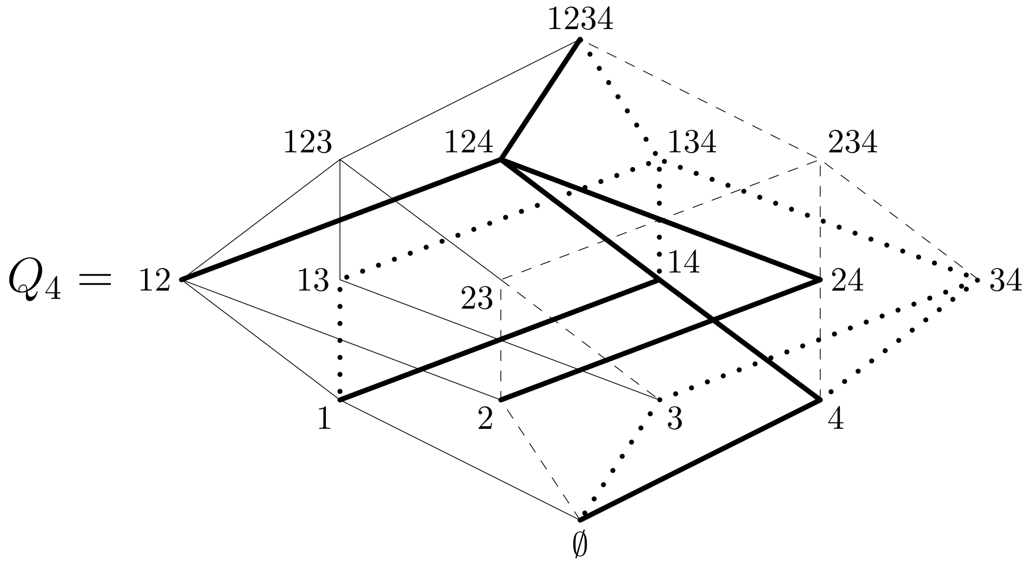
Let us resume the discussion of our trees T_n and put $n = 3, \pi = (1,2,3)$. Then we have



where for simplicity we wrote e.g. 23 instead of $\{2,3\}$. The edge-disjoint decomposition of Q_3 into the isomorphic rooted trees $T_3,\ T_3 \cdot \pi$, and $T_3 \cdot \pi^2$ is now apparent:



The corresponding decomposition of Q_4 into the trees $T_4 \cdot \pi^i$ $(0 \leq i < 4,\ \pi := (1,2,3,4))$ is already more surprising:



Let us now define T_n in general. For any positive integer n, we set $V(T_n) = \mathcal{P}([n-1]) \cup [n]$, i.e. the vertices are all subsets of $[n-1]$ and the *root* $[n]$. It is obvious from this definition that $|V(T_n)| = 2^{n-1} + 1$ for all n. Furthermore, we assign a *parent vertex* $p(v)$ to every vertex $v \neq [n]$, namely the set $p(v) = v \cup \{x(v)\}$, where

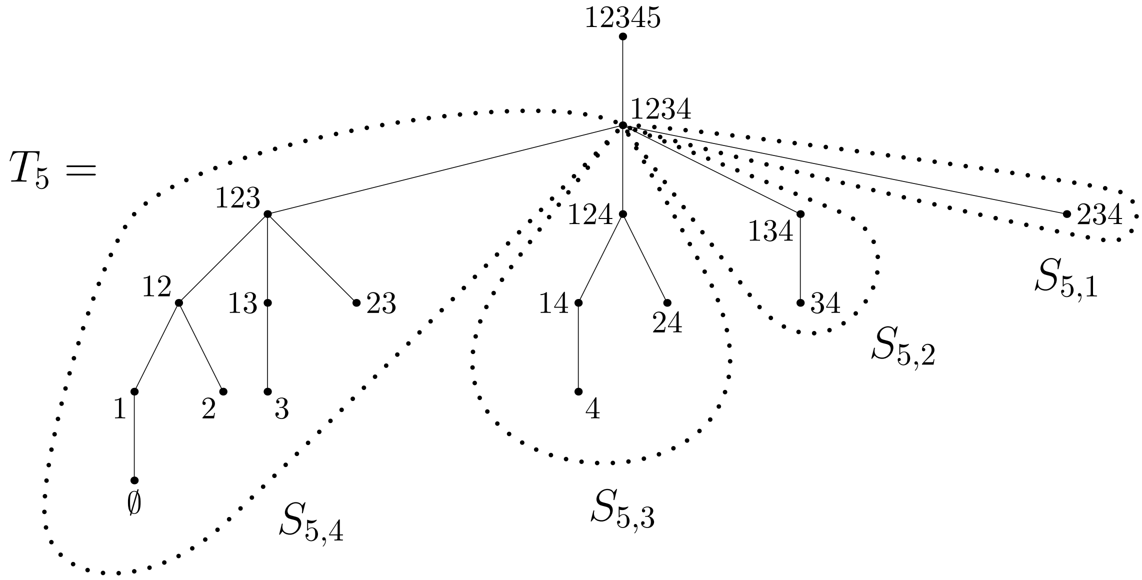
$$x(v) \quad = \quad \min(\mathbb{N} \setminus v). \tag{1}$$

In other words, to obtain the parent vertex, we add the smallest positive integer that is not yet contained in the set. If $v \neq [n-1]$, this is an element of $[n-1]$; if $v = [n-1]$, then $x(v) = n$ and $p(v) = [n]$. Now the edge set $E(T_n)$ consists of all pairs $(v, p(v))$. The fact that every set (= vertex) $v \neq [n]$ is adjacent to a *unique* superset, implies that the graph $(V(T_n), E(T_n))$ has no cycles, i.e. is a forest. Since every vertex $v \neq [n]$ is connected to $[n-1]$ via $p(v), p^2(v), \cdots, p^k(v) = [n-1]$, the forest is in fact a tree. The levels (which consist of all those vertices whose distance from the root is some given integer) correspond exactly to the subsets of $[n-1]$ of fixed cardinality.

It should be noted that T_n can be constructed in a recursive manner as well. Consider the subforest $S_{n,i}$ of T_n $(1 \leq i \leq n-1)$ that is induced by the vertex set

$$V(S_{n,i}) \quad = \quad \{v \in \mathcal{P}([n-1]) : \{i+1, i+2, \ldots, n-1\} \subseteq v,\ i \notin v\} \quad \cup \quad [n-1] \tag{2}$$

which consists of all those sets that contain $i+1, i+2, \ldots, n-1$ but do not contain i, together with the set $[n-1]$. The picture for $n = 5$ is as follows:



Notice that

(3) $\quad S_{5,4} = T_4, \quad S_{5,3} \simeq T_3, \quad S_{5,2} \simeq T_2, \quad S_{5,1} \simeq T_1$

Proposition 1: The subforest $S_{n,i}$ of T_n $(1 \le i \le n-1)$ is a tree isomorphic to T_i. It follows that T_n is isomorphic to the tree that is obtained by gluing $T_1, \cdots, T_{n-1}$ at their roots and then attaching a new root.

Proof. The second statement of Proposition 1 follows from the first because $\mathcal{P}([n-1]) \setminus \{[n-1]\}$ is by (2) partitioned by the sets $V(S_{n,1}) \setminus \{[n-1]\}$ up to $V(S_{n,n-1}) \setminus \{[n-1]\}$. As to the first statement, fix n and $i \in \{1, 2, \cdots, n-1\}$.

<u>Case 1:</u> $i = n-1$. As a special case of (2),

(4) $\quad V(S_{n,n-1}) \ = \ \{v \in \mathcal{P}([n-1]) : \ n-1 \notin v\} \cup [n-1],$

and so $V(S_{n,n-1})$ *coincides* with $V(T_{n-1})$. Thus $S_{n,n-1}$ is not just isomorphic to but actually equals T_{n-1} (cf. (3)).

<u>Case 2:</u> $i \in \{1, 2, \cdots, n-2\}$. Then

(5) $\quad V(S_{i+1,i}) \ = \ \{v \in \mathcal{P}([i]) : \ i \notin v\} \cup [i],$

and so the map f from $V(S_{n,i})$ to $V(S_{i+1,i})$ defined by

$$f(v) := v \setminus \{i+1, \ i+2, \cdots, n-1\}$$

is again bijective. If v and $v \cup \{x(v)\}$ are adjacent in $S_{n,i}$, then their images are $f(v)$ and $f(v) \cup \{x(v)\}$ (since $x(v) < i$), which are adjacent in $S_{i+1,i}$. Analogously, adjacent vertices w and $w \cup \{x(w)\}$ in $S_{i+1,i}$ yield adjacent vertices $f^{-1}(w)$ and $f^{-1}(w) \cup \{x(w)\}$ in $S_{n,i}$. Hence $S_{n,i} \simeq S_{i+1,i} \simeq T_i$ where the second isomorphism follows by Case 1. $\square$

We mention that the recursive construction was the way the second author discovered T_n. The equivalent view in terms of parent vertices, which enabled to properly prove the speculated theorem below, is due to the first author.

3 Main result

Our main result states that the edges of the n-dimensional hypercube Q_n can be partitioned into isomorphic copies of T_n:

Theorem 2: Let $\pi = (1, 2, \cdots, n)$. If $\{X, Y\} \in E(T_n)$, then it follows that $\{X, Y\} \notin E(T_n \cdot \pi^i)$ for all $0 < i < n$. Therefore, the edge sets of the trees $T_n,\ T_n \cdot \pi,\ \cdots, T_n \cdot \pi^{n-1}$ form a partition of the edge set of Q_n.

Proof: Let i be an integer with $0 < i < n$. We show the following: if a set X belongs to the vertex sets of both T_n and $T_n \cdot \pi^i$, then the parent vertices of X in T_n and $T_n \cdot \pi^i$ are distinct. Since parent vertices are unique, this implies immediately that T_n and $T_n \cdot \pi^i$ have no common edges. The vertex set of $T_n \cdot \pi^i$ is precisely

(6) $$(\mathcal{P}([n-1]) \cup [n]) \cdot \pi^i \quad = \quad \mathcal{P}([n] \setminus \{i\}) \cup [n].$$

Hence, a set X can only belong to the vertex sets of both T_n and $T_n \cdot \pi^i$ if $i, n \notin X$ or if $X = [n]$. Since $X = [n]$ has no parent vertex, it suffices to consider the case that $i, n \notin X$. Let $X \cup \{x\}$ and $X \cup \{y\}$ be the parent vertices of X in T_n and $T_n \cdot \pi^i$ respectively.

Since $i \notin X$, we have $1 \le x \le i$ by the definition of T_n. Let $Z \in V(T_n)$ be the preimage of $X \in V(T_n \cdot \pi^i)$. From $n \notin X$ follows $(n-i) \notin Z$, and so the parent vertex $Z \cup \{z\}$ of Z satisfies $z \in \{1, 2, \cdots, n-i\}$. But this forces $y = z \cdot \pi^i \in \{i+1, \cdots, n\}$. Altogether, this shows that $x \neq y$, and so the parents $X \cup \{x\}$ and $X \cup \{y\}$ are distinct, as claimed.

Therefore, the edge sets of T_n and $T_n \cdot \pi^i$ are disjoint for all $0 < i < n$. This also implies that the edge sets of $T_n \cdot \pi^i$ and $T_n \cdot \pi^j$ are disjoint for all $0 \le i < j < n$: if they were not disjoint, the edge sets of T_n and $T_n \cdot \pi^{j-i}$ would not be disjoint either, a contradiction.

Since T_n has 2^{n-1} edges and the isomorphic copies $T_n,\ T_n \cdot \pi,\ \cdots, T_n \cdot \pi^{n-1}$ are pairwise edge-disjoint by the above argument, it follows immediately that they partition the edge set of Q_n that comprises of $n2^{n-1}$ edges. $\square$

4 Related matters

Let E be a subset of $E(Q_n)$. Following Ramras call E a *fundamental set* for Q_n with group G , if G is a subgroup of $\mathrm{Aut}(Q_n)$ such that $\{g(E)|\ g \in G\}$ is an edge decomposition of Q_n. It is shown in [5] that if $|E| = n$ and the graph induced by E is connected with at most one cycle (e.g. a tree), then E is a fundamental set for Q_n. Results about fundamental $2n$-element sets are contained in [6]. Our tree T_n fits into this framework in that $E(T_n)$ is a fundamental 2^{n-1}-set for the group $G \subseteq \mathrm{Aut}(Q_n)$ that is induced by the cyclic group $\langle \pi \rangle \subseteq S_n$.

In another vein, every rooted tree T (e.g. $T = T_n$) becomes a unique partially ordered set $(T, \leq)$ when the root is postulated as largest element. It is an open problem to find necessary or sufficient conditions for a rooted tree $(T, \leq)$ to be cover preserving order embeddable into $(Q_n, \subseteq)$. That is, we want a map $\phi : T \to Q_n$ that satisfies

(7) $\quad (\forall x, y \in T) \quad x \leq y \quad \Leftrightarrow \quad \phi(x) \subseteq \phi(y)$

(8) $\quad (\forall x, y \in T) \quad x \prec y \quad \Rightarrow \quad \phi(x) \prec \phi(y)$

Here $x \prec y$ means that $x < y$ and $(x < z \leq y \Rightarrow z = y)$. We mention that for $0,1$-posets $(P, \leq)$ the problem is settled in [8] in terms of the chromatic number of some auxiliary graph. Notice that e.g. $(T_4, \leq)$ is *not* order embedded in Q_4: while $\{2\} \subseteq \{2, 3\}$ in Q_4, the corresponding vertices are not comparable in $(T_4, \leq)$.

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