Imaginaries in dense pairs of real-closed fields

by

Tsinjo Odilon Rakotonarivo

Thesis presented in partial fulfilment of the requirements for the degree of Master of Science in Mathematics in the Faculty of Science at Stellenbosch University

Department of Mathematical Sciences,
University of Stellenbosch,
Private Bag X1, Matieland 7602, South Africa.

Supervisor: Dr. Gareth Boxall

March 2017

Declaration

By submitting this thesis electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the sole author thereof (save to the extent explicitly otherwise stated), that reproduction and publication thereof by Stellenbosch University will not infringe any third party rights and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Signature: ....

Tsinjo Odilon Rakotonarivo

Date: March 2017

Abstract

Imaginaries in dense pairs of real-closed fields

Tsinjo Odilon Rakotonarivo

Department of Mathematical Sciences,
University of Stellenbosch,
Private Bag X1, Matieland 7602, South Africa.

Thesis: MSc

March 2017

Imaginaries are definable equivalence classes, which play an important role in model theory. In this thesis, we are interested in imaginaries of dense pairs of real-closed fields. More precisely, we consider the following problem: is acl^{eq} equal to dcl^{eq} in dense pairs of real-closed fields? To answer this question, we first present some results about real-closed fields, which are basically completeness, quantifier elimination and elimination of imaginaries. Then, we concentrate on the completeness and near model-completeness for the theory of dense pairs of real-closed fields. And finally, we present the key point of the thesis. Namely, we demonstrate that $acl^{eq}(\emptyset) = dcl^{eq}(\emptyset)$ but there exists A such that $acl^{eq}(A) \neq dcl^{eq}(A)$.

Uittreksel

Imaginêres in dig pare van reël-geslote liggame

Tsinjo Odilon Rakotonarivo

Departement Wiskundige Wetenskappe,
Universiteit van Stellenbosch,
Privaatsak X1, Matieland 7602, Suid Afrika.

Tesis: MSc

Maart 2017

Imaginêres is definiëerbare ekwivalensieklasse, wat 'n belangrike rol in modelteorie speel. In hierdie tesis stel ons belang in imaginêres in dig pare van reël-geslote liggame. Meer spesifiek beskou ons die volgende probleem: is acl^{eq} gelyk aan dcl^{eq} in dig pare van reël-geslote liggame? Om hierdie vraag te beantwoord, begin ons met 'n paar resultate oor reël-geslote liggame, namelik volledigheid, kwantoreliminasie en eliminasie van imaginêres. Daarna behandel ons die volledigheid en byna-model-volledigheid vir die teorie van dig pare van reël-geslote liggame. Uiteindelik behandel ons die hoofresultat van hierdie tesis, d.w.s. ons bewys dat $acl^{eq}(\emptyset) = dcl^{eq}(\emptyset)$ maar dat daar A bestaan sodat $acl^{eq}(A) \neq dcl^{eq}(A)$.

Acknowledgements

I wish to express my profound gratitude to my supervisor, Dr Gareth Boxall, for his valuable advice and suggestions, as well as his support throughout this thesis.

I also address my thanks to the National Research Foundation (NRF) and the African Institute for Mathematical Sciences (AIMS-Ghana) for funding this project.

I do appreciate all the staff members and colleagues at the Department of Mathematical Sciences at the Stellenbosch University for their permanent collaboration. I extend my special thanks to Prof. Breuer for translating the abstract into Afrikaans.

My greatest thanks goes to my wife, my son and my family for their unconditional love and encouragement.

Dedications

To Fanomezantsoa,

Contents

Chapter 1

Introduction

Model theory is a branch of mathematical logic which studies classes of structures. A structure may be defined as a set together with a collection of symbols which are interpreted in it. We call this collection a language. We study the first-order theory of a structure with respect to its language.
The language plays an important role in the study of structures and theories. Depending on its choice, a specific theory might or might not have certain properties, such as quantifier elimination. For example, the theory of real-closed fields with the language of rings does not have quantifier elimination, while with the language of ordered rings, it admits quantifier elimination.
For some theory, having quantifier elimination is also giving control over definable sets. For example, if an o-minimal theory T has quantifier elimination, then it helps to prove o-minimality. A theory T is o-minimal if any definable subsets of the underlying set are finite unions of intervals and points.
Another important property in model theory is the elimination of imaginaries. Imaginaries are definable equivalence classes, namely they are quotient of elements of M^n by $\emptyset$-definable equivalence relations. Imaginaries are elements of the structure M^{eq}. However these imaginaries can be eliminated for some theories. In fact, elimination of imaginaries helps in determining canonical parameters for the identification of definable sets.
In this thesis, we focus on the theory of dense pairs of real-closed fields, and study its different properties, including the relative quantifier elimination, the completeness property and the imaginaries. Our main contribution is to answer the following question: is $acl^{eq} = dcl^{eq}$ for dense pairs of real-

closed fields? This question was raised in 2015 at a LYMOTS meeting in Manchester.

The remainder of this document is organised as follows.

In chapter 2, some basic definitions and important properties in model theory are reviewed. The notion of imaginaries and definable sets are outlined accordingly.

The quantifier elimination as well as the completeness of the theory of real-closed fields are reviewed in chapter 3. The elimination of imaginaries of this theory is presented as well.

In chapter 4, we first present the near model-completeness for the theory of $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$, using back and forth property. Then, we generalize this result for the theory of dense pairs of real-closed fields and prove completeness of that theory.

The last chapter contains the main results of this thesis, which concern the imaginaries of dense pairs of real-closed fields. Results in [3] are reviewed first, in order to understand the behaviour of imaginaries in the case of dense pairs. Thereafter, the results related to the relation between acl^{eq} and dcl^{eq} are shown. We first show that $acl^{eq}(\emptyset) = dcl^{eq}(\emptyset)$, and then demonstrate that there exists $A \subseteq M^{eq}$ such that $acl^{eq}(A) \neq dcl^{eq}(A)$ for any dense pair of real-closed fields.

Chapter 2

Basic Model Theory

In this chapter, we are reviewing basic model theory. We are giving some definitions, propositions and theorems that will be useful in the rest of the chapters. Most of the properties are well-known. They can be found in any standard book on model theory. For more details about this chapter, refer to [10] and [13].

2.1 Structures

The first definitions allow us to be more familiar with the technical terms of model theory. We begin by introducing the definitions of languages and structures.

Definition 2.1.1. A *language* or *signature* L is a collection of constant symbols, function symbols and relation symbols where functions and relations are equipped with arity.

Definition 2.1.2. An *L-structure* is a pair (M, I) where M is a non-empty set that is called the underlying set or domain and I is the interpretation of the language L in the set M. A function with n-arity is interpreted as a function from M^n to M, and a relation of m-arity is interpreted as a subset of M^m.

Let us illustrate the two first definitions by an example.

Example 2.1.3. Let $L = \{0, 1, \cdot, +\}$. The constant symbols of L are $0, 1$, the function symbols are $\cdot, +$ with arity 2, then $(\mathbb{R}, 0, 1, \cdot, +)$ is an L-structure where $\mathbb{R}$ is the underlying set and $0, 1, \cdot, +$ are interpreted in $\mathbb{R}$ as usual. In this case, L is called the language of rings.

We also need to know about many sorted structures, which plays an important role in the studies of imaginaries.

Definition 2.1.4. Given a language L, a *many sorted L-structure* is just an L-structure with many underlying sets. Fix a non-empty set J where each element $j \in J$ corresponds to one sort, and let $\mathcal{M}$ be the many sorted structure associated to J, which is defined as follows:

- For each $j \in J$, a non empty set M_j is the underlying set associated to sort j.
- Each constant symbol c of the sort j is interpreted as an element $c^{\mathcal{M}}$ of M_j.
- Each function symbol f of the sorts $(j_1, \cdots, j_n, j)$ is interpreted as a function
$$f^{\mathcal{M}} : M_{j_1} \times \cdots \times M_{j_n} \longrightarrow M_j,$$
- Each relation symbol R of the sorts $(j_1, \cdots, j_n)$ is interpreted as a relation
$$R^{\mathcal{M}} \subseteq M_{j_1} \times \cdots \times M_{j_n}.$$

Example 2.1.5. Let $\mathcal{M} = (K, V, 0_K, 1_K, +_K, \cdot_K, 0_V, +_V, \cdot_{K,V})$ be a structure where K is a field and V is a vector space over the field K. In fact, $\{0_K, 1_K, +_K, \cdot_K\}$ is the language related to the field K, $\{0_V, +_V\}$ is the language for vector spaces and $\{\cdot_{K,V}\}$ is the part of the language involving both.

From now on, we are using one sorted structure but all the results extend naturally to the many sorted setting. Let us continue with the definition of definable sets.

Definition 2.1.6. Let L be a language and $\mathcal{M}$ be an L-structure with underlying set M. We say that $X \subseteq M^n$ is *definable* in $\mathcal{M}$ if and only if there exist a first-order L-formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $(b_1, \ldots, b_m) \in M^m$ such that $X = \{(a_1, \ldots, a_n) \in M^n : \mathcal{M} \models \varphi(a_1, \ldots, a_n, b_1, \ldots, b_m)\}$.
Furthermore, let A be a fixed subset of M. We say that $X \subseteq M^n$ is definable in $\mathcal{M}$ with parameters from A, or A-definable, if and only if there exist a

first-order L-formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $(b_1, \ldots, b_m) \in A^m$ such that $X = \{(a_1, \ldots, a_n) \in M^n : \mathcal{M} \models \varphi(a_1, \ldots, a_n, b_1, \ldots, b_m)\}$. The set X is $\emptyset$-definable if $m = 0$.

It is also natural to relate structures to one another. We are going to see some of these relations in the following definitions.

Definition 2.1.7. Let $\mathcal{M}, \mathcal{N}$ be two L-structures with underlying set M, N respectively. We say $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ if $M \subseteq N$ and the interpretation of each symbol in M is the restriction of each symbol in N, or equivalently, the quantifier free formulas with parameters in M have the same interpretation in both structures. We denote it by $\mathcal{M} \leq \mathcal{N}$.
We say that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$, and denote it by $\mathcal{M} \prec \mathcal{N}$, if $\mathcal{M}$ is a substructure of $\mathcal{N}$ and for all first order L-formulas $\phi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$, we have $\mathcal{M} \models \phi(a_1, \ldots, a_n)$ if and only if $\mathcal{N} \models \phi(a_1, \ldots, a_n)$.

Definition 2.1.8. Let $\mathcal{M}$ be an L-structure. The *full theory of* $\mathcal{M}$ is the set of all first-order L-sentences which are true of the L-structure $\mathcal{M}$. Denote by $Th(\mathcal{M})$ the full theory of $\mathcal{M}$. Recall that L-sentences are L-formulas without free variables.

Definition 2.1.9. Let $\mathcal{M}$ and $\mathcal{N}$ be two L-structures. If $Th(\mathcal{M}) = Th(\mathcal{N})$, then $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* and we denote $\mathcal{M} \equiv \mathcal{N}$.

Example 2.1.10. Let $L = \{0, 1, +, \cdot, <\}$. The set of all real algebraic numbers $\mathbb{R} \cap \overline{\mathbb{Q}}$ is an elementary substructure of $\mathbb{R}$ considered as L-structures. This is due to quantifier elimination that we will define later. Recall that $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$.

We keep using one-sorted structures so, from now on, structure and underlying set are denoted as the same.

2.2 Saturation and homogeneity

This section is dedicated to the notions of saturation and homogeneity. These are tools that we will strongly use later. Definitions and facts can be found in all basic model theory books, such as [10], [13] and [15].

Definition 2.2.1. A *type* over $A \subseteq M$ is a set $p(\overline{x})$ of formulas in the language $L(A)$ such that for every finite subset $p_0(\overline{x}) \subseteq p(\overline{x})$ there is some $\overline{b} \in M^n$, with $M \models p_0(\overline{b})$. We also say that $p_0(\overline{x})$ is realised by $\overline{b}$.

Definition 2.2.2. An L-structure M is *κ-saturated* for a cardinal κ if, for every subset $A \subseteq M$ such that $|A| < \kappa$ and type $\Sigma(\overline{x})$ of $L(A)$- formulas, Σ is realised in M. Note that $L(A)$ is L together with a new constant symbol for each $a \in A$.

Definition 2.2.3. An L-structure M is said to be *strongly κ-homogeneous* if whenever $\overline{a}, \overline{b}$ are tuples from M of length less than κ having the same type over the empty set, then there is some automorphism of M which sends $\overline{a}$ to $\overline{b}$.

Lemma 2.2.4. For every cardinal κ, every structure has an elementary extension which is both κ-saturated and strongly κ-homogeneous, see [10] and [13].

This lemma is proved using the compactness theorem.

Definition 2.2.5. Let Σ be a set of L-sentences. Then Σ is consistent if there is some structure M such that $M \models \Sigma$.
Let T be a set of L-sentences, T is finitely consistent if any finite subset of T has a model.

Now we are stating the compactness theorem. One of the basic theorems in model theory. We are not giving any proof of the theorem here but more details can be seen in [6], [7] and [10].

Theorem 2.2.6 (Compactness Theorem)**.** *Let Σ be a set of L-sentences. Then Σ is consistent if and only if Σ is finitely consistent.*

2.3 Quantifier elimination and completeness

In this section, we define quantifier elimination and completeness for structures.

Definition 2.3.1. An *L-theory* is a consistent set of L-sentences.

The next definition is a formal definition of quantifier elimination. There will be an equivalent definition that we will state as a proposition.

Definition 2.3.2. Let T be an L-theory. The theory T has *quantifier elimination* if for every $\phi(\bar{x})$ there is a quantifier free L-formula $\psi(\bar{x})$ such that $T \models \forall \bar{x}\,(\phi(\bar{x}) \longleftrightarrow \psi(\bar{x}))$.

Definition 2.3.3. An L-theory T is *complete* if for any models M and N of T then $M \equiv N$.

Completeness allows us to study any model of T and deduce the same properties for all models.

Definition 2.3.4. Let T be a complete theory. T is *model complete* if any substructure of a model M of T is an elementary substructure of M.

Proposition 2.3.5. Let T be a theory. Assume T is complete and has quantifier elimination. Then it is model-complete.

Proof. This proof is from [10]. Suppose T has quantifier elimination and let M, N be two models of T such that M is substructure of N. We have to show that M is an elementary substructure of N. By quantifier elimination of T, for a first-order L-formula $\phi(\bar{x})$, there is a quantifier free formula $\psi(\bar{x})$ such that $M \models \forall \bar{x}(\phi(\bar{x}) \longleftrightarrow \psi(\bar{x}))$. Since ψ is quantifier free then for $\bar{a} \in M$, $M \models \psi(\bar{a}) \iff N \models \psi(\bar{a})$. Then we have

$$M \models \phi(\bar{a}) \iff M \models \psi(\bar{a}) \iff N \models \psi(\bar{a}) \iff N \models \phi(\bar{a}).$$

Therefore M is an elementary substructure of N. □

In fact, a complete theory T is model complete if and only if every formula is equivalent to an existential formula.

Definition 2.3.6. A theory T is called *near model complete* if every formula is equivalent to a boolean combination of existential formulas in any model of T.

Definition 2.3.7. Let M and N be two L-structures. We say M, N have the *back and forth property* if, for any finite tuples $\overline{a}$ from M and $\overline{b}$ from N such that $qftp(\overline{a}) = qftp(\overline{b})$ and $c \in M$, then there is $d \in N$ such that $qftp(\overline{a}\ c) = qftp(\overline{b}\ d)$, and dually.

Note that $qftp$ is quantifier free type, and the quantifier free type over a set A is the set of quantifier free formulas that are true using only parameters from the set A.

The following proposition is a strong result in model theory since it allows to determine completeness for some theories. It is as well one of the most important propositions used in this thesis. For more information about it, see [10] and [13].

Proposition 2.3.8. Let T be a theory. T is complete and has quantifier elimination if and only if for any two sufficiently saturated models M, N of T

(i) $qftp_M(\emptyset) = qftp_N(\emptyset)$ and

(ii) M and N have the back and forth property.

Note that a theory T which is complete does not necessarily have quantifier elimination.

Example 2.3.9. The theory of real closed fields in the language of rings does not have quantifier elimination but is complete, see [9] and [13].

Example 2.3.10. The theory of real closed ordered fields in the language of ordered rings has quantifier elimination and is complete.

Example 2.3.11. The theory of algebraically closed fields has quantifier elimination and is complete for a given characteristic, see [13].

Before we move to next section, let us introduce the notion of algebraic closure and definable closure in the two following definitions.

Definition 2.3.12. Let $A \subseteq M$ and $a \in M$. We say a is *algebraic* over A if there is an L-formula $\phi(x)$ with parameters from A satisfied by a and only finitely many other elements of M.
The *algebraic closure* of A, denoted by $acl(A)$, is the set of all $a \in M$ such that a is algebraic over A.

Definition 2.3.13. Let $A \subseteq M$. An element $a \in M$ is in the *definable closure* of A if $\{a\}$ is definable over A.

2.4 Imaginaries and M^{eq} construction

In this section, we consider a language, a complete theory of that language and a model of this theory. By defining some equivalence relations, we

will be able to obtain new structures that are called the equivalence relation structures. Here we will give the construction and some results about the equivalence relation structures. Notions of imaginaries were first introduced by Shelah in [18] in 1979. Here we closely follow the note [11] of Rahim Moosa.
Let L be a language, let T be a complete L-theory and let $M \models T$. We describe the many sorted language L^{eq} in the following way:

- Each $\emptyset$-definable equivalence relation E corresponds to a sort S_E in L^{eq}.
- The symbols of L are interpreted as relations, functions and constants to the sort $S_=$.
- For any n-ary $\emptyset$-definable equivalence relation E, there is a function f_E which sends an n-tuple of the sort $S_=$ to elements of the sort S_E.

We define the theory T^{eq} to be the L^{eq}-theory which consists of all the T-sentences and the following axioms:

- The function f_E is surjective.
- $f_E(\overline{x}) = f_E(\overline{y})$ iff $E(\overline{x}, \overline{y})$.

The L^{eq}-structure M^{eq} can be obtained from M as follows:

- For each $\emptyset$-definable equivalence relation, the elements corresponding to the sort S_E in M^{eq} are equivalence classes $\overline{a}/_E$ where $\overline{a} \in M^n$ and n is the arity of E. These elements are called *imaginaries*.
- $S_=$ in M^{eq} is M and the symbols of L^{eq} that are from L are interpreted in M^{eq} on the sort $S_=$ as in M.
- f_E is interpreted as the quotient map $M^n \longrightarrow M^n/_E$ and it is surjective.

Remark 2.4.1. If $M \models T$ then $M^{eq} \models T^{eq}$, moreover if $N \models T^{eq}$ then $N \equiv M^{eq}$. So T^{eq} is complete.

Let M be a “monster" model of T. It means κ-saturated and strongly κ-homogeneous for some large cardinal κ. There will be many propositions that we are going to prove here. First, let us see how the formulas are related between a structure and its equivalence relation structure.

Proposition 2.4.2. Let $\phi(\overline{x})$ be an L-formula and $\overline{a}$ be a tuple from M. Then $M \models \phi(\overline{a})$ is equivalent to $M^{eq} \models \phi(\overline{a})$.

Proof. By the construction of M^{eq}, M is embedded in M^{eq} as the sort $S_=$ which is M itself and the symbols of L are exactly interpreted in M^{eq} as they are in M. $\square$

The next proposition will be one of the properties of automorphisms that an equivalence relation structure keeps from the original structure.

Proposition 2.4.3. Any automorphism of M has a unique extension to an automorphism of M^{eq}.

Proof. Uniqueness. Let $a \in M^{eq}$. Then for some $\emptyset$-definable equivalence relation E and for some tuple $\overline{a}$ from M, $a = \overline{a}/_E$. Let σ be an automorphism of M and let σ_1, σ_2 be two extensions of σ to M^{eq}. Then $\sigma_1(a) = \sigma_1(\overline{a}/_E) = \sigma_1(\overline{a})/_E = \sigma(\overline{a})/_E = \sigma_2(\overline{a})/_E = \sigma_2(\overline{a}/_E) = \sigma_2(a)$, so $\sigma_1 = \sigma_2$.
Existence. Let $a = \overline{a}/_E = f_E(\overline{a}) \in M^{eq}$ and define $\sigma'(a) = \sigma(\overline{a})/_E$. We first need to prove that σ' is well defined. We have $a = \overline{a}/_E = \overline{b}/_E = b \iff \overline{a}E\overline{b} \iff \sigma(\overline{a})E\sigma(\overline{b}) \iff \sigma(\overline{a})/_E = \sigma(\overline{b})/_E \iff \sigma'(a) = \sigma'(b)$. We proved that σ' is well defined and one to one at the same time. By construction σ' is surjective and preserves the function f_E and maps any sort to itself. It is also clear that σ' extends σ and so preserves all the functions and relations of L^{eq} that are from L. Thus σ' is an automorphism of M^{eq} and σ' extends σ. $\square$

Previously we have seen properties of formulas and automorphisms. Now using some of these properties and the property of strong homogeneity, we have the following.

Proposition 2.4.4. Let $\overline{a_1}, \dots, \overline{a_n}, \overline{b_1}, \dots, \overline{b_n}$ be tuples from M and let $E_1, \dots, E_n$ be $\emptyset$-definable equivalence relations such that $\overline{a_1}/_{E_1}, \dots, \overline{a_n}/_{E_n}, \overline{b_1}/_{E_1}, \dots, \overline{b_n}/_{E_n} \in M^{eq}$. If $tp_L(\overline{a_1} \cdots \overline{a_n}) = tp_L(\overline{b_1} \dots \overline{b_n})$, then

$$tp_{L^{eq}}(\overline{a_1}/_{E_1}, \dots, \overline{a_n}/_{E_n}) = tp_{L^{eq}}(\overline{b_1}/_{E_1}, \dots, \overline{b_n}/_{E_n}).$$

Proof. Using the fact that M is strongly κ-homogeneous, there is an automorphism σ of M such that $\sigma(\overline{a_1}, \dots, \overline{a_n}) = (\overline{b_1}, \dots, \overline{b_n})$. By Proposition 2.4.3, there is an automorphism σ' of M^{eq} which extends σ and so

$\sigma'(\overline{a_1}/_{E_1}, \ldots, \overline{a_n}/_{E_n}) = (\overline{b_1}/_{E_1}, \ldots, \overline{b_n}/_{E_n})$. Since σ' preserves the type then $tp_{L^{eq}}(\overline{a_1}/_{E_1}, \ldots, \overline{a_n}/_{E_n}) = tp_{L^{eq}}(\overline{b_1}/_{E_1}, \ldots, \overline{b_n}/_{E_n})$. □

Definition 2.4.5. Let M be a structure which is sufficiently saturated and strongly homogeneous. A tuple $\bar{e}$ from M^{eq} is a *canonical parameter* or *code* for a definable set X if and only if for all automorphisms σ of M^{eq},

$$\sigma(X) = X \iff \sigma(\bar{e}) = \bar{e}.$$

In fact, every definable set has a code in M^{eq} but sometimes a definable set has one in M^n for some n.
The following example illustrates obviously the existence of codes for certain structures.

Example 2.4.6. For a field K, every finite subset has a canonical parameter. In fact, let $X = \{a_1, \ldots, a_n\}$ and consider the polynomial

$$(x - a_1) \cdots (x - a_n) = x^n + e_{n-1}x^{n-1} + \cdots + e_1 x + e_0,$$

then $e = (e_0, \ldots, e_{n-1}) \in K^n$ is a canonical parameter for X.

Definition 2.4.7. A theory T has *elimination of imaginaries* if and only if every definable set has a canonical parameter in M^n for some n.

We are now giving some examples of theories that have elimination of imaginaries.

Example 2.4.8. The theory of real closed fields has elimination of imaginaries. We will give more details about it in the next chapter.

Example 2.4.9. The theory of algebraically closed fields for a given characteristic has elimination of imaginaries.

Chapter 3

Real-closed Fields

This chapter concerns the study of the theory of real-closed fields. First of all, we will give some definitions illustrated by helpful examples. Then, we will prove that the theory of real-closed fields with the language of ordered rings has quantifier elimination and is complete using Proposition 2.3.8. In the final part of the chapter, we consider notions of o-minimality and sketch the proof of elimination of imaginaries for real-closed fields.

3.1 Definitions and examples

Let us start with some definitions from algebra.

Definition 3.1.1. An *ordered field* $(F, \leq)$ is a field with a total ordering satisfying the following axioms:

- If $x \leq y$ then $x + z \leq y + z$.
- If $0 \leq x$ and $0 \leq y$ then $0 \leq xy$.

Example 3.1.2. The rational numbers and the real numbers are ordered fields, the ordering is the usual inequality.

Definition 3.1.3. A field F is *real-closed* if it is an ordered field satisfying the following properties:

- Every positive element of F has a square root in F.
- Every odd degree polynomial has a zero in F.

The next properties are equivalents for an ordered field, see [12].

(1) F is a real-closed field.

(2) $F(\sqrt{-1})$ is algebraically closed.

(3) For any polynomial $p(x)$ over F, if $a, b \in F$, $a < b$ and $p(a) < 0$ and $p(b) > 0$, then there is $c \in F$ in the interval (a, b) such that $p(c) = 0$.

Example 3.1.4. The real algebraic numbers and the real numbers are real-closed fields.

We are now ready to state the axioms of the theory of *real-closed fields* in the language of ordered rings.

- Axioms for ordered fields.
- $\forall x \exists y (x > 0 \implies y^2 = x)$
- For each $n \geq 0$, $\forall x_0 \cdots x_{2n} \exists y (y^{2n+1} + \sum_{i=0}^{2n} x_i y^i = 0)$.

We define the theory of real-closed ordered fields ($RCOF$) to be the deductive closure of these axioms.
Before going into further details on $RCOF$, we will require two more notions. The first is that of pregeometry, which will highlight nice properties of closure, especially the exchange property that is useful to prove our results. The second is o-minimality, which will help to describe definable sets.

Definition 3.1.5. Let M be a set and let $cl : \mathcal{P}(M) \to \mathcal{P}(M)$ be an operator. Then (M, cl) is a *pregeometry* if:

(i) $A \subseteq M$ implies $A \subseteq cl(A)$,

(ii) for $A, B \subseteq M$ such that $A \subseteq B$, then $cl(A) \subseteq cl(B)$,

(iii) $cl(cl(A)) = cl(A)$,

(iv) for $A \subseteq M$ and $a, b \in M$, if $a \in cl(\{b\} \cup A) \setminus cl(A)$ then $b \in cl(\{a\} \cup A)$. This is called the exchange property.

(v) If $A \subseteq M$ and $a \in cl(A)$, then there is a finite subset B of A such that $a \in cl(B)$.

Proposition 3.1.6. For real-closed fields, $a \in acl(A)$ means there is some non-zero polynomial $p(x)$ with coefficients from the field generated by A such that $p(a) = 0$.

This definition is equivalent to the definition of algebraic closure in chapter 2. The proof of this equivalence uses quantifier elimination.

Example 3.1.7. Algebraic closure in *RCOF* has the exchange property, so it defines a pregeometry, see [19] and [20].

Definition 3.1.8. Let K be an ordered field. A real-closed field R is a *real closure* of K if R is algebraic over K and the ordering of R extends the ordering of K, see [8], [10] and [17].

Definition 3.1.9. Let $(M, <, \dots)$ be a structure where $<$ defines a total ordering. The structure $(M, <, \dots)$ is o-*minimal* if the only definable subsets of the underlying set are finite unions of intervals and points.

O-minimality is a very nice tool for generalizing ideas from real algebraic geometry. In fact, it is inspired by the properties of semialgebraic sets on real-closed fields. More details can be found in [5].

Example 3.1.10. $(\mathbb{R}, <, 0, 1, \cdot, +)$ is o-minimal. This is because it has quantifier elimination. We will see quantifier elimination of *RCOF* in general in the next section.
$(\mathbb{R}, <, 0, 1, \cdot, +, sin)$ is not o-minimal. In fact, the set of solutions to $sin(x) = 0$ is infinite and discrete.

3.2 Quantifier elimination of RCOF

The following theorem is one of the most important theorems that was proved in the theory of real-closed fields. In fact, many more results have been deduced from quantifier elimination. To know more about these results, see [10].
Before introducing the theorem, let us state some well known algebraic facts, that we can find in Chapter XI of Lang's Algebra [8].
Fact 1 : (*Artin-Schreier*) Let $(F, <)$ be an ordered field. Then there exists a real closure R of F.

Moreover if R and R' are two real closures of F, then they are isomorphic over F.
Fact 2 : Let $(F,<)$ be an ordered field and let $(S,<)$ be a real-closed field such that $(F,<) \leq (S,<)$. Then there is a unique real closure $(R,<)$ of $(F,<)$ such that $(F,<) \leq (R,<) \leq (S,<)$. Furthermore F is dense in R.
Now denote by $rcl_S(F)$ the real closure of F in the field S. Remark that

$$rcl_S(F) = acl_{S(\sqrt{-1})}(F) \cap S.$$

Theorem 3.2.1. *The theory of real-closed fields has quantifier elimination and is complete in the language of ordered rings.*

Proof. To prove this result, we present the standard back and forth argument. See, for example, Section 3 of [13].
We will use Proposition 2.3.8 to show quantifier elimination and completeness for real-closed fields.
Let M and N be two sufficiently saturated models of $RCOF$. Let us show first that $qftp_M(\emptyset) = qftp_N(\emptyset)$. By definition, $qftp_M(\emptyset)$ and $qftp_N(\emptyset)$ are the set of all quantifier-free sentences that can be formed from the language $\{0, 1, \cdot, +, <\}$ and are true respectively in M and N. These are just statements about rational numbers. Besides $\mathbb{Q} \subseteq M$ and $\mathbb{Q} \subseteq N$. So $qftp_M(\emptyset) = qftp_N(\emptyset)$.
Let $\bar{a}, \bar{b}$ be finite tuples, respectively from M and N, and let $c \in M$. Assume $qftp(\bar{a}) = qftp(\bar{b})$. We want to show that there is some $d \in N$ such that $qftp(\bar{a}c) = qftp(\bar{b}d)$. Two cases can be considered.
Case $(\mathbf{1})$: $c \in acl_M(\bar{a})$
Let $c \in acl_M(\bar{a})$. We want $d \in acl_N(\bar{b})$ such that $qftp(\bar{a}c) = qftp(\bar{b}d)$. Let $\mathbb{Q}(\bar{a})$ be the field generated by $\bar{a}$. By the Fact 2 there is a real closure of $\mathbb{Q}(\bar{a})$ and denote it by $rcl_M(\mathbb{Q}(\bar{a}))$ and such that $\mathbb{Q}(\bar{a}) \leq rcl_M(\mathbb{Q}(\bar{a})) \leq M \leq M(\sqrt{-1})$. Recall $M(\sqrt{-1})$ is an algebraically closed field. By assumption, $qftp(\bar{a}) = qftp(\bar{b})$, then $\mathbb{Q}(\bar{a}) \simeq \mathbb{Q}(\bar{b})$ via an isomorphism which sends $\bar{a}$ to $\bar{b}$.
We can extend this isomorphism to the respective real closures by Fact 1. Then we have $rcl_M(\mathbb{Q}(\bar{a})) \simeq rcl_N(\mathbb{Q}(\bar{b}))$.
So, by Fact 2, $acl_{M(\sqrt{-1})}(\mathbb{Q}(\bar{a})) \cap M \simeq acl_{N(\sqrt{-1})}(\mathbb{Q}(\bar{b})) \cap N$ via an isomorphism which sends $\bar{a}$ to $\bar{b}$. Therefore, there is some $d \in acl_N(\bar{b})$ such that $qftp(\bar{a}c) = qftp(\bar{b}d)$.
Case $(\mathbf{2})$: $c \notin acl_M(\bar{a})$

Let $c \notin acl_M(\bar{a})$. We want $d \notin acl_N(\bar{b})$ such that $qftp(\bar{a}c) = qftp(\bar{b}d)$. Let $\mathbb{Q}(\bar{a})$ be the field generated by $\bar{a}$. By saturation, we can find $d \in N$ such that the cut of c over $\mathbb{Q}(\bar{a})$ corresponds to the cut of d over $\mathbb{Q}(\bar{b})$. Since $\mathbb{Q}(\bar{a})$ is dense in $rcl_M(\mathbb{Q}(\bar{a}))$ then by saturation we can find $d \in N$ such that the cut of c over $rcl_M(\mathbb{Q}(\bar{a}))$ corresponds to the cut of d over $rcl_N(\mathbb{Q}(\bar{b}))$. Then $d \notin acl_N(\bar{b})$ and $qftp(\bar{a}c) = qftp(\bar{b}d)$.
To conclude, M and N have the back and forth property and $qftp_M(\emptyset) = qftp_N(\emptyset)$. By Proposition 2.3.8, *RCOF* has quantifier elimination and is complete. □

Note that for *RCOF*, definable sets are boolean combinations of polynomial equalities and inequalities. This is due to quantifier elimination. So *RCOF* is o-minimal.

3.3 Elimination of imaginaries

This section will be allocated to elimination of imaginaries for *RCOF*. These results are already known. See [4] , [6] and [19] for more details.

Definition 3.3.1. A theory T has *definable Skolem functions* if for every formula $\phi(\bar{x},\bar{y})$ there is a $\emptyset$-definable function $f(\bar{y})$ such that whenever $M \models T$, $\bar{b} \in M$ and $\phi(M^n,\bar{b})$ is non-empty, then $f(\bar{b}) \in \phi(M^n,\bar{b})$. Recall that $\phi(M^n,\bar{b}) = \{\bar{a} \in M^n : M \models \phi(\bar{a},\bar{b})\}$.
Note that functions are said to be definable if their graphs are definable sets.

Theorem 3.3.2. [4] *RCOF has definable Skolem functions.*

Claim 3.3.3. [19] Let $M \models RCOF$ and let $X \subseteq M^{n+m}$ be a $\emptyset$-definable set. There is a definable function $f : M^n \longrightarrow M^m$ such that $f(\bar{a}) \in X_{\bar{a}}$ if $X_{\bar{a}} \neq \emptyset$, where $X_{\bar{a}} = \{\bar{b} \in M^m : (\bar{a},\bar{b}) \in X\}$, for all $\bar{a} \in M^n$.

Proof of Claim 3.3.3. The proof follows the one in [19]. It is an induction proof. Assume that $m = 1$.Then by o-minimality, $X_{\bar{a}}$ is a finite union of points and intervals. If $X_{\bar{a}} = \emptyset$ or $X_{\bar{a}} = M$ then we can take $f(\bar{a}) = 0$.

Now assume $X_{\bar{a}}$ is non-empty. We define $f(\bar{a})$ as follows:

$$\begin{aligned} f(\bar{a}) &= b && \text{if} \quad X_{\bar{a}} = \{b\}, \\ f(\bar{a}) &= b-1 && \text{if} \quad X_{\bar{a}} = (-\infty, b), \\ f(\bar{a}) &= b+1 && \text{if} \quad X_{\bar{a}} = (b, \infty), \\ f(\bar{a}) &= \frac{b+c}{2} && \text{if} \quad X_{\bar{a}} = (b, c). \end{aligned}$$

If the set $X_{\bar{a}}$ is a finite disjoint union of such cases, then we could take the interval with the lowest upperbound.
Now assume the claim is true for m and let $X \subseteq M^{n+m+1}$. By induction hypothesis there is a definable function $f : M^{n+1} \longrightarrow M^m$ such that if $a_1, \ldots, a_n, b \in M$ and $X_{a_1 \ldots a_n b} \neq \emptyset$ then we have $(\bar{a}, b, f(\bar{a}, b)) \in X$. By the base case there is a definable function $g : M^n \longrightarrow M$ such that $(\bar{a}, g(\bar{a}))$ is in the projection $\pi(X)$ of X to the $n+1$ first coordinates, provided $\pi(X)_{\bar{a}} \neq \emptyset$. Let $h : M^n \longrightarrow M^{m+1}$ such that $h(\bar{x}) = (g(\bar{x}), f(\bar{x}, g(\bar{x})))$. If $\bar{a} \in M^n$ then $(\bar{a}, h(\bar{a})) = (\bar{a}, g(\bar{a}), f(\bar{a}, g(\bar{a})))$. By construction g, f are $\emptyset$-definable and $(\bar{a}, g(\bar{a}),\ f(\bar{a}, g(\bar{a}))) \in X$, provided $X_{\bar{a}} \neq \emptyset$. Then h is $\emptyset$-definable and $(\bar{a}, h(\bar{a})) \in X$, provided $X_{\bar{a}} \neq \emptyset$. □

In [14], Poizat defines uniform elimination of imaginaries as follows. We state it now and later we will prove that uniform elimination implies elimination of imaginaries as defined in Chapter 2.

Definition 3.3.4. Let T be an L-theory with at least two constant symbols. Let $M \models T$. T admits *uniform elimination of imaginaries* if for every $\emptyset$-definable equivalence relation E on M^n, there is a $\emptyset$-definable function $f : M^n \longrightarrow M^m$ such that $\bar{a}E\bar{b}$ if and only if $f(\bar{a}) = f(\bar{b})$.

The next results and proofs are taken from [6].

Theorem 3.3.5. *Assume $T \models RCOF$. T has Definable Skolem function implies T has uniform elimination of imaginaries.*

Proof. Let $E \subseteq M^{n+n}$ be a $\emptyset$-definable equivalence relation. Let f be a $\emptyset$-definable function as in Claim 3.3.3. So for $\bar{a} \in M^n$, $(\bar{a}, f(\bar{a})) \in E$.
We may assume f is such that $f(\bar{a})$ depends on $X_{\bar{a}}$ and not the choice of $\bar{a}$. This is clear in the base case of the proof of Claim 3.3.3 and can be preserved in the induction.
So $f(\bar{a}) = f(\bar{b})$ if and only if $\bar{a}E\bar{b}$. □

Theorem 3.3.6. *Uniform elimination of imaginaries implies elimination of imaginaries.*

Proof. Let X be a definable set, defined by the formula $\phi(\overline{x}, \overline{a})$. Let E be a $\emptyset$-definable equivalence relation such that $\overline{y}E\overline{y'} \equiv \forall\overline{x}(\phi(\overline{x}, \overline{y}) \longleftrightarrow \phi(\overline{x}, \overline{y'}))$. There is a $\emptyset$-definable function f such that $\overline{y}E\overline{y'}$ if and only if $f(\overline{y}) = f(\overline{y'})$. We can now define X by the formula $\exists\overline{y}(\phi(\overline{x}, \overline{y}) \wedge f(\overline{y}) = f(\overline{a}))$.
Let $\sigma \in Aut(M)$, if $\sigma(f(\overline{a})) = f(\overline{a})$ then we get $\sigma(X) = X$.
Now, let $\sigma \in Aut(M)$ such that $\sigma(X) = X$. $\sigma(X)$ is defined by $\phi(\overline{x}, \sigma(\overline{a}))$. Since $\sigma(X) = X$, then $\overline{a}E\sigma(\overline{a})$. Thus $f(\overline{a}) = f(\sigma(\overline{a}))$, and then $f(\overline{a}) = \sigma(f(\overline{a}))$.
So $f(\overline{a})$ is a code for X.
Since any definable set of (M, B) has a code, if we have uniform elimination of imaginaries, then we obtain elimination of imaginaries. $\square$

Chapter 4

Completeness of dense pairs of real-closed fields

In 1958, the completeness of the theory of pairs (M, B), where M is a real-closed field and B is a proper dense real-closed subfield, has been proved by Abraham Robinson in [16]. His method consists of constructing a sequence of extension fields and uses algebra, i.e. properties of real-closed fields, especially the notions of algebraic independence and basis.
In 1998, Lou Van den Dries in his paper [4] generalised the work of Robinson and gave descriptions of definable sets. This chapter is based on this paper. Most of the study of the theory depends on quantifier elimination using the back and forth property that is defined in Definition 2.3.7. More precisely, we are using Proposition 2.3.8 to prove the completeness of the theory of dense pairs of real-closed fields.

4.1 Definitions

In this section, we give a definition of cut. We also describe the notion of dense pair of real-closed fields.

Definition 4.1.1. Let M be $RCOF$. If B is a substructure of M, $p(x)$ is a *cut* over B if there exist two sets A, C such that:

(i) $A \cap C = \emptyset$,

(ii) $A \cup C = B$,

(iii) $a < c$ for all $a \in A$ and $c \in C$,

(iv) $p(x) = \{a < x < c : a \in A, c \in C\}$.

Definition 4.1.2. Let A and B be two models of the theory of real-closed ordered fields. A *dense pair* is a pair (B, A) such that A is an elementary substructure of B, $A \neq B$ and A is dense in B.

4.2 Quantifier elimination

Let $L = \{+, \cdot, 0, 1, <\}$ be the language of ordered rings, and $L^* = \{+, \cdot, 0, 1, <, U, D_\varphi\}$ the language of the structure $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$ where U is an unary predicate such that $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}}) \models U(a)$ if and only if $a \in \mathbb{R} \cap \overline{\mathbb{Q}}$ and, for each L-formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_k)$, $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}}) \models D_\varphi(a_1, \ldots, a_n)$ if and only if there exist $b_1, \ldots, b_k \in \mathbb{R} \cap \overline{\mathbb{Q}}$ such that $\mathbb{R} \models \varphi(a_1, \ldots, a_n, b_1, \ldots, b_k)$. Note that the underlying set of the structure $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$ is $\mathbb{R}$.
Let us start by some lemmas concerning the set of real numbers which will be useful later to prove theorems.

Lemma 4.2.1. For all $i, j \in \mathbb{R}$, $i < j$, and any finite $C \subseteq \mathbb{R}$, the interval (i, j) is not contained in $acl_L(C \cup (\mathbb{R} \cap \overline{\mathbb{Q}}))$. Recall acl_L to be the algebraic closure in sense of L that we have defined in the previous chapter.

Proof. Assume the interval (i, j) is contained in $acl_L(C \cup (\mathbb{R} \cap \overline{\mathbb{Q}}))$. Since C is finite and $\mathbb{R} \cap \overline{\mathbb{Q}}$ is countable then $C \cup (\mathbb{R} \cap \overline{\mathbb{Q}})$ is countable. Hence $acl_L(C \cup (\mathbb{R} \cap \overline{\mathbb{Q}}))$ is also countable. However the interval (i, j) is not countable. We have contradiction. Then the interval (i, j) is not contained in $acl_L(C \cup (\mathbb{R} \cap \overline{\mathbb{Q}}))$. □

Let (M, B) be such that $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}}) \prec (M, B)$ and (M, B) is sufficiently saturated.

Lemma 4.2.2. For all $i, j \in M$, $i < j$, and any finite $C \subseteq M$, the interval (i, j) is not contained in $acl_L(C \cup B)$.

Proof. We are going to write the Lemma 4.2.1 in terms of formulas.
Let $F = \{P_1, \ldots, P_k\}$ be a set of polynomials over $\mathbb{Z}$ in the variables $x, \overline{y}$ and $\overline{z_i}$.
Let $\varphi_{P_i}(x, \overline{y}, \overline{z_i})$ be $\big(P_i(x, \overline{y}, \overline{z_i}) = 0 \wedge \exists w(P_i(w, \overline{y}, \overline{z_i}) \neq 0) \wedge U(z_{j_1}) \cdots \wedge U(z_{j_i})\big)$ where $z_{j_1}, \ldots, z_{j_i}$ are the components of the tuple $\overline{z_i}$. Now we can translate the Lemma 4.2.1 as follows:

Let C be the set $\{c_1, \ldots, c_n\}$ and let $\bar{c}$ be the tuple $c_1, \ldots, c_n$.
Consider

$$\theta_F(x) \equiv \Big(i < x < j \wedge \neg\exists\overline{z_1}\varphi_{P_1}(x, \bar{c}, \overline{z_1}) \wedge \neg\exists\overline{z_2}\varphi_{P_2}(x, \bar{c}, \overline{z_2}) \wedge \ldots \quad \wedge \quad \neg\exists\overline{z_k}\varphi_{P_k}(x, \bar{c}, \overline{z_k})\Big).$$

Since (M, B) is an elementary extension of $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$ and by Lemma 4.2.1, each such $\theta_F(x)$ is realised in M. Then there is $a \in M$ such that $M \models \theta_F(a)$. Then by saturation there is $a \in M$ such that $M \models \theta_F(a)$ for every such $\theta_F(a)$. This concludes the proof of the Lemma 4.2.2. □

The following result is partly from [16] where model-completeness is shown. The proof we present is based on the back and forth argument given in [4].

Theorem 4.2.3. *The L^*-structure $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$ has quantifier elimination.*

Proof. Let us first remind ourselves that a subset $A \subseteq M$ is independent if for every $a \in A$, $a \notin dcl_L(A \setminus \{a\})$. Any subset A of M contains a maximal independent subset and each such set is called basis of A.
Let (M, B) be such that $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}}) \prec (M, B)$ and (M, B) is sufficiently saturated.
Let $\overline{a'}$ be acl_L-independent over B, and $\overline{a_B}$ a tuple from B. Similarly, let $\overline{b'}$ be acl_L-independent over B, and $\overline{b_B}$ a tuple from B.

Claim 4.2.4. If $c \in B$ then there is some $d \in B$ such that if $tp_L(\overline{a'}\ \overline{a_B}) = tp_L(\overline{b'}\ \overline{b_B})$ then

$$tp_L(\overline{a'}\ \overline{a_B}\ c) = tp_L(\overline{b'}\ \overline{b_B}\ d).$$

Moreover, by repeated application of this process, we get

$$qftp_{L^*}(\overline{a'}\ \overline{a_B}) = qftp_{L^*}(\overline{b'}\ \overline{b_B}).$$

Proof of Claim 4.2.4. This claim is proved by considering the following two cases.
Case (i) : $c \in acl_L(\overline{a'} \cup \overline{a_B})$
Suppose $c \in acl_L(\overline{a'} \cup \overline{a_B})$ then $c \in dcl_L(\overline{a'} \cup \overline{a_B})$.
By quantifier elimination of M as an L-structure, there exists $d \in M$ such that $tp_L(\overline{a'}\ \overline{a_B}\ c) = tp_L(\overline{b'}\ \overline{b_B}\ d)$ and d is unique. Now we need to show that $d \in B$.

First let us show that $c \in acl_L(\overline{a_B})$. Suppose $c \notin acl_L(\overline{a_B})$ and let $\overline{a''}$ be a minimal length subtuple of $\overline{a'}$ such that $c \in acl_L(\overline{a''} \cup \overline{a_B})$.
Say $\overline{a''} = a''_1, \dots, a''_k$ such that $c \in acl_L(\overline{a_B} \cup \{ a''_1, \dots, a''_{k-1}, a''_k\}) \setminus acl_L(\overline{a_B} \cup \{a''_1, \dots, a''_{k-1}\})$. Then by the exchange property,

$$a''_k \in acl_L(\overline{a_B} \cup \{ a''_1, \dots, a''_{k-1}\} \cup \{c\}).$$

This is in contradiction with the fact that $\overline{a''}$ is acl_L-independent over B.
Therefore $c \in acl_L(\overline{a_B}) = dcl_L(\overline{a_B})$, and thus $d \in dcl_L(\overline{b_B}) \subseteq B$, because $B \prec M$ is an L-structures. Then we have $d \in B$ such that

$$tp_L(\overline{a'}\ \overline{a_B}\ c) = tp_L(\overline{b'}\ \overline{b_B}\ d).$$

Case (ii) : $c \notin acl_L(\overline{a'} \cup \overline{a_B})$
Suppose $c \notin acl_L(\overline{a'} \cup \overline{a_B})$ and recall $tp_L(\overline{a'}\ \overline{a_B}) = tp_L(\overline{b'}\ \overline{b_B})$. By saturation and the denseness property, we can find $d \in B$ realising a cut over $dcl_L(\overline{b'} \cup \overline{b_B})$ which corresponds to the cut over $dcl_L(\overline{a'} \cup \overline{a_B})$ realised by c. Then

$$tp_L(\overline{a'}\ \overline{a_B}\ c) = tp_L(\overline{b'}\ \overline{b_B}\ d).$$

This completes the proof of the Claim 4.2.4. □

We are now able to prove the Theorem 4.2.3.
Let $\overline{a}, \overline{b} \in M^n$ such that $qftp_{L^*}(\overline{a}) = qftp_{L^*}(\overline{b})$. Let $\overline{a'}$ be an acl_L-basis for $\overline{a}$ over B and $\overline{b'}$ the corresponding subtuple acl_L-basis of $\overline{b}$ over B. We let $\overline{a_B} = \overline{a} \cap B$ and $\overline{b_B} = \overline{b} \cap B$.
Since we only have one structure (M, B) in this section then we are using the same structure to apply the back and forth argument.
Let $c \in M$. We want to show that there is $d \in M$ such that $qftp_{L^*}(\overline{a}\ c) = qftp_{L^*}(\overline{b}\ d)$.
Three cases can be considered.
Case (1): $c \in B$
Suppose $c \in B$. From the definition of $\overline{a'}$, each element of $\overline{a}$ is in $dcl_L(\overline{a'} \cup B)$. Let $\overline{c} \in B^m$ such that each element of $\overline{a}$ is in $dcl_L(\overline{a'} \cup \overline{c})$. Using the Claim 4.2.4 m times, we obtain $\overline{d} \in B^m$ such that $qftp_{L^*}(\overline{a'}\ \overline{a_B}\ \overline{c}) = qftp_{L^*}(\overline{b'}\ \overline{b_B}\ \overline{d})$. Therefore

$$qftp_{L^*}(\overline{a}\ \overline{c}) = qftp_{L^*}(\overline{b}\ \overline{d}).$$

We may assume $c \in \overline{c}$. Then there is a corresponding $d \in \overline{d}$ with

$$qftp_{L^*}(\overline{a}\ c) = qftp_{L^*}(\overline{b}\ d).$$

Case (**2**) : $c \notin acl_L(\overline{a} \cup B)$
Suppose $c \notin acl_L(\overline{a} \cup B)$. Let $\overline{c}$, $\overline{d}$ be as in Case (1). We want to find $d \notin acl_L(\overline{b} \cup B)$ such that $tp_L(\overline{a}\ \overline{c}\ c) = tp_L(\overline{b}\ \overline{d}\ d)$.
By saturation and Lemma 4.2.2, we can find $d \notin acl_L(\overline{b} \cup B)$ and which realises the cut over $dcl_L(\overline{b}\ \overline{d})$ corresponding to the cut over $dcl_L(\overline{a}\ \overline{c})$ realised by c, which implies $tp_L(\overline{a}\ \overline{c}\ c) = tp_L(\overline{b}\ \overline{d}\ d)$. Since $c \notin acl_L(\overline{a} \cup B)$ and $d \notin acl_L(\overline{b} \cup B)$, it follows from Claim 4.2.4 that $qftp_{L^*}(c\ \overline{a'}\ \overline{a_B}\ \overline{c}) = qftp_{L^*}(d\ \overline{b'}\ \overline{b_B}\ \overline{d})$.
We then have

$$qftp_{L^*}(\overline{a}\ c) = qftp_{L^*}(\overline{b}\ d).$$

Case (**3**): $c \notin B$ but $c \in acl_L(\overline{a} \cup B)$
Suppose $c \notin B$ and $c \in acl_L(\overline{a} \cup B)$. Then $c \in dcl_L(\overline{a} \cup B)$. Let $\overline{c} \in B^m$ be such that $\overline{a}\ c \in acl_L(\overline{a'} \cup \overline{a_B} \cup \overline{c}) = dcl_L(\overline{a'} \cup \overline{a_B} \cup \overline{c})$.
By Case (1), there exists $\overline{d} \in B^m$ such that $qftp_{L^*}(\overline{a}\ \overline{c}) = qftp_{L^*}(\overline{b}\ \overline{d})$. By quantifier elimination for M as L-structure, there is a unique $d \in M$ such that $tp_L(\overline{a}\ \overline{c}\ c) = tp_L(\overline{b}\ \overline{d}\ d)$. By Claim 4.2.4, $qftp_{L^*}(\overline{a'}\ \overline{a_B}\ \overline{c}) = qftp_{L^*}(\overline{b'}\ \overline{b_B}\ \overline{d})$. Therefore

$$qftp_{L^*}(\overline{a}\ c) = qftp_{L^*}(\overline{b}\ d).$$

This concludes the proof of quantifier elimination. □

Remark 4.2.5. Let $L' = \{0, 1, \cdot, +, <, U\}$. It follows from the quantifier elimination of the theory $Th_{L^*}(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$ that the theory $Th_{L'}(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$ is near-model complete. In fact without the D_φ predicates, the theory of dense pairs of real-closed fields does not have quantifier elimination but any formula can be written as a boolean combination of existential formulas.

We have proved that the theory of $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$ has quantifier elimination in the language L^*. Using a similar argument, we are going to give a general proof of the completeness of the theory of dense pairs of real-closed fields in the language of dense pairs L'.
Let T be the theory of $RCOF$ and L the language associated with it. Denote by T^* the theory of dense pairs of $RCOF$, and L^* its language, where L^*

is an extension of L defined earlier. Remark that in this study, we are still using a one-sorted approach.

4.3 Completeness

Let T^* be the set of all logical consequences of axioms expressible as L^*-sentences which say:

- B and M are $RCOF$,
- $B \leq M$,
- $B \neq M$,
- $\forall i,\ j \in M,\ i < j, \exists b \in B$ such that $i < b < j$,
- U and D_φ are interpreted in (M, B) as in the previous section.

In order to prove the next theorem, we need a variant of Lemma 4.2.2. In this section, we state again the lemma and give different proof. In fact we use the axioms of dense pairs of real-closed fields to prove the lemma instead of using an elementary extension of $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$. This result and proof are taken from van den Dries paper [4] and we omit some details.

Lemma 4.3.1. For all $i, j \in M,\ i < j$, and any finite $C \subseteq M$, the interval (i, j) is not contained in $acl_L(C \cup B)$.

Proof of Lemma 4.3.1. Let us prove the following claim first.

Claim 4.3.2. Let B, M be two models of $RCOF$. Let $B \subseteq M$, let $f : M^{n+1} \longrightarrow M$ be B-definable in M, and let $a \in M \setminus B$. Then there exist $b_0, \dots, b_n \in B$ such that

$$b_0 + b_1 a + \cdots + b_n a^n \notin f(B^n \times \{a\}).$$

Proof of Claim 4.3.2. We are not giving a full proof of this claim but one can prove it using dimension. In fact, the set $\{f(b_1, \dots, b_n, a) : b_1, \dots, b_n \in B\}$ is at most n-dimensional and the set $\{b_0 + b_1 a + \cdots + b_n a^n : b_0, \dots, b_n \in B\}$ is an $n + 1$-dimensional vector space. $\square$

In proving the lemma, it is enough to show that if $g : M^n \longrightarrow M$ is definable in M, then $g(B^n) \neq M$.
Let $g : M^n \longrightarrow M$ be definable in M, then there is $h : M^{n+p} \longrightarrow M$ that is B-definable in M and $(a_1, \ldots, a_p) \in M^p$ such that $g(x) = h(x, a_1, \ldots, a_p)$ for all $x \in M^n$.
By enlarging B if necessary, we may assume $M \subseteq acl_L(\{a\} \cup B)$.
For all i, since $a_i \in acl_L(\{a\} \cup B)$, then $a_i = l_i(a)$ for some function $l_i : M \longrightarrow M$ that is B-definable in M. Adopting the notations in the previous claim, let $f : M^{n+1} \longrightarrow M$ such that $f(x,y) = h(x, l_1(y), \ldots, l_p(y))$. It follows that

$$f(B^n, a) = h(B^n, l_1(a), \ldots, l_p(a)) \neq M.$$

So,

$$g(B^n) \neq M.$$

□

As with Theorem 4.2.3, the following result has also been taken from Section 2 of [4], although completeness and model-completeness are shown in [16]. Proofs are still based on the proofs in [4].

Theorem 4.3.3. *The theory T^* has quantifier elimination and is complete.*

Proof. The proof of this theorem is very similar to the proof of quantifier elimination that we have seen in Section 4.2. Instead of working directly with some saturated elementary extension of $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$, we use two models saturated and homogeneous of the theory of dense pairs of real-closed fields. It will then follow that $T^* = Th(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$.
Let (K, A) and (M, B) be two models of the theory T^* such that they are both κ-saturated for some large κ. We know that K, M, A and B are real-closed fields. Moreover A and B are respectively dense in K and M. We have $K \equiv M$ as L-structures, K and M have quantifier elimination as L-structures.
The following claims will be used to prove completeness of dense pairs of real-closed fields.

Claim 4.3.4. $qftp^K_{L^*}(\emptyset) = qftp^M_{L^*}(\emptyset)$.

Proof of Claim 4.3.4. By definition, $qftp^M_{L^*}(\emptyset)$ and $qftp^K_{L^*}(\emptyset)$ are just the set of all sentences that can be formed from the language $\{0, 1, \cdot, +, <, U, D_\varphi\}$, and are true respectively in (K, A) and (M, B). These are equivalent to L-sentences in the L-structures A and B. Since $A \equiv B$, we have

$$qftp^K_{L^*}(\emptyset) = qftp^M_{L^*}(\emptyset).$$

□

Let $\overline{a'}$ be acl^K_L-independent over A, and $\overline{a_A}$ a tuple from A. Similarly, let $\overline{b'}$ be acl^M_L-independent over B, and $\overline{b_B}$ a tuple from B. The next claim is basically the same as Claim 4.2.4.

Claim 4.3.5. If $c \in A$ then there is some $d \in B$ such that if $tp^K_L(\overline{a'}\ \overline{a_A}) = tp^M_L(\overline{b'}\ \overline{b_B})$ then

$$tp^K_L(\bar{a}\ \overline{a_A}\ c) = tp^M_L(\bar{b}\ \overline{b_B}\ d).$$

Furthermore, note that by repeated application of the first part of this claim, we obtain

$$qftp^K_{L^*}(\overline{a'}\ \overline{a_A}) = qftp^M_{L^*}(\overline{b'}\ \overline{b_B}).$$

Proof of claim 4.3.5. The proof of this claim contains two cases.
Case (i) : $c \in acl^K_L(\overline{a'} \cup \overline{a_A})$
Suppose $c \in acl^K_L(\overline{a'} \cup \overline{a_A})$ then $c \in dcl^K_L(\overline{a'} \cup \overline{a_A})$. By quantifier elimination of K and M as L-structures, there exists $d \in M$ such that $tp^K_L(\overline{a'}\ \overline{a_A}\ c) = tp^M_L(\overline{b'}\ \overline{b_B}\ d)$ and d is unique. We need to show that $d \in B$.
First we are showing that $c \in acl^K_L(\overline{a_A})$.
Suppose $c \notin acl^K_L(\overline{a_A})$ and let $\overline{a''}$ be a minimal length subtuple of $\overline{a'}$ such that $c \in acl^K_L(\overline{a''} \cup \overline{a_A})$. Say $\overline{a''} = a''_1, \ldots, a''_k$ such that $c \in acl^K_L(\overline{a_A} \cup \{a''_1, \ldots, a''_{k-1}, a''_k\}) \setminus acl^K_L(\overline{a_A} \cup \{a''_1, \ldots, a''_{k-1}\})$. Then by the exchange property, we have

$$a''_k \in acl^K_L(\overline{a_A} \cup \{a''_1, \ldots, a''_{k-1}\} \cup \{c\}).$$

This is in contradiction with the fact that $\overline{a''}$ is acl^K_L-independent over A. Therefore $c \in acl^K_L(\overline{a_A}) = dcl^K_L(\overline{a_A})$, and so $d \in dcl^M_L(\overline{b_B}) \subseteq B$, because $B \prec M$ as L-structures.
Then we have $d \in B$ such that

$$tp^K_L(\overline{a'}\ \overline{a_A}\ c) = tp^M_L(\overline{b'}\ \overline{b_B}\ d).$$

Case (ii) : $c \notin acl^K_L(\overline{a'} \cup \overline{a_A})$
Suppose $c \notin acl^K_L(\overline{a'} \cup \overline{a_A})$ and $tp^K_L(\overline{a'}\ \overline{a_A}) = tp^M_L(\overline{b'}\ \overline{b_B})$.

By saturation, we can find $d \in B$ realising a cut over $dcl_L^M(\overline{b'} \cup \overline{b_B})$ which corresponds to the cut over $dcl_L^K(\overline{a'} \cup \overline{a_A})$ realised by c.
Then

$$tp_L^K(\overline{a'}\ \overline{a_A}\ c) = tp_L^M(\overline{b'}\ \overline{b_B}\ d).$$

This concludes the proof of the Claim 4.3.5. $\square$

The next step is to prove that (K, A) and (M, B) have the back and forth property.
Let $\overline{a} \in K^n$, $\overline{b} \in M^n$ such that $qftp_{L^*}^K(\overline{a}) = qftp_{L^*}^M(\overline{b})$. Let $\overline{a'}$ an acl_L^K-basis for $\overline{a}$ over A and $\overline{b'}$ the corresponding subtuple acl_L^M-basis of $\overline{b}$ over B. We let $\overline{a_A} = \overline{a} \cap A$ and $\overline{b_B} = \overline{b} \cap B$.
Let $c \in M$. We want to show that there is some $d \in M$ such that

$$qftp_{L^*}^K(\overline{a}\ c) = qftp_{L^*}^M(\overline{b}\ d).$$

Three cases can be considered as before.
Case $(\mathbf{1})$: $c \in A$
Suppose $c \in A$. We know that each element of $\overline{a}$ is in $dcl_L^K(\overline{a'} \cup A)$. Let $\overline{c} \in A^m$ such that each element of $\overline{a}$ is in $dcl_L^K(\overline{a'} \cup \overline{c})$. Using Claim 4.3.5 m times, we obtain $\overline{d} \in B^m$ such that $qftp_{L^*}^K(\overline{a'}\ \overline{a_A}\ \overline{c}) = qftp_{L^*}^M(\overline{b'}\ \overline{b_B}\ \overline{d})$.
Therefore

$$qftp_{L^*}^K(\overline{a}\ \overline{c}) = qftp_{L^*}^M(\overline{b}\ \overline{d}).$$

We may assume $c \in \overline{c}$. Then there is a corresponding $d \in \overline{d}$ such that

$$qftp_{L^*}^K(\overline{a}\ c) = qftp_{L^*}^M(\overline{b}\ d).$$

Case $(\mathbf{2})$: $c \notin acl_L^K(\overline{a} \cup A)$
Suppose $c \notin acl_L^K(\overline{a} \cup A)$. Let $\overline{c}$, $\overline{d}$ be as in Case (1). We want to find $d \notin acl_L^M(\overline{b} \cup B)$ such that $tp_L^K(\overline{a}\ \overline{c}\ c) = tp_L^M(\overline{b}\ \overline{d}\ d)$.
By saturation and Lemma 4.3.1, we can find $d \notin acl_L^M(\overline{b} \cup B)$ and which realises the cut over $dcl_L^M(\overline{b}\ \overline{d})$ corresponding to the cut over $dcl_L^K(\overline{a}\ \overline{c})$ realised by c, which implies $tp_L^K(\overline{a}\ \overline{c}\ c) = tp_L^M(\overline{b}\ \overline{d}\ d)$. Since $c \notin acl_L^K(\overline{a} \cup A)$ and $d \notin acl_L^M(\overline{b} \cup B)$, it follows from Claim 4.3.5 that $qftp_{L^*}^K(c\ \overline{a'}\ \overline{a_A}\ \overline{c}) = qftp_{L^*}^M(d\ \overline{b'}\ \overline{b_B}\ \overline{d})$.
Then we get

$$qftp_{L^*}^K(\overline{a}\ c) = qftp_{L^*}^M(\overline{b}\ d).$$

Case $(\mathbf{3})$: $c \notin A$ but $c \in acl_L^K(\overline{a} \cup A)$

Suppose $c \notin A$ but $c \in acl_L^K(\overline{a} \cup A) = dcl_L^K(\overline{a} \cup A)$.
Let $\overline{c} \in A^m$ be such that $\overline{a}\ c \in acl_L^K(\overline{a'} \cup \overline{a_A} \cup \overline{c}) = dcl_L^K(\overline{a'} \cup \overline{a_A} \cup \overline{c})$.
By Case (1), there exists $\overline{d} \in B^m$ such that $qftp_{L^*}^K(\overline{a}\ \overline{c}) = qftp_{L^*}^M(\overline{b}\ \overline{d})$. By quantifier elimination for K, M as L-structures, there is a unique $d \in M$ such that $tp_L^K(\overline{a}\ \overline{c}\ c) = tp_L^M(\overline{b}\ \overline{d}\ d)$. By Claim 4.3.5, $qftp_{L^*}(\overline{a'}\ \overline{a_B}\ \overline{c}) = qftp_{L^*}(\overline{b'}\ \overline{b_B}\ \overline{d})$. Therefore

$$qftp_{L^*}(\overline{a}\ c) = qftp_{L^*}(\overline{b}\ d).$$

We have proved that the theory T^*, the theory of dense pairs of real-closed fields in the language L^*, has back and forth property. Claim 4.3.4 and back and forth property ensure completeness and quantifier elimination. $\square$

Remark 4.3.6. Let T' be the theory of dense pairs of real-closed fields in the language $L' = \{+, \cdot, 0, 1, <, U\}$. The theory T' is complete.
This is because the D_φ predicates are $\emptyset$-definable in any model of T'.

Chapter 5

Imaginaries in dense pairs of real-closed fields

In this chapter we will study imaginaries in dense pairs of real-closed fields. There are already interesting results about imaginaries, which can mostly be found in [1], [2] and [3]. In fact we will introduce first some of these results in Section 5.2. Then, we will focus on the relations between acl^{eq} and dcl^{eq}.

5.1 Definable sets

To better understand the concept of imaginaries, a good comprehension of definable sets in dense pairs is required. From now on, we are going to work in a saturated elementary extension (M, B) of $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$.
Let L be the language of ordered rings $\{0, 1, +, \cdot, <\}$ and L' the extension of L with a unary predicate U defined in chapter 4. Recall L^* to be the language of dense pairs of real-closed fields as in chapter 4, having quantifier elimination.
Let us start with some results taken from [4], which help to describe definable sets.

Lemma 5.1.1. Let $\bar{c}$ be a tuple such that if $\overline{c'}$ is a basis for $\bar{c}$ over B and $c_B = \bar{c} \cup B$ then $\bar{c} \in acl_L(\overline{c'} \cup \overline{c_B})$. Let $\bar{a}, \bar{b} \in B^n$ such that $tp_{L'}(\bar{c}\,\bar{a}) \neq tp_{L'}(\bar{c}\,\bar{b})$ where $L^{'} = \{0, 1, +, \cdot, U, <\}$. Then $tp_L(\bar{c}\,\bar{a}) \neq tp_L(\bar{c}\,\bar{b})$.

Proof. Assume that $tp_L(\bar{c}\,\bar{a}) = tp_L(\bar{c}\,\bar{b})$. By quantifier elimination of (M, B) as an L^*-structure and by Claim 4.2.4, we have $tp_{L^*}(\bar{c}\,\bar{a}) = tp_{L^*}(\bar{c}\,\bar{b})$. This

implies that $tp_{L'}(\bar{c}\,\bar{a}) = tp_{L'}(\bar{c}\,\bar{b})$, which is in contradiction to our hypothesis.
Therefore $tp_L(\bar{c}\,\bar{a}) \neq tp_L(\bar{c}\,\bar{b})$. □

Proposition 5.1.2. Let $X \subseteq B^n$ be definable in (M, B). Then there exists $Y \subseteq M^n$ definable in M such that $X = Y \cap B^n$.

Proof. Let $\bar{c} \in M^m$ be a finite tuple such that X is defined over $\bar{c}$, and assume $\bar{c}$ is as in Lemma 5.1.1. Let $\bar{a} \in X$ and $\bar{b} \in B^n \setminus X$. Then Lemma 5.1.1 says that there is a set $Y_{\bar{a}\bar{b}}$ definable over $\bar{c}$ in M such that $\bar{a} \in Y_{\bar{a}\bar{b}}$ and $\bar{b} \notin Y_{\bar{a}\bar{b}}$.
We fix $\bar{a} \in X$. We have $\bigcap\{Y_{\bar{a}\bar{b}} \cap B^n : \bar{b} \in B^n \setminus X\} \subseteq X$. Since there are only countably many $Y_{\bar{a}\bar{b}}$, then by saturation there is a finite set $G_{\bar{a}} \subseteq B^n \setminus X$ such that $\bigcap\{Y_{\bar{a}\bar{b}} \cap B^n : \bar{b} \in G_{\bar{a}}\} \subseteq X$. Let $Y_{\bar{a}} = \bigcap\{Y_{\bar{a}\bar{b}} : \bar{b} \in G_{\bar{a}}\}$, then $\bar{a} \in Y_{\bar{a}} \cap B^n \subseteq X$ and $Y_{\bar{a}}$ is definable over $\bar{c}$ in M.
Consequently $X = \bigcup\{Y_{\bar{a}} \cap B^n : \bar{a} \in X\}$. By saturation again $X = \bigcup\{Y_{\bar{a}} \cap B^n : \bar{a} \in F\}$ for a finite set $F \subseteq X$.
Now let $Y = \bigcup\{Y_{\bar{a}} : \bar{a} \in F\}$, then Y is definable over $\bar{c}$ in M and $X = Y \cap B^n$. □

The following definition is about small closure. It is originally described in [4], however we use notations from [3]. More properties about it can be found in [1], [2] and [3].
Let (M, B) be a dense pair and $A \subseteq M$.

Definition 5.1.3. Let $b \in M$. We say that b is in the *small closure* of A, denoted by $scl(A)$, if $b \in acl_L(A \cup B)$. That is $scl(A) = acl_L(A \cup B)$.

Proposition 5.1.4. *scl* satisfies the exchange property.

Proof. If $a \in acl(A \cup \{b\}) \setminus acl(A)$, then $b \in acl(A \cup \{a\})$ by the exchange property for acl. So for $a \in acl_L(A \cup B \cup \{b\}) \setminus acl_L(A \cup B)$, we have $b \in acl_L(A \cup B \cup \{a\})$. Since $scl(A) = acl_L(A \cup B)$ then scl satisfies the exchange property. □

We conclude this section by standard definitions that we may find in model theory books, such as [10].
Now, we give the definition of interdefinability for real tuples. It can be extended to imaginary elements.

Definition 5.1.5. Tuples $\bar{a}$ and $\bar{b}$ of length m are said to be *interdefinable* over a set A if $\bar{a} \in (dcl(\bar{b} \cup A))^m$ and $\bar{b} \in (dcl(\bar{a} \cup A))^m$.

The next two definitions concern the notions of small set and conjugate elements.

Definition 5.1.6. Let $a \in M^{eq}$ and $b \in M^{eq}$. Let $A \subseteq M^{eq}$. Then b is a *conjugate* of a over A if there is an automorphism σ of M^{eq}, which fixes A point-wise, such that $\sigma(a) = b$.

Definition 5.1.7. Let $Y \subseteq M^n$ be definable in (M, B). We say that Y is *small* if there is a finite $A \subseteq M$ such that $Y \subseteq (scl(A))^n$.

We define definable closure in the equivalence relation structure as follows. It is well known that the next definition is equivalent to the definition of definable closure in Chapter 2 but for M^{eq}.

Definition 5.1.8. Assume M is sufficiently saturated and strongly homogeneous. Let $A \subseteq M$ and denote by $Aut(M^{eq}/A)$ the set of automorphisms of M^{eq} fixing A pointwise. Then

$$dcl^{eq}(A) = \{x \in M^{eq} : \sigma(x) = x \text{ for all } \sigma \in Aut(M^{eq}/A)\}.$$

5.2 Known results about imaginaries

As we mentioned at the beginning of this chapter, this section is centered on some of the results about imaginaries from [1], [2] and [3], which will be helpful for the rest of the chapter. We are just using them to obtain new results. The proofs can be found in [3].
We extend the notion of small closure to allow imaginary parameters. Since it is still for real elements, we call it real small closure.

Definition 5.2.1. Let $e \in (M, B)^{eq}$. The *real small closure* of e is the set $\{a \in M$: there exists a set X small and definable over e such that $a \in X\}$, and we denote it by $scl_M(e)$.

The following theorem is one of the first results on imaginaries in dense pairs of real-closed fields. One can prove it by using Proposition 5.1.2.

Theorem 5.2.2. [3] *Let $Z \subseteq B^n$ be definable in (M, B). Then the canonical parameter for Z in $(M, B)^{eq}$ is interdefinable with an element of M^{eq}.*

Theorem 5.2.3. [3] *Let $e \in (M, B)^{eq}$ and let $\overline{b}$ be an acl_L-basis for $scl_M(e)$ over B. Then e is interdefinable over $\overline{b}$ with an element of M^{eq}.*

These two theorems will be used to prove the main results of this thesis in the last section. The following theorem is interesting and worth mentioning.

Theorem 5.2.4. [1] *Let $e \in (M, B)^{eq}$. Then e is a canonical parameter for a small definable set.*

Stronger version of Lemma 4.2.1

From the previous chapter, the theory of dense pairs of real-closed fields is complete and so any dense pair of real-closed fields is elementarily equivalent to the structure $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}})$. Let (M, B) be such that $(\mathbb{R}, \mathbb{R} \cap \overline{\mathbb{Q}}) \prec (M, B)$ and (M, B) is sufficiently saturated. For more appropriate results, let us improve the Lemma 4.2.1.

Lemma 5.2.5. For all $i_1 < i_2 < \cdots < j_2 < j_1$ in M and any finite $C \subseteq M$, the convex set $\bigcap_{n \in \mathbb{N}} (i_n, j_n)$ is not contained in $acl_L(C \cup B)$.

Proof. By adopting the notation used in the proof of Lemma 4.2.2, for some polynomials $P_1, \ldots, P_k$, we consider

$$\theta_F(x) \equiv \forall y_1, \ldots, y_n \exists x \Big(i_1 < x < j_1 \wedge \cdots \wedge i_n < x < j_n \wedge \neg \exists \overline{z_1} \varphi_{P_1}(x, \overline{y}, \overline{z_1}) \wedge$$
$$\neg \exists \overline{z_2} \varphi_{P_2}(x, \overline{y}, \overline{z_2}) \wedge \cdots \wedge \quad \neg \exists \overline{z_k} \varphi_{P_k}(x, \overline{y}, \overline{z_k}) \Big).$$

Since M is an elementary extension of $\mathbb{R}$ then there is $a \in M$ such that $M \models \theta_F(a)$. Then by saturation there is $a \in M$ such that $M \models \theta_F(a)$ for any set of polynomials F. This concludes the proof of the Lemma 5.2.5. □

The above lemma plays a significant role in proving the next two theorems.

5.3 Relations between acl^{eq} and dcl^{eq}

This section contains the principal results of this thesis. In fact, the question about the relation between acl^{eq} and dcl^{eq} was raised during an international LYMOTS meeting in Manchester in 2015, and here, we are going to

answer this question by the following theorems. On the one hand, we prove that for any dense pair of real-closed fields, we have $acl^{eq}(\emptyset) = dcl^{eq}(\emptyset)$. However, there is $A \subseteq (M, B)^{eq}$ such that $acl^{eq}(A) \neq dcl^{eq}(A)$.
In order to prove the equality stated above, we need the following lemma.

Lemma 5.3.1. Let $e \in acl^{eq}(\emptyset)$ such that $scl_M(e) \subseteq B$ then $e \in dcl^{eq}(\emptyset)$.

Proof. Let $e \in acl^{eq}(\emptyset)$ and $\overline{b}$ be an acl_L-basis for $scl_M(e)$ over B. By Theorem 5.2.3, e is interdefinable over $\overline{b}$ with some $\alpha \in M^{eq}$.
Since $scl_M(e) \subseteq B$ then the basis $\overline{b} = \emptyset$ and so e is interdefinable with α. Thus $\alpha \in dcl^{eq}(e) \subseteq acl^{eq}(e)$. Moreover, we have $e \in acl^{eq}(\emptyset)$, then $acl^{eq}(e) \subseteq acl^{eq}(\emptyset)$, which implies that $\alpha \in acl^{eq}(\emptyset)$. By elimination of imaginaries in M, α is interdefinable with some $\overline{a} \in M^n$. Say $\overline{a} = a_1 \dots a_n$. Since M is totally ordered then $a_i \in acl(\emptyset) = dcl(\emptyset)$ and therefore $\alpha \in dcl^{eq}(\emptyset)$. Since e is interdefinable with α then $e \in dcl^{eq}(\emptyset)$. □

Theorem 5.3.2. *Let (M, B) be a dense pair of real-closed fields, then $acl^{eq}(\emptyset) = dcl^{eq}(\emptyset)$.*

Proof. We may assume that (M, B) is sufficiently saturated. Let $e \in acl^{eq}(\emptyset)$. By using Lemma 5.3.1, it is sufficient to prove that $scl_M(e) \subseteq B$.
Let $a \in scl_M(e)$ and $a \in M \setminus B$. By Definition 5.2.1, $a \in scl_M(e)$ if and only if there exists a set X small definable over e such that $a \in X$. The set X is small if there is a finite set $A \subseteq M$ such that $X \subseteq scl(A) = acl_L(A \cup B)$.
Let $\{e_1, \dots, e_n\}$ be the conjugates of e and $X_1, \dots, X_n$ be the conjugates of X associated respectively to $\{e_1, \dots, e_n\}$. Then $a \in X_1 \cup \dots \cup X_n$ and the set $X_1 \cup \dots \cup X_n$ is small and definable over $\emptyset$. By definition of small set there is a finite set $\tilde{A} \subseteq M$ such that $X_1 \cup \dots \cup X_n \subseteq acl_L(\tilde{A} \cup B)$.
We want to find a contradiction.
Define two sequences of rational numbers $(i_n)_{n\in\mathbb{N}}$ and $(j_n)_{n\in\mathbb{N}}$ such that $i_n < a < j_n$ for all $n \in \mathbb{N}$. This is possible if a is finite. If a is infinite then we can use the fact that B is dense in M to find some $b \in B$ such that $a + b$ is finite and adapt the following argument accordingly.
Moreover we assume that $(i_n)_{n\in\mathbb{N}}$ increases, $(j_n)_{n\in\mathbb{N}}$ decreases and $j_n - i_n$ goes to 0 when n goes to infinity.
By Lemma 5.2.5 $\bigcap\limits_{n\in\mathbb{N}} (i_n, j_n) \not\subseteq acl_L(\tilde{A} \cup B)$. It implies that there exists $a' \in \bigcap\limits_{n\in\mathbb{N}} (i_n, j_n)$ such that $a' \notin acl_L(\tilde{A} \cup B)$.

Furthermore, for any $c, d \in \mathbb{Q}$ with $c < a < d$, we can always find a large $N \in \mathbb{N}$ such that $c < i_N < a, a' < j_N < d$.
Therefore for all $c, d \in \mathbb{Q}$, we have the following equivalence:

$$c < a' < d \iff c < a < d,$$

which implies that $tp_L(a/\emptyset) = tp_L(a'/\emptyset)$ by Theorem 4.2.3. Thus $a' \notin B$ implies that $tp(a/\emptyset) = tp(a'/\emptyset)$. Consequently $a' \in X_1 \cup \cdots \cup X_n$ which tells us that $X_1 \cup \cdots \cup X_n \not\subseteq acl_L(\tilde{A} \cup B)$. This is contradiction, therefore $scl_M(e) \subseteq B$. □

We are going to prove that generally acl^{eq} is different from dcl^{eq}. For that, we will construct a definable set X such that an imaginary element of the $acl^{eq}(\ulcorner X \urcorner)$ does not belong to $dcl^{eq}(\ulcorner X \urcorner)$. Here $\ulcorner X \urcorner$ denotes a code for X. The construction will be as follows. First, we let $a \notin B$ and prove that we can choose $b \notin acl_L(\{a\} \cup B)$ such that $tp(a) = tp(b)$. Second, we show that for all $c \in B$, $a + c \notin acl_L(\{b\} \cup B)$. Finally, we have to prove that we can choose $c \in B$ such that $tp(b, a) = tp(a + c, b)$. This will complete the proof of the theorem.

Proposition 5.3.3. Let $a \notin B$. We can choose $b \notin acl_L(\{a\} \cup B)$ such that $tp(a) = tp(b)$.

Proof. Let $(i_n)_{n \in \mathbb{N}}$ be an increasing sequence from $\mathbb{Q}$ and $(j_n)_{n \in N}$ be a decreasing sequence from $\mathbb{Q}$ such that $j_n - i_n$ goes to 0 when n tends to infinity and such that a is contained in the interval (i_n, j_n) for all $n \in \mathbb{N}$. (Otherwise, since B is dense in M, there exists $b' \in B$ such that $i_n + b' < a < j_n + b'$).
By Lemma 5.2.5, $\bigcap_{n \in \mathbb{N}} (i_n, j_n) \not\subseteq acl_L(A \cup B)$ for any finite subset A of M. We can choose A to be $\{a\}$. So, $\bigcap_{n \in \mathbb{N}} (i_n, j_n) \not\subseteq acl_L(\{a\} \cup B)$.
Then there exists $b \in \bigcap_{n \in \mathbb{N}} (i_n, j_n)$ such that $b \notin acl_L(\{a\} \cup B)$.
For any $c, d \in \mathbb{Q}$ such that $c < a < d$, we can always find a large $N \in \mathbb{N}$ such that $c \leq i_N \leq a, b \leq j_N \leq d$. Then

$$tp_L(a) = tp_L(b).$$

By assumption $a \notin B$, and since $b \notin acl_L(\{a\} \cup B)$, then $b \notin B$. Therefore

$$tp(a) = tp(b).$$

by Theorem 4.2.3. □

Fix $b \notin acl_L(\{a\} \cup B)$ as in Proposition 5.3.3. We have the following proposition.

Proposition 5.3.4. For all $c \in B$, we have $a + c \notin acl_L(\{b\} \cup B)$.

Proof. We are going to prove this proposition by contradiction. For that we need the following claim.

Claim 5.3.5. For such b in Proposition 5.3.3, $a \notin acl_L(\{b\} \cup B)$.

Proof of Claim 5.3.5. Since $a \notin B$ and B is closed, then $a \notin acl_L(B)$. Assume $a \in acl_L(\{b\} \cup B)$, then $a \in acl_L(\{b\} \cup B) \setminus acl_L(B)$. By the exchange property we have $b \in acl_L(\{a\} \cup B)$, which contradicts the fact that $b \notin acl_L(\{a\} \cup B)$. Therefore $a \notin acl_L(\{b\} \cup B)$. □

Now, let $c \in B$ and assume that $a + c \in acl_L(\{b\} \cup B)$. So there is a nonzero polynomial $p(x) = e_n x^n + e_{n-1} x^{n-1} + \cdots + e_1 x + e_0$ with coefficients from $\mathbb{Q}(\{b\} \cup B)$ such that $p(a + c) = 0$.
Let n be the degree of the polynomial $p(x)$ and let e_n be the coefficient of x^n. Of course, $e_n \neq 0$. Then we can write

$$p(a+c) = e_n(a+c)^n + e_{n-1}(a+c)^{n-1} + \cdots + e_1(a+c) + e_0.$$

Using the binomial theorem,

$$(a+c)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} c^k,$$

we have

$$\begin{aligned}
p(a+c) = e_n a^n &+ \left[e_n \binom{n}{1} c + e_{n-1} \binom{n-1}{0} \right] a^{n-1} \\
&+ \left[e_n \binom{n}{2} c^2 + e_{n-1} \binom{n-1}{1} c + e_{n-2} \binom{n-2}{0} \right] a^{n-2} + \ldots \\
&+ \left[e_n \binom{n}{n-1} c^{n-1} + e_{n-1} \binom{n-1}{n-2} c^{n-2} + \cdots + e_1 \binom{1}{0} \right] a \\
&+ e_n \binom{n}{n} c^n + e_{n-1} \binom{n-1}{n-1} c^{n-1} + \cdots + e_1 c + e_0.
\end{aligned}$$

Let $p_c(x) = c_n x^n + \cdots + c_1 x + c_0$ where

$$\begin{aligned}
c_n &= e_n \\
c_{n-1} &= e_n \binom{n}{1} c + e_{n-1} \binom{n-1}{0} \\
&\vdots \\
c_1 &= e_n \binom{n}{n-1} c^{n-1} + e_{n-1} \binom{n-1}{n-2} c^{n-2} + \cdots + e_1 \binom{1}{0} \\
c_0 &= e_n \binom{n}{n} c^n + e_{n-1} \binom{n-1}{n-1} c^{n-1} + \cdots + e_1 c + e_0.
\end{aligned}$$

So $p_c(x) = p(x+c)$.
The polynomial $p_c(x)$ is not a zero polynomial for any $c \in B$. In fact, the degree of the polynomial p is equal to n and the coefficient of x^n does not depend on c and it is not zero.
It is also clear that the coefficients of $p_c(x)$ are elements of $\mathbb{Q}(\{b\} \cup B)$. We have $p_c(a) = 0$ for some $c \in B$. Then $a \in acl_L(\{b\} \cup B)$ and we have a contradiction with the Claim 5.3.5. Therefore $a + c \notin acl_L(\{b\} \cup B)$. □

Theorem 5.3.6. *There exists $e \in (M, B)^{eq}$ such that $acl^{eq}(e) \neq dcl^{eq}(e)$.*

Proof. We are proving that there is an element of $acl^{eq}(e)$ which is not an element of $dcl^{eq}(e)$. We may assume $(M, B)^{eq}$ is sufficiently saturated and strongly homogeneous.
Let E be the $\emptyset$-definable equivalence relation such that for any $a, b \in M$, aEb if and only if $a = b + c$ for some $c \in B$.

Claim 5.3.7. E is an equivalence relation.

Proof of Claim 5.3.7. Let a, b and c be elements of M. We know that $a = a + 0$ and $0 \in B$. Then we have reflexivity.
By definition aEb if and only if $a = b + u$ for some $u \in B$, then $b = a + d$ where $d = -u$ and $d \in B$ because B is closed. Thus we have bEa, which proves the symmetry property.
Assume aEb and bEc. We want aEc. We have $a = b + u$ and $b = c + v$ for some $u, v \in B$ which imply $a = c + (u + v)$. Since B is closed then $u + v \in B$ and aEc. We then get transitivity. □

Let a, b be two elements of M as in Proposition 5.3.3. Note that a and b are not related by the equivalence relation E. Let $X = [a] \cup [b]$ where $[a]$ and

$[b]$ denote respectively the equivalence classes of a and b. By the definition of the equivalence relation, $[a] \cap [b] = \emptyset$, so $[a]$ and $[b]$ form a partition of X.

Lemma 5.3.8. $\ulcorner [a] \urcorner \in acl^{eq}(\ulcorner X \urcorner)$.

Proof of Lemma 5.3.8. Let $\sigma \in Aut(M, B)$ such that $\sigma(X) = X$. We know that $[a] \subset X$ then

$$\sigma([a]) \subseteq \sigma(X) = X$$
$$\sigma([a]) \subseteq [a] \cup [b].$$

Since σ preserves the equivalence relation E, then $\sigma([a]) \subseteq [a]$ or $\sigma([a]) \subseteq [b]$. But for any $c \in [a], [a] = [c]$ (same reasoning for b). Therefore $\sigma([a]) = [a]$ or $\sigma([a]) = [b]$ and so $\sigma(\ulcorner [a] \urcorner) = \ulcorner [a] \urcorner$ or $\sigma(\ulcorner [a] \urcorner) = \ulcorner [b] \urcorner$. $\square$

Lemma 5.3.9. $\ulcorner [a] \urcorner \notin dcl^{eq}(\ulcorner X \urcorner)$.

Proof of Lemma 5.3.9. Now we need to show that $tp(a, b) = tp(b, a + c)$ for some $c \in B$.
Since $tp(a) = tp(b)$, then there is an automorphism σ of (M, B), such that $\sigma(a) = b$. By Theorem 4.2.3, the type of b over a, denoted by $tp(b/a)$, is determined by the cut realised by b over the field generated by a, $\mathbb{Q}(a)$.
Let $I = \{i \in \mathbb{Q}(a) : i < b\}$ and $J = \{j \in \mathbb{Q}(a) : b < j\}$. Then the cut is $\{(i, j) : i \in I, j \in J\}$. By saturation, the intersection $\cap(i, j)$ contains an interval.
We also have $\sigma(i), \sigma(j) \in \mathbb{Q}(b)$ for all $i \in I$ and $j \in J$. Consider the set $K = \{(\sigma(i), \sigma(j)) : i \in I, j \in J\}$. We can choose $c \in B$ such that $a + c$ realises K. Note that by denseness property, we can always find this c.
We then have that $tp(a + c/b)$ corresponds to $tp(b/a)$. Since $tp(a) = tp(b)$. Using Theorem 4.2.3 and Proposition 5.3.4, we then have

$$tp(b, a) = tp(a + c, b)$$

and hence

$$tp(a, b) = tp(b, a + c).$$

Then there is an automorphism σ such that $\sigma(a) = b$ and $\sigma(b) = a + c$. By Proposition 2.4.3, any automorphism of (M, B) can be extended to an automorphism of $(M, B)^{eq}$. Then $\sigma([a]) = [b]$ and $\sigma([b]) = [a + c] = [a]$.

Thus $\sigma(X) = X$ which is equivalent to $\sigma(\ulcorner X \urcorner) = \ulcorner X \urcorner$. However $\sigma(\ulcorner [a] \urcorner) = \ulcorner [b] \urcorner \neq \ulcorner [a] \urcorner$.
Therefore $\ulcorner [a] \urcorner \notin dcl^{eq}(\ulcorner X \urcorner)$ $\square$

Lemma 5.3.8 and Lemma 5.3.9 say that there is an element of $acl^{eq}(\ulcorner X \urcorner)$ which is not in $dcl^{eq}(\ulcorner X \urcorner)$. We proved the theorem. $\square$

List of References

[1] Alexander Berenstein, Clifton Ealy, and Ayhan Günaydın. Thorn independence in the field of real numbers with a small multiplicative group. *Annals of Pure and Applied Logic*, 150(1):1–18, 2007.

[2] Alexander Berenstein and Evgueni Vassiliev. On lovely pairs of geometric structures. *Annals of Pure and Applied Logic*, 161(7):866–878, 2010.

[3] Gareth John Boxall. *Lovely pairs and dense pairs of real closed fields*. PhD thesis, University of Leeds, 2009.

[4] Lou van den Dries. Dense pairs of o-minimal structures. *Fundamenta Mathematicae*, 157(1):61–78, 1998.

[5] Lou van den Dries. *Tame topology and o-minimal structures*, volume 248. Cambridge University Press, Cambridge, 1998.

[6] Wilfrid Hodges. *Model theory*, volume 42. Cambridge University Press, Cambridge, 1993.

[7] Wilfrid Hodges. *A shorter model theory*. Cambridge University Press, Cambridge, 1997.

[8] Serge Lang. *Algebra*, volume 211. GTM, Springer-Verlag, New York, 2002.

[9] David Marker. Introduction to the model theory of fields. In *Model theory of Fields*, pages 1–37. Springer-Verlag, Berlin, 1996.

[10] David Marker. *Model theory: an introduction*, volume 217. GTM, Springer-Verlag, New York, 2002.

[11] Rahim Moosa. Some elementary facts about M^{eq}. *Unpublished note*.

[12] Victoria Lynn Noquez. *Model Theory of Real Closed Fields*. Senior thesis, Carnegie Mellon University, 2008.

[13] Anand Pillay. Lecture notes-model theory. Available at `https://www3.nd.edu/~apillay/pdf/lecturenotes_modeltheory.pdf`, 2002.

[14] Bruno Poizat. Une théorie de Galois imaginaire. *The Journal of Symbolic Logic*, 48(04):1151–1170, 1983.

[15] Bruno Poizat. Cours de théorie des modèles. Une introduction à la logique mathématique contemporaine. *Bruno Poizat, Lyon*, 1985.

[16] Abraham Robinson. Solution of a problem of Tarski. *Fundamenta Mathematicae*, 47(2):179–204, 1959.

[17] Tomas Sander. Existence and uniqueness of the real closure of an ordered field without Zorn's lemma. *Journal of Pure and Applied Algebra*, 73(2):165–180, 1991.

[18] Saharon Shelah. *Classification theory: and the number of non-isomorphic models. Second Edition*. North-Holland, Amsterdam, 1990.

[19] Marcus Tressl. Introduction to o-minimal structures and an application to neural network learning. Available at `http://www1.maths.leeds.ac.uk/school/modeltheory/tresslnotes.pdf`, 2010.

[20] Martin Ziegler. An exposition of Hrushovski's new strongly minimal set. *Annals of Pure and Applied Logic*, 164(12):1507–1519, 2013.