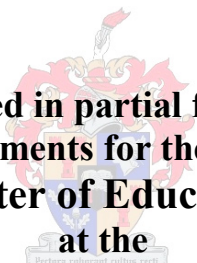


Developing proportional reasoning in Mathematical Literacy students

by
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Declaration

By submitting this thesis/dissertation electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the owner of the copyright thereof (unless to the extent explicitly otherwise stated) and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

March 2010

Abstract

The aim of this research is three-fold. Firstly I aimed to show the difficulty of the concept of proportional reasoning through empirical research. Several researchers have shown the degree of difficulty learners experience with proportional reasoning and have even indicated that many university students (and adults) do not have sound proportional reasoning skills. Piaget's controversial developmental levels classify proportional reasoning as a higher order thinking skill in his highest level of development, formal operational thought, and claims that most people do not reach this level. The difficulty of proportional reasoning and the fact that it is a skill needed within all Learning Outcomes of Mathematical Literacy creates a predicament in terms of the difficulty of the subject in general. Is it then fair to classify Mathematical Literacy as an inferior subject in the way it has been done over the last few years if it is a subject that requires learners to operate at such a high level of thought through proportional reasoning?

Secondly, I would like to confirm with the use of a baseline assessment that learners entering Grade 10 Mathematical Literacy have poor proportional reasoning skills and have emotional barriers to Mathematics and therefore Mathematical Literacy. The research will be done in three private schools located in the West Coast District of the Western Cape in South Africa. If learners in these educationally ideal environments demonstrate poor proportional reasoning skills even though they were privileged enough to have all the possible support since their formative years, then results from overcrowded government schools may be expected to be even worse.

The learners in Mathematical Literacy classes often lack motivation, interest and enthusiasm when it comes to doing mathematics. Through the baseline assessment I confirm this and also suggest classroom norms and values that will help these learners to become involved in classroom activities and educational discourse.

Thirdly and finally this research will focus on the design of activities that will aim to build on learners' prior knowledge and further develop their proportional reasoning skills. I argue that activities to develop proportional reasoning should take equivalence of fractions as basis to work from. The activities will aim to help learners to set up questions in such a way that they can solve it with techniques with which they are familiar.

Interconnectivity will form a vital part to this investigation. Not only do I indicate the interconnectivity between concepts in the Mathematical Literacy Learning Outcomes of the National Curriculum Statement, but I would like to make these links clear to learners when working through the proposed activities. Making links between concepts is seen as a higher order thinking skill and is part of meta-cognition which involves reflection on thoughts and processes.

In short, this research can be summarised as the design of activities (with proposed activities) that aims to develop proportional reasoning by making connections between concepts and requires of learners to be active participants in their own learning.

Opsomming

Die doel van hierdie navorsing is drieledig. Eerstens wil ek die probleme met die konsep van proporsionele denke uitlig deur eksperimentele ontwerp navorsing. Verskeie navorsers verwys na die moeilikheidsgraad van probleme wat leerders ondervind met proporsionele denke. Sommige van hierdie navorsers het ook bevind dat verskeie universiteitstudente (en ander volwassenes) nie oor die vaardigheid van proporsionele denke beskik nie. Piaget se kontroversiële ontwikkelingsvlakke klassifiseer proporsionele denke as 'n hoër orde denkvaardigheid in sy hoogste vlak van ontwikkeling, formele operasionele denke, en noem dat meeste mense nooit hierdie vlak bereik nie. Die hoë moeilikheidsgraad van proporsionele denke en die feit dat dit 'n vaardigheid is wat binne al die Leerruitkomst van Wiskundige Geletterdheid benodig word veroorsaak 'n dilemma as mens dit vergelyk met die moeilikheidsgraad van die vak oor die algemeen.

Tweedens wil ek met behulp van 'n grondfase assessering bewys dat leerders wat Graad 10 Wiskunde Geletterdheid betree swak proporsionele denkvaardighede het, gepaardgaande met emosionele weerstand teenoor Wiskunde en Wiskunde Geletterdheid. Die navorsing sal gedoen word in drie privaatskole in die Weskus distrik van die Wes-Kaap van Suid-Afrika. Indien leerders in hierdie ideale opvoedkundige omstandighede swak proporsionele denkvaardighede ten toon stel, ten spyte van die feit dat hulle bevoorreg was om sedert hulle vormingsjare alle moontlike opvoedkundige ondersteuning te geniet, dan kan verwag word dat resultate komende van oorvol staatskole selfs swakker mag wees.

By leerders in Wiskunde Geletterdheid klasse kan daar gereeld 'n gebrek aan motivering, belangstelling en entoesiasme ten opsigte van Wiskunde bespeur word. Deur gebruik van die grondfase assessering wil ek hierdie stelling bewys en ook voorstelle maak vir klaskamernorme en waardes wat sal help om die leerders meer betrokke te maak by klaskameraktiwiteite en opvoedkundige gesprekke.

Derdens sal hierdie navorsing fokus op die ontwikkeling van aktiwiteite wat ten doel sal hê om leerders se huidige kennis te versterk en hul proporsionele denkvaardighede verder te ontwikkel. Ek wil dit stel dat aktiwiteite vir die ontwikkeling van proporsionele denkvaardighede gebaseer moet wees op die konsep van gelykheid of ekwivalensie van breuke. Die aktiwiteite sal leerders help om probleme op so 'n wyse te struktureer dat dit opgelos kan word deur die gebruik van ekwivalensie tegnieke waarmee hulle reeds vertrou is.

'n Belangrike aspek van hierdie navorsing is interkonnektiwiteit. Ek wil die interkonnektiwiteit tussen aspekte in die Wiskunde Geletterdheid Leer Uitkomst van die Nasionale Kurrikulum Verklaring uitwys en hierdie verbande aan die leerders duidelik maak deur die voorgestelde aktiwiteite. Die begrip van interkonnektiwiteit word ook beskou as a hoër-orde denkvaardigheid en maak deel uit van meta-kognisie wat handel oor refleksie oor gedagtes en prosesse.

In kort kan hierdie navorsing opgesom word as die ontwikkeling van aktiwiteite wat ten doel het om die proporsionele denkvaardighede van leerders te ontwikkel en konneksies te maak tussen verskeie konsepte terwyl daar van leerders vereis word om aktief deel te neem aan hul eie leer van vaardighede en kennis.

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Chapter one: Introduction

1.1 Overview

This investigation recounts research into proportionality as higher order thinking skill in the Mathematical Literacy classroom and the problems encountered in developing these skills within three private schools on the West Coast of the Western Cape in South Africa. It is an account of activities designed from in-depth analysis of learners' understanding of proportional reasoning that can potentially help to develop proportional thinking skills with learners entering the field of Mathematical Literacy. These learners often enter the Mathematical Literacy class with a number of emotional and cognitive barriers towards Mathematics as I will show through a baseline assessment.

Mathematics as being taught in schools is generally perceived as a subject to make children mathematically literate, and although this is a major outcome of the subject, Mathematics is also a *mental discipline* (Schoenfeld 1992: 35). Ideally teachers want their learners to be inquisitive problem solvers by the time they leave school. We want them to take whatever information is available, see what is applicable and use it to solve intricate real-world problems, very much the process being followed when for example solving geometry questions at school: use the theorems we know, see which ones are applicable to the situation and use them to find the unknown. Unfortunately this idea is not always the reality in schools.

Mathematics as mental discipline dates back as far as Plato. The idea that mathematicians are good thinkers and those trained in Mathematics become good thinkers can only be true if teachers give their learners opportunities to think. So often teachers use a Pavlovian technique with their learners where learners have to spot a stimulus and give a certain automatic response. Although giving an automatic response requires some thought, it is not the thought processes of an autonomic, self-thinking and individual that will develop high order thinking skills.

When the South African Department of Education enforced Mathematical Literacy as compulsory alternative to Mathematics for all students within the National Curriculum Statement in 2006, it placed both teachers and students in unfamiliar terrain. This new initiative aims to empower *all* learners to be mathematically literate by the time they leave school, meaning that learners should be able to identify, interpret and understand mathematics in their daily lives. This includes comparing prices, calculating discounts, interpreting map scales, understanding taxes, inflation and interest, to mention but a few.

There has been some controversy about the meaning of mathematics in the “daily lives” of teenagers. Certainly not all teenagers are active investors, car buyers or the like (Julie 2006: 67). Although this is true, and many may argue that these teens might need this knowledge once they become independent young adults, I believe that the emphasis should be on the use and appreciation of Mathematics, rather than forcing realistic but unfamiliar context upon them. In Mathematical Literacy, like in many other subjects, context is used as vehicle to convey information. When learners write newspaper articles in the language classrooms it does not make them journalists, and since few young people still read newspapers the context may not even be that familiar to them at all. Yet, whilst writing these articles, they are learning about writing styles and techniques. When using a context in Mathematical Literacy, it is mostly used as a problem solving approach where learners must evaluate and reflect upon answers using the context. Using a financial context, does not mean that we want learners to be financial advisers, it is used to spark thought and educational discourse. When for example we refer to a choice between two investments, the context guides learners to come to a conclusion after they have completed the necessary calculations. Without a context the calculations would be meaningless. It is important that teachers are empowered to teach in such a way that the intention of *meaning* and appreciation for Mathematics may materialise. This emphasises the great need for research into Mathematical Literacy.

What is of great concern, is the lack of qualified Mathematical Literacy teachers and the lack of interest by Mathematics teachers to teach this new subject. Julie (1996) reports that Mathematical Literacy is “more difficult to teach than the ‘normal’

school-going Mathematics”. Although no reasons were offered for this finding during his research I would like to think that it is a combination of the following factors:

1. Often learners taking Mathematical Literacy have a negative attitude towards Mathematics.
2. Some textbooks present Mathematical Literacy as watered down Mathematics and this is de-motivating to both teachers and learners since it is perceived as lower grade Mathematics.
3. Learners taking Mathematical Literacy and even teachers teaching Mathematical Literacy are *perceived* to be less capable than learners taking and teachers teaching Mathematics.
4. For a teacher to make Mathematical Literacy a challenging and enjoyable subject, many additional resources need to be gathered which is time consuming for already overburdened teachers.

I have had the experience more than once that learners taking Mathematics are hesitant to ask Mathematical Literacy teachers for help since these learners perceive teachers teaching Mathematical Literacy as less competent.

Learners taking Mathematical Literacy are also often labelled by learners taking Mathematics as being “stupid” or “not clever enough” to take Mathematics. This does not contribute towards building a positive Mathematics-specific self-esteem.

Teaching Mathematical Literacy places a heavy emotional and time management burden upon teachers. The teacher in the Mathematical Literacy classroom cannot only teach learners to be mathematically literate, but must build a sound self-esteem in learners taking the subject so that they may too experience Mathematics as a wonderful tool to interpret the world around them. They have to create an appreciation for the subject by finding resources that will excite and entice their learners – a difficult task indeed. Research into Mathematical Literacy is thus not only to empower learners to discover the wonder of Mathematics, but to also empower teachers to be enthusiastic about the possibilities of the subject.

This chapter is an introduction to the inquiry by first examining the background, context and rationale for the investigation. The research problem is also set out before looking at the research design and methodology. Finally the scope, importance and limitations will be discussed before outlying the rest of the inquiry.

1.2 Background and context of the inquiry

Mathematical Literacy as compulsory alternative to Mathematics has always appealed to me as a deserving initiative. The thought of empowering all learners with basic Mathematical knowledge is a worthy cause. I have become increasingly passionate about the possibilities of the new subject. There is a strong thread of proportionality running through the Mathematical Literacy curriculum which is a hallmark of the formal operational stage of development – the highest level of thought according to Piaget's levels of cognitive development (Inhelder & Piaget, 1958). Proportional reasoning skills are essential in understanding a range of related concepts such as rate, functions, percentages, appreciation, depreciation, trigonometry, enlargements and reductions, to mention but a few.

Many researchers have reported on the difficulties that students have with proportional reasoning and have identified some of the variables that affect problem difficulty (Abramowitz 1974; Karplus, Pulos & Stage 1983; Hart 1981, 1984; Noelling 1980a, 1980b; Rupley 1981). More than a mere lesson or two must be spent on ratios for learners to master the concept and accompanying proportional reasoning skills, yet many Mathematical Literacy have very limited sections on this concept.

It is important to realise that although the learners taking Mathematical Literacy may not be talented in manipulating algebraic expressions or recognise theorems in a complex geometric drawing, they are not mathematically illiterate. Many skills being developed in Mathematics classes are in fact of little particular use to the average adult and I am doubtful if many highly successful doctors, accountants or even lawyers can still remember how to factorise a trinomial. What would have been of value to them, would have been the development of specific thought patterns that would aid them in drawing up a budget, calculating the effect of higher interest rates

and inflation, reading a map and calculating distance or even something simple as manipulating a recipe. But, as mentioned earlier, these concepts all have proportional reasoning as foundation and to fully understand these concepts sound proportional reasoning skills are necessary.

In my grade 12 Mathematical Literacy class of 2008, I had 23 learners of who 22 took either Dance, Drama, Visual Arts or a combination of these subjects. It made me realise that the learners in my class may not be as capable as I am in solving intricate Mathematics problems, but that they are talented far beyond my own capabilities in other interest fields. Each of them bring a unique set of experiences and interests to the classroom and it is my responsibility to use these interests as window through which learners can see the wonder of Mathematics. Would it be fair to judge these learners' intelligence based on their ability to manipulate algebraic or trigonometry expressions?

In 2008 a grade 11 girl taking Mathematics requested to move to Mathematical Literacy since she felt that she was not coping in Mathematics. I provided her with the work she missed out on and scheduled tutorials for her to catch up. After about a month, and poor progress, she requested to move back to the Mathematics class. She explained that Mathematical Literacy was just beyond her grasp and she found Mathematics much easier. Learners who failed grade 9 Mathematics, unlike this girl who did fairly well in grade 9, and have been in Mathematical Literacy ever since the start of grade 10, were able to easily score between 60% and 80% on the same tests she was not able to pass.

This made me even more curious as to the nature of Mathematical Literacy. To what extent am I empowering the learners in my Mathematical Literacy classes? What skills have they acquired that this girl lacked? Is it possible to change the perception that currently exists around Mathematical Literacy?

1.3 Rationale of the inquiry

1.3.1 Why Mathematical Literacy?

Mathematical Literacy is defined as a subject driven by life-related applications of mathematics. It enables learners to develop the ability and confidence to think numerically and spatially in order to interpret and critically analyse everyday situations and to solve problems (Department of Education 2006a: 9).

Little work has been done in the new field of Mathematical Literacy and many Mathematics teachers believe that it will also die a silent death as other “Applied Mathematics” subjects of the past, such as . To keep this initiative alive it is of the utmost importance that research must be conducted that will show the subject’s worth and the valuable position it has in the curriculum.

Unlike the heavily loaded curriculum of Mathematics, Mathematical Literacy has a less intensive work load. The approach in Mathematical Literacy is more on developing logic and problem-solving techniques than on the manipulation of expressions as we find so often in Mathematics. Mathematical Literacy provides the teacher with time to develop thought and investigate Mathematics and Mathematical procedures in all aspects of life. It is important to have teachers teaching Mathematical Literacy that will encourage thought and that are aware of the Mathematics around them.

North (2008) has remarked how Mathematical Literacy has changed his outlook on life. He uses the example of going to Mugg and Bean for a bottomless coffee. He was given a big mug instead of a standard size cup for his coffee and started wondering why the restaurant would do this? On investigation, he realised that people who are given a cup, tend to drink two cups of coffee, and people who were given a mug would normally have just one helping. He asked the waitress to supply him with a cup, a mug and a big glass of water. After filling the cup with water twice and filling the mug using the water in the cups, he found that two cups could hold more coffee than one mug. To save on their bottomless coffee, it is better for the management of Mugg and Bean to give their clients a mug rather than a cup. Other teachers of Mathematical Literacy could relate similar stories. If Mathematical Literacy teachers

could develop this investigative outlook with their students, may change the way Mathematical Literacy is currently perceived.

The didactical strategies involved in teaching Mathematical Literacy are complex. It follows a strong Learning Support (remedial) approach to teaching, not only in the content but also in attitudes. If teachers are not excited and interested themselves in the subject, their learners will not be excited and involved either.

1.3.2 Why proportional reasoning?

“Proportionality underlies key aspects of number, algebra, shape, space and measures, and handling data¹. It is also central in applications of mathematics in subjects such as science, technology, geography and art” (Department of Education & Employment 1999: 5). Proportional reasoning underlies nearly all content in the Mathematical Literacy National Curriculum Statement. If we look at an example of comparing prices:

If one apple costs 15c, how much would five apples cost? We work with the equivalent ratios of 1:15 and 5:75

We also use proportional reasoning in map scales, functions, calculus, enlargements and reductions, probabilities, percentages, tax and especially trigonometry. Proportional reasoning can be seen as the foundation of all learning content in Mathematical Literacy and it is thus important that teachers help learners in building a strong foundation for their learning structures.

Many leading researchers (such as Carpenter, Fennema, Franke, Levi & Empson 1999; Tournaire & Pulos 1985) have identified the difficulty that learners experience with proportional reasoning and state that many of them, even as adults, never fully

¹ There is a strong correlation between the UK Framework for Teaching and South Africa's five Learning Outcomes in Mathematics and Mathematical Literacy: LO1 : Numbers, Operations and Relationships, LO2 : Patterns, Functions and Algebra, LO3 : Space and Shape, LO4 : Measurement and LO5 : Data Handling

develop the ability. If there are adults who are not able to reason proportionally, how do Mathematical Literacy teachers develop proportional reasoning with learners who have predominant negative experiences in Mathematics? How do teachers engage them in activities that will make them *think proportionally* if they associate Mathematics (and Mathematical Literacy) with feelings of failure and confusion?

Although researchers have identified the problems associated with proportional reasoning and defined levels of developing proportional reasoning (such as Noelting 1980; Carpenter et al. 1999), little has been done on solutions to the problem. In the words of Lamon (1993) “we need to move beyond the level of identifying a litany of task variables that affect problem difficulty, toward the identification of components that offer more explanatory power for children's performances in the domain.”

Guidelines towards teaching and developing proportional reasoning are available, but researchers (e.g., Lamon 1999) agree that proportional reasoning entails much more than setting up a proportion and cross-multiplying. In particular, a proportional reasoner should be able to do the following:

1. *solve a variety of problem types* (Carpenter et al. 1999; Cramer, Post, & Currier 1993; Behr & Lesh 1989; Karplus et al. 1983b; Lamon 1993b; Noelting 1980; Post, Lesh, & Behr 1988)
2. *discriminate proportional from non-proportional situations* (Cramer et al. 1993; Lamon 1995)
3. *understand the mathematical relationships embedded in proportional situations* (Cramer et al. 1993; Lesh et al. 1988).

It should be clear from the three identified learning outcomes that, unlike what many teachers and learners might believe, proportional reasoning entails a lot more than merely the manipulation of ratio problems. Many textbook chapters labelled “Ratio and Proportion” present a few ratio word problems as the extent of the concept. Proportionality is a much wider concept and not seeing it as such is in my opinion a cause for great concern. As the research above suggests, to reason proportionally (and reach higher levels of thought), one must be able to think of proportionality in its

wider sense of meaning. To think that it is merely the manipulation of ratios is to sell oneself short of the wonder of Mathematics and its interconnectivity. Wu (2002) uses the following analogy for the problem of transferring knowledge in Mathematics: “We endeavour to make students see the individual trees clearly but, in the process, we short-change them by not calling their attention to the forest.” Using one’s knowledge of ratios to better understand functions, rate and percentages is fundamental in developing sound proportional reasoning. This is why the researchers say that children need to *solve a variety of problem types*. This variety must include direct, indirect and proportion with a constant, such as $y = mx + c$, so that learners can *distinguish between proportional and non-proportional situations*. This all helps in *understanding the mathematical relationships embedded in proportional situations*. The aim of this study will be to focus on proportionality (not merely ratio) and how we can develop proportional skills in such a way that learners will be able to see proportionality in a variety of Mathematical concepts.

1.4 Problem statement

How should Mathematical Literacy teachers design activities that will help learners to develop proportional reasoning skills?

Key questions that will arise from this research problem:

1. How can existing research on proportional reasoning help learners to develop proportional reasoning?
2. How important is proportional reasoning within the Mathematical Literacy National Curriculum Statement and how can the concept of proportional reasoning be developed as an interconnected knowledge structure within the National Curriculum Statement?
3. How can proportional reasoning activities be designed and sequenced so that learners will build on existing knowledge structures as well as interconnected knowledge structure between related concepts?

It is a misconception that higher order skills, such as thinking proportionally, can be taught. These skills need to be developed through carefully constructed activities to

initiate thought and engage learners in problem solving. It is the task of the Mathematical Literacy teacher to develop these thinking patterns so that learners may develop a better understanding of proportionality and how it links to other concepts.

This also raises the question of how a better understanding of proportionality can aid learners' understanding of related concepts, if indeed it does. If this be the case, then how can teachers help learners in making the mental links between related concepts?

Through Person's (2004: 23) study he has found that "techniques and number operations are given higher priority than concept clarity. It may also throw students in an abstract and technical world where connections between concepts are harder to make but where the correct answer is found."

As mentioned earlier, teaching for mere reproduction is not the aim of Mathematical Literacy or of the National Curriculum Statement as such. With teachers who do not fully comprehend proportionality, it is nearly impossible for learners to develop a sound understanding of the concept. This study hopes to provide teachers with an understanding of proportionality and methods of teaching that will help to develop a more comprehensive understanding of proportionality and the associated thought patterns with learners.

The concept of interconnectivity forms a vital part of this study and will show how proportional reasoning is prevalent in all four learning outcomes of Mathematical Literacy within the National Curriculum Statement. This prevalence supports the importance of research into didactical practices that will support the development of proportional reasoning in Mathematical Literacy. It also poses a challenge to Mathematical Literacy teachers who must guide learners in developing cognitive structures that are able to connect knowledge from different concepts through a range of higher order thinking processes. Piaget, as in Louw, Van Ede and Louw (1998: 75) describe these thinking processes as adaptation where learners either have to integrate new knowledge into existing knowledge structures, known as assimilation, or adapt existing knowledge structures according to the new knowledge, known as accommodation. Activities must therefore be designed, sequenced and structured in such

a way that it will assist learners in making these cognitive adaptations in order to build a network of interconnected knowledge structures.

1.5 Research approach

Coe (2007: 17) states the “goal of research is to provide a coherent model to explain the possible ways a person may be thinking.” The type of research that I will be using during this investigation is experimental in nature and is classified as “design research” according to Edelson (2002).

The design of the research will be based on experimenting with designed activities for classroom teaching to form a better understanding of proportional reasoning and how it links to other subject content. I will thus be designing activities based on theoretical and field research and present the activities in a researched didactical way that supports the activity. Data collected on learners’ thinking patterns and motivation for answers will aid analysis and interpretation on learners’ response, the didactics used and the activities itself (Mouton 1996: 24).

During this study I will investigate cognitive processes involved in developing proportional reasoning. Piaget classified proportional reasoning as a higher order thinking skill, and although his work has sparked strong critique and controversy, research has shown that few people are able to think proportionally. Kolodity, as in McLaughlin (2003: 1) states “that a majority of college freshmen do not function at this level and Lawson (1975) cites that 40% to 75% of post-secondary students do not operate at Piaget’s formal level.”

This evidence supports Piaget’s claims of the high level of thought that is involved in proportional reasoning and emphasises the great need of more insight into proportional reasoning teaching and learning within Mathematical Literacy. Many researchers have done research into children’s understanding, and problem of understanding, ratios and proportions. I shall be using this as basis for further study into proportionality and its interconnectivity to other concepts within the subject of Mathematical Literacy.

I shall also be conducting a baseline assessment to determine the level of understanding of proportionality that a learner possesses when entering the subject of Mathematical Literacy at grade 10 level. The analysis of the baseline assessment will be an indication of learners' pre-existing knowledge and misconceptions from which further activities can be developed. This is in line with Outcomes Based Education where continuous formative assessment forms a central part of teaching. This investigation will use formative assessment to monitor the learner's progress and to ensure that the strategies used in the designed activities are in fact being learnt by learners.

The main focus of this study will however fall on designing didactical material to help learners in forming a better understanding of proportional reasoning, and I will therefore mainly use *design research*. Design research takes the problem statement to merely direct the research process. The process is experimental in nature and after design and implementation the process and the hypotheses needs to be analysed to make adaptations and elaborations on the original hypotheses. The design and implementation is central to design research as researchers engage in teaching and learning experiences in order to develop a better and meaningful research understanding on didactical practices. Information yielded from the research can thus be used by teachers in the classroom to better an understanding of the research concept (Edelson 2002: 107, 118-119).

The existing research into ratios and proportional reasoning will help to formulate certain set hypotheses which will either be supported or proved insufficient after the baseline assessment. From here, if necessary, a new set of hypotheses can be formulated and activities can be designed accordingly. Evaluation of the process will happen continuously in the form of formative assessment. The directing hypothesis will be to design activities in such a manner that it will better learners' understanding of proportionality and its interconnectivity in the Mathematical Literacy National Curriculum Statement.

1.6 Scope

For the purpose of this study, I shall direct my focus on grade 10 learners entering the subject of Mathematical Literacy in three private schools in the West Coast District of the Western Cape. At this stage of their high school careers they have been through the General Education and Training strand and should have a basic understanding of most Mathematical concepts. We need to assume that most of these learners have a fairly wealthy home environment and have been enriched by a privileged school setting.

All three of these schools offer easy access to computers and related teaching aids to enhance teaching and learning in the classroom. Classes are kept small with no more than 25 learners per class. Individual attention and support to learners is a high priority of both teachers and parents. Parents sending their child to any of these three schools see their child's schooling as an investment rather than a mere obligation and for this reason also expect results. This gives an interesting slant to teaching practises. Teaching and learning in a private school environment differs in many regards to teaching and learning in most government schools where classrooms are crowded, technology limited and parent involvement is often inadequate. These factors should not be overlooked during this study for two main reasons:

1. The teaching and learning environment can mostly be described as ideal in comparison to the environment found in most government schools.
2. If the development of proportional reasoning as a high order thinking skill proves to be problematic in such an "ideal" environment, how much more difficult must it then not be in an overcrowded, technology poor classroom?

I have decided to focus my attention on grade 10 Mathematical Literacy since the National Curriculum Statement is rather thin and many of the schools I have dealt with complain that they finish it too quickly. Except for the fact that teachers will feel less concerned about their teaching time being used for this research, I will also be looking at ways in which the grade 10 National Curriculum Statement could be used as better foundation for grades 11 and 12 Mathematical Literacy.

The National Curriculum Statement is not heavily loaded for Mathematical Literacy in grade 10 and teachers should caution that learners do not start to think that Mathematical Literacy is indeed the substandard subject that many make it out to be. Although easy tests and resultant high test scores by learners may boost learner's self-esteem, it is meaningless if learners know that they are writing ridiculously easy tests. Mathematical Literacy at grade 10 level should build on grade 9 and should not be substandard to it. I am of the opinion that grade 10 can be used as a remedial year during which learners' problems can be addressed and self-confidence built whilst still challenging their abilities. Learners cannot rise to the occasion if no occasion is provided. Grade 10 should not be a calculator number punching year, it should be the start of level four reasoning and reflecting according to the Mathematical Literacy taxonomy (Department of Education 2007: 14).

1.7 Limitations

The timeframe of this research was restricted to one term only. Classroom activities done in this time period and information yielded from these activities were experimental in nature and was used to help design activities rather than to examine the successfulness of the final product. This process formed part of the design research approach described under research methodology. Constant reflection was needed on the process and the activities to ensure that it was in line with the needs of learners. Activities that showed to be deficient after reflection on the process supplied the process of opportunities of improvement. Since this is a continuous process, it is difficult to establish when the activities are near perfected. The design research process is thus limiting due to its continuous nature.

Although it would be possible to conduct a post-test in these three schools to test the successfulness of the sequence of activities designed, results would be biased since my role as both teacher and researcher could influence the outcomes. To have objective results, this program should be given to a representative sample group of schools with teachers that are willing to engage in and give criticism on the program.

1.8 Importance of the research

Although there are currently some controversy on the creditability of Mathematical Literacy when applying for certain university courses since it is seen as lower grade Mathematics and not as a valuable subject in itself, I argue that the subject lends itself towards becoming an intensive remedial programme in which teachers can address problems and work towards developing proportional thinking patterns that is of a higher level thinking order. Mathematical Literacy could thus be taught and ultimately assessed in grade 12 in such a manner that learners who developed these high order thinking skills could be distinguished from learners with a basic understanding of the content being covered. Through this study I would like to argue that learners who display higher levels of thought in Mathematical Literacy must be considered for tertiary studies (excluding fields of study which require calculus), but most definitely in financial fields.

1.9 Outline and organisation of this report

I start this research with a literature review on proportional reasoning in chapter two. This will include the teachers' perspective as well as learners' development of proportional reasoning. It would not be possible to design a baseline assessment without understanding the concept of proportional reasoning and having insight into both teacher's and learners' understanding of the concept.

Chapter three contains all information regarding the baseline assessment to test learners' existing knowledge of proportional reasoning as well as their attitude towards Mathematical Literacy. This will include the theoretical analysis of the baseline assessment as well as detailed results of the assessment.

The theory regarding the didactical practices needed for the design research and teaching process is described in chapter four. This will include both the way in which the activities will be presented as well as the context within which the activities will be presented. It is also in chapter four where the importance of proportional reasoning throughout the Mathematical Literacy National Curriculum Statement is described and the importance of the interconnectivity of these different learning fields.

Chapter 5 describes the rationale for the designed activities as well as the sequential progression of the presentation of these activities.

Chapter two: Literature Review

Before designing activities for baseline assessment or for the development of proportional reasoning, it is important to reflect on literature done on people's understanding of the concept. I will firstly look at the work of Clark, Berenson and Laurie (2003) who have researched the comparison of fractions and ratios and teachers' understanding of the relation between these two concepts. To have insight into teacher's perspective of proportional reasoning might help in understanding learners' understanding of the concept. Since I will also point to the importance of understanding fractions in order to understand ratio, considering the work of Clark, Berenson and Laurie (2003) is essential.

Secondly, I will look at learners' understanding of proportional reasoning through the work of Noelting (1980) and Thompson and Thompson (1994). Noelting's research on the developmental stages of acquiring proportional reasoning skills played a vital role in the design of the baseline assessment activities.

2.1 The teacher's perspective and the importance of fractions

One cannot assume that all teachers teaching Mathematical Literacy are proficient in proportionality. "Although a lot of research has been conducted on students' understanding of proportional reasoning, few publications actually focus on teachers' conceptual understanding of ratios and proportions, especially at a high school level" (Person 2004: 4).

Teachers bring a unique view of proportionality to the classroom based on the way they have been taught years ago and made sense of the concept. "But traditional mathematics lessons have consisted of the demonstration (sometimes with explanation) of a single method followed by practice with a variety of different numbers. Converting fractions to decimals or percentages, performing operations on directed numbers, and solving proportion problems have all been dealt with in this way" (Bell 1993: 7). Many teachers qualified before the introduction of Outcomes

Based Education, many newly qualified teachers still revert back to the way they have been taught at school and so the cycle of instructional teaching keeps on repeating itself. “The notation and strategies that teachers use in class to solve proportion-related problems has an impact on the conceptions and activity of individual students, while the students’ preferences may influence the teacher’s pedagogical strategy, and therefore the classroom mathematical practices” (Clarke, Berenson, & Cavey 2003: 302).

Because of the broad extent of proportionality, teachers take a variety of views when thinking about the concept. Clarke et al. (2003) distinguish between five models:

Model 1: ratios as a subset of fractions

Model 2: fractions as a subset of ratios

Model 3: ratios and fractions as distinct sets

Model 4: ratios and fractions as overlapping sets

Model 5: ratios and fractions as identical sets

According to Clarke et al. (2003) and Person (2004) teachers choosing Model 1, do so on the assumption that all ratios are rational numbers that can be written in a fraction format. This however excludes irrational numbers as we encounter so often in trigonometry ratios (the ratio of the hypotenuse to leg length of an isosceles right triangle ($\sqrt{2}$)) or working with π when doing calculations in geometry. It also excludes ratios that imply more than two numbers, for example an inheritance that is divided between three siblings according to their age.

Model 2 implies that all fractions can be interpreted as ratios, but that not all ratios are fractions. In Clarke et al.’s (2003) research this model proved to be the most popular amongst participating teachers and mathematicians, since this model includes the scenarios that have been excluded in model 1. It can however be argued that fractions encountered in pure number related context, for example measuring, cannot be considered a ratio, although some argue that it is still a part-whole relationship (as in 20 mm of 1 m).

Teachers supporting Model 3 did so by describing a fraction as a part-whole relationship and a ratio as a part-part relationship with no connection between the two (Clark et al. 2003: 299). If we think of mixing juice, we can represent the situation in a variety of ways: 2 parts concentrate to 8 parts water (2:8) or 2 parts of the 10 part liquid are concentrate ($\frac{2}{10}$). Surely the part-whole (fraction) relationship is also a ratio 2:10?

Model 4 considers the queries about model 2 and implies that “some, but not all, ratios are fractions, and some, but not all, fractions are ratios” (Clarke et al. 2003: 300). This model seems to accommodate the views of most teachers and the broad extent of proportionality.

In Clarke et al.’s (2003) research they have not identified any textbooks that claim that fractions and ratios are the same, as Model 5 suggests. Yet, the concepts are sometimes presented as so similar, that it is difficult to distinguish.

Clarke et al. (2003) defend their view of Model 4 with the concern that ratio as an all-inclusive concept “loses its power of discrimination”. They argue that ratio is attenuated when the term is used so loosely. This is however exactly what I would like to argue: teachers need to convey the broad extent of proportionality. I believe that in contrast to the belief of Clark and his fellow authors, model 2 rather intensifies the concept of ratio. The distinct relationship between fractions and ratios as set out in Model 2 could be a handy starting point to initiate discussion in class and could act as concept for debate.

Teachers should have a clear understanding of proportionality and how it can be presented before they try to convey this broad and powerful idea to learners as a mere simple “comparison of numbers”. Clarke et al. (2003: 315) also emphasise the importance of mathematical terminology together with notation that “can act as either pathway for students as they grow in their understanding or as obstacles to that growth.” It is important to note that the introduction of ratio and proportion does not have to include the colon or fraction representation. In fact, no mathematical notation

is needed when developing proportional reasoning. Karplus et al. (1983) found that “when students did employ proportional reasoning, they usually chose the type of comparison that allowed them to use integral ratios among the given data...” and tried to avoid working with fractions as far as possible. Unless we are thus sure that learners are secure in their fraction understanding, fraction notation could prove daunting to learners. Although learners might have seen the colon notation on maps or juice mixtures, the meaning of the notation might be more inhibiting than helpful.

Adjidage and Pluvinae (2007) have conducted extensive research into the concept of ratio and proportion and found that “processing fractions is well accorded to the valid treatments in proportionality, so that pupils that master fractions should better master proportionality”. If teachers can make this all important link between fractions and ratios, Adjidage and Pluvinae (2007: 170) argues that it will not only lead to a better understanding of proportionality, but also of Algebra. Many fraction type questions can also be considered as ratios and vice versa. Let us look at the following example. Would this be categorised as fraction, ratio or rate?

Three boys share two pizzas and nine girls share seven pizzas. Did the boys or the girls have more pizza each? Explain your answer.

Thinking in terms of fractions:

Sharing 2 pizza's between 3 boys, is $2 \div 3 = \frac{2}{3}$

Sharing 7 pizza's between 9 girls, is $7 \div 9 = \frac{7}{9}$

If there were 9 boys, they would have had to get 6 pizzas ($\frac{2}{3} = \frac{6}{9}$), thus meaning that the girls had more.

Thinking in terms of ratio:

Boys: 2:3 Girls: 7:9

With-in strategy²:

Boys: $2 \times 1,5 = 3$

² The with-in and between strategy will be discussed under the work done by Gerald Noelting.

Girls: $7 \times 1,5 = 10,5$

Between strategy:

Boys: $2 (x3) : 3 (x3) = 6:9$

Girls: 7:9

The girls had more.

Thinking in terms of rate:

How much pizza did 1 boy have? $\frac{2}{3}$ or 0.67

How much pizza did 1 girl have? $\frac{7}{9}$ or 0.78

The girls had more.

Ratio problems can thus be a natural extension to fractions and learners can be left to explore informal ways of representation.

Comprehensive research has been done by Noelting (1980) who tested a large number of children between the ages of six and sixteen on their concept of fractions versus ratios. He gave them the following four questions:

1. The orange juice problem where learners have to compare the strength of the two mixtures: (2:3) vs. (7:9)
2. The pizza problem where 3 boys shared 2 pizza and 9 boys shared 7 pizza and learners had to determine which group had more pizza per person.
3. A normal comparison of fractions problem: $\frac{2}{3}$ vs. $\frac{7}{9}$
4. A normal addition of fractions problem: $\frac{2}{3} + \frac{7}{9}$

His findings showed that pupils had few problems in solving calculations involving adding and comparing fractions, but that pupils found ratio questions problematic. This leads us to the question: Can the pizza problem really be seen as purely a ratio problem? In the way the question was asked, it could surely be classified as a rate problem. Furthermore, this type of question should, according to Murray, Olivier and Human (1999), be used as part of the problem-solving approach in fractions. Learners

could possibly find this problem difficult due to the wording being used or a lack of exposure to problem-solving questions when dealing with fractions. Previous teaching experiences might have been mostly instrumental in nature, where learners were given strict methods and procedures to follow when confronted with fraction questions. This automatic technique does not guarantee that learners understand fractions. I am sure that learners who were faced with problem-solving type questions when dealing with fractions would have recognised the pizza problem as a fraction problem and would have dealt with it accordingly.

Learners are exposed to fractions from as early as grade four and might have been continuously exposed to problems similar to the normal addition and comparison of fractions, whereas their knowledge of problem-solving type questions or ratios might have been limited. For learners who have never been exposed to ratios, the notation in itself might be problematic.

Wu (2002) supports the link between ratios and fractions when he says that if “Euclid in his *Elements*...had the mathematical understanding of the real numbers as we do now, he would have said outright: ‘the ‘ratio of A to B’ means the quotient $\frac{A}{B}$ ’. For this reason, the long tradition of the inability to make sense of ratio continues to encroach on school textbooks even after human beings came to a complete understanding of the real numbers round 1870.”

If learners are comfortable in solving fraction problems, then a logical deduction would be that if we can stress the important link between fractions and ratios, then learners could represent the ratio problem in terms of fractions and solve it. As an example: 2 apples cost R3, what is the price of 5 apples? Learners could represent it as $\frac{2}{3} = \frac{5}{?}$ rather than calculating the cost per apple. Wu (2002) warns however “that the central issue is *why* these enigmatic “ratios” should be equal, but the explanation of this issue is usually not forthcoming in a typical classroom.” Learners who were taught that fractions are a part-whole situation will have trouble buying into the fact that 2 *apples* is a part of 3 *Rand*. This fraction is capturing the *rate*: 2 apples per R3, which can first be simplified to 1 apple per R1,50 before further calculation. This is

the joy of writing ratios as fractions that we simplify (*within*) before we make calculations (*between*). This does not necessarily mean that we find the *rate*, but merely that we simplify to more manageable set of numbers. For example: 21:18 = 28: ? can be calculated as:

$$\frac{21}{18} = \frac{28}{?} \text{ simplified to } \frac{7}{6} = \frac{28}{?}$$

With the simplification it is easy to see that the multiplicative *between* factor is 4.

$$7 \times 4 = 28 \text{ thus } 6 \times 4 = 24$$

Some researchers hardly distinguish between proportional reasoning and fractions. Whilst discussing the difficulties learners experience in proportional reasoning Schwartz (1998) gives an example of comparing $\frac{2}{5}$ to $\frac{1}{2}$ and goes on by explaining how learners have to compare both *denominators* and *numerators* and the relation between and with-in each fraction.

Lo and Watanabe (1997: 225) conducted research on a fifth grader's developing understanding of ratio and proportion. The following problem was given to a fifth grader named Bruce. His strategy in solving it clearly supports the importance of fractions in developing proportional reasoning schemes:

A house was 24 feet tall and had a window that was 12 feet above the ground. This house became 18 feet tall after a certain amount of magic liquid was applied. How tall would the window be above the ground after the magic liquid was applied?

Bruce's solution of this task clearly indicated his intention to find the multiplicative relationship between the new height of the house and the old height of the house, which he knew needed to be preserved between the new height of the window and the old height of the window. He identified 6 as a common factor between 24 and 18. The number 24 was reconceptualised as 4 sixes, and the number 18 was reconceptualised as 3 sixes. Then it appeared that there was another level of reconceptualisation, similar to what Lamon

(1993a) described as "norming": the 4 sixes were reconceptualised as "one," but the 3 sixes were reconceptualised as "three fourths." Bruce's last statements, "Nine is one half of 18. Twelve is one half of 24," did not appear to be part of his original thinking, but rather another way to explain his solution. Implicitly, Bruce seemed to be saying, "See, all these equivalent relationships prove that my answer is correct."

The concept of equivalence and equivalent fractions seems to be the foundation of Bruce's approach. Although Bruce might not be able to express his scheme in formal Mathematical language, he is reasoning proportionally and finding equivalent proportions to solve the problem.

I would like to stress at this point that I am analysing fractions as a method that can be employed to help learners create proportional reasoning schemes. They should not be seen as methods that are followed without any discussion of *why* they are followed. The reason why I suggest the use of fractions in developing proportional reasoning, is because learners *should* have a firm understanding of fractions by the time they enter into grade 10. Noelting's tests support the fact that they feel comfortable with fractions, but not with ratios, and if the link can be made to show learners the connection between the two concepts it might be a doorway to a better understanding of ratios and ultimately proportionality.

2.2 Understanding learners' understanding of proportionality

It is important that teachers not only have a clear understanding of the concept to be taught, but also the learners' understanding and possible misconceptions that may arise. When teachers anticipate certain common mistakes or misconceptions they can use these as learning experiences. It was Shulman (1986) who introduced this notion of *pedagogical content knowledge*.

When learners, for example, think additively instead of multiplicatively during proportional problems, it would be a good idea to give counter problems to prove them wrong. Placing the problem into a context and using the context to give meaning

to answers is the aim of context in Mathematical Literacy and should be used as such. If the context is used merely to mask a set of calculations, the context has no meaning.

To make Mix-On® cool drink, one needs 2 litres of water on 3 sachets of Mix-On®.
How many sachets does one need for 3 litres of water?

Children reasoning additively would instinctively say $2 + 1 = 3$ litres, thus $3 + 1 = 4$ sachets. Mixing the cool drink in class would be a visual and “tasty” activity to prove the learners wrong, since only adding 4 sachets would give a weaker taste. A counter argument that is also helpful is the following: “In other words, if I want to know how many sachets I need for 100 litres, I must say $2 + 98 = 100$ litres, thus $3 + 98 = 101$ sachets. Do you think this will taste the same than a mix of 2:3?”

Another idea might be to work with rate by asking learners how many sachets would be needed for one litre of water. Through calculating the sachets needed per litre, they are determining the *rate*. Linking the Mix-On® question to that of rate and price as in the following example might also give learners some insight into the Mix-On® ratio:

If I buy 2 apples for R3, how much would I pay for 3 apples?

Learners normally have a better “feel” for price than for ratios although both questions are asking them to make the exact same calculation.

Proportional reasoning is a high order thinking skill and so is transferring knowledge from one concept to the next as in ratio to rate and vice versa. This is one of biggest concerns in the Mathematical Literacy class: Linking knowledge of fractions, to ratios, to rate, to scale or any other related concept. When teaching Mathematical Literacy teachers must thus not teach these concepts in isolation since this will not lead to internalisation of knowledge. If we want learners to truly understand a concept, it is essential to link knowledge structures so that they may be able to use strategies that they are familiar with and have used in previous related concepts. Most learners entering Mathematical Literacy in Grade 10 know that $\frac{2}{3}$ and $\frac{3}{4}$ are not

equivalent, yet they do not make this link to ratios. This link will help them to realise that the ratio 2 : 3 and 3 : 4 are not the same.

Teachers need to ask conflicting questions to initiate thought about the procedures they used. “Does this strategy you used mean that $\frac{2}{3}$ and $\frac{3}{4}$ are the same? Or would the price of 3 apples be R4 if 2 apples cost R3?” When learners reflect on their thoughts, they are using essential higher order thinking skills.

Creating conflict to initiate thought is one of the most powerful approaches we can use when teaching Mathematical Literacy. Mathematical Literacy teachers need to get their learners into a habit of thinking about the *process* of getting to the answer, rather than merely producing the answer. Teachers need to press upon learners to stop and think about what they are doing rather than to produce answers in order to receive praise. They must establish a classroom culture where reasoning and reflection is praised instead of only correct answers. Teachers need to realise they need to develop skills such as transfer of knowledge and meta-cognition (reflection on thought) rather than teaching procedures which learners must be able to apply in different context based questions. Riedesel (1969: 428) found that “many students believe they are dealing with a new discipline with its own operations and rules. It is no wonder that they stumble through by memorising the new game rather than building firmly on past experiences.”

Teaching proportional reasoning is much more than manipulating numbers around a colon. Taking all these intricacies into consideration, Mathematical Literacy can not be seen as watered down Mathematics. In fact, it is an opportunity to explore all these high level thinking skills in a less packed curriculum. It is essential that teachers teaching Mathematical Literacy are empowered to give learners the support they need to develop these skills. This opportunity has a low priority in the heavily loaded curriculum of the Mathematics class.

2.2.1 Noeiting's pedagogical approach to ratio

Noeiting (1980) focused his attention on learners' understanding of two comparative ratios. This study enabled him to look at strategies used by learners in the development of proportional reasoning. Two major thoughts stemming from his research is that if the "between" strategy, referring to relations between a term in the first ratio and a term of the second ratio, and the "within" strategy, referring to the relation between terms in the same ratio. His research can be described as developmental stages of understanding of proportionality and can thus guide teachers in making sense of their learners' level of understanding. If teachers know *why* learners make certain mistakes, it is easier for them to design activities in such a way that it will address these problems and create conflicting thoughts.

Noeiting distinguishes between three broad strands, each with a couple of sub-stages. Stage I, subset A, has a typical structure of (1:4) vs. (4:1). The within strategies would here be of less value, but the between strategy, comparing for example the amount of concentrate in ratio one to ratio two, proved efficient. This method is however only possible if the second term remains constant or if two ratios are inversely related. Children often find it difficult to distinguish between inverse ratios and tend to say that the taste of the drinks would be the same. This is typical of stage 1A which only entails the comparison between the first terms of the two ratios.

Subset IB keeps the first terms of the two ratios the same ($a = c$ where $b > d$, for example 1: 4 vs. 1: 2) but have different terms for the second terms in the ratios. Subset B thus includes the strategy of 1A by first comparing the first terms in the ratios, finding them the same and then focussing on the relation between the second numbers.

To progress to stage IC, learners have to choose the with-in strategy as the appropriate strategy for the circumstance. According to Noeiting this stage focuses on with-in ratios first and between ratios second to that. A typical structure of this stage would be ($a = b$, $c < d$, for example 1: 1 vs. 2: 3). The thought process involved in choosing a with-in ratio first, is what distinguishes level IC from level 1A and B. Although this problem could be solved by using between ratios, a quick, logical way would be to see

that the two parts in the first ratio are exactly the same whilst it is not the case in the second ratio. All learners at stage IC still experience problems with the order of numbers within the ratio. They would be able to recognise that the ratios are different, but if the ratios represented concentrate and water, they would not be able to say which has a stronger or weaker taste. They would also not be able to see the difference between ratios 1 : 1 and 2 : 2.

Stage II requires that learners must be able to compare two with-in ratios, thus looking at between ratios of with-in ratios. Stage II is also the start of the use of calculations, and specifically multiplicative strategies, to compare ratios. At stage IIA, computation are not really necessary, since the structure is kept simple ($a = b$; $c = d$). The strategy that was used in IC is similar to that of IIA.

Stage IIB requires learners to first use with-in calculations, before comparing the with-in results, thus the between strategy. A typical structure of stage IIB is that of ($a \neq b$; $c \neq d$), but kept to equivalent ratios of (2:3 vs. 4:6) where the second ratio is a multiple of the first. Noelting (1980) describes the difference between stage IIA to IIB as “the independence of first and second terms inside the equivalence class”. The results of IIA and IIB are thus the same, since both compare equivalent ratios, but stage IIB requires the recognition of both terms in the second ratio being multiples of the terms in the first ratio.

The structure of stage IIIA is also ($a \neq b$; $c \neq d$), but only the first terms in each ratio are multiples of each other, whilst the second ones are not. The structure of the stage is thus of a non-multiplicative strategy within the ratios, but a multiplicative strategy in one between ratio. It is at this level where learners have to know to use both with-in and between strategies to compare the ratios.

The last stage is that of IIIB, which has a general structure of ($a \neq c$; $b \neq d$ and $a \neq b$; $c \neq d$) where no multiple relation is found between terms within ratios or between ratios. As in stage IIIA, both with-in and between strategies need to be used as well as multiplicative and additive strategies to transform the ratios into comparable forms. These multiplicative and additive strategies are the same strategies used when

comparing fractions with different denominators, where we first need to find a common denominator before making judgements about values.

It is clear, that stage IIIB would be rather difficult to reach if a sound understanding of fractions is not in place. Teachers seem to underestimate the importance of fractions as foundation for proportionality and algebra. I want to argue that without a clear understanding of equivalent fractions, learners would not be able to comprehend the thoughts underlying ratio and proportion as well as certain key concepts in Algebra.

Although Noelting's research gives us an indication of learners' developing thoughts surrounding proportional reasoning, his work mainly focuses on comparison problems. Although the comparison of ratios are important concept in developing proportional reasoning, more than half of the proportional reasoning situations encountered in Mathematical Literacy are missing value problems. The following concepts, which I will explain in terms of proportional reasoning in chapter 4.4, are all missing value type questions: VAT, appreciation, inflation, consumer price index (CPI), depreciation, scale, conversions and trigonometry.

2.2.2 Thompson and Thompson's research in the mental process in moving from ratio to rate

Thompson and Thompson (1994) acknowledge the importance of ratio in understanding other related concepts, such as rate. Rate is of course of great importance in understanding functional mathematics (and later calculus). Several researchers have made the same claims about the importance of ratio and rate in understanding higher level mathematics: Monk 1987; Thompson 1994a; Zandieh 2000; Carlson, Jacobs, Coe, Larsen, and Hsu 2002 as in Coe 2007:13. The question I would like to ask is, if problems in these concepts could be traced back to a poor understanding of ratio?

As Noelting, Thompson and Thompson (1994: 8) have also identified levels of development in children's understanding of ratio to rate. They distinguish between

four levels of development in the mental process of moving from an understanding of ratio to an understanding of rate:

1. The first level is that of *ratio*, in which children are able to compare two ratios without the need to change them. At this level they do not use any computations to solve the ratio, they make judgments merely on face-value. Children at this level would recognise that the ratios 1:4 and 1:3 are not equivalent and would be able to reason that the second ratio has less. In a problem with numbers that are not the same, children functioning at this level would not be able to compare ratio. If the ratios of 1:3 and 2:6 are presented, they will not be able to recognise the numbers in the second ratio as multiples of the first.
2. At the second level, *internalised ratio*, children compare ratios through basic additive strategies without realising that additive strategies will change the ratio between numbers. These learners will assume that 2:3 is equivalent to 3:4 since $2 + 1 = 3$ and $3 + 1 = 4$.
3. At the third level, *interiorised ratio*, children are able to use appropriate additive strategies to compare ratios or to find missing value. These strategies are also referred to as “building-up” strategies. For these children to calculate if 2:3 and 3:4 are equivalent, they will reason that if half of 2, namely 1, is added onto 2 to get 3, then the half of 3, namely 1.5, must be added onto 3 to get 4.5. Since this method did produce an answer of 4, the two ratios are not equivalent. Another example would be: 300g sugar for 4 people, 600g sugar for 8 people, another 2 people is half of 4 people, so that we need 750g sugar for 10 people. Although the “build-up” strategy delivers correct answers and can be seen as primitive multiplicative reasoning, researchers agree that is “*is a relatively weak indicator of proportional reasoning*” (Lesh et al., 1988, pp. 104–105).
4. It is only at level four that learners use the concept of *rate* as multiplicative strategy to solve ratio problems. Thompson et al. (1994) refers to this level as “reflectively-abstracted conception of constant ratio.” Children that are able to

operate at this level will be able to recognise that in the ratio 2:3, three is 1.5 times more than two.

2.3 Summary

Thompson's levels have the same progression to that of Noelting, but are not restricted to comparison problems only. The use of both level systems is important since each bring a different perspective to light. Noelting focus a lot of attention on the use of with-in and between strategies. Karplus, Pulos and Stage (1983) found that learners who were able to reason proportionally "exploited integral ratios within or between relationships, and... recommend the emphasis of these approaches to proportionality problems." Since these strategies are largely linked to the nature of the problem, problems should be given that will draw attention to both strategies. It is important that learners are made aware of different approaches to a question since this initiates thought on their own thoughts, also known as meta-cognitive skills.

These levels of development are important to teacher researchers if they want to determine the level of understanding of their own students. It is an indication of where they are, where they are going to and what we as teachers can do to get them to where we want them to go. With this in mind we can set activities accordingly to support the growth process. I will also be using this information in setting a comprehensive baseline assessment and to henceforth develop activities that will help my Mathematical Literacy students in forming proportional reasoning schemes.

Chapter three: Research

This chapter will give a theoretical background to the baseline assessment. This theoretical background will be the rationale for activities that were used and will be based on the research of Noelting (1980) as well as Thompson and Thompson (1994).

The data collected from the three independent schools will then be analysed to determine learners' existing proportional reasoning knowledge as well as their feelings and attitudes towards Mathematics. The data collected on their proportional reasoning skills will be analysed according to the work of Noelting (1980) and Thomson and Thompson (1994) to highlight different levels of understanding as well as different strategies used to solve proportional reasoning questions. Concerning factors will be summarised before analysing the baseline questionnaire results to determine learners' perceptions of Mathematical Literacy.

3.1 The theoretical approach to the baseline assessment

Noelting (1980) has distinguished between three broad strands of developing proportional thoughts in acquiring knowledge about ratios. We can assume with relative certainty that most learners entering Mathematical Literacy in Grade 10 will be confident in handling stage I questions. They should even find stage II questions relatively easy, but as can be expected the problem occurs with stage III questions. In the baseline assessment I will aim to determine learners' understanding of proportionality according to these strands. I will furthermore also include related proportional questions which will also be structured according to the degree of difficulty as Noelting (1980) set it out in his research on ratios. Questions will be structured in such a manner that it does not promote any strategies. Learners will thus be able to use either the with-in or between strategies.

The baseline assessment acts as a base-line from where activities will be designed that could potentially address any problem areas identified in the assessment. Through the baseline assessment I will also try and identify learners' level of understanding

according to Noelting's levels and if learners can relate their knowledge of one concept to similar situations. I would also like to determine which strategies they follow when working with ratios and if these are the same strategies they employ when working with related concepts. Questions are set up in such a manner, as Noelting also intended, to favour several strategies such as the with-in or between strategy or a combination of the two.

The baseline assessment will include a questionnaire that will contain questions on learners' perception of Mathematics and Mathematical Literacy. With these questions I would like to establish learners' experiences in the Mathematics class – if it was constituted with feelings of confusion, persistent failure or low self-worth. I would also like to determine the reason why they chose to take Mathematical Literacy and what they expect from the subject. Questions on what they perceive as difficult in Mathematics and what they normally found problematic might give teachers some insight into ways to approach these concepts in the Mathematical Literacy class.

3.2 Analysis of the baseline assessment

The idea with the baseline assessment was to test learners on several concepts involving proportional reasoning including fractions, probability, ratios and scale.

In question 1 the focus was on fractions:

At the athletics days the tuck-shop sells hot potato chips. To make one small bag of chips they use $\frac{3}{4}$ of a kilogram of potatoes.

- 1) If the school has 120 kg of potatoes, how many small packets of chips can they make?
- 2) The tuck-shop ladies estimate that they will need to make 200 small packets of chips. How many kilograms of potatoes will they need for 200 packets of chips?

- 3) If the price of potatoes is R6,75 per kilogram, what is the cost of making a small bag of potatoes?

Learners had to think in terms of the following ratio:

1 packet of chips : 0,75 kg of potatoes

Learners had to realise that they either had to multiply by $\frac{3}{4}$ to calculate the amount of kilograms or divide by $\frac{3}{4}$ to obtain the number of packets. This implies that learners have to realise that dividing by a fraction gives a bigger answer and multiplying by fraction gives a smaller answer.

In question 1.1 learners were given the amount of available potatoes in kilogram and they had to calculate, by means of division, how many packets of potato chips can be made. In question 1.2 they had to calculate the exact opposite. Question 1.3 required similar calculations to that of question 1.2 but dealt with the *price* per kilogram to calculate the *price* per packet. Learners that were able to calculate question 1.2 should thus be able to also calculate question 1.3.

We assume that learners should be comfortable with fractions by the time they get to grade 10 since they have been doing fractions since grade 4. They should also have been introduced to Algebraic fractions in grade 9 which requires a firm understanding and knowledge of fractions.

Look at the following tables in which the times for the 100 m and the 200 m races have been summarised. All four these learners ran in both races.

100 m		200 m	
Sam	0:15	Sam	0:40
Alex	0:23	Alex	0:45
Charlie	0:18	Charlie	0:36
Jordan	0:21	Jordan	0:45

- 1) Did Sam run at the same average speed for both races? Explain your answer.
- 2) Which of the runners kept the same average speed for both races?
- 3) Which of the runners managed to run at a faster average speed for the 200 m race?
- 4) If Charlie can manage to run at the same average speed for 400 m, what will his time be for completing the 400 m race?

Question 2 was set up in accordance with research done by Noelting, with special reference to stage IIB and IIIA.

Stage IIB requires learners to first use with-in calculations before comparing the with-in results, thus the between strategy. A typical structure of stage IIB is that of ($a \neq b$; $c \neq d$), but kept to equivalent of (2:3 vs. 4:6) where the second ratio is a multiple of the first. The structure of stage IIIA is also ($a \neq b$; $c \neq d$), but only the first terms in each ratio are multiples of each other, whilst the second ones are not. The structure of the stage is thus of a non-multiplicative strategy within the ratios, but a multiplicative strategy in one between ratio.

The question required learners to realise a doubling in distance from 100 m to 200 m from where they could compare the times for the races. If the time was also doubled, the athlete ran at the same average speed for both races. If the time was less than double, then the athlete ran faster and if the time was more than double, the athlete ran slower. Learners merely had to compare the running times without any calculations of average speed involved. Realising that both the distance and the time has been doubled is, according Noelting, at stage IIB, but realising that it is more or less than doubled is at stage IIIA. It is only in question 2.4 where the comparative nature of

this question changes to that of a missing value type question. Here learners have to see the multiplicative relationship between the distances as being multiplied by 4, so that they may be able to also multiply the time by 4.

Question 3

The school has organised soft drinks for all the athletes that finish an event. They ordered 500 soft drinks and received 240 tins of Coke-Cola®, 140 tins of Fanta® Orange, 84 tins of Fanta® Grape and 36 tins of Sprite®. The school organises that a couple of parents will hand out the soft drinks as the athletes finish.

- 1) What is the chance that an athlete will receive a Coke-Cola®?
- 2) What is the chance that an athlete will not receive a Coke-Cola®?

When working with probability learners have to see the fractional relationship within the situation, for example, 240 Coke-Colas® out of 500 soft drinks or $\frac{240}{500}$. This part-

whole fraction that has to be converted to a percentage is another form of proportional reasoning. Look at the following set-up:

$$\frac{240}{500} = \frac{?}{100} \quad \text{or} \quad 240 : 500 = ? : 100 \quad \text{a typical missing value problem}$$

When asked what the chance is that an athlete will *not* receive a Coke-Cola®, learners will need to recognise that the remainder of the fraction that makes up a whole will be the chance on not receiving a Coke-Cola®. If 240 of the 500 tins are Coke-Cola®, then the remaining 260 tins will not be Coke-Cola®s.

$$\frac{260}{500} = \frac{?}{100} \quad \text{or} \quad 260 : 500 = ? : 100$$

Question 4

Sam and Alex both brought post race energy drinks. Sam has Energade® concentrate and Alex has a Game® sachet. On Alex's Game® sachet, it says that 1 sachet makes 1 litre of Game®.

- 1) How many sachets does she need to make 2 litres of Game®?
- 2) How many sachets does she need to make 500 mℓ of Game®?

Question 4 is similar in terms of difficulty to question 2 where learners were asked to compare times of athletes over 100 m and 200 m. The big difference between question 4, where learners need to calculate the amount of Game® sachets needed, and question 2, where learners had to compare the times of athletes, is that question 2 asks of learners to find the missing value instead of comparing the ratios as in question 2. The work of Noelting is mostly done on comparison problems, but these problems can surely be classified as stage IIA where computation is not really necessary, since the structure is kept simple ($a = b$; $c = d$). If 1 sachet makes 1 litre, then 2 sachets will make 2 litres. The same when asked how many sachets for 500 mℓ. If learners knew that 500 mℓ is the same as half a litre, then they would be able to make the same deduction as for question 4.1. If 1 sachet makes 1 litre, then $\frac{1}{2}$ a sachet would make $\frac{1}{2}$ a litre.

Sam's Energade® bottle has mixing instructions in the form of a ratio. It says that you must always mix the Energade® concentrate and the water in the ratio of 1:3.

- 3) If Sam wants to make 1 litre (or 1000 mℓ) of Energade®, how many millilitres of concentrate and how many millilitres of water is needed for 1 litre?
- 4) What percentage of the Energade® will always be concentrate if it is mixed in the ratio 1:3?
- 5) What percentage of the Energade® will always be water if it is mixed in the ratio 1:3?
- 6) If Sam mixed 75 mℓ of concentrate with 300 mℓ of water, was the Energade® mixed according to the ratio 1:3?
- 7) Which will have the stronger taste: Energade® made from 75 mℓ of concentrate and 300 mℓ of water or 45 mℓ of concentrate and 270 mℓ of water?

What makes the remainder of question 4 more difficult than the first two questions, is that learners need to realise that the ratio 1:3 has a part-part relationship. In question 4.3 the “whole” is given and learners are asked to calculate the parts of the whole.

Learners thus need to realise that 1 part concentrate is 1 part of a *total* of 4 parts and that 3 parts water are 3 parts of the *total* of 4 parts. It is imperative that learners can make the link between fractions and ratios in order to solve this missing-value question. Learners must be able to change the part-part relationship to a part-whole relationship, especially since the whole (or total) is given in question 4.3 and learners must calculate the parts.

The part-whole notation is also important when learners are asked in questions 4.4 and 4.5 to calculate the percentage of concentrate and the percentage of water respectively. As with question 3, they will have to set it up as follows:

$$\frac{1}{4} = \frac{?}{100} \quad \text{or} \quad 1 : 4 = ? : 100$$

Question 4.6 can be classified as a Noelting level IIIA. Learners are asked to see the multiplicative relationship ratio between 3 and 300 as dissimilar to the relationship between 1 and 75. They will have to reason in either of the following ways:

If the ratio 1 : 3 is made hundred times bigger so that three will become three hundred, then one will become hundred and not seventy five as in the given ratio 75 : 300.

OR

In the ratio 1 : 3, three is three times bigger than one, but in the ratio 75 : 300, three hundred is *not* three times bigger than seventy five.

Question 4.7 takes the comparison problem one step further by asking to compare two ratios where there is no obvious with-in or between ratio between the numbers in the ratios. This is at Noelting's level IIIB, the highest of Noelting's proportional reasoning levels. There are several ways in which learners could approach this question either by using with-in or between strategies, but for all approaches it is imperative to make one of the quantities (either the concentrate or the water) the same so that the other quantities (either the water or the concentrate) can be compared.

When using the between strategy, learners can either realise that 75 and 45 are both multiples of five or that 300 and 270 are both multiples of three. When following this strategy, learners can then solve it in either of the following two ways:

Working with the multiplicative relationship between the concentrates:

If we have to multiply 75 by $\frac{9}{15}$ to obtain 45 then we also have multiply 300 by $\frac{9}{15}$, but this equals 180 and not 270. The ratios are thus 75 : 300 and 45 : 180. This means that a ratio of 45 : 270 contains a lot more water and will thus not taste as strong as the ratio 75 : 300.

Working with the multiplicative relationship between the waters:

If we have to multiply 300 by $\frac{9}{10}$ to obtain 270 then we also have multiply 75 by $\frac{9}{10}$, but this equals 67.5 and not 45. The ratios are thus 75 : 300 and 67.5 : 270. This means that a ratio of 45 : 270 contains less concentrate and will thus not taste as strong as the ratio 75 : 300.

It is not always easy for learners to spot the multiplicative relationship between quantities and many learners rather opt for different strategies.

Working with the multiplicative relationship with-in the ratios:

75 : 300 can be simplified to 1 : 4

45 : 270 can be simplified to 1 : 6

Since the ratio 45 : 270 contains two more parts water to the one part concentrate it will not taste as strong as the ratio 75 : 300.

This last method has, once again, several similarities with the calculations involved in fractions. The colon notation can thus be substituted for fraction notation if learners should find it more familiar:

$$\frac{75}{300} = \frac{1}{4} \quad \text{and} \quad \frac{45}{270} = \frac{1}{6}$$

This fraction is of course not a part-whole fraction as in questions 4.3 to 4.5, but a part-part relationship where the numerator is the concentrate and the denominator the water. Should learners however use a part-whole relationship, it would be set up as:

$$\frac{75}{375} = \frac{1}{5} \text{ of the Energade® in concentrate versus}$$

$$\frac{45}{315} = \frac{1}{6} \text{ of the Energade® is concentrate}$$

Since $\frac{1}{5}$ is bigger than $\frac{1}{6}$, the Energade® with $\frac{1}{5}$ of concentrate tastes stronger than the Energade® with $\frac{1}{6}$ of concentrate.

There are thus several methods which learners can follow to solve question 4.7, but firm proportional reasoning skills need to be in place. Learners need to be able to think either in terms of fractions: “does this part equal that part when either the denominators or numerators are made the same” or in terms of multiplicative relationships: “does this quantity equal that quantity if multiplied by the same ratio”

Alex builds a scale model of the Athletics Stadium.

1. If her scale model measures 30 cm from the start of the 100 m mark to the finish line (of the 100 m mark), what is the scale of the model? (100 cm = 1 m)
2. Using the scale calculated in 1 above, if the width of the field (the grass) is 70m in real life, what will the width of the field be on the model?
3. According to Alex’s research, the height of the stadium pavilion is 12 m. If she makes the height of her model 4 cm, will this be to scale (as in question 1)?

Here learners are asked to simplify a given ratio so that it can read as a scale of drawing. They need to recognise that both units need to be the same for this special type of ratio called scale. They thus need to make 30 cm to 100 m into 30 (cm) : 10 000 (cm). This ratio can first be simplified to 3 : 1000 and further to 1 : 333,3. Because the final answer is not a whole number, learners could find this question difficult.

Once learner found the ratio for question 5.1 it would have been easy to find the missing value in the question 5.2. If the real life measurements are 333,3 times bigger than the scale, then the scale model would 333,3 times smaller than the real life stadium. Once again learners first had to convert all measurements to the same unit. The *between* method of calculation will thus look as follow:

$$70 \text{ m} = 7000 \text{ cm}$$

$$7000 \text{ cm} \div 333,3 = 21 \text{ cm}$$

Learners could also work with the un-simplified ratio to solve the missing value problem with a *with-in* ratio strategy.

$$\begin{array}{ccc} \times \frac{3}{10} & & \times \frac{3}{10} \\ \swarrow & & \swarrow \\ 30 \text{ cm} : 100 \text{ m} & \text{thus} & ? \text{ cm} : 70 \text{ m} \end{array} \quad 70 \text{ m} \times \frac{3}{10} = 21 \text{ cm}$$

Question 5.3 is a comparison problem and could be solved using any of the methods above. Learners have to determine if the ratio 4 cm : 12 m is the same as 30 cm : 100 m. Here the with-in strategy seems to be the most obvious method to follow since 4 can be multiplied by 3 to get 12, but 30 multiplied by 3 is not 90. Learners using the with-in strategy in question 5.2 will be more likely to use this strategy again in question 5.3.

Another strategy would be to determine the scale of the drawing 4 cm : 12 m. Simplified it would come to 1 : 300 which is clearly not the same as 1 : 333,3. The measurements are thus not correct since it is not to scale (of 1:333,3).

There seem to be a considerable number of methods that can be used to solve these questions, but in fact there are only two: determining the relationship between the numbers with-in a ratio and determining the relationship of the numbers between the ratios. We can also distinguish between two types of problems: missing value and

comparison. Although all of these questions covered several different concepts in the Mathematics curriculum, they were all either a missing value or a comparison problem.

3.3 Baseline assessment results

When marking the baseline assessment only one mark was given for every correct answer since it was only important for the purpose of the research to see which questions learners could and could not do. It is interesting to note the similarities between the responses of learners in school 1 and school 3. These two schools conducted the baseline assessment during the first week of teaching whilst school 2, which started its year a week earlier, conducted the baseline assessment after a week of teaching ratios. It is for this reason that results of the three schools were kept separate to see if a week of teaching would make any substantial difference to learners' understanding of ratios. A graphical representative summary of this information is included in Appendix B.

Although learners in school 2 seemed to score better and on average answered more questions, the overall group average was merely 1% more than that of school 1. Learners in school 2 also seemed to score lower in question 2, a question in which the other two schools scored really well.

The averages achieved by the three schools are as follows:

School 1: 47% out of a total of 5 learners

School 2: 48% out of a total of 19 learners

School 3: 37% out of a total of 6 learners

3.3.1 Results for Question 1

Almost 60% of students in school 2 that obtained the wrong answer in question 1.1, used multiplication instead of division to calculate the answer. Instead of saying $120 \text{ kg} \div \frac{3}{4}$, they said $120 \text{ kg} \times \frac{3}{4}$. None of the learners in school 1 and 3 made this

mistake. Interestingly enough only 16% of learners in school 2 that multiplied instead of divided, used the same method to obtain the correct answer in question 1.2.

Other incorrect methods for question 1.1 include a learner that divided by 3.4 (instead of 0.75), thinking that it equals $\frac{3}{4}$. Two learners divided 120 by 4, but did not multiply by 3 to get the final answer and left the answer as 30. Two more learners somehow obtained an answer of 10 packets when dividing 120 by $\frac{3}{4}$.

Answers obtained through “alternative” methods includes one learner from school 1 that first converted $\frac{3}{4}$ kg into 750 grams and 120 kg to 120 000 grams before dividing 120 000 by 750 to obtain 160 bags.

The wrong answers to question 1.2 were diverse with few learners obtaining the same wrong answers. Three learners divided by $\frac{3}{4}$ instead of multiplying by $\frac{3}{4}$ of which two of them set the calculation out as $\square \times \frac{3}{4} = 200$. The same two learners that only divided by 4 in question 1.1, did the same for question 1.2 and calculated an answer of 50 kg. A further two learners subtracted 40 from 200 to get an answer of 160 kg which seems arbitrary.

The answers to question 1.3 were equally haphazard. Learners from schools 1 and 3 that managed to answer question 1.2 correctly were also able to correctly calculate question 1.3, but in school 2 the picture looks slightly different and are summarised in table 1.

16% of learners in school 2 managed to calculate the answer to question 1.3 without being able to calculate the answers to the previous questions which also required a multiplicative procedure. Only 21% of learners in school 2 calculated the answers to both questions 1.2 and 1.3 correctly.

Table 1

Question	School 1	School 2	School 3	average
3 1.1 If $\frac{3}{4}$ of a kg potatoes make one small packet of chips, how many packets of chips does 120 kg of potatoes make?	60%	16%	50%	42%
1.2 How many kilograms of potatoes are needed for 200 packets of chips?	20%	47%	17%	28%
1.3 What is the cost of a bag of chips if the price of potatoes is R6,75?	20%	37%	17%	25%

The overall performance of learners on questions 1.2 and 1.3 was low with performance figures of 28% and 25% respectively.

3.3.2 Results for Question 2

The results for question 2 was by far the highest of all the questions and are summarised in table 2:

Table 2

Question	School 1	School 2	School 3	Average
2.1 Did Sam run at the same average speed for both races if she ran 15 s for the 100 m and 40 s for the 200 m race?	100%	68%	100%	89%
2.2 Which runner ran at the same average speed for both races?	100%	84%	83%	89%
2.3 Which runner ran at a faster average speed for the 200 m race?	80%	58%	83%	74%
2.4 If Charlie can manage to run at the same average speed for the 400 m race, what will his speed be?	100%	63%	83%	82%

Once again school 2's results were not in line with that of school 1 and 2 although it also showed a lower success rate for question 2.3.

For question 2.1 several learners in school 2 said that Sam ran faster in the 200 m race. A number of learners also wrote incoherent answers to this question, such as:

“In the first race there was two times below 20 and 2 above 20, so it’s a tie. In the second race she was in the average time.”

Another learner calculated Sam’s speed in metres per second (but wrote seconds per metre) without drawing any conclusion.

“100m / 15 seconds $\rightarrow 100 \div 15 = 6,7$ seconds/m

200m / 40 seconds $\rightarrow 200 \div 40 = 5$ seconds/m”

80% of learners in school 1 and 50% of learners in school 3 gave a multiplicative reasoning answer for question 2.1 that read mostly as:

“No, because 15×2 isn’t 40, it’s 30 and he would have to run exactly the same amount just double.”

Only 16% of students in school 2 noticed the multiplicative relationship. Most learners in school 2 calculated the speed of each athlete and then drew a conclusion based on the speed in metres per second. None of the learners in the other two schools followed this method.

One would think that if learners were successful in question 2.1, that they would achieve the same success in question 2.2. If learners were able to see that the average speed at which Sam completed the two races were not the same in question 2.1, then surely they could find which learner kept his/her average speed for both races and vice versa. Although this was the case for school 1, school 2 and 3 showed different results. Where school 3 scored 17% lower for question 2.2, school 2 scored 16% higher. The learner responsible for the lower results for school 3 must have misread the question since she swopped the answers for question 2.2 and 2.3 around. Almost all the learners in school 2 that obtained the incorrect answer did so by calculating the

average speed of each runner in metres per second and due to their rounding techniques they calculated that Alex ran at the same average speed in both races. One learner scratched out the decimals to obtain an answer of 4 m/s (or in her case seconds/m).

$$100\text{m} / 23 \text{ seconds} \rightarrow 100 \div 23 = 4,35 \text{ seconds/m}$$

$$200\text{m} / 45 \text{ seconds} \rightarrow 200 \div 45 = 4,44 \text{ seconds/m}$$

Question 2.3 seemed to have obtained the lowest scores of question 2. 55% of learners that obtained the incorrect answer did so by saying that Sam and/or Jordan ran faster because their time was more. One learner explained it as running “higher for 200 m”.

One learner came to the conclusion that “no body” ran faster and a further 36% of learners said that Charlie ran at a faster average speed for the 200 m race. One of these learners even gave the correct answer for question 2.2 by saying that Charlie ran at the same average speed for both races.

A third of the learners that obtained the incorrect answer for question 2.4 left the question blank. An additional third obtained the incorrect answer by trying to work with average speed in metres per second, but incorrectly multiplying the answer by 400 m. The remainder of the learners all worked with the wrong person’s time and calculated the time for Sam or Jordan instead of for Charlie.

One learner in school 2 clearly used a build-up strategy during question 2. It is not clear how much insight there is into her calculations since the explanations that accompanied her calculation were quite puzzling. Her methods are as follows:

- 2.1 *He did in a way because another 15 is added to 15 gives you 30 and he ran 10 s after that, so he did basically keep the same average: $100 + 100 = 200$ and $15 + 15 = 30$*
- 2.2 *They all did basically. Some just over*
Sam: $15 + 15 = 30 / 40$
Alex: $23 + 23 = 46 / 45$

Charlie: $18 + 18 = 36 / 36$

Jordan: $21 + 21 = 42 / 45$

2.3 *Alex*

2.4 $36 + 36 = 72$ $36 + 18 + 18 = 72$

One can almost follow her trail of thought in question 2.4 as follows:

200 m + 200 m is 400 m, therefore I must add 36 s and 36 s which equals 72 s. But I can also say 200 m + 100 m, which is 300 m, so I need another 100 m to get 400 m, thus: $36 + 18 + 18$ also equals 72 s.

3.3.3 Results for Question 3

Learners did not obtain the intended results in question 3 and in retrospect this question should have read:

What percentage of Coke-Cola®'s were bought? or What fraction of the soft drinks are Coke-Cola®?

Learners that were not familiar with probabilities found this question difficult and some learners wrote long involved explanations without putting values to their argument:

“Most of the 500 soft drinks is Cokes so theres a good chance of getting Coke”

“There is a good chance that an athlete will receive a Coke although Coke-Cola is very common and my be a better options for everyone to take”

In general, the results were poor as shown in table 3. Only one learner from school 3 managed to obtain the correct answers to these questions. In all cases learners who obtained the correct answer for question 3.1 also obtained the correct answer to question 3.2.

Table 3

Question	School 1	School 2	School 3	average
3.1 What is the probability of receiving a Coke-Cola if 240 out of 500 tins are Coke-Cola?	40%	53%	17%	36%
3.2 What is the probability of <i>not</i> receiving a Coke-Cola?	40%	53%	17%	36%

An undetected error crept in with question 3 but interestingly enough the learners that followed the expected method were not bothered by this error at all. If the tins of Coke-Cola®, Fanta® and Sprite® are added together the total is only 480 tins and not 500. Learners who calculated the chance of *not* getting a Coke-Cola® by subtracting the number of Coke-Colas® from 500, obtained the correct answer of 260 in 500 chance or 52% did not notice this discrepancy. Learners who added all the Fantas® and Sprites® came to an answer of 240 in 500 or 48% - this was also taken as correct since learners showed that they had insight into the problem and could reason proportionally. Some learners even expressed the answer as 240 in 480 thus equalling 50% for both chance on Coke-Cola® or not getting Coke-Cola®.

One learner recognised that the chance of receiving a Coke-Cola® at the end would be less than 50% but included the following calculation:

“ $500 \div 240 = 2,08$ the chance would be less than half”

3.3.4 Results for Question 4

Learners achieved the best results in question 4.1 and 4.2. with all learners obtaining the correct answer to question 4.1 and only four learners (11%) incorrectly answering question 4.2. The remainder of question 4 was poorly answered with only learners from school 2 attempting all questions. No learners from school 3 obtained any correct answer for questions 4.3, 4.4, 4.5, 4.5 and 4.7. In school 1 some learners were only able to answer questions 4.3 and 4.6, but there were no correct answers for questions 4.4, 4.5 and 4.7 as is clear from table 4:

Table 4

Question	School 1	School 2	School 3	Average
4.1 If 1 sachet of Game makes 1 ℓ then how many sachets are needed to make 2 ℓ?	100%	100%	100%	100%
4.2 How many sachets are needed for 500 mℓ of Game?	100%	84%	83%	89%
4.3 Energade is mixed in the ratio 1:3, how many mℓ of concentrate and water is needed for make 1 ℓ of Energade.	20%	21%	0%	14%
4.4 What percentage of the Energade is concentrate?	0%	26%	0%	9%
4.5 What percentage of the Energade is water?	0%	26%	0%	9%
4.6 Is 75 mℓ of concentrate and 300 mℓ of water mixed in the ratio 1:3?	60%	74%	50%	61%
4.7 Which will have the stronger taste: 75 mℓ of concentrate to 300 mℓ or 45 mℓ of concentrate and 270 mℓ of water?	0%	58%	0%	19%

In question 4.1 learners had to recognise the doubling of values. If 1 sachet of Game® is needed to make 1 litre of Game® energy drink, then to make 2 litres of energy drink – double the amount of energy drink – one would need double the amount of Game® sachets, thus 2 sachets.

In question 4.2 learners had to recognise the halving of values. If 1 sachet of Game® is needed for 1 litre of Game® energy drink, then to make 500 mℓ of energy drink – half the amount of energy drink – one would need half the amount of Game® sachets, thus $\frac{1}{2}$ a sachet. The incorrect answers to these questions were quite arbitrary with answers ranging from 500 mℓ to 2, 3 and 4 sachets.

Question 4.3 saw learners really confused with the ratio 1:3. A third of learners that obtained the incorrect answer for question 4.3 worked with a fraction of $\frac{1}{3}$ instead of

$\frac{1}{4}$. Learners incorrectly thought of the ratio 1:3 as a part-whole ratio where one part is concentrate of the total of three parts energy drink and the remaining two parts of the

whole is concentrate. For this reason they obtained the answer of 333,3 ml concentrate and 666,6 ml water.

7% of learners obtained the answer of 300 ml of concentrate and 700 ml of water with no calculations to substantiate their answer. This still seemed to make more Mathematical sense than the following three answers:

“1 millilitres of Energade® and 3 millilitres of water is needed = 1:3”

“13 millilitres of concentrate and 500 ml of water is needed for 1 litre”

“0.5 ml : 1.5 ml “

“3.9”

One really wonders how much thought went into answers like these and how long it will take learners like these to adapt from an answer driven approach to Mathematics to a thought driven approach to Mathematics – an essential quality needed to make sense of Mathematical Literacy.

Two thirds of learners that calculated question 4.3's answer as 333,3 ml and 666,7 ml, gave the answers of 33,3% and 66,7% for questions 4.4 and 4.5 respectively. One learner that also obtained the answer of 333,3 ml and 666,7 ml made the following interesting calculation:

“ $333 \times 3\% = 10\%$ of the Energade® will always be concentrate

$667 \times 3\% = 20\%$ of the mix is water”

One would think that a learner that was able to calculate the correct answer for question 4.3 would use the same strategy to solve questions 4.4 and 4.5. From the mere 9% of learners that obtained the correct answers for questions 4.4 and 4.5, only two learners also obtained the correct answer for question 4.3. The rest of these learners only answered question 4.4 and 4.5 correctly.

61% of learners obtained the correct answer for question 4.6. Only 35% gave an explanation for their answer which was disappointing since giving no explanation can merely indicate a lucky guess on the learners' part. This argument is strengthened by two learners that correctly answered "no" to this question but reasoned that the ratio contained too much concentrate and too little water. Explanations that were given were not always clear and two learners correctly simplified the ratio of 75 : 300 to the ratio 1 : 4 from which they deduced that the mixture contained too much water. Three of the learners worked with the with-in ratio of 1 : 3 to calculate that for the ratios to be the same one would either need 100 mL ($300 \div 3$) of concentrate or 225 mL of water (75×3).

One learner used the between ratio strategy and explain it as:

$$\begin{array}{ccc} 1 & : & 3 \\ \swarrow & & \searrow \\ \times 75 & & \times 100 \\ 75 & : & 300 \end{array} \quad \text{no}$$

Without any further explanation it was clear how this learner thought about the calculation. The multiplicative relationship between the first terms and between the second terms is not the same and therefore the ratios are not equivalent.

In question 4.7 learners achieved less successful results and only learners from school 2 managed to answer this question correctly. In this question it was clear that learners from school 2 had spent some time on ratios before the baseline assessment. 60% of learners in school 1 and 17% of learners in school 3 thought that the two ratios were the same which could indicate towards additive instead of multiplicative strategies. No learners in school 2 made this mistake. The reason why learners could possibly think that the two ratios were equivalent could be explained as:

$45 + 30 = 75$ and $270 + 30 = 300$ thus the ratio 45 : 270 and 75 : 300 must be equivalent. Learners reason that if you could add equal amounts of water and concentrate to the mixture it would remain in the same ratio.

Four learners explained their answers to question 4.7. Only one learner simplified each ratio to obtain the answers of 1 : 4 and 1 : 6 from where it was easy to explain that the second ratio contains more water.

The remaining three learners gave explanations which used a combination of multiplicative and additive strategies.

“45 ml x 3 = 135 ml 270 – 235 = 135 ml more water

75 ml x 3 = 225 ml 300 – 225 = 75 ml more water

45 ml mixed with 270 ml is less stronger”

Interestingly enough, although these three learners all used the multiplicative relationship of “multiply by three”, none of them seem to calculate the answer to 4.3 and 4.5 correctly. Although this method ensured the correct results for this question, learners might not be so successful when given the following two ratios:

75 : 300 and 40 : 160

$40 \text{ ml} \times 3 = 120 \text{ ml}$ $160 - 120 = 40 \text{ ml}$ more water

$75 \text{ ml} \times 3 = 225 \text{ ml}$ $300 - 225 = 75 \text{ ml}$ more water

A learner will say that the ratio of 75 : 300 contains more water, whilst in fact both ratios can be simplified to 1 : 4.

3.3.5 Result for Question 5

Table 5

Question	School 1	School 2	School 3	Average
5.1 What is the scale of a model, if 30 cm on a scale model represents 100 m?	20%	0%	0%	7%
5.2 If the real life width of the track is 70, what will the width of the track on the model be?	0%	0%	0%	0%
5.3 If the real height of the stadium is 12 m and Alex makes the height on the model 4 cm, is the height in the same scale as 5.1?	40%	37%	0%	26%

In Question 5.1 no learners were able to simplify the scale of the drawing to 1 : 333,3. Only one learner managed to determine the un-simplified scale of 3 : 1000 and wrote it in fraction form as $\frac{30}{10000} = \frac{3}{1000}$.

50% of learners in school 3 obtained the answer of 1 : 3. In itself this mistake is excusable, but what is of concern is that their teacher marked this answer correct. If his knowledge on scale (and ratio) is of such a nature that he interprets this answer as correct, then it will reflect in his learners understanding.

With the poor results of question 5.1 it was no surprise that no learners managed to calculate the correct answer to question 5.2.

Question 5.3 was answered in a similar fashion to that of question 4.6 where one can expect “educated guesses” instead of calculations. Only learner gave an explanation to her answer of 4 : 12 which can be simplified to 1 : 3 and made no conclusion since she was unable to find the ratio to question 5.1 to compare it to. No other learner gave explanations to their answers.

3.3.6 Concerning factors

While school 1 and 3 conducted this assessment during the first week of school, school 2 conducted this assessment at the end of the second week of teaching. During this teaching period they covered the concept of ratio and proportion. Conducting this assessment at different stages of teaching proportional reasoning gave valuable insight into learners’ thinking. Learners in school 2 where clearly more experienced in answering questions such as “which will have the stronger taste: 75 ml of concentrate to 300 ml or 45 ml of concentrate and 270 ml of water?”, and yet they performed worst on presumably easier questions according to Noelting’s levels. For question 2 on the comparison of running time, school 1 scored an average of 95%, school 3 score 88% and school 2 scored a mere 68%. This raises questions about the foundations on which teaching for proportional reasoning is happening.

When it comes to Mathematics one is always faced with learners memorising methods rather than using intuition and logical thought processes. When looking at the answers given by school 2, especially for question 2, one wonders how learners reasoned about the situation to obtain such unmethodical answers and if methods they had been taught were not merely misused. As was shown in the results of question 1 where learners had to calculate the number of packets chips or the amount of kilograms needed, learners often get confused between having to multiply or divide. They either have a poor understanding of the fractions and ratio or they are not making sense of what is being asked. Many of the responses to other questions were equally confusing and it would seem to appear that learners are not trying to make sense of what is being asked. They do not seem to engage in problems and seem to write down a set of calculations for the sake of doing answering the question. This attitude is supported by answers given to the questionnaire below.

3.4 Section B: The questionnaire

The questionnaire, as set out in Addendum C, was given to learners as section B of baseline assessment and aimed to capture some of the attitudes that learners bring to the Mathematical Literacy classroom. I compiled this questionnaire with the goal of gaining some insight into learners' attitude towards Mathematical Literacy and their conception about Mathematics, Mathematical Literacy and ratios, so that it may help to design activities accordingly.

3.4.1 Analysis of the questionnaire results

The response captured here are the combined responses of Mathematical Literacy students in the three different private schools in the West Coast District. There were no remarkable difference in the responses between the schools except for question 3 and 4 where learners were asked about their favourite concepts in Mathematics and what they find the easiest in Mathematics. 67% of learners in one of the schools were the only learners who said they like geometry and found it easy. Interestingly enough, even the remaining 33% of learners in this specific school said they dislike Geometry and find it the most difficult concept. Most learners within the other two schools said they dislike Geometry and find it the most difficult concept in Mathematics.

Question 1: There were a variety of reasons why learners chose Mathematics over Mathematical Literacy. This could be due to the way the subject was “sold” to learners but the leading perception was still that Mathematical Literacy is easier than Mathematics. Few learners chose Mathematical Literacy over Mathematics because they dislike Mathematics although most of them expressed their emotions as “tense and/or irritated” in question 7.

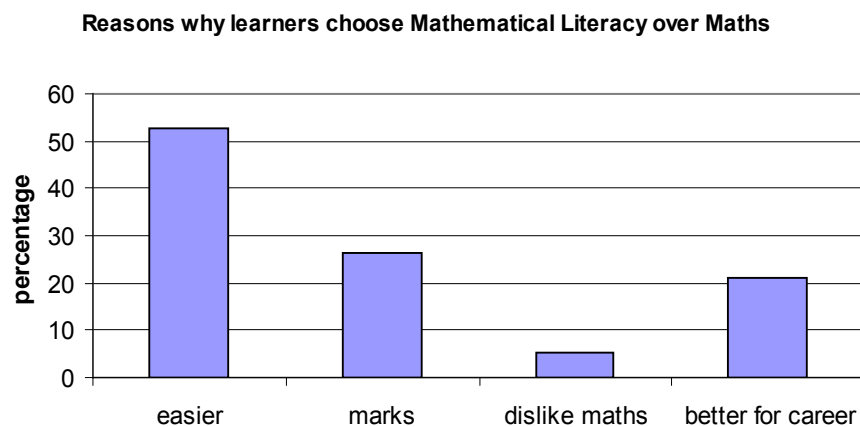
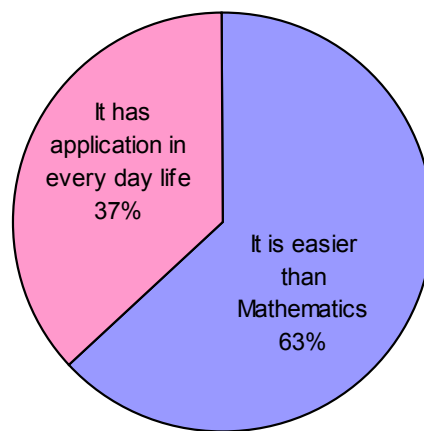
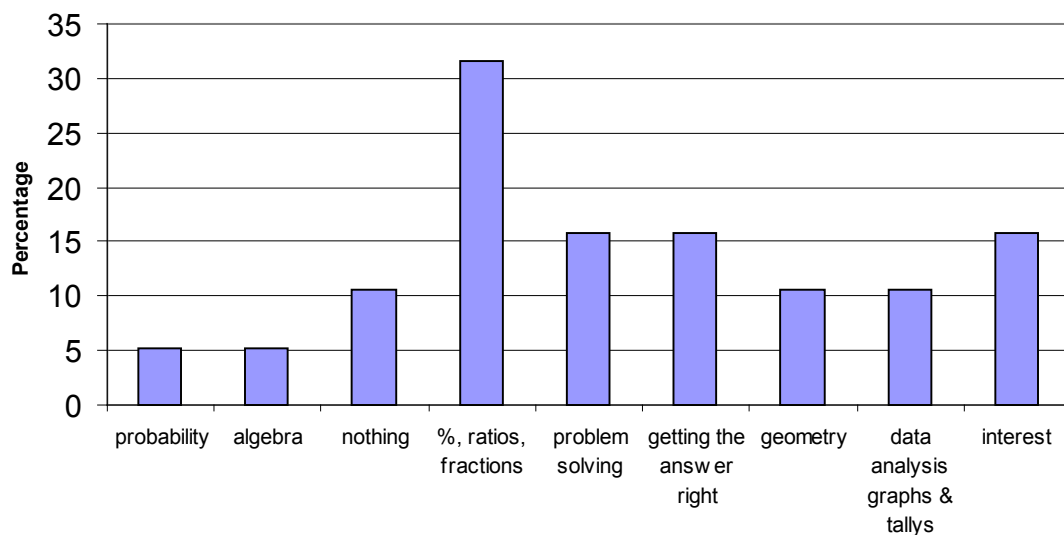


Figure 1

Question 2: When asked how learners perceive the difference between Mathematical Literacy and Mathematics, the overwhelming response was once again that they perceived Mathematical Literacy as being easier. They did however recognise that the objective of the subject is different to that of Mathematics and focuses on Mathematics in everyday life rather than the manipulation of numbers.

Perceptions of Mathematical Literacy**Figure 2**

Question 3: When asked what learners enjoy the most in Mathematics the responses varied from “nothing” to “getting the answer right” to even “problem solving”. The learners who enjoy Geometry are from one specific school whilst the rest of the responses were varied with no particular favourites within a particular school.

Favourite topics in Mathematics**Figure 3**

Question 4: It is interesting to note that although there is a strong connection between what learners like and what they perceive as easy, it is not exactly the same. Look for example at the percentage of learners that enjoy problem solving – it is not necessarily

easy for them, but yet they find it enjoyable. We can speculate why learners enjoy interest, and find it easy, as that it can be attributed to the “follow the formula and get the answer” nature of interest as it is presented in grade 9. In general the responses to this question were diverse with no definite favourite under participants. Once again the geometry responses only came from one particular school.

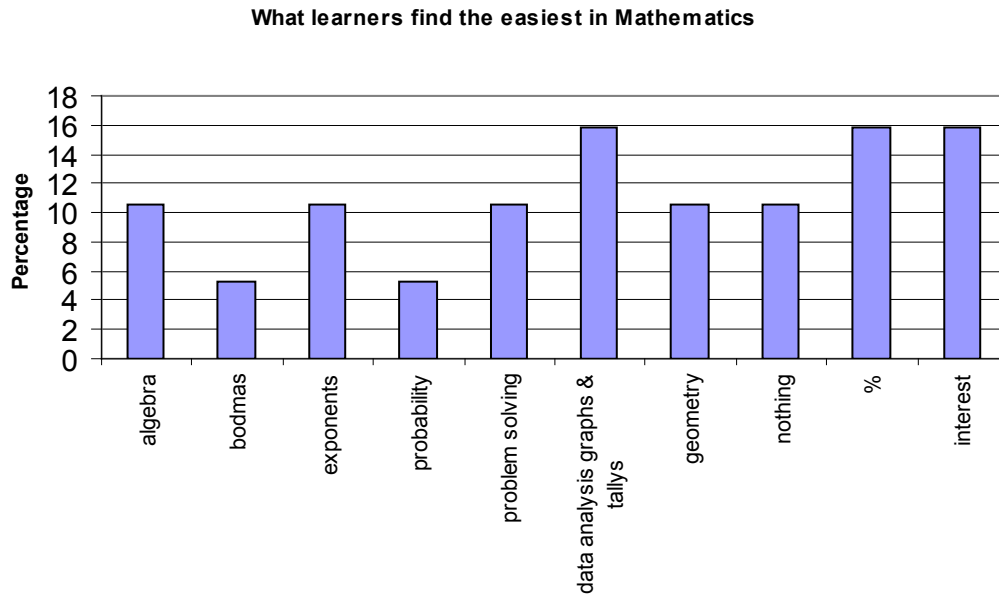
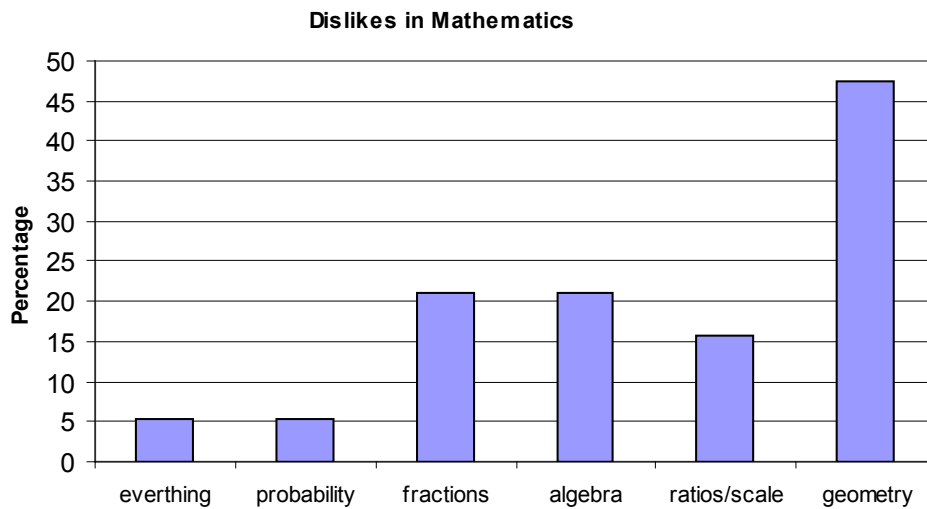


Figure 4

Question 5: It was clear from the gathered information that almost 50% of the participants strongly disliked Geometry. One normally anticipates that most learners would dislike Algebra, but only 21% of participants indicated a dislike in Algebra in comparison to a dislike in Geometry of 47%. Where 32% of participants indicated fractions as a favourite concept, 21% showed a dislike in the concept.

**Figure 5**

Question 6: The responses to this question are not in line with the answers from question 5 as was the case with question 3 and 4. Although some learners seemed to dislike fractions, they did not necessarily find it difficult. The percentage of learners that dislike ratio and scale, 16%, is not the same as those that find it difficult, 11%, but an additional 5% of learners find conversions difficult which could directly be linked to ratio and scale. For example: if there are 100 000 cm in 1 km, then we are working with a ratio (or scale) of 1 : 100 000. Although most learners seem to find Geometry difficult it is still only 32% of participants compared to 47% of participants that dislike the concept. The percentage of learners that dislike Algebra is much more in line with the percentage that finds it difficult. None of the participants said that they dislike problem solving – on the contrary – 16% of learners said that they enjoy the concept, yet learners listed it as a concept which they find difficult. This is a clear indication that learners taking Mathematical Literacy do not necessarily dislike challenges which they may find difficult. 16% of learners answered that “getting the answer right” as the best part of Mathematics and although a lot can be read into this statement in terms of classroom norms where answers are more important than methods, I am sure that all mathematicians have felt that wonderful sense of satisfaction when solving an intricate problem.

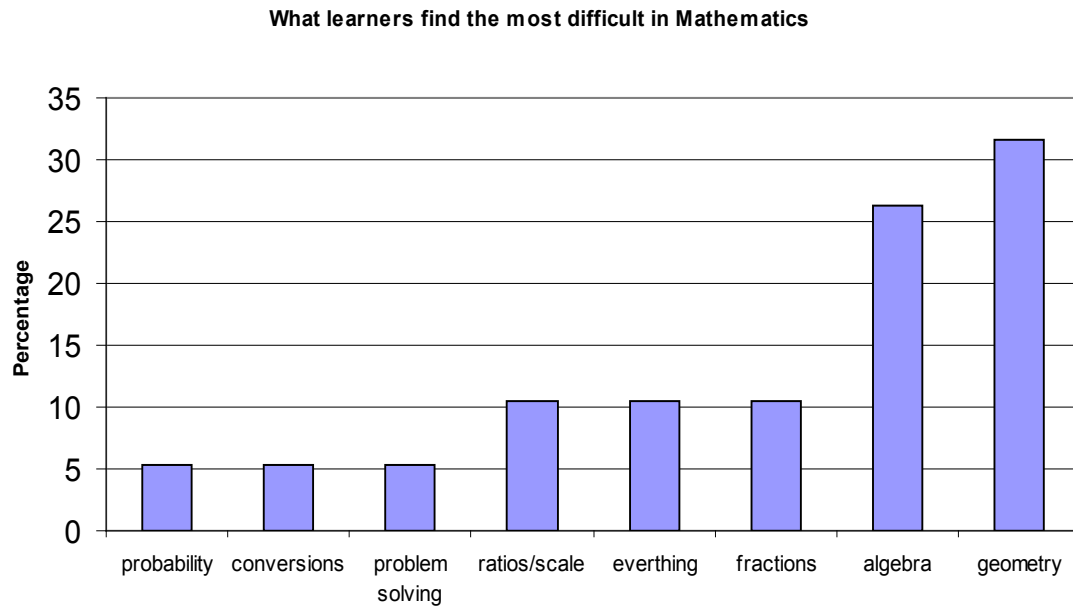


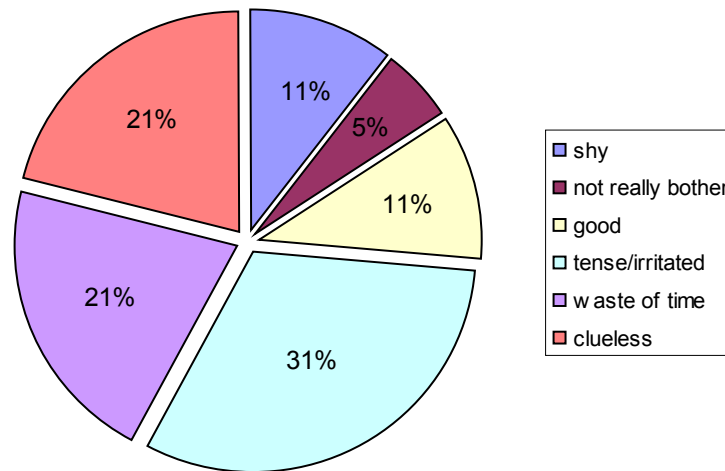
Figure 6

Question 7: What learners like and dislike and the type of answers given to these questions like “I enjoyed nothing” and “everything is difficult” already conveyed messages of the type of emotions that these learners experienced in the Mathematics class. Except that these are not the type of emotions that a teacher would like to evoke in children, it is also these attitudes that the learners bring to the Mathematical Literacy class. Most participants in this study said they felt tense or irritated in class and that they struggle to keep up with everything that was happening. Some of the statements include:

“confussed, the teacher never helped me”

“I feel bored, stupid and under constant pressure”

“Stressed, because it moves so fast and it takes a long time for me to understand.”

Emotions experienced in the Mathematics classroom**Figure 7**

Questions 8 and 9: Learners really struggled to explain what they knew about ratios and its application. Most learners (36%) did not know that it had any application in real life and one learner thought it meant the same as “radius”. 31% of learners said that it had to do with division while another 11% and a further 5% gave more specific examples of distance, scale and exchange rates. The 15% of learners that said that we use it “everywhere” did not supply any examples and we can thus not assume that they can substantiate their answer.

Some of the answers to question 8 include:

“Makes the bigger proportions of numbers into their simplest form.”

From this statement we can gather that the learners have experienced simplifying ratios but saying that you simplify the “bigger proportion” almost sounds as if the learner thinks the ratios are not equivalent.

“That its comparing 2 or more points in as specific form.”

Although it is also a clumsy explanation, this learner's answer was the closest to the nature of ratios. Ratios can indeed be seen as comparisons of quantities: Boys to girls, measurements on a map to measurement in reality etc.

Learners' thoughts on where ratio's are used in our daily lives

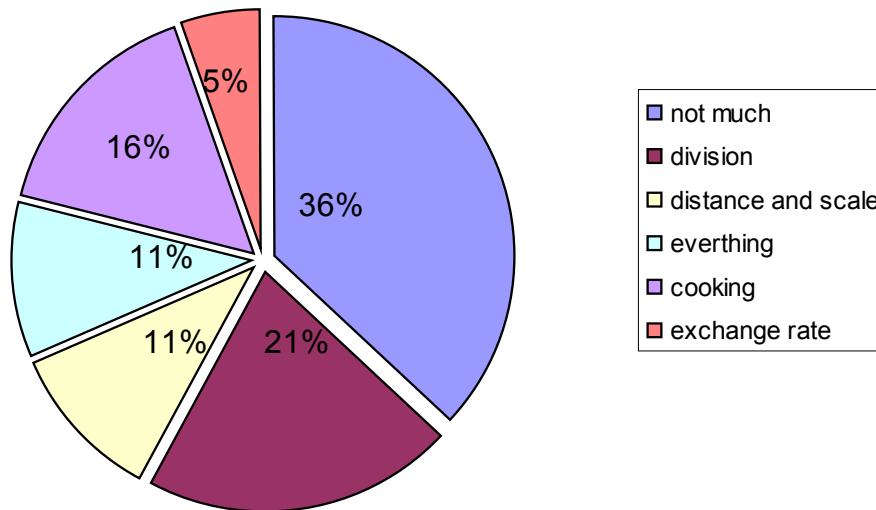


Figure 8

3.5 Conclusion

Most learners chose to take Mathematical Literacy since it is perceived as “easier” than Mathematics. Although the baseline assessment showed very poor results on learners’ proportional reasoning skills, as well as their knowledge of where ratios are used in real life, many of them indicated percentages, ratios and fractions as favourite concepts in Mathematics. A close second was “getting the answer right” which show these learners’ need for success and appreciation. 84% of learners felt shy, tense, irritated or clueless in the Mathematics class or that Mathematics was a waste of time.

The highest scoring concepts that learners rated as the easiest in Mathematics included interest and data analysis – two concepts that require, in most cases, merely

the substitution of numbers into a given formula. This supports the claim that learners prefer not to actively engage in problems. This is further supported by almost 50% of learners that dislike geometry since geometry requires learners to actively analyse questions in order to find solutions. It cannot be solved by using a set formula.

Chapter four: Proposed didactical strategies for teaching proportional reasoning

This chapter contains the theoretical rationale for the didactics used during this research process. These didactics are influenced both by learners' proportional reasoning skills identified in the baseline assessment test, as well their attitudes identified by the baseline assessment questionnaire. The theoretical approach to didactical strategies will include the meaning of proportional reasoning and extent of the concept in the Mathematical Literacy National Curriculum Statement. It will also look at the principles of Realistic Mathematics Education and especially how the principle of *intertwinement* or interconnectivity must be employed to link concepts that require proportional reasoning.

The psychological approach will link to this by exploring the type of thought patterns that learners will need in order to make sense of proportional reasoning. This will include the work of Skemp on relational versus instrumental understanding. This all forms part of the classroom culture that needs to be created in the Mathematical Literacy class.

4.1 A theoretical approach to didactical strategies

There are several problems that hinder the successful teaching of proportional reasoning in Mathematical Literacy. As we discussed earlier, the nature of the proportional reasoning itself is quite problematic and requires that the teacher has a firm understanding of the concept him/herself as well as a clear understanding of learners' thoughts. What is also of great concern, is that teachers not only have to facilitate transfer of proportional reasoning knowledge between the different fields in Mathematical Literacy that require proportional reasoning, (such as ratio, rate, percentages), but also facilitate transfer from abstract concepts to real life situations. Although Mathematical Literacy has a strong focus on real-world problems and by its nature follows a problem-solving approach, learners still find it difficult to connect their primitive strategies to the formal strategies suggested by teachers. It thus becomes a fine balancing act to include different context that will include different

concepts in such a way that teachers enable learners to transfer knowledge within context and concept whilst dealing with the difficult concept of proportional reasoning.

Cramer et al. (1993: 13) describes the ability to reason proportionally as follows:

Proportional reasoning also involves the ability to solve a variety of problem types. Missing value problems, numerical comparison problems, and two types of qualitative situations are among the types of problems that are important for children to understand. Proportional reasoning involves the ability to discriminate proportional from nonproportional situations. A proportional reasoner ultimately should not be influenced by context nor numerical complexity. That is, students should be able to overcome the effects of unfamiliar settings and cumbersome numbers.

Cramer also emphasises the importance of transfer by saying that learners should not be affected by context and numerical differences. By teaching proportional reasoning we are not only focussing on the high order thinking skills involved in proportionality, but also on the high order thinking skill of transfer of knowledge between one proportional reasoning problem to the next.

4.1.1 Defining proportionality

Wu (2002) places a lot of emphasis on giving meaning to Mathematics. He stresses the following three points which should be the cornerstone of any Mathematics teachers' didactical approach:

1. that precise definitions form the basis of any mathematical explanation, and without explanations mathematics becomes difficult to learn,
2. that logical reasoning is the lifeblood of mathematics, and one must always ask why as well as find out the answer, and finally,
3. that concepts and facts in mathematics are tightly organised as part of a coherent whole so that the understanding of any fact or concept requires also the understanding of its interconnections

It is point three that is the essence of this investigation: “the understanding of any fact or concept requires also the *understanding of its interconnections*.” Interconnectivity is such an important concept in Mathematics teaching but it is not an easy task to focus learners’ attention on the bigger picture whilst teaching the smaller parts that make up this intricate network of knowledge. I will therefore be focussing my attention on the teaching of proportional reasoning and its interconnectivity in the next chapter.

Mathematics teachers seem to focus their attention on the manipulation of numbers rather than the meaning and the wording behind their actions. Children seem to think that the words being used in the Mathematics classroom have nothing to do with what they learn in the language class. Take for example the words binomial and trinomial. Few children make the link to *bicycle*, meaning two, and *tricycle*, meaning three. They have learnt the prefixes and their meanings in the language class, yet they do not connect it to what is being taught in the Mathematics class. Teachers can blame learners for this, but it is ultimately the responsibility of the teacher to make Mathematics a sensible subject that is, in contrast to what learners might believe, connected to the world around them. This is in fact what Mathematical Literacy is all about: Mathematics in the world around us. But if learners cannot make the link between the language being used outside the Mathematics class to the language being used *inside* the Mathematics class, what is the chance that they will make the all important connections between concepts teachers would like for them to make?

Wu (2002) places a lot of emphasis on definition. Whilst reading his article I looked up the Oxford Dictionary’s definition of the word “proportionality”. It gives us the following explanations:

1. Part of Whole: a part or share of a whole: Water covers a large proportion of the earth’s surface
2. Relationship: the relationship of one thing to another in size, amount, etc.
Ratio: The proportion of men to women in the college has changed dramatically over the years.
3. *Size/Shape*: the measurement of something; it’s size and shape

4. *Mathematics*: the equal relationship between two pairs of numbers, as in the statement “4 is to 8 as 6 is to 12”. (Oxford Advanced Learner’s Dictionary, 2005: 1166)

What is interesting to note is that even in the dictionary the meaning of the word has a different description in terms of Mathematics. Yet, if we look at all four the descriptions, they are all true to the Mathematical meaning. The second meaning of “relationship”, which immediately brings thought of functions and rate to mind is then also in their opinion synonymous with ratio. If linguists can see the links between these Mathematical concepts, then why do teachers of Mathematics seem to have such a difficult time in making the link ourselves and helping our learners to make the link?

By opening the dictionary and exploring the meaning of a word, teachers can initiate complex discussions on the nature of a concept in Mathematics. By doing this, teachers are addressing all three cornerstones of Wu’s approach to teaching Mathematics. They are explaining and defining a concept. They are initiating logical thought and exploring the interconnectivity of the proportionality. And although this discussion will not turn all students into immediate masters of proportionality, it is a way to open the door for exploration. It can act as platform to work from and refer back to whilst busy with proportionality and also related concepts. Teachers must aim to create a web of interconnected pieces of information and skills in Mathematics rather than carefully structured chunks of information that learners need to memorise in order to pass exams.

What remains the most important and defining factor of proportional reasoning, is that it is multiplicative in nature. To develop proportional reasoning, learners will thus have to move away from additive strategies to multiplicative strategies. Instead of asking *how much more*, they will have to ask *how many times more*?

Whilst working on this research, a Mathematical Literacy teacher happened to ask me if I had any proportion problems she could use. Enthusiastically I replied that I had several and wanted to know exactly what she was looking for whilst handing her a set of activities. She looked confused and asked again as if she had not made herself clear

the first time. It was only after a while that I realised that she was referring to inverse proportions only. She interpreted the content of the chapter of *ratio and proportion* as directly proportional situations being *ratio* and indirectly proportional questions being *proportion*.

She was convinced of her opinion and ensured me that it was how her subject head, an experienced Mathematical Literacy teacher and textbook writer, had designed the work-schedule.

Her argument has its merit since I have often wondered what the “proportion” in the chapter *ratio and proportion* refers to. What concerns me though is this teacher’s ability to teach Mathematical Literacy in such a way that it has meaning to learners if she is not even able to make sense of the word proportion herself. Surely she has used the word “proportion” outside its mathematical context and has a specific meaning she attaches to the word. Are the mathematical and the non-mathematical meanings mutually exclusive as the dictionary had tried to explain?

4.1.2 Proportional reasoning through-out the Mathematical Literacy National Curriculum Statement

Transfer plays a vital role in the Mathematical Literacy classroom: not only between content and context, but also between content concepts. As mentioned earlier, few teachers, and ultimately learners, make the connection between different proportional situations. Hence, I will briefly point out the importance of proportional reasoning throughout the Mathematical Literacy curriculum and Subject Assessment Guidelines as published by the Department of Education (2007).

Learning Outcome One

Learning outcome one centralises around numbers, operations and relationships. The learner must be able to recognise, describe and represent numbers and their relationships, and to count, estimate, calculate and check with competence and confidence in solving problems.

Assessment standard 1.1 includes the following explanation of the range of problem types that is included: “percentage, ratio, rate and proportion (direct and inverse), simple and compound growth, calculations with very small and very large numbers in decimal and scientific notation” (Department of Education 2007: 14).

Once again the term “proportion” is used freely alongside ratio and rate as to suggest that it is an independent concept linked to ratio and rate and not related to percentages, growth and fractions at all.

A sharing problem often given to primary school learners as part of fractions can also be seen as a ratio problem. I would like to refer back to the pizza problem as example of this where 3 boys shared 2 pizzas and 9 boys shared 7 pizzas and learners have to decide which group had more pizza. The calculations involved are:

$$2 \div 3 = \frac{2}{3} \text{ (or } 0,67)$$

$$7 \div 9 = \frac{7}{9} \text{ (or } 0,78)$$

What teachers want learners to achieve during this question is to compare the fractions of $\frac{2}{3}$ and $\frac{7}{9}$ by either using decimals or equivalent fractions depending on what has been specified in the question. In this specific situation learners can make use of a build-up strategy: If the three children have 2 pizza, then six children will have 4 and nine children will have 6. Thus, the 9 boys who have 7 pizza have more.

This build-up strategy could be an indication of either a ratio or a fraction approach. The fractions being used in this instance would be $\frac{2}{3}$ or $\frac{6}{9}$ compared to $\frac{7}{9}$ whilst the ratios would be 2 : 3 or 6 : 9 compared to 7 : 9. A learner using the build-up strategy would not make use of either of these notations but yet reasons proportionally, although not on a formal level. I would like to argue that the equivalence of fractions is vital in making sense of ratios and comparing ratios and will be using this hypothesis in my didactical strategies.

The concept of percentages is another way in which proportional reasoning can be addressed. If 58% of learners in a school are girls and there are a total of 1200 learners, how many girls are in the school? This is in fact a missing value ratio problem:

58 : 42	totals 100
? : ?	totals 1200

It is unlikely that teachers set this question out in such a way or relates to a question such as the following:

The ratio of boys to girls in a specific school is 5 : 7. If there are 1200 learners, how many are boys and how many are girls?

The question is basically the same:

5 : 7	totals 12
? : ?	totals 1200

A learner who knows how to calculate 58% of 1200 will be able to make use of the same strategy for the ratio question. In both cases the learner will have to reason proportionally: $\frac{58}{100} \times 1200 = 696$ and $\frac{7}{12} \times 1200 = 700$. As mentioned before the confusion normally centres around part-part and part-whole representations, not only for teachers, as in the work by Clarke et al. (2003), but ultimately also for learners. Where 58% is part of a whole 5 : 7 are two parts (that make up a whole of 12). The two different representations will also be addressed during the chapter on didactical practises and is an important distinguishing factor when dealing with proportional reasoning in the classroom.

Percentages form a vital part of Financial Mathematics – a concept that spreads broadly across the Mathematical Literacy National Curriculum Statement. Percentages in Financial Mathematics includes: converting currency, VAT, price increases, appreciation, price decreases, depreciation, inflation and interest.

As simple as it may sound, the conversion of currency is not an easy concept for learners. They struggle to make sense of a situation where the exchange rate of \$1 = R7,80 is given and they are asked to calculate how many dollars one can buy from R200. Learners want to jump in and make calculations without reasoning. Often their reasoning is incorrect: instead of understanding that a lot of Rand is needed for few dollars, they reason that because the Dollar is stronger, it needs to be more. They make the following calculation:

$$200 \times 7,80 = \$1560 \text{ instead of } 200 \div 7,80 = \$25,64$$

Another aspect which seems to influence learners' choice of operation is the nature of the answer. Learners who multiply and obtain an answer of \$1560 will assume it is correct since the answer of \$25,64102564 is not as visually attractive as \$1560. Since most learners were seldom confronted with "unsightly" numbers in the junior grades, they assume that the calculation giving them a whole number must be correct.

Murray, Olivier and Human (1999) confirmed this in their research when asking the apple tart question: $\frac{3}{4}$ of an apple is used to make one apple tart. If there are 20 apples available, how many apple tarts can be made? Many learners immediately make the calculation of $\frac{3}{4} \times 20 = 15$ and feel satisfied with their answer. It takes some probing to make learners question their answer: if less than one apple is needed to make an apple tart, how can 20 apples only give 15 tarts?

It is this same type of probing that Mathematical Literacy teachers need to do to initiate reflection on answers: How can R200 give me \$1560? Is \$1560 dollars not worth a lot more than R200?

Most learners find it fairly easy to calculate the amount of VAT that needs to be added onto a product. If 14% VAT needs to be added onto an amount of R550, most learners will make the following calculation effortlessly: $\frac{14}{100} \times 550 = 77$. Yet, if asked what amount of VAT was charged on R456, few learners can make the calculation. If structured as ratios, the two questions would look as follow:

VAT that has to be charged on R550:

14 : 100

? : 550

VAT that has been charged on R456:

14 : 114

? : 456

Of course one can argue that the reason that most learners cannot solve the latter question is due to their inability to recognise that R456 is representative of (or equivalent to) 114%. Regardless of the fact that the learners might or might not recognise VAT inclusive amounts as representative of 114%, the reasoning involved in solving this type of question remains proportional in nature. Most learners calculating VAT would first calculate the VAT amount and add it onto the original price. Although it is a totally acceptable method, it does not allow learners insight into the “make-up” of the new price. They calculate the 14% VAT, without reasoning that the new price is comprised of the original price, of 100%, plus the VAT, of 14%, to make 114%.

The same principle applies when calculating price increases or assets that have appreciated in value. If the price of milk has increased with 50% to that of R16,50 then the R16,50 is representative of 150% and the original price which is representative of 100%. Setting the question up in ratio format would look as follow:

150 : 100

16,50 : ?

This is another example of a typical missing value type of problem. Once again learners would find this question much more difficult than calculating the price *after* an increase.

When working with price decreases or the depreciation of an item, it would also be easier to calculate the price *after* the decrease than *before* the decrease. Once again many learners follow a two step method during which they first calculate the decrease amount after which they subtract the decrease amount from the original amount. For example: A car valued at R100 000 depreciated with 5%. What is the car worth now?

Learners who follow a two step method would make the following calculation:

$$100000 \times \frac{5}{100} = 5000$$
$$100000 - 5000 = 95000$$

These learners would not necessarily recognise that the new price is represented by 95%. Although it might seem a rather simple calculation to make to find that the new price is 95% of the original price, learners do not seem to make the link between this calculation and their two-step method.

$$94050 \times \frac{5}{95} = 4950$$

The proportional set-up to this question would look as follow:

$$95 : 5$$

$$94050 : ?$$

Price changes due to inflation work on the same principles as price increases. The Consumer Price Index (CPI), associated with inflation, requires of learners to find representative values of prices and cannot be solved by using the two step method mentioned earlier unless learners first calculate the index difference to calculate the price difference so that they can calculate the new price of an item. For example: If the CPI for 2001 was 106 and the CPI for 2002 was 115, then what would a basket of groceries that was R100 in 2001 amount to in 2002?

One would need to realise that the groceries of R100 are represented by an index of 106, whilst the groceries for 2002 are represented by 115. The situation could be set-up as follow:

106 : 115

100 : ?

When solving this problem one would have to say: $100 \times \frac{115}{106} = 108,49$

Interest in terms of the growth of money in an account, at either simple or compound interest, is an excellent way of exploring constant growth versus non-constant growth. According to Cramer et al. (1993) the ability to reason proportionally “involves the ability to discriminate proportional from non-proportional situations”. What is important to note is that the amount of interest over time in a simple interest account will grow *directly* proportional, whilst the total amount of money in the account over time will grow merely proportionally. A further discrimination can be made between this proportional growth (either directly or not) and exponential growth of compound interest.

Learning Outcome Two

Learning Outcome two: Patterns, Functions and Algebra - The learner will be able to recognise, describe and represent patterns and relationships, as well as to solve problems using algebraic language and skills.

It is clear from this description that this outcome focuses on rate and growth rate. Direct and indirect proportions are included in this learning outcome as well as the distinction between directly proportional growth ($y = mx$) and proportional growth that includes a constant ($y = mx + c$). As Olivier (1992) and Riedesel (1969) suggest, functional mathematics and presenting ratios as a “special” function or direct proportion are essential in developing proportional reasoning.

Once again I would like to draw the attention to the use of equivalence, learnt from fractions, when teaching for proportional reasoning using rate. If 3 boys share 2 pizzas, then 6 boys share 4 and 9 boys share 6 pizzas. Only if the question specifically asks how much pizza did *each person* get, are we implying that learners have to calculate the (gradient) value of $\frac{2}{3}$. Learners will need to realise that they employ the

same strategies when filling in a table indicating “rate of change” as when they calculate equivalent fractions.

The counter argument to this statement will of course be that fractions have a limiting representation. If the numbers in the following table have to be written as fractions, it would become rather unsightly:

People:	1	2	3	4	5	6
Pizza:	$\frac{2}{3}$	$1\frac{1}{3}$	2	$2\frac{2}{3}$	$3\frac{1}{3}$	4

As fractions: $\frac{1}{2}$; $\frac{2}{1\frac{1}{3}}$; $\frac{3}{2}$; $\frac{4}{2\frac{2}{3}}$; $\frac{5}{3\frac{1}{3}}$; $\frac{6}{4}$

Although I would not for one moment expect from learners to represent the situation as fractions, I feel that learners (hopefully) already have knowledge on multiplicative strategies used in equivalent fractions. This knowledge of multiplicative strategies in equivalence should be used as reference from where rate, and other concepts involving proportional reasoning, should be introduced.

Learning Outcome Three

Learning Outcome three: Space and Shape (Geometry) - The learner will be able to describe and represent characteristics and relationships between two-dimensional shapes and three-dimensional objects in a variety of orientations and positions. Assessment standards 3.2 and 3.3 specifically refer to conversions and scale drawings which are proportional concepts. Understanding that 70 m is equivalent to 0,07 km or calculating that $1 \text{ m}^2 = 10000 \text{ cm}^2$ is for some reason not an easy concept for learners and could most likely be attributed to the proportional nature of the problem. Proportionally, the problem look as follows:

1000 m : 1 km

70 m : ?

or

1 m : 100 cm

1 m² : ?

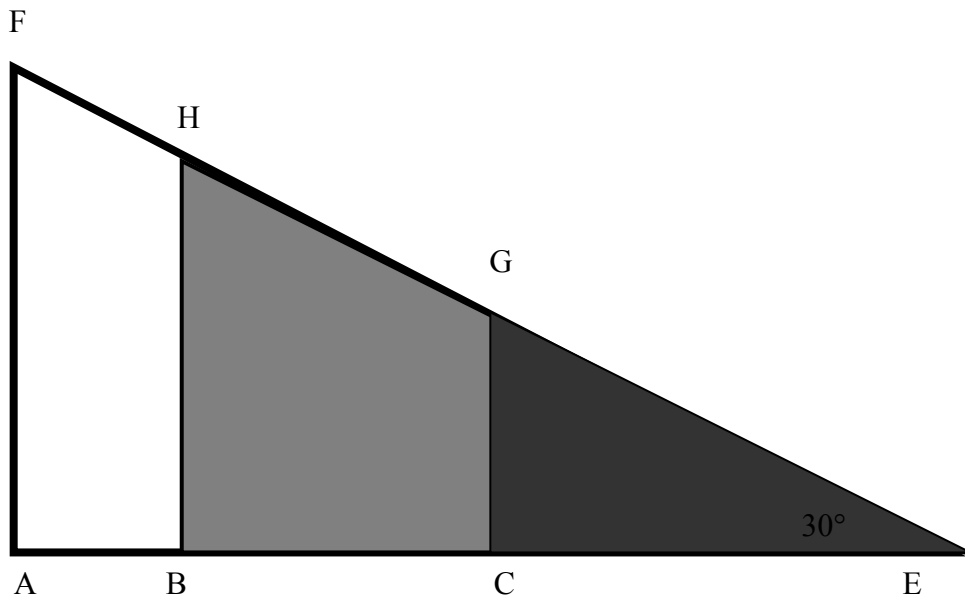
In the latter question learners would need to realise that $1 \text{ m}^2 = 1 \text{ m} \times 1 \text{ m}$, so that they will be able to calculate that $100 \text{ cm} \times 100 \text{ cm}$ equals 10000 cm^2 .

Scale, as rate, is a simplified ratio. Just as rate is for example distance per *one* hour, so scale is the real world measurement to a single unit on the map. On a map with a scale of 1 : 50000, a single unit represent 50000 of those specific units in real life. What has been interesting in my experience is that learners often ask what unit the scale is in. The fact that it could be any unit is mind-boggling to them. When working with scale it is best to use terminology such as “the map is 50000 times smaller than the real life”, to make learners accustomed to the multiplicative idea of a certain amount of “times” bigger or “times” smaller.

Although trigonometry is not currently examined at the end of grade 12 in the two obligatory papers, the National Curriculum Statement originally included this section and many textbooks have included it as well. I do believe that simple trigonometry questions of sin, cos and tan where learners must find either missing sides or missing angles is well within the reach of Mathematical Literacy students. It is also a way of explaining the special ratios of sides that exists between similar triangles – an activity which is often used as introduction to triangles.

The link between proportional reasoning and trigonometry is not always as clear as the links to ratio, rate, fractions and percentages and the importance of proportion in trigonometry lies in similar triangles. Take for example $\sin 30^\circ = \frac{1}{2}$. This means that

the ratio of the side opposite to the 30° angle and the hypotenuse will always be 1:2. The following is an example this relationship within a 30-60-90 triangle:



If angle E is 30° , then the ratio of $CG:EG$ will equal $BH:EH$ will equal $AF:EF$ will equal 1:2.

Learning Outcome Four

Learning Outcome four: Data Handling - The learner will be able to collect, summarise, display and analyse data and to apply knowledge of statistics and probability to communicate, justify, predict and critically interrogate findings and draw conclusions.

An aspect of proportional reasoning which is also often overlooked is that of probability. In the questions of the probability that you will draw a spade from a pack of cards, we are working with the fraction of $\frac{13}{52}$ (or $\frac{1}{4}$) or 25% or even a ratio of 13 spades to 52 cards (13:52). The wording teachers use when teaching probability is often accidental and support the link between fractions, ratios and percentages. People would generally say things such as: “He has a 50% chance of survival” or “you have half a chance of being right” or “the odds of him winning is 2 to 1”.

To see how all of these concepts are linked is not difficult for someone familiar with the content, although Sowder et al. (1998) and Person et al. (2004) agree that teachers

do not always make these links on their own. For starters, teachers have to make these links themselves so that they can devise teaching activities that will support learners in making these links. For this purpose social interaction (Boaler 1993: 13) and Realistic Mathematics Education can be most helpful.

4.1.3 The theory of Realistic Mathematics Education

Social interaction is the heart of constructivism. Von Glasersfeld (1987) talks of constructivism as *a theory of knowledge*. The constructivist view involves two principles:

1. Knowledge is actively constructed by the learner, not passively received from the environment.
2. Coming to know is a process of adaptation based on and constantly modified by a learner's experience of the world.

This entails that the child must be an active participator in the learning experience. The didactical method of teaching must enable the child to dig into the essence of the work and not merely listen passively to an explanation by the teacher. Constructivism links to the outcomes of Outcomes-based Education and places strong emphasis on a learner centred approach rather than a “talk and chalk” approach. The involvement of the child is thus crucial. This concept was already mentioned by both Piaget and Vygotsky who emphasised the active involvement of the learner in the learning process (Green 2001: 84). Heller et al., as in McLaughlin (2003) “support this constructivist suggestion seeing value in developing the mathematics to deal with situations rather than look at the situations as an afterthought application.”

A study of Mathematical Literacy would not be complete without looking at research done into Realistic Mathematics Education or RME. This stresses Mathematics as *meaningful* or real activity. Bakker (2004) clarifies that “this does not necessarily mean that the problem situations are always encountered in daily life. Students can experience an abstract mathematical problem as real when the mathematics of that problem is meaningful to them.”

On the basis of earlier projects in mathematics education, Treffers (1987) has defined five general principles of Realistic Mathematics Education:

1. *Phenomenological exploration.* A rich and meaningful context or phenomenon, concrete or abstract, should be explored to develop intuitive notions that can be the basis for concept formation.
2. *Using models and symbols for progressive mathematisation.* The development from intuitive, informal, context-bound notions towards more formal mathematical concepts is a gradual process of progressive mathematisation. A variety of models, schemes, diagrams, and symbols can support this process, provided these instruments are meaningful for the students and have the potential for generalisation and abstraction.
3. *Using students' own constructions and productions.* It is assumed that what students make on their own is meaningful for them. Hence, using students' constructions and productions is promoted as an essential part of instruction.
4. *Interactivity.* Students' own contributions can then be used to compare and reflect on the merits of the different models or symbols. Students can learn from each other in small groups or in whole-class discussions.
5. *Intertwinement.* It is important to consider an instructional sequence in its relation to other domains. Mathematics education should lead to useful integrated knowledge. This means, for instance, that theory and applications are not taught separately, but that theory is developed from solving problems (Bakker 2004: 6)

As the term Realistic Mathematics Education suggests, the mathematics that are being dealt with must be “real” to students. Since “realistic mathematics” is also the underlying theory of Mathematical Literacy, the principles of RME cannot be overlooked when dealing with the subject of Mathematical Literacy.

“The approach that needs to be adopted in developing Mathematical Literacy is to engage with context rather than applying Mathematics already learned to the context” Department of Education (2003a: 42). The context referred to in Mathematical Literacy documentation is not the same than “word problems” or “story sums” where

learners need to recognise the mathematical concept they have been taught. The context that is referred to is a problem-solving approach where learners are not asked to do a “sum”, but to engage in a problem using logical thought rather than manipulating a set of numbers.

During this investigation I will be using context for problems within which learners can *explore* the nature of proportional reasoning. The context will also help learners to verify answers and engage in educational discourse with other students that might disagree with them.

The *models and symbols* that I will be using to help learners to make sense of proportional problems, will be in the form a structural “set-up” of the questions. For this set-up I will be using a ratio notation although other symbols could be used instead of the colon. (This set up will be discussed in further detail in Chapter 5.) When teaching learners how to analyse and make sense of a situation by using this set-up, I aim to draw their attention to the proportional nature of different concepts that require proportional reasoning skills. The model for setting up proportional reasoning problems will thus aid in *intertwinement*.

When helping learners to set-up question, it is not to teach a specific method. The set-up is to help learners to *construct their own* methods and procedures. If learners cannot make sense of a situation, they also not be able to construct methods to solve the problem. Models and symbols are thus there to help learners to construct their own methods. It is this process that Von Glasersfeld (1987) refers to as *constructivism*.

Constructivism also entails interaction with others to reflect on one’s own methods and methods used by others. This *interactivity* is also referred to as educational discourse. When sharing their views with peers, learners are not only passing information to others, they are also actively reflecting on their own strategies.

The interconnectivity of concepts through proportional reasoning, as set out in the previous section, is linked to Treffer’s notion of *intertwinement*. This concept entails connections between concepts as well as connections between mathematics in the

classroom and mathematics in real life. Further connections could even be made to other Learning Areas.

These principles of Realistic Mathematics Education, and especially the concept of intertwinement or interconnectivity, will be central to didactical practises described in chapter 5.

The concept of Realistic Mathematics Education is also in line with Bell's (1993) implications for teaching when stating that "the pupils' main lesson experience should be of genuine and substantial mathematical activities, which bring into play general mathematical strategies such as abstracting, representing, symbolising, generalising, proving, and formulating new questions." This sentence is laden with responsibilities for the Mathematical Literacy teachers teaching proportional reasoning. "Genuine" (real life or authentic) activities make up the essence of Mathematical Literacy, and teachers should aim to give "substance" to these genuine problems.

Keeping the numbers in Mathematics problems simple they are helping their learners. Especially in proportional problems this may lead to stagnation. If learners are always able to use build-up strategies to find the answer in a missing value question, they will never start thinking multiplicatively. Even if numbers are of such a nature that learners have to use calculators, all is well as long as they know, and understand, the multiplicative methods involved. Instead of given problems with ratios such as $2:3 = 8: ?$, rather give problems such as $2:3 = 7:?$. Learners who always used build-up strategies to solve the ratio problem by saying $2:3$ is the same as $4:6$ and is the same as $8:12$, will now have to rethink their methods. This will initiate higher level of thought even though they might use a calculator.

Bell also refers to "abstracting, representing, symbolising, generalising, proving, and formulating new questions" – qualities that are all part of advanced levels of cognitive development. One may think that these qualities are only developed in the Mathematics, but abstraction is also of great importance in the Mathematical Literacy class. A simple activity such as finding a formula for the total cost or total income is a form of abstracting, representing and symbolising. This alone does not constitute that

learners have higher order thinking skills that include proportional reasoning. Proportional reasoning still needs to be developed by means of the qualities that Bell refers to.

Sowder, Armstrong, Lamon, Simon, Sowder and Thompson (1998) argues that there are several considerations when preparing teachers to teach multiplicative strategies for proportional reasoning. The following two strategies which he suggests must be included in teacher training are also strategies that must be followed when teaching for proportional reasoning.

1. Teachers need to encounter problems that involve enough complexity to require sense making with the quantities involved.
2. Teachers should be given extensive opportunities to use notation (conventional and unconventional) to express their reasoning about quantitative relationships in given contexts and to describe their own invented solution methods.

As mentioned before, we can mainly distinguish between two types of proportional reasoning questions: the missing value problem and the comparison problem. These two types of problems should be given to learners simultaneously. Context play an important role in Mathematical Literacy due to its application based nature, but caution should be taken that the emphasis on context for the content does not overshadow the content. This does not imply that the context is not important, but that learners also need an opportunity to implement the Mathematical concepts they “extracted” from a context. The context can at no point overshadow the powerful Mathematics behind it.

4.1.4 Interconnectivity

Tourniare and Pulas (1985: 200) recognises that “proportional reasoning is not a unitary construct (and) it is therefore difficult to conceive of a linear teaching sequence. On the contrary, proportional reasoning should be considered as a multi-faceted activity, and presented as such.” It is this multi-faced nature of Mathematical Literacy that I will be taking as my directing paradigm in this investigation. This interconnectivity, or multi-faced nature of proportional reasoning, is also a wonderful

way of giving meaning, proof and logic to Mathematics, but it is a difficult concept to convey to learners when it comes to didactical practises. Two main problems arise.

1. Where does one start teaching in this web of knowledge?
2. How does one *teach* or develop interconnectivity?

The word *interconnectivity* is a way of expression the connections and interrelatedness between concepts in the Mathematical Literacy National Curriculum Statement that require proportional reasoning. Bell (1993: 9) refers to this concept as “*connectedness*” whilst Treffers (1987), refers to it as “*intertwinement*”.

Literature on Outcomes Based Education all stress the importance of *prior knowledge* as a basis from which to work. It is for this reason that we use assessment tools to determine our learners’ pre-existing knowledge on the concept. For the purpose of this study I compiled a fairly comprehensive baseline assessment covering almost all concepts of proportionality on Noelting’s levels of proportional development, but for the non researching teacher this can be in the form of informal discussions on the concept or a less comprehensive baseline assessment. My aim with the baseline assessment is to determine what learners already know, be it correct or incorrect, from which I would be able to build on the correct strategies and try to remediate the incorrect strategies or misconceptions. The Mathematical Literacy teachers often have not taught the learners in grade 9 and cannot formulate an opinion on the learners’ proportional knowledge and skills. The concept might not have been addressed at all or it was covered only briefly.

It is also a good idea to test learners on related concepts. Should a teacher wish to start teaching proportionality from a fraction perspective, for example, he/she need to know that learners have a sound understanding of fractions. If the foundation for our teaching is not firm, we cannot build a solid understanding of proportionality. As Riedesel (1969: 428) suggest, we cannot isolate concepts in Mathematics since learners will interpret it as new operations with its own set of rules. Bell (1993: 9) supports this by emphasising that “richly connected bodies of knowledge are well retained; isolated elements are quickly lost.” Since proportional reasoning skills are

needed in such a wide range of concepts in the Mathematical Literacy National Curriculum Statement, as will be discussed earlier in this chapter, it is essential to help learners to grasp the extent of the concept.

Some researchers (such as Olivier 1992, Riedesel 1969) suggest that in developing proportional reasoning teachers have to focus their attention on functional relationships and present ratios as a “special” function, i.e. a direct proportion with the format of $y = mx$. These researchers support the importance of equivalence in ratios and place emphasis on the use of tables where learners have to complete the table with the use of the specific constant ratio. The importance of equivalence is thus evident in their research and I would like to argue that learners that are able to do calculations involving equivalent fractions, would find the notion of equivalent ratios, as found in functional relationships, easier. It is for this reason that I will start my proposed activities for development of proportional reasoning with fractions to not only build on learners’ prior knowledge, but to insure that learners have a firm understanding of equivalence.

Once again I would like to stress the difference between ratios and proportional reasoning. Whilst ratio is but one concept in the mathematics curriculum that requires proportional reasoning it cannot be seen as proportional reasoning in itself.

Lovell and Butterworth (1966: 5) states that “proportion involves a relation between relations and the ability to handle it necessitates that the child recognise the equivalence of two ratios.” Although this explanation of proportion tends to focus one’s attention to relational or functional mathematics, I argue that we cannot overlook the importance of equivalence. If a learner is able to set up the ratio pairs and compare them, either with with-in or between strategies, he/she still has to have a basic understanding of equivalence as taught in fractions. Learners will have to know that in order to compare the two sets they have to be in a comparable format – similar to having common denominators.

The interconnectivity of proportional reasoning is always visible, though problematic, when teaching this concept. When taking a functional approach to proportional

reasoning, learners are essentially determining the “gradient” or rate at which the one number changes to give the other. For example: If 2 apples cost R3, what is the price of 5 apples? A child will have to determine the value of m in the expression $2m = 3$ in order to solve $5m$. Once they know that the rate is R1.50 or $\frac{3}{2}$, they can solve the equation by substitution: $5 \times \frac{3}{2} = \text{R}7.50$. In this example it might be quite easy to calculate the rate or “price per apple”, but what if we worked with similar triangles (enlargements or reductions) or VAT? Would learners know that they need to find the simplified ratio, or rate, to solve the proportional reasoning problem? Functional Mathematics and rate is a powerful idea in Mathematics and eventually leads to calculus. It is without doubt a natural part of proportional reasoning, but the problem remains: is it an adequate starting point to work from?

Interconnectivity can also be seen as transfer of knowledge between concepts and Boaler (1993) makes the link by referring to the same principles as set discussed in Realistic Mathematics Education.

“The problems of transfer may be attributed to a mechanised learning environment which ignores the needs of students and the cultural and social setting of the mathematics classroom. It is the individuals who negotiate, integrate and makes sense of the mathematics in their setting. They do this through interacting with the environment, each student arrives with a multitude of experiences and these influence and go toward the social dynamic of the mathematics classroom which in turn informs understanding” (Boaler 1993: 13).

4.2 A psychological approach to Mathematical Literacy

4.2.1 Logical structured thought

It is my strong belief that the aim of Mathematical Literacy is perfectly summarised in Freudenthal’s ideal that mathematical learning should be an augmentation of common sense (Gravemeijer and Terwel 2000: 785) Unfortunately it is true that what teachers perceive as common sense is not always common to learners. For learners taking

Mathematical Literacy logically structured thought does not necessarily come naturally at all, whilst it seems to be in the genetic make-up of any mathematician or mathematics teacher. Almost all the learners I teach in my Mathematical Literacy classes are highly talented artists, dancers or musicians with a strong creative side. Although we cannot generalise and say that all creative people lack structure and order in their lives, psychologists have made strong links with the arts and right brain thinking. Caine (1992) discusses Herrmann's Whole Brain Model and how it can be used to build learning experiences to enhance learning and make it more memorable for all participants.

I do not intend to explore the Whole Brain Theory in any detail, but believe that the concept can be of great value in the Mathematical Literacy class. In short the theory centralises around the idea that teaching focuses, to a great extent, on left brain characteristics such as logical, sequential, rational, analytical and objective thoughts that look at parts and not the whole in contrast to right brain thinking which involves intuition and a holistic approach.

Many learners cannot analyse a problem and break it down into smaller pieces of information so that they can deal with it in order to solve the bigger problem. Learners, who find it difficult to approach a mathematical problem in a logical step by step manner, often tend to memorise steps taught by the teacher. Yet, teachers who focus their attention on the smaller parts of the information should not neglect to see the bigger relations between concepts, such as the interconnectivity of proportional reasoning discussed in this study. As Sowder et al. (1998) and Person, Berenson and Greenspon (2004) found, teachers struggle to make the *correct* connection between fractions, ratios, rate and trigonometry as proportional reasoning themes. They tend to focus their attention on the computations involved in solving the fractions, ratios, rate and trigonometry questions, so that they fail to see the strong proportional reasoning theme present in all of them. Mathematical Literacy teachers need to see the web of interconnected knowledge themselves before they attempt to convey this web of knowledge to learners.

How the brain works has a significant impact on what kinds of learning activities are most effective. Teachers need to create appropriate experiences and capitalise on

those experiences. According to Caine (1992: 113) there are three interactive elements that are essential in helping learners to make connections:

1. Teachers must immerse learners in complex, interactive experiences that are both rich and real.
2. Students must have a personally meaningful challenge. Such challenges stimulate a student's mind to the desired state of alertness.
3. In order for a student to gain insight about a problem, there must be intensive analysis of the different ways to approach it, and about learning in general. This is what's known as the "active processing of experience."

Caine (1992) also stresses that designers of educational tools *must be artistic* in their creation of brain-friendly environments. Instructors need to realise that the best way to learn is not through lecture, but by participation in realistic environments that will enable learners to experience and explore new situations in the safe environment of the classroom. If learners in the Mathematical Literacy class tend to be artistic in nature, teachers cannot keep lessons confined to explanation on the board and pen and paper textbook activities. Lessons need to be colourful and interactive and even without the availability of data projectors (and PowerPoint's®), there is no reason for a Mathematical Literacy teacher to teach in a traditional structured manner.

4.2.2 Understanding

Skemp (1972) differentiates between Instrumental understanding and Relational understanding. Instrumental understanding refers to the old school way of teaching and learning. Formulas and procedures are given to learners to memorise and reproduce in examination situations. When confronted with a problem, the learners must merely pick a method and follow the procedures to produce an answer. The focus falls on the correct answer or performance of the student, and not on the reason or meaning of the calculations. When teachers use an instrumental way of teaching, learners have trouble recalling the work after a period of time. The reason for this being that learners never really fully comprehended it. They memorised rules and procedures, not mathematics. They were merely following procedures and manipulating numbers. Truly understanding mathematics means to get to the core of

the subject: constructing relations, asking why and finding the how. Although instrumental teaching allows for quick and easy results, there are a number of things that are of concern. Learners have no idea why they are following certain procedures, the concept has no value to them and they tend to develop a negative attitude towards the subject as a result of this.

If the teacher follows a *relational* way of teaching, the focus falls on the meaning of the word: building *relationships*. Learners are given the opportunity to make sense of what they are doing. They are encouraged to ask *why* a certain method is being used and what other methods could be used. When confronted with a problem, they must try to make sense of what is being asked and how it could be solved in a logical way. Each learner is allowed to choose his/her own way of conceptualising the problem.

When learners use their “own”³ methods, and understand the basic concepts of what and why they are doing something, it makes it easier to remember. It is thus not a memorising act, but a comprehensive understanding. In years to come, it will be easier for a learner with a relational understanding of Mathematics to build on his/her prior knowledge and concepts, than for a learner who memorised a list of procedures. Boaler (1993: 12) supports this notion that learners’ own self-generated methods when he says that “students transfer (methods) from one situation to another because these methods are meaningful to (them); the methods learned in school often are not.”

Bell (1993: 7) also states that the traditional didactics used in the Mathematics class was always just of “the demonstration (sometimes with explanation) of a single method followed by practice with a variety of different numbers”. There are no opportunities for learners to explore the concept and experiment with different methods. They are forced to memorise given methods, often with limited insight into the concept and are resultantly unable to apply the concepts they have learnt to similar situations which Bell (1993: 7) refers to as “the extension or adaptation of the basic idea”. Even a situation as simple as changing the given and required information seems mind-boggling to these learners and this is of course where many educators

³ “Own” methods refers to the methods that learners have chosen as their own and not necessarily their own creations.

falter when it comes to proportional reasoning. What is important to note is that learners often become frustrated by this relational way of teaching, since there is now no quick and easy way to solve a problem and the reasoning behind this should be explained.

4.2.3 Classroom culture

One of the most important tasks of a teacher is to create an atmosphere of trust, respect, willingness and questioning in the class that is a culture vital to relational teaching. It has no meaning to present problems that lead to exploration and questioning, if the learners in the class are used to passively listening for instructions. We must try to change the attitudes of learners in such a way that they are willing to dig into the essence of a problem. Some learners perceive mathematics as only within the capacity of a selected few. Some parents tell their children that they are not capable of doing mathematics. Teachers must let learners experience the joys of mathematics and assure them that everyone has a valuable contribution to make. Classrooms should be playgrounds of experimenting with mathematical ideas, taking chances, making mistakes and not feeling embarrassed by attempts. This atmosphere is called a positive classroom culture.

Creating a positive classroom culture is probably one of the most difficult dimensions to teaching, since so many learners are used to an instrumental way of teaching and since relational teaching involves trust and respect between all role-players. A classroom where learners feel safe to express their ideas and communicate their thoughts is vital for teaching relational understanding. Both the constructivism and the problem-solving approach place emphasis on communicating ideas and reflecting on thought. This is of great importance if we want learners to make sense of a situation. By sharing their ideas, learners also reflect on their thoughts in a different way and sometimes make sense of a problem without even listening to the thoughts of others.

Without learners having respect for others, teachers experience the problem of learners not listening to the ideas of their classmates. Learners are prejudiced in the sense that they will listen to some learners and not to others on various grounds, being it popularity or a perceived idea of competence. Teachers must enforce the idea that

all learners have the right to express their thoughts. As mentioned by Mathematical Literacy learners in the questionnaire, they have all had generally unpleasant experiences in the junior grade Mathematics classes in the past. These learners would not necessarily feel comfortable with sharing ideas and actively searching for solutions since they perceive themselves as less competent in Mathematics. It could have happened that these learners had the rest of the class react negatively towards them in the past.

Through creating social norms and standards, teachers can improve the quality of reflection and communication. Hiebert (1992) calls this, learning through cognitive psychology and social cognition: Reflection has its emphasis on internal mental operations, cognitive psychology, and communication has its emphasis on the context of learning and social interaction, social cognition.

It should be clear that by classroom culture we do not only refer to mathematical norms and values, but also to social norms and values. “In classrooms that promote (relational) understanding, the norms indicate that tasks are viewed as problems to be solved, not exercises to be completed using specific procedures” (Carpenter et al., 1999) Especially in senior grades, the task of creating an accepting atmosphere is quite difficult since teenagers are much more aware of the image they hold than junior learners. It is however imperative that teachers ask learners to reflect on their methods and ask them if they could relate the new concepts to things they have learnt in the past. This is the essence of relational understanding: building a network of knowledge.

4.3 Conclusion

What should be clear from this chapter is the importance of both a theoretical as well as a psychological approach to teaching Mathematical Literacy. The research on Realistic Mathematics Education and concept of interconnectivity or *intertwinement* will be a directing theory in the design of activities to develop proportional reasoning. Proportional reasoning is evident in every Learning Outcomes of Mathematical

Literacy within the National Curriculum Statement and in the next chapter I will be showing how the different concepts employing proportional reasoning skills in each Learning Outcomes can be connected. This needs to be structured in realistic contexts and implemented in classrooms that support constructivism.

Chapter five: Towards a theory of teaching and learning proportional reasoning

Gravemeijer, Dolk and Den Hertog (2002: 170) distinguish between three phases of design research that occur and reoccur within the cycle of this research approach. The first is the preliminary phase where a hypothetical learning trajectory of an instructional theory is presented. In the second phase this trajectory is either extended or amended in a cyclic process of making a new hypothetical learning trajectory and reflecting on the improved trajectory. The third phase is the result of information gathered throughout the process that can be “used to reconstruct an optimal instructional sequence” (Gravemeijer et al. 2002: 170).

The chapter will start by describing the original hypothetical trajectory of a non-linear approach in developing proportional reasoning in Mathematical Literacy and how and why this trajectory failed to work in the classroom. The change to a linear approach and the theoretical and structural analysis of the order of the progression of activities are subsequently described as an amended trajectory. The order of this amended trajectory and linear approach can in itself be seen as the optimal instructional sequence.

The manner in which these activities were designed was based on both the baseline assessment and on empirical research. The principles of Realistic Mathematics Education (Treffers 1987) is a common thread throughout these activities both in the way the activities were designed and was conducted in the classroom. As discussed earlier, on page 83, the principles of Realistic Mathematics Education are closely linked to the principles of Mathematical Literacy.

In the activities discussed in this chapter, I aimed to keep the context meaningful to learners so that the context supports the mathematics rather than distract from it. The linear approach discussed in the most of this chapter starts with an informal, context supported manner of dealing with proportionality and builds up towards the notion of

discrimination between proportional and non-proportional situations. Cramer et al. (1993) and Lamon (1995), as referred to on page 16, describe the ability to discriminate between proportional and non-proportional situations as one of the characteristic qualities of a proportional reasoner and it is therefore the goal and final stage of the activities described here.

The baseline assessment gave some insight into learners' constructions, methods and thinking patterns. This information was essential in conducting the activity to engage learners in class discussion and challenge them on their methods of reasoning. As Treffers (1987) suggests, learners that are actively involved in making sense of what they are doing, make the mathematics their own. This is strengthened by peer discussions and interaction, as the constructivism theory by Von Glasersfeld (1987) suggest, since learners have to reflect on their own understanding before orally expressing their ideas to others.

The last principle of Realistic Mathematics Education and the main theme of this chapter, is the idea of intertwinement, or as I am referring to it, interconnectivity. A great deal of research has been done on proportional reasoning. To my knowledge, none of the researchers have suggested a sequential progression of proportional activities that will take learners from an informal understanding of proportionality to a discriminative understanding of proportionality (Cramer et al. 1993; Lamon 1995) where they can solve a variety of problems types (Carpenter et al. 1999; Cramer, Post, Currier 1993; Behr & Lesh 1989; Karplus et al. 1983b; Lamon 1993b; Noelting 1980; Post, Lesh & Behr 1988) and understand the relationships embedded in proportionality (Cramer et al. 1993; Lesh et al. 1988). These qualities of a proportional reasoner are high standards to obtain, especially when setting these goals for grade 10 Mathematical Literacy students. Yet, when analysing the curriculum content, as in chapter 4.1.2, proportionality is strongly prominent and learners must therefore be supported into becoming proportional reasoners.

5.1 The didactical approach

This research was done during the first term of 2009 with grade 10 learners entering the subject of Mathematical Literacy. Included in the baseline assessment, a questionnaire was given to determine learners' attitude towards Mathematics and Mathematical Literacy.

As was evident in the questionnaire results where 83% of learners experienced negative emotions in the Mathematics class, learners enter the subject of Mathematical Literacy with a range of emotional barriers. When asked in the baseline assessment questionnaire *why* learners have chosen Mathematical Literacy over Mathematics, the answers were saddening. Answers ranged from a dislike in Mathematics, to hate of the subject as well as learners being forced to take Mathematical Literacy because, according to teachers, they were to “dumb” to take Mathematics. Although I am doubtful that any teachers used the word “dumb”, this was how learners perceived the recommendation to take Mathematical Literacy. Without intervention these learners will most likely carry this emotional burden with them throughout their high school careers. This places a big responsibility on Mathematical Literacy teachers. Except for teaching Mathematical Literacy students basic mathematics that will empower them in their future lives, these teachers need to rebuild these learners' mathematical self-confidence.

Many schools have strict policies that do not allow learners to change to from Mathematics to Mathematical Literacy in the middle of a year or any later than the end of grade 10. These schools are therefore very strict in allowing grade 10 learners into Mathematics. If teachers have any doubt that learners will pass Mathematics, they are “strongly recommended” to take Mathematical Literacy.

The questionnaire and baseline assessment results were used to *direct* the nature of activities and the nature in which the activities must be conducted in order to help develop proportional reasoning skills. The Mathematical Literacy student and the Mathematical Literacy classroom must be distinguished from a normal Mathematics classroom. In this environment mathematical self-esteems are being developed first and content second. These learners need to hear that they “can do it”, not by giving

them substandard work that will lead to high marks, but challenging them in calm and reassuring way and showing them what they are capable of. These learners know very well when they are being challenged and when success opportunities are given to them on a silver platter.

As mentioned throughout this study and supported by researchers such as McLaughlin (2003) and Lawson (1975), proportional reasoning is a higher order thinking skill and is therefore not a simple concept to teach. The further interconnectivity between proportional concepts within the Mathematical Literacy curriculum is another challenging concept to teach. It would have been unwise to conduct this investigation without taking into account the type of learners involved. For this learner the cognitive processes of acquiring knowledge might not come as natural as to a learner taking Mathematics, since this learner might have created several emotional barriers towards learning Mathematics throughout their school career.

Piaget, as in Louw, Van Ede and Louw (1998: 75), distinguishes between two invariant cognitive functions: organisation and adaptation. In the cognitive field of adaption, Piaget further distinguishes between assimilation, a process where new knowledge is integrated into existing knowledge structures, and accommodation, where existing knowledge structures need to be adapted and structured according to the new knowledge. When showing learners the similarities and differences between the concepts and the common procedures that can be followed in order to obtain answers to different types of proportion questions, learners will use either of these cognitive processes. Only when learners are made aware of the (correct) similarities and differences, can they formulate and develop their knowledge structures to include the new information they have acquired. When learners are able to do think and reflect on these processes by themselves, they have made the step from assisted thought to metacognitive thought processes and metacognition is seen as higher order cognitive skill.

The process of designing activities that will aid in developing proportional reasoning was experimental in nature. The principle of interconnectivity between concepts that involve proportional reasoning was the directing paradigm, but the manner in which

activities should be structured or conducted was a trial and error process. The final structure was the result of several less successful approaches to the principle.

5.2 A Non-linear approach to developing proportional reasoning

Since Tourniare and Pulas (1985: 200) argued that “proportional reasoning is not a unitary construct (and) it is therefore difficult to conceive of a linear teaching sequence”, my initial thought when designing activities to develop proportional reasoning was to adopt a non-linear approach with a strong emphasis on interconnectivity. The interconnectedness, or intertwinement as Treffers (1987) refers to this concept in the principles of Realistic Mathematics Education, of proportional reasoning became more and more visible within every learning outcome as this research progressed. For this reason my initial strategy was to design activities that covered all learning outcomes with proportional reasoning as the main theme. One worksheet would contain as many concepts as possible, but would have one dimension of proportional reasoning as theme. The approach could almost be described as a mind-map approach to proportional reasoning. An example of such a dimension of proportional reasoning would be missing value – a type of proportional reasoning question that could be asked in almost all of the Learning Outcomes of the National Curriculum Statement but all requiring proportional reasoning. The following questions formed part of such a worksheet covering all several concepts requiring proportional thought, such as ratio, interest, percentages and scale:

1. Jay and Silent Bob are two brothers. Their parents have decided to give them monthly allowances in the ratio of their ages.
 - a) If Jay is 17 and Silent Bob 13, what is the ratio of their allowance?
 - b) If Jay receives R255 per month, how much will Silent Bob get?
2. Angelina decides to open a savings account for her children’s schooling. How much money will she have in her account if she receives 12% interest on her investment of R100 000?

3. Brad and Angelina flew to Namibia for their twins' birth. The flight took them 22 hours at an average cruising speed of 500 km/h. How far was their journey?
4. Sean bought a Lamborghini three years ago that have depreciated with 25% to R360 000.
 - a) What was the price of the car when he bought it three years ago?
 - b) What is car worth now in \$ if 1\$ = R8,25?
5. Scale models are often used in the design and planning of movie sets. If a building of 25 meters high needs to be made using a scale of 1:50, what will the height of the model be?

Although this approach emphasised the idea of interconnectivity and the importance of proportional reasoning, learners found this a too general approach to developing proportional reasoning. In the above example, all questions were missing value type questions and required similar strategies, yet it covered several concepts with which learners were not necessarily familiar. Learners could not focus on the mathematics (and multiplicative reasoning) behind proportionality since each concept brought “a new context” and “new set of rules” all within one activity.

During this activity learners dealt with the concepts of ratio, percentages, interest, rate, exchange rate depreciation, and scale. Several learners did not know that scale is not unit bound. They wanted to know what unit the scale was in. They did not comprehend that the scale merely refers to the ratio of the model to that of real life. This is scale-specific knowledge that is important in understanding the scale.

Another learner did not make sense of speed. He wanted the formula to calculate speed so that he could calculate the distance. On explaining that “kilometres per hour” gives you a ratio of distance travelled in one hour and that this ratio could be used to find the missing value, he argued that km/h is merely a unit. It took him quite a while

to realise that the unit is descriptive of the concept. In his mind km/h, was a unit such as metre or litre with no meaning.

When following such a general approach to proportional reasoning, specific concept knowledge is second to the concept of interconnectivity. For learners to understand the essence of each concept it was imperative to interrupt the broad discussion to first teach specific concept knowledge in order for learners to understand the broader discussion. Learners' superficial knowledge of a specific concept was not enough to answer questions within all inclusive activities. In-depth knowledge of each concept was needed. As mentioned before, teaching Mathematical Literacy is vastly different to teaching Mathematics. The aim of the subject is to develop mathematical self-confidence whilst developing mathematical content. If learners are faced with so many different conceptual knowledge which they have to understand in order to identify differences and similarities they are hardly feeling self-confident.

It was for this reason that I decided on a linear approach to develop proportional reasoning. The concepts which needed to be addressed were the same as in the general approach, but needed to be structured in such a way that it would form a logical sequential progression. To find this sequential progression the work of Noelting and Thompson and Thompson described in chapter two, the baseline assessment results in chapter three and the didactical strategies described in chapter four, was of utmost importance. It was also essential to analyse each concept so that the key concepts in each could be clear. When referring back to Tourniare and Pulas (1985: 200) thoughts of a non-linear approach to proportional reasoning where "proportional reasoning is (described as) not a unitary construct (and) therefore difficult to conceive of a linear teaching sequence," it does not necessarily mean that a linear approach is inadequate. Tourniare and Pulas are referring the difficulty of a linear approach. I argue that with a detailed analysis of each concept requiring proportional reasoning skills, a progressive sequence of activities can be designed that will link concepts appropriately.

5.3 The didactics used for a linear approach

As discussed in 4.2.2 on page 73, Skemp (1972) differentiates between instrumental and relational understanding. From responses during class activities it became evident that many learners that have struggled with Mathematics in the junior grades have adopted an instrumental understanding to cope with difficult content. From the answers to questions in the baseline assessment test and questionnaire (as discussed in 3.5 on page 66), one could deduce that these learners entering the subject of Mathematical Literacy are not active participants in their own learning. They seem to prefer formulas and recipes to solve problems to taking time to think logically about what is being asked. This approach is not in line with the definition of Mathematical Literacy since it is a subject which “provides learners with an awareness and understanding of the role that mathematics plays in the modern world. It enables learners to develop the ability and confidence to think numerically and spatially in order to interpret and critically analyse the everyday situations and to solve problems” (Department of Education 2003a: 9).

One of the objectives of Mathematical Literacy is to create an appreciation for Mathematics with students and to enable them to become critical thinkers. Whilst teaching Mathematical Literacy I have had two instances where learners changed back to Mathematics since they could not get used to the context and application based questions in Mathematical Literacy. They wanted to follow a formula and get an answer without much reading and interpretation.

It is the concept of interconnectivity that forms the core of this investigation into developing proportional reasoning. My aim with these activities is to use learners’ prior knowledge, link it to new knowledge and emphasise the common strategies between the different concepts which were difficult to do in the non-linear approach. With the linear approach we scaffold the information and allow learners to continue once their foundational understanding is secure.

Whilst helping individual learners during class time and tutorials on the concept of ratios, it became evident that many of them had trouble “setting” up ratio and proportion equations when asked as word problems. Many learners in Mathematical

Literacy have not yet mastered the skill of asking reflective questions to help answer problems: When mixing Energade® in the ratio 1:3, it means I can use 1 part concentrate and 3 parts water which gives me 4 parts juice. So if I would like to make a litre of juice, the litre represents 4 parts of which 1 part is concentrate and 3 parts water.

To make it even more complicated when working with ratios, there are problems that ask of learners to work with parts of wholes (as in the Energade® question), or only parts. Then they can follow either a with-in or a between strategy and these strategies are also determined by the structure of the problem.

The “setting up” of equations refers to the way one structures the given information in such a way that it can easily be solved. Here are several ways in which people would structure the Energade® question from the baseline assessment:

$$\begin{array}{lcl}
 1 : 3 & = & 4 \\
 \square : \square & = & 1000 \text{ ml} \\
 1 \times 250 : 3 \times 250 & = & 250 : 750
 \end{array}
 \quad
 \begin{array}{c}
 \nearrow \times 250 \\
 \searrow
 \end{array}$$

$$1000 \text{ ml} \div 4 \times 1 = 250$$

$$1000 \text{ ml} \div 4 \times 3 = 750$$

$$1000 \text{ ml} \times \frac{1}{4} = 250$$

$$1000 \text{ ml} \times \frac{3}{4} = 750$$

Although these strategies might all look similar they are representative of different thinking patterns. The first of these strategies explain, in my opinion, exactly what is described by the given problem since it acknowledges exactly what the 4 represents as well as “equivalence” between the 4 and the 1000 millilitres. The other two more formal strategies assume that the learner has mentally completed the first representation.

Teaching learners how to set out a question does not necessarily mean that one is coaching methods instrumentally. It merely helps learners to analyse the numerical relationships without the words that make up the problem. I conjecture that this set-up of a proportional situation will help learners to identify what is being given and what is being asked. Would a total, for example be applicable to the specific situation to help in finding the part-whole representation or is situation where a total (or whole) is not applicable. Although there are several factors to take into consideration when faced with a proportional situation, structuring the given numbers in the “A is to B, as C is to D” structure, will help in identifying the type of proportional problem.

We can distinguish between four types of proportionality problems: missing value, comparison, rate and sharing. Although these concepts problem types are themselves closely linked, they each require a slightly different approach. Where missing value and comparison problems have a strong emphasis on equivalence, sharing problems have a strong focus on the part-whole representation. The key concept in rate is that of simplifying to a quantity per *one* unit, as in 100 kilometre per (*one*) hour. I will distinguish between these four types of questions and also the different strategies that can be used to solve each type. To distinguish between these four types of questions I will give some examples of questions:

5.3.1 *Missing value*

The heights of Jane and Jess are in the ratio 7:9. The shorter one of the two is 154 cm. Who is taller and what is her height?

When explaining a question such as this one, I helped learners to set up the question as follows:

Jane : Jess

7 : 9

154 : □

This representation helps learners to realise that the 7 represents Jane’s height of 154 centimetres and that the question asks for the height of Jess, represented as 9. From this learners can reason in a number of ways. In this instance it is a part-part scenario

and it would not make sense to find a total for the two girls' length. If learners understand the meaning of the ratio they would know that if Jane was (ridiculously speaking) 7 centimetres tall, then Jess would be 9 centimetre tall, but now Jane is 154 centimetre. Then they can ask themselves: How many times bigger is 154 than 7? Once they know this, they could do the same to 9. This is very much the same principles we use when solving equivalent fractions:

$$\frac{7}{9} = \frac{154}{?}$$

Representing the situation as a fraction reminds learners of the multiplicative calculations involved in equivalent fractions. Unfortunately this method has a strong emphasis on the relationship between ratios and the numbers being used also support this notion.

This missing value question does not require learners to find a total and is thus a part-part question. A part-whole missing value question would be a question like the Energade® question (where learners will have to find how many millilitres of water and concentrate is needed to make a litre of Energade®) described earlier.

The use of the colon to represent the information is not essential in this set-up. Arrows, or any other symbol that has a similar meaning, can be used with equal success:

Jane → Jess

7 → 9

154 →

5.3.2 Comparison questions

Which of the following ratios will give the lightest shade of lilac (contains the most white paint). Red: Blue: White in a mix of 1:2:5 or in a mix of 2:3:5?

When setting up this question it would look as follow:

Red : Blue : White	Total
1 : 2 : 5	8
2 : 3 : 5	10

The easiest and most logical approach to this question would be to see the with-in ratio of red and blue to white. In both mixtures the white paint is represented by 5, but in the first ratio the red and blue make up 3 parts, while it makes up 5 parts in the second ratio. In comparison to the other colours, there is more white paint in the first ratio (since there are less red and blue.)

A fraction method in this scenario will have to take into account the *total* in each case. Learners will have to find the fraction of paint in each ratio which is white paint:

$\frac{5}{8}$ in the first ratio versus $\frac{5}{10}$ in the second ratio. Some learners will immediately recognize that $\frac{5}{8}$ is bigger than $\frac{5}{10}$ whilst others will have to find a common denominator. The big question is however if learners will know to pick the bigger or smaller fraction and what each means? It is here where the “literacy” comes into Mathematical Literacy. Although learners taking Mathematics have to evaluate their answers, they seldom do so, unless the instruction specifically asks for it to be done.

Learners will have to be conscious of the fact that the numerator is the white paint whilst the denominator is the whole paint mix. Learners who can see that $\frac{5}{8}$ is a bigger fraction will need to know that this bigger fraction means more white paint.

It is not wrong for learners to make fractions of $\frac{\text{white paint}}{\text{red and blue paint}}$, but it does understandably make less sense. Once again learners will have to keep in mind exactly what they use as numerator and denominator to make sense of their answers.

The numbers in this question were convenient to work with and most learners can clearly see that the first ratio contains more white. If the amounts of white paint were not the same, it would involve a lot more calculations:

Which of the following ratios will give the lightest shade of lilac (contains the most white paint). Red: Blue: White in a mix of 1:2:5 or in a mix of 2:3:6?

Red : Blue : White	Total
1 : 2 : 5	8
2 : 3 : 6	11

$\frac{5}{8}$ of the mix is white in the first ratio versus $\frac{6}{11}$ in the second ratio. In this scenario it is not so easy to merely see the answer. Equivalent fractions provide the answer:

From $\frac{5}{8} = \frac{55}{88}$ and $\frac{6}{11} = \frac{48}{88}$, it is clear that the first ratio has more white paint.

I have used both routine problems (with limited context), to focus learners' attention on the Mathematics involved in the problem, and problem-solving questions to tap into learners logical thinking and reasoning skills. Problem-solving enables learners to confirm their answers by using the context – something not possible when working with routine problems. Yet, routine problems are often useful to practise concepts learnt from problem-solving questions. An example of this would be the part-part versus part-whole problem which arose from the Energade® problem where learners thought that the ratio 1:3 means that $\frac{1}{3}$ of the mix is concentrate. After a discussion on this concept and working through similar problem-solving questions, it would be a good idea to give a few “conversion” questions where learners need to convert ratios to fractions and vice versa. These routine problems will help learners in practising the

skills learnt within a context-driven problem and give them an opportunity to focus on the Mathematics involved without being influenced by context.

From the baseline assessment it was evident that learners had some multiplicative idea when it came to proportional reasoning but that the nature of the numbers and the context influenced their understanding to a great extent.

The pizza problem: 3 boys shared 2 pizzas and 9 girls shared 7 pizzas.

Learners in the senior grades seem to interpret these type of questions the way teachers intend to ask them. Caution should however be taken with the wording of this type of question since it can greatly influence the answer. When asking learners to compare the two situations (or ratios) by saying: *Which group had more pizza?* The correct answer would be the girls, since they had 7 pizzas and the boys had 2. 7 is more than 2. This is not necessarily the answer intended by this question.

If the question is asked: *Which group had more pizza per person?* It is implied that learners should use *rate*, leading to calculations such as:

$$2 \div 3 = \frac{2}{3} \text{ or } 0,67$$

$$7 \div 9 = \frac{7}{9} \text{ or } 0,78$$

It can also be asked: *What percentage of the pizza did each boy and each girl get. Can you tell who had more?* Here we are guiding them to use percentage - another logical approach to this problem. Learners will thus have to extend on the rate approach to get 67% and 78% respectively.

Yet, this problem is often used as a fraction question due to the nature of the integers used. Learners are thus expected to compare the fractions $\frac{2}{3}$ and $\frac{7}{9}$.

Certain problems lend themselves more towards one approach than another, but the way in which the question is formulated, should give an indication of what is expected.

5.4 A linear approach to proportional reasoning

From the data collected on the baseline assessment, it was clear that learners have poor proportional reasoning skills and strategies employed, such as the build-up strategy, were weak. It was thus important to approach this concept almost as if learners had never done it before. From the research drawn from both the baseline assessment and researchers such as Adjidage and Pluvinae (2007), Clarke et al. (2003), Noelting (1980) and Treffers (1987), the following linear approach was prearranged.

- The link between fractions and ratios
- Distinguishing between missing value, sharing and comparison
- Percentages
- Ratio, Scale and enlargements and reductions
- Looking at rate and the role of ratios
- Conversions
- Distinguishing between proportion and direct and indirect proportion

Interconnectivity forms a vital part of this research and deciding on the most appropriate linear approach to connecting all concepts required a thorough analysis of each concept. The reasoning behind this approach is as follows:

When designing the linear structure during this research the basis from where to start this linear program was always clear. Adjidage and Pluvinae (2007: 170) argued that “processing fractions is well accorded to the valid treatments in proportionality, so that pupils that master fractions should better master proportionality”.

When introducing a new concept it is important to link the new knowledge with learners’ prior knowledge. It was clear from the baseline assessment that learners had a basic understanding of ratios but that there were several areas of concern. Noelting’s findings on learners’ understanding of fractions and ratios, as discussed on page 17,

showed that pupils had few problems in solving calculations involving adding and comparing fractions, but that pupils found ratio questions problematic. The importance of fractions and the similarities between the multiplicative strategies used when solving equivalent fraction problems and those involved in solving equivalent ratio problems, was used as basis from which further concepts could be introduced. When learners were asked to calculate the amount of packets of chips in question 1.1 of the baseline assessment, 60% of learners used multiplication instead of division. This is an indication that learners do not have a clear understanding of the nature of fractions. They were able to do the multiplicative calculation correctly, but lacked an understanding of what they were doing. Since both fractions and ratios employ the same multiplicative strategies, supported by the research of Clark et al. (chapter 2), fractions and ratio will be the start of the sequential progression of activities.

The goal of the linear program was equally well apparent in the definition of proportional reasoning: a learner that can reason proportionally has a discriminative understanding of proportionality (Cramer et al. 1993; Lamon 1995). As the first principle of Realistic Mathematics Education suggests, there needs to be a growth from an intuitive understanding of the concept to an abstract understanding. In the case of proportional reasoning, learners need to be able to think in terms of functions and the accompanying formulas and graphs and distinguish direct proportions from proportions and indirect proportions.

By defining a starting point and objective to work towards with the linear approach, it directed the structure of the activities and concepts that should make up the path towards this goal. Conversion was initially an introductory concept, having functions as objective brought to mind the functional nature of conversions and more specifically conversions, such as temperatures, which are not directly proportional. For this reason conversions could be a defining factor between directly and non-directly proportional situations. By having an objective of functions, it was possible to fill the other concepts in like puzzle pieces either from the objective backwards or from the starting point of fractions forward.

To link with fraction and the idea of equivalence in the concept of fractions fractions and ratio as discussed earlier in this chapter, it was important to follow fraction with

ratios. The different structures of questions that involve proportional reasoning, namely missing value, comparison, sharing and rate had to follow accordance. The research of Noelting focused a great deal of attention on the use of with-in and between strategies. Karplus, Pulos and Stage (1983: 232) found that learners who were able to reason proportionally “exploited integral ratios within or between relationships, and (therefore) recommend the emphasis of these approaches to proportionality problems.” Together with the different structures of proportional reasoning, different strategies in solving proportional reasoning must be investigated. For this reason I included questions that either preference with-in or between strategies so that learners would be able to use both and discuss the differences.

Since percentages are merely another form of fractions, it is important to extend learners’ knowledge of fractions and ratio to that of percentages. Although it may seem a relatively easy concept, proportional reasoning is essential in making sense of appreciation, depreciation, VAT and inflation, especially when learners need to find values before increases or decreases.

Scale and enlargements and reductions was placed before rate since scale is similar to rate in the sense that it is a simplified ratio. Scale is also an important concept in understanding proportional reasoning since it explains the concept of “representation”. On a map with a scale of 1 : 50 000, one unit represents 50 000 units in real life. This concept is similar to, for example, a 20% increase, where 100% represents the original price and 120% represents the increased amount.

Rate, as scale, requires of learners to simplify. When, for example, they need to compare speed or price, they need to find the kilometres travelled per *one* hour and the price per *one* unit. This strategy could also be employed in other situations and links closely to that of conversions. A concept such as exchange rate for example, can be classified as both rate and a conversion. It is for this reason that conversions were the sixth concept in this sequence.

When working with conversions most, but not all conversions are directly proportional, which means that 0 metres would be 0 kilometres and \$0 would be R0.

When converting degrees Celsius to degrees Fahrenheit, it is not directly proportional and this situation is ideal for initiating conversation on directly proportional situations and non-directly proportional situation. The formula for calculating degrees Fahrenheit has a proportional structure of $y = mx + c$ structure whilst most other conversions are direct proportions with a $y = mx$ structure. Compare the following:

$$\text{Fahrenheit} = \frac{9}{5}C + 32$$

$$\text{Kilometres} = 1000 \text{ } m$$

To distinguish between different types of proportionality is the last stage of this set of activities. Since learners should have an understanding of rate, introducing gradient or rate of change should be a natural progression. Both Cramer et al. (1993) and Lamon, (1995) emphasise that to be able to reason proportionally one must be able to “discriminate proportional from non-proportional situations”. Converting degrees Celsius to degrees Fahrenheit, for example, must be recognized as a proportional question, unlike Rands to dollars which is a directly proportional question.

5.5 Activities used in the classroom

Proportionality is in most cases represented in textbooks as the concept “Ratio and Proportion” where proportion is hardly defined or spoken of after its use in the title of the chapter. It is assumed that proportion is synonymous with “ratios” and a set of ratio questions are thus presented. Although ratios form a mayor part of proportional reasoning, not synonymous terms and learners that might be able to answer ratio questions are hardly proportional reasoners. The aim of these activities is to introduce learners to the widespread use of proportional reasoning throughout the Learning Outcomes of Mathematical Literacy within the National Curriculum Statement. These activities will not merely put a few proportional reasoning questions in context, but will aim to link all concepts which require proportional reasoning and discuss how the same thought processes can be applied to solve problems within all these concepts.

All activities described below were used in the classroom as part of a grade 10 Mathematical Literacy work schedule.

5.5.1 The link between ratios and fractions

The multiplicative reasoning involved in equivalent fractions is similar to that of equivalent ratios and I believe that the link between fraction and ratios are vital in developing proportional reasoning. Showing learners that they can take their pre-existing basic knowledge of fractions and apply it to the less familiar concept of ratios is a strategy which aims to help learners to make sense of the new concept. As Noelting (1980) showed in his investigation of learners' fraction and ratio knowledge, learners had few problems in solving calculations involving adding and comparing fractions, but that pupils found ratio questions problematic. One would think that if learners can simplify $\frac{21}{28}$ they would be able to simplify 21 : 28. Or if they are able to

calculate the missing value of $\frac{21}{28} = \frac{18}{?}$, they should also be able to calculate the missing value of 21 : 28 which is equivalent to 18 : ?. In the light of design research, more attention could have been focussed on the computations involved in fraction and ratios within the baseline assessment. Results from these questions could then have been used to support Noelting's findings on fractions and ratios.

Several learners in my class had trouble seeing the similarities between ratios and fractions. They were comfortable answering questions where they had to find the missing denominator and numerator, but seemed confused when asked ratio questions which required the same operations. To make the link between multiplicative strategies needed for both ratios and fractions, it was helpful to compare the two concepts by writing it in two columns on the whiteboard as in the following example:

<p>Sally wants to convert her test mark out of 40 to a mark out of 10 for her portfolio. If she received $\frac{32}{40}$ for her test, what will her mark out of 10 be?</p> $\frac{32}{40} = \frac{\square}{10}$	<p>A poster has with to length ratio of 32 : 40. If this poster must be reduced so that the length is 10 cm, what will the width be?</p> <p>32 : 40</p> <p>\square : 10</p>
---	--

Examples like these seemed to have helped learners in making the connection (or adaptation) between fractions and ratios.

To further explain the reasoning behind the link between fractions and ratios, situation must be presented where it is essential for learners to be able to distinguish between the two concepts. Clark et al. (2003) distinguish between the part-part and part-whole representation and it is the confusion between these two concepts that resulted in learners being confused about the ratio 1:3 in the baseline assessment. Learners could not distinction between ratios and fraction when asked to mix Energade® in the ratio 1:3. A third of learners that obtained the incorrect answer for the Energade® question worked with a fraction of $\frac{1}{3}$ instead of $\frac{1}{4}$. They assumed that $\frac{1}{3}$ must be concentrate and $\frac{2}{3}$ water instead of one *part* concentrate to three *parts* water.

Fractions are more commonly known as part-whole scenarios where the numerator shows the parts and the denominator the whole. If one third of a one litre mixture of Energade® is concentrate, it means that 333 millilitres would be concentrate and 667 millilitres would be water. But the ratio on the bottle is written as a part-part scenario which means that one part of a one litre of Energade® would be 250 millilitres and three parts would be 750 millilitres. It is important that learners are able to distinguish between a part-part and part-whole scenario since it is often helpful to switch between the two. For example in the ratio 1:3 learners need to realise that there are *four* parts in total. The concentrate is thus $\frac{1}{4}$ of the Energade® and the water is $\frac{3}{4}$. If learners are able to realise this important concept, it would be easy for them to realise that to make one litre of Energade® they need one part, $1000 \text{ millilitres} \div 4$ or 250 millilitres of concentrate and three parts, $250 \text{ millilitres} \times 3$ or 750 millilitres of water.

To help learners to discriminate between ratios and fraction I used visual representation of the Energade® questions to initiate discussion on the fraction of water and the fraction of concentrate used in a specific mix.

Water
Water
Water
Concentrate

To initiate education discourse, which forms part of both principle three and four of Realistic Mathematics Education (Treffers 1987), I created conflict in the class by writing both the correct (part-part interpretation) as well as the incorrect (part-whole interpretation) answers to the Energade® question on the board. I asked learners to decide which answer is correct and which is incorrect and to discuss the two different approaches. I also encourage learners to make a drawing of the situations (as the one above) to motivate their choice of answer or to use the Energade® bottle in class examine the markings on the bottle. When confronted with both these possible solutions learners soon realised their incorrect thinking patterns when being convinced by peers, and themselves, of the correct part-whole method. Learners are thus both activity involved in reflective processes of their learning and sharing ideas with others. By creating conflict in the classroom, learners are forced to interact with the mathematics and their peers.

The following activity was used as extension on the baseline assessment Energade® question. It focuses on the concept of part-part representation versus part-whole. Learners are asked specific questions related to the *total* amount of liquid in each mix to emphasize the part-whole representation that could be made from the part-part (concentrate-water) representation.

Julio and Craig are on a summer camp and have to make orange juice for all the campers. They plan to make the juice by mixing water and orange-juice concentrate. They are not sure how to mix the juice and make a couple of mixtures from which they can have a taste:

<p>Mix A</p> <hr/> <table> <tr> <td>2 cups</td> <td>3 cups</td> </tr> <tr> <td>concentrate</td> <td>cold water</td> </tr> </table> <hr/>	2 cups	3 cups	concentrate	cold water	<p>Mix B</p> <hr/> <table> <tr> <td>5 cups</td> <td>9 cups</td> </tr> <tr> <td>concentrate</td> <td>cold water</td> </tr> </table> <hr/>	5 cups	9 cups	concentrate	cold water
2 cups	3 cups								
concentrate	cold water								
5 cups	9 cups								
concentrate	cold water								
<p>Mix C</p> <hr/> <table> <tr> <td>1 cup</td> <td>2 cups</td> </tr> <tr> <td>concentrate</td> <td>cold water</td> </tr> </table> <hr/>	1 cup	2 cups	concentrate	cold water	<p>Mix D</p> <hr/> <table> <tr> <td>3 cups</td> <td>5 cups</td> </tr> <tr> <td>concentrate</td> <td>cold water</td> </tr> </table> <hr/>	3 cups	5 cups	concentrate	cold water
1 cup	2 cups								
concentrate	cold water								
3 cups	5 cups								
concentrate	cold water								

1. Which mix will give a stronger taste? Explain your answer.
2. Which mix will give the weakest taste? Explain your answer.
3. Which comparison is correct?

$\frac{5}{9}$ of mix B is concentrate or $\frac{5}{14}$ of mix B is concentrate

Explain your answer.

4. Assume that each camper will get $\frac{1}{2}$ a cup of juice, how many batches of each mix needs to be made for 240 campers?
5. For each mix, how much water and how much concentrate is needed to make juice for 240 campers?
6. For each mix, how much water and how much concentrate is needed to make 1 cup of juice ?

In this activity learners are consciously made aware of the *total* of a ratio. In Mix B 5 parts of the *whole* 14 parts are concentrate. Saying that $\frac{5}{14}$ of the mixture is concentrate is correct, but many learners still tend to use comparisons like $\frac{2}{3} ; \frac{5}{9} ; \frac{1}{2} ; \frac{3}{5}$ that they convert to $\frac{60}{90} ; \frac{50}{90} ; \frac{45}{90} ; \frac{54}{90}$. This representation is not a wrong way of comparing the ratios. It is merely incorrect to say that $\frac{5}{9}$ of the mixture

is concentrate. Learners who compare fractions like $\frac{2}{3}; \frac{5}{9}; \frac{1}{2}; \frac{3}{5}$ need to be reminded that they are not comparing part-whole fraction, but that they are comparing part-part fractions. They can therefore only compare the ratios if one of the “ingredients” (water or concentrate) is made the same for every mixture. Between 3, 9, 2 and 5 they find a common denominator (90) that will ensure that all the mixtures have the same amount of water, but resultantly have different quantities of concentrate: $\frac{60}{90}; \frac{50}{90}; \frac{45}{90}; \frac{54}{90}$. With this set of equivalent fractions one can conclude that the ratio 2:3 converts to the biggest fraction where there are 90 units of water to 60 units of concentrate.

Several learners, mostly girls, employed the visual representation method to analyse the question before making any further calculations. This approach helped them to use part-whole fractions with which they compared the amounts of concentrate. In this specific example, the common denominator between 5, 14, 3 and 8 amounted to a rather big common denominator of 840, which resulted in a few calculation issues. Learners using part-part calculations in this situation seemed to make fewer calculation errors due to the nature of the numbers. Although the methodologies surrounding this question were good, the numbers should be revised for future use. If the ratio of 5:9 were changes to 5:7, the common denominator of 120 would be easier to work with.

One learner used percentages to solve the mixture question. He explained that it was easier to calculate the percentage of concentrate in each mix than to find the common denominator. His calculations resulted in the following percentages: 67%, 56%, 50% and 60%. This learner recognized the connection between fractions and percentages and was able to use knowledge to apply this knowledge to ratio questions.

I used this opportunity to draw the rest of the class’s attention to the link between fractions, ratios and percentages. This specific learner preferred to think in terms of percentages instead of fractions. Percentages seem to feel more “natural” to him than fractions. This could be ascribed to the use of percentages on the end of term report, in the media and have even people’s way of speaking.

Learners however still need the part-whole fraction format to calculate the percentage on their calculators and must thus be comfortable with the notion. I used the following activity to help learners in making the link between ratios, fractions and percentages and also focuses their attention on the difference between part-whole and part-part representations.

1. Change to a Ratio

If possible, change each comparison of concentrate to water into a ratio. If it is not possible, explain why:

- a) The mix is 60% concentrate
- b) The fraction of the mix that is water is $\frac{3}{5}$.
- c) The difference between the amount of concentrate and water is 4 cups

2. Change to a Percentage

If possible, change each comparison of red paint to white paint into a percentage. If it is not possible, explain why:

- a) The fraction of the mix that is red paint is $\frac{1}{4}$.
- b) The ratio of red to white paint in a different mix is 2 to 5.

3. Change to a Fraction

If possible, change each comparison to a fraction comparison. If it is not possible, explain why:

- a) The nut mix has 30% peanuts.
- b) The ratio of almonds to other nuts in the mix is 1 to 7.

It is important to constantly remind learners of the part-part versus the part-whole representation which they are confronted with in transforming ratios to fractions and vice versa. Teachers need to create conflicting thoughts by asking learners if the part-part and part-whole representations are equivalent. This will enable learners to develop the habit of asking this question themselves when faced with a similar problem. By enabling the process of *self reflection*, we are not only helping learners

with proportional reasoning, but also creating valuable meta-cognition skills which form a vital part of higher order thinking skills. Questioning one's answers and reflecting on methods and calculations is a skill vital for making sense of mathematics. Mathematical Literacy is but a tool in developing logical thinking and problem-solving skills.

As discussed in chapter 2, Clark et al. (2003) argues against the use of model two that implies that all fractions can be interpreted as ratios, but that not all ratios are fractions. Their view is that this model is not a powerful discrimination between the concepts. As I argued on page 25, model two is in fact a better basis for discrimination. At this stage of the sequential progression of activities, it would be a good idea to confront learners with this idea and ask them what link they identified between the two concepts.

To start this discussion, learners should be made aware of the fact that it is not always possible to write ratio questions as part-whole scenarios. If we take the question of the heights of two people being in a specific ratio, say 7:9, it is meaningless to say that the shorter one of the two has a height of $\frac{7}{16}$ of their combined height. Saying the shorter

person is $\frac{7}{9}$ of the height of the taller person is indeed much more meaningful. The

latter fraction will also help in calculating the height of the shorter person if the height of the taller person is given. Say the taller person is 198 cm, then the shorter person will be $\frac{7}{9}$'s of 198 cm which amounts to 154 cm. Teachers must also keep in mind

that asking the height of the taller person if the height of the shorter person is given, may not be as simple for learners since they will have to work with an improper fraction or mixed fraction which somehow does not seem as natural to them. Saying

that Sally is $\frac{7}{9}$'s of David's height seems much more natural than saying that David is

$\frac{9}{7}$'s of Sally's height.

It is difficult for learners to distinguish between part-part and part-whole situations and they should be encouraged to think of situations where a total (or whole) would be meaningful and where it would not be.

Another question arising from the baseline assessment apart from the part-part versus part-whole representative problem was that of additive strategies. 60% of learners in school 1 and 17% of learners in school 3 used incorrect additive strategies and clearly operated on Thompson and Thomson's level 2. A number of students also used additive build-up strategies in the baseline assessment and although this strategy is not incorrect, it is seen as "weak" proportional reasoning (Lesh et al. 1988: 105).

In exploring the reasons behind incorrect methods, the emphasis falls on the reasoning behind the correct methods. Learners who "instinctively" knew not to use additive strategies might not have realised that proportional problems are multiplicative in nature. By exploring incorrect additive strategies the multiplicative nature of proportionality might become more apparent to them.

I gave learners the following question to discuss the incorrect additive strategies employed by learners:

Are the following two ratios the same or different? Motivate your answer:
2:3 and 3:4

Two learners in the class were convinced that the ratios were equivalent. Even after using visual representation they were not convinced otherwise. A learner in the class gave the following counter argument to the problem:

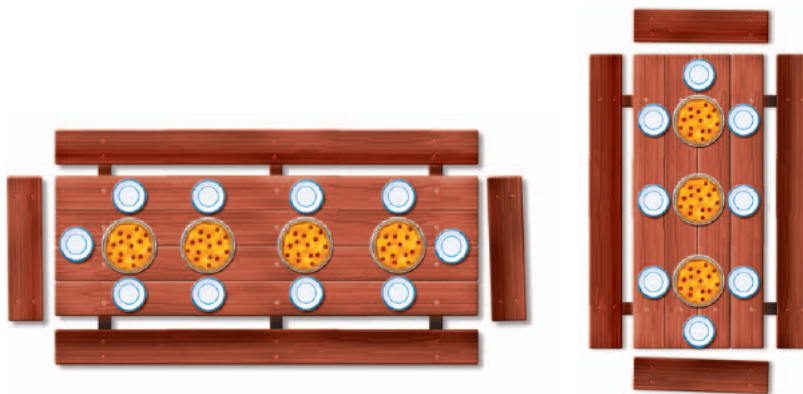
"So you are saying they're the same because two plus one is three and three plus one is four. That means that two to three must be the same as 1001 to 1002 because you added a thousand. But this can't be right because in two to three, three is half of two more, but 1001 and 1002 are almost the same."

His explanation did not manage to explain a correct method of comparison to the two “adders” in the class, but did cause them to realise that their additive strategy was incorrect.

The learner’s explanation contained more than one argument. Firstly he showed how the ratios cannot be equivalent when applying additive strategies. He supported this referring to the with-in relationship between the two and the three. Although he explained this as “*half of two more*”, is reasoned that the three was 50% or 150% of two. His explanation of “*half of two more*” can in fact be classified as an additive strategy as well since he utilised a build-up method. Although it might not be evident of strong proportional reasoning skills, his convincing argument does deserve merit.

The following activity focuses on the distinctive differences between additive and multiplicative strategies to build on this class discussion. In question C learners are also asked to investigate an incorrect additive (or subtractive) strategy and have to explain *why* it is incorrect.

The dining hall at a school campsite has two different kinds of tables. There are larger tables seating ten persons and smaller tables seating eight persons. On pizza night the students serving dinner put three pizzas on each small table and four pizzas on each large table.



- A) Suppose the pizzas are shared equally by everyone seated at the table. Does a person seated at one of the small tables get the same amount of pizza as a person seated at one of the large tables? Explain your reasoning.

- B) Which table relates to the fraction $\frac{3}{8}$? What do the 3 and the 8 mean? Is $\frac{3}{8}$ a part-to-part comparison or a part-to-whole comparison?
- C) John thinks he can calculate at which table a person gets the most pizza. He uses the following reasoning:
- $10 - 4 = 6$ and $8 - 3 = 5$ so it is better to be seated at a large table.
- 1) What does the 6 mean and what does the 5 mean in John's reasoning?
 - 2) Do you agree or disagree with John's method?
 - 3) Suppose you put 9 pizzas on a large table.
 - 4) What answer does John's method give?
 - 5) Does this answer make sense?
 - 6) What can you now say about John's method?

On asking learners about the meaning of $\frac{3}{8}$, a learner gave the following explanation:

"I think if we went out for pizza we would order different pizzas so that we could all taste different ones. If there was eight of us and we ordered three pizzas I would ask them to make sure they divided the pizza into eight. That way I could get a slice from each pizza. That means I would have three slices and each one is an eighth of the pizza – that makes three eighths!"

On asking this learner how much she would get at the bigger table, she gave the following explanation:

"I am not sure if they can cut pizzas in ten? It would be nice because then I could taste four different pizzas."

I asked her if these smaller slices would be more than the three bigger slices.

"I will have to work it out. I'm not sure."

Since this learner made sense of the situation and identified the amount of pizza she would get at each table as $\frac{3}{8}$ and $\frac{4}{10}$, she was able to compare the fractions by using her knowledge of equivalent fraction to obtain the correct answer.

I conjecture that a firm understanding of fraction will lead to a better understanding of proportionality. The specific aspect of fractions I am referring to here is equivalence. Let us consider the following primary school question:

Bella ate $\frac{2}{3}$ of her pizza whilst Tess finished $\frac{3}{4}$ of her pizza. Who ate the biggest portion?

Although many high school learners still have an unsatisfactory understanding of fractions, I am convinced that most learners will remember that they have to find a common denominator and then compare the portions. Is this not similar to the following ratio question:

Bella mixes juice by adding concentrate and water in the ratio 2:3 whilst Tess mixes it in the ratio 3:4. Who of Bella or Tess mixed the stronger tasting juice?

The ratio question does not seem as easy as the fraction question and it is largely due to the part-part versus part-whole dilemma. It was clear in the fraction question that the girls had a fraction of one pizza. In the ratio question it is not clear if both girls made the same amount of juice or if Bella for example made 5 litres (2 parts concentrate plus three parts water) and Tess 7 litres (3 parts concentrate and 4 parts water). Although it is possible to compare the two ratios by comparing $\frac{2}{3}$ and $\frac{3}{4}$, one would need to realise that when making the fractions equivalent, the *water* parts of the ratio (the denominator) would be equivalent and therefore the bigger numerator would be the mixture with more concentrate. What would sound more logical, would be to say that 2 parts of Bella's 5 part mixture is concentrate, thus $\frac{2}{5}$, and 3 parts of Tess's

7 part mixture is concentrate, thus $\frac{3}{7}$. This means that one can compare the portion of concentrate per mixture.

However one approaches the question, there will be equivalence involved in order to compare the ratios. The problem with ratios is that learners need to recognise the ratios as fractions before they can use fraction strategies to solve the problem. But as Clarke et al. (2003) explained, teachers themselves are unsure about the connection between fractions and ratios.

The idea of exploring ratio and fractions out of context is to investigate learners' specific methods that they follow when dealing with ratio and fraction. Learners often use a set of instructions without fully comprehending what they are doing. Without a context, learners have the chance to solve fraction and ratio problems using the methods they have been taught and it helps to highlight these methods for research purposes. Just as context can be helpful in solving problems it can also cloud the problem for learners. Mathematical Literacy students tend to struggle to make the transfer from the largely number-driven subject of Mathematics to the context-driven subject of Mathematical Literacy. It seems a great effort to them to extract the necessary information and use it to solve the given problem. Yet, in the words of Cramer et al. (1993: 13): "A proportional reasoner ultimately should not be influenced by context nor numerical complexity". It is the function of the Mathematical Literacy teacher to help learners to develop these proportional reasoning skills that are independent of context or number complexity. By giving learners a chance to experiment with strategies outside of context, I am hoping that the experience will open their minds to the range of methods discussed in this chapter to solve a problem within context.

5.5.2 Missing value, sharing and comparison

Whilst the previous section have devoted much attention to the similarities between concepts and methods in solving different types of proportional reasoning questions, this section will focus on the different types of proportional questions and the importance of being able to distinguish between part-part and part-whole situation.

Learners should at this stage have some idea of the difference between part-part and part-whole situations and should also have decided on a method of solving proportional problems with which they feel comfortable.

A “missing value” question takes a single ratio and states its equivalence to another incomplete ratio. The following question would be an example of a missing value question. The ratio of the heights of the two people is given as well as the actual height of one person. The missing value to be calculated is the height of the shorter person:

Sally and David’s heights are in the ratio 5:7. If David is 189 cm, how tall is Sally?

Learners will need to recognize that 189 cm is 27 times more than 7 and therefore 5 must be multiplied by 27 as well to obtain 135 cm. The multiplicative reasoning in this question is the same reasoning as utilized for equivalent fractions:

$$\frac{5}{7} = \frac{135}{189}$$

Most proportional questions are missing value type of questions as I will show in the following sections of this sequential progression of activities. Below is an example of a slightly more complex or multi-step procedure and level three question according to the knowledge level taxonomy (Department of Education 2007: 14):

Lindi works as a hairdresser. She uses shampoo and conditioner in the ratio 5:2. At the start of the day she opens a 500 mℓ bottle of conditioner. During the day she uses 1 litre of shampoo.

- a) How much conditioner is left over at the end of the day?
- b) If she used the whole 500 mℓ of conditioner, how much shampoo would she have used?

It is important that learners know exactly what is being asked in this question. They would need to realise that they will have to find the amount of conditioner used for 1 litre of shampoo before they can calculate how much conditioner is left over. The 500

ml conditioner is thus not used in the initial proportion. It would also be advisable that learners convert the 1 litre to millilitres since the unit for the conditioner is given in millilitres. This conversion is in itself a ratio, but should be general knowledge to learners rather than a calculation that needs to be made.

$$5 : 2$$

$$1000 : ?$$

As mentioned before, learners can either solve the situation with the use of fractions, or they could see that 1000 is 200 times more than 5, which will mean that the amount of conditioner must be 200 times more than 2, equalling 400 millilitres. Only once learners have calculated this amount can they subtract it from 500 millilitres to obtain the 100 millilitres that is left over.

The second question asks of learners to calculate the amount of shampoo for 500 millilitres of conditioner which is thus asking to find “the other” missing value:

$$5 : 2$$

$$? : 500$$

After employing the structure for quite some time, a girl remarked that she has finally realized how straightforward ratio type questions are:

“You divide the two ones under each other and then times it to find the one you want. It’s like finding out what one is.”

This girl realized that she has to find the multiplicative relationship between ratios. In the question above, she would have made the following calculation:

$$500 \div 2 = 250$$

$$5 \times 250 = 1250$$

When I asked this girl how she made sense of the situation, she explained that she was first getting the value of one unit (as one would do in rate) and then used this value to find the missing value by multiplying.

Another typical proportion problem is that of *sharing* which is in fact a “missing values” question. Sharing implies that the whole is given and that it must be distributed in a specific ratio. The following question would be an example of a sharing question:

A grandfather leaves an inheritance to his three grandsons of R4500 which must be divided proportionally according to their age of 3 years, 5 years and 7 years. How much money must each child get?

Setting up the question and including a total would help to make sense of what is being asked.

3 : 5 : 7	totals 15
? : ? : ?	totals 4500

By using this set-up learners can either see that the amount of money is 300 times more than the total of their ages or that the sons will each receive a specific fraction of R4500. With the knowledge of part-part and part-whole representations, learners could also conclude that each child will get a fraction of the money equivalent to their age as part of the three boys' *total* age. This approach seems to be logical in terms of the given context. The youngest child will thus get $\frac{3}{15}$ of R4500, the middle child will get $\frac{5}{15}$ of R4500 and the eldest will receive $\frac{7}{15}$ of R4500. If learners are able to reason in such a way that they can verbalise these methods they would know that $\frac{3}{15}$ of R4500 is the calculation of $\frac{3}{15} \times 4500$ which leads to the answer of R900. The inheritance of R1500 and R2100 could then be calculated using the same strategy.

Since learners are constantly reminded of ratios as either part-part or part-whole situations throughout this sequential progression of activities, they become increasingly aware of the nature of problems. On asking learners about the number of pieces the inheritance needs to be shared in, most learners respond with that of three.

On making the calculation of $4500 \div 3 = R1500$ several learners remarked that the method was incorrect.

“It’s shared between the three, but not in three. The older ones get more.”

To help learners in obtaining the correct fractions, I changed the multi-step procedure broke it down into a few short questions as to guide learners to the answer. An example of such a pyramid question would be: What fraction of the inheritance would the youngest child receive? Although these “guiding” questions are suitable when introducing these types of questions to learners, we ultimately want learners to perform these guiding questions without being asked for it. By giving learners this question in pyramid format, we are asking them to operate on a lower thinking level. Although there is nothing wrong with asking level two or three questions, according to the knowledge level taxonomy (Department of Education 2007: 14), it is important to keep in mind that teaching is not merely assessment preparation, but for development of learners into individuals with an appreciation of mathematics and the ability to reason for themselves.

At this stage there is no need to ask learners to simplify the fractions. Saying that the middle child receives a third of the inheritance could lead learners to believe that the money is divided equally among the boys. Keeping the fractions in terms of the age as part of the sum of the boys’ ages makes much more sense than writing the fractions in their simplest form as $\frac{1}{3}$ and $\frac{1}{5}$. The one, three and five has no meaning in the context and could lead to misinterpretations. Simplifying ratios might have its place when working with scale, but if it does not impact on the calculation there is no need to force learners into simplifying the ratios.

Noelting (1980) focussed his attention on *comparison* questions in his quest to distinguish between different stages of development of proportional reasoning. Comparison questions are mainly used in ratios and rate and lesser so in financial mathematics, conversions, scales and geometry. The following question is an example of comparing ratios:

The colour lilac is mixed using red, blue and white paint.

- 1) Which of the following ratios will give the lightest shade of lilac (contains the most white paint). Red: Blue: White in a mix of 1:2:5 or in a mix of 2:3:5?
Explain your answer
- 2) If you mix 1 litre of lilac using red : blue : white in the ratio 2:3:5. What percentage of the mix would be red paint?
- 3) If you mix 1 litre of lilac using red : blue : white in the ratio 2:3:5, how many millilitres would be red paint?
- 4) Which of the following ratios would give the lightest shade of lilac (contains the most white paint) red : blue : white in a mix of 1:2:5 or in a mix of 2:3:6.
Explain your answer.

As with the mixing Energade® question in the baseline assessment, this question is difficult in the sense that a part-part ratio is given and the question requires learners to calculate a part of the whole. Learners will have to find which fraction or percentage of each mix contains the most white or red paint. There is build-up in terms of difficulty which is meant to guide learners in using the correct strategy. In question 1 it is easy to compare which of the mixtures contain the most white since the part of the ratios that represent the white paint are in both cases five. Learners merely have to realise that there are more of the other colours in the second ratio since two parts red and three parts blue are mixed with the white, whereas one part red and two parts blue are mixed with the white in the first ratio. This question is an extension of Noelting's subset IB which keeps the first terms of the two ratios the same ($a = c$ where $b > d$). Although this question works with a ratio containing three terms where we kept the last terms the same, learners can still use the same strategy to solve this question. Learners are asked to explain their answer which will hopefully lead to a class discussion on different methods used. Not all learners recognized that the numbers representing the white paint were the same in both ratios and some learners used calculations to solve their problem. Explaining the calculations behind this notion and how it might be linked to calculations done by other learners, can enhance the learning experience.

Most learners in class were able to compare the ratio of 1:2:5 and 2:3:5 without any calculations. These learners recognised that the amounts of white paint in both ratios were the same but that there were “*more other colours*” in the 2:3:5 ratio than in the 1:2:5 ratio. This realisation does not conclude that learners have a sound proportional understanding of the question. Comparison without calculation is in fact on Thompson and Thompson’s first level of proportional reasoning. Yet, the method employed by learners in solving this question involved both between and within strategies.

I structured questions 2 and 3 in such a way that it will force learners to look at the fraction (or percentage) of red paint before comparing more difficult ratios in question 4. This was to lead learners’ thinking to a part-whole approach. They must calculate what percentage (or part) the red paint is of the whole mix. Numbers are kept simple as to encourage this way of thinking. The idea is that learners who are able to calculate question 2 and 3 correctly, would ultimately also calculate question 4 correctly using the same part-whole strategies in their calculation.

A number of learners still employed visual representations to analyse the question before calculating the percentage. A few learners were not able to see the connection between the percentage of red paint and the amount of red paint needed for a litre of paint.

In question 4 the numbers are more difficult and it is assumed that learners will follow similar strategies of part-whole representations to calculate their answer. This question is set at Noelting’s stage IIIB where there is no obvious with-in or between ratio between the terms in the ratios. Learners are encouraged to find the part-whole relationships from these part-part ratios: out of the 8 parts, 5 parts are white and out of the 11 parts, 6 parts are white. Learners can either calculate the percentage in each case (62,5% and 54,4%) or work with equivalent fractions:

$$\frac{5}{8} = \frac{55}{88} \quad \text{vs} \quad \frac{6}{11} = \frac{48}{88}$$

From either of these calculations it should be clear that the first ratio contains more white paint than the second ratio.

5.5.3 Percentages as fractions and ratios

The concept of percentages was touched upon during the previous sections and conversions between fractions, ratios and percentages were already discussed. The nature of percentages that I will be dealing with in the section, is mainly that of percentage increases and decrease in the form of appreciation, inflation, VAT, depreciation and Consumer Price Index.

I used the context of the upcoming April 2009 elections as simple percentage activity for learners to start this section:

In a recent survey researchers found that approximately 59% of South Africans will be voting for the ANC. If South Africa has approximately 35 million registers voters, how many of them would be voting for the ANC?

Many learners have been taught in primary school already that the word “of” means times or multiply. With this in mind they should know that 59% *of* 35 000 000 means 59% *times* 35 000 000. As was evident in the first question in the baseline assessment regarding the amount of kilograms of potatoes and the number of packets of chips that can be made, there is the possibility that learners will confuse multiply and divide operations. Learners should however have some intuition of how many people of the 35 million will be voting for the specific party. Learners also tend to be more familiar with percentages since they relate it to (their) test scores. On asking learners how they could check their answer, several learners remarked that the answers should be a “*bit more than*” half of 35 million. Although a “*bit more*” in this case amounts to 3 150 000, learners’ reasoning implied that they were making sense of the situation and not merely manipulating a set of numbers.

I tried to give my learners a range of different types of percentage question so that the importance of proportional reasoning can be more and more visible to them. The next question is an example of a “percentage *more*” instead of asking for the “percentage *of*” as in the previous question.

Wholesale quotations are often given excluding VAT. If a product is quoted as R900 excluding VAT, what would the VAT inclusive price be if VAT is calculated at 14% (of the wholesale price).

This can be solved by using percentage *of* and then adding it on to the original amount. From classroom experience I have seen that learners are relatively comfortable with this calculation. Although I would like learners to understand the concept of the VAT inclusive amount being representative of 114%, it is not an incorrect method followed by learners. Working backwards to find the original amount after a percentage has been added, however requires this concept of “more than a 100%”. Learners using a two-step method to add a percentage onto the original amount find working backwards problematic as in the next question:

A promotional bottle of cooldrink contains 10% more than a normal bottle of cooldrink. If the promotional bottle contains 660 mℓ of cooldrink, how much did the original bottle contain?

Learners have to comprehend that the promotional bottle is representative of 110%. I normally write the following layout on the board and ask learners if the question looks familiar to them:

100 : 110

? : 660

Several learners recognised that the answer to this question was 600 mℓ but struggled to give a reason to their answer. The set-up used above helped learners to see that a multiplicative ratio of 6 existed between 110% and 660 mℓ.

For learners that struggle to understand the concept of 100% and 110%, the question could be asked: If the promotional bottle is 110 ~~fr~~ what was the original bottle? It should be explained that the 100 and 110 are merely ways of representing the relationship between the original and the promotional bottle, similar to saying that the length of two people are in the ratio 5:7. The amounts in this specific question where

however kept simple so that learners will most likely “see” that the promotional bottle containing 660 ml after a 10% increase had originally only 600 ml content. Learners that are able to “see” the answer do not necessarily know how to solve it when numbers in the question are more complex, which is an indication that they might operate on a lower level of proportional reasoning skill. It is a way of initiating didactical discourse where learners have to find and discuss methods that could be used to find the answer they can “see” so clearly.

Another example of this type of question is when learners are asked to calculate the VAT *exclusive* price from a given VAT *inclusive* price when analysing a till slip.

Learners are thus asked to calculate the VAT exclusive amount from the VAT inclusive amount. Learners often tend to try and solve this question by also trying a percentage *of* method where they calculate 14% of the VAT inclusive amount and then subtract it from the (VAT inclusive) amount. In proportional terms they are reasoning that the exclusive and inclusive amounts are in the ratio 86 : 100 instead of 100 : 114. It is at this stage where one can distinguish between learners who think proportionally and those who try and find methods that have been taught to them in seemingly similar scenarios. Learners who think proportionally will recognise the VAT inclusive amount as representative of 114% and the exclusive amount of 100% and will use appropriate *multiplicative* strategies to solve the problem. They would most likely also use a similar strategy to find the VAT inclusive amount from the exclusive amount by multiplying the exclusive amount by 1,14 instead of finding the value of VAT and then adding it onto the exclusive amount. Learners not thinking proportionally will try to use *additive* (or subtractive) strategies to solve the problem which would prove somewhat problematic in this case. As with the “promotional bottle” question, easy numbers such as 100 and 114 or 200 and 228 can be used to introduce questions involving VAT. Also ask learners to produce their own set of questions where they know the VAT exclusive amount, have calculated the VAT inclusive amount from that, and have to work backwards to get their original answer.

An activity that worked really well in the class was to use till slips to analyse the VAT that has been charged on items. Since the amount of VAT is given on the till slip,

learners were asked to show the *method* of calculation to obtain the correct answer. On doing this activity one learner remarked:

“But what about just dividing by $\frac{114}{100}$? If you have to times to get the price with the VAT then you must do the opposite to get the price without VAT?”

This led to an interesting discussion on multiplying with the inverse fraction when dividing. Most learners had no idea why they were multiplying when dividing by fractions. This indicates to Skemp’s (1972) instrumental understanding as discussed in 4.2.2 on page 73.

Although it is not expected from learners at grade 10 level, the concepts of appreciation and depreciation are linked to the concept of percentage increases and decreases. Many schools do not include the calculation of VAT exclusive amounts from VAT inclusive amounts. These concepts are however touched in the grade 9 Mathematical National Curriculum Statement and not including it in grade 10 would be to underestimating the ability of learners taking Mathematical Literacy. Appreciation type questions could be kept simple as in the following examples:

1. A small apartment of R500 000 increased in value to R700 000. With what percentage did it increase?
2. A house of R1 200 000 increased in value with 80%. What is the house worth after the increase?
3. A house has increased in price by 50% to a value of R900 000. What was the price of the house before the increase?

Solving the first question with a proportional approach would look as follow:

500 000 : 700 000 (or simplified to 5:7)

100 : 140

or $\frac{7}{5} = \frac{?}{100}$

Both methods would lead to an answer of 140, from which learners could then see that the house has increase with 40%. Learners can also compare the increase, of R200 000 to the original amount:

$$500\ 000 : 200\ 000 \text{ (or simplified to 5:7)}$$

$$100 : \boxed{40}$$

To calculate the increase most learners use a two-step method to find the answer o

$$1200000 \times \frac{80}{100} = 960000$$

$$1200000 + 960000 = 2160000$$

As explained earlier, learners have trouble realising that the new price of the house would be 180%. Since VAT (and the idea of 114%) was discussed before appreciation, learners were more comfortable with the concept of “more than 100%”, but still preferred to use the two-step method.

Even though learners might use their own set of successful methods to calculate the answer, the proportional nature of the question should be highlighted through the following method:

$$1\ 200\ 000 : ?$$

$$100 : 180$$

$$\text{or} \quad \frac{100}{180} = \frac{1200000}{?}$$

Once again, both methods will lead to an answer of R2 160 000.

Question 3 employed the same concept of a percentage more than 100% when asking learners to calculate the original price of the house before the increase.

$$100 : 150 \quad \text{(or simplified to 2:3)}$$

$$? : 900\ 000$$

$$\text{or} \quad \frac{100}{150} = \frac{2}{3} = \frac{?}{900000}$$

If teachers can get learners used to these types of questions in grade 10 already, it will make it much easier for students to deal with concepts such as Consumer Price Index and Inflation when they get to grade 11 and 12.

To link with the concept of financial mathematics, I also include the following type of questions:

Tabi saves 15% of her monthly salary. If she saves the amount of R975, what is her monthly salary?

It is a seemingly easy question but also contains a strong proportional quality. Learners need to realise that R975 is representative of 15% and that they are asked for 100%. We are thus working with the ratio 15 : 100. They would either have to recognise that 100 is $6\frac{2}{3}$ times more than 15, as a within strategy, or they would have to calculate the value of 1%, by dividing 975 by 15, as a between strategy. When using the latter method, learners are in fact calculating the rate by finding the value of 1% and will then have to calculate 100% by multiplying by 100. However learners choose to approach this question, they will work with either of the following representations.

15 : 100

975 : ?

or

$$\frac{15}{100} = \frac{975}{?}$$

5.5.4 Ratio, scale and enlargements and reductions

Scale, enlargements and reduction offer a range of didactical possibilities that include reading maps, building models and drawing house plans. Firstly learners need to realise that the scale is a mere ratio indicating how many times bigger the real world is to that of the representation, being it on a map or a scale model. 1 : 50000 can mean

that 1 cm on the map is 50000 cm in real life, but it can also mean that 1 mm on the map is representative of 50000 mm in real life. The scale is not unit-bound. It is this concept of scale that is the most difficult for learners to comprehend. They often ask: *"But what's the unit?"*

Learners also need to understand that scale is customarily written as *one* to a specific number. If 2 cm represent 5 m, not only do the units have to be made the same, but it also needs to be simplified to 1, thus:

2 cm : 5 m

2 : 500 units are made the same

1 : 250 simplified to one

When working with scale the conversion of units is often involved which makes scale a multi-step procedure, and thus a taxonomy level three question.

A project often given to grade 10 learners is the lay-out of a room. They have to find an appropriate scale for their room which will also apply to the furniture. They can for example make a scale drawing of the classroom and make desks to the same scale that they can arrange with-in the given space. From here they can make a range of calculations to see if the floor area of the drawing and the real life floor area would be in the same ratio as the lengths of the walls on the drawings to the lengths of the walls in real life. It is also an opportunity to discuss the use of scale models and scale drawings in real life.

Enlargements and reductions are natural extension to scale. When enlarging an A4 page to A3, there is a specific ratio involved just as reducing an A4 page to A5 size. Many school copiers work with enlargement and reduction percentages which is a starting point for discussions around the concept.

When a teacher reduces a double page out of a textbook to A4 size, she sets the copier on 70%.

1) What does the 70% mean?

- 2) Use the size of an A4 page to calculate the size of the original double spread in the textbook?
- 3) What are the dimensions of the book?

It is fairly easy to understand that the 70% means 70% of the original size, but having the measurements of the A4 page and working backwards to find the original size is slightly more complicated. If we are looking for the measurements of the original 100%, and we have 70% and its measurements, the set-up would look as follow:

70% : 100%

30 cm : ? length

20 cm : ? width

Learners would first have to find the value of 1%, which could be rounded to 0,43 cm, from where they can calculate the full 100% of 43 cm. It is the same strategy used when calculating how many times 30 is bigger than 70 and then multiplying 100 by the same amount.

5.5.5 Looking at rate and the role of ratio

The pizza question mentioned earlier is a powerful comparison problem. It involved fractions, division and also rate (as will be discussed in this section.) When comparing the amount of pizza *each* boy will get, when 3 boys share 2 pizzas, and the amount of pizza *each* girls will get, when 9 girls share 7 pizzas we are referring to rate: “How much *per* boy” and “how much *per* girl?” Yet, it is also a division problem and involves fractions.

In this specific problem, the question was phrased as a rate question referring to pizza *per* child. It is also possible to look at this question in terms of equivalence (as learners use it within fraction calculations).

If two more groups of 3 boys joined the group of three boys and each group bought two more pizzas, the 9 boys will have 6 pizzas. Since the 9 girls have 7 pizzas, it is clear that the girls have more pizzas than they boys.

Although this method can be labelled as equivalence, it is also a form of rate. The rate *per* nine boys is 6 pizzas, whilst the rate *per* nine girls is seven pizzas. The big advantage to this method is that we are working with whole pizzas instead of $\frac{2}{3}$ and $\frac{7}{9}$ of a pizza which also first needs to be made into equivalent fractions before being compared.

One of the types of questions which are commonly used in Mathematical Literacy to explain rate is that of comparing prices.

Which is the better buy: 2 ℓ of milk for R15,95 or 1 ℓ of milk R7,85.

One must of course be careful when wording these type of questions not to ask “which is cheaper” since the answer to this question would be that R7,45 since it is less than R15,95. Teachers can either phrase the question as “which one is cheaper per litre” or “is it better to buy two one litre bottles or one two litre bottle”. Depending on the question, learners would either have to calculate the price per litre or per two litre:

1 : 7,45 thus

2 : (7,45 x 2 = 14,90)

2 : 15,95

or

2 : 15,95 thus

1 : (15,95 ÷ 2 = 7,975 ≈ 7,98)

1 : 7,45

In both cases it is clear that the one litre milk container works out to be cheaper. Teachers need to vary the way in which they ask these types of comparison rate problems so that learners may become sensitive to the methods being asked.

Answering the question of which container of milk is cheaper per litre by doing the first set of calculations would be wrong although the correct answer might have been given. This comes back to classroom norms of calculations being more important than answers. Learners using this incorrect method in a test or examination should not receive the marks since they did not answer the question.

Exchange rate is another form of rate. If it is R7,89 to a Dollar, then the multiplicative relationship between Rand and Dollar would be 7,89. The concept of exchange rate is also associated with conversions discussed in the next section.

Another concept in Mathematical Literacy is that of function mathematics. When introducing functions one normally thinks of speed questions: kilometre *per* hour, metres *per* second. The constant rate (ratio or gradient) is of course multiplicative in nature. Few learners will make the mistake of saying that a person driving at 100 km/h will drive 101 km in 2 hours. Yet, additive strategies are prevalent when working with ratios and rate can help to prove those strategies wrong. Unlike the comparison of prices, function questions are normally missing value type questions.

I am sure that most learners have asked the following question whilst travelling in car on holiday: Are we there yet? A range of powerful proportional reasoning problems could be structured around this tongue in cheek question, for example:

If the family is travelling at 120 km/h and they are 30 km away from their destination, how long will it take them to reach their destination?

This specific example can explore both with-in and between strategies:

120 : 60

30 : ?

It seems that in most cases it is easier to use between strategies to solve a missing value question since numbers in a specific ratio are not necessarily multiplicatively linked by a whole number. In this specific case the with-in relationship is almost calling out for use since it could easily be simplified to 2:1. Realizing that the distance

is twice the time in minutes would lead to the answer of 15 minutes without almost any calculations. The problem of course in this regard would be to realise that 120 km/h means that you cover 120 km in 1 hour and one would also have to know that 1 hour equals 60 minutes. Once again I would like to stress the fact that learners need to be taught how to set up a question since this is a way of looking at what is given and what needs to be calculated.

It is not essential that learners convert hours to minutes. They could also reason that 30 is a $\frac{1}{4}$ of 120 and therefore the time must be $\frac{1}{4}$ of an hour.

As mentioned in the beginning of this chapter, a specific learner in my class could not make sense of kilometres per hour as a method of calculation. He saw km/h as a unit and nothing more. On asking him how far his dad will drive in 1 hour if he drives at 100 km/h, he replied with a confused expression:

“I don’t know... Sixty?”

Except for the fact that his peers found it a rather amusing answer, it was saddening to see that this learner did not make sense of speed at all. He wanted to know if he can have the formula to work it out:

“Isn’t it something like distance over time? Then I just work backwards!”

He seemed oblivious the explanations of both myself and his peers in trying to convince him of the logic of the situation.

The importance of rate questions is that they are introductory to functional mathematics. It is normally the start of linear graphs and the concept of constant growth and gradient. As mentioned earlier, Riedesel (1969) and later Olivier (1992) suggested that rate is the key factor in proportional reasoning. Although rate and functional mathematics play an important role in developing proportional reasoning, learners still have to understand basic concepts such as equivalence, and the

multiplicative reasoning involved in equivalence, before they can understand the intricacies surrounding functions. To be able to plot a speed graph, learners will most likely use a table such as the one below:

Distance	0	100	200	300	600	900
Hours	0	1				

If a distance of 100 km was covered in 1 hour, then learners should see that:

1 : 100 equals 2 : 200 equals 9 : 900

or

$$\frac{1}{100} = \frac{2}{200} = \frac{9}{900}$$

Both researchers valued the concept of equivalence when focusing on the use of rate in proportion and used the table format extensively so that learners were able to see the ratio (or constant gradient) between the two variables. Although they worked with a structure of $y = mx$, and not with the equivalent fraction notation shown above, their contribution to the importance of equivalence in proportional situation supports the view of the importance of learners having a sound foundation in equivalent fractions in order to understand related proportional concepts.

The following question on ratio, scale and rate is given to adults in the Map Work section of the Trial Walking Guide Qualification (Tourism Guide Qualification 19, Unit standard 9284.) It embodies the interconnectivity of proportional reasoning within a real life situation. Due to the authentic nature of the questions in terms of Realistic Mathematics Education, this question was included in this sequential progression of activities:

If a person hiking with a heavy backpack walk at 3 km/h on a section of a mountain trail represented by 8 cm on a 1:50 000 map and gains 200 m in height, how long did he walk for? (10 minutes is added to the time for every 100 m gained in height.)

This question employs several proportional reasoning concepts and it is not surprising that many prospective mountain walking guides struggle with this question. First they would need to calculate the distance covered by using the *scale* and *conversions* (from cm to km).

$$1 : 50\,000$$

$$8 : 50\,000 \times 8$$

$$8 : 400\,000$$

$$400\,000 \text{ cm} = 4 \text{ km}$$

Secondly, they would need to work with the walking *rate* (or speed) of the hiker:

$$3 \text{ km} : 1 \text{ h}$$

or

$$3 \text{ km} : 60 \text{ min}$$

$$4 \text{ km} : \mathbf{80 \text{ min}}$$

Thirdly, they would need to find the extra time needed when gaining 200 m.

$$100 \text{ m} : 10 \text{ min}$$

$$200 \text{ m} : \mathbf{20 \text{ min}}$$

Only after three proportional reasoning calculations could they determine the total amount of time needed for the 4 km hike, amounting to 100 minutes or 1 h 40 minutes.

Not only did this question cover ratio, scale, conversions and rate, it also suggested both with-in and between strategies. Scale suggests a between strategies, since 8 centimetres is 8 times bigger than 1, then 50 000 should also be 8 times bigger. Yet, the missing value in the speed ratio is easily calculated by using with-in strategies since the numbers used, especially when converted to minutes, are multiples of each other. If $3 \times 20 = 60$, then $4 \times 20 = 80$. This method would be much easier than calculating that 4 is $\frac{4}{3}$ times more than 3 and therefore the missing value will be $\frac{4}{3}$ times more than 60.

5.5.6 Conversions

For some reason conversions are not as easy for learners as one would expect. There are a range of conversions that could be looked at: converting between units within the metric system, converting between the metric and imperial system, conversions between currencies (although this could also be seen as rate), time and time zone conversions and even British shoe sizes to American or European shoe sizes.

It was interesting to see that learners seem to be comfortable with the first conversion, but not with the second:

- 1) If 10 litres equals 2,2 gallons, how many gallons will 15 litres be?
- 2) How many litres will 15 gallons be?

Learners tend to use a build-up strategy to solve the first question, by saying that since 5 litres will be 1,1 gallons, that will mean that 15 litres will equal 3,3 gallons. It is not as simple to use this strategy to solve the second question.

$$10 : 2,2$$

$$? : 15$$

Learners that have used this set-up method with continuous success, are learners that realise that they have to divide by 2,2 and multiply by 10. When dividing by 2,2 they are calculating how many times bigger 15 is than 2,2 and if they have this value, they can also multiply the 10 by this figure, as shown below:

$$\begin{array}{l} 10 : 2,2 \\ ? : 15 \end{array} \quad \begin{array}{c} \curvearrowright \\ \times 6,82 \end{array}$$

$$\text{Thus } 10 \times 6,82 = 68,2$$

Learners that are not able to calculate the second question are thus not yet fully developed proportional reasoners and are functioning on Thompson and Thompson's level three since they are employing build-up strategies.

Employing *rate* strategies by first finding the value of *one* gallon or *one* litre will aid in making sense of conversions. Learners that simplified the ratio of litres to gallons to 1 : 0,22, were able to either multiply the amount of litres by 0,22 to calculate the gallons or divide the gallons by 0,22 to calculate the litres.

Not all conversions are directly proportional in nature. When converting degrees Celsius to degrees Fahrenheit, the formula suggest a proportional scenario rather than a directly proportional scenario. The formula below suggests a proportional $y = mx + c$ structure rather than a directly proportional $y = mx$ structure.

$$\text{Fahrenheit} = \frac{9}{5}C + 32$$

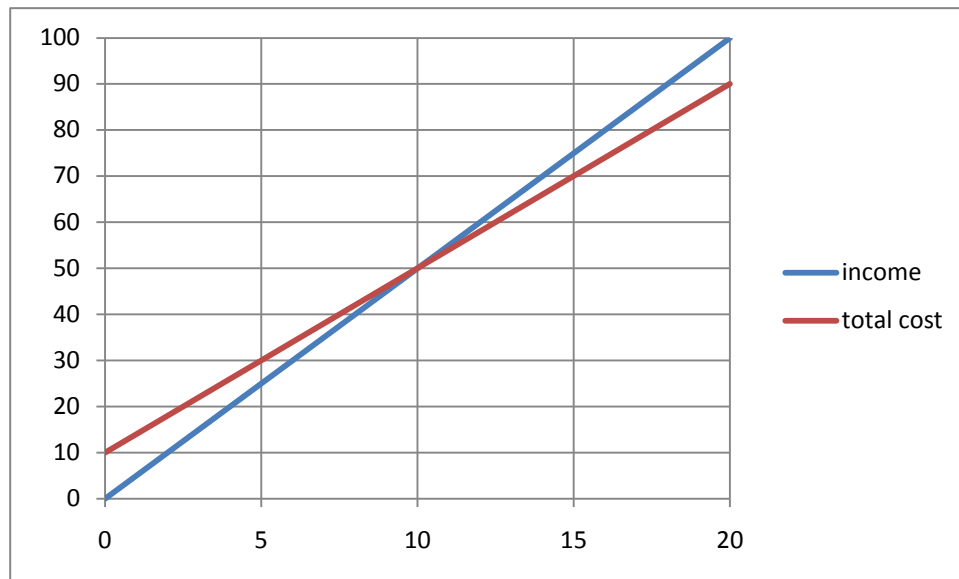
On investigating different types of conversions and their proportional or directly proportional structure, it is advisable to fill in the tables and draw linear representations. This helped learners to see that proportional situations could have value other than zero when one of the variables were zero. Although the 0 centimetre would be equal to 0 metre, 0 degrees Celsius would be equal to 32 degrees Fahrenheit. Unlike directly proportional conversions, proportional conversions will also not have a constant ratio between variables due this constant value when one of the variable equal zero.

5.5.7 Distinguishing between proportion and direct and indirect proportion

Distinguishing between different types of proportional situations is a natural extension to conversions since it also touched on the differences between directly proportional and proportional situations. Cramer et al. (1993) and Lamon (1995) describe the ability of a proportional reasoner, to, except solving a variety of problems and understanding mathematical relationships involved in proportion, also be able to *discriminate proportional from nonproportional situations*. Comparing directly proportional problems of $y = mx$ to proportional problems of $y = mx + c$, seems better distinguished when presented visually by means of linear graphs.

According to the Mathematical Literacy assessment standards, learners in grade 10 merely have to plot values on linear graphs, but they must be able to calculate input

value from output values and output values from input values. These assessment standards should have been covered in grade 9 already and learners should have some idea of functions when entering Mathematical Literacy in grade 10. The idea of “a fixed cost”, is visually well explained when learners investigate graphs of total income and total cost. When discussing the following graph, learners can “see” that the total cost is a constant that remains unchanged and influences the relationship between the variables. This links to the difference of proportional conversions (such as temperature conversions) and directly proportional conversions (such as between metric units) discussed in the previous sections



A didactic that almost always seems to work in helping learners to distinguish between directly proportional situations and proportional situations, is asking learners to create their own context for each type of situation. Learners are encouraged to think of situations that have not been discussed in class and to be as creative as possible. Discussing these situations in class exposes learners to a variety of context that can be used to distinguish between directly proportional situations and proportional situations.

Asking learners to create posters of the situation and giving them opportunities to draw their graphs on Microsoft Excel®, helps them to buy into the activity. Teachers must not think that putting learner work up in class or giving rewards for the most creative situation is reserved for use by primary school teachers only. High school learners seem to appreciate these gestures and even plead for stickers in their books.

The power of motivation cannot be misjudged and it is often these silly extrinsic motivational tricks that will aid learners in creating intrinsic motivation. These motivational actions help to create a classroom atmosphere of trust and appreciation especially in the light of overwhelming negative attitudes of learners towards Mathematics (and Mathematical Literacy) as shown in the baseline assessment questionnaire.

To be able to distinguish between direct proportions and other types of proportionality, situations must be given that are also indirectly proportional. The following questions are examples of such indirect proportions.

- 1) It costs R156 each for 7 workers to hire a minibus for a day. How much will each person have to pay if 12 workers hire a minibus?
- 2) A farmer estimates that it takes 12 workers 10 days to harvest his crop. How many workers would it take to harvest his crop in 8 days?
- 3) Three people can deliver 1000 brochures in 30 minutes. How long will it take five people to deliver 1000 brochures?

Before learners can begin to make calculations, they need to realise that unlike with direct proportions, the two variables do not grow with the same factor. With indirect proportions the one unknown decreases as the other increases: the more workers, the less days; the more people, the less each one has to pay; the more people, the less the amount of time to deliver the brochures.

The first question is relatively easy since the context makes it easy to find the product of the two variables. If 7 workers each pay R156, then they pay R1092 in total. If 12 workers have to share this cost, the cost per person would be $R1092 \div 12 = R91$.

The second question is not as simple as the first since calculating the product of the variables does not fit into the context. To help learners in answering the question of how many workers are needed to harvest the crop in 8 days, the following questions could be asked: If 12 workers take 10 days, then how long will 24 workers take?

Learners should be able to interpret from the context that *double* the number of workers, would take *half* the time. Also ask learners how many workers would be needed to finish the project in 20 days. Since it is *double* the number of days, the work must have been done by *half* the amount of workers. Learners can write a range of ratios to see if they can find a pattern. It can be represented in a table or as a set of ratios:

Workers	3	6	12	24	48
Days	40	20	10	5	2.5

Learners can reason in one of two ways. Either they can realise that every time we multiply the one variable by two, we divide the other variable by two. Thus, if we want to know how many workers it will take to finish the project in 8 days, we must see what we have to multiply or divide 10 by to get to 8 and do the exact opposite to 12.

Working with whole numbers: if $10 \times 8 \div 10 = 8$ then $12 \div 8 \times 10 = 15$

Or as fractions: if $10 \times \frac{8}{10} = 8$ then $12 \div \frac{8}{10} = 15$

Learners can also realise that the multiplicative relationship between the variables is always equal to 120 ($24 \times 5 = 12 \times 10 = 6 \times 20 = 3 \times 40$). With this knowledge, they can calculate:

$120 \div 8 = 15$ because the product of 8 and 15 also equals 120.

What makes the third question more difficult than the previous two is the fact that like question two, the product of the variable are not a natural extension to the context. There is also redundant information, of 1000 brochures, included in this problem which learners need to discard in order to make sense of what is being asked:

3 people take 30 minutes, that will mean that

6 people take 15 minutes (twice the amount of people, half the time)

9 people take 10 minutes (triple the amount of people, a third of the time)

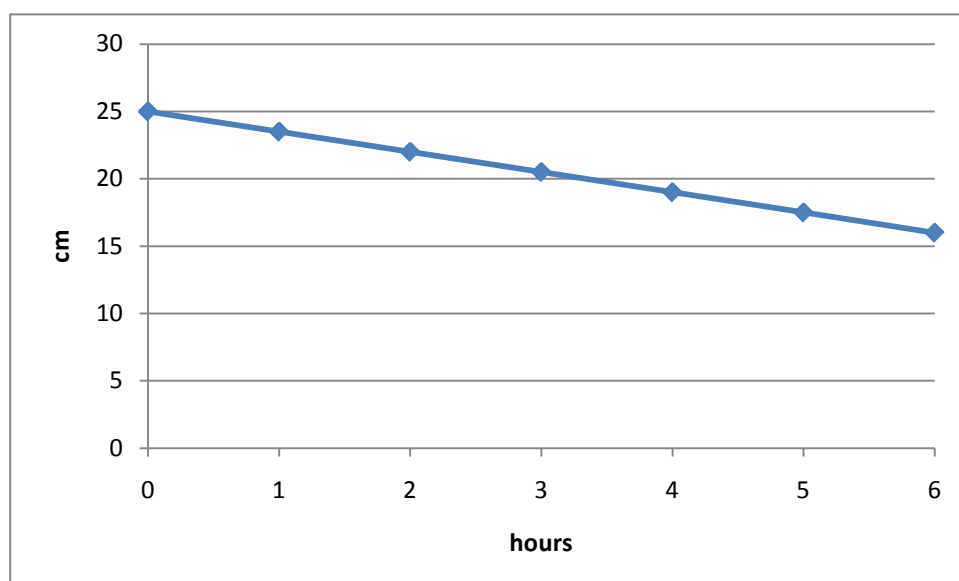
Learners must be encouraged to find their own formula to calculate inverse proportions. To supply learners with the formula is to deny them the opportunity to engage with the situation. Understanding that 5 is $\frac{5}{3}$ time more than 3 and that the amount of minutes must thus be $\frac{5}{3}$ times less than 30, is a powerful concept.

Substituting the values into the formula of $y = \frac{c}{x}$ requires little proportional reasoning and learners using this method of substitution does not necessarily understand the reasoning behind it.

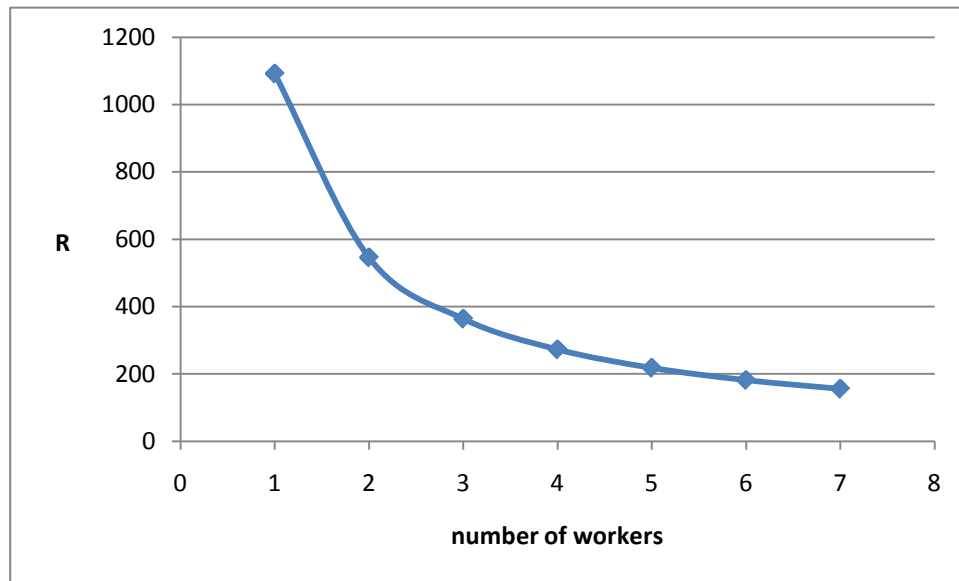
Learners need to be made aware of the difference between inversely proportional problems and that of proportions with a negative growth or gradient. Take the example of a candle burning:

If a candle with a length of 25 cm burns at a rate of 1,5 cm per hour, what will its length be after 4 hours?

Learners will need to realise that the candle gets shorter as time increases, but that it is not an inverse proportion. The reason for this being that it is a *constant* negative growth. Learners should be asked to plot graphs of the different situations to see the difference in growth patterns. The candles graph would look as follow:



Whereas the graph of the workers renting a taxi would look as follows:



These representations should help learners in distinguishing between negative growth and inverse proportions.

5.6 Conclusion

Throughout this sequential progression of activities the interconnectivity between concepts should form clear links between concepts. These links are shortly summarised below.

The activities start off with building on learners' knowledge of equivalent fractions and distinguishing between part-part and part-whole ratio situations. Throughout this progression of activities learners are encouraged to set-up questions in a ratio format in order to analyse the structure of what is being asked. Since there can be distinguished between missing value, sharing, comparison and later rate, learners can use the set-up structure to help them to identify the nature of each problem.

Once learners are able to distinguish between different types of proportional problems and employ multiplicative strategies similar to that of equivalent fraction, they can use these strategies in a variety of context. Since percentages are merely another form of fractions, this section is included first. Percentages include the notion of percentages more, as in inflation and appreciation, and percentages less, as in depreciation. These

missing value type of questions can be solved by using similar strategies to that of ratio questions discussed in the previous sections.

Scale and enlargements and reductions are the start of the idea of simplifying to the simplest form to calculate the missing value. Learners have trouble understanding that a scale is not unit specific and that it is merely a representative ratio of “how many times bigger or smaller” the representation is. This is the same strategy used when working with rate when the amount per one unit is worked with. It is for this reason that the concept of rate follows that of scale. If the rate of a proportional problem is calculated it is simple to calculate the missing values by merely multiplying or dividing by the rate. Rate can thus be used as a strategy in solving proportional problems.

Exchange rate is a special form of rate that works with the conversions between currencies. It can however also be linked to the concept of conversions and therefore conversions were used as an extension to rate. Not all conversion are directly proportional in nature and some conversions, such as temperature conversions, have a proportional structure of $y = mx + c$.

To be able to reason proportionally one must be able to “discriminate proportional from nonproportional situations” (Cramer et al. 1993; and Lamon 1995). Distinguishing between proportion and direct and indirect proportion links to different structures of conversions covered in the previous section. At this final stage of the sequence of activities, learners should have a better understanding of the different types of directly proportional situations as well as strategies to solve them. Discriminating direct proportional situations from non-direct proportional situation is thus the final stage in becoming a proportional reasoner.

Although this sequential progression of activities will not guarantee that learners will become proportional reasoners that operate on Piaget’s highest level of cognitive development, it should help learners along the process of becoming proportional reasoners.

Conclusion

In this research I have focused on the great extent of research that has been done on the different stages of acquiring proportional reasoning. Yet few researchers have offered solutions to the problem of acquiring proportional reasoning skill. Researchers also limited the concept of proportionality to ratios, rate and indirect proportions. As I have set out in Chapter four, the concept of proportional reasoning extends far beyond these concepts and it is evident in all the Mathematical Literacy learning outcomes within the National Curriculum Statement. The concept of proportional reasoning cannot merely be brushed over in Mathematical Literacy textbooks. It is a powerful set of activities that runs through the subject. This supports the claim of the importance of research into proportional reasoning in Mathematical Literacy since it is a concept that is needed within all learning outcomes in the subject.

Proportional reasoning is not a cognitive skill that learners will acquire with age. It is a skill that needs to be developed. With the use of baseline assessment I was able to show the poor proportional reasoning skills of learners in Grade 10 Mathematical Literacy classes as well as their mostly negative attitudes towards Mathematics and Mathematical Literacy. Learners attending government schools would most likely have even weaker proportional reasoning skills. What is also noteworthy is the fact that most government schools have overcrowded classrooms where individual attention and support is hardly possible. This supports the great need for activities that will help learners to develop proportional reasoning skills in Mathematical Literacy classrooms.

The baseline assessment also points out the great anxiety and emotional barriers that learners bring to the classroom. This places a great responsibility on the Mathematical Literacy teacher to create an environment which will be uplifting and motivating to learners. If learners do not feel comfortable in the Mathematical Literacy classroom, they will not engage in learning activities and educational discourse.

Since there is such overwhelming evidence of the difficulty of proportional reasoning by several researchers since the beginning of the previous century and further supported in this research, and since I have shown the strong thread of proportional reasoning throughout the Mathematical Literacy learning outcomes of the National Curriculum Statement, the views of Mathematical Literacy as lower grade mathematics need to be revised. This subject lends itself towards the development of a range of high order thinking skills that is not only restricted to the Mathematical Literacy classroom. Mathematical Literacy teachers have the opportunity to help develop learners' critical and analytical thinking skills, transfer of knowledge skills and reflection on thinking methods skills, whilst developing proportional reasoning skills. It is a task that should not be taken lightly.

The activities suggested in this research to develop proportional reasoning focused on the development of these skills high order thinking skills. For learners to develop sound proportional reasoning skills, they will need to make clear links between interconnective concepts. The activities take the concept of equivalence in the concept of fractions, and the methods involved in solving it, as the basis from which to reason proportionally. It aimed to make sequential links between concepts that all require proportional reasoning skills. I have argued that the methods learners employ to solve missing value and comparison questions involving fractions are the same methods learners must tap into to solve ratios, percentages, conversions and other proportional situations.

Implementing the aspects of the proposed activities will help to deliver the intended Learning Outcomes of the National Curriculum Statement within the subject of Mathematical Literacy. It is a field of study that lends itself to further research and development so that teachers may be empowered to teach Mathematical Literacy in such a way that it will also empower their learners.

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Addendum A

Mathematical Literacy Grade 10

Proportional Reasoning Worksheet

Please show all your calculations and reasoning. Don't hesitate to explain answers in words rather than in numbers. Your methods for doing these activities are far more important than your answers.

Question 1

At the athletics days the tuck-shop sells hot potato chips. To make one small bag of chips they use $\frac{3}{4}$ of a kilogram of potatoes.

1. If the school has 120 kg of potatoes, how many small packets of chips can they make?
2. The tuck-shop ladies estimate that they will need to make 200 small packets of chips. How many kilograms of potatoes will they need for 200 packets of chips?
3. If the price of potatoes is R6,75 per kilogram, what is the cost of making a small bag of potatoes?

Question 2

Look at the following tables in which the times for the 100 m and the 200 m races have been summarised. All four of these learners ran in both races.

100 m	
Sam	0:15
Alex	0:23
Charlie	0:18
Jordan	0:21

200 m	
Sam	0:40
Alex	0:45
Charlie	0:36
Jordan	0:45

1. Did Sam run at the same average speed for both races? Explain your answer.
2. Which of the runners kept the same average speed for both races?
3. Which of the runners managed to run at a faster average speed for the 200 m race?
4. If Charlie can manage to run at the same average speed for 400 m, what will his time be for completing the 400 m race?

Question 3

The school has organised soft drinks for all the athletes that finish an event. They ordered 500 soft drinks and received 240 tins of Coke-Cola®, 120 tins of Fanta® Orange, 84 tins of Fanta® Grape and 36 tins of Sprite®. The school organises that a couple of parents will hand out the soft drinks as the athletes finish.

1. What is the chance that an athlete will receive a Coke-Cola®?
2. What is the chance that an athlete will *not* receive a Coke-Cola®?

Question 4

Sam and Alex both brought post race energy drinks. Sam has Energade® concentrate and Alex has a Game® sachet. On Alex's Game® sachet, it says that 1 sachet makes 1 litre of Game®.

1. How many sachets does she need to make 2 litres of Game®?
2. How many sachets does she need to make 500 ml of Game®?



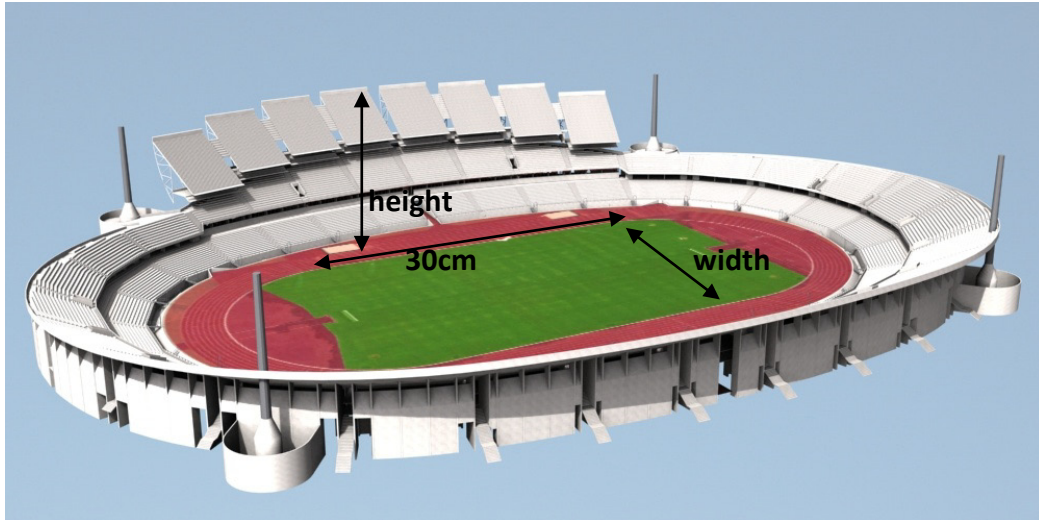
Sam's Energade® bottle has mixing instructions in the form of a ratio. It says that you must always mix the Energade® concentrate and the water in the ratio of 1:3.

3. If Sam wants to make 1 litre (or 1000 ml) of Energade®, how many millilitres of concentrate and how many millilitres of water is needed for 1 litre?
4. What percentage of the Energade® will always be concentrate if it is mixed in the ratio 1:3?
5. What percentage of the Energade® will always be water if it is mixed in the ratio 1:3?
6. If Sam mixed 75 ml of concentrate with 300 ml of water, was the Energade® mixed according to the ratio 1:3?
7. Which will have the stronger taste: Energade® made from 75 ml of concentrate and 300 ml of water or 45 ml of concentrate and 270 ml of water?

Question 5

Alex builds a scale model of the Athletics Stadium.

1. If her scale model measures 30 cm from the start of the 100 m mark to the finish line (of the 100 m mark), what is the scale of the model? (100 cm = 1 m)
2. Using the scale calculated in 1 above, if the width of the field (the grass) is 70 m in real life, what will the width of the field be on the model?
3. According to Alex's research, the height of the stadium pavillion is 12 m. If she makes the height of her model 4 cm, will this be to scale (as in question 1)?



Addendum B

Table of analysis of results:

	5 learners		19 learners		6 learners		
	School 1	%	School 2	%	School 3	%	average
³ 1.1 If $\frac{3}{4}$ of a kg potatoes make one small packet of chips, how many packets of chips does 120 kg of potatoes make?	3	60	3	16	3	50	42
1.2 How many kilograms of potatoes are needed for 200 packets of chips?	1	20	9	47	1	17	28
1.3 What is the cost of a bag of chips if the price of potatoes is R6,75?	1	20	7	37	1	17	25
2.1 Did Sam run at the same average speed for both races if she ran 15 s for the 100 m and 40 s for the 200 m race?	5	100	13	68	6	100	89
2.2 Which runner ran at the same average speed for both races?	5	100	16	84	5	83	89
2.3 Which runner ran at a faster average speed for the 200 m race?	4	80	11	58	5	83	74
2.4 If Charlie can manage to run at the same average speed for the 400 m race, what will his speed be?	5	100	12	63	5	83	82
3.1 What is the probability of receiving a Coke-Cola if 240 out of 500 tins are Coke-Cola?	2	40	10	53	1	17	36
3.2 What is the probability of <i>not</i> receiving a Coke-Cola?	2	40	10	53	1	17	36
4.1 If 1 sachet of Game makes 1 ℓ then how many sachets are needed to make 2 ℓ?	5	100	19	100	6	100	100
4.2 How many sachets are needed for 500 mL of Game?	5	100	16	84	5	83	89
4.3 Energade is mixed in the ratio 1:3, how many mL of concentrate and water is needed for make 1 ℓ of Energade.	1	20	4	21	0	0	14

4.4 What percentage of the Energade is concentrate?	0	0	5	26	0	0	9
4.5 What percentage of the Energade is water?	0	0	5	26	0	0	9
4.6 Is 75 mℓ of concentrate and 300 mℓ of water mixed in the ratio 1:3?	3	60	14	74	3	50	61
4.7 Which will have the stronger taste: 75 mℓ of concentrate to 300 mℓ or 45 mℓ of concentrate and 270 mℓ of water?	0	0	11	58	0	0	19
5.1 What is the scale of a model, if 30 cm on a scale model represents 100 m?	1	20	0	0	0	0	7
5.2 If the real life width of the track is 70, what will the width of the track on the model be?	0	0	0	0	0	0	0
5.3 If the real height of the stadium is 12 m and Alex makes the height on the model 4 cm, is the height in the same scale as 5.1?	2	40	7	37	0	0	26
		47		48		37	44

Addendum C

Research into Mathematical Literacy

A M.ED STUDY DONE BY MRS ELMARIE MEYER AT THE UNIVERISTY OF STELLENBOSH

Please note that there are no right or wrong answers when completing this questionnaire. It is a way in which teachers want to determine what your fears and ideas are about Mathematics and Mathematical Literacy. The only reason why you are asked to write your name on this questionnaire is so that your teacher can ask you questions about some of your answers if he/she is not clear on what you meant. Your name will not be used in any research documentations. The goal of this study is to better understand learners taking Mathematical Literacy so that teachers can be better at teaching the subject.

Mathematical Literacy QUESTIONNAIRE

Name: _____

School: _____

1. Why have you chosen to take Mathematical Literacy instead of Mathematics?
2. How do you think Mathematical Literacy will be different to Mathematics?
3. What do you like most in Mathematics?
4. What do you find the easiest in Mathematics?
5. What do you dislike most in Mathematics?

6. What do you find most difficult in Mathematics?
7. How did you normally feel in the Mathematics class and why?
8. What do you know about ratios?
9. Where do we use ratios in our every day lives?