Positive Weighted Koopman Semigroups on Banach lattice modules

by

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Abstract

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In this thesis, we introduce the notion of a positive weighted semigroup representation on a Banach lattice module over a group representation on a commutative Banach lattice algebra. One main theme of this work is the following: for topological dynamics, we obtain the abstract representation of the lattice of continuous sections vanishing at infinity of a *topological* Banach lattice bundle (over a locally compact space Ω) as a structure which we call an AM m-lattice module over $C_0(\Omega)$ on which every positive weighted semigroup representation over the Koopman group representation on $C_0(\Omega)$ is isomorphic to a positive weighted Koopman semigroup representation induced by a unique positive semiflow on the underlying *topological* Banach lattice bundle (over the continuous flow on the base space Ω). And as a result, every positive dynamical Banach lattice bundle can be assigned uniquely to a certain positive dynamical m-lattice module and vice versa, which is the Gelfand-type theorem that we proved. In order to do this, we, in particular, establish the following two categories of (i) Banach lattice modules and their dynamics; and (ii) Banach lattice bundles and their dynamics. We pay special attention to the case of a *topological* positive $\mathbb{R}_+$-dynamical Banach lattice bundle by which we obtain the corresponding C_0-semigroup of positive weighted Koopman operators, and using the theory of strongly continuous semigroup of positive operators, we obtain results pertaining to properties of the generator, and spectral theory of this positive semigroup.

Uittreksel

Positive Weighted Koopman Semigroups on Banach lattice modules

(“ Positive Weighted Koopman Semigroups on Banach lattice modules ”)

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In hierdie tesis stel ons die idee van ’n positiewe geweegde halfgroepvoorstelling op ’n Banach-roostermodule oor die groepvoorstelling op ’n kommutatiewe Banach-roosteralgebra bekend. Een hooftema van hierdie tesis is die volgende: vir topologiese dinamika verkry ons die abstrakte voorstelling van die rooster van kontinue snitte wat verdwyn by oneindig van ’n *topologiese* Banach-roosterbundel (oor ’n lokaal-kompakte ruimte Ω) as ’n struktuur wat ons ’n AM-m-roostermodule oor $C_0(\Omega)$ noem, waarop elke positiewe geweegde halfgroepvoorstelling oor die Koopman-groepvoorstelling op $C_0(\Omega)$ isomorfies is aan ’n positiewe geweegde Koopman-halfgroepvoorstelling geïnduseer deur ’n unieke positiewe halfvloei op die onderliggende *topologiese* Banach-roosterbundel (oor die kontinue vloei op die basisruimte Ω). Gevolglik kan elke positiewe dinamiese Banach-roosterbundel uniek aan ’n sekere positiewe dinamiese m-roostermodule toegeken word en omgekeerd, wat die Gelfand-tipe stelling is wat ons bewys het. Om dit te doen, stel ons veral die volgende twee kategorieë van (i) Banach-roostermodules en hul dinamika; en (ii) Banach-roosterbundels en hul dinamika bekend. Ons gee besondere aandag aan die geval van ’n *topolo-*

giese positiewe $\mathbb{R}_+$-dinamiese Banach-roosterbundel waardeur ons die ooreenstemmende C_0-halfgroep van positiewe geweegde Koopman-operatore verkry en deur die teorie van sterk-kontinue halfgroepe van positiewe operatore te gebruik, verkry ons resultate wat betrekking het op eienskappe van die generator, en spektraalteorie van hierdie positiewe halfgroep.

Acknowledgements

I have seen something else under the sun: The race is not to the swift or the battle to the strong, nor ... Ecc.9:11. So, then, it's not of him that willeth, nor of him that runneth, but of God that showeth mercy. Rom.9:16.

If I have seen further, it is by standing on the shoulders of giants. - Sir Isaac Newton, 1675.

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Ultimately, my gratitude goes unto God, for His mercy and privilege of life.

Dedications

To my parents.

Contents

Notation

Ω	A locally compact (Hausdorff) topological space.
G	A locally compact (Hausdorff) topological group with neutral element e.
K	A compact (Hausdorff) topological space.
$\mathbb{K}$	The field of real $\mathbb{R}$ or complex $\mathbb{C}$ scalars.
X	A complete σ-finite positive measure space $X = (\Omega_X, \Sigma_X, \mu_X)$.
$C_0(\Omega)$	The lattice algebra of continuous $\mathbb{K}$-valued functions vanishing at infinity.
$L^\infty(X)$	The (quotient) lattice algebra of essentially-bounded measurable $\mathbb{K}$-valued functions.
$\mathscr{L}(Z)$	The algebra of bounded (linear) operators on a Banach space Z.
$\sigma(T)$	The spectrum of a bounded (linear) operator T on a Banach space.
$\sigma_p(T)$	The point spectrum of a bounded (linear) operator T on a Banach space.
$\sigma_{ap}(T)$	The approximate point spectrum of a bounded (linear) operator T on a Banach space.
$r(T)$	The spectral radius of a bounded (linear) operator T on a Banach space.
Id_Q	The identity map on any space Q.
$\Gamma_0(\Omega, E)$	The lattice of continuous sections vanishing at infinity of a *topological* Banach lattice bundle E over Ω.
$\Gamma^1(X, E)$	The (quotient) lattice of integrable measurable sections of a *measurable* Banach lattice bundle E over X.
$\text{lin}\,\{W\}$	The linear span of a set of vectors W.
WLOG	Without Loss Of Generality.
B_+	The positive cone of an ordered Banach space B.
$IntB_+$	The interior of the positive cone B_+ of an ordered Banach space B.
$\mathscr{Z}(Z)$	The lattice algebra of central operators of a Banach lattice Z.

Introduction

This thesis is about positive weighted Koopman semigroups on a Banach lattice module. This work was inspired by Sita Siewert's PhD dissertation ([29]), where the notion of weighted Koopman semigroups for topological or measurable dynamics were studied. Therefore, this thesis forms part of the broader field named "Operator Theoretic Aspects of Ergodic theory" ([10]).
The idea comes from assigning to a nonlinear cocycle on a (topological or measurable) *state space* the corresponding linear operators, now called weighted Koopman operators, on an *observable space*, i.e., a linear space of vector-valued functions on the state space. A semigroup consisting of weighted Koopman operators is called *a weighted Koopman semigroup*. A typical discrete-time instance of this concept, for topological dynamics, is the following:

Consider a *continuous* cocycle $\{\phi(x) \in \mathscr{L}(Z) : x \in K\}$ associated with a homeomorphism $\varphi : K \longrightarrow K$ on a compact space K with Banach space Z, i.e.,

$$\begin{cases} \phi^0(x) = Id_Z \text{ for } x \in K; \\ \phi^{n+m}(x) = \phi^n(\varphi^m(x))\phi^m(x) \text{ for } x \in K \text{ and } n, m \in \mathbb{N}; \text{and} \\ K \times Z \longrightarrow Z; (x, v) \mapsto \phi(x)v \text{ is continuous,} \end{cases}$$

and the Koopman operator $T_\varphi : C(K) \longrightarrow C(K);\ f \mapsto f \circ \varphi^{-1}$.

Then, the *continuous* cocycle induces a weighted Koopman operator

$$\mathcal{T}_\phi : C(K, Z) \longrightarrow C(K, Z);\ s \mapsto \phi(\varphi^{-1}(\cdot)) \circ s \circ \varphi^{-1}$$

i.e., $\mathcal{T}_\phi \in \mathscr{L}(C(K, Z))$ and $\mathcal{T}_\phi fs = T_\varphi f \mathcal{T}_\phi s$ for $f \in C(K)$ and $s \in C(K, Z)$ with $(fs)(x) := f(x)s(x)$ for all $x \in K$.

The generalisation of these concepts, as treated in the paper [21] by H. Kreidler and S. Siewert, is the introduction of *non-linear dynamics* on the so-called (topological or measurable) Banach bundle E over a dynamical base space (i.e., topological dynamics for a locally compact space Ω or measurable dynamics for a complete σ-finite measure space X) and consider the corresponding weighted Koopman operators (now, *linear dynamics*) on the observable space, i.e., the Banach space $\Gamma_0(\Omega, E)$ of continuous sections of E vanishing at infinity for topological dynamics or the (quotient) Banach space $\Gamma^1(X, E)$ of integrable measurable sections of E for measurable dynamics. The following is of importance: (i) the observable spaces, i.e., the Banach spaces $\Gamma_0(\Omega, E)$ and $\Gamma^1(X, E)$, are, in addition, Banach modules over commutative Banach algebras $C_0(\Omega)$ and $L^\infty(X)$ respectively, and (ii) a weighted Koopman semigroup is an instance of the (general) notions of weighted semigroup representations on these Banach modules, i.e., dynamical Banach modules.

These weighted Koopman semigroups arise naturally in many applications, for example when modelling a dynamical system using an evolution semigroup or in smooth ergodic theory. On the one hand, we note that the notion of a cocycle over a (continuous) flow, giving rise to a certain evolution semigroup, has become a useful tool: (i) in modelling non-autonomous problems, e.g., dissipative partial differential equations, in particular, the Navier-Stokes equation; (ii) as an abstract framework for the study of random dynamical systems; (iii) in the general theory of ideal fluid dynamics; (iv) in detecting the existence of the so-called exponential dichotomy (or hyperbolicity), which is of practical importance; and (v) in the generalisation of the classical notion of a two-parameter evolution family ([1, 26]). On the other hand, see [29, Chapter 6] - an example from differential geometry - for a systematic treatment of smooth ergodic theory associated with the space of continuous sections of the tangent bundle of a compact smooth manifold. Moreover, these kinds of evolution operators, which we call weighted Koopman operators, are also known in the literature as weighted composition or weighted shift operators in general operator theory, as transfer operators in dynamical systems theory, and as push-forward operators in the theory of differentiable manifolds. We refer to the monographs [5] and [11] and the references therein for the general theory of evolution equations.

Now as it has been said, *In many concrete problems solvable by semigroups, there is a natural notion of "positivity" and only "positive" solutions make sense* [12, Chapter VI, first paragraph, p.205]. So, in our work, we study the scenario where each weighted Koopman operator is, in addition, a positive operator, which of course requires additional lattice structure on the space of sections in our consideration. This led us, in an attempt to adapt these concepts and standardise terminologies, to introduce the notion of a positive weighted semigroup representation on a Banach lattice module over a group representation on a commutative Banach lattice algebra. For the general theory of positive operators and positive operator semigroups on Banach lattices, we refer to the monographs [28] and [2].

We now briefly discuss the structure of the present thesis and refer to each of the chapters for a detailed introduction.

In Chapter 1, we motivate the notion of a positive weighted Koopman operator on a certain Banach lattice as a functional-analytic approach to the study of a certain continuous linear skew-product, associated with a positive continuous cocycle, on the trivial *topological* Banach lattice bundle.

Chapter 2 consists of our introduction and contributions to the notion of a Banach lattice module over a commutative Banach lattice algebra. We find that, in particular, our definition of a *Banach lattice module* (see Definition 2.2.1.1) can be seen to be a generalisation of a certain Banach lattice $L^\infty(\mathcal{G})$-module as defined by K-T. Eisele and S. Taieb (see [9, Definition 5.3(iii), p.531-532]). In Section 2.3, we introduce the concept of a (positive) weighted semigroup representation on a Banach lattice module (BLM) over a commutative Banach lattice algebra (BLA). In doing so, we, in particular, establish the category of Banach lattice modules and their dynamics. In Section 2.4, inspired by the work of H. Kreidler and S. Siewert (see [21, Section 4, p.15 and Section 5, p.20]) about AM-modules and AL-modules over the Banach algebra $C_0(\Omega)$, we introduce in a natural way the analogous concepts of an *AM m-lattice module* (see Definition 2.4.1.2) and an *AL m-lattice module* (see Definition 2.4.2.2) over an m-Banach lattice algebra $C_0(\Omega)$, respectively. In the last Section 2.5, we further our consideration on Banach lattice modules wherein we introduce, with examples, several concepts, many of which are proven useful in this thesis. Such concepts include: a *Banach sub-lattice algebra* (see Definition 2.5.1.1), a *closed lattice algebra ideal* (see Definition 2.5.1.2), a *Banach lattice submodule* (see Definition 2.5.2.1), a *closed ideal submodule* (see Definition 2.5.2.3); and a *dual Banach lattice module* (see Section 2.5.4).

In Chapter 3, we consider positive semiflows on *topological* Banach lattice bundles. We introduce the concepts of a (positive) *topological* dynamical Banach lattice bundle and the so-called (positive) weighted Koopman semigroup on the Banach lattice of continuous sections vanishing at infinity (see Definition 3.4.0.5 and Remark 3.5.0.1(v)). More importantly, we present our first Gelfand-type theorem for dynamical AM m-lattice modules (see Theorem 3.5.0.11 and Corollary 3.5.0.12). In addition, we further our consideration, by introducing several notions in the last Section 3.6 on the Banach lattice of continuous sections (vanishing at infinity) associated with a *topological* Banach lattice bundle over a (locally) compact space which yields interesting results of which many are subsequently needed. Such notions include: a *Banach lattice subbundle* (see Definition 3.6.1.1), a *closed ideal subbundle* (see Definition 3.6.3.1), the *direct sum of Banach lattice bundles* (see Subsection 3.6.2); and the *direct sum of positive weighted Koopman semigroups* (see Subsection 3.6.4). Other considerations are the characterisations of certain order structures (see Subsection 3.6.5) and the so-called centre (see Subsection 3.6.6) of this Banach lattice, which are also of independent interest.

Chapter 4 is about one-parameter C_0-semigroups of positive weighted Koopman operators. In particular, we consider some order structures of an AM m-lattice module (see Section 4.2), the lattice C_0-semigroup of weighted Koopman operators (see Section 4.3) and the positive C_0-semigroup of weighted Koopman operators (see Section 4.4) associated to a *topological* Banach lattice bundle over a compact space. Based on certain assumptions on the Banach lattice of continuous sections concerning its order structures, we obtain some results related to the *generation problem* of (positive) weighted Koopman semigroups (see Proposition 4.3.0.3, Corollary 4.3.0.4, Remark 4.3.0.5 and Proposition 4.4.0.3). Moreover, in Section 4.5, we present certain spectral properties of a positive C_0-semigroup of weighted Koopman operators (see Proposition 4.5.0.4).

In Appendix A we recall the definition and concept of a Banach lattice algebra as presented by Wickstead ([31]). In Appendix B, we recall certain definitions and concepts about an ordered Banach space as presented by O.

Bratteli, et. al ([4]) and the conditions under which it becomes a Banach lattice.

We also include more of our work which is added for the sake of comprehensiveness, but put in the appendices C, D, E and F because the scope of this thesis has become very large.

In Appendix C, we consider positive semiflows on *measurable* Banach lattice bundles. We introduce the concepts of (positive) *measurable* dynamical Banach lattice bundles and the so-called (positive) weighted Koopman semigroup on the (quotient) Banach lattice of integrable measurable sections (see Definition C.4.0.5 and Remark C.5.0.1(v)). More important, we present our second Gelfand-type theorem for dynamical separable $L^1(X)$-normed m-lattice modules (see Theorem C.5.0.9 and Corollary C.5.0.10).

In Appendix D, we present one central theme of our study: "When is a positive weighted semigroup representation on a Banach lattice module a (or isomorphic to) positive weighted Koopman semigroup representation?"

In Appendix E, we consider asymptotics of positive weighted Koopman semigroups associated with a *topological* Banach lattice bundle over a compact space. We, in particular, investigate irreducibility in Section E.2 and exponential dichotomy in Section E.3 of positive weighted Koopman semigroups. Moreover, based on some results already obtained in Chapter 3 (Section 3.6), we obtain several correspondences and characterisations of the asymptotic behaviour under consideration (see Propositions E.2.0.4 and E.3.0.4).

In Appendix F, we introduce the Markovian weighted Koopman group associated with a Banach lattice of continuous sections whose positive cone has a non-empty interior. In Section F.3, we introduce the concept of a *Markovian weighted Koopman operator* (see Definition F.3.0.1) and we show that every bijective Markovian weighted Koopman operator (resp. Markovian weighted Koopman group) is isomorphic to a bijective Koopman operator (resp. Koopman group) which extends the original bijective Koopman operator (resp. Koopman group) (see Proposition F.3.0.6, resp. Proposition F.3.0.7). This result, in particular, generalises the situation in [29, Example 3.11, p.65](iv) which is also included. That is, a C_0-group of weighted Koopman operators is isomorphic to an extended Koopman group if and only if it is Markovian in our sense. Furthermore, under this identification, we study the spectral property of an invertible Markovian weighted Koopman operator and extend the result to the Markovian weighted Koopman group in Section F.4.

Chapter 1

On the trivial *topological* Banach lattice bundle

In this chapter, we set the scene for our study of positive weighted Koopman operators using the example of the trivial *topological* Banach lattice bundle. Moreover, to keep the reading flow without distraction, we try to avoid a lot of definitions and several concepts at this stage; and we also consider only discrete-time dynamics.

Throughout, we take K to be a compact space, which we assume to be Hausdorff by definition. Z will be a Banach lattice, which we can assume to be real WLOG.

1.1 The trivial *topological* Banach lattice bundle and its lattice of continuous sections

By a trivial *topological* Banach lattice bundle over K, we mean the product space $E := K \times Z$ together with the natural projection $p : E \longrightarrow K$ onto the first factor, identifying each fiber $p^{-1}(x)$ with the Banach lattice Z for all $x \in K$.

As such, there is a one-to-one correspondence between the Banach lattice $C(K, Z)$ of continuous Z-valued functions, equipped with the sup-norm, and the corresponding lattice of continuous sections of the trivial *topological* Banach lattice bundle $E = K \times Z$. One thing that is important here is that the (bilinear) pairing

$$C(K) \times C(K,Z) \longrightarrow C(K,Z);\ (f,s) \mapsto fs := [x \mapsto f(x)s(x)]$$

turns the Banach lattice $C(K,Z)$ into a Banach module over the commutative Banach lattice algebra $C(K)$, i.e., it is a $C(K)$-module such that $||fs|| \leq ||f||||s||$ for all $f \in C(K), s \in C(K,Z)$, since

$$\begin{aligned} ||fs|| &= \sup_{x\in K} ||f(x)s(x)||_Z \\ &\leq \sup_{x\in K} |f(x)| \sup_{x\in K} ||s(x)||_Z \\ &= ||f||||s||. \end{aligned}$$

Observing, in addition, that $\mathsf{I} fs \mathsf{I} = |f| \mathsf{I} s \mathsf{I}$ for all $f \in C(K), s \in C(K,Z)$, since $\mathsf{I} fs \mathsf{I}(x) = \mathsf{I} f(x)s(x) \mathsf{I} = |f|(x) \mathsf{I} s \mathsf{I}(x)$ for all $x \in K$, we see that we have obtained an example of such Banach lattices which we later refer to as a Banach lattice module, and in particular an m-Banach lattice module over $C(K)$ in this case.

1.2 Positive cocycles, skew-products on the trivial Banach lattice bundle and positive weighted Koopman operators

From now on, by a trivial Banach lattice bundle, we mean a trivial *topological* Banach lattice bundle as introduced in Section 1.1 above.

Consider a *positive* continuous cocycle $\{\phi(x) : Z \longrightarrow Z \text{ positive operator} \mid x \in K\}$ associated with a homeomorphism $\varphi : K \longrightarrow K$, i.e.,

$$\begin{cases} \phi(x) \text{ is a positive } \textit{linear} \text{ operator for } x \in K; \\ \phi^0(x) = Id_Z \text{ for } x \in K; \\ \phi^{n+m}(x) = \phi^n(\varphi^m(x))\phi^m(x) \text{ for } x \in K \text{ and } n, m \in \mathbb{N}; \text{and} \\ K \times Z \longrightarrow Z;\ (x,v) \mapsto \phi(x)v \text{ is continuous,} \end{cases}$$

and the Koopman operator $T_\varphi : C(K) \longrightarrow C(K);\ f \mapsto f \circ \varphi^{-1}$.

Then, the *positive* continuous cocycle induces a *positive* weighted operator

$$\mathcal{T}_\phi : C(K,Z) \longrightarrow C(K,Z);\ s \mapsto \phi(\varphi^{-1}(\cdot)) \circ s \circ \varphi^{-1}$$

$$\text{i.e., } (\mathcal{T}_\phi s)(x) := \phi(\varphi^{-1}(x))s(\varphi^{-1}(x)) \text{ for } s \in C(K,Z), x \in K$$

over the Koopman operator $T_\varphi : C(K) \longrightarrow C(K)$, which we call a *positive* weighted Koopman operator. By this, we mean that

(i) $\mathcal{T}_\phi$ is a positive operator, since

$$\begin{aligned} |\mathcal{T}_\phi s|(x) &= |\phi(\varphi^{-1}(x))s(\varphi^{-1}(x))| \\ &\leq \phi(\varphi^{-1}(x))|s|(\varphi^{-1}(x)) \\ &= \mathcal{T}_\Phi |s|(x) \end{aligned}$$

for all $x \in K$ and each $s \in C(K,Z)$; and

(ii) $\mathcal{T}_\phi f s = T_\varphi f \mathcal{T}_\phi s$, since

$$\begin{aligned} \mathcal{T}_\phi f s(x) &= \phi(\varphi^{-1}(x))f(\varphi^{-1}(x))s(\varphi^{-1}(x)) \\ &= f(\varphi^{-1}(x))\phi(\varphi^{-1}(x))s(\varphi^{-1}(x)) \\ &= T_\varphi f(x)\mathcal{T}_\phi s(x) \end{aligned}$$

for $f \in C(K), s \in C(K,Z)$ and all $x \in K$.

As we said in Section 1.1 above, there is a one-to-one correspondence between the Banach lattice $C(K,Z)$ and the lattice of continuous sections of the trivial Banach lattice bundle $E = K \times Z$, denoted as $\Gamma(K,E)$. Though, this will be introduced later in general, but in this case $\Gamma(K,E)$ is comprised of functions $\tilde{s} : K \longrightarrow E;\ x \mapsto (x, s(x))$ where $s \in C(K,Z)$, and thus $\tilde{s}$ is uniquely determined by s. Therefore, the *corresponding* positive weighted Koopman operator is defined by

$$\mathcal{T}_\Phi : \Gamma(K,E) \longrightarrow \Gamma(K,E);\ \tilde{s} \mapsto \Phi \circ \tilde{s} \circ \varphi^{-1}$$

$$\text{i.e., } (\mathcal{T}_\Phi \tilde{s})(x) := \Phi(\varphi^{-1}(x), s(\varphi^{-1}(x))) \text{ for } s \in C(K,Z), x \in K$$

where

$$\Phi : E \longrightarrow E;\ (x,v) \mapsto (\varphi(x), \phi(x)v)$$

is the *continuous* skew-product on the trivial Banach lattice bundle $E = K \times Z$ associated with the positive cocycle. We note that $(f\tilde{s})(x) := (x, (fs)(x))$ for all $x \in K$, $f \in C(K)$ and $s \in C(K, Z)$ defines the lattice module structure such that $\Gamma(K, E) \cong C(K, Z)$ in this case.

We note that if we define $||\Phi|| := \sup_{x\in K} ||\phi(x)||_{\mathscr{L}(Z)}$, then $||\Phi|| < \infty$. More so, $\mathcal{T}_\phi$ being positive implies that it is also bounded, i.e., $\mathcal{T}_\phi \in \mathscr{L}(C(K, Z))$. And in fact, it follows that $||\Phi|| = ||\mathcal{T}_\Phi||$, where $||\mathcal{T}_\Phi|| := ||\mathcal{T}_\phi||$, i.e.,

$$\begin{aligned}
||\Phi|| &= \sup_{x\in K} ||\phi(x)||_{\mathscr{L}(Z)} \\
&= \sup_{x\in K} \sup \{||\phi(x)s(x)||_Z : \ s \in C(K, Z), \ ||s(x)||_Z \le 1\} \\
&= \sup_{x\in K} \sup \left\{||\phi(\varphi^{-1}(x))s(\varphi^{-1}(x))||_Z : \ s \in C(K, Z), ||s(\varphi^{-1}(x))||_Z \le 1\right\} \\
&= \sup_{x\in K} \sup \left\{||\mathcal{T}_\phi s(x)||_Z : \ s \in C(K, Z), \ ||s(\varphi^{-1}(x))||_Z \le 1\right\} \\
&= \sup \left\{\sup_{x\in K} ||\mathcal{T}_\phi s(x)||_Z : \ s \in C(K, Z), \ ||s|| \le 1\right\} \\
&= \sup \{||\mathcal{T}_\phi s|| : \ s \in C(K, Z), \ ||s|| \le 1\} \\
&= ||\mathcal{T}_\phi||.
\end{aligned}$$

By the cocycle property, we see that $\Phi^n(x, v) = (\varphi^n(x), \phi^n(x)v)$ for all $(x, v) \in K \times Z = E$, and $n \in \mathbb{N}_0$. As such, the positive cocycle induces certain $\mathbb{N}_0$-dynamics $\{\Phi^n : E \longrightarrow E | \ n \in \mathbb{N}_0\}$ on the trivial Banach lattice bundle $E = K \times Z$ over the $\mathbb{Z}$-dynamics $\{\varphi^n : K \longrightarrow K | \ n \in \mathbb{Z}\}$. And such positive dynamics is what we later call positive $\mathbb{N}_0$-dynamical Banach lattice bundle on $E = K \times Z$ over that of $\mathbb{Z}$-dynamical on K. And in particular, we call $(\Phi^n)_{n\in\mathbb{N}_0}$ a positive semiflow on $E = K \times Z$ over the flow $(\varphi^n)_{n\in\mathbb{Z}}$ on K. By this, we mean that

(i) $\varphi^n \circ p = p \circ \Phi^n$ for each $n \in \mathbb{N}_0$ i.e., the following diagram commutes,

$$\begin{array}{ccc}
E = K \times Z & \xrightarrow{\Phi^n} & K \times Z = E \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
K & \xrightarrow[\varphi^n]{} & K
\end{array}$$

(ii) $\Phi^n|_Z := \phi^n(x) : Z \longrightarrow Z$ are positive operators for $x \in K$ and each $n \in \mathbb{N}_0$, and

(iii) $||\Phi^n|| := \sup_{x\in K} ||\phi^n(x)||_{\mathscr{L}(Z)} < \infty$ for each $n \in \mathbb{N}_0$.

On the side of the trivial Banach lattice bundle, we have seen certain positive dynamics from above. On the other side, i.e., on the Banach lattice module $\Gamma(K, E) \cong C(K, Z)$ over $C(K)$, one also obtain the corresponding positive weighted dynamics as follows.

We note that $\mathcal{T}^n_\Phi = \mathcal{T}_{\Phi^n}$, since $\mathcal{T}^n_\Phi \tilde{s} = \Phi^n \circ \tilde{s} \circ \varphi^{-n} = \mathcal{T}_{\Phi^n}\tilde{s}$ for $s \in C(K, Z)$ and each $n \in \mathbb{N}_0$. More so, $\mathcal{T}^n_\Phi f\tilde{s} = T^n_\varphi f \mathcal{T}^n_\Phi \tilde{s}$ for $f \in C(K), s \in C(K, Z)$ where $T^n_\varphi f = f \circ \varphi^{-n} = T_{\varphi^n} f$ for $f \in C(K)$ and each $n \in \mathbb{N}_0$. As such, an $\mathbb{N}_0$-dynamics $\left\{\mathcal{T}^n_\Phi : \Gamma(K, E) \longrightarrow \Gamma(K, E) |\ n \in \mathbb{N}_0\right\}$ is induced by the positive weighted Koopman operator $\mathcal{T}_\Phi$ on the Banach lattice module $\Gamma(K, E)$ over the $\mathbb{Z}$-dynamics $\left\{T^n_\varphi : C(K) \longrightarrow C(K) |\ n \in \mathbb{Z}\right\}$. And such positive dynamics give rise to what we later call positive $\mathbb{N}_0$-dynamical Banach lattice module on $\Gamma(K, E)$ over that of $\mathbb{Z}$-dynamical on $C(K)$. And in particular, we call $(\mathcal{T}^n_\Phi)_{n\in\mathbb{N}_0}$ a positive weighted Koopman semigroup on $\Gamma(K, E)$ over the Koopman group $(T^n_\varphi)_{n\in\mathbb{Z}}$ on $C(K)$. And by this, we mean that

(i) $\mathcal{T}^n_\Phi : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ is a positive operator for each $n \in \mathbb{N}_0$, and

(ii) $\mathcal{T}^n_\Phi f\tilde{s} = T^n_\varphi f \mathcal{T}^n_\Phi \tilde{s}$ for $f \in C(K), s \in C(K, Z)$ and each $n \in \mathbb{N}_0$.

1.3 Is every positive weighted operator induced by a positive cocycle on the trivial Banach lattice bundle?

What we have seen so far is that positive cocycle and the corresponding linear skew-product on the trivial Banach lattice bundle $E = K \times Z$ induces a certain positive weighted Koopman operator on the Banach lattice module $\Gamma(K, E) \cong C(K, Z)$ over $C(K)$. And what is important is that this assignment provides a functional-analytic toolbox for the study of this positive cocycle. And as such, it may be interesting to ask the following question.

Question 1.3.0.1. *Is every positive weighted operator on $\Gamma(K, E)$ over the Koopman operator induced by a unique positive cocycle?*

In other word, given a homeomorphism $\varphi : K \longrightarrow K$, and the Koopman operator $T_\varphi : C(K) \longrightarrow C(K)$; $f \mapsto f \circ \varphi^{-1}$. If $\mathcal{T} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ is a linear map such that

(i) $\mathcal{T} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ *is a positive operator, and*

(ii) $\mathcal{T} f\tilde{s} = T_\varphi f \mathcal{T}\tilde{s}$ *for* $f \in C(K), s \in C(K, Z)$*,*

then, can we find a unique positive cocycle $\{\phi(x) : Z \longrightarrow Z$ positive operator $| \; x \in K\}$ such that $\mathcal{T} = \mathcal{T}_\Phi$, where $\Phi : E \longrightarrow E$ denoted the corresponding continuous linear skew-product on the trivial Banach lattice bundle $E = K \times Z$?

We give a positive answer to this question as we close this chapter. The idea is that, there is a correspondence between the skew-product and the cocycle, which one could take note of.

To start with, for each $x \in K$, we let $e_x : \Gamma(K, E) \longrightarrow Z$; $\tilde{s} \mapsto s(x)$ for $s \in C(K, Z)$ be the evaluation map, which immediately can be seen to be a quotient lattice homomorphism.

Now, let $\mathcal{T} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ be a positive weighted operator over the Koopman operator $T_\varphi : C(K) \longrightarrow C(K)$, i.e., satisfying the two conditions stated in the above question. Then, we claim that the collection of mappings $\{\phi(x) : Z \longrightarrow Z \mid x \in K\}$ for which the following diagram

$$\begin{tikzcd} \Gamma(K,E) \arrow[r, "\tilde{s}\mapsto\mathcal{T}\tilde{s}"] \arrow[d, "e_x"'] & \Gamma(K,E) \arrow[d, "e_{\varphi(x)}"] \\ Z \arrow[r, dashed, "\phi(x)"'] & Z \end{tikzcd}$$

commutes, for each $x \in K$, are precisely the operators that define the required positive cocycle.

That is, the *linear* operators

$$\phi(x) : Z \longrightarrow Z;\ e_x(\tilde{s}) \mapsto e_{\varphi(x)}(\mathcal{T}\tilde{s}) \quad x \in K, \text{ for all } s \in C(K, Z)$$

is the *continuous* positive cocycle over the homeomorphism $\varphi : K \longrightarrow K$ such that $e_x(\mathcal{T}\tilde{s}) = \phi(\varphi^{-1}(x))s(\varphi^{-1}(x))$; and so $\mathcal{T}\tilde{s} = \Phi \circ \tilde{s} \circ \varphi^{-1} = \mathcal{T}_\Phi \tilde{s}$ where

$$\Phi : E \longrightarrow E;\ (x, s(x)) \mapsto (\varphi(x), \phi(x)s(x)) \quad \text{for all } s \in C(K, Z)$$

denote the continuous linear skew-product on $E = K \times Z$ associated with the positive cocycle. We sketch the following outline from which the claim immediately follows:

First we note that, by definition, $\phi(x)s(x) = e_{\varphi(x)}(\mathcal{T}\tilde{s})$ implies that $\phi(\varphi^{-1}(x))s(\varphi^{-1}(x)) = e_x(\mathcal{T}\tilde{s})$ for each $s \in C(K, Z)$ and all $x \in K$.

(i) $\phi(x)$ is linear for each $x \in K$.

(ii) $\phi(x)$ is positive for each $x \in K$.

(iii) $K \times Z \longrightarrow Z; (x, s(x)) \mapsto \phi(x)s(x)$ is continuous for all $s \in C(K, Z)$.

(iv) $\{\phi(x) : Z \longrightarrow Z \mid x \in K\}$ satisfy the cocycle rule, i.e., $\phi^0(x) = Id_Z$ and $\phi^{n+m}(x) = \phi^n(\varphi^m(x))\phi^m(x)$ for $x \in K$ and $n, m \in \mathbb{N}$.

Chapter 2

On Banach lattice modules over a Banach lattice algebra

2.1 Introduction

In this chapter, we introduce a mathematical object, namely a *Banach lattice module* over a commutative Banach lattice algebra. In particular, our definition of a Banach lattice module can be seen to be a generalisation of a certain Banach lattice $L^\infty(\mathcal{G})$-module as defined by K-T. Eisele and S. Taieb (see [9, Definition 5.3(iii), p.531-532]). They consider this in their context of representation theorems for conditional and multi-period risk measures in stochastic analysis. Moreover, this chapter consists of our contributions to the notion of an *m-Banach lattice module* over a commutative *m-Banach lattice algebra*.

In Section 2.2, we propose a definition for a Banach lattice module (see Definition 2.2.1.1) and identify some examples of these mathematical objects (see Example 2.2.1.4). Moreover, we introduce a special class of Banach lattice algebras, namely *m-Banach lattice algebras* (see Definition 2.2.2.1) and some examples (see Remark 2.2.2.2(i)). Consequently, we also introduce a special class of Banach lattice modules over a commutative m-Banach lattice algebra, namely *m-Banach lattice modules* (see Definition 2.2.2.4) and identify some examples (see Example 2.2.2.6).

In Section 2.3, we introduce the concept of a (positive) weighted semigroup representation on a Banach lattice module (BLM) over a commutative Banach lattice algebra (BLA). In doing so and by answering question 2.3.0.1, we, in particular, establish the category of Banach lattice modules and their dynamics.

In Section 2.4, inspired by the work of H. Kreidler and S. Siewert (see [21, Section 4, p.15 and Section 5, p.20]) about AM-modules and AL-modules over the Banach algebra $C_0(\Omega)$, we introduce, in a natural way, the analogous concepts of an *AM m-lattice module* (see Definition 2.4.1.2) and of an *AL m-lattice module* (see Definition 2.4.2.2), respectively, over the m-Banach lattice algebra $C_0(\Omega)$.

It is known that, for a *topological* Banach bundle E over a locally compact space Ω, the Banach space $\Gamma_0(\Omega, E)$ of its continuous sections vanishing at infinity is an AM-module (equivalently $U_0(\Omega)$-normed module) over the Banach algebra $C_0(\Omega)$ (see [21, Example 4.4, p.16, and Example 5.2, p.21]). We obtain, in our situation, that for a *topological* Banach lattice bundle E over a locally compact space Ω, the Banach lattice $\Gamma_0(\Omega, E)$ of its continuous sections vanishing at infinity is an AM m-lattice module (equivalently $U_0(\Omega)$-normed m-lattice module) over the Banach lattice algebra $C_0(\Omega)$ (see Examples 2.4.1.4 and 2.4.1.11).

Analogously, it is known that, for a *measurable* Banach bundle E over a complete σ-finite positive measure space X, the (quotient) Banach space $\Gamma^1(X, E)$ of its integrable measurable sections is an AL-module (equivalently $L^\infty(X)'$-normed module) over the Banach algebra $L^\infty(X)$ (see [21, Example 4.14, p.20, and second paragraph on p.24]) and in particular $\Gamma^1(X, E)$ is an $L^1(X)$-normed module (see [21, Definition 5.11, p.24]). We obtain, in our situation, that for a *measurable* Banach lattice bundle E over a complete σ-finite positive measure space X, the (quotient) Banach lattice $\Gamma^1(X, E)$ of its integrable measurable sections is an AL m-lattice module (equivalently $L^\infty(X)'$-normed m-lattice module) over the Banach lattice algebra $L^\infty(X)$ (see Examples 2.4.2.5 and 2.4.2.12) and in particular $\Gamma^1(X, E)$ is an $L^1(X)$-normed m-lattice module (see Definition 2.4.2.15).

In the last Section 2.5, we further our consideration on Banach lattice modules. We start with a commutative Banach lattice algebra L, and introduce the notions of a *sub-lattice algebra* (see Definition 2.5.1.1) and a *lattice algebra ideal* (see Definition 2.5.1.2) in a natural way. Moreover, for a Banach lattice module Γ over the commutative Banach lattice algebra L, we introduce the notions of *N-lattice submodule* (see Definition 2.5.2.1) and *N-ideal submodule* (see Definition 2.5.2.3) over a subspace $N \subseteq L$. In Propositions 2.5.1.4 and 2.5.2.5 we note special instances of these concepts, which are also very important for our study. In Subsection 2.5.3, we identify certain quotient spaces of a Banach lattice module Γ over a commutative Banach lattice algebra L, which give rise to special (quotient) Banach lattice modules over (quotient) lattice algebras (see Corollary 2.5.3.2). We study, in Subsection 2.5.4, the Banach dual Γ' of a Banach lattice module Γ over $L = C_0(\Omega)$, and obtain that the Banach dual Γ' is, in particular, *an order complete Banach lattice module* over $C_0(\Omega)$ which we call *the dual Banach lattice module* (see Proposition 2.5.4.1). By this, we state certain duality results (see Proposition 2.5.4.3) between AM- and AL- lattice modules over $C_0(\Omega)$ similar to those of (general) Banach modules ([21, Proposition 4.17, p.21]). Other concepts considered include: *Banach lattice modules and lattice isomorphisms* (see Subsection 2.5.5) and *Banach lattice modules and lattice ideals* (see Subsection 2.5.6).

2.2 Banach lattice modules

Definition 2.2.0.1. *(Banach module) [29, Definition 2.1, p.23] Let A be a commutative Banach algebra. Then a Banach module over A is a Banach space Γ which is also an A-module such that the outer norm is sub-multiplicative, i.e., $||f \cdot s|| \leq ||f||||s||$ for all $f \in A$ and $s \in \Gamma$.*

On the other hand, we refer to Appendix A (see e.g., Definition A.1.0.3 for the definition of a Banach lattice algebra) for the concept of a Banach lattice algebra (BLA), including examples. Moreover, we consider commutative Banach lattice algebras.

Next, we need to clarify what a Banach lattice module is.

2.2.1 What is a Banach lattice module?

Since there does not seem to be a concise concept of a Banach lattice being a Banach module over a commutative Banach lattice algebra (BLA), we propose the following definition. Moreover, our definition can be seen to be a generalisation of a certain Banach lattice $L^\infty(\mathcal{G})$-modules as defined by K-T. Eisele and S. Taieb ([9]) viz: Let $\mathcal{G}$ be a sub-σ-algebra of some probability space. An ordered $L^\infty(\mathcal{G})$-module is a (Banach) $L^\infty(\mathcal{G})$-module Γ with a partial order ($\leq$) such that for $s, s_2, s_2 \in \Gamma$ and $f \in L^\infty(\mathcal{G})$ (i) $s_1 \leq s_2$ implies $s_1 + s \leq s_2 + s$; and (ii) $0 \leq s$, $0 \leq f$ implies $0 \leq f \cdot s$. A normed lattice $L^\infty(\mathcal{G})$-module is an ordered $L^\infty(\mathcal{G})$-module Γ whose partial order is, in particular, a lattice ordering and equipped with a lattice norm. Finally, a normed lattice $L^\infty(\mathcal{G})$-module Γ which is (topologically) complete is called a *Banach lattice $L^\infty(\mathcal{G})$-module* (see [9, Definition 5.3(iii), p.531-532]).

Definition 2.2.1.1. *Let L be a commutative BLA. We call a Banach lattice Γ which is also an L-module a Banach lattice module (BLM) if the outer norm is sub-multiplicative and the "module" product of positive elements is positive. Moreover, in the complex case, we require that the "module" product of real elements is real.*

In other words, a BLM is a Banach lattice Γ which is also a Banach module over L, such that $L_+ \cdot \Gamma_+ \subseteq \Gamma_+$ and $L_\mathbb{R} \cdot \Gamma_\mathbb{R} \subseteq \Gamma_\mathbb{R}$, i.e., $f \in L_+$, and $s \in \Gamma_+$ always implies $f \cdot s \in \Gamma_+$. Moreover, in the complex case, $f \in L_\mathbb{R}$, and $s \in \Gamma_\mathbb{R}$ always implies $f \cdot s \in \Gamma_\mathbb{R}$.

Note 2.2.1.2. *We can also deduce that if Γ is a Banach lattice module over the BLA L, then we have that $|f \cdot s| \leq |f| \cdot |s|$ for all $f \in L$ and $s \in \Gamma$. This can also be taken as a characterisation of a Banach lattice module by the following proposition, since the other implication would be trivial. That is, if L is a commutative BLA, then a Banach lattice Γ which is also a Banach module over L is a "Banach lattice module" if and only if $|f \cdot s| \leq |f| \cdot |s|$ for all $f \in L$ and $s \in \Gamma$.*

Proposition 2.2.1.3. *Let Γ be a Banach lattice module over a commutative Banach lattice algebra L. Then $|f \cdot s| \leq |f| \cdot |s|$ for all $f \in L$ and $s \in \Gamma$.*

Proof. WLOG, we will assume both Γ and L are real Banach lattices. It is, therefore, sufficient to show that $\pm(f \cdot s) \leq |f| \cdot |s|$ for all $f \in L$ and $s \in \Gamma$. Now, for every $f \in L$ and $s \in \Gamma$, we note first that

$$f \cdot s = f^+ \cdot s^+ - f^+ \cdot s^- + f^- \cdot s^- - f^- \cdot s^+, \text{ and}$$

$$|f| \cdot |s| = f^+ \cdot s^+ + f^+ \cdot s^- + f^- \cdot s^- + f^- \cdot s^+.$$

Therefore,

$$|f| \cdot |s| + f \cdot s = 2(f^+ \cdot s^+ + f^- \cdot s^-) \geq 0, \text{ and}$$
$$|f| \cdot |s| - f \cdot s = 2(f^+ \cdot s^- + f^- \cdot s^+) \geq 0.$$

□

We identify the following examples of Banach lattice modules.

Example 2.2.1.4. *(i) Let L be a commutative Banach lattice algebra. Then, the (bilinear) algebra paring $L \times L \longrightarrow L; (f,g) \mapsto f \cdot g := f \star g$ turns L into a Banach lattice module over itself. Indeed, for all $f, g_1, g_2 \in L$;*

(a) $f \cdot (g_1 + g_2) = f \star (g_1 + g_2) = f \star g_1 + f \star g_2 = f \cdot g_1 + f \cdot g_2$

(b) $(g_1 + g_2) \cdot f = (g_1 + g_2) \star f = g_1 \star f + g_2 \star f = g_1 \cdot f + g_2 \cdot f$

(c) $(g_1 \star g_2) \cdot f = (g_1 \star g_2) \star f = g_1 \star (g_2 \star f) = g_1 \cdot (g_2 \cdot f)$

(d) $||g_1 \cdot g_2|| = ||g_1 \star g_2|| \leq ||g_1|| ||g_2||$

imply that L is a Banach module over itself. In addition,

(e) $|g_1 \cdot g_2| = |g_1 \star g_2| \leq |g_1| \star |g_2| = |g_1| \cdot |g_2|$

implies the "module" product of positive elements is again positive. Hence L is a Banach lattice module over itself.

We note that, in this case, the (closed) ideal submodules of L (see Definition 2.5.2.3) correspond to (closed) lattice algebra ideals of L, as introduced in Definition 2.5.1.2.

(ii) As in Appendix A [Example A.1.0.4(iii)], let Γ be a Banach lattice, and $L := \mathcal{Z}(\Gamma)$ its center. Then, the (bilinear) paring $L \times \Gamma \longrightarrow \Gamma; (T,s) \mapsto T \cdot s := Ts$ turns Γ into a Banach lattice module over L.

We note that, in this case, the (closed) ideal submodules of Γ (see Definition 2.5.2.3) correspond to (closed) lattice ideals of Γ.

(iii) As in Appendix A [Example A.1.0.4(vi)], let Γ be an (order complete) Banach lattice, and L a uniformly bounded closed lattice algebra of (commuting) regular operators on Γ. Then, the (bilinear) paring $L \times \Gamma \longrightarrow \Gamma; (T,s) \mapsto T \cdot s := Ts$ turns Γ into a Banach lattice module over L.

We note that, in this case, the (closed) ideal submodules of Γ (see Definition 2.5.2.3) correspond to (closed) lattice ideals of Γ which are L-invariant.

2.2.2 m-Banach lattice modules over m-Banach lattice algebras

Here, we first introduce a special class of Banach lattice algebras in which the lattice structure is multiplicative, namely *m-Banach lattice algebras* (see Definition 2.2.2.1). Consequently, we also introduce a special class of Banach lattice modules over commutative m-Banach lattice algebra in which the module structure is "lattice multiplicative", namely *m-Banach lattice modules* (see Definition 2.2.2.4).

2.2.2.1 m-Banach lattice algebras

Here we introduce a special class of Banach lattice algebras in which the lattice structure is multiplicative.

Definition 2.2.2.1. *We call a Banach lattice L with an associative algebraic structure, multiplication $\star$ and lattice modulus $|\cdot|$, an m-Banach lattice algebra (m-BLA) if the following hold for all $f, g \in L$;*

(i) $|f \star g| = |f| \star |g|$ *and* $\overline{f \star g} = \overline{f} \star \overline{g}$, *and*

(ii) $||f \star g|| \leq ||f||||g||$.

It is clear from our definition that an m-Banach lattice algebra (m-BLA) is also a Banach lattice algebra (BLA) (see Appendix A, Note A.1.0.2 and Definition A.1.0.3) . The second property in (i) above is essentially required if L is a complex Banach lattice which, in particular, implies that the multiplication of real elements is again a real element.

Remark 2.2.2.2.

(i) Typical examples of commutative m-BLAs are Examples A.1.0.4 (i), (ii) and (iii) in Appendix A.

(ii) Moreover, by the classical theorems of Gelfand or Kakutani; it follows that if a BLA is either a $*$-algebra with unit or an AM-space with unit, then it must be of the form $C(Q)$ for some compact space Q.

(iii) There are interesting properties of our m-BLA which follow by definition. For instance, if L is m-BLA, then for all $f \in L_+$ and $g, g_1, g_2 \in L_{\mathbb{R}}$ we have that

(a) $f \star (g_1 \vee g_2) = (f \star g_1) \vee (f \star g_2)$,

(b) $f \star (g_1 \wedge g_2) = (f \star g_1) \wedge (f \star g_2)$,

(c) $f \star g^+ = (f \star g)^+$, *and*

(d) $f \star g^- = (f \star g)^-$.

Moreover, we have that $||f \star |g|\,|| = ||f \star g|| = ||f \star \overline{g}||$ *for all* $f, g \in L$.

For this class of Banach lattice algebras we can answer the following question raised by Wickstead ([31]).

Question 2.2.2.3. *[31, Question 2.3 , p.807] If a Banach lattice algebra has an approximate identity* (e_i) *with all* $||e_i|| \leq 1$, *must it have an approximate identity composed of positive elements?*

If L is a (commutative) m-BLA, by setting setting $b_i := |e_i|$, then (b_i) is an approximate identity for L_+ composed of positive elements, since for each $f \in L$,

$$\lim_i e_i \star f = f = \lim_i f \star e_i$$

implies

$$\lim_i b_i \star |f| = |f| = \lim_i |f| \star b_i.$$

2.2.2.2 m-Banach lattice module

Next, we introduce a special class of Banach lattice modules over a commutative m-BLA in which the module structure is "lattice multiplicative".

Definition 2.2.2.4. *Let L be a commutative m-BLA. A Banach lattice* Γ *which is also an L-module is called an m-Banach lattice module (m-BLM) over L if the following hold for all* $f \in L$ *and* $s \in \Gamma$*:*

(i) $|f \cdot s| = |f| \cdot |s|$ *and* $\overline{f \cdot s} = \overline{f} \cdot \overline{s}$; *and*

(ii) $||f \cdot s|| \leq ||f|| ||s||$.

Remark 2.2.2.5.

(i) It is clear from our definition that an m-Banach lattice module is also a Banach lattice module (see Definition 2.2.1.1 and Note 2.2.1.2). The second property in (i) in the above definition is essentially required if either L or Γ *is a complex Banach lattice which, in particular, implies that the "module" product of real elements is again a real element.*

(ii) There are interesting properties of our m-Banach lattice module Γ over the m-BLA L which are immediate from the definition. For instance, for any $f \in L$ and $s, s_1, s_2 \in \Gamma$:

(a) $|f \cdot s| = |f| \cdot s$ *if* $s \in \Gamma_+$,

(b) $|f \cdot s| = f \cdot |s|$ *if* $f \in L_+$.

So, in the real case, we even have:

(c) $f \cdot (s_1 \vee s_2) = (f \cdot s_1) \vee (f \cdot s_2)$ *if* $f \in L_+$,

(d) $f \cdot (s_1 \wedge s_2) = (f \cdot s_1) \wedge (f \cdot s_2)$ *if* $f \in L_+$,

(e) $(f \cdot s)^+ = f \cdot s^+$ *if* $f \in L_+$,

(f) $(f \cdot s)^+ = f^+ \cdot s$ *if* $s \in \Gamma_+$.

(iii) Another immediate, and very important, observation about our m-Banach lattice module is that, for any $f \in L$, and $s \in \Gamma$, we have

$$||f \cdot |s|\,|| = ||f \cdot s|| = ||f \cdot \bar{s}||.$$

The following examples of m-Banach lattice modules serve as motivation for our definition.

Example 2.2.2.6. *(i) Let L be a commutative m-Banach lattice algebra (see Definition 2.2.2.1). Then, L is an m-Banach lattice module over itself. Indeed, as in Example 2.2.1.4(i), the (bilinear) algebra paring $L \times L \longrightarrow L; (f, g) \mapsto f \cdot g := f \star g$ turns L into Banach lattice module over itself. Moreover, by definition, for any $f, g \in L$,*

(a) $|f \cdot g| = |f \star g| = |f| \star |g| = |f| \cdot |g|$*; and*

(b) $\overline{f \cdot g} = \overline{f \star g} = \overline{f} \star \overline{g} = \overline{f} \cdot \overline{g}$

imply that L is an m-Banach lattice module over itself.

(ii) Let Γ be any Banach lattice, and $L := \mathcal{Z}(\Gamma)$ its center (see Example 2.2.1.4(ii)). Then, Γ is an m-Banach lattice module over L. That is, in fact, every Banach lattice is an m-Banach lattice module over its center.

(iii) Let E be a topological Banach lattice bundle over a locally compact space Ω, then the lattice of its continuous sections $\Gamma_0(\Omega, E)$ vanishing at infinity is an m-Banach lattice module over $C_0(\Omega)$ (see Chapter 3, Remark 3.5.0.1(i)).

(iv) Let E be a measurable Banach lattice bundle over a measure space X, then the (quotient) lattice of its integrable measurable sections $\Gamma^1(X, E)$ is an m-Banach lattice module over $L^\infty(X)$ (see Appendix C, Remark C.5.0.1(i)) .

2.2.3 Banach lattice modules with property (m_+)

As a result of the above observation in Remark 2.2.2.5(iii), and to give room for a wide range of the so-called Banach lattice modules, we introduce the following class which generalises our m-Banach lattice modules.

Definition 2.2.3.1. *Let L be a commutative m-Banach lattice algebra (m-BLA). Then we say a Banach lattice module Γ over L "satisfies property (m_+)" if the following holds for all $f \in L_+$, and $s \in \Gamma$,*

$$||f \cdot |s||| = ||f \cdot s|| = ||f \cdot \bar{s}||.$$

2.2.4 Non-degenerate Banach lattice modules

Definition 2.2.4.1. *(Non-degenerate Banach lattice module) Let L be a commutative Banach lattice algebra. A Banach lattice module Γ over L is called non-degenerate if $\Gamma = \overline{lin}\{fs : f \in L, s \in \Gamma\}$[1], in other words, the submodule $lin\{fs : f \in L, s \in \Gamma\}$ is norm-dense in Γ.*

In our study, we assume that any Banach lattice module Γ over a commutative Banach lattice algebra L is non-degenerate. Furthermore, in the context of the work of W. Paravicini ([25]), this type of Banach lattice module would be called a $C_0(\Omega)$-Banach lattice in the particular case when $L = C_0(\Omega)$ for a locally compact space Ω.

We also note that if either L is, in addition, a C^*- algebra (over $\mathbb{C}$) or considering its self-adjoint part (over $\mathbb{R}$), then Γ is non-degenerate if and only if $\lim_i e_i s = s$ for every $s \in \Gamma$, where (e_i) represents an approximate unit (or identity) of L. (cf. [29, note after Definition 2.1, p.23-24]).

In the case where L is an m-Banach lattice algebra with approximate unit (e_i), since we obtain a positive approximate unit (b_i) of L_+ by setting $b_i := |e_i|$ for each i (see Question 2.2.2.3) non-degeneracy of an m-Banach lattice module Γ over L also implies Γ_+ is non-degenerate over L_+. That is, $\Gamma_+ = \overline{\text{lin}}\{fs : f \in L_+, s \in \Gamma_+\}$ which is the case if and only if $\lim_i b_i s = s$ for every $s \in \Gamma_+$. Indeed, for any $s \in \Gamma$;

$$\lim_i e_i \cdot s = s$$

[1] $\overline{\text{lin}}\{W\}$ denotes the norm-closure of the linear span $\text{lin}\{W\}$ of the set of vectors W.

implies

$$\lim_i b_i \cdot |s| = |s|.$$

Remark 2.2.4.2. *In the case where L is a 1-Banach lattice algebra (i.e., there exists an identity $e \in L$ with $||e|| = 1$) and $e \cdot s = s$ for all $s \in \Gamma$, we call Banach lattice module Γ over L "unitary".*

Immediate examples of unitary Banach lattice modules are:

(i) all 1-Banach lattice algebras (see Example 2.2.1.4(i));

(ii) Examples 2.2.1.4(ii) and (iii); and

(iii) Example 2.2.2.6(ii) if $\Omega = K$ is compact.

Moreover, a unitary Banach lattice module is necesarilly non-degenerate. So, one can assume WLOG that every Banach lattice module is non-degenerate, as we have done.

Throughout our study, by an m-BLM over L, we always mean an m-Banach lattice module over a commutative m-BLA L.

Furthermore, as in above, we will use juxtaposition for the algebra multiplication $\star$ in L and the "module" product $\cdot$ in Γ if no confusion arises.

2.3 Positive weighted semigroup representations on Banach lattice modules

In this Section, we introduce the concept of a (positive) weighted semigroup representation on a Banach lattice module (BLM) over a commutative Banach lattice algebra (BLA). Classical examples are the motivation for this abstract construction. See [29, Definition 2.12, p.28] for the case of Banach modules over commutative Banach algebras which inspired this work.

We note that, in this Section, we establish the category of Banach lattice modules and their dynamics.

More important, we seek to answer, in this Section, the following questions in a systematic manner.

We will see that each notion describes a certain dynamical system on the underlying space.

Question 2.3.0.1. *(i) What are morphisms and isomorphisms of BLAs?*

(ii) What are (positive) morphisms, isomorphisms and (positive) weighted morphisms of BLMs over a fixed commutative BLA?

(iii) What is a group representation on a commutative BLA?

(iv) What is a positive weighted semigroup representation on a BLM?

(v) What are homomorphisms and isomorphisms between positive weighted semigroup representations on BLMs?

2.3.1 Morphisms and isomorphisms of BLAs

First, we introduce morphisms and isomorphisms of BLAs (Banach lattice algebras), most naturally, by combining the notions of algebra homomorphisms and lattice homomorphisms.

Definition 2.3.1.1. *Let L and M be BLAs. A linear map $T : L \longrightarrow M$ is called a morphism if the following hold for all $f, g \in L$:*

(i) $Tfg = Tf \cdot Tg$, and

(ii) $|Tf| = T|f|$.

Since by (ii), T is a lattice homomorphism, it is necessarily bounded; and hence $T \in \mathscr{L}(L, M)$ is also an algebra homomorphism.

A morphism $T : L \longrightarrow M$ of BLAs is an isomorphism if T is bijective; hence $T^{-1} : M \longrightarrow L$ is also a morphism; i.e., $T^{-1}pq = T^{-1}p \cdot T^{-1}q$, and $|T^{-1}p| = T^{-1}|p|$ for all $p, q \in M$. The group of self-isomorphisms (automorphisms) $T : L \longrightarrow L$ of a BLA L will be denoted by $Aut(L)$.

Remark 2.3.1.2. *It should be noted that a BLA is simultaneously a Banach algebra and a Banach lattice; and so in the category of BLAs, morphisms are linear maps which are simultaneously algebra homomorphisms and lattice homomorphisms.*

A motivation for this definition arises from the following examples.

Example 2.3.1.3. *(i) Let $\varphi : \Omega \longrightarrow \Omega$ be a homeomorphism of a locally compact space Ω. Then the Koopman operator $T_\varphi : C_0(\Omega) \longrightarrow C_0(\Omega)$; $f \mapsto f \circ \varphi^{-1}$ is a self-isomorphism of the BLA $C_0(\Omega)$, and hence an automorphism.*

(ii) Let $\varphi : X \longrightarrow X$ be an automorphism of the measure space X. Then the Koopman operator $T_\varphi : L^\infty(X) \longrightarrow L^\infty(X)$; $f \mapsto f \circ \varphi^{-1}$ is a self-isomorphism of the BLA $L^\infty(X)$, and hence an automorphism.

2.3.2 (Positive) Morphisms, isomorphisms and (positive) weighted morphisms of BLMs

Next, we introduce the concept of a (positive) morphism between Banach lattice modules (BLMs) over a fixed BLA L. This we do naturally by combining the notions of Banach module homomorphisms (see [29, paragraph after Definition 2.1, p.23]) over a commutative Banach algebra with that of positivity and lattice homomorphisms in the Banach lattice setting.
Moreover, for important examples of a Banach lattice module Γ over a commutative Banach lattice algebra L (see Examples 2.2.1.4 and 2.2.2.6).

Definition 2.3.2.1. *Let Γ and Λ be BLMs over a commutative BLA L. A linear map $\mathcal{T} : \Gamma \longrightarrow \Lambda$ is called*

(i) a positive module homomorphism (i.e., positive morphism) if:

(a) $\mathcal{T}$ is positive, i.e., $|\mathcal{T}s| \leq \mathcal{T}|s|$; and

(b) $\mathcal{T}fs = f \cdot \mathcal{T}s$

for all $f \in L$ and $s \in \Gamma$.

(ii) a lattice module homomorphism (i.e., morphism) if:

(a) $\mathcal{T}$ *is a lattice homomorphism, i.e.,* $|\mathcal{T}s| = \mathcal{T}|s|$*; and*

(b) $\mathcal{T}fs = f \cdot \mathcal{T}s$

for all $f \in L$ *and* $s \in \Gamma$.

A (positive) morphism $\mathcal{T} : \Gamma \longrightarrow \Lambda$ *is called a (positive) isometry if* $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$ *is isometric.*

Remark 2.3.2.2. *(i) First, we note that* $\mathcal{T}$ *being positive implies that* $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$*; and hence each positive module homomorphism is also a Banach module homomorphism (see [29, paragraph after Definition 2.1, p.23]). Moreover, we note that every lattice module homomorphism is also a positive one.*

(ii) In the category of BLMs over a (fixed) commutative BLA L*, morphisms are linear operators which are simultaneously module homomorphisms and lattice homomorphisms.*

(iii) Furthermore, two Banach lattice modules Γ *and* Λ *over* L *are said to be isometrically isomorphic, denoted by* $\Gamma \cong \Lambda$*, if there exists a surjective isometric morphism (i.e., a surjective and isometric lattice module homomorphism) between them.*

We are interested in (positive) weighted morphisms, and so we introduce the following definitions. See [29, Definition 2.5, p.25] for the case of Banach modules over a commutative Banach algebra. Moreover, we note that every positive T-homomorphism over a commutative Banach lattice algebra L is, in particular, a T-homomorphism over a commutative Banach algebra L, in this situation.

Definition 2.3.2.3. *Let* L *be a commutative BLA, and* $T : L \longrightarrow L$ *a morphism on* L*. Moreover, let* Γ *and* Λ *be BLMs over* L*. A linear map* $\mathcal{T} : \Gamma \longrightarrow \Lambda$ *is called*

(i) a positive T*-homomorphism (i.e., a positive weighted morphism) if:*

(a) $\mathcal{T}$ *is positive, i.e.,* $|\mathcal{T}s| \leq \mathcal{T}|s|$*; and*

(b) $\mathcal{T}fs = Tf \cdot \mathcal{T}s$

for all $f \in L$ *and* $s \in \Gamma$.

(ii) a lattice T-homomorphism (i.e., a weighted morphism) if:

(a) $\mathcal{T}$ is a lattice homomorphism, i.e $|\mathcal{T}s| = \mathcal{T}|s|$; and

(b) $\mathcal{T}fs = Tf \cdot \mathcal{T}s$

for all $f \in L$ and $s \in \Gamma$.

A positive T-homomorphism $\mathcal{T} : \Gamma \longrightarrow \Lambda$ is called a positive isometry if $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$ is an isometry. Similarly, a lattice T-homomorphism $\mathcal{T} : \Gamma \longrightarrow \Lambda$ is called an isometry if $\mathcal{T} \in \mathscr{L}(\Gamma, \Lambda)$ is an isometry.

The following two examples are the motivation for the above definitions (see Chapter 3, Remark 3.5.0.1(iv) and Appendix C, Remark C.5.0.1(iii)).

Example 2.3.2.4. *(i) Let $\varphi : \Omega \longrightarrow \Omega$ be a homeomorphism of a locally compact space Ω, and*

$$T_\varphi : C_0(\Omega) \longrightarrow C_0(\Omega); f \mapsto f \circ \varphi^{-1}$$

the Koopman operator on $C_0(\Omega)$. Moreover, let E be a topological Banach lattice bundle over Ω (see Chapter 3, Definition 3.2.0.1), and $\Phi : E \longrightarrow E$ a (positive) Banach lattice bundle morphism over φ (see Chapter 3, Definition 3.4.0.1). The Banach lattice $\Gamma_0(\Omega, E)$ of continuous sections of E vanishing at infinity is an m-Banach lattice module over $C_0(\Omega)$ (see Chapter 3, Remark 3.5.0.1(i)); and the induced operator given by

$$\mathcal{T}_\Phi : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, E);\ s \mapsto \Phi \circ s \circ \varphi^{-1}$$

is a (positive) lattice T_φ-homomorphism on $\Gamma_0(\Omega, E)$ over $C_0(\Omega)$. We will call $\mathcal{T}_\Phi$ a (positive) weighted Koopman operator on $\Gamma_0(\Omega, E)$ over T_φ.

(ii) Let $\varphi : X \longrightarrow X$ be an automorphism of a measure space X, and

$$T_\varphi : L^\infty(X) \longrightarrow L^\infty(X); f \mapsto f \circ \varphi^{-1}$$

the Koopman operator on $L^\infty(X)$. Moreover, let E be a measurable Banach lattice bundle over X (see Appendix C, Definition C.2.0.1), and $\Phi : E \longrightarrow E$ a (positive) Banach lattice bundle morphism over φ (see Appendix C Definition C.4.0.2). The (quotient) Banach lattice $\Gamma^1(X, E)$ of integrable measurable sections of E is an m-Banach lattice module over $L^\infty(X)$ (see Appendix C, Remark C.5.0.1(ii)); and the induced operator given by

$$\mathcal{T}_\Phi : \Gamma^1(X, E) \longrightarrow \Gamma^1(X, E);\ s \mapsto \Phi \circ s \circ \varphi^{-1}$$

is a (positive) lattice T_φ-homomorphism on $\Gamma^1(X,E)$ over $L^\infty(X)$. We will call $\mathcal{T}_\Phi$ a (positive) weighted Koopman operator on $\Gamma^1(X,E)$ over T_φ.

2.3.3 Group Representations on Banach lattice algebras

We introduce the concept of a group representation on a commutative Banach lattice algebra L; and give a special type of this notion, the so-called Koopman group representation on L (where $L \in \{C_0(\Omega), L^\infty(X)\}$). See, for instance, [29, second paragraph on p.28] for the case of a dynamical Banach algebra. We note that every group representation on a Banach lattice algebra L is, in particular, a group representation on the Banach algebra L, in this situation.

Note 2.3.3.1. *In the sequel, we will use the following notation.*

(i) We let G be a locally compact group, and S a closed subsemigroup of G containing the neutral element e, i.e., a closed "submonoid" of G. For instance, we can take $G = \mathbb{R}$, $S = \mathbb{R}_{\geq 0}$ or $G = \mathbb{Z}$, $S = \mathbb{N}_0$.

(ii) We let L be a commutative Banach lattice algebra, for instance $C_0(\Omega)$ and $L^\infty(X)$ where Ω is a locally compact space and $X := (\Omega_X, \Sigma_X, \mu_X)$ is a complete positive σ-finite measure space, respectively. For other examples of Banach lattice algebras (see Appendix A, Example A.1.0.4).

(iii) We let $Aut(L)$ be the group of automorphisms on L, i.e., $T \in Aut(L)$ if and only if $T \in \mathscr{L}(L)$ is a bijective morphism of BLAs, and so T^{-1} is also a morphism of BLAs (see Definition 2.3.1.1).

Definition 2.3.3.2. *(G-dynamical Banach lattice algebra)*
A G-dynamical BLA is a pair $(L, \mathbf{T})$ where

$$\mathbf{T} : G \longrightarrow Aut(L),\ g \mapsto T_g$$

defines a strongly continuous[2] group action on the Banach lattice algebra L. Such a pair $(L, \mathbf{T})$ will be called a group representation on L, and $\mathbf{T} = (T_g)_{g\in G}$ a flow on L.

[2] i.e., $G \longrightarrow L$; $g \mapsto T_g f$ is continuous for each $f \in L$.

The following two examples motivate this definition, which naturally extends situations in Example 2.3.1.3 to group actions. These are well-known and always our starting points for topological and measurable dynamics, respectively.

Example 2.3.3.3. *(i) Let $(\Omega, (\varphi_g)_{g\in G})$ be a topological G-dynamical system over a locally compact space Ω (i.e., $(\varphi_g)_{g\in G}$ is a continuous flow on Ω [3]), then the group representation given by*

$$\mathbf{T}_\varphi : G \longrightarrow Aut(C_0(\Omega)),\ g \mapsto T_\varphi(g) := \left[f \mapsto f \circ \varphi_{g^{-1}}\right]$$

is a strongly continuous group action on $C_0(\Omega)$. Hence $(C_0(\Omega), \mathbf{T}_\varphi)$ is a G-dynamical system of an m-Banach lattice algebra, and we call $\mathbf{T}_\varphi = (T_\varphi(g))_{g\in G}$ the Koopman group representation on $C_0(\Omega)$.

(ii) Let $(X, (\varphi_g)_{g\in G})$ be a G-dynamical measure-preserving system over a measure space X (i.e., $(\varphi_g)_{g\in G}$ is a measurable flow on X [4]) and G is discrete; then the group representation given by

$$\mathbf{T}_\varphi : G \longrightarrow Aut(L^\infty(X)),\ g \mapsto T_\varphi(g) := \left[f \mapsto f \circ \varphi_{g^{-1}}\right]$$

is a strongly continuous group action on $L^\infty(X)$. Hence $(L^\infty(X), \mathbf{T}_\varphi)$ is a G-dynamical system of an m-Banach lattice algebra, and we call $\mathbf{T}_\varphi = (T_\varphi(g))_{g\in G}$ the Koopman group representation on $L^\infty(X)$.

2.3.4 Positive Weighted Semigroup Representations on Banach lattice modules

We now introduce the concept of a positive weighted semigroup representation on a Banach lattice module Γ over a fixed group representation $\mathbf{T} = (T_g)_{g\in G}$ on L. See [29, Definition 2.12, p.28] for the case of a weighted semigroup representation on a Banach module. We note that every positive S-dynamical BLM over $(L, \mathbf{T})$ is, in particular, an S-dynamical Banach module over $(L, \mathbf{T})$ in this situation.

[3] i.e., $\varphi : G \longrightarrow Aut(\Omega)$; $g \mapsto \varphi_g$ is a *continuous* group homomorphism, where $Aut(\Omega)$ denotes the group of automorphisms (homeomorphisms) on Ω.

[4] i.e., $\varphi : G \longrightarrow Aut(X)$; $g \mapsto \varphi_g$ is a group homomorphism, where $Aut(X)$ denotes the group of automorphisms on X (see also Appendix C, Definition C.4.0.1 and Note C.4.0.4)

In what follows, for a commutative BLA L, we fix a group representation $(L, \mathbf{T})$ on L, i.e., $\mathbf{T} = (T_g)_{g \in G}$ is a flow on L (see Definition 2.3.3.2).

Definition 2.3.4.1. *(Positive S-dynamical Banach lattice module)*
A positive S-dynamical BLM over $(L, \mathbf{T})$ is a pair $(\Gamma, \boldsymbol{\mathcal{T}})$ consisting of a Banach lattice module Γ over L and a monoid representation[5]

$$\boldsymbol{\mathcal{T}} : S \longrightarrow \Gamma^{\Gamma};\ g \mapsto \mathcal{T}(g)$$

such that

(i) $\mathcal{T}(g) : \Gamma \longrightarrow \Gamma$ *is a positive T_g-homomorphism for each $g \in S$; and*

(ii) $\boldsymbol{\mathcal{T}}$ *is strongly continuous, i.e.,*

$$S \longrightarrow \Gamma,\ g \mapsto \mathcal{T}(g)s$$

is continuous for every $s \in \Gamma$.

We call $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g \in S}$ a positive weighted semigroup representation on Γ over $(L, \mathbf{T})$ (or simply over $\mathbf{T} = (T_g)_{g \in G}$ on L).
If $S = G$, then we have a positive G-dynamical BLM over $(L, \mathbf{T})$; and we call $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g \in G}$ a positive weighted group representation on Γ over $\mathbf{T} = (T_g)_{g \in G}$ on L.

In the following remark, we mention additional properties of a positive S-dynamical BLM $(\Gamma, \boldsymbol{\mathcal{T}})$ over $(L, \mathbf{T})$ from Definition 2.3.4.1 above.

Remark 2.3.4.2. *(i) If $\mathcal{T}(g)$ is a positive isometry for all $g \in S$, then we call $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g \in S}$ a positive isometry.*

(ii) If $\mathcal{T}(g)$ is a lattice T_g-homomorphism for all $g \in S$, then we have an S-dynamical BLM over $(L, \mathbf{T})$; and we call $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g \in S}$ a weighted semigroup representation on Γ over $(L, \mathbf{T})$.

(iii) If $\mathcal{T}(g)$ is an isometric lattice T_g-homomorphism for all $g \in S$, then we call $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g \in S}$ an isometry.

[5] i.e., $\mathcal{T}(gh) = \mathcal{T}(g)\mathcal{T}(h)$ for all $g, h \in S$ and $\mathcal{T}(e) = Id_{\Gamma}$ for the neutral element $e \in S$.

It is clear that every S-dynamical BLM $(\Gamma, \mathcal{T})$ is also a positive S-dynamical BLM over $(L, \mathbf{T})$, since every lattice T-homomorphism is also a positive T-homomorphism. Moreover, by the above remark, if $S = G$, we can also speak of a G-dynamical BLM and a weighted group representation on Γ over $(L, \mathbf{T})$; and being an isometry.

The following two special examples are the motivation for the above formulations, which naturally extend the situations in Example 2.3.2.4 to group actions over to those of Example 2.3.3.3 (see also Chapter 3, Remark 3.5.0.1(v) and Appendix C, Remark C.5.0.1(iv)).

Example 2.3.4.3. *(i) Let $(\Omega, (\varphi_g)_{g\in G})$ be a topological G-dynamical system over a locally compact space Ω, and let $(E, \mathbf{\Phi})$ be a (positive) S-dynamical topological Banach lattice bundle over $(\Omega, (\varphi_g)_{g\in G})$ (see Chapter 3, Definition 3.4.0.5). The monoid representation given by*

$$\mathcal{T}_{\mathbf{\Phi}} : S \longrightarrow \mathscr{L}(\Gamma_0(\Omega, E)), g \mapsto \mathcal{T}_{\Phi}(g) := \left[s \mapsto \Phi_g \circ s \circ \varphi_{g^{-1}}\right]$$

defines a (positive) S-dynamical BLM $(\Gamma_0(\Omega, E), \mathcal{T}_{\mathbf{\Phi}})$ over the Koopman group representation $(C_0(\Omega), \mathbf{T}_{\varphi})$; and we call $\mathcal{T}_{\Phi}(g)_{g\in S}$ the (positive) weighted Koopman semigroup representation on $\Gamma_0(\Omega, E)$ over $(T_{\varphi}(g))_{g\in G}$ induced by $(E, \mathbf{\Phi})$.

(ii) Let $(X, (\varphi_g)_{g\in G})$ be a (discrete) G-dynamical measure-preserving system over a measure space X, and a let $(E, \mathbf{\Phi})$ be a (positive) S-dynamical measurable separable Banach lattice bundle over $(X, (\varphi_g)_{g\in G})$ (see Appendix C, Definition C.4.0.5). The monoid representation given by

$$\mathcal{T}_{\mathbf{\Phi}} : S \longrightarrow \mathscr{L}(\Gamma^1(X, E)), g \mapsto \mathcal{T}_{\Phi}(g) := \left[s \mapsto \Phi_g \circ s \circ \varphi_{g^{-1}}\right]$$

defines a (positive) S-dynamical BLM $(\Gamma^1(X, E), \mathcal{T}_{\mathbf{\Phi}})$ over the Koopman group representation $(L^\infty(X), \mathbf{T}_{\varphi})$; and we call $\mathcal{T}_{\Phi}(g)_{g\in S}$ the (positive) weighted Koopman semigroup representation on $\Gamma^1(X, E)$ over $(T_{\varphi}(g))_{g\in G}$ induced by $(E, \mathbf{\Phi})$.

2.3.5 Homomorphisms and isomorphisms between positive weighted semigroup representations on BLMs

In what follows we introduce the concept of homomorphism between positive S-dynamical BLMs over a fixed $(L, \mathbf{T})$.

Definition 2.3.5.1. *A homomorphism from a positive S-dynamical BLM $(\Gamma, \boldsymbol{\mathcal{T}})$ over $(L, \mathbf{T})$ to a positive S-dynamical BLM $(\Lambda, \boldsymbol{\mathcal{S}})$ over $(L, \mathbf{T})$ is a lattice module homomorphism (see Definition 2.3.2.1(ii))*

$$\Theta : \Gamma \longrightarrow \Lambda$$

such that the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\Theta} & \Lambda \\ {\scriptstyle \mathcal{T}(g)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{S}(g)} \\ \Gamma & \xrightarrow{\Theta} & \Lambda \end{array}$$

commutes for all $g \in S$, i.e., $\Theta \circ \mathcal{T}(g) = \mathcal{S}(g) \circ \Theta$ for all $g \in S$.
It is called a positive isometry if $\Theta \in \mathscr{L}(\Gamma, \Lambda)$ is isometric.

Remark 2.3.5.2. *(i) We note that a homomorphism between S-dynamical BLMs over $(L, \mathbf{T})$ can be defined similarly. It will be called an isometry if Θ is an isometry.*

(ii) In the category of positive S-dynamical BLMs over $(L, \mathbf{T})$; an isometric homomorphism here is just a positive isometry. So, we can speak of, for instance, unique up to positive isometry for an isometry in the category. We choose this terminology and reserve the word "isometry" for S-dynamical BLMs over $(L, \mathbf{T})$.

(iii) Furthermore, two positive S-dynamical BLMs $(\Gamma, \boldsymbol{\mathcal{T}})$ and $(\Lambda, \boldsymbol{\mathcal{S}})$ over $(L, \mathbf{T})$ are said to be isomorphic if there exists a surjective positive isometry (i.e., a surjective isometric homomorphism) between them. In this situation, we write $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g \in S} \cong (\mathcal{S}(g))_{g \in S} = \boldsymbol{\mathcal{S}}$ on $\Gamma \cong \Lambda$.

2.4 AM- and AL- lattice modules

Inspired by the work of S. Siewert ([29, Section 2.1, p.31], see also [21, Section 4, p.15 and Section 5, p.20]), we now introduce the concepts of AM lattice modules and AL lattice modules over the commutative Banach lattice algebra $C_0(\Omega)$. We do so in a natural way by combining the notions of AM-modules and AL-modules in the Banach module setting with the lattice structure.

2.4.1 AM lattice modules

This type of Banach lattice module is based on the concept of an AM-space in the Banach module setting as introduced in [29, Definition 2.18, p.33]. Before introducing this, we recall certain results.

In [29, Proposition 2.17, p.31], it was shown that if Γ is a Banach module over $C_0(\Omega)$, then for each $s \in \Gamma$, the closed submodule $\Gamma_s := \overline{C_0(\Omega) \cdot s}$ is a Banach lattice. More important, we take note of the following lemma.

Lemma 2.4.1.1. *[29, Definition 2.18, and Remark 2.19, p. 33] Let Γ be a Banach module over $C_0(\Omega)$. Then the following are equivalent.*

(i) *Γ is an AM-module, i.e., the submodule $\Gamma_s = \overline{C_0(\Omega) \cdot s}$ is an AM-space for each $s \in \Gamma$.*

(ii) $||(f \vee g)s|| = \max(||fs||, ||gs||)$ *for all $f, g \in C_0(\Omega)_+$ and $s \in \Gamma$.*

We note that AM-modules over $C_0(\Omega)$ are also called *locally convex* Banach $C_0(\Omega)$-modules (see [29, Remark 2.21 (i), p.33]); and for further characterisations of these *locally convex* Banach $C_0(\Omega)$-modules, we refer to works of F. Cunningham ([6, Theorem 2, p.618]) and W. Paravicini ([25, Proposition 2.1, p.272]). Using these concepts, we introduce the following natural definition, namely an *AM lattice module* over $C_0(\Omega)$ which can, therefore, also be called a *locally convex* Banach $C_0(\Omega)$-lattice module.

Definition 2.4.1.2. *Let Γ be a Banach lattice module over $C_0(\Omega)$. We call Γ an AM lattice module if Γ is an AM-module, i.e., the submodule $\Gamma_s = \overline{C_0(\Omega) \cdot s}$ is an AM-space for each $s \in \Gamma$.*
If, in addition, Γ is an m-Banach lattice module, we call it an AM m-lattice module over $C_0(\Omega)$.

The following is an immediate corollary using Lemma 2.4.1.1 from above.

Corollary 2.4.1.3. *Let Γ be a Banach lattice module over $C_0(\Omega)$. If Γ has property (m_+), then $\Gamma_{|s|}$ is an AM-space if and only if Γ_s is an AM-space if and only if $\Gamma_{\bar{s}}$ is an AM-space for every $s \in \Gamma$.*

Proof. Since Γ is a Banach lattice module over $C_0(\Omega)$ with property (m_+) (see Definition 2.2.3.1), the following holds for all $f \in C_0(\Omega)_+$ and $s \in \Gamma$:

$$||f \cdot |s|\,|| = ||f \cdot s|| = ||f \cdot \bar{s}||.$$

This implies that for all $f, g \in C_0(\Omega)_+$ and $s \in \Gamma$

(i) $||(f \vee g)s|| = ||(f \vee g)|s|\,|| = ||(f \vee g)\bar{s}||$; and

(ii) $\max(||fs||, ||gs||) = \max(||f|s|\,||, ||g|s|\,||) = \max(||f\bar{s}||, ||g\bar{s}||)$.

Hence, the assertion follows from Lemma 2.4.1.1.

□

The following is an example of such AM m-lattice modules serving as motivation for our formation.

Example 2.4.1.4. *If E is a topological Banach lattice bundle over Ω, then the Banach lattice $\Gamma_0(\Omega, E)$ of its continuous sections vanishing at infinity is an AM m-lattice module over $C_0(\Omega)$ (see also Example 2.4.1.11).*

2.4.1.1 $U_0(\Omega)$-normed lattice module

It is known that an AM-module over $C_0(\Omega)$ admits an additional lattice norm structure; namely, an AM- module over $C_0(\Omega)$ is also a $U_0(\Omega)$-normed module (see [29, Section 2.2.1., p.39-40]). We recall these results and introduce an analogous definition for the case of Banach lattice modules.

For a locally compact space Ω, we write;
$U(\Omega) := \{g : \Omega \longrightarrow \mathbb{R} \mid g \text{ is upper semicontinuous}\}$,
$U_0(\Omega) := \{g \in U(\Omega) \mid \forall \varepsilon > 0\ \exists K \subseteq \Omega \text{ compact with } |g(x)| \leq \varepsilon\ \forall x \notin K\}$, and
$U_0(\Omega)_+ := \{g \in U_0(\Omega) \mid g \geq 0\}$.

Definition 2.4.1.5. *($U_0(\Omega)$-normed module) [29, Definition 2.34, p. 39] Let Ω be a locally compact space and Γ a Banach module over $C_0(\Omega)$. A mapping*

$$|\cdot| : \Gamma \longrightarrow U_0(\Omega)_+$$

is a $U_0(\Omega)$-valued norm if:

(i) $|| |s| || = ||s||$;

(ii) $|fs| = |f| \cdot |s|$; *and*

(iii) $|s_1 + s_2| \leq |s_1| + |s_2|$ *for all $s, s_1, s_2 \in \Gamma$ and $f \in C_0(\Omega)$.*

A Banach module over $C_0(\Omega)$ together with a $U_0(\Omega)$-valued norm is called a $U_0(\Omega)$-normed module.

It readily follows that a $U_0(\Omega)$-normed module is an AM-module over $C_0(\Omega)$. A more important result about a $U_0(\Omega)$-normed module is that the converse also holds, as the following lemma shows.

Lemma 2.4.1.6. *[29, Proposition 2.36, p.40] Let Ω be a locally compact space. For a Banach module Γ over $C_0(\Omega)$, the following are equivalent.*

(i) Γ is an AM-module over $C_0(\Omega)$.

(ii) Γ is a $U_0(\Omega)$-normed module.

Moreover, if these assertions hold, then the $U_0(\Omega)$-valued norm is unique and given by

$$|s|(x) := \inf\{||fs|| : f \in C_0(\Omega)_+ \text{ with } f(x) = 1\} \quad (x \in \Omega, s \in \Gamma).$$

Remark 2.4.1.7. *(i) For a topological Banach bundle E over a locally compact space Ω, setting $|s|(x) := ||s(x)||$ for $x \in \Omega$ and $s \in \Gamma_0(\Omega, E)$ turns $\Gamma_0(\Omega, E)$ into a $U_0(\Omega)$-normed module (see also [29, Example 2.35, p.40]).*

(ii) However, for a topological Banach lattice bundle E over Ω, since $|| \mathsf{I} s(x) \mathsf{I} || = ||s(x)|| = ||\overline{s(x)}||$ for $x \in \Omega$ and $s \in \Gamma_0(\Omega, E)$, we see that the $U_0(\Omega)$-valued norm "agrees" with the "lattice" structure on $\Gamma_0(\Omega, E)$. Inspired by this, we introduce the concept of a $U_0(\Omega)$-normed m-lattice module in the next definition.

Definition 2.4.1.8. *Let Ω be a locally compact space and Γ a Banach lattice module over $C_0(\Omega)$. A mapping*

$$|\cdot|:\Gamma\longrightarrow U_0(\Omega)_+$$

is a lattice $U_0(\Omega)$-valued norm if:

(i) $|\,|s|\,| = |s| = |\bar{s}|$;

(ii) $|||s||| = ||s||$;

(iii) $|fs| = |f|\cdot|s|$; *and*

(iv) $|s_1+s_2|\leq|s_1|+|s_2|$ *for all* $s, s_1, s_2\in\Gamma$ *and* $f\in C_0(\Omega)$.

A Banach lattice module over $C_0(\Omega)$ together with a lattice $U_0(\Omega)$-valued norm is called a $U_0(\Omega)$-normed lattice module. If Γ is, in addition, an m-Banach lattice module, we call it a $U_0(\Omega)$-normed m-lattice module.

By the definition above, it is clear that a $U_0(\Omega)$-*normed lattice module* necessarily satisfies property (m_+). Furthermore, using Lemma 2.4.1.6, we obtain the following equivalence.

Proposition 2.4.1.9. *Let Ω be a locally compact space. For a Banach lattice module Γ over $C_0(\Omega)$, the following are equivalent.*

(i) *Γ is an AM lattice module over $C_0(\Omega)$ with property (m_+).*

(ii) *Γ is a $U_0(\Omega)$-normed lattice module.*

Moreover, if this assertion holds, then the lattice $U_0(\Omega)$-valued norm is unique and given by

$$|s|(x):=\inf\{||fs||: f\in C_0(\Omega)_+ \text{ with } f(x)=1\}\quad (x\in\Omega, s\in\Gamma).$$

Proof. $(ii)\implies(i)$: That a $U_0(\Omega)$-normed lattice module Γ is an AM lattice module readily follows from definition. Indeed, for all $f,g\in C_0(\Omega)_+$ and $s\in\Gamma$ we have that

$$\begin{aligned}|(f\vee g)s| &= |f\vee g|\cdot|s|\\ &= f|s|\vee g|s|.\end{aligned}$$

Furthermore, since $C_0(\Omega) \subseteq U_0(\Omega)$ is an AM-space, it follows that for all $f, g \in C_0(\Omega)_+$ and $s \in \Gamma$

$$\begin{aligned} ||(f \vee g)s|| &= ||f|s| \vee g|s||| \\ &= \max(||fs||, ||gs||), \end{aligned}$$

and the first assertion follows from Lemma 2.4.1.1.

Moreover, Γ satisfies property (m_+) (see Definition 2.2.3.1). Indeed, by the definition of the lattice $U_0(\Omega)$-valued norm, we see that if $f \in C_0(\Omega)_+$ and $s \in \Gamma$, then

$$|f \,|\, s \,|\,| = f|s| = |fs| \qquad \text{and} \qquad |f\bar{s}| = f|s| = |fs|.$$

And so, for all $f \in C_0(\Omega)_+$ and $s \in \Gamma$ we have that

$$||f \cdot \,|\, s \,|\,|| = ||f \cdot s|| = ||f \cdot \bar{s}||.$$

$(i) \implies (ii)$: Since an AM-module admits a $U_0(\Omega)$-valued norm (by Lemma 2.4.1.6) given by

$$|s|(x) = \inf\{||fs|| : f \in C_0(\Omega)_+ \text{ with } f(x) = 1\} \quad (x \in \Omega, s \in \Gamma),$$

it suffices to show this $U_0(\Omega)$-valued norm "agrees" with the "lattice" structure on Γ.

Now, since Γ is a Banach lattice module over $C_0(\Omega)$ with property (m_+) (see Definition 2.2.3.1), we have that, for all $f \in C_0(\Omega)_+$ and $s \in \Gamma$,

$$||f \cdot \,|\, s \,|\,|| = ||f \cdot s|| = ||f \cdot \bar{s}||.$$

In particular, this implies that, for every $s \in \Gamma$ and $x \in \Omega$,

$W_s := \{||fs|| : f \in C_0(\Omega)_+ \text{ with } f(x) = 1\} = W_{|\,s\,|} = W_{\bar{s}}$

and so, we have that $\inf W_{|\,s\,|} = \inf W_s = \inf W_{\bar{s}}$ for all $s \in \Gamma$ and $x \in \Omega$. This proves the assertion.

□

The following is an immediate corollary from the above Proposition 2.4.1.9.

Corollary 2.4.1.10. *Let Ω be a locally compact space. For a Banach lattice module Γ over $C_0(\Omega)$, the following are equivalent.*

(i) *Γ is an AM m-lattice module over $C_0(\Omega)$.*

(ii) *Γ is a $U_0(\Omega)$-normed m-lattice module.*

Moreover, if these assertions hold, then the lattice $U_0(\Omega)$-valued norm is unique and given by

$$|s|(x) := \inf\{||fs|| : f \in C_0(\Omega)_+ \textit{ with } f(x) = 1\} \quad (x \in \Omega, s \in \Gamma).$$

Now, we can state an example of such a $U_0(\Omega)$-normed m-lattice module serving as motivation.

Example 2.4.1.11. *For a topological Banach lattice bundle E over Ω, setting $|s|(x) := ||s(x)||$ for $x \in \Omega$ and $s \in \Gamma_0(\Omega, E)$ turns $\Gamma_0(\Omega, E)$ into a $U_0(\Omega)$-normed m-lattice module (see also Example 2.4.1.4).*

2.4.2 AL lattice modules

This type of Banach lattice module is based on the concept of an AL-space in the Banach module setting as introduced in [29, Definition 2.28, p.38]. Before introducing this, we also recall certain results.

The following lemma, which states an equivalent definition of an AL-module over $C_0(\Omega)$, will be useful.

Lemma 2.4.2.1. *[29, Definition 2.28, and Remark 2.29, p. 38] Let Γ be a Banach module over $C_0(\Omega)$, then the following are equivalent.*

(i) *Γ is an AL-module i.e., the submodule $\Gamma_s = \overline{C_0(\Omega) \cdot s}$ is an AL-space for each $s \in \Gamma$.*

(ii) *$||(f+g)s|| = ||fs|| + ||gs||$ for all $f, g \in C_0(\Omega)_+$ and $s \in \Gamma$.*

Using these concepts, we introduce the following natural definition.

Definition 2.4.2.2. *Let Γ be a Banach lattice module over $C_0(\Omega)$. We call Γ an AL lattice module if Γ is an AL-module, i.e., the submodule $\Gamma_s = \overline{C_0(\Omega) \cdot s}$ is an AL-space for each $s \in \Gamma$.*
If Γ is, in addition, an m-Banach lattice module, we call it an AL m-lattice module.

Analogous to Corollary 2.4.1.3, the following corollary is immediate using Lemma 2.4.2.1 from above.

Corollary 2.4.2.3. *Let Γ be a Banach lattice module over $C_0(\Omega)$. If Γ has property (m_+), then $\Gamma_{|s|}$ is an AL-space if and only if Γ_s is an AL-space if and only if $\Gamma_{\overline{s}}$ is an AL-space for every $s \in \Gamma$.*

Remark 2.4.2.4. *(i) As in the work of H. Kreidler and S. Siewert, see [21, second paragraph on p.20], the space $L^\infty(X)$ can be identified (as a Banach lattice algebra) with $C_0(\Omega)$ for some locally compact space Ω. Indeed, $L^\infty(X)$ can be seen as a commutative C^*-algebra and WLOG as a Banach lattice with $IntL^\infty(X)_+ \neq \emptyset$; hence by classical theorems of Gelfand([7, Theorem 1.4.1, p.11]) and Kakutani (see Appendix B, Theorem B.0.0.6), we can obtain this representation.*

(ii) So, as in their work, we can also speak of an AM lattice module or an AL lattice module over $L^\infty(X)$. In this direction, the following is an example of such an AL m-lattice module serving as motivation for obtaining our representation.

Example 2.4.2.5. *If E is a measurable Banach lattice bundle over X, then the (quotient) Banach lattice $\Gamma^1(X, E)$ of its integrable measurable sections is an AL m-lattice module over $L^\infty(X)$ (see also Example 2.4.2.12).*

2.4.2.1 $C_0(\Omega)'$-normed lattice module

It is known that an AL-module over $C_0(\Omega)$ admits an additional lattice norm structure; namely, an AL-module over $C_0(\Omega)$ is also a $C_0(\Omega)'$-normed module (see [29, Section 2.2.2., p.42]). We recall these results and introduce an analogous definition for the case of Banach lattice modules.

Definition 2.4.2.6. *($C_0(\Omega)'$-normed module) [29, Definition 2.41, p. 43] Let Ω be a locally compact space and Γ a Banach module over $C_0(\Omega)$. A mapping*

$$|\cdot| : \Gamma \longrightarrow C_0(\Omega)'_+$$

is a $C_0(\Omega)'$-valued norm if:

(i) $|||s||| = ||s||$;

(ii) $|fs| = |f| \cdot |s|$; *and*

(iii) $|s_1 + s_2| \leq |s_1| + |s_2|$ *for all* $s, s_1, s_2 \in \Gamma$ *and* $f \in C_0(\Omega)$.

A Banach module over $C_0(\Omega)$ together with a $C_0(\Omega)'$-valued norm is called a $C_0(\Omega)'$-normed module.

Since $C_0(\Omega)'$ is an AL-space, it readily follows that a $C_0(\Omega)'$-normed module is an AL- module over $C_0(\Omega)$. A more important result about a $C_0(\Omega)'$-normed module is that the converse also holds, as the following lemma shows.

Lemma 2.4.2.7. *[29, Proposition 2.42, p.43] Let Ω be a locally compact space. For a Banach module Γ over $C_0(\Omega)$, the following are equivalent.*

(i) Γ *is an AL-module over* $C_0(\Omega)$.

(ii) Γ *is a* $C_0(\Omega)'$*-normed module.*

Moreover, if these assertions hold, then the $C_0(\Omega)'$-valued norm is unique and given by

$$|s|(f) := ||fs|| \quad (f \in C_0(\Omega)_+,\ s \in \Gamma).$$

Remark 2.4.2.8. *(i) For a measurable Banach bundle E over Ω, setting $|\cdot| : \Gamma^1(X, E) \longrightarrow L^1(X);\ s \mapsto ||s(\cdot)||$ turns $\Gamma^1(X, E)$ into a $L^\infty(X)'$-normed module. Here we note that this is so since $L^1(X)$ is canonically embedded in $L^\infty(X)'$ (see also [29, paragraph two, p.44]).*

(ii) However, for a measurable Banach lattice bundle E over Ω, since $||\,\mathbf{|}s(\cdot)\mathbf{|}\,|| = ||s(\cdot)|| = ||\overline{s(\cdot)}||$ for all $s \in \Gamma^1(X, E)$; we see that the $L^\infty(X)'$-valued norm "agrees" with the "lattice" structure on $\Gamma^1(X, E)$. Inspired by this, we introduce the concept that gives rise to what we call an $L^\infty(X)'$-normed m-lattice module in the next definition.

Definition 2.4.2.9. *Let Ω be a locally compact space and Γ a Banach lattice module over $C_0(\Omega)$. A mapping*

$$|\cdot| : \Gamma \longrightarrow C_0(\Omega)'_+$$

is a lattice $C_0(\Omega)'$-valued norm if:

(i) $|\,|s|\,| = |s| = |\bar{s}|$;

(ii) $|||s||| = ||s||$;

(iii) $|fs| = |f| \cdot |s|$; *and*

(iv) $|s_1 + s_2| \leq |s_1| + |s_2|$ *for all* $s, s_1, s_2 \in \Gamma$ *and* $f \in C_0(\Omega)$.

A Banach lattice module over $C_0(\Omega)$ together with a lattice $C_0(\Omega)'$-valued norm is called a $C_0(\Omega)'$-normed lattice module. And if Γ is, in addition, an m-Banach lattice module, we call it $C_0(\Omega)'$-normed m-lattice module.

By definition, it follows that a $C_0(\Omega)'$-*normed lattice module* necessarily satisfies property (m_+), and using Lemma 2.4.2.7, we also obtain the following equivalence.

Proposition 2.4.2.10. *Let Ω be a locally compact space. For a Banach lattice module Γ over $C_0(\Omega)$, the following are equivalent.*

(i) Γ *is an AL lattice module over* $C_0(\Omega)$ *with property* (m_+).

(ii) Γ *is a* $C_0(\Omega)'$*-normed lattice module.*

Moreover, if these assertions hold, then the lattice $C_0(\Omega)'$-valued norm is unique and given by

$$|s|(f) := ||fs|| \quad (f \in C_0(\Omega)_+,\ s \in \Gamma).$$

Proof. $(ii) \implies (i)$: That a $C_0(\Omega)'$-normed lattice module Γ is an AL module over $C_0(\Omega)$ readily follows from the definition of a lattice $C_0(\Omega)'$-valued norm. Indeed, for all $f, g \in C_0(\Omega)_+$ and $s \in \Gamma$ we have that

$$\begin{aligned} |(f+g)s| &= |f+g| \cdot |s| \\ &= f|s| + g|s|. \end{aligned}$$

And since $C_0(\Omega)'$ is an AL-space, it follows that for all $f, g \in C_0(\Omega)_+$ and $s \in \Gamma$

$$\begin{aligned} ||(f+g)s|| &= ||f|s| + g|s||| \\ &= ||fs|| + ||gs||. \end{aligned}$$

So, the first assertion follows from Lemma 2.4.2.1.

Moreover, Γ satisfies property (m_+). Indeed, by the definition of the lattice $C_0(\Omega)'$-valued norm, we see that if $f \in C_0(\Omega)_+$ and $s \in \Gamma$, then

$$|f\,\mathsf{I}\,s\,\mathsf{I}\,| = f|s| = |fs| \qquad \text{and} \qquad |f\bar{s}| = f|s| = |fs|.$$

So, for all $f \in C_0(\Omega)_+$ and $s \in \Gamma$ we have that

$$||f \cdot \mathsf{I}\,s\,\mathsf{I}\,|| = ||f \cdot s|| = ||f \cdot \bar{s}||.$$

$(i) \implies (ii)$: Since an AL-module admits a $C_0(\Omega)'$-valued norm (by Lemma 2.4.2.7) given by

$$|s|(f) := ||fs|| \quad (f \in C_0(\Omega)_+,\ s \in \Gamma),$$

it suffices to show this $C_0(\Omega)'$-valued norm "agrees" with the "lattice" structure on Γ.

Now, since Γ is a Banach lattice module over $C_0(\Omega)$ with property (m_+) (see Definition 2.2.3.1), we have that, for all $f \in C_0(\Omega)_+$ and $s \in \Gamma$,

$$||f \cdot \mathsf{I}\,s\,\mathsf{I}\,|| = ||f \cdot s|| = ||f \cdot \bar{s}||$$

which proves the assertion.

$\square$

The following is an immediate corollary from Proposition 2.4.2.10 above.

Corollary 2.4.2.11. *Let Ω be a locally compact space. For a Banach lattice module Γ over $C_0(\Omega)$, the following are equivalent.*

(i) Γ *is an AL m-lattice module over* $C_0(\Omega)$.

(ii) Γ *is a* $C_0(\Omega)'$*-normed m-lattice module.*

Moreover, if these assertions hold, then the lattice $C_0(\Omega)'$*-valued norm is unique and given by*

$$|s|(f) := ||fs|| \quad (f \in C_0(\Omega)_+,\ s \in \Gamma).$$

Now, we can state an important example serving as motivation.

Example 2.4.2.12. *For a measurable Banach lattice bundle E over X, setting $|\cdot|: \Gamma^1(X,E) \longrightarrow L^1(X);\ s \mapsto ||s(\cdot)||$ turns the Banach lattice $\Gamma^1(X,E)$ into an $L^\infty(X)'$-normed m-lattice module. Here, we note again that, this is so since $L^1(X)$ is embedded in $L^\infty(X)'$ (see also Example 2.4.2.5).*

2.4.2.2 $L^1(X)$-normed lattice modules

Remark 2.4.2.13. *As noted in [29, paragraph after Proposition 2.42, p.43] (see also [21, Example 5.10 p.24]), an AL-module over $L^\infty(X)$ can only be isometrically isomorphic to $\Gamma^1(X,E)$ for some measurable Banach bundle E over a measure space X if the $L^\infty(X)'$-valued norm takes values in (the canonical image of) $L^1(X)$. This observation leads to the following definition.*

Definition 2.4.2.14. *($L^1(X)$-normed module) [29, Definition 2.44, p.45] Let X be a measure space. An $L^\infty(X)'$-normed module Γ is called an $L^1(X)$-normed module if $|s| \in L^1(X)$ for every $s \in \Gamma$.*

Since, for a measurable Banach bundle E over the measure space X, the Banach space $\Gamma^1(X,E)$ is actually an $L^1(X)$-normed module with respect to the above definition, we introduce the following natural definition for the case of a measurable Banach lattice bundle.

Definition 2.4.2.15. *Let X be a measure space. An $L^\infty(X)'$-normed lattice module Γ is called an $L^1(X)$-normed lattice module if $|s| \in L^1(X)$ for every $s \in \Gamma$.*
If Γ is, in addition, an m-Banach lattice module, then we call it an $L^1(X)$-normed m-lattice module.

An immediate example is the following serving as motivation.

Example 2.4.2.16. *For a measurable Banach lattice bundle E over X, setting $|\cdot|: \Gamma^1(X,E) \longrightarrow L^1(X);\ s \mapsto ||s(\cdot)||$ turns the Banach lattice $\Gamma^1(X,E)$ into an $L^1(X)$-normed m-lattice module.*

Similar to [21, Definition 5.8, p.23] in the case of Banach modules; we make the following definition in order to capture the general notion of $C_0(\Omega)'$-*normed lattice module* which we introduce in Definition 2.4.2.9.

Definition 2.4.2.17. *Let Γ be a Banach lattice module over a commutative Banach lattice algebra L. Moreover, let L'_+ be the positive cone of the Banach dual L' of L. A mapping*

$$n : \Gamma \longrightarrow L'_+$$

is a lattice L'-valued norm if:

(i) $n(|s|) = n(s) = n(\bar{s})$;

(ii) $||n(s)|| = ||s||$;

(iii) $n(f \cdot s) = |f| \cdot n(s)$; *and*

(iv) $n(s_1 + s_2) \leq n(s_1) + n(s_2)$ *for all* $s, s_1, s_2 \in \Gamma$ *and* $f \in L$.

A Banach lattice module over L together with a lattice L'-valued norm will be called an *L'-normed lattice module*. And if Γ is, in addition, m-Banach lattice module, we call it an *L'-normed m-lattice module*.

2.5 More on Banach lattice modules over a Banach lattice algebra

In this Section, we introduce and consider several notions about a Banach lattice module over a Banach lattice algebra. Many of these notions which are also important for our study include: *Sub-lattice algebras and lattice algebra ideals* (see Subsection 2.5.1), *Lattice submodules and Ideal submodules* (see Subsection 2.5.2), *Quotient spaces of Banach lattice module* (see Subsection 2.5.3), *Dual Banach lattice modules* (see Subsection 2.5.4), *Banach lattice modules and lattice isomorphisms* (see Subsection 2.5.5), and *Banach lattice modules and lattice ideals* (see Subsection 2.5.6).

Throughout this Section, by a Banach lattice algebra L, we mean a commutative Banach lattice algebra, Ω a locally compact space, K will be a compact space and $X := (\Omega_X, \Sigma_X, \mu_X)$ will be a complete positive σ-finite measure space. As always, by m-Banach lattice module over L, we mean an m-Banach lattice module over the m-BLA L (see Definition 2.2.2.4).

2.5.1 Sub-lattice algebras and lattice algebra ideals

In this Subsection, we introduce the notions of a sub-lattice algebra and a lattice algebra ideal of L, a Banach lattice algebra (BLA).

Definition 2.5.1.1. *Let L be a Banach lattice algebra. A subspace $M \subseteq L$ will be called a sub-lattice algebra if the following hold; for all $f, g \in M$*

(i) $fg \in M$, *and*

(ii) $|f|, \overline{f} \in M$.

It is clear that a sub-lattice algebra is both a subalgebra and a sublattice. Moreover M is again a BLA if and only if $M \subseteq L$ is a closed subspace; and in that case, we call M a Banach sub-lattice algebra of L.

Definition 2.5.1.2. *Let L be a Banach lattice algebra. A subspace $N \subseteq L$ will be called a lattice algebra ideal if the following hold; for all $f \in N$, and $m \in L$*

(i) $mf \in N$, *and*

(ii) $|m| \leq |f| \implies m \in N$.

Also, we see here that a lattice algebra ideal is both an algebra ideal and a lattice ideal. Furthermore, in the case where $N \subseteq L$ is closed, we call N a closed lattice algebra ideal of L.

Example 2.5.1.3. *(i) If $L = C(K)$; we know that every closed lattice ideal is precisely a closed algebra ideal, i.e., of the form $I_A := \left\{ f \in C(K) : f_{|A} \equiv 0 \right\}$ for some closed subspace $A \subseteq K$ (see [10, Remark 7.11 (iii), p.124]).*

(ii) If $L = L^\infty(X)$; a set of the form

$$I_A := \{ f \in L^\infty(X) : \mathbb{1}_A \wedge |f| = 0 \iff |f| \wedge \mathbb{1} \leq \mathbb{1}_{A^c} \iff A \subseteq [f = 0] \}$$

for some $A \in \Sigma_X$ is a closed lattice algebra ideal. However, it should be noted that even a lattice ideal in $L^\infty(X)$ is not always of this form, unless $L^\infty(X)$ is finite-dimensional (see [10, Remark 7.11 (i), p.124]).

In the following proposition, we note an instance of a closed lattice algebra ideal when $L = C_0(\Omega)$.

Proposition 2.5.1.4. *Let $L = C_0(\Omega)$. For each $x \in \Omega$, the set of the form*

$$A_x := \{f \in C_0(\Omega) : f(x) = 0\}$$

is a Banach sub-lattice algebra. In particular, A_x is a closed lattice algebra ideal of $C_0(\Omega)$.

Proof. Let $x \in \Omega$ be fixed. Since $\{x\}$ is closed in Ω, by Example 2.5.1.3(i), it follows that A_x is a closed lattice ideal and closed algebra ideal of $C_0(\Omega)$, hence the assertion holds. Indeed, clearly A_x is a closed sublattice and subalgebra and for all $g \in C_0(\Omega)$ and $f \in A_x$; we have that

(i) $gf \in A_x$ since $(gf)(x) = g(x)f(x) = 0$, and

(ii) $|g| \leq |f| \implies g \in A_x$ since $|g|(x) \leq |f|(x) = 0$.

□

2.5.2 Lattice submodules and Ideal submodules

In this Subsection, we introduce the concept of lattice submodules and ideal submodules of a Banach lattice module Γ over a subspace of a Banach lattice algebra L.

Definition 2.5.2.1. *Let $N \subseteq L$ be a subspace. A lattice submodule over N is a subspace $\Lambda \subseteq \Gamma$ such that the following hold: for all $f \in N$ and $s \in \Lambda$*

(i) $fs \in \Lambda$, and

(ii) $|s|, \bar{s} \in \Lambda$.

A lattice submodule over N will also be called an N-lattice submodule of Γ. Moreover, in the case where $N = L$ and $\Lambda \subseteq \Gamma$ is a closed subspace, Λ is called a Banach lattice submodule of Γ.

Remark 2.5.2.2. *(i) It is clear that, if Λ is a Banach lattice submodule of Γ, then it is a Banach lattice module over L.*

(ii) Furthermore, if N is a Banach sub-lattice algebra, then a closed N-lattice submodule of Γ is also a Banach lattice module over N.

Definition 2.5.2.3. *Let $N \subseteq L$ be a subspace. An ideal submodule over N is a subspace $I_\Gamma \subseteq \Gamma$ such that the following holds. For all $f \in N, s \in I_\Gamma$ and $r \in \Gamma$,*

(i) $fs \in I_\Gamma$, *and*

(ii) $\mid r \mid \leq \mid s \mid \implies r \in I_\Gamma$.

An ideal submodule over N will also be called an N-ideal submodule of Γ. Moreover, in the case where $N = L$, and $I_\Gamma \subseteq \Gamma$ is a closed subspace, I_Γ is called a closed ideal submodule of Γ.

Remark 2.5.2.4. *(i) We note here that, since every lattice ideal is also a sublattice, if I_Γ is an N-ideal submodule of Γ, then it is, in particular, an N-lattice submodule of Γ.*

(ii) So, every closed ideal submodule of Γ is also a Banach lattice submodule of Γ.

We note an instance of a Banach lattice submodule of Γ in the following proposition in the case where $L = C_0(\Omega)$.

Proposition 2.5.2.5. *Let Γ be an m-BLM over $C_0(\Omega)$ and let $x \in \Omega$ be fixed. Then, the following hold.*

(A) The set of the form

$$J_x := \overline{lin}\,\{fs : f \in C_0(\Omega) \text{ with } f(x) = 0 \text{ and } s \in \Gamma\}$$

is a Banach lattice submodule of Γ. In particular, J_x is A_x-lattice submodule of Γ and $J_x = \overline{lin}\,\{fs : f \in A_x \text{ and } s \in \Gamma\}$ where $A_x := \{f \in C_0(\Omega) : f(x) = 0\}$.

(B) If, in addition, Γ is a $U_0(\Omega)$-normed lattice module, then the set of the form

$$\Gamma_x := \{s \in \Gamma : |s|(x) = 0\}$$

is a closed ideal submodule of Γ. In this case, $J_x \subseteq \Gamma_x$ and Γ_x is, in particular, an A_x-ideal submodule of Γ.

Proof. (A) For fixed $x \in \Omega$, if we take $A_x := \{f \in C_0(\Omega) : f(x) = 0\}$ as in Proposition 2.5.1.4, we first show that J_x is a lattice submodule over the Banach lattice algebra A_x, i.e., for any $g \in A_x$ and $r \in J_x$, we have that $gr \in J_x$ and $\overline{r}, \mid r \mid \in J_x$. Indeed, if, WLOG, $r = fs$ for some $f \in A_x$, and $s \in \Gamma$, we have that

(i) $gr = (gf)s \in J_x$ since $gf \in A_x$, and

(ii) $\mid r \mid = \mid fs \mid = \mid f \mid \mid s \mid \in J_x$ and $\overline{r} = \overline{f}\overline{s} \in J_x$ since $\mid f \mid, \overline{f} \in A_x$.

This shows that J_x is a closed lattice submodule over A_x, since J_x is closed by definition. Moreover, $J_x = \overline{\text{lin}}\{fs : f \in A_x \text{ and } s \in \Gamma\}$ also by definition.

Now, since A_x is, in particular, a closed lattice algebra ideal of $C_0(\Omega)$, we can conclude that J_x is a Banach lattice submodule of Γ.

(B) If, in addition, Γ is a $U_0(\Omega)$-normed lattice module (see Definition 2.4.1.8), i.e., Γ is a $U_0(\Omega)$-normed m-lattice module, we first show that Γ_x is lattice submodule over $C_0(\Omega)$. Indeed, Γ_x is a subspace of Γ, and for any $f \in C_0(\Omega)$ and $s \in \Gamma_x$, we have that

(i) $fs \in \Gamma_x$ since $|fs|(x) = |f|(x)|s|(x) = 0$, and

(ii) $\mid s \mid, \overline{s} \in \Gamma_x$, since $|\mid s \mid|(x) = |\overline{s}|(x) = |s|(x) = 0$.

Moreover, Γ_x is a closed subspace of Γ. Indeed, if $(s_n)_{n\in\mathbb{N}} \subseteq \Gamma_x$ is a sequence converging to $s \in \Gamma$, which is the case if and only if the sequence $(|s_n|)_{n\in\mathbb{N}} \subseteq U_0(\Omega)_+$ converges to $|s| \in U_0(\Omega)_+$, then $|s_n|(x) = 0$ for all $n \in \mathbb{N}$ implies that $|s|(x) = 0$, i.e., $s \in \Gamma_x$.

Next, we show that Γ_x is a lattice ideal of Γ. Indeed, if $s_o \in \Gamma$ and $s \in \Gamma_x$, then $\mid s_o \mid \leq \mid s \mid$ implies that $|s_o|(x) = |\mid s_0 \mid|(x) \leq |\mid s \mid|(x) = |s|(x) = 0$, i.e., $s_o \in \Gamma_x$.

Thus, Γ_x is a closed ideal submodule of Γ.

Finally, if, WLOG, $r = fs \in J_x$ for some $f \in A_x$, and $s \in \Gamma$, then $|r|(x) = |f|(x)|s|(x) = 0$ implies that $r \in \Gamma_x$. Hence, $J_x \subseteq \Gamma_x$ and the assertion is proved.

$\square$

2.5.3 Quotient Spaces of a Banach lattice module

In this Subsection, we identify certain quotient spaces of a Banach lattice module Γ over a commutative BLA L.

Proposition 2.5.3.1. *Let $N \subseteq L$ be a closed algebra ideal, and $\Lambda \subseteq \Gamma$ a closed N-submodule of Γ. Then the following hold.*

(i) The Banach space Γ/Λ is a Banach module over the quotient algebra L/N; where the action of L/N on Γ/Λ is characterised by the identity

$$(f+N)\cdot(s+\Lambda) := fs+\Lambda$$

for all $f \in L$ and $s \in \Gamma$.

(ii) The Banach module Γ/Λ over L/N is a Banach lattice module over the quotient lattice algebra L/N if, in addition, N and Λ are lattice ideals of L and Γ respectively, i.e., $N \subseteq L$ is a closed lattice algebra ideal, and Λ is a closed N-ideal submodule of Γ.

In this case, the quotient maps $p : L \longrightarrow L/N$ and $q : \Gamma \longrightarrow \Gamma/\Lambda$ are lattice homomorphisms. In addition, $p\overline{f} = \overline{pf}$ and $q\overline{s} = \overline{qs}$ for all $f \in L$ and $s \in \Gamma$.

Moreover, if Γ is an m-Banach lattice module over L, then Γ/Λ is also an m-Banach lattice module over L/N,

Proof. (i) Since Γ is non-degenerate (see Subsection 2.2.4) we have that $\Lambda = \overline{lin}\,\{fs : f \in N \text{ and } s \in \Lambda\}$. By [20, Proposition 2.2, p.144] we obtain that the Banach space Γ/Λ is a Banach module over the quotient algebra L/N as claimed. This, in particular, implies that

$$||fs+\Lambda|| \leq ||f+N||||s+\Lambda||$$

for all $f \in L$ and $s \in \Gamma$.

(ii) Since N and Λ are lattice ideals of L and Γ, respectively; the quotient lattices L/N and Γ/Λ are, in addition, Banach lattices, where the lattice structures are defined canonically such that quotient maps $p : L \longrightarrow L/N$ and $q : \Gamma \longrightarrow \Gamma/\Lambda$ are lattice homomorphisms and $p\overline{f} = \overline{pf}$, and $q\overline{s} = \overline{qs}$ for all $f \in L$ and $s \in \Gamma$ (cf. [28, Proposition 5.4, p.85]). That is, we have that

$$|f+N| := |f| + N \text{ and } \overline{f+N} := \overline{f}+N, \text{ and}$$

$$|s+\Lambda| := |s| + \Lambda \text{ and } \overline{s+\Lambda} := \overline{s}+\Lambda$$

for all $f \in L$ and $s \in \Gamma$.

It follows immediately that L/N is a Banach lattice algebra, since

$$|(f+N)(g+N)| = |fg| + N \leq |f||g| + N = |f+N||g+N|$$

for all $f, g \in L$ (see Appendix A, Note A.1.0.2 and Definition A.1.0.3). Moreover, Γ/Λ is a Banach lattice module over the Banach lattice algebra L/N, since

$$|(f+N)\cdot(s+\Lambda)| = |fs| + \Lambda \leq |f||s| + \Lambda = |f+N| \cdot |s+\Lambda|$$

for all $f \in L$ and $s \in \Gamma$ (see Definition 2.2.1.1 and Note 2.2.1.2).

Finally, from the above consideration, it follows that Γ/Λ is an m-Banach lattice module over L/N if Γ is an m-Banach lattice module over L.

□

Combining Propositions 2.5.2.5, 2.5.3.1 and 2.4.1.10 we have the following.

Corollary 2.5.3.2. *Let Γ be an m-BLM over $C_0(\Omega)$ and let $x \in \Omega$ be fixed. Then the following hold.*

(A) Let $J_x := \overline{lin}\,\{fs : f \in A_x$ and $s \in \Gamma\}$ where $A_x := \{f \in C_0(\Omega) : f(x) = 0\}$. Then, the Banach space Γ/J_x is a Banach module over the quotient algebra $C_0(\Omega)/A_x$; where the action of $C_0(\Omega)/A_x$ on Γ/J_x is given by the identity

$$(f + A_x)\cdot(s + J_x) := fs + J_x$$

for all $f \in C_0(\Omega)$ and $s \in \Gamma$.

Moreover, $||fs + J_x|| \leq ||f + A_x||||s + J_x||$ and $||f + A_x|| = |f|(x)$ for all $f \in C_0(\Omega)$ and $s \in \Gamma$.

(B) If, in addition, Γ is a $U_0(\Omega)$-normed lattice module, and

$$\Gamma_x := \{s \in \Gamma : |s|(x) = 0\},$$

then the quotient lattice Γ/Γ_x is an m-Banach lattice module over the quotient lattice algebra $C_0(\Omega)/A_x$.

Moreover, the quotient maps $p_x : C_0(\Omega) \longrightarrow C_0(\Omega)/A_x$ and $q_x : \Gamma \longrightarrow \Gamma/\Gamma_x$ are lattice homomorphisms, and $p_x\overline{f} = \overline{p_x f}$ and $q_x\overline{s} = \overline{q_x s}$ for all $f \in C_0(\Omega)$ and $s \in \Gamma$.

In this case, $||s + \Gamma_x|| = |s|(x) = ||s + J_x||$ *for all* $s \in \Gamma$*, and the mapping*

$$\pi_x : \Gamma/J_x \longrightarrow \Gamma/\Gamma_x;\ s + J_x \mapsto s + \Gamma_x$$

is a surjective isometry of Banach modules over $C_0(\Omega)/A_x$*.*

Proof. (A) By Proposition 2.5.2.5(A), J_x is, in particular, an A_x-lattice submodule of Γ. Since A_x is a closed algebra ideal of $C_0(\Omega)$, by Proposition 2.5.3.1(i), we obtain that the Banach space Γ/J_x is a Banach module over the quotient algebra $C_0(\Omega)/A_x$; where the action of $C_0(\Omega)/A_x$ on Γ/J_x is given by the identity

$$(f + A_x) \cdot (s + J_x) := fs + J_x$$

which implies that $||fs + J_x|| \leq ||f + A_x||||s + J_x||$ for all $f \in C_0(\Omega)$ and $s \in \Gamma$. Finally, since A_x is the kernel of dirac functional $\delta_x : C_0(\Omega) \longrightarrow \mathbb{K}; f \mapsto f(x)$, by [20, Lemma 1.3, p.139], we have that $||f + A_x|| = |f|(x)$ for all $f \in C_0(\Omega)$.

(B) By Proposition 2.5.2.5(B), Γ_x is, in particular, a closed A_x-ideal submodule of Γ. Since A_x is a closed lattice algebra ideal of $C_0(\Omega)$, by Proposition 2.5.3.1(ii), we obtain that the Banach module Γ/Γ_x over the quotient lattice algebra $C_0(\Omega)/A_x$ is an m-Banach lattice module over $C_0(\Omega)/A_x$. It follows that the quotient maps $p_x : C_0(\Omega) \longrightarrow C_0(\Omega)/A_x$ and $q_x : \Gamma \longrightarrow \Gamma/\Gamma_x$ are lattice homomorphisms. Moreover, $p_x\overline{f} = \overline{p_x f}$ and $q_x\overline{s} = \overline{q_x s}$ for all $f \in C_0(\Omega)$ and $s \in \Gamma$.

To prove the last assertion, we proceed in the following manner. We first note that $J_x \subseteq \Gamma_x$ (see Proposition 2.5.2.5(B)).

(a) The mapping $\pi_x : \Gamma/J_x \longrightarrow \Gamma/\Gamma_x;\ s + J_x \mapsto s + \Gamma_x$ is a surjective $C_0(\Omega)/A_x$-module homomorphism. Indeed, we first show that it is well-defined. For $s_1, s_2 \in \Gamma$, $s_1 + J_x = s_2 + J_x \iff s_2 - s_1 \in J_x \subseteq \Gamma_x$ implies that $s_1 + \Gamma_x = s_2 + \Gamma_x$ as required. Moreover, it is clearly a surjective and linear mapping, and we have that $\pi_x(fs + J_x) = (f + A_x) \cdot \pi_x(s + J_x)$ for all $f \in C_0(\Omega)$, and $s \in \Gamma$.

(b) We claim that for each $s \in \Gamma$, $|s|(x) = ||s + \Gamma_x||$. Indeed, for each $s \in \Gamma$, we have that

$$|s|(x) \leq \inf \left\{ \sup_{y \in \Omega} |s + r|(y) : r \in \Gamma_x \right\} = \inf \{||s + r|| : r \in \Gamma_x\} = ||s + \Gamma_x||$$

which implies that $|s|(x) = 0$ if and only if $||s + \Gamma_x|| = 0$. Hence, the assertion holds.

(c) Finally, we claim that $||s + \Gamma_x|| = ||s + J_x||$ for each $s \in \Gamma$. Indeed, since

(i) $\operatorname{lin}\{fs_o : f \in A_x \text{ and } s_o \in \Gamma\} \subseteq J_x \subseteq \Gamma_x$, and

(ii) $\{||gs|| : g \in C_0(\Omega)_+,\ g(x) = 1\} \subseteq \{||s + fs_o|| : f \in A_x,\ s_o \in \Gamma\}$

for each $s \in \Gamma$, we have that

$$\begin{aligned}
|s|(x) &= ||s + \Gamma_x|| \\
&= \inf\{||s + r|| : r \in \Gamma_x\} \\
&\leq \inf\{||s + r|| : r \in J_x\} \\
&= ||s + J_x|| \\
&\leq \inf\{||s + fs_o|| : f \in A_x \text{ and } s_o \in \Gamma\} \\
&\leq \inf\{||gs|| : g \in C_0(\Omega)_+, \text{ with } g(x) = 1\}
\end{aligned}$$

for each $s \in \Gamma$, which, by the uniqueness of lattice $U_0(\Omega)$-normed value (see Proposition 2.4.1.10) implies that

$$|s|(x) = ||s + \Gamma_x|| = ||s + J_x||.$$

Thus, the mapping $\pi_x : \Gamma/J_x \longrightarrow \Gamma/\Gamma_x;\ s + J_x \mapsto s + \Gamma_x$ is a surjective isometry of Banach modules over $C_0(\Omega)/A_x$.

□

2.5.4 Dual Banach lattice modules

We seek here to obtain a certain duality between a Banach lattice module Γ over $C_0(\Omega)$ and the corresponding dual Banach lattice module Γ' over $C_0(\Omega)$ analogous to the situation in [21, Proposition 4.17, p.21] for Banach modules over $C_0(\Omega)$. In this direction, we first show that the *continuous* Banach dual space Γ' of a Banach lattice module over $C_0(\Omega)$ is again a Banach lattice module over $C_0(\Omega)$ (see Proposition 2.5.4.1) which we call the *dual Banach lattice module*.

Furthermore, using their result (see Lemma 2.5.4.2), we obtain certain duality results between a Banach lattice module Γ over $C_0(\Omega)$ and its dual Banach lattice module Γ' over $C_0(\Omega)$ (see Proposition 2.5.4.3).

2.5.4.1 Dual Banach lattice module over $C_0(\Omega)$

Here, we consider the *continuous* Banach dual space Γ' of a Banach lattice module Γ over $C_0(\Omega)$. We use the notation B_Γ to represent the unit ball of a Banach space Γ. (c.f. [2, C.I. 3, p.238-239] ; [29, Definition 2.32, p.39] ; [28, II. Section 11, p.133-137]).
We now list a few observations.

(A) Since Γ is a Banach lattice, Γ' coincides with its order dual which is an (order complete) Banach lattice with positive cone (consisting of positive continuous functionals)

$$\Gamma'_+ := \left\{ s' \in \Gamma' : s' \geq 0 \overset{def}{\iff} s'(s) \geq 0 \, \forall s \in \Gamma_+ \right\}.$$

In the real case, for any $s' \in \Gamma'$, its absolute value $\mid s' \mid \in \Gamma'_+$ is defined as

$$\mid s' \mid (s) := \sup \left\{ s'(h) : \mid h \mid \leq s, \, h \in \Gamma \right\} \qquad (s \in \Gamma_+).$$

Moreover, if $\Gamma = \Gamma_{\mathbb{R}} \oplus i\Gamma_{\mathbb{R}}$ is complex, then we can identify $\Gamma' = \Gamma'_{\mathbb{R}} \oplus i\Gamma'_{\mathbb{R}}$ with the real ordered closed subspace (consisting of real continuous functionals)

$$\Gamma'_{\mathbb{R}} := \left\{ s' \in \Gamma' : s' \text{ is real } \overset{def}{\iff} s'(s) \in \mathbb{R} \, \forall s \in \Gamma_{\mathbb{R}} \right\}.$$

Now, for any $s' \in \Gamma'$, its modulus $\mid s' \mid \in \Gamma'_+$ can be defined as

$$\mid s' \mid (s) := \sup \left\{ |s'(h)| : \mid h \mid \leq s, \, h \in \Gamma \right\} \qquad (s \in \Gamma_+).$$

More important, we can observe the following 'duality';

(i) $\Gamma_+ = \{ s \in \Gamma : s'(s) \geq 0 \, \forall \, s' \in \Gamma'_+ \}$, and

(ii) $\Gamma_{\mathbb{R}} = \{ s \in \Gamma : s'(s) \in \mathbb{R} \, \forall \, s' \in \Gamma'_{\mathbb{R}} \}$.

(B) Since Γ is a Banach module over $C_0(\Omega)$, the Banach lattice Γ' can canonically be turned into a module over $C_0(\Omega)$, via

$$C_0(\Omega) \times \Gamma' \longrightarrow \Gamma'; (f, s') \mapsto f \cdot s' := \left[s \mapsto s'(f \cdot s) \right].$$

It is clear that Γ' is a $C_0(\Omega)$- module. Moreover, $||f \cdot s|| \leq ||f||||s||$ for all $f \in C_0(\Omega)$ and $s \in \Gamma$ implies that if $s \in B_\Gamma$, then $f \cdot s \in ||f||B_\Gamma$; so that

$$||f \cdot s'|| = \sup_{s \in B_\Gamma} ||s'(f \cdot s)|| \leq \sup_{s \in ||f||B_\Gamma} ||s'(s)|| \leq ||f||||s'||.$$

Hence, Γ' is also a Banach module over $C_0(\Omega)$.

(C) Now, since Γ is a Banach lattice module over $C_0(\Omega)$, we claim that Γ' is also a Banach lattice module over $C_0(\Omega)$. This can be seen by observing the following.

(i) If $f \in C_0(\Omega)_+$ and $s' \in \Gamma'_+$, then $(f \cdot s')(s) = s'(f \cdot s) \geq 0$ for all $s \in \Gamma_+$ since $f \cdot s \in \Gamma_+$. This implies $f \cdot s' \in \Gamma'_+$, i.e., module 'product' of positive elements is again positive.

(ii) If $f \in C_0(\Omega, \mathbb{R})$ and $s' \in \Gamma'_\mathbb{R}$, then $(f \cdot s')(s) = s'(f \cdot s) \in \mathbb{R}$ for all $s \in \Gamma_\mathbb{R}$ since $f \cdot s \in \Gamma_\mathbb{R}$. And this also implies $f \cdot s' \in \Gamma'_\mathbb{R}$, i.e., module 'product' of real elements is again real.

Hence, Γ' is a Banach lattice module over $C_0(\Omega)$ (see Definition 2.2.1.1).

We put all the preceding information together in the following proposition.

Proposition 2.5.4.1. *Let Γ be a Banach lattice module over $C_0(\Omega)$. Then the following hold.*

(i) The Banach dual Γ' is an (order complete) Banach lattice with positive cone (consisting of positive continuous functionals)

$$\Gamma'_+ := \left\{ s' \in \Gamma' : s' \geq 0 \overset{def}{\iff} s'(s) \geq 0 \, \forall s \in \Gamma_+ \right\}.$$

(ii) The Banach lattice Γ' becomes a Banach module over $C_0(\Omega)$ under the canonical mapping $(f \cdot s')(s) := s'(f \cdot s)$ for all $f \in C_0(\Omega), s \in \Gamma$ and $s' \in \Gamma'$.

(iii) $f \in C_0(\Omega)_+$ and $s' \in \Gamma'_+$ implies $f \cdot s' \in \Gamma'_+$; and also, $f \in C_0(\Omega, \mathbb{R})$ and $s' \in \Gamma'_\mathbb{R}$ implies $f \cdot s' \in \Gamma'_\mathbb{R}$.

In particular, the Banach dual Γ' is an order complete Banach lattice module over $C_0(\Omega)$, called the dual Banach lattice module.

Proof. Assertions (i), (ii) and (iii) follow from preceding considerations (A), (B) and (C), respectively, of Subsection 2.5.4.1, which conclusively implies that the Banach dual Γ' is an order complete Banach lattice and also a Banach lattice module over $C_0(\Omega)$, which we call *the dual Banach lattice module.* □

2.5.4.2 Duality between AM- and AL- lattice modules over $C_0(\Omega)$

The following lemma will be important for this consideration (see also [6, Theorem 5, p.622]).

Lemma 2.5.4.2. *[21, Proposition 4.17, p.21] Let Ω be a locally compact space and Γ a Banach module over $C_0(\Omega)$. Furthermore, let Γ' be the dual Banach module over $C_0(\Omega)$. Then the following hold.*

(i) *Γ is an AM-module if and only if Γ' is an AL-module.*

(ii) *Γ is an AL-module if and only if Γ' is an AM-module.*

Combining Proposition 2.5.4.1, Lemma 2.5.4.2 with Definitions 2.4.1.2 and 2.4.2.2, we present our duality results.

Proposition 2.5.4.3. *Let Ω be a locally compact space and Γ a Banach lattice module over $C_0(\Omega)$. Furthermore, let Γ' be the dual Banach lattice module over $C_0(\Omega)$. Then the following hold.*

(A) *The following are equivalent:*

(i) *Γ is an AM lattice module.*

(ii) *Γ' is an order complete AL lattice module.*

(B) *The following are equivalent:*

(i) *Γ is an AL lattice module.*

(ii) *Γ' is an order complete AM lattice module.*

Proof. (A) Since, by Proposition 2.5.4.1, the dual Banach lattice module Γ' is always an order complete Banach lattice, the equivalence $(i) \Leftrightarrow (ii)$ follows from Lemma 2.5.4.2(i).

(B) Similar to (A) above, by Lemma 2.5.4.2(ii), the equivalence $(i) \Leftrightarrow (ii)$ follows .

□

2.5.5 Banach lattice modules and lattice isomorphisms

In this Subsection, we consider lattice isomorphisms of a Banach lattice module Γ over $C_0(\Omega)$ with Ω a locally compact space. In particular, we claim that every lattice isomorphism determines a unique lattice module isomorphism in the following way.

Proposition 2.5.5.1. *Let Γ be a Banach lattice module over $C_0(\Omega)$. Furthermore, let Λ be an arbitrary Banach lattice. Then, the following are equivalent:*

(i) $i : \Gamma \longrightarrow \Lambda$ *is an isomorphism of Banach lattices.*

(ii) $i : \Gamma \longrightarrow \Lambda$ *is an isomorphism of Banach lattice modules over* $C_0(\Omega)$.

Moreover, if this assertion holds, Γ is an m-Banach lattice module over $C_0(\Omega)$ if and only if Λ is an m-Banach lattice module over $C_0(\Omega)$.

Proof.

Clearly $(ii) \implies (i)$ (see Remark 2.3.2.2(iii)).

Now assume (i) holds. We claim that Λ can canonically be turned into a Banach module over $C_0(\Omega)$ for which $i : \Gamma \longrightarrow \Lambda$ becomes module isomorphism. To do this, we consider the following.

(a) Since $i : \Gamma \longrightarrow \Lambda$ is a bijection; for each $s \in \Gamma$, the pairing

$$C_0(\Omega) \times \Lambda \longrightarrow \Lambda;\ (f, is) \mapsto f \cdot is := i(f \cdot s)$$

turn Λ into $C_0(\Omega)$-module, and $i : \Gamma \longrightarrow \Lambda$ into module homomorphism.

(b) Since $i : \Gamma \longrightarrow \Lambda$ is a surjective isometry, for every $f \in C_0(\Omega)$ and $s \in \Gamma$,

$$||f \cdot is|| = ||i(f \cdot s)|| = ||f \cdot s|| \leq ||f||||s|| = ||f||||is||$$

implies that Λ is a Banach module over $C_0(\Omega)$.

(c) Since the restriction $i : \Gamma_+ \longrightarrow \Lambda_+$ is also a bijection, it follows that if $f \in C_0(\Omega)_+$ and $s \in \Gamma_+$, then $f \cdot is = i(f \cdot s) \in \Lambda_+$, i.e., the module 'product' of positive elements is again positive. In the complex case, since the restriction $i : \Gamma_{\mathbb{R}} \longrightarrow \Lambda_{\mathbb{R}}$ is again a bijection, it follows that if $f \in C_0(\Omega, \mathbb{R})$ and $s \in \Gamma_{\mathbb{R}}$, then $f \cdot is = i(f \cdot s) \in \Lambda_{\mathbb{R}}$, i.e., the module 'product' of real elements is again real.

Hence, Λ is a Banach lattice module over $C_0(\Omega)$ (see Definition 2.2.1.1). This implies (ii).

Moreover, since $i : \Gamma \longrightarrow \Lambda$ is a lattice module isomorphism, and observing that, for each $f \in C_0(\Omega)$ and $s \in \Gamma$,

$$|f \cdot is| = |f| \cdot |is| \iff |f \cdot s| = |f| \cdot |s|$$

implies that the last assertion holds. □

From the last assertion of the above proposition, the following is an immediate corollary. That is, our concept of an m-Banach lattice module is invariant under lattice module isomorphism.

Corollary 2.5.5.2. *Let Γ and Λ be Banach lattice modules over $C_0(\Omega)$. If $i : \Gamma \longrightarrow \Lambda$ is lattice module isomorphism, then Γ is an m-Banach lattice module over $C_0(\Omega)$ if and only if Λ is an m-Banach lattice module over $C_0(\Omega)$.*

Using the above result in Proposition 2.5.5.1 and the duality between AM-spaces with unit and AL-spaces (see [28, Proposition 9.1 p.121]), we obtain the following result.

Proposition 2.5.5.3. *Let Γ be a unitary Banach lattice module over $C(K)$ with K compact, and Γ' the dual Banach lattice module over $C(K)$. Then, the following are equivalent.*

(i) *The interior $Int\Gamma_+$ is non-empty.*

(ii) *Γ is isometrically lattice module isomorphic to $C(Q)$ for some compact space Q.*

(iii) *Γ' is isometrically lattice module isomorphic to $L^1(Y)$ for some σ-finite measure space Y.*

Proof. $(i) \implies (ii)$: Assume the interior $Int\Gamma_+ \neq \emptyset$. By Appendix B (Theorem B.0.0.6), WLOG we can find a compact space Q such that $i : \Gamma \longrightarrow C(Q)$ is an isomorphism of Banach lattices. Furthermore, by Proposition 2.5.5.1, we obtain that $C(Q)$ is a Banach lattice module over $C(K)$, and $i : \Gamma \longrightarrow C(Q)$ is an isomorphism of Banach lattice modules over $C(K)$. In particular, $i : \Gamma \longrightarrow C(Q)$ is an isometric lattice module isomorphism.

$(ii) \implies (iii)$: That $\Gamma \cong C(Q)$ as Banach lattices, in particular, implies that Γ is an AM-space with unit. As such the dual Banach lattice Γ' is an AL-space, i.e., $||s' + r'|| = ||s'|| + ||r'||$ for every $s', r' \in \Gamma'_+$. Now, by the representation theorem of AL-space (see [28, , Theorem 8.5, p.114]) we can find a σ-finite measure space Y, such that $j : \Gamma' \longrightarrow L^1(Y)$ is an isomorphism of Banach lattices. By Proposition 2.5.5.1, we obtain that $L^1(Y)$ is a Banach lattice module over $C(K)$, and $j : \Gamma' \longrightarrow L^1(Y)$ is an isomorphism of Banach lattice modules over $C(K)$. Thus, $j : \Gamma' \longrightarrow L^1(Y)$ is an isometric lattice module isomorphism.

$(iii) \implies (i)$: That $\Gamma' \cong L^1(Y)$ as Banach lattices, in particular, implies that Γ' is an AL-space. As such, the double dual Banach lattice Γ'' is an AM-space with unit. Since $\Gamma \hookrightarrow \Gamma''$ is an isometry embedding, it necessarily follows that Γ is also an AM-space with unit. Hence, the interior $Int\Gamma_+$ is non-empty.

□

2.5.6 Banach lattice modules and lattice ideals

In this Subsection, we consider lattice ideals of a *unitary* m-Banach lattice module Γ over $C(K)$ with K compact (see Remark 2.2.4.2). In particular, we claim that, every (closed) lattice ideal of Γ is a (closed) ideal submodule of Γ in the next proposition. To this end, the following lemma will be useful.

Lemma 2.5.6.1. *Let Γ be an m-BLM over L. Then, the following hold, for any $f, g \in L_+$ and $s, r \in \Gamma_+$.*

(i) $f \leq g$ *implies* $fs \leq gs$.

(ii) $s \leq r$ *implies* $fs \leq fr$.

(iii) $f \leq g$ *and* $s \leq r$ *implies* $fs \leq gr$.

Proof. (i) Considering $|gs - fs| = |(g - f)s| = |g - f|s$, it follows that $gs - fs \geq 0$ if $g - f \geq 0$.

(ii) Similarly, considering $|fr - fs| = |f(r-s)| = \mathrm{f}|r-s|$, it also follows that $fr - fs \geq 0$ if $r - s \geq 0$.

(iii) Combining (i) and (ii), it follows that $fs \leq gs \leq gr$.

$\square$

Proposition 2.5.6.2. *Let Γ be a unitary m-Banach lattice module over $C(K)$ with K compact. Then, the following are equivalent.*

(i) $I_\Gamma \subseteq \Gamma$ is a (closed) lattice ideal of Γ.

(ii) $I_\Gamma \subseteq \Gamma$ is a (closed) ideal submodule of Γ .

Proof. Clearly $(ii) \implies (i)$ by definition (see Definition 2.5.2.3).

Now assume (i) holds. We claim that $fs \in I_\Gamma$ for all $f \in C(K)$ and $s \in I_\Gamma$. Indeed, for any $f \in C(K)$ and $s \in I_\Gamma$,

$|f| \leq ||f||\mathbb{1}_K$ implies that $|fs| = |f||s| \leq ||f||\mathbb{1}_K|s|$ by Lemma 2.5.6.1. Now, since Γ is unitary (see Remark 2.2.4.2) and I_Γ is a lattice ideal,

$|fs| \leq ||f|||s| \in I_\Gamma$ implies that $fs \in I_\Gamma$ as required.

$\square$

Chapter 3

Positive semiflows on *topological* Banach lattice bundles

3.1 Introduction

In this chapter, we introduce the notions of *topological* Banach lattice bundles (see Section 3.2), the spaces of their continuous sections (see Section 3.3), morphisms of these Banach lattice bundles (see Section 3.4), and their representation theories (see Section 3.5). Moreover, in Section 3.6, we further our consideration on the Banach lattice of continuous sections vanishing at infinity associated with a *topological* Banach lattice bundle over a locally compact space.

We start in Section 3.2 with our definition of a *topological* Banach lattice bundle E over a locally compact space Ω (see Definition 3.2.0.1). In Section 3.3, we study the Banach space $\Gamma_0(\Omega, E)$ of its continuous sections vanishing at infinity, which becomes a Banach lattice (see Proposition 3.3.0.2).

In Section 3.4, we introduce two notions of morphisms between two Banach lattice bundles E and F over Ω: namely, a positive Banach lattice bundle morphism over a continuous map $\varphi : \Omega \longrightarrow \Omega$ (see Definition 3.4.0.1) and a Banach lattice bundle morphism over a continuous map $\varphi : \Omega \longrightarrow \Omega$ (see Remark 3.4.0.3(iii)). In (Definition 3.4.0.5) Remark 3.4.0.6(iii), we introduce the notion of a (positive) S-dynamical Banach lattice bundle $(E, \mathbf{\Phi})$ over a *topological* G-dynamical system (Ω, φ), and we call $\mathbf{\Phi} = (\Phi_g)_{g \in S}$ a (positive) semiflow on E over the flow $\varphi = (\varphi_g)_{g \in G}$ on Ω.

In Section 3.5, we obtain that every (positive) S-dynamical *topological* Banach lattice bundle $(E, \mathbf{\Phi})$ ove a G-dynamical system (Ω, φ) induces a (positive) S-dynamical Banach lattice module $(\Gamma_0(\Omega, E), \mathcal{T}_{\mathbf{\Phi}})$ over the Koopman group representation $(C_0(\Omega), \boldsymbol{T}_\varphi)$ (see Remark 3.5.0.1(v) and also Chapter 2, Example 2.3.4.3(i)). We call $\mathcal{T}_{\mathbf{\Phi}} = \mathcal{T}_\Phi(g)_{g \in S}$ the (positive) weighted Koopman semigroup representation on $\Gamma_0(\Omega, E)$ over the Koopman group $\boldsymbol{T}_\varphi = (T_\varphi(g))_{g \in G}$ induced by $(E, \mathbf{\Phi})$. Moreover, we obtain the abstract representation of the Banach lattice $\Gamma_0(\Omega, E)$ of continuous sections vanishing at infinity of a *topological* Banach lattice bundle E over a locally compact space Ω as what we call an AM m-lattice module over the Banach lattice algebra $C_0(\Omega)$ (see Appendix D, Question D.1.0.2, Remark 3.5.0.1(ii) and Proposition 3.5.0.9). Moreover, as one major result of our study, every (positive) S-dynamical AM m-lattice module over the Koopman group $(C_0(\Omega), \boldsymbol{T}_\varphi)$ can be assigned uniquely to a (positive) S-dynamical *topological* Banach lattice bundle $(E, \mathbf{\Phi})$ over a G-dynamical system (Ω, φ) and vice versa (see also Appendix D Proposition D.1.0.3). This is our Gelfand-type theorem for dynamical AM m-lattice modules (see Theorem 3.5.0.11 and Corollary 3.5.0.12).

In the last Section 3.6, we further our consideration on the Banach lattice $\Gamma_0(\Omega, E)$ of continuous sections (vanishing at infinity) of a *topological* Banach lattice bundle E over a (locally) compact space Ω. We introduce the notions of a (Banach) lattice subbundle and a (closed) ideal subbundle of E (see Definitions 3.6.1.1 and 3.6.3.1 respectively). As major results, we obtain a certain correspondence between the notions of Banach lattice subbundles of E and Banach lattice submodules of $\Gamma(K, E)$ (see Proposition 3.6.1.3) and as well as a correspondence between the notions of closed ideal subbundles of E and closed ideal submodules of $\Gamma(K, E)$ (see Corollary 3.6.3.4). In Subsection 3.6.4, we introduce and obtain a certain correspondence between the notions of direct sums (see Subsection 3.6.2) and decompositions of positive semiflows on two *topological* Banach lattice bundles E and F over a compact space K, and the direct sum of two positive weighted Koopman semigroups on the direct sum of the two AM m-lattice modules $\Gamma(K, E)$ and $\Gamma(K, F)$. These concepts, in particular, are proven very useful in our consideration of asymptotics of positive weighted Koopman semigroups treated in Appendix E. We conclude the Section with two other considerations. One is about certain order structures: namely, non-emptiness of the positive cone,

(σ-) order completeness, and order continuity of the norm of the Banach lattice $\Gamma_0(\Omega, E)$ in Subsection 3.6.5, which also appeared in Chapter 4; and the other, which is also of independent interest, is the characterisation of the so-called centre of this Banach lattice (see Subsection 3.6.6).

3.2 *Topological* Banach lattice bundles

In this Section, we introduce our definition of a *topological* Banach lattice bundle. Every locally compact space is assumed to be Hausdorff. See [29, Chapter 1; Section 1.1, p.11-12] for the notion of a *topological* Banach bundle. See also [16, note on p.299][1] and [23, Section 1 and 2, p.41-42].

Definition 3.2.0.1. *Let E be a topological space (the total space), Ω a locally compact space (the base space), and $p_E : E \longrightarrow \Omega$ a continuous, open, and surjective mapping (bundle projection). Then the triple (E, Ω, p_E), denoted by $p_E : E \longrightarrow \Omega$, is called a topological Banach lattice bundle over Ω if the following conditions are satisfied.*

(i) For each $x \in \Omega$, the fiber $E_x := p_E^{-1}(x)$ is a Banach lattice.

(ii) The mappings

$$+ : E \times_\Omega E \longrightarrow E, \quad (u, v) \mapsto u +_{E_{p_E(v)}} v,$$

$$\cdot : \mathbb{K} \times E \longrightarrow E, \quad (\lambda, v) \mapsto \lambda ._{E_{p_E(v)}} v$$

are continuous where $E \times_\Omega E := \cup_{x \in \Omega} E_x \times E_x \subseteq E \times E$ is equipped with the subspace topology.

(iii) The mapping (bundle norm)

$$|| \cdot || : E \longrightarrow \mathbb{R}_+, \quad v \mapsto ||v||_{E_{p_E(v)}}$$

is upper semicontinuous.

[1]We note that there is a slight difference in their definition (of a *topological* bundle of Banach lattices, which we call a *topological* Banach lattice bundle for brevity) to ours, but each of our Banach lattice bundles canonically defines a *topological* bundle of Banach lattices according to their definition having the same space of continuous sections.

(iv) The mapping (bundle modulus)

$$|\cdot| : E \longrightarrow E, \quad v \mapsto |v|_{E_{p_E(v)}}$$

is continuous.

(v) For each $x \in \Omega$ and each open set $W \subseteq E$ containing zero $0_x \in E_x$, there exist $\varepsilon > 0$ and an open set $U \subseteq \Omega$ such that

$$\left\{ v \in p_E^{-1}(U) \mid ||v||_{E_{p_E(v)}} \leq \varepsilon \right\} \subseteq W.$$

If, in addition, the bundle norm $||\cdot||$ in (iii) above is continuous, then $p_E : E \longrightarrow \Omega$ is called a continuous topological Banach lattice bundle over Ω.

Note 3.2.0.2. *(i) If no confusion arises, we say E is a topological Banach lattice bundle over Ω, while we mean $p_E : E \longrightarrow \Omega$ or simply $p : E \longrightarrow \Omega$.*

(ii) Moreover, the only additional properties we added to the definition of a topological Banach bundle as defined in [29, Chapter 1, Definition 1.1, p.11] in our setting of topological Banach lattice bundle is that:

(a) each fiber is a Banach lattice instead of a Banach space; and

(b) the continuity of the "bundle modulus".

Thus every topological Banach lattice bundle is, in particular, a topological Banach bundle in our situation.

3.3 The space of continuous sections

A *topological* Banach lattice bundle induces a natural ordered vector space.

Definition 3.3.0.1. *Let $p_E : E \longrightarrow \Omega$ be a topological Banach lattice bundle over Ω. The vector space of its continuous sections is given by*

$$\Gamma(\Omega, E) := \{ s : \Omega \longrightarrow E \mid p_E \circ s = Id_\Omega, \ s \text{ is continuous} \}$$

endowed with pointwise addition and pointwise scalar multiplication. Now, consider its linear subspace of continuous sections vanishing at infinity, defined by

$$\Gamma_0(\Omega, E) := \{ \, s \in \Gamma(\Omega, E) | \, \forall \, \varepsilon > 0, \, \exists \text{ compact subset } K_\varepsilon \subseteq \Omega \\ \text{such that } ||s(x)||_{E_x} < \varepsilon \, \forall \, x \in \Omega \setminus K_\varepsilon \, \}$$

which becomes a Banach space when equipped with the sup-norm
$||\cdot|| : \Gamma_0(\Omega, E) \longrightarrow \mathbb{R}_+; \ s \mapsto \sup_{x \in \Omega} ||s(x)||_{E_x}$.

Defining for each $s \in \Gamma_0(\Omega, E)$ the mapping

$$\mathsf{I}\, s\, \mathsf{I} : \Omega \longrightarrow E;\ x \mapsto \mathsf{I}\, s(x)\, \mathsf{I}_{E_x^+}$$

it follows from the continuity of the "bundle modulus" that this mapping is continuous, and hence it is a continuous section. Moreover, it easily follows that $||\,\mathsf{I}\, s\, \mathsf{I}\,|| = ||s||$ for any $s \in \Gamma_0(\Omega, E)$.

If $\mathbb{K} = \mathbb{R}$, by defining the set

$$\Gamma_0(\Omega, E)_+ := \left\{ s \in \Gamma_0(\Omega, E) : s \geq 0 \overset{def}{\iff} \mathsf{I}\, s\, \mathsf{I} = s \right\}$$

we obtain a partial order on $\Gamma_0(\Omega, E)$ by saying $s_1 \leq s_2 \overset{def}{\iff} s_2 - s_1 \geq 0$.
If $\mathbb{K} = \mathbb{C}$, defining the set

$$\Gamma_0(\Omega, E)_{\mathbb{R}} := \{ s \in \Gamma_0(\Omega, E) : Re\ s = s \}$$

we obtain a natural ordering on $\Gamma_0(\Omega, E)_{\mathbb{R}}$ as above.

More importantly, we claim the following.

Proposition 3.3.0.2. *For a topological Banach lattice bundle E over a locally compact space Ω; we have the following properties on the Banach space $\Gamma_0(\Omega, E)$ of its continuous sections vanishing at infinity.*

(i) If $\mathbb{K} = \mathbb{R}$, then $\Gamma_0(\Omega, E)$ is an ordered Banach space with normal positive cone $\Gamma_0(\Omega, E)_+$.

(ii) If $\mathbb{K} = \mathbb{C}$, then $\Gamma_0(\Omega, E)_{\mathbb{R}}$ is an ordered closed subspace of $\Gamma_0(\Omega, E)$.

In particular, $\Gamma_0(\Omega, E)$ is a Banach lattice such that:

(a) if $\mathbb{K} = \mathbb{R}$, then $\Gamma_0(\Omega, E) = \Gamma_0(\Omega, E)_+ - \Gamma_0(\Omega, E)_+$ is a real Banach lattice, and

(b) if $\mathbb{K} = \mathbb{C}$, then $\Gamma_0(\Omega, E) = \Gamma_0(\Omega, E)_{\mathbb{R}} \oplus i\Gamma_0(\Omega, E)_{\mathbb{R}}$ is a complex Banach lattice.

For the proof of Proposition 3.3.0.2, we will require some more results. We include its proof in the proof of Proposition 3.3.0.5.

In the following lemma, we state certain important properties associated with a (general) *topological* Banach bundle over Ω.

Lemma 3.3.0.3. *Let E be a Banach bundle over Ω. Then, the following hold.*

(A) Let $f_1 : E \times_\Omega E \longrightarrow E; (u,v) \mapsto f_1(u,v)_{E_{p_E(v)}}$ and $f_2 : E \times_\Omega E \longrightarrow E; (u,v) \mapsto f_2(u,v)_{E_{p_E(v)}}$ be mappings. Then,

(i) the mappings $f_a : E \longrightarrow E; u \mapsto f_1(u,v)_{E_{p_E(v)}}$ and $f_b : E \longrightarrow E; v \mapsto f_1(u,v)_{E_{p_E(v)}}$, for each fixed $v \in E$ and $u \in E$ respectively, are both continuous if f_1 is continuous.

(ii) the mapping $f_1 + f_2 : E \times_\Omega E \longrightarrow E; (u,v) \mapsto f_1(u,v)_{E_{p_E(v)}} + f_2(u,v)_{E_{p_E(v)}}$ is continuous if f_1 and f_2 are both continuous.

(B) Let $h : \mathbb{K} \times E \longrightarrow E; (\lambda, v) \mapsto h(\lambda, v)_{E_{p_E(v)}}$ be mapping. Then,

the mappings $h_a : E \longrightarrow E; v \mapsto h(\lambda, v)_{E_{p_E(v)}}$ and $h_b : \mathbb{K} \longrightarrow E; \lambda \mapsto h(\lambda, v)_{E_{p_E(v)}}$, for each fixed $\lambda \in \mathbb{K}$ and $v \in E$ respectively, are both continuous if h is continuous.

(C) Let $g_1 : E \longrightarrow E; v \mapsto g_1(v)_{E_{p_E(v)}}$ and $g_2 : E \longrightarrow E; v \mapsto g_2(v)_{E_{p_E(v)}}$ be mappings. Then,

(i) the mapping $\lambda g_a : \mathbb{K} \times E \longrightarrow E; (\lambda, v) \mapsto \lambda g_1(v)_{E_{p_E(v)}}$ is continuous if g_1 is continuous.

(ii) the mapping $g_1 + g_2 : E \longrightarrow E; v \mapsto g_1(v)_{E_{p_E(v)}} + g_2(v)_{E_{p_E(v)}}$ is continuous if g_1 and g_2 are both continuous.

Proof. (A) (i) Let $\pi_1 : E \times_\Omega E \longrightarrow E; (u,v)_{E_{p_E(v)}} \mapsto u$ and $\pi_2 : E \times_\Omega E \longrightarrow E; (u,v)_{E_{p_E(v)}} \mapsto v$ be projections onto the first and second factor respectively, and consider the following commutative diagrams.

$$\begin{tikzcd}
E \times_\Omega E \arrow[r, "\pi_1"] & E \\
E \times_\Omega E \arrow[u, "(u,v)\mapsto(f_1(u,v),v)"] \arrow[ur, "f_1"] & E \arrow[l, "(u,v)E_{p_E(v)} \leftarrow\!\shortmid u"'] \arrow[u, dashed, "f_a"']
\end{tikzcd}$$

$$\begin{tikzcd}
E \times_\Omega E \arrow[r, "\pi_2"] & E \\
E \times_\Omega E \arrow[u, "(u,v)\mapsto(u,f_1(u,v))"] \arrow[ur, "f_1"] & E \arrow[l, "(u,v)E_{p_E(v)} \leftarrow\!\shortmid v"'] \arrow[u, dashed, "f_b"']
\end{tikzcd}$$

Since $E \times_\Omega E = \bigcup_{x\in\Omega} E_x \times E_x \subseteq E \times E$ is equipped with the (subspace) product topology, it follows that if f_1 is continuous, then f_a and f_b are both continuous.

This is because the projections π_1 and π_2 are both continuous (surjective open) mappings.

(ii) Consider the following commutative diagram.

$$\begin{tikzcd}
E \times_\Omega E \arrow[r, "(u,v) \overset{f}{\longmapsto} (f_1(u,v),f_2(u,v))"] \arrow[dr, dashed, "f_1+f_2"'] & E \times_\Omega E \arrow[d, "(u,v)\mapsto u+_{E_{p_E(v)}} v"] \\
& E
\end{tikzcd}$$

It follows that $f_1 + f_2$ is continuous if f is continuous if f_1 and f_2 are both continuous.

(B) Let $\pi_E : \mathbb{K} \times E \longrightarrow E; (\lambda, v)_{E_{p_E(v)}} \mapsto v$ be a projection onto the second factor, and consider the following commutative diagrams.

$$\begin{tikzcd}
\mathbb{K} \times E \arrow[r, "\pi_E"] & E \\
\mathbb{K} \times E \arrow[u, "(\lambda,v)\mapsto(\lambda,h(\lambda,v))"] \arrow[ur, "h"'] & E \arrow[l, "(\lambda,v)E_{p_E(v)} \mapsfrom v"] \arrow[u, dashed, "h_a"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K} \times E \arrow[r, "\pi_E"] & E \\
\mathbb{K} \times E \arrow[u, "(\lambda,v)\mapsto(\lambda,h(\lambda,v))"] \arrow[ur, "h"'] & \mathbb{K} \arrow[l, "(\lambda,v)E_{p_E(v)} \mapsfrom \lambda"] \arrow[u, dashed, "h_b"']
\end{tikzcd}$$

Since $\mathbb{K} \times E$ is equipped with the product topology, π_E is a continuous (surjective open) mapping, and so it follows that if h is continuous, then h_a and h_b are both continuous.

(C) (i) Consider the following commutative diagram.

$$\begin{tikzcd}
\mathbb{K} \times E \arrow[rr, "(\lambda,u)\mapsto\lambda_{E_{p_E(v)}} v"] & & E \\
\mathbb{K} \times E \arrow[u, "(\lambda,v)\mapsto(\lambda,g_1(v))"] \arrow[urr, dashed, "\lambda g_a"'] & & E \arrow[u, "g_1"'] \arrow[ll, "(\lambda,v)E_{p_E(v)} \leftarrow\!\shortmid v"]
\end{tikzcd}$$

We see that if g_1 is continuous, then λg_a is continuous.

(ii) Consider the following commutative diagram.

$$\begin{tikzcd}
E \times_\Omega E \arrow[rr, "(u,v)\mapsto u+_{E_{p_E(v)}} v"] & & E \\
E \times_\Omega E \arrow[u, "(u,v)\mapsto(g_1(v),g_2(v))"] \arrow[urr, dashed, "q"'] & &
\end{tikzcd}$$

It follows that if g_1 and g_2 are both continuous, then q is continuous. From $(A)(i)$ above we see that if q is continuous, then, in particular, the mapping $E \longrightarrow E; v \mapsto q(v,v)$ is continuous. And since $q(v,v) = g_1(v) + g_2(v)$ the assertion follows.

$\square$

In what follows, by a Banach lattice bundle over Ω, we always mean a *topological* Banach lattice bundle over a locally compact space Ω.

Using the above Lemma 3.3.0.3, we can now state and prove the following properties of a *topological* Banach lattice bundle.

Proposition 3.3.0.4. *Let E be a Banach lattice bundle over Ω. Then the following hold.*

(A) If $\mathbb{K} = \mathbb{R}$, i.e., E_x is a real Banach lattice for all $x \in \Omega$, then

(i) the mappings (bundle join, bundle meet)

$$E \times_\Omega E \longrightarrow E; (u,v) \mapsto u \vee_{E_{p_E(v)}} v,$$

$$E \times_\Omega E \longrightarrow E; (u,v) \mapsto u \wedge_{E_{p_E(v)}} v$$

are continuous; and

(ii) the mappings

$$E \longrightarrow E; v \mapsto v^{+}_{E_{p_E(v)}},$$
$$E \longrightarrow E; v \mapsto v^{-}_{E_{p_E(v)}}$$

are continuous.

In addition, the set $E_+ := \left\{v \in E : |v|_{E_{p_E(v)}} = v\right\}$ *is a closed subspace of* E.

(B) If $\mathbb{K} = \mathbb{C}$, *i.e.,* $E_x = E_x^{\mathbb{R}} \oplus iE_x^{\mathbb{R}}$ *is complex Banach lattice for all* $x \in \Omega$, *then*

(i) the mappings

$$E \longrightarrow E; v \mapsto Re\ v_{E_{p_E(v)}},$$
$$E \longrightarrow E; v \mapsto Im\ v_{E_{p_E(v)}},$$
$$E \longrightarrow E; v \mapsto \overline{v}_{E_{p_E(v)}}$$

are continuous; and

(ii) the mappings

$$E \times_\Omega E \longrightarrow E; (u,v) \mapsto u \vee_{E_{p_E(v)}} v,$$
$$E \times_\Omega E \longrightarrow E; (u,v) \mapsto u \wedge_{E_{p_E(v)}} v$$

are continuous.

Moreover, the set $E_{\mathbb{R}} := \left\{v \in E : Re\ v_{E_{p_E(v)}} = v\right\}$ *is a closed subspace of* E.

Proof. (A) (i) By defining the mappings $f_1, f_2 : E \times_\Omega E \longrightarrow E; f_1(u,v) = \frac{1}{2}(u+v)_{E_{p_E(v)}}$ and $f_2(u,v) = \frac{1}{2}\,|\,u - v\,|\,_{E_{p_E(v)}}$, we see that $u \vee_{E_{p_E(v)}} v = f_1(u,v) + f_2(u,v)$ and $u \wedge_{E_{p_E(v)}} v = f_1(u,v) - f_2(u,v)$. Hence, by Lemma 3.3.0.3 $A(ii)$, it suffices to show that both f_1 and f_2 are continuous.

That f_1 is continuous is immediate, and for f_2 consider the following commutative diagram.

$$\begin{tikzcd}
E \times_\Omega E \arrow[rr, "(u,v)_{E_{p_E(v)}} \mapsto u-v"] \arrow[drr, dashed, "f_2"'] & & E \arrow[d, "v_{E_{p_E(v)}} \mapsto \frac{1}{2}\,|\,v\,|"] \\
& & E
\end{tikzcd}$$

And we see that f_2 is a composition of two continuous mappings, and hence continuous.

(ii) By setting $o := o_{E_{p_E(v)}}$, it follows from (i) above that the mappings $E \times_\Omega E \longrightarrow E; (o,v)_{E_{p_E(v)}} \mapsto v \vee o$ and $E \times_\Omega E \longrightarrow E; (o,v)_{E_{p_E(v)}} \mapsto v \wedge o$ are both continuous. By Lemma 3.3.0.3 $A(i)$ and $C(i)$, this, in particular, implies that the mappings (with $\lambda = -1$)

$E \longrightarrow E; v_{E_{p_E(v)}} \mapsto v \vee o$ and $E \longrightarrow E; v_{E_{p_E(v)}} \mapsto -(v \wedge o)$ are both continuous. Since $v^+_{E_{p_E(v)}} = v \vee o$ and $v^-_{E_{p_E(v)}} = -(v \wedge o)$, the assertion follows.

Moreover, by setting $E^c_+ := \left\{ v \in E : |v| \neq v_{E_{p_E(v)}} \right\}$, the complement of E_+, it suffices to show that E^c_+ is an open subspace. Indeed, for $v \in E^c_+$, since $v \neq o_{E_{p_E(v)}}$ we can find an open set $U \subseteq \Omega$ containing $p_E(v)$ and $\varepsilon > 0$ (small enough) such that the open set

$S(s, U, \varepsilon) := \left\{ w \in p_E^{-1}(U) : ||w - s(p_E(w))||_{E_{p_E(w)}} < \varepsilon \right\} \subseteq E^c_+$,
for all/some $s \in \Gamma_0(\Omega, E)$ with $s(p_E(v)) = v$.

This is the case since the set of the form $S(s, U, \varepsilon)$ forms a base for the topology on E. (cf. [29, Lemma 1.4, p.13]).

From here, it is clear that for each $v \in E_+$, the set of the form $S(s, U, \varepsilon)_+ := \{w \in S(s, U, \varepsilon) : w \in E_+\}$ for some open set $U \subseteq \Omega$ containing $p_E(v)$ and $\varepsilon > 0$, is a base for a topology on E_+; which coincides with its subspace topology from E.

(B) (i) Since for $v \in E_{p_E(v)}$, $Re\ v = (Re\ v)^+ - (Re\ v)^-$, we can infer from (A) (ii) above that the mappings $E \longrightarrow E; Re\ v \mapsto (Re\ v)^+$ and $E \longrightarrow E; Re\ v \mapsto (Re\ v)^-$ are both continuous.

Now, setting $g_1 : E \longrightarrow E; v_{E_{p_E(v)}} \mapsto (Re\ v)^+$ and $g_2 : E \longrightarrow E; v_{E_{p_E(v)}} \mapsto (Re\ v)^-$; by Lemma 3.3.0.3 $C(ii)$, it suffices to show that g_1 and g_2 are both continuous, since $g_1(v) + g_2(v) = Re\ v$.

As a result, we will only show that g_1 is continuous, since the proof would be essentially the same for g_2.

For $v \in E$, let $W \subseteq E$ be an open set containing $(Re\ v)^+$. We can choose an open set $W_0 \subseteq E$ containing $|v| = \sup_{t \in \mathbb{Q}} \left\{ Re\ e^{\pi i t} v \right\}$ which also contains the set

$$\left\{ (Re\ e^{\pi i t} v)^+ : \text{ for some/all } t \in [0, \delta] \cap \mathbb{Q} \text{ and small } \delta > 0 \right\}.$$

In this way $W_0 \cap W \neq \emptyset$, and by the continuity of the mappings $E \longrightarrow E; v_{E_{p_E(v)}} \mapsto |v|$ and $E \longrightarrow E; Re\ v \mapsto (Re\ v)^+$, we can find an open set $W_1 \subseteq E$ containing v, such that the set

$\{(Re\ e^{\pi i t} w)^+ : \forall w \in W_1$ for some/all $t \in [0, \delta] \cap \mathbb{Q}$ and small $\delta > 0\} \subseteq W_0 \cap W$.

In particular, $|w| \in W_0$ and $(Re\ w)^+ \in W$ for all $w \in W_1$.

By a similar argument, we can show that the mapping $E \longrightarrow E; v \mapsto Im\ v_{E_{p_E(v)}}$ is continuous. So, by Lemma 3.3.0.3B and $C(i)$, it follows that the mapping (with $\lambda = i$)

$$E \longrightarrow E;\ v_{E_{p_E(v)}} \mapsto iIm\ v$$

is continuous. Hence, by Lemma 3.3.0.3 $C(ii)$, the mapping

$$E \longrightarrow E;\ v_{E_{p_E(v)}} \mapsto \overline{v} = Re\ v - iIm\ v$$

is continuous.

(ii) The continuity of the two mappings follows immediately from $A(i)$ above, since "join" and "meet" are always meant for the real parts in a complex vector lattice.

Moreover, similar to the case of E_+ in (A) above, by setting $E^c_{\mathbb{R}} := \left\{v \in E : Re\ v \neq v_{E_{p_E(v)}}\right\}$, the complement of $E_{\mathbb{R}}$, it suffices to show that $E^c_{\mathbb{R}}$ is an open subspace. Indeed, for $v \in E^c_{\mathbb{R}}$, we can find an open set $U \subseteq \Omega$ containing $p_E(v)$ and $\varepsilon > 0$ (small enough) such that the open set

$S(s, U, \varepsilon) := \left\{w \in p_E^{-1}(U) : ||w - s(p_E(w))||_{E_{p_E(w)}} < \varepsilon\right\} \subseteq E^c_{\mathbb{R}}$,
for all/some $s \in \Gamma_0(\Omega, E)$ with $s(p_E(v)) = v$.

Similarly, we can infer that, for $v \in E_{\mathbb{R}}$ the set of the form $S(s, U, \varepsilon)_{\mathbb{R}} := \{w \in S(s, U, \varepsilon) : w \in E_{\mathbb{R}}\}$ for some open set $U \subseteq \Omega$ containing $p_E(v)$ and $\varepsilon > 0$ is a base for a topology on $E_{\mathbb{R}}$, which coincides with its subspace topology of E.

□

Using the above result in Proposition 3.3.0.4 we claim the following, which, in particular, proves Proposition 3.3.0.2.

Proposition 3.3.0.5. *Let E be a Banach lattice bundle over Ω, and $\Gamma_0(\Omega, E)$ the Banach space of its continuous sections vanishing at infinity.*

(A) If $\mathbb{K} = \mathbb{R}$, then we have the following.

(i) For $s_1, s_2 \in \Gamma_0(\Omega, E)$, the mappings

$$s_1 \vee s_2 : \Omega \longrightarrow E; x \mapsto s_1(x) \vee_{E_x} s_2(x),$$

$$s_1 \wedge s_2 : \Omega \longrightarrow E; x \mapsto s_1(x) \wedge_{E_x} s_2(x)$$

are continuous.

(ii) The set

$$\Gamma_0(\Omega, E)_+ = \left\{ s \in \Gamma_0(\Omega, E) : s \geq 0 \overset{def}{\iff} |s| = s \right\}$$

satisfies the following:

(a) $\Gamma_0(\Omega, E)_+$ is norm-closed in $\Gamma_0(\Omega, E)$,

(b) $\Gamma_0(\Omega, E)_+ + \Gamma_0(\Omega, E)_+ \subseteq \Gamma_0(\Omega, E)_+$,

(c) $\Gamma_0(\Omega, E)_+ \cap \big(- \Gamma_0(\Omega, E)_+ \big) = \{0\}$,

(d) $\lambda \Gamma_0(\Omega, E)_+ \subseteq \Gamma_0(\Omega, E)_+$ for all $0 \leq \lambda \in \mathbb{R}$, and

(e) for $s_1, s_2 \in \Gamma_0(\Omega, E)_+$; $0 \leq s_1 \leq s_2$ implies that $||s_1|| \leq ||s_2||$.

In particular, $\Gamma_0(\Omega, E)_+$ is a normal convex cone of $\Gamma_0(\Omega, E)$, and $\Gamma_0(\Omega, E) = \Gamma_0(\Omega, E)_+ - \Gamma_0(\Omega, E)_+$ is a real Banach lattice.

(B) If $\mathbb{K} = \mathbb{C}$, then we have the following.

(i) For each $s \in \Gamma_0(\Omega, E)$, the mappings

$$Re\ s : \Omega \longrightarrow E; x \mapsto Re\ s(x)_{E_x},$$

$$Im\ s : \Omega \longrightarrow E; x \mapsto Im\ s(x)_{E_x},$$

$$\bar{s} : \Omega \longrightarrow E; x \mapsto \overline{s(x)}_{E_x}$$

are continuous.

(ii) The set

$$\Gamma_0(\Omega, E)_{\mathbb{R}} := \{ s \in \Gamma_0(\Omega, E) : Re\ s = s \}$$

is a closed subspace of $\Gamma_0(\Omega, E)$, such that

(a) $\Gamma_0(\Omega, E)_{\mathbb{R}} = \Gamma_0(\Omega, E)^+_{\mathbb{R}} - \Gamma_0(\Omega, E)^+_{\mathbb{R}}$ is a real vector lattice, and

(b) the set

$$\Gamma_0(\Omega, E)^+_{\mathbb{R}} := \{s \in \Gamma_0(\Omega, E)_{\mathbb{R}} : |s| = s\}$$

is the positive cone for $\Gamma_0(\Omega, E)$.

In particular, $\Gamma_0(\Omega, E)_{\mathbb{R}}$ *is a real Banach lattice, and* $\Gamma_0(\Omega, E) = \Gamma_0(\Omega, E)_{\mathbb{R}} \oplus i\Gamma_0(\Omega, E)_{\mathbb{R}}$ *is a complex Banach lattice.*

Proof. (A) (i) For $s_1, s_2 \in \Gamma_0(\Omega, E)$, by considering the following commutative diagram,

$$\begin{tikzcd}
E \times_\Omega E \arrow[rr, "(u,v)\mapsto u\vee_{E_{p_E(v)}} v"] & & E & & E \times_\Omega E \arrow[ll, "u\wedge_{E_{p_E(v)}} v \,\leftarrow\!\shortmid (u,v)"'] \\
\Omega \arrow[u, "x\mapsto(s_1(x),s_2(x))"] \arrow[urr, dashed, "s_1\vee s_2"'] & & & & \Omega \arrow[u, "x\mapsto(s_1(x),s_2(x))"'] \arrow[ull, dashed, "s_1\wedge s_2"]
\end{tikzcd}$$

it follows from Proposition 3.3.0.4 A(i) that both $s_1 \wedge s_2$ and $s_1 \wedge s_2$ are continuous sections. Hence $s_1 \wedge s_2, s_1 \vee s_2 \in \Gamma_0(\Omega, E)$.

(ii) It is clear, by definition, that the set $\Gamma_0(\Omega, E)_+$ consists of positive continuous sections, i.e., $s \in \Gamma_0(\Omega, E)_+ \iff s \in \Gamma_0(\Omega, E)$ and $0_x \leq s(x) \in E^+_x$ for all $x \in \Omega$.

In this case it follows, from (i) above, or even by Proposition 3.3.0.4 A(ii), that for any $s \in \Gamma_0(\Omega, E)$, setting $s^+ := s \wedge 0$ and $s^- := -(s \wedge 0)$ we have that $s^+, s^- \in \Gamma_0(\Omega, E)_+$ and $s = s^+ - s^-$ is the unique representation as a difference of two positive continuous sections with modulus $|s| = s^+ + s^-$.

(a) Since $E_+ \subseteq E$ is a closed subspace by Proposition 3.3.0.4 A, we can immediately conclude that $\Gamma_0(\Omega, E)_+$ is norm-closed in $\Gamma_0(\Omega, E)$. Equivalently, if $(s_n)_{n\in\mathbb{N}} \subseteq \Gamma_0(\Omega, E)_+$ is a sequence of positive continuous sections converging to a continuous section $s \in \Gamma_0(\Omega, E)$; this, in particular, implies that the sequence $(s_n(x))_{n\in\mathbb{N}} \subseteq E^+_x$ converges to $s(x) \in E_x$ for each $x \in \Omega$. Now, since $E^+_x \subseteq E_x$ is norm-closed for each $x \in \Omega$, we have that $s(x) \in E^+_x$ for each $x \in \Omega$, hence $s \in \Gamma_0(\Omega, E)_+$ as required.

Assertions (b), (c), and (d) immediately follow, since $E^+_x + E^+_x \subseteq E^+_x$, $E^+_x \cap (-E^+_x) = \{0_x\}$ and $\lambda E^+_x \subseteq E^+_x$ for each $x \in \Omega, 0 \leq \lambda \in \mathbb{R}$.

From here, using even only (ii)(a - d), we can conclude that $\Gamma_0(\Omega, E)$ is an ordered Banach space with positive cone $\Gamma_0(\Omega, E)_+$ as defined in [4] (see Appendix B, Definition B.0.0.1).

(e) For $s_1, s_2 \in \Gamma_0(\Omega, E)_+, 0 \leq s_1 \leq s_2$ implies $0_x \leq s_1(x) \leq s_2(x)$ for each $x \in \Omega$. Thus $||s_1(x)||_{E_x} \leq ||s_2(x)||_{E_x}$ for each $x \in \Omega$, which also implies that $||s_1|| \leq ||s_2||$.

From here, using (i) and (ii)(a - e) and the fact that $||s|| = ||\,|s|\,||$ for any $s \in \Gamma_0(\Omega, E)$, we can conlude that $\Gamma_0(\Omega, E)_+$ is a normal positive cone (see Appendix B, Proposition B.0.0.2)(by taking $\beta = 1$) for $\Gamma_0(\Omega, E)$, and in addition $\Gamma_0(\Omega, E) = \Gamma_0(\Omega, E)_+ - \Gamma_0(\Omega, E)_+$ is a real Banach lattice (see also Appendix B, Proposition B.0.0.5).

(B) (i) For $s \in \Gamma_0(\Omega, E)$, by considering the following commutative diagram,

$$\begin{tikzcd} E \arrow[r, "v\mapsto Re\ v_{E_{p_E(v)}}"] & E & E \arrow[l, "Im\ v_{E_{p_E(v)}} \mapsfrom v"'] \\ \Omega \arrow[u, "x\mapsto s(x)"] \arrow[ur, dashed, "Re\ s"'] & & \Omega \arrow[u, "x\mapsto s(x)"'] \arrow[ul, dashed, "Im\ s"] \end{tikzcd}$$

it follows from Proposition 3.3.0.4 B(i) that both $Re\ s$ and $Im\ s$ are continuous sections. Hence $Re\ s, iIm\ s \in \Gamma_0(\Omega, E)$, which also implies that the conjugate $\bar{s} = Re\ s - iIm\ s$ is also a continuous section.

(ii) We note that, by definition, the set $\Gamma_0(\Omega, E)_{\mathbb{R}}$ consists of real continuous sections, i.e., $s \in \Gamma_0(\Omega, E)_{\mathbb{R}} \iff s \in \Gamma_0(\Omega, E)$ and $s(x) \in E_x^{\mathbb{R}}$ for all $x \in \Omega$.

Since $E_{\mathbb{R}} \subseteq E$ is a closed subspace by Proposition 3.3.0.4 B, we can immediately conclude that $\Gamma_0(\Omega, E)_{\mathbb{R}}$ is a closed subspace of $\Gamma_0(\Omega, E)$. Equivalently, if $(s_n)_{n\in\mathbb{N}} \subseteq \Gamma_0(\Omega, E)_{\mathbb{R}}$ is a sequence of real continuous sections converging to a continuous section $s \in \Gamma_0(\Omega, E)$, this, in particular, implies that the sequence $(s_n(x))_{n\in\mathbb{N}} \subseteq E_x^{\mathbb{R}}$ converges to $s(x) \in E_x$ for each $x \in \Omega$. And since $E_x^{\mathbb{R}} \subseteq E_x$ is a closed subspace for each $x \in \Omega$, we have that $s(x) \in E_x^{\mathbb{R}}$ for each $x \in \Omega$, hence, $s \in \Gamma_0(\Omega, E)_{\mathbb{R}}$ as required.

For assertions (a) and (b), by the argument in (A) above, we can conclude that $\Gamma_0(\Omega, E)_{\mathbb{R}}$ is a real Banach lattice with positive cone

$\Gamma_0(\Omega, E)^+_{\mathbb{R}} := \{s \in \Gamma_0(\Omega, E)_{\mathbb{R}} : |s| = s\}$.

In this situation, for any $s \in \Gamma_0(\Omega, E)$, it follows that $Re\ s, Im\ s \in \Gamma_0(\Omega, E)_{\mathbb{R}}$, and $s = Re\ s + iIm\ s$ is the unique representation of each complex continuous section. Hence, $\Gamma_0(\Omega, E) = \Gamma_0(\Omega, E)_{\mathbb{R}} \oplus i\Gamma_0(\Omega, E)_{\mathbb{R}}$ is a complex Banach lattice.

□

The following are two important examples of Banach lattice bundles serving as motivation. See [29, Examples 1.5, p.13] for the case of a Banach bundle.

Example 3.3.0.6. *(i) Let Z be a Banach lattice and Ω a locally compact space. Then $E := \Omega \times Z$ is a continuous Banach lattice bundle over Ω, which we call the trivial topological Banach lattice bundle with fiber Z if $p : E \longrightarrow \Omega$ is the projection onto the first component and E is equipped with the product topology. Here, the space $\Gamma_0(\Omega, E)$ is lattice isomorphic to $C_0(\Omega, Z)$ the Banach lattice of continuous functions $s : \Omega \longrightarrow Z$ vanishing at infinity (see also Chapter 1, Section 1.1).*

(ii) Let $\pi : L \longrightarrow K$ be a continuous surjection between the compact spaces L and K. For each $k \in K$, let $L_k := \pi^{-1}(k)$ be the associated fiber. We define

$$E := \dot{\bigcup}_{k \in K} C(L_k)$$
$$p : E \longrightarrow K, v \in C(L_k) \mapsto k$$

and endow E with the topology generated by the sets

$$W(s, U, \varepsilon) := \left\{ v \in p^{-1}(U) \mid ||v - s|_{L_{p(v)}}||_{C(L_{p(v)})} < \varepsilon \right\}$$

where $U \subseteq K$ is open, $s \in C(L)$, and $\varepsilon > 0$. Then, E is a Banach lattice bundle over K and the corresponding lattice of continuous sections $\Gamma(K, E)$ is lattice isomorphic to $C(L)$. We refer to Appendix F, and in particular to Proposition F.2.0.1 for a certain generalisation of this situation.

Moreover, E is a continuous Banach lattice bundle if and only if π is an open map.

3.4 Morphisms of *topological* Banach lattice bundles

Here, we investigate the notion of a (positive) Banach lattice bundle morphism between Banach lattice bundles over Ω, and their dynamics. We also establish the category of *topological* Banach lattice bundles and their dynamics.

Once again, if no confusion arises, by a Banach lattice bundle over Ω, we always mean a *topological* Banach lattice bundle over a locally compact space Ω (see Definition 3.3.0.2).

Definition 3.4.0.1. *Let $p_E : E \longrightarrow \Omega$ and $p_F : F \longrightarrow \Omega$ be Banach lattice bundles over Ω, and $\varphi : \Omega \longrightarrow \Omega$ continuous. A continuous mapping $\Phi : E \longrightarrow F$ is called a positive Banach lattice bundle morphism over φ if:*

(i) $\varphi \circ p_E = p_F \circ \Phi$, *i.e., the following diagram commutes,*

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ {\scriptstyle p_E}\downarrow & & \downarrow{\scriptstyle p_F} \\ \Omega & \xrightarrow{\varphi} & \Omega \end{array}$$

(ii) $\Phi(x) := \Phi|_{E_x} : E_x \longrightarrow F_{\varphi(x)}$ *is a positive operator for each* $x \in \Omega$, *and*

(iii) $||\Phi|| := \sup_{x\in\Omega} ||\Phi(x)||_{\mathcal{L}(E_x, F_{\varphi(x)})} < \infty$.

Remark 3.4.0.2. *(i) It should be noted that a positive operator between two Banach lattices N and M is a linear operator $T : N \longrightarrow M$ such that $TN_+ \subseteq M_+$, i.e., $n \geq 0$ in N always implies that $Tn \geq 0$ in M for all $n \in N$;* or, *equivalently, $|Tn| \leq T|n|$ for all $n \in N$. Moreover, positive operators are always bounded and $\mathbb{C}$-linear (see [10, Lemma 7.5, p.121]).*

(ii) Furthermore, a linear operator $T : N \longrightarrow M$ between two Banach lattices N and M is a called lattice homomorphism if $|Tn| = T|n|$ for all $n \in N$. It follows that a lattice homomorphism is positive, and hence bounded. Furthermore, $T(n \vee m) = Tn \vee Tm$ and $T(n \wedge m) = Tn \wedge Tm$ for all $n, m \in N_{\mathbb{R}}$ (see [10, Lemma 7.5, p.121 and paragraph afterwards]).

Remark 3.4.0.3. *From Definition 3.4.0.1, we identify the following additional properties of a positive Banach lattice bundle morphism Φ over φ .*

(i) If $\Phi(x)$ is an isometry for each $x \in \Omega$, we call Φ a positive isometry.

(ii) If $\Phi(x)$ is an isometric lattice homomorphism for each $x \in \Omega$, we call Φ an isometry.

(iii) If $\Phi(x)$ is a lattice homomorphism for each $x \in \Omega$, we call Φ a Banach lattice bundle morphism over φ.

(iv) If $\varphi = Id_\Omega$, we call Φ a positive Banach lattice bundle morphism.

(v) If $\varphi = Id_\Omega$, and $\Phi(x)$ is a lattice homomorphism for each $x \in \Omega$, we call Φ a Banach lattice bundle morphism.

We are interested in positive dynamical Banach lattice bundles over a topological system induced by groups. And so, we introduce the notion of a positive S-dynamical Banach lattice bundle in the next definition. See [29, Definition 1.8, p.15] for the case of a dynamical Banach bundle. Moreover, we note that every positive S-dynamical Banach lattice bundle is, in particular, an S-dynamical Banach bundle.

Note 3.4.0.4. *In the sequel, we will use the following notation.*

1. *We let G be a locally compact group, and S a closed subsemigroup of G containing the neutral element e, i.e., a closed "submonoid" of G. For instance, we can take $G = \mathbb{R}$, $S = \mathbb{R}$ or $G = \mathbb{Z}$, $S = \mathbb{N}_0$.*

2. *We let (Ω, φ) be a topological G-dynamical system over a locally compact space Ω, i.e., $\varphi = (\varphi_g)_{g\in G}$ defines a continuous group action[2], called a continuous flow on Ω.*

[2] i.e., $\varphi : G \longrightarrow Aut(\Omega)$; $g \mapsto \varphi_g$ is a *continuous* group homomorphism, where $Aut(\Omega)$ denotes the group of automorphisms (homeomorphisms) on Ω.

Definition 3.4.0.5. *A positive S-dynamical Banach lattice bundle over (Ω, φ) is a pair $(E, \mathbf{\Phi})$ of a Banach lattice bundle E over Ω, and a monoid representation*[3]

$$\mathbf{\Phi} : S \longrightarrow E^E,\; g \mapsto \Phi_g$$

such that

(i) $\Phi_g : E \longrightarrow E$ *is a positive Banach lattice bundle morphism over* φ_g *for each* $g \in S$,

(ii) $\mathbf{\Phi}$ *is jointly continuous, i.e., the mapping*

$$S \times E \longrightarrow E, \quad (g, v) \mapsto \Phi_g v$$

is continuous, and

(iii) $\mathbf{\Phi}$ *is locally bounded, i.e.,* $\sup_{g\in K} ||\Phi_g|| < \infty$ *whenever* $K \subseteq S$ *is compact.*

We call $\mathbf{\Phi} = (\Phi_g)_{g\in S}$ *a positive semiflow on E over the flow* $(\varphi_g)_{g\in G}$. *If* $S = G$, *then we call* $\mathbf{\Phi} = (\Phi_g)_{g\in G}$ *a positive flow on E over the flow* $(\varphi_g)_{g\in G}$, *and* $(E, \mathbf{\Phi})$ *a positive G-dynamical Banach lattice bundle over* (Ω, φ).

Remark 3.4.0.6. *From Definition 3.4.0.5 above we identify the following additional properties of a positive S-dynamical Banach lattice bundle* $(E, \mathbf{\Phi})$ *over* (Ω, φ) *using Remark 3.4.0.3.*

(i) If Φ_g *is a positive isometry for each* $g \in S$, *we call* $\mathbf{\Phi}$ *a positive isometry.*

(ii) If Φ_g *is an isometry for each* $g \in S$, *we call* $\mathbf{\Phi}$ *an isometry.*

(iii) If Φ_g *is a Banach lattice bundle morphism over* φ_g *for each* $g \in S$, *we call* $(E, \mathbf{\Phi})$ *an S-dynamical Banach lattice bundle over* (Ω, φ); *and* $\mathbf{\Phi} = (\Phi_g)_{g\in G}$ *a semiflow on E over the flow* $(\varphi_g)_{g\in G}$.

[3] i.e., $\Phi_{gh} = \Phi_h \circ \Phi_g, \forall g, h \in S$ and $\Phi_e = Id_E$, the identity on E^E

Remark 3.4.0.7. *As in [29, Remark 1.9, p.16], and as is evident in Example 3.4.0.8(i) below, we note here that our concept of a positive dynamical Banach lattice bundle is closely related to the notions of positive cocycles and linear skew-product flows, as motivated in Chapter 1. In fact, if $(E, \mathbf{\Phi})$ is a positive S-dynamical Banach lattice bundle over (Ω, φ), then the family of positive operators $(\Phi_g(x))_{x\in\Omega}$ for $g \in S$, satisfies the so-called cocycle rule, i.e.,*

$$\Phi_{g_1 g_2}(x) = \Phi_{g_1}(\varphi_{g_2}(x)) \circ \Phi_{g_2}(x)$$

for all $g_1, g_2 \in S$ and $x \in \Omega$, where $\Phi_g(x) := \Phi_{g|_{E_x}} \in \mathscr{L}(E_x, E_{\varphi_g(x)})$. Indeed, since $\Phi_{g_1 g_2} = \Phi_{g_1} \circ \Phi_{g_2}$ for all $g_1, g_2 \in S$, it follows that, for $x \in \Omega$,

$$\begin{aligned}
\Phi_{g_1 g_2}(x) &:= \Phi_{g_1 g_2}|_{E_x} \\
&= \Phi_{g_1} \circ (\Phi_{g_2}|_{E_x}) \\
&= \Phi_{g_1}|_{E_{\varphi_{g_2}(x)}} \circ \Phi_{g_2}|_{E_x} \\
&=: \Phi_{g_1}(\varphi_{g_2}(x)) \circ \Phi_{g_2}(x).
\end{aligned}$$

Now, we introduce instances of (positive) S-dynamical Banach lattice bundles in Examples 3.3.0.6 (i) and (ii). See [29, Examples 1.12, p.16] for examples of S-dynamical Banach bundles. Moreover, we note that every (positive) S-dynamical Banach lattice bundle is, in particular, an S-dynamical Banach bundle.

Example 3.4.0.8. *(i) Assume that $G = \mathbb{R}$, $S = \mathbb{R}_+$, Z a Banach lattice, and $E = \Omega \times Z$ the corresponding trivial Banach lattice bundle. Moreover, let $\varphi = (\varphi_t)_{t\in\mathbb{R}}$ be a continuous flow on Ω.*
Now, suppose that $\{\Phi^t(x) : Z \longrightarrow Z \text{ a positive operator} \mid x \in \Omega, t \geq 0\}$ is a positive strongly continuous exponentially bounded cocycle over $(\varphi_t)_{t\in\mathbb{R}}$, meaning that:

(a) $\Phi^t(x)Z_+ \subseteq Z_+$ for all $t \geq 0$, $x \in \Omega$,

(b) $\Phi^{t+r}(x) = \Phi^t(\varphi_r(x)) \circ \Phi^r(x)$ and $\Phi^0(x) = Id_Z$ for all $t, r \geq 0$, $x \in \Omega$,

(c) the mapping $\mathbb{R}_+ \times \Omega \longrightarrow Z; (t, x) \mapsto \Phi^t(x)v$ is continuous for all $v \in Z$, and

(d) for each $w \in \mathbb{R}$, there exist $M \geq 1$ such that $||\Phi^t(x)|| \leq Me^{wt}$ for all $t \geq 0$ and $x \in \Omega$.

Then the continuous skew-product (linear) flow $\Phi_t : \Omega \times Z \longrightarrow \Omega \times Z$ given by

$$\Phi_t(x, v) := (\varphi_t(x), \Phi^t(x)v)$$

for $x \in \Omega, v \in Z$, and $t \geq 0$ defines a positive $\mathbb{R}_+$-dynamical Banach lattice bundle $(E, \mathbf{\Phi})$ over (Ω, φ), i.e., $\mathbf{\Phi} = (\Phi_t)_{t\geq 0}$ is a positive semiflow on E over $(\varphi_t)_{t\in\mathbb{R}}$ on Ω.

Conversely, every positive $\mathbb{R}_+$-dynamical Banach lattice bundle $(\Omega \times Z, \mathbf{\Phi})$ over (Ω, φ) defines a positive strongly continuous cocycle over $(\varphi_t)_{t\in\mathbb{R}}$ by setting

$$\Phi^t(x)v := Pr_2(\Phi_t(x, v))$$

for each $x \in \Omega, v \in Z$ and $t \geq 0$, where $Pr_2 : \Omega \times Z \longrightarrow Z$ is the projection onto the second factor. We note that Chapter 1 (Section 1.2) is a discrete-time instance of the above situation.

(ii) Assume $\Omega = K$ and L are compact, and $\pi : (L, \psi) \longrightarrow (K, \varphi)$ is an extension of topological G-dynamical systems, i.e., $(\psi_g)_{g\in G}$ and $(\varphi_g)_{g\in G}$ are continuous flows on L and K, respectively; and $\pi : L \longrightarrow K$ is a continuous surjection such that the diagram

$$\begin{tikzcd} L \arrow[r, "\psi_g"] \arrow[d, "\pi"'] & L \arrow[d, "\pi"] \\ K \arrow[r, "\varphi_g"'] & K \end{tikzcd}$$

commutes for each $g \in G$, i.e., $\pi \circ \psi_g = \varphi_g \circ \pi$ for all $g \in G$. Also assume that E is the Banach lattice bundle over K as defined in Example 3.3.0.6 (ii). For each $g \in G$, consider the mapping

$$\Phi_g : E \longrightarrow E;\ v \in C(L_k) \mapsto v \circ \psi_{g^{-1}} \in C(L_{\varphi_g(k)}).$$

This defines a (positive) G-dynamical Banach lattice bundle $(E, \mathbf{\Phi})$ over (K, φ). So, $\mathbf{\Phi} = (\Phi_g)_{g\in G}$ is a (positive) flow on E over the flow $\varphi = (\varphi_g)_{g\in G}$ on K.

We refer to Appendix F, and in particular to Proposition F.3.0.7 for a certain generalisation of this situation for the case where $G = \mathbb{R}$.

Next, we introduce morphisms between positive S-dynamical Banach lattice bundles over (Ω, φ).

Definition 3.4.0.9. *A morphism from a positive S-dynamical Banach lattice bundle $(E, \mathbf{\Phi})$ over (Ω, φ) to a positive S-dynamical Banach lattice bundle $(F, \mathbf{\Psi})$ over (Ω, φ) is a Banach lattice bundle morphism (see Remark 3.4.0.3 (v))*

$$\Theta : E \longrightarrow F$$

such that the diagram

$$\begin{tikzcd} E \arrow[r, "\Theta"] \arrow[d, "\Phi_g"'] & F \arrow[d, "\Psi_g"] \\ E \arrow[r, "\Theta"] & F \end{tikzcd}$$

commutes for each $g \in S$, i.e., $\Theta \circ \Phi_g = \Psi_g \circ \Theta$ for each $g \in S$.
It is called a positive isometry if Θ is an isometry.

Remark 3.4.0.10. *(i) We note that, similarly, a morphism between S-dynamical Banach lattice bundles over (Ω, φ) can be defined. It will be called isometry if Θ is an isometry.*

(ii) It is clear that in the category of positive S-dynamical Banach lattice bundles over (Ω, φ), an isometric morphism is just a positive isometry. We choose this terminology and reserve the word "isometry" for S-dynamical Banach lattice bundles over (Ω, φ).

(iii) Furthermore, two positive S-dynamical Banach lattice bundles $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$ over (Ω, φ) are said to be isomorphic if there exists a homeomorphic positive isometry (i.e., a homeomorphic isometric morphism) between them. In this situation, we write $\mathbf{\Phi} = (\Phi_g)_{g \in S} \cong (\Psi_g)_{g \in S} = \mathbf{\Psi}$ on $E \cong F$.

3.5 On a representation of the space of continuous sections

Throughout this Section, by Banach lattice bundle E over Ω, we always mean a *topological* Banach lattice bundle $p_E : E \longrightarrow \Omega$ over a locally compact space Ω (see Definition 3.2.0.1).

Remark 3.5.0.1. *(i) Let E be a Banach lattice bundle over Ω. Then, the Banach space $\Gamma_0(\Omega, E)$ of its continuous sections vanishing at infinity is an m-Banach lattice module over $C_0(\Omega)$. Indeed, $\Gamma_0(\Omega, E)$ is a Banach lattice (see Proposition 3.3.0.2); and also a $C_0(\Omega)$-module given by*

$$C_0(\Omega) \times \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, E); (f, s) \mapsto f \cdot s := \big[x \mapsto f(x)s(x)\big]$$

and, for all $f \in C_0(\Omega)$ and $s \in \Gamma_0(\Omega, E)$, we have that

(a) $|f \cdot s| = |f| \cdot |s|$, since $|f \cdot s|(x) = |f(x)s(x)| = (|f| \cdot |s|)(x)$ for each $x \in \Omega$;

(b) $\overline{f \cdot s} = \overline{f} \cdot \overline{s}$, since $\overline{f \cdot s}(x) = \overline{f(x)s(x)} = (\overline{f} \cdot \overline{s})(x)$ for each $x \in \Omega$; and

(c) $||f \cdot s|| \leq ||f||||s||$, since $||f \cdot s|| = \sup_{x\in\Omega} ||f(x)s(x)|| \leq \sup_{x\in\Omega} |f(x)| \sup_{x\in\Omega} ||s(x)|| = ||f||||s||$.

(ii) In particular, if E is a Banach lattice bundle over Ω, then the m-Banach lattice module $\Gamma_0(\Omega, E)$ is a $U_0(\Omega)$-normed m-lattice module, or, equivalently, an AM m-lattice module over $C_0(\Omega)$ (see Chapter 2, Example 2.4.1.4, Proposition 2.4.1.9 and Example 2.4.1.11).

(iii) Let E be a Banach lattice bundle over Ω. For each $x \in \Omega$, the evaluation (quotient) map $e_x : \Gamma_0(\Omega, E) \longrightarrow E_x; s \mapsto s(x)$ is a lattice homomorphism, i.e., $|e_x s| = e_x|s|$ for all $s \in \Gamma_0(\Omega, E)$ and $x \in \Omega$. Moreover, $\overline{e_x s} = e_x \overline{s}$ for all $s \in \Gamma_0(\Omega, E)$ and $x \in \Omega$. In addition, $e_x : \Gamma(K, E) \longrightarrow E_x$ is order continuous, i.e., $s_\gamma \downarrow 0$ in $\Gamma(K, E)$ implies $s_\gamma(x) \downarrow 0_x$ in E_x for all $x \in \Omega$.

(iv) Let E be a Banach lattice bundle over Ω. For each (positive) Banach lattice bundle morphism $\Phi : E \longrightarrow E$ over a homeomorphism $\varphi : \Omega \longrightarrow \Omega$, we define an operator

$$\mathcal{T}_\Phi : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, E); s \mapsto \Phi \circ s \circ \varphi^{-1}$$

called the (positive) weighted Koopman operator over the Koopman operator $T_\varphi : C_0(\Omega) \longrightarrow C_0(\Omega); f \mapsto f \circ \varphi^{-1}$ induced by Φ. It follows immediately that the (positive) weighted Koopman operator $\mathcal{T}_\Phi$ is a (positive) lattice T_φ-homomorphism (see also Chapter 2 (Example 2.3.2.4)). We show this in (a) and (b) below.

(a) If $\Phi : E \longrightarrow E$ is a positive Banach lattice bundle morphism over a homeomorphism $\varphi : \Omega \longrightarrow \Omega$ (see Definition 3.4.0.1), then it must be that

$|\Phi \circ s| \leq \Phi \circ |s|$ for each $s \in \Gamma_0(\Omega, E)$. So, $|\mathcal{T}_\Phi s| = |\Phi \circ s \circ \varphi^{-1}| \leq \Phi \circ |s| \circ \varphi^{-1} = \mathcal{T}_\Phi |s|$ for each $s \in \Gamma_0(\Omega, E)$, i.e., $\mathcal{T}_\Phi : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, E)$ is a positive operator.

Moreover, $\mathcal{T}_\Phi f s = \Phi \circ f s \circ \varphi^{-1} = \Phi \circ [f \circ \varphi^{-1} \cdot s \circ \varphi^{-1}] = f \circ \varphi^{-1} \cdot \Phi \circ s \circ \varphi^{-1} = T_\varphi f \cdot \mathcal{T}_\Phi s$ for every $f \in C_0(\Omega)$ and $s \in \Gamma_0(\Omega, E)$ implies that $\mathcal{T}_\Phi : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, E)$ is, in addition, a module homomorphism.

(b) If $\Phi : E \longrightarrow E$ is a Banach lattice bundle morphism over a homeomorphism $\varphi : \Omega \longrightarrow \Omega$ (see Remark 3.4.0.3(iii)), then it must be that $|\Phi \circ s| = \Phi \circ |s|$ for each $s \in \Gamma_0(\Omega, E)$. So, $|\mathcal{T}_\Phi s| = |\Phi \circ s \circ \varphi^{-1}| = \Phi \circ |s| \circ \varphi^{-1} = \mathcal{T}_\Phi |s|$ for each $s \in \Gamma_0(\Omega, E)$, i.e., $\mathcal{T}_\Phi : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, E)$ is a lattice homomorphism.

Moreover, by the above argument, $\mathcal{T}_\Phi : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, E)$ is, in addition, a module homomorphism.

Hence, the assertion holds (see Chapter 2 (Definition 2.3.2.3)).

(v) Now, a (positive) S-dynamical Banach lattice bundle $(E, \mathbf{\Phi})$ over a G-dynamical system (Ω, φ), induces a (positive) S-dynamical m-Banach lattice module $(\Gamma_0(\Omega, E), \mathcal{T}_{\mathbf{\Phi}})$ over the Koopman group representation $(C_0(\Omega), \boldsymbol{T}_\varphi)$ (see also Chapter 2, Example 2.3.4.3 (i)). We show this in (a) and (b) below.

(a) Suppose that $(E, \mathbf{\Phi})$ is a positive S-dynamical Banach lattice bundle over a G-dynamical system (Ω, φ) (see Definition 3.4.0.5), i.e., $\mathbf{\Phi} = (\Phi_g)_{g \in S}$ is a positive semiflow on E over the flow $(\varphi_g)_{g \in G}$ on Ω. By [29, Proposition 2.14 (i), p.29], we obtain that $\mathcal{T}_{\mathbf{\Phi}} = (\mathcal{T}_\Phi(g))_{g \in S}$ is a weighted semigroup representation on $\Gamma_0(\Omega, E)$ over the Koopman group representation $(C_0(\Omega), \boldsymbol{T}_\varphi)$ in the sense of S-dynamical Banach module (see [29, Definition 2.5, p.25]).

Since by (iv) above, $\mathcal{T}_\Phi(g) : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, E)$ is, in addition, a positive operator for each $g \in S$, the assertion holds (see Chapter 2, Definition 2.3.4.1).

(b) Suppose that $(E, \mathbf{\Phi})$ is an S-dynamical Banach lattice bundle over a G-dynamical system (Ω, φ) (see Remark 3.4.0.6(iii)), i.e., $\mathbf{\Phi} = (\Phi_g)_{g \in S}$ is a semiflow on E over the flow $(\varphi_g)_{g \in G}$ on Ω. It follows from (a) above and since $\mathcal{T}_\Phi(g) : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, E)$ is, in addition, a lattice homomorphism for each $g \in S$, that $\mathcal{T}_{\mathbf{\Phi}} = (\mathcal{T}_\Phi(g))_{g \in S}$ is a weighted semigroup rep-

resentation on the Banach lattice module $\Gamma_0(\Omega, E)$ over the Koopman group representation $(C_0(\Omega), \boldsymbol{T_\varphi})$ (see Chapter 2, Remark 2.3.4.2(ii)).

The following simple observation about a Banach lattice bundle over Ω will be useful. See [29, Lemma 2.23, p.34] for the case of a Banach bundle.

Lemma 3.5.0.2. *Let Ω be a locally compact space, $\varphi : \Omega \longrightarrow \Omega$ a homeomorphism, and $p_E : E \longrightarrow \Omega$ a Banach lattice bundle. Then $p_\varphi : E_\varphi \longrightarrow \Omega$ with $E_\varphi := E$ and $p_\varphi := \varphi^{-1} \circ P_E$ is a Banach lattice bundle over Ω which has the following properties.*

(i) The identity mapping $Id_E : E \longrightarrow E_\varphi$ is a Banach lattice bundle morphism over φ^{-1}.

(ii) If $p_F : F \longrightarrow \Omega$ is a Banach lattice bundle over Ω, then a mapping $\Phi : F \longrightarrow E$ is a (positive) Banach lattice bundle morphism over φ if and only if $\Phi : F \longrightarrow E_\varphi$ is a (positive) Banach lattice bundle morphism over Id_Ω.

Proof. It is clear that $p_\varphi : E_\varphi \longrightarrow \Omega$ is a Banach lattice bundle over Ω, with fiber $p_\varphi^{-1}(x) = p_E^{-1}(\varphi(x)) = E_{\varphi(x)}$ for each $x \in \Omega$.

(i) Now, $Id_E : E \longrightarrow E_\varphi$ satisfies the following commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{Id_E} & E_\varphi \\ {\scriptstyle p_E}\downarrow & & \downarrow{\scriptstyle p_\varphi} \\ \Omega & \xrightarrow[\varphi^{-1}]{} & \Omega \end{array}$$

since $p_\varphi \circ Id_E = p_\varphi = \varphi^{-1} \circ P_E$.

In addition, $E_x = E_{\varphi^{-1}(\varphi(x))}$ and $Id_E(x) := Id_{E_{|E_x}} : E_x \longrightarrow E_x$ is a lattice homomorphism, from which it follows that $Id_E : E \longrightarrow E_\varphi$ is indeed a Banach lattice bundle morphism over φ^{-1}.

(ii) For a mapping $\Phi : F \longrightarrow E$; we see that $\varphi \circ p_F = p_E \circ \Phi$ if and only if $Id_\Omega \circ P_F = p_\varphi \circ \Phi$, i.e., the diagram

$$\begin{array}{ccc} F & \xrightarrow{\Phi} & E \\ {\scriptstyle p_F}\downarrow & & \downarrow{\scriptstyle p_E} \\ \Omega & \xrightarrow[\varphi]{} & \Omega \end{array}$$

commutes if and only if the diagram

$$\begin{tikzcd} F \arrow[r, "\Phi"] \arrow[d, "p_F"'] & E_\varphi \arrow[d, "p_\varphi"] \\ \Omega \arrow[r, "Id_\Omega"'] & \Omega \end{tikzcd}$$

commutes.

Moreover, in either case observing that $\Phi(x) = \Phi_{|_{F_x}} : F_x \longrightarrow E_{\varphi(x)}$ is a mapping proves the assertion.

□

In the following lemma and its corollary, we prove that every (positive) lattice T_φ-homomorphism on the lattice of continuous sections is equal to a unique (positive) weighted Koopman operator over T_φ. See [29, Lemma 2.24, p.34-35] for the case of Banach bundles.

Lemma 3.5.0.3. *Let E and F be Banach lattice bundles over Ω. Moreover, let $\varphi : \Omega \longrightarrow \Omega$ be a homeomorphism and an operator $\mathcal{T} : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, F)$ a lattice T_φ-homomorphism. Then,*

there exists a unique Banach lattice bundle morphism $\Phi : E \longrightarrow F$ over φ with $\mathcal{T} = \mathcal{T}_\Phi$.

Moreover, $||\Phi|| = ||\mathcal{T}||$ and $\mathcal{T}$ is an isometry if and only if Φ is an isometry.

Proof. We follow the proof of [29, Lemma 2.24, p.34-35] which is for the case of Banach bundles. So, WLOG, we can assume $\Omega = K$ is compact. Consider the Banach lattice bundle F_φ induced by φ as in Lemma 3.5.0.2 above.

(i) The operator $V : \Gamma(K, F) \longrightarrow \Gamma(K, F_\varphi)$ defined by $Vs = s \circ \varphi$ is an isometric and surjective lattice $T_{\varphi^{-1}}$-homomorphism. Indeed, for $s_1, s_2 \in \Gamma(K, F)$ and $\lambda \in \mathbb{K}$ we have

$$\begin{aligned} V(s_1 + s_2) &= (s_1 + s_2) \circ \varphi \\ &= s_1 \circ \varphi + s_2 \circ \varphi \\ &= Vs_1 + Vs_2 \end{aligned}$$

and

$$\begin{aligned} V(\lambda s_1) &= (\lambda s_1) \circ \varphi \\ &= \lambda(s_1 \circ \varphi) \\ &= \lambda V s_1 \end{aligned}$$

i.e., V is $\mathbb{K}$-linear.

Moreover, for $s \in \Gamma(K, F)$,

$$\begin{aligned} |Vs| &= |s \circ \varphi| \\ &= |s| \circ \varphi \\ &= V|s|, \end{aligned}$$

i.e., V is a lattice homomorphism, and hence $V \in \mathscr{L}(\Gamma(K, F), \Gamma(K, F_\varphi))$.

Now, for $r \in \Gamma(K, F_\varphi)$, $r(x) \in F_{\varphi(x)}$ for all $x \in K$; and since φ is surjective, we can find $s \in \Gamma(K, F)$ such that $s(\varphi(x)) = r(x)$ for all $x \in K$. This implies that $Vs = s \circ \varphi = r$, i.e., V is surjective.

Since φ is bijective, we have that

$$\begin{aligned} ||Vs|| &= \sup_{x \in K} ||s(\varphi(x))|| \\ &= \sup_{y \in K} ||s(y)|| \\ &= ||s||, \end{aligned}$$

i.e., V is an isometry.

Moreover, for $f \in C(K)$, and $s \in \Gamma(K, F)$, we have

$$\begin{aligned} Vfs &= fs \circ \varphi \\ &= f \circ \varphi \cdot s \circ \varphi \\ &= T_{\varphi^{-1}} \cdot Vs, \end{aligned}$$

i.e., V is $T_{\varphi^{-1}}$- module homomorphism.

(ii) The operator $V\mathcal{T} : \Gamma(K, E) \longrightarrow \Gamma(K, F_\varphi)$ is a homomorphism of Banach lattice modules (i.e., lattice module homomorphism) over $C(K)$. Indeed, for $s_1, s_2 \in \Gamma(K, E)$ and $\lambda \in \mathbb{K}$,

$$\begin{aligned} V\mathcal{T}(s_1 + s_2) &= \mathcal{T}(s_1 + s_2) \circ \varphi \\ &= \mathcal{T}s_1 \circ \varphi + \mathcal{T}s_2 \circ \varphi \\ &= V\mathcal{T}s_1 + V\mathcal{T}s_2 \end{aligned}$$

and

$$\begin{aligned} V\mathcal{T}(\lambda s_1) &= (\mathcal{T}\lambda s_1) \circ \varphi \\ &= \lambda \mathcal{T}s_1 \circ \varphi \\ &= \lambda V\mathcal{T}s_1, \end{aligned}$$

i.e., $V\mathcal{T}$ is $\mathbb{K}$-linear.

Moreover, for $s \in \Gamma(K,E)$ and $f \in C(K)$,

$$\begin{aligned} |V\mathcal{T}s| &= |\mathcal{T}s \circ \varphi| \\ &= |\mathcal{T}s| \circ \varphi \\ &= \mathcal{T}|s| \circ \varphi \\ &= V\mathcal{T}|s| \end{aligned}$$

and

$$\begin{aligned} V\mathcal{T}(fs) &= \mathcal{T}fs \circ \varphi \\ &= T_\varphi f \cdot \mathcal{T}s \circ \varphi \\ &= f \cdot \mathcal{T}s \circ \varphi \\ &= f \cdot V\mathcal{T}s, \end{aligned}$$

which implies that $V\mathcal{T}$ is a lattice module homomorphism.

(iii) As in [29, Lemma 2.24, p.34-35], we obtain a unique Banach bundle morphism $\Phi : E \longrightarrow F_\varphi$ over Id_K with

$$V\mathcal{T}s = \Phi \circ s$$

for each $s \in \Gamma(K,E)$. That is, the mapping $\Phi : E \longrightarrow F_\varphi$ given by $\Phi(x) := \Phi_{|E_x} : E_x \longrightarrow F_{\varphi(x)}; s(x) \mapsto V\mathcal{T}s(x) := \mathcal{T}s(\varphi(x))$ for every $s \in \Gamma(K,E)$, $x \in K$ defines a (bounded) Banach bundle morphism. See [29, Definition 1.6, p.14].

We claim that Φ is the unique Banach lattice bundle morphism over Id_K. Indeed, for $s \in \Gamma(K,E)$ the equality

$$\begin{aligned} |\Phi \circ s| &= |V\mathcal{T}s| \\ &= V\mathcal{T}|s| \\ &= \Phi \circ |s| \end{aligned}$$

implies that $\Phi(x) : E_x \longrightarrow F_{\varphi(x)}$ is a lattice homomorphism for each $x \in K$. Hence, by Lemma 3.5.0.2 above, $\Phi : E \longrightarrow F$ is the unique Banach lattice bundle morphism over φ, with

$$\mathcal{T}s = V^{-1}(\Phi \circ s) = \Phi \circ s \circ \varphi^{-1}$$

for every $s \in \Gamma(K,E)$, i.e., $\mathcal{T} = \mathcal{T}_\Phi$.

(iv) Moreover, we have that $||\Phi|| = ||V\mathcal{T}|| = ||\mathcal{T}||$. Indeed, V being an isometry implies that $||V\mathcal{T}s|| = ||\mathcal{T}s||$ for all $s \in \Gamma(K,E)$. Moreover, $V\mathcal{T}s = \Phi \circ s$ implies that $||V\mathcal{T}s|| = ||\Phi \circ s||$, so that

$$\begin{aligned} ||\mathcal{T}|| &= \sup\{||\mathcal{T}s|| : s \in \Gamma(K,E), ||s|| \leq 1\} \\ &= \sup\{||V\mathcal{T}s|| : s \in \Gamma(K,E), ||s|| \leq 1\} \\ &= \sup\{||\Phi \circ s|| : s \in \Gamma(K,E), ||s|| \leq 1\} \\ &= ||\Phi||. \end{aligned}$$

From this, it follows that Φ is an isometry if and only if $V\mathcal{T}$ is an isometry if and only if $\mathcal{T}$ is an isometry. The case where Ω is locally compact readily follows, but which we omit the details (see [29, Remark 1.2, p.12] and [21, Lemma 3.11, p.12]).

□

The following is an immediate corollary of the above Lemma 3.5.0.3. We note that this also answers Question 1.3.0.1 in Chapter 1 in a more general sense.

Corollary 3.5.0.4. *Let E and F be Banach lattice bundles over Ω. Moreover, let $\varphi : \Omega \longrightarrow \Omega$ be a homeomorphism and an operator $\mathcal{T} : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, F)$ a positive T_φ-homomorphism. Then,*

(i) there exists a unique positive Banach lattice bundle morphism $\Phi : E \longrightarrow F$ over φ with $\mathcal{T} = \mathcal{T}_\Phi$.

(ii) Moreover, $||\Phi|| = ||\mathcal{T}||$ and $\mathcal{T}$ is a positive isometry if and only if Φ is a positive isometry.

In the following proposition, we represent every (positive) S-dynamical m-Banach lattice module on $\Gamma_0(\Omega, E)$ over the Koopman group $(C_0(\Omega), \boldsymbol{T_\varphi})$ as a (positive) weighted Koopman semigroup representation induced by a unique (positive) S-dynamical Banach lattice bundle over the G-dynamical system (Ω, φ). This is due to [29, Lemma 2.25, p.35] for the case of Banach bundles.

Proposition 3.5.0.5. *Let G be a locally compact group, $S \subseteq G$ a closed submonoid, and (Ω, φ) a topological G-dynamical system. Moreover, let E be a Banach lattice bundle over Ω and let*

$$\boldsymbol{\mathcal{T}} : S \longrightarrow \Gamma_0(\Omega, E)^{\Gamma_0(\Omega,E)}; g \mapsto \mathcal{T}(g)$$

be a strongly continuous representation such that $(\Gamma_0(\Omega, E), \boldsymbol{\mathcal{T}})$ is a (positive) S-dynamical m-Banach lattice module over $(C_0(\Omega), \boldsymbol{T_\varphi})$. Then there is a unique (positive) S-dynamical Banach lattice bundle $(E, \boldsymbol{\Phi})$ over (Ω, φ) such that $\mathcal{T}_{\boldsymbol{\Phi}} = \boldsymbol{\mathcal{T}}$.
Moreover, $||\mathcal{T}(g)|| = ||\Phi_g||$ for each $g \in S$, and $\boldsymbol{\mathcal{T}}$ is a (positive) isometry if and only if Φ is a (positive) isometry.

Proof. We follow the proof of [29, Lemma 2.25, p.35] which is for the case of Banach bundles. Since every S-dynamical m-Banach lattice module on $\Gamma_0(\Omega, E)$ over $(C_0(\Omega), T_\varphi)$ defines an S-dynamical Banach module over $(C_0(\Omega), T_\varphi)$ for the Banach bundle E; we obtain a unique S-dynamical Banach bundle $(E, \boldsymbol{\Phi})$ over (Ω, φ) such that $\mathcal{T}_{\boldsymbol{\Phi}} = \boldsymbol{\mathcal{T}}$ in the sense of a dynamical Banach module (see [29, Definition 2.12, p. 28]).

We claim that $(E, \boldsymbol{\Phi})$ is the unique (positive) S-dynamical Banach lattice bundle over (Ω, φ).
In particular :

(i) Φ_g is a (positive) Banach lattice bundle morphism over φ_g for each $g \in S$. Indeed, for each $g \in S$, $\mathcal{T}(g) : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, E)$ is a (positive) lattice $T_\varphi(g)$-homomorphism; and by (Corollary 3.5.0.4) Lemma 3.5.0.3, it follows that for each $g \in S$, $\Phi_g : E \longrightarrow E$ is the unique (positive) Banach lattice bundle morphism over φ_g; and

(ii) $\boldsymbol{\mathcal{T}}$ is a (positive) isometry if and only if $\boldsymbol{\Phi}$ is a (positive) isometry. Indeed, also by (Corollary 3.5.0.4) Lemma 3.5.0.3 we have that

$$||\mathcal{T}(g)|| = ||\Phi_g||$$

and so for each $g \in S$, $\mathcal{T}(g)$ is a (positive) isometry if and only if Φ_g is an (positive) isometry.

□

As noted in Remark 3.5.0.1(v) and from above Proposition 3.5.0.5, each (positive) S-dynamical Banach lattice bundle $(E, \mathbf{\Phi})$ over a G-dynamical system (Ω, φ) induces a (positive) S-dynamical m-Banach lattice module $(\Gamma_0(\Omega, E), \mathcal{T}_\Phi)$ via

$$\mathcal{T}_{\mathbf{\Phi}} : S \longrightarrow \mathscr{L}(\Gamma_0(\Omega, E)); g \mapsto \mathcal{T}_\Phi(g) := \left[s \mapsto \Phi_g \circ s \circ \varphi_{g^{-1}}\right]$$

over $(C_0(\Omega), \boldsymbol{T_\varphi})$, the Koopman group representation on $C_0(\Omega)$. We call $(\mathcal{T}_\Phi(g))_{g \in S}$ a (positive) weighted Koopman semigroup representation on $\Gamma_0(\Omega, E)$ over the Koopman group representation $(T_\varphi(g))_{g \in G}$ on $C_0(\Omega)$.

By the following lemma and its corollaries, we show that, for a fixed G-dynamical system (Ω, φ), a morphism of (positive) S-dynamical Banach lattice bundles (see Definition 3.4.0.9) is uniquely determined by the homomorphism of the corresponding induced (positive) S-dynamical m-Banach lattice modules (see Chapter 2, Definition 2.3.5.1).

Lemma 3.5.0.6. *Let (Ω, φ) be a G-dynamical system. Moreover, let $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$ be S-dynamical Banach lattice bundles over (Ω, φ). Furthermore, let $(\Gamma_0(\Omega, E), \mathcal{T}_{\mathbf{\Phi}})$ and $(\Gamma_0(\Omega, F), \mathcal{T}_{\mathbf{\Psi}})$ be S-dynamical m-Banach lattice modules over the Koopman group $(C_0(\Omega), \boldsymbol{T_\varphi})$ induced by $(E, \mathbf{\Phi})$ and$(F, \mathbf{\Psi})$, respectively.*
Then, for a mapping $\Theta : E \longrightarrow F$, the following are equivalent.

(i) $\Theta : E \longrightarrow F$ *is a morphism between $(E, \mathbf{\Phi})$ and$(F, \mathbf{\Psi})$.*

(ii) $V_\Theta : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, F); s \mapsto \Theta \circ s$ *is a homomorphism between* $(\Gamma_0(\Omega, E), \mathcal{T}_{\mathbf{\Phi}})$ *and* $(\Gamma_0(\Omega, E), \mathbf{\Psi})$.

Moreover, if these assertions hold, then $||\Theta|| = ||V_\Theta||$, and Θ is an isometry if and only if V_Θ is an isometry.

Proof. (a) First, we observe that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\Theta} & F \\ {\scriptstyle \Phi_g}\downarrow & & \downarrow{\scriptstyle \Psi_g} \\ E & \xrightarrow{\Theta} & F \end{array}$$

commutes for each $g \in S$ if and only if the diagram

$$\begin{tikzcd}
\Gamma_0(\Omega, E) \arrow[r, "V_\Theta"] \arrow[d, "\mathcal{T}_\Phi(g)"'] & \Gamma_0(\Omega, F) \arrow[d, "\mathcal{T}_\Psi(g)"] \\
\Gamma_0(\Omega, E) \arrow[r, "V_\Theta"] & \Gamma_0(\Omega, F)
\end{tikzcd}$$

commutes for each $g \in S$. Indeed, for each $g \in S$,

$$\begin{aligned}
&\Psi(g) \circ \Theta = \Theta \circ \Phi(g) \\
\Longleftrightarrow\ &\Psi(g) \circ \Theta \circ s = \Theta \circ \Phi(g) \circ s \ \forall s \in \Gamma_0(\Omega, E) \\
\Longleftrightarrow\ &\Psi(g) \circ \Theta \circ s \circ \varphi_{g^{-1}} = \Theta \circ \Phi(g) \circ s \circ \varphi_{g^{-1}} \ \forall s \in \Gamma_0(\Omega, E) \\
\Longleftrightarrow\ &\mathcal{T}_\Psi(g) V_\Theta s = V_\Theta \mathcal{T}_\Phi(g) s \ \forall s \in \Gamma_0(\Omega, E) \\
\Longleftrightarrow\ &\mathcal{T}_\Psi(g) \circ V_\Theta = V_\Theta \circ \mathcal{T}_\Phi(g).
\end{aligned}$$

(b) In addition, $\Theta : E \longrightarrow F$ is a Banach lattice bundle morphism if and only if $V_\Theta : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, F)$ is a lattice module homomorphism. Indeed, for each $s \in \Gamma_0(\Omega, E)$

$$|\Theta \circ s| = \Theta \circ |s| \iff |V_\Theta s| = V_\Theta |s|.$$

Hence, the assertion $(i) \Leftrightarrow (ii)$ is proved.
Moreover, by Lemma 3.5.0.3, $V_\Theta : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, F)$ being a Banach lattice module homomorphism implies that $\Theta : E \longrightarrow F$ is the unique Banach bundle morphism over Id_Ω, such that $||\Theta|| = ||V_\Theta||$, and Θ is an isometry if and only if V_Θ is an isometry. □

By Lemma 3.5.0.3, we immediately obtain the following corollary, which states that the converse result of Lemma 3.5.0.6 above holds.

Corollary 3.5.0.7. *Let (Ω, φ) be a G-dynamical system. Moreover, let $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$ be S-dynamical Banach lattice bundles over (Ω, φ). Furthermore, let $(\Gamma_0(\Omega, E), \mathcal{T}_{\mathbf{\Phi}})$ and $(\Gamma_0(\Omega, F), \mathcal{T}_{\mathbf{\Psi}})$ be S-dynamical m-Banach lattice modules over the Koopman group $(C_0(\Omega), \boldsymbol{T}_\varphi)$ induced by $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$, respectively. Then for a mapping $V : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, F)$ the following are equivalent.*

(i) $V : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, F)$ is a homomorphism between $(\Gamma_0(\Omega, E), \mathcal{T}_{\mathbf{\Phi}})$ and $(\Gamma_0(\Omega, F), \mathcal{T}_{\mathbf{\Psi}})$.

(ii) There exists a unique morphism $\Theta : E \longrightarrow F$ between $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$ such that $V = V_\Theta$.

Moreover, if these assertions hold, then $||\Theta|| = ||V||$, and Θ is an isometry if and only if V is an isometry.

The following is also an immediate corollary which is obtained by combining Lemma 3.5.0.6 and Corollary 3.5.0.4 above.

Corollary 3.5.0.8. *Let (Ω, φ) be a G-dynamical system. Moreover, let $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$ be positive S-dynamical Banach lattice bundles over (Ω, φ), respectively. Furthermore, let $(\Gamma_0(\Omega, E), \mathcal{T}_{\mathbf{\Phi}})$ and $(\Gamma_0(\Omega, F), \mathcal{T}_{\mathbf{\Psi}})$ be positive S-dynamical m-Banach lattice modules over the Koopman group $(C_0(\Omega), \boldsymbol{T}_\varphi)$ induced by $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$, respectively.*
Then, for a mapping $V : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, F)$, the following are equivalent.

(i) $V : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, F)$ is a homomorphism between $(\Gamma_0(\Omega, E), \mathcal{T}_{\mathbf{\Phi}})$ and $(\Gamma_0(\Omega, E), \mathcal{T}_{\mathbf{\Psi}})$.

(ii) There exists a unique morphism $\Theta : E \longrightarrow F$ between $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$ such that $V = V_\Theta$.

Moreover, if this assertion holds, $||\Theta|| = ||V||$, and Θ is a positive isometry if and only if V is a positive isometry.

In the following proposition, we represent the lattice of continuous sections vanishing at infinity of a Banach lattice bundle as an AM m-lattice module or, equivalently, a $U_0(\Omega)$-normed m-lattice module (see Chapter 2, Corollary 2.4.1.10). This is essentially due to [21, Proposition 4.10, p.19], in the case of Banach bundles. See also [15, Corollary 7.28, p.78-79]. By this, we also answer Question D.1.0.2 raised in Appendix D.

Proposition 3.5.0.9. *Let Ω be a locally compact space and Γ an AM m-lattice module over $C_0(\Omega)$. Then,*

there is a Banach lattice bundle E over Ω such that $\Gamma_0(\Omega, E)$ is isometrically isomorphic to Γ as m-Banach lattice modules over $C_0(\Omega)$.

Moreover, this Banach lattice bundle is unique up to isometric isomorphism.

Proof. We follow the proof of [21, Proposition 4.10, p.19-21] which is for the case of Banach bundles. Since Γ is, in particular, an AM-module over $C_0(\Omega)$, we obtain a unique Banach bundle F over Ω such that the Banach space $\Gamma_0(\Omega, F)$ of its continuous sections vanishing at infinity is isometrically isomorphic to Γ as Banach modules over $C_0(\Omega)$. That is, there exists an operator $\mathcal{P} \in \mathscr{L}(\Gamma, \Gamma_0(\Omega, F))$ such that $\mathcal{P}$ is an isometric and surjective module homomorphism.
Hence, it suffices to show that F is a Banach lattice bundle over Ω; the operator $\mathcal{P}$ is a lattice isomorphism; and that F is unique up to isometric isomorphism.

We thus, claim the following:

(i) F is isometrically embedded in a bundle of Banach lattices. Indeed, by their proof, we see that $F := \dot{\bigcup}_{x\in\Omega} F_x$, where, for each $x \in \Omega$, we define the Banach space $F_x := \Gamma / J_x$ with

$$J_x := \overline{\text{lin}} \{ fs : f \in C_0(\Omega) \text{ with } f(x) = 0 \text{ and } s \in \Gamma \} .$$

Now, we define a new bundle $E := \dot{\bigcup}_{x\in\Omega} E_x$ over Ω, by setting

$$E_x := \Gamma / \Gamma_x$$

with

$$\Gamma_x := \left\{ s \in \Gamma : \inf \{ ||fs|| : f \in C_0(\Omega)_+ \text{ with } f(x) = 1 \} = 0 \right\}.$$

By the uniqueness of the lattice $U_0(\Omega)$-normed value (see Chapter 2, Corollary 2.4.1.10), we have that $\Gamma_x = \{ s \in \Gamma : |s|(x) = 0 \}$ which implies, by Chapter 2 (Corollary 2.5.3.2), that E_x is, in particular, a Banach lattice and $F_x \cong E_x$ (isomorphic Banach spaces) for each $x \in \Omega$.

Thus, the assertion holds as we obtain that $F \cong E$ is an isomorphism of Banach bundles over Ω, and $\Gamma_0(\Omega, F) \cong \Gamma_0(\Omega, E)$ is an isometric isomorphism of Banach modules over $C_0(\Omega)$.

(ii) The bundle modulus

$$E \longrightarrow E;\ v_{E_{p_E(v)}} \mapsto |v|$$

is continuous. Indeed, this is derived from the continuity of the modulus $\Gamma \longrightarrow \Gamma;\ s \mapsto |s|$ which "descends"[4] to the quotient lattice $E_x = \Gamma/\Gamma_x$ for each $x \in \Omega$.

This shows that E is a Banach lattice bundle over Ω, and hence the Banach space $\Gamma_0(\Omega, E)$ is, in particular, a Banach lattice (see Definition 3.2.0.1 and Proposition 3.3.0.2).

(iii) There exists a lattice isomorphism $\mathcal{T} : \Gamma \longrightarrow \Gamma_0(\Omega, E)$. Indeed, by defining the (linear) bijection $\mathcal{T} : \Gamma \longrightarrow \Gamma_0(\Omega, E)$ such that $\mathcal{T}s(x) = s + \Gamma_x$ for each $x \in \Omega$, let $e_x : \Gamma_0(\Omega, E) \longrightarrow E_x$ be the evaluation map and $I_x : \Gamma/\Gamma_x \longrightarrow E_x$ be the identity map; and consider the following commutative diagram.

$$\begin{tikzcd} \Gamma \arrow[rr, shift left, "s \mapsto \mathcal{T}s"'] \arrow[d, "q_x"'] & & \Gamma_0(\Omega, E) \arrow[ll, shift left, "\mathcal{T}^{-1}r \leftarrow r"'] \arrow[d, "e_x"] \\ \Gamma/\Gamma_x \arrow[rr, leftrightarrow, "I_x"'] & & E_x \end{tikzcd}$$

This implies that, for each $s \in \Gamma$, and $x \in \Omega$,

(a) $\mathcal{T}s(x) = e_x(\mathcal{T}s) = q_x(s)$; and

(b) $|\mathcal{T}s(x)| = e_x|\mathcal{T}s| = q_x|s| = e_x\mathcal{T}|s| = \mathcal{T}|s|(x)$

since e_x and q_x are both lattice homomorphisms for each $x \in \Omega$.

So, for each $s \in \Gamma$, the canonical mapping

$$|\mathcal{T}s| : \Omega \longrightarrow E;\ x \mapsto |\mathcal{T}s(x)|$$

is a continuous section vanishing at infinity, which coincides with $\mathcal{T}|s|$ for each $s \in \Gamma$. This shows that $\mathcal{T}$ is a lattice homomorphism, and since $\mathcal{T}$ is a bijection, it follows that $\mathcal{T}^{-1} : \Gamma_0(\Omega, E) \longrightarrow \Gamma$ is also a lattice homomorphism; and hence the assertion holds.

This shows that $\Gamma \cong \Gamma_0(\Omega, E)$ as m-Banach lattice modules over $C_0(\Omega)$.

(iv) E is unique up to isometric isomorphism of Banach lattice bundles over Ω. This follows from Lemma 3.5.0.3. Indeed, suppose H is another Banach lattice bundle over Ω, such that the Banach lattice $\Gamma_0(\Omega, H)$

[4] i.e., implies $\Gamma \longrightarrow E_x;\ s \mapsto |s| + \Gamma_x$ is continuous for each $x \in \Omega$.

of its continuous sections vanishing at infinity is isometrically isomorphic to Γ as m-Banach lattice modules over $C_0(\Omega)$, i.e., there exists an operator $\mathcal{S} : \Gamma \longrightarrow \Gamma_0(\Omega, H)$ such that $\mathcal{S}$ is an isometric and surjective lattice module homomorphism.

Then, the operator $\mathcal{S} \circ \mathcal{T}^{-1} : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, H)$ is an isometric and surjective lattice module homomorphism. So, by Lemma 3.5.0.3, we obtain a unique isometric and surjective Banach lattice bundle morphism $\Phi_1 : E \longrightarrow H$ over Id_Ω such that

$$\mathcal{S} \circ \mathcal{T}^{-1} s = \Phi_1 \circ s$$

for all $s \in \Gamma_0(\Omega, E)$. This, in particular, implies that the Banach lattices E_x and H_x are isometrically lattice isomorphic for each $x \in \Omega$.

Moreover, the inverse operator $\mathcal{T} \circ \mathcal{S}^{-1} : \Gamma_0(\Omega, H) \longrightarrow \Gamma_0(\Omega, E)$ is also an isometric lattice module homomorphism. Similarly, by Lemma 3.5.0.3, we also obtain a unique isometric and surjective Banach lattice bundle morphism $\Phi_2 : H \longrightarrow E$ over Id_Ω such that

$$\mathcal{T} \circ \mathcal{S}^{-1} r = \Phi_2 \circ r$$

for all $r \in \Gamma_0(\Omega, H)$. It then follows that

$$\Phi_2 \circ \Phi_1 = Id_E \text{ and } \Phi_1 \circ \Phi_2 = Id_H$$

and

$$\Phi_2^{-1} = \Phi_1 \text{ and } \Phi_1^{-1} = \Phi_2,$$

i.e., E and H are isomorphic (=homeomorphic).

□

Remark 3.5.0.10. *We note, in particular, that, if Γ is a complex AM m-lattice module over the complex Banach algebra $C_0(\Omega)$, then each fiber E_x of the constructed Banach lattice bundle E over Ω is a complex Banach lattice for each $x \in \Omega$, and the isomorphism $\Gamma \cong \Gamma_0(\Omega, E)$ can be seen to be $\mathbb{C}$-linear.*

We are now able to state our first representation result for dynamical m-Banach lattice modules. See [29, Theorem 2.22, p.34] for the case of dynamical Banach modules. This is our Gelfand-type theorem for dynamical AM m-lattice modules over $C_0(\Omega)$. We note that its corollary and remark, in particular, proves Proposition D.1.0.3 stated in Appendix D.

Theorem 3.5.0.11. *Let G be a locally compact group, $S \subseteq G$ a closed submonoid, and (Ω, φ) a topological G-dynamical system. Then the assignments*

$$(E, \boldsymbol{\Phi}) \longmapsto (\Gamma_0(\Omega, E), \mathcal{T}_{\boldsymbol{\Phi}})$$

$$\Theta \longmapsto V_\Theta$$

define an essentially surjective, fully faithful functor from the category of S-dynamical topological Banach lattice bundles over (Ω, φ) to the category of S-dynamical AM m-lattice modules over $(C_0(\Omega), \boldsymbol{T}_\varphi)$.

Proof. Combining Proposition 3.5.0.9 and Corollary 3.5.0.7 proves the theorem. □

The following is an immediate corollary of the above theorem.

Corollary 3.5.0.12. *Let G be a locally compact group, $S \subseteq G$ a closed submonoid, and (Ω, φ) a topological G-dynamical system. Then the assignments*

$$(E, \boldsymbol{\Phi}) \longmapsto (\Gamma_0(\Omega, E), \mathcal{T}_{\boldsymbol{\Phi}})$$

$$\Theta \longmapsto V_\Theta$$

define an essentially surjective, fully faithful functor from the category of positive S-dynamical topological Banach lattice bundles over (Ω, φ) to the category of positive S-dynamical AM m-lattice modules over $(C_0(\Omega), \boldsymbol{T}_\varphi)$.

Proof. Combining Proposition 3.5.0.9 and Corollary 3.5.0.8 proves the assertion. □

In the following remark, we note how the unique positive semiflow $\mathbf{\Phi} = (\Phi_g)_{g\in S}$ can be obtained canonically on the Banach lattice bundle E associated with a positive S-dynamical AM m-lattice module $(\Gamma, \boldsymbol{\mathcal{T}})$. By this, we construct the inverse functors for the above theorems (see also [29, Remark 3.9, p.64]).

Remark 3.5.0.13. *For an AM m-lattice module Γ over $C_0(\Omega)$, let $E := \dot{\bigcup}_{x\in\Omega} E_x$, with $E_x := \Gamma/\Gamma_x$ for each $x \in \Omega$, be the unique (up to isometric isomorphism) topological Banach lattice bundle such that $\Gamma \cong \Gamma_0(\Omega, E)$ as in the proof of Proposition 3.5.0.9.*
Now, let $(\Omega, (\varphi_g)_{g\in G})$ be a topological G-dynamical system and $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g\in S}$ a positive weighted semigroup representation on Γ over the Koopman group $(C_0(\Omega), \boldsymbol{T_\varphi})$. Moreover, for each $x \in \Omega$, let $q_x : \Gamma \longrightarrow E_x$ be the corresponding (quotient) lattice homomorphism. Then, for each $g \in S$, the positive operators $\Phi_g(x) := \Phi_g|_{E_x} : E_x \longrightarrow E_{\varphi_g(x)}$ are precisely the operators for which the diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "s\mapsto\mathcal{T}(g)s"] \arrow[d, "q_x"'] & \Gamma \arrow[d, "q_{\varphi_g(x)}"] \\
E_x \arrow[r, dashed, "\Phi_g(x)"'] & E_{\varphi_g(x)}
\end{tikzcd}$$

commutes for all $x \in \Omega$. That is, for each $g \in S$, the mappings $\Phi_g(x) : E_x \longrightarrow E_{\varphi_g(x)}; s + \Gamma_x \mapsto \mathcal{T}(g)s + \Gamma_{\varphi_g(x)}$ for all $x \in \Omega$ and $s \in \Gamma$ (uniquely) defines a positive morphism $\Phi_g : E \longrightarrow E$ over φ_g. From these observations, we have that

$$(\mathcal{T}(g))_{g\in S} \cong (\mathcal{T}_\Phi(g))_{g\in S} \quad on \quad \Gamma \cong \Gamma_0(\Omega, E).$$

3.6 More on the Banach lattice of continuous sections

In this Section, given a *topological* Banach lattice bundle E over a locally compact space Ω, we introduce and consider several notions about the Banach lattice $\Gamma_0(\Omega, E)$ of its continuous sections vanishing at infinity. While many of these concepts are proven very useful in this thesis, we note that, others will become handy for further studies.

3.6.1 On lattice subbundles and lattice submodules

Inspired by the work of S. Siewert [29, A.1, p.109-112], we now introduce the notions of *Banach lattice subbundles*, and we seek to obtain similar duality with Banach lattice submodules as introduced in Chapter 2 (Subsection 2.5.2).

Throughout, we take $\Omega = K$ to be a compact space, and $p : E \longrightarrow K$ a Banach lattice bundle over K.

Definition 3.6.1.1. *A subspace $F \subseteq E$ is called a lattice subbundle if the following properties are satisfied.*

(i) For each $x \in K$, the set $F_x := p^{-1}(x) \cap F$ is a sublattice of the fiber E_x, i.e., $F_x \subseteq E_x$ is a subspace such that $v \in F_x$ implies $|v|, \overline{v} \in F_x$; and

(ii) the restriction of the bundle projection $p|_F : F \longrightarrow K$ is open.

If, in addition, $F_x \subseteq E_x$ is a closed subspace for each $x \in K$, then we call $F \subseteq E$ a Banach lattice subbundle.

More importantly, we claim the following.

Proposition 3.6.1.2. *A Banach lattice subbundle $F \subseteq E$ is a Banach lattice bundle over K with the bundle projection, bundle norm, and bundle modulus restricted to F.*
Moreover, F induces an AM m-lattice module $\Gamma(K, F)$ over $C(K)$ which is also a Banach lattice submodule of $\Gamma(K, E)$.

Proof. By [29, Proposition A.2, p.110], we obtain that $p|_F : F \longrightarrow K$ is a Banach bundle over K with the bundle projection and bundle norm restricted to F. Since, by definition, F_x is a Banach lattice for each $x \in K$, it is sufficient to show that the restriction of the bundle modulus

$$|\cdot|_F : F \longrightarrow F;\ v \mapsto |v|_{F_{p|_F(v)}}$$

is continuous. Indeed, for $v \in F$, if $W \subseteq F$ is an an open set containing $|v|$, then we find an open set $W_0 \subseteq E$ containing v such that $|w_0| \in U$ for all $w_0 \in W_0$. It follows that $W_1 := F \cap W_0$ is also an open set in F containing v such that $|w| \in W$ for all $w \in W_1$.
Hence, the first assertion is proved.

Since, F is a Banach lattice bundle over K, the lattice of its continuous sections $\Gamma(K, F)$ is also an AM m-lattice module over $C(K)$ (see Remark 3.5.0.1(ii)). The fact that $\Gamma(K, F) \subseteq \Gamma(K, E)$ is a Banach lattice submodule follows readily.

□

Now, if $\Gamma \subseteq \Gamma(K, E)$ is a Banach lattice submodule, it readily follows that Γ is also an AM m-lattice module over $C(K)$. So, by Proposition 3.5.0.9, there exists, up to isometric isomorphism, a unique Banach lattice bundle F over K such that Γ is isometrically isomorphic to $\Gamma(K, F)$.
We claim this Banach lattice bundle F can be identified with a Banach lattice subbundle of E. More so, we also obtain a correspondence between Banach lattice subbundles and Banach lattice submodules. See [29, Proposition A.4, p.110] for the case of Banach subbundles and Banach submodules.

Proposition 3.6.1.3. *The following statements hold true.*

(i) For each Banach lattice subbundle $F \subseteq E$, the induced AM m-lattice module $\Gamma(K, F)$ is a Banach lattice submodule of $\Gamma(K, F)$.

(ii) For each Banach lattice submodule $\Gamma \subseteq \Gamma(K, E)$, the induced Banach lattice bundle $F := \bigcup_{x \in K} e_x(\Gamma)$ over K is a Banach lattice subbundle of E, where $e_x : \Gamma \longrightarrow E_x$; $s \mapsto s(x)$, $x \in K$ is the evaluation map.

Moreover, the assignment $F \mapsto \Gamma(K, F)$ is a bijection of Banach lattice subbundles and Banach lattice submodules. The inverse is given by $\Gamma \mapsto \bigcup_{x \in K} e_x(\Gamma)$.

Proof. (i) This follows from Proposition 3.6.1.2 from above.

(ii) By [29, Proposition A.4 (ii), p.110], we obtain that $F := \bigcup_{x \in K} e_x(\Gamma)$ is a Banach bundle over E, such that $F \subseteq E$ is also a Banach subbundle.

We claim that F is the unique Banach lattice bundle over E.

In particular:

(a) $e_x(\Gamma) \subseteq E_x$ is a Banach sublattice for every $x \in K$. Indeed, by Remark 3.5.0.1(iii), we note that the evaluation map $e_x : \Gamma \longrightarrow E_x$; $s \mapsto s(x)$, $x \in K$ is a lattice homomorphism, and moreover $\overline{e_x s} = e_x \overline{s}$ for all $s \in \Gamma$ and $x \in K$. This implies that the closed subspace $e_x(\Gamma) \subseteq E_x$ is a sublattice for every $x \in K$.

(b) The restriction of the bundle modulus to F is continuous. Indeed, this immediately follows from the proof of Proposition 3.6.1.2.

Hence, $F \subseteq E$ is a Banach lattice subbundle, and again by Proposition 3.6.1.2, it is a Banach lattice bundle over K.

Moreover, it is clear that, the assignment $F \mapsto \Gamma(K, F)$ is an injective map of Banach lattice subbundles of E and Banach lattice submodules of $\Gamma(K, E)$. Now, to show surjectivity, suppose $\Gamma \subseteq \Gamma(K, E)$ is a Banach lattice submodule such that $\Gamma \cong \Gamma(K, F_1)$ defines an isometric Banach lattice module isomorphism for a (unique up to isometric isomorphism) Banach lattice bundle F_1 over K. Since, by the proof of Proposition 3.5.0.9, each fiber $F_{1x}, x \in K$ can be identified with a quotient space of Γ, it follows that $F_{1x} \subseteq E_x$ is Banach sublattice for every $x \in K$. Hence, from (ii) above, we can conclude that $F_{1x} = e_x(\Gamma)$ for every $x \in K$, and $F_1 = \bigcup_{x \in K} e_x(\Gamma)$ is the unique Banach lattice subbundle of E.

□

As in [29, Remark A.5. p.110], for the case of Banach modules, we also note the following properties concerning the kernel and the image of a lattice module homomorphism of Banach lattice modules.

Proposition 3.6.1.4. *Let $\mathcal{T} : \Gamma(K, E_1) \longrightarrow \Gamma(K, E_2)$ be a lattice module homomorphism. Then Ker $\mathcal{T} \subseteq \Gamma(K, E_1)$ and the closure of rg $\mathcal{T} \subseteq \Gamma(K, E_2)$ are Banach lattice submodules.*
Moreover, Ker$\mathcal{T}$ is a closed ideal submodule of $\Gamma(K, E_1)$.

Proof. (i) Let $s \in Ker\mathcal{T}$. Then $\mathcal{T} f \cdot s = f \cdot \mathcal{T} s = 0$ for any $f \in C(K)$ implies that $Ker\mathcal{T}$ is a Banach submodule. Now, $s \in Ker\mathcal{T}$ also implies that $\mathcal{T} | s | = | \mathcal{T} s | = 0$, i.e., $| s | \in Ker\mathcal{T}$.

In the complex case, since $\mathcal{T}$ is $\mathbb{C}$-linear; $s \in Ker\mathcal{T}$ implies that $Re\, s, Im\, s \in Ker\mathcal{T}$. So, we can conclude that $\bar{s} \in Ker\mathcal{T}$.

Moreover, if $s \in Ker\mathcal{T}$, $r \in \Gamma(K, E_1)$, with $| r | \leq | s |$, then $| \mathcal{T} r | \leq | \mathcal{T} s | = 0$ implies that $r \in Ker\mathcal{T}$.

Hence, $Ker\mathcal{T}$ is a closed ideal submodule of $\Gamma(K, E_1)$, and, in particular, a Banach lattice submodule.

(ii) Let $s \in rg\mathcal{T}$. Then $f \cdot s = \mathcal{T}(f \cdot r)$ for some $r \in \Gamma(K, E_1)$ with $\mathcal{T}r = s$, for any $f \in C(K)$, implies that the closure of $rg\mathcal{T}$ is a Banach submodule. Now, $s \in rg\mathcal{T}$ also implies $|s| = \mathcal{T}|r|$ for some $r \in \Gamma(K, E_1)$ with $\mathcal{T}r = s$, i.e., $|s| \in rg\mathcal{T}$.

Furthermore, in the complex case, since $\mathcal{T}$ is $\mathbb{C}$-linear, $s \in rg\mathcal{T}$ implies that $Re\, s = \mathcal{T}(Re\, r)$ and $Im\, s = \mathcal{T}(Re\, r)$ for for some $r \in \Gamma(K, E_1)$ with $\mathcal{T}r = s$, i.e., $Re\, s, Im\, s \in rg\mathcal{T}$. So, we can also conclude that $\bar{s} \in rg\mathcal{T}$.

Hence, the closure of rg $\mathcal{T} \subseteq \Gamma(K, E_2)$ is a Banach lattice submodule.

□

3.6.2 Direct sum of Banach lattice bundles and Banach lattice modules

Following the discussion as in [29, A.2, p.111-112], we now consider the direct sum of two Banach lattice bundles E and F over the compact space K, as well as the direct sum of their lattices of continuous sections.

For each $x \in K$, let $E_x \oplus F_x$ be the direct sum of Banach spaces E_x and F_x equipped canonically with the lattice structure $|(u,v)| := (|u|, |v|)$ and $\overline{(u,v)} := (\bar{u}, \bar{v})$; and norm $||(u,v)|| := \max(||u||, ||v||)$ for $(u,v) \in E_x \oplus F_x$, which induces the product topology of E_x and F_x on $E_x \oplus F_x$ as a Banach lattice with natural ordering.
We then endow the direct sum

$$E \oplus F := \bigcup_{k \in K} E_x \oplus F_x \subseteq E \times F$$

with the subspace topology induced by the product topology on $E \times F$.

Equipped with the canonical projection, addition, scalar multiplication and modulus, the direct sum $E \oplus F$ of two Banach lattice bundles E and F is, again, a Banach lattice bundle over K.
Indeed, by [29, Construction A.6, p.111], we obtain that $E \oplus F$ is a Banach bundle over K, but each fiber $E_x \oplus F_x$ is a Banach lattice for $x \in K$. Hence, it suffices to show that the bundle modulus

$$|(\cdot,\cdot)| : E \oplus F \longrightarrow E \oplus F; (u,v) \mapsto (|u|, |v|)_{E_x \oplus F_x}$$

is continuous. This follows immediately, since $E \oplus F$ is equipped with the product topology.

Now, for two AM m-lattice modules $\Gamma(K, E)$ and $\Gamma(K, F)$ over $C(K)$ we equip the Banach space direct sum $\Gamma(K, E) \oplus \Gamma(K, F)$ canonically with the lattice structure $|(s_1, s_2)| := (|s_1|, |s_2|)$ and $\overline{(s_1, s_2)} := (\overline{s_1}, \overline{s_2})$; $C(K)$-module structure $f(s_1, s_2) := (fs_1, fs_2)$ and norm $||(s_1, s_2)|| := \max(||s_1||, ||s_2||)$ for $s_1 \in \Gamma(K, E), s_2 \in \Gamma(K, F)$, $f \in C(K)$; which induces the product topology of $\Gamma(K, E)$ and $\Gamma(K, F)$ on $\Gamma(K, E) \oplus \Gamma(K, F)$ as a Banach lattice module with natural ordering.

From this construction, we also claim that the direct sum $\Gamma(K, E) \oplus \Gamma(K, F)$ of two AM m-lattice modules $\Gamma(K, E)$ and $\Gamma(K, F)$ over $C(K)$ is, again, an AM m-lattice module over $C(K)$.

Proposition 3.6.2.1. *In the situation above, the mapping*

$$\Gamma(K, E) \oplus \Gamma(K, F) \longrightarrow \Gamma(K, E \oplus F); \ (s_1, s_2) \mapsto s_1 \oplus s_2$$

with $(s_1 \oplus s_2)(x) := (s_1(x), s_2(x))$, $s_1 \in \Gamma(K, E)$, $s_2 \in \Gamma(K, F)$, *and* $x \in K$ *defines an isometric isomorphism of AM m-lattice modules (see Chapter 2, Definition 2.3.2.1 and Remark 2.3.2.2).*

Proof. (i) First we note that, since $E \oplus F$ is a Banach lattice bundle over K, its lattice of continuous sections $\Gamma(K, E \oplus F)$ is an AM m-lattice module over $C(K)$.

(ii) The Banach lattice module $\Gamma(K, E) \oplus \Gamma(K, F)$ over $C(K)$ is an m-Banach lattice module. Indeed, for any $f \in C(K), s_1 \in \Gamma(K, E), s_2 \in \Gamma(K, F)$, we have that

$$|f(s_1, s_2)| = (|fs_1|, |fs_2|) = (|f||s_1|, |f||s_2|) = |f||(s_1, s_2)|.$$

(iii) Now, the fact that the mapping $\Gamma(K, E) \oplus \Gamma(K, F) \longrightarrow \Gamma(K, E \oplus F); (s_1, s_2) \mapsto s_1 \oplus s_2$ is a lattice module isomorphism is clear. Indeed,

(a) $p \in \Gamma(K, E \oplus F)$ if and only if $p(x) \in E_x \oplus F_x$ for every $x \in K$ if and only $p(x) = (s_1(x), s_2(x))$ for every $x \in K$ and $s_1 \in \Gamma(K, E), s_2 \in \Gamma(K, F)$ if and only if $p = s_1 \oplus s_2$. That is, the mapping is surjective.

(b) For $s_1, r_1 \in \Gamma(K, E), s_2, r_2 \in \Gamma(K, F)$, we have that $s_1 \oplus s_2 = r_1 \oplus r_2$ if and only if $(s_1(x), s_2(x)) = (r_1(x), r_2(x))$ for all $x \in K$ if and only if $(s_1, s_2) = (r_1, r_2)$. That is, the mapping is injective.

(c) For every $f \in C(K), s_1 \in \Gamma(K, E), s_2 \in \Gamma(K, F)$, we have that

$$(fs_1 \oplus fs_2)(x) = (f(x)s_1(x), f(x)s_2(x)) = f(x)(s_1 \oplus s_2)(x)$$

for all $x \in K$. That is, the mapping is a module homomorphism.

(d) For every $s_1 \in \Gamma(K, E), s_2 \in \Gamma(K, F)$, we have that

$$|(s_1 \oplus s_2)|(x) = (|s_1(x)|, |s_2(x)|) = (|s_1| \oplus |s_2|)(x)$$

for all $x \in K$. That is, the mapping is a lattice homomorphism.

Furthermore, since the mapping is a bijection, it is also a module isomorphism and a lattice isomorphism. Hence, the assertion follows.

(iv) The mapping $\Gamma(K, E) \oplus \Gamma(K, F) \longrightarrow \Gamma(K, E \oplus F); (s_1, s_2) \mapsto s_1 \oplus s_2$ is an isometry. Indeed, for $s_1 \in \Gamma(K, E), s_2 \in \Gamma(K, F)$ we have that

$$\begin{aligned} ||(s_1 \oplus s_2)|| &= \sup_{x \in K}(\max(||s_1(x)||, ||s_2(x)||)) \\ &= \max(\sup_{x \in K} ||s_1(x)||, \sup_{x \in K} ||s_2(x)||) \\ &= \max(||s_1||, ||s_2||) \\ &= ||(s_1, s_2)|| \end{aligned}$$

Finally, by the isometry, we obtain that the Banach m-lattice module $\Gamma(K, E) \oplus \Gamma(K, F)$ over $C(K)$ is an AM m-lattice module, i.e.,

$$||(f \vee g)(s_1, s_2)|| = \max(||f(s_1, s_2)||, ||g(s_1, s_2)||)$$

for all $f, g \in C(K)_+$, $s_1 \in \Gamma(K, E)$, $s_2 \in \Gamma(K, F)$.

□

The result above yields a correspondence between decompositions of Banach lattice bundles and decompositions of AM m-lattice modules. See [29, Proposition A.8, p.112] for the case of Banach bundles and AM-modules.

Proposition 3.6.2.2. *Let E be a Banach lattice bundle over K and $\Gamma(K, E)$ the corresponding AM m-lattice module over $C(K)$. Then, the following hold.*

(i) If there are two non-trivial Banach lattice subbundles $E_1, E_2 \subseteq E$ such that $E \cong E_1 \oplus E_2$, then we have that $\Gamma(K, E_1)$ and $\Gamma(K, E_2)$ are non-trivial Banach lattice submodules of $\Gamma(K, E)$ and $\Gamma(K, E) \cong \Gamma(K, E_1) \oplus \Gamma(K, E_2)$.

(ii) If there are two non-trivial Banach lattice submodules $\Gamma_1, \Gamma_2 \subseteq \Gamma(K, E)$ such that $\Gamma(K, E) \cong \Gamma_1 \oplus \Gamma_2$, then there are two non-trivial Banach lattice subbundles $E_1, E_2 \subseteq E$ such that $E \cong E_1 \oplus E_2$ and $\Gamma_1 \cong \Gamma(K, E_1)$ and $\Gamma_2 \cong \Gamma(K, E_2)$.

Proof. (i) Since $E_1, E_2 \subseteq E$ are non-trivial Banach lattice subbundles and $E \cong E_1 \oplus E_2$, we can assume that there is a nontrivial decomposition of the Banach lattice $E_x = E_{(1,x)} \oplus E_{(2,x)}$ for every $x \in K$. It follows immediately from Proposition 3.6.1.2 that the induced AM m-lattice modules $\Gamma(K, E_1)$ and $\Gamma(K, E_2)$ over $C(K)$ are non-trivial Banach lattice submodules of $\Gamma(K, E)$. Hence, by Proposition 3.6.2.1, the mappings

$$\begin{array}{ccccc} \Gamma(K, E_1) \oplus \Gamma(K, E_2) & \longrightarrow & \Gamma(K, E_1 \oplus E_2) & \longrightarrow & \Gamma(K, E) \\ (s_1, s_2) & \longmapsto & s_1 \oplus s_2 & \longmapsto & s_1 \oplus s_2 \end{array}$$

define isometric isomorphisms of AM m-lattice modules.

(ii) Since $\Gamma_1, \Gamma_2 \subseteq \Gamma(K, E)$ are non-trivial Banach lattice submodules, by Proposition 3.6.1.3, we can find two non-trivial Banach lattice subbundles $E_1, E_2 \subseteq E$ such that $\Gamma_1 \cong \Gamma(K, E_1)$ and $\Gamma_2 \cong \Gamma(K, E_2)$.

So, by Proposition 3.6.2.1, we obtain that

$$\Gamma(K, E) \cong \Gamma_1 \oplus \Gamma_2 \cong \Gamma(K, E_1) \oplus \Gamma(K, E_2) \cong \Gamma(K, E_1 \oplus E_2)$$

are isometric isomorphic AM m-lattice modules. Hence, we can conclude that $E \cong E_1 \oplus E_2$.

□

3.6.3 On ideal subbundles and ideal submodules

Inspired by Proposition 3.6.1.3, obtained in Section 3.6.1, which gives us a correspondence between Banach lattice subbundles and Banach lattice modules, one is prompted naturally to expect a similar result for ideal submodules (see Chapter 2, Definition 2.5.2.3) and what we now call "ideal subbundles".

We give the definition of an ideal subbundle below, which is analogous to the definition of a lattice subbundle (see Definition 3.6.1.1) and we seek to obtain similar correspondence.

As before, K is compact and $p : E \longrightarrow K$ a Banach lattice bundle over K.

Definition 3.6.3.1. *A subspace $I_E \subseteq E$ is called an ideal subbundle if the following properties are satisfied.*

(i) For each $x \in K$, the set $I_{Ex} := p^{-1}(x) \cap I_E$ is a lattice ideal of the fiber E_x, i.e., $I_{Ex} \subseteq E_x$ is a subspace such that if $v \in I_{Ex}, w \in E_x$ then $\mid w \mid \leq \mid v \mid$ implies $w \in I_{Ex}$.

(ii) The restriction of the bundle projection $p|_{I_E} : I_E \longrightarrow K$ is open.

If, in addition, $I_{Ex} \subseteq E_x$ is a closed subspace for each $x \in K$, then we call $I_E \subseteq E$ a closed ideal subbundle.

Remark 3.6.3.2. *(i) Since every lattice ideal is a sublattice, it follows that if $I_E \subseteq E$ is an ideal subbundle, then it is also a lattice subbundle.*

(ii) So, if $I_E \subseteq E$ is a closed ideal subbundle, then it is also a Banach lattice subbundle.

The following is an immediate corollary of Proposition 3.6.1.2 and the definition above.

Corollary 3.6.3.3. *A closed ideal subbundle $I_E \subseteq E$ is a Banach lattice bundle over K with the bundle projection, bundle norm, and bundle modulus restricted to F. Moreover, I_E induces an AM m-lattice module $\Gamma(K, I_E)$ over $C(K)$ which is also a closed ideal submodule of $\Gamma(K, E)$.*

Proof. Since $I_E \subseteq$ is a closed ideal subbundle, it is, in particular, a Banach lattice subbundle; and it follows from Proposition 3.6.1.2 that $p|_{I_F} : I_F \longrightarrow K$ is a Banach lattice bundle over K. Moreover, the lattice of its continuous sections $\Gamma(K, I_E)$ is an AM m-lattice module over $C(K)$, which is also a Banach lattice submodule of $\Gamma(K, E)$.

So, it suffices to show that $\Gamma(K, I_E)$ is a lattice ideal of $\Gamma(K, E)$. Now, let $s \in \Gamma(K, I_E)$ and $r \in \Gamma(K, E)$ be such that $\mid r \mid \leq \mid s \mid$. This implies that, for every $x \in K$,

$$\mid r(x) \mid_{E_x} \leq \mid s(x) \mid_{I_{Ex}}.$$

Now, since $I_{Ex} \subseteq E_x$ is a lattice ideal for every $x \in K$, it follows that $r(x) \in I_{Ex}$ for every $x \in K$. Hence, $r \in \Gamma(K, I_E)$.
This shows that $\Gamma(K, I_E)$ is also a closed ideal submodule of $\Gamma(K, E)$.

□

Now, if $I_\Gamma \subseteq \Gamma(K, E)$ is a closed ideal submodule, it readily follows that I_Γ is also an AM m-lattice module over $C(K)$. So, by Proposition 3.5.0.9, there exists, up to isometric isomorphism, a unique Banach lattice bundle F over K such that I_Γ is isometrically isomorphic to $\Gamma(K, F)$.
We claim this Banach lattice bundle F can be identified with a closed ideal subbundle of E. Moreover, we also obtain a correspondence between closed ideal subbundles and closed ideal submodules.

This can be seen as a corollary of Proposition 3.6.1.3 for the case of Banach lattice subbundle and Banach lattice submodule.

Corollary 3.6.3.4. *The following statements hold true.*

(i) For each closed ideal subbundle $I_E \subseteq E$ the induced AM m-lattice module $\Gamma(K, I_E)$ is a closed ideal submodule of $\Gamma(K, E)$.

(ii) For each closed ideal submodule $I_\Gamma \subseteq \Gamma(K, E)$, the induced Banach lattice bundle $I_E := \bigcup_{x \in K} e_x(I_\Gamma)$ over K is a closed ideal subbundle of E, where $e_x : I_\Gamma \longrightarrow E_x;\ s \mapsto s(x),\ x \in K$ is the evaluation map.

Moreover, the assignment $I_E \mapsto \Gamma(K, I_E)$ is a bijection of closed ideal subbundles and closed ideal submodules. The inverse is given by $I_\Gamma \mapsto \bigcup_{x \in K} e_x(I_\Gamma)$.

Proof. (i) This follows from Corollary 3.6.3.3 above.

(ii) Since $I_\Gamma \subseteq \Gamma(K, E)$ is a closed ideal submodule it is, in particular, a Banach lattice submodule, and so by Proposition 3.6.1.3 (ii) the induced Banach lattice bundle $I_E := \bigcup_{x \in K} e_x(I_\Gamma)$ over K is a Banach subbundle of E.

So, it suffices to show that for each (fixed) $x \in K$, $I_{E_x} = e_x(I_\Gamma) \subseteq E_x$ is a lattice ideal. Now, let $v \in I_{E_x}$ and $w \in E_x$ be such that $|w| \leq |v|$. Since $I_\Gamma \subseteq \Gamma(K, E)$ is lattice ideal, and $e_x : \Gamma(K, E) \longrightarrow E_x$ is a quotient map, we can find $s \in I_\Gamma$, $r \in \Gamma(K, E)$ with $|r| \leq |s|$ such that $e_x(r) =$

w and $e_x(s) = v$, which implies that $r \in I_\Gamma$ and $w = e_x(r) \in I_{E_x}$. See also [28, Proposition 3.1, p.65].

This implies that $I_E = \bigcup_{x \in K} e_x(I_\Gamma)$ is a closed ideal subbundle of E.

Moreover, the fact that the assignment $I_E \mapsto \Gamma(K, I_E)$ is a bijective map from closed ideal subbundles of E to closed ideal submodules of $\Gamma(K, E)$ immediately follows from Proposition 3.6.1.3 by a similar argument as in the proof of the conclusion.

□

3.6.4 Direct sum of positive semiflows and positive weighted Koopman semigroups

In Subsection 3.6.2, we introduced the concepts of the direct sum $E \oplus F$ of two Banach lattice bundles E and F over K, as well as the direct sum $\Gamma(K, E) \oplus \Gamma(K, F)$ of the two AM m-lattice modules $\Gamma(K, E)$ and $\Gamma(K, F)$ over the Banach lattice algebra $C(K)$. In particular, $E \oplus F$ is again a Banach lattice bundle over K, and $\Gamma(K, E) \oplus \Gamma(K, F) \cong \Gamma(K, E \oplus F)$ is again an AM m-lattice module over $C(K)$, see Proposition 3.6.2.1.

Moreover, earlier in Subsection 3.6.1, we introduced the concepts of Banach lattice subbundles and we obtained a correspondence between Banach lattice subbundles and Banach lattice submodules (see Proposition 3.6.1.3). Again, in Section 3.6.2, we also obtained a correspondence between decompositions of Banach lattice bundles and decompositions of spaces of continuous sections (see Proposition 3.6.2.2).

In this direction, following [29, Section 5.2 , p.87-90], we will also introduce the concepts of direct sums of (positive) semiflows on Banach lattice bundles and as well as direct sums of (positive) weighted Koopman semigroups on the spaces of continuous sections.

As before, K is compact and $(\varphi_t)_{t\in\mathbb{R}}$ is a flow on K.

We take two (positive) semiflows $(\Phi_t)_{t\geq 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on a Banach lattice bundle E over K, and $(\Psi_t)_{t\geq 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on Banach lattice bundle F over K. We claim that, setting

$$(\Phi \oplus \Psi)_t(u, v) := (\Phi_t u, \Psi_t v) \text{ for all } t \geq 0, (u, v) \in E \oplus F,$$

defines a (positive) semiflow $(\Phi \oplus \Psi)_{t\geq 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on the Banach lattice bundle $E \oplus F$ over K. Indeed, since $E \oplus F$ is equipped with the product topology, we obtain that the mapping

$$\Phi \oplus \Psi : \mathbb{R}_+ \longrightarrow E \oplus F^{E\oplus F};\ t \mapsto (\Phi \oplus \Psi)_t$$

is a monoid representation which is jointly continuous, locally bounded,

$$\begin{tikzcd} E \oplus F \arrow[r, "(\Phi\oplus\Psi)_t"] \arrow[d, "p_{E\oplus F}"'] & E \oplus F \arrow[d, "p_{E\oplus F}"] \\ K \arrow[r, "\varphi_t"] & K \end{tikzcd}$$

and the diagram above commutes for each $t \geq 0$, where $p_{E\oplus F} : E \oplus F \longrightarrow K$ is the natural associated projection map (see also Definitions 3.4.0.1 and 3.4.0.5). Hence, it suffices to show that the mapping $(\Phi \oplus \Psi)_t : E \oplus F \longrightarrow E \oplus F; (u, v)_{E\oplus F} \mapsto (\Phi_t u, \Psi_t v)$ is a (positive) morphism of Banach lattice bundles over φ_t for each $t \geq 0$. This follows from (i) and (ii) below.

(i) If $(\Phi_t)_{t\geq 0}$ and $(\Psi_t)_{t\geq 0}$ are semiflows over $(\varphi_t)_{t\in\mathbb{R}}$ on E and F, respectively, then

$|(\Phi \oplus \Psi)_t(x)(u, v)| = |(\Phi_t(x)u, \Psi_t(x)v)| = (\Phi_t(x)|u|, \Psi_t(x)|v|) = (\Phi \oplus \Psi)_t(x)|(u, v)|$ for all $x \in K, (u, v) \in E_x \oplus F_x$, $t \geq 0$ and where $(\Phi \oplus \Psi)_t(x) := (\Phi \oplus \Psi)_{t_{|E_x\oplus F_x}} : E_x \oplus F_x \longrightarrow E_{\varphi_t(x)} \oplus F_{\varphi_t(x)}$ are the restriction maps on the fibers $E_x \oplus F_x$.

Hence, $(\Phi \oplus \Psi)_{t\geq 0}$ is a semiflow on $E \oplus F$ over $(\varphi_t)_{t\in\mathbb{R}}$ on K.

(ii) If $(\Phi_t)_{t\geq 0}$ and $(\Psi_t)_{t\geq 0}$ are positive semiflows over $(\varphi_t)_{t\in\mathbb{R}}$ on E and F, respectively, then it follows from the argument in (i) above that $(\Phi \oplus \Psi)_{t\geq 0}$ is a positive semiflow on $E \oplus F$ over $(\varphi_t)_{t\in\mathbb{R}}$ on K.

Now, if $(\Phi_t)_{t\geq 0}$ and $(\Psi_t)_{t\geq 0}$ are two (positive) semiflows over $(\varphi_t)_{t\in\mathbb{R}}$ on E and F, respectively, then we consider the two (positive) weighted Koopman semigroups $(\mathcal{T}_\Phi(t))_{t\geq 0}$ on $\Gamma(K, E)$ over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$ and $(\mathcal{T}_\Psi(t))_{t\geq 0}$ on $\Gamma(K, F)$ over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$. We claim that setting

$$(\mathcal{T}_\Phi \oplus \mathcal{T}_\Psi)(t) := \mathcal{T}_\Phi(t) \oplus \mathcal{T}_\Psi(t) \text{ for all } t \geq 0$$

defines a (positive) weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0} \oplus (\mathcal{T}_\Psi)(t))_{t\geq 0}$ on $\Gamma(K,E) \oplus \Gamma(K,F)$ over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$. Indeed, since $\Gamma(K,E) \oplus \Gamma(K,F)$ is equipped with product topology, we obtain that the mapping

$$\mathcal{T}_\Phi \oplus \mathcal{T}_\Psi : \mathbb{R}_+ \longrightarrow \Gamma(K,E) \oplus \Gamma(K,F)^{\Gamma(K,E)\oplus\Gamma(K,F)};\ t \mapsto \mathcal{T}_\Phi(t) \oplus \mathcal{T}_\Psi(t)$$

is a strongly continuous semigroup representation on $\Gamma(K,E) \oplus \Gamma(K,F)$ (see also Chapter 2 (Definition 2.3.4.1)) such that $(\mathcal{T}_\Phi \oplus \mathcal{T}_\Psi)(t)f(s_1,s_2) = T_\varphi(t)f \cdot (\mathcal{T}_\Phi \oplus \mathcal{T}_\Psi)(s_1,s_2)$ for all $t \geq 0, f \in C(K), s_1 \in \Gamma(K,E), s_2 \in \Gamma(K,F)$. Hence, it suffices to show that the mapping $(\mathcal{T}_\Phi \oplus \mathcal{T}_\Psi)(t) : \Gamma(K,E) \oplus \Gamma(K,F) \longrightarrow \Gamma(K,E) \oplus \Gamma(K,F); (s_1,s_2)_{\Gamma(K,E)\oplus\Gamma(K,F)} \mapsto \mathcal{T}_\Phi(t) \oplus \mathcal{T}_\Psi(t)(s_1,s_2)$ is a (positive) lattice $T_\varphi(t)$-homomorphism for every $t \geq 0$. This follows from (i) and (ii) below.

(i) If $(\mathcal{T}_\Phi(t))_{t\geq 0}$ and $(\mathcal{T}_\Psi(t))_{t\geq 0}$ are weighted Koopman semigroups over $T_\varphi(t)_{t\in\mathbb{R}}$ on $\Gamma(K,E)$ and $\Gamma(K,F)$, respectively, which is the case if and only if $(\Phi_t)_{t\geq 0}$ and $(\Psi_t)_{t\geq 0}$ are semiflows over $(\varphi_t)_{t\in\mathbb{R}}$ on E and F, respectively, then

   $\mathsf{I}(\mathcal{T}_\Phi \oplus \mathcal{T}_\Psi)(t)(s_1,s_2)\mathsf{I} = \mathsf{I}(\mathcal{T}_\Phi(t)s_1, \mathcal{T}_\Psi(t)s_2)\mathsf{I} = (\mathcal{T}_\Phi(t)\mathsf{I}s_1\mathsf{I}, \mathcal{T}_\Psi(t)\mathsf{I}s_2\mathsf{I}) = (\mathcal{T}_\Phi \oplus \mathcal{T}_\Psi)(t)\mathsf{I}(s_1,s_2)\mathsf{I}$ for all $t \geq 0, s_1 \in \Gamma(K,E), s_2 \in \Gamma(K,F)$.

   Hence, $(\mathcal{T}_\Phi(t))_{t\geq 0} \oplus (\mathcal{T}_\Psi)(t))_{t\geq 0}$ is a weighted Koopman semigroup on $\Gamma(K,E) \oplus \Gamma(K,F)$ over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$.

(ii) If $(\mathcal{T}_\Phi(t))_{t\geq 0}$ and $(\mathcal{T}_\Psi(t))_{t\geq 0}$ are positive weighted Koopman semigroups over $T_\varphi(t)_{t\in\mathbb{R}}$ on $\Gamma(K,E)$ and $\Gamma(K,F)$, respectively, which is the case if and only if $(\Phi_t)_{t\geq 0}$ and $(\Psi_t)_{t\geq 0}$ are positive semiflows over $(\varphi_t)_{t\in\mathbb{R}}$ on E and F, respectively, then it follows from the argument in (i) above that $(\mathcal{T}_\Phi(t))_{t\geq 0} \oplus (\mathcal{T}_\Psi)(t))_{t\geq 0}$ is a positive weighted Koopman semigroup on $\Gamma(K,E) \oplus \Gamma(K,F)$ over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$.

With the construction above, and using Proposition 3.6.2.1, we also claim that the direct sum $(\mathcal{T}_\Phi(t))_{t\geq 0} \oplus (\mathcal{T}_\Psi(t))_{t\geq 0}$ on $\Gamma(K,E) \oplus \Gamma(K,F)$ of two (positive) weighted Koopman semigroups $(\mathcal{T}_\Phi(t))_{t\geq 0}$ and $(\mathcal{T}_\Psi(t))_{t\geq 0}$ over $T_\varphi(t)_{t\in\mathbb{R}}$ on $\Gamma(K,E)$ and $\Gamma(K,F)$, respectively, can uniquely be identified with the (positive) weighted Koopman semigroup $(\mathcal{T}_{\Phi\oplus\Psi}(t))_{t\geq 0}$ on $\Gamma(K, E \oplus F)$ over $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$.

Proposition 3.6.4.1. *In the situation above, if $(\Phi_t)_{t\geq 0}$ and $(\Psi_t)_{t\geq 0}$ are (positive) semiflows over $(\varphi_t)_{t\in\mathbb{R}}$ on E and F, respectively, then*

$$(\mathcal{T}_{\Phi}(t))_{t\geq 0} \oplus (\mathcal{T}_{\Psi})(t))_{t\geq 0} \cong (\mathcal{T}_{\Phi\oplus\Psi}(t))_{t\geq 0}.$$

In particular, the mapping

$$\Theta : \Gamma(K,E) \oplus \Gamma(K,F) \longrightarrow \Gamma(K,E\oplus F); (s_1,s_2) \mapsto s_1 \oplus s_2$$

with $(s_1 \oplus s_2)(x) := (s_1(x), s_2(x))$ $s_1, \in \Gamma(K,E), s_2 \in \Gamma(K,F)$, and $x \in K$ is an isometric isomorphism of (positive) $\mathbb{R}_+$-dynamical AM m-lattice modules $(\Gamma(K,E) \oplus \Gamma(K,F), \mathcal{T}_{\Phi} \oplus \mathcal{T}_{\Psi})$ and $(\Gamma(K,E\oplus F), \mathcal{T}_{\Phi\oplus\Psi})$ over $T_{\varphi}(t)_{t\in\mathbb{R}}$ on $C(K)$ (see Chapter 2, Definition 2.3.5.1 and Remark 2.3.5.2).

Moreover, $||(\mathcal{T}_{\Phi} \oplus \mathcal{T}_{\Psi})(t)|| = ||\mathcal{T}_{\Phi\oplus\Psi}(t)|| = ||(\Phi \oplus \Psi)(t)||$ for all $t \geq 0$.

Proof. (i) First, we note that, by Proposition 3.6.2.1, $\Theta : \Gamma(K,E) \oplus \Gamma(K,F) \longrightarrow \Gamma(K,E\oplus F)$ is an isometric lattice module isomorphism, i.e., $\Gamma(K,E) \oplus \Gamma(K,F) \cong \Gamma(K,E\oplus F)$ as AM m-lattice modules over $C(K)$.

(ii) If $(\Phi_t)_{t\geq 0}$ and $(\Psi_t)_{t\geq 0}$ are (positive) semiflows over $(\varphi_t)_{t\in\mathbb{R}}$ on E and F, respectively, then the direct sum $(\Phi \oplus \Psi)_{t\geq 0}$ is a (positive) semiflow on $E \oplus F$ over $(\varphi_t)_{t\in\mathbb{R}}$ on K. So, the induced weighted semigroup $(\mathcal{T}_{\Phi\oplus\Psi}(t))_{t\geq 0}$ on $\Gamma(K,E\oplus F)$ is a (positive) weighted Koopman semigroup over $T_{\varphi}(t)_{t\in\mathbb{R}}$ on $C(K)$.

(iii) Also, if $(\Phi_t)_{t\geq 0}$ and $(\Psi_t)_{t\geq 0}$ are (positive) semiflows over $(\varphi_t)_{t\in\mathbb{R}}$ on E and F, respectively, then the direct sum $(\mathcal{T}_{\Phi}(t))_{t\geq 0} \oplus (\mathcal{T}_{\Psi})(t))_{t\geq 0}$ on $\Gamma(K,E) \oplus \Gamma(K,F)$ of the induced (positive) weighted Koopman semigroups $(\mathcal{T}_{\Phi}(t))_{t\geq 0}$ and $(\mathcal{T}_{\Psi}(t))_{t\geq 0}$ over $T_{\varphi}(t)_{t\in\mathbb{R}}$ on $\Gamma(K,E)$ and $\Gamma(K,F)$, respectively, is a (positive) weighted Koopman semigroup on $\Gamma(K,E) \oplus \Gamma(K,F)$ over $T_{\varphi}(t)_{t\in\mathbb{R}}$.

(iv) Hence, in either case (i.e., positive or lattice homomorphism) it suffices to show that the diagram

$$\begin{tikzcd}
\Gamma(K,E) \oplus \Gamma(K,F) \arrow[r, "\Theta"] \arrow[d, "(\mathcal{T}_{\Phi}\oplus\mathcal{T}_{\Psi})(t)"'] & \Gamma(K,E\oplus F) \arrow[d, "\mathcal{T}_{\Phi\oplus\Psi}(t)"] \\
\Gamma(K,E) \oplus \Gamma(K,F) \arrow[r, "\Theta"] & \Gamma(K,E\oplus F)
\end{tikzcd}$$

commutes for all $t \geq 0$. This follows immediately, since

$$\begin{aligned}\Theta(\mathcal{T}_\Phi \oplus \mathcal{T}_\Psi)(t)(s_1, s_2) &= \Theta(\mathcal{T}_\Phi(t)s_1, \mathcal{T}_\Psi(t)s_2)\\ &= \mathcal{T}_\Phi(t)s_1 \oplus \mathcal{T}_\Psi(t)s_2\\ &= \mathcal{T}_{\Phi\oplus\Psi}(t)(s_1 \oplus s_2)\\ &= \mathcal{T}_{\Phi\oplus\Psi}(t)\Theta((s_1, s_2))\end{aligned}$$

for all $t \geq 0, s_1 \in \Gamma(K, E), s_2 \in \Gamma(K, F)$.

This shows that $(\mathcal{T}_\Phi(t))_{t\geq 0} \oplus (\mathcal{T}_\Psi)(t))_{t\geq 0} \cong (\mathcal{T}_{\Phi\oplus\Psi}(t))_{t\geq 0}$.

(v) Moreover, by Proposition 3.5.0.5, we have that $||\mathcal{T}_{\Phi\oplus\Psi}(t)|| = ||(\Phi \oplus \Psi)(t)||$ for all $t \geq 0$, and since $\Theta : \Gamma(K, E) \oplus \Gamma(K, F) \longrightarrow \Gamma(K, E \oplus F)$ is a surjective-isometry, by (iv) above it follows that $||(\mathcal{T}_\Phi \oplus \mathcal{T}_\Psi)(t)|| = ||\mathcal{T}_{\Phi\oplus\Psi}(t)|| = ||(\Phi \oplus \Psi)(t)||$ for all $t \geq 0$.

□

The result above in Proposition 3.6.4.1 combined with Proposition 3.6.2.2, yields a correspondence between the decomposition of (positive) semiflows on a Banach lattice bundle and the decomposition of the corresponding (positive) weighted Koopman semigroups on the lattice of continuous sections.

Proposition 3.6.4.2. *Let E be Banach lattice bundle over K and $\Gamma(K, E)$ the corresponding AM m-lattice module over $C(K)$. Moreover, let $(\Phi_t)_{t\geq 0}$ be a positive semiflow on E over $(\varphi_t)_{t\in\mathbb{R}}$ on K, and $(\mathcal{T}_\Phi(t))_{t\geq 0}$ the associated positive weighted Koopman semigroup on $\Gamma(K, E)$ over $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$. Then, the following hold true.*

(i) If there are two non-trivial Banach lattice subbundles $E_1, E_2 \subseteq E$ and two positive semiflows $(\Phi^1_t)_{t\geq 0}$ on E_1 and $(\Phi^2_t)_{t\geq 0}$ on E_2 over $(\varphi_t)_{t\in\mathbb{R}}$ on K such that

$$(\Phi_t)_{t\geq 0} \cong (\Phi^1_t)_{t\geq 0} \oplus (\Phi^2_t)_{t\geq 0} \quad on \quad E \cong E_1 \oplus E_2,$$

then $\Gamma(K, E_1)$ and $\Gamma(K, E_2)$ are non-trivial Banach lattice submodules of $\Gamma(K, E)$, and

$$(\mathcal{T}_\Phi(t))_{t\geq 0} \cong (\mathcal{T}_{\Phi^1}(t))_{t\geq 0} \oplus (\mathcal{T}_{\Phi^2}(t)) \quad on \quad \Gamma(K, E) \cong \Gamma(K, E_1) \oplus \Gamma(K, E_2).$$

Moreover, $||\Phi^1_t|| = ||\mathcal{T}_{\Phi^1}(t)||$ and $||\Phi^2_t|| = ||\mathcal{T}_{\Phi^2}(t)||$ for all $t \geq 0$.

(ii) *If there are two non-trivial Banach lattice submodules $\Gamma_1, \Gamma_2 \subseteq \Gamma(K, E)$, and two positive weighted semigroups $(\mathcal{T}_1(t))_{t\geq 0}$ on Γ_1 and $(\mathcal{T}_2(t))_{t\geq 0}$ on Γ_2 over $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$ such that*

$$(\mathcal{T}_\Phi(t))_{t\geq 0} \cong (\mathcal{T}_1(t))_{t\geq 0} \oplus (\mathcal{T}_2(t))_{t\geq 0} \quad on \quad \Gamma(K, E) \cong \Gamma_1 \oplus \Gamma_2,$$

then there are two non-trivial Banach lattice subbundles $E_1, E_2 \subseteq E$ and two positive semiflows $(\Phi_t^1)_{t\geq 0}$ on E_1 and $(\Phi_t^2)_{t\geq 0}$ on E_2 over $(\varphi_t)_{t\in\mathbb{R}}$ on K such that

$$(\Phi_t)_{t\geq 0} \cong (\Phi_t^1)_{t\geq 0} \oplus (\Phi_t^2)_{t\geq 0} \quad on \quad E \cong E_1 \oplus E_2.$$

Moreover, $||\mathcal{T}_1(t)|| = ||\Phi_t^1||$ and $||\mathcal{T}_2(t)|| = ||\Phi_t^2||$ for all $t \geq 0$.

Proof. (i) By Proposition 3.6.2.2 (i), if $E_1, E_2 \subseteq E$ are two non-trivial Banach lattice subbundles such that $E \cong E_1 \oplus E_2$, it follows that $\Gamma(K, E_1)$ and $\Gamma(K, E_2)$ are two non-trivial Banach lattice submodules of $\Gamma(K, E)$ such that $\Gamma(K, E) \cong \Gamma(K, E_1) \oplus \Gamma(K, E_2)$.

Since, E_1 and E_2 are also Banach lattice bundles over K, if $(\Phi_t^1)_{t\geq 0}$ and $(\Phi_t^2)_{t\geq 0}$ are positive semiflows over $(\varphi_t)_{t\in\mathbb{R}}$ on E_1 and E_2, respectively, such that $(\Phi_t)_{t\geq 0} \cong (\Phi_t^1)_{t\geq 0} \oplus (\Phi_t^2)_{t\geq 0}$, it follows from Proposition 3.6.4.1 that

$$(\mathcal{T}_\Phi(t))_{t\geq 0} \cong (\mathcal{T}_{\Phi^1}(t))_{t\geq 0} \oplus (\mathcal{T}_{\Phi^2}(t)) \quad \text{on} \quad \Gamma(K, E) \cong \Gamma(K, E_1) \oplus \Gamma(K, E_2).$$

Moreover, by Proposition 3.5.0.5, we have that $||\Phi_t^1|| = ||\mathcal{T}_{\Phi^1}(t)||$ and $||\Phi_t^2|| = ||\mathcal{T}_{\Phi^2}(t)||$ for all $t \geq 0$.

(ii) By Proposition 3.6.2.2 (ii), if $\Gamma_1, \Gamma_2 \subseteq \Gamma(K, E)$ are two non-trivial Banach lattice submodules such that $\Gamma(K, E) \cong \Gamma_1 \oplus \Gamma_2$, then we can find two non-trivial lattice subbundles $E_1, E_2 \subseteq E$ such that $E \cong E_1 \oplus E_2$, $\Gamma_1 \cong \Gamma(K, E_1)$ and $\Gamma_2 \cong \Gamma(K, E_2)$.

Since $\Gamma_1 \cong \Gamma(K, E_1)$ and $\Gamma_2 \cong \Gamma(K, E_2)$ are also AM m-lattice modules over $C(K)$, if $(\mathcal{T}_1(t))_{t\geq 0}$ and $(\mathcal{T}_2(t))_{t\geq 0}$ are positive weighted semigroups over $T_\varphi(t)_{t\in\mathbb{R}}$ on Γ_1 and on Γ_2, respectively, by Proposition 3.5.0.5, we obtain unique (up to isometric isomorphism) positive semiflows $(\Phi_t^1)_{t\geq 0}$ on E_1 and $(\Phi_t^2)_{t\geq 0}$ on E_2 over $(\varphi_t)_{t\in\mathbb{R}}$ on K

such that

$$(\mathcal{T}_1(t))_{t\geq 0} \cong (\mathcal{T}_{\Phi^1}(t))_{t\geq 0} \quad \text{on} \quad \Gamma_1 \cong \Gamma(K, E_1), \text{ and}$$

$$(\mathcal{T}_2(t))_{t\geq 0} \cong (\mathcal{T}_{\Phi^2}(t))_{t\geq 0} \quad \text{on} \quad \Gamma_2 \cong \Gamma(K, E_2).$$

Therefore, since $(\mathcal{T}_\Phi(t))_{t\geq 0} \cong (\mathcal{T}_1(t))_{t\geq 0} \oplus (\mathcal{T}_2(t))_{t\geq 0}$ on $\Gamma(K,E) \cong \Gamma_1 \oplus \Gamma_2$ by assumption, it follows that

$$\begin{aligned}
(\mathcal{T}_\Phi(t))_{t\geq 0} &\cong (\mathcal{T}_1(t))_{t\geq 0} \oplus (\mathcal{T}_2(t))_{t\geq 0} \qquad \text{on } \Gamma(K,E) \cong \Gamma_1 \oplus \Gamma_2 \\
&\cong (\mathcal{T}_{\Phi^1}(t))_{t\geq 0} \oplus (\mathcal{T}_{\Phi^2}(t))_{t\geq 0} \quad \text{on } \Gamma(K,E) \cong \Gamma(K,E_1) \oplus \Gamma(K,E_2) \\
&\cong (\mathcal{T}_{\Phi^1 \oplus \Phi^2}(t))_{t\geq 0} \quad \text{on } \Gamma(K,E) \cong \Gamma(K, E_1 \oplus E_2)
\end{aligned}$$

where we have used Proposition 3.6.4.1 in the last step above.

Therefore, we can uniquely (up to isometric isomorphism) identify the positive semiflows $(\Phi_t)_{t\geq 0} \cong (\Phi_t^1)_{t\geq 0} \oplus (\Phi_t^2)_{t\geq 0}$ on the Banach lattice bundles $E \cong E_1 \oplus E_2$.

Moreover, by Proposition 3.5.0.5, we have that $||\mathcal{T}_1(t)|| = ||\Phi_t^1||$ and $||\mathcal{T}_2(t)|| = ||\Phi_t^2||$ for all $t \geq 0$.

□

3.6.5 Banach lattices of continuous sections and order structures

In this Subsection, we consider certain order structures of the Banach lattice $\Gamma_0(\Omega, E)$ of continuous sections vanishing at infinity of a *topological* Banach lattice bundle E over a locally compact space Ω. The order structures under consideration are; non-emptiness of the positive cone, (σ-) order completeness, and order continuity of the norm. We note that, by Proposition 3.5.0.9, we are actually considering these order structures for an AM m-lattice module over $C_0(\Omega)$ (see Chapter 4, Section 4.2).

We start with the special situation of Example 3.3.0.6(i). That is, given a locally compact space Ω and a Banach lattice Z, under what conditions does the Banach lattice $C_0(\Omega, Z)$ satisfy any of the order structures under consideration? In this direction, we collect the following results due to Ercan and Wickstead ([13]).

Proposition 3.6.5.1. *Let Z be a Banach lattice, Ω a locally compact space, and $E := \Omega \times Z$ the so-called trivial Banach lattice bundle in Example 3.3.0.6(i). Moreover, let $\Gamma_0(\Omega, E) \cong C_0(\Omega, Z)$ be the associated AM m-lattice module over $C_0(\Omega)$. Then we have the following.*

(A) Consider the following statements.

(i) The interior $IntC_0(\Omega, Z)_+ \neq \emptyset$*.*

(ii) $\Omega = K$ *is compact and the interior* $IntZ_+ \neq \emptyset$*.*

Then $(i) \iff (ii)$.

(B) Consider the following statements.

(i) $C_0(\Omega, Z)$ *is* σ*-order complete.*

(ii) Ω *is discrete and* Z *is* σ*-order complete.*

(iii) $C_0(\Omega)$ *is* σ*-order complete and* Z *has compact order intervals.*

(iv) $\Omega = K$ *is infinite, compact, and* $C(K, Z)$ *is* σ*-order complete.*

(v) Z *has compact order intervals.*

(vi) If $U \subseteq K$ *is an open* F_σ *subset and* $s : U \longrightarrow Z$ *is a continuous order bounded function then* s *has a continuous extension,* $\bar{s} : K \longrightarrow Z$*.*

Then $(i) \iff (ii) \iff (iii)$*. Moreover, if* K *is totally disconnected (i.e., quasi-Stonian), then* $(iv) \iff (v) \iff (vi)$*.*

(C) Consider the following statements.

(i) $C_0(\Omega, Z)$ *is order complete.*

(ii) Ω *is discrete and* Z *is order complete.*

(iii) $C_0(\Omega)$ *is order complete and* Z *has compact order intervals.*

(iv) $\Omega = K$ *is infinite, compact, and* $C(K, Z)$ *is order complete.*

(v) Z *has compact order intervals.*

(vi) If $U \subseteq K$ *is an open subset and* $s : U \longrightarrow Z$ *is a continuous order bounded function then* s *has a continuous extension,* $\bar{s} : K \longrightarrow Z$*.*

Then $(i) \iff (ii) \iff (iii)$*. Moreover, if* K *is extremally disconnected (i.e., Stonian), then* $(iv) \iff (v) \iff (vi)$*.*

(D) The following are equivalent.

(i) $C_0(\Omega, Z)$ *has order continuous norm.*

(ii) Ω *is discrete and* Z *has order continuous norm.*

Proof. We note again that, throughout our study, a locally compact space is assumed to be Hausdorff by definition.

(A) By [13, Proposition 2.1, p.123], we obtain that: the mapping $C_0(\Omega) \longrightarrow C_0(\Omega, Z); f \mapsto f \otimes z$, for each (fixed) positive element $z \in Z_+$, and the mapping $Z \longrightarrow C_0(\Omega, Z); z \mapsto f \otimes z$, for each (fixed) positive function $f \in C_0(\Omega)$, respectively, are order continuous lattice homomorphisms, with $f \otimes z(x) := f(x)z$.

So, if $z_0 \in Z_+$ is a (fixed) positive element such that $||z_0|| = 1$, then the mapping $C_0(\Omega) \longrightarrow C_0(\Omega, Z); f \mapsto f \otimes z_0$ is, in addition, an isometry since $||f \otimes z_0|| = \sup_{x\in\Omega} ||f(x)z_0|| = \sup_{x\in\Omega} |f(x)| = ||f||$ for all $f \in C_0(\Omega)$. Similarly, if $f_0 \in C_0(\Omega)$ is a (fixed) positive function such that $||f_0|| = 1$, the mapping $Z \longrightarrow C_0(\Omega, Z); z \mapsto f_0 \otimes z$ is also an isometry.

From these identifications, it follows that if the interior $IntC_0(\Omega, Z)_+ \neq \emptyset$, then it follows necessarily that the interior $IntC_0(\Omega)_+ \neq \emptyset$ which is the case only if $\Omega = K$ is compact (see Appendix B, Theorem B.0.0.6); and also the interior $IntZ_+ \neq \emptyset$. Thus, $(i) \implies (ii)$.

Now, assume (ii) holds. WLOG, we identify $Z = C(M)$ for some compact space M (see also Appendix B, Theorem B.0.0.6), and define the mapping $T : C(K, C(M)) \longrightarrow C(K \times M); s \mapsto Ts(x, q) := s(x)q$ for $(x, q) \in K \times M$. It is clearly linear, and since $|Ts|(x, q) = |s(x)q| = |s|(x)q = T|s|(x, q)$ for all $(x, q) \in K \times M$ and $s \in C(K, C(M))$, it is a lattice homomorphism. Moreover, it is an isometry, since

$$||s|| = \sup_{x\in K} ||s(x)|| = \sup_{x\in K} \sup_{q\in M} |s(x)q| = \sup_{(x,q)\in K\times M} |s(x)q| = ||Ts||$$

for each $s \in C(K, C(M))$. To show surjectivity, suppose $w \in C(K \times M)$, i.e., the mapping $K \times M \longrightarrow \mathbb{K}; (x, q) \mapsto w(x, q)$ is continuous. For each $x \in K$, we can define the continuous mapping $w_x : M \longrightarrow \mathbb{K}; q \mapsto w(x, q)$, and choose a mapping $s \in C(K, C(M))$ such that $s(x) = w_x$ for each $x \in K$. It follows that $Ts(x, q) = w_x(q) = w(x, q)$ for $(x, q) \in K \times M$. Thus, $T : C(K, C(M)) \longrightarrow C(K \times M)$ is an isometric lattice isomorphism which implies the interior $IntC(K, Z)_+ \neq \emptyset$.

(B) The equivalence of (i), (ii) and (iii) follows from [13, Theorem 3.2, p.126]. Moreover, the equivalence of (iv), (v) and (vi) follows also from [13, Theorem 5.13, p.133].

(C) The equivalence of (i), (ii) and (iii) follows from [13, Theorem 3.3, p.126]. Moreover, the equivalence of (iv), (v) and (vi) follows also from [13, Theorem 5.12, p.133].

(D) Similarly, by [13, Theorem 4.1, p.126] this equivalence holds.

□

Proposition 3.6.5.1 has provided the necessary conditions for the order structures under consideration. Using these results and some of their arguments, we extend the result to the general case of Banach lattices of continuous sections in the next proposition. To this end, the following lemma will be useful.

Lemma 3.6.5.2. *Let E be a topological Banach lattice bundle over a locally compact space Ω, and $\Gamma_0(\Omega, E)$ the associated AM m-lattice module over $C_0(\Omega)$.*

(i) For each (fixed) positive continuous section $s \in \Gamma_0(\Omega, E)_+$, the linear mapping

$$C_0(\Omega) \longrightarrow \Gamma_0(\Omega, E); f \mapsto f \cdot s := [\, x \mapsto f(x)s(x)]$$

is an order continuous lattice homomorphism.

(ii) Moreover, if $\Omega = K$ is compact, and we choose a (fixed) positive continuous section $s \in \Gamma(K, E)_+$ with $||s(x)||_{E_x} = 1$ for all $x \in K$, then the mapping in (i) above is, in addition, an isometry, i.e., $||f \cdot s|| = ||f||$ for all $f \in C(K)$.

Proof. (i) It is clear that the linear mapping is a lattice homomorphism, since $|f \cdot s| = |f| \cdot s$ for all $f \in C_0(\Omega)$. Now, we show that it is order continuous. By [13, Proposition 2.1, p.123], as in the proof of Proposition 3.6.5.1 A(i), for $x \in \Omega$, the mapping, $C_0(\Omega) \longrightarrow C_0(\Omega, E_x); f \mapsto f \otimes v_x$ for each (fixed) positive element $v_x \in E_x^+$ is an order continuous lattice homomorphism, with $f \otimes v_x(x) := f(x)v_x$.

Now, suppose $f_\gamma \downarrow 0$ in $C_0(\Omega)$. By setting $v_x := s(x)$, it follows from the above that $f_\gamma \otimes s(x) \downarrow 0$ in $C_0(\Omega, E_x)$, for each $x \in \Omega$, which then implies $f_\gamma \cdot s \downarrow 0$ in $\Gamma_0(\Omega, E)$ as claimed.

(ii) Observing that, for any $f \in C(K)$,

$$||f \cdot s|| = \sup_{x \in K} ||f(x)s(x)||_{E_x} = \sup_{x \in K} |f(x)| = ||f|,|$$

proves the assertion.

□

Proposition 3.6.5.3. *Let E be a topological Banach lattice bundle over a locally compact space Ω, and $\Gamma_0(\Omega, E)$ the associated AM m-lattice module over $C_0(\Omega)$. Then we have the following.*

(A) Consider the following statements.

(i) The interior $Int\Gamma_0(\Omega, E)_+ \neq \emptyset$.

(ii) $\Omega = K$ is compact and the interior $IntE_x^+ \neq \emptyset$ for each $x \in K$.

Then $(i) \implies (ii)$.

(B) Consider the following statements.

(i) $\Gamma_0(\Omega, E)$ is σ-order complete.

(ii) Ω is discrete and E_x is σ-order complete for each $x \in \Omega$.

(iii) $C_0(\Omega)$ is σ-order complete and E_x has compact order intervals for each $x \in \Omega$.

(iv) $\Omega = K$ is infinite, compact, and $\Gamma(K, E)$ is σ-order complete.

(v) E_x has compact order intervals for each $x \in K$.

(vi) If $U \subseteq K$ is an open F_σ-subset and $s : U \longrightarrow E$ is a continuous order bounded local section, i.e., $s : U \longrightarrow E$ is a continuous local section with $\pm s \leq s_{o|_U}$ for some $s_o \in \Gamma(K, E)_+$, then s has a continuous extension, $\bar{s} : K \longrightarrow E$.

Then $(i) \iff (ii) \iff (iii)$. Moreover, if K is totally disconnected (i.e., quasi-Stonian), then $(iv) \iff (v) \implies (vi)$.

(C) Consider the following statements.

(i) $\Gamma_0(\Omega, E)$ is order complete.

(ii) Ω is discrete and E_x is order complete for each $x \in \Omega$.

(iii) $C_0(\Omega)$ is order complete and E_x has compact order intervals for each $x \in \Omega$.

(iv) $\Omega = K$ *is infinite, compact, and* $\Gamma(K, E)$ *is order complete.*

(v) E_x *has compact order intervals for each* $x \in K$.

(vi) If $U \subseteq K$ *is an open subset and* $s : U \longrightarrow E$ *is a continuous order bounded local section then* s *has a continuous extension,* $\bar{s} : K \longrightarrow E$.

Then $(i) \iff (ii) \iff (iii)$. *Moreover, if K is extremally disconnected (i.e., Stonian), then* $(iv) \iff (v) \implies (vi)$.

(D) The following are equivalent.

(i) $\Gamma_0(\Omega, E)$ *has order continuous norm.*

(ii) Ω *is discrete and* E_x *has order continuous norm for each* $x \in \Omega$.

Proof. (A) $(i) \Rightarrow (ii)$: This implication follows from Proposition 3.6.5.1(A) and Lemma 3.6.5.2. Indeed, assume $r \in \mathrm{Int}\Gamma_0(\Omega, E)_+$, and, on the contrary, suppose there exists $x_0 \in \Omega$ such that the interior $\mathrm{Int}E^+_{x_0}$ is empty. Since $r(x_0) \in E^+_{x_0}$, it follows that, for all open sets $U \subseteq \Omega$ containing x_0, and for all $\varepsilon > 0$, the set difference $S(s, U, \varepsilon)_+ \setminus r(x_0) \nsubseteq E^+_{x_0}$, for all $s \in \Gamma(K, E)_+$ with $s(x_0) = r(x_0)$. This is the case, since, by Proposition 3.3.0.4, the sets of the form $S(s, U, \varepsilon)_+$ form a base for the subspace topology on the set of positive elements E_+ of E. This is, however, a contradiction to the assumption.

(B) By Proposition 3.6.5.1(B), the argument here is essentially similar to (C) below.

(C) Combining Lemma 3.6.5.2 and Proposition 3.6.5.1(C), the equivalent conditions (ii) and (iii) necessarily hold whenever (i) holds. Thus, it suffices to show that (ii) implies (i) for the first part of the assertion.

Now assume (ii) holds. Let $\boldsymbol{\mathcal{A}} \subseteq \Gamma_0(\Omega, E)$ be an order bounded subset, i.e., $\boldsymbol{\mathcal{A}} \subseteq [s_1, s_2] := \{s \in \Gamma_0(\Omega, E) : s_1 \leq s \leq s_2\}$ for some $s_1, s_2 \in \Gamma_0(\Omega, E)$. Since, for each $x \in \Omega$, the (quotient) evaluation map $e_x : \Gamma_0(\Omega, E) \longrightarrow E_x$ is an order continuous lattice homomorphism (see Remark 3.5.0.1(iii)), we have that

$$e_x(\boldsymbol{\mathcal{A}}) := \{s(x) : s \in \boldsymbol{\mathcal{A}}\} \subseteq [s_1(x), s_2(x)] := \{v \in E_x : s_1(x) \leq v \leq s_2(x)\}.$$

Given that E_x is order complete for each $x \in \Omega$, we can find $v_x \in E_x$ such that $\sup e_x(\boldsymbol{\mathcal{A}}) = v_x \in [s_1(x), s_2(x)]$, and define a section

$$s_o : \Omega \longrightarrow E;\ x \mapsto v_x$$

which is continuous since Ω is discrete. It thus follows that $s_o \in \Gamma_0(\Omega, E)$ and $\sup \boldsymbol{\mathcal{A}} = s_o \in [s_1, s_2]$, as required.

We now assume that $\Omega = K$ is an infinite compact Stonian space. Since this, in particular, implies that $C_0(\Omega) = C(K)$ is order complete, the equivalence of conditions (iv) and (v) holds by equivalent condition (iii). Thus, it suffices to show that (iv) implies (vi) for the second part of the assertion.

WLOG, let $s : U \longrightarrow E$ be a continuous local section with $0 \leq s \leq s_{o|_U}$ for some $s_o \in \Gamma(K, E)_+$ and $U \subseteq K$ an open subset. If we take a maximal family $(U_\gamma)_{\gamma \in I}$ of disjoint open and closed subsets of U, and define sections $s_\gamma : K \longrightarrow E$ by

$$s_\gamma(x) := \begin{cases} s(x) & \text{if } x \in U_\gamma \\ 0_x & \text{if } x \notin U_\gamma, \end{cases}$$

then $\{s_\gamma : \gamma \in I\}$ is a subset of $\Gamma(K, E)$ which is bounded above by, say $s_o \in \Gamma(K, E)_+$. Since $\Gamma(K, E)$ is order complete, we can find a continuous section $\bar{s} : K \longrightarrow E$ such that $\sup_{\gamma \in I} s_\gamma = \bar{s} \leq s_o$. Then we must have that $\bar{s}_{|_U} = s$, and $\bar{s} : K \longrightarrow E$ can thus be seen to be the desired extension.

(D) As in (C) above; combining Lemma 3.6.5.2 and Proposition 3.6.5.1(C) ascertains the necessity of equivalent conditions (ii) and (iii) whenever (i) holds. Thus, it suffices also to show that (ii) implies (i).

Now assume (ii) holds. Since order continuity of the norm of E_x also implies it is order complete, for each $x \in \Omega$, (C) above implies that $\Gamma_0(\Omega, E)$ is order complete. Hence, as in the proof of Proposition 3.6.5.1(D) in [13, Theorem 4.1, p.126], it suffices to show that every order bounded disjoint sequence [5] of positive continuous sections must converge in norm to zero.

Therefore, let $(s_n)_{n \in \mathbb{N}} \subseteq \Gamma_0(\Omega, E)$ be a disjoint sequence of positive continuous sections with $0 \leq s_n \leq s$ for some $s \in \Gamma_0(\Omega, E)$. Given $\varepsilon > 0$, for each $n \in \mathbb{N}$, we have that the set

$$\{x \in \Omega : \varepsilon \leq ||s_n(x)||_{E_x}\} \subseteq \{x \in \Omega : \varepsilon \leq ||s(x)||_{E_x}\} = \{x_1, \cdots, x_k\}$$

[5] a sequence $(s_n)_{n \in \mathbb{N}}$ of pairwise lattice disjoint elements, i.e., $s_n \perp s_m \ \forall n \neq m$.

is compact in Ω, and hence finite, for some $k \in \mathbb{N}$, since Ω is discrete.

For each $1 \leq j \leq k$, it follows that $((s_n)(x_j))_{n\in\mathbb{N}} \subseteq E_{x_j}$ is a disjoint sequence of positive elements which is bounded above by $s(x_j) \in E_{x_j}$. Since E_x has order continuous norm for each $x \in \Omega$, we have that, for each $1 \leq j \leq k$, $||(s_n)(x_j)||_{E_{x_j}} \to 0$ as $n \to \infty$. We can choose $n_j \in \mathbb{N}$ such that $||(s_n)(x_j)||_{E_{x_j}} < \varepsilon$ whenever $n > n_j$, and set $n_0 := \max\{n_1, \cdots, n_k\}$.

Now, for each $x \in \Omega$, either $x \notin \{x_1, \cdots, x_k\}$ in which case we have that $||s_n(x)||_{E_x} < \varepsilon$ for all $n \in \mathbb{N}$, or $x \in \{x_1, \cdots, x_k\}$ which also implies $||s_n(x)||_{E_x} < \varepsilon$ whenever $n > n_0$. As a result, we have that $||s_n|| < \varepsilon$ for all $n > n_0$, and thus $\Gamma_0(\Omega, E)$ has order continuous norm.

□

3.6.6 Banach lattices of continuous sections and their centres

In this Subsection, we characterise the space $\mathcal{Z}(\Gamma_0(\Omega, E))$ of central operators (see Appendix A, Example A.1.0.4(iii)) of the Banach lattice $\Gamma_0(\Omega, E)$ of continuous sections vanishing at infinity of a *topological* Banach lattice bundle E over a locally compact space Ω. We also note that, by Proposition 3.5.0.9, we are actually characterising the centre of an AM m-lattice module over $C_0(\Omega)$. Although this is of independent interest, it will be a useful tool in the study of the so-called multiplication operators on these Banach lattices.

We also start with the special situation of Example 3.3.0.6(i). That is, given a locally compact space Ω and a Banach lattice Z, how can we characterise the centre $\mathcal{Z}(C_0(\Omega, Z))$ of the Banach lattice $C_0(\Omega, Z)$? In this direction, we also collect certain results due to Ercan and Wickstead ([13]). We, in particular, claim that its centre $\mathcal{Z}(C_0(\Omega, Z))$ is again an AM m-lattice module over the centre $\mathcal{Z}(C_0(\Omega)) \cong C_b(\Omega)$.

Proposition 3.6.6.1. *Let Z be a Banach lattice, Ω a locally compact space, $E := \Omega \times Z$ the trivial Banach lattice bundle in Example 3.3.0.6(i), and let $\Gamma_0(\Omega, E) \cong C_0(\Omega, Z)$ be the associated AM m-lattice module over $C_0(\Omega)$. Furthermore, if we endow the centre $\mathcal{Z}(Z) \subseteq \mathscr{L}(Z)$ of Z by the strong operator topology and denote* *it by $\mathcal{Z}(Z)_s$, we let $E_z := \Omega \times \mathcal{Z}(Z)_s$ be the trivial Banach lattice bundle with fiber $\mathcal{Z}(Z)$.*

Then, we have the following.

(A) *The mapping $C_b(\Omega, \mathcal{Z}(Z)_s) \longrightarrow \mathcal{Z}(C_0(\Omega, Z))$; $\phi \mapsto T_\phi$ defined by*

$$T_\phi s := \phi \circ s,$$

i.e., $T_\phi s(x) = \phi(x)s(x)$ for all $x \in \Omega$, $\phi \in C_b(\Omega, \mathcal{Z}(Z)_s)$ and $s \in C_0(\Omega, Z)$, is an isomorphism of commutative 1-Banach lattice algebras.

(B) *Moreover, the mapping $C_b(\Omega, \mathcal{Z}(Z)_s) \longrightarrow \Gamma_b(\Omega, E_z); \phi \mapsto \tilde{\phi}$ defined by*

$$\tilde{\phi}(x) := (x, \phi(x)) \quad \text{for all } x \in \Omega, \text{ and } \phi \in C_b(\Omega, \mathcal{Z}(Z)_s)$$

is:

(i) an isomorphism of AM m-lattice modules over $C_b(\Omega)$; and

(ii) an isomorphism of commutative 1-Banach lattice algebras

from $C_b(\Omega, \mathcal{Z}(Z)_s)$ onto the Banach lattice of bounded continuous sections $\Gamma_b(\Omega, E_z)$ associated with E_z.

Proof. (A) This follows immediately from [13, Theorem 6.2, p.135].

(B) We proceed in the following manner (a) - (b) by which we prove (i) and then (ii) afterwards.

(a) First, we note that, the (bilinear) pairing

$$C_b(\Omega) \times C_b(\Omega, \mathcal{Z}(Z)_s) \longrightarrow C_b(\Omega, \mathcal{Z}(Z)_s); \; (f, \phi) \mapsto f \cdot \phi := [x \mapsto f(x)\phi(x)]$$

turns $C_b(\Omega, \mathcal{Z}(Z)_s)$ into a Banach module over $C_b(\Omega)$, since

$$||f \cdot \phi|| = \sup_{x \in \Omega} ||f(x)\phi(x)||_{\mathscr{L}(Z)} \leq \sup_{x \in \Omega} |f(x)| \sup_{x \in \Omega} ||\phi(x)||_{\mathscr{L}(Z)} = ||f|| ||\phi||$$

for all $f \in C_b(\Omega)$ and $\phi \in C_b(\Omega, \mathcal{Z}(Z)_s)$. Moreover, for any $f \in C_b(\Omega)$ and $\phi \in C_b(\Omega, \mathcal{Z}(Z)_s)$,

$$|f \cdot \phi|(x) = |f(x)||\phi(x)| = (|f| \cdot |\phi|)(x)$$

for all $x \in \Omega$, which implies that $|f \cdot \phi| = (|f| \cdot |\phi|)$, so that $C_b(\Omega, \mathcal{Z}(Z)_{\mathbf{s}})$ is indeed an m-Banach lattice module over the commutative m-Banach lattice algebra $C_b(\Omega)$ (see also Chapter 2, Definition 2.2.2.4).

Similarly, since $p_{\mathbf{z}} : E_{\mathbf{z}} \longrightarrow \Omega$ is a trivial *topological* Banach lattice bundle, with fiber $p_{\mathbf{z}}^{-1}(x) := \mathcal{Z}(Z)$ for all $x \in \Omega$, the Banach lattice of its bounded continuous sections

$$\Gamma_b(\Omega, E_{\mathbf{z}}) := \Big\{ s : \Omega \longrightarrow E_{\mathbf{z}} \text{ continuous }, \; p_{\mathbf{z}} \circ s = Id_{\Omega} \text{ and } ||s|| := \sup_{x \in \Omega} ||s(x)||_{\mathscr{L}(Z)} < \infty \Big\}$$

is, in particular, a Banach lattice. Moreover, by our consideration as above, $\Gamma_b(\Omega, E_{\mathbf{z}})$ is also an m-Banach lattice module over $C_b(\Omega)$.

(b) It is clear from (a) above that the mapping $\phi \mapsto \tilde{\phi}$ from $C_b(\Omega, \mathcal{Z}(Z)_{\mathbf{s}})$ into $\Gamma_b(\Omega, E_z)$ is linear and bijective, i.e., $s \in \Gamma_b(\Omega, E_{\mathbf{z}})$ if and only if there exists a unique $\phi \in C_b(\Omega, \mathcal{Z}(Z)_{\mathbf{s}})$ such that $s(x) = (x, \phi(x)) = \tilde{\phi}(x)$ for all $x \in \Omega$.

Since $|\tilde{\phi}|(x) = (x, |\phi(x)|) = (x, |\phi|(x)) = |\tilde{\phi}|(x)$ for all $x \in \Omega$ and $\phi \in C_b(\Omega, \mathcal{Z}(Z)_{\mathbf{s}})$, the mapping $\phi \mapsto \tilde{\phi}$ is also a lattice homomorphism. Moreover, it is a $C_b(\Omega)$-module homomorphism, since, for any $f \in C_b(\Omega)$ and $\phi \in C_b(\Omega, \mathcal{Z}(Z)_{\mathbf{s}})$,

$$\tilde{f\phi}(x) = (x, f(x)\phi(x)) = (f \cdot \tilde{\phi})(x)$$

for all $x \in \Omega$.

Finally, the lattice module homomorphism $\phi \mapsto \tilde{\phi}$ is an isometry, since

$$||\tilde{\phi}|| = \sup_{x \in \Omega} ||\phi(x)||_{\mathscr{L}(Z)} = ||\phi||$$

for all $\phi \in C_b(\Omega, \mathcal{Z}(Z)_{\mathbf{s}})$.

Hence, we obtain that $C_b(\Omega, \mathcal{Z}(Z)_{\mathbf{s}}) \cong \Gamma_b(\Omega, E_{\mathbf{z}})$ is an isomorphism of AM m-lattice modules over $C_b(\Omega)$ (see Chapter 2, Remark 2.3.2.2(iii) and also Proposition 3.5.0.9). This proves (B)(i).

To prove (B)(ii), we note that $p_{\mathbf{z}} : E_{\mathbf{z}} \longrightarrow \Omega$ is, in addition, a *topological* bundle of commutative unital Banach algebras with (fixed) fiber $p_{\mathbf{z}}^{-1}(x) = \mathcal{Z}(Z)$ for all $x \in \Omega$, which we can call trivial (see [20, Proposition 1.1, p.136]). As such, its space of bounded continuous sections

$\Gamma_b(\Omega, E_{\mathbf{z}})$ is, in addition, a commutative unital Banach algebra. Indeed, we have that the mapping (*bundle product*)

$$E_{\mathbf{z}} \vee_\Omega E_{\mathbf{z}} \longrightarrow E_{\mathbf{z}};\ (q_1, q_2)_{\mathcal{Z}(Z)} \mapsto q_1 \circ q_2$$

is continuous, where $E_{\mathbf{z}} \vee_\Omega E_{\mathbf{z}} := \{(q_1, q_2) \in \mathcal{Z}(Z) \times \mathcal{Z}(Z)\} \subseteq E_{\mathbf{z}} \times E_{\mathbf{z}}$ is equipped with the subspace topology.

Moreover, $\phi_1 \circ \phi_2 = \phi_2 \circ \phi_1 \in C_b(\Omega, \mathcal{Z}(Z)$ for any $\phi_1, \phi_2 \in C_b(\Omega, \mathcal{Z}(Z)$ if and only if $\tilde{\phi}_2 \circ \tilde{\phi}_2 = \tilde{\phi}_2 \circ \tilde{\phi}_1 \in \Gamma_b(\Omega, E_{\mathbf{z}})$ implies that $\Gamma_b(\Omega, E_{\mathbf{z}})$ is indeed a commutative unital Banach algebra, and the mapping $\phi \mapsto \tilde{\phi}$ is, in particular, a bijective morphism of Banach lattice algebras preserving the units.

□

Proposition 3.6.6.1 has provided necessary indications for the characterisation of the centre of a (general) AM m-lattice module, by obtaining properties of the centre for the case where the AM m-lattice module is obtained by starting with the trivial Banach lattice bundle. Using these results and some of their arguments, we extend to the general AM m-lattice modules in the next proposition.

Proposition 3.6.6.2. *Let E be a topological Banach lattice bundle over a locally compact space Ω, $\Gamma_0(\Omega, E)$ the associated AM m-lattice module over $C_0(\Omega)$, and $\mathcal{Z}(\Gamma_0(\Omega, E))$ the centre of the Banach lattice $\Gamma_0(\Omega, E)$. Furthermore, for each $x \in \Omega$, let $\mathcal{Z}(E_x)_{\mathbf{s}} \subseteq \mathscr{L}(E_x)$ be the centre of the Banach lattice E_x equipped with the strong operator topology. Then we have the following.*

(A) *For each $x \in \Omega$, the mapping $q_x : \mathcal{Z}(\Gamma_0(\Omega, E)) \longrightarrow \mathcal{Z}(E_x); \mathcal{T} \mapsto q_x(\mathcal{T})$ defined by*

$$q_x(\mathcal{T})s(x) := \mathcal{T}s(x) \quad \text{for all } s \in \Gamma_0(\Omega, E)$$

is a morphism of Banach lattice algebras (see Chapter 2, Definition 2.3.1.1). In particular,

(i) $q_x(\mathcal{I}) = I_x$, where $\mathcal{I} \in \mathcal{Z}(\Gamma_0(\Omega, E))$ and $I_x \in \mathcal{Z}(E_x)$ denote the respective identity operators; and

(ii) $\sup_{x\in\Omega} ||q_x(\mathcal{T})||_{\mathscr{L}(E_x)} = ||\mathcal{T}||$ for every $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$.

(B) Keeping the notation introduced in (A) above, let

$$E_z := \dot{\bigcup}_{x\in\Omega} \mathcal{Z}(E_x)_s,$$

$$p_z : E_z \longrightarrow \Omega,\ v \in \mathcal{Z}(E_x)_s \mapsto x,$$

and endow E_z with the topology generated by the sets

$$S(\mathcal{T}, U, \varepsilon) := \left\{ v \in p_z^{-1}(U) \mid ||v - q_{p_z(v)}(\mathcal{T})||_{\mathscr{L}(E_{p_z(v)})} < \varepsilon \right\}$$

where $U \subseteq \Omega$ is open, $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$, and $\varepsilon > 0$.

Then $p_z : E_z \longrightarrow \Omega$ is the (unique up to isometric isomorphism) topological bundle of commutative 1-Banach lattice algebras over Ω, such that

the mapping $\Gamma_b(\Omega, E_z) \longrightarrow \mathcal{Z}(\Gamma_0(\Omega, E)); \phi \mapsto T_\phi$ defined by

$$T_\phi s := \phi \circ s$$

i.e., $T_\phi s(x) = \phi(x)s(x)$ for all $x \in \Omega$, $\phi \in \Gamma_b(\Omega, E_z)$ and $s \in \Gamma_0(\Omega, E)$

is:

(i) an isomorphism of commutative 1-Banach lattice algebras; and

(ii) an isomorphism of AM m-lattice modules over $C_b(\Omega)$

from $\Gamma_b(\Omega, E_z)$, the Banach lattice of bounded continuous sections associated with E_z onto the centre $\mathcal{Z}(\Gamma_0(\Omega, E))$.

Proof. (A) First, we note that, if $\mathcal{T} : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, E)$ is a bounded operator, then $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$ if and only if $\pm\mathcal{T} \leq ||\mathcal{T}||\mathcal{I}$ which is the case if and only if $|\mathcal{T}s| \leq ||\mathcal{T}||\,|s|$ for all $s \in \Gamma_0(\Omega, E)$. Now let $x \in \Omega$ be fixed, and identify E_x with a quotient lattice of $\Gamma_0(\Omega, E)$. Then, for each $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$, $|\mathcal{T}s(x)| \leq ||\mathcal{T}||\,|s(x)|$ for all $s \in \Gamma_0(\Omega, E)$, which implies that the mapping

$$q_x(\mathcal{T}) : E_x \longrightarrow E_x;\ s(x) \mapsto \mathcal{T}s(x) \text{ for all } s \in \Gamma_0(\Omega, E)$$

is well-defined and linear. Moreover, $|q_x(\mathcal{T})s(x)| = |\mathcal{T}s(x)| \leq ||\mathcal{T}||\,|s(x)|$ for all $s \in \Gamma_0(\Omega, E)$ which implies that $q_x(\mathcal{T}) \in \mathcal{Z}(E_x)$ and

$$||q_x(\mathcal{T})||_{\mathscr{L}(E_x)} = \sup\left\{||\mathcal{T}s(x)||_{E_x} :\ s \in \Gamma_0(\Omega, E),\ ||s(x)||_{E_x} = 1\right\} \leq ||\mathcal{T}||.$$

Now we show that the linear mapping $\mathcal{T} \mapsto q_x(\mathcal{T})$ is a morphism of Banach lattice algebra (see Chapter 2, Definition 2.3.1.1), i.e., $q_x(\mathcal{T}_1\mathcal{T}_2) = q_x(\mathcal{T}_1)q_x(\mathcal{T}_2)$ and $|q_x(\mathcal{T})| = q_x(|\mathcal{T}|)$ for every $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2 \in \mathcal{Z}(\Gamma_0(\Omega, E))$. Indeed,

$$q_x(\mathcal{T}_1\mathcal{T}_2)s(x) = (\mathcal{T}_1\mathcal{T}_2)s(x) = \mathcal{T}_1(\mathcal{T}_2 s(x)) = q_x(\mathcal{T}_1)q_x(\mathcal{T}_2)s(x)$$

for every $s \in \Gamma_0(\Omega, E)$, which implies that $q_x(\mathcal{T}_1\mathcal{T}_2) = q_x(\mathcal{T}_1)q_x(\mathcal{T}_2) \in \mathcal{Z}(E_x)$ for every $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{Z}(\Gamma_0(\Omega, E))$. Similarly, for every $s \in \Gamma_0(\Omega, E)_+$,

$$\begin{aligned} |q_x(\mathcal{T})|s(x) &= \sup\{q_x(\mathcal{T})|r(x)| : r \in \Gamma_0(\Omega, E), |r(x)| \leq s(x)\} \\ &= \sup\{\mathcal{T}|r(x)| : r \in \Gamma_0(\Omega, E), |r(x)| \leq s(x)\} \\ &= \sup\{\mathcal{T}|r| : r \in \Gamma_0(\Omega, E), |r| \leq s\}(x) \\ &= q_x(|\mathcal{T}|)s(x), \end{aligned}$$

which implies that $|q_x(\mathcal{T})| = q_x(|\mathcal{T}|)$ for every $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$.

(i) It is clear that, if $\mathcal{I} \in \mathcal{Z}(\Gamma_0(\Omega, E))$ is the identity operator on $\Gamma_0(\Omega, E)$, then the central operator $q_x(\mathcal{I}) : E_x \longrightarrow E_x;\ s(x) \mapsto s(x)$, for all $s \in \Gamma_0(\Omega, E)$, coincides with the identity operator $I_x : E_x \longrightarrow E_x; w \mapsto w$ on E_x, for each $x \in \Omega$.

(ii) Since, from the preceding argument, if $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$, then $||q_x(\mathcal{T})||_{\mathscr{L}(E_x)} \leq ||\mathcal{T}||$ for each $x \in \Omega$, it follows that

$$\sup_{x\in\Omega} ||q_x(\mathcal{T})||_{\mathscr{L}(E_x)} \leq ||\mathcal{T}||$$

for every $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$.

On the other hand, for every $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$,

$$\begin{aligned} ||\mathcal{T}|| &= \sup\{||\mathcal{T}s|| : s \in \Gamma_0(\Omega, E), ||s|| \leq 1\} \\ &= \sup\left\{\sup_{x\in\Omega} ||\mathcal{T}s(x)||_{E_x} : s \in \Gamma_0(\Omega, E), ||s(x)||_{E_x} \leq 1\right\} \\ &\leq \sup_{x\in\Omega}\sup\{||\mathcal{T}s(x)||_{E_x} : s \in \Gamma_0(\Omega, E), ||s(x)||_{E_x} \leq 1\} \\ &= \sup_{x\in\Omega} ||q_x(\mathcal{T})||_{\mathscr{L}(E_x)}. \end{aligned}$$

Hence, $\sup_{x\in\Omega} ||q_x(\mathcal{T})||_{\mathscr{L}(E_x)} = ||\mathcal{T}||$ for every $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$.

(B) To prove the uniqueness of $E_{\mathbf{z}}$, it suffices to show that the commutative 1-Banach lattice algebra $\mathcal{Z}(\Gamma_0(\Omega, E))$ can be represented uniquely as the space of bounded continuous sections of some *topological* bundle $p_F : F \longrightarrow \Omega$ of commutative 1-Banach lattice algebras which is isometrically embedded in $E_{\mathbf{z}}$. We first show this in (a)-(e) below. Then we will show that (i) and (ii) hold.

(a) For $x \in \Omega$, let $Q_x := \{\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E)) : q_x(\mathcal{T}) = 0_x \in E_x\} = \operatorname{Ker} q_x$. That is, for each $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$, $||q_x(\mathcal{T})||_{\mathscr{L}(E_x)} = 0$ if and only if $\mathcal{T} \in Q_x$. Now, since, $q_x : \mathcal{Z}(\Gamma_0(\Omega, E)) \longrightarrow \mathcal{Z}(E_x); \mathcal{T} \mapsto q_x(\mathcal{T})$ is a morphism of 1-Banach lattice algebras, it follows that $\{Q_x : x \in \Omega\}$ is a family of closed lattice algebra ideals of $\mathcal{Z}(\Gamma_0(\Omega, E)$ (see Chapter 2, Definition 2.5.1.2).

(b) From (a) above, it follows that the quotient space $\mathcal{Z}(\Gamma_0(\Omega, E))/Q_x$, equipped with the quotient norm is a commutative 1-Banach lattice algebra for each $x \in \Omega$. Moreover, for $x \in \Omega$, $||\mathcal{T} + Q_x|| = 0$ if and only if $\mathcal{T} \in Q_x$ for every $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$ implies that

$$||\mathcal{T} + Q_x|| = ||q_x(\mathcal{T})||_{\mathscr{L}(E_x)} \text{ for every } \mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E)).$$

From this we obtain that $\mathcal{Z}(\Gamma_0(\Omega, E))/Q_x \hookrightarrow \mathcal{Z}(E_x)$ is an isometric morphism of commutative 1-Banach lattice algebras, for each $x \in \Omega$.

(c) We claim that the mapping $\Omega \longrightarrow \mathbb{R}_{\geq 0}; x \mapsto ||\mathcal{T} + Q_x|| = ||q_x(\mathcal{T})||_{\mathscr{L}(E_x)}$ is upper semicontinuous for every $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$. Indeed, since the mapping $\Omega \longrightarrow \mathbb{R}_{\geq 0};\ x \mapsto ||s(x)||_{E_x}$ is upper semicontinuous for every $s \in \Gamma_0(\Omega, E)$, and $\mathcal{Z}(\Gamma_0(\Omega, E)$ is equipped with the strong operator topology, it follows that the mapping $\Omega \longrightarrow \mathbb{R}_{\geq 0}; x \mapsto ||\mathcal{T}s(x)||_{E_x}$, and hence the mapping $\Omega \longrightarrow \mathbb{R}_{\geq 0};\ x \mapsto ||q_x(\mathcal{T})||_{\mathscr{L}(E_x)} = \sup\{||\mathcal{T}s(x)||_{E_x} :\ s \in \Gamma_0(\Omega, E), ||s(x)||_{E_x} \leq 1\}$ is upper semicontinuous for every $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$.

(d) Consider the bundle $F := \dot{\bigcup}_{x\in\Omega} F_x$, with $F_x := \mathcal{Z}(\Gamma_0(\Omega, E))/Q_x$ and $x \in \Omega$. Note that each $\tilde{\mathcal{T}} \in \Gamma_b(\Omega, F)$, its bounded continuous sections, is given by a unique $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$ such that $\tilde{\mathcal{T}}(x) = \mathcal{T} + Q_x$ for every $x \in \Omega$, so that

$$\Gamma_b(\Omega, F) := \{\, \tilde{\mathcal{T}} : \Omega \longrightarrow F \text{ continuous },\ p_F \circ \tilde{\mathcal{T}} = Id_\Omega \\ \text{and } ||\tilde{\mathcal{T}}|| := \sup_{x\in\Omega} ||\mathcal{T} + Q_x|| < \infty \,\}.$$

By combining [19, Proposition 1, p.817] and [20, Proposition 1.2, p.137], we obtain that the bundle F is the unique *topological* bundle $p_F : F \longrightarrow \Omega$ of commutative 1-Banach lattice algebras such that the space of its bounded continuous sections, $\Gamma_b(\Omega, F)$, is isometrically isomorphic to the centre $\mathcal{Z}(\Gamma_0(\Omega, E))$ as commutative 1-Banach lattice algebras.

(e) Since $F_x = \mathcal{Z}(\Gamma_0(\Omega, E))/Q_x \hookrightarrow \mathcal{Z}(E_x)$ for each $x \in \Omega$, we may thus conclude that $F \hookrightarrow E_z$, and consequently $\mathcal{Z}(\Gamma_0(\Omega, E)) \cong \Gamma_b(\Omega, F) \hookrightarrow \Gamma_b(\Omega, E_{\mathbf{z}})$.

Now, we can show that the mapping $\Gamma_b(\Omega, E_{\mathbf{z}}) \longrightarrow \mathcal{Z}(\Gamma_0(\Omega, E)); \phi \mapsto T_\phi$ defined by

$$T_\phi s := \phi \circ s,$$

$$\text{i.e., } T_\phi s(x) = \phi(x)s(x) \text{ for all } x \in \Omega,\ \phi \in \Gamma_b(\Omega, E_{\mathbf{z}}) \text{ and } s \in \Gamma_0(\Omega, E),$$

satisfies assertions (i) and (ii). The fact that $E_{\mathbf{z}}$ is unique with these properties then follows immediately.

(i) Since for $\phi \in \Gamma_b(\Omega, E_{\mathbf{z}})$ and every $s \in \Gamma_0(\Omega, E)$,

$$\begin{aligned} |T_\phi s(x)| &= |\phi(x)s(x)| \\ &\leq ||\phi(x)||_{\mathscr{L}(E_x)} |s(x)| \\ &\leq ||\phi|| \, |s(x)|, \end{aligned}$$

for all $x \in \Omega$, which implies that $|T_\phi s| \leq ||\phi|| \, |s|$ for every $s \in \Gamma_0(\Omega, E)$, it follows that $T_\phi \in \mathcal{Z}(\Gamma_0(\Omega, E))$. Moreover, it is linear and an isometry, since clearly $||T_\phi|| \leq ||\phi||$, and

$$\begin{aligned} ||\phi|| &= \sup_{x \in \Omega} ||\phi(x)||_{\mathscr{L}(E_x)} \\ &= \sup_{x \in \Omega} \sup \{||\phi(x)s(x)||_{E_x} : s \in \Gamma_0(\Omega, E), ||s(x)||_{E_x} \leq 1\} \\ &\leq \sup \{||T_\phi s|| : s \in \Gamma_0(\Omega, E), ||s|| \leq 1\} \\ &= ||T_\phi|| \end{aligned}$$

for every $\phi \in \Gamma_b(\Omega, E_{\mathbf{z}})$. Furthermore, it easy to see that $T_{\phi_1 \circ \phi_2} = T_{\phi_1} \circ T_{\phi_2}$ and $T_{|\phi|} = |T_\phi|$ for every $\phi, \phi_1, \phi_2 \in \Gamma_b(\Omega, E_{\mathbf{z}})$. In particular, the identity bounded continuous section $e : \Omega \longrightarrow E_{\mathbf{z}}; x \mapsto I_x$ corresponds to the identity operator $\mathcal{I} : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, E); s \mapsto s$, i.e., $T_e = \mathcal{I}$.

Thus, it suffices to show that the mapping $\phi \mapsto T_\phi$ is surjective. Indeed, if $\mathcal{T} \in \mathcal{Z}(\Gamma_0(\Omega, E))$, then as in part (A) above, we can define a section $\phi : \Omega \longrightarrow E_{\mathbf{z}}$ by setting

$$\phi(x) := q_x(\mathcal{T}) \text{ for every } x \in \Omega,$$

which can immediately be seen to be continuous and bounded. This implies that $\phi(x)s(x) = q_x(\mathcal{T})s(x) = \mathcal{T}s(x)$ for all $x \in \Omega$ and every $s \in \Gamma_0(\Omega, E)$, so that $\mathcal{T} = T_\phi$ as claimed.

(ii) It is clear that the (bilinear) pairing

$$C_b(\Omega) \times \Gamma_b(\Omega, E_{\mathbf{z}}) \longrightarrow \Gamma_b(\Omega, E_{\mathbf{z}});\ (f, \phi) \mapsto f \cdot \phi := [x \mapsto f(x)\phi(x)]$$

turns $\Gamma_b(\Omega, E_{\mathbf{z}})$ into an m-Banach lattice module, and, in particular, an AM m-lattice module over $C_b(\Omega)$. This implies that $|f| \cdot \mathsf{I}\phi\mathsf{I} = \mathsf{I} f \cdot \phi \mathsf{I}$ and $(f \cdot \phi) \circ s = \phi \circ fs$ for all $f \in C_b(\Omega)$, $\phi \in \Gamma_b(\Omega, E_{\mathbf{z}})$ and every $s \in \Gamma_0(\Omega, E)$.

As a result, the (bilinear) pairing

$$C_b(\Omega) \times \mathcal{Z}(\Gamma_0(\Omega, E)) \longrightarrow \mathcal{Z}(\Gamma_0(\Omega, E));\ (f, T_\phi) \mapsto f \cdot T_\phi := [s \mapsto T_\phi(fs)]$$

also turns $\mathcal{Z}(\Gamma_0(\Omega, E)$ into an m-Banach lattice module over $C_b(\Omega)$. Indeed, for any $f \in C_b(\Omega)$ and $\phi \in \Gamma_b(\Omega, E_{\mathbf{z}})$,

$$\begin{aligned} \mathsf{I}(f \cdot T_\phi)s(x)\mathsf{I} &= \mathsf{I}(\phi \circ fs)(x)\mathsf{I} \\ &= \mathsf{I}(f \cdot \phi)(x)s(x)\mathsf{I} \\ &\leq ||f \cdot \phi||\,\mathsf{I}s(x)\mathsf{I} \end{aligned}$$

for all $x \in \Omega$ and every $s \in \Gamma_0(\Omega, E)$, which implies that $f \cdot T_\phi \in \mathcal{Z}(\Gamma_0(\Omega, E))$ and $||f \cdot T_\phi|| \leq ||f \cdot \phi|| \leq ||f||||\phi|| = ||f||||T_\phi||$. Moreover, for any $f \in C_b(\Omega)$ and $\phi \in \Gamma_b(\Omega, E_{\mathbf{z}})$,

$$T_{f \cdot \phi}s = (f \cdot \phi) \circ s = \phi \circ fs = (f \cdot T_\phi)s$$

for every $s \in \Gamma_0(\Omega, E)$, which implies that $f \cdot T_\phi = T_{f \cdot \phi}$. It folllows that $||f \cdot T_\phi|| = ||T_{f \cdot \phi}|| = ||f \cdot \phi||$ for any $f \in C_b(\Omega)$ and $\phi \in \Gamma_b(\Omega, E_{\mathbf{z}})$, and

$$\mathsf{I} f \cdot T_\phi \mathsf{I} = \mathsf{I} T_{f \cdot \phi} \mathsf{I} = T_{\mathsf{I} f \cdot \phi \mathsf{I}} = T_{|f| \cdot \mathsf{I}\phi\mathsf{I}} = |f| \cdot T_{\mathsf{I}\phi\mathsf{I}} = |f| \cdot \mathsf{I} T_\phi \mathsf{I},$$

which implies that $\mathcal{Z}(\Gamma_0(\Omega, E))$ is indeed an m-Banach lattice module over $C_b(\Omega)$.

Finally, the consideration above implies that the isometric lattice isomorphism $\phi \mapsto T_\phi$ is, in addition, a $C_b(\Omega)$-module homomorphism. Thus, we obtain that

$$\Gamma_b(\Omega, E_{\mathbf{z}}) \cong \mathcal{Z}(\Gamma_0(\Omega, E)),$$

given by an isomorphism of AM m-lattice modules over $C_b(\Omega)$, as claimed (see Chapter 2, Remark 2.3.2.2(iii) and also Proposition 3.5.0.9).

$\square$

Chapter 4

One-parameter C_0-semigroups of positive weighted Koopman operators

4.1 Introduction

In this chapter, we consider the order structure of an AM m-lattice module (see Section 4.2), lattice C_0-semigroups of weighted Koopman operators (see Section 4.3) and positive C_0-semigroups of weighted Koopman operators (see Section 4.4). Moreover, in Section 4.5, we consider certain spectral properties of a positive C_0-semigroup of weighted Koopman operators.

In Section 4.2, we first state our general characterisation of an AM m-lattice module (see Proposition 4.2.0.1). We further identify and characterise implications of some of its order structures: namely, non-emptiness of the positive cone, (σ-) order completeness, and order continuity of the norm (see Proposition 4.2.0.2).

We consider, in Section 4.3, the lattice C_0-semigroup induced by a semiflow $(\Phi_t)_{t\geq 0}$ on a *topological* Banach lattice bundle E over a flow $(\varphi_t)_{t\in\mathbb{R}}$ on a compact space K. We give certain properties of this lattice C_0-semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ of weighted Koopman operators on $\Gamma(K,E)$, with generator $(\mathcal{A}, D(\mathcal{A}))$ a *local δ-derivation*, over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$ with its generator $(\delta, D(\delta))$ (see Proposition 4.3.0.1 and Remark 4.3.0.2). Furthermore, under the assumption that the Banach lattice of continuous sections $\Gamma(K,E)$

has order continuous norm, we obtain a complete characterisation of a lattice C_0-semigroup of weighted Koopman operators through its generator $(\mathcal{A}, D(\mathcal{A}))$ as what we call a *Kato δ-derivation* (see Proposition 4.3.0.3, Corollary 4.3.0.4 and Remark 4.3.0.5).

In Section 4.4, we turn our attention to the notion of a positive C_0-semigroup induced by a positive semiflow $(\Phi_t)_{t\geq 0}$ on E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on a compact space K. In Proposition 4.4.0.1, we give certain properties of this positive C_0-semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ of weighted Koopman operators on $\Gamma(K, E)$. Moreover, under the assumption that $\Gamma(K, E)$ is a real σ-order complete Banach lattice, we obtain a characterisation of a positive C_0-semigroup of weighted Koopman operators through its generator in certain sense (see Proposition 4.4.0.3).

Finally, in Section 4.5, we consider certain spectral properties of a positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ on $\Gamma(K, E)$ with generator $(\mathcal{A}, D(\mathcal{A}))$. Based on certain assumptions, we obtain that the boundary spectrum $\sigma^b(\mathcal{A})$ of the generator $(\mathcal{A}, D(\mathcal{A}))$ is an imaginary additively cyclic set (see Proposition 4.5.0.4).

4.2 The order structure of an AM m-lattice module over $C_0(\Omega)$

In this Section, we consider some order structures of an AM m-lattice module over $C_0(\Omega)$ where Ω is a locally compact space. To this end, we first restate our general characterisation of an AM m-lattice module over $C_0(\Omega)$ (see Chapter 2, Corollary 2.4.1.10 and Chapter 3, Proposition 3.5.0.9).

Proposition 4.2.0.1. *Let Γ be a Banach lattice module over the Banach lattice algebra $C_0(\Omega)$. Then the following are equivalent.*

(i) Γ *is an AM m-lattice module.*

(ii) Γ *is a $U_0(\Omega)$-normed m-lattice module.*

(iii) Γ *is isometrically isomorphic to the Banach lattice $\Gamma_0(\Omega, E)$ of continuous sections vanishing at infinity of a (unique up to isometric isomorphism) topological Banach lattice bundle E over Ω.*

Moreover, if these assertions hold, then the lattice $U_0(\Omega)$-valued norm is unique and given by

$$|s|(x) := \inf\{||fs|| : f \in C_0(\Omega)_+ \text{ with } f(x) = 1\} \quad (x \in \Omega, s \in \Gamma).$$

We identify, in the next proposition, implications of some order structures of an AM m-lattice module over $C_0(\Omega)$: namely, non-emptiness of the positive cone, (σ-) order completeness, and order continuity of the norm (see also Chapter 3, Subsection 3.6.5). Throughout, Ω is a locally compact space, which we assume to be Hausdorff by definition.

Proposition 4.2.0.2. *Let Γ be an AM m-lattice module over the Banach lattice algebra $C_0(\Omega)$. Furthermore, let E be the unique (up to isometric isomorphism) Banach lattice bundle over Ω such that $\Gamma \cong \Gamma_0(\Omega, E)$ as an isomorphism of m-Banach lattice modules over $C_0(\Omega)$. Then we have the following.*

(A) Consider the following statements.

(i) The interior $Int\Gamma_+$ is non-empty.

(ii) $\Omega = K$ is compact and the interior $IntE_x^+$ is non-empty for each $x \in K$.

Then $(i) \implies (ii)$.

(B) Consider the following statements.

(i) Γ is σ-order complete.

(ii) Ω is discrete and E_x is σ-order complete for each $x \in \Omega$.

(iii) $C_0(\Omega)$ is σ-order complete and E_x has compact order intervals for each $x \in \Omega$.

(iv) $\Omega = K$ is infinite, compact, and Γ is σ-order complete.

(v) E_x has compact order intervals for each $x \in K$.

Then $(i) \iff (ii) \iff (iii)$. Moreover, if K is totally disconnected (i.e., quasi-Stonian), then $(iv) \iff (v)$.

(C) Consider the following statements.

(i) Γ is order complete.

(ii) Ω is discrete and E_x is order complete for each $x \in \Omega$.

(iii) $C_0(\Omega)$ *is order complete and* E_x *has compact order intervals for each* $x \in \Omega$.

(iv) $\Omega = K$ *is infinite, compact, and* Γ *is order complete.*

(v) E_x *has compact order intervals for each* $x \in K$.

Then $(i) \iff (ii) \iff (iii)$. *Moreover, if* K *is extremally disconnected (i.e., Stonian), then* $(iv) \iff (v)$.

(D) *The following are equivalent.*

(i) Γ *has order continuous norm.*

(ii) Ω *is discrete and* E_x *has order continuous norm for each* $x \in \Omega$.

Proof. First, we note that for each $x \in \Omega$, the Banach lattice E_x can be identified uniquely with a quotient lattice of $\Gamma_0(\Omega, E)$ via the evaluation map $e_x : \Gamma_0(\Omega, E) \longrightarrow E_x;\ s \mapsto s(x)$, which is a surjective and order continuous lattice homomorphism (see also Chapter 3, Remark 3.5.0.1(iii)). From this observation, the assertions readily follow from Chapter 3 (Proposition 3.6.5.3).

(A) Suppose the interior $Int\Gamma_+ \neq \emptyset$ (see Appendix B(Proposition B.0.0.3)). Since this is the case if and only if the interior $Int\Gamma_0(\Omega, E)_+ \neq \emptyset$, then the assertion follows from Chapter 3[Proposition 3.6.5.3(A)].

In this situation, if $u \in Int\Gamma(K, E)_+$, i.e., u is an order unit, then $e_x(u) = u(x) \in IntE_x^+$ is an order unit for each $x \in K$.

(B) Similarly, since Γ is σ-order complete (see [28, Definition 1.8, p.54]) if and only if $\Gamma_0(\Omega, E)$ is σ-order complete, by Chapter 3 [Proposition 3.6.5.3(B)] the assertion follows.

(C) Moreover, if Γ is order complete (see [28, Definition 1.8, p.54]) which is the case if and only if $\Gamma_0(\Omega, E)$ is order complete, then the assertion follows from Chapter 3 [Proposition 3.6.5.3(C)].

(D) Finally, since Γ has order continuous norm (see [28, Definition 5.12, p.92]) if and only if $\Gamma_0(\Omega, E)$ has order continuous norm, by Chapter 3 [Proposition 3.6.5.3(D)] the assertion follows (see also [28, Section 11.1 Corollary, p.209]).

□

Remark 4.2.0.3. *(i) We note that a locally compact space Ω is discrete if and only if the Banach lattice $C_0(\Omega)$ has order continuous norm. So, by Proposition 4.2.0.2(B) above, whenever AM m-lattice module $\Gamma_0(\Omega, E)$ is σ-order complete, the Banach lattice algebra $C_0(\Omega)$ will always have order continuous norm unless $\Omega = K$ is infinite, compact and quasi-Stonian.*

(ii) If E is a topological Banach lattice bundle over a discrete space Ω, then every mapping $\Omega \longrightarrow E$ is continuous. So, every section $s : \Omega \longrightarrow E$ is continuous, and $s \in \Gamma_0(\Omega, E)$ if and only if for every $\varepsilon > 0$, the set

$$\{x \in \Omega : \varepsilon \leq ||s(x)||_{E_x}\}$$

is compact, and hence finite, since Ω is discrete. In addition, for each $s \in \Gamma_0(\Omega, E)$, the mapping

$$\Omega \longrightarrow \mathbb{R}_{\geq 0} : x \xmapsto{|s|} ||s(x)||_{E_x}$$

is always continuous as Ω is discrete. This is so, since $U_0(\Omega) \subseteq C_0(\Omega)$.

Hence, E is a continuous Banach lattice bundle over a locally compact space Ω, see Chapter 3(Definition 3.2.0.1), whenever AM m-lattice module $\Gamma_0(\Omega, E)$ is (σ-) order complete unless maybe $\Omega = K$ is infinite, compact and Stonian (see also [29, Proposition 3.2(iii), p.59]).

(iii) Moreover, we note that $\Omega = K$ is a compact discrete space if and only if K is finite.

4.3 Lattice C_0-semigroups of weighted Koopman operators

In what follows, we set $G = \mathbb{R}$, $S = \mathbb{R}_{\geq 0}$ and $\Omega = K$ where K is a compact space. Furthermore, we let $(\Phi_t)_{t\geq 0}$ be a semiflow on Banach lattice bundle E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K (see Chapter 3, Remark 3.4.0.6(iii)).

First, we note that the Koopman group representation $(C(K), \mathbf{T}_\varphi)$ induces a Markovian lattice C_0-group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$ (see [2, Part B-II Definition 3.3, p.144]), and its generator $(\delta, D(\delta))$ which is a derivation on $C(K)$ (see [29, Theorem 3.1, p.58]). For a generalisation of this Markovian group, we refer to the work of W. M. Priestley ([27]).

Moreover, the induced weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ on $\Gamma(K, E)$ over the Koopman group $(C(K), \mathbf{T}_\varphi)$ is a lattice C_0-semigroup. We collect this information in the following proposition, which immediately follows from combining [29, Proposition 3.6, p.61] with Remark 3.5.0.1(v) in Chapter 3.

Proposition 4.3.0.1. *Let $(\Phi_t)_{t\geq 0}$ be a a semiflow on Banach lattice bundle E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K, and $(\mathcal{T}_\Phi(t))_{t\geq 0}$ the induced weighted Koopman semigroup on $\Gamma(K, E)$. Moreover, let $(\delta, D(\delta))$ be the generator of the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$. Then the following hold.*

(i) The family $(\mathcal{T}_\Phi(t))_{t\geq 0}$ is a lattice C_0-semigroup on $\Gamma(K, E)$.

(ii) Each $\mathcal{T}_\Phi(t)$ is a lattice $T_\varphi(t)$-homomorphism for each $t \geq 0$.

(iii) The generator $(\mathcal{A}, D(\mathcal{A}))$ of $(\mathcal{T}_\Phi(t))_{t\geq 0}$ is a δ-derivation on $\Gamma(K, E)$, i.e., $D(\mathcal{A})$ is a $D(\delta)$-submodule of $\Gamma(K, E)$ and

$$\mathcal{A}(fs) = \delta f \cdot s + f \cdot \mathcal{A}s$$

for all $f \in D(\delta)$ and $s \in D(\mathcal{A})$.

(iv) $(\mathcal{A}, D(\mathcal{A}))$ is a local operator i.e., $s \perp r$ implies $\mathcal{A}s \perp r$ for all $s \in D(A)$ and $r \in \Gamma(K, E)$.

Proof. Since $(\Phi_t)_{t\geq 0}$ is a semiflow on E over the flow $(\varphi_t)_{t\in\mathbb{R}}$, we have that, for each $t \geq 0$, the induced weighted Koopman operator $\mathcal{T}_\Phi(t)$ is, in particular, a lattice homomorphism, i.e., $|\mathcal{T}_\Phi(t)s| = \mathcal{T}_\Phi(t)|s|$ for all $s \in \Gamma(K, E)$ and $t \geq 0$ (see also Chapter 3, Remark 3.5.0.1(v)).

(i) By [29, Proposition 3.6, p.61](i), we obtain that $(\mathcal{T}_\Phi(t))_{t\geq 0}$ is a C_0-semigroup on $\Gamma(K, E)$. That is,

$$\lim_{t\downarrow 0} \mathcal{T}_\Phi(t)s - s = \lim_{t\downarrow 0} \Phi_t \circ s \circ \varphi_{-t} - s = 0$$

for all $s \in \Gamma(K, E)$. Hence, it is a C_0-semigroup on $\Gamma(K, E)$ comprised of lattice homomorphisms, i.e., a lattice C_0-semigroup.

(ii) Since, for each $t \geq 0$, $\mathcal{T}_\Phi(t)fs = T_\varphi(t)f \cdot \mathcal{T}_\Phi(t)s$ for all $f \in C(K)$ and $s \in \Gamma(K, E)$, the assertion follows.

(iii) This immediately follows from [29, Proposition 3.6, p.61](iii), since the operator $(\mathcal{A}, D(\mathcal{A}))$ is, in particular, a generator of a weighted Koopman semigroup on the Banach module $\Gamma(K, E)$ in this situation. That is,

$$\begin{aligned}\lim_{t\downarrow 0} \frac{\mathcal{T}_\Phi(t)(fs) - fs}{t} &= \lim_{t\downarrow 0} \frac{T_\varphi(t)f - f}{t} \cdot \mathcal{T}_\Phi(t)s \; + \; \lim_{t\downarrow 0} f \cdot \frac{\mathcal{T}_\Phi(t)s - s}{t} \\ &= \delta f \cdot s + f \cdot \mathcal{A}s\end{aligned}$$

for all $f \in D(\delta)$ and $s \in D(\mathcal{A})$.

(iv) Since, for all $t \geq 0$, the lattice homomorphisms $\mathcal{T}_\Phi(t)$ are also disjointness-preserving, i.e., $s_1 \perp s_2$ implies $\mathcal{T}_\Phi(t)s_1 \perp \mathcal{T}_\Phi(t)s_2$ for all $s_1, s_2 \in \Gamma(K, E)$ and $t \geq 0$, then [2, Proposition 5.4, p.282] implies that $(\mathcal{A}, D(\mathcal{A}))$ is a local operator, i.e., $s \perp r$ implies $\mathcal{A}s \perp r$ for all $s \in D(A)$ and $r \in \Gamma(K, E)$.

□

Remark 4.3.0.2. *(i) We will call a generator $(\mathcal{A}, D(\mathcal{A}))$ of an arbitrary C_0-semigroup on the AM m-lattice module $\Gamma(K, E)$ a "local δ-derivation" if it satisfies conditions (iii) and (iv) of Proposition 4.3.0.1.*

(ii) In general, as evident in the previous Proposition 4.3.0.1, nothing more can be said concerning the generator $(\mathcal{A}, D(\mathcal{A}))$ of a lattice C_0-semigroup of weighted Koopman operators on $\Gamma(K, E)$ other than being a local δ-derivation. However, under the assumption that the Banach lattice of continuous sections $\Gamma(K, E)$ has order continuous norm, we can give a complete characterisation.

Proposition 4.3.0.3. *Let $\mathcal{T}(t)_{t\geq 0}$ be a C_0-semigroup on $\Gamma(K, E)$ with generator $(\mathcal{A}, D(\mathcal{A}))$. Now, consider the following statements.*

(i) $\mathcal{T}(t)$ is a lattice $T_\varphi(t)$-homomorphism for every $t \geq 0$.

(ii) There exists a unique semiflow $(\Phi_t)_{t\geq 0}$ on the Banach lattice bundle E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K such that $\mathcal{T}(t) = \mathcal{T}_\Phi(t)$ for every $t \geq 0$.

(iii) $D(\mathcal{A})$ is a $D(\delta)$-lattice submodule of $\Gamma(K, E)$ such that

(a) $\mathcal{A}(fs) = \delta f \cdot s + f \cdot \mathcal{A}s$, and

(b) $Re((sign\,\overline{s})\mathcal{A}s) = \mathcal{A}\,|s|$

for all $f \in D(\delta)$ and $s \in D(\mathcal{A})$.

Then $(i) \iff (ii)$*. If the complex Banach lattice* $\Gamma(K,E)$ *has order continuous norm (see Chapter 3, Proposition 3.6.5.3(D)), then* $(i) \iff (ii) \iff (iii)$*.*

Moreover, if these assertions hold, then the semiflow $(\Phi_t)_{t\geq 0}$ *in (ii) is unique, satisfies* $||\mathcal{T}_\Phi(t)|| = ||\Phi_t||$ *for all* $t \geq 0$*, and* $(\mathcal{T}_\Phi(t))_{t\geq 0}$ *is an isometry if and only if* $(\Phi_t)_{t\geq 0}$ *is an isometry.*

Proof. The equivalence $(i) \iff (ii)$ readily follows from Chapter 3 (Proposition 3.5.0.5).

Now, assume that the complex Banach lattice $\Gamma(K,E)$ has order continuous norm.

$(i) \implies (iii)$: First, since the C_0-semigroup $\mathcal{T}(t)_{t\geq 0}$ is, in particular, a lattice C_0-semigroup on $\Gamma(K,E)$, and the Banach lattice $\Gamma(K,E)$ has order continuous norm, then [2, Corollary 5.8, p.285] implies that its generator $(\mathcal{A}, D(\mathcal{A}))$ must satisfy the so-called Kato's equality. That is,

$$s \in D(\mathcal{A}) \text{ implies } |s|, \bar{s} \in D(\mathcal{A}) \text{ and } Re((sign\, \bar{s})\mathcal{A}s) = \mathcal{A}|s|$$

where, for each $s \in \Gamma(K,E)$, the mapping $(sign\ s) : \Gamma(K,E) \longrightarrow \Gamma(K,E)$ is the (unique) associated signum linear operator such that that $(sign\, \bar{s})s = |s|$ (see also [2, Proposition 2.1, p.256-257]).

Furthermore, for $f \in D(\delta)$ and $s \in D(\mathcal{A})$,

$$\begin{aligned}\lim_{t\downarrow 0} \frac{\mathcal{T}(t)(fs) - fs}{t} &= \lim_{t\downarrow 0} \frac{T_\varphi(t)f - f}{t} \cdot \mathcal{T}(t)s \;+\; \lim_{t\downarrow 0} f \cdot \frac{\mathcal{T}(t)s - s}{t} \\ &= \delta f \cdot s + f \cdot \mathcal{A}s\end{aligned}$$

implies that the generator $(\mathcal{A}, D(\mathcal{A}))$ is, in addition, a δ-derivation on $\Gamma(K,E)$. That is, $D(\mathcal{A})$ is a $D(\delta)$-submodule of $\Gamma(K,E)$ and

$$\mathcal{A}(fs) = \delta f \cdot s + f \cdot \mathcal{A}s$$

for all $f \in D(\delta)$ and $s \in D(\mathcal{A})$.

Combining these two results we obtain that $D(\mathcal{A})$ is a $D(\delta)$-lattice submodule (see Chapter 2, Definition 2.5.2.1) satisfying the stated properties (a) and (b).

$(iii) \implies (i)$: Since the generator $(\mathcal{A}, D(\mathcal{A}))$, in particular, satisfies the Kato's equality, then [2, Corollary 5.8, p.285] implies it is a generator of a lattice C_0-semigroup on $\Gamma(K,E)$. Furthermore, since $(\mathcal{A}, D(\mathcal{A}))$ is also a δ-derivation on $\Gamma(K,E)$, it follows from [29, Theorem 3.8, p.63] that $\mathcal{T}(t)$ is a lattice $T_\varphi(t)$-homomorphism for every $t \geq 0$.

Moreover, the last claim follows from Chapter 3(Proposition 3.5.0.5).

□

In the real case, that is $\Gamma(K,E)$ and hence E_x is a real Banach lattice for each $x \in K$, we can reformulate the above characterisation in Proposition 4.3.0.3.

Corollary 4.3.0.4. *Let $\mathcal{T}(t)_{t\geq 0}$ be a C_0-semigroup on $\Gamma(K,E)$ with generator $(\mathcal{A}, D(\mathcal{A}))$. Now, consider the following statements.*

(i) $\mathcal{T}(t)$ is a lattice $T_\varphi(t)$-homomorphism for every $t \geq 0$.

(ii) There exists a unique semiflow $(\Phi_t)_{t\geq 0}$ on Banach lattice bundle E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K such that $\mathcal{T}(t) = \mathcal{T}_\Phi(t)$ for every $t \geq 0$.

(iii) $D(\mathcal{A})$ is a $D(\delta)$-lattice submodule of $\Gamma(K,E)$ such that

(a) $\mathcal{A}(fs) = \delta f \cdot s + f \cdot \mathcal{A}s$ for all $f \in D(\delta)$ and $s \in D(\mathcal{A})$, and

(b) $(\mathcal{A}, D(\mathcal{A}))$ is a local operator.

Then $(i) \iff (ii)$. If the real Banach lattice $\Gamma(K,E)$ has order continuous norm (see Chapter 3, Proposition 3.6.5.3(D)), then $(i) \iff (ii) \iff (iii)$.

Moreover, if these assertions hold, then the semiflow $(\Phi_t)_{t\geq 0}$ in (ii) is unique, satisfies $||\mathcal{T}_\Phi(t)|| = ||\Phi_t||$ for all $t \geq 0$, and $(\mathcal{T}_\Phi(t))_{t\geq 0}$ is an isometry if and only if $(\Phi_t)_{t\geq 0}$ is an isometry.

Proof. The result follows from combining [2, Corollary 5.9, p.285] with the previous Proposition 4.3.0.3. □

Remark 4.3.0.5. *(i) We will call a generator $(\mathcal{A}, D(\mathcal{A}))$ of an arbitrary C_0-semigroup on the AM m-lattice module $\Gamma(K,E)$ satisfying condition (iii) in either Proposition 4.3.0.3 (in complex case) or Corollary 4.3.0.4 (in real case) a "Kato δ-derivation" on $\Gamma(K,E)$.*

(ii) Now, our result implies that, if an AM m-lattice module Γ over $C(K)$ has order continuous norm (see Proposition 4.2.0.2(D)), then there is a one-to-one correspondence between:

(a) C_0-semigroups of weighted morphisms over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$; and

(b) Kato δ-derivations on Γ.

4.4 Positive C_0-semigroups of weighted Koopman operators

In this Section, we let $(\Phi_t)_{t\geq 0}$ be a positive semiflow on a *topological* Banach lattice bundle E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K (see Chapter 3, Definition 3.4.0.5). Now, consider the induced family $(\mathcal{T}_\Phi(t))_{t\geq 0}$ of positive weighted Koopman operators on the AM m-lattice module $\Gamma(K, E)$ which gives rise to a positive C_0-semigroup. We collect this information in the following proposition which follows from [29, Proposition 3.6, p.61] as well as from Proposition 4.3.0.1.

Proposition 4.4.0.1. *Let $(\Phi_t)_{t\geq 0}$ be a positive semiflow on Banach lattice bundle E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K, and $(\mathcal{T}_\Phi(t))_{t\geq 0}$ the induced positive weighted Koopman semigroup on $\Gamma(K, E)$. Moreover, let $(\delta, D(\delta))$ be the generator of the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$. Then the following hold.*

(i) The family $(\mathcal{T}_\Phi(t))_{t\geq 0}$ is a positive C_0-semigroup on $\Gamma(K, E)$.

(ii) Each $\mathcal{T}_\Phi(t)$ is a positive $T_\varphi(t)$-homomorphism for each $t \geq 0$.

(iii) The generator $(\mathcal{A}, D(\mathcal{A}))$ of $(\mathcal{T}_\Phi(t))_{t\geq 0}$ is a δ-derivation on $\Gamma(K, E)$, i.e., $D(\mathcal{A})$ is a $D(\delta)$-submodule of $\Gamma(K, E)$ and

$$\mathcal{A}(fs) = \delta f \cdot s + f \cdot \mathcal{A}s$$

for all $f \in D(\delta)$ and $s \in D(\mathcal{A})$.

Proof. Since $(\Phi_t)_{t\geq 0}$ is a positive semiflow on E over the flow $(\varphi_t)_{t\in\mathbb{R}}$, we have that, for each $t \geq 0$, the induced weighted Koopman operator

$\mathcal{T}_\Phi(t)$ is, in particular, a positive operator, i.e., $|\mathcal{T}_\Phi(t)s| \leq \mathcal{T}_\Phi(t)|s|$ for all $s \in \Gamma(K,E)$ and $t \geq 0$ (see also Chapter 3, Remark 3.5.0.1(v)).

From this observation, assertions (i), (ii) and (iii) follow from Proposition 4.3.0.1.

□

Remark 4.4.0.2. *(i) Unlike the case of a lattice C_0-semigroup of weighted Koopman operators (see Proposition 4.3.0.1), we could not provide direct properties of a generator $(\mathcal{A}, D(\mathcal{A}))$ of a positive C_0-semigroup of weighted Koopman operators other than being a δ-derivation on $\Gamma(K,E)$. However, we can collect the following (additional) information associated with a generator of a positive C_0-semigroup on certain Banach lattices.*

(a) If $\Gamma(K,E)$ is σ-order complete (see Chapter 3, Proposition 3.6.5.3(B)), then, by [2, Theorem 2.4, p.258], $(\mathcal{A}, D(\mathcal{A}))$ satisfies the so-called Kato's inequality. That is,

$$< Re(sign\,\bar{s})\mathcal{A}s, s' > \leq < |s|, \mathcal{A}'s' > \qquad (s \in D(\mathcal{A}),\ 0 \leq s' \in D(\mathcal{A}'))$$

where $(\mathcal{A}', D(\mathcal{A}'))$ is the adjoint operator (of generator $\mathcal{A}$) on the dual Banach lattice module $\Gamma(K,E)'$ (see Chapter 2, Subsection 2.5.4), and $< s, s' >:= s'(s)$ is a paring of $(s, s') \in \Gamma(K,E) \times \Gamma(K,E)'$.

(b) If $\Gamma(K,E)$ is real Banach lattice, then by [2, Proposition 3.5, p.261] there exists a strictly positive non-empty subset $M' \subseteq \Gamma(K,E)'$ of subeigenvectors of $(\mathcal{A}', D(\mathcal{A}'))$. See [2, Definition 3.2, and Definition 3.4, p.261] for concepts of strictly positive set and subeigenvectors.

(ii) With the setup in (i) above, we can, to some extent, give a characterisation of a positive C_0-semigroup of weighted Koopman operators, through its generator, under the assumption that the real Banach lattice $\Gamma(K,E)$ is σ-order complete.

Proposition 4.4.0.3. *Let $\mathcal{T}(t)_{t\geq 0}$ be a C_0-semigroup on $\Gamma(K,E)$ with generator $(\mathcal{A}, D(\mathcal{A}))$. Now, consider the following statements.*

(i) $\mathcal{T}(t)$ is a positive $T_\varphi(t)$-homomorphism for every $t \geq 0$.

(ii) There exists a unique positive semiflow $(\Phi_t)_{t\geq 0}$ on the Banach lattice bundle E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K such that $\mathcal{T}(t) = \mathcal{T}_\Phi(t)$ for every $t \geq 0$.

(iii) There exists a core $(\mathcal{A}, D_o)$ of the generator $(\mathcal{A}, D(\mathcal{A}))$ which is a $D(\delta)$-submodule of $\Gamma(K, E)$ and a strictly positive non-empty subset $M' \subseteq \Gamma(K, E)'$ of subeigenvectors of $(\mathcal{A}', D(\mathcal{A}'))$ such that

(a) $\mathcal{A}(fs) = \delta f \cdot s + f \cdot \mathcal{A}s$, and

(b) $< Re(sign\ s)\mathcal{A}s, s' > \leq < |s|, \mathcal{A}'s' >$

for all $f \in D(\delta)$, $s \in D_o$ and $s' \in M'$.

Then $(i) \iff (ii)$. If the real Banach lattice $\Gamma(K, E)$ is σ-order complete (see Chapter 3, Proposition 3.6.5.3(B)), then $(i) \iff (ii) \iff (iii)$.

Moreover, if these assertions hold, then the positive semiflow $(\Phi_t)_{t\geq 0}$ in (ii) is unique, satisfies $||\mathcal{T}_\Phi(t)|| = ||\Phi_t||$ for all $t \geq 0$, and $(\mathcal{T}_\Phi(t))_{t\geq 0}$ is a positive isometry if and only if $(\Phi_t)_{t\geq 0}$ is a positive isometry.

Proof. The equivalence $(i) \iff (ii)$ readily follows from Chapter 3 (Proposition 3.5.0.5).

Now, assume that the real Banach lattice $\Gamma(K, E)$ is σ-order complete.

$(i) \implies (iii)$: First, since $\mathcal{T}(t)_{t\geq 0}$ is, in particular, a positive C_0-semigroup on real Banach lattice $\Gamma(K, E)$ which is σ-order complete, our consideration in Remark 4.4.0.2(i) implies that its generator $(\mathcal{A}, D(\mathcal{A}))$ satisfies the Kato's inequality for which there exists a strictly positive non-empty subset $M' \subseteq \Gamma(K, E)'$ of subeigenvectors of the adjoint operator $(\mathcal{A}', D(\mathcal{A}'))$. This, in particular, implies that

$$< Re(sign\ s)\mathcal{A}s, s' > \leq < |s|, \mathcal{A}'s' > \text{ for all } s \in D_o, \text{ and } s' \in M'$$

for any core $(\mathcal{A}, D_o)$ of the generator $(\mathcal{A}, D(\mathcal{A}))$ and any positive subset M' of $D(\mathcal{A}')$.

Futhermore, $\mathcal{T}(t)$ being a $T_\varphi(t)$-homomorphism for every $t \geq 0$ implies that, $(\mathcal{A}, D(\mathcal{A}))$ is, in addition, a δ-derivation $\Gamma(K, E)$, i.e., it is a $D(\delta)$-submodule of $\Gamma(K, E)$ and $\mathcal{A}(fs) = \delta f \cdot s + f \cdot \mathcal{A}s$ for all $f \in D(\delta)$ and $s \in D(\mathcal{A})$; a property which must trivially hold for any core $(\mathcal{A}, D_o)$ of generator $(\mathcal{A}, D(\mathcal{A}))$.

Hence, assertion (iii) holds by combining these two results.

$(iii) \implies (i)$: Since the generator $(\mathcal{A}, D(\mathcal{A}))$ satisfies the following condition:

there exists a core $(\mathcal{A}, D_o)$ of the generator $(\mathcal{A}, D(\mathcal{A}))$ and a strictly positive non-empty subset $M' \subseteq \Gamma(K,E)'$ of subeigenvectors of $(\mathcal{A}', D(\mathcal{A}'))$ such that

$$< Re(sign\ s)\mathcal{A}s, s' > \leq < |s|, \mathcal{A}'s' > \text{ for all } s \in D_o, \text{ and } s' \in M',$$

we see that [2, Theorem 3.8, p.262] implies that it is a generator of a positive C_0-semigroup on $\Gamma(K,E)$. Moreover, since $(\mathcal{A}, D(\mathcal{A}))$ is also a δ-derivation on its core $(\mathcal{A}, D_o)$, we can conclude that $(\mathcal{A}, D(\mathcal{A}))$ is, in addition, a δ-derivation on $\Gamma(K,E)$ by which [29, Theorem 3.8, p.63] implies that $\mathcal{T}(t)$ is a positive $T_\varphi(t)$-homomorphism for every $t \geq 0$.

Furthermore, the last assertion follows from Chapter 3 (Proposition 3.5.0.5).

□

4.5 Spectral theory for Positive C_0-semigroups of weighted Koopman operators

The spectral theory for *non-weighted* Koopman operators $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$ induced by a unique flow $(\varphi_t)_{t\in\mathbb{R}}$ on K has been described, for instance, in [29, Section 4.1, p.72-75], and also its generator $(\delta, D(\delta))$. Furthermore, in [29, Section 4.2, p.72-75] the spectral theory of *weighted* Koopman operators on an AM-module with its generator was investigated, and sufficient conditions for their spectral mapping theorems were obtained (see [29, Proposition 4.6, p.75] and [29, Theorem 4.13, p.79]).

In our situation of a positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ on an AM m-lattice module $\Gamma(K,E)$ over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$, we build on these existing results and provide further implications for the positivity.

To start, we recall certain definitions and concepts of periodicity and (strictly) aperiodicity of a flow on a compact space (see Remark 4.5.0.1). Moreover, we collect certain results in Lemma 4.5.0.2 which are important for our consideration.

Remark 4.5.0.1. *([29, Definition 4.2, p.73] and [29, Definition 4.5, p.74])*

(i) *Let $\varphi : K \longrightarrow K$ be a homeomorphism on a compact space K. We call a point $x \in K$ a periodic point if there exists $n \in \mathbb{N}$ such that $\varphi^n(x) = x$. It is called aperiodic if $\varphi^n(x) \neq x$ for all $n \in \mathbb{N}$. We consider the prime function $v : K \longrightarrow \mathbb{N} \cup \{\infty\}$ of φ defined by*

$$v(x) := \begin{cases} \inf\{n \in \mathbb{N} \mid \varphi^n(x) = x\}, & x \text{ periodic} \\ \infty, & x \text{ aperiodic} \end{cases}$$

and the set $B(K) := \{x \in K \mid v \text{ is bounded in some neighbourhood of } x\}$.

The homeomorphism φ is called aperiodic if $B(K) = \emptyset$. It is called strictly aperiodic if each point $x \in K$ is aperiodic. If $v(x) < \infty$ for all $x \in K$, then φ is called periodic.

(ii) *Let $(\varphi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact space K. We call a point $x \in K$ periodic point if there exists $t > 0$ such that $\varphi_t(x) = x$. It is called an aperiodic point if $\varphi_t(x) \neq x$ for all $t > 0$. We consider the prime function $v : K \longrightarrow [0, \infty]$ of $(\varphi_t)_{t\in\mathbb{R}}$ defined by*

$$v(x) := \begin{cases} \inf\{t > 0 \mid \varphi_t(x) = x\}, & x \text{ periodic} \\ \infty, & x \text{ aperiodic} \end{cases}$$

and the set $B(K) := \{x \in K \mid v \text{ is bounded in some neighbourhood of } x\}$.

The flow $(\varphi_t)_{t\in\mathbb{R}}$ is called aperiodic if $B(K) = \emptyset$. It is called strictly aperiodic if each point $x \in K$ is aperiodic. If $v(x) < \infty$ for all $x \in K$, then $(\varphi_t)_{t\in\mathbb{R}}$ is called periodic.

Lemma 4.5.0.2. *([29, Proposition 4.3, p.73], [29, Proposition 4.4, p.74] and [29, Proposition 4.7, p.75])*

(A) *Let $\varphi : K \longrightarrow K$ be a homeomorphism of the compact space K, and $T_\varphi : C(K) \longrightarrow C(K)$ the associated invertible Koopman operator.*

(i) If φ is aperiodic, we have that $\sigma(T_\varphi) = \sigma_{ap}(T_\varphi) = \mathbb{T}$.

(ii) If φ is periodic, i.e., $v(x) < \infty$ for all $x \in K$, then

$$\sigma(T_\varphi) = \overline{\bigcup_{x\in K} P_{v(x)}},$$

where $P_n := \{z \in \mathbb{C} \mid z^n = 1\}$ is the group of n-roots of unity for $n \in \mathbb{N}$.

If $v(x) = n$ for all $x \in K$ and a fixed $n \in \mathbb{N}$, then $\sigma(T_\varphi) = P_n$.

(B) Let $(\varphi_t)_{t\in\mathbb{R}}$ be a flow on the compact space K, and $T_\varphi(t)_{t\in\mathbb{R}}$ the associated Koopman group on $C(K)$ with generator $(\delta, D(\delta))$. For each $t \geq 0$,

$$\sigma(T_\varphi(t)) = \sigma_{ap}(T_\varphi(t)) \subseteq \mathbb{T}$$

is the union of subgroups of $\mathbb{T}$. Furthermore,

$$\sigma(\delta) = \sigma_{ap}(\delta) \subseteq i\mathbb{R}$$

is the union of additive subgroups of $i\mathbb{R}$.

(C) Let $(\varphi_t)_{t\in\mathbb{R}}$ be a flow on compact space K, and $T_\varphi(t)_{t\in\mathbb{R}}$ the associated Koopman group on $C(K)$ with generator $(\delta, D(\delta))$.

(i) If the flow is aperiodic, then

$$\sigma(T_\varphi(t)) = \mathbb{T} \ \text{for all} \ t \in \mathbb{R}; \ \text{and}$$

$$\sigma(\delta) = i\mathbb{R}.$$

Moreover, the spectral mapping theorem holds, i.e.,

$$\sigma(T_\varphi(t)) = e^{t\sigma(\delta)} \ \text{for all} \ t \in \mathbb{R}.$$

(ii) If the flow is periodic with $0 < v(x) < \infty$ for all $x \in K$, then the weak spectral mapping theorem holds, i.e.,

$$\sigma(T_\varphi(t)) = \overline{e^{t\sigma(\delta)}} \ \text{for all} \ t \in \mathbb{R}.$$

In the following, we consider a (general) Banach bundle E over a compact space K, and the Banach space of its continuous sections $\Gamma(K, E)$, the associated AM-module over $C(K)$ (see [29, Proposition 3.2, p.59]).

Lemma 4.5.0.3. *([29, Proposition 4.10, p.78] and [29, Theorem 4.13, p.79])*

(A) Let $\varphi : K \longrightarrow K$ be a homeomorphism, and $\mathcal{T} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ a T_φ-homomorphism. Assume that one of the following conditions holds.

(i) φ is aperiodic and E is a continuous Banach bundle.

(ii) φ strictly aperiodic.

Then, $\mathbb{T} \cdot \sigma_{ap}(\mathcal{T}) \subseteq \sigma_{ap}(\mathcal{T})$ and $\mathbb{T} \cdot \sigma(\mathcal{T}) \subseteq \sigma(\mathcal{T})$. Thus $\sigma(\mathcal{T})$ is invariant under rotation by a complex number of modulus one.

(B) Let $(\varphi_t)_{t\in\mathbb{R}}$ be a flow on K, and $(\mathcal{T}(t))_{t\geq 0}$ a weighted Koopman semigroup on AM module $\Gamma(K,E)$ with generator $(\mathcal{A}, D(\mathcal{A}))$. Assume that one of the following conditions holds.

(i) $(\varphi_t)_{t\in\mathbb{R}}$ is aperiodic and E is a continuous Banach bundle.

(ii) $(\varphi_t)_{t\in\mathbb{R}}$ strictly aperiodic.

Then, the spectral mapping theorem holds, i.e.,

$$\sigma(\mathcal{T}(t)) \setminus \{0\} = e^{t\sigma(\mathcal{A})} \ \textit{for each} \ \ t \geq 0$$

Moreover, $\sigma(\mathcal{A}) = \sigma(\mathcal{A}) + i\mathbb{R}$, and $\sigma(\mathcal{T}(t)) = \mathbb{T}\cdot\sigma(\mathcal{T}(t))$, $t \geq 0$.

Using the above result in Lemma 4.5.0.3, we claim the following additional properties hold in our setting. We now consider a Banach lattice bundle E over a compact space K, and the Banach lattice of its continuous sections $\Gamma(K,E)$, the associated AM m-lattice module over Banach lattice algebra $C(K)$ (see also Chapter 3, Remark 3.5.0.1(ii)).

Proposition 4.5.0.4. *(A) Let $\varphi : K \longrightarrow K$ be a homeomorphism, and $\mathcal{T} : \Gamma(K,E) \longrightarrow \Gamma(K,E)$ a positive T_φ-homomorphism. Assume that one of the following conditions holds.*

(i) φ is aperiodic and E is a continuous Banach lattice bundle.

(ii) φ strictly aperiodic.

Then, $\mathbb{T}\cdot\sigma_{ap}(\mathcal{T}) \subseteq \sigma_{ap}(\mathcal{T})$ and $\mathbb{T}\cdot\sigma(\mathcal{T}) \subseteq \sigma(\mathcal{T})$.

Moreover, $\mathbb{T}\cdot Per\sigma(\mathcal{T}) \subseteq Per\sigma(\mathcal{T})$. That is, the peripheral spectrum $Per\sigma(\mathcal{T}) := \{\lambda \in \sigma(\mathcal{T}) : |\lambda| = r(\mathcal{T})\}$ is invariant under rotation by a complex number of modulus one.

(B) Let $(\varphi_t)_{t\in\mathbb{R}}$ be a flow on K, and $(\mathcal{T}(t))_{t\geq 0}$ a positive weighted Koopman semigroup on AM m-lattice module $\Gamma(K,E)$ with generator $(\mathcal{A}, D(\mathcal{A}))$. Assume that one of the following conditions holds.

(i) $(\varphi_t)_{t\in\mathbb{R}}$ is aperiodic and E is a continuous Banach lattice bundle..

(ii) $(\varphi_t)_{t\in\mathbb{R}}$ strictly aperiodic.

Then, the spectral mapping theorem holds, i.e.,

$$\sigma(\mathcal{T}(t)) \setminus \{0\} = e^{t\sigma(\mathcal{A})} \ \textit{for each} \ \ t \geq 0.$$

Moreover, $\sigma(\mathcal{A}) = \sigma(\mathcal{A}) + i\mathbb{R}$ *and* $\sigma(\mathcal{T}(t)) = \mathbb{T} \cdot \sigma(\mathcal{T}(t))$, $t \geq 0$.

In addition:

(a) The growth bound w_0 *of* $(\mathcal{T}(t))_{t\geq 0}$ *coincides with the spectral bound* $s(A) := \sup\{Re\,\mu : \mu \in \sigma(\mathcal{A})\}$ *of the generator* $(\mathcal{A}, D(\mathcal{A}))$.

(b) $\sigma^b(\mathcal{A}) + i\mathbb{R} \subseteq \sigma^b(\mathcal{A})$. *That is, the boundary spectrum* $\sigma^b(\mathcal{A}) := \{\mu \in \sigma(\mathcal{A}) : Re\,\mu = s(\mathcal{A})\}$ *is an imaginary additively cyclic subset of* $\mathbb{C}$.

Proof. (A) By Lemma 4.5.0.3(A), it suffices to show that the peripheral spectrum $Per\sigma(\mathcal{T})$ is rotation invariant by $\mathbb{T}$. This, however, follows since the peripheral spectrum of a positive operator is contained in its approximate point spectrum $\sigma_{ap}(\mathcal{T})$. Indeed, since $\sigma(\mathcal{T})$ is a compact subset of $\mathbb{C}$, we have that there exists $\lambda \in \sigma(\mathcal{T})$ such that $|\lambda| = r(\mathcal{T})$. Now, since $\sigma(\mathcal{T})$ is rotation invariant by $\mathbb{T}$, we see that $r(\mathcal{T}) \in \sigma(\mathcal{T})$.

(B) Similarly, by Lemma 4.5.0.3(B), it suffices to show the additional properties (a) and (b).

(a) We note that, in general, we have that $s(\mathcal{A}) \leq w_0$ and $r(\mathcal{T}(t)) = e^{w_o t}$ for all $t \geq 0$ for any generator of a C_0-semigroup (see [12, V. Proposition 1.22, p.168]). Since, in our case, $r(\mathcal{T}(t)) \in \sigma(\mathcal{T}(t))$ for each $t \geq 0$, the spectral mapping theorem implies that $e^{w_o t} \in e^{t\sigma(\mathcal{A})}$ for each $t > 0$ and $w_0 \in \sigma(\mathcal{A})$ whenever $r(\mathcal{T}(1)) \neq 0$. Moreover, since $(\mathcal{A}, D(\mathcal{A}))$ is a generator of positive C_0-semigroup, we have that $s(A) \in \sigma(\mathcal{A})$ (see [2, Corollary 1.4, p.294]).

Combining these results, it follows that $e^{w_o} \leq e^{s(A)} \in e^{\sigma(\mathcal{A})}$ and consequently, $s(A) = w_0$ since $s(\mathcal{A}) < w_0$ is not possible in $\sigma(\mathcal{A})$.

(b) We note that, $\sigma(\mathcal{A}) = \sigma(\mathcal{A}) + i\mathbb{R}$, in particular, implies $\sigma_{ap}(\mathcal{A}) + i\mathbb{R} \subseteq \sigma_{ap}(\mathcal{A})$ for the approximate point spectrum of the generator $(\mathcal{A}, D(\mathcal{A}))$. Now, since the boundary spectrum $\sigma^b(\mathcal{A})$ of a generator of positive C_0-semigroup is always contained in $\sigma_{ap}(\mathcal{A})$, the assertion follows. Indeed, if $\mu \in \sigma^b(\mathcal{A})$, then $\mu + i\lambda \in \mu + i\mathbb{R} \subseteq \sigma(\mathcal{A})$ for any $\lambda \in \mathbb{R}$, which implies that $\mu + i\lambda \in \mu + i\mathbb{R} \subseteq \sigma^b(\mathcal{A})$ since $Re(\mu + i\lambda) = Re\,\mu = s(\mathcal{A})$.

Thus, $\sigma^b(\mathcal{A})$ is an imaginary additively cyclic subset of $\mathbb{C}$ as claimed.

∎

4.6 Note

We refer to the Bonus Chapters:

(i) C for Positive semiflows on *measurable* Banach lattice bundles,

(ii) D for Positive weighted Koopman semigroup everywhere,

(iii) E for Asymptotics of positive weighted Koopman semigroup, and

(iv) F for Markovian weighted Koopman group,

which are added for the sake of comprehensiveness, but put in the appendices because the scope of this thesis has become very large.

Appendices

Appendix A

Banach lattice algebras

A.1 What is a Banach lattice algebra?

We observe, as reported by Wickstead ([31]), that there does not appear to be a consensus on a definition of this mathematical object.
However, from [31, p. 805-806], we note the following about a Banach lattice algebra.

Remark A.1.0.1. *(i) What is agreed is that a Banach lattice algebra should be a Banach lattice, an associative algebra with a sub-multiplicative norm and the product of positive elements should be positive.*

(ii) To cover as wide a range of examples as possible and in an effort to standardise terminology we propose that a Banach lattice algebra simply be at the same time a Banach lattice, an associative algebra with sub-multiplicative norm and with the product of positive elements being positive. If there is an identity which has norm one we call it a 1-Banach lattice algebra.

(iii) [31, Proposition 2.1] In any Banach lattice algebra L (whether real or complex), with multiplication $\star$, $|f \star g| \leq |f| \star |g|$ for all $f, g \in L$.

Note A.1.0.2. *We note that condition (iii) of Remark A.1.0.1 above can precisely be taken to be a characterisation of the so-called Banach lattice algebra. That is, a Banach lattice $(L, |\cdot|)$ which is also a Banach algebra $(L, \star)$ is a "Banach lattice algebra" $(L, |\cdot|, \star)$ if and only if $|f \star g| \leq |f| \star |g|$ for all $f, g \in L$.*

Definition A.1.0.3. *A Banach lattice algebra* $(L, |\cdot|, \star)$ *is a Banach lattice* $(L, |\cdot|)$ *which is also a Banach algebra* $(L, \star)$ *such that* $|f \star g| \leq |f| \star |g|$ *for all* $f, g \in L$.

We list here some examples of Banach lattice algebras, and refer to [31, Example 1.1 p.804] for some details.

Example A.1.0.4. *(i) Let* Ω *be a locally compact space, then the Banach space* $C_0(\Omega)$ *is a commutative Banach lattice algebra. Moreover,* $C_0(\Omega)$ *is a 1-Banach lattice algebra if and only if* $\Omega = K$ *is compact.*

(ii) Let $X := (\Omega_X, \Sigma_X, \mu_X)$ *be a complete* σ*-finite (positive) measure space, then the Banach space* $L^\infty(X)$ *is a commutative Banach lattice algebra.*

(iii) Let E *be a Banach lattice, then the space*

$$\mathcal{Z}(E) := \{T \in \mathscr{L}(E) : \exists\, \lambda > 0 \text{ such that } |Tf| \leq \lambda |f| \;\forall\, f \in E\}$$

of central operators, the so-called centre of E *equipped with the operator norm, is a commutative 1-Banach lattice algebra.*

(iv) Let G *be a locally compact topological group, and* μ *the left Haar measure on* G*, then the Banach space* $L^1(G)$ *of integrable* $\mathbb{K}$*-valued functions is a Banach lattice algebra.* $L^1(G)$ *is a 1-Banach lattice algebra if and only if* G *is discrete. Moreover,* $L^1(G)$ *is commutative if and only if* G *is abelian.*

(v) Let G *be a locally compact topological group, then the Banach space* $M(G)$ *of bounded regular Borel measures on* G *is a Banach lattice algebra.*

(vi) Let E *be an order complete Banach lattice (i.e., a Dedekind complete Banach lattice), then the space* $L^r(E)$ *of regular operators, equipped with regular-norm, is a 1-Banach lattice algebra.*

Throughout we will use "BLA" to mean a Banach lattice algebra, if no confusion arises.

Appendix B

Ordered Banach spaces

Here, we recall certain definitions and concepts about an ordered Banach space as presented by O. Bratteli et al ([4]) and the conditions under which it becomes a Banach lattice (see [4, Section 1, p.372-376]).

Definition B.0.0.1. *Let B be a real Banach space. A subset "B_+" of B is defined to be a proper closed convex cone in B if:*

(i) B_+ *is norm closed;*

(ii) $B_+ + B_+ \subseteq B_+$*;*

(iii) $\lambda B_+ \subseteq B_+$ *for all* $\lambda \geq 0$*; and*

(iv) $B_+ \cap (-B_+) = \{0\}$.

Each B_+ determines a partial order $\geq$ on B by defining $b \geq c$ whenever $b - c \in B_+$. Thus $b \geq 0$ is equivalent to $b \in B_+$. Elements of B_+ are referred to as positive elements of B.
Then B is said to be an ordered Banach space, with positive cone B_+.

The following lemma states the equivalence of the 'normality' of a positive cone of an ordered Banach space.

Proposition B.0.0.2. *For a positive cone B_+ of B, the following are equivalent.*

(i) B_+ *is normal, i.e., there exists an* $\alpha > 0$ *such that* $\alpha \leq ||b + c||$ *for all* $b, c \in B_+$ *with* $||b|| = 1 = ||c||$.

(ii) There exists $\beta > 0$ *such that* $0 \leq b \leq c$ *always implies* $\beta||b|| \leq ||c||$ *for all* $b, c \in B_+$.

(iii) There exists a $\gamma > 0$ *such that* $b \leq c \leq d$ *always implies* $\gamma||b|| \leq ||c|| \vee ||d||$ *for all* $b, c, d \in B$.

The following states the equivalence for a positive cone of an ordered Banach space having a non-empty 'interior'.

Proposition B.0.0.3. *For a positive cone* B_+ *of* B*, the following are equivalent.*

(i) $u \in B_+$ *is an interior point of* B_+*, i.e., there exists an* $\varepsilon > 0$ *such that* $\{b : ||u - b|| \leq \varepsilon\} \subset B_+$.

(ii) $u \in B_+$ *is an order unit, i.e., for each* $b \in B$*, there is a* $\lambda \geq 0$ *such that* $b \leq \lambda u$.

(iii) $B = B_u$*, where* $B_u := \{b \in B : -\delta u \leq b \leq \delta u \text{ for some } \delta \geq 0\}$.

There are some connections between normality, interior of a positive cone $IntB_+$ and 'topologies' on an ordered Banach space.

Proposition B.0.0.4. *For a positive cone* B_+ *of* B*, let* $b(B)$ *and* $o(B)$ *represent the norm-bounded and order-bounded sets in* B *respectively. Then the following hold.*

(i) B_+ *is normal implies* $o(B) \subseteq b(B)$.

(ii) $IntB_+ \neq \emptyset$ *if and only if* $b(B) \subseteq o(B)$.

(iii) If $IntB_+ \neq \emptyset$*, then* B_+ *is normal if and only if* $o(B) = b(B)$.

Therefore, B_+ *is normal and* $IntB_+ \neq \emptyset$ *if and only if* $o(B) = b(B)$.

The following is a comment on when an ordered Banach space B with positive cone B_+ becomes a Banach lattice.

Proposition B.0.0.5. *For a positive cone* B_+ *of* B*, if:*

(i) the partial order $\geq$ *associated with* B_+ *is a lattice ordering, i.e., each pair* $b, c \in B$*, has a least upper bound* $b \vee c$ *and a greatest lower bound* $b \wedge c$*; and*

(ii) the norm on B *is a lattice norm, i.e., for* $b, c \in B$*,* $||\,|b|\,|| = ||b||$ *and* $0 \leq b \leq c$ *always implies* $||b|| \leq ||c||$*, where* $|b| := b_+ + b_-$ *is the 'modulus' of* b *and* $b_\pm := (\pm b) \vee 0$ *represent the positive/negative part of* b*,*

then $(B, ||\cdot||)$ *is a Banach lattice.*

The following theorem is essentially due to the theorem of Kakutani which represents an AM-space with unit as a space of scalar-valued continuous functions on some compact space.

Theorem B.0.0.6. *For a Banach lattice $(B, ||\cdot||)$, the following are equivalent.*

(i) $IntB_+ \neq \emptyset$

(ii) *B is isometrically lattice isomorphic to $C(Q)$ for some compact space Q.*

Proof. That $(ii) \implies (i)$ is evident.
$(i) \implies (ii)$: Take $u \in IntB_+$. Since $B = B_u$ (see Proposition B.0.0.2), the norm $||\cdot||_u$ associated with $u \in B_+$, defined as $||b||_u := \max\{N_u(b), N_u(-b)\}$ for any $b \in B_u = B$, where $N_u(b) := \inf\{\lambda \geq 0 : b \leq \lambda u\}$, is a lattice norm on B.
Observing that $||b \vee c||_u = \max(||b||_u, ||c||_u)$ for any $b, c \in B_+$ implies that $(B, ||\cdot||_u)$ is an AM-space with order unit $u \in B_+$. And since the two norms $||\cdot||_u$ and $||\cdot||$ are equivalent, we see that $(B, ||\cdot||)$ is also an AM-space with order unit. By Kakutani's theorem (see [28, Theorem 7.4, p.104]), it follows that the Banach lattice $(B, ||\cdot||)$ is isometrically lattice isomorphic to $(C(Q), ||\cdot||_\infty)$ for some compact space Q.

□

Appendix C

Bonus Chapter: Positive semiflows on *measurable* Banach lattice bundles

C.1 Introduction

In this chapter, we introduce the notions of *measurable* Banach lattice bundles (see Section C.2), the vector lattices of measurable sections (see Section C.3), morphisms of these *measurable* Banach lattice bundles (see Section C.4) and their representation theories (see Section C.5).

We start in Section C.2 with our definition of a *measurable* Banach lattice bundle E over a complete σ-finite measure space X (see Definition C.2.0.1). In Section C.3, we consider the vector lattice $\mathcal{M}_E$ of measurable sections of E and obtain that, the (quotient) vector lattices $\Gamma^\infty(X, E)$ and $\Gamma^1(X, E)$ of essentially-bounded and integrable measurable sections are, respectively, m-Banach lattice modules over the commutative Banach lattice algebra $L^\infty(X)$ (see Corollary C.3.0.7). Moreover, if E is separable over a separable measure space X, then $\Gamma^1(X, E)$ is, in addition, a separable Banach lattice (see Proposition C.3.0.8(iv)).

In Section C.4, we introduce two notions of morphisms between two *measurable* Banach lattice bundles E and F over a measure space X: namely, a *positive morphism* of measurable Banach lattice bundles over $\varphi : X \longrightarrow X$, a morphism of measure space X (see Definition C.4.0.2) and a *morphism* of measurable Banach lattice bundles over a morphism $\varphi : X \longrightarrow X$ (see Remark C.4.0.3(iii)). In (Definition C.4.0.5) Remark C.4.0.6(iii), we introduce the notion of a (positive) S-dynamical *measurable* Banach lattice bundle $(E, \mathbf{\Phi})$ over a *measure-preserving* G-dynamical system (X, φ), and we call $\mathbf{\Phi} = (\Phi_g)_{g \in S}$ a (positive) semiflow on E over the flow $\varphi = (\varphi_g)_{g \in G}$ on X.

In the last Section C.5, we obtain that, for a discrete group G, every (positive) S-dynamical *measurable* Banach lattice bundle $(E, \mathbf{\Phi})$ over a *measure-preserving* G-dynamical system (X, φ) induces a (positive) S-dynamical Banach lattice module $(\Gamma^1(X, E), \mathcal{T}_{\mathbf{\Phi}})$ over the Koopman group representation $(L^\infty(X), \boldsymbol{T}_\varphi)$ (see Remark C.5.0.1(v), and also Chapter 2, Example 2.3.4.3(ii)). Furthermore, we call $\mathcal{T}_{\mathbf{\Phi}} = \mathcal{T}_\Phi(g)_{g \in S}$ the (positive) weighted Koopman semigroup representation on $\Gamma^1(X, E)$ over the Koopman group $\boldsymbol{T}_\varphi = (T_\varphi(g))_{g \in G}$ induced by $(E, \mathbf{\Phi})$. Moreover, we obtain the abstract representation of the (quotient) vector lattice $\Gamma^1(X, E)$ of integrable measurable sections of a separable *measurable* Banach lattice bundle E over a separable measure space X as what we call a separable $L^1(X)$-normed m-lattice module over $L^\infty(X)$ (see Appendix D Question D.2.0.2), Remark C.5.0.1(ii) and Proposition C.5.0.8(ii).

As one major result of our study, for a discrete group G, every (positive) S-dynamical separable $L^1(X)$-normed m-lattice module over the Koopman group $(L^\infty(X), \boldsymbol{T}_\varphi)$ can be assigned uniquely to a separable (positive) S-dynamical *measurable* Banach lattice bundle $(E, \mathbf{\Phi})$ over a *measure-preserving* G-dynamical system (X, φ) with X separable and vice versa (see also Appendix D, Proposition D.2.0.3). This is our Gelfand-type theorem for dynamical separable $L^1(X)$-normed m-lattice modules (see Theorem C.5.0.9 and Corollary C.5.0.10).

C.2 *Measurable* Banach lattice bundles

In this Section, we introduce the notion of a *measurable* Banach lattice bundle.

Here, any measure space $X := (\Omega_X, \Sigma_X, \mu_X)$ is assumed to be complete[1] and

[1] i.e., subsets of null sets are measurable.

with a positive σ-finite measure. See [29, Chapter 1, Section 1.2, p.18-21] for the case of measurable Banach bundle.

Definition C.2.0.1. *Let E be a set (total space), X a measure space (base space), and $p_E : E \longrightarrow \Omega_X$ a surjective mapping (bundle projection). Then the triple $(E, p_E, \mathcal{M}_E)$ is called a measurable Banach lattice bundle over X, where $\mathcal{M}_E$ is a vector sublattice of $\mathcal{S}_E := \{s : \Omega_X \longrightarrow E \mid p_E \circ s = Id_{\Omega_X}\}$, if the following conditions are satisfied.*

(i) *For each $x \in \Omega_X$, the fiber $E_x := p_E^{-1}(x)$ is a Banach lattice.*

(ii) *If $f : \Omega_X \longrightarrow \mathbb{K}$ is measurable and $s \in \mathcal{M}_E$, then $fs \in \mathcal{M}_E$, where fs is defined pointwisely.*

(iii) *For each $s \in \mathcal{M}_E$ the mapping*

$$|s| : \Omega_X \longrightarrow \mathbb{R}_+,\ x \mapsto ||s(x)||_{E_x}$$

is measurable.

(iv) *If $\left(s_n\right)_{n\in\mathbb{N}} \in \mathcal{M}_E$ is a sequence converging almost everywhere to $s \in \mathcal{S}_E$, then $s \in \mathcal{M}_E$.*

Elements of $\mathcal{S}_E$ are called sections while that of $\mathcal{M}_E$ are measurable sections of the measurable Banach lattice bundle $(E, p_E, \mathcal{M}_E)$.

And if, in addition,

(v) *there exists a sequence $\left(s_n\right)_{n\in\mathbb{N}} \in \mathcal{M}_E$ such that the linear span $lin\{s_n(x) \mid n \in \mathbb{N}\}$ is dense in E_x for almost every $x \in \Omega_X$. then $(E, p_E, \mathcal{M}_E)$ is said to be separable.*

Note C.2.0.2. *(i)* *If no confusion arises, we will say E is a measurable Banach lattice bundle over X, while we mean $(E, p_E, \mathcal{M}_E)$.*

(ii) *We note that, in our setting of a measurable Banach lattice bundle (which may as well be called a measurable bundle of Banach lattices), the only additional property we added to the definition of measurable Banach bundle as defined in [29, Definition 1.12 p.18] is that*

(a) *each fiber is a Banach lattice instead of a Banach space; and*

(b) *we choose a vector sublattice as the space of its measurable sections instead of a vector subspace.*

And so every (separable) measurable Banach lattice bundle is, in particular, a (separable) measurable Banach bundle in this situation.

We mention here a situation whereby a measurable Banach lattice bundle can be generated. See [29, Remark 1.14, p.18] for the case measurable Banach bundle.

Remark C.2.0.3. *Let X be a measure space and (E, p) is a pair of a set E and a surjective map $p : E \longrightarrow \Omega_X$ satisfying condition (i) of Definition C.2.0.1 above. For any vector sublattice $\mathcal{M}_E$ of $\mathcal{S}_E := \{s : \Omega_X \longrightarrow E \mid p \circ s = Id_{\Omega_X}\}$ satisfying condition (iv) of Definition C.2.0.1, we have that*

(i) $\mathcal{M}_E$ generates a measurable Banach lattice bundle i.e., there exists a smallest vector sublattice $\tilde{\mathcal{M}}_E$ of $\mathcal{S}_E$ containing $\mathcal{M}_E$ such that $(E, p, \tilde{\mathcal{M}}_E)$ is a measurable Banach lattice bundle over X, and, moreover,

(ii) $\tilde{\mathcal{M}}_E$ consists precisely of all almost everywhere limits of sequences in $lin\{\mathbb{1}_A s \,|A \in \Sigma_X, s \in \mathcal{M}_E\}$.

Similar to the case of a measurable Banach bundle (see [29, Example 1.16, p.19]) we identify the following two important examples of measurable Banach lattice bundles.

Example C.2.0.4.

(i) Let Z be a Banach lattice and X a measure space. Consider $E := \Omega_X \times Z$ and the projection $p : E \longrightarrow \Omega_X$ onto the first factor. The space of sections $\mathcal{S}_E$ is identified with the space of all functions $s : \Omega_X \longrightarrow Z$. The set of all strongly measurable functions then defines a subset $\mathcal{M}_E$ of $\mathcal{S}_E$ which turns E into a measurable Banach lattice bundle called the trivial Banach lattice bundle with fiber Z. Moreover, this coincides with the measurable Banach lattice bundle generated by constant sections, i.e., $\mathcal{M}_E$ consists precisely of all almost everywhere limits of sequences in $lin\{\mathbb{1}_A s \,|A \in \Sigma_X,\ s(x) = z \in Z$ for all $x \in \Omega_X\}$.

(ii) Let E be a topological Banach lattice bundle over a locally compact space Ω, μ a σ-finite regular Borel measure on Ω, and $B(\Omega)$ the Borel σ-algebra of Ω. Then the vector lattice $\Gamma_0(\Omega, E)$ generates a measurable Banach lattice bundle E_μ over the completion of the measure space $(\Omega, B(\Omega), \mu)$. We say E_μ is the induced measurable Banach lattice bundle over a measure space induced by Ω.

C.3 The space of measurable sections

A measurable Banach lattice bundle $(E, p_E, \mathcal{M}_E)$ over a measure space X induces a natural ordered vector space.
Indeed, on the space of sections $\mathcal{S}_E := \{s : \Omega_X \longrightarrow E \mid p_E \circ s = Id_{\Omega_X}\}$, we have for each $s \in \mathcal{S}_E$, a "modulus" map

$$\mathcal{S}_E \longrightarrow \mathcal{S}_E;\ s \mapsto |s| := \left[x \mapsto |s(x)|\right].$$

If $\mathbb{K} = \mathbb{R}$, defining the set

$$\mathcal{S}_E^+ := \left\{ s \in \mathcal{S}_E : s \geq 0 \overset{def}{\iff} |s| = s \right\},$$

we obtain a partial order on $\mathcal{S}_E$ by saying $s_1 \leq s_2 \overset{def}{\iff} s_2 - s_1 \geq 0$.
And if $\mathbb{K} = \mathbb{C}$, defining the set

$$\mathcal{S}_E^{\mathbb{R}} := \{s \in \mathcal{S}_E : Re\, s = s\},$$

we obtain a partial order on $\mathcal{S}_E^{\mathbb{R}}$ as above.

The following lemma will be useful, which states important order structures associated with a measurable Banach lattice bundle.

Lemma C.3.0.1. *Let E be a measurable Banach lattice bundle over X. Then the following hold on the space of its sections $\mathcal{S}_E$.*

(A) If $\mathbb{K} = \mathbb{R}$, then $\mathcal{S}_E$ is a real vector lattice where $\mathcal{S}_E^+$ denotes the sets of its positive elements such that

(i) if $(s_n)_{n\in\mathbb{N}} \in \mathcal{S}_E^+$ is a sequence converging (almost everywhere) to $s \in \mathcal{S}_E$, then $s \in \mathcal{S}_E^+$ (almost everywhere),

(ii) $\mathcal{S}_E^+ \cap (-\mathcal{S}_E^+) = \{0\}$,

(iii) $\mathcal{S}_E^+ + \mathcal{S}_E^+ \subseteq \mathcal{S}_E^+$,

(iv) $\lambda \mathcal{S}_E^+ \subseteq \mathcal{S}_E^+$ for any $\lambda \geq 0$, and

(v) For $s_1, s_2 \in \mathcal{S}_E^+$, $0 \leq s_1 \leq s_2 \implies ||s_1(x)||_{E_x} \leq ||s_2(x)||_{E_x}$ for all $x \in \Omega_X$.

(B) If $\mathbb{K} = \mathbb{C}$, then $\mathcal{S}_E^{\mathbb{R}}$ is a real vector lattice where

$$\mathcal{S}_E^+ := \left\{ r \in \mathcal{S}_E^{\mathbb{R}} : r \geq 0 \overset{def}{\iff} |r| = r \right\}$$

denotes the set of its positive elements such that

(i) if $(s_n)_{n\in\mathbb{N}} \in \mathcal{S}_E^{\mathbb{R}}$ is a sequence converging (almost everywhere) to $s \in \mathcal{S}_E$, then $s \in \mathcal{S}_E^{\mathbb{R}}$ (almost everywhere), and

(ii) $\mathcal{S}_E = \mathcal{S}_E^{\mathbb{R}} \oplus i\mathcal{S}_E^{\mathbb{R}}$ is a complex vector lattice.

Proof. (A) If $\mathbb{K} = \mathbb{R}$, i.e., E_x is a real Banach lattice for each $x \in \Omega_X$. In this situation, the lattice operations in $\mathcal{S}_E$ are defined pointwise via $(s_1 \wedge s_2)(x) = s_1(x) \wedge s_2(x)$ and $(s_1 \vee s_2)(x) = s_1(x) \vee s_2(x)$ for any $s_1, s_2 \in \mathcal{S}_E$ and every $x \in \Omega_X$. Moreover, for any $s \in \mathcal{S}_E$ setting $s^+ := s \vee 0$ and $s^- := -(s \wedge 0)$, it follows that $s^+, s^- \in \mathcal{S}_E^+$ and $s = s^+ - s^-$ is the unique representation as a difference of positive sections with modulus $|s| = s^+ + s^-$.

It is clear that, $\mathcal{S}_E$ is a real vector lattice, with positive sections $\mathcal{S}_E^+$. That is $s \in \mathcal{S}_E^+ \iff s \in \mathcal{S}_E$ and $0_x \leq s(x) \in E_x^+$ for each $x \in \Omega_X$.

(i) Let $(s_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}_E^+$ be a sequence of positive sections converging (almost everywhere) to a section $s \in \mathcal{S}_E$. This, in particular, implies that $(s_n(x))_{n\in\mathbb{N}} \subseteq E_x^+$ is a sequence converging to $s(x) \in E_x$ for (almost) every $x \in \Omega_X$. And since $E_x^+ \subseteq E_x$ is norm-closed for each $x \in \Omega_X$; then we have that $s(x) \in E_x^+$ for (almost) every $x \in \Omega_X$.

Assertions (ii), (iii) and (iv) follow, respectively, since $E_x^+ + E_x^+ \subseteq E_x^+$, $E_x^+ \cap (-E_x^+) = \{0_x\}$ and $\lambda E_x^+ \subseteq E_x^+$ for each $x \in \Omega_X$, $0 \leq \lambda \in \mathbb{R}$.

(v) For $s_1, s_2 \in \mathcal{S}_E^+$, $0 \leq s_1 \leq s_2$ implies that $0_x \leq s_1(x) \leq s_2(x)$ for each $x \in \Omega_X$. And this also implies that $||s_1(x)||_{E_x} \leq ||s_2(x)||_{E_x}$ for all $x \in \Omega_X$.

(B) If $\mathbb{K} = \mathbb{C}$, i.e., $E_x = E_x^{\mathbb{R}} \oplus iE_x^{\mathbb{R}}$ is a complex Banach lattice for each $x \in \Omega_X$, then we see that $\mathcal{S}_E^{\mathbb{R}}$ is the set of real sections, i.e., $s \in \mathcal{S}_E^{\mathbb{R}} \iff s \in \mathcal{S}_E$ and $s(x) \in E_x^{\mathbb{R}}$ for each $x \in \Omega_X$. And as in (A) above we obtain that $\mathcal{S}_E^{\mathbb{R}}$ is a real vector lattice.

(i) Let $(s_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}_E^{\mathbb{R}}$ be a sequence of real sections converging (almost everywhere) to a section $s \in \mathcal{S}_E$. This, in particular, implies that $(s_n(x))_{n\in\mathbb{N}} \subseteq E_x^{\mathbb{R}}$ is a sequence converging to $s(x) \in E_x$ for (almost) every $x \in \Omega_X$. And since $E_x^{\mathbb{R}} \subseteq E_x$ is a closed subspace for each $x \in \Omega_X$, we have that $s(x) \in E_x^{\mathbb{R}}$ for (almost) every $x \in \Omega_X$.

(ii) Moreover, for any $s \in \mathcal{S}_E$, we see that the mappings $Re\ s : \Omega_X \longrightarrow E; x \mapsto Re\ s(x)$ and $Im\ s : \Omega_X \longrightarrow E; x \mapsto Im\ s(x)$ are real sections, i.e., $Re\, s, Im\, s \in \mathcal{S}_E^{\mathbb{R}}$, and so $s = Re\, s + iIm\, s$ is the unique representation of complex section with modulus $|\,s\,| = \sup_{t\in\mathbb{Q}} \left\{Re\ e^{\pi i t}s\right\}$. It thus follows that $\mathcal{S}_E = \mathcal{S}_E^{\mathbb{R}} \oplus i\mathcal{S}_E^{\mathbb{R}}$ is a complex vector lattice.

□

The following is an immediate corollary of Lemma C.3.0.1 above combined with condition (iv) of Definition C.2.0.1.

Corollary C.3.0.2. *Let E be a measurable Banach lattice bundle over X. Then the following hold, on the space of its measurable sections $\mathcal{M}_E$.*

(A) If $\mathbb{K} = \mathbb{R}$, then $\mathcal{M}_E$ is a real vector lattice where $\mathcal{M}_E^+ := \mathcal{S}_E^+ \cap \mathcal{M}_E$ denotes the sets of its positive elements such that

(i) $\mathcal{M}_E^+$ is (sequentially) closed in $\mathcal{S}_E$ in the sense that, if $\left(s_n\right)_{n\in\mathbb{N}} \in \mathcal{M}_E^+$ is a sequence converging almost everywhere to $s \in \mathcal{S}_E$, then $s \in \mathcal{M}_E^+$,

(ii) $\mathcal{M}_E^+ \cap (-\mathcal{M}_E^+) = \{0\}$,

(iii) $\mathcal{M}_E^+ + \mathcal{M}_E^+ \subseteq \mathcal{M}_E^+$,

(iv) $\lambda\mathcal{M}_E^+ \subseteq \mathcal{M}_E^+$ for any $\lambda \geq 0$, and

(v) For $s_1, s_2 \in \mathcal{M}_E^+$, $0 \leq s_1 \leq s_2 \implies ||s_1(x)||_{E_x} \leq ||s_2(x)||_{E_x}$ for all $x \in \Omega_X$.

(B) If $\mathbb{K} = \mathbb{C}$, then $\mathcal{M}_E^{\mathbb{R}} := \mathcal{S}_E^{\mathbb{R}} \cap \mathcal{M}_E$ is a real vector lattice where $\mathcal{M}_E^+ := \mathcal{S}_E^+ \cap \mathcal{M}_E^{\mathbb{R}}$ denotes the set of its positive elements, such that

(i) $\mathcal{M}_E^{\mathbb{R}}$ is (sequentially) closed in $\mathcal{S}_E$ in the sense that, if $\left(s_n\right)_{n\in\mathbb{N}} \in \mathcal{M}_E^{\mathbb{R}}$ is a sequence converging almost everywhere to $s \in \mathcal{S}_E$, then $s \in \mathcal{M}_E^{\mathbb{R}}$, and

(ii) $\mathcal{M}_E = \mathcal{M}_E^{\mathbb{R}} \oplus i\mathcal{M}_E^{\mathbb{R}}$ is a complex vector lattice.

To be able to state further results, we first recall some important properties of (quotient) spaces of measurable sections in the (general) case of *measurable* Banach bundle (see also [29, Example 2.4., p.24]).

Remark C.3.0.3. *Let $(E, p_E, \mathcal{M}_E)$ be a measurable Banach bundle over X. Then, the following hold.*

(i) Let $\Gamma^0(X, E) := \mathcal{M}_E/\sim$ be the space of equivalence classes of measurable sections which coincide almost everywhere, i.e., $s \sim r$ if $s(x) = r(x)$ for a.e. $x \in \Omega_X$. Under the natural operation, $\Gamma^0(X, E)$ is a vector space, and with the (lattice)-norm $||\boldsymbol{s}|| := |s| \in L^0(X)$ for some (hence for all) $s \in \boldsymbol{s}$; $\Gamma^0(X, E)$ is a LNS (lattice normed space) over $L^0(X)$. In particular $\Gamma^0(X, E)$ is a module over the commutative ring $L^0(X)$.

(ii) Let $\Gamma^\infty(X, E) := \{s \in \Gamma^0(X, E) \mid |s| \in L^\infty(X)\}$, the (quotient) space of essentially bounded measurable sections. It follows that under the natural operations, $\Gamma^\infty(X, E)$ is a Banach module over the commutative Banach algebra $L^\infty(X)$.

(iii) Let $\Gamma^1(X, E) := \{s \in \Gamma^\infty(X, E) : |s| \in L^1(X)\}$ the (quotient) space of integrable measurable sections. Then, $\Gamma^1(X, E)$ is also a Banach module over the commutative Banach algebra $L^\infty(X)$.

In what follows, as in Remark C.3.0.3 above, we identify an equivalence class of measurable section in $\Gamma^\infty(X, E)$ or $\Gamma^1(X, E)$ with its representative in $\mathcal{M}_E$, and we speak of just measurable sections if no confusion arises.

Using this Corollary C.3.0.2 and Remark C.3.0.3, we claim the following results.

Proposition C.3.0.4. *For a measurable Banach lattice bundle E over a measure space X; we have the following properties on the Banach space $\Gamma^\infty(X, E)$ of its essentially-bounded measurable sections.*

(i) If $\mathbb{K} = \mathbb{R}$, then $\Gamma^\infty(X, E)$ is an ordered Banach space with normal positive cone $\Gamma^\infty(X, E)_+ := \{s \in \Gamma^\infty(X, E) : s \in \mathcal{M}_E^+\}$.

(ii) If $\mathbb{K} = \mathbb{C}$, then $\Gamma^\infty(X, E)_\mathbb{R} := \{s \in \Gamma^\infty(X, E) : s \in \mathcal{S}_E^\mathbb{R}\}$ is an ordered closed subspace of $\Gamma^\infty(X, E)$.

In particular, $\Gamma^\infty(X, E)$ is a Banach lattice such that

(a) if $\mathbb{K} = \mathbb{R}$, then $\Gamma^\infty(X, E) = \Gamma^\infty(X, E)_+ - \Gamma^\infty(X, E)_+$ is a real Banach lattice, and

(b) if $\mathbb{K} = \mathbb{C}$, then $\Gamma^\infty(X,E) = \Gamma^\infty(X,E)_{\mathbb{R}} \oplus i\Gamma^\infty(X,E)_{\mathbb{R}}$ is a complex Banach lattice.

Proof.

(i) In the case where $\mathbb{K} = \mathbb{R}$, i.e., E_x is a real Banach lattice for each $x \in \Omega_X$, it follows from Corollary C.3.0.2 (A) that the set $\Gamma^\infty(X,E)_+$ of essentially-bounded positive measurable sections satisfies the following properties:

(a) $\Gamma^\infty(X,E)_+$ is norm-closed in $\Gamma^\infty(X,E)$,

(b) $\Gamma^\infty(X,E)_+ \cap (-\Gamma^\infty(X,E)_+) = \{0\}$,

(c) $\Gamma^\infty(X,E)_+ + \Gamma^\infty(X,E)_+ \subseteq \Gamma^\infty(X,E)_+$, and

(d) for $s_1, s_2 \in \Gamma^\infty(X,E)_+, 0 \leq s_1 \leq s_2 \implies ||s_1|| \leq ||s_2||$.

Hence, $\Gamma^\infty(X,E)_+$ is normal positive cone (see Appendix B, Proposition B.0.0.2) (by taking $\beta = 1$) of ordered Banach space $\Gamma^\infty(X,E)$. And since, $\mathcal{M}_E$ and hence $\Gamma^\infty(X,E)$ is a real vector lattice and the fact that $||s|| = ||\,|s|\,||$ for any $s \in \Gamma^\infty(X,E)$, we can conclude that $\Gamma^\infty(X,E) = \Gamma^\infty(X,E)_+ - \Gamma^\infty(X,E)_+$ is a real Banach lattice (see also Appendix B, Proposition B.0.0.5).

(ii) Similarly, in the case where $\mathbb{K} = \mathbb{C}$, i.e., E_x is a complex Banach lattice for each $x \in \Omega_X$, it follows from Corollary C.3.0.2 (B) and the argument in (i) above that the set $\Gamma^\infty(X,E)_{\mathbb{R}}$ of essentially-bounded real measurable sections is a real Banach lattice. And since, $\mathcal{M}_E$ and hence $\Gamma^\infty(X,E)$ is a complex vector lattice, we can conclude that $\Gamma^\infty(X,E) = \Gamma^\infty(X,E)_{\mathbb{R}} \oplus i\Gamma^\infty(X,E)_{\mathbb{R}}$ is a complex Banach lattice. □

The following is an immediate corollary of Proposition C.3.0.4 above.

Corollary C.3.0.5. *For a measurable Banach lattice bundle E over a measure space X, we have the following properties on the Banach space $\Gamma^1(X,E)$ of its integrable measurable sections.*

(i) If $\mathbb{K} = \mathbb{R}$, then $\Gamma^1(X,E)$ is an ordered Banach space with normal positive cone $\Gamma^1(X,E)_+ := \Gamma^\infty(X,E)_+ \cap \Gamma^1(X,E)$.

(ii) If $\mathbb{K} = \mathbb{C}$, then $\Gamma^1(X,E)_{\mathbb{R}} := \Gamma^\infty(X,E)_{\mathbb{R}} \cap \Gamma^1(X,E)$ is an ordered closed subspace of $\Gamma^1(X,E)$.

In particular, $\Gamma^1(X,E)$ is a Banach lattice such that *(a) if* $\mathbb{K} = \mathbb{R}$*, then* $\Gamma^1(X,E) = \Gamma^1(X,E)_+ - \Gamma^1(X,E)_+$ *is a real Banach lattice, and*

(b) if $\mathbb{K} = \mathbb{C}$*, then* $\Gamma^1(X,E) = \Gamma^1(X,E)_{\mathbb{R}} \oplus i\Gamma^1(X,E)_{\mathbb{R}}$ *is a complex Banach lattice.*

The following is an important observation about the vector lattice $\mathcal{S}_E$ of sections of a measurable Banach lattice bundle.

Proposition C.3.0.6. *Let* E *be a measurable Banach lattice bundle over* X*. Then the following holds for the vector lattice of its sections* $\mathcal{S}_E$*.*
For any function $f : \Omega_X \longrightarrow \mathbb{K}$*, and section* $s \in \mathcal{S}_E$*, we have that*

(i) $|fs| = |f||s|$*, and*

(ii) $\overline{fs} = \overline{f}\overline{s}$*.*

Proof. In the case where $\mathbb{K} = \mathbb{R}$, which implies E_x is a real Banach lattice for each $x \in \Omega_X$, the assertion follows since $|fs|$(x)= $|f(x)s(x)|=$ $|f|$(x)$|s|$(x) for all $x \in \Omega_X$.
Now suppose $\mathbb{K} = \mathbb{C}$, which implies E_x is a complex Banach lattice for each $x \in \Omega_X$. Here $|s| = \sup_{t\in\mathbb{Q}} \{Re\, e^{\pi it}s\}$, which is defined pointwise. And so for a function $f : \Omega_X \longrightarrow \mathbb{C}$, and $s \in S_E$ we have that

$$\begin{aligned} |fs|(x) &= \sup_{t\in\mathbb{Q}} \left\{Re\, e^{\pi it} fs(x)\right\} \\ &= \sup_{t\in\mathbb{Q}} \left\{Re\, e^{\pi it} f(x)s(x)\right\} \\ &= |f|(x) \sup_{t\in\mathbb{Q}} \left\{Re\, e^{\pi it} s(x)\right\} \\ &= |f|(x)|s|(x) \end{aligned}$$

for all $x \in \Omega_X$. And this proves assertion (i).
Moreover, $\overline{fs}(x) = \overline{f(x)s(x)} = \overline{f}(x)\overline{s}(x)$ for all $x \in \Omega_X$ proves assertion (ii). □

The following is an immediate corollary combining Propositions C.3.0.6 and C.3.0.4 with Remark C.3.0.3.

Corollary C.3.0.7. *For a measurable Banach lattice bundle E over a measure space X; we have the following properties on the Banach lattices $\Gamma^\infty(X,E)$ and $\Gamma^1(X,E)$ of its essentially-bounded and integrable measurable sections respectively.*

(i) If $\mathbb{K}=\mathbb{R}$, then $\Gamma^\infty(X,E)$ and hence $\Gamma^1(X,E)$ is a real Banach lattice module over $L^\infty(X)$.

(ii) If $\mathbb{K}=\mathbb{C}$, then $\Gamma^\infty(X,E)$ and hence $\Gamma^1(X,E)$ is a complex Banach lattice module over $L^\infty(X)$.

In particular, $\Gamma^\infty(X,E)$ and hence $\Gamma^1(X,E)$ is an m-Banach lattice module over $L^\infty(X)$.

In the following proposition, we state several results about a *separable* measurable Banach lattice bundle. Moreover, if the measure space $X=(\Omega_X,\Sigma_X,\mu_X)$ is separable, then the (quotient) lattice of its integrable measurable sections is, in particular, a separable Banach lattice. This is essentially due to [29, Lemma 2.48, p.46 and Lemma 4.46, p.45] in the case of *separable* measurable Banach bundle.

Proposition C.3.0.8. *For a separable measurable Banach lattice bundle E over a measure space X, let $(s_n)_{n\in\mathbb{N}}\in\mathcal{M}_E$ be sequence such that lin$\{s_n(x)\mid n\in\mathbb{N}\}$ is dense in E_x for almost every $x\in\Omega_X$. Then the following hold.*

(i) There exists a sequence $(s_n^+)_{n\in\mathbb{N}}\in\mathcal{M}_E^+$ such that lin$\{s_n^+\mid n\in\mathbb{N}\}$ generates $E_+:=\left\{v\in E: v\in|v|=v_{E_{p_E(v)}}\right\}$, i.e., every $s^+\in\mathcal{M}_E^+$ is almost everywhere limit of a sequence in lin$\{\mathbb{1}_A s_n^+\mid A\in\Sigma_X, n\in\mathbb{N}\}$.

(ii) There exists a sequence $(s_n^+)_{n\in\mathbb{N}}\in\mathcal{M}_E^+$ such that

(a) lin$\{s_n^+(x)\mid n\in\mathbb{N}\}$ is dense in the positive cone E_x^+ for almost every $x\in\Omega_X$,

(b) $\mu_X(\{|s_n^+|\neq 0\})<\infty$ for every $n\in\mathbb{N}$, and

(c) $|s_n^+|=\mathbb{1}_{\{|s_n^+|\neq 0\}}$ almost everywhere for every $n\in\mathbb{N}$.

(iii) If $(s_n^+)_{n\in\mathbb{N}}\in\mathcal{M}_E^+$ is a sequence satisfying conditions (ii) (a) and (b) above then

$$lin\left\{\mathbb{1}_A s_n^+\mid A\in\Sigma_X,\ n\in\mathbb{N}\right\}\subseteq\Gamma^1(X,E)_+$$

is dense in $\Gamma^1(X,E)_+$.

(iv) If X is a separable measure space, i.e., there is a sequence $(A_n)_{n\in\mathbb{N}}$ of measurable subsets of Ω_X such that for every $B \in \Sigma_X$ and $\varepsilon > 0$ there is an $n \in \mathbb{N}$ with $\mu_X(A_n \Delta B) < \varepsilon$; then $\Gamma^1(X, E)_+$ is separable. In particular, $\Gamma^1(X, E)$ is a separable Banach lattice.

Proof. As the vector space $\operatorname{lin}\{s_n(x) \mid n \in \mathbb{N}\}$ is dense in E_x for almost every $x \in \Omega_X$, we may assume that it is a vector sublattice for almost every $x \in \Omega_X$, i.e., $v \in \operatorname{lin}\{s_n(x) \mid n \in \mathbb{N}\}$ implies that $|v|, \overline{v} \in \operatorname{lin}\{s_n(x) \mid n \in \mathbb{N}\}$ for almost every $x \in \Omega_X$.

(i) Since E is, in particular, a separable Banach bundle over X, by [29, Lemma 2.47, p.45], it follows that $\operatorname{lin}\{s_n \mid n \in \mathbb{N}\}$ generates E, i.e., every $s \in \mathcal{M}_E$ is almost everywhere limit of a sequence in $\operatorname{lin}\{\mathbb{1}_A s_n \mid A \in \Sigma_X, n \in \mathbb{N}\}$. Now, we may also assume WLOG that $\operatorname{lin}\{s_n \mid n \in \mathbb{N}\} \subseteq \mathcal{M}_E$ is a vector sublattice, i.e., $r \in \operatorname{lin}\{s_n \mid n \in \mathbb{N}\}$ implies that $|r|, \overline{r} \in \operatorname{lin}\{s_n \mid n \in \mathbb{N}\}$. And as such, we can set $s_n^+ := |s_n|$ for each $n \in \mathbb{N}$. And observing that, for each $s \in \mathcal{M}_E$, if

$$(r_n)_{n\in\mathbb{N}} \in \operatorname{lin}\{\mathbb{1}_A s_n \mid A \in \Sigma_X, n \in \mathbb{N}\}$$

is a sequence converging almost everywhere to $s \in \mathcal{M}_E$, then

$$(|r_n|)_{n\in\mathbb{N}} \in \operatorname{lin}\{\mathbb{1}_A s_n^+ \mid A \in \Sigma_X, n \in \mathbb{N}\} \subseteq \mathcal{M}_E^+$$

is a sequence converging almost everywhere to $|s| \in \mathcal{M}_E^+$ proves the assertion.

(ii) Similarly, since E is also a separable Banach bundle over X, we can choose a sequence $(s_n)_{n\in\mathbb{N}} \in \mathcal{M}_E$ as in [29, Lemma 2.48, p.46]. And as in (i) above, setting $s_n^+ := |s_n|$ for each $n \in \mathbb{N}$, we may also assume that $\operatorname{lin}\{s_n^+(x) \mid n \in \mathbb{N}\} \subseteq \mathcal{M}_E^+$ is a vector sublattice, and thus the assertions [(a) to (c)] follows again by [29, Lemma 2.48, p.46][(i) to (iii)] .

(iii) With the same consideration as in (ii) above, if $(s_n^+)_{n\in\mathbb{N}} \in \mathcal{M}_E^+$ is a sequence satisfying conditions (ii) (a) and (b) it satisfies conditions (i) and (ii) of [29, Lemma 2.48, p. 46] and again realising $\operatorname{lin}\{s_n^+(x) \mid n \in \mathbb{N}\} \subseteq \mathcal{M}_E^+$ as a vector sublattice implies that

$$\operatorname{lin}\{\mathbb{1}_A s_n^+ \mid A \in \Sigma_X,\ n \in \mathbb{N}\} \subseteq \Gamma^1(X, E)_+$$

is also a vector sublattice which is dense in $\Gamma^1(X, E)_+$.

(iv) Let X be a separable measure space. By choosing a sequence $(s_n^+)_{n\in\mathbb{N}} \in \mathcal{M}_E^+$ satisfying conditions (ii) (a) and (b) from above, and by realising $\{\mathbb{1}_{A_m}s_n^+ \mid n,m \in \mathbb{N}\} \subseteq \Gamma^1(X,E)_+$ as a vector sublattice, it follows from the proof of [29, Lemma 2.46, p. 47] that

$$\{\mathbb{1}_{A_m}s_n^+ \mid n,m \in \mathbb{N}\} \subseteq \Gamma^1(X,E)_+$$

is total in $\Gamma^1(X,E)_+$.

Now, if $\mathbb{K} = \mathbb{R}$, it immediately follows that $\Gamma^1(X,E)$ is a separable Banach lattice.

In the case $\mathbb{K} = \mathbb{C}$, it follows from the above consideration that if the vector lattice $\operatorname{lin}\{s_n \mid n \in \mathbb{N}\}$ generates E, then by setting $s_n^r := Re\ s_n$ for each $n \in \mathbb{N}$ we obtain that $\operatorname{lin}\{s_n^r \mid n \in \mathbb{N}\}$ generates $E_{\mathbb{R}} := \left\{v \in E : Re\ v = v_{E_{p_E(v)}}\right\}$, i.e., every $s^r \in \mathcal{M}_E^{\mathbb{R}}$ is almost everywhere limit of a sequence in $\operatorname{lin}\{\mathbb{1}_A s_n^r \mid A \in \Sigma_X, n \in \mathbb{N}\}$. Moreover, if we choose $(s_n)_{n\in\mathbb{N}} \in \mathcal{M}_E$ as in [29, Lemma 2.48, p.46], setting $s_n^r := Re\ s_n$ for each $n \in \mathbb{N}$ implies that $(s_n^r)_{n\in\mathbb{N}} \in \mathcal{M}_E^{\mathbb{R}}$ is a sequence satisfying again conditions (i) and (ii) of [29, Lemma 2.48, p.46] and so we obtain that

$$\operatorname{lin}\{\mathbb{1}_A s_n^r \mid A \in \Sigma_X,\ n \in \mathbb{N}\} \subseteq \Gamma^1(X,E)_{\mathbb{R}}$$

is also a vector sublattice which is dense in $\Gamma^1(X,E)_{\mathbb{R}}$. And if X is a separable measure space, we also obtain, again by the proof of [29, Lemma 2.46, p.47], that

$$\{\mathbb{1}_{A_m}s_n^r \mid n,m \in \mathbb{N}\} \subseteq \Gamma^1(X,E)_{\mathbb{R}}$$

is total in $\Gamma^1(X,E)_{\mathbb{R}}$.

Thus, $\Gamma^1(X,E) = \Gamma^1(X,E)_{\mathbb{R}} \oplus i\Gamma^1(X,E)_{\mathbb{R}}$ is a separable Banach lattice.

□

C.4 Morphisms of *measurable* Banach lattice bundles

Here, we introduce the concept of (positive) semiflows on measurable Banach lattice bundles and their dynamics. We essentially follow the work of

S. Siewert ([29]) for the case of measurable Banach bundles (see [29, Section 1.2., p.17-21]). We note that, in this Section, we establish the category of measurable Banach lattice bundles and their dynamics.

Before introducing dynamics on measurable Banach lattice bundles, we recall the definitions of a morphism and an automorphism of measure spaces.

Definition C.4.0.1. *(Morphism of Measure spaces)*
Let X and Y be two measure spaces; a premorphism $\varphi : X \longrightarrow Y$ is a measurable and measure-preserving mapping $\varphi : \Omega_X \longrightarrow \Omega_Y$. And for any two premorphisms φ and ψ, setting $\varphi \sim \psi$ if $\varphi(x) = \psi(x)$ for almost every $x \in \Omega_X$ defines an equivalence relation on the set of premorphisms from X to Y. An equivalence class $[\varphi]$ is called a morphism from X to Y, which will be denoted by φ again.

If $\varphi : X \longrightarrow Y$ is a morphism such that $\varphi^{-1} : Y \longrightarrow X$ is also a morphism with $\varphi \circ \varphi^{-1} = Id_{\Omega_Y}$ and $\varphi^{-1} \circ \varphi = Id_{\Omega_X}$, then φ is called an isomorphism. The set of automorphisms (i.e., self-isomorphisms) from X to X will be denoted as $Aut(X)$.

Definition C.4.0.2. *(Positive morphism of Measurable Banach lattice bundle)*
Let $\varphi : X \longrightarrow X$ be a morphism on a measure space X; while $(E, p_E, \mathcal{M}_E)$ and $(F, p_F, \mathcal{M}_F)$ are measurable Banach lattice bundles over X. A positive premorphism Φ from E to F over φ is a mapping $\Phi : E \longrightarrow F$ such that

(i) $\Phi \circ \mathcal{M}_E \subseteq \mathcal{M}_F \circ \varphi$,

(ii) $p_F \circ \Phi = \varphi \circ p_E$ *almost everywhere, i.e., the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ {\scriptstyle p_E}\big\downarrow & & \big\downarrow{\scriptstyle p_F} \\ \Omega_X & \xrightarrow{\varphi} & \Omega_X \end{array}$$

commutes almost everywhere,

(iii) $\Phi_x := \Phi|_{E_x} : E_x \longrightarrow F_{\varphi(x)}$ *is a positive operator for almost every* $x \in \Omega_X$, *and*

(iv) Φ *is essentially bounded, i.e.,* $||\Phi|| := \text{ess}\sup_{x\in\Omega_X} ||\Phi_x||_{\mathscr{L}(E_x, F_{\varphi(x)})} < \infty$.

As in the case of morphism of measurable Banach bundle, we also identify positive premorphisms that agree up to a null set.
So, let $pPremor_{\varphi}(E,F) := \{\Phi : E \longrightarrow F$ *is a positive premorphism over* $\varphi\}$, *and* $\mathcal{N}_{\varphi}(E,F) := \Big\{\Phi \in pPremor_{\varphi}(E,F) \,|\, \Phi = 0$ *almost everywhere*$\Big\}$,
An equivalence class $[\Phi] \in Mor_{\varphi}(E,F) := pPremor_{\varphi}(E,F)/\mathcal{N}_{\varphi}(E,F)$ *is called a positive morphism from E to F over* φ, *which we denote as* Φ *again.*

Remark C.4.0.3. *From above, as in the case of topological Banach lattice bundles (see Chapter 3, Remark 3.4.0.3), we also identify the following additional properties of a positive morphism* Φ *of measurable Banach lattice bundles over morphism* φ *on* X.

(i) *If* Φ_x *is an isometry for almost every* $x \in \Omega_X$, *then we call* Φ *a positive isometry.*

(ii) *If* Φ_x *is an isometric lattice homomorphism for almost every* $x \in \Omega_X$, *then we call* Φ *an isometry.*

(iii) *If* Φ_x *is a lattice homomorphism for almost every* $x \in \Omega_X$, *we call* Φ *a morphism of measurable Banach lattice bundle over* φ.

(iv) *If* $\varphi = Id_{\Omega_X}$, *we call* Φ *a positive morphism of measurable Banach lattice bundle.*

(v) *If* $\varphi = Id_{\Omega_X}$, *and* Φ_x *is a lattice homomorphism for almost every* $x \in \Omega_X$, *we call* Φ *a morphism of measurable Banach lattice bundle.*

Next, we introduce positive dynamical *measurable* Banach lattice bundles over a measure-preserving dynamical system induced by groups.
From now on, by a Banach lattice bundle over X, we always mean a measurable Banach lattice bundle over a measure space X. And by a (positive) morphism over a morphism φ on X, we always mean a (positive) morphism of measurable Banach lattice bundle over φ, if no confusion arises.

Note C.4.0.4. *In the remainder of this appendix, we will use the following notation.*

(i) *We let G be a locally compact group, and S a closed subsemigroup of G containing the neutral element e, i.e., a closed "submonoid" of G. For instance, we can take* $G = \mathbb{R}$, $S = \mathbb{R}_+$ *or* $G = \mathbb{Z}$, $S = \mathbb{N}_0$.

(ii) *We let (X, φ) be a measure-preserving G-dynamical system over a measure space X, i.e., $\varphi = (\varphi_g)_{g\in G}$ defines a group action*[2] *on X, called a (measurable) flow on X.*

Definition C.4.0.5. *A positive S- dynamical Banach lattice bundle over the measure-preserving G-dynamical system (X, φ) is a pair $(E, \mathbf{\Phi})$ of a measurable Banach lattice bundle E over X and a monoid representation*[3]

$$\mathbf{\Phi} : S \longrightarrow E^E,\ g \mapsto \Phi_g$$

such that
$\Phi_g : E \longrightarrow E$ is a positive morphism over φ_g for each $g \in S$.

We call $\mathbf{\Phi} = (\Phi_g)_{g\in S}$ a positive semiflow on E over the flow $(\varphi_g)_{g\in G}$ on X. If $S = G$, then we call $\mathbf{\Phi} = (\Phi_g)_{g\in S}$ a positive flow on E over the flow $(\varphi_g)_{g\in G}$ on X, and $(E, \mathbf{\Phi})$ a positive G-dynamical Banach lattice bundle over (X, φ).

And if E is, in addition, separable, then $(E, \mathbf{\Phi})$ is called separable.

Remark C.4.0.6. *From Definition C.4.0.5 above, using Remark C.4.0.3, we also identify the following additional properties of a positive S-dynamical Banach lattice bundle $(E, \mathbf{\Phi})$ over (X, φ).*

(i) *If Φ_g is a positive isometry for each $g \in S$, we call $\mathbf{\Phi}$ a positive isometry.*

(ii) *If Φ_g is an isometry for each $g \in S$, we call $\mathbf{\Phi}$ an isometry.*

(iii) *If Φ_g is a morphism over φ_g for each $g \in S$, we call $(E, \mathbf{\Phi})$ an S-dynamical Banach lattice bundle over (X, φ); and $\mathbf{\Phi} = (\Phi_g)_{g\in S}$ a semiflow on E over the flow $(\varphi_g)_{g\in G}$ on X.*

Now, we introduce instances of (positive) S-dynamical *measurable* Banach lattice bundles on Examples C.2.0.4 (i) and (ii). See [29, Example 1.19, p.21] for the case of dynamical *measurable* Banach bundles.

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[2] i.e., $\varphi : G \longrightarrow Aut(X)$; $g \mapsto \varphi_g$ is a group homomorphism, where $Aut(X)$ denote the group of automorphisms on X.

[3] $\Phi_{gh} = \Phi_h \circ \Phi_g, \forall g, h \in S$ and $\Phi_e = Id_E$, the identity on E^E.

Example C.4.0.7. *(i) Let X be a measure space, Z a Banach lattice and $E = \Omega_X \times Z$ the trivial Banach lattice bundle with fiber Z. Moreover, let $\varphi = (\varphi_g)_{g\in G}$ be a (measurable) flow on X. Then a positive S-dynamical measurable Banach lattice bundle $(E, \mathbf{\Phi})$ corresponds to a positive measurable cocycle, i.e., a mapping*

$$\boldsymbol{\phi} : S \times \Omega_X \longrightarrow Z^Z;\ (g, x) \mapsto \boldsymbol{\phi}(g, x) := \phi_g(x)$$

such that

(a) $\phi_g(x) : Z \longrightarrow Z$ is a positive operator for almost every $x \in \Omega_X$ and for all $g \in S$,

(b) $\phi_{gh}(x) = \phi_g(\varphi_h(x)) \circ \phi_h(x)$ for almost every $x \in \Omega_X$ and for all $g, h \in S$,

(c) $\phi_e(x) = Id_Z$ for almost every $x \in \Omega_X$,

(d) for every $g \in S$, the mapping $\Omega_X \longrightarrow Z; x \mapsto \phi_g(x)v$ is strongly measurable for all $v \in Z$, and

(e) for every $g \in S$, $||\phi_g|| :=$ ess $\sup_{x\in\Omega_X} ||\phi_g(x)||_Z < \infty$.

In this situation, we may identify $\Phi_g(x, v) = (\varphi_g(x), \phi_g(x)v)$ for all $g \in S$ and $(x, v) \in \Omega_X \times Z = E$.

(ii) Let G be a (discrete) group, $(E, \mathbf{\Phi})$ a topological S-dynamical Banach lattice bundle over a topological G-dynamical system (Ω, φ) and μ be a σ-finite regular Borel measure on Ω. Moreover, let E_μ be the induced measurable Banach lattice bundle. Then $(E_\mu, \mathbf{\Phi})$ is an S-dynamical measurable Banach lattice bundle over the measure-preserving G-dynamical system induced by (Ω, φ).

Next, we introduce morphism between positive S-dynamical *measurable* Banach lattice bundles over a G-dynamical *measure-preserving* system (X, φ).

Definition C.4.0.8. *A morphism from a positive S-dynamical measurable Banach lattice bundle $(E, \mathbf{\Phi})$ over (X, φ) to a positive S-dynamical measurable Banach lattice bundle $(F, \mathbf{\Psi})$ over (X, φ) is a morphism (see Remark C.4.0.3 (v))*

$$\Theta : E \longrightarrow F$$

such that the following diagram

$$\begin{tikzcd} E \arrow[r, "\Theta"] \arrow[d, "\Phi_g"'] & F \arrow[d, "\Psi_g"] \\ E \arrow[r, "\Theta"] & F \end{tikzcd}$$

commutes for each $g \in S$, i.e., $\Theta \circ \Phi_g = \Psi_g \circ \Theta$ for each $g \in S$. It is called a positive isometry if Θ is a positive isometry.

Just as the case in the topological setting (see Chapter 3, Remark 3.4.0.10), we also note the following.

Remark C.4.0.9. *(i) In a similar manner, a morphism between S-dynamical measurable Banach lattice bundles over (X, φ) can be defined. It will be called an isometry if Θ is an isometry.*

(ii) In the category of positive S-dynamical measurable Banach lattice bundles over (X, φ); an isometric morphism is just a positive isometry. We choose this terminology and reserve the word "isometry" for S-dynamical measurable Banach lattice bundles over (X, φ).

(iii) Furthermore, two positive S-dynamical measurable Banach lattice bundles $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$ over (X, φ) are said to be isomorphic if there exists a surjective positive isometry (i.e., a surjective isometric morphism) between them. In this situation, we write $\mathbf{\Phi} = (\Phi_g)_{g \in S} \cong (\Psi_g)_{g \in S} = \mathbf{\Psi}$ on $E \cong F$.

C.5 On a representation of the space of integrable measurable sections

Throughout this Section, by a Banach lattice bundle E over X, we mean a *measurable* Banach lattice bundle $p_E : E \longrightarrow \Omega_X$ over a measure space X (see Definition C.2.0.1). And by a (positive) morphism over a morphism φ on X, we mean a (positive) morphism of measurable Banach lattice bundle over φ, if no confusion arises (see Definition C.4.0.2 and Remark C.4.0.3).

Remark C.5.0.1. *(i) Let E be a Banach lattice bundle over a measure space X. Then the (quotient) lattices $\Gamma^\infty(X, E)$ and $\Gamma^1(X, E)$ of its essentially bounded and integrable measurable sections are, respectively, m-Banach lattice modules over commutative Banach lattice algebra $L^\infty(X)$ (see Corollary C.3.0.7).*

(ii) In particular, if E is a Banach lattice bundle over a measure space X, then the m-Banach lattice module $\Gamma^1(X, E)$ is an $L^1(X)$-normed m-lattice module over $L^\infty(X)$ (see Chapter 2, Definition 2.4.2.15 and Example 2.4.2.16). And if E is separable over a separable measure space X, then by Proposition C.3.0.8(iv), $\Gamma^1(X, E)$ is, in addition, a separable Banach lattice.

(iii) Let E be a Banach lattice bundle over a measure space X. With a (positive) morphism $\Phi : E \longrightarrow E$ of Banach lattice bundle over an automorphism $\varphi : X \longrightarrow X$; we associate a map

$$\mathcal{T}_\Phi : \Gamma^1(X, E) \longrightarrow \Gamma^1(X, E);\ s \mapsto \Phi \circ s \circ \varphi^{-1}$$

called the (positive) weighted Koopman operator over the Koopman operator $T_\varphi : L^\infty(X) \longrightarrow L^\infty(X); f \mapsto f \circ \varphi^{-1}$ induced by Φ. It immediately follows that the (positive) weighted Koopman operator $\mathcal{T}_\Phi$ is a (positive) lattice T_φ-homomorphism (see Chapter 2, Definition 2.3.2.3 and Example 2.3.2.4(ii)).

(iv) And so, if G is a discrete group, a pair $(E, \boldsymbol{\Phi})$ of (positive) S-dynamical Banach lattice bundle over a measure-preserving G-dynamical system (X, φ) (see Definition C.4.0.5) induces a (positive) S-dynamical m-Banach lattice module $(\Gamma^1(X, E), \boldsymbol{\mathcal{T}_\Phi})$ over the Koopman group representation $(L^\infty(X), \boldsymbol{T_\varphi})$ (see also Chapter 2, Example 2.3.4.3 (ii)). And we call $\boldsymbol{\mathcal{T}_\Phi} = \mathcal{T}_\Phi(g)_{g\in S}$ the (positive) weighted Koopman semigroup representation on $\Gamma^1(X, E)$ over $\boldsymbol{T_\varphi} = (T_\varphi(g))_{g\in G}$ induced by $(E, \boldsymbol{\Phi})$.

The following lemma will be useful, which gives a certain characterisation of the weighted Koopman operator on the space of integrable sections over the Koopman operator T_φ on $L^\infty(X)$ in the (general) case of measurable Banach bundle over a measure space X (see [29, Proposition 2.49, p.47]).

Lemma C.5.0.2. *Let X be a measure space and $\varphi : X \longrightarrow X$ an automorphism. Moreover, let E and F be measurable Banach bundles over X, and $\mathcal{T} : \Gamma^1(X, E) \longrightarrow \Gamma^1(X, F)$ be a bounded operator. If E is separable, the following are equivalent.*

(a) $\mathcal{T}$ is a T_φ-homomorphism, i.e., $\mathcal{T} fs = T_\varphi f \cdot \mathcal{T} s$ for all $f \in L^\infty(X)$ and $s \in \Gamma^1(X, E)$.

(b) $|\mathcal{T}s| \leq ||\mathcal{T}|| \cdot T_\varphi |s|$ *for every* $s \in \Gamma^1(X, E)$.

(c) There exists a unique measurable bundle morphism $\Phi : E \longrightarrow F$ *over* φ *such that* $\mathcal{T} = \mathcal{T}_\Phi$.

Moreover, if this assertion holds we have the following.

(i) $|\Phi| : \Omega_X \longrightarrow [0, \infty);\ x \mapsto ||\Phi_x||$ *defines an element of* $L^\infty(X)$.

(ii) $\sup\{|\mathcal{T}_\Phi s| : s \in \Gamma^\infty(X, E) \text{ with } |s| \leq \mathbb{1}\} = T_\varphi |\Phi| \in L^\infty(X)$.

(iii) $||\Phi|| = ||\mathcal{T}_\Phi||_{\Gamma^\infty(X,E)} = ||\mathcal{T}_\Phi||_{\Gamma^1(X,E)}$.

(iv) Φ *is an isometry if and only if* $\mathcal{T}_\Phi$ *is an isometry.*

Using Lemma C.5.0.2 above, we, in particular, claim the following result in our setting.

Proposition C.5.0.3. *Let* X *be a measure space and* $\varphi : X \longrightarrow X$ *an automorphism. Moreover, let* E *and* F *be measurable Banach lattice bundles over* X*, and* $\mathcal{T} : \Gamma^1(X, E) \longrightarrow \Gamma^1(X, F)$ *a lattice homomorphism. If* E *is separable, the following are equivalent.*

(i) $\mathcal{T}$ *is a lattice* T_φ*-homomorphism, i.e.,* $\mathcal{T} fs = T_\varphi f \cdot \mathcal{T}s$ *for all* $f \in L^\infty(X)$ *and* $s \in \Gamma^1(X, E)$.

(ii) There exists a unique morphism $\Phi : E \longrightarrow F$ *of Banach lattice bundles over* φ *such that* $\mathcal{T} = \mathcal{T}_\Phi$.

Moreover, $||\Phi|| = ||\mathcal{T}||$ *and* Φ *is an isometry if and only if* $\mathcal{T}$ *is an isometry.*

Proof. That $(ii) \implies (i)$ immediately follows from definition. Indeed, for all $f \in L^\infty(X)$ and $s \in \Gamma^1(X, E)$, we have that

$$\begin{aligned}
\mathcal{T}_\Phi fs &= \Phi \circ fs \circ \varphi^{-1} \\
&= \Phi \circ [f \circ \varphi^{-1} \cdot s \circ \varphi^{-1}] \\
&= f \circ \varphi^{-1} \cdot \Phi \circ s \circ \varphi^{-1} \\
&= T_\varphi f \cdot \mathcal{T}_\Phi s.
\end{aligned}$$

$(i) \implies (ii)$: Since $\mathcal{T} : \Gamma^1(X, E) \longrightarrow \Gamma^1(X, F)$ is, in particular, a bounded operator, by Lemma C.5.0.2 we obtain a unique morphism $\Phi : E \longrightarrow F$ of measurable Banach bundles over φ such that $\mathcal{T} = \mathcal{T}_\Phi$.

We claim that $\Phi : E \longrightarrow F$ is the unique morphism of Banach lattice bundles over φ.

To this end, we follow the proof of Lemma C.5.0.2[$(b) \implies (c)$] as in [29, Proposition 2.49, p.47] by which Φ is obtained.

(a) Since E is a separable Banach lattice bundle over X, if $(s_n)_{n\in\mathbb{N}} \in \mathcal{M}_E$ is a sequence satisfying [29, Lemma 2.48, p.46], by our consideration as in the proof of Proposition C.3.0.8, we obtain that the Q-vector space

$$H_x := \operatorname{lin}_Q \{s_k(x) \mid k \in \mathbb{N}\} \subseteq E_x$$

is a Q-vector lattice for each $x \in \Omega_X$, where $Q = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ or $Q = \mathbb{Q} + i\mathbb{Q}$ if $\mathbb{K} = \mathbb{C}$, i.e., for each $x \in \Omega_X$, $v \in H_x$ implies $|v|$, $\overline{v} \in H_x$.

(b) And since $\mathcal{T}$ is a lattice homomorphism, if $(r_n)_{n\in\mathbb{N}} \in \mathcal{M}_F$ is a representative of $(\mathcal{T}s_n)_{n\in\mathbb{N}} \in \Gamma^1(X, F)$, then $(|r_n|)_{n\in\mathbb{N}} \in \mathcal{M}_F$ can be taken as a representative of $(\mathcal{T}|s_n|)_{n\in\mathbb{N}} \in \Gamma^1(X, F)$. And so the Q-linear map $\Phi_x : H_x \longrightarrow F_{\varphi(x)}$ given by

$$\Phi_x(s_n(x)) := (r_n)(\varphi(x))$$

for every $n \in \mathbb{N}$ is a Q-vector lattice homomorphism for every $x \in \Omega_X$. Indeed, for each $x \in \Omega_X$, we have that

$$\begin{aligned} |\Phi_x(s_n(x))| &= |(r_n)(\varphi(x))| \\ &= (|r_n|)(\varphi(x)) \\ &= \Phi_x|(s_n(x))| \end{aligned}$$

for every $n \in \mathbb{N}$. And this implies that for each $x \in \Omega_X$, $|\Phi_x v| = \Phi_x|v|$ for every $v \in H_x$.

(c) And so, the unique extension of $\Phi_x : H_x \longrightarrow F_{\varphi(x)}$ to $\Phi_x : E_x \longrightarrow F_{\varphi(x)}$ is a lattice homomorphism for almost every $x \in \Omega_X$.

And since $\Phi_x := 0 : E_x \longrightarrow F_{\varphi(x)}$, which is trivially a lattice homomorphism, was set for the remaining points $x \in \Omega_X$, we can conclude that, the obtained mapping

$$\Phi : E \longrightarrow F;\ v \mapsto \Phi_{E_{p_E(v)}} v$$

is the unique morphism of measurable Banach lattice bundles over φ.

Moreover, by Lemma C.5.0.2, we have that $||\Phi|| = ||\mathcal{T}_\Phi|| = ||\mathcal{T}||$ and so Φ is an isometry if and only if $\mathcal{T}$ is an isometry.

□

The following is an immediate corollary of Proposition C.5.0.3 above.

Corollary C.5.0.4. *Let X be a measure space, $\varphi : X \longrightarrow X$ an automorphism. Moreover, let E and F be measurable Banach lattice bundles over X, and $\mathcal{T} : \Gamma^1(X,E) \longrightarrow \Gamma^1(X,F)$ be a linear operator. If E is separable, the following are equivalent.*

(i) $\mathcal{T}$ is a positive T_φ-homomorphism, i.e., $|\mathcal{T}s| \leq \mathcal{T}|s|$ and $\mathcal{T}fs = T_\varphi f \cdot \mathcal{T}s$ for all $f \in L^\infty(X)$ and $s \in \Gamma^1(X,E)$.

(ii) There exists a unique positive morphism $\Phi : E \longrightarrow F$ of Banach lattice bundles over φ such that $\mathcal{T} = \mathcal{T}_\Phi$.

Moreover, $||\Phi|| = ||\mathcal{T}||$ and Φ is a positive isometry if and only if $\mathcal{T}$ is a positive isometry.

The following proposition is a consequence of Proposition C.5.0.3, by which we represent every (positive) S-dynamical m-Banach lattice module on $\Gamma^1(X,E)$ over the Koopman group $(L^\infty(X), \boldsymbol{T}_\varphi)$ as (positive) weighted Koopman semigroup representation induced by a unique (positive) S-dynamical measurable Banach lattice bundle over a *measure-preserving* G-dynamical system (X,φ). See [29, Corollary 2.50, p.51] for the case of measurable Banach bundle.

Proposition C.5.0.5. *Let G be a (discrete) group, $S \subseteq G$ a submonoid, and (X,φ) a measure-preserving G-dynamical system. Moreover, let E be a separable measurable Banach lattice bundle over X, and let*

$$\boldsymbol{\mathcal{T}} : S \longrightarrow \Gamma^1(X,E)^{\Gamma^1(X,E)}; g \mapsto \mathcal{T}(g)$$

be a strongly continuous representation such that $(\Gamma^1(X,E), \boldsymbol{\mathcal{T}})$ is an (positive) S-dynamical m-Banach lattice module over $(L^\infty(X), \boldsymbol{T}_\varphi)$. Then, there is a unique (positive) S-dynamical Banach lattice bundle $(E, \boldsymbol{\Phi})$ over (X,φ) such that $\mathcal{T}_{\boldsymbol{\Phi}} = \boldsymbol{\mathcal{T}}$.
Moreover, $||\mathcal{T}(g)|| = ||\Phi_g||$ for each $g \in S$ and so, $\boldsymbol{\mathcal{T}} = \mathcal{T}(g)_{g\in S}$ is an (positive) isometry if and only if $\boldsymbol{\Phi} = (\Phi_g)_{g\in S}$ is an (positive) isometry.

Proof. We follow the proof of [29, Corollary 2.50, p.51] in the case of measurable Banach bundle. And since every (positive) S-dynamical m-Banach lattice module on $\Gamma^1(X,E)$ over $(L^\infty(X), \boldsymbol{T}_\varphi)$ defines S-dynamical Banach module over $(L^\infty(X), \boldsymbol{T}_\varphi)$, we obtain a unique S-dynamical Banach bundle $(E, \boldsymbol{\Phi})$ over (X, φ) such that $\mathcal{T}_{\boldsymbol{\Phi}} = \boldsymbol{\mathcal{T}}$ in the sense of dynamical Banach module (see [29, Definition 2.12, p. 28]).

We claim that $(E, \boldsymbol{\Phi})$ is the unique (positive) S-dynamical Banach lattice bundle over (X, φ).
In particular :

(i) Φ_g is a (positive) Banach lattice bundle morphism over φ_g for each $g \in S$. Indeed, for each $g \in S$, $\mathcal{T}(g) : \Gamma^1(X,E) \longrightarrow \Gamma^1(X,E)$ is a (positive) lattice $T_\varphi(g)$-homomorphism; and by Proposition C.5.0.3(CorollaryC.5.0.4), it follows that, $\Phi_g : E \longrightarrow E$ is the unique (positive) Banach lattice bundle morphism over φ_g for each $g \in S$.

(ii) $\boldsymbol{\mathcal{T}}$ is an (positive) isometry if and only if $\boldsymbol{\Phi}$ is an (positive) isometry. Indeed, also by Proposition C.5.0.3(CorollaryC.5.0.4) we have that

$$||\mathcal{T}(g)|| = ||\Phi_g||$$

and so for each $g \in S$, $\mathcal{T}(g)$ is an (positive) isometry if and only if Φ_g is an (positive) isometry.

□

As noted in Remark C.5.0.1(iv) and also from Proposition C.5.0.5 above, if G is a discrete group, each separable (positive) S-dynamical Banach lattice bundle $(E, \boldsymbol{\Phi})$ over a *measure-preserving* G-dynamical system (X, φ) induces a (positive) S-dynamical m-Banach lattice module $(\Gamma^1(X,E), \mathcal{T}_{\boldsymbol{\Phi}})$ via

$$\mathcal{T}_{\boldsymbol{\Phi}} : S \longrightarrow \mathscr{L}(\Gamma^1(X,E)), g \mapsto \mathcal{T}_\Phi(g) := \left[s \mapsto \Phi_g \circ s \circ \varphi_{g^{-1}}\right]$$

over $(L^\infty(X), \boldsymbol{T}_\varphi)$, the Koopman group representation on $L^\infty(X)$. And we call $(\mathcal{T}_\Phi(g))_{g \in S}$ a (positive) weighted Koopman semigroup representation on $\Gamma^1(X,E)$ over the Koopman group representation $(T_\varphi(g))_{g \in G}$ on $L^\infty(X)$.

The following lemma shows that, for a fixed *measure-preserving* G-dynamical system (X, φ), a morphism of separable (positive) S-dynamical Banach lattice bundles (see Definition C.4.0.8) is uniquely determined by a homomorphism of the corresponding induced (positive) S-dynamical m-Banach lattice modules (see Chapter 2, Definition 2.3.5.1).

Lemma C.5.0.6. *Let G be a (discrete) group, $S \subseteq G$ a submonoid, and (X, φ) a measure-preserving G-dynamical system. Moreover, let $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$ be separable S-dynamical Banach lattice bundles over (X, φ). Furthermore, let $(\Gamma^1(X, E), \mathcal{T}_{\mathbf{\Phi}})$ and $(\Gamma^1(X, F), \mathcal{T}_{\mathbf{\Psi}})$ be S-dynamical m-Banach lattice modules over the Koopman group $(L^\infty(X), \boldsymbol{T}_\varphi)$ induced by $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$, respectively.*
Then, we have the following.

(A) For a mapping $\Theta : E \longrightarrow F$ the following are equivalent.

(i) $\Theta : E \longrightarrow F$ is a morphism between $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$.

(ii) $V_\Theta : \Gamma^1(X, E) \longrightarrow \Gamma^1(X, F); s \mapsto \Theta \circ s$ is an homomorphism between $(\Gamma^1(X, E), \mathcal{T}_{\mathbf{\Phi}})$ and $(\Gamma^1(X, F), \mathcal{T}_{\mathbf{\Psi}})$.

Moreover, if this assertion holds, $||\Theta|| = ||V_\Theta||$, and Θ is an isometry if and only if V_Θ is an isometry.

(B) For a mapping $V : \Gamma^1(X, E) \longrightarrow \Gamma^1(X, F)$ the following are equivalent.

(i) $V : \Gamma^1(X, E) \longrightarrow \Gamma^1(X, F)$ is an homomorphism between $(\Gamma^1(X, E), \mathcal{T}_{\mathbf{\Phi}})$ and $(\Gamma^1(X, F), \mathcal{T}_{\mathbf{\Psi}})$.

(ii) There exists a unique morphism $\Theta : E \longrightarrow F$ between $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$ such that $V = V_\Theta$.

Moreover, if this assertion holds, $||\Theta|| = ||V||$, and Θ is an isometry if and only if V is an isometry.

Proof. (A) $(i) \Leftrightarrow (ii)$:

(a) First, we observe that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\Theta} & F \\ {\scriptstyle \Phi_g}\downarrow & & \downarrow{\scriptstyle \Psi_g} \\ E & \xrightarrow{\Theta} & F \end{array}$$

commutes for each $g \in S$ if and only if the diagram

$$\begin{tikzcd} \Gamma^1(X,E) \arrow[r, "V_\Theta"] \arrow[d, "\mathcal{T}_\Phi(g)"'] & \Gamma^1(X,F) \arrow[d, "\mathcal{T}_\Psi(g)"] \\ \Gamma^1(X,E) \arrow[r, "V_\Theta"] & \Gamma^1(X,F) \end{tikzcd}$$

commutes for each $g \in S$. Indeed, for each $g \in S$

$$\begin{aligned} &\Psi(g) \circ \Theta = \Theta \circ \Phi(g) \\ \iff &\Psi(g) \circ \Theta \circ s = \Theta \circ \Phi(g) \circ s \ \forall s \in \Gamma^1(X,E) \\ \iff &\Psi(g) \circ \Theta \circ s \circ \varphi_{g^{-1}} = \Theta \circ \Phi(g) \circ s \circ \varphi_{g^{-1}} \ \forall s \in \Gamma^1(X,E) \\ \iff &\mathcal{T}_\Psi(g) V_\Theta s = V_\Theta \mathcal{T}_\Phi(g) s \ \forall s \in \Gamma^1(X,E) \\ \iff &\mathcal{T}_\Psi(g) \circ V_\Theta = V_\Theta \circ \mathcal{T}_\Phi(g). \end{aligned}$$

(b) Moreover, $\Theta : E \longrightarrow F$ is a morphism of measurable Banach lattice bundles if and only if $V_\Theta : \Gamma_0(\Omega, E) \longrightarrow \Gamma_0(\Omega, F)$ is a lattice module homomorphism. Indeed, for each $s \in \Gamma^1(X,E)$

$$|\Theta \circ s| = \Theta \circ |s| \iff |V_\Theta s| = V_\Theta |s|$$

Hence, the assertion is proved.

Moreover, since $V_\Theta : \Gamma^1(X,E) \longrightarrow \Gamma^1(X,F)$ is a lattice module homomorphism, Proposition C.5.0.3 implies that $\Theta : E \longrightarrow F$ is the unique morphism over Id_{Ω_X} such that $||\Theta|| = ||V_\Theta||$, and Θ is an isometry if and only if V_Θ is an isometry.

(B) That $(ii) \implies (i)$ follows immediately from (A) above.

$(i) \implies (ii)$: Since $V : \Gamma^1(X,E) \longrightarrow \Gamma^1(X,F)$ is a lattice module homomorphism by Proposition C.5.0.3, we obtain a unique morphism $\Theta : E \longrightarrow F$ of Banach lattice bundle over Id_{Ω_X} such that $Vs = \Theta \circ s$ for all $s \in \Gamma^1(X,E)$, i.e., $V = V_\Theta$.

Moreover, $||\Theta|| = ||V_\Theta|| = ||V||$, implies Θ is an isometry if and only if V is an isometry.

□

The following is an immediate corollary of the above Lemma C.5.0.6 combined with Corollary C.5.0.4.

Corollary C.5.0.7. *Let G be a (discrete) group, $S \subseteq G$ a submonoid, and (Ω, φ) a measure-preserving G-dynamical system. Moreover, let $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$ be separable positive S-dynamical Banach lattice bundles over (X, φ). Furthermore, let $(\Gamma^1(X,E), \mathcal{T}_{\mathbf{\Phi}})$ and $(\Gamma^1(X,F), \mathcal{T}_{\mathbf{\Psi}})$ be positive S-dynamical m-Banach lattice modules over the Koopman group $(L^\infty(X), \boldsymbol{T}_\varphi)$ induced by $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$, respectively. Then the following are equivalent.*

(i) $V : \Gamma^1(X,E) \longrightarrow \Gamma^1(X,F)$ is a homomorphism between $(\Gamma^1(X,E), \mathcal{T}_{\mathbf{\Phi}})$ and $(\Gamma^1(X,F), \mathcal{T}_{\mathbf{\Psi}})$.

(ii) There exists a unique morphism $\Theta : E \longrightarrow F$ between $(E, \mathbf{\Phi})$ and $(F, \mathbf{\Psi})$ such that $V = V_\Theta$.

Moreover, if this assertion holds, $||\Theta|| = ||V||$, and Θ is a positive isometry if and only if V is a positive isometry.

In the following proposition, we obtain a representation of the (quotient) lattice of integrable measurable sections of a measurable Banach lattice bundle as what we call an $L^1(X)$-normed m-lattice module (see Chapter 2, Definition 2.4.2.15). This is analogous to [29, Proposition 2.51, p.51] for the case of $L^1(X)$-normed modules. And by this, we, in particular, answer Question D.2.0.2 raised in Appendix D.

Proposition C.5.0.8. *Let X be a measure space and Γ an $L^1(X)$-normed m-lattice module. The following assertions hold.*

(i) There is a measurable Banach lattice bundle E over X such that $\Gamma^1(X,E)$ is isometrically isomorphic to Γ as $L^1(X)$-normed m-lattice modules.

(ii) If Γ is a separable Banach lattice, then there is a separable Banach lattice bundle E over X such that $\Gamma^1(X,E)$ is isometrically isomorphic to Γ as $L^1(X)$-normed m-lattice modules. Moreover, this separable Banach lattice bundle is unique up to isometric isomorphism.

Proof. (i) As in the proof of [29, Proposition 2.51, p.51], in the real case and considering Γ just as $L^1(X)$-normed module, we obtain that Γ is a Banach-Kantorivich space over $L^1(X)$. Now since Γ is, in particular, a Banach lattice, Γ is a Banach-Kantorovich lattice (see [22, Remark 2.25, p.788]). And as a result due to Ganiev ([14]), as stated in [22, Theorem

2.9., p.789], we find a measurable Banach lattice bundle[4] E such that the (quotient) lattice of its integrable measurable sections $\Gamma^1(X, E)$ is isometrically lattice isomorphic to Γ as lattice normed space.

If we start with a complex $L^1(X)$-normed m-lattice module, we can also conclude that the constructed Banach lattice bundle E is canonically a measurable bundle of complex Banach lattices for which the isomorphism of Γ and $\Gamma^1(X, E)$ can be seen to be $\mathbb{C}$-linear.

Now, let $\mathcal{T} : \Gamma \longrightarrow \Gamma^1(X, E)$ be the $\mathbb{K}$-linear isometric isomorphism of Banach-Kantorovich lattices over $L^1(X)$ (see [22, Definition 4.5, p793]). Since both Γ and $\Gamma^1(X, E)$ are $L^1(X)$-normed modules and $|\mathcal{T}s| = |s|$ for every $s \in \Gamma$, by [29, Proposition 2.49, p.47] equivalences $[(a) \Leftrightarrow (b) \Leftrightarrow (c)]$, we obtain that $\mathcal{T} : \Gamma \longrightarrow \Gamma^1(X, E)$ is a module homomorphism. Hence, $\mathcal{T} : \Gamma \longrightarrow \Gamma^1(X, E)$ is an isometric lattice module isomorphism.

(ii) Now assume Γ and hence $\Gamma^1(X, E)$ is a separable Banach lattice. Let $(s_n)_{n\in\mathbb{N}} \in \Gamma^1(X, E)$ be a sequence such that $\operatorname{lin}\{s_n \mid n \in \mathbb{N}\}$ is dense in $\Gamma^1(X, E)$. By our consideration as in the proof of Proposition C.3.0.8, we may assume WLOG that $\operatorname{lin}\{s_n \mid n \in \mathbb{N}\}$ is a vector sublattice, and choose a representative in $\mathcal{M}_E$ for each s_n which we also denote as s_n. By realising $\operatorname{lin}\{s_n(x) \mid n \in \mathbb{N}\} \subseteq E_x$ as a vector sublattice for every $x \in \Omega_X$, we define a new measurable Banach lattice bundle $F := \bigcup_{x\in\Omega_X} F_x$ over X, by setting

$$F_x := \overline{\operatorname{lin}\{s_n(x) \mid n \in \mathbb{N}\}}$$

for every $x \in \Omega_X$ and

$$\mathcal{M}_F := \{s \in \mathcal{M}_E \mid s(x) \in F_x \text{ for every } x \in \Omega_X\}.$$

It then follows that, the mapping

$$V : \Gamma^1(X, F) \longrightarrow \Gamma^1(X, E);\ s \mapsto s$$

is an isometric lattice module isomorphism. Moreover, F is a separable Banach lattice bundle over X by its definition.

[4] We note that the definition of a measurable bundle of Banach lattices by Ganiev, as in [22, Definition 2.2, p.787], slightly differs from ours. However, every measurable bundle of Banach lattices in sense of Ganiev canonically defines a measurable Banach lattice bundle in our sense having the same (quotient) lattice of integrable measurable sections $\Gamma^1(X, E)$.

From this construction, F is unique up to isometric isomorphism.

Indeed, suppose F_1 is another separable Banach lattice bundle over X, such that $V_1 : \Gamma^1(X, F_1) \longrightarrow \Gamma^1(X, E)$ is also an isometric lattice module isomorphism. Then, the operator $V^{-1} \circ V_1 : \Gamma^1(X, F_1) \longrightarrow \Gamma^1(X, F)$ is an isometric lattice module isomorphism. And by Proposition C.5.0.3, we obtain a unique isometry and surjective morphism $\Theta : F_1 \longrightarrow F$ of Banach lattice bundles over Id_{Ω_X} such that $V^{-1} \circ V_1 s = \Theta \circ s$ for all $s \in \Gamma^1(X, F_1)$. This, in particular, implies that the Banach lattices F_{1x} and F_x are isometrically lattice isomorphic for almost every $x \in \Omega_X$. Furthermore $\Theta : F_1 \longrightarrow F$ is surjective implies that F_1 and F are essentially isomorphic (=bijective).

□

We are now able to state our second representation result for separable dynamical m-Banach lattice modules. See [29, Theorem 2.45, p.45] for the case of separable dynamical Banach modules. This is our Gelfand-type theorem for dynamical separable $L^1(X)$-normed m-lattice modules over $L^\infty(X)$. We note that its corollary and remark, in particular, prove Proposition D.2.0.3 stated in Appendix D.

Theorem C.5.0.9. *Let G be a (discrete) group, $S \subseteq G$ a submonoid, and (X, φ) a measure-preserving G-dynamical system with X separable. Then the assignments*

$$(E, \mathbf{\Phi}) \longmapsto (\Gamma^1(X, E), \mathcal{T}_{\mathbf{\Phi}})$$

$$\Theta \longmapsto V_\Theta$$

define an essentially surjective, fully faithful functor from the category of S-dynamical separable measurable Banach lattice bundles over (X, φ) to the category of S-dynamical separable $L^1(X)$-normed m-lattice modules over $(L^\infty(X), \boldsymbol{T}_\varphi)$.

Proof. Combining Proposition C.3.0.8(iv), Proposition C.5.0.8(ii) and Lemma C.5.0.6 proves the theorem. □

The following is an immediate corollary of the above theorem.

Corollary C.5.0.10. *Let G be a (discrete) group, $S \subseteq G$ a submonoid, and (X, φ) a measure-preserving G-dynamical system with X separable. Then the assignments*

$$(E, \mathbf{\Phi}) \longmapsto (\Gamma^1(X, E), \mathcal{T}_{\mathbf{\Phi}})$$

$$\Theta \longmapsto V_\Theta$$

define an essentially surjective, fully faithful functor from the category of positive S-dynamical separable measurable Banach lattice bundles over (X,φ) to the category of positive S-dynamical separable $L^1(X)$-normed m-lattice modules over $(L^\infty(X), \boldsymbol{T}_\varphi)$.

Proof. Combining Proposition C.3.0.8(iv), Proposition C.5.0.8(ii) and Corollary C.5.0.7 proves the assertion. $\square$

In the following remark, we note how the unique positive semiflow $\boldsymbol{\Phi} = (\Phi_g)_{g\in S}$ can be [5] (*canonically*) obtained on the separable measurable Banach lattice bundle E associated to a positive S-dynamical separable $L^1(X)$-normed m-lattice module $(\Gamma, \boldsymbol{\mathcal{T}})$. By this, we construct the inverse functor for the above theorems.

Remark C.5.0.11. *For Γ a separable $L^1(X)$-normed m-lattice module over $L^\infty(X)$ with $\overline{lin\{s_n \in \Gamma \mid n \in \mathbb{N}\}} = \Gamma$, let $E := \bigcup_{x\in\Omega_X} E_x$ be the unique (up to isometric isomorphism) separable measurable Banach lattice bundle such that $\Gamma \overset{i}{\cong} \Gamma^1(X,E)$ as in proof of Proposition C.5.0.8(ii).*

Now, with G a discrete group, let $(X, (\varphi_g)_{g\in G})$ be measure-preserving G-dynamical system and $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g\in S}$ a positive weighted semigroup representation on Γ over the Koopman group $(L^\infty(X), \boldsymbol{T}_\varphi)$. Moreover, for almost every $x \in \Omega_X$, let $q_x : \Gamma \longrightarrow E_x; s \mapsto is(x)$ be the corresponding (quotient) lattice homomorphism. Then, for each $g \in S$, the positive operators $\Phi_g(x) := \Phi_g|_{E_x} : E_x \longrightarrow E_{\varphi_g(x)}$ are precisely the unique extensions of operators for which the following diagram

$$\begin{tikzcd} \Gamma \arrow[r, "s_n \mapsto \mathcal{T}(g)s_n"] \arrow[d, "q_x"'] & \Gamma \arrow[d, "q_{\varphi_g(x)}"] \\ E_x \arrow[r, dashed, "\Phi_g(x)"'] & E_{\varphi_g(x)} \end{tikzcd}$$

[5] In contrast to the topological setting, the representing separable measurable Banach lattice bundle constructed is not canonical and involves choices. See also [21, Remark 5.19, p.31].

commutes for all $n \in \mathbb{N}$ and almost every $x \in \Omega_X$.
That is, for each $g \in S$, $\Phi_g(x) : E_x \longrightarrow E_{\varphi_g(x)}; is_n(x) \mapsto i\mathcal{T}(g)s_n(\varphi_g(x))$ for all $n \in \mathbb{N}$ and almost every $x \in \Omega_X$ extends uniquely to positive morphism $\Phi_g : E \longrightarrow E$ over φ_g. From these, we have that

$$(\mathcal{T}(g))_{g\in S} \cong (\mathcal{T}_\Phi(g))_{g\in S} \quad on \quad \Gamma \overset{i}{\cong} \Gamma^1(X, E).$$

Appendix D

Bonus Chapter: Positive weighted Koopman semigroups everywhere

Here, we present one central theme of our study. That is, the following question.

Question D.0.0.1. *When is a positive weighted semigroup representation on a Banach lattice module a (or isomorphic to) positive weighted Koopman semigroup representation?*

D.1 In topological dynamics

In this Section, we motivate and present an answer to question D.0.0.1 above in topological dynamics.

Resulting from Chapter 3 [see Remark 3.5.0.1(v) and Proposition 3.5.0.5], each pair $(E, \mathbf{\Phi})$ comprising a positive S-dynamical Banach lattice bundle over a G-dynamical system $(\Omega, (\varphi_g)_{g\in G})$ induces a positive S-dynamical m-Banach lattice module $(\Gamma_0(\Omega, E), \mathcal{T}_{\mathbf{\Phi}})$; the positive weighted Koopman semigroup representation on AM m-lattice module $\Gamma_0(\Omega, E)$ over the Koopman group $(C_0(\Omega), \mathbf{T}_\varphi)$.

> By an AM m-lattice module Γ over $C_0(\Omega)$ (see also Chapter 2, Definition 2.4.1.2) we mean the following:
>
> (a) Γ is an m-BLM over $C_0(\Omega)$; and
>
> (b) For each $s \in \Gamma$, the closed submodule $\Gamma_s := \overline{C_0(\Omega) \cdot s}$ of Γ is an AM-space (represented) as a Banach lattice.

We present the following situation, which in particular implies that, every positive S-dynamical *topological* Banach lattice bundle can be uniquely assigned to a certain positive S-dynamical AM m-lattice module and vice-versa.

Fix $(\Omega, (\varphi_g)_{g\in G})$ a *topological* G-dynamical system over a locally compact space Ω, and $(C_0(\Omega), \mathbf{T}_\varphi)$ the associated Koopman group representation (see also Chapter 2, Example 2.3.3.3 (i)).

We start with the classical situation of an AM m-lattice module over $C_0(\Omega)$, i.e., the Banach lattice $\Gamma_0(\Omega, E)$ of continuous sections vanishing at infinity of a *topological* Banach lattice bundle E over a locally compact space Ω (see Chapter 3, Remark 3.5.0.1(iii)). In this situation, every positive weighted semigroup representation on $\Gamma_0(\Omega, E)$ over the Koopman group representation $\mathbf{T}_\varphi = T_\varphi(g)_{g\in G}$ on $C_0(\Omega)$ is a positive weighted Koopman semigroup over $\mathbf{T}_\varphi = T_\varphi(g)_{g\in G}$ on $C_0(\Omega)$. We restate this result (see Chapter 3, Proposition 3.5.0.5) in the present situation.

Proposition D.1.0.1. *Let E be a topological Banach lattice bundle over a locally compact space Ω, and $\Gamma_0(\Omega, E)$ the associated AM m-lattice module over $C_0(\Omega)$. Then, the following are equivalent.*

(i) $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g\in S}$ *is a positive weighted semigroup representation on* $\Gamma_0(\Omega, E)$ *over* $(C_0(\Omega), \mathbf{T}_\varphi)$.

(ii) There exists a unique pair $(E, \mathbf{\Phi})$ comprising a positive S-dynamical topological Banach lattice bundle over $(\Omega, (\varphi_g)_{g\in G})$, such that $\mathcal{T}(g) = \mathcal{T}_\Phi(g)$ and $||\mathcal{T}(g)|| = ||\Phi_g||$ for all $g \in S$.

Thus, $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g\in S}$ *is a positive isometry if and only if* $\mathbf{\Phi} = (\Phi_g)_{g\in S}$ *is a positive isometry.*

By the work of H. Kreidler and S. Siewert (see [21, Theorem 4.10, p.19]), while using result essentially due to Dupré and Gillete ([8]), it is shown that an AM-module Γ over $C_0(\Omega)$ is isometrically isomorphic to the Banach space $\Gamma_0(\Omega, E)$ of continuous sections vanishing at infinity of a (unique up to isometric isomorphism) *topological* Banach bundle E over Ω.

A natural question arises.

Question D.1.0.2. *Is an AM m-lattice module over $C_0(\Omega)$ isometrically isomorphic to the Banach lattice $\Gamma_0(\Omega, E)$ of continuous sections vanishing at infinity of a (unique up to isometric isomorphism) topological Banach lattice bundle E over Ω ?*

We give a positive answer to this question (see Chapter 3, Proposition 3.5.0.9) by showing that, starting with an AM m-lattice module over $C_0(\Omega)$, each fiber obtained in this process is canonically a Banach lattice, and that an isometric isomorphism will be an isometric isomorphism in the setting of topological Banach lattice bundles. This abstract representation of the lattice of continuous sections of a topological Banach lattice bundle is what is obtained in Chapter 3 (Section 3.5).

Using this abstract representation, the following result immediately follows, which is a generalisation of Proposition D.1.0.1 above. We refer to Chapter 3 (Corollary 3.5.0.12) for the (functorial) proof (see also Chapter 3, Remark 3.5.0.13).

Proposition D.1.0.3. *Let Γ be an AM m-lattice module over $C_0(\Omega)$, then any positive weighted semigroup representation $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g\in S}$ on Γ over $(C_0(\Omega), \mathbf{T}_\varphi)$ is (unique up to isometric isomorphism) a positive weighted Koopman semigroup representation over $(C_0(\Omega), \mathbf{T}_\varphi)$.*
More precisely, there exists a unique (up to isometric isomorphism) pair $(E, \boldsymbol{\Phi})$ comprising a positive S-dynamical topological Banach lattice bundle over $(\Omega, (\varphi_g)_{g\in G})$ such that

$$(\mathcal{T}(g))_{g\in S} \cong (\mathcal{T}_\Phi(g))_{g\in S} \quad on \quad \Gamma \cong \Gamma_0(\Omega, E).$$

Moreover, $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g\in S}$ is a positive isometry if and only if $\boldsymbol{\Phi} = (\Phi_g)_{g\in S}$ is a positive isometry.

D.2 In measurable dynamics

In this Section, similar to the case of *topological* dynamics (see Section D.1), we present an answer to question D.0.0.1 for measurable dynamics.

With G a discrete group, by Appendix C [see Remark C.5.0.1(iv) and Proposition C.5.0.5], every pair $(E, \boldsymbol{\Phi})$ comprising a separable positive S-dynamical

Banach lattice bundle over a *measure-preserving* G-dynamical system $(X,(\varphi_g)_{g\in G})$ induces a positive S-dynamical m-Banach lattice module $(\Gamma^1(X,E),\mathcal{T}_{\mathbf{\Phi}})$; the positive weighted Koopman semigroup representation on $L^1(X)$-normed m-lattice module $\Gamma^1(X,E)$ over the Koopman group $(L^\infty(X),\mathbf{T}_\varphi)$.

By a separable $L^1(X)$-normed m-lattice module Γ over $L^\infty(X)$ (see also Chapter 2, Definition 2.4.2.15) we mean the following:

(a) Γ is a separable m-BLM over $L^\infty(X)$; and

(b) Γ is an $L^\infty(X)'$-normed space such that $|s|\in L^1(X)$ for all $s\in\Gamma$.

We also present the following situation, which in particular implies that, every separable positive S-dynamical *measurable* Banach lattice bundle (over a separable measure space) can be uniquely assigned to a certain positive S-dynamical separable $L^1(X)$-normed m-lattice module and vice-versa.

Here, we fix a *measure-preserving* G-dynamical system $(X,(\varphi_g)_{g\in G})$ over a separable measure space X, and $(L^\infty(X),\mathbf{T}_\varphi)$ the associated Koopman group representation, where G is discrete (see also Chapter 2, Example 2.3.3.3 (ii)).

We also start with the classical situation of a separable $L^1(X)$-normed m-lattice module over $L^\infty(X)$, i.e., the (quotient) Banach lattice $\Gamma^1(X,E)$ of integrable measurable sections of a separable measurable Banach lattice bundle E over a separable measure space X (see Appendix C, Remark C.5.0.1(ii)). In this situation, every positive weighted semigroup representation on $\Gamma^1(X,E)$ over the Koopman group representation $\mathbf{T}_\varphi = T_\varphi(g)_{g\in G}$ on $L^\infty(X)$ is a positive weighted Koopman semigroup over $\mathbf{T}_\varphi = T_\varphi(g)_{g\in G}$ on $L^\infty(X)$. We restate this result in the present situation (see Appendix C, Proposition C.5.0.5).

Proposition D.2.0.1. *Let E be a separable measurable Banach lattice bundle over a separable measure space X, and $\Gamma^1(X,E)$ the associated separable $L^1(X)$-normed m-lattice module over $L^\infty(X)$. Then, the following are equivalent.*

(i) $\mathcal{T}=(\mathcal{T}(g))_{g\in S}$ *is a positive weighted semigroup representation on* $\Gamma^1(X,E)$ *over* $(L^\infty(X),\mathbf{T}_\varphi)$.

(ii) There exists a unique pair $(E, \mathbf{\Phi})$ comprising a positive S-dynamical measurable Banach lattice bundle over $(X, (\varphi_g)_{g\in G})$, such that $\mathcal{T}(g) = \mathcal{T}_\Phi(g)$ and $||\mathcal{T}(g)|| = ||\Phi_g||$ for all $g \in S$.

Thus, $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g\in S}$ is a positive isometry if and only if $\mathbf{\Phi} = (\Phi_g)_{g\in S}$ is a positive isometry.

On the other hand, by the work of H. Kreidler and S. Siewert (see [21, Proposition 5.18 (ii), p.30]), while using results essentially due to Gutmann ([17, 18]), it is shown that a separable $L^1(X)$-normed module over $L^\infty(X)$ is isometrically isomorphic to the (quotient) Banach space $\Gamma^1(X, E)$ of integrable measurable sections of a (unique up to isometric isomorphism) separable measurable Banach bundle E over the measure space X.

Similarly, a natural question arises.

Question D.2.0.2. *Is a separable $L^1(X)$-normed m-lattice module over $L^\infty(X)$ isometrically isomorphic to the (quotient) Banach lattice $\Gamma^1(X, E)$ of integrable measurable sections of a (unique up to isometric isomorphism) separable measurable Banach lattice bundle E over the measure space X ?*

We also give a positive answer to this question (see Appendix C, Proposition C.5.0.8 (ii)), while we make use of the result essentially due to Ganiev ([14]) as stated in [22, Theorem 2.9., p.789]. Appendix C (Section C.5) is devoted to this abstract representation of the (quotient) lattice of integrable sections of a measurable Banach lattice bundle.

By this abstract representation, the following result follows which can also be seen as a generalisation of Proposition D.2.0.1 above. We refer to Appendix C (Corollary C.5.0.10) for the (functorial) proof (see also Appendix C, Remark C.5.0.11). We recall that G is discrete, in this situation.

Proposition D.2.0.3. *Let Γ be a separable $L^1(X)$-normed m-lattice module over $L^\infty(X)$, then any positive weighted semigroup representation $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g\in S}$ on Γ over $(L^\infty(X), \mathbf{T}_\varphi)$ is (unique up to isometric isomorphism) a positive weighted Koopman semigroup representation over $(L^\infty(X), \mathbf{T}_\varphi)$.*

More precisely, there exists a unique (up to isometric isomorphism) pair $(E, \mathbf{\Phi})$ *comprising a separable positive S-dynamical measurable Banach lattice bundle over* $(X, (\varphi_g)_{g \in G})$ *such that*

$$(\mathcal{T}(g))_{g\in S} \cong (\mathcal{T}_{\Phi}(g))_{g\in S} \quad \textit{on} \quad \Gamma \cong \Gamma^1(X, E).$$

Moreover, $\boldsymbol{\mathcal{T}} = (\mathcal{T}(g))_{g\in S}$ *is a positive isometry if and only if* $\mathbf{\Phi} = (\Phi_g)_{g\in S}$ *is a positive isometry.*

Appendix E

Bonus Chapter: Asymptotics of positive weighted Koopman semigroups

E.1 Introduction

In this chapter, we investigate certain long-term behaviour types: namely, irreducibility in Section E.2 and exponential dichotomy in Section E.3 of a positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ on the Banach lattice of continuous sections $\Gamma(K,E)$ over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$ induced by the unique positive semiflow $(\Phi_t)_{t\geq 0}$ on a *topological* Banach lattice bundle E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K.

In particular, using our results in Chapter 3 (Section 3.6), the correspondence between closed ideal subbundles of E and closed ideal submodules of $\Gamma(K,E)$ (see Chapter 3, Corollary 3.6.3.4); and the correspondence between decompositions of positive semiflow $(\Phi_t)_{t\geq 0}$ on E and decompositions of positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ on $\Gamma(K,E)$ (see Chapter 3, Proposition 3.6.4.2), we obtain equivalence of the concepts of irreducibility of positive semiflow $(\Phi_t)_{t\geq 0}$ on E and irreducibility of the positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ on $\Gamma(K,E)$ (see Proposition E.2.0.4) and as well as equivalence of exponential dichotomy of the positive semiflow $(\Phi_t)_{t\geq 0}$ on E and exponential dichotomy of the positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ on $\Gamma(K,E)$ (see Proposition E.3.0.4).

Furthermore, in both cases, under the assumption that the Banach lattice of continuous sections $\Gamma(K, E)$ is either σ-order complete or has order continuous norm, we give further geometric characterisations.

E.2 Irreducibility of positive weighted Koopman semigroups

In this Section, we introduce the concepts of irreducibility of positive semiflows on a Banach lattice bundle, and that of positive weighted Koopman semigroups on the lattice of continuous sections. Furthermore, we seek to obtain a certain correspondence between these concepts.

As before, K is compact, E a Banach lattice bundle over K and $\Gamma(K, E)$ the associated AM m-lattice module over $C(K)$. Moreover, $(\varphi_t)_{t\in\mathbb{R}}$ is a flow on K, and $T_\varphi(t)_{t\in\mathbb{R}}$ the Koopman group on $C(K)$.

Definition E.2.0.1. *A positive semiflow $(\Phi_t)_{t\geq 0}$ on E over $(\varphi_t)_{t\in\mathbb{R}}$ on K is said to be irreducible (or ergodic) if E contains no non-trivial $(\Phi_t)_{t\geq 0}$- invariant closed ideal subbundle; i.e.,*
if $I_E \subseteq E$ is a closed ideal subbundle (see Chapter 3, Definition 3.6.3.1) and $\Phi_t I_E \subseteq I_E$ for all $t \geq 0$, then either $I_E = \emptyset$ or $I_E = E$.

Remark E.2.0.2. *(i) From above definition, it follows that if $(\Phi_t)_{t\geq 0}$ is an irreducible positive semiflow on E over $(\varphi_t)_{t\in\mathbb{R}}$ on K, then the associated positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ satisfies the property that,*

if $I_E \subseteq E$ is a non-empty closed ideal subbundle, then

$\mathcal{T}_\Phi(t)\Gamma(K, I_E) \subseteq \Gamma(K, I_E)$ for all $t \geq 0$ implies that $\Gamma(K, I_E) = \Gamma(K, E)$, i.e., the positive weighted Koopman semigroup is not invariant under any closed ideal submodule of the form $\Gamma(K, I_E)$.

(ii) Inspired by the above result in (i), we introduce the concept of irreducibility (or ergodicity) of a positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ on the lattice of continuous sections $\Gamma(K, E)$ in the next definition.

Definition E.2.0.3. *A positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ on $\Gamma(K,E)$ over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$ is said to be irreducible (or ergodic) if $\Gamma(K,E)$ contains no non-trivial $(\mathcal{T}_\Phi(t))_{t\geq 0}$- invariant closed ideal submodule; i.e., if $I_\Gamma \subseteq \Gamma(K,E)$ is a closed ideal submodule (see Chapter 2, Definition 2.5.2.3) and $\mathcal{T}_\Phi(t)I_\Gamma \subseteq I_\Gamma$ for all $t \geq 0$, then either $I_\Gamma = \{0\}$ or $I_\Gamma = \Gamma(K,E)$.*

Using Corollary 3.6.3.4 in Chapter 3, we obtain equivalence of irreducibility (or ergodicity) of positive semiflows and that of positive weighted Koopman semigroups. Moreover, under the assumption that the Banach lattice $\Gamma(K,E)$ is either σ-order complete or has order continuous norm, we give a further geometric characterisation.

Proposition E.2.0.4. *Let $(\Phi_t)_{t\geq 0}$ be a positive semiflow on E over $(\varphi_t)_{t\in\mathbb{R}}$ on K, and $(\mathcal{T}_\Phi(t))_{t\geq 0}$ the associated positive weighted Koopman semigroup on $\Gamma(K,E)$ over $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$. Now, consider the following statements.*

(i) *$(\Phi_t)_{t\geq 0}$ is an irreducible positive semiflow over $(\varphi_t)_{t\in\mathbb{R}}$.*

(ii) *$(\mathcal{T}_\Phi(t))_{t\geq 0}$ is an irreducible positive weighted Koopman semigroup over $T_\varphi(t)_{t\in\mathbb{R}}$.*

(iii) *For every band projection $\mathcal{P} : \Gamma(K,E) \longrightarrow \Gamma(K,E)$ such that*

(a) the restriction $\mathcal{P}_{|_{rg\mathcal{P}}} : rg\mathcal{P} \longrightarrow rg\mathcal{P}$ is a module homomorphism, and

(b) the restrictions $\mathcal{T}_\Phi(t)_{|_{rg\mathcal{P}}} : rg\mathcal{P} \longrightarrow \Gamma(K,E)$ commutes with $\mathcal{P}$ for all $t \geq 0$,

we have that, either $\mathcal{P} = 0$ or $\mathcal{P} = \mathcal{I}$.

(iv) *For every positive projection $\mathcal{P} : \Gamma(K,E) \longrightarrow \Gamma(K,E)$ such that*

(a) the restriction $\mathcal{P}_{|_{rg\mathcal{P}}} : rg\mathcal{P} \longrightarrow rg\mathcal{P}$ is a module homomorphism, i.e., $\mathcal{P}f\mathcal{P}s = f \cdot \mathcal{P}s$ for all $f \in C(K); s \in \Gamma(K,E)$, and

(b) the restrictions $\mathcal{T}_\Phi(t)_{|_{rg\mathcal{P}}} : rg\mathcal{P} \longrightarrow \Gamma(K,E)$ commutes with $\mathcal{P}$ for all $t \geq 0$, i.e., $\mathcal{P}\mathcal{T}_\Phi(t)\mathcal{P} = \mathcal{T}_\Phi(t)\mathcal{P}$ for every $t \geq 0$,

we have that, either $\mathcal{P} = 0$ or $\mathcal{P} = \mathcal{I}$.

Then $(i) \iff (ii)$. If the Banach lattice $\Gamma(K,E)$ is σ-order complete (see Chapter 3, Proposition 3.6.5.3(B)), then $(i) \iff (ii) \iff (iii)$.
Moreover, if the Banach lattice $\Gamma(K,E)$ has order continuous norm (see Chapter 3, Proposition 3.6.5.3(D)), then $(i) \iff (ii) \iff (iii) \iff (iv)$.

Proof. $(i) \Rightarrow (ii)$: Let $I_\Gamma \subseteq \Gamma(K,E)$ be a non-zero $(\mathcal{T}_\Phi(t))_{t\geq 0}$-invariant closed ideal submodule.

By Chapter 3 [Corollary 3.6.3.4(ii)], we find a non-empty closed ideal subbundle $I_E \subseteq E$, such that $I_\Gamma \cong \Gamma(K, I_E)$. We claim that, I_E is $(\Phi_t)_{t\geq 0}$-invariant. Indeed, $\mathcal{T}_\Phi(t)I_\Gamma \subseteq I_\Gamma$ for all $t \geq 0$, if and only if $\mathcal{T}_\Phi(t)s \in \Gamma(K, I_E)$ for all $s \in \Gamma(K, I_E)$ and $t \geq 0$, implies that $\Phi_t I_E \subseteq I_E$ for all $t \geq 0$.

And by the irreducibility of the positive semiflow $(\Phi_t)_{t\geq 0}$, we have that $I_E = E$, i.e., $I_\Gamma \cong \Gamma(K, I_E) = \Gamma(K,E)$. Hence, the positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ is irreducible.

$(ii) \Rightarrow (i)$: Let $I_E \subseteq E$ be a non-empty $(\Phi_t)_{t\geq 0}$-invariant closed ideal subbundle.

By Chapter 3 [Corollary 3.6.3.4(i)] $\Gamma(K, I_E)$ is a non-zero closed ideal submodule of $\Gamma(K,E)$. We claim that, this closed ideal is $(\mathcal{T}_\Phi(t))_{t\geq 0}$-invariant. Indeed, $\Phi_t I_E \subseteq I_E$ for all $t \geq 0$ implies that $\Phi_t(x)v \subseteq I_{E_{\varphi_t(x)}}$ for all $v \in I_{E_x}$ $x \in K$ and $t \geq 0$, where $\Phi_t(x) := \Phi_t|_{E_x}$. And as such it follows that $\mathcal{T}_\Phi(t)s = \Phi_t \circ s \circ \varphi_{-t} \in \Gamma(K, I_E)$ for all $s \in \Gamma(K, I_E)$ and $t \geq 0$.

And by the irreducibility of the positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$, we have that $\Gamma(K, I_E) = \Gamma(K,E)$. Hence, $I_E = E$, i.e., the positive semiflow $(\Phi_t)_{t\geq 0}$ is irreducible.

$(iii) \Rightarrow (ii)$: Let $I_\Gamma \subseteq \Gamma(K,E)$ be a non-zero $(\mathcal{T}_\Phi(t))_{t\geq 0}$-invariant closed ideal submodule.

As we assumed that the Banach lattice $\Gamma(K,E)$ is σ-order complete, WLOG, we may identify every of its closed lattice ideal with the range of a band projection. Indeed, setting $I_B := I_\Gamma^{\perp\perp}$, i.e., I_B is the (projection) band generated by closed ideal submodule I_Γ, we can find its associated band projection $\mathcal{P} : \Gamma(K,E) \longrightarrow \Gamma(K,E)$ such that the range $rg\mathcal{P} = I_B$ (see also [28, Theorem 2.10 p.62]).

And since the band projection is necessarily positive (see [28, Proposition 2.7, p.61]) and satisfies conditions (a) and (b), the assertion follows from similar argument as in the implication $(iv) \Rightarrow (ii)$ below.

$(ii) \Rightarrow (iii)$: Since, every band projection is necessarily positive, the assertion follows from a similar argument as in the implication $(ii) \Rightarrow (iv)$ below.

$(ii) \Rightarrow (iv)$: We assumed that the Banach lattice $\Gamma(K,E)$ has order continuous norm. By [28, Proposition 11.1, p.208], this is the case if and only if each closed ideal of $\Gamma(K,E)$ is the range of a positive projection.

Now let $\mathcal{P} : \Gamma(K,E) \longrightarrow \Gamma(K,E)$ be a positive projection as in the hypothesis. We claim that the range $rg\mathcal{P} \subseteq \Gamma(K,E)$ is $(\mathcal{T}_\Phi(t))_{t\geq 0}$-invariant closed ideal submodule. Indeed, property (iv) (a) implies that the range $rg\mathcal{P}$ is a closed ideal submodule, i.e., $rg\mathcal{P} \subseteq \Gamma(K,E)$ is a closed ideal such that $f \cdot r \in rg\mathcal{P}$ for all $f \in C(K)$ and $r \in rg\mathcal{P}$. Furthermore, property (iv) (b) implies that the range $rg\mathcal{P}$ is $(\mathcal{T}_\Phi(t))_{t\geq 0}$-invariant, i.e., $\mathcal{T}_\Phi(t)r \in rg\mathcal{P}$ whenever $r \in rg\mathcal{P}$, and for all $t \geq 0$.

Hence, by the irreducibility of the positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$, we have either $rg\mathcal{P} = \{0\}$ or $rg\mathcal{P} = \Gamma(K,E)$.

And so, we have that, either $\mathcal{P} = 0$ or $\mathcal{P} = \mathcal{I}$ as required.

$(iv) \Rightarrow (ii)$: Let $I_\Gamma \subseteq \Gamma(K,E)$ be a non-zero $(\mathcal{T}_\Phi(t))_{t\geq 0}$-invariant closed ideal submodule.

We will show that I_Γ is a range of a non-zero positive projection satisfying the hypothesis in (iv).

Since, the Banach lattice $\Gamma(K,E)$ has order continuous norm, by [28, Proposition 11.1, p.208], we find a non-zero positive projection $\mathcal{P} : \Gamma(K,E) \longrightarrow \Gamma(K,E)$ such that $rg\mathcal{P} = I_\Gamma$.

And since, the range $rg\mathcal{P} = I_\Gamma$ is a submodule; i.e., $fr \in rg\mathcal{P}$ for all $f \in C(K); r \in rg\mathcal{P}$, it follows that $\mathcal{P}f\mathcal{P}s = f \cdot \mathcal{P}s$ for all $f \in C(K)$ and $s \in \Gamma(K,E)$. Thus, the restriction $\mathcal{P}_{|_{rg\mathcal{P}}} : rg\mathcal{P} \longrightarrow rg\mathcal{P}$ is a module homomorphism.

Furthermore, since the range $rg\mathcal{P} = I_\Gamma$ is $(\mathcal{T}_\Phi(t))_{t\geq 0}$-invariant, i.e., $\mathcal{T}_\Phi(t)r \subseteq rg\mathcal{P}$ for all $r \in rg\mathcal{P}$ and $t \geq 0$, it follows that $\mathcal{P}\mathcal{T}_\Phi(t)\mathcal{P}s = \mathcal{T}_\Phi(t)\mathcal{P}s$ for all $s \in \Gamma(K,E)$ and $t \geq 0$. Thus, the restrictions $\mathcal{T}_\Phi(t)_{|_{rg\mathcal{P}}} : rg\mathcal{P} \longrightarrow \Gamma(K,E)$ commutes with $\mathcal{P}$ for all $t \geq 0$.

Hence, as $\mathcal{P} \neq 0$, we have that $\mathcal{P} = \mathcal{I}$ which implies $I_\Gamma = rg\mathcal{P} = \Gamma(K, E)$, i.e., the positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ is irreducible.

□

We can apply the result in the previous Proposition E.2.0.4, in particular, to the situation in Chapter 3 [Example 3.4.0.8(i)].

Example E.2.0.5. *Let Z be a Banach lattice, and $E = K \times Z$ the trivial Banach lattice bundle. Moreover, let $(\varphi_t)_{t\in\mathbb{R}}$ be a continuous flow on K, and $\{\Phi^t(x) : x \in K, t \geq 0\}$ of positive operators comprising a strongly continuous exponentially bounded cocycle on Z over $(\varphi_t)_{t\in\mathbb{R}}$, as in Example 3.4.0.8(i) in Chapter 3.*

Furthermore, let $(\mathcal{T}_\Phi(t))_{t\geq 0}$ be the associated positive weighted Koopman semigroup on $\Gamma(K, E) \cong C(K, Z)$ over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$ induced by the continuous skew-product (linear) flow $(\Phi_t)_{t\geq 0}$ on E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K, which we can call positive evolution semigroup on $C(K, Z)$ in our situation (see [29, Example 3.11(ii), p.65]). Then, any of the following two equivalent conditions implies irreducibility of the continuous skew-product (linear) flow $(\Phi_t)_{t\geq 0}$ on E.

(i) The positive cocycle $\{\Phi^t(x) : x \in K, t \geq 0\}$ satisfies the condition;

$$\Phi^t(x)I_Z \subseteq I_Z \text{ for all } x \in K \text{ and } t \geq 0 \text{ implies either}$$
$$I_Z = \{0\} \text{ or } I_Z = Z \text{ for every closed lattice ideal } I_Z \text{ of } Z.$$

(ii) The positive evolution semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ satisfies the condition;

$$\mathcal{T}_\Phi(t)I_\Gamma \subseteq I_\Gamma \text{ for all } t \geq 0 \text{ implies either } I_\Gamma = \{0\} \text{ or } I_\Gamma = C(K, Z)$$
$$\text{for every closed ideal submodule } I_\Gamma \text{ of } C(K, Z)$$

where $\mathcal{T}_\Phi(t)s(x) = \Phi^t(\varphi_{-t}(x))s(\varphi_{-t}(x))$ for every $s \in C(K, Z)$, $t \geq 0$ and $x \in K$.

(iii) If the Banach lattice $C(K, Z)$ is σ-order complete (see Chapter 3, Proposition 3.6.5.1(B)), then (ii) is equivalent to;

$$\mathcal{P}f\mathcal{P}s = f \cdot \mathcal{P}s \text{ and } \mathcal{P}\mathcal{T}_\Phi(t)\mathcal{P} = \mathcal{T}_\Phi(t)\mathcal{P}$$
$$\forall f \in C(K),\ s \in C(K, Z) \text{ and } t \geq 0 \text{ implies either } \mathcal{P} = 0 \text{ or } \mathcal{P} = \mathcal{I}$$
$$\text{for every band projection } \mathcal{P} : C(K, Z) \longrightarrow C(K, Z).$$

(iv) If the Banach lattice $C(K,Z)$ has order continuous norm (see Chapter 3, Proposition 3.6.5.1(D)), then (ii) is equivalent to;

$$\mathcal{P} f \mathcal{P} s = f \cdot \mathcal{P} s \text{ and } \mathcal{P}\mathcal{T}_{\Phi}(t)\mathcal{P} = \mathcal{T}_{\Phi}(t)\mathcal{P}$$
$$\forall\, f \in C(K),\ s \in C(K,Z) \text{ and } t \geq 0 \text{ implies either } \mathcal{P} = 0 \text{ or } \mathcal{P} = \mathcal{I}$$
$$\text{for every positive projection } \mathcal{P} : C(K,Z) \longrightarrow C(K,Z).$$

E.3 Exponential dichotomy of positive weighted Koopman semigroups

Following the consideration in [29, Section 5.3, p.89-92] about exponential dichotomy (or hyperbolicity) of weighted Koopman semigroup on Banach modules and that of the exponential dichotomy of semiflows on Banach bundles; we introduce here the analogous scenario for the case of a positive weighted Koopman semigroup on a Banach lattice module and positive semiflows on a Banach lattice bundle. More importantly, we also seek to obtain similar correspondence in this setting.

As before, K is compact, E a Banach lattice bundle over K and $\Gamma(K,E)$ the associated AM m-lattice module over $C(K)$. Moreover, $(\varphi_t)_{t\in\mathbb{R}}$ is a flow on K, and $T_{\varphi}(t)_{t\in\mathbb{R}}$ the Koopman group on $C(K)$.

Definition E.3.0.1. *A positive semiflow $(\Phi_t)_{t\geq 0}$ on E over $(\varphi_t)_{t\in\mathbb{R}}$ on K is said to have exponential dichotomy if there are $(\Phi_t)_{t\geq 0}$-invariant closed ideal subbundles $I_E^s, I_E^u \subseteq E$ such that*

$$E = I_E^s \oplus I_E^u$$

and the restricted positive semiflows $(\Phi_t^s)_{t\geq 0}$ on I_E^s and $(\Phi_t^u)_{t\geq 0}$ on I_E^u satisfy the following.

(i) The positive semiflow $(\Phi_t^s)_{t\geq 0}$ is uniformly exponentially stable on I_E^s, i.e., there are constants $M \geq 1, \varepsilon > 0$ such that $||\Phi_t^s|| \leq Me^{-\varepsilon t}$ for all $t \geq 0$.

(ii) The positive semiflow $(\Phi_t^u)_{t\geq 0}$ extends to a positive flow $(\Phi_t^u)_{t\in\mathbb{R}}$ on I_E^u, such that $(\Phi_{-t}^u)_{t\geq 0}$ is uniformly exponentially stable on I_E^u.

We call I_E^s the stable closed ideal subbundle and I_E^u the unstable closed ideal subbundle of E under $(\Phi_t)_{t\geq 0}$ while $(\Phi_t^s)_{t\geq 0}$ is the stable part and $(\Phi_t^u)_{t\geq 0}$ the unstable part of $(\Phi_t)_{t\geq 0}$.

Remark E.3.0.2. *(i) From the above definition, if $(\Phi_t)_{t\geq 0}$ is an exponential dichotomic positive semiflow on E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K, it immediately follows that the associated positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ on $\Gamma(K,E)$ over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$ satisfies the following.*

(a) the restricted positive weighted Koopman semigroup $(\mathcal{T}_{\Phi^s}(t))_{t\geq 0}$ on (its invariant) closed ideal submodule $\Gamma(K,I_E^s)$ is uniformly exponentially stable, since $||\mathcal{T}_{\Phi^s}(t)|| = ||\Phi_t^s||$ for each $t\geq 0$.

(b) the restricted positive weighted Koopman semigroup $(\mathcal{T}_{\Phi^u}(t))_{t\geq 0}$ extends to a positive group $(\mathcal{T}_{\Phi^u}(t))_{t\in\mathbb{R}}$ on (its invariant) closed ideal submodule $\Gamma(K,I_E^u)$ such that $(\mathcal{T}_{\Phi^u}(-t))_{t\geq 0}$ is uniformly exponentially stable, since $||\mathcal{T}_{\Phi^u}(-t)|| = ||\Phi_{-t}^u||$ for each $t\geq 0$.

(c) Moreover, we have that $(\mathcal{T}_\Phi(t))_{t\geq 0} \cong (\mathcal{T}_{\Phi^s}(t))_{t\geq 0} \oplus (\mathcal{T}_{\Phi^u}(t))_{t\geq 0}$ on $\Gamma(K,E) \cong \Gamma(K,I_E^s) \oplus \Gamma(K,I_E^u)$ (see also Chapter 3, Proposition 3.6.4.2(i)).

(ii) Inspired by the result above in (i), we introduce the concept of exponential dichotomy (or hyperbolicity) of positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ on lattice of continuous sections $\Gamma(K,E)$ in the next definition.

Definition E.3.0.3. *A positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ on $\Gamma(K,E)$ over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$ is said to have exponential dichotomy (or is hyperbolic) if there are $(\mathcal{T}_\Phi(t))_{t\geq 0}$-invariant closed ideal submodules $I_\Gamma^s, I_\Gamma^u \subseteq \Gamma(K,E)$ such that*

$$\Gamma(K,E) = I_\Gamma^s \oplus I_\Gamma^u$$

and the restricted positive weighted semigroups $(\mathcal{T}_s(t))_{t\geq 0}$ on I_Γ^s and $(\mathcal{T}_u(t))_{t\geq 0}$ on I_Γ^u satisfy the following.

(i) The positive semigroup $(\mathcal{T}_s(t))_{t\geq 0}$ is uniformly exponentially stable on I_Γ^s, i.e., there are constants $M\geq 1, \varepsilon > 0$ such that $||\mathcal{T}_s(t)|| \leq Me^{-\varepsilon t}$ for all $t\geq 0$.

(ii) Each positive $T_\varphi(t)$-homomorphism $\mathcal{T}_u(t)$ is invertible and the positive semigroup $(\mathcal{T}_u(-t))_{t\geq 0}$ is uniformly exponentially stable on I_Γ^u.

We call I_Γ^s the stable closed ideal submodule and I_Γ^u the unstable closed ideal submodule of $\Gamma(K,E)$ under $(\mathcal{T}_\Phi(t))_{t\geq 0}$ while $(\mathcal{T}_s(t))_{t\geq 0}$ is the stable part and $(\mathcal{T}_u(t))_{t\geq 0}$ the unstable part of $(\mathcal{T}_\Phi(t))_{t\geq 0}$.

Combining Corollary 3.6.3.4 and Proposition 3.6.4.2 in Chapter 3, we obtain equivalence between the exponential dichotomy of positive semiflows and the exponential dichotomy (or hyperbolicity) of positive weighted Koopman semigroups. Moreover, under the assumption that the Banach lattice $\Gamma(K, E)$ is either σ-order complete or has order continuous norm, we give a further geometric characterisation.

Proposition E.3.0.4. *Let $(\Phi_t)_{t\geq 0}$ be a positive semiflow on E over $(\varphi_t)_{t\in\mathbb{R}}$ on K, and $(\mathcal{T}_\Phi(t))_{t\geq 0}$ the associated positive weighted Koopman semigroup on $\Gamma(K, E)$ over $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$. Consider the following statements.*

(i) $(\Phi_t)_{t\geq 0}$ *has exponential dichotomy.*

(ii) $(\mathcal{T}_\Phi(t))_{t\geq 0}$ *has exponential dichotomy.*

(iii) There exists a band projection $\mathcal{P} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$, which is also a module homomorphism, commuting with $(\mathcal{T}_\Phi(t))_{t\geq 0}$ and $\mathcal{T}_\Phi(t) Ker\mathcal{P} = Ker\mathcal{P}$; and there are constants $M \geq 1$, $\varepsilon > 0$ such that

   (a) $||\mathcal{T}_\Phi(t)s|| \leq Me^{-\varepsilon t}||s||$ *for all* $t \geq 0$, $s \in rg\mathcal{P}$, *and*

   (b) $||\mathcal{T}_\Phi(t)s|| \geq \frac{1}{M}e^{+\varepsilon t}||s||$ *for all* $t \geq 0$, $s \in Ker\mathcal{P}$.

(iv) There exists a positive module homomorphism projection[1] $\mathcal{P} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ *commuting*[2] *with $(\mathcal{T}_\Phi(t))_{t\geq 0}$ and $\mathcal{T}_\Phi(t) Ker\mathcal{P} = Ker\mathcal{P}$; and there are constants $M \geq 1$, $\varepsilon > 0$ such that*

   (a) $||\mathcal{T}_\Phi(t)s|| \leq Me^{-\varepsilon t}||s||$ *for all* $t \geq 0$, $s \in rg\mathcal{P}$, *and*

   (b) $||\mathcal{T}_\Phi(t)s|| \geq \frac{1}{M}e^{+\varepsilon t}||s||$ *for all* $t \geq 0$, $s \in Ker\mathcal{P}$.

(v) $\sigma(\mathcal{T}_\Phi(t)) \cap \mathbb{T} = \emptyset$ *for one/all* $t > 0$.

Then $(i) \iff (ii)$. If the Banach lattice $\Gamma(K, E)$ is σ-order complete (see Chapter 3, Proposition 3.6.5.3(B)), then $(i) \iff (ii) \iff (iii)$.
Moreover, if the Banach lattice $\Gamma(K, E)$ has order continuous norm (see Chapter 3, Proposition 3.6.5.3(D)), then $(i) \iff (ii) \iff (iii) \iff (iv) \iff (v)$.

[1] i.e., $\mathcal{P} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ is a positive projection such that $\mathcal{P}fs = f \cdot \mathcal{P}s$ for every $f \in C(K)$ and $s \in \Gamma(K, E)$

[2] i.e., $\mathcal{P}\mathcal{T}_\Phi(t) = \mathcal{T}_\Phi(t)\mathcal{P}$ for every $t \geq 0$

Proof. $(i) \Rightarrow (ii)$: If the positive semiflow $(\Phi_t)_{t\geq 0}$ on E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K has exponential dichotomy, as in consideration of Remark E.3.0.2, it follows that, the associated positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ has exponential dichotomy. Indeed, by setting $I^s_\Gamma := \Gamma(K, I^s_E)$, $I^u_\Gamma := \Gamma(K, I^u_E)$, $(\mathcal{T}_s(t))_{t\geq 0} := (\mathcal{T}_{\Phi^s}(t))_{t\geq 0}$ and $(\mathcal{T}_u(t))_{t\geq 0} := (\mathcal{T}_{\Phi^u}(t))_{t\geq 0}$ we have that $\Gamma(K,E) \cong I^s_\Gamma \oplus I^u_\Gamma$ and

(a) I^s_Γ is a $(\mathcal{T}_\Phi(t))_{t\geq 0}$-invariant closed ideal submodule, and the positive weighted semigroup $(\mathcal{T}_s(t))_{t\geq 0}$ is uniformly exponentially stable on I^s_Γ since $||\mathcal{T}_s(t)||| = ||\mathcal{T}_{\Phi^s}(t)|| = ||\Phi^s_t||$ for each $t \geq 0$.

(b) I^u_Γ is a $(\mathcal{T}_\Phi(t))_{t\geq 0}$-invariant closed ideal submodule, and since $(\Phi^u_t)_{t\geq 0}$ extends to a group $(\Phi^u_t)_{t\in\mathbb{R}}$ on I^u_E, each positive $T_\varphi(t)$-homomorphism $\mathcal{T}_u(t)$ is invertible for each $t \in \mathbb{R}$. Moreover, the positive weighted semigroup $(\mathcal{T}_u(-t))_{t\geq 0}$ is uniformly exponentially stable on I^u_Γ since $||\mathcal{T}_u(-t)||| = ||\mathcal{T}_{\Phi^u}(-t)|| = ||\Phi^u_{-t}||$ for each $t \geq 0$.

$(ii) \Rightarrow (i)$: Assume the positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ has exponential dichotomy. By Corollary 3.6.3.4(ii) and Proposition 3.6.4.2(ii) in Chapter 3, we find two $(\Phi_t)_{t\geq 0}$-invariant closed ideal subbundles $I^s_E, I^u_E \subseteq E$ and two positive semiflows $(\Phi^s_t)_{t\geq 0}$ and $(\Phi^u_t)_{t\geq 0}$ over $(\varphi_t)_{t\in\mathbb{R}}$ on I^s_E and I^u_E respectively, such that

$$(\Phi_t)_{t\geq 0} \cong (\Phi^s_t)_{t\geq 0} \oplus (\Phi^u_t)_{t\geq 0} \text{ on } E \cong I^s_E \oplus I^u_E$$

$$(\mathcal{T}_s(t))_{t\geq 0} \cong (\mathcal{T}_{\Phi^s}(t))_{t\geq 0} \text{ on } I^s_\Gamma \cong \Gamma(K, I^s_E)$$
$$(\mathcal{T}_u(t))_{t\geq 0} \cong (\mathcal{T}_{\Phi^u}(t))_{t\geq 0} \text{ on } \quad I^u_\Gamma \cong \Gamma(K, I^u_E)$$

Moreover,

(a) $||\Phi^s_t|| = ||\mathcal{T}_s(t)||$ for each $t \geq 0$, implies that the positive semiflow $(\Phi^s_t)_{t\geq 0}$ is uniformly exponentially stable on I^s_E.

(b) that each positive $T_\varphi(t)$-homomorphism $\mathcal{T}_u(t)$ is invertible on $I^u_\Gamma \cong \Gamma(K, I^u_E)$ for each $t \in \mathbb{R}$, implies that each Φ^u_t is invertible (i.e., homeomorphic) on I^u_E over φ_t for each $t \in \mathbb{R}$, so that the positive semiflow $(\Phi^u_t)_{t\geq 0}$ extends to a positive flow $(\Phi^u_t)_{t\in\mathbb{R}}$ on I^u_E. Moreover, $||\Phi^u_{-t}|| = ||\mathcal{T}_u(-t)||$ for each $t \geq 0$, implies that the positive semiflow $(\Phi^u_{-t})_{t\geq 0}$ is uniformly exponentially stable on I^u_E.

$(ii) \Rightarrow (iii)$: Similar to the situation in the proof of implication $(iii) \Rightarrow (ii)$ in Proposition E.2.0.4 above, as the Banach lattice $\Gamma(K,E)$ is assumed to be σ-order complete, we may identify every of its closed lattice ideal with the range of a band projection. And since the band projection is necessarily positive (see [28, Proposition 2.7, p.61]), the assertion follows from the same argument as in the implication $(ii) \Rightarrow (iv)$ below.

$(ii) \Rightarrow (iv)$: We assumed the Banach lattice $\Gamma(K,E)$ has order continuous norm, and by the proof of [28, Proposition 11.1, p.208] every closed ideal of $\Gamma(K,E)$ is a projection band (see also [28, Theorem 5.14 p.94]). So, starting with closed ideal I^s_Γ, we can find its associated band projection, i.e., there exists a positive projection $\mathcal{P} : \Gamma(K,E) \longrightarrow \Gamma(K,E)$ such that

$$\Gamma(K,E) = rg\mathcal{P} \oplus Ker\mathcal{P} \quad \text{and} \quad rg\mathcal{P} = I^s_\Gamma$$

as decomposition into closed ideals of $\Gamma(K,E)$.

It follows immediately that we can identify $Ker\mathcal{P} = I^u_\Gamma$ as a closed ideal of $\Gamma(K,E)$.

Now let $\mathcal{Q} := \mathcal{I} - \mathcal{P}$, i.e., $rg\mathcal{Q} = Ker\mathcal{P}$. Then, we have $\mathcal{P}s + \mathcal{Q}s = s \in \Gamma(K,E)$ as the unique representation.

(a) We claim that $\mathcal{P} : \Gamma(K,E) \longrightarrow \Gamma(K,E)$ is a module homomorphism. Indeed, since both $rg\mathcal{P} = I^s_\Gamma$ and $rg\mathcal{Q} = I^u_\Gamma$ are submodule, i.e., $f\mathcal{P}s \in rg\mathcal{P}$ and $f\mathcal{Q}s \in rg\mathcal{Q}$ for every $f \in C(K)$ and $s \in \Gamma(K,E)$, we have that

$$\begin{aligned} fs &= f\mathcal{P}s + f\mathcal{Q}s \\ &= \mathcal{P}f\mathcal{P}s + \mathcal{Q}f\mathcal{Q}s \end{aligned}$$

for every $f \in C(K)$ and $s \in \Gamma(K,E)$. So that

$$\begin{aligned} \mathcal{P}fs &= \mathcal{P}f\mathcal{P}s + \mathcal{P}f\mathcal{Q}s \\ &= \mathcal{P}f\mathcal{P}s + \mathcal{P}\mathcal{Q}f\mathcal{Q}s \\ &= f\mathcal{P}s \end{aligned}$$

for every $f \in C(K)$ and $s \in \Gamma(K,E)$.

(b) We claim that $\mathcal{P} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ commutes with $(\mathcal{T}_\Phi(t))_{t\geq 0}$. Indeed, since both $rg\mathcal{P} = I^s_\Gamma$ and $rg\mathcal{Q} = I^u_\Gamma$ are $(\mathcal{T}_\Phi(t))_{t\geq 0}$-invariant, i.e., $\mathcal{T}_\Phi(t)\mathcal{P}s \in rg\mathcal{P}$ and $\mathcal{T}_\Phi(t)\mathcal{Q}s \in rg\mathcal{Q}$ for every $t \geq 0$, $s \in \Gamma(K, E)$, we have that

$$\begin{aligned}\mathcal{T}_\Phi(t)s &= \mathcal{T}_\Phi(t)\mathcal{P}s + \mathcal{T}_\Phi(t)\mathcal{Q}s \\ &= \mathcal{P}\mathcal{T}_\Phi(t)\mathcal{P}s + \mathcal{Q}\mathcal{T}_\Phi(t)\mathcal{Q}s\end{aligned}$$

for every $t \geq 0$, $s \in \Gamma(K, E)$. So that

$$\begin{aligned}\mathcal{P}\mathcal{T}_\Phi(t)s &= \mathcal{P}\mathcal{T}_\Phi(t)\mathcal{P}s + \mathcal{P}\mathcal{T}_\Phi(t)\mathcal{Q}s \\ &= \mathcal{P}\mathcal{T}_\Phi(t)\mathcal{P}s + \mathcal{P}\mathcal{Q}\mathcal{T}_\Phi(t)\mathcal{Q}s \\ &= \mathcal{T}_\Phi(t)\mathcal{P}s\end{aligned}$$

for every $t \geq 0$, $s \in \Gamma(K, E)$.

(c) We claim that $\mathcal{T}_\Phi(t)Ker\mathcal{P} = Ker\mathcal{P}$, i.e., $\mathcal{T}_\Phi(t)rg\mathcal{Q} = rg\mathcal{Q}$ for each $t \geq 0$. Indeed, clearly $\mathcal{T}_\Phi(t)rg\mathcal{Q} \subseteq rg\mathcal{Q}$ for each $t \geq 0$. And since each positive $T_\varphi(t)$-homomorphism $\mathcal{T}_u(t)$ is invertible on $rg\mathcal{Q}$ for each $t \in \mathbb{R}$, it follows that for any $r := \mathcal{Q}s \in rg\mathcal{Q}$, setting $s_o := \mathcal{T}_u(-t)r \in rg\mathcal{Q}$ we have that $\mathcal{T}_\Phi(t)s_o = r \in rg\mathcal{Q}$, for each $t \geq 0$ and some $s \in \Gamma(K, E)$. Hence, $rg\mathcal{Q} \subseteq \mathcal{T}_\Phi(t)rg\mathcal{Q}$.

(d) Finally, since the positive weighted semigroups $(\mathcal{T}_s(t))_{t\geq 0}$ and $(\mathcal{T}_u(-t))_{t\geq 0}$ are uniformly exponentially stable on $rg\mathcal{P} = I^s_\Gamma$ and $rg\mathcal{Q} = Ker\mathcal{P} = I^u_\Gamma$ respectively, we find constants $M_1, M_2 \geq 1$ and $\varepsilon_1, \varepsilon_2 > 0$ such that

$$||\mathcal{T}_\Phi(t)s|| = ||\mathcal{T}_s(t)s|| \leq M_1 e^{-\varepsilon_1 t}||s|| \text{ for all } t \geq 0,\ s \in rg\mathcal{P}$$

and

$$||\mathcal{T}_\Phi^{-1}(t)s|| = ||\mathcal{T}_u(-t)s|| \leq M_2 e^{-\varepsilon_2 t}||s|| \text{ for all } t \geq 0,\ s \in rg\mathcal{Q} = Ker\mathcal{P}.$$

Setting $M := \max\{M_1, M_2\}$ and $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$, and since $||\mathcal{T}_u(t)|| \geq ||\mathcal{T}_u(-t)||^{-1}$ for $t \geq 0$; it immediately follows that $M \geq 1$ and $\varepsilon > 0$ are constants such that

$$||\mathcal{T}_\Phi(t)s|| = ||\mathcal{T}_s(t)s|| \leq M e^{-\varepsilon t}||s|| \text{ for all } t \geq 0,\ s \in rg\mathcal{P}$$

and

$$||\mathcal{T}_\Phi(t)s|| = ||\mathcal{T}_u(t)s|| \geq \frac{1}{M} e^{+\varepsilon t}||s|| \text{ for all } t \geq 0,\ s \in rg\mathcal{Q} = Ker\mathcal{P}.$$

$(iii) \Rightarrow (ii)$: Since, every band projection is necessarily positive, the assertion follows from a similar argument as in the implication $(iv) \Rightarrow (ii)$ below.

$(iv) \Rightarrow (ii)$: Let $\mathcal{P} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ be a positive module homomorphism projection as in the hypothesis. We claim that the positive weighted Koopman semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ has exponential dichotomy.

(a) Since the Banach lattice $\Gamma(K, E)$ has order continuous norm, by the proof of [28, Proposition 11.1, p.208], the range of positive projection $rg\mathcal{P}$ is a closed ideal of $\Gamma(K, E)$, and WLOG we may assume that it is a projection band (with $\mathcal{P}$ the associated band projection) (see also [28, Theorem 2.10 p.62] and [28, Theorem 5.14 p.94]). And as such, we have complementary decomposition into closed ideals

$$\Gamma(K, E) = rg\mathcal{P} \oplus Ker\mathcal{P}.$$

Now, let $\mathcal{Q} := \mathcal{I} - \mathcal{P}$, i.e., $rg\mathcal{Q} = Ker\mathcal{P}$.

(b) Since $\mathcal{P} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ is a module homomorphism, i.e., $\mathcal{P}fs = f\mathcal{P}s$ for all $f \in C(K), s \in \Gamma(K, E)$; we also have that $\mathcal{Q}fs = f\mathcal{Q}s$ for all $f \in C(K), s \in \Gamma(K, E)$. This implies that both $rg\mathcal{P}$ and $rg\mathcal{Q} = Ker\mathcal{P}$ are closed ideal submodules.

(c) Since $\mathcal{P} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ commutes with $(\mathcal{T}_\Phi(t))_{t\geq 0}$, i.e., $\mathcal{P}\mathcal{T}_\Phi(t) = \mathcal{T}_\Phi(t)\mathcal{P}$ for every $t \geq 0$; we also have that $\mathcal{Q}\mathcal{T}_\Phi(t) = \mathcal{T}_\Phi(t)\mathcal{Q}$ for every $t \geq 0$. This implies that both $rg\mathcal{P}$ and $rg\mathcal{Q} = Ker\mathcal{P}$ are $(\mathcal{T}_\Phi(t))_{t\geq 0}$-invariant closed ideal submodules of $\Gamma(K, E)$.

(d) Since $\mathcal{T}_\Phi(t)Ker\mathcal{P} = Ker\mathcal{P}$ for each $t \geq 0$, i.e., for any $r := \mathcal{Q}s \in rg\mathcal{Q} = Ker\mathcal{P}$ there exists a unique $s_o \in rg\mathcal{Q}$ such that $\mathcal{T}_\Phi(t)s_o = r \in rg\mathcal{Q}$ for each $t \geq 0$ and some $s \in \Gamma(K, E)$. This implies that each positive $T_\varphi(t)$-homomorphism $\mathcal{T}_\Phi(t)_{|_{rg\mathcal{Q}}} : rg\mathcal{Q} \longrightarrow rg\mathcal{Q}$ is invertible for each $t \in \mathbb{R}$.

(e) Finally, given there are constants $M \geq 1, \varepsilon > 0$ such that

$$||\mathcal{T}_\Phi(t)s|| \leq Me^{-\varepsilon t}||s|| \text{ for all } t \geq 0,\ s \in rg\mathcal{P}$$

and

$$||\mathcal{T}_\Phi(t)s|| \geq \frac{1}{M}e^{+\varepsilon t}||s|| \text{ for all } t \geq 0,\ s \in Ker\mathcal{P} = rg\mathcal{Q},$$

and since $||\mathcal{T}_{\Phi}^{-1}(t)|| \geq ||\mathcal{T}_{\Phi}(t)||^{-1}$ on $rg\mathcal{Q} = Ker\mathcal{P}$, it follows immediately that $I_{\Gamma}^{s}, I_{\Gamma}^{u} \subseteq \Gamma(K,E)$ are $(\mathcal{T}_{\Phi}(t))_{t\geq 0}$-invariant closed ideal submodules such that

$$\Gamma(K,E) = I_{\Gamma}^{s} \oplus I_{\Gamma}^{u}$$

$$||\mathcal{T}_{s}(t)s|| \leq Me^{-\varepsilon t}||s|| \text{ for all } t \geq 0,\ s \in I_{\Gamma}^{s}$$

and

$$||\mathcal{T}_{u}^{-1}(t)s|| \leq Me^{-\varepsilon t}||s|| \text{ for all } t \geq 0,\ s \in I_{\Gamma}^{u}$$

by setting $I_{\Gamma}^{s} := rg\mathcal{P}$, $I_{\Gamma}^{u} := Ker\mathcal{P}$, $(\mathcal{T}_{s}(t))_{t\geq 0} := (\mathcal{T}_{\Phi}(t)_{|_{rg\mathcal{P}}})_{t\geq 0}$ and $(\mathcal{T}_{u}(t))_{t\geq 0} := (\mathcal{T}_{\Phi}(t)_{|_{Ker\mathcal{P}}})_{t\geq 0}$.

$(iv) \implies (v)$: Clearly, (iv), in particular, implies that $(\mathcal{T}_{\Phi}(t))_{t\geq 0}$ is a hyperbolic C_0- semigroup of bounded operators on Banach space $\Gamma(K,E)$, which is if and only if $\sigma(\mathcal{T}_{\Phi}(t)) \cap \mathbb{T} = \emptyset$ for one/all $t > 0$. (c.f. [29, Proposition 1.3 p.87]).

$(v) \implies (iv)$: Assume $\sigma(\mathcal{T}_{\Phi}(t_0)) \cap \mathbb{T} = \emptyset$ for some $t_0 > 0$. We follow the proof of [29, Theorem 3.8, p.90][$(c) \Rightarrow (a)$], for the case of Banach bundle; and as such we find a spectral projection $\mathcal{P} : \Gamma(K,E) \longrightarrow \Gamma(K,E)$ which is also a module homomorphism associated to the decomposition of the spectrum $\sigma(\mathcal{T}_{\Phi}(t_0)) := K_1 \dot{\cup} K_2$ with

$$K_1 := \sigma(\mathcal{T}_{\Phi}(t_0) \cap \{z \in \mathbb{C} : |z| < 1\}$$

$$K_2 := \sigma(\mathcal{T}_{\Phi}(t_0) \cap \{z \in \mathbb{C} : |z| > 1\}$$

such that $\mathcal{P}$ commutes with $(\mathcal{T}_{\Phi}(t))_{t\geq 0}$; $\mathcal{T}_{\Phi}(t)Ker\mathcal{P} = Ker\mathcal{P}$ and there are constants $M \geq 1, \varepsilon > 0$ such that

(a) $||\mathcal{T}_{\Phi}(t)s|| \leq Me^{-\varepsilon t}||s||$ for all $t \geq 0,\ s \in rg\mathcal{P}$;

(b) $||\mathcal{T}_{\Phi}(t)s|| \geq \frac{1}{M}e^{+\varepsilon t}||s||$ for all $t \geq 0,\ s \in Ker\mathcal{P}$.

Hence, it suffices to show that $\mathcal{P} : \Gamma(K,E) \longrightarrow \Gamma(K,E)$ is, in addition, a positive operator. This, however, immediately follow, since $\mathcal{P}$ is, in particular, a spectral projection associated with a decomposition of the spectrum of a positive operator $\mathcal{T}_{\Phi}(t_0)$.

□

Similar to the situation in Example E.2.0.5, we can also apply the result in the previous Proposition E.3.0.4 to the continuous skew-product (linear) flow in Chapter 3 [Example 3.4.0.8(i)].

Example E.3.0.5. *Let Z be a Banach lattice, and $E = K \times Z$ the trivial Banach lattice bundle. Moreover, let $(\varphi_t)_{t\in\mathbb{R}}$ be a continuous flow on K, and $\left\{\Phi^t(x) : x \in K, t \geq 0\right\}$ a family of positive operators comprising a strongly continuous exponentially bounded cocycle on Z over $(\varphi_t)_{t\in\mathbb{R}}$ as in Example 3.4.0.8(i) in Chapter 3.*

Furthermore, let $(\mathcal{T}_\Phi(t))_{t\geq 0}$ be the associated positive weighted Koopman semigroup on $\Gamma(K, E) \cong C(K, Z)$ over the Koopman group $T_\varphi(t)_{t\in\mathbb{R}}$ on $C(K)$ induced by the continuous skew-product (linear) flow $(\Phi_t)_{t\geq 0}$ on E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K, which we also call positive evolution semigroup on $C(K, Z)$. Then, any of the following two equivalent conditions implies exponential dichotomy of the continuous skew-product (linear) flow $(\Phi_t)_{t\geq 0}$ on E.

(i) The positive cocycle $\left\{\Phi^t(x) : x \in K, t \geq 0\right\}$ satisfies the condition;

there exist two closed lattice ideals I_{Z_1} and I_{Z_2} of Z with $Z = I_{Z_1} \oplus I_{Z_2}$, $\Phi^t(x)I_{Z_i} \subseteq I_{Z_i}$ for all $t \geq 0$, $x \in K$, $i = 1, 2$; and there are constants $M_i \geq 1, \varepsilon_i > 0$, $i = 1, 2$ such that

(a) $||\Phi^t(x)v|| \leq M_1 e^{-\varepsilon_1 t}||v||$ for all $t \geq 0$, $v \in I_{Z_1}$, $x \in K$, and

(b) $\Phi^t(x)$ is invertible on I_{Z_2} for each $t \geq 0$ and $x \in K$; and $||\Phi^{-t}(x)v|| \leq M_2 e^{-\varepsilon_2 t}||v||$ for all $t \geq 0$, $v \in I_{Z_2}$, $x \in K$.

(ii) The positive evolution semigroup $(\mathcal{T}_\Phi(t))_{t\geq 0}$ satisfies the condition:

there are two closed ideal submodules I_{Γ_1} and I_{Γ_2} of $C(K, Z)$ with $C(K, Z) = I_{\Gamma_1} \oplus I_{\Gamma_2}$, $\mathcal{T}_\Phi(t)I_{\Gamma_i} \subseteq I_{\Gamma_i}$ for all $t \geq 0$, $i = 1, 2$; and there are constants $M_i \geq 1, \varepsilon_i > 0$, such that $(\mathcal{T}_i(t))_{t\geq 0} := (\mathcal{T}_\Phi(t)_{|_{I_{\Gamma_i}}})_{t\geq 0}$, $i = 1, 2$ implies

(a) $||\mathcal{T}_1(t)s|| \leq M_1 e^{-\varepsilon_1 t}||s||$ for all $t \geq 0$, $s \in I_{\Gamma_1}$, and

(b) $\mathcal{T}_2(t)$ is invertible on I_{Γ_2} for each $t \geq 0$; and $||\mathcal{T}_2(-t)s|| \leq M_2 e^{-\varepsilon_2 t}||s||$ for all $t \geq 0$, $s \in I_{\Gamma_2}$.

If the Banach lattice $C(K, Z)$ is σ-order complete (see Chapter 3, Proposition 3.6.5.1(B)), then the (i) and (ii) are equivalent to the condition;

(iii) there exists a band projection $\mathcal{P} : C(K, Z) \longrightarrow C(K, Z)$, which is also a module homomorphism commuting with $(\mathcal{T}_\Phi(t))_{t\geq 0}$, $\mathcal{T}_\Phi(t)Ker\mathcal{P} = Ker\mathcal{P}$; and there are constants $M \geq 1, \varepsilon > 0$ such that

(a) $||\mathcal{T}_\Phi(t)s|| \leq Me^{-\varepsilon t}||s||$ *for all* $t \geq 0,\ s \in rg\mathcal{P}$

(b) $||\mathcal{T}_\Phi(t)s|| \geq \frac{1}{M}e^{+\varepsilon t}||s||$ *for all* $t \geq 0,\ s \in Ker\mathcal{P}$.

If the Banach lattice $C(K, Z)$ has order continuous norm (see Chapter 3, Proposition 3.6.5.1(D)), then the (i), (ii) and (iii) are equivalent to any of the conditions:

(iv) There exists a positive module homomorphism projection $\mathcal{P} : C(K, Z) \longrightarrow C(K, Z)$ commuting with $(\mathcal{T}_\Phi(t))_{t\geq 0}$, $\mathcal{T}_\Phi(t)Ker\mathcal{P} = Ker\mathcal{P}$; and there are constants $M \geq 1, \varepsilon > 0$ such that

(a) $||\mathcal{T}_\Phi(t)s|| \leq Me^{-\varepsilon t}||s||$ *for all* $t \geq 0,\ s \in rg\mathcal{P}$, *and*

(b) $||\mathcal{T}_\Phi(t)s|| \geq \frac{1}{M}e^{+\varepsilon t}||s||$ *for all* $t \geq 0,\ s \in Ker\mathcal{P}$.

(v) $\sigma(\mathcal{T}_\Phi(t)) \cap \mathbb{T} = \emptyset$ *for one/all* $t > 0$.

Appendix F

Bonus Chapter: Markovian weighted Koopman groups

F.1 Introduction

In this chapter, we are particularly interested in an AM m-lattice module Γ over the 1-Banach lattice algebra $C(K)$ whose positive cone Γ_+ has a non-empty interior, i.e., the Banach lattice $\Gamma(K, E)$ of continuous sections of a *topological* Banach lattice bundle E over a compact space K, such that $\Gamma(K, E)_+$ has order unit (see also Chapter 4, Proposition 4.2.0.2(A)).

In Section F.2, we specialise this situation and show that for a Banach lattice bundle E, the Banach lattice $\Gamma(K, E)$ of its continuous sections has order unit only if it is a bundle of AM-spaces with order units (see Proposition F.2.0.1). This result, in particular, verifies and generalises the situation in Chapter 3 [Example 3.3.0.6 (ii)].

In Section F.3, we introduce a special class of weighted Koopman operators on $\Gamma(K, E)$, namely, Markovian (positive) weighted Koopman operators (see Remark F.3.0.2(i)), and then Markovian (positive) morphisms on E (see Definition F.3.0.3), and we obtain a certain correspondence between these concepts (see Proposition F.3.0.5). In this direction, we show that every bijective Markovian weighted Koopman operator (resp. Markovian weighted Koopman group) is isomorphic to a bijective Koopman operator (resp. Koopman group) which extends the original bijective Koopman operator (resp. Koopman group) (see Proposition F.3.0.6, resp. Proposition F.3.0.7). This result, in particular, generalises the situation in [29, Example 3.11, p.65](iv), which is also included. That is, a C_0-group of weighted Koopman operators is isomorphic to an extended Koopman group if and only if it is Markovian in our sense.

As a by-product, every Markovian flow on such a Banach lattice bundle can be assigned uniquely to an extended Koopman group and vice versa. Under this identification, we study the spectral property of an invertible Markovian weighted Koopman operator and extend the result to the Markovian weighted Koopman group in Section F.4.

F.2 Banach lattice of continuous sections with order unit

Throughout this chapter, by a Banach lattice bundle E over K, we mean a *topological* Banach lattice bundle $p : E \longrightarrow K$ over a compact space K (see Chapter 3, Definition 3.2.0.1), and $\Gamma(K, E)$ the Banach lattice of its continuous sections.

The result in the next proposition, in particular, verifies and generalises the lattice isomorphism obtained in Chapter 3 [Example 3.3.0.6 (ii)] on the lattice of its continuous sections, while we use the result obtained in Chapter 2 (Proposition 2.5.5.3).

Proposition F.2.0.1. *Let $p : E \longrightarrow K$ be a Banach lattice bundle over a compact space K, and $\Gamma(K, E)$ the Banach lattice of its continuous sections.*

(A) The following are equivalent.

(i) The interior $\mathrm{Int}\Gamma(K, E)_+$ is non-empty.

(ii) $\Gamma(K, E)$ is isometrically lattice module isomorphic to $C(Q)$ for some compact space Q.

Moreover, if this assertion holds, we have the following:

(a) for each $x \in K$, the Banach lattice E_x is isometrically lattice isomorphic to $C(Q_x)$, for some compact space Q_x; and

(b) there is a continuous surjection $\pi : Q \longrightarrow K$ such that the fiber $\pi^{-1}(x) \cong Q_x$ for each $x \in K$.

(B) Conversely, in the situation above, let $\pi^{-1}(x) := Q_x$ for each $x \in K$;

$$F := \dot{\bigcup}_{x \in K} C(Q_x)$$

$$\tilde{p} : F \longrightarrow K,\ v \in C(Q_x) \mapsto x$$

and endow F with the topology generated by the sets

$$S(\tilde{s}, U, \varepsilon) := \left\{ v \in \tilde{p}^{-1}(U) \mid ||v - \tilde{s}_{|Q_{\tilde{p}(v)}}||_{C(Q_{\tilde{p}(v)})} < \varepsilon \right\}$$

where $U \subseteq K$ is open, $\tilde{s} \in C(Q)$, and $\varepsilon > 0$.

Then $\tilde{p} : F \longrightarrow K$ is a Banach lattice bundle over K, and the Banach lattice $\Gamma(K, F)$ of its continuous sections satisfies the following.

(i) The interior $Int\Gamma(K, F)_+$ is non-empty.

(ii) $\Gamma(K, F)$ is isometrically lattice module isomorphic to $C(Q)$.

Either way $\Gamma(K, E) \cong C(Q) \cong \Gamma(K, F)$ as AM m-lattice modules over $C(K)$, and $E \cong F$ as Banach lattice bundles over K.

Proof. (A)

The equivalence $(ii) \iff (i)$ follows immediately by Chapter 2 (Proposition 2.5.5.3). In this situation, we write $\Gamma(K, E) \overset{i}{\cong} C(Q)$, and WLOG, we identify $iu = \mathbb{1}_Q \in C(Q)_+$, where $u \in Int\Gamma(K, E)_+ \neq \emptyset$ is the order unit. More generally, we write $is = \tilde{s} \in C(Q)$ for a unique $s \in \Gamma(K, E)$.

(a) As also in Chapter 4 [Proposition 4.2.0.2(A)], we have that $e_x(u) = u(x) \in IntE_x^+ \neq \emptyset$, where $e_x : \Gamma(K, E) \longrightarrow E_x;\ s \mapsto s(x)$ denotes the (quotient) evaluation map for each $x \in K$. As such by Appendix B (Theorem B.0.0.6), we find, for each $x \in K$, a compact space Q_x such that $E_x \overset{i_x}{\cong} C(Q_x)$ as Banach lattices, i.e., $i_x : E_x \longrightarrow C(Q_x)$ is an isometric lattice isomorphism for each $x \in K$. In this situation, we also identify $i_x u(x) = \mathbb{1}_{Q_x} \in C(Q_x)_+$ for each $x \in K$.

(b) Setting, for each $x \in K$, $\tilde{e}_x := i_x e_x i^{-1}$, it follows from the following commutative diagram

$$\begin{tikzcd}
\Gamma(K,E) \arrow[r, "e_x"] \arrow[d, shift right, "i"'] & E_x \arrow[d, shift right, "i_x"'] \\
C(Q) \arrow[r, dashed, "\tilde{e}_x"'] \arrow[u, shift right, "i^{-1}"'] & C(Q_x) \arrow[u, shift right, "i_x^{-1}"']
\end{tikzcd}$$

that the evaluation $\tilde{e}_x : C(Q) \longrightarrow C(Q_x)$; $\tilde{s} \mapsto \tilde{e}_x(\tilde{s})$ is a quotient map. Moreover, we see that $|\tilde{e_x}\tilde{s}| = \tilde{e_x}|\tilde{s}|$, i.e., $\tilde{e}_x : C(Q) \longrightarrow C(Q_x)$ is also a lattice homomorphism for each $x \in K$, and more so, $\tilde{e}_x \mathbb{1}_Q = \mathbb{1}_{Q_x}$, for each $x \in K$. Now, for each $x \in K$, that $\tilde{e}_x : C(Q) \longrightarrow C(Q_x)$ is a surjective lattice homomorphism satisfying $\tilde{e}_x \mathbb{1}_Q = \mathbb{1}_{Q_x}$ implies that, there exists a unique injective continuous mapping $\pi_x : Q_x \longrightarrow Q$ such that $\tilde{e}_x(\tilde{s}) = \tilde{s} \circ \pi_x$ for all $\tilde{s} \in C(Q)$ (cf. [28, Theorem 9.1, p.195]; [10, Lemma 4.14, p.55]).

Thus, for each $x \in K$, by [10, Proposition A.4, p.486], we obtain that $Q_x \cong \pi_x(Q_x)$ is an homeomorphism of compact spaces, which also implies that $\dot{\bigcup}_{x \in K} Q_x \overset{m}{\cong} \dot{\bigcup}_{x \in K} \pi_x(Q_x)$ as disjoint unions of compact spaces; where the dense subspace $\dot{\bigcup}_{x \in K} \pi_x(Q_x) \subseteq Q$ is equipped with the subspace topology.

We obtain, naturally, a continuous injective mapping $w : \dot{\bigcup}_{x \in K} Q_x \longrightarrow Q$ with $w_{|Q_x} = \pi_x$ for each $x \in K$; and similarly a continuous surjective mapping $k : \dot{\bigcup}_{x \in K} Q_x \longrightarrow K$ by setting $k(q) := x$ for all $q \in Q_x$. This yields a continuous surjective mapping $\pi : \dot{\bigcup}_{x \in K} \pi_x(Q_x) \longrightarrow K$ such that $\pi_{|\pi_x(Q_x)} \cong k_{|Q_x}$ for each $x \in K$. All of this can be illustrated by the following commutative diagram.

$$\begin{tikzcd}
K & \dot{\bigcup}_{x \in K} Q_x \arrow[l, "k(Q_x)=\{x\}"'] \arrow[r, dashed, "w(Q_x)=\pi_x(Q_x)"] \arrow[d, shift right, "m"'] & Q \\
& \dot{\bigcup}_{x \in K} \pi_x(Q_x) \arrow[ul, dashed, "\pi(\pi_x(Q_x))=\{x\}"] \arrow[ur, hook] \arrow[u, shift right, "m^{-1}"'] &
\end{tikzcd}$$

As a result due to Taĭmov ([30]), as stated in [3, Theorem 1A, p.355], the continuous surjective mapping $\pi : \dot{\bigcup}_{x \in K} \pi_x(Q_x) \longrightarrow K$ extends continuously to a (unique) continuous surjective mapping $\pi : Q \longrightarrow K$ such that $\pi^{-1}(x) = \pi_x(Q_x) \cong Q_x$ for each $x \in K$.

(B) Conversely, we first note that $\tilde{p} : F \longrightarrow K$ is a Banach bundle over K (see [29, Example 1.5(iii), p.13]). And in the situation above, if $\tilde{s} \in C(Q)$, then for each $x \in K$, the restriction $\tilde{s}_{|Q_x} \in C(Q_x)$ coincides with the value of the evaluation map $\tilde{e}_x : C(Q) \longrightarrow C(Q_x)$.

And since $C(Q)$ is a Banach lattice, the continuity of its modulus $\tilde{s} \mapsto |\tilde{s}|$ induces the continuity of the "bundle modulus"

$$|\cdot| : F \longrightarrow F; v_{C(Q_{\tilde{p}(v)})} \mapsto |v|$$

by the following commutative diagram,

$$\begin{tikzcd} C(Q) \arrow[r, "\tilde{s} \longmapsto |\tilde{s}|"] \arrow[d, "\tilde{e}_x"'] & C(Q) \arrow[d, "\tilde{e}_x"] \\ C(Q_x) \arrow[r, "\tilde{s}_{|Q_x} \longmapsto |\tilde{s}|_{|Q_x}"'] & C(Q_x) \end{tikzcd}$$

for every $x \in K$. Indeed, for $x \in K$, if $v = \tilde{s_o}_{|Q_x} \in C(Q_x)$ for some $\tilde{s_o} \in C(Q)$, and $W \subseteq F$ is an open set containing the modulus $|v| = |\tilde{s_o}|_{|Q_x}$. Then by the topology on F, we find an $\varepsilon > 0$ and an open set $U \subseteq K$ containing $x \in K$, such that the open set

$$S(\tilde{s_o}, U, \varepsilon) = \left\{ w \in \tilde{p}^{-1}(U) \mid ||w - |\tilde{s_o}|_{|Q_x}||_{C(Q_x)} < \varepsilon \right\} \subseteq W$$

for some $\tilde{s_o} \in C(Q)$. It then follows immediately that $|w| \in W$ for all $w \in S(\tilde{s_o}, U, \varepsilon)$ for some $\tilde{s_o} \in C(Q)$.

Hence, $\tilde{p} : F \longrightarrow K$ is a Banach lattice bundle over K.

Every $\tilde{s} \in C(Q)$ can uniquely be identified with the continuous section $x \longmapsto \tilde{s}_{|Q_x}$ in $\Gamma(K, F)$, the Banach lattice of continuous sections of F. It follows immediately that this assignment is a surjective isometry of normed lattices.

The order unit $\mathbb{1}_Q \in C(Q)_+$ is identified with the continuous section $x \longmapsto \mathbb{1}_{Q_x}$ in $\Gamma(K, F)_+$, which can immediately be seen to be the order unit. Hence, the interior $\mathrm{Int}\Gamma(K, F)_+$ is non-empty, and we obtain $\Gamma(K, F) \cong C(Q)$ as an isomorphism of Banach lattices. Furthermore, by the first part of (A) above, we obtain that $\Gamma(K,F) \cong C(Q)$ is an isomorphism of AM m-lattice module.

Hence, in either way $\Gamma(K,E) \cong C(Q) \cong \Gamma(K,F)$ as AM m-lattice modules over $C(K)$, and $E \cong F$ as Banach lattice bundles over K by the consideration from above.

□

F.3 Markovian weighted Koopman group on Banach lattice of continuous sections with order unit

In this Section, we introduce a special class of (positive) weighted Koopman operators, namely Markovian (positive) weighted Koopman operators (see Remark F.3.0.2). Moreover, we will show that, in this situation, every bijective Markovian weighted Koopman operator is isomorphic to a bijective Koopman operator which is an extension of the original Koopman operator (see Proposition F.3.0.6). Furthermore, we extend this result to show that, in this situation, every Markovian weighted Koopman group is isomorphic to a Koopman group which is an extension of the original Koopman group (see Proposition F.3.0.7).

Definition F.3.0.1. *Let $p : E \longrightarrow K$ be a Banach lattice bundle over K, and $\Gamma(K,E)$ the Banach lattice of its continuous sections. Furthermore assume $u \in Int\Gamma(K,E)_+ \neq \emptyset$. For a homeomorphism $\varphi : K \longrightarrow K$, we call an operator $\mathcal{T} : \Gamma(K,E) \longrightarrow \Gamma(K,E)$ Markovian (positive) T_φ-homomorphism if:*

(i) $\mathcal{T}$ is (positive) lattice T_φ-homomorphism; and

(ii) $\mathcal{T}u = u$.

And if $\varphi = Id_K$, we simply call $\mathcal{T}$ a Markovian (positive) module homomorphism.

Remark F.3.0.2. *(a) From the above definition we see that, if $\mathcal{T}$ is a Markovian (positive) T_φ-homomorphism on $\Gamma(K, E)$; (i) implies that $\mathcal{T}$ is a (positive) weighted operator on $\Gamma(K, E)$ over the Koopman operator T_φ on $C(K)$. And as such, by Lemma 3.5.0.3 (Corollary 3.5.0.4) in Chapter 3, we find a unique (positive) morphism $\Phi : E \longrightarrow E$ of a Banach lattice bundle over φ, such that $\mathcal{T} = \mathcal{T}_\Phi$, and $||\mathcal{T}|| = ||\Phi||$. In this situation, we will also say the (positive) weighted Koopman operator $\mathcal{T}_\Phi$ on $\Gamma(K, E)$ over T_φ is Markovian.*

(b) By (ii) we have $\Phi \circ u = u \circ \varphi$, which also implies that, for each $x \in K$, the (positive operator) lattice homomorphism $\Phi(x) : E_x \longrightarrow E_{\varphi(x)}$ satisfies

$$\Phi(x)u(x) = u(\varphi(x)) \qquad \textit{where} \qquad \Phi(x) := \Phi_{|E_x}.$$

Since $u(x) \in IntE_x^+ \neq \emptyset$ for each $x \in K$, this implies that the associated (positive) morphism $\Phi : E \longrightarrow E$ "sends order unit to order unit over φ". This observation led us to introduce the following class of (positive) morphism on a Banach lattice bundle, namely Markovian (positive) morphism.

Definition F.3.0.3. *Let $p : E \longrightarrow K$ be Banach lattice bundle over K. Furthermore assume $u_x \in IntE_x^+ \neq \emptyset$ for each $x \in K$. For a continuous map $\varphi : K \longrightarrow K$, we call a map $\Phi : E \longrightarrow E$ a Markovian (positive) morphism over φ if:*

(i) Φ is a (positive) morphism of Banach lattice bundle over φ; and

(ii) for each $x \in K$, the (positive operator) lattice homomorphism $\Phi(x) : E_x \longrightarrow E_{\varphi(x)}$ satisfies

$$\Phi(x)u_x = u_{\varphi(x)} \qquad \textit{where} \qquad \Phi(x) := \Phi_{|E_x}.$$

And if $\varphi = Id_K$, we simply call Φ a Markovian (positive) morphism.

Remark F.3.0.4. *(i) Similarly, if $\Phi : E \longrightarrow E$ is a Markovian (positive) morphism over φ, we will also say the (positive) morphism Φ over φ is Markovian.*

(ii) Now, one would expect certain correspondence between Markovian (positive) morphism and Markovian (positive) weighted Koopman operator. We clarify this in the next proposition in certain sense.

Proposition F.3.0.5. *Let $p : E \longrightarrow K$ be a Banach lattice bundle over K, and $\Gamma(K, E)$ the Banach lattice of its continuous sections. Furthermore assume $u \in Int\Gamma(K, E)_+ \neq \emptyset$. For a homeomorphism $\varphi : K \longrightarrow K$, the following are equivalent.*

(i) $\mathcal{T} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ is a Markovian (positive) T_φ-homomorphism.

(ii) There exists a unique Markovian (positive) morphism $\Phi : E \longrightarrow E$ over φ, such that $\mathcal{T} = \mathcal{T}_\Phi$, and $||\mathcal{T}|| = ||\Phi||$.

In this situation, a (positive) weighted Koopman operator $\mathcal{T}_\Phi$ is Markovian over T_φ if and only if the (positive) morphism $\Phi : E \longrightarrow E$ is Markovian over φ.

Proof. $(i) \implies (ii)$: By Remark F.3.0.2, it follows that the unique associated (positive) morphism $\Phi : E \longrightarrow E$ over φ for which $\mathcal{T} = \mathcal{T}_\Phi$, and $||\mathcal{T}|| = ||\Phi||$ is Markovian. That is, if a (positive) weighted Koopman operator $\mathcal{T}_\Phi$ is Markovian over T_φ, then the associated (positive) morphism Φ is necessarily Markovian over φ.

$(ii) \implies (i)$: First we note that $u \in Int\Gamma(K, E)_+ \neq \emptyset$ implies that $u(x) \in IntE_x^+ \neq \emptyset$ for each $x \in K$. And so if $\Phi : E \longrightarrow E$ is a Markovian (positive) morphism over φ, we have that, for each $x \in K$, the (positive operator) lattice homomorphism $\Phi(x) : E_x \longrightarrow E_{\varphi(x)}$ satisfies

$$\Phi(x)u(x) = u(\varphi(x)) \qquad \text{where} \qquad \Phi(x) := \Phi_{|E_x}.$$

This, in particular, implies that $\Phi \circ u = u \circ \varphi$, and hence $\mathcal{T}_\Phi u = \Phi \circ u \circ \varphi^{-1} = u$, i.e., the (positive) weighted Koopman operator $\mathcal{T}_\Phi$ on $\Gamma(K, E)$ over T_φ is Markovian.

The last assertion is just the conclusion of the equivalence $(i) \iff (ii)$.

□

In the next proposition, we demonstrate that an invertible weighted Koopman operator is isomorphic to a bijective Koopman operator if and only if it is Markovian in our sense (see Definition F.3.0.1). Moreover, we see that the newly obtained Koopman operator is necessarily an extension of the original one.

Proposition F.3.0.6. *Let $p : E \longrightarrow K$ be a Banach lattice bundle over K, and $\Gamma(K, E)$ the Banach lattice of its continuous sections. Furthermore assume $Int\Gamma(K, E)_+ \neq \emptyset$, and $\varphi : K \longrightarrow K$ is a homeomorphism. The following are equivalent for a mapping $\mathcal{T} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$.*

(i) *$\mathcal{T}$ is a bijective Markovian T_φ-homomorphism.*

(ii) *$\mathcal{T}$ is a bijective Markovian weighted Koopman operator over T_φ, i.e., there exists a unique homeomorphic Markovian morphism $\Phi : E \longrightarrow E$ over φ such that $\mathcal{T} = \mathcal{T}_\Phi$, and $||\mathcal{T}|| = ||\Phi||$.*

(iii) *There exists a unique homeomorphism $\psi : Q \longrightarrow Q$ on a compact space Q, and a continuous surjection $\pi : Q \longrightarrow K$ with $\pi \circ \psi = \varphi \circ \pi$ such that*

$$\mathcal{T} \cong T_\psi \quad on \quad \Gamma(K, E) \cong C(Q)$$

where $T_\psi \tilde{s} = \tilde{s} \circ \psi^{-1}$ for each $\tilde{s} \in C(Q)$.

Moreover, if this assertion holds, the mapping $i_\pi : (C(K), T_\varphi) \longrightarrow (C(Q), T_\psi); f \in C(K) \mapsto f \circ \pi \in C(Q)$ is an isometric embedding, in the sense that it is a continuous linear map such that $i_\pi \circ T_\varphi = T_\psi \circ i_\pi$ and $||i_\pi f|| = ||f||$ for all $f \in C(K)$.

Proof. We will use notation essentially as in the proof of Proposition F.2.0.1.

The equivalence $(i) \iff (ii)$ follows immediately from Proposition F.3.0.5.

$(i) \iff (iii)$: We systematically prove this equivalence.

(a) As in Proposition F.2.0.1(A), we find a compact space Q, and a continuous surjection $\pi : Q \longrightarrow K$ with $\pi^{-1}(x) \cong Q_x$ for each $x \in K$ such that $\Gamma(K, E) \overset{i}{\cong} C(Q)$. Moreover, we identify $iu = \mathbb{1}_Q \in C(Q)_+$ where $u \in \Gamma(K, E)_+$ is the order unit, and $E_x \overset{i_x}{\cong} C(Q_x)$ with $i_x u(x) = \mathbb{1}_{Q_x} \in C(Q_x)_+$. More generally we write $is = \tilde{s} \in C(Q)$ for a unique $s \in \Gamma(K, E)$. Furthermore, for each $x \in K$, $\tilde{e}_x : C(Q) \longrightarrow C(Q_x)$; $\tilde{s} \mapsto \tilde{s} \circ \pi_x$ denote the corresponding quotient map, where $\pi_x : Q_x \longrightarrow Q$ is the unique continuous injection such that $\pi^{-1}(x) = \pi_x(Q_x) \cong Q_x$ for each $x \in K$.

(b) Now, since $\Gamma(K, E) \overset{i}{\cong} C(Q)$ is an isometric lattice module isomorphism, it follows that a mapping $\mathcal{T} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ is a lattice T_φ-homomorphism if and only if the mapping $\tilde{\mathcal{T}} := i\mathcal{T}i^{-1} : C(Q) \longrightarrow C(Q)$ is a lattice T_φ-homomorphism, as the following commutative diagram shows.

$$\begin{tikzcd} \Gamma(K,E) \arrow[r, "s\mapsto \mathcal{T}s"] \arrow[d, shift right, "i"'] & \Gamma(K,E) \arrow[d, shift left, "i"] \\ C(Q) \arrow[u, shift right, "i^{-1}"'] \arrow[r, dashed, "\tilde{s}\mapsto \tilde{\mathcal{T}}\tilde{s}"'] & C(Q) \arrow[u, shift left, "i^{-1}"] \end{tikzcd}$$

This, in particular, implies that for $\tilde{s} \in C(Q)$ and each $x \in K$, we have that

$$\tilde{s} \circ \pi_x \in C(Q_x) \longmapsto \tilde{\mathcal{T}}\tilde{s} \circ \pi_{\varphi(x)} \in C(Q_{\varphi(x)})$$

for every $x \in K$.

That is, for each $x \in K$, the diagram

$$\begin{tikzcd} \Gamma(K,E) \arrow[r, "s\mapsto \mathcal{T}s"] \arrow[d, "e_x"'] & \Gamma(K,E) \arrow[d, "e_{\varphi(x)}"] \\ E_x \arrow[r, dashed, "\Phi(x)"'] & E_{\varphi_t(x)} \end{tikzcd}$$

commutes if and only if the corresponding diagram

$$\begin{tikzcd} C(Q) \arrow[r, "\tilde{s}\mapsto \tilde{\mathcal{T}}\tilde{s}"] \arrow[d, "\tilde{e}_x"'] & C(Q) \arrow[d, "\tilde{e}_{\varphi(x)}"] \\ C(Q_x) \arrow[r, dashed, "\tilde{\Phi}(x)"'] & C(Q_{\varphi(x)}) \end{tikzcd}$$

commutes for each $x \in K$, where $\Phi(x) := \Phi_{|E_x} : E_x \longrightarrow E_{\varphi(x)}$ is the unique invertible lattice homomorphism in (ii), corresponding to $\tilde{\Phi}(x) := i_{\varphi(x)}\Phi(x)i_x^{-1} : C(Q_x) \longrightarrow C(Q_{\varphi(x)})$ for each $x \in K$.

(c) And also $\mathcal{T}$ is bijective and Markovian over T_φ if and only if $\tilde{\mathcal{T}}$ is, in particular, a bijective Markov lattice homomorphism on $C(Q)$. As such we find a unique homeomorphism $\psi : Q \longrightarrow Q$ such that $\tilde{\mathcal{T}}\tilde{s} = \tilde{s} \circ \psi^{-1}$ for each $\tilde{s} \in C(Q)$ (cf. [28, Theorem 9.1, p.195]; [10, Lemma 4.14, p.55]). So, if $\tilde{\mathcal{T}}\tilde{s} = \tilde{s} \circ \psi^{-1}$ for every $\tilde{s} \in C(Q)$ and unique homeomorphism $\psi : Q \longrightarrow Q$, it follows from (b) above that for $\tilde{s} \in C(Q)$ and each $x \in K$,

$$\tilde{s} \circ \pi_x \in C(Q_x) \longmapsto \tilde{\mathcal{T}}\tilde{s} \circ \pi_{\varphi(x)} = \tilde{s} \circ \psi^{-1} \circ \pi_{\varphi(x)} \in C(Q_{\varphi(x)}) \quad \cdots \text{L.H.S}$$

$$\Longleftrightarrow$$

$$\pi \circ \psi = \varphi \circ \pi. \quad \cdots \text{R.H.S}$$

Indeed, since $\tilde{s} \in C(Q)$ is arbitrary, the L.H.S holds if and only if $\pi_x = \psi^{-1} \circ \pi_{\varphi(x)}$ for every $x \in K$, which is the case if and only if $\pi^{-1}(x) = \pi_x(Q_x) = \psi^{-1}(\pi_{\varphi(x)}(Q_{\varphi(x)})) = \psi^{-1}(\pi^{-1}(\varphi(x)))$ for every $x \in K$, which is the case if and only if $\pi^{-1} \circ \varphi^{-1} = \psi^{-1} \circ \pi^{-1}$, which is true if and only if the R.H.S holds. That is, the diagram

$$\begin{tikzcd} Q \arrow[r, "\pi"] \arrow[d, "\psi"'] & K \arrow[d, "\varphi"] \\ Q \arrow[r, "\pi"'] & K \end{tikzcd}$$

necessarily commutes. Thus, $\pi : (Q, \psi) \longrightarrow (K, \varphi)$ is an extension of invertible *topological* dynamical systems[1] (see [24, Definition 1.2.1 (a) p.51]).

(d) Finally setting $T_\psi := \tilde{\mathcal{T}}$, we have that $\mathcal{T} \cong T_\psi$ on $\Gamma(K, E) \cong C(Q)$ if and only if $\mathcal{T}$ is a bijective Markovian T_φ-homomorphism.

Moreover, since $\pi : (Q, \psi) \longrightarrow (K, \varphi)$ is an extension of invertible *topological* dynamical systems, it follows immediately that $(C(K), T_\varphi)$ is a Koopman subsytem of the invertible Koopman system $(C(Q), T_\psi)$ given by the mapping $i_\pi : C(K) \longrightarrow C(Q); f \mapsto f \circ \pi$ (see [24, Theorem 12.4 (a), p.54]).
That is, the mapping $i_\pi : (C(K), T_\varphi) \longrightarrow (C(Q), T_\psi); f \in C(K) \mapsto f \circ \pi \in C(Q)$ is continuous, $i_\pi \circ T_\varphi = T_\psi \circ i_\pi$, i.e., the diagram

$$\begin{tikzcd} C(K) \arrow[r, "i_\pi"] \arrow[d, "T_\psi"'] & C(Q) \arrow[d, "T_\varphi"] \\ C(K) \arrow[r, "i_\pi"'] & C(K) \end{tikzcd}$$

commutes, and is an isometry, i.e., $||i_\pi f|| = ||f||$ for all $f \in C(K)$.

□

[1] discrete-time

We can extend the previous result in Proposition F.3.0.6 to group actions. In particular, this implies that a weighted Koopman group is isomorphic to a Koopman group if and only if it is Markovian in our sense. Moreover, the newly obtained Koopman group is an extension of the original Koopman group. This result verifies and generalises the result stated in [29, Example 3.11, p.65](iv), which we can take as a corollary. We, however, add this as part (B) for the sake of completeness.

For a given flow $(\varphi_t)_{t\in\mathbb{R}}$ on K, $T_\varphi(t)_{t\in\mathbb{R}}$ as always will denote the Koopman group on 1-Banach lattice algebra $C(K)$. And by a Markovian flow $(\Phi_t)_{t\in\mathbb{R}}$ on Banach lattice bundle E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K we mean $(\Phi_t)_{t\in\mathbb{R}}$ is a flow on E such that Φ_t is Markovian over φ_t for every $t\in\mathbb{R}$ (see Definition F.3.0.3).

Combining Propositions F.2.0.1 and F.3.0.6 we obtain the following.

Proposition F.3.0.7. *Let $p : E \longrightarrow K$ be a Banach lattice bundle over K, and $\Gamma(K,E)$ the Banach lattice of its continuous sections. Furthermore, assume $Int\Gamma(K,E)_+ \neq \emptyset$, and $(\varphi_t)_{t\in\mathbb{R}}$ is a flow on K.*

(A) The following are equivalent for a C_0-group $\mathcal{T}(t)_{t\in\mathbb{R}}$ on $\Gamma(K,E)$.

(i) $\mathcal{T}(t)_{t\in\mathbb{R}}$ is a Markovian weighted group representation over $T_\varphi(t)_{t\in\mathbb{R}}$, i.e., $\mathcal{T}(t)$ is a Markovian $T_\varphi(t)$-homomorphism for every $t\in\mathbb{R}$.

(ii) $\mathcal{T}(t)_{t\in\mathbb{R}}$ is a Markovian weighted Koopman group over $T_\varphi(t)_{t\in\mathbb{R}}$, i.e., there exists a unique Markovian flow $(\Phi_t)_{t\in\mathbb{R}}$ on E over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K, such that $\mathcal{T}(t) = \mathcal{T}_\Phi(t)$ and $||\mathcal{T}(t)|| = ||\Phi_t||$ for every $t\in\mathbb{R}$.

(iii) There exists a unique flow $(\psi_t)_{t\in\mathbb{R}}$ on compact space Q, and $\pi : (Q,(\psi_t)_{t\in\mathbb{R}}) \longrightarrow (K,(\varphi_t)_{t\in\mathbb{R}})$ an extension of topological $\mathbb{R}$-dynamical systems, i.e., $\pi : Q \longrightarrow K$ is a continuous surjection with $\pi\circ\psi_t = \varphi_t\circ\pi$ for every $t\in\mathbb{R}$; such that

$$\mathcal{T}(t)_{t\in\mathbb{R}} \cong T_\psi(t)_{t\in\mathbb{R}} \quad \text{on} \quad \Gamma(K,E)\cong C(Q)$$

where $T_\psi(t)_{t\in\mathbb{R}}$ denotes the Koopman group on $C(Q)$ associated with the flow $(\psi_t)_{t\in\mathbb{R}}$ on Q.

(B) Conversely, in the situation above, let $\pi^{-1}(x) := Q_x$ for each $x\in K$;

$$F := \dot{\bigcup}_{x\in K} C(Q_x)$$

$$\tilde{p} : F \longrightarrow K,\ v \in C(Q_x) \mapsto x;$$

and endow F with the topology generated by the sets

$$S(\tilde{s}, U, \varepsilon) := \left\{ v \in \tilde{p}^{-1}(U) \mid ||v - \tilde{s}_{|Q_{\tilde{p}(v)}}||_{C(Q_{\tilde{p}(v)})} < \varepsilon \right\}$$

where $U \subseteq K$ is open, $\tilde{s} \in C(Q)$, and $\varepsilon > 0$. Then, we have the following.

(i) $\tilde{p} : F \longrightarrow K$ is a Banach lattice bundle over K, and $\Gamma(K, F)$ the Banach lattice of its continuous sections is such that the interior $Int\Gamma(K, F)_+ \neq \emptyset$ and $\Gamma(K, F) \cong C(Q)$.

(ii) For each $t \in \mathbb{R}$, the mapping

$$\tilde{\Phi}_t : F \longrightarrow F,\ v \in C(Q_x) \mapsto v \circ \psi_{-t} \in C(Q_{\varphi_t(x)})$$

defines a Markovian flow $(\tilde{\Phi}_t)_{t\in\mathbb{R}}$ on F over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K, such that

$$\mathcal{T}_{\tilde{\Phi}}(t)_{t\in\mathbb{R}} \cong T_\psi(t)_{t\in\mathbb{R}} \quad on \quad \Gamma(K, F) \cong C(Q)$$

where $\mathcal{T}_{\tilde{\Phi}}(t)_{t\in\mathbb{R}}$ denote the Markovian weighted Koopman group over $T_\varphi(t)_{t\in\mathbb{R}}$ induced by $(\tilde{\Phi}_t)_{t\in\mathbb{R}}$ on F.

Either way, we have that

$$\mathcal{T}_\Phi(t)_{t\in\mathbb{R}} \cong T_\psi(t)_{t\in\mathbb{R}} \cong \mathcal{T}_{\tilde{\Phi}}(t)_{t\in\mathbb{R}} \quad on \quad \Gamma(K, E) \cong C(Q) \cong \Gamma(K, F)$$

$$(\Phi_t)_{t\in\mathbb{R}} \cong (\tilde{\Phi}_t)_{t\in\mathbb{R}} \quad on \quad E \cong F$$

and the mapping $i_\pi : (C(K), T_\varphi(t)_{t\in\mathbb{R}}) \longrightarrow (C(Q), T_\psi(t)_{t\in\mathbb{R}}); f \in C(K) \mapsto f \circ \pi \in C(Q)$ is an isometric embedding of Koopman group, in the sense that it is a continuous linear map such that $i_\pi \circ T_\varphi(t) = T_\psi(t) \circ i_\pi$ for each $t \in \mathbb{R}$ and $||i_\pi f|| = ||f||$ for all $f \in C(K)$.

Proof. (A) The equivalences $(i) \iff (ii) \iff (iii)$ follow immediately from Proposition F.3.0.6.

(B) The assertion (i) follows from Proposition F.2.0.1 (B). To prove assertion (ii), we note first that $(\tilde{\Phi}_t)_{t\in\mathbb{R}}$ is a flow on F over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K as in Chapter 3 [Example 3.4.0.8(ii)]. And since, for each $x \in K$, the lattice homomorphism $\tilde{\Phi}_t(x) : C(Q_x) \longrightarrow C(Q_{\varphi_t(x)})$ satisfies

$\tilde{\Phi}_t(x)\mathbb{1}_{Q_x} = \mathbb{1}_{Q_{\varphi_t(x)}}$ that is $\mathbb{1}_{Q_x} \circ \psi_{-t} = \mathbb{1}_{Q_{\varphi_t(x)}}$, where $\tilde{\Phi}_t(x) := \tilde{\Phi}_t|_{C(Q_x)}$

for every $t \in \mathbb{R}$, it follows that $\tilde{\Phi}_t$ is a Markovian morphism over φ_t (see Definition F.3.0.3) for each $t \in \mathbb{R}$. Hence, $(\tilde{\Phi}_t)_{t\in\mathbb{R}}$ is a Markovian flow on F over the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K.

And since the interior $Int\Gamma(K,F)_+ \neq \emptyset$, it follows from Part (A) above that the induced weighted Koopman group $\mathcal{T}_{\tilde{\Phi}}(t)_{t\in\mathbb{R}}$ is Markovian on $\Gamma(K,F)$ over $T_\varphi(t)_{t\in\mathbb{R}}$.

Finally, since $\Gamma(K,F) \cong C(Q)$, it follows immediately that $(\psi_t)_{t\in\mathbb{R}}$ is the unique flow on the compact space Q such that

$$\mathcal{T}_{\tilde{\Phi}}(t)_{t\in\mathbb{R}} \cong T_\psi(t)_{t\in\mathbb{R}} \cong \mathcal{T}_\Phi(t)_{t\in\mathbb{R}} \quad \text{on} \quad \Gamma(K,F) \cong C(Q) \cong \Gamma(K,E)$$

$$(\Phi_t)_{t\in\mathbb{R}} \cong (\tilde{\Phi}_t)_{t\in\mathbb{R}} \quad \text{on} \quad E \cong F.$$

And so, the remaining assertions follow by the conclusion of Propositions F.2.0.1 and F.3.0.6.

□

F.4 Spectral theory for Markovian weighted Koopman groups on Banach lattices of continuous sections with order unit

In Section F.2, we demonstrated that if Γ is an AM m-lattice module over the 1-Banach lattice algebra $C(K)$ such that the interior $Int\Gamma_+ \neq \emptyset$, then it must be isomorphic to a Banach lattice of continuous sections of a bundle of AM-spaces with order units (see Proposition F.2.0.1). Moreover, in this situation, we introduce the concept of a Markovian weighted Koopman operator (see Definition F.3.0.1), and we show that every bijective Markovian weighted Koopman operator (resp. Markovian weighted Koopman group) is isomorphic to a bijective Koopman operator (resp. Koopman group) with extends the original bijective Koopman operator (resp. Koopman group) (see Proposition F.3.0.6, resp. Proposition F.3.0.7).

In this situation, every spectral property, say, of a bijective Markovian weighted Koopman operator can be characterised by that of the representing extended bijective Koopman operator. We start with the spectral properties of a single bijective Markovian weighted Koopman operator and extend the results to the Markovian weighted Koopman group. We essentially make use of results in [29, Section 4.1.1. p.72-73] and [29, Section 4.1.2 , p.74-75] and we refer to Chapter 4 (Remark 4.5.0.1) for concepts related to aperiodic homeomorphisms and aperiodic flows on a compact space.

Throughout, $p : E \longrightarrow K$ is a Banach lattice bundle over K, and $\Gamma(K, E)$ is the Banach lattice of its continuous sections. Furthermore, we assume $u \in Int\Gamma(K, E)_+ \neq \emptyset$, and so we can make use of results (and notation) as in Propositions F.2.0.1 and F.3.0.6 as needed.

F.4.1 Bijective Markovian weighted Koopman operators

We start with a homeomorphism $\varphi : K \longrightarrow K$, and consider a bijective Markovian T_φ-homomorphism $\mathcal{T} : \Gamma(K, E) \longrightarrow \Gamma(K, E)$ on $\Gamma(K, E)$.

Proposition F.4.1.1. *The spectral radius $r(\mathcal{T}) = 1$, even $1 \in \sigma_p(\mathcal{T})$, and the spectrum $\sigma(\mathcal{T}) \subseteq D := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. In fact, the whole spectrum $\sigma(\mathcal{T})$, the point spectrum $\sigma_p(\mathcal{T})$, the approximate point spectrum $\sigma_{ap}(\mathcal{T})$ and the peripheral spectrum $Per\sigma(\mathcal{T}) := \{\lambda \in \sigma(\mathcal{T}) : |\lambda| = r(\mathcal{T})\}$ are cyclic, i.e.,*

$$\lambda = |\lambda|\gamma \in \sigma(\mathcal{T}) \implies |\lambda|\gamma^k \in \sigma(\mathcal{T}) \quad \textit{for} \quad k \in \mathbb{Z},$$

$$\lambda = |\lambda|\gamma \in \sigma_p(\mathcal{T}) \implies |\lambda|\gamma^k \in \sigma_p(\mathcal{T}) \quad \textit{for} \quad k \in \mathbb{Z},$$

$$\lambda = |\lambda|\gamma \in \sigma_{ap}(\mathcal{T}) \implies |\lambda|\gamma^k \in \sigma_{ap}(\mathcal{T}) \quad \textit{for} \quad k \in \mathbb{Z},$$

$$\lambda = |\lambda|\gamma \in Per\sigma(\mathcal{T}) \implies |\lambda|\gamma^k \in Per\sigma(\mathcal{T}) \quad \textit{for} \quad k \in \mathbb{Z}.$$

And as $\mathcal{T}$ is bijective we have

$$Per\sigma(\mathcal{T}) = \sigma(\mathcal{T}) = \sigma_{ap}(\mathcal{T}) \subseteq \mathbb{T}.$$

Furthermore, as $\mathcal{T}$ is bijective if we, in particular, identify $\mathcal{T} \cong T_\psi$ on $\Gamma(K, E) \cong C(Q)$, where $\psi : Q \longrightarrow Q$ is the unique homeomorphism with $\pi \circ \psi = \varphi \circ \pi$, then we have the following.

(a) *$\sigma_p(\mathcal{T}) \subseteq \mathbb{T}$ is a group and the fixed space $fix(\mathcal{T})$ is one-dimesional if $\psi : Q \longrightarrow Q$ is topologically ergodic, i.e., the fixed space $fix(T_\psi) \subseteq C(Q)$ is one-dimesional. This is the case, for instance, if $\psi : Q \longrightarrow Q$ is minimal*[2].

In this situation, the homeomorphism $\varphi : K \longrightarrow K$ is also topologically ergodic, i.e., the fixed space $fix(T_\varphi) \subseteq C(K)$ is one-dimensional, and so the point spectrum $\sigma_p(T_\varphi) \subseteq \mathbb{T}$ is a group. Moreover, $fix(\mathcal{T})$ is the closed vector-lattice generated by $lin\{fu : f \in \mathbb{C}\mathbb{1}_K\} \subseteq \Gamma(K,E)$.

(b) *$\sigma(\mathcal{T}) = \mathbb{T}$ if $\psi : Q \longrightarrow Q$ is an aperiodic homeomorphism.*

In this situation, the homeomorphism $\varphi : K \longrightarrow K$ is also aperiodic, and $\sigma_{ap}(T_\varphi) = \sigma(T_\varphi) = \mathbb{T}$.

Proof. We identify $\mathcal{T} \cong T$ on $\Gamma(K,E) \cong C(Q)$, and as such T is, in particular, a bijective Markov lattice homomorphism on $C(Q)$. So, all the spectral properties of bijective Markov lattice homomorphism T transfer directly to that of $\mathcal{T}$, since for instance, $||\mathcal{T}|| = ||T|| = 1$ and, for $\lambda \in \mathbb{C}$, the operator $\lambda - \mathcal{T}$ is invertible (injective, surjective) on $\Gamma(K,E)$ if and only if the operator $\lambda - T$ is invertible (injective, surjective) on $C(Q)$, i.e., $\sigma(\mathcal{T}) = \sigma(T)$, and as well as all other parts of the spectrum. And so the spetral radii coincide, i.e., $r(\mathcal{T}) = r(T) = 1$.

And even $1 \in \sigma_p(\mathcal{T}) = \sigma_p(T)$, which implies that, $\sigma(\mathcal{T}) = \sigma(T) \subseteq D := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

Now, the cyclicity of the spectrum (and its respective subsets) of $\mathcal{T}$ follows from that of the bijective Markov lattice homomorphism T on $C(Q)$. We note that the peripheral spectrum $Per\sigma(\mathcal{T}) = \{\lambda \in \sigma(T) : |\lambda| = 1\}$, and as T is bijective, we even have $Per\sigma(T) = \sigma(T) = \sigma_{ap}(T) \subseteq \mathbb{T}$. (c.f. [29, Section 4.1.1. p.72]).

(a) We note that $fix(\mathcal{T}) \cong fix(T_\psi)$ on $\Gamma(K,E) \cong C(Q)$. And so, WLOG, if we assume $\psi : Q \longrightarrow Q$ is minimal, then $\sigma_p(\mathcal{T}) = \sigma_p(T_\psi) \subseteq \mathbb{T}$ is a group and $fix(T_\psi)$ is one-dimensional, also implies that $fix(\mathcal{T})$ is also one-dimensional.

Since $\pi : (Q,\psi) \longrightarrow (K,\varphi)$ is an extension, $\psi : Q \longrightarrow Q$ is minimal, implies that $\varphi : K \longrightarrow K$ is also minimal, so that the fix space $fix(T_\varphi) \subseteq C(K)$ is one-dimensional, and the point spectrum $\sigma_p(T_\varphi) \subseteq$

[2] i.e., Q contains no non-trivial ψ-invariant closed subset.

$\mathbb{T}$ is a group. Alternatively, since the mapping $i_\pi : (C(K), T_\varphi) \longrightarrow (C(Q), T_\psi); f \in C(K) \mapsto f \circ \pi \in C(Q)$ is an isometric embedding, it follows that $i_\pi(fix(T_\varphi)) \subseteq fix(T_\psi)$, so that if $fix(T_\psi) \subseteq C(Q)$ is one dimensional, then necessarily $fix(T_\varphi) \subseteq C(K)$ is also one-dimensional.

Moreover, if $s \in fix(\mathcal{T}) \subseteq \Gamma(K, E)$, then $\mathcal{T} fs = T_\varphi fs = fs$ if and only if $f \in fix(T_\varphi) = \mathbb{C}\mathbb{1}_K$. This implies that $fix(\mathcal{T})$ is the closed vector-lattice generated by $\text{lin}\{fu : f \in \mathbb{C}\mathbb{1}_K\} \subseteq \Gamma(K, E)$ which is one-dimensional.

(b) If $\psi : Q \longrightarrow Q$ is an aperiodic homeomorphism, then $\sigma_{ap}(T_\psi) = \sigma(T_\psi) = \mathbb{T}$, which also implies that $\sigma(\mathcal{T}) = \sigma(T_\psi) = \mathbb{T}$ (see also Chapter 4, Lemma 4.5.0.2(A)).

Since $\pi : (Q, \psi) \longrightarrow (K, \varphi)$ is an extension, $\psi : Q \longrightarrow Q$ being aperiodic implies that $\varphi : K \longrightarrow K$ is also an aperiodic homeomorphism, so that $\sigma_{ap}(T_\varphi) = \sigma(T_\varphi) = \mathbb{T}$. (c.f. [29, Proposition 4.3. p.73]).

□

F.4.2 Markovian weighted Koopman groups

As in above, $p : E \longrightarrow K$ is a Banach lattice bundle over K, and $\Gamma(K, E)$ the Banach lattice of its continuous sections is such that $u \in Int\Gamma(K, E)_+ \neq \emptyset$. Moreover, for a given flow $(\varphi_t)_{t \in \mathbb{R}}$ on K, $T_\varphi(t)_{t \in \mathbb{R}}$ will denote the associated Koopman group on $C(K)$ with generator $(\delta, D(\delta))$.

Now, we consider a Markovian weighted Koopman group $\mathcal{T}(t)_{t \in \mathbb{R}}$ on $\Gamma(K, E)$ over $T_\varphi(t)_{t \in \mathbb{R}}$ with generator $(\mathcal{A}, D(\mathcal{A}))$.

Proposition F.4.2.1. *For the generator $(\mathcal{A}, D(\mathcal{A}))$ of Markovian weighted Koopman group $\mathcal{T}(t)_{t \in \mathbb{R}}$ we have the following;*
The spectral bound $s(\mathcal{A}) = 0$. Moreover, the whole spectrum $\sigma(\mathcal{A})$, the approximate point spectrum $\sigma_{ap}(\mathcal{A})$, the point spectrum $\sigma_p(\mathcal{A})$ and the boundary spectrum $\sigma^b(\mathcal{A}) := \{\lambda \in \sigma(\mathcal{A}) : Re\lambda = s(\mathcal{A})\}$ are imaginary additively cyclic subsets of $\mathbb{C}$; i.e.,

$$\lambda \in \sigma(\mathcal{A}) \implies Re\lambda + ikIm\lambda \in \sigma(\mathcal{A}) \quad \textit{for} \quad k \in \mathbb{Z},$$

$$\lambda \in \sigma_{ap}(\mathcal{A}) \implies Re\lambda + ikIm\lambda \in \sigma_{ap}(\mathcal{A}) \quad for \quad k \in \mathbb{Z},$$

$$\lambda \in \sigma_{p}(\mathcal{A}) \implies Re\lambda + ikIm\lambda \in \sigma_{p}(\mathcal{A}) \quad for \quad k \in \mathbb{Z},$$

$$\lambda \in \sigma^{b}(\mathcal{A}) \implies Re\lambda + ikIm\lambda \in \sigma^{b}(\mathcal{A}) \quad for \quad k \in \mathbb{Z}.$$

As each $\mathcal{T}(t)$ is bijective, we even have $\sigma(\mathcal{A}) \subseteq i\mathbb{R}$ and

$$\sigma^{b}(\mathcal{A}) = \sigma_{ap}(\mathcal{A}) = \sigma(\mathcal{A}).$$

Furthermore, if we, in particular, identify $\mathcal{T}(t)_{t\in\mathbb{R}} \cong T_{\psi}(t)_{t\in\mathbb{R}}$ on $\Gamma(K,E) \cong C(Q)$ for a unique flow $(\psi_t)_{t\in\mathbb{R}}$ on Q with $\pi : (Q, (\psi_t)_{t\in\mathbb{R}}) \longrightarrow (K, (\varphi_t)_{t\in\mathbb{R}})$ an extension of topological $\mathbb{R}$-dynamical systems such that,

$$(\mathcal{A}, D(\mathcal{A})) \cong (\tilde{\mathcal{A}}, D(\tilde{\mathcal{A}}))$$

where $(\tilde{\mathcal{A}}, D(\tilde{\mathcal{A}}))$ is the generator of the Koopman group $T_{\psi}(t)_{t\in\mathbb{R}}$ on $C(Q)$, then

$\sigma(\mathcal{A}) = i\mathbb{R}$ if $(\psi_t)_{t\in\mathbb{R}}$ is an aperiodic flow on Q.

In this situation, the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K is also aperiodic, and $\sigma(\delta) = i\mathbb{R}$.

Moreover, $\sigma(T_{\varphi}(t)) = \sigma(\mathcal{T}(t)) = \mathbb{T}$ for each $t \in \mathbb{R}$, and the spectral mapping theorems holds i.e.,

$$\sigma(\mathcal{T}(t)) = e^{t\sigma(\mathcal{A})} \quad and \quad \sigma(T_{\varphi}(t)) = e^{t\sigma(\delta)} \quad for\ all\ t \in \mathbb{R}.$$

Proof. By identifying $\mathcal{A} \cong \tilde{\mathcal{A}}$, i.e., $\tilde{\mathcal{A}} := i\mathcal{A}i^{-1}$ so that $D(\tilde{\mathcal{A}}) := \{\tilde{s} \in C(Q) : i^{-1}\tilde{s} = s \in D(\mathcal{A})\}$ where $\Gamma(K,E) \overset{i}{\cong} C(Q)$, it follows immediately that $(\tilde{\mathcal{A}}, D(\tilde{\mathcal{A}}))$ is the generator of a unique Markovian lattice C_0-group $T(t)_{t\in\mathbb{R}}$ on $C(Q)$ such that $\mathcal{T}(t)_{t\in\mathbb{R}} \cong T(t)_{t\in\mathbb{R}}$.

So all spectral properties of $\tilde{\mathcal{A}}$ transfer directly to that of $\mathcal{A}$, since for instance, $\sigma(\mathcal{A}) = \sigma(\tilde{\mathcal{A}})$ and as well all other parts of the spectrum coincide; and more so the spectral bounds coincide i.e., $s(\mathcal{A}) = s(\tilde{\mathcal{A}})$. And since $T(t)_{t\in\mathbb{R}}$ is, in particular, C_0-group of isometries, we have that the spectral bound $s(\mathcal{A}) = s(\tilde{\mathcal{A}}) = 0$.

Now, the additive cyclicity of the spectrum of $\mathcal{A}$ (and its respective subsets) follows from that of $\tilde{\mathcal{A}}$. We note that the boundary spectrum

$\sigma^b(\mathcal{A}) = \sigma^b(\tilde{\mathcal{A}}) = \{\lambda \in \sigma(\tilde{\mathcal{A}}) : Re\lambda = 0\}$, and as each $T(t)$ is bijective, we even have $\sigma^b(\tilde{\mathcal{A}}) = \sigma_{ap}(\tilde{\mathcal{A}}) = \sigma(\tilde{\mathcal{A}}) \subseteq i\mathbb{R}$. (cf. [29, Section 4.1.2 , p.74]).

Thus, $\sigma^b(\mathcal{A}) = \sigma_{ap}(\mathcal{A}) = \sigma(\mathcal{A}) \subseteq i\mathbb{R}$.

Now, we identify $\mathcal{T}(t)_{t\in\mathbb{R}} \cong T_\psi(t)_{t\in\mathbb{R}}$ on $\Gamma(K, E) \cong C(Q)$ as in above, for a unique flow $(\psi_t)_{t\in\mathbb{R}}$ on Q with $\pi : (Q, (\psi_t)_{t\in\mathbb{R}}) \longrightarrow (K, (\varphi_t)_{t\in\mathbb{R}})$ an extension of *topological* $\mathbb{R}$-dynamical systems.

If $(\psi_t)_{t\in\mathbb{R}}$ is an aperiodic flow on Q, then we have that $\sigma(\mathcal{A}) = \sigma(\tilde{\mathcal{A}}) = i\mathbb{R}$ and $\sigma(\mathcal{T}(t)) = \sigma(T_\psi(t)) = \mathbb{T}$ for each $t \in \mathbb{R}$ (see also Chapter 4, Lemma 4.5.0.2(C)).

And since, $\pi : (Q, (\psi_t)_{t\in\mathbb{R}}) \longrightarrow (K, (\varphi_t)_{t\in\mathbb{R}})$ is an extension of *topological* $\mathbb{R}$-dynamical systems, it follows that the flow $(\varphi_t)_{t\in\mathbb{R}}$ on K is also aperiodic. And so, we have $\sigma(\delta) = i\mathbb{R}$ and $\sigma(T_\varphi(t)) = \mathbb{T}$ for each $t \in \mathbb{R}$.

Hence, in this situation, the spectral mapping theorems hold, i.e.,

$$\sigma(\mathcal{T}(t)) = e^{t\sigma(\mathcal{A})} \quad \text{and} \quad \sigma(T_\varphi(t)) = e^{t\sigma(\delta)} \quad \text{for all } t \in \mathbb{R}.$$

(c.f [29, Proposition 4.6, p.75]).

□

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