On the Regularity of Refinable Functions

by

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Thesis presented in partial fulfilment of the requirements for
the award of Master of Science degree in Physical and Mathematical Analysis
at the University of Stellenbosch, South Africa.

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April 2006

Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and has not previously, in its entirety or in part, been submitted at any university for a degree.

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Summary

This work studies the regularity (or smoothness) of continuous finitely supported refinable functions which are mainly encountered in multiresolution analysis, iterative interpolation processes, signal analysis, etc. Here, we present various kinds of sufficient conditions on a given mask to guarantee the regularity class of the corresponding refinable function.

First, we introduce and analyze the cardinal B-splines $N_m,\ m \in \mathbb{N}$. In particular, we show that these functions are refinable and belong to the smoothness class $C^{m-2}(\mathbb{R})$. As a generalization of the cardinal B-splines, we proceed to discuss refinable functions with positive mask coefficients. A standard result on the existence of a refinable function in the case of positive masks is quoted. Following [13], we extend the regularity result in [25], and we provide an example which illustrates the fact that the associated symbol to a given positive mask need not be a Hurwitz polynomial for its corresponding refinable function to be in a specified smoothness class. Furthermore, we apply our regularity result to an integral equation.

An important tool for our work is Fourier analysis, from which we state some standard results and give the proof of a non-standard result. Next, we study the Hölder regularity of refinable functions, whose associated mask coefficients are not necessarily positive, by estimating the rate of decay of their Fourier transforms. After showing the embedding of certain Sobolev spaces into a Hölder regularity space, we proceed to discuss sufficient conditions for a given refinable function to be in such a Hölder space. We specifically express the minimum Hölder regularity of refinable functions as a function of the spectral radius of an associated transfer operator acting on a finite dimensional space of trigonometric polynomials.

We apply our Fourier-based regularity results to the Daubechies and Dubuc-Deslauriers refinable functions, as well as to a one-parameter family of refinable functions, and then compare our regularity estimates with those obtained by means of a subdivision-based result from [28]. Moreover, we provide graphical examples to illustrate the theory developed.

Opsomming

Hierdie werk bestudeer die regulariteit (of gladheid) van kontinue eindig-ondersteunde verfynbare funksies wat meestal teëgekom word in multi-resolusie analise, iteratiewe interpolasie prosesse, seinanalise, ens. Ons bied hier verskeie soorte voldoende voorwaardes op 'n gegewe masker aan om te waarborg dat die ooreenkomstige verfynbare funksie aan 'n sekere regulariteitsklas behoort.

Eerstens stel ons voor, en analiseer ons, die kardinale B-latfunksies, N_m, $m \in \mathbb{N}$. In die besonder wys ons dat hierdie funksies verfynbaar is, en dat hulle aan die gladheidsklas $C^{m-2}(\mathbb{R})$ behoort. As 'n veralgemening van die kardinale B-latfunksies gaan ons dan voort om verfynbare funksies met positiewe maskerkoëffisiënte te ondersoek. 'n Standaardresultaat oor die bestaan van 'n verfynbare funksie in die geval van positiewe maskers word aangehaal. Soos gedoen is in [13], brei ons die regulariteitsresultaat in [25] uit, en verskaf ons 'n voorbeeld wat die feit illustreer dat die ooreenkomstige simbool van 'n gegewe positiewe masker nie nodig het om 'n Hurwitz polinoom te wees vir die ooreenstemmende verfynbare funksie om aan 'n gespesifiseerde gladheidsklas te behoort nie. Verder pas ons ons regulariteitsresultaat toe op 'n integraalvergelyking.

'n Belangrike stuk gereedskap in ons werk is Fourier analise, waarvan ons sekere standaardresultate aanhaal, en die bewys van 'n nie-standaard resultaat gee. Vervolgens bestudeer ons die Hölder regulariteit van verfynbare funksies, waarvan die ooreenstemmende maskerkoëffisiënte nie noodwendig positief is nie, deur middel van die afskatting van die vervaltempo van hulle Fourier transforms. Nadat ons die inbedding van sekere Sobolevruimtes in 'n Hölder regulariteitsruimte aangetoon het, gaan ons voort om voldoende voorwaardes vir 'n gegewe verfynbare funksie om in so 'n Hölderruimte te wees, te bespreek. Spesifiek druk ons die minimum Hölder regulariteit van verfynbare funksies uit as 'n funksie van die spektraalradius van 'n ooreenkomstige oorgangsoperator wat inwerk op 'n eindig-dimensionele ruimte van trigonometriese polinome.

Ons pas ons Fourier-gebaseerde regulariteitsresultate toe op die Daubechies en Dubuc-Deslauriers verfynbare funksies, asook op 'n een-parameter familie van verfynbare funksies, en dan vergelyk ons ons regulariteitsafskattings met dié wat verkry is deur middel van 'n subdivisie-gebaseerde resultaat in [28]. Daarby verskaf ons grafiese voorbeelde ter illustrasie van die ontwikkelde teorie.

Acknowledgements

I would like to express my profound gratitude to my supervisor Prof J. M. de Villiers for his encouragement and kindness to me throughout the period of this work; without his keen error-spotting eyes, this work would not exist in its present form. I am also greatly indebted to the African Institute for Mathematical Sciences (AIMS) and the Faculty of Science, University of Stellenbosch for providing me with the funds for the programme. I must not fail to appreciate the entire staff of the Department of Mathematics, University of Stellenbosch, especially Prof B. W. Green, Prof P. Maritz, Dr C. H. Rohwer, Dr S. Mouton, Dr H.P. Mashele, Mrs L. Adams and Mrs O. Marais for creating an enabling environment for me to study in the department.

Many thanks to my wonderful mentors – Rev. & Rev. (Mrs) U. A. Onwunta, Dr N. Ogbonna, Dr E. Ukeje, and Mr B. I. Oruh – for allowing themselves to be used by God to influence my life a great deal. Finally, my family, friends, and colleagues deserve a special place in my heart for their prayers and support at all times.

Contents

Preface

For more than a decade now, wavelets and subdivision have developed into important mathematical tools in such applications as signal analysis, smoothing, digital image processing, computer-aided graphics design (CAGD), medical imaging, solution of differential equations, etc. Broadly speaking, a wavelet can be defined as a finitely supported function ψ whose dilations and translations $\{\psi(2^r \cdot -j) : r, j \in \mathbb{Z}\}$ form a basis for the space $L^2(\mathbb{R})$ of all square-integrable functions on $\mathbb{R}$. On the other hand, a subdivision scheme is an iterative process in which, for a given initial control point sequence, each new point is expressed as a linear combination of its neighbouring points, thereby generating, if the subdivision scheme is convergent, increasingly dense point sequences converging to a smooth limit curve Φ.

Underlying the analysis of both wavelets and subdivision schemes are the issues of existence, uniqueness, regularity (or smoothness), and numerical evaluation of refinable functions; that is, functions ϕ which satisfy a refinement equation of the form $\phi = \sum_k a_k \phi(2 \cdot -k)$, where the bi-infinite sequence $a = \{a_k : k \in \mathbb{Z}\}$ is called the corresponding mask. In fact, a wavelet ψ and the limit function Φ for a subdivision scheme are expressible as linear combinations of the integer shifts of some refinable function ϕ. Consequently, the functions ψ and Φ naturally inherit the properties of ϕ. In particular, the regularity of ϕ is preserved by both the functions ψ and Φ. Hence the regularity of ϕ substantially influences the efficiency of the associated wavelet decomposition algorithm, as well as that of the corresponding subdivision scheme.

Since a refinable function is, in many circumstances, not known analytically, the analysis of its properties is based on the explicitly known mask. Of great practical relevance is the case where finitely many mask coefficients are non-zero, and we shall restrict our discussion in this thesis only to this case. In subdivision, this corresponds to finite masks whereas in wavelet construction, it corresponds to compactly supported scaling functions and wavelets. In this thesis, our main focus is to investigate sufficient conditions on the mask coefficients to guarantee the global regularity of the associated refinable function. For results about local (pointwise) regularity and their connection with fractals, we refer to [10].

First, in Chapter 1, we motivate the introduction of a general theory of refinable functions and their smoothness analysis by means of two simple examples. We then proceed to identify the cardinal B-spline N_m of order m as a very special refinable function, in the sense that it can be expressed explicitly in closed form, possesses positive mask coefficients, and belongs to the regularity class $C^{m-2}(\mathbb{R})$. Recall that, for each $m \in \mathbb{N}$, the piecewise polynomial N_m is a polynomial of degree $\leq (m-1)$ on any integer interval $[k, k+1)$, $k \in \mathbb{Z}$, and vanishes outside the interval $[0, m]$. We also review the concepts of wavelets and subdivision; here, we briefly discuss the cardinal B-spline wavelets and the Lane-Riesenfeld subdivision scheme. Moreover, we make the observation that the Lane-Riesenfeld subdivision scheme provides an efficient computational algorithm for the cardinal B-spline if the initial control sequence is chosen as the delta sequence.

In Chapter 2, we study the regularity of refinable functions with positive mask coefficients, which can be interpreted as a generalization of the cardinal B-spline case. Here, we extend the regularity result in [25, Theorem 2.7], which states that if a mask a has strictly positive coefficients (or elements) and is such that its corresponding symbol $A(z) = \frac{1}{2^l}(1+z)^{l+1}C(z)$, where C is a Hurwitz polynomial, satisfies certain conditions, then $\phi \in C^l(\mathbb{R})$. Following [13], we argue that this regularity result remains true if we merely demand that C be a polynomial of degree $d \geq 1$, with $C(1) = 1$, and with the polynomial $B(z) = (1+z)C(z)$ having positive coefficients, (so that B, and therefore also C, are not necessarily Hurwitz polynomials), and, at the same time, replace the term $(1+z)^{l+1}$ by $(1+z)\prod_{j=1}^{l}(1+z^{2^{\mu_j}})$, with $\{\mu_1, \mu_2, \ldots, \mu_l\}$ denoting a sequence in $\mathbb{Z}_+$ satisfying specified recursive bounds. We also provide numerical and graphical illustrations to corroborate this claim.We show that the result of Theorem 2.4 can be used to determine the smoothness of an approximation to the solution of an integral equation which was studied in [1].

From Chapter 3 till end of the thesis, we lay emphasis on refinable functions whose associated mask coefficients are not necessarily positive in the support of the mask. The main thrust of Chapters 3 and 4 is the use of Fourier analysis to study the Hölder regularity of such refinable functions by estimating the rate of decay of their Fourier transforms. To achieve this, we first prove in Theorem 3.10 the embedding of certain Sobolev spaces into a Hölder space, which then leads in a natural way to the formulation and proof of a sufficient condition for regularity of a general class of functions in Theorem 3.11. Relying on the results of Theorems 3.10 and 3.11 in Chapter 3, we proceed to prove in Theorems 4.5 and 4.11 sufficient conditions for regularity of refinable functions. We specifically study the Hölder regularity of refinable functions in Theorem 4.11 as a function of the spectral radius of an associated transfer operator acting on a finite dimensional space of trigonometric polynomials.

In Chapter 5, we apply our regularity results of Theorem 4.5 and 4.11 to the Daubechies orthornomal refinable functions, the Dubuc-Deslauriers interpolatory refinable functions, as well as to a certain one-parameter family of refinable functions. Moreover, we compare these regularity results with those obtained from a result which was proved in [28] in the framework of subdivision.

Finally, in Chapter 6, we draw the conclusion that Fourier-based results of Theorems 4.5 and 4.11 generally yield less optimal Hölder regularity results than does the subdivision-based result of Theorem 5.3. However, the result of Theorem 4.5 is particularly suitable for the investigation of the regularity of a class of refinable functions as a function of a continuous parameter, whereas the result of Theorem 4.11 particularly enjoys the advantage of being easier to implement on the computer than, and also compares favourably with, the subdivision-based result of Theorem 5.3.

List of symbols

Symbol	Definition
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Z}_+$	the set of nonnegative integers
$\mathbb{C}$	the set of complex numbers
$\sum\limits_k$	the sum $\sum\limits_{k\in\mathbb{Z}}$
$\lfloor x \rfloor$	the largest integer $\leq x$
$M(\mathbb{Z})$	the linear space of bi-infinite real-valued sequences
$M_0(\mathbb{Z})$	the subspace of $M(\mathbb{Z})$ consisting of those sequences in $M(\mathbb{Z})$ with finite support, i.e. sequences which possess a finite number of non-zero elements
$M(\mathbb{R})$	the linear space of real-valued (or complex-valued, from Chapter 3) functions on $\mathbb{R}$
$M_0(\mathbb{R})$	the subspace of $M(\mathbb{R})$ consisting of finitely supported functions in $M(\mathbb{R})$
$M_u(\mathbb{R})$	the subspace of functions in $M(\mathbb{R})$ which are bounded in the uniform norm
a	(refinement) mask in $M_0(\mathbb{Z})$
A	the mask symbol, defined by $\sum\limits_k a_k(\cdot)^k$ (a Laurent polynomial corresponding to the mask $a \in M_0(\mathbb{Z})$, or a polynomial, if $a_k = 0,\ k < 0$)
N	the order of the zero at -1 of A
B	the Laurent polynomial satisfying $A = \frac{1}{2^{N-1}}(1+\cdot)^N B$
$\sup\limits_t$	the supremum over all $t \in \mathbb{R}$
$\sup\limits_k$	the supremum over all $k \in \mathbb{Z}$
$l^\infty(\mathbb{Z})$	the subspace of bounded sequences in $M(\mathbb{Z})$
Δc	the backward difference sequence defined by $(\Delta c)_k = c_k - c_{k-1}$, $k \in \mathbb{Z}$, if $c \in M(\mathbb{Z})$
$\Delta^\infty(\mathbb{Z})$	the subspace of $M(\mathbb{Z})$ consisting of those bi-infinite sequences $c \in M(\mathbb{Z})$ which are such that $\Delta c \in l^\infty(\mathbb{Z})$
$C(\mathbb{R})$	the linear space of continuous functions in $M(\mathbb{R})$
$C_0(\mathbb{R})$	$C(\mathbb{R}) \cap M_0(\mathbb{R})$

Symbol	Definition
$C^k(\mathbb{R})$	for $k \in \mathbb{Z}_+,\ C^k(\mathbb{R}) := \{f \in M(\mathbb{R}) : f^{(k)} \in C(\mathbb{R}),\ j = 0, 1, \ldots, k\}$, with the convention $f^{(0)} = f$
$C_0^k(\mathbb{R})$	$C^k(\mathbb{R}) \cap M_0(\mathbb{R})$
$C^{-1}(\mathbb{R})$	the subspace of $M(\mathbb{R})$ consisting of piecewise continuous functions
$C_u(\mathbb{R})$	$C(\mathbb{R}) \cap M_u(\mathbb{R})$
$\|\| \cdot \|\|_\infty$	the sup norm of both the linear spaces $l^\infty(\mathbb{Z})$ and $C_u(\mathbb{R})$
$S_m(\mathbb{Z})$	cardinal spline space of order m
N_m	cardinal B-spline of order m
$a^{(m)}$	$\left\{\frac{1}{2^{m-1}}\binom{m}{j}\right\}$, the refinement mask of N_m
$A^{(m)}$	$\frac{1}{2^{m-1}}(1+\cdot)^m$, i.e, the mask symbol associated with $a^{(m)}$
ϕ_N^D	the Daubechies refinable function of order N
ϕ_N^{DD}	the Dubuc-Deslauriers refinable function of order N
δ_j	the Kroneker delta, equal to zero for all $j \in \mathbb{Z}$, except for $\delta_0 = 1$
$\delta_{j,k}$	the Kroneker delta, equal to zero for all $j, k \in \mathbb{Z}$, except for $\delta_{j,j} = 1$
δ	the sequence $\{\delta_j : j \in \mathbb{Z}\}$
$(\cdot)_+^k$	the truncated power, where $t_+^k = t^k$ if $t \geq 0$, and $t_+^k = 0$ if $t < 0$, and with $0^0 = 1$
π_n	the linear space of polynomials of degree $\leq n$
$L^p(\mathbb{R})$	for $p \in [1, \infty),\ \ L^p(\mathbb{R}) := \{f \in M(\mathbb{R}) : \int_{-\infty}^{\infty} \|f(t)\|^p\, dt < \infty\}$
$\|\| \cdot \|\|_p$	for $p \in [1, \infty)$, the norm $\left(\int_{-\infty}^{\infty} \|f(t)\|^p\, dt\right)^{\frac{1}{p}},\ f \in L^p(\mathbb{R})$
$\langle \cdot, \cdot \rangle$	the inner product $\int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt,\ f, g \in L^2(\mathbb{R})$
$\mathcal{F}f$	the Fourier transform $\hat{f}(\omega) := \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt,\ \omega \in \mathbb{R},\ f \in L^1(\mathbb{R})$
$\mathcal{F}^{-1}\hat{f}$	the inverse Fourier transform $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\hat{f}(\omega)\, d\omega,\ t \in \mathbb{R},\ \hat{f} \in L^1(\mathbb{R})$
$\mathcal{A}$	the set $\{f \in M(\mathbb{R}) : f \in L^1(\mathbb{R}) \cap C(\mathbb{R}); \hat{f} \in L^1(\mathbb{R})\}$
$\mathcal{A}_0$	$\mathcal{A} \cap M_0(\mathbb{R})$
Q	for a mask symbol A, the trigonometric mask symbol $Q(\omega) = \frac{1}{2}A(e^{-i\omega}),\ \omega \in \mathbb{R}$
R	for a mask symbol $A = \frac{1}{2^{N-1}}(1+\cdot)^N B$, the trigonometric mask symbol $R(\omega) = B(e^{-i\omega}),\ \omega \in \mathbb{R}$
DD	Dubuc-Deslauriers
Q_N^D, R_N^D	the Daubechies trigonometric mask symbols

Symbol	Definition
Q_N^{DD}, R_N^{DD}	the Dubuc-Deslauriers trigonometric mask symbols
$\mathcal{L}^\beta(\mathbb{R})$	for $\beta \in (0,1]$, the Lipschitz space $\mathcal{L}^\beta(\mathbb{R}) := \{f \in M(\mathbb{R}) : \lvert f(t+h) - f(t)\rvert \leq c\lvert h\rvert^\beta,\ t,h \in \mathbb{R}\}$
$\mathcal{L}_u^\beta(\mathbb{R})$	$\mathcal{L}^\beta(\mathbb{R}) \cap M_u(\mathbb{R})$
$\mathcal{L}_0^\beta(\mathbb{R})$	$\mathcal{L}^\beta(\mathbb{R}) \cap M_0(\mathbb{R})$
$C^\alpha(\mathbb{R})$	for $\alpha = \mathbb{R}_+ \backslash \mathbb{Z}_+$, the Hölder space $C^\alpha(\mathbb{R}) := \{f \in M(\mathbb{R}) : f \in C^n(\mathbb{R});\ f^{(n)} \in \mathcal{L}^\beta(\mathbb{R}),\ n = \lfloor \alpha \rfloor,\ \beta \in (0,1)\}$
$C_u^\alpha(\mathbb{R})$	$C^\alpha(\mathbb{R}) \cap M_u(\mathbb{R})$
$C_0^\alpha(\mathbb{R})$	$C^\alpha(\mathbb{R}) \cap M_0(\mathbb{R})$
$H^{p,q}(\mathbb{R})$	for $p \in [1,\infty),\ q \in [0,\infty)$, the Sobolev space $H^{p,q}(\mathbb{R}) := \{f \in L^1(\mathbb{R}) : \int_{-\infty}^{\infty} (1 + \lvert\omega\rvert^p)^q \lvert\hat{f}(\omega)\rvert^p \, d\omega < \infty\}$
$C_{2\pi}(\mathbb{R})$	the 2π-periodic space $\{f \in C(\mathbb{R}) : f = f(\cdot + 2\pi)\}$
$\mathcal{Q}_n$	for $n \in \mathbb{Z}_+$, the $(2n+1)$-dimensional trigonometric polynomial space $\left\{ f \in M(\mathbb{R}) : f(\omega) = \sum_{k=-n}^{n} c_k e^{-i\omega k},\ \omega \in \mathbb{R},\ c_k \in \mathbb{C} \right\}$
T	for $f, P \in \mathcal{Q}_n$, the transfer operator, $Tf := P\left(\frac{\cdot}{2}\right) f\left(\frac{\cdot}{2}\right) + P\left(\frac{\cdot}{2} + \pi\right) f\left(\frac{\cdot}{2} + \pi\right)$
M_T	the matrix for the linear operator T
$\sigma(T)$	the spectrum of the linear operator T
$\rho(T)$	the spectral radius of the linear operator T

Chapter 1

Introduction

The purpose of this thesis is to analyze the regularity (or smoothness) of finitely supported continuous refinable functions. The dependence of the regularity of a given refinable function on its associated mask has been studied recently by many authors, both with varying motivations and definitions of regularity. In the construction of finitely supported orthonormal or biorthogonal wavelet bases for $L^2(\mathbb{R})$, refinement equations are satisfied by the scaling function of the underlying multiresolution analysis (see e.g. [7], [8]). The wavelet is then a finite linear combination of the translates of the scaling function. When studying the convergence properties of subdivision schemes for curve design, refinement equations also arise in a natural way (see e.g. [11], [15], [16], [20], [24]).

Several conditions have been shown to hold in order for a given finitely supported continuous refinable function to have continuous derivatives or to be in a specified Hölder class. These conditions are either sufficient [10], or necessary and sufficient [6], [26], [28]. These results (except in [28]) are often formulated in terms of joint spectral properties of two matrices defined from the mask coefficients.

By methods based on the Fourier transform, conditions that are either sufficient or necessary for the Hölder regularity have been found e.g. in [7], [8], [11]. However, in terms of Sobolev spaces, precise results can be obtained e.g. in [17], [32]. It was also shown in [9] that finitely supported, infinitely differentiable refinable functions are impossible.

Before proceeding with our discussion on the general theory of refinable functions and their smoothness analysis, we first study two basic examples. Our goal is to identify some important features that will be studied with more details, in the general theory developed in the subsequent chapters.

1.1 Refinability

Consider the following two related problems of finding real-valued functions ϕ_1 and ϕ_2 on $\mathbb{R}$ such that

$$\phi_1 = \phi_1(2\cdot) + \phi_1(2\cdot -1); \tag{1.1}$$

and

$$\phi_2 = \frac{1}{2}\phi_2(2\cdot) + \phi_2(2\cdot -1) + \frac{1}{2}\phi_2(2\cdot -2). \tag{1.2}$$

It is easy to verify that the piecewise continuous function ϕ_1 defined by

$$\phi_1(t) = \begin{cases} 1, & t \in [0,1), \\ 0, & t \notin [0,1), \end{cases} \tag{1.3}$$

satisfies (1.1), since

$$\phi_1(2t) = \begin{cases} 1, & t \in [0,1/2), \\ 0, & t \notin [0,1/2); \end{cases}$$

and

$$\phi_1(2t-1) = \begin{cases} 1, & t \in [1/2,1), \\ 0, & t \notin [1/2,1). \end{cases}$$

Moreover, the continuous function

$$\phi_2(t) = \begin{cases} t, & t \in [0,1), \\ 2-t, & t \in [1,2), \\ 0, & t \notin [0,2), \end{cases} \tag{1.4}$$

solves (1.2), since

$$\frac{1}{2}\phi_2(2t) = \begin{cases} t, & t \in [0,1/2), \\ 1-t, & t \in [1/2,1), \\ 0, & t \notin [0,1); \end{cases}$$

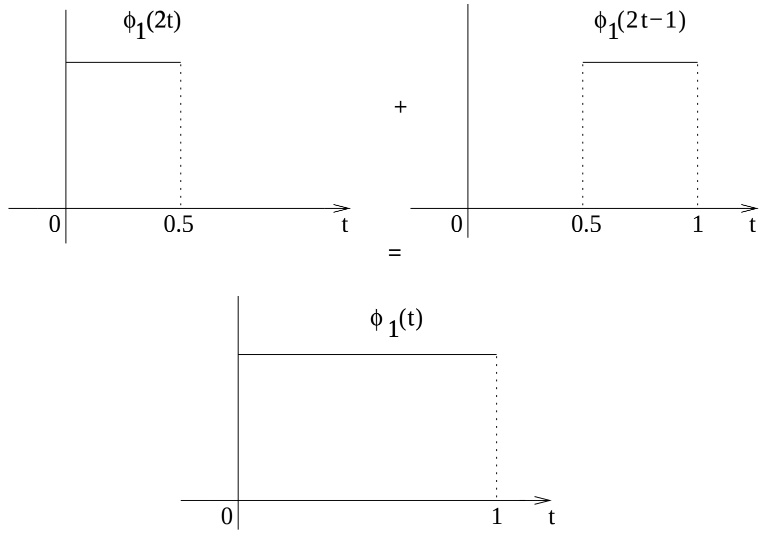

Figure 1.1: *Graphical illustration of equation (1.1).*

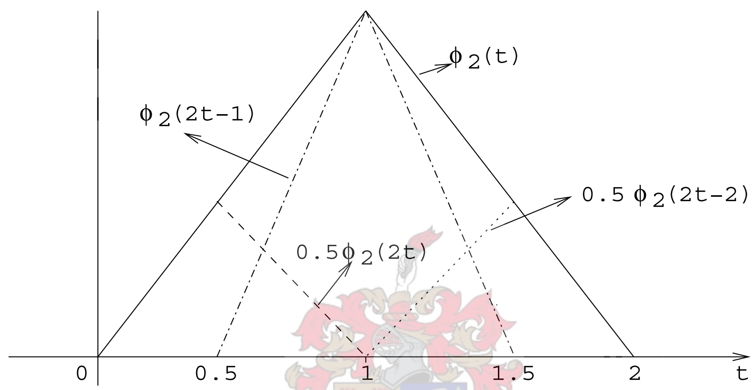

Figure 1.2: *Graphical illustration of the equation (1.2).*

$$\phi_2(2t-1) = \begin{cases} 2t-1, & t \in [1/2, 1), \\ 3-2t, & t \in [1, 3/2), \\ 0, & t \notin [1/2, 3/2); \end{cases}$$

and

$$\frac{1}{2}\phi_2(2t-2) = \begin{cases} t-1, & t \in [1, 3/2), \\ 2-t, & t \in [3/2, 2), \\ 0, & t \notin [1, 2). \end{cases}$$

Graphical illustrations of (1.1) and (1.2) are shown in Figures 1.1 and 1.2.

For $k \in \{-1, 0, 1, \ldots\}$, we write $C^k(\mathbb{R})$ for the space of real-valued functions f on $\mathbb{R}$ such that $f^{(j)} \in C(\mathbb{R}),\ j = 0, 1, \ldots, k$, if $k \geq 0$. Note that then $f^{(0)} = f$ and $C^0(\mathbb{R}) =$

$C(\mathbb{R})$. Also, $C^{-1}(\mathbb{R})$ denotes the space of piecewise continuous functions on $\mathbb{R}$. Observe from (1.3) and (1.4) that $\phi_1 \in C^{-1}(\mathbb{R})\backslash C(\mathbb{R})$, whereas $\phi_2 \in C(\mathbb{R})\backslash C^1(\mathbb{R})$. With a finitely supported function $f : \mathbb{R} \to \mathbb{R}$ defined as a function for which there exists an interval $[\alpha, \beta]$ such that $f(t) = 0,\ t \notin [\alpha, \beta]$, we note that both ϕ_1 and ϕ_2 are finitely supported.

In an attempt to formalize and generalize the above observations, we first introduce the symbol $M(\mathbb{Z})$ to denote the space of bi-infinite real-valued sequences, whereas the symbol $M_0(\mathbb{Z})$ denotes the subspace of $M(\mathbb{Z})$ where $a = \{a_j : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ if and only if the set $\{j : a_j \neq 0\}$ has a finite number of elements, i.e, the sequence a is finitely supported. We write $M(\mathbb{R})$ for the space of real-valued functions on $\mathbb{R}$, whereas the symbol $M_u(\mathbb{R})$ denotes the subspace of $M(\mathbb{R})$ consisting of functions which are bounded in the uniform norm; that is, $f \in M_u(\mathbb{R})$ if and only if $\sup_t |f(t)| < \infty$, where we have adopted the notation $\sup_t = \sup_{t\in\mathbb{R}}$. We write $M_0(\mathbb{R})$ for the subspace of finitely supported functions in $M(\mathbb{R})$. Also, we define $C_0(\mathbb{R}) := C(\mathbb{R}) \cap M_0(\mathbb{R})$, $C_0^k(\mathbb{R}) := C^k(\mathbb{R}) \cap M_0(\mathbb{R})$ and $C_u(\mathbb{R}) := C(\mathbb{R}) \cap M_u(\mathbb{R})$.

A function $\phi \in M(\mathbb{R})$ is called a *refinable function* if there exists a sequence $a \in M_0(\mathbb{Z})$ such that

$$\phi = \sum_j a_j \phi(2 \cdot - j), \tag{1.5}$$

where we have adopted the notation $\sum_j = \sum_{j\in\mathbb{Z}}$. The equation (1.5) is called a *refinement equation,* and the sequence $a \in M_0(\mathbb{Z})$ in (1.5) is known as the *mask* of the refinable function ϕ. Hence, according to (1.1) and (1.3), ϕ_1 is a refinable function in $C_0^{-1}(\mathbb{R})\backslash C(\mathbb{R})$ with respect to the mask $a \in M_0(\mathbb{Z})$ defined by

$$a_j = \begin{cases} 1, & j = 0, 1, \\ 0, & j \notin \{0, 1\}, \end{cases} \tag{1.6}$$

whereas, according to (1.2) and (1.4), ϕ_2 is a refinable function in $C_0(\mathbb{R})\backslash C^1(\mathbb{R})$ with respect to the mask $a \in M_0(\mathbb{Z})$ given by

$$a_j = \begin{cases} \frac{1}{2}, & j = 0, 2, \\ 1, & j = 1, \\ 0, & j \notin \{0, 1, 2\}. \end{cases} \tag{1.7}$$

1.2 Cardinal B-splines

Approximation methods based on piecewise polynomials are very important in many applications. We consider here cardinal spline functions which are piecewise polynomials with uniformly spaced breakpoints and satisfying a certain smoothness condition. The specific properties of these functions make them particularly useful for both subdivision and wavelet analysis.

Recall that our main focus is to investigate, for a given $k \in \mathbb{N}$, the construction of masks $a \in M_0(\mathbb{Z})$ for which the corresponding refinement equation (1.5) has a solution ϕ in $C_0^k(\mathbb{R})$. With a view to a partial result in this direction, we next introduce, for $m \in \mathbb{N}$, the space $S_m(\mathbb{Z})$ of cardinal spline functions of degree $(m-1)$ by

$$S_m(\mathbb{Z}) \quad = \quad \{s \in M(\mathbb{R}) : s|_{[j,j+1)} \in \pi_{m-1}, \quad j \in \mathbb{Z}; \quad s \in C^{m-2}(\mathbb{R})\}, \tag{1.8}$$

where, for any non-negative integer k, the symbol π_k denotes the space of polynomials of degree $\leq k$.

Next, we introduce a finitely supported function in $S_m(\mathbb{Z})$, the sequence of integer shifts of which provides a basis for $S_m(\mathbb{Z})$. To this end, we define the sequence $\{N_m : m \in \mathbb{N}\} \subset M(\mathbb{R})$ by

$$N_m \quad = \quad \frac{1}{(m-1)!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(\cdot - j)_+^{m-1}, \quad m \in \mathbb{N}, \tag{1.9}$$

where the binomial coefficients

$$\binom{m}{j} = \begin{cases} \dfrac{m!}{j!(m-j)!}, & j \in \{0,1,\cdots,m\}, \\ 0, & j \notin \{0,1,\cdots,m\}, \end{cases} \tag{1.10}$$

with the convention $0! = 1$, and where the truncated power $(\cdot)_+^{m-1} \in M(\mathbb{R})$ is defined by

$$t_+^{m-1} = \begin{cases} t^{m-1}, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{1.11}$$

with the convention $0^0 = 1$. It is then immediately clear from (1.8), (1.9), (1.10), and (1.11) that $N_m(\cdot - j) \in S_m(\mathbb{Z}), \quad j \in \mathbb{Z}$. We call N_m the *cardinal B-spline* of order m.

It can be shown (see e.g. [3, Theorem 4.3]) that the cardinal B-splines N_m possess the following properties:

$$(i)\ \ N_m(t) = 0, \quad t \notin \begin{cases} [0,1), & m = 1, \\ (0,m), & m \geq 2; \end{cases} \tag{1.12}$$

$$(ii)\ \ N_m(t) > 0, \quad t \in (0,m), \quad t \in \mathbb{R}; \tag{1.13}$$

$$(iii)\ \ \sum_j N_m(t-j) = 1, \quad t \in \mathbb{R}; \tag{1.14}$$

$$(iv)\ \ N_m' = N_{m-1} - N_{m-1}(\cdot - 1), \quad m \geq 3; \tag{1.15}$$

$$(v)\ \ N_{m+1} = \int_0^1 N_m(\cdot - x)\, dx; \tag{1.16}$$

$$(vi)\ \ \int_{-\infty}^{\infty} N_m(t)\, dt = 1; \tag{1.17}$$

$$(vii)\ \ N_m(t) = \frac{t}{m-1} N_{m-1}(t) + \frac{m-t}{m-1} N_{m-1}(t-1), \quad t \in \mathbb{R}, \quad m \geq 2; \tag{1.18}$$

$$(viii)\ \ N_m(m - \cdot) = N_m, \quad m \geq 2, \tag{1.19}$$

or, alternatively,

$$N_m\left(\frac{m}{2} - \cdot\right) = N_m\left(\frac{m}{2} + \cdot\right), \quad m \geq 2. \tag{1.20}$$

Moreover, as proved in [25, Theorem 2.1], the integer shift space $\{N_m(\cdot - j) : j \in \mathbb{Z}\}$ is, for each $m \in \mathbb{N}$, a basis for $S_m(\mathbb{Z})$ in the sense that for each $s \in S_m(\mathbb{Z})$ there exists a unique sequence $c = \{c_j : j \in Z\} \in M(\mathbb{Z})$ such that

$$s = \sum_j c_j N_m(\cdot - j). \tag{1.21}$$

Using (1.9), (1.10), and (1.11), we can now deduce that $N_1 = \phi_1$, as in (1.3), whereas $N_2 = \phi_2$, as in (1.4). Furthermore, using the defining formula (1.9), or the recurrence relation (1.18), we obtain the expressions

$$N_3(t) = \begin{cases} \frac{1}{2}t^2, & t \in [0,1), \\ \frac{1}{2}(-2t^2 + 6t - 3), & t \in [1,2), \\ \frac{1}{2}(3-t)^2, & t \in [2,3), \\ 0, & t \notin [0,3), \end{cases} \tag{1.22}$$

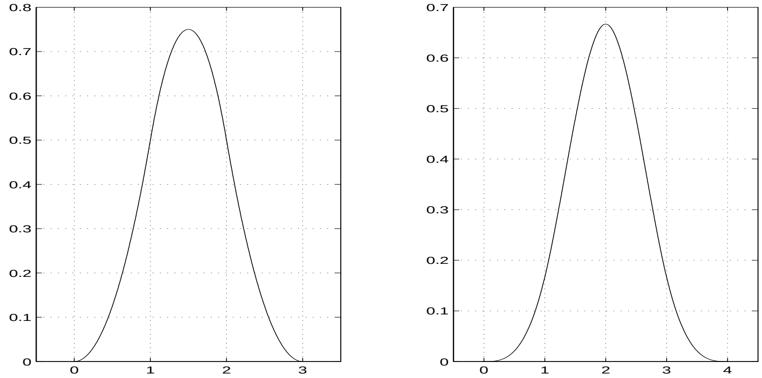

Figure 1.3: *Plots of N_3 (left) and N_4 (right)*

and

$$N_4(t) = \begin{cases} \frac{1}{6}t^3, & t \in [0,1), \\ \frac{1}{6}(-3t^3 + 12t^2 - 12t + 4), & t \in [1,2), \\ \frac{1}{6}(3t^3 - 24t^2 + 60t - 44), & t \in [2,3), \\ \frac{1}{6}(4-t)^3, & t \in [3,4), \\ 0, & t \notin [0,4), \end{cases} \tag{1.23}$$

as graphically illustrated in Figure 1.3.

For every $m \in \mathbb{N}$, the cardinal B-spline N_m is, in the sense of (1.5), a refinable function with respect to the mask $a = a^{(m)} \in M_0(\mathbb{Z})$ defined by

$$a_j^{(m)} = \frac{1}{2^{m-1}} \binom{m}{j}, \quad j \in \mathbb{Z}, \tag{1.24}$$

as is immediately evident from the following identity, the proof of which is from [12, Theorem 4.4].

Theorem 1.1 *The cardinal B-splines $\{N_m : m \in \mathbb{N}\}$, as defined by (1.9), satisfy the refinement equation*

$$N_m = \sum_{j=0}^{m} a_j^{(m)} N_m(2 \cdot -j), \quad m \in \mathbb{N}, \tag{1.25}$$

with the sequence $a^{(m)} = \{a_j^{(m)} : j \in \mathbb{Z}\} \in M_0(\mathbb{Z})$ defined by (1.24).

Proof: Let $m \in \mathbb{N}$. From (1.8), we see that $s \in S_m(\mathbb{Z})$ implies $s(\frac{\cdot}{2}) \in S_m(\mathbb{Z})$; hence, $N_m \in S_m(\mathbb{Z})$ implies $N_m(\frac{\cdot}{2}) \in S_m(\mathbb{Z})$. But then, from (1.21), there exists a unique sequence $a^{(m)} \in M(\mathbb{Z})$ such that

$$N_m\left(\frac{\cdot}{2}\right) = \sum_j a_j^{(m)} N_m(\cdot - j), \tag{1.26}$$

or, equivalently,

$$N_m = \sum_j a_j^{(m)} N_m(2\cdot - j). \tag{1.27}$$

It now remains to show that the sequence $a^{(m)} = \{a_j^{(m)} : j \in \mathbb{Z}\}$ is indeed given by the formula (1.24). To prove (1.24), we first fix $m \in \mathbb{N}$, and use (1.26) and (1.16) to deduce that, for $t \in \mathbb{R}$,

$$\begin{aligned}
\sum_j a_j^{(m+1)} N_{m+1}(t-j) &= N_{m+1}\left(\frac{t}{2}\right) \\
&= \int_0^1 N_m\left(\frac{t}{2} - x\right) dx \\
&= \int_0^1 \left[\sum_j a_j^{(m)} N_m(t - 2x - j)\, dx\right] \\
&= \sum_j a_j^{(m)} \left[\int_0^1 N_m(t - 2x - j)\, dx\right] \\
&= \sum_j a_j^{(m)} \left[\int_0^{\frac{1}{2}} N_m(t - 2x - j)\, dx + \int_{\frac{1}{2}}^1 N_m(t - 2x - j)\, dx\right] \\
&= \frac{1}{2} \sum_j a_j^{(m)} \left[\int_0^1 N_m(t - j - x)\, dx + \int_0^1 N_m(t - j - 1 - x)\, dx\right] \\
&= \frac{1}{2} \sum_j a_j^{(m)} \left[N_{m+1}(t-j) + N_{m+1}(t-j-1)\right] \\
&= \frac{1}{2} \sum_j a_j^{(m)} N_{m+1}(t-j) + \frac{1}{2} \sum_j a_{j-1}^{(m)} N_{m+1}(t-j) \\
&= \sum_j \frac{1}{2}\left[a_j^{(m)} + a_{j-1}^{(m)}\right] N_{m+1}(t-j).
\end{aligned}$$

Thus

$$\sum_j \left\{a_j^{(m+1)} - \frac{1}{2}\left[a_j^{(m)} + a_{j-1}^{(m)}\right]\right\} N_{m+1}(t-j) = 0, \quad t \in \mathbb{R}, \quad m \in \mathbb{N}, \tag{1.28}$$

from which, together with the the fact that the sequence c in (1.21) is unique, it follows that the sequence $\{a_j^{(m)} : j \in \mathbb{Z}, m \in \mathbb{N}\}$ satisfies the recursion formula

$$a_j^{(m+1)} = \frac{1}{2}\left[a_j^{(m)} + a_{j-1}^{(m)}\right], \quad j \in \mathbb{Z}. \tag{1.29}$$

Since $N_1 = \phi_1$, as given by (1.3), we see from (1.1) and (1.25) that (1.24) holds for $m = 1$. Suppose next that (1.24) holds for a fixed $m \in \mathbb{N}$. Then, by virtue of the standard combinatorial identity

$$\binom{m}{j-1} + \binom{m}{j} = \binom{m+1}{j}, \quad j \in \mathbb{Z}, \quad m \in \mathbb{Z}_+, \tag{1.30}$$

and (1.29), we get, for $j \in \mathbb{Z}$,

$$\begin{aligned} a_j^{(m+1)} &= \frac{1}{2}\left[\frac{1}{2^{m-1}}\binom{m}{j} + \frac{1}{2^{m-1}}\binom{m}{j-1}\right] \\ &= \frac{1}{2^m}\left[\binom{m}{j} + \binom{m}{j-1}\right] \\ &= \frac{1}{2^m}\binom{m+1}{j}. \end{aligned}$$

The general validity of (1.24) follows by mathematical induction. ∎

So far, we have therefore identified the refinable functions ϕ_1 and ϕ_2 in (1.3) and (1.4) as the first two members of the family $\{N_m : m \in \mathbb{Z}\}$ of cardinal B-splines. Moreover, we have shown that the function $\phi = N_m$ satisfies the refinement equation (1.5) if the mask $a \in M_0(\mathbb{Z})$ is chosen as $a = a^{(m)}$, as defined by (1.24). Observe also that, by virtue of (1.8) and the fact $N_m(\cdot - j) \in S_m(\mathbb{Z}),\ j \in \mathbb{Z}$, we have

$$\phi = N_m \in C^{m-2}(\mathbb{R}), \quad m \in \mathbb{N}; \tag{1.31}$$

that is, the regularity (smoothness) of ϕ increases with m. A natural question is therefore: in general, which class of masks $a \in M_0(\mathbb{Z})$ has a corresponding refinable function ϕ in a prescribed smoothness class? We shall investigate this question in subsequent chapters.

Before introducing, in Section 1.4, the concepts of wavelets and subdivision, in the course of which the significance of the smoothness class of a refinable function with respect to a given mask $a \in M_0(\mathbb{Z})$ will become evident, we first consider the issues of the existence and uniqueness of refinable functions for the case of a positive mask sequence a.

1.3 Existence and Uniqueness of Refinable Functions

We shall focus our attention in this chapter and the next on refinable functions with masks $a \in M_0(\mathbb{Z})$ satisfying, for a given integer $n \geq 2$, the conditions

$$a_j = 0, \quad j \notin \{0, 1, \dots, n\}; \tag{1.32}$$

$$a_j > 0, \quad j \in \{0, 1, \dots, n\}; \tag{1.33}$$

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1. \tag{1.34}$$

Define, for a given mask $a \in M_0(\mathbb{Z})$, the Laurent polynomial A as given by

$$A(z) = \sum_j a_j z^j, \quad z \in \mathbb{C}\backslash\{0\}. \tag{1.35}$$

Then, $A(z)$ is referred to as the *mask symbol* corresponding to the mask a in (1.5). Note that, if $a_j = 0,\ j < 0$, as is, for example, the case when (1.32) holds, the mask symbol is in fact a polynomial. Also, observe in particular that the choice $a = a^{(m)} \in M_0(\mathbb{Z})$, as appearing (1.24) and (1.25), satisfies the conditions (1.32) and (1.33) with $n = m$. Moreover, from (1.35) and (1.24), we find that the corresponding symbol $A = A^{(m)}$ is given by

$$A^{(m)}(z) = \sum_j a_j^{(m)} z^j = \frac{1}{2^{m-1}}(1+z)^m, \quad z \in \mathbb{C}. \tag{1.36}$$

Note from (1.36) that

$$2 = A^{(m)}(1) = \sum_j a_j = \sum_j a_{2j} + \sum_j a_{2j+1},$$

and

$$0 = A^{(m)}(-1) = \sum_j a_j (-1)^j = \sum_j a_{2j} - \sum_j a_{2j+1},$$

and thus,

$$\sum_j a_{2j}^{(m)} = \sum_j a_{2j+1}^{(m)} = 1,$$

which shows that the mask $a = a^{(m)}$ also satisfies the condition (1.34). Hence the conditions (1.32), (1.33), and (1.34) characterise a class of masks containing the cardinal B-spline mask $a = a^{(m)}$ as a special case.

For a given mask $a \in M_0(\mathbb{Z})$ satisfying conditions (1.32), (1.33), and (1.34), a constructive existence theorem for a corresponding refinable function $\phi \in C(\mathbb{R})$ is based on the *cascade operator* $T_a : M(\mathbb{R}) \to M(\mathbb{R})$ as defined by

$$T_a f = \sum_j a_j f(2 \cdot -j), \quad f \in M(\mathbb{R}).$$

We now introduce the *cascade algorithm* which, for a given initial function $\psi \in M(\mathbb{R})$, generates the sequence $\{\phi_r : r \in \mathbb{Z}_+\} \subset M(\mathbb{R})$ recursively by means of

$$\phi_0 = \psi, \qquad \phi_{r+1} = T_a \phi_r = \sum_j a_j \phi_r(2 \cdot -j), \quad r \in \mathbb{Z}_+, \tag{1.37}$$

where we use the symbol $\mathbb{Z}_+$ to denote the set of non-negative integers. Recall that $C_u(\mathbb{R})$ is a Banach space with respect to the norm $||f||_\infty = \sup_t |f(t)|$, $f \in C_u(\mathbb{R})$.

The following convergence and existence result appears in [13, Theorem 2.2].

Theorem 1.2 *For an integer $n \geq 2$, suppose the mask $a \in M_0(\mathbb{Z})$ satisfies (1.32), (1.33), and (1.34). If, for $m \in \{2, 3, \ldots, n\}$, we choose $\psi = N_m$ in the cascade algorithm (1.37), then the resulting sequence $\{\phi_r : r \in \mathbb{Z}_+\}$ converges uniformly on $\mathbb{R}$, and at a geometric rate, to a solution $\phi \in C_0(\mathbb{R})$ of (1.5), in the sense that, with the positive number $\rho = \rho(a)$ defined by*

$$\rho = \frac{1}{2} \sup \left\{ \sum_l |a_{j-2l} - a_{k-2l}| : j, k \in \mathbb{Z} \right\}, \tag{1.38}$$

we have

$$\frac{1}{2} \leq \rho \leq 1 - \min\{a_0, a_1, \ldots, a_n\} < 1, \tag{1.39}$$

and

$$||\phi - \phi_r||_\infty \leq \frac{\rho^r}{1-\rho} \to 0, \quad r \to \infty. \tag{1.40}$$

Regarding the properties of the refinable function ϕ of Theorem 1.2, the following result was proved in [25, pp.76-83].

Theorem 1.3 *Let ϕ be as in Theorem 1.2. Then*

$$\phi(t) = 0, \quad t \notin (0, n); \tag{1.41}$$

$$\phi(t) > 0, \quad t \in (0, n); \tag{1.42}$$

$$\sum_j \phi(t-j) = 1, \quad t \in \mathbb{R}; \tag{1.43}$$

$$\int_{-\infty}^{\infty} \phi(t)\, dt = 1. \quad t \in \mathbb{R}. \tag{1.44}$$

Note in particular that, for $m \geq 2$, the properties (1.12), (1.13), (1.14), and (1.17) of the (refinable) cardinal B-spline N_m correspond to, respectively, the properties (1.41), (1.42), (1.43), and (1.44) above of ϕ. The following uniqueness result for the refinable function ϕ of Theorem 1.2 was proved in [25, pp.80-81].

Theorem 1.4 *In Theorem 1.2, the function ϕ is the unique solution in $C_0(\mathbb{R})$ of (1.5) such that (1.43) also holds.*

1.4 Refinable Functions, Wavelets and Subdivision

We now proceed to briefly discuss the fundamental role played by refinable functions in both wavelet and subdivision analysis.

1.4.1 Wavelets

As defined in the standard textbooks on wavelets (see e.g. [3], [4], [8]), a *wavelet* ψ, for some integer $k \in \{-1, 0, 1, 2, \ldots\}$, is a function in $C_0^k(\mathbb{R})$ such that, for every $f \in M(\mathbb{R})$ satisfying $\int_{-\infty}^{\infty} [f(t)]^2\, dt < \infty$, there exists a sequence $\{d^{(r)} : r \in \mathbb{Z}\} \subset M(\mathbb{Z})$, with $\sum_j [d_j^{(r)}]^2 < \infty,\ r \in \mathbb{Z}$, satisfying

$$f = \sum_r \sum_j d_j^{(r)} \psi(2^r \cdot - j). \tag{1.45}$$

The right hand side of (1.45) is then called the wavelet series of f, whereas the sequence $\{d_j^{(r)} : j, r \in \mathbb{Z}\}$ is called the discrete wavelet transform of f, and provides localised information on f at different resolution levels r. The multiresolution analysis (MRA) method of constructing a wavelet ψ uses as main building block a given refinable function $\phi \in C_0(\mathbb{R})$, and proceeds as follows.

For a given refinement mask $a \in M_0(\mathbb{Z})$, suppose that $\phi \in C_0(\mathbb{R})$ satisfies (1.5), (1.43), as well as the Riesz-stability condition according to which there exists a positive constant

K such that

$$\int_{-\infty}^{\infty} \left[\sum_j c_j \phi(t-j) \right]^2 dt \geq K \sum_j c_j^2 \tag{1.46}$$

for every sequence $c \in M(\mathbb{Z})$ with $\sum_j c_j^2 < \infty$, and where K is independent of the choice of the sequence c. Such a function ϕ is called a *scaling function.* A wavelet ψ can then be constructed from the function ϕ by finding the sequence $q \in M_0(\mathbb{Z})$ of the smallest possible support such that the function ψ defined by

$$\psi = \sum_j q_j \phi(2 \cdot -j), \tag{1.47}$$

satisfies the orthogonality condition

$$\int_{-\infty}^{\infty} \phi(t-j)\psi(t)dt = 0, \quad j \in \mathbb{Z}.$$

The simplest example of a wavelet is the *Haar wavelet*, namely $\psi^H = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, where χ_E denotes the characteristic function of the set E. The associated Haar scaling function is $\phi = \phi^H = \chi_{[0,1)}$, and the basis $\{\psi^H(2^j \cdot -k),\ j,k \in \mathbb{Z}\}$ determined by ψ^H is called the *Haar system.* Although the Haar system has the desirable property that the wavelet ψ^H is finitely supported, the fact that ψ^H is discontinuous limits its usefulness. An important issue in wavelet theory is therefore the construction of smooth, finitely supported wavelets. By (1.47), it suffices to construct smooth, finitely supported scaling functions. To this end, we briefly discuss here, out of the many such wavelets that appear in the literature, the cardinal B-spline wavelets.

For any integer $m \geq 2$, the mth order cardinal B-spline wavelet is given by

$$\psi_m = \sum_{k=0}^{3m-2} q_{m,k} N_m(2 \cdot -k), \tag{1.48}$$

where

$$q_{m,k} = \frac{(-1)^k}{2^{m-1}} \sum_{l=0}^{m} N_{2m}(k-l+1), \quad k = 0, 1, \ldots, 3m-2.$$

The values $N_{2m}(k-l+1)$ can be computed by applying the recursive formula (1.18), together with the condition

$$N_2(k) = \delta_{k,1}, \quad k \in \mathbb{Z},$$

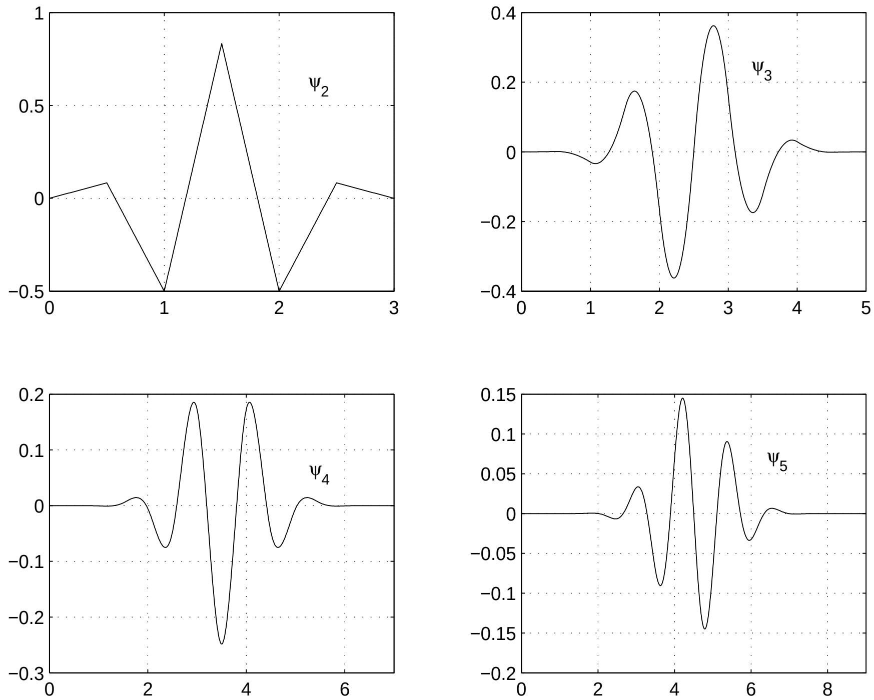

Figure 1.4: *Plots of the cardinal B-spline wavelets* ψ_m *for* $m = 2, 3, 4,$ *and* 5

as is obtained from (1.4) and the fact that $N_2 = \phi_2$.

Illustrating graphs of the cardinal B-spline wavelets ψ_m for $m = 2, 3, 4$, and 5 are shown in Figure 1.4. Using (1.48) and (1.12), we find that $\psi_m(t) = 0,\ t \notin (0, 2m-1)$, whereas (1.20) can be used to prove that ψ_m is symmetric if m is even, and antisymmetric if m is odd, with respect to its centre $t = t_m = \frac{2m-1}{2}$. Observe in particular from (1.48) and (1.31) that $\psi_m \in C^{m-2}(\mathbb{R})$. For details on the construction and other properties of cardinal B-spline wavelets, we refer to [3] and [4].

1.4.2 Subdivision

Next, recall that, in subdivision, we are given a sequence of data or control points in the plane from which we compute a denser sequence of new control points by means of repeated applications of a rule which expresses each new control point as a linear combination of the initial control points. Such a rule is known as a *subdivision scheme*. The general subdivision scheme (see e.g [25]) is given, for a given mask $a \in M_0(\mathbb{Z})$, and an initial sequence $c \in M(\mathbb{Z})$, by

$$c^{(0)} = c, \qquad c_j^{(r+1)} = \sum_k a_{j-2k} c_k^{(r)}, \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \tag{1.49}$$

Out of the many subdivision schemes, we shall consider here particularly the Lane-Riesenfeld subdivision scheme, which, for a given sequence $\{c_j : j \in \mathbb{Z}\} \in M(\mathbb{Z})$, provides an efficient computational algorithm for the cardinal spline $\Phi \in S_m(\mathbb{Z})$ defined by

$$\Phi = \sum_j c_j N_m(\cdot - j). \tag{1.50}$$

For $m \in \mathbb{N}$, and for a given initial sequence $c = \{c_j : j \in \mathbb{Z}\} \in M(\mathbb{Z})$, the Lane-Riesenfeld subdivision scheme recursively generates the sequence $\{c^{(r)} : r \in \mathbb{N}\} \subset M(\mathbb{Z})$ by means of

$$c^{(0)} = c, \qquad c_j^{(r+1)} = \frac{1}{2^{m-1}} \sum_k \binom{m}{j-2k} c_k^{(r)}, \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \tag{1.51}$$

Observe also from (1.49) and (1.51) that the Lane-Riesenfeld subdivision scheme has the mask $a = a^{(m)}$ as defined by (1.24), and with corresponding mask symbol $A(z) = A^{(m)}(z)$ as given by (1.36). Recall also that the Lane-Riesenfeld mask $a = a^{(m)}$ satisfies the conditions (1.32), (1.33), and (1.34).

A natural question which now arises is whether the sequence $\{c^{(r)} : r \in \mathbb{Z}_+\}$ of sequences in $M(\mathbb{Z})$ are increasingly dense sets of points as r increases, and in the process approaches some smooth limit function. The result of Theorem 1.5 below aptly addresses this question.

First, however, we need to introduce the following notation and definitions. Let the backward difference operator $\triangle : M(\mathbb{Z}) \to M(\mathbb{Z})$ be defined, for a given sequence $c = \{c_j : j \in \mathbb{Z}\} \in M(\mathbb{Z})$, by $(\triangle c)_j = c_j - c_{j-1}, \quad j \in \mathbb{Z}$. Also, let $l^\infty(\mathbb{Z})$ denote the subspace $\{c : \sup_j |c_j| < \infty, \quad j \in \mathbb{Z}\} \subset M(\mathbb{Z})$, and write $\triangle^\infty(\mathbb{Z})$ for the subspace of $M(\mathbb{Z})$ consisting of bi-infinite sequences c which are such that $\triangle c \in l^\infty(\mathbb{Z})$.

We say that the subdivision scheme (1.49) is *convergent on a subset M of $M(\mathbb{Z})$* if, for every initial sequence $c \in M$, we have

$$c^{(r)} \in \triangle^\infty(\mathbb{Z}), \quad r \in \mathbb{Z}_+, \tag{1.52}$$

with

$$|| \triangle c^{(r)} ||_\infty \to 0, \quad r \to \infty;$$

and there exists a function $\Phi \in C(\mathbb{R})$ such that

$$\Phi\left(\frac{\cdot}{2^r}\right) - c^{(r)} \in l^\infty(\mathbb{Z}), \quad r \in \mathbb{Z}_+, \tag{1.53}$$

with

$$||\Phi\left(\frac{\cdot}{2^r}\right) - c^{(r)}||_\infty \to 0, \quad r \to \infty.$$

Here, we have used the norm $|| \cdot ||_\infty : l^\infty(\mathbb{Z}) \to \mathbb{R}$ as defined by

$$||c||_\infty = \sup_j |c_j|, \quad c \in l^\infty(\mathbb{Z}).$$

The function Φ is called the limit function of the subdivision scheme (1.49). We can now state the following convergence result (see e.g. [12, Theorem 5.3]) for the Lane-Riesenfeld subdivision scheme .

Theorem 1.5 *The Lane-Riesenfeld subdivision scheme (1.51) is convergent on $\triangle^\infty(\mathbb{Z})$, with limit function Φ given by (1.50), and where*

$$||\Phi\left(\frac{\cdot}{2^r}\right) - \gamma^{(r)}||_\infty \leq \frac{m-2}{2^{r+1}} || \triangle c||_\infty, \quad r \in \mathbb{Z}_+, \tag{1.54}$$

with

$$\gamma_j^{(r)} = \begin{cases} c_{j-k}^{(r)}, & m = 2k, \\ \\ c_{j-k-1}^{(r)}, & m = 2k+1, \end{cases} \qquad k \in \mathbb{N}, \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \tag{1.55}$$

Observe in particular from (1.50) that the limit function Φ has the same regularity as the refinable function N_m, so that, from (1.8), $\Phi \in C^{m-2}(\mathbb{R})$; i.e. the degree of smoothness of Φ increases with m. A special case of Theorem 1.5 is obtained if we choose $c \in M(\mathbb{Z})$ as

$$c = \delta = \{\delta_{j,0} : j \in \mathbb{Z}\}, \tag{1.56}$$

where $\delta_{j,k}$ is the Kroneker delta defined by

$$\delta_{j,k} = \begin{cases} 1, & j = k, \\ & \qquad\qquad j,k \in \mathbb{Z}. \\ 0, & j \neq k, \end{cases} \tag{1.57}$$

But then, from (1.56) and (1.57), we have

$$(\triangle c)_j = (\triangle \delta)_j = \begin{cases} 1, & j = 0, \\ -1, & j = 1, \\ 0, & j \notin \{0,1\}, \end{cases} \tag{1.58}$$

and thus $c = \delta \in \triangle^\infty(\mathbb{Z})$, with

$$|| \triangle c||_\infty = || \triangle \delta||_\infty = 1. \tag{1.59}$$

Moreover, we observe from (1.50), (1.56), and (1.57) that $\Phi = N_m$, so that, from (1.54) and (1.59), we get

$$||N_m \left(\frac{\cdot}{2^r}\right) - \gamma^{(r)}||_\infty \leq \frac{m-2}{2^{r+1}}, \quad r \in \mathbb{Z}_+.$$

Hence, the Lane-Riesenfeld scheme provides an efficient computational algorithm for computing N_m at the dyadic numbers $\{\frac{j}{2^r} : j \in \mathbb{Z}, r \in \mathbb{Z}_+\}$, which are dense in $\mathbb{R}$.

For the convergence of the subdivision scheme (1.49) when the mask a is positive, we quote the following result from [12, Theorem 6.8].

Theorem 1.6 *For a given $n \geq 2$, the subdivision scheme (1.49) corresponding to the mask a of Theorem 1.2 is convergent on $\triangle^\infty(\mathbb{Z})$, with limit function $\Phi \in C(\mathbb{R})$ defined by*

$$\Phi = \sum_j c_j \phi(\cdot - j), \tag{1.60}$$

where

$$||\Phi \left(\frac{\cdot}{2^r}\right) - c^{(r)}||_\infty \leq || \triangle c||_\infty \rho^r \to 0, \quad r \to \infty, \tag{1.61}$$

and with ϕ and ρ as in Theorem 1.2.

Observe that Theorem 1.5 is a special case of Theorem 1.6, but with the geometric constant ρ in (1.61), as bounded in (1.39), replaced in (1.54) by the lower bound $\frac{1}{2}$ in

(1.39).

As is the case for the special case of the cardinal B-spline N_m in the context of the Lane-Riesenfeld subdivision, we see that a refinable function ϕ can be computed efficiently by choosing $c_j = \delta_{j,0}$ as defined in (1.56) and (1.57), and then employing the subdivision scheme (1.49). Hence, we have established in Theorem 1.6 that the refinable function ϕ plays a fundamental role in subdivision, in the sense that the limit curve Φ has the explicit representation (1.60) in terms of ϕ. Moreover, the formula (1.60) shows that Φ belongs to the same smoothness class as ϕ. We therefore proceed in the rest of the thesis to establish sufficient conditions on a given mask $a \in M_0(\mathbb{Z})$ which guarantee a prescribed regularity class for ϕ.

Chapter 2

Regularity Results for Positive Masks

In this chapter we study how, for a given positive mask $a \in M_0(\mathbb{Z})$, the properties of the corresponding positive mask symbol $A(z)$ control the regularity (or degree of smoothness) of the associated refinable function ϕ. Moreover, we show that our regularity result can be used to investigate the regularity of the approximation to the solution of an integral equation. First, however, we introduce the following important concept.

2.1 Preliminaries

A polynomial $p \in \pi_n$ as given by

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad z \in \mathbb{C}, \tag{2.1}$$

with $a_n \neq 0$, is called a *Hurwitz polynomial* if all its zeros have strictly negative real part; i.e. if $z_0 \in \mathbb{C}$ is such that $p(z_0) = 0$, then $Re(z_0) < 0$.

The coefficients $a_0, a_1, \cdots, a_n$ of a Hurwitz polynomial p are necessarily of the same sign. In particular, as proved in [12, Proposition 7.5], if p in (2.1) is a Hurwitz polynomial with $p(1) > 0$, then $a_j > 0, j = 0, 1, \ldots, n$. It should be noted that the converse of this result is not necessarily true; as we will soon see, there does indeed exist a polynomial p, with $a_j > 0, j = 0, 1, \ldots, n$, and such that p is not a Hurwitz polynomial.

Next, we establish the following equivalent formulation of the condition (1.34) on the mask a in terms of the corresponding mask symbol $A(z)$ as defined by (1.35).

Proposition 2.1 *Let $a \in M_0(\mathbb{Z})$ denote a mask, and suppose the corresponding mask symbol $A(z)$ is defined by (1.35). Then (1.34) holds if and only if*

$$A(1) = 2, \tag{2.2}$$

and

$$A(-1) = 0. \tag{2.3}$$

Proof: From (1.34) we obtain

$$A(1) = \sum_j a_j = \sum_j a_{2j} + \sum_j a_{2j+1},$$

and

$$A(-1) = \sum_j a_j(-1)^j = \sum_j a_{2j} - \sum_j a_{2j+1},$$

from which it follows that (1.34) holds if and only if (2.2) and (2.3) are satisfied. ■

Hence, the mask symbol $A(z)$ corresponding to the mask a of Theorem 1.2 has a zero at -1. Observe from (1.36) that the cardinal B-spline mask symbol $A = A^{(m)}$ is in accordance with this result.

2.2 Sufficient Conditions for Regularity

As regards the regularity of the refinable function ϕ of Theorem 1.2, the result of Theorem 2.2 below, as proved in [25, Theorem 2.7], shows that, with further restrictions on the mask a, the minimum smoothness class of ϕ increases with the order of zero at -1 of the mask symbol $A(z)$.

Theorem 2.2 *For an integer $n \geq 3$, suppose that the mask $a \in M_0(\mathbb{Z})$ is such that the conditions (1.32), (1.33) and (1.34) are satisfied. Moreover, suppose that there is an integer l, with $1 \leq l \leq n-2$, such that the corresponding symbol $A(z)$, as defined by (1.35), and for which (2.2) and (2.3) hold, satisfies*

$$A(z) = \frac{1}{2^l}(1+z)^{l+1}C(z), \quad z \in \mathbb{C}, \tag{2.4}$$

where $C(z)$ is a Hurwitz polynomial. Then the refinable function ϕ of Theorem 1.2 satisfies $\phi \in C_0^l(\mathbb{R})$.

Note from (1.35) and (1.32) that the mask symbol $A(z)$ in Theorem 2.2 is a polynomial of degree n; hence, from (2.4), we deduce that the polynomial $C(z)$ is of degree $(n-l-1) \geq [n-(n-2)-1] = 1$. Thus the conditions of Theorems 1.2 and 2.2 imply the inequality deg(C) $\geq$ 1. Also, observe that the hypothesis in Theorem 2.2 that $C(z)$ is a Hurwitz polynomial imply, together with (2.4), that $A(z)$ is also necessarily a Hurwitz polynomial. Observe furthermore that (2.2) in Proposition 2.1, together with (2.4), imply the condition

$$C(1) = 1. \tag{2.5}$$

As a special case of Theorem 2.2, we can therefore choose $C(z) = \frac{1+z}{2}, z \in \mathbb{C}$, which is a Hurwitz polynomial of degree 1, and such that (2.5) holds. Also, if we choose the integer l in Theorem 2.2 as $l = n-2$, then, from (2.4), we have $A(z) = \frac{1}{2^{n-1}}(1+z)^n, z \in \mathbb{C}$. But then, replacing the symbol n by m, we see from (1.36) that $A(z) = A^{(m)}(z), z \in \mathbb{C}$; i.e. the mask $a \in M_0(\mathbb{Z})$ in Theorem 1.2 is given by $a = a^{(m)}$, with the corresponding (unique) refinable function $\phi = N_m$. According to Theorem 2.2, and since $l = n-2 = m-2$, we have $\phi = N_m \in C^{m-2}(\mathbb{R})$, which is consistent with the fact that $N_m \in S_m(\mathbb{Z}) \subset C^{m-2}(\mathbb{R})$, from the definition (1.8).

We now proceed to illustrate Theorem 2.2 graphically in Figures 2.1, 2.2, and 2.3 by means of the refinable functions ϕ_0, ϕ_1, and ϕ_2, corresponding, respectively, to the mask symbols given, for $z \in \mathbb{C}$, by

$$(i)\;\; A_0(z) = (1+z)\left(\frac{z^2+4z+1}{6}\right) = \frac{1}{6}(1+5z+5z^2+z^3), \tag{2.6}$$

$$(ii)\;\; A_1(z) = \frac{1}{2}(1+z)^2\left(\frac{z^2+4z+1}{6}\right) = \frac{1}{12}(1+6z+10z^2+6z^3+z^4), \tag{2.7}$$

$$(iii)\;\; A_2(z) = \frac{1}{4}(1+z)^3\left(\frac{z^2+4z+1}{6}\right) = \frac{1}{24}(1+7z+16z^2+16z^3+7z^4+z^5). \tag{2.8}$$

In all of the graphical examples in this and subsequent chapters, the refinable function ϕ is computed by means of the cascade algorithm, as described in Theorem 1.2, and with $m = 2$, whereas the derivative functions ϕ' and ϕ'' are, respectively, approximated by

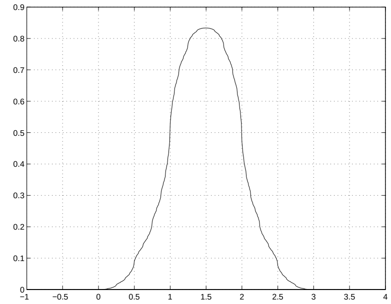

Figure 2.1: ϕ_0, *with* $A = A_0$ *as in (2.6)*

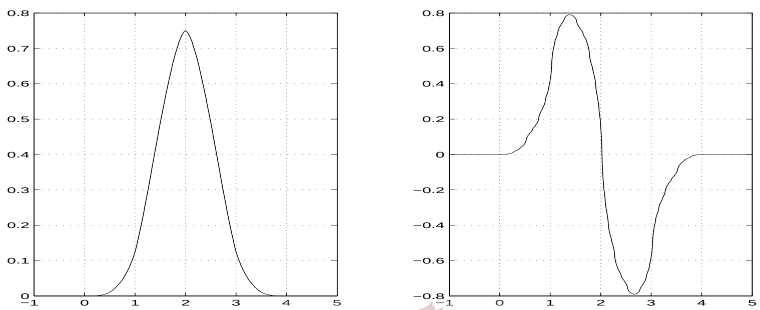

Figure 2.2: ϕ_1 *(left) and* $\phi_1^{'}$ *(right), with* $A = A_1$ *as in (2.7)*

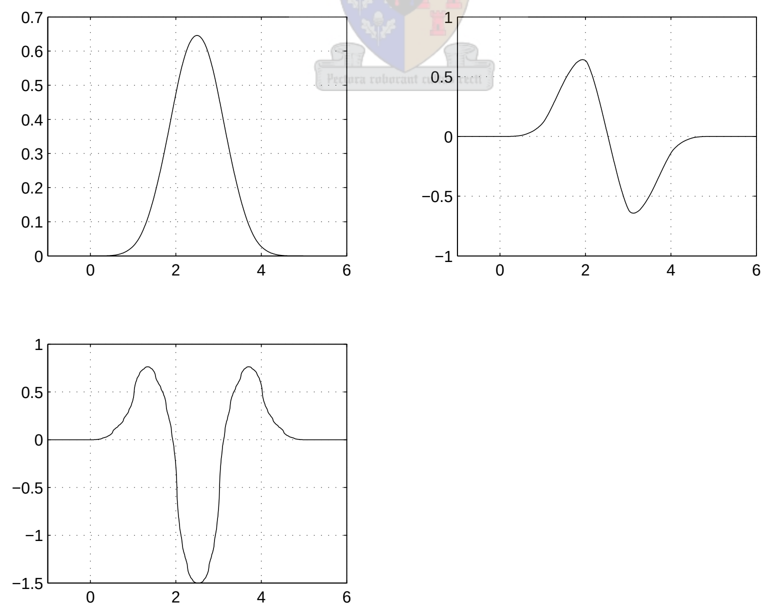

Figure 2.3: ϕ_2 *(top left),* $\phi_2^{'}$ *(top right), and* $\phi_2^{''}$ *(bottom left), with* $A = A_2$ *as in (2.8)*

using the difference quotients

$$\phi'(t) \approx \frac{\phi(t+h)-\phi(t)}{h}, \quad h \neq 0,$$

$$\phi''(t) \approx \frac{\phi(t-h)-2\phi(t)+\phi(t+h)}{h^2}, \quad h \neq 0,$$

for an appropriately small positive value of the step size h.

Observe in particular that the polynomial $(z^2+4z+1)/6,\ z \in \mathbb{C}$, as chosen in (2.6), (2.7) and (2.8), is a Hurwitz polynomial. It follows from Theorems 1.2 and 2.2 that

$$\phi_k \in C_0^k(\mathbb{R}), \qquad k = 0,1,2. \tag{2.9}$$

The graphs in Figures 2.1, 2.2, and 2.3 clearly suggest that (2.9) does indeed hold.

Following [13, Lemma 4.1 and Theorem 4.2], as well as [12, Theorem 7.3], we next extend the regularity result of Theorem 2.2 by showing that result $\phi \in C^l(\mathbb{R})$ remains true if we merely demand that $C(z)$ be a polynomial of degree $d \geq 1$ satisfying (2.5), and with the polynomial $B(z) = (1+z)C(z)$ having positive coefficients, (so that B, and therefore also C, are not necessarily Hurwitz polynomials), and, at the same time, replace the term $(1+z)^{l+1}$ in Theorem 2.2 by a more general term of the form

$$(1+z)\prod_{j=1}^{l}(1+z^{2^{\mu_j}}),$$

with $\{\mu_1, \mu_2, \ldots, \mu_l\}$ denoting a sequence in $\mathbb{Z}_+$ satisfying specified recursive upper bounds.

The following result is fundamental for the proof of our main result in Theorem 2.4 below. We shall, in particular, rely on the uniqueness result of Theorem 1.4.

Theorem 2.3 *Suppose C is a polynomial, with $deg(C) = d \geq 1$, and satisfying (2.5). Let $b \in M_0(\mathbb{Z})$ denote the mask with corresponding symbol $B(z) = \sum_j b_j z^j = \sum_{j=0}^{d+1} b_j z^j, \quad z \in \mathbb{C}$, where*

$$B(z) = (1+z)C(z), \quad z \in \mathbb{C}, \tag{2.10}$$

and suppose that $b_j > 0,\ j = 0,1,\ldots,d+1$. Furthermore, let ψ denote, according to Theorems 1.2, 1.3 and 1.4 (with $n = d+1$), the unique function in $C_0(\mathbb{R})$ satisfying

$$\psi = \sum_j b_j \psi(2\cdot -j), \tag{2.11}$$

and

$$\sum_j \psi(\cdot - j) = 1. \tag{2.12}$$

Let $\mu \in \mathbb{Z}_+$ be such that

$$\mu \leq \log_2(d+1), \tag{2.13}$$

and denote by $a \in M_0(\mathbb{Z})$ the mask with corresponding symbol $A(z)$, as given by (1.35), where

$$A(z) = \frac{1}{2}(1+z)(1+z^{2^\mu})C(z), \quad z \in \mathbb{C}. \tag{2.14}$$

Also, denote by ϕ, according to Theorems 1.2, 1.3 and 1.4 (with $n = d + 2^\mu + 1$), the unique function in $C_0(\mathbb{R})$ satisfying (1.5) and (1.43). Then

$$\phi(t) = \frac{1}{2^\mu} \int_0^{2^\mu} \psi(t-x)\, dx, \quad t \in \mathbb{R}. \tag{2.15}$$

[Observe in particular from (2.13) and (2.14), together with the fact that the polynomial $B(z)$ has positive coefficients, that (1.33) holds with $n = d + 2^\mu + 1$].

Proof: According to Theorem 1.4, the desired result (2.15) would follow if we can show that the function $\theta \in C_0(\mathbb{R})$ defined by

$$\theta(t) = \frac{1}{2^\mu} \int_0^{2^\mu} \psi(t-x)\, dx, \quad t \in \mathbb{R}, \tag{2.16}$$

satisfies the properties

$$\theta = \sum_j a_j \theta(2 \cdot - j), \tag{2.17}$$

and

$$\sum_j a_j \theta(\cdot - j) = 1. \tag{2.18}$$

To prove (2.17), we first observe from (2.10) and (2.14) that, for $z \in \mathbb{C}$,

$$\begin{aligned} A(z) &= \sum_j a_j z^j \\ &= \frac{1}{2}(1+z)(1+z^{2^\mu})C(z) \\ &= \frac{1}{2}(1+z^{2^\mu})B(z) \\ &= \frac{1}{2}(1+z^{2^\mu})\sum_j b_j z^j \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\sum_j b_j z^j + \sum_j b_j z^{j+2^\mu}\right] \\
&= \frac{1}{2}\left[\sum_j b_j z^j + \sum_j b_{j-2^\mu} z^j\right] \\
&= \sum_j \frac{1}{2}(b_j + b_{j-2^\mu}) z^j,
\end{aligned}
$$

and thus,

$$
a_j \;=\; \frac{1}{2}(b_j + b_{j-2^\mu}), \quad j \in \mathbb{Z}. \tag{2.19}
$$

Now, using (2.16), (2.19), and (2.11), we obtain, for $t \in \mathbb{R}$,

$$
\begin{aligned}
\sum_j a_j \theta(2t-j) &= \sum_j \frac{1}{2}(b_j + b_{j-2^\mu}) \left(\frac{1}{2^\mu}\int_0^{2^\mu} \psi(2t-j-x)\,dx\right) \\
&= \frac{1}{2^{\mu+1}} \sum_j (b_j + b_{j-2^\mu}) \int_0^{2^\mu} \psi(2t-x-j)\,dx \\
&= \frac{1}{2^{\mu+1}} \left(\int_0^{2^\mu} \sum_j b_j \psi(2t-x-j)\,dx + \int_0^{2^\mu} \sum_j b_{j-2^\mu} \psi(2t-x-j)\,dx\right) \\
&= \frac{1}{2^{\mu+1}} \left(\int_0^{2^\mu} \sum_j b_j \psi(2t-x-j)\,dx + \int_0^{2^\mu} \sum_j b_j \psi(2t-2^\mu-x-j)\,dx\right) \\
&= \frac{1}{2^{\mu+1}} \left(\int_0^{2^\mu} \psi(t-\frac{x}{2})\,dx + \int_0^{2^\mu} \psi(t-2^{\mu-1}-\frac{x}{2})\,dx\right) \\
&= \frac{1}{2^\mu} \left(\int_0^{2^{\mu-1}} \psi(t-x)\,dx + \int_{2^{\mu-1}}^{2^\mu} \psi(t-x)\,dx\right) \\
&= \frac{1}{2^\mu} \int_0^{2^\mu} \psi(t-x)\,dx = \theta(t),
\end{aligned}
$$

thereby yielding (2.17).

To prove (2.18), we note from (2.16) and (2.12) that, for $t \in \mathbb{R}$,

$$
\begin{aligned}
\sum_j \theta(t-j) &= \frac{1}{2^\mu} \sum_j \int_0^{2^\mu} \psi(t-j-x)\,dx \\
&= \frac{1}{2^\mu} \int_0^{2^\mu} \sum_j \psi(t-x-j)\,dx \\
&= \frac{1}{2^\mu} \int_0^{2^\mu} dx = 1,
\end{aligned}
$$

thereby completing the proof of the Theorem. ■

Note that if, in Theorem 2.3, we choose, for an integer $m \geq 2$, the polynomial $C(z) = \frac{1}{2^{m-1}}(1+z)^{m-1}$, $z \in \mathbb{C}$, so that $C(1) = 1$ and $d = m-1 \geq 1$, and if we choose $\mu = 0 \leq \log_2(d+1)$, so that (2.13) is also satisfied, then, from (2.10) and (2.14), we have

$$B(z) = \frac{1}{2^{m-1}}(1+z)^m, \quad z \in \mathbb{C},$$

and

$$A(z) = \frac{1}{2^m}(1+z)^{m+1}, \quad z \in \mathbb{C}.$$

But then,

$$b_j = \frac{1}{2^{m-1}}\binom{m}{j} = a_j^{(m)}, \quad j \in \mathbb{Z},$$

and

$$a_j = \frac{1}{2^m}\binom{m+1}{j} = a_j^{(m+1)}, \quad j \in \mathbb{Z},$$

and it follows from (1.27) and (1.14), together with the uniqueness result of Theorem 1.4, that $\psi = N_m$ and $\phi = N_{m+1}$. It then follows from (2.15) in Theorem 2.3 that

$$N_{m+1}(t) = \int_0^1 N_m(t-x)\,dx, \quad t \in \mathbb{R},$$

which is precisely the property (1.16) of cardinal B-splines.

Our main result is now as follows.

Theorem 2.4 *Suppose in Theorem 1.2 we have $n \geq 3$, and let the polynomial C be as in Theorem 2.3. If, for an integer $l \in \mathbb{N}$, there exists a sequence $\{\mu_1, \mu_2, \ldots, \mu_l\} \subset \mathbb{Z}_+$ satisfying*

$$\mu_1 \leq \mu_2 \leq \ldots \leq \mu_l, \tag{2.20}$$

and

$$\mu_1 \leq \log_2(d+1), \quad \mu_{r+1} \leq \log_2\left(d+1+\sum_{j=1}^{r} 2^{\mu_j}\right), \quad r = 1, 2, \ldots, l-1, \tag{2.21}$$

such that the corresponding mask symbol $A(z)$ defined by (1.35) is given by

$$A(z) \;=\; \frac{1}{2^l}(1+z)\prod_{r=1}^{l}(1+z^{2^{\mu_r}})C(z), \quad z \in \mathbb{C}, \tag{2.22}$$

then $\phi \in C_0^l(\mathbb{R})$.

Proof. Note from (2.2) in Proposition 2.1, together with (2.22), that the condition $C(1) = 1$ in Theorem 2.3 necessarily holds. Hence, noting also (2.22), we can apply Theorem 2.3 with the choice $\mu = \mu_1$ to deduce that

$$\phi_1(t) = \frac{1}{2^{\mu_1}} \int_0^{2^{\mu_1}} \phi_0(t-x)\,dx = \frac{1}{2^{\mu_1}} \int_{t-2^{\mu_1}}^{t} \phi_0(x)\,dx, \tag{2.23}$$

with ϕ_0 and ϕ_1 denoting, as in Theorem 2.3, the refinable functions with respect to the masks corresponding to the symbols given, respectively, by B_0 and B_1, where

$$B_0(z) = (1+z)C(z), \quad z \in \mathbb{C},$$

and

$$B_1(z) = \frac{1}{2}(1+z)(1+z^{2^{\mu_1}})C(z), \quad z \in \mathbb{C}.$$

Since $\phi_0 \in C_0(\mathbb{R})$ from Theorem 1.2, it then follows from (2.23), together with the fundamental theorem of integral calculus, that $\phi_1 \in C_0^1(\mathbb{R})$. Repeating the above procedure by successively setting $\mu = \mu_j, \quad j = 2, \ldots, l$ in Theorem 2.3, and noting that the inequalities (2.21) imply that the corresponding masks have positive coefficients at each step, we construct a sequence $\{\phi_j : j = 1, 2, \ldots, l\}$ of refinable functions with $\phi_j \in C_0^j(\mathbb{R}), \quad j = 1, \ldots, l$, and where $\sum_k \phi_k(t-k) = 1, \quad t \in \mathbb{R}, \quad j = 1, \ldots, l$. Hence, from the uniqueness result of Theorem 1.4, we eventually obtain $\phi = \phi_l \in C_0^l(\mathbb{R})$. ■

Remarks

(a) Observe that if we choose $\mu_1 = \mu_2 = \cdots = \mu_l = 0$ in Theorem 2.4, then the result is exactly the same as that of Theorem 2.2.

(b) Note also that the choice

$$\mu_r = r, \quad r = 1, 2, \ldots, l,$$

satisfies the condition (2.21), since then, using the fact that $d \geq 1$, we have

$$\mu_1 = 1 = \log_2 2 \leq \log_2(d+1),$$

whereas, for $r \geq 1$,

$$\begin{aligned}
\mu_{r+1} &= r+1 \\
&= \log_2(2^{r+1}) \\
&= \log_2\left(2 + \sum_{j=1}^{r} 2^j\right) \\
&\leq \log_2\left(d + 1 + \sum_{j=1}^{r} 2^j\right).
\end{aligned}$$

Next, we illustrate graphically the result of Theorem 2.4 by increasing the value of l for a given polynomial $A(z)$ in (2.22) with positive coefficients. To this end, we consider, for $z \in \mathbb{C}$, the following mask symbols:

$$(i)\ \ A_0(z) = (1+z)\left(\frac{1+z+z^2}{3}\right) = \frac{1}{3}(1+2z+2z^2+z^3); \tag{2.24}$$

$$(ii)\ \ A_1(z) = \frac{1}{2}(1+z)(1+z^2)\left(\frac{1+z+z^2}{3}\right) = \frac{1}{6}(1+2z+3z^2+3z^3+2z^4+z^5); \tag{2.25}$$

$$\begin{aligned}
(iii)\ \ A_2(z) &= \frac{1}{4}(1+z)(1+z^2)(1+z^4)\left(\frac{1+z+z^2}{3}\right) \\
&= \frac{1}{12}(1+2z+3z^2+3z^3+3z^4+3z^5+3z^6+3z^7+2z^8+z^9).
\end{aligned} \tag{2.26}$$

The graphs in Figures 2.4, 2.5 and 2.6 clearly suggest that $\phi_k \in C_0^k(\mathbb{R}),\ k = 0, 1, 2$, all of which are consistent with Theorems 1.2 and 2.4.

The following example, the graphical illustration of which is shown in Figure 2.7, illustrates the fact that in spite of $A(z)$ not being a Hurwitz polynomial (as demanded in Theorem 2.2) we still have (according to Theorem 2.3) that $\phi \in C_0^1(\mathbb{R})$. The particular illustrating mask symbol A chosen here is given, for $z \in \mathbb{C}$, by

$$\begin{aligned}
A(z) &= \frac{1}{50}(10+22z+19z^2+16z^3+17z^4+12z^5+4z^6) \\
&= \frac{1}{50}(1+z)^2(10+2z+5z^2+4z^3+4z^4) \\
&= \frac{1}{50}(1+z)^2(2+2z+z^2)(5-4z+4z^2).
\end{aligned} \tag{2.27}$$

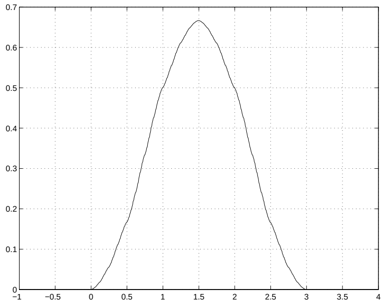

Figure 2.4: ϕ_0 *with* $A = A_0$ *as in (2.24)*

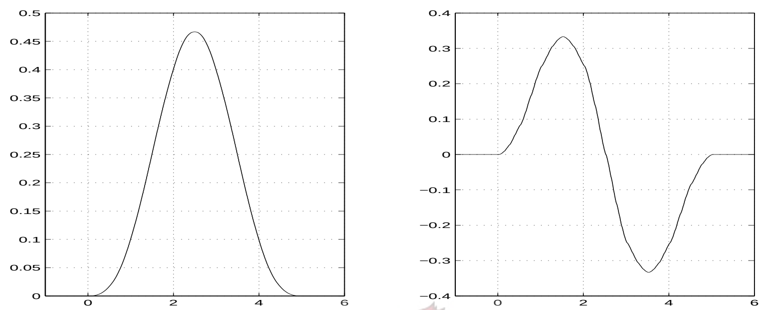

Figure 2.5: ϕ_1 *(left) and* ϕ_1' *(right) with* $A = A_1$ *as in (2.25)*

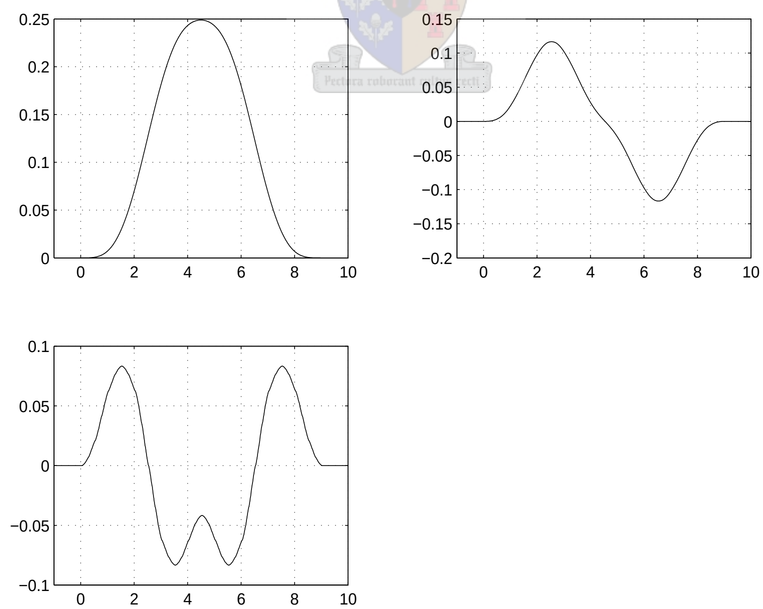

Figure 2.6: ϕ_2 *(top left),* ϕ_2' *(top right), and* ϕ_2'' *(bottom left), with* $A = A_2$ *as in (2.26)*

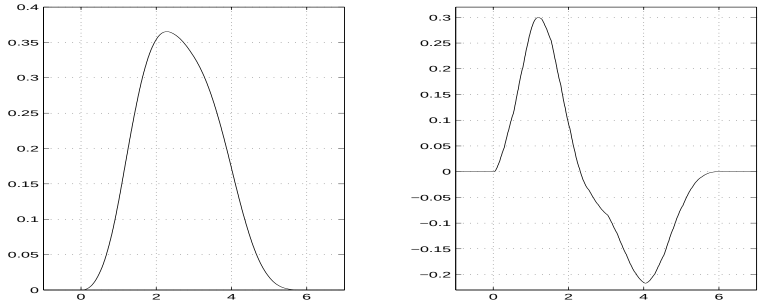

Figure 2.7: ϕ *(left) and* ϕ' *(right), with* $A(z)$ *as in (2.27)*

So far in this chapter, we have shown that the existence of every irreducible factor of the kind $(1+z^{2^\mu})$ in a symbol $A(z)$ corresponding to a positive mask a, with μ denoting a non-negative integer satisfying a mild condition, and always including the case $\mu = 0$, guarantees an additional degree of smoothness in the corresponding refinable function ϕ.

If at least one of the mask coefficients a_j is non-positive, then we cannot appeal to Theorems 2.2 and 2.4 to determine the regularity of the associated refinable function. In Chapters 3 and 4, we shall develop a regularity theory for refinable functions associated with masks of arbitrary sign.

2.3 Application to an Integral Equation

Refinement equations appear as approximation of certain integral equations. We show in this section that the result of Theorem 2.4 can be used to determine the regularity of an approximation to the solution of the integral equation which was studied in [1]. To this end, consider the problem of finding a solution $f \in C(\mathbb{R})$ of the integral equation

$$f\left(\frac{x}{2}\right) = 2\int_{x-1}^{x} f(t)dt, \quad x \in \mathbb{R}. \tag{2.28}$$

Now, recall that the trapezoidal rule for n subintervals for any integral $\displaystyle\int_a^b y(x)dx$ is given by

$$\frac{h}{2}\left[y(a) + y(b) + 2\sum_{j=1}^{n-1} y(t_j)\right], \tag{2.29}$$

where $h = \frac{b-a}{n}, t_j = a + jh$ or $t_j = b - jh,\ j = 0, 1, \ldots, n$. Thus, applying (2.29) to the integral in (2.28), we consider the possibility of constructing, for $n \in \mathbb{N},\ n \geq 2$, an approximate solution $g = g_n$ of (2.28), in the sense that g satisfies the equation

$$g\left(\frac{x}{2}\right) = \frac{1}{n}\left[g(x) + g(x-1) + 2\sum_{j=1}^{n-1} g\left(x - \frac{j}{n}\right)\right], \quad x \in \mathbb{R}. \tag{2.30}$$

Setting $x = \dfrac{t}{n}$ and $\phi(t) = g\left(\dfrac{t}{n}\right)$ in (2.30), we get

$$\phi\left(\frac{t}{2}\right) = \frac{1}{n}\left[\phi(t) + \phi(t-n) + 2\sum_{j=1}^{n-1} \phi(t-j)\right], \quad t \in \mathbb{R}. \tag{2.31}$$

Recalling that the refinement equation (1.5) has the equivalent form $\phi(\frac{\cdot}{2}) = \sum_j a_j \phi(\cdot - j)$, we find that (2.31) is a refinement equation with mask coefficients given by

$$a_j = \begin{cases} \frac{1}{n}, & \text{if} \quad j \in \{0, n\}, \\ \frac{2}{n}, & \text{if} \quad j \in \{1, 2, \ldots, n-1\}. \end{cases}$$

Hence, the corresponding mask symbol A is given, for $z \in \mathbb{C}$, by

$$\begin{aligned} A(z) &= \frac{1}{n}\left[(1 + z^n) + 2\sum_{j=1}^{n-1} z^j\right] \\ &= \frac{1}{n}[1 + 2z + 2z^2 + 2z^3 + \cdots + 2z^{n-1} + z^n], \end{aligned}$$

and thus,

$$A(z) = \frac{1}{n}\frac{(1+z)(1-z^n)}{(1-z)}, \quad z \in \mathbb{C}\backslash\{1\}. \tag{2.32}$$

In the special case $n = 2^k,\ k \in \mathbb{Z}_+$, one gets

$$A(z) = A_k(z) = \frac{1}{2^k}\frac{(1+z)(1-z^{2^k})}{(1-z)}, \quad z \in \mathbb{C}\backslash\{1\}. \tag{2.33}$$

Now, using the identity

$$(1 - z^{2^k}) = (1-z)(1+z)(1+z^2)(1+z^{2^2})\cdots(1+z^{2^{k-1}}),\ k \in \mathbb{N},$$

as can be proved inductively, we find that (2.33) yields, for $z \in \mathbb{C}$,

$$A_k(z) = \frac{1}{2^k}(1+z)^2(1+z^2)(1+z^4)\cdots(1+z^{2^{k-1}})$$

$$= \frac{1}{2^{k-1}}(1+z)\prod_{r=1}^{k-1}(1+z^{2^r})\left[\frac{1+z}{2}\right]. \tag{2.34}$$

Recalling also Remark (b) after the proof of Theorem 2.4, we observe that the mask symbol in (2.34) satisfies the conditions of Theorem 2.4, with $l = k-1, \quad \mu_r = r, \quad r = 1, \ldots, k-1$, and $C(z) = \dfrac{1+z}{2},\ z \in \mathbb{C}$. Thus, appealing to Theorem 2.4, we find that the associated refinable function ϕ_k satisfies $\phi_k \in C_0^{k-1}(\mathbb{R})$. Note also that, for each $k,\ \phi_k(t) = 0,\ t \notin (0, 2^k)$, and $\phi_k(t) > 0,\ t \in (0, 2^k)$.

It seems reasonable to conjecture that there exists a function $f \in C^\infty(\mathbb{R})$ such that

$$||f - \phi_k||_\infty \to 0,\ k \to \infty,$$

and with f satisfying the integral equation (2.28). We do not pursue this particular issue further here.

Chapter 3

A General Regularity Theory based on Fourier Transforms

Henceforth, we investigate the regularity of refinable functions with respect to masks that do not necessarily satisfy the condition that the non-zero mask coefficients are all positive, as was demanded in the first two chapters (see e.g (1.33)). For the existence of continuous refinable functions associated with such masks, we refer to e.g. [1], [3], [7], [8], [9], [11], [14], [18], [24] . To develop our regularity theory, we shall employ Fourier transform techniques to show the dependence of the minimum regularity class of functions $f \in C_0(\mathbb{R})$ on the decay rate of their Fourier transforms. First, we present results from Fourier analysis as will be needed in the rest of the chapter.

3.1 Results from Fourier Analysis

In what follows, we extend the definitions of the spaces $M(\mathbb{R}), M_0(\mathbb{R}), M_u(\mathbb{R}), C(\mathbb{R}), C_0(\mathbb{R})$ and $C_u(\mathbb{R})$ to include also functions $f : \mathbb{R} \to \mathbb{C}$. For $p \in [1, \infty)$, we define the linear space

$$L^p(\mathbb{R}) = \left\{ f \in M(\mathbb{R}) : \int_{-\infty}^{\infty} |f(t)|^p \, dt < \infty \right\}, \tag{3.1}$$

where the integral is taken in the sense of Lebesgue. Then $L^p(\mathbb{R})$ is a Banach space with respect to the norm

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(t)|^p \, dt \right)^{\frac{1}{p}}, \tag{3.2}$$

whereas $L^2(\mathbb{R})$ is also a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt. \tag{3.3}$$

The Fourier transform $\hat{f} \in M(\mathbb{R})$, as also denoted by $\mathcal{F}f$, of a function $f \in L^1(\mathbb{R})$ is defined by

$$\hat{f}(\omega) = (\mathcal{F}f)(\omega) := \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt, \quad \omega \in \mathbb{R}. \tag{3.4}$$

The operator $\mathcal{F}$ is called the Fourier transform operator. Fourier transforms satisfy the following properties, the proof of which can be found in [3, Chapter 2].

Theorem 3.1 *Let the Fourier transform of a function $f \in L^1(\mathbb{R})$ be as defined in (3.4). Then we have the following:*

(i) *$\hat{f}$ is uniformly continuous on $\mathbb{R}$, with*

$$\hat{f} \in C_u(\mathbb{R}), \quad \textit{and} \quad \|\hat{f}\|_\infty \leq \|f\|_1; \tag{3.5}$$

(ii) *the inversion formula: if $\hat{f} \in L^1(\mathbb{R})$, then*

$$f(t) = (\mathcal{F}^{-1}\hat{f})(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\omega} \hat{f}(\omega)\, d\omega \tag{3.6}$$

at every point t where f is continuous;

(iii) *the Parseval identity: if $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then*

$$\langle f, g \rangle = \frac{1}{2\pi}\left\langle \hat{f}, \hat{g} \right\rangle, \tag{3.7}$$

and thus,

$$\|f\|_2 = \frac{1}{\sqrt{2\pi}}\|\hat{f}\|_2; \tag{3.8}$$

(iv) *the Riemann-Lebesgue Lemma:*

$$\hat{f}(\omega) \to 0, \quad |\omega| \to \infty. \tag{3.9}$$

The operator $\mathcal{F}^{-1}$ defined in (3.6) is called the inverse Fourier transform.

In our analysis, we give specific attention to the subspace $\mathcal{A}$ of $L^1(\mathbb{R})$ as defined by

$$\mathcal{A} = \{f \in M(\mathbb{R}) : f \in L^1(\mathbb{R}) \cap C(\mathbb{R}); \hat{f} \in L^1(\mathbb{R})\}. \tag{3.10}$$

Also, we define $\mathcal{A}_0 = \mathcal{A} \cap M_0(\mathbb{R})$.

Proposition 3.2 *The space $\mathcal{A}$ in (3.10) satisfies $\mathcal{A} \subset L^2(\mathbb{R})$.*

Proof. Suppose $f \in \mathcal{A}$. Then $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, which implies that $|f(t)| \to 0$, $|t| \to \infty$. But, since $|f(t)|^2 \leq |f(t)|$ for all $t \in \mathbb{R}$ in the set $\{t \in \mathbb{R} : |f(t)| \leq 1\}$, it follows that $f \in L^2(\mathbb{R})$. ∎

The following proposition was established in [18, Theorem 1.1] and [29, Theorem 2.1].

Proposition 3.3 *The Fourier transform operator $\mathcal{F}$ is a one-to-one map of $\mathcal{A}$ onto itself, where the inverse Fourier transform $\mathcal{F}^{-1}$ is defined as in (3.6).*

Proof. We first show that $\mathcal{F} : \mathcal{A} \to \mathcal{A}$ is injective. Since, from (3.5) and (3.10), $f \in \mathcal{A}$ implies $\hat{f} \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, we have to show that

$$\mathcal{F}\hat{f} \in L^1(\mathbb{R}), \quad f \in \mathcal{A}. \tag{3.11}$$

Suppose therefore that $f \in \mathcal{A}$. Then, using (3.4) and (3.6), one obtains

$$(\mathcal{F}\hat{f})(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\hat{f}(\omega)\,d\omega = 2\pi f(-t), \quad t \in \mathbb{R},$$

and thus,

$$\int_{-\infty}^{\infty} |(\mathcal{F}\hat{f})(t)|\,dt = 2\pi \int_{-\infty}^{\infty} |f(-t)|\,dt = 2\pi \int_{-\infty}^{\infty} |f(t)|\,dt,$$

and it follows that (3.11) holds, since $f \in \mathcal{A} \subset L^1(\mathbb{R})$. Hence, $\mathcal{F}$ maps $\mathcal{A}$ into itself.

It remains to show that, for a given $g \in \mathcal{A}$, there exists a unique $f \in \mathcal{A}$ such that $\hat{f} = g$. To this end, we define the function $g_- : \mathbb{R} \to \mathbb{C}$ by $g_-(t) := g(-t),\ t \in \mathbb{R}$. Then, since $g \in \mathcal{A}$, and since it can easily be verified from (3.4) that $\hat{g}_-(\omega) = \hat{g}(-\omega),\ \omega \in \mathbb{R}$, we also have $g_- \in \mathcal{A}$. Hence, if we define $f = \frac{1}{2\pi}\hat{g}_-$, we also have $f \in \mathcal{A}$. Moreover, (3.4) and (3.6) yield

$$\hat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\omega}\hat{g}_-(\omega)\,d\omega = g_-(-t) = g(t),\ t \in \mathbb{R},$$

and thus, $\hat{f} = g$. To prove the uniqueness of f, suppose the function $h \in \mathcal{A}$ is such that $\hat{h} = g$. Then, with the function u defined by $u = f - h$, we get $u \in \mathcal{A}$ and $\hat{u} = \hat{f} - \hat{h} = 0$, and thus $\|\hat{u}\|_2 = 0$. Hence, by (3.8), and keeping in mind also Proposition 3.2, we get $\|u\|_2 = 0$, so that $u = 0$; that is, $f = h$. ■

We shall rely on the following immediate consequence of Proposition 3.3 and the inversion formula in Theorem 3.1(ii).

Corollary 3.4

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\omega} \hat{f}(\omega)\, d\omega, \quad t \in \mathbb{R},\ f \in \mathcal{A}. \tag{3.12}$$

3.2 The Cardinal B-spline Case

As a special case of the function f in Section 3.1, we calculate here the Fourier transform of the function $f = N_m$, the cardinal B-spline of order m, as introduced in Section 1.2, and we investigate the decay rates of $|\hat{N}_m(\omega)|$ as $|\omega| \to \infty$.

Proposition 3.5 *For $m \in \mathbb{N}$, we have*

$$(a)\ \ \hat{N}_m(\omega) = \begin{cases} \left(\dfrac{1 - e^{-i\omega}}{i\omega}\right)^m, & \omega \in \mathbb{R}\backslash\{0\}, \\ 1, & \omega = 0. \end{cases} \tag{3.13}$$

$$(b)\ \ |\hat{N}_m(\omega)| = \begin{cases} \left|\dfrac{\sin(\omega/2)}{\omega/2}\right|^m, & \omega \in \mathbb{R}\backslash\{0\}, \\ 1, & \omega = 0. \end{cases} \tag{3.14}$$

$$(c)\ \ |\hat{N}_m(\omega)| \leq \frac{2^{2m}}{(1 + |\omega|)^m}, \quad \omega \in \mathbb{R}. \tag{3.15}$$

$$(d)\ \ |\hat{N}_m(\omega)| \leq \frac{(2\sqrt{2})^m}{(1 + \omega^2)^{\frac{m}{2}}}, \quad \omega \in \mathbb{R}. \tag{3.16}$$

(e) $N_m \in \mathcal{A}$ if and only if $m \geq 2$.

Proof. (a) First, since $N_1 = \phi_1$ as given by (1.3), we have from (3.4) that

$$\hat{N}_1(\omega) = \int_0^1 e^{-i\omega t}\, dt, \quad \omega \in \mathbb{R},$$

which yields (3.13) for $m = 1$.

Suppose now that the first line of (3.13) holds for a fixed $m \in \mathbb{N}$. Then (3.4), (1.12), and (1.16) give, for $\omega \in \mathbb{R}$,

$$\begin{aligned}
\hat{N}_{m+1}(\omega) &= \int_0^{m+1} e^{-i\omega t} N_{m+1}(t)\,dt, \\
&= \int_0^{m+1} e^{-i\omega t} \left[\int_0^1 N_m(t-x)\,dx\right] dt, \\
&= \int_0^1 \left[\int_0^{m+1} e^{-i\omega t} N_m(t-x)\,dt\right] dx, \\
&= \int_0^1 \left[\int_{-x}^{m+1-x} e^{-i\omega(t+x)} N_m(t)\,dt\right] dx, \\
&= \int_0^1 e^{-i\omega x} \left[\int_{-\infty}^{\infty} e^{-i\omega t} N_m(t)\,dt\right] dx, \\
&= \left[\int_0^1 e^{-i\omega x}\,dx\right] \hat{N}_m(\omega),
\end{aligned}$$

thereby completing the inductive proof of (3.13).

(b) Since

$$\left|\frac{1-e^{-i\omega}}{i\omega}\right| = \frac{2}{|\omega|}\left|\frac{e^{i\omega/2}-e^{-i\omega/2}}{2i}\right| = 2\frac{|\sin(\omega/2)|}{|\omega|}, \quad \omega \in \mathbb{R}\backslash\{0\}, \tag{3.17}$$

we see that that (3.13) implies (3.14).

(c) The second line of (3.14) shows that (3.15) holds for $\omega = 0$. If $|\omega| \in (0,1]$, we use the inequality

$$|\sin x| \leq |x|, \quad x \in \mathbb{R}, \tag{3.18}$$

to deduce from the first line of (3.14) that

$$|\hat{N}_m(\omega)| \leq 1 \leq 2^m \leq \frac{2^{2m}}{(1+|\omega|)^m},$$

whereas if $|\omega| \in (1,\infty)$, then the first line of (3.14) gives

$$|\hat{N}_m(\omega)| \leq \frac{2^m}{|\omega|^m} \leq \frac{2^{2m}}{(1+|\omega|)^m},$$

thereby completing the proof of (3.15).

(d) The proof of (3.16) is similar to the proof of (3.15).

(e) Suppose first that $m \geq 2$. Then $N_m \in C_0(\mathbb{R}) \subset C(\mathbb{R}) \cap L^1(\mathbb{R})$. Also, the bound (3.15) shows that $\hat{N}_m \in L^1(\mathbb{R})$. Hence, $N_m \in \mathcal{A}$. If $m = 1$, then we have $N_1 \notin C(\mathbb{R})$. Also, (3.14) and (3.15) show that $|\hat{N}_1| \notin L^1(\mathbb{R})$, and therefore $\hat{N}_1 \notin L^1(\mathbb{R})$. Thus, $N_1 \notin \mathcal{A}$. ∎

Observe from (3.15) that $\hat{N}_m(\omega) \to 0, \ |\omega| \to \infty$, which is in accordance with the Riemann-Lebesgue Lemma (3.9).

3.3 A Sufficient Condition for Hölder Regularity

Our regularity theory in this chapter will be formulated in terms of the following linear spaces. For $\beta \in (0, 1]$, we say that a function $f \in M(\mathbb{R})$ is *Lipschitz continuous (or Lipschitz regular) of order* β if and only if there exists a non-negative number c such that

$$|f(t+h) - f(t)| \leq c|h|^\beta, \quad t, h \in \mathbb{R}, \tag{3.19}$$

where c is independent of t and h. The linear space of all such functions is denoted by $\mathcal{L}^\beta(\mathbb{R})$. We define $\mathcal{L}_u^\beta(\mathbb{R}) = \mathcal{L}^\beta(\mathbb{R}) \cap M_u(\mathbb{R})$, whereas $\mathcal{L}_0^\beta(\mathbb{R}) = \mathcal{L}^\beta(\mathbb{R}) \cap M_0(\mathbb{R})$. For $k \in \mathbb{Z}_+$, we also define

$$C_u^k(\mathbb{R}) = \{f : f^{(j)} \in C_u(\mathbb{R}), \quad j = 0, 1, \cdots, k\}.$$

The following inclusions then hold.

Proposition 3.6

$$C_u^1(\mathbb{R}) \subset \mathcal{L}_u^\alpha(\mathbb{R}) \subset \mathcal{L}_u^\beta(\mathbb{R}) \subset C_u(\mathbb{R}), \quad 0 < \beta \leq \alpha \leq 1. \tag{3.20}$$

Proof. The inclusion $\mathcal{L}_u^\beta(\mathbb{R}) \subset C_u(\mathbb{R})$ is an immediate consequence of (3.19). To prove, for $0 < \beta \leq \alpha \leq 1$, the inclusion $\mathcal{L}_u^\alpha(\mathbb{R}) \subset \mathcal{L}_u^\beta(\mathbb{R})$, suppose $f \in \mathcal{L}_u^\alpha(\mathbb{R})$. It follows that there exists a non-negative constant k such that

$$|f(t+h) - f(t)| \leq k|h|^\alpha \leq k|h|^\beta, \quad t \in \mathbb{R}, \quad |h| \leq 1. \tag{3.21}$$

Since $f \in \mathcal{L}_u^\alpha(\mathbb{R}) \subset C_u(\mathbb{R})$, we also have

$$|f(t+h) - f(t)| \leq 2||f||_\infty \leq 2||f||_\infty |h|^\beta, \quad t \in \mathbb{R}, \quad |h| \geq 1. \tag{3.22}$$

It follows from (3.21) and (3.22) that (3.19) holds with $c = \max\{k, 2||f||_\infty\}$, and thus $f \in \mathcal{L}_u^\beta(\mathbb{R})$.

Finally, to prove the inclusion $C_u^1(\mathbb{R}) \subset \mathcal{L}_u^\alpha(\mathbb{R})$, suppose $f \in C_u^1(\mathbb{R})$. Then the Mean-Value Theorem gives

$$|f(t+h) - f(t)| \leq ||f'||_\infty |h| \leq ||f'||_\infty |h|^\alpha, \quad t \in \mathbb{R}, \quad |h| \leq 1, \tag{3.23}$$

whereas (3.22), with β replaced by α, also holds in this case. Hence, the Lipschitz condition (3.19) is satisfied with $c = \max\{||f'||_\infty, 2||f||_\infty\}$, and thus $f \in \mathcal{L}_u^\alpha(\mathbb{R})$. ■

Note that $C_u^1(\mathbb{R}) \subset \mathcal{L}_u^\alpha(\mathbb{R})$ is a proper inclusion for $\alpha \in (0,1]$, since, for example, the linear cardinal B-spline N_2 belongs to the set $\mathcal{L}_u^1(\mathbb{R}) \backslash C_u^1(\mathbb{R})$. Next, for $\alpha \in [0, \infty)$, and with the definition $k := \lfloor \alpha \rfloor$, $\beta := \alpha - k$, we define the linear spaces $C^\alpha(\mathbb{R})$ and $C_u^\alpha(\mathbb{R})$ by

$$C^\alpha(\mathbb{R}) = \begin{cases} C^k(\mathbb{R}), & \alpha = k \in \mathbb{Z}_+, \\ \{f : f \in C^k(\mathbb{R}); \quad f^{(k)} \in \mathcal{L}^\beta(\mathbb{R})\}, & \alpha \in \mathbb{R}_+ \backslash \mathbb{Z}_+; \end{cases} \tag{3.24}$$

$$C_u^\alpha(\mathbb{R}) = \begin{cases} C_u^k(\mathbb{R}), & \alpha = k \in \mathbb{Z}_+, \\ \{f : f \in C_u^k(\mathbb{R}); \quad f^{(k)} \in \mathcal{L}_u^\beta(\mathbb{R})\}, & \alpha \in \mathbb{R}_+ \backslash \mathbb{Z}_+. \end{cases} \tag{3.25}$$

If $f \in C^\alpha(\mathbb{R})$, $\alpha \in \mathbb{R}_+ \backslash \mathbb{Z}_+$, then we say that f is *Hölder continuous of order* α. Also, we define $C_0^\alpha(\mathbb{R}) := C^\alpha(\mathbb{R}) \cap M_0(\mathbb{R})$. Observe that $C_0^\alpha(\mathbb{R}) \subset C_u^\alpha(\mathbb{R})$, and

$$\left. \begin{array}{l} C^\alpha(\mathbb{R}) = \mathcal{L}^\alpha(\mathbb{R}) \\ C_u^\alpha(\mathbb{R}) = \mathcal{L}_u^\alpha(\mathbb{R}) \end{array} \right\}, \ \alpha \in (0,1).$$

It follows immediately from Proposition 3.6 that the definition (3.25) yields the following inclusion.

Corollary 3.7 *For* $\alpha, \gamma \in [0, \infty)$, *we have*

$$C_u^\alpha(\mathbb{R}) \subset C_u^\gamma(\mathbb{R}), \quad 0 \leq \gamma \leq \alpha. \tag{3.26}$$

For $p \in [1, \infty)$ and $q \in [0, \infty)$, we define the subspace $H^{p,q}(\mathbb{R})$ of $L^1(\mathbb{R})$ by

$$H^{p,q}(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) : \int_{-\infty}^{\infty} (1 + |\omega|^p)^q |\hat{f}(\omega)|^p \, d\omega < \infty \right\}. \tag{3.27}$$

The space $H^{p,q}(\mathbb{R})$ is sometimes referred to as a *Sobolev space.* The following inclusion holds.

Proposition 3.8 *For* $p \in [1, \infty)$, *we have*

$$H^{p,q}(\mathbb{R}) \subset H^{p,r}(\mathbb{R}), \quad 0 \leq r \leq q < \infty. \tag{3.28}$$

Proof. Let $0 \leq r \leq q < \infty$ and suppose $f \in H^{p,q}(\mathbb{R})$. Since $q \geq r$, we have

$$(1 + |\omega|^p)^r \leq (1 + |\omega|^p)^q, \quad \omega \in \mathbb{R},$$

and thus

$$\int_{-\infty}^{\infty} (1 + |\omega|^p)^r |\hat{f}(\omega)|^p \, d\omega \leq \int_{-\infty}^{\infty} (1 + |\omega|^p)^q |\hat{f}(\omega)|^p \, d\omega < \infty.$$

Hence $f \in H^{p,r}(\mathbb{R})$. ■

The result of Proposition 3.8 can now be used to prove the following inclusion.

Proposition 3.9

$$H^{1,q}(\mathbb{R}) \cap C_0(\mathbb{R}) \subset \mathcal{A}_0, \quad q \in [0, \infty). \tag{3.29}$$

Proof. For $q \in [0, \infty)$, suppose $f \in H^{1,q}(\mathbb{R}) \cap C_0(\mathbb{R})$. We see from the definition (3.27) that $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$. Moreover, since (3.28) yields $H^{1,q}(\mathbb{R}) \subset H^{1,0}(\mathbb{R})$, we also have $\hat{f} \in L^1(\mathbb{R})$, so that, from (3.10), we have $f \in \mathcal{A}$, and thus also $f \in \mathcal{A}_0$. ■

Our main result of this section, as given by Theorem 3.10 below, states that, for $p \in \{1, 2\}$, and for a given $q \in [0, \infty)$, we have the embedding result

$$H^{p,q}(\mathbb{R}) \cap C_0(\mathbb{R}) \subset C_0^{\alpha}(\mathbb{R}) \tag{3.30}$$

for $\alpha \in I_q$, where I_q is an interval that has zero as its left endpoint, and where the length of I_q grows linearly with q. Keeping in mind also the inclusion result of Corollary 3.7, our eventual regularity theory will therefore be based on the principle of, for a given function

$f \in C_0(\mathbb{R})$, and for $p \in \{1,2\}$, finding the largest possible positive number q such that $f \in H^{p,q}(\mathbb{R})$.

Our fundamental embedding result is as follows.

Theorem 3.10 *The embedding result (3.30) holds for*

$$\alpha \in \begin{cases} [0,q], & \text{if} \quad p=1, \quad q \in (0,\infty), \\ [0,q-\frac{1}{2}), & \text{if} \quad p=2, \quad q \in (\frac{1}{2},\infty). \end{cases} \tag{3.31}$$

[Our proof below is based on those given in [3, Lemma 7.21] and [21, p. 105], but we provide here considerably more details.]

Proof of Theorem 3.10. Suppose $f \in H^{1,q}(\mathbb{R}) \cap C_0(\mathbb{R})$, with $q \in \mathbb{N}$. We shall prove inductively that

$$f^{(j)} \in C_0(\mathbb{R}), \quad j = 0,1,\ldots,q, \tag{3.32}$$

with

$$f^{(j)}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (i\omega)^j e^{i\omega t}\hat{f}(\omega)\,d\omega, \quad t \in \mathbb{R},\ j=0,1,\ldots,q, \tag{3.33}$$

which will then, according to the first line of (3.24) (or (3.25)), yield the first line of (3.31) for integer values of q.

First, observe from Proposition 3.9 that $f \in \mathcal{A}_0$, so that the inversion formula (3.12) in Corollary 3.4 gives (3.33) for $j=0$. Hence, (3.32) and (3.33) hold for $j=0$. Suppose therefore that (3.32) and (3.33) hold for a fixed $j \in \{0,1,\ldots,q-1\}$. Now fix $t \in \mathbb{R}$ and let $h \neq 0$. Then, from (3.33), we have

$$\frac{f^{(j)}(t+h)-f^{(j)}(t)}{h} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\omega)^j\frac{e^{i\omega(t+h)}-e^{i\omega t}}{h}\hat{f}(\omega)\,d\omega. \tag{3.34}$$

As in (3.17), we have

$$\left|\frac{e^{i\omega(t+h)}-e^{i\omega t}}{h}\right| = \left|\frac{e^{i\omega h}-1}{h}\right| = 2\frac{|\sin(\omega h/2)|}{|h|}, \quad \omega \in \mathbb{R}, \tag{3.35}$$

and thus, from (3.35) and (3.18),

$$\left|\frac{e^{i\omega(t+h)}-e^{i\omega t}}{h}\right| \leq |\omega|. \tag{3.36}$$

It follows from (3.36) that

$$\left|(i\omega)^j \frac{e^{i\omega(t+h)} - e^{i\omega t}}{h} \hat{f}(\omega)\right| \leq |\omega|^{j+1} |\hat{f}(\omega)| \leq (1+|\omega|)^q |\hat{f}(\omega)|, \quad \omega \in \mathbb{R}, \tag{3.37}$$

since $j \leq q-1$. Since $f \in H^{1,q}(\mathbb{R})$, we know that

$$\int_{-\infty}^{\infty} (1+|\omega|)^q |\hat{f}(\omega)| \, d\omega < \infty.$$

Hence we may appeal to the Lebesgue Dominated Convergence Theorem (see e.g. [27, Theorem 8.11]) to deduce from (3.34) and (3.37), together with the limit

$$\lim_{h\to 0} \frac{e^{i\omega(t+h)} - e^{i\omega t}}{h} = \frac{d}{dt}(e^{i\omega t}) = (i\omega)e^{i\omega t},$$

that $f^{(j)}$ is differentiable at t, with

$$f^{(j+1)}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega)^{j+1} e^{i\omega t} \hat{f}(\omega) \, d\omega, \quad t \in \mathbb{R}. \tag{3.38}$$

Thus, (3.33) is true with j replaced by $(j+1)$.

Next, from (3.38), we see, for a fixed $t \in \mathbb{R}$ and $h \neq 0$, that

$$f^{(j+1)}(t+h) - f^{(j+1)}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega)^{j+1} \left[e^{i\omega(t+h)} - e^{i\omega t}\right] \hat{f}(\omega) \, d\omega. \tag{3.39}$$

Note as in (3.35) that

$$\left|e^{i\omega(t+h)} - e^{i\omega t}\right| = 2\left|\sin\frac{\omega h}{2}\right| \leq 2, \quad \omega \in \mathbb{R},$$

which gives, for $\omega \in \mathbb{R}$,

$$\begin{aligned} \left|(i\omega)^{j+1} \left[e^{i\omega(t+h)} - e^{i\omega t}\right] \hat{f}(\omega)\right| &\leq 2|\omega|^{j+1} |\hat{f}(\omega)| \\ &\leq 2(1+|\omega|)^{j+1} |\hat{f}(\omega)| \\ &\leq 2(1+|\omega|)^q |\hat{f}(\omega)|, \end{aligned}$$

since $j \leq q-1$. As before, we may appeal to the Lebesgue Dominated Convergence Theorem to deduce from (3.39) that

$$\lim_{h\to 0}\left[f^{(j+1)}(t+h)-f^{(j+1)}(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\omega)^{j+1}e^{i\omega t}\lim_{h\to 0}\left(e^{i\omega h}-1\right)\hat{f}(\omega)\,d\omega = 0.$$

Hence, $f^{(j+1)} \in C(\mathbb{R})$. Also, since $f^{(j)} \in \mathcal{A}_0 \subset C_0(\mathbb{R})$, we have $f^{(j+1)} \in C_0(\mathbb{R})$, so that also (3.32) holds with j replaced by $j+1$, thereby completing our inductive proof of (3.32) and (3.33) for $q \in \mathbb{N}$.

Next, suppose $f \in H^{1,q}(\mathbb{R})\cap C_0(\mathbb{R})$, with $q \notin \mathbb{N}$, and define $k := \lfloor q \rfloor$, $\beta := q-k$, so that $\beta \in (0,1)$. Since $k<q$, it follows from (3.28) in Proposition 3.8 that $f \in H^{1,k}(\mathbb{R})\cap C_0(\mathbb{R})$, and thus, since the first line of (3.31) has been shown to hold if $q \in \mathbb{N}$, we know that $f \in C_0^k(\mathbb{R})$. From (3.25) and (3.26), it therefore remains to prove that $f^{(k)} \in \mathcal{L}^{\beta}(\mathbb{R})$. To prove this, we shall rely on the inequality

$$|\sin x| \leq |x|^{\beta}, \quad x \in \mathbb{R}, \tag{3.40}$$

which is proved by noting that if $|x| \leq 1$, then (3.18) gives

$$|\sin x| \leq |x| \leq |x|^{\beta},$$

whereas, if $|x| \geq 1$, we have

$$|\sin x| \leq 1 \leq |x|^{\beta}.$$

Now, use (3.33), with $j=k$, together with (3.35) and (3.40) to obtain, for $t,h \in \mathbb{R}$,

$$\begin{aligned}
\left|f^{(k)}(t+h)-f^{(k)}(t)\right| &= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty}(i\omega)^k\left[e^{i\omega(t+h)}-e^{i\omega t}\right]\hat{f}(\omega)\,d\omega\right| \\
&\leq \frac{1}{\pi}\int_{-\infty}^{\infty}|\omega|^k\left|\sin\frac{\omega h}{2}\right||\hat{f}(\omega)|\,d\omega \\
&\leq \frac{1}{\pi}\int_{-\infty}^{\infty}|\omega|^k\left|\frac{\omega h}{2}\right|^{\beta}|\hat{f}(\omega)|\,d\omega \\
&= \left[\frac{1}{2^{\beta}\pi}\int_{-\infty}^{\infty}|\omega|^q|\hat{f}(\omega)|\,d\omega\right]|h|^{\beta} \\
&\leq \left[\frac{1}{2^{\beta}\pi}\int_{-\infty}^{\infty}(1+|\omega|)^q|\hat{f}(\omega)|\,d\omega\right]|h|^{\beta},
\end{aligned}$$

and it follows that (3.19) holds with

$$c = \frac{1}{2^\beta \pi} \int_{-\infty}^{\infty} (1+|\omega|)^q |\hat{f}(\omega)|\, d\omega < \infty,$$

from the fact that $f \in H^{1,q}(\mathbb{R})$. Hence $f^{(k)} \in \mathcal{L}^\beta(\mathbb{R})$.

It remains to prove the second line of (3.31). To this end, suppose that $f \in H^{2,q}(\mathbb{R}) \cap C_0(\mathbb{R})$ for some fixed $q \in (\frac{1}{2}, \infty)$. Since $(1-|\omega|)^2 \geq 0,\ \omega \in \mathbb{R}$, it follows that

$$(1+|\omega|)^2 \leq 2(1+\omega^2), \quad \omega \in \mathbb{R},$$

and thus

$$(1+|\omega|)^{2q} \leq 2^q (1+\omega^2)^q, \quad \omega \in \mathbb{R}. \tag{3.41}$$

Now, choose $\varepsilon \in (0, q - \frac{1}{2})$. Using the Cauchy-Schwarz inequality, together with (3.41), we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} (1+|\omega|)^{q-\frac{1}{2}-\varepsilon} |\hat{f}(\omega)|\, d\omega &= \int_{-\infty}^{\infty} (1+|\omega|)^{-\frac{1}{2}-\varepsilon} \left[(1+|\omega|)^q |\hat{f}(\omega)|\right] d\omega \\
&\leq \left[\int_{-\infty}^{\infty} (1+|\omega|)^{-1-2\varepsilon}\, d\omega\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} (1+|\omega|)^{2q} |\hat{f}(\omega)|^2\, d\omega\right]^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{\varepsilon}} \left[\int_{-\infty}^{\infty} (1+|\omega|)^{2q} |\hat{f}(\omega)|^2\, d\omega\right]^{\frac{1}{2}} \\
&\leq \frac{2^{\frac{q}{2}}}{\sqrt{\varepsilon}} \left[\int_{-\infty}^{\infty} (1+\omega^2)^q |\hat{f}(\omega)|^2\, d\omega\right]^{\frac{1}{2}} < \infty,
\end{aligned} \tag{3.42}$$

since $f \in H^{2,q}(\mathbb{R})$. It follows from (3.42) that $f \in H^{1,q-\frac{1}{2}-\varepsilon}(\mathbb{R})$, $\varepsilon \in (0, q-\frac{1}{2})$, which, together with the first line of (3.31), implies the second line of (3.31). ■

The result of Theorem 3.10 can now be used to prove our main result of this chapter, which yields the following sufficient condition for regularity.

Theorem 3.11 *Suppose $f \in C_0(\mathbb{R})$. If, for $p \in \{1,2\}$, there exist a number $\alpha \in (1,\infty)$ and a non-negative number k such that*

$$|\hat{f}(\omega)|^p \leq \frac{k}{(1+|\omega|^p)^\alpha}, \quad \omega \in \mathbb{R}, \tag{3.43}$$

and with k independent of ω, then

$$f \in C_0^\gamma(\mathbb{R}), \quad \gamma \in (0, \alpha - 1). \tag{3.44}$$

Proof. Suppose first $p = 1$ and let $q \in (0, \alpha - 1)$. Then, from (3.43), we have

$$(1 + |\omega|)^q |\hat{f}(\omega)| \leq \frac{k}{(1 + |\omega|)^{\alpha - q}}, \quad \omega \in \mathbb{R},$$

which, together with the definition (3.27) for $p = 1$ and the fact that $\alpha - q > 1$, shows that

$$f \in H^{1,q}(\mathbb{R}) \cap C_0(\mathbb{R}), \quad q \in (0, \alpha - 1). \tag{3.45}$$

It follows from (3.45) and the first line of (3.31) that

$$f \in C_0^\gamma(\mathbb{R}), \quad \gamma \in [0, q], \quad q \in (0, \alpha - 1), \tag{3.46}$$

which immediately yields (3.44).

Next, suppose (3.43) holds for $p = 2$ and let $q \in (1/2, \alpha - 1/2)$. Then we see from (3.43) and (3.41) that, for $\omega \in \mathbb{R}$,

$$(1 + \omega^2)^q |\hat{f}(\omega)|^2 \leq \frac{k}{(1 + \omega^2)^{\alpha - q}} \leq \frac{k 2^{\alpha - q}}{(1 + |\omega|)^{2\alpha - 2q}}, \tag{3.47}$$

But then, $q \in (1/2, \alpha - 1/2)$ also implies that $2\alpha - 2q > 1$, which, together with (3.47) and the definition (3.27) for $p = 2$, shows that

$$f \in H^{2,q}(\mathbb{R}) \cap C_0(\mathbb{R}), \quad q \in (1/2, \alpha - 1/2). \tag{3.48}$$

Observe now from (3.48) and the second line of (3.31) that

$$f \in C^\gamma(\mathbb{R}), \quad \gamma \in (0, q - 1/2), \quad q \in (1/2, \alpha - 1/2), \tag{3.49}$$

from which, since also $f \in C_0(\mathbb{R})$, it then follows that (3.44) holds. ∎

Observe from (3.15) and (3.16) that, for $m \geq 2$, the function $N_m \in C_0(\mathbb{R})$ satisfies the condition (3.43) with $\alpha = m$ for both $p = 1$ and $p = 2$. Hence, according to Theorem 3.11, we have $N_m \in C_0^\gamma(\mathbb{R})$, $\gamma \in (0, m-1)$, which is consistent with the facts that $N_m \in S_m(\mathbb{Z}) \subset$

$C^{m-2}(\mathbb{R})$, and the derivative $N_m^{(m-2)}$ is a piecewise linear continuous function on $\mathbb{R}$. We proceed in Chapter 4 to apply the regularity results of Theorems 3.10 and 3.11 to the case when f is a refinable function.

Chapter 4

The Hölder Regularity of a Refinable Function

In this chapter, we study the Hölder regularity of a given refinable function ϕ using the regularity results of Theorems 3.10 and 3.11. In particular, we shall, for the case $p = 2$, rely on a special operator called the transfer operator. First, however, we discuss the Fourier transform of ϕ and some of its properties.

4.1 The Fourier Transform of a Refinable Function

Throughout this chapter, we shall assume that

[A] the sequence $a = \{a_j : j \in \mathbb{Z}\}$ is a mask in $M_0(\mathbb{Z})$ such that

$$\sum_k a_{2k} = \sum_k a_{2k+1} = 1, \tag{4.1}$$

or, equivalently, in terms of the corresponding mask symbol

$$A(z) = \sum_k a_k z^k, \quad z \in \mathbb{C}\backslash\{0\}, \tag{4.2}$$

and according to Proposition 2.1, such that

$$A(1) = 2, \quad A(-1) = 0; \tag{4.3}$$

[B] there exists a non-trivial solution $\phi \in C_0(\mathbb{R})$ of the corresponding refinement equation

$$\phi = \sum_k a_k \phi(2 \cdot -k). \tag{4.4}$$

To simplify our notation, we introduce the trigonometric polynomial $Q \in M(\mathbb{R})$ by means of the definition

$$Q(\omega) = \frac{1}{2}A(e^{-i\omega}) = \frac{1}{2}\sum_k a_k e^{-i\omega k}, \quad \omega \in \mathbb{R}, \tag{4.5}$$

in terms of which we note that our assumption in (4.3) is equivalent to

$$Q(0) = 1, \quad Q(\pi) = 0. \tag{4.6}$$

Since $\phi \in C_0(\mathbb{R})$ implies $\phi \in L^1(\mathbb{R})$, we know that the Fourier transform $\hat{\phi}$ exists. The following identity then holds.

Proposition 4.1

$$\hat{\phi}(\omega) = \left[\prod_{k=1}^{j} Q(\omega/2^k)\right] \hat{\phi}(\omega/2^j), \quad \omega \in \mathbb{R}, \quad j \in \mathbb{N}. \tag{4.7}$$

Proof. Using (3.4), (4.4), and (4.5), we obtain, for $\omega \in \mathbb{R}$,

$$\begin{aligned}
\hat{\phi}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t}\left[\sum_k a_k \phi(2t-k)\right] dt \\
&= \sum_k a_k \left[\int_{-\infty}^{\infty} e^{-i\omega t} \phi(2t-k)\, dt\right] \\
&= \frac{1}{2}\sum_k a_k \left[\int_{-\infty}^{\infty} e^{-i\omega\frac{(t+k)}{2}} \phi(t)\, dt\right] \\
&= \frac{1}{2}\left[\sum_k a_k e^{-i\omega k/2}\right] \int_{-\infty}^{\infty} e^{-i\omega t/2}\phi(t)\, dt \\
&= Q(\omega/2)\hat{\phi}(\omega/2), \qquad (4.8)
\end{aligned}$$

and thus also

$$\hat{\phi}(\omega) = Q(\omega/2)Q(\omega/4)\hat{\phi}(\omega/4) = \cdots = \left[\prod_{k=1}^{j} Q(\omega/2^k)\right] \hat{\phi}(\omega/2^j), \quad j \in \mathbb{N}. \qquad \blacksquare$$

Using Proposition 4.1, we can now prove the following property of $\hat{\phi}$.

Proposition 4.2

$$\hat{\phi}(2j\pi) = 0, \quad j \in \mathbb{Z}\backslash\{0\}. \tag{4.9}$$

Proof. Since (4.6) gives $Q(\pi) = 0$, and since (4.5) gives

$$Q(\omega + 2\pi j) = Q(\omega), \quad j \in \mathbb{Z}, \quad \omega \in \mathbb{R}, \tag{4.10}$$

we find that

$$Q((2j+1)\pi) = 0, \quad j \in \mathbb{Z}. \tag{4.11}$$

Let $j \in \mathbb{N}$. Then, there exist integers $l \in \mathbb{N}$ and $r \in \mathbb{Z}_+$ such that $2j = 2^l(2r+1)$. Now use (4.7) with $j = l$ to obtain

$$\hat{\phi}(2j\pi) = \hat{\phi}(2^l(2r+1)\pi) = \left[\prod_{k=1}^{l} Q\left(2^{l-k}(2r+1)\pi\right)\right]\hat{\phi}((2r+1)\pi) = 0,$$

since $Q((2r+1)\pi) = 0$ by virtue of (4.11).

If $j \in \mathbb{Z}\backslash\{0\}$ is such that $-j \in \mathbb{N}$, we write $-2j = 2^l(2r+1)$ for $l \in \mathbb{N},\ r \in \mathbb{Z}_+$, and again use (4.7) with $j = l$ to obtain

$$\hat{\phi}(2j\pi) = \left[\prod_{k=1}^{l} Q\left(-2^{l-k}(2r+1)\pi\right)\right]\hat{\phi}(-(2r+1)\pi) = 0,$$

since $Q(-(2r+1)\pi) = Q((2(-r-1)+1)\pi) = 0$, from (4.11), after having noted also that $-1-r \in \mathbb{Z}$. ■

Observe from (3.13) that $\hat{N}_m(2j\pi) = 0,\ j \in \mathbb{Z}\backslash\{0\}$, which is consistent with (4.9) above.

We next show that the Fourier transform $\hat{\phi}$ can be formulated explicitly in terms of an infinite product.

Theorem 4.3 *Suppose $a \in M_0(\mathbb{Z})$ and $\phi \in C_0(\mathbb{R})$ are such that the conditions [A] and [B] are satisfied. Then the infinite product*

$$\prod_{k=1}^{\infty} Q\left(\frac{\omega}{2^k}\right) \tag{4.12}$$

converges for each fixed $\omega \in \mathbb{R}$. Moreover,

$$\hat{\phi}(\omega) = \left[\prod_{k=1}^{\infty} Q\left(\frac{\omega}{2^k}\right)\right] \hat{\phi}(0), \quad \omega \in \mathbb{R}, \tag{4.13}$$

with

$$\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(t)\, dt \neq 0. \tag{4.14}$$

Proof. Let $\omega \in \mathbb{R}$ be fixed. Suppose first that there exists an integer $J \in \mathbb{N}$ such that

$$Q\left(\frac{\omega}{2^J}\right) = 0. \tag{4.15}$$

Then the infinite product (4.12) trivially converges to zero. Also, choosing $j = J$ in (4.7), we find that $\hat{\phi}(\omega) = 0$. Hence the result (4.13) holds with both sides equal to zero.

Suppose next that there does not exist an integer $J \in \mathbb{N}$ such that (4.15) holds, i.e,

$$Q\left(\frac{\omega}{2^k}\right) \neq 0, \quad k \in \mathbb{N}, \tag{4.16}$$

which is equivalent to

$$\left|Q\left(\frac{\omega}{2^k}\right)\right| > 0, \quad k \in \mathbb{N}. \tag{4.17}$$

Next, we use (4.5) and (4.1), as well as (3.17) and (3.18), to obtain

$$\begin{aligned} |Q(\omega) - 1| &= \frac{1}{2}\left|\sum_k a_k(e^{-i\omega k} - 1)\right| \\ &\leq \frac{1}{2}\sum_k |a_k| \left|e^{-i\omega k} - 1\right| \\ &= \sum_k |a_k| \left|\sin(\omega k/2)\right| \\ &\leq \frac{1}{2}\sum_k |a_k||\omega k| \\ &= \left[\frac{1}{2}\sum_k |k a_k|\right] |\omega| = c|\omega|, \end{aligned} \tag{4.18}$$

where $c = \frac{1}{2}\sum_k |k a_k|$. It then follows from (4.18) that

$$\left|\left|Q\left(\frac{\omega}{2^k}\right)\right| - 1\right| \leq \left|Q\left(\frac{\omega}{2^k}\right) - 1\right| \leq \frac{c|\omega|}{2^k}, \quad k \in \mathbb{N}, \tag{4.19}$$

which implies $\left|Q\left(\frac{\omega}{2^k}\right)\right| \to 1,\ k \to \infty$, so that there exists an integer $M \in \mathbb{N}$ such that

$$\left|\left|Q\left(\frac{\omega}{2^k}\right)\right| - 1\right| \leq \frac{1}{2}, \quad k \geq M, \tag{4.20}$$

and thus,

$$\left|Q\left(\frac{\omega}{2^k}\right)\right| \geq \frac{1}{2}, \quad k \geq M. \tag{4.21}$$

Next, we use (4.17), (4.18) and (4.21) to obtain, for $k \geq M$,

$$\begin{aligned}
\left|\ln\left|Q\left(\frac{\omega}{2^k}\right)\right|\right| &= \left|\int_1^{\left|Q\left(\frac{\omega}{2^k}\right)\right|} \frac{1}{t}\,dt\right| \\
&\leq \frac{1}{\min\{1, \left|Q\left(\frac{\omega}{2^k}\right)\right|\}} \left|\left|Q\left(\frac{\omega}{2^k}\right)\right| - 1\right| \\
&\leq 2\left|Q\left(\frac{\omega}{2^k}\right) - 1\right| \\
&\leq \frac{c|\omega|}{2^{k-1}}.
\end{aligned} \tag{4.22}$$

Since (4.22) gives

$$\left|\ln\left|Q\left(\frac{\omega}{2^k}\right)\right|\right| \leq (c|\omega|)\frac{1}{2^{k-1}}, \quad k \geq M,$$

and since $\sum\limits_{k=1}^{\infty} \dfrac{1}{2^{k-1}}$ $(= 2)$ is a convergent series, we deduce that the series

$$\sum_{k=1}^{\infty} \ln\left|Q\left(\frac{\omega}{2^k}\right)\right| \tag{4.23}$$

is absolutely convergent, and therefore also convergent. For $K \in \mathbb{N}$, we therefore have

$$\begin{aligned}
\prod_{k=1}^{K}\left|Q\left(\frac{\omega}{2^k}\right)\right| &= \exp\left(\ln\left[\prod_{k=1}^{K}\left|Q\left(\frac{\omega}{2^k}\right)\right|\right]\right) \\
&= \exp\left(\sum_{k=1}^{K}\ln\left|Q\left(\frac{\omega}{2^k}\right)\right|\right) \\
&\to \exp\left(\sum_{k=1}^{\infty}\ln\left|Q\left(\frac{\omega}{2^k}\right)\right|\right) < \infty, \quad K \to \infty,
\end{aligned}$$

since the exponential function $\exp(x)$ is continuous on $\mathbb{R}$, and (4.23) is a convergent series. Hence the infinite product (4.12) also converges if (4.16) holds, and therefore converges for every fixed $\omega \in \mathbb{R}$. Also, we know from (3.5) in Theorem 3.1 that $\hat{\phi} \in C(\mathbb{R})$. Hence, by letting $j \to \infty$ in the right hand side of (4.7) in Proposition 4.1, we find that the formula

(4.13) is satisfied.

Finally, suppose $\hat{\phi}(0) = 0$. Then (4.13) gives $\hat{\phi} = 0$. But the zero function 0 belongs to the set $\mathcal{A}$, so that we may appeal to the inversion formula (3.12) in Corollary 3.4 to deduce that $\phi = 0$, which contradicts the assumption *[B]*, according to which ϕ is non-trivial. Hence, (4.14) holds true. ■

Observe from Theorem 1.1 that the choice $\phi = N_m$ satisfies the assumptions *[A]* and *[B]*, with $A(z) = \frac{1}{2^{m-1}}(1+z)^m, \quad z \in \mathbb{C}$. Thus, we can combine the results of Proposition 3.5 and Theorem 4.3, together with the fact that (3.4) and (1.17) give $\hat{N}_m(0) = 1$, to obtain the following result on which we shall later rely.

Corollary 4.4

$$|\hat{N}_m(\omega)| = \left| \prod_{k=1}^{\infty} \left(\frac{1+e^{-i\omega/2^k}}{2} \right)^m \right| \leq \begin{cases} \dfrac{2^{2m}}{(1+|\omega|)^m}, \\[2ex] \dfrac{(2\sqrt{2})^m}{(1+\omega^2)^{m/2}}, \end{cases} \qquad \omega \in \mathbb{R}, \quad m = 2, 3, \ldots \tag{4.24}$$

4.2 Applying the Case $p = 1$ of Theorem 3.11

Noting the condition $A(-1) = 0$ in (4.3), we denote by $N \in \mathbb{N}$ the order of the zero at -1 of $A(z)$. It follows that there exists a Laurent polynomial B such that

$$A(z) = \frac{1}{2^{N-1}}(1+z)^N B(z), \quad z \in \mathbb{C}\backslash\{0\}, \tag{4.25}$$

where

$$B(1) = 1, \quad B(-1) \neq 0, \tag{4.26}$$

since also $A(1) = 2$ from (4.3).

With the trigonometric polynomial $R \in M(\mathbb{R})$ defined by

$$R(\omega) = B(e^{-i\omega}), \quad \omega \in \mathbb{R}, \tag{4.27}$$

it follows from (4.5), (4.25), and (4.26) that

$$Q(\omega) = \left(\frac{1+e^{-i\omega}}{2} \right)^N R(\omega), \quad \omega \in \mathbb{R}, \tag{4.28}$$

with

$$R(0) = 1, \quad R(\pi) \neq 0. \tag{4.29}$$

The following regularity result (see e.g.[8, Lemmas 7.1.1 and 7.1.2]) for ϕ is based on the case $p = 1$ of Theorem 3.11. Our proof is considerably more detailed than the one given in [8].

Theorem 4.5 *Suppose $a \in M_0(\mathbb{Z})$ and $\phi \in C_0(\mathbb{R})$ are such that the conditions [A] and [B] of Section 4.1 are satisfied. For $N \in \mathbb{N}$, let the mask symbol A be given by (4.25) and (4.26). If, moreover, there exists an integer $l \in \mathbb{N}$ such that*

$$M_l := \sup_{\omega\in\mathbb{R}} \prod_{k=0}^{l-1} \left| R\left(\frac{\omega}{2^k}\right)\right| < 2^{l(N-1)}, \tag{4.30}$$

then

$$\phi \in C_0^\gamma(\mathbb{R}), \quad \gamma \in \left(0, N - 1 - \frac{1}{l}\log_2 M_l\right). \tag{4.31}$$

Remark

Since $R(0) = 1$ from (4.29), we see that $\prod\limits_{k=0}^{l-1} |R(0)| = 1,\ l \in \mathbb{N}$, which, together with (4.30), shows that

$$0 \leq \frac{1}{l}\log_2 M_l < N - 1, \quad l \in \mathbb{N}. \tag{4.32}$$

Proof of Theorem 4.5. Since the conditions *[A]* and *[B]* are satisfied, we may appeal to Theorem 4.3, together with (4.28), to deduce that

$$\hat{\phi}(\omega) = \left[\prod_{k=1}^{\infty}\left(\frac{1+e^{-i\omega/2^k}}{2}\right)^N\right]\left[\prod_{k=1}^{\infty} R\left(\frac{\omega}{2^k}\right)\right]\hat{\phi}(0), \quad \omega \in \mathbb{R}. \tag{4.33}$$

Substituting the first line of (4.24) into (4.33) then yields the bound

$$|\hat{\phi}(\omega)| \leq \frac{2^{2N}}{(1+|\omega|)^N}\prod_{k=1}^{\infty}\left|R\left(\frac{\omega}{2^k}\right)\right| |\hat{\phi}(0)|, \quad \omega \in \mathbb{R}. \tag{4.34}$$

According to the case $p = 1$ of Theorem 3.11, if we can show that there exists a non-negative number K such that

$$\prod_{k=1}^{\infty}\left|R\left(\frac{\omega}{2^k}\right)\right| \leq K(1+|\omega|)^{\frac{1}{l}\log_2 M_l}, \quad \omega \in \mathbb{R}, \tag{4.35}$$

and with K independent of ω, then (4.34) and (4.35) would imply (3.43) with $f = \phi$, $k = 2^{2N}K|\hat{\phi}(0)|$, $p = 1$, and $\alpha = N - \frac{1}{l}\log_2 M_l$, and it follows from (3.44) in Theorem 3.11 that the desired result (4.31) holds.

To prove (4.35), we introduce the notation $B(z) = \sum_j b_j z^j$, $z \in \mathbb{C}\backslash\{0\}$, and use (4.26), (4.27), (3.17) and (3.18) to deduce, analogous to the proof of (4.18), that, for $\omega \in \mathbb{R}$,

$$\begin{aligned}
|R(\omega) - 1| &= |\sum_j b_j(e^{-i\omega j} - 1)| \\
&\leq \sum_j |b_j||1 - e^{-i\omega j}| \\
&= 2\sum_j |b_j|\,|\sin(\omega j/2)| \\
&\leq \sum_j |jb_j||\omega| = c'|\omega|, \qquad (4.36)
\end{aligned}$$

where $c' = \sum_j |jb_j|$.

Next, using (4.36) and the inequality $1 + x \leq e^x$, $x \geq 0$, we obtain, for $\omega \in \mathbb{R}$,

$$\begin{aligned}
|R(\omega)| &= |1 + (R(\omega) - 1)| \\
&\leq 1 + |R(\omega) - 1| \\
&\leq 1 + c'|\omega| \leq e^{c'|\omega|}. \qquad (4.37)
\end{aligned}$$

Suppose now that $\omega \in \mathbb{R}\backslash\{0\}$, and denote by r the (unique) integer such that

$$2^r \leq |\omega| < 2^{r+1}. \qquad (4.38)$$

Let $J_l \in \mathbb{N}$ be such that $J_l \geq l\lfloor \frac{r}{l} \rfloor + 1$, and observe that

$$\prod_{k=1}^{J_l}\left|R\left(\frac{\omega}{2^k}\right)\right| = \left[\prod_{k=1}^{\lfloor \frac{r}{l} \rfloor}\prod_{j=1}^{l}\left|R\left(\frac{\omega}{2^{(k-1)l+j}}\right)\right|\right]\prod_{k=l\lfloor \frac{r}{l} \rfloor+1}^{J_l}\left|R\left(\frac{\omega}{2^k}\right)\right|. \qquad (4.39)$$

Using (4.30) and (4.38), and keeping in mind also the fact that $\log_2 M_l \geq 0$, as noted before in (4.32), we obtain

$$\prod_{k=1}^{\lfloor \frac{r}{l} \rfloor}\prod_{j=1}^{l}\left|R\left(\frac{\omega}{2^{(k-1)l+j}}\right)\right| = \prod_{k=1}^{\lfloor \frac{r}{l} \rfloor}\prod_{j=0}^{l-1}\left|R\left(\frac{\omega}{2^{(k-1)l+j+1}}\right)\right|$$

$$
\begin{aligned}
&= \prod_{k=1}^{\lfloor \frac{r}{l} \rfloor} \prod_{j=0}^{l-1} \left| R\left(\frac{\omega/2^{(k-1)l+1}}{2^j}\right)\right| \\
&\leq (M_l)^{\lfloor \frac{r}{l} \rfloor} \\
&= 2^{\lfloor \frac{r}{l} \rfloor \log_2 M_l} \\
&\leq 2^{\frac{r}{l} \log_2 M_l} \\
&= (2^r)^{\frac{1}{l} \log_2 M_l} \\
&\leq |\omega|^{\frac{1}{l} \log_2 M_l} \\
&< (1+|\omega|)^{\frac{1}{l} \log_2 M_l}. \qquad (4.40)
\end{aligned}
$$

Next, we use (4.37) and (4.38) to obtain

$$
\begin{aligned}
\prod_{k=l\lfloor \frac{r}{l} \rfloor+1}^{J_l} \left| R\left(\frac{\omega}{2^k}\right)\right| &\leq \prod_{k=l\lfloor \frac{r}{l} \rfloor+1}^{J_l} \exp\left(\frac{c'|\omega|}{2^k}\right) \\
&= \exp\left(c'|\omega| \sum_{k=l\lfloor \frac{r}{l} \rfloor+1}^{J_l} \frac{1}{2^k}\right) \\
&= \exp\left(\frac{c'|\omega|}{2^{l\lfloor \frac{r}{l} \rfloor+1}} \sum_{k=0}^{J_l-l\lfloor \frac{r}{l} \rfloor-1} \frac{1}{2^k}\right) \\
&< \exp\left(c' 2^{r-l\lfloor \frac{r}{l} \rfloor} \sum_{k=0}^{\infty} \frac{1}{2^k}\right) \\
&= \exp(c' 2^{r-l\lfloor \frac{r}{l} \rfloor+1}). \qquad (4.41)
\end{aligned}
$$

Since there exist (unique) integers q and s, with $0 \leq s \leq l-1$, such that $r = ql+s$, we get

$$
r - l\left\lfloor \frac{r}{l} \right\rfloor = ql + s - l\left\lfloor q + \frac{s}{l} \right\rfloor = ql + s - lq = s \leq l-1.
$$

Also, $r - l\lfloor \frac{r}{l} \rfloor \geq r - l(\frac{r}{l}) = 0$, so that the inequality

$$
0 \leq r - l\left\lfloor \frac{r}{l} \right\rfloor \leq l-1 \qquad (4.42)
$$

holds. It follows from (4.41) and (4.42) that

$$
\prod_{k=l\lfloor \frac{r}{l} \rfloor+1}^{J_l} \left| R\left(\frac{\omega}{2^k}\right)\right| < e^{2^l c'}. \qquad (4.43)
$$

Combining (4.39), (4.40), and (4.43) then gives

$$\prod_{k=1}^{J_l} \left| R\left(\frac{\omega}{2^k}\right) \right| \leq e^{2^l c'} (1+|\omega|)^{\frac{1}{l}\log_2 M_l}, \quad \omega \in \mathbb{R}\backslash\{0\}, \quad J_l \geq l \left\lfloor \frac{r}{l} \right\rfloor + 1.$$

and thus,

$$\prod_{k=1}^{\infty} \left| R\left(\frac{\omega}{2^k}\right) \right| \leq e^{2^l c'} (1+|\omega|)^{\frac{1}{l}\log_2 M_l}, \quad \omega \in \mathbb{R}\backslash\{0\}. \tag{4.44}$$

Since, moreover, (4.29) gives $R(0) = 1$, so that

$$\prod_{k=1}^{\infty} \left| R\left(\frac{\omega}{2^k}\right) \right| = 1 \quad \text{if} \quad \omega = 0, \tag{4.45}$$

we deduce from (4.44) and (4.45) that the desired result (4.35) holds with $K = e^{2^l c'}$, which is a positive number independent of ω. ■

Before discussing the Hölder regularity of ϕ for the case $p = 2$, we first proceed to introduce some results from linear algebra, and also the transfer (or transition) operator, all of which we shall rely on in the remaining part of this chapter.

4.3 Results from Linear Algebra

Let V denote an n-dimensional normed linear space, with basis $S = \{v_1, v_2, \ldots, v_n\} \subset V$, and suppose $L : V \to V$ is a linear operator. If $u = \sum_{k=1}^{n} c_k v_k$, we write $[u]_S = (c_1, c_2, \ldots, c_n)^t \in \mathbb{R}^n$ for the co-ordinate (column) vector of u with respect to the basis S. Then the $n \times n$ matrix M_L defined by

$$M_L = ([Lv_1]_S \mid [Lv_2]_S \mid \cdots \mid [Lv_n]_S) \tag{4.46}$$

gives

$$Lu = \sum_{k=1}^{n} (M_L[u]_S)_k \, v_k, \quad u \in V, \tag{4.47}$$

and is called the matrix of the operator L. The *spectrum* $\sigma(L)$ of L is defined by

$$\sigma(L) = \{\lambda \in \mathbb{C} : \lambda \ \text{ is an eigenvalue of } L\}, \tag{4.48}$$

in terms of which the *spectral radius* $\rho(L)$ of L is defined by

$$\rho(L) = \max_{\lambda \in \sigma(L)} |\lambda|. \tag{4.49}$$

For a square matrix M, the spectrum $\sigma(M)$ and the spectral radius $\rho(M)$ of M are defined, respectively, as in (4.48) and (4.49), with L replaced by M. We then have $\sigma(L) = \sigma(M_L)$, so that, in particular,

$$\rho(L) = \rho(M_L). \tag{4.50}$$

Observe from (4.49) that

$$\rho(\alpha L) = |\alpha|\rho(L). \tag{4.51}$$

Since L is a linear operator on a finite dimensional space V, we know from a standard result (see e.g. [22, pp. 92,96]) that L is a bounded operator in sense that

$$||L|| := \sup\left\{\frac{||Lu||}{||u||} : u \in V,\ u \neq 0\right\} = \sup\{||Lu|| : u \in V,\ ||u|| = 1\} < \infty.$$

Observe also that

$$||Lu|| \leq ||L||\,||u||, \quad u \in V. \tag{4.52}$$

We shall later rely on the following result from [23, pp. 617-618].

Theorem 4.6 *If* $\rho(L) < 1$, *then* $||L^k|| \to 0,\ k \to \infty$.

4.4 The Transfer Operator

Let the space of 2π-periodic continuous complex-valued functions on $\mathbb{R}$ be denoted by $C_{2\pi}(\mathbb{R})$; that is,

$$C_{2\pi}(\mathbb{R}) := \{f \in C(\mathbb{R}) : f = f(\cdot + 2\pi)\}. \tag{4.53}$$

For $n \in \mathbb{Z}_+$, define the $(2n+1)$-dimensional linear space $\mathcal{Q}_n$ of trigonometric polynomials by

$$\mathcal{Q}_n = \left\{ f \in M(\mathbb{R}) : f(\omega) = \sum_{k=-n}^{n} c_k e^{-i\omega k}, \quad \omega \in \mathbb{R}, \quad c_k \in \mathbb{C} \right\}. \tag{4.54}$$

Observe that $\mathcal{Q}_n$ is a subspace of $C_{2\pi}(\mathbb{R})$. For $N \in \mathbb{N}$, let the Laurent polynomials A and B be as in (4.25) and (4.26). Also, for the trigonometric polynomial R in (4.27), define

$$P(\omega) = |R(\omega)|^2, \quad \omega \in \mathbb{R}, \tag{4.55}$$

where

$$R(\omega) = B(e^{-i\omega}) = \sum_k b_k e^{-i\omega k}, \quad \omega \in \mathbb{R}. \tag{4.56}$$

Then the following result holds.

Proposition 4.7 *Let $N \in \mathbb{N}$, and suppose $a \in M_0(\mathbb{Z})$ is the mask sequence associated with the mask symbol A in (4.25). Also, denote by $\mu, \nu \in \mathbb{Z}$ the (unique) integers such that*

$$a_j = 0,\ j \notin \{\mu, \mu+1, \ldots, \nu\}, \tag{4.57}$$

with $a_\mu \neq 0,\ a_\nu \neq 0$. Then $P \in \mathcal{Q}_\tau$, where $\tau = \nu - \mu - N$, and $P(\omega) = P(-\omega),\ \omega \in \mathbb{R}$.

Proof. For $\omega \in \mathbb{R}$, we have, from (4.55), (4.56), (4.57), and (4.25),

$$\begin{aligned} P(\omega) = |R(\omega)|^2 &= R(\omega)\overline{R(\omega)} \\ &= \sum_j b_j e^{-i\omega j} \sum_k b_k e^{i\omega k} \\ &= \sum_j b_j \sum_k b_k e^{-i\omega(j-k)} \\ &= \sum_j b_j \sum_k b_{j-k} e^{-i\omega k} \\ &= \sum_k \left(\sum_j b_j b_{j-k} \right) e^{-i\omega k}, \end{aligned} \tag{4.58}$$

and thus,

$$P(\omega) = \sum_{k=-\tau}^{\tau} \left(\sum_{j=\mu}^{\nu-N} b_j b_{j-k} \right) e^{-i\omega k}, \quad \omega \in \mathbb{R},$$

which implies $P \in \mathcal{Q}_\tau$.

Also, for $\omega \in \mathbb{R}$, we deduce from (4.58) that

$$P(-\omega) \;=\; \sum_k \left(\sum_j b_j b_{j-k} \right) e^{i\omega k}$$

$$= \sum_k \left(\sum_j b_j b_{j+k} \right) e^{-i\omega k}$$

$$= \sum_k \left(\sum_j b_{j-k} b_j \right) e^{-i\omega k} = P(\omega).$$

■

Next, define the operator $T : \mathcal{Q}_\tau \to C_{2\pi}(\mathbb{R})$ by

$$Tf = P\left(\frac{\cdot}{2}\right) f\left(\frac{\cdot}{2}\right) + P\left(\frac{\cdot}{2}+\pi\right) f\left(\frac{\cdot}{2}+\pi\right), \quad f \in \mathcal{Q}_\tau. \tag{4.59}$$

The operator T as defined in (4.59) is called the *transfer operator* corresponding to P; it is also known as the *Perron-Frobenius operator* or the *transition operator* in the literature.

The following result can now be established.

Proposition 4.8

$$Tf \in \mathcal{Q}_\tau, \quad f \in \mathcal{Q}_\tau.$$

Proof. Let $\{c_{-\tau}, \dots, c_\tau\}$ and $\{d_{-\tau}, \dots, d_\tau\}$ be complex-valued sequences such that

$$f(\omega) = \sum_{k=-\tau}^{\tau} c_k e^{-i\omega k}, \quad \omega \in \mathbb{R}, \tag{4.60}$$

$$P(\omega) = \sum_{k=-\tau}^{\tau} d_k e^{-i\omega k}, \quad \omega \in \mathbb{R}, \tag{4.61}$$

and define $c_k = d_k = 0$, $|k| \geq \tau + 1$. Then, for $\omega \in \mathbb{R}$, we have, from (4.59), (4.60) and (4.61),

$$\begin{aligned}
(Tf)(\omega) &= \sum_k d_k e^{-i\frac{\omega}{2}k} \sum_j c_j e^{-i\frac{\omega}{2}j} + \sum_k d_k e^{-i(\frac{\omega}{2}+\pi)k} \sum_j c_j e^{-i(\frac{\omega}{2}+\pi)j} \\
&= \sum_k d_k \sum_j c_j e^{-i\frac{\omega}{2}(j+k)} + \sum_k d_k \sum_j c_j e^{-i(\frac{\omega}{2}+\pi)(j+k)} \\
&= \sum_k d_k \sum_j c_{j-k} e^{-i\frac{\omega}{2}j} + \sum_k d_k \sum_j c_{j-k} e^{-i(\frac{\omega}{2}+\pi)j} \\
&= \sum_j \left[\sum_k c_{j-k} d_k \right] e^{-i\frac{\omega}{2}j} + \sum_j (-1)^j \left[\sum_k c_{j-k} d_k \right] e^{-i\frac{\omega}{2}j} \\
&= \sum_j [1 + (-1)^j] \left[\sum_k c_{j-k} d_k \right] e^{-i\frac{\omega}{2}j}
\end{aligned}$$

$$= \quad 2\sum_j \left[\sum_k c_{2j-k} d_k\right] e^{-i\omega j}, \tag{4.62}$$

and thus,

$$(Tf)(\omega) = 2\sum_{j=-\tau}^{\tau} \left[\sum_{k=-\tau}^{\tau} c_{2j-k} d_k\right] e^{-i\omega j}, \quad \omega \in \mathbb{R},$$

which implies $Tf \in \mathcal{Q}_\tau$. ∎

Since, according to Proposition 4.8, the linear operator T maps the finite dimensional trigonometric polynomial space $\mathcal{Q}_\tau$ into itself, we proceed next to calculate the corresponding $(2\tau+1) \times (2\tau+1)$ matrix of the operator T.

Theorem 4.9 *The $(2\tau+1)\times(2\tau+1)$ matrix $M_T = [M_{j,k} : -\tau \leq j, k \leq \tau]$ of the transfer operator T is given by*

$$M_{j,k} = 2\sum_l b_l b_{l+j-2k}, \; -\tau \leq j, k \leq \tau. \tag{4.63}$$

Proof. Let f be as in (4.60), and suppose $\{q_{-\tau}, \ldots, q_\tau\}$ is the complex-valued sequence such that

$$(Tf)(\omega) = \sum_{k=-\tau}^{\tau} q_k e^{-i\omega k}, \quad \omega \in \mathbb{R}. \tag{4.64}$$

Then, setting $q_k = 0, \; |k| \geq \tau + 1$, we use (4.64), (4.62), and (4.58) to obtain, for $\omega \in \mathbb{R}$,

$$\begin{aligned}
\sum_k q_k e^{-i\omega k} &= 2\sum_k \left[\sum_j c_{2k-j} \left(\sum_l b_l b_{l-j}\right)\right] e^{-i\omega k} \\
&= \sum_k \left[2\sum_l b_l \sum_j b_{l-j} c_{2k-j}\right] e^{-i\omega k} \\
&= \sum_k \left[2\sum_l b_l \sum_j b_{l+j-2k} c_j\right] e^{-i\omega k} \\
&= \sum_k \left[\sum_j \left(2\sum_l b_l b_{l+j-2k}\right) c_j\right] e^{-i\omega k},
\end{aligned}$$

and thus,

$$q_k = \sum_{j=-\tau}^{\tau} \left(2\sum_l b_l b_{l+j-2k}\right) c_j, \; k = -\tau, \ldots, \tau,$$

thereby completing the proof of the theorem. ∎

The matrix M_T in (4.63) is called the *transfer matrix* (or *transition matrix*) associated with the operator T.

We shall need the following result.

Proposition 4.10 *Let the trigonometric polynomial P be defined as in (4.55). Then we have*

$$\int_{-2^r\pi}^{2^r\pi} \prod_{j=1}^{r} P\left(\frac{\omega}{2^j}\right) d\omega \leq 2\pi ||T^r||_\infty, \quad r \in \mathbb{N}. \tag{4.65}$$

Proof. Following [8, Lemma 7.1.10], we show that

$$\int_{-2^r\pi}^{2^r\pi} \left[\prod_{j=1}^{r} P\left(\frac{\omega}{2^j}\right)\right] f\left(\frac{\omega}{2^r}\right) d\omega = \int_{-\pi}^{\pi} (T^r f)(\omega)\, d\omega, \quad r \in \mathbb{N},\ f \in \mathcal{Q}_\tau. \tag{4.66}$$

The desired result (4.65) would therefore follow immediately by first choosing $f \in \mathcal{Q}_\tau$ in (4.66) as $f(t) = 1,\ t \in \mathbb{R}$, and then using the inequality (4.52).

Our proof of (4.66) is by induction. First, from the definition (4.59), together with the fact that Proposition 4.7 gives $P \in \mathcal{Q}_\tau \subset C_{2\pi}(\mathbb{R})$, we obtain, for $f \in \mathcal{Q}_\tau$,

$$\begin{aligned}
\int_{-\pi}^{\pi} (Tf)(\omega)\, d\omega &= \int_{-\pi}^{\pi} \left[P\left(\frac{\omega}{2}\right) f\left(\frac{\omega}{2}\right) + P\left(\frac{\omega}{2}+\pi\right) f\left(\frac{\omega}{2}+\pi\right)\right] d\omega \\
&= 2\int_{-\pi/2}^{\pi/2} [P(\omega)f(\omega) + P(\omega+\pi)f(\omega+\pi)]\, d\omega \\
&= 2\int_{-\pi/2}^{\pi/2} P(\omega)f(\omega)\, d\omega + 2\int_{\pi/2}^{3\pi/2} P(\omega)f(\omega)\, d\omega \\
&= 2\int_{-\pi/2}^{3\pi/2} P(\omega)f(\omega)\, d\omega \\
&= 2\int_{-\pi}^{\pi} P(\omega)f(\omega)\, d\omega \\
&= \int_{-2\pi}^{2\pi} P\left(\frac{\omega}{2}\right) f\left(\frac{\omega}{2}\right) d\omega,
\end{aligned}$$

which proves that (4.66) holds for $r = 1$. Suppose now that (4.66) holds for a fixed $r \in \mathbb{N}$. Then, once again from (4.59), and exploiting also the facts that $T^r f \in \mathcal{Q}_\tau \subset C_{2\pi}(\mathbb{R})$, from Proposition 4.8, and $P \in C_{2\pi}(\mathbb{R})$, we obtain

$$\begin{aligned}
\int_{-\pi}^{\pi} (T^{r+1} f)(\omega)\, d\omega &= \int_{-\pi}^{\pi} (T^r T f)(\omega)\, d\omega \\
&= \int_{-2^r\pi}^{2^r\pi} \left[\prod_{j=1}^{r} P\left(\frac{\omega}{2^j}\right)\right] (Tf)\left(\frac{\omega}{2^r}\right) d\omega
\end{aligned}$$

$$
\begin{aligned}
&= 2^{r+1} \int_{-\pi/2}^{\pi/2} \left[\prod_{j=1}^{r} P(2^{r+1-j}\omega) \right] (Tf)(2\omega)\, d\omega \\
&= 2^{r+1} \int_{-\pi/2}^{\pi/2} \left[\prod_{j=1}^{r} P(2^{r+1-j}\omega) \right] [P(\omega)f(\omega) + P(\omega+\pi)f(\omega+\pi)]\, d\omega \\
&= 2^{r+1} \left[\int_{-\pi/2}^{\pi/2} \left[\prod_{j=0}^{r} P(2^{j}\omega) \right] f(\omega)\, d\omega + \int_{\pi/2}^{3\pi/2} \left[\prod_{j=1}^{r} P(2^{j}(\omega-\pi)) \right] P(\omega)f(\omega)\, d\omega \right] \\
&= 2^{r+1} \int_{-\pi/2}^{3\pi/2} \left[\prod_{j=0}^{r} P(2^{j}\omega) \right] f(\omega)\, d\omega \\
&= 2^{r+1} \int_{-\pi}^{\pi} \left[\prod_{j=0}^{r} P(2^{j}\omega) \right] f(\omega)\, d\omega \\
&= \int_{-2^{r+1}\pi}^{2^{r+1}\pi} \left[\prod_{j=0}^{r} P\left(\frac{\omega}{2^{r+1-j}}\right) \right] f\left(\frac{\omega}{2^{r+1}}\right)\, d\omega \\
&= \int_{-2^{r+1}\pi}^{2^{r+1}\pi} \left[\prod_{j=1}^{r+1} P\left(\frac{\omega}{2^{j}}\right) \right] f\left(\frac{\omega}{2^{r+1}}\right)\, d\omega.
\end{aligned}
$$

so that (4.66) also holds with r replaced by $r+1$, and thereby completing our inductive proof of (4.66) . ■

4.5 Applying the Case $p = 2$ of Theorem 3.10

We proceed in this section to prove a Hölder regularity result for ϕ based on the case $p = 2$ of Theorem 3.10. Here, our approach relies on the calculation of the spectral radius of the associated matrix, called the transfer matrix, of the transfer operator defined in the previous section.

Our result is as follows.

Theorem 4.11 *Suppose $a \in M_0(\mathbb{Z})$ and $\phi \in C_0(\mathbb{R})$ are such that the conditions [A] and [B] of Section 4.1 are satisfied. For $N \in \mathbb{N}$, let the mask symbol A be given by (4.25) and (4.26). If the spectral radius $\rho(M_T)$ of the matrix M_T in Theorem 4.9 satisfies the inequality*

$$1 < \rho(M_T) < 4^{N-1/2}, \tag{4.67}$$

then

$$\phi \in C_0^{\gamma}(\mathbb{R}), \quad \gamma \in (0, N - 1/2 - \log_4 \rho(M_T)). \tag{4.68}$$

Proof. We shall prove that

$$\phi \in H^{2,q}(\mathbb{R}), \quad q \in (0, N - \log_4 \rho(T)), \tag{4.69}$$

which, together with the case $p = 2$ of Theorem 3.10 and (4.50), will then yield the desired result (4.68).

To this end, we choose $\varepsilon > 0$, and define the linear operator $L_\varepsilon : \mathcal{Q}_\tau \to \mathcal{Q}_\tau$ by $L_\varepsilon = \frac{T}{\rho(T)+\varepsilon}$, with T denoting the transfer operator defined by (4.59). Then (4.51) implies $\rho(L_\varepsilon) = \left[\frac{1}{\rho(T)+\varepsilon}\right]\rho(T) < 1$. It therefore follows from Theorem 4.6 that

$$\frac{||T^r||_\infty}{(\rho(T)+\varepsilon)^r} = ||L_\varepsilon^r||_\infty \to 0,\ r \to \infty.$$

Hence there exists a number $K > 0$ such that

$$\frac{||T^r||_\infty}{(\rho(T)+\varepsilon)^r} = ||L_\varepsilon^r||_\infty \leq K, \quad r \in \mathbb{N},$$

or equivalently,

$$||T^r||_\infty \leq K(\rho(T)+\varepsilon)^r, \quad r \in \mathbb{N}, \tag{4.70}$$

with K independent of r.

Since the conditions *[A]* and *[B]* of Section 4.1 are satisfied, we may appeal to Theorem 4.3 to deduce that (4.13) holds. But then (4.28) and (4.13) yield (4.33), which, together with the second line of (4.24), gives

$$|\hat{\phi}(\omega)|^2 \leq \frac{8^N}{(1+\omega^2)^N} \prod_{k=1}^{\infty} \left|R\left(\frac{\omega}{2^k}\right)\right|^2 |\hat{\phi}(0)|^2, \quad \omega \in \mathbb{R}. \tag{4.71}$$

Let r be fixed, and suppose $\omega \in \mathbb{R}\backslash\{0\}$ is such that (4.38) is satisfied. Then we can argue as in the steps (4.36) to (4.39) and (4.41) to (4.43), with $l = 1$ in the proof of Theorem 4.5, to obtain

$$\prod_{k=1}^{\infty} \left|R\left(\frac{\omega}{2^k}\right)\right|^2 \leq e^{4c'} \prod_{k=1}^{r} \left|R\left(\frac{\omega}{2^k}\right)\right|^2. \tag{4.72}$$

where $c' = \sum\limits_j |jb_j|$. Noting that (4.67) implies $\log_4 \rho(T) = \log_4 \rho(M_T) < N - 1/2 < N$, we now let $q \in (0, N - \log_4 \rho(T))$, so that also $q \in (0, N)$, since (4.50) and (4.67) imply that $\log_4 \rho(T) = \log_4 \rho(M_T) > 0$. It follows that

$$\frac{1}{(1+\omega^2)^{N-q}} < \frac{1}{\omega^{2N-2q}} \leq \frac{1}{4^{r(N-q)}}, \tag{4.73}$$

from (4.38). By inserting (4.72) and (4.73) into (4.71), we obtain the inequality

$$(1+\omega^2)^q |\hat{\phi}(\omega)|^2 < \frac{8^N e^{4c'} |\hat{\phi}(0)|^2}{(4^{N-q})^r} \prod_{k=1}^{r} \left| R\left(\frac{\omega}{2^k}\right)\right|^2, \quad r \in \mathbb{N}. \tag{4.74}$$

Then, with the definition $C = 8^N e^{4c'} |\hat{\phi}(0)|^2$, we have, from (4.74) and (4.38), and for $r \in \mathbb{N}$,

$$\begin{aligned} \int_{2^r}^{2^{r+1}} (1+\omega^2)^q |\hat{\phi}(\omega)|^2 \, d\omega \; &< \; \frac{C}{(4^{N-q})^r} \int_{2^r}^{2^{r+1}} \prod_{k=1}^{r} \left| R\left(\frac{\omega}{2^k}\right)\right|^2 \, d\omega \\ &< \; \frac{C}{(4^{N-q})^r} \int_{-2^r\pi}^{2^r\pi} \prod_{k=1}^{r} \left| R\left(\frac{\omega}{2^k}\right)\right|^2 \, d\omega, \end{aligned} \tag{4.75}$$

having used also the fact that $\pi > 2$, and similarly

$$\begin{aligned} \int_{-2^{r+1}}^{-2^r} (1+\omega^2)^q |\hat{\phi}(\omega)|^2 \, d\omega \; &< \; \frac{C}{(4^{N-q})^r} \int_{-2^{r+1}}^{-2^r} \prod_{k=1}^{r} \left| R\left(\frac{\omega}{2^k}\right)\right|^2 \, d\omega \\ &< \; \frac{C}{(4^{N-q})^r} \int_{-2^r\pi}^{2^r\pi} \prod_{k=1}^{r} \left| R\left(\frac{\omega}{2^k}\right)\right|^2 \, d\omega. \end{aligned} \tag{4.76}$$

Next, we combine the bound (4.65), in Proposition 4.10, and (4.70) to deduce that

$$\int_{-2^r\pi}^{2^r\pi} \prod_{k=1}^{r} \left| R\left(\frac{\omega}{2^k}\right)\right|^2 \, d\omega \leq 2\pi K (\rho(T) + \varepsilon)^r, \quad r \in \mathbb{N}, \tag{4.77}$$

which, together with (4.75) and (4.76), yields

$$\int_{2^r}^{2^{r+1}} (1+\omega^2)^q |\hat{\phi}(\omega)|^2 \, d\omega < 2\pi C K \left(\frac{\rho(T)+\varepsilon}{4^{N-q}}\right)^r, \quad r \in \mathbb{N}, \tag{4.78}$$

and

$$\int_{-2^{r+1}}^{-2^r} (1+\omega^2)^q |\hat{\phi}(\omega)|^2 \, d\omega < 2\pi C K \left(\frac{\rho(T)+\varepsilon}{4^{N-q}}\right)^r, \quad r \in \mathbb{N}. \tag{4.79}$$

Since $q \in (0, N - \log_4 \rho(T))$, we have $\log_4 \rho(T) < N - q$, and thus $\rho(T) < 4^{N-q}$. It follows that if we choose $\varepsilon = \frac{1}{2}[4^{N-q} - \rho(T)]$, we get $\varepsilon > 0$ and

$$0 < \beta := \frac{\rho(T) + \varepsilon}{4^{N-q}} = \frac{1}{2}\left(1 + \frac{\rho(T)}{4^{N-q}}\right) < 1. \tag{4.80}$$

Hence, from (4.78) and (4.80), we have, for $p \in \mathbb{N}$,

$$\begin{aligned}\int_{2}^{2^{p+1}} (1+\omega^2)^q |\hat{\phi}(\omega)|^2 \, d\omega &= \sum_{r=1}^{p} \int_{2^r}^{2^{r+1}} (1+\omega^2)^q |\hat{\phi}(\omega)|^2 \, d\omega \\ &< 2\pi C K \sum_{r=0}^{p} \beta^r \\ &< 2\pi C K \sum_{r=0}^{\infty} \beta^r = \frac{2\pi C K}{1-\beta},\end{aligned}$$

so that

$$\int_{2}^{\infty} (1+\omega^2)^q |\hat{\phi}(\omega)|^2 \, d\omega = \lim_{p\to\infty} \int_{2}^{2^{p+1}} (1+\omega^2)^q |\hat{\phi}(\omega)|^2 \, d\omega \leq \frac{2\pi C K}{1-\beta} < \infty. \tag{4.81}$$

Similarly, from (4.79) and (4.80), we get

$$\int_{-\infty}^{-2} (1+\omega^2)^q |\hat{\phi}(\omega)|^2 \, d\omega = \lim_{p\to\infty} \int_{-2^{p+1}}^{-2} (1+\omega^2)^q |\hat{\phi}(\omega)|^2 \, d\omega \leq \frac{2\pi C K}{1-\beta} < \infty. \tag{4.82}$$

Furthermore, since $\hat{\phi} \in C_u(\mathbb{R})$, from Theorem 3.1(i), we also have

$$\int_{-2}^{2} (1+\omega^2)^q |\hat{\phi}(\omega)|^2 \, d\omega < \infty. \tag{4.83}$$

The desired result (4.69) then follows from (3.27), (4.81), (4.82), and (4.83). ■

Remarks

(i) The use of transfer operator to study the smoothness of a refinable function appears in [5], [8], [17], [21], and [32]. Our proof of Theorem 4.11 above builds rigorously on the ideas from these references.

(ii) Note that one could define the transfer operator T by means of the trigonometric polynomial $|Q(\omega)|^2$ (instead of $|R(\omega)|^2$) so that the associated transfer matrix M_T is then determined by mask sequence $a \in M_0(\mathbb{Z})$. This option would, however, lead to the less efficient numerical procedure of having to compute the spectral radius of a larger matrix.

(iii) The equation (4.33) plays an essential role in the proof of both Theorems 4.5 and 4.11. It is therefore clear that our determination of the smoothness of ϕ is based on our ability to control the growth of the infinite product $\prod_{k=1}^{\infty} |R(2^{-k}\omega)|^p$ as $|\omega| \to \infty$, for $p = \{1, 2\}$.

(iv) Note that, whereas we proved Theorem 4.5 by using the case $p = 1$ of Theorem 3.11, our proof of Theorem 4.11 appealed directly to the fundamental embedding result for $p = 2$ in Theorem 3.10. We have not investigated the possibility of proving Theorem 4.11 by using the case $p = 2$ of Theorem 3.11.

Chapter 5

Application to Specific Refinable Functions

In this chapter, we apply the results of Theorems 4.5 and 4.11 to the Daubechies and Dubuc-Deslauriers refinable functions, as well as to a one-parameter family of refinable functions. Moreover, we compare our Fourier-based regularity results with a subdivision-based regularity result derived in [28].

5.1 Daubechies Refinable Functions

Recall (see e.g [3], [7], [8]) that the Daubechies refinable function $\phi_N^D \in C_0(\mathbb{R})$ of order $N \geq 2$ satisfies the following properties:

(i) $\phi_N^D(t) = 0,\ t \notin (0, 2N-1)$;

(ii) $\phi_N^D = \sum_j a_{N,j}\phi_N^D(2\cdot -j) = \sum_{j=0}^{2N-1} a_{N,j}\phi_N^D(2\cdot -j)$,
where, in the notation

$$A_N^D(z) = \sum_j a_{N,j} z^j,\ z \in \mathbb{C},$$

we have that (4.25) – (4.29) are satisfied with $A = A_N^D, B = B_N^D, Q = Q_N^D,$ and $R = R_N^D$;

(iii) $\left\langle \phi_N^D(\cdot - j), \phi_N^D(\cdot - k)\right\rangle = \delta_{j,k}, \quad j,k \in \mathbb{Z}$,
i.e., ϕ_N^D is an orthonormal refinable function.

Moreover,

$$|R_N^D(\omega)|^2 = f_N\left(\sin^2 \frac{\omega}{2}\right),\ \omega \in \mathbb{R}, \tag{5.1}$$

where the polynomial f_N of degree $N-1$, as given explicitly by the formula

$$f_N(z) = \sum_{k=0}^{N-1} \binom{N-1+k}{k} z^k, \quad z \in \mathbb{C}, \tag{5.2}$$

is the unique polynomial in π_{N-1} satisfying the Bezout identity

$$z^N f_N(1-z) + (1-z)^N f_N(z) = 1, \quad z \in \mathbb{C}. \tag{5.3}$$

For the cases $N = 2$ and $N = 3$, the graphs of ϕ_N^D are shown in Figures 5.1 and 5.2, and the explicit formulas for R_2^D and R_3^D are given, for $\omega \in \mathbb{R}$, by

$$R_2^D(\omega) = \frac{1}{2}[(1+\sqrt{3}) + (1-\sqrt{3})e^{-i\omega}], \tag{5.4}$$

$$R_3^D(\omega) = \frac{1}{4}\left[1+\sqrt{10}+\sqrt{5+2\sqrt{10}} + 2(1-\sqrt{10})e^{-i\omega} + (1+\sqrt{10}-\sqrt{5+2\sqrt{10}})e^{-2i\omega}\right]. \tag{5.5}$$

Using (5.4), (5.5), (4.25), (4.28) and (4.5), we obtain the mask sequences

$$\left.\begin{aligned} a_{2,0} &= \frac{1+\sqrt{3}}{4}, \\ a_{2,1} &= \frac{3+\sqrt{3}}{4}, \\ a_{2,2} &= \frac{3-\sqrt{3}}{4}, \\ a_{2,3} &= \frac{1-\sqrt{3}}{4}, \\ a_{2,k} &= 0, \quad k \notin \{0,1,2,3\}; \end{aligned}\right\} \tag{5.6}$$

and

$$\left.\begin{aligned} a_{3,0} &= \frac{1+\sqrt{10}+\sqrt{5+2\sqrt{10}}}{16}, \\ a_{3,1} &= \frac{5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}}{16}, \\ a_{3,2} &= \frac{10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}}}{16}, \\ a_{3,3} &= \frac{10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}}}{16}, \\ a_{3,4} &= \frac{5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}}{16}, \\ a_{3,5} &= \frac{1+\sqrt{10}-\sqrt{5+2\sqrt{10}}}{16}, \\ a_{3,k} &= 0, \quad k \notin \{0,1,2,3,4,5\}. \end{aligned}\right\} \tag{5.7}$$

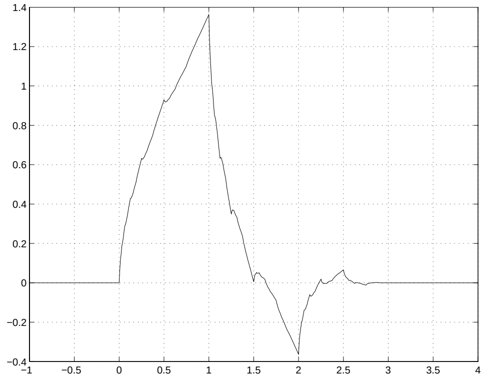

Figure 5.1: *The refinable function* ϕ_2^D.

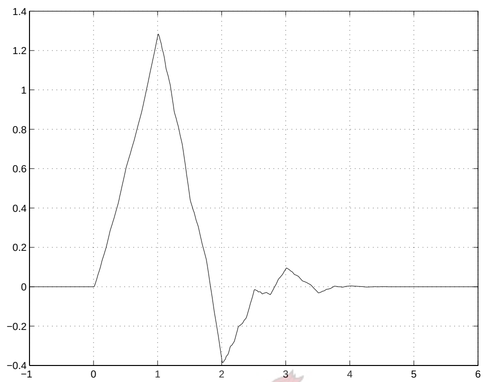

Figure 5.2: *The refinable function* ϕ_3^D.

Note that $a_{2,3} < 0$ and $a_{3,3}, a_{3,4} < 0$, so that Theorem 2.4 is not applicable in these cases. In fact, according to [8, p. 195], the positivity condition (1.33), with $n = 2N-1$, does not hold for the Daubechies mask sequence a_N for $2 \leq N \leq 10$. Observe also in particular that the conditions *[A]* and *[B]* of Section 4.1 are satisfied by the mask sequence $a = a_N \in M_0(\mathbb{Z})$, and by the refinable function $\phi = \phi_N^D \in C_0(\mathbb{R})$, so that we may appeal to Theorems 4.5 and 4.11 to investigate the regularity of ϕ_N^D.

5.1.1 Applying the case $l = 1$ of Theorem 4.5

We first investigate the regularity of ϕ_N^D by means of the case $l = 1$ of Theorem 4.5. To this end, we observe from (5.1) and (5.2) that

$$\sup_{\omega \in \mathbb{R}} |R_N^D(\omega)| = \sqrt{f_N(\sin^2 \frac{\pi}{2})} = \sqrt{\sum_{k=0}^{N-1} \binom{N-1+k}{k}}. \tag{5.8}$$

Since, moreover,

$$\begin{aligned}\sum_{k=0}^{N}\binom{N+k}{k} &= \binom{2N}{N}+\sum_{k=0}^{N-1}\binom{N+k}{k}\\ &= \binom{2N}{N}+\sum_{k=0}^{N-1}\left[\binom{N+k-1}{k}+\binom{N+k-1}{k-1}\right]\\ &= \binom{2N}{N}+\sum_{k=0}^{N-1}\binom{N+k-1}{k}+\left[\sum_{k=0}^{N}\binom{N+k}{k}-\binom{2N}{N}-\binom{2N-1}{N-1}\right],\end{aligned}$$

we get the identity

$$\sum_{k=0}^{N-1}\binom{N+k-1}{k}=\binom{2N-1}{N-1},$$

which, together with (5.8), yields

$$\sup_{\omega\in\mathbb{R}}|R_N(\omega)|=\sqrt{\binom{2N-1}{N-1}},$$

so that, in the notation of Theorem 4.5, we have

$$M_1=\sqrt{\binom{2N-1}{N-1}}. \tag{5.9}$$

It follows from the case $l=1$ of Theorem 4.5, together with (5.9), that if the inequality

$$\sqrt{\binom{2N-1}{N-1}}<2^{N-1} \tag{5.10}$$

is satisfied, then

$$\phi_N^D\in C_0^\gamma(\mathbb{R}),\quad \gamma\in\left(0,N-1-\frac{1}{2}\log_2\binom{2N-1}{N-1}\right). \tag{5.11}$$

Hence we need to investigate the function

$$g(N)=N-1-\frac{1}{2}\log_2\binom{2N-1}{N-1},\quad N=2,3,\ldots, \tag{5.12}$$

or, equivalently,

$$g(N)=N-1-\frac{\ln\binom{2N-1}{N-1}}{\ln 4},\quad N=2,3,\ldots$$

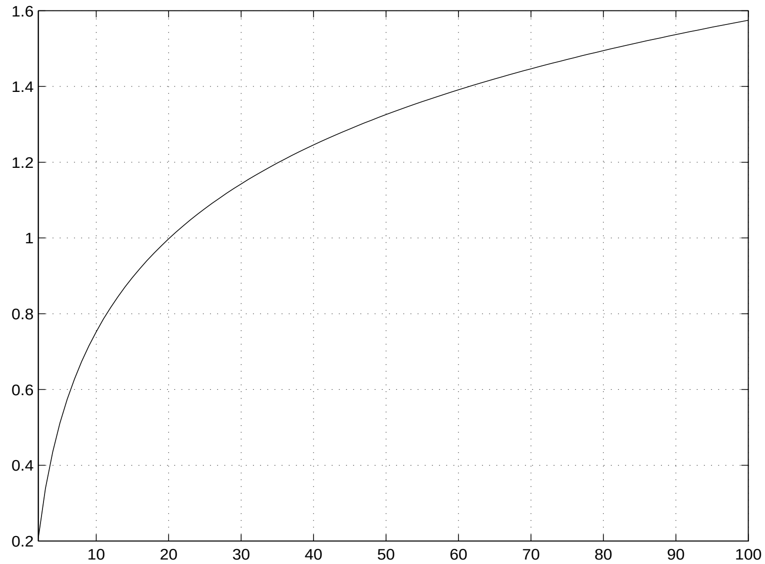

Figure 5.3: *The plot of the function g, as defined by (5.12).*

The following two properties are obtained.

Proposition 5.1

$$g(N+1) > g(N), \quad N \in \mathbb{N}, \tag{5.13}$$

$$g(N+1) - g(N) \to 0, \quad N \to \infty. \tag{5.14}$$

Proof. We see from (5.12), that, for $N \in \mathbb{N}$,

$$g(N+1) - g(N) = 1 + \frac{1}{2}\log_2\left(\frac{\binom{2N-1}{N-1}}{\binom{2N+1}{N}}\right) = \frac{1}{2}\left[1 + \log_2\left(\frac{N+1}{2N+1}\right)\right].$$

Now the function $h : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$h(x) = \frac{x+1}{2x+1}, \quad x \geq 0,$$

satisfies $h(0) = 1$,

$$h'(x) = -\frac{1}{(2x+1)^2} < 0, \quad x \geq 0,$$

and

$$h(x) = \frac{1+1/x}{2+1/x} \to \frac{1}{2}, \quad x \to \infty,$$

from which we then deduce that (5.13) and (5.14) hold. ■

Since $g(2) = 1 - \log_2\sqrt{3} > 0$, it is clear from (5.13) that $g(N) > 0$, $N \geq 2$; that is,

Table 5.1: $\phi_N^D \in C^\gamma(\mathbb{R})$, *as obtained from (5.11)*

N	2	3	4	10	20	21	100
$\gamma <$	0.2075	0.3390	0.4354	0.7524	0.9979	1.0152	1.5747

the inequality (5.10) holds for all $N \geq 2$, so that we may indeed appeal to Theorem 4.5, with $l = 1$. The graph of g is shown in Figure 5.3, whereas Table 5.1 lists some minimum regularity results for ϕ_N^D, as obtained from (5.11). It follows from Table 5.1, together with (5.13), that $\phi_N^D \in C_0^1(\mathbb{R}),\ N \geq 21$.

5.1.2 Applying the case $l = 2$ of Theorem 4.5

Next, we investigate the regularity of ϕ_N^D by means of the case $l = 2$ of Theorem 4.5. First, observe from (5.1) that

$$|R_N^D(\omega)R_N^D(\omega/2)|^2 = f_N\left(\sin^2\frac{\omega}{2}\right) f_N\left(\sin^2\frac{\omega}{4}\right), \quad \omega \in \mathbb{R},$$

and thus, from (4.30),

$$(M_2)^2 = \sup_{\omega\in\mathbb{R}}\left[f_N\left(\sin^2\frac{\omega}{2}\right) f_N\left(\sin^2\frac{\omega}{4}\right)\right]. \tag{5.17}$$

Since also

$$\sin^2\frac{\omega}{2} = \left(2\sin\frac{\omega}{4}\cos\frac{\omega}{4}\right)^2 = 4\sin^2\frac{\omega}{4}\left(1-\sin^2\frac{\omega}{4}\right), \quad \omega \in \mathbb{R}, \tag{5.18}$$

it follows from (5.17) and (5.18) that

$$(M_2)^2 = \max_{0\leq x\leq 1}\left[f_N(x)f_N(4x(1-x))\right]. \tag{5.19}$$

We shall need the following result from [8, Lemma 7.1.4].

Proposition 5.2 *The polynomial f_N, as defined by (5.2), satisfies the following properties:*

(i) $y^{-N+1}f_N(y) \leq x^{-N+1}f_N(x), \quad 0 \leq x \leq y;$ (5.20)

(ii) $$f_N(x) \leq \begin{cases} 2^{N-1}, & 0 \leq x \leq \frac{1}{2}, \\ 2^{2N-2}x^{N-1}, & \frac{1}{2} \leq x \leq 1. \end{cases} \tag{5.21}$$

Proof. (*i*) To prove (5.20), suppose $0 \leq x \leq y$. Then, $x^{N-1-k} \leq y^{N-1-k}$, $k = 0, \ldots N-1$, and thus,

$$\begin{aligned} y^{-N+1} f_N(y) &= \sum_{k=0}^{N-1} \binom{N-1+k}{k} y^{-(N-1-k)} \\ &\leq \sum_{k=0}^{N-1} \binom{N-1+k}{k} x^{-(N-1-k)} = x^{-N+1} f_N(x). \end{aligned}$$

(*ii*) Next, to prove (5.21), observe that since f_N satisfies the Bezout identity (5.3), we find, by setting $z = \frac{1}{2}$ in (5.3), that $f_N(\frac{1}{2}) = 2^{N-1}$. Thus, if $0 \leq x \leq \frac{1}{2}$, then we get $f_N(x) \leq f_N(\frac{1}{2}) = 2^{N-1}$, since (5.2) shows that f_N is increasing for $x \geq 0$, and thereby proving the first line of (5.21). If $x \geq \frac{1}{2}$, then it follows from (5.20) that $f_N(x) \leq x^{N-1} 2^{N-1} f_N(\frac{1}{2})$, thereby completing the proof of (5.21). ■

We proceed to bound the positive number M_2 by using Proposition 5.2 (ii). First, observe that the polynomial h defined by $h(x) = 4x(1-x)$, $x \in \mathbb{R}$, satisfies

$$0 \leq h(x) \leq \frac{1}{2}, \quad x \in \left[0, \frac{2-\sqrt{2}}{4}\right] \cup \left[\frac{2+\sqrt{2}}{4}, 1\right], \tag{5.22}$$

and

$$\frac{1}{2} \leq h(x) \leq 1, \quad x \in \left[\frac{2-\sqrt{2}}{4}, \frac{2+\sqrt{2}}{4}\right], \tag{5.23}$$

where also

$$0 < \frac{2-\sqrt{2}}{4} < \frac{1}{2} < \frac{2+\sqrt{2}}{4} < 1. \tag{5.24}$$

It follows from (5.21), (5.22), (5.23), and (5.24) that

$$f_N(x) f_N(4x(1-x)) \leq \begin{cases} 2^{2N-2}, & 0 \leq x \leq \frac{2-\sqrt{2}}{4}, \\ 2^{5N-5}[x(1-x)]^{N-1}, & \frac{2-\sqrt{2}}{4} \leq x \leq \frac{1}{2}, \\ 2^{6N-6}[x^2(1-x)]^{N-1}, & \frac{1}{2} \leq x \leq \frac{2+\sqrt{2}}{4}, \\ 2^{3N-3} x^{N-1}, & \frac{2+\sqrt{2}}{4} \leq x \leq 1. \end{cases} \tag{5.25}$$

Thus, since for the polynomials p and q defined by $p(x) = [x(1-x)]^{N-1}$, $x \in \mathbb{R}$, and $q(x) = [x^2(1-x)]^{N-1}$, $x \in \mathbb{R}$, we have

$$\max_{\frac{2-\sqrt{2}}{4} \leq x \leq \frac{1}{2}} p(x) = p(1/2) = 2^{-2N+2}, \tag{5.26}$$

and

$$\max_{\frac{1}{2}\leq x\leq\frac{2+\sqrt{2}}{4}} q(x) = q(2/3) = \frac{2^{2N-2}}{3^{3N-3}}, \tag{5.27}$$

we can now substitute (5.26) and (5.27) into (5.25) to obtain

$$f_N(x)f_N(4x(1-x)) \leq \begin{cases} 2^{2N-2}, & 0 \leq x \leq \frac{2-\sqrt{2}}{4}, \\ 2^{3N-3}, & \frac{2-\sqrt{2}}{4} \leq x \leq \frac{1}{2}, \\ \frac{2^{8N-8}}{3^{3N-3}}, & \frac{1}{2} \leq x \leq \frac{2+\sqrt{2}}{4}, \\ 2^{3N-3}, & \frac{2+\sqrt{2}}{4} \leq x \leq 1. \end{cases}$$

But,

$$\frac{2^{8N-8}}{3^{3N-3}} > 2^{3N-3} > 2^{2N-2}, \quad N \geq 2,$$

and thus,

$$f_N(x)f_N(4x(1-x)) \leq \frac{2^{8N-8}}{3^{3N-3}}, \quad 0 \leq x \leq 1, \tag{5.28}$$

which, together with (5.19), yields the bound

$$M_2 \leq \frac{2^{4(N-1)}}{3^{3(N-1)/2}}. \tag{5.29}$$

It follows from (5.29) that the choice $l = 2$ in Theorem 4.5 will be applicable if the inequality

$$\frac{2^{4(N-1)}}{3^{3(N-1)/2}} < 2^{2(N-1)} \tag{5.30}$$

holds, for then the inequality (4.30) does indeed hold for $l = 2$.

But, (5.30) is equivalent to the inequality

$$4^{N-1} < (3^{3/2})^{N-1}, \quad N \geq 2,$$

which holds, since $3^{3/2} \simeq 5.2 > 4$. Hence we may use the case $l = 2$ of Theorem 4.5 to deduce that

$$\phi_N^D \in C_0^\gamma(\mathbb{R}), \quad \gamma \in \left(0, N-1-\frac{1}{2}\log_2\left[2^{4(N-1)}3^{-3(N-1)/2}\right]\right), \tag{5.31}$$

Table 5.2: $\phi_N^D \in C^\gamma(\mathbb{R})$, *as obtained from (5.32)*

N	2	3	4	5	6	7	8	9	10
$\gamma <$	0.189	0.378	0.566	0.755	0.943	1.132	1.321	1.510	1.698

having also used (5.29). Now observe in (5.31) that

$$\begin{aligned}\frac{1}{2}\log_2\left[2^{4(N-1)}3^{-3(N-1)/2}\right] &= \frac{1}{2}\left[4(N-1)-\frac{3}{2}(N-1)\log_2 3\right]\\ &= (N-1)\left[2-\frac{3}{4}\log_2 3\right]\end{aligned}$$

so that (5.31) can be rewritten as

$$\phi_N^D \in C_0^\gamma(\mathbb{R}), \quad \gamma \in \left(0, \left(\frac{3}{4}\log_2 3 - 1\right)(N-1)\right). \tag{5.32}$$

In (5.32), we have $\frac{3}{4}\log_2 3 - 1 \simeq 0.1887$.

We see from Table 5.2 that $\phi_N^D \in C_0^1(\mathbb{R})$ for $N \geq 7$, which is a significant improvement on the regularity results for ϕ_N^D shown in Table 5.1. In fact, it was shown in [33] that $\lim\limits_{N\to\infty} \gamma_N/N = 1 - \log_4 3 \simeq 0.2075$, where γ_N is the optimal Hölder regularity of ϕ_N^D.

5.1.3 Applying Theorem 4.11 for $2 \leq N \leq 10$

Next, we apply the result of Theorem 4.11 to the Daubechies refinable functions of order $2 \leq N \leq 10$. Since, in Proposition 4.7, we have here $\mu = 0,\ \nu = 2N-1$, and therefore $\tau = N-1$, it follows that the corresponding transfer matrix M_T is a $(2N-1)\times(2N-1)$ matrix. Using (4.56), (4.63), (5.4), and (5.5), we obtain

$$M_T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad N = 2, \tag{5.33}$$

Table 5.3: $\phi_N^D \in C^\gamma(\mathbb{R})$, *as obtained from (5.35)*

N	2	3	4	5	6	7	8	9	10
$\gamma <$	0.500	0.915	1.275	1.596	1.888	2.158	2.415	2.661	2.902

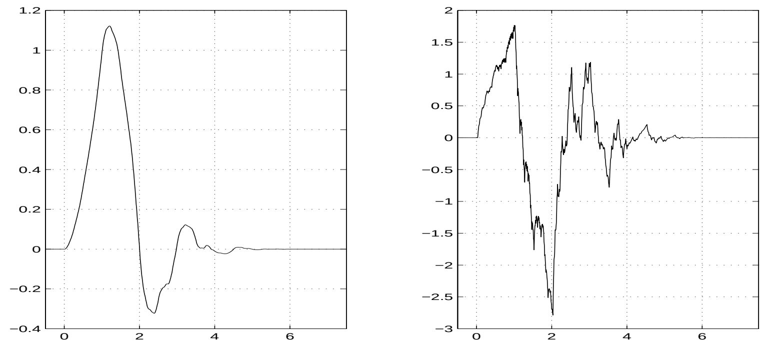

Figure 5.4: *The refinable function ϕ_4^D (left) and its derivative (right)*

and

$$M_T = \begin{pmatrix} 0.75 & 9.5 & 0.75 & 0 & 0 \\ 0 & -4.5 & -4.5 & 0 & 0 \\ 0 & 0.75 & 9.5 & 0.75 & 0 \\ 0 & 0 & -4.5 & -4.5 & 0 \\ 0 & 0 & 0.75 & 9.5 & 0.75 \end{pmatrix}, \quad N = 3, \tag{5.34}$$

etc, where, due to their $(2N-1)\times(2N-1)$ size, we do not display the matrices M_T for $4 \leq N \leq 10$ here. Our numerical results show that the condition (4.67) holds for $2 \leq N \leq 10$, so that, from Theorem 4.11, we have

$$\phi_N^D \in C_0^\gamma(\mathbb{R}), \quad \gamma \in (0, N - 1/2 - \log_4 \rho(M_T)), \quad 2 \leq N \leq 10. \tag{5.35}$$

From Table 5.3, we see that the regularity results for ϕ_N^D as obtained with Theorem 4.11 improve on those in Tables 5.1 and 5.2, as obtained from respectively the cases $l = 1$ and $l = 2$ of Theorem 4.5. Observe in particular from Table 5.3 that $\phi_N^D \in C_0^1(\mathbb{R}),\ 4 \leq N \leq 10$. The plots of ϕ_4^D and its derivative are shown in Figure 5.4, and numerically illustrate the fact that $\phi_4^D \in C_0^1(\mathbb{R})$.

5.2 Dubuc-Deslauriers (DD) Refinable Functions

As was proved in [24] (see also [18, Theorem 4.2]), the Dubuc-Deslauriers (DD) refinable function $\phi_N^{DD} \in C_0(\mathbb{R})$ of order $N \geq 1$ satisfies the following properties:

(i) $\phi_N^{DD}(t) = 0,\ t \notin (-2N+1, 2N-1)$;

(ii) $\phi_N^{DD} = \sum_j d_{N,j}\phi_N^{DD}(2\cdot -j) = \sum_{j=-2N+1}^{2N-1} d_{N,j}\phi_N^{DD}(2\cdot -j)$;

(iii) $\phi_N^{DD}(j) = \delta_j, \quad j \in \mathbb{Z}$;

(iv) $\sum_j \phi_N^{DD}(t-j) = 1, \quad t \in \mathbb{R}$.

The associated DD symbol $A = D_N$ is defined by

$$D_N(z) = \sum_j d_{N,j} z^j, \quad z \in \mathbb{C} \setminus \{0\}, \tag{5.36}$$

where, for a given $N \in \mathbb{N}$, the corresponding DD mask $a = d_N = \{d_{N,j} : \ j \in \mathbb{Z}\}$ is given by

$$\left.\begin{array}{lcl} d_{N,2j} & = & \delta_j, \quad j \in \mathbb{Z}, \\ d_{N,1-2j} & = & \dfrac{N}{2^{4N-3}}\dbinom{2N-1}{N}\dfrac{(-1)^{j+1}}{2j-1}\dbinom{2N-1}{N-j}, \quad j \in \{-N+1, \cdots, N\}, \\ d_{N,j} & = & 0, \quad |j| \geq 2N. \end{array}\right\} \tag{5.37}$$

For example, using (5.36) and (5.37), we obtain, for $z \in \mathbb{C} \setminus \{0\}$, the formulas

$$D_1(z) = \frac{1}{2}(z^{-1} + 2 + z); \tag{5.38}$$

$$D_2(z) = \frac{1}{16}(-z^{-3} + 9z^{-1} + 16 + 9z - z^3); \tag{5.39}$$

$$D_3(z) = \frac{1}{256}(3z^{-5} - 25z^{-3} + 150z^{-1} + 256 + 150z - 25z^3 + 3z^5). \tag{5.40}$$

Note from (5.37) that the mask coefficients $\{d_{N,j} : \ j \in \mathbb{Z}\}$ are symmetric in the sense that $d_{N,j} = d_{N,-j}$. Moreover, according to [18, Eqn (5.28)], the DD mask coefficients $\{d_{N,1-2j} : j = -N+1, \ldots, 0\}$ alternate in sign; that is, the regularity results for positive masks are not applicable here.

It is shown in [31] that the Daubechies trigonometric mask symbol $Q = Q_N^D$ and the DD trigonometric mask symbol $Q = Q_N^{DD}$ are related by

$$|Q_N^{DD}(\omega)| = |Q_N^D(\omega)|^2, \quad \omega \in \mathbb{R}. \tag{5.41}$$

Also, as shown in [18], $Q = Q_N^{DD}$ satisfies (4.28) with N replaced by $2N$. In particular, for $N \in \{2, 3\}$, we can use (5.39), (5.40), (4.5) and (4.28) to obtain the formulas

$$Q_2^{DD}(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^4 \left(\frac{-1+4e^{-i\omega}-e^{-2i\omega}}{2e^{-3i\omega}}\right), \quad \omega \in \mathbb{R}, \tag{5.42}$$

and

$$Q_3^{DD}(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^6 \left(\frac{3-18e^{-i\omega}+38e^{-2i\omega}-18e^{-3i\omega}+3e^{-4i\omega}}{8e^{-5i\omega}}\right), \quad \omega \in \mathbb{R}. \tag{5.43}$$

Observe in particular that the conditions *[A]* and *[B]* of Section 4.1 are satisfied by the mask sequence $a = d_N \in M_0(\mathbb{Z})$ and by the refinable function $\phi = \phi_N^{DD} \in C_0(\mathbb{R})$, so that we may appeal to Theorems 4.5 and 4.11 to investigate the regularity of ϕ_N^{DD}.

5.2.1 Applying the case $l = 1$ of Theorem 4.5

By virtue of (5.41), we find that the smoothness analysis of ϕ_N^{DD} based on Theorem 4.5 is quite similar to that of ϕ_N^D. To apply Theorem 4.5 for $l = 1$, we first note, from (5.9), and since (5.41) implies $|R_N^{DD}(\omega)| = |R_N^D(\omega)|^2$, $\omega \in \mathbb{R}$, we have in (4.30), for the case $\phi = \phi_N^{DD}$, that

$$M_1 = \binom{2N-1}{N-1}, \quad N \in \mathbb{N}. \tag{5.44}$$

It follows from the case $l = 1$ of Theorem 4.5, together with (5.44), and the fact that Q_N^{DD} satisfies (4.28) with N replaced by $2N$, that if the inequality

$$\binom{2N-1}{N-1} < 2^{2N-1} \tag{5.45}$$

is satisfied, then

$$\phi_N^{DD} \in C_0^{\gamma}(\mathbb{R}), \quad \gamma \in \left(0, 2N-1-\log_2\binom{2N-1}{N-1}\right). \tag{5.46}$$

Hence, analogous to the situation in the Daubechies case, we need to investigate the

Table 5.4: $\phi_N^{DD} \in C^{\gamma}(\mathbb{R})$, as obtained from (5.46)

N	2	3	4	5	6	7	8	9	10
$\gamma <$	1.415	1.678	1.871	2.023	2.148	2.255	2.348	2.431	2.505

function

$$\tilde{g}(N) = 2N - 1 - \log_2 \binom{2N-1}{N-1}, \quad N = 2, 3, \ldots,$$

or, equivalently,

$$\tilde{g}(N) = 2N - 1 - \frac{\ln \binom{2N-1}{N-1}}{\ln 2}, \quad N = 2, 3, \ldots \tag{5.47}$$

Note from (5.12) and (5.47) that

$$\tilde{g}(N) = 1 + 2g(N), \quad N = 2, 3, \cdots$$

so that (5.13) and (5.14) in Proposition 5.1 hold, with g replaced by $\tilde{g}$. Thus, since $\tilde{g}(2) = 3 - \log_2 \sqrt{3} > 0$, it is clear from (5.13), with g replaced by $\tilde{g}$, that $\tilde{g}(N) > 0, N \geq 2$; that is, the inequality (5.45) holds for all $N \geq 2$, so that we may indeed appeal to Theorem 4.5, with $l = 1$. Table 5.4 lists some minimum regularity results for ϕ_N^{DD}, as obtained from (5.46). It follows from Table 5.4 that $\phi_N^{DD} \in C_0^1(\mathbb{R}),\ N \geq 2$, whereas $\phi_N^{DD} \in C_0^2(\mathbb{R}),\ N \geq 5$.

5.2.2 Applying the case $l = 2$ of Theorem 4.5

To apply the case $l = 2$ of Theorem 4.5 to the DD refinable function ϕ_N^{DD}, we first observe from (5.29), together with the fact that (5.41) and (4.28) imply $|R_N^{DD}(\omega)| = |R_N^D(\omega)|^2$, $\omega \in \mathbb{R}$, that here

$$M_2 \leq 2^{8(N-1)} 3^{-3(N-1)}. \tag{5.48}$$

It follows from (5.48) that the choice $l = 2$ in Theorem 4.5 will be applicable if the inequality

$$\frac{2^{8(N-1)}}{3^{3(N-1)}} < 2^{2(2N-1)}, \quad N \geq 2, \tag{5.49}$$

holds, for then the inequality (4.30) does indeed hold for $l = 2$.

Table 5.5: $\phi_N^{DD} \in C^{\gamma}(\mathbb{R})$, *as obtained from (5.51)*

N	2	3	4	5	6	7	8	9	10
$\gamma <$	1.377	1.755	2.132	2.510	2.887	3.265	3.642	4.020	4.397

But, (5.49) is equivalent to the inequality

$$\left(\frac{16}{27}\right)^{N-1} < 4, \quad N \geq 2,$$

which holds, since $16/27 < 1$. Hence we may appeal to the case $l = 2$ of Theorem 4.5 to deduce that

$$\phi_N^{DD} \in C_0^{\gamma}(\mathbb{R}), \quad \gamma \in \left(0, 2N - 1 - \frac{1}{2}\log_2\left[2^{8(N-1)}3^{-3(N-1)}\right]\right), \tag{5.50}$$

having also used (5.48). Now observe in (5.50) that

$$\begin{aligned} 2N - 1 - \frac{1}{2}\log_2\left[2^{8(N-1)}3^{-3(N-1)}\right] &= 2N - 1 - \frac{1}{2}\left[8(N-1) - 3(N-1)\log_2 3\right] \\ &= N + (N-1)\left[\frac{3}{2}\log_2 3 - 3\right] \\ &= \left(\frac{3}{2}\log_2 3 - 2\right)N + \left(3 - \frac{3}{2}\log_2 3\right) \\ &\simeq 0.3774N + 0.6226, \end{aligned}$$

and thus,

$$\phi_N^{DD} \in C_0^{\gamma}(\mathbb{R}), \quad \gamma \in \left(0, \left(\frac{3}{2}\log_2 3 - 2\right)N + \left(3 - \frac{3}{2}\log_2 3\right)\right), \tag{5.51}$$

from which we deduce that $\phi_N^{DD} \in C_0^1(\mathbb{R})$ for $N \geq 2$, which agrees with the result from (5.46), whereas $\phi_N^{DD} \in C_0^2(\mathbb{R})$, $N \geq 4$, which improves on (5.46). Table 5.5 shows the regularity results for ϕ_N^{DD} using Theorem 4.5 with $l = 2$. Observe that these results improve on those obtained in Table 5.4, as were obtained from the case $l = 1$ of Theorem 4.5.

5.2.3 Applying Theorem 4.11 for $2 \leq N \leq 10$

We next apply the result of Theorem 4.11 to the DD refinable functions ϕ_N^{DD} of orders $2 \leq N \leq 10$. Since, in Proposition 4.7, we have here $\mu = -2N + 1$, $\nu = 2N - 1$, and

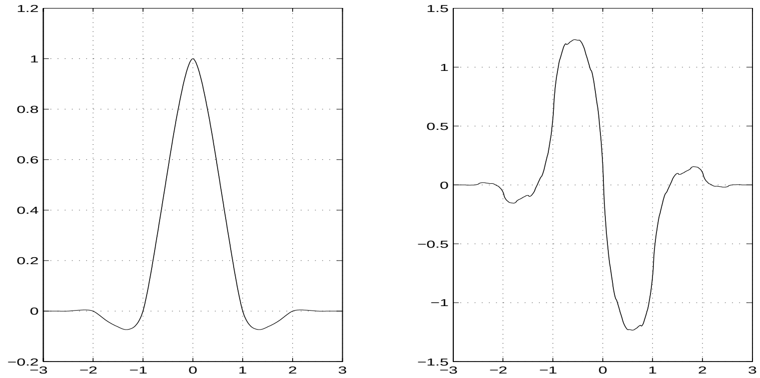

Figure 5.5: *Plots of the refinable function ϕ_2^{DD} (left) and its derivative (right).*

Table 5.6: $\phi_N^{DD} \in C^{\gamma}(\mathbb{R})$, *as obtained from (5.52)*

N	2	3	4	5	6	7	8	9	10
$\gamma <$	1.941	2.675	3.293	3.844	4.362	4.863	5.353	5.835	6.311

therefore $\tau = (2N-1)-(-2N+1)-2N = 2N-2$, having recalled also that $Q = Q_N^{DD}$ is given by (4.28) with N replaced by $2N$, it follows that the corresponding transfer matrix is a $(4N-3)\times(4N-3)$ matrix. Using (4.56), (4.63), and (5.42), we obtain

$$M_T \quad = \quad \begin{pmatrix} 0.5 & 9 & 0.5 & 0 & 0 \\ 0 & -4 & -4 & 0 & 0 \\ 0 & 0.5 & 9 & 0.5 & 0 \\ 0 & 0 & -4 & -4 & 0 \\ 0 & 0 & 0.5 & 9 & 0.5 \end{pmatrix}, \quad N = 2,$$

etc, where, due to their $(4N-3)\times(4N-3)$ size, we do not display the matrices M_T for $3 \leq N \leq 10$ here. Our numerical results show that the condition (4.67) of Theorem 4.11, which in this case, since N is replaced by $2N$ in (4.28), is given by $1 < \rho(M_T) < 4^{2N-1/2}$, does indeed hold for $2 \leq N \leq 10$. Hence, we may appeal to Theorem 4.11 to obtain the regularity result

$$\phi_N^{DD} \in C_0^{\gamma}(\mathbb{R}), \quad \gamma \in (0, 2N - 1/2 - \log_4 \rho(M_T)). \tag{5.52}$$

From Table 5.6 we find, just as in the Daubechies case, that the regularity results for ϕ_N^{DD} as obtained with Theorem 4.11 improve on those in Tables 5.4 and 5.5 obtained

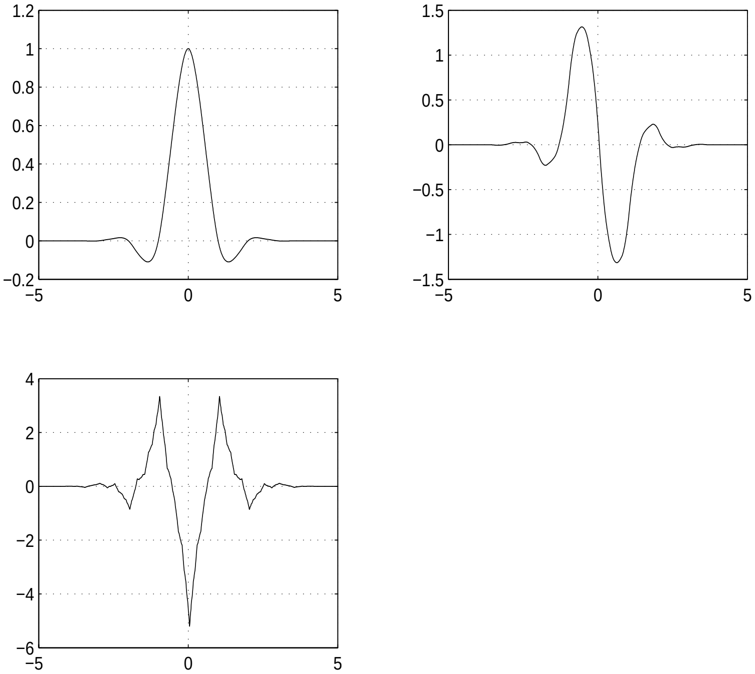

Figure 5.6: *Plots of* ϕ_3^{DD} *(top left) and its first (top right) and second (bottom left) derivatives.*

with, respectively, the cases $l = 1$ and $l = 2$ of Theorem 4.5. In particular, we see that $\phi_N^{DD} \in C_0^2(\mathbb{R})$, $3 \leq N \leq 10$. The facts that $\phi_2^{DD} \in C_0^1(\mathbb{R})$ and $\phi_3^{DD} \in C_0^2(\mathbb{R})$ are graphically illustrated in Figures 5.5 and 5.6 respectively.

5.3 A One-parameter Family of Refinable Functions

In this section, we investigate the regularity of refinable functions with respect to the one-parameter family of mask symbols given, for an integer $N \geq 2$, and for $\alpha \notin \{-1, 0\}$, by

$$A(z) = A_N(z|\alpha) = \sum_j a_{N,j}(\alpha) z^j = \frac{2}{1+\alpha}\left(\frac{1+z}{2}\right)^N (z+\alpha), \quad z \in \mathbb{C}. \tag{5.53}$$

Note that the mask sequence $a_N(\alpha) = \{a_{N,j}(\alpha) : j \in \mathbb{Z}\}$ then satisfies

$$a_{N,j}(\alpha) = 0, \quad j \notin \{0, \ldots, N+1\}.$$

According to a recent result in [14], the cascade algorithm can be used as in Theorem 1.2 to prove that, for the parameter range

$$\alpha \in I := (-\infty, -3) \cup (-1/3, 0) \cup (0, \infty), \tag{5.54}$$

there exists a refinable function $\phi = \phi_N(\cdot|\alpha) \in C_0(\mathbb{R})$ with respect to the mask symbol $A = A_N(\cdot|\alpha)$ as defined by (5.53), and where

$$\phi_N(\cdot|\alpha) = \sum_j a_{N,j}(\alpha)\phi_N(2 \cdot -j|\alpha). \tag{5.55}$$

This result thus extends the existence result for $\alpha > 0$, for which (1.33) holds, with $n = N + 1$, as obtained from Theorem 1.2. We henceforth, in this section, assume that (5.54) holds, since then the conditions *[A]* and *[B]* of Section 4.1 are satisfied, and we may appeal to our regularity results of Chapter 4 to study the regularity of the corresponding refinable functions.

5.3.1 Applying Theorem 4.5

From (5.53) and (4.28), we find that here

$$Q(\omega) = \frac{1}{1+\alpha}\left(\frac{1+e^{-i\omega}}{2}\right)^N (e^{-i\omega} + \alpha), \quad \omega \in \mathbb{R}, \quad \alpha \in I,$$

so that

$$|R(\omega)| = \left|\frac{\alpha + e^{-i\omega}}{\alpha + 1}\right| = \frac{\sqrt{1 + \alpha^2 + 2\alpha \cos\omega}}{|\alpha + 1|}, \quad \omega \in \mathbb{R}, \quad \alpha \in I. \tag{5.56}$$

It follows from (5.56) that if $\alpha > 0$, then

$$\sup_{\omega \in \mathbb{R}} |R(\omega)| = |R(0)| \leq \frac{\sqrt{(\alpha+1)^2}}{|\alpha + 1|} = 1,$$

whereas, if $\alpha \in (-\infty, -3) \cup (-1/3, 0)$, then

$$\begin{aligned}
\sup_{\omega \in \mathbb{R}} |R(\omega)| = |R(\pi)| &\leq \frac{\sqrt{1 + \alpha^2 - 2\alpha}}{|\alpha + 1|} \\
&= \frac{\sqrt{1 + \alpha^2 + 2|\alpha|}}{|\alpha + 1|} \\
&= \frac{\sqrt{(|\alpha| + 1)^2}}{|\alpha + 1|} = \frac{|\alpha| + 1}{|\alpha + 1|}.
\end{aligned}$$

Hence, we have, in the notation of (4.30) in Theorem 4.5, that

$$M_1 = \begin{cases} \dfrac{|\alpha|+1}{|\alpha+1|}, & \alpha \in (-\infty,-3) \cup (-1/3,0), \\ 1, & \alpha > 0. \end{cases} \tag{5.57}$$

Also, noting that $\sup_{\omega\in\mathbb{R}} |R(\omega)R(\omega/2)| = \sup_{\omega\in\mathbb{R}} |R(\omega)R(2\omega)|$, we obtain, for $\omega \in \mathbb{R}$ and $\alpha \in I$,

$$\begin{aligned} |R(\omega)R(2\omega)| &= \frac{|(\alpha+e^{-i\omega})(\alpha+e^{-2i\omega})|}{(1+\alpha)^2} \\ &= \frac{\sqrt{(1+\alpha^2+2\alpha\cos\omega)(1+\alpha^2+2\alpha\cos 2\omega)}}{(1+\alpha)^2} \\ &= \frac{\sqrt{(1+\alpha^2+2\alpha\cos\omega)(1+\alpha^2+2\alpha(2\cos^2\omega-1))}}{(1+\alpha)^2} \\ &= \frac{\sqrt{(1+\alpha^2+2\alpha\cos\omega)(1-2\alpha+\alpha^2+4\alpha\cos^2\omega)}}{(1+\alpha)^2} \\ &= \frac{\sqrt{(1+\alpha^2+2\alpha\cos\omega)((1-\alpha)^2+4\alpha\cos^2\omega)}}{(1+\alpha)^2}. \end{aligned} \tag{5.58}$$

It follows from (5.58) that if $\alpha > 0$, then

$$\sup_{\omega\in\mathbb{R}} |R(\omega)R(2\omega)| = |R(0)|^2 = 1,$$

whereas if $\alpha \in (-\infty,-3) \cup (-1/3,0)$, then

$$\begin{aligned} \sup_{\omega\in\mathbb{R}} |R(\omega)R(2\omega)| &= \max_{-1\le x\le 1} \frac{\sqrt{(1+\alpha^2+2\alpha x)((1-\alpha)^2+4\alpha x^2)}}{(1+\alpha)^2} \\ &\le \left[\max_{-1\le x\le 1} \frac{\sqrt{1+\alpha^2+2\alpha x}}{|1+\alpha|}\right]\left[\max_{-1\le x\le 1} \frac{\sqrt{(1-\alpha)^2+4\alpha x^2}}{|1+\alpha|}\right] \\ &= \left[\frac{\sqrt{(1-\alpha)^2}}{|\alpha+1|}\right]\left[\frac{\sqrt{(1-\alpha)^2}}{|\alpha+1|}\right] \\ &= \left(\frac{1-\alpha}{|\alpha+1|}\right)^2 = \left(\frac{|\alpha|+1}{|\alpha+1|}\right)^2, \end{aligned}$$

and thus

$$\left.\begin{aligned} M_2 &\le \left(\frac{|\alpha|+1}{|\alpha+1|}\right)^2, \quad \alpha \in (-\infty,-3) \cup (-1/3,0) \\ M_2 &= 1, \qquad\qquad \alpha > 0. \end{aligned}\right\} \tag{5.59}$$

Now observe from (5.57) and (5.59) that, for both the cases $l = 1$ and $l = 2$, the condition (4.30) of Theorem 4.5 is satisfied if, for $N \geq 2$,

$$\frac{|\alpha| + 1}{|\alpha + 1|} < 2^{N-1}, \quad \alpha \in I. \tag{5.60}$$

If $\alpha > 0$, then we see that (5.60) holds, since then

$$\frac{|\alpha| + 1}{|\alpha + 1|} = 1 < 2^{N-1}.$$

Suppose next that $\alpha \in (-1/3, 0)$, for which (5.60) is given by

$$\frac{1 - \alpha}{1 + \alpha} < 2^{N-1}. \tag{5.61}$$

Since for the function u defined by

$$u(x) = \frac{1 - x}{1 + x}, \quad x \in \mathbb{R} \backslash \{-1\}, \tag{5.62}$$

we have

$$u'(x) = \frac{-2}{(1 + x)^2} < 0, \quad x \in \mathbb{R} \backslash \{-1\},$$

we deduce that

$$\frac{1 - x}{1 + x} < u(-1/3) = 2 \leq 2^{N-1}, \quad \alpha \in (-1/3, 0),$$

since $N \geq 2$. Hence (5.61), and therefore (5.60), is satisfied for $\alpha \in (-1/3, 0)$.

Finally, if $\alpha \in (-\infty, -3)$, then the inequality (5.60) is given by

$$\frac{1 - \alpha}{1 + \alpha} > -2^{N-1}. \tag{5.63}$$

But then,

$$\frac{1 - \alpha}{1 + \alpha} > u(-3) = -2 \geq -2^{(N-1)},$$

since $N \geq 2$, so that (5.63), and therefore also (5.60), holds for $\alpha \in (-\infty, -3)$. We have therefore proved that the inequality (5.60), and therefore also the condition (4.30) of Theorem 4.5, is satisfied for all $\alpha \in I$. Hence, noting also the fact that $\dfrac{|\alpha| + 1}{|\alpha + 1|} = 1, \ \alpha > 0$, we conclude from Theorem 4.5 that the refinable function $\phi_N(\cdot|\alpha)$ corresponding to the mask symbol $A_N(\cdot|\alpha)$ in (5.53) satisfies

$$\phi_N \in C_0^\gamma(\mathbb{R}), \quad \gamma \in (0, \gamma_N^*(\alpha)), \quad \alpha \in I, \tag{5.64}$$

where

$$\gamma_N^*(\alpha) = N - 1 - \log_2\left(\frac{|\alpha|+1}{|\alpha+1|}\right), \quad \alpha \in I. \tag{5.65}$$

It then follows from (5.64) and (5.65) that

$$\phi_N \in C_0^\gamma(\mathbb{R}), \quad \gamma \in (0, N-1), \quad \alpha > 0. \tag{5.66}$$

Note that, according to (1.36), $A_N(\cdot|1) = A^{(N+1)}$, the mask symbol for the cardinal B-spline of order $N+1$, and thus $\phi_N(\cdot|1) = N_{N+1} \in C_0^{N-1}(\mathbb{R})$. Moreover, if $\alpha > 0$, then we can appeal to Theorem 2.4 with $C(z) = \dfrac{\alpha+z}{\alpha+1}$, $z \in \mathbb{C}$, to deduce that $\phi_N(\cdot|\alpha) \in C_0^{N-1}(\mathbb{R})$, $\alpha > 0$, which is a slight improvement on the regularity result (5.66), as obtained from Theorem 4.5. Hence the more significant part of the regularity result (5.64) is for $\alpha \in (-\infty, -3) \cup (-1/3, 0)$; i.e, when there is at least one negative mask coefficient.

From (5.64), (5.65), and the analysis above of the function u as defined by (5.62), we deduce that the regularity exponent $\gamma_N^*(\alpha)$ in (5.64) satisfies

$$\left.\begin{array}{ll} \gamma_N^*(\alpha) \to (N-1)^-, \ \alpha \to 0^-, & \gamma_N^*(\alpha) \to (N-2)^+, \ \alpha \to -1/3^+; \\ \gamma_N^*(\alpha) \to (N-2)^+, \ \alpha \to -3^-, & \gamma_N^*(\alpha) \to (N-1)^-, \ \alpha \to -\infty; \end{array}\right\} \tag{5.67}$$

where all the arrows indicate strictly monotone behaviour. Observe from (5.67) that

$$\gamma_N^*(\alpha) \in (N-2, N-1), \quad \alpha < 0. \tag{5.68}$$

5.3.2 Applying Theorem 4.11

We next apply Theorem 4.11. From Proposition 4.7, we have here $\mu = 0$, $\nu = N+1$, so that $\tau = 1$, and the transfer matrix M_T is a 3×3 matrix in this case. Using (4.63) and (5.53), we find that the corresponding M_T is given by

$$M_T = \frac{2}{(1+\alpha)^2}\begin{pmatrix} \alpha & \alpha & 0 \\ 0 & \alpha^2+1 & 0 \\ 0 & \alpha & \alpha \end{pmatrix}, \quad \alpha \in I,$$

with eigenvalues

$$\lambda_1 = \lambda_2 = \frac{2\alpha}{(\alpha+1)^2}, \quad \lambda_3 = \frac{2(\alpha^2+1)}{(\alpha+1)^2},$$

so that $\rho(M_T) = \rho(\alpha) = 2(\alpha^2+1)/(\alpha+1)^2$, after having noted that $\alpha^2-\alpha+1 > 0,\ \alpha \in \mathbb{R}$, and thus $2(\alpha^2+1) > 2\alpha,\ \alpha \in I$. Hence the condition (4.67) of Theorem 4.11 is given by

$$1 < \frac{2(\alpha^2+1)}{(\alpha+1)^2} < 4^{N-1/2}. \tag{5.69}$$

To investigate whether the inequality (5.69) holds, we define the function

$$w(x) = \frac{2(x^2+1)}{(x+1)^2}, \quad x \in \mathbb{R}\backslash\{-1\},$$

for which

$$w'(x) = \frac{4(x^2-1)}{(x+1)^4}, \quad x \in \mathbb{R}\backslash\{-1\}.$$

Hence

$$w'(x) \begin{cases} > 0, & x < -1, \\ < 0, & -1 < x < 1, \\ = 0, & x = 1, \\ > 0, & x > 1. \end{cases} \tag{5.70}$$

Since also $\rho(\alpha) = w(\alpha),\ \alpha \in I$, we therefore have

$$\left.\begin{aligned} &\lim_{\alpha\to-\infty} \rho(\alpha) = 2, \quad \rho'(\alpha) > 0,\ \alpha \in (-\infty, -3), \\ &\lim_{\alpha\to-3^-} \rho(\alpha) = w(-3) = 5, \quad \lim_{\alpha\to-1/3^+} \rho(\alpha) = w(-1/3) = 5, \\ &\rho'(\alpha) < 0,\ \alpha \in (-1/3, 0), \quad \lim_{\alpha\to0^-} \rho(0) = w(0) = 2, \\ &\lim_{\alpha\to0^+} \rho(0) = w(0) = 2, \quad \rho'(\alpha) < 0,\ \alpha \in (0,1), \quad \rho'(1) = 0, \\ &\rho(1) = w(1) = 1, \quad \rho'(\alpha) > 0,\ \alpha \in (1, \infty), \\ &\lim_{\alpha\to+\infty} \rho(\alpha) = 2, \end{aligned}\right\} \tag{5.71}$$

and thus, for $N \geq 2$, we get

$$1 < \rho(\alpha) \leq 5 < 8 \leq 4^{N-1/2}, \quad \alpha \in I\backslash\{1\}. \tag{5.72}$$

As noted before, $\phi_N(\cdot|1) = N_{N+1} \in C_0^{N-1}(\mathbb{R})$. By (5.72), we deduce from Theorem 4.11

that

$$\phi_N \in C_0^\gamma(\mathbb{R}), \quad \gamma \in (0, \tilde{\gamma}_N(\alpha)), \quad \alpha \in I, \tag{5.73}$$

where

$$\tilde{\gamma}_N(\alpha) = N - 1/2 - \log_4 \rho(\alpha) = N - 1 - \log_4 \left(\frac{\alpha^2+1}{(\alpha+1)^2} \right), \quad \alpha \in I, \tag{5.74}$$

from which, together with (5.71), we deduce that the regularity index $\tilde{\gamma}_N(\alpha)$ satisfies

$$\left.\begin{array}{l}
\tilde{\gamma}_N(\alpha) \to (N-1)^-, \quad \alpha \to -\infty, \\
\tilde{\gamma}_N(\alpha) \to (N - \log_4 10)^+ \simeq (N-1.661)^+, \quad \alpha \to -3^-, \\
\tilde{\gamma}_N(\alpha) \to (N - \log_4 10)^+ \simeq (N-1.661)^+, \quad \alpha \to -1/3^+, \\
\tilde{\gamma}_N(\alpha) \to (N-1)^-, \ \alpha \to 0^-, \qquad \tilde{\gamma}_N(\alpha) \to (N-1)^+, \ \alpha \to 0^+, \\
\tilde{\gamma}_N(\alpha) \to (N-1/2)^-, \ \alpha \to 1^-, \qquad \tilde{\gamma}_N(\alpha) \to (N-1/2)^-, \ \alpha \to 1^+, \\
\tilde{\gamma}_N(\alpha) \to (N-1)^+, \quad \alpha \to +\infty,
\end{array}\right\} \tag{5.75}$$

where all the arrows indicate strictly monotone behaviour. We see from (5.75) that

$$\tilde{\gamma}_N(\alpha) \in (N - \log_4 10, N - 1/2) \simeq (N - 1.661, N - 1/2), \quad \alpha \in I. \tag{5.76}$$

By comparing (5.68) and (5.76), we see that Theorem 4.11 improves on Theorem 4.5. As noted before in Section 5.3.1, an application of Theorem 2.4 to the positive mask case $\alpha > 0$ yields the regularity result $\phi_N(\cdot|\alpha) \in C_0^{N-1}(\mathbb{R}),\ \alpha > 0$. But, according to (5.71), we have $\rho(\alpha) \in [1,2),\ \alpha > 0$, and thus

$$N - 1/2 - \log_4 \rho(\alpha) > N - 1, \quad \alpha > 0.$$

We therefore deduce from (5.74) that (5.73) actually improves on the result $\phi_N(\cdot|\alpha) \in C_0^{N-1}(\mathbb{R}),\ \alpha > 0$, as obtained from Theorem 2.4, in the sense that

$$\phi_N^{(N-1)}(\cdot|\alpha) \in \mathcal{L}_0^\beta(\mathbb{R}), \quad \alpha > 0,$$

where $\beta \in (0, 1/2 - \log_4 \rho(\alpha)) = \left(0, \log_4\left(\frac{(\alpha+1)^2}{\alpha^2+1}\right)\right), \quad \alpha > 0.$

For $\alpha < 0$, our results (5.73), (5.74), and (5.75) only guarantee that $\phi_2(\cdot|\alpha) \in C_0(\mathbb{R})$, and not $\phi_2(\cdot|\alpha) \in C_0^1(\mathbb{R})$. Hence, our regularity results for $\alpha < 0$ are better illustrated

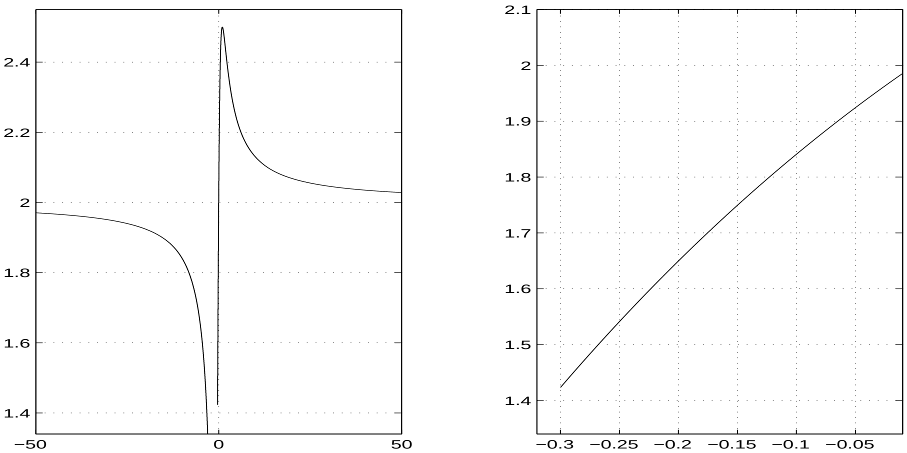

Figure 5.7: *Plots of* $\tilde{\gamma}_3(\alpha)$ *for* $\alpha \in I\backslash(-\infty,-50)\cup(50,\infty)$ *and* $-1/3 < \alpha < 0$

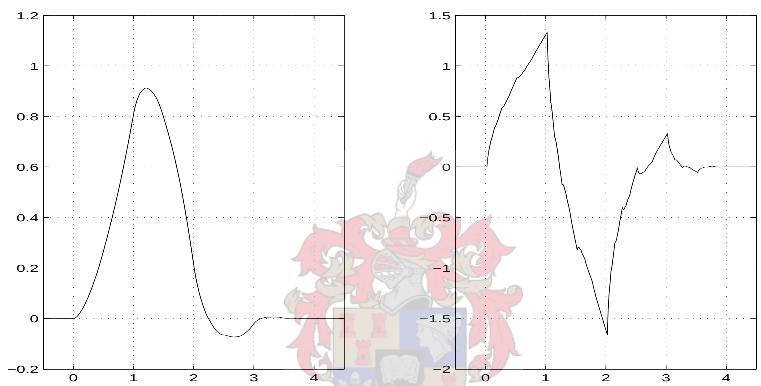

Figure 5.8: *Plots of* $\phi_3(\cdot|-1/4)$ *(left) and* $\phi_3'(\cdot|-1/4)$ *(right)*

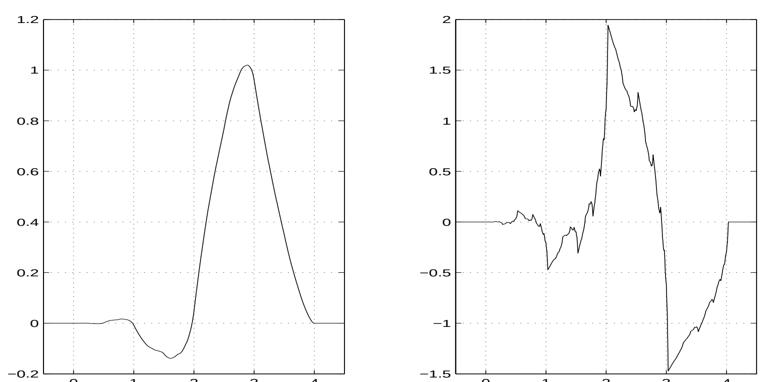

Figure 5.9: *Plots of* $\phi_3(\cdot|-31/10)$ *(left) and* $\phi_3'(\cdot|-31/10)$ *(right)*

with $N \geq 3$, since (5.73) and (5.76) imply that $\phi_3(\cdot|\alpha) \in C_0^1(\mathbb{R}),\ \alpha < 0$. Figure 5.7 shows the graphs of $\tilde{\gamma}_3(\alpha)$. If we set $N = 3,\ \alpha = -1/4$ in (5.53), then we obtain the mask sequence

$$\left.\begin{array}{rcl} a_{3,0}(-1/4) & = & -1/12, \\ a_{3,1}(-1/4) & = & 1/12, \\ a_{3,2}(-1/4) & = & 9/12, \\ a_{3,3}(-1/4) & = & 11/12, \\ a_{3,4}(-1/4) & = & 1/3, \\ a_{3,k}(-1/4) & = & 0, \quad k \notin \{0,1,2,3,4\}, \end{array}\right\} \tag{5.77}$$

whereas $N = 3,\ \alpha = -31/10$ yield the mask sequence

$$\left.\begin{array}{rcl} a_{3,0}(-31/10) & = & 31/84, \\ a_{3,1}(-31/10) & = & 83/84, \\ a_{3,2}(-31/10) & = & 63/84, \\ a_{3,3}(-31/10) & = & 1/84, \\ a_{3,4}(-31/10) & = & -5/42, \\ a_{3,k}(-31/10) & = & 0, \quad k \notin \{0,1,2,3,4\}. \end{array}\right\} \tag{5.78}$$

Observe from (5.77) and (5.78) that $a_{3,0}(-1/4) < 0$ and $a_{3,4}(-31/10) < 0$. Figures 5.8 and 5.9 graphically suggest that both $\phi_3(\cdot| - 1/4)$ and $\phi_3(\cdot| - 31/10)$ have continuous derivatives, which are confirmed by the regularity results $\phi_3(\cdot| - 1/4) \in C_0^{1.54}(\mathbb{R})$ and $\phi_3(\cdot| - 31/10) \in C_0^{1.37}(\mathbb{R})$, from (5.73) and (5.74).

5.4 Comparison with a Subdivision-based Regularity Result

Generally speaking, the Hölder regularity results obtained through the Fourier transform techniques above are not optimal. A result which does yield optimal Hölder regularity results for refinable functions which are Riesz-stable in the sense of (1.46), but whose proof is beyond the scope of this thesis, was proved by Rioul in [28, Theorem 11.1] in the framework of subdivision. In this section, we proceed to quote the said result and compare its Hölder regularity estimates with those obtained through Fourier analysis. To this end, suppose the symbol A is as in (4.25); that is, for $N \geq 2$,

$$A(z) = \frac{1}{2^{N-1}}(1+z)^N B(z), \quad z \in \mathbb{C}\backslash\{0\},$$

where

$$B(1) = 1, \quad B(-1) \neq 0.$$

Now define

$$B^r(z) = \sum_j b_j^{(r)} z^j = \prod_{j=1}^{r} B(z^{2^{j-1}}), \quad z \in \mathbb{C}\backslash\{0\}, \quad r \in \mathbb{N},$$

and

$$\beta_r = -\frac{1}{r}\log_2\left[\max_{0\leq j\leq 2^r-1}\sum_k |b_{j+2^r k}^{(r)}|\right], \quad r \in \mathbb{N},$$

and let

$$\beta = \sup_r \beta_r.$$

Then the following result holds.

Theorem 5.3 *Suppose that* $a \in M_0(\mathbb{Z})$ *and* $\phi \in C_0(\mathbb{R})$ *satisfy conditions [A] and [B] of Section 4.1. If there exists an integer* $r \in \mathbb{N}$ *such that* $N + \beta_r > 0$*, then* $\phi \in C_0^{N+\beta_r}(\mathbb{R})$*, and therefore*

$$\phi \in C_0^{\gamma}(\mathbb{R}), \quad \gamma \in (0, N+\beta). \tag{5.79}$$

Moreover, (5.79) is the optimal regularity result for ϕ *if* ϕ *is Riesz-stable in the sense of (1.46).*

Since both the Daubechies and the DD refinable functions have been shown to be Riesz-stable in [12, p. 131] and [18, Theorem 4.2], respectively, we proceed to apply the result of Theorem 5.3 to them. Tables 5.7 and 5.8 show the Hölder regularity results obtained for the Daubechies and DD refinable functions respectively. Observe from Tables 5.7 and 5.8 that the subdivision-based results improve on the Fourier-based results. For instance, whereas the subdivision-based approach reveals that $\phi_3^D \in C_0^1(\mathbb{R})$, as illustrated in Figure 5.10, the same result was not obtained through Fourier analysis. However, the estimates obtained by means of Theorem 4.11 compare more favourably with the subdivision-based results of Theorem 5.3 than do those obtained with Theorem 4.5. It is particularly remarkable, from Table 5.8 and Figure 5.6, that $\phi_3^{DD} \in C_0^2(\mathbb{R})$, a result which Theorem 4.5 could not guarantee.

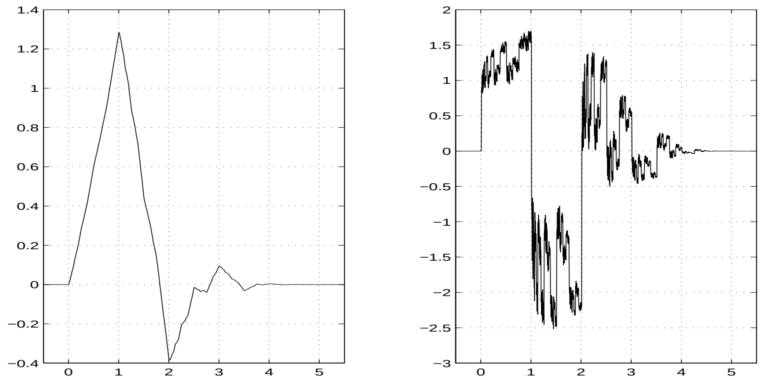

Figure 5.10: *Plots of the refinable function ϕ_3^D (left) and its derivative (right).*

Table 5.7: *A comparison of the regularity results γ for ϕ_N^D*

N	Thm 4.5, $l=1$, $\gamma <$	Thm 4.5, $l=2$, $\gamma <$	Thm 4.11, $\gamma <$	Thm 5.3, $\gamma <$
2	0.207	0.189	0.500	0.550
3	0.339	0.378	0.915	1.083
4	0.435	0.566	1.275	1.606
5	0.511	0.755	1.596	1.942
6	0.574	0.943	1.888	2.164
7	0.628	1.132	2.158	2.434
8	0.674	1.321	2.415	2.735
9	0.715	1.510	2.661	3.043
10	0.752	1.698	2.902	3.310

Table 5.8: *A comparison of the regularity results γ for ϕ_N^{DD}*

N	Thm 4.5, $l=1$, $\gamma <$	Thm 4.5, $l=2$, $\gamma <$	Thm 4.11, $\gamma <$	Thm 5.3, $\gamma <$
2	1.415	1.377	1.941	2.000
3	1.678	1.755	2.675	2.830
4	1.871	2.132	3.293	3.551
5	2.023	2.510	3.844	4.194
6	2.148	2.887	4.362	4.777
7	2.255	3.265	4.863	5.315
8	2.348	3.642	5.353	5.829
9	2.431	4.020	5.835	6.323
10	2.505	4.397	6.311	6.805

Table 5.9: *A comparison of the regularity results γ for $\phi_3(\cdot|\alpha)$ for some values of $\alpha \in I$*

α	Thm 4.5, $l=1$, $\gamma <$	Thm 4.5, $l=2$, $\gamma <$	Thm 4.11, $\gamma <$	Thm 5.3, $\gamma <$
-1/10	1.711	1.711	1.841	1.848
-1/5	1.415	1.415	1.650	1.678
-1/4	1.263	1.263	1.541	1.585
-31/10	1.035	1.035	1.367	1.560
-27/4	1.569	1.569	1.753	1.912
-76/5	1.810	1.810	1.899	1.975

Note also from Table 5.8 that ϕ_2^{DD} is “almost” in $C_0^2(\mathbb{R})$ in the sense that its first derivative satisfies (3.19) with $|h|^\beta$ replaced by $|h \log |h||$, according to a result which was proved in [15] (see also [11]). In fact, Daubechies and Lagarias proved in [10], using a completely different but more rigorous approach, that the first derivative of ϕ_2^{DD} is not differentiable at the dyadic rationals in its support. Observe from Figure 5.5 that ϕ_2^{DD}, looks mostly “smooth,” but that its derivative exhibits “bumps” which repeat themselves at different scales. With the same method, they also obtained the regularity result 0.55 in Table 5.7 for ϕ_2^D, and showed that this result is optimal.

Table 5.9 shows the regularity results for the refinable functions $\phi_3(\cdot|\alpha)$ associated with the mask symbols $A_3(\cdot|\alpha)$ in (5.53), for some selected negative values of $\alpha \in I$. Observe from Table 5.9 that, as in the Daubechies and DD cases, the subdivision-based results improve on the Fourier-based results. However, we cannot guarantee the optimality of the subdivision-based results in this case, since we did not investigate the Riesz-stability of $\phi_3(\cdot|\alpha)$ for each fixed α.

Chapter 6

Conclusions

In this thesis, our main focus has been on the global regularity as opposed to the local (pointwise) regularity of a given refinable function. We achieved this through one of the three major techniques that appear in the literature namely, the Fourier transform technique. Although the Fourier-based Hölder regularity results are not as sharp as their subdivision-based counterparts, we find, unlike the subdivision-based results which are indeed computationally expensive, that the Fourier-based results discussed in this thesis lend themselves to simple calculations. Furthermore, the spectral result in Theorem 4.11 yields Hölder regularity estimates which compare favourably with those obtained through a subdivision technique. We also note that the Fourier-based results of Theorems 4.5 and 4.11 can often be applied to analyze a continuum of masks e.g. the one-parameter family of masks as discussed in Section 5.3. This is difficult, if not impossible, with Theorem 5.3, since β_r is not expressible in terms of a parameter.

Apart from the Fourier transform and subdivision techniques, the third major approach is the matrix method. Here, the regularity of a refinable function is studied by computing the joint spectral radius (see e.g. [19] and [30]) of two square matrices defined from the mask coefficients. In practice, this technique becomes quickly impractical if the length of the mask is not small. However, if carefully implemented, it yields optimal regularity results like the subdivision technique, and can also be used to investigate the local regularity of refinable functions. For details of this method, we refer to [10]. It should be noted that the use of matrices to study the regularity of refinable functions first appeared in [26]. Other recent articles on the topic include [2] and [6].

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