Towards Projective Set Theory

by

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Abstract

Towards Projective Set Theory

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In this thesis an axiomatic framework is presented which extends the projective group theory introduced by Z Janelidze to also hold for sets. The isomorphism theorems are reformulated so that they hold for sets. Interestingly, the theorems do not hold for a number of *null* cases, which in this sense makes it a point-free approach to set theory—that is, singletons cannot be selected as abstract images of morphisms, but they can be studied by factorisation properties. In particular, this aspect is explained in the last chapter, where a comparison is drawn between the isomorphism theorems here and those for regular categories presented in Tholen's doctoral thesis. The proofs are done by means of chasing elements of ΣX, here called $\mathcal{A}$-subobjects, forwards and backwards, where ΣX is the fibre at an object X in $\mathcal{C}$ for which the functor $G\colon \mathcal{C} \longrightarrow \mathbf{Gal}$ is the central object of study in the axiomatic setting; moreover, the axioms are functorially self-dual for this functor. A minor result on bounded morphisms is included: when a bounded morphism is the left adjoint of a Galois connection with meets and joins it is equivalent to the Frobenius property for Galois connections.

Uittreksel

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In hierdie tesis word 'n aksiomatiese raamwerk ontwikkel en uiteengesit om die projektiewe groepsleer van Z Janelidze uit te brei om ook versamelings in the sluit. Die isomorfismestellings word hergeformuleer sodat dit ook vir versamelings geldig is. Hierdie proses het die interessante gevolg dat die stellings vir 'n klas van *nul* gevalle nie geldig is nie en in daardie sin kan mens hiérdie benadering sien as 'n puntvrye versamelingsleer. 'n Enkele punt kan nie vasgevang word as die abstrakte beeld van 'n morfisme nie, maar kan wel bestudeer word op grond van faktoriseringseienskappe. In besonder word hierdie aspek verduidelik in die laaste hoofstuk, waar 'n vergelyking getref word met die isomorfismestellings vir reëlmatige kategorieë voorgesit in Tholen se doktorale tesis. Die bewyse van die stellings word by wyse van elemente van ΣX, hier genoem $\mathcal{A}$-subvoorwerpe, vorentoe en agtertoe aan te volg, waar ΣX die beeld van 'n voorwerp X in 'n kategorie $\mathcal{C}$ is vir die funktor $G\colon \mathcal{C} \longrightarrow \mathbf{Gal}$, 'n sentrale struktuur waarvoor die aksiomas funktoriaal self-duaal is. 'n Kort resultaat vir begrensde morfismes word ingesluit wat sê dat wanneer 'n begrensde morfisme 'n linker adjunk van 'n Galois konneksie met infima en suprema is, is dit ekwivalent aan die Frobenius eienskap vir Galois konneksies.

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This body of work is an elaboration to a general case of work done by Professor Z Janelidze for group-like structures, with whom there were many inspiring discussion sessions. Furthermore, I would like to thank my other two supervisors, Dr Gray and Professor Rewitzky for the opportunity to let my mind explore the creativity inherent to the spirit of mathematical investigation.

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Dedications

Opgedra aan my ouers, familie en vriende, én aan Helené.

Contents

Nomenclature

Gal The category of posets with galois connections between them.

Grp The category of groups.

Set The category of sets.

form A faithful, amnestic functor. That is, a functor for which the fibres are posets.

concrete set A set in the usual set theoretic sense.

abstract set An object in the category $\mathcal{C}$, where $F\colon \mathcal{E} \longrightarrow \mathcal{C}$ is the form in the axiomatic setting with the corresponding Grothendieck construction $\mathcal{C} \longrightarrow$ **Gal**.

ΣX The poset of fibres in **Gal** corresponding to an abstract set X, i.e., an object X in $\mathcal{C}$.

$\mathcal{A}$-subobject An element of ΣX for some abstract set X in the axiomatic setting.

abstract subobject The same as an $\mathcal{A}$-subobject. Depending on context this can be for example an equivalence relation in **Set** or a subgroup in **Grp**.

concrete subset A subset of a set in the usual set theoretic sense.

abstract direct image map The left adjoint of $G(f)$ for some morphism f for the form with construction $G\colon \mathcal{C} \longrightarrow$ **Gal**.

abstract inverse image map The corresponding right adjoint.

concrete direct image map The usual direct image map of subsets for some function f.

concrete inverse image map The usual inverse image map of subsets for some function f.

fX The abstract direct image of f applied to an $\mathcal{A}$-subobject X.

Xf The abstract inverse image of f applied to an $\mathcal{A}$-subobject X.

Chapter 1

Introduction

Mathematics can be seen as the study of those statements that follow from some chosen basic but formal assumptions in a given language or context. During this process one may discover that many patterns repeat, in both the individual's approach towards the subject and the objective conclusions therein. However, such observations are not always precisely formalised. One such repeating concept is what we may perhaps refer to as *symmetry*.

Indeed, the study of groups can be seen as the study of geometric symmetry. Considering the ways in which one may manipulate a rectangle or other shape invariantly in physical space leads to the group of symmetries of that object. And yet, within groups themselves there are concepts of a structural level of symmetry, such as between elements and their inverses. When one represents a group as a one object category, then the morphisms conveniently exhibit this symmetry—in the sense that the opposite category is isomorphic to the original.

This invites the question: to which extent can one exhibit a mathematical structure in the most symmetric way? This may start out as a rather informal question and yet, precisely formalising this question has lead to the development of many different branches of mathematics.

In this body of work, the spirit is very much in this same direction of investigating the most self-similar way to present structures. For example, a group is a category; but the category of groups is also a category, and they both exhibit the basic axioms for being a category.

1.1 The Context for Self-dual Set Theory

The main approach in the context of projective group theory [8, 12] is to have the symmetry exhibited on different levels to adhere to a style of self-duality within a precisely defined context. In other words, the objects of study are described in such a way that properties have dual properties within the context, and these dual properties also hold.

Duality, then, can perhaps be described as symmetry in two directions, or symmetry between duals. This concept arises in many different contexts. For example, in lattice theory, the least upper bound would be dual to the greatest lower bound.

In this thesis a mathematical context is presented with a set of self-dual axioms on its specified structures, and this context is shown to hold for any model of set theory. The axioms are new for sets and the language itself is also new, as will be explained in more detail later. The intention is not to replace the usual axioms of set theory, indeed, the axioms here are not sufficient to encapsulate all of set theory; rather, the motivation is to look at **Set** from the perspective of specific properties that are known to hold for **Grp**. These axioms are developed from axioms introduced by Z Janelidze [8] and others [9, 10, 14, 15]. Moreover, Mac Lane was especially interested in formalising invariance under categorical duality [18] and formulated axioms for abelian groups that also hold in the op-category; and indirectly, so too was G Janelidze and others [11] with the introduction of semi-abelian categories. As is shown by Z Janelidze in [13], self-dual conditions on a form is a suitable structure for expressing that a category is protomodular or semi-abelian. In the present text, a basic investigation is done in a similar style for sets with a focus specifically on functorial duality of an underlying form. In addition to this, for each abstract subobject of a set, a family of outmorphisms and a family of inmorphisms is introduced. Once this structure and the axioms on it are fixed, one then develops the corresponding isomorphism theorems in **Set** that one has for **Grp**.

One of the main points of interest is to appropriately handle empty and constant functions. In groups, there is always exactly one constant homomorphism between any two groups, sending all elements to unity in the codomain. Sets exhibit a different phenomenon where there are typically many constant functions between two sets. Moreover, since groups cannot be empty, there are also not any empty homomorphisms. The effect of this is that the isomorphism theorems in the abstract setting for sets do not hold in these null cases, i.e., where constant or empty maps form part of the diagram. They may hold for the category of sets, but they cannot be proven from the self-dual axioms without adding restraints that are not self-dual. It is possible to modify the approach to prove the isomorphism theorems for some examples of constant maps, but the approach used in this present work is to exclude all constant maps in **Set** from diagrams for the isomorphism theorems. For the specialisation to **Grp**, however, the constant homomorphisms are handled.

1.2 Structures within the Axiomatic Context

Before we can go further, we need to introduce structure in the context that we are working with. In projective group theory [12, 8], one studies the self-dual properties of the Grothendieck bifibration of subgroups, which turns out to be a form in the sense of Z Janelidze [13, 14]. To each subgroup S of a group G one assigns the extra structure of an *inmorphism* corresponding to the inclusion of S into G, and an *outmorphism*, the quotient by the smallest normal subgroup containing S. Furthermore, each homomorphism $f\colon G \longrightarrow H$ gives rise to a Galois connection between the lattice of subgroups of G and H.

In order to adapt this structure to sets, one considers the bifibration of quotients, where quotients translate exactly to equivalence relations on sets. Instead of a single inmorphism and outmorphism, they are replaced by a family of inmorphisms (injections selecting each equivalence class) and a family of outmorphisms (the surjection whose kernel is the equivalence relation). Although for sets there is only one outmorphism up to isomorphism, one needs the theory to be self-dual. Hence, the general case allows for a family of outmorphisms.

Furthermore, we also have for each function $f\colon A \longrightarrow B$ a Galois connection between $f(-)\colon \Sigma A \longrightarrow \Sigma B$ and $(-)f\colon \Sigma B \longrightarrow \Sigma A$, where ΣX is the lattice of equivalence relations on a set X. Finally, we have a class $\mathcal{N}$ of null morphisms, which we will call $\mathcal{N}$-null morphisms.

Thus the full context that we work with in the abstract setting is as follows. We have a form $F\colon \mathcal{E} \longrightarrow \mathcal{C}$, which can be seen as a functor $G\colon \mathcal{C} \longrightarrow \mathbf{Gal}$ into the category of posets with Galois connections between them. Hence, for each abstract set X, we have a lattice of abstract subobjects $G(X) = \Sigma X$; furthermore, for each $S \in \Sigma X$ we have an index set I and a family $\{f_i\colon L_i \longrightarrow X \mid i \in I\}$ of inmorphisms. Dually, we have for S an index set J and a family of outmorphisms $\{g_j\colon X \longrightarrow R_j \mid j \in J\}$. The axioms to fully describe the context that we work in are introduced on this structure.

1.3 Outline of the Thesis

1.3.1 Contents of the Chapters

Chapter 2: Preliminaries is a general introduction to some useful concepts in category theory. In *Chapter 3: Towards Self-dual Axioms for Sets* the structure briefly described above is stated and axioms are introduced on this structure. The main departure point from projective group theory is the change in Axiom 3, which handles the families of inmorphisms and outmorphisms, and the addition of a class of $\mathcal{N}$-null morphisms. Axiom 4 holds only for morphisms that are not $\mathcal{N}$-null. Basic immediate propositions are stated and proven, followed by verifying lattice properties and proving that a bounded morphism on an adjoint of a Galois connection is equivalent to the Frobenius property; finally, examples that satisfy the abstract theory are given and proven, the prototypical example being the category of sets.

In *Chapter 4: Homomorphism Theorems* we introduce *Pyramid Lemmas*, which entails a technical procedure to ensure an induced universal morphism and to show when such a morphism is an isomorphism. From here, the analogues of the (self-dual) isomorphism theorems for the category of groups [12] are introduced, but now with a focus on the category of sets. Then, we briefly revisit the doctoral work of Tholen [22] and show how it relates to this text. Tholen's version of the isomorphism theorems hold for any regular category and hence also for **Set**. In order for his version of the isomorphism theorems to hold in this new axiomatic context, one needs to relax some of the duality properties, but only for the $\mathcal{N}$-null cases. As an example, consider the smallest equivalence relation on the set of natural numbers $\mathbb{N}$, which will have $\mathbb{N}$ many inmorphisms—each choosing an equivalence class corresponding to $n \in \mathbb{N}$. The image (of the only equivalence relation on a singleton $S \simeq \{\emptyset\}$) of each of these inmorphisms will be equal. They are distinguishable by factorisation, but not by images. Hence when one formulates a statement which involves inmorphisms one cannot equate images with inmorphisms, whereas in contrast when we use the usual direct image map on subsets, function images and injections uniquely identify each other up to isomorphism. One of the differences between this work and the regular category case is that in the latter setting one works with both (concrete) images and quotients, that is, two Galois connections. In this sense the approach presented in this thesis is simpler, even though the axioms are more involved than the definition of a regular category.

An alternative to handle the $\mathcal{N}$-null cases for *Section 4.2: Isomorphism Theorems* and *Section 4.3: Comparison to Tholen's Approach* is briefly stated in the concluding Section 4.4.3 at the end of the thesis: to induce a morphism, the $\mathcal{N}$-null functions can be handled in the **Set** case by requiring, instead of the present approach elaborated in Section 4.1.1, that the composite forward relation of a base sequence is not just a relation, but a function.

1.3.2 Outline of Contributions

- The synthesis of the 5 axioms contained in this thesis together with the axiom for $\mathcal{N}$-null morphisms to the **Set** context is a new result. In particular, Axiom 3 is new, and was generalised from the specialisation of the axiom to the **Grp** case. The $\mathcal{N}$-null axiom is also new, and captures properties of null morphisms needed for the isomorphism theorems. The use here of a family of inmorphisms and a family of outmorphisms for each $\mathcal{A}$-subobject replaces the notion a single isomorphism class for an inmorphism and a single isomorphism class for an outmorphism.

- The conditions for constructibility of a pyramid for the **Set** case is new. The concept of the pyramid itself was introduced by Z Janelidze. In the **Grp** case such a pyramid is always constructible and hence constructibility falls away as unnecessary.

- Many of the propositions that follow from the axioms were taken and modified from the propositions in the notes for projective group theory [12]. Some important propositions were added, such as Proposition 3.2.14, concerning now the family of inmorphisms or outmorphisms.

- The lattice theoretic properties coincide with those for the group case and the proofs were adapted from the notes for projective group theory [12], with steps and comments added. The equivalence between the Frobenius property for Galois connections and the left adjoint being a bounded morphism is new.

- The isomorphism theorems for the axiomatic context and the order and format in which they are introduced mirror those of the projective group theory notes. The sketches of the proofs in the notes were used and adapted to the **Set** case and extra notes, assumptions and steps were added. The main logical changes are to exclude certain $\mathcal{N}$-null cases in order to ensure constructibility of the pyramids for the different zigzags that arise in the formulation of the theorems.

- The final section compares this new context that may be called projective set theory to Tholen's doctoral thesis and presents an outline of how to approach the isomorphism theorems for regular categories in the specialisation to **Set** from the perspective of the abstract projective set theory context.

1.3.3 On the Different Notions of Duality

In this thesis, and in the work of those before, duality, invariance under duality and self-duality is defined and used in specific formal mathematics.

The main departure point here is to move from the notion of categorical duality to functorial duality. But that of course does not mean that there may not be more options of how to precisely define and use the approach. Whether any mathematical structure or statement can elegantly be formulated in some self-dual framework is an interesting question to ask. But this question can be taken further: perhaps a deep level of insight into a structure or concept within a given framework or language could be defined to be the knowledge of how to move from a structure to its self-dual form. That is, to fully understand how to express self-duality or duality in a given context can perhaps be seen as to fully understand the relevant structure itself. Indeed, formulating such a question rigorously has lead to the notion of functorial duality.

Hence, we would expect that formulating alternative notions of self-duality in different branches of mathematics may lead to insightful mathematical investigation. An objective behind the contents of this thesis is to illustrate to the reader the insight of this topic in more general mathematics by focusing on a foundational branch: that of set theory.

Chapter 2

Preliminaries

2.1 Image and Preimage Maps

Definition 2.1.1. Consider a function $f\colon A \longrightarrow B$. This induces two functions, $f_*\colon \mathscr{P}(A) \longrightarrow \mathscr{P}(B)$ and $f^*\colon \mathscr{P}(A) \longrightarrow \mathscr{P}(B)$, where $\mathscr{P}(C)$ denotes the powerset of a set C. These functions are defined as follows:

$$f_*(X) = \{b \in B \mid \exists x(x \in X \wedge f(x) = b)\}$$

$$f^*(Y) = \{a \in A \mid f(a) \in Y\}$$

By convention, f_* is called the *direct image map* and f^* is called the *inverse image map*. For brevity, the inverse image map can be called the *preimage map*. Many authors simply write *image map* when they refer to the direct image map. We shall use both terms.

2.1.1 A Remark on Notation

One of the aims of Chapter 3 will be to establish an abstraction of the two induced maps f_* and f^*. The convention will be to write:

$$fX = f_*(X)$$

$$Yf = f^*(Y)$$

Brackets may be used for clarity. For instance, $f(Xf)$ is the same as $(f_* \circ f^*)(X) = f_*(f^*(X))$. Most often we will write fg for the composite $f \circ g$. It should usually be clear from the context whether fx refers to a composite $f \circ x$ or whether it is the function value at x, $f(x)$.

In this text, we will have different notational conventions for image maps and preimage maps. This is mainly to differentiate between the development of the abstract concept as opposed to the concrete case. Sometimes a particular notation is used for elegance and at other times the notation is used to prevent confusion. For example, for the concrete case, we shall also use the notation:

$$f(X) = f_*(X)$$

$$f^{-1}(X) = f^*(X)$$

This is ambiguous, however, when f is a bijection. In that case, f^{-1} may be used to refer to the inverse of f. It will either be stated explicitly or it will be clear from the context which meaning is intended.

When $f\colon A \longrightarrow B$ is a function and $a \in A$, then we will write the direct image of $\{a\}$ as $f(\{a\})$ or simply $f\{a\}$, or at times even as fa—as long as the intended meaning is clear. Moreover, $f(A)$ will be called the *image* of f and will sometimes be written as $im(f)$. Since $\mathscr{P}(A)$ is a boolean algebra, we have a top element $1 = A$. Hence, we can write the image as $im(f) = f(1)$. For brevity, we write $f(1)$ interchangeably as $f1$. Usually $f1$ will be used to refer to the *abstract image*. The liberal use of notation is both for convenience and in an attempt to make the intuitive approach clear, that is, the abstraction from direct image and inverse maps to a general Galois connection.

When 1 is the top element in ΣX, it will either be clear from context, or it will be stated as $1 \in \Sigma X$. Notation such as $1^{\Sigma X}$ or similar presentations will be avoided in an attempt at fluidity within the text and to avoid confusion with identity maps such as 1_X or a set of functions, such as 2^X.

2.2 Groups

Consider a group G. There are two dual ways of creating a substructure from G. The first notion is that of a subgroup of G.

Proposition 2.2.1. *Let $f\colon G \longrightarrow H$ be a group homomorphism. Let S be a subgroup of G. Then fS is a subgroup of H. Hence, $im(f) = fG$ is a subgroup of H.*

Proof. It suffices to show that fS is closed under multiplication and inverses. Let $x, y \in fS$. Then for some $s, t \in S$ we have $x = f(s)$ and $y = f(t)$. Hence, $xy = f(s)f(t) = f(st)$. Since S is a subgroup, $st \in S$ and hence $f(st) \in f(S)$. This shows that fS is closed under multiplication. $x^{-1} = (f(s))^{-1} = f(s^{-1})$. But again, $s^{-1} \in S$, and hence $f(s^{-1}) \in fS$. Hence, fS is closed under inverses. $\square$

Proposition 2.2.2. *A subgroup of a group G corresponds exactly to an injective group homomorphism $S \hookrightarrow G$, up to isomorphism of S.*

Proof. Let S be a subgroup of G. There is a unique inclusion function

$$S \hookrightarrow G$$

which maps each element of S to itself in G. Conversely, let $f\colon S \hookrightarrow G$ be a injective group homomorphism. Then $im(f)$ is a subgroup of G. Suppose there is another injective homomorphism $g\colon S' \hookrightarrow G$ with $im(g) = im(f)$. Then there is a bijection $i\colon S \longrightarrow S'$ defined by $i(s) = g_{inv}f(s)$, where $g_{inv}\colon im(f) = im(g) \longrightarrow S'$ is defined as follows:

$$g_{inv}(x) = t \quad \text{where } g(t) = x.$$

Since g is an injection, t is unique. Note that this means that $g(g_{inv}(x)) = x$ when $x \in im(g)$ and $g_{inv}(g(y)) = y$ for any $y \in G$. The brackets are necessary, since strictly speaking g and g_{inv} do not compose, however, they do compose *element wise*. Now, i is injective since both of g' and f are injective. Suppose s' is an element of S'. Then $i(f_{inv}(g(s'))) = g_{inv}(ff_{inv}(g(s'))) = g_{inv}(g(s'))$ since $im(g) = im(f)$, and finally $g_{inv}(g(s')) = s'$. Thus, i is surjective and hence i is bijective.

$$\begin{tikzcd}
S \arrow[r, hook, "f"] \arrow[d, "i"', "\wr"] & G \\
S' \arrow[ur, hook, "g"'] &
\end{tikzcd}$$

Thus, injective homomorphisms $S \hookrightarrow G$ from a group S to a group G correspond to subgroups of G up to isomorphism. □

The second notion is that of quotients of G.

Definition 2.2.3. Suppose we have a group homomorphism $f\colon G \longrightarrow H$. Then we write $0f$ for the kernel of f, $ker(f)$. That is, $0f = (\{0\})f$, where 0 is the identity in H. We use this notation, because the trivial group containing only 0 is the bottom element of the subgroup lattice of a group.

Proposition 2.2.4. *A quotient of a group G corresponds exactly to a surjective group homomorphism $G \longrightarrow H$ up to isomorphism of H.*

Proof. Consider a quotient of a group G by a normal subgroup N. Then $N = 0f$ for the surjective homomorphism $f\colon G \longrightarrow G/N$ defined by $f(x) = [x]$ where $[x]$ is the equivalence class of x in G/N.

Conversely, suppose we have a surjective group homomorphism, $g\colon G \longrightarrow H$. We have a factorisation of g through the quotient group $G/0f$ as $g = hf$, where $h([x]_\sim) = g(x)$ and $x \sim y$ if and only if $g(x) = g(y)$. But since g is surjective, we also have the surjection followed by injection factorisation $g = 1_H g$. By applying the first isomorphism theorem to the diagram, there is an isomorphism: $i: G/0f \longrightarrow H$ with $h = 1_H i = i$.

$$\begin{tikzcd}
G \arrow[r, two heads, "g"] \arrow[dr, two heads, "f"'] & H \arrow[dr, "1_H"] & \\
& G/0f \arrow[u, "i", "\wr"'] \arrow[r, hook, "h"'] & H
\end{tikzcd}$$

So a surjective group homomorphism $f: G \longrightarrow H$ corresponds exactly to a quotient, namely $G/0f$. □

2.3 Substructures Determined by Morphisms

We can now ask the question, which further structures correspond to certain types of group homomorphisms? We can see that every group homomorphism $f: G \longrightarrow H$ specifies a normal subgroup of G, namely $0f$.

Also, a normal subgroup can be viewed as a group homomorphism $f: N \hookrightarrow G$. However, not every such injective f specifies a subgroup that is normal.

2.3.1 Rings

For rings (separately for both **Ring** and **Rng**), we also have subrings corresponding to injective ring homomorphisms and we have quotients corresponding to surjective ring homomorphisms. Moreover, we have the following:

Proposition 2.3.1. *A surjective homomorphism $f: R \longrightarrow D$ into a integral domain corresponds exactly to a prime ideal of R.*

Proof. To see this, let $P = 0f$, for a surjective homomorphism f. Suppose $ab \in P$. Then

$$f(ab) = 0 \Rightarrow f(a)f(b) = 0$$

and thus $f(a) = 0$ or $f(b) = 0$ and hence $a \in P$ or $b \in P$. Thus P is prime.

Conversely, suppose P is a prime ideal. Suppose we have $[x][y] = [xy] = 0$, where $[x], [y] \in R/P$. Then $xy \in P$. Since P is prime, $x \in P$ or $y \in P$. Thus, $[x] = 0$ or $[y] = 0$ and thus R/P has no dividers of zero and hence is an integral domain. The kernel of the canonical surjective homomorphism $f: R \longrightarrow R/P$ is P. □

2.3.2 Boolean algebras

Proposition 2.3.2. *A Boolean algebra homomorphism $f: B \longrightarrow B'$ determines a filter.*

Proof. Let 1 be the top element of B'. Then $F = (\{1\})f$ (we can write this as $1f$ by slightly abusing the notation) is up-closed since for $a \in F$ we have $f(a) = 1$ and $a \leqslant b$ implies $f(a) = 1 \leqslant f(b)$. Hence, $b \in F$. F is closed under meet since

$f(a) = 1 = f(b)$ implies that $f(a \wedge b) = f(a) \wedge f(b) = 1 \wedge 1 = 1$. Hence, F is a filter. □

Proposition 2.3.3. *If $B' = 2$, the two-element Boolean algebra, then f determines and is determined by a prime filter of B.*

Proof. To see that $F = 1f$ is prime, note that if $a \vee b \in F$, then $f(a \vee b) = 1$. Since 2 only contains 1 and 0, this means that $f(a) = 1$ or $f(b) = 1$ in order to get the join equal to 1. Hence $a \in F$ or $b \in F$ and thus F is prime.

Given a prime filter F, f can be constructed as expected, by letting $f(a) = 1$ if and only if $a \in F$. □

2.3.3 Subsets

Proposition 2.3.4. *A subset S of a set X corresponds up to isomorphism to an injection $S \hookrightarrow X$ and also corresponds exactly to a function $X \longrightarrow 2$, where 2 is a set with two elements.*

Proof. For an injective function $f\colon S \hookrightarrow X$, define $\chi_S\colon X \longrightarrow 2$ as

$$\chi_S(x) = \begin{cases} 1 & \text{if } \exists s(fs = x) \\ 0 & \text{otherwise} \end{cases}$$

Conversely, given $g\colon X \longrightarrow 2$, define $S = \{x \in X \mid gx = 1\}$ and since $S \subseteq X$, it has inclusion $S \hookrightarrow X$. □

The idea that a subset can be seen in these two ways is used to define the concept of a subobject classifier as a condition for a category to be a topos.

2.4 Categorical Approaches

2.4.1 The Definition of a Category

A category $\mathcal{C}$ can be defined as consisting of the following data. A class $\mathcal{C}_0$ consisting of objects and a class $\mathcal{C}_1$ of morphisms. Also, it has the maps $i\colon \mathcal{C}_0 \longrightarrow \mathcal{C}_1$, $dom\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_0$ and $cod\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_0$. When, for morphisms f and g, we have $cod(f) = dom(g)$, then there is the composite $g \circ f\colon dom(f) \longrightarrow cod(g)$. These data are subject to an identity axiom and an associativity axiom.

1. For all $Y \in \mathcal{C}_0$ and arrows $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$, we have

$$i(Y) \circ f = f$$

and

$$g \circ i(Y) = g.$$

The convention is to write $i(Y)$ as 1_Y.

2. For arrows f, g and h in the setup $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, we have that

$$(g \circ h) \circ f = g \circ (h \circ f).$$

For more information, one may consult references such as Borceux [2, pp. 1–10] and Mac Lane [17, pp. 7–12]. In particular, a perhaps more elegant way to define a category is to define it consisting only of morphisms. The identity morphisms then fulfil the role of objects.

For a treatment of some of the foundations of the definition of a category, one may also consult Bénabou [1].

2.4.2 The Definition of a Functor

A functor fulfils the role of a structure preserving morphism between categories. Indeed, in the category of categories, where objects are exactly categories, morphisms are exactly functors. Specifically, a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ consists of two maps:

$$F_0\colon \mathcal{C}_0 \longrightarrow \mathcal{D}_0$$

and

$$F_1\colon \mathcal{C}_1 \longrightarrow \mathcal{D}_1.$$

These maps are subject to conditions that ensure that composable morphisms are also composable when the functor is applied to them: $dom(F_1(f)) = F_0(dom(f))$ and $cod(F_1(f)) = F_0(cod(f))$. Furthermore, it satisfies functoriality:

$$F_0(f \circ g) = F_0(f) \circ F_0(g).$$

The convention is to write $F_0(X)$ simply as $F(X)$ and $F_1(f)$ simply as $F(f)$.

2.4.3 Monomorphisms

Definition 2.4.1. Let $\mathcal{C}$ be a category with objects $\mathcal{C}_0$ and morphisms $\mathcal{C}_1$.

A morphism $f\colon B \longrightarrow C$ in $\mathcal{C}_1$ is a *monomorphism* if for any two $g, h\colon A \longrightarrow B$,

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & B \arrow[r, "f"] & C \end{tikzcd}$$

we have that $fg = fh \Rightarrow g = h$.

Proposition 2.4.2. *If $\mathcal{C} = \mathbf{Set}$, then monomorphisms are exactly injective functions.*

Proof. Suppose $f \colon B \hookrightarrow C$ is an injection. Then, for any $a \in A$, we have that $f(g(a)) = f(h(a)) \Rightarrow g(a) = h(a)$ and hence f is a monomorphism.

Conversely, suppose f is a monomorphism. Then the condition holds for any g and f, and hence holds when $A = \{0\}$. Suppose $f(a) = f(b)$. Let $g(0) = a$ and $h(0) = b$. Then we have that $fg = fh \Rightarrow g = h$ and hence $g(0) = a = b = h(0)$. Hence, f is injective. □

2.4.4 Epimorphisms

Definition 2.4.3. A morphism $f \colon A \longrightarrow B$ in $\mathcal{C}_1$ is an *epimorphism* if for any two $g, h \colon B \longrightarrow C$,

$$A \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C$$

we have that $gf = hf \Rightarrow g = h$.

Proposition 2.4.4. *If $\mathcal{C}$ =* **Set***, then epimorphisms are exactly surjective functions.*

Proof. Suppose $f \colon A \longrightarrow B$ is a surjection. Then we have that for any $b \in B$ there is some $a \in A$ with $f(a) = b$. If $gf(a) = hf(a)$ for each a, then $g(b) = h(b)$ for each b and hence $h = g$.

Conversely, suppose f is an epimorphism. Define two functions $h, g \colon B \longrightarrow C$ as follows. Let $C = \{1, 0\}$. For all $b \in B$, let $h(b) = 1$. Define g as:

$$g(b) = \begin{cases} 1 & \text{if } b \in im(f) \\ 0 & \text{else} \end{cases}$$

Now we have that $hf = gf$ and hence $h = g$. This means that $g(b) = 1$ for all b and hence $im(f) = B$, meaning that f is surjective. □

For a morphism $f \colon A \longrightarrow B$, a monomorphism is indicated as $f \colon A \hookrightarrow B$ and an epimorphism as $f \colon A \twoheadrightarrow B$, but if explicitly stated, they indicate injections and surjections (which may at the same time be homomorphisms) instead.

There are examples of categories where the objects are structures with underlying sets and where the morphisms are functions, but where monomorphisms and epimorphisms don't correspond to injections and surjections respectively. In the category of rings, the inclusion $\mathbb{Z}$ into $\mathbb{Q}$ is an epimorphism, but it is not surjective.

2.5 Dual Structures

2.5.1 The Dual of a Category

Given a category $\mathcal{C}$, another category can be defined, $\mathcal{C}^{op}$. For objects, $\mathcal{C}_0^{op} = \mathcal{C}_0$. A morphism $f^{op} \colon B \longrightarrow A$ in C_1^{op} is exactly the morphism $f \colon A \longrightarrow B$ in C_1, i.e.,

$dom(f^{op}) = cod(f)$ and $cod(f^{op}) = dom(f)$. Given morphisms in $\mathcal{C}_1^{op}$, $f^{op} \colon B \longrightarrow A$ and $g^{op} \colon C \longrightarrow B$, we define $f^{op} \circ g^{op} \colon C \longrightarrow B$ as $(g \circ f)^{op}$.

Proposition 2.5.1. *Given a category $\mathcal{C}$, then the identity and associative rules hold in $\mathcal{C}^{op}$.*

Proof.

1. For morphisms $f \colon X \longrightarrow Y$ and $g \colon Y \longrightarrow Z$ we have:

$$f^{op} \circ 1_Y^{op} = (1_Y \circ f)^{op} = f^{op}$$

and

$$1_Y^{op} \circ g^{op} = (g \circ 1_Y)^{op} = g^{op}.$$

Thus the identity law holds.

2. Let $f \colon A \longrightarrow B, g \colon B \longrightarrow C$ and $h \colon C \longrightarrow D$ be morphisms in C_1. Then,

$$\begin{aligned}(f^{op} \circ g^{op}) \circ h^{op} &= ((g \circ f)^{op}) \circ h^{op} \\ &= (h \circ (g \circ f))^{op} \\ &= ((h \circ g) \circ f)^{op} \\ &= f^{op} \circ (h \circ g)^{op} \\ &= f^{op} \circ (g^{op} \circ h^{op})\end{aligned}$$

This proves associativity and hence, $\mathcal{C}$ is a category.

□

2.5.2 The Dual of a Functor

Suppose we have a functor $F \colon \mathcal{C} \longrightarrow \mathcal{D}$. Then, another functor can be defined, $F^{op} \colon \mathcal{C}^{op} \longrightarrow \mathcal{D}^{op}$, where

$$F^{op}(C) = F(C)$$

and

$$F^{op}(f^{op}) = (F(f))^{op}.$$

2.6 Limits

In this section we will primarily use uppercase letters for objects of a category and specific uppercase letters to denote functors. Again, the reader can consult Mac Lane [17, pp. 31–76] for more. The definition of a natural transformation, which is the notion of a structure preserving map between parallel functors, is found in Mac Lane [17, pp. 13–19].

Definition 2.6.1. Given a category $\mathcal{C}$ and index category $\mathcal{D}$, the *diagonal functor* is a functor $\Delta\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{D}}$ that sends each $C \in \mathcal{C}$ to the constant functor, i.e.,

$$\begin{aligned} \Delta(C)(X) &= C \\ \Delta(C)(f) &= 1_C \end{aligned}$$

An $f\colon C \longrightarrow C'$ with $f \in \mathcal{C}_1$ is sent to the natural transformation $\Delta(f) = \tau$ where $\tau\colon \Delta(C) \longrightarrow \Delta(C')$ consists of the map $f_D\colon \Delta(C)(D) \longrightarrow (C')(D) = f\colon C \longrightarrow C'$ at each $D \in \mathcal{D}_0$.

A functor $F\colon \mathcal{D} \longrightarrow \mathcal{C}$ in this context is usually called a diagram. A natural transformation $\tau\colon \Delta(C) \longrightarrow F$ is called a cone from Δ to F. The collection of all such τ for a C forms a category, called the comma category $\Delta \downarrow F$, with the morphisms forming commutative squares, explained in the following definition.

Definition 2.6.2. Given functors F, G as in the picture:

$$\begin{tikzcd} \mathcal{C} & \mathcal{A} \arrow[l, "F"'] \\ \mathcal{B} \arrow[u, "G"] \end{tikzcd}$$

the *comma category* $F \downarrow G$ has

- as objects all triples (A, B, f) with $A \in \mathcal{A}_0$, $B \in \mathcal{B}_0$ and $f\colon F(A) \longrightarrow G(B)$

$$\begin{tikzcd} F(A) \arrow[d, "f"] \\ G(B) \end{tikzcd}$$

- and as morphisms $(A, B, f) \longrightarrow (A', B', f')$ all pairs $(g\colon A \longrightarrow A', h\colon B \longrightarrow B')$ such that the diagram commutes.

$$\begin{tikzcd} F(A) \arrow[r, "F(g)"] \arrow[d, "f"] & F(A') \arrow[d, "f'"] \\ G(B) \arrow[r, "G(h)"] & G(B') \end{tikzcd}$$

When F or G are certain special functors, there is a notational convention. If $\mathcal{A} = \bullet$, the one point category, and $F(\bullet) = X$, the functor value on the one object, the convention is to write $X \downarrow G$. When $\mathcal{A} = \mathcal{C}$ and $F = 1_{\mathcal{C}}$, the convention is to write $\mathcal{C} \downarrow G$.

$\Delta \downarrow F$ is thus comma category with the diagram above replaced by

$$\begin{tikzcd} \mathcal{C}^{\mathcal{D}} & \mathcal{C} \arrow[l, "\Delta"'] \\ \bullet \arrow[u, "G"] \end{tikzcd}$$

and where $G(\bullet) = F, G(1_\bullet) = 1_F$ (the identity natural transformation). Here $1_\bullet$ is the identity on the one object of $\bullet$.

Objects in $\Delta \downarrow F$ are natural transformations, for $C \in \mathcal{C}$, $\tau\colon \Delta(C) \longrightarrow F$, called cones. A cone is thus a collection of morphisms, for each $D \in \mathcal{D}$, we have a morphism, $\tau_D\colon \Delta(C)(D) = C \longrightarrow F(D)$. From the definition above, morphisms between cones are pairs $(g\colon C \longrightarrow C', 1_\bullet)$ such that the diagram commutes:

$$\begin{tikzcd} \Delta(C) \arrow[r, "\Delta(g)"] \arrow[d, "\tau"] & \Delta(C') \arrow[d, "\tau'"] \\ F \arrow[r, "1_F"] & F \end{tikzcd}$$

i.e., for each $D \in \mathcal{D}$, the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "\tau_D"] & C' \arrow[d, "\tau'_D"] \\ F(D) \arrow[r, equal, "1_{F(D)}"] & F(D) \end{tikzcd}$$

since $\Delta(g)(D) = g$, $\Delta(C)(D) = C$ and $\Delta(C')(D) = C'$.

Definition 2.6.3. The *limit* of a functor F is the terminal object of the category $\Delta \downarrow F$.

Definition 2.6.4. Dually, the *colimit* of a functor F is the initial object of the category $F \downarrow \Delta$.

Proposition 2.6.5. *Let $F\colon \mathcal{D} \longrightarrow$ **Set** be a functor where $\mathcal{D} = \{0, 1\}$, the discrete two object category with $F(0) = C$ and $F(1) = C'$. Then the limit of F is the cartesian product $C \times C'$.*

Proof. A cone in $\Delta \downarrow F$ is a set W in **Set** with a pair of functions, $f\colon W \longrightarrow C$ and $g\colon W \longrightarrow C'$. To show that $(C \times C', \pi_1, \pi_2)$ is the limit, we show that it is the terminal object in $\Delta \downarrow F$, i.e., there is a unique morphism from any cone (W, f, g) to $(C \times C', \pi_1, \pi_2)$. Such a morphism is, from our definition, a pair $(h\colon W \longrightarrow C \times C', 1_\bullet\colon \bullet \longrightarrow \bullet)$. The morphism $1_\bullet\colon \bullet \longrightarrow \bullet$ carries no information, so we require that the h as drawn in the diagram makes the two triangles commute.

$$\begin{tikzcd} & W \arrow[dl, "g"'] \arrow[d, dashed, "\exists!\, h"] \arrow[dr, "f"] & \\ C & C \times C' \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & C' \end{tikzcd}$$

For h to be defined, we need that $h(w) = (x, y)$ for some $x \in C$ and $y \in C'$. For the diagram to commute, we require that $\pi_1 h(w) = \pi_1(x, y) = x = g(w)$ and $\pi_2 h(w) = \pi_2(x, y) = y = f(w)$. Thus h must be defined as $h(w) = (g(w), f(w))$ to make the diagram commute. Hence, there is only one morphism of cones from (W, f, g) to $(C \times C', \pi_1, \pi_2)$ and hence $(C \times C', \pi_1, \pi_2)$ is the terminal object in $\Delta \downarrow F$. $\square$

Definition 2.6.6. A *pullback* is the limit of a functor $F \colon \mathcal{D} \longrightarrow \mathbf{Set}$, where $\mathcal{D}$ consists of two morphisms with a common codomain. Dually, a pushout is a colimit of such a functor where $\mathcal{D}$ consists of two morphisms with a common domain.

The standard way to define a pushout is to define it as a quotient of a coproduct. In **Set**, the pushout of $f \colon A \longrightarrow B$ and $g \colon A \longrightarrow C$ is a minimal quotient of the coproduct $B + C$ such that, for $b \in B$ and $c \in C$, $b \sim c$ when there is some a such th1at $c = g(a)$ and $b = f(a)$, which consequently gives $\iota_1 g(a) = \iota_1(c) = [c]_\sim = [b]_\sim = \iota_2(b) = \iota_2 f(a)$.

Definition 2.6.7. For a function $f \colon A \longrightarrow B$ we call the elements of B which are not in the image of f the *isolated points* of f.

Proposition 2.6.8. *In* **Set**, *another way of computing the pushout of $f \colon A \longrightarrow B$ and $g \colon A \longrightarrow C$ is as follows. We firstly construct an equivalence relation Q on A. Let $(a_1, a_2) \in Q$ if and only if $a_1 \in f^{-1}fg^{-1}g \ldots f^{-1}fg^{-1}g\{a_2\}$. Then define $C +_A B$ as the set partition corresponding to Q disjoint union the isolated points of B under f and the isolated points of C under g. The two maps $\iota_1 \colon C \longrightarrow C +_A B$ and $\iota_2 \colon B \longrightarrow C +_A B$ are defined as follows: for points in the images of f and g, $\iota_1(c) = [g^{-1}\{c\}]$ and $\iota_2(b) = [f^{-1}\{b\}]$ and $\iota_1(c) = c$, $\iota_2(b) = b$ for the isolated points. Note that $[g^{-1}\{c\}]$ is the equivalence class of Q containing $g^{-1}\{c\}$.*

Proof. We need to show the universal property of the pushout

$$
\begin{tikzcd}
 & & W \\
B \arrow[r, "\iota_2"] \arrow[urr, bend left, "q_2"] & C +_A B \arrow[ur, dashed, "\exists!\, u"] & \\
A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, "\iota_1"'] \arrow[uur, bend right, "q_1"'] &
\end{tikzcd}
$$

Suppose we have $q_2 f = q_1 g$ as in the diagram. This means that for all $a \in A$, $q_2 f(a) = q_1 g(a)$. We define u as $u([a]) = q_1 g(a)$, and for the isolated points $b \in B$ and $c \in C$, $u(c) = q_1(c)$ and $u(b) = q_2(b)$. Suppose that $[a_i] = [a_j]$, then we have $a_j \in f^{-1}fg^{-1}g \ldots f^{-1}fg^{-1}g\{a_i\}$, which implies that

$$a_j \in f^{-1}q_2^{-1}q_2 f g^{-1}q_1^{-1}q_1 g \ldots f^{-1}q_2^{-1}q_2 f g^{-1}q_1^{-1}q_1 g\{a_i\}$$

since $X \subseteq q_1^{-1}q_1X$ and $Y \subseteq q_2^{-1}q_2Y$. If for some a_1 and a_2, $q_1g(a_1) = q_1g(a_2)$, it is equivalent to $q_2f(a_2) = q_1g(a_1)$. This means that there is some possibly infinite chain

$$q_1g(a_i) = q_1g(a_{i_2}) = q_2f(a_{i_2}) = q_2f(a_{i_3}) = q_1g(a_{i_3}) = \ldots = q_2f(a_j) = q_1g(a_j)$$

and hence $q_1g(a_i) = q_1g(a_j)$. And thus u is well-defined. Furthermore, $u\iota_1(c) = u([g^{-1}\{c\}]) = u([a]) = q_1g(a) = q_1(c)$ where $g(a) = c$ is not an isolated point, and $u\iota_1(c) = u(c) = q_1(c)$ when it is an isolated point. Hence, $u\iota_1 = q_1$; similarly, $u\iota_2 = q_2$.

Suppose there is another u' such that $u'\iota_1 = q_1$ and $u'\iota_1 = q_1$. Then, for isolated points $b \in B$ and $c \in C$, $u(c) = q_1(c) = u'\iota_2(c) = u'(c)$ and $u(b) = q_2(b) = u'\iota_2(b) = u'(b)$ and for an equivalence class in Q, $u([a]) = u([g^{-1}g(\{a\})]) = u([g^{-1}\{g(a)\}]) = u'\iota_1g(a) = u'([a])$.

This proves that $u(x) = u'(x)$ for every possible $x \in C +_A B$, and hence $u = u'$, which proves uniqueness of u. □

2.7 Adjunctions

Definition 2.7.1. An *adjunction* between two categories consists of the data of a pair of functors, $(F\colon \mathcal{X} \longrightarrow \mathcal{A}, G\colon \mathcal{A} \longrightarrow \mathcal{X})$ together with a family of bijections,

$$\alpha_{X,A}\colon \hom(F(X), A) \simeq \hom(X, G(A))$$

which is natural in $X \in \mathcal{X}$ and $A \in \mathcal{A}$. F is called the left adjoint of the adjunction and G is called the right adjoint. Left and right adjoints to a functor are unique up to isomorphism.

Proposition 2.7.2. *The functors, $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}^{op}$ and $G\colon \mathbf{Set}^{op} \longrightarrow \mathbf{Set}$, where F sends a set to its powerset and a function to its inverse image map, and $G = F^{op}$, form an adjunction.*

Proof. Let us write $F(f)^{op}$ to mean $(F(f))^{op}$. The functor F works as follows. Given a function $f\colon X' \longrightarrow X$, we have $F(f)\colon 2^{X'} \longrightarrow 2^X$ as a morphism in $\mathbf{Set}^{op}$, where 2^X is the powerset of X. But such a morphism is a function in $\mathbf{Set}$, the inverse image $(F(f))^{op}\colon 2^X \longrightarrow 2^{X'}$. Now, given an element of the powerset of X, $q\colon X \longrightarrow 2$, we have $F(f)^{op}(q) = q \circ f$, visually:

$$F(f)^{op}(X \xrightarrow{q} 2) = X' \xrightarrow{f} X \xrightarrow{q} 2 \tag{2.1}$$

We then get the dual definition for $G = F^{op}$, i.e., given a morphism $s^{op}\colon A \longrightarrow A'$ in $\mathbf{Set}$, we have

$$G(s^{op})(A \xrightarrow{r} 2) = A' \xrightarrow{s} A \xrightarrow{r} 2 \tag{2.2}$$

Consider a morphism in $\mathbf{Set}^{op}$ of the form $f\colon F(X) \longrightarrow A$. This is a function, $A \xrightarrow{f} 2^X$. From this, we need to construct a morphism, $\alpha_{X,A}(f)\colon X \longrightarrow G(A)$ in **Set**, which is a function $X \xrightarrow{\alpha_{X,A}(f)} 2^A$.

Let $\alpha_{X,A}(f)(x) = g$ where $g(a) = f(a)(x)$. In other words, we have the equality,

$$(\alpha_{X,A}(f))(x)(a) = f(a)(x). \tag{2.3}$$

We need to show now that $\alpha_{X,A}\colon \hom(A, 2^X) \longrightarrow \hom(X, 2^A)$ is natural in A and X. That is, given $g^{op}\colon A \longrightarrow A'$ in $\mathbf{Set}^{op}$ and $f\colon X' \longrightarrow X$ in **Set**, we need to show that the following diagram commutes. (The notation is rewritten such that we work only in **Set**).

$$\begin{tikzcd}
\hom_{\mathbf{Set}}(A, 2^X) \arrow[r, "\alpha_{X,A}"] \arrow[d, "\hom_{\mathbf{Set}}(g, F(f)^{op})"'] & \hom_{\mathbf{Set}}(X, 2^A) \arrow[d, "\hom_{\mathbf{Set}}(f, G(g^{op}))"] \\
\hom_{\mathbf{Set}}(A', 2^{X'}) \arrow[r, "\alpha_{X',A'}"] & \hom_{\mathbf{Set}}(X', 2^{A'})
\end{tikzcd}$$

We can show this by chasing an arbitrary element around the diagram.

$$\begin{tikzcd}
A \xrightarrow{h} 2^X \arrow[r, mapsto, "\alpha_{X,A}"] \arrow[dd, mapsto, "\hom_{\mathbf{Set}}(g, F(f)^{op})"'] & X \xrightarrow{\alpha_{X,A}(h)} 2^A \arrow[d, mapsto, "\hom_{\mathbf{Set}}(f, G(g^{op}))"] \\
 & X' \xrightarrow{f} X \xrightarrow{\alpha_{X,A}(h)} 2^A \xrightarrow{G(g^{op})} 2^{A'} \arrow[d, equal] \\
A' \xrightarrow{g} A \xrightarrow{h} 2^X \xrightarrow{F(f)^{op}} 2^{X'} \arrow[r, mapsto, "\alpha_{X',A'}"] & X' \xrightarrow{\alpha_{X',A'}(F(f)^{op} \circ h \circ g)} 2^{A'}
\end{tikzcd} \tag{2.4}$$

The equality in (2.4) is computed as follows, for any $x' \in X'$ and $a' \in A'$, we have:

$$\begin{aligned}
&\alpha_{X',A'}(F(f)^{op} \circ h \circ g)(x')(a') \\
&= (F(f)^{op} \circ h \circ g)(a')(x') \quad \text{by the equality in (2.3)} \\
&= F(f)^{op}((h \circ g)(a'))(x') \quad \text{expanding brackets} \\
&= ((h \circ g)(a') \circ f)(x') \quad \text{from the definition in (2.1)} \\
&= (h \circ g)(a')(f(x')) \\
&= h(g(a'))(f(x')) \\
&= \alpha_{X,A}(h)(f(x'))(g(a')) \quad \text{by the equality in (2.3)} \\
&= (\alpha_{X,A}(h)(f(x')) \circ g)(a') \quad \text{writing as a composite} \\
&= G(g^{op})(\alpha_{X,A}(h)(f(x')))(a') \quad \text{from the definition in (2.2)} \\
&= G(g^{op})((\alpha_{X,A}(h) \circ f)(x'))(a') \\
&= (G(g^{op}) \circ \alpha_{X,A}(h) \circ f)(x')(a')
\end{aligned}$$

Thus, we have that

$$G(g^{op}) \circ \alpha_{X,A}(h) \circ f = \alpha_{X',A'}(F(f)^{op} \circ h \circ g).$$

This completes the proof for naturality. We now need to show that each $\alpha_{X,A}$ for some X, A is indeed a bijection. We define

$$\alpha^{-1}_{X,A}(p)(a)(x) = p(x)(a). \tag{2.5}$$

To show

$$\alpha^{-1}_{X,A}(\alpha_{X,A}(h)) = h$$

means that we need to show it is equal at each argument,

$$\alpha^{-1}_{X,A}(\alpha_{X,A}(h))(a) = h(a).$$

And to show this resulting functions are equal, we again need to show they are equal at each argument,

$$\alpha^{-1}_{X,A}(\alpha_{X,A}(h))(a)(x) = h(a)(x).$$

But we immediately have from (2.3) and (2.5) that

$$\begin{aligned} &\alpha^{-1}_{X,A}(\alpha_{X,A}(h))(a)(x) \\ &= \alpha_{X,A}(h)(x)(a) \\ &= h(a)(x) \end{aligned}$$

And similarly, we have $\alpha_{X,A}(\alpha^{-1}_{X,A}(p)) = p$. This concludes the proof. □

2.7.1 Adjunction by Unit and Counit

Proposition 2.7.3. *An adjunction determines and is determined by two natural transformations,* $\eta\colon 1_X \longrightarrow GF$ *and* $\varepsilon\colon FG \longrightarrow 1_A$ *such that* $\eta_{G(A)} \circ G(\varepsilon_A) = 1_{G(A)}$ *and* $\varepsilon_{F(X)} \circ F(\eta_X) = 1_{F(X)}$. *That is, the following two diagrams commute:*

$$\begin{tikzcd} F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, equal, "1_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\ & F(X) \end{tikzcd} \qquad \begin{tikzcd} GFG(A) \arrow[d, "\eta_{G(A)}"'] & G(A) \arrow[l, "G(\varepsilon_A)"'] \arrow[dl, equal, "1_{G(A)}"] \\ G(A) & \end{tikzcd} \tag{2.6}$$

Proof. Let us show that the first triangle condition holds given a family of bijections α that are natural in their arguments, as in definition 2.7.1. Define the unit and counit as

$$\begin{aligned} \eta_X &= \alpha_{X,F(X)}(1_{F(X)}) \\ \varepsilon_A &= \alpha^{-1}_{G(A),A}(1_{G(A)}) \end{aligned}$$

Then we need to show that the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(\alpha_{X,F(X)}(1_{F(X)}))"] \arrow[dr, equal, "1_{F(X)}"'] & FGF(X) \arrow[d, "\alpha^{-1}_{GF(X),F(X)}(1_{GF(X)})"] \\ & F(X) \end{tikzcd} \tag{2.7}$$

We prove this using naturality of α. Given $\alpha_{X,F(X)}(1_{F(X)}) \colon X \longrightarrow GF(X)$, we have that the following diagram commutes:

$$\begin{tikzcd} \hom_{\mathcal{X}}(GF(X), GF(X)) \arrow[r, "\alpha^{-1}_{GF(X),F(X)}"] \arrow[d, "\hom_{\mathcal{X}}(\alpha_{X,F(X)}(1_{F(X)}), G(1_{F(X)}))"'] & \hom_{\mathcal{A}}(FGF(X), F(X)) \arrow[d, "\hom_{\mathcal{A}}(F(\alpha_{X,F(X)}(1_{F(X)})), 1_{F(X)})"] \\ \hom_{\mathcal{X}}(X, GF(X)) \arrow[r, "\alpha^{-1}_{X,F(X)}"] & \hom_{\mathcal{A}}(F(X), F(X)) \end{tikzcd}$$

Note that since G is a functor, $G(1_{F(X)}) = 1_{GF(X)}$. Now, by chasing $1_{GF(X)}$ around the diagram, we get:

$$\begin{tikzcd} 1_{GF(X)} \arrow[r, mapsto] \arrow[dd, mapsto] & \alpha^{-1}_{GF(X),F(X)}(1_{GF(X)}) \arrow[d, mapsto] \\ & \alpha^{-1}_{GF(X),F(X)}(1_{GF(X)}) \circ F(\alpha_{X,F(X)}(1_{F(X)})) \arrow[d, equal] \\ \alpha_{X,F(X)}(1_{F(X)}) \arrow[r, mapsto] & \alpha^{-1}_{X,F(X)}(\alpha_{X,F(X)}(1_{F(X)})) = 1_{F(X)} \end{tikzcd}$$

Thus, we have that

$$\alpha^{-1}_{GF(X),F(X)}(1_{GF(X)}) \circ F(\alpha_{X,F(X)}(1_{F(X)})) = 1_{F(X)}$$

which is what we wanted to show. Hence the first triangle condition in (2.7) holds and hence the first triangle condition in (2.6) holds. A dual argument proves that the second triangle condition in (2.6) holds. Now, we also need to show naturality of ε and η.

Naturality of ε means that for $f \colon A \longrightarrow A'$, one needs to show that

$$f \circ \varepsilon_A = \varepsilon_{A'} \circ FG(f)$$

which translates to

$$f \circ \alpha^{-1}_{G(A),A}(1_{G(A)}) = \alpha^{-1}_{G(A'),A'}(1_{G(A')}) \circ FG(f). \tag{2.8}$$

The latter can be shown by chasing along two diagrams. Chasing $1_{G(A')}$ along:

$$\begin{tikzcd}
\hom_{\mathcal{X}}(G(A'),G(A')) \arrow[r, "\alpha^{-1}_{G(A'),A'}"] \arrow[d, "\hom_{\mathcal{X}}(G(f),G(1_{A'}))"'] & \hom_{\mathcal{A}}(FG(A'),A') \arrow[d, "\hom_{\mathcal{A}}(FG(f),1_{A'})"] \\
\hom_{\mathcal{X}}(G(A),G(A')) \arrow[r, "\alpha^{-1}_{G(A),A'}"] & \hom_{\mathcal{A}}(FG(A),A')
\end{tikzcd}$$

yields

$$\alpha^{-1}_{G(A'),A'}(1_{G(A')}) \circ FG(f) = \alpha^{-1}_{G(A'),A}(G(f)). \tag{2.9}$$

Now, chasing $1_{G(A)}$ along:

$$\begin{tikzcd}
\hom_{\mathcal{X}}(G(A),G(A)) \arrow[r, "\alpha^{-1}_{G(A),A}"] \arrow[d, "\hom_{\mathcal{X}}(1_{G(A)},G(f))"'] & \hom_{\mathcal{A}}(FG(A),A)) \arrow[d, "\hom_{\mathcal{A}}(F(1_{G(A)}),f))"] \\
\hom_{\mathcal{X}}(G(A),G(A')) \arrow[r, "\alpha^{-1}_{G(A),A'}"] & \hom_{\mathcal{A}}(FG(A),A')
\end{tikzcd}$$

yields

$$f \circ \alpha^{-1}_{G(A),A}(1_{G(A)}) = \alpha^{-1}_{G(A'),A}(G(f)). \tag{2.10}$$

Together, (2.9) and (2.10) give that (2.8) holds. Hence, ε is indeed a natural transformation. Again, a dual argument shows that η is natural.

The reverse implication, that the definition by unit and counit in 2.7.3 implies the definition by homsets in 2.7.1, can be proven by setting

$$\alpha_{X,A}(f) = G(f) \circ \eta_X$$

and by setting

$$\alpha^{-1}_{X,A}(g) = \varepsilon_A \circ F(g).$$

□

2.7.2 Galois Connections

Definition 2.7.4. A *poset category* is a category where there is at most one morphism between any two objects. Moreover, all isomorphisms are identities. This coincides exactly with the notion of a poset in set theory.

To see how a poset category $\mathcal{X}$ forms a poset, consider $(X = \mathcal{X}_0, \leqslant)$, where $x \leqslant y$ if and only if there is a morphism $x \longrightarrow y$. Reflexivity follows from the identity axiom and transitivity follows from the associativity axiom. Anti-symmetry follows since isomorphisms are identities.

Definition 2.7.5. A *Galois connection* is an adjunction between two poset categories.

Proposition 2.7.6. *The data of a Galois connection are exactly a pair of order preserving maps $f\colon X \longrightarrow A$ and $g\colon A \longrightarrow X$ between posets such that*

$$fx \leqslant y \Leftrightarrow x \leqslant gy.$$

Proof. Suppose that f and g form an adjunction of poset categories. Then the functors f and g are, equivalently, functions between posets. Let $fx \leqslant y$, then it implies that $\hom(fx, y) \simeq \hom(x, gy)$ are bijective, and the first contains the single morphism $fx \longrightarrow y$ and the latter hence contains the single morphism $x \longrightarrow gy$, i.e. $x \leqslant yf$. The converse is similar, if we suppose $x \leqslant gy$, then it implies that $fx \leqslant y$. Finally, given the condition

$$fx \leqslant y \Leftrightarrow x \leqslant gy$$

one can form an adjunction of poset categories, since $\hom(fx, y) \simeq \hom(x, gy)$ implies either both contain one element or are both empty. □

Notation 2.7.7. As an abstraction of the notions of direct image and preimage maps, we may write the Galois condition as:

$$fx \leqslant y \Leftrightarrow x \leqslant yf$$

where $f(-)$ and $(-)f$ are understood to refer to two different maps, having the same values as the left adjoint f and right adjoint g, respectively.

Remark 2.7.8. For order preserving maps $f(-)\colon X \longrightarrow A$ and $(-)f$, the adjunction by unit and co-unit definition gives an equivalent condition for a Galois connection: for all $x \in X$, $x \leqslant (fx)f$, and for all $a \in A$, $f(af) \leqslant a$.

Proposition 2.7.9. *The concrete direct image map and concrete inverse image map of sets form a Galois connection.*

Proof. Let $f\colon A \longrightarrow B$ be a function. Then, $f(-)$ and $(-)f$ are monotone and for all $X \subseteq A$ and $Y \subseteq B$, we have $X \subseteq (fX)f$ and $f(Yf) \subseteq Y$ and it immediately follows that the two maps form a Galois connection. □

2.8 Equivalence of Categories

Definition 2.8.1. Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* when there exists an adjunction between them, (F, G, α), such that the unit η and counit ε of the adjunction are natural isomorphisms.

Proposition 2.8.2. *The category of finite topological spaces is equivalent to the category of finite preordered sets.*

Proof. We present a sketch of the proof by explaining how the functors for the equivalence work. For a finite topological space, (X, τ), the corresponding partial order is defined as $(X, \leqslant)$ with

$$x \leqslant y \quad \Longleftrightarrow \quad \textit{all open sets containing } x \textit{ contain } y$$

Conversely, given a preorder, $(X, \leqslant)$, we have the topological space (X, τ), where τ consists of the upsets of $\leqslant$.

It needs to be shown that continuous maps correspond to order preserving maps and the other way around. Suppose $f\colon (X, \leqslant) \longrightarrow (Y, \leqslant')$ is a order preserving map. We want to show $f\colon (X, \tau) \longrightarrow (Y, \tau')$ between the corresponding topological spaces is continuous. Suppose U is open in τ' and hence upclosed under $\leqslant'$. Then we want to show that $(U)f$ is open in τ, i.e., upclosed under $\leqslant$. Suppose $x \leqslant y$ where $f(x) \in U$. By monotonicity, $f(x) \leqslant' f(y)$ and since U is upclosed we have that $f(y) \in U$ and hence $y \in (U)f$ and hence $(U)f$ is upclosed and by implication open in τ.

Conversely, suppose $f\colon (X, \tau) \longrightarrow (Y, \tau')$ is a continuous map. Suppose further that $x \leqslant y$. Then $(\uparrow f(x))f$ must be open in τ, i.e., upclosed in $\leqslant$. Since $x \in (\uparrow f(x))f$, $y \in (\uparrow f(x))f$ and hence $f(y) \in \uparrow f(x)$ and $f(x) \leqslant f(y)$. This shows the how the functors act between the two categories. What remains is to show that the unit and counit are isomorphisms. □

2.9 Forms and Fibrations

Definition 2.9.1. A *form* is a faithful functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$. In order to have posets and not preorders for preimage categories, one assumes that F is also amnestic. Of course, moving from a preorder to a poset can be done by simply replacing equivalent elements under the preorder with their equivalence class.

Definition 2.9.2. The *fibres* of a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ refer to the preimage categories. That is, for each $D \in \mathcal{D}_0$ we have the fibre $F^{-1}(D) = \mathcal{X}$ where $\mathcal{X}_0$ consists of all $C \in \mathcal{C}_0$ such that $F(C) = D$ and $\mathcal{X}_1$ consists of all $f \in \mathcal{C}_1$, such that $F(f) = 1_D$.

Proposition 2.9.3. *The fibres of a functor are categories.*

Proof. Let $\mathcal{X} = F^{-1}(D)$. Identities map to identities and hence $F^{-1}(D)$ contains all the identities of objects in $\mathcal{X}$. Then it suffices to show that composites are in $\mathcal{X}_1$. Suppose $F(f\colon X \longrightarrow Y) = 1_D = F(g\colon Y \longrightarrow Z)$. Then $F(gf) = F(g)F(f) = 1_D \circ 1_D = 1_D$. Furthermore, $\mathcal{X}$ retains the associativity and unit rules from $\mathcal{C}$ and hence forms a category. □

Proposition 2.9.4. *The fibres of a form are posets.*

Proof. Let $\mathcal{X} = F^{-1}(D)$.

- *Reflexivity.* Identities in $\mathcal{X}_1$ have $F(1_X) = 1_D$. Since F is faithful, any other $F(f\colon X \longrightarrow X) = 1_D$ implies that $f = 1_X$ and hence there is only one morphism $X \longrightarrow X$.
- *Transitivity.* Suppose $F(f\colon X \longrightarrow Y) = 1_D = F(g\colon Y \longrightarrow Z)$. The composite gf is in $\mathcal{X}_1$. Since F is faithful, any other $F(h\colon X \longrightarrow Z)$ is in $\mathcal{X}_1$ if and only if $F(h) = 1_D$ and since F is faithful, this forces $h = gf$, so there is only one morphism $X \longrightarrow Z$.
- *Antisymmetry.* Suppose some $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ are in $\mathcal{X}_1$. Since F is faithful this forces $fg = 1_X$ and $gf = 1Y$. Hence, f and g are isomorphisms and $X \simeq Y$. But F is amnestic, so since $F(f) = F(g) = 1_D$, this forces f and g to be identities and hence $X = Y$.

$\square$

Definition 2.9.5. A functor $F\colon \mathcal{E} \longrightarrow \mathcal{C}$ is a *Grothendieck fibration* if the following holds.

- For each $f'\colon C \longrightarrow F(E)$ there exists a morphism $f\colon E' \longrightarrow E$ with $F(f) = f'$.

$$\begin{tikzcd} E & & F(E) \\ & {} \arrow[r, mapsto, "F"] & {} \\ E' \arrow[uu, "\exists\, f"] & & C = F(E') \arrow[uu, "f'=F(f)"'] \end{tikzcd}$$

- Additionally, if there is a $g\colon E'' \longrightarrow E$ with $g' = F(g)$ factoring as $f'h' = g'$ for some $h'\colon F(E'') \longrightarrow F(E')$, then there is a unique $h\colon E'' \longrightarrow E'$ with $F(h) = h'$ and such that $fh = g$. We shall refer to this as the *path lifting property*.

$$\begin{tikzcd} & E & & & F(E) \\ & & {} \arrow[r, mapsto, "F"] & {} & \\ E'' \arrow[ruu, "g"] \arrow[r, dashed, "\exists!\, h"'] & E' \arrow[uu, "f"'] & & F(E'') \arrow[ruu, "g'"] \arrow[r, "h'"'] & F(E') \arrow[uu, "f'"'] \end{tikzcd}$$

An *opfibration* $F\colon \mathcal{E} \longrightarrow \mathcal{C}$ is a fibration in the opfunctor $F^{op}\colon \mathcal{E}^{op} \longrightarrow \mathcal{C}^{op}$. A *bifibration* is a functor which is both a fibration and an opfibration.

2.9.1 The Fundamental Fibration

Given a category $\mathcal{C}$, one constructs the arrow category $\mathcal{C}^{\rightarrow}$ as follows:

- Objects in $\mathcal{C}_0^{\rightarrow}$ are morphisms in $\mathcal{C}_1$.
- A morphism in $\mathcal{C}_1^{\rightarrow}$ is a pair of morphisms (h, q) between two morphisms $f\colon C \longrightarrow C'$ and $g\colon D \longrightarrow D'$ in $\mathcal{C}_1$ such that $qf = gh$:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[d, "h"'] & C' \arrow[d, "q"] \\
D \arrow[r, "g"'] & D'
\end{tikzcd}$$

Thus, the category $\mathcal{C}^{\rightarrow}$ is the comma category $1_{\mathcal{C}} \downarrow 1_{\mathcal{C}}$.

Note that when we say **a pair** (h, q) **between** f **and** g, we can equally represent this with a 4-tuple, (h, q, f, g). The important thing to note is that the pair retains the information of its domain and codomain, just like how a function $f\colon X \longrightarrow Y$ should be seen as a pair $(R, (X, Y))$, where $R \subseteq X \times Y$ is the graph of the function and hence $f = (R, (X, Y))$ retains its domain and codomain data.

Proposition 2.9.6. *The functor $F\colon \mathcal{C}^{\rightarrow} \longrightarrow \mathcal{C}$ which sends an object in $\mathcal{C}^{\rightarrow}$ (i.e. a morphism in $\mathcal{C}_0$) to its codomain and a morphism (i.e. a pair (h, q) of morphisms forming a commutative square with two other morphisms f and g) to its second projection, q, is an opfibration. If $\mathcal{C}$ has pullbacks, then F is a bifibration.*

Proof. Suppose we have $f'\colon F(e) \longrightarrow C$ where e is a morphism, $D' \longrightarrow D$ and $D = F(e)$. The pair $(1_{D'}, f')$ between $e' = f'e$ and e maps to f' under F. Now we need to show that $(1_{D'}, f')$ satisfies the path lifting property. Suppose there is some pair (g, g') from e to some e''. Suppose further that in the image we have a factorisation $h'f' = g'$ through some $h'\colon C \longrightarrow B$. An h that completes the commutative diagram must satisfy $h \circ 1_{D'} = g$, i.e., we must have $h = g$. Also, we need that $e''h = h'f'e$, which holds since $h'f'e = g'e = e''g = e''h$. Hence, F is opfibration.

$$\begin{tikzcd}
B' \arrow[dd, "e''"'] & & & & D' \arrow[llll, "g"'] \arrow[dll, "1_{D'}"] \arrow[dd, "e"] & & & & \\
& & D' \arrow[ull, dashed, "\exists!\, h"] \arrow[dd, "e'=f'e"] & & & & & & \\
B & & & & D \arrow[llll, "g'"'] \arrow[dll, "f'"] & & B & & D \arrow[ll, "g'"'] \arrow[dl, "f'"] \\
& & C \arrow[ull, "h'"] & & & {} \arrow[r, mapsto, "F"] & {} & C \arrow[ul, "h'"] &
\end{tikzcd}$$

Assume that $\mathcal{C}$ has pullbacks. Suppose we have $f'\colon C \longrightarrow F(e)$. The pair (π_2, f') between e and $\pi_1\colon C \times_D D' \longrightarrow C$ maps to f' under F. Again, we need to show that the pair as a morphism in $\mathcal{C}^{\rightarrow}$ satisfies the path lifting property. Suppose we have a pair (g, g') between e'' and e, where there is a factorisation in the image $f'h' = g'$. Then $eg = g'e'' = f'h'e''$. By the universal property of the pullback, there is a unique h with $\pi_1 h = h'e''$, meaning the pair (h, h') forms a commutative square with e'' and e'. Also, $\pi_2 h = g$, thus giving a commutative triangle of morphisms in $\mathcal{C}^{\rightarrow}$. From this, F is a fibration.

$$\begin{tikzcd}
B' \arrow[rr, "g"] \arrow[dr, dashed, "\exists!\, h"] \arrow[dd, "e''"'] & & D' \arrow[dd, "e"] & & & & \\
& C \times_d D' \arrow[ur, "\pi_2"'] \arrow[dd, "\pi_1 = e'"] & & & & & \\
B \arrow[rr, "g'"] \arrow[dr, "h'"'] & & D & & B \arrow[rr, "g'"] \arrow[dr, "h'"'] & & D \\
& C \arrow[ur, "f'"'] & & {} \arrow[r, mapsto, "F"] & {} & C \arrow[ur, "f'"'] &
\end{tikzcd} \tag{2.11}$$

□

2.9.2 The Subobject Bifibration

Certain restrictions of the domain functor and codomain functor $F, G \colon \mathcal{C}^{\rightarrow} \longrightarrow \mathcal{C}$ are also fibrations or opfibrations. Consider the category of groups **Grp**. Let $\mathcal{C}$ be the restriction of $\mathbf{Grp}^{\rightarrow}$ to monomorphisms. Let $F \colon \mathcal{C} \longrightarrow \mathbf{Grp}$ be the resulting restriction of the codomain functor.

Proposition 2.9.7. $F \colon \mathcal{C} \longrightarrow \mathbf{Grp}$ *is a bifibration.*

Proof. The lifting of some $f' \colon F(e) = D \longrightarrow C$ is defined to be the commutative square composed of e, f' and the epi-mono factorisation of $f'e$. Such a factorisation always exists in **Grp** by the first isomorphism theorem.

$$\begin{tikzcd}
& D' \arrow[dl, two heads, "f"] \arrow[dd, hook, "e"] & & & \\
im(f'e) \arrow[dd, hook, "e'"'] & & & & \\
& D \arrow[dl, "f'"] & {} \arrow[r, mapsto, "F"] & {} & D \arrow[dl, "f'"] \\
C & & & C &
\end{tikzcd}$$

Again, we need to show the path lifting property. Suppose we have a pair (g, g') between e and e''. Suppose the pair factorises in the image under F as $g' = h'f'$ for some h'. Define h as follows: $h(x) \in (\{h'e'(x)\})e''$, i.e. the preimage under e'' of the set containing one element, $h'e'(x)$. We need to show that this actually defines an appropriate value for $h(x)$. The above set, $(\{h'e'(x)\})e''$, contains at most one element since e'' is an injection (since we are in **Grp**). Now, f is a surjection so $x = f(y)$ for some y and

$$\begin{aligned} h'e'(x) &= h'e'f(y) \\ &= h'f'e(y) \\ &= g'e(y) \\ &= e''g(y) \end{aligned}$$

and hence, since $h'e'(x)$ is in the image of e'', h maps to at least one element and thus exactly one element. So h is well defined.

$$\begin{tikzcd}
B' \arrow[dd, hook, "e''"'] & & D' \arrow[ll, "g"'] \arrow[dl, two heads, "f"] \arrow[dd, hook, "e"] & & & & \\
& im(f'e) \arrow[ul, dashed, "\exists!\, h"] \arrow[dd, hook, "e'"] & & & & & \\
B & & D \arrow[ll, "g'"'] \arrow[dl, "f'"] \arrow[r, phantom, "\overset{F}{\longmapsto}"] & {} & B & & D \arrow[ll, "g'"'] \arrow[dl, "f'"] \\
& C \arrow[ul, "h'"] & & & & C \arrow[ul, "h'"] &
\end{tikzcd}$$

Now we need to show that h makes the diagram commute.

$$\begin{aligned} hf(y) &= x \in (\{h'e'f(y)\})e'' \\ &= x \in (h'e'f(\{y\}))e'' \\ &= x \in (e''g(\{y\}))e'' \quad \text{since } h'e'f(y) = e''g(y) \\ &= (e''g(y))e'' \\ &= g(y) \quad \text{since } e'' \text{ is surjective} \end{aligned}$$

So, the upper triangle commutes.

$$\begin{aligned} e''h(z) &= e''(x) \quad \text{where } x \in (\{h'e'(z)\})e'' \\ &= h'e'(z) \end{aligned}$$

So, the left most square commutes. The uniqueness of h follows from e'' being a monomorphism. Hence, F is an opfibration. Now we wish to show that F is a fibration.

Suppose we have $f' \colon C \longrightarrow F(e)$. Then e is a monomorphism, $e \colon D' \hookrightarrow D = F(e)$. Again, we form the pullback, $C \times_D D'$. Monomorphisms are preserved along pullbacks, which means that π_1 is a mono and thus in the category of monomorphisms $\mathcal{C}$. Now the rest of the proof follows as in diagram 2.11. Here the e'' must be a monomorphism, but that does not influence the proof.

$$
\begin{tikzcd}
B' \arrow[rr, "g"] \arrow[dr, dashed, "\exists!\, h"] \arrow[dd, hook, "e''"'] & & D' \arrow[dd, hook, "e"] \\
& C \times_D D' \arrow[ur, "\pi_2"'] \arrow[dd, hook, "\pi_1 = e'"] & \\
B \arrow[rr, "g'"] \arrow[dr, "h'"'] & & D \\
& C \arrow[ur, "f'"'] &
\end{tikzcd}
\quad \overset{F}{\longmapsto} \quad
\begin{tikzcd}
B \arrow[rr, "g'"] \arrow[dr, "h'"'] & & D \\
& C \arrow[ur, "f'"'] &
\end{tikzcd}
\tag{2.12}
$$

□

2.10 Substructures as Functorial Preimages

Let us revisit the idea of substructures that are determined by morphisms as briefly mentioned in the start of this chapter in Section 2.3.

Consider **CRing**, the category of commutative rings. Let $\mathcal{C}$ be the restriction of $\mathbf{CRing}^{\rightarrow}$ to surjective homomorphisms into integral domains. That means that $\mathcal{C}_0$ consists of surjective homomorphisms of the form

$$g\colon R \twoheadrightarrow D$$

where D is an integral domain, and $\mathcal{C}_1$ consists of commutative squares of the form

$$
\begin{tikzcd}
R \arrow[r, two heads, "g"] \arrow[d, "f"'] & D \arrow[d, "f'"] \\
R' \arrow[r, two heads, "g'"] & D'
\end{tikzcd}
$$

We now construct the the restriction of the domain functor $F\colon \mathcal{C} \longrightarrow \mathbf{CRing}$ as follows. A surjective homomorphism in $\mathcal{C}_0$ gets mapped to its domain, and a commutative square consisting of a pair (f, f') between g and g' gets mapped to the pair's first projection, f.

$$
\begin{tikzcd}
R \arrow[r, two heads, "g"] \arrow[d, "f"'] & D \arrow[d, "f'"] & & R \arrow[d, "f"] \\
R' \arrow[r, two heads, "g'"] & D' & & R'
\end{tikzcd}
\quad \text{with} \quad \overset{F}{\longmapsto}
$$

Proposition 2.10.1. *The functor $F\colon \mathcal{C} \longrightarrow$ **CRing** is a form.*

Proof. The functor F is faithful. To see this, suppose we have another $F((f, f'')) = f$ where (f, f'') forms a commutative square from g to g'. Then $f''g = g'f = f'g$. And since surjective homomorphisms are epimorphisms, we have $f' = f''$.

F can be made into a form by choosing a representative D for every equivalence class of isomorphisms. Then we have only one commutative diagram of the form

$$\begin{tikzcd} R \arrow[r, two heads, "g"] \arrow[d, "1_R"'] & D \arrow[d, "i", "\wr"'] \\ R \arrow[r, two heads, "g'"] & D' \end{tikzcd}$$

the one in which $D' = D$ and $i = 1_D$. That would make F amnestic and hence a form. $\square$

Proposition 2.10.2. *The functor $F\colon \mathcal{C} \longrightarrow$ **CRing** is a fibration.*

Proof. Let $e\colon R_1 \longrightarrow D_1$ be a surjective ring homomorphism in **CRing**, i.e., an object in $\mathcal{C}_0$. Suppose now that we have a morphism $f'\colon R_2 \longrightarrow R_1 = F(e)$. Then we need to show that there is a suitable lifting of f'. Let us define the lifting of f' as the commutative square

$$\begin{tikzcd} R_2 \arrow[r, two heads, "e'"] \arrow[d, "f'"'] & R_2/0(ef') \arrow[d, hook, "f"] \\ R_1 \arrow[r, two heads, "e"] & D_1 \end{tikzcd} \tag{2.13}$$

where e' is the canonical surjection to the quotient group sending $r \mapsto [r]$ and we define f as $f([r]) = ef'(r)$. So, e' and f in fact form the epi-mono factorisation of ef'. Also, $0(ef')$ is a prime ideal since it is the kernel of a function into an integral domain and so $R_2/0(ef')$ is an integral domain since it is a quotient by a prime ideal.

Now, we need to show the path lifting property. Suppose we have another commutative square, from some e'' to e,

$$\begin{tikzcd} R_3 \arrow[r, two heads, "e''"] \arrow[d, "g'"'] & D_3 \arrow[d, "g"] \\ R_1 \arrow[r, two heads, "e"] & D_1 \end{tikzcd} \tag{2.14}$$

and suppose we have a factorisation in the image of (g', g) through f' as $g' = f'h'$ for some h'.

$$\begin{tikzcd} R_2 \arrow[dr, "f'"'] & R_3 \arrow[l, "h'"'] \arrow[d, "g'"] \\ & R_1 \end{tikzcd} \tag{2.15}$$

Then, we need to construct a unique function $h\colon D_3 \longrightarrow R_2/0(ef')$ such that the full diagram commutes:

$$\begin{tikzcd}
R_2 \arrow[rr, two heads, "e'"] \arrow[dd, "f'"'] & & R_2/0(ef') \arrow[dd, hook, "f"] & \\
& R_3 \arrow[ul, "h'"] \arrow[dl, "g'"] \arrow[rr, two heads, "e''"] & & D_3 \arrow[ul, dashed, "h"'] \arrow[dl, "g"] \\
R_1 \arrow[rr, two heads, "e"] & & D_1 &
\end{tikzcd} \tag{2.16}$$

We define $h(i) = e'(x)$ where $x \in h'((\{i\})e'')$, h' applied to the preimage of $\{i\}$ under e''.

We need to show that h is well defined. Suppose we have $x_1, x_2 \in h'((\{i\})e'')$. Then we want to show that $e'(x_1) = e'(x_2)$. Now,

$$\begin{aligned}
x_1, x_2 \in h'(\{i\}e'') \Rightarrow \exists q_1 \exists q_2 (e''(q_1) &= i \\
\text{and } x_1 &= h'(q_1) \\
\text{and } e''(q_2) &= i \\
\text{and } x_2 &= h'(q_2))
\end{aligned}$$

and then we have:

$$\begin{aligned}
ef'(x_1) &= ef'h'(q_1) \\
&= eg'(q_1) \\
&= ge''(q_1) \\
&= g(i) \\
&= ge''(q_2) \\
&= eg'(q_2) \\
&= ef'h'(q_2) \\
&= ef'(x_2)
\end{aligned}$$

Which gives

$$\begin{aligned}
x_1, x_2 \in h'(\{i\}e'') &\Rightarrow ef'(x_1) = ef'(x_2) \\
&\Rightarrow fe'(x_1) = fe'(x_2) \\
&\Rightarrow e'(x_1) = e'(x_2) \quad \text{since } f \text{ is injective}
\end{aligned}$$

Note that $(\{i\})e''$ is non-empty since e'' is surjective. Hence, h is well-defined. We need to show that the square containing h commutes, $e'h' = he''$. As a preliminary result, we wish to show that

$$e''(i_1) = e''(i_2) \Rightarrow e'h'(i_1) = e'h'(i_2). \tag{2.17}$$

Note that:

$$\begin{aligned} ge''(i) &= eg'(i) \\ &= ef'h'(i) \\ &= fe'h'(i) \end{aligned}$$

Now suppose that $e''(i_1) = e''(i_2)$. Then $ge''(i_1) = ge''(i_2)$ and thus, $fe'h'(i_1) = fe'h'(i_2)$. But f is injective and hence $e'h'(i_1) = e'h'(i_2)$, which proves (2.17).

The pre-image of an image of $i \in R_3$ under e'' is all i' such that $e''(i') = e''(i)$, i.e.

$$(\{e''(i)\})e'' = \{i' \in R_3 \mid e''(i') = e''(i)\}.$$

But any such i' also satisfies $e'h'(i') = e'h'(i)$ by (2.17) and hence,

$$e'h'((\{e''(i)\})e'') = \{e'h'(i)\}$$

as singleton sets. We have for any $i \in R_3$:

$$he''(i) = e'(x) \quad \text{where } x \in h'((\{e''(i)\})e'')$$

hence, as direct and inverse image maps:

$$\begin{aligned} he''(\{i\}) &= e'h'((e''(\{i\}))e'') \\ &= e'h'(\{i\}) \end{aligned}$$

Which proves that the original function values are also equal, that is:

$$he''(i) = e'h'(i).$$

Finally, we need to show that $fh = g$, which would then make the rest of the diagram commute. For any $i \in D_3$,

$$\begin{aligned} fh(i) &= fe'(x) \quad \text{where } x \in h'(\{i\}e'') \\ &= ef'(x) \end{aligned}$$

But this holds for all such $x \in h'(\{i\}e''$ and thus for direct and inverse image maps we have:

$$\begin{aligned} ef'(\{x\}) &= ef'(h'(\{i\}e'')) \\ &= ef'h'(\{i\}e'') \\ &= eg'(\{i\}e'') \\ &= ge''(\{i\}e'') \\ &= g(\{i\}) \end{aligned}$$

This means that the equality also holds for the function values, $fh(i) = g(i)$, and hence $fh = g$. Furthermore, h is unique since f is a mono. The path lifting property holds and $F\colon \mathcal{C} \longrightarrow \mathbf{CRing}$ is a fibration. □

2.11 Fibrations as Pseudofunctors

Let **Cat** be the category of categories. There is an equivalence between fibrations (for varying $\mathcal{D}$) $F\colon \mathcal{D} \longrightarrow \mathcal{C}$ and pseudofunctors $P\colon \mathcal{C} \longrightarrow \mathbf{Cat}^{op}$. A pseudofunctor into **Cat** requires only up to natural isomorphism: $P(1_X) \simeq 1_{P(X)}$ and $P(f \circ g) \simeq P(f) \circ P(g)$ as natural transformations.

A full treatment is given by Borceux [3, pp. 387–394], from where the following paragraphs are adapted. Suppose we have a fibration $F\colon \mathcal{D} \longrightarrow \mathcal{C}$. Then one needs to construct a pseudofunctor $P\colon \mathcal{C} \longrightarrow \mathbf{Cat}^{op}$.

Given $f\colon D \longrightarrow E$ in $\mathcal{C}_1$, we need to define a functor between two categories, $P(f)\colon F^{-1}(E) \longrightarrow F^{-1}(D)$. Choose for each $X \in F^{-1}(E)$ a corresponding lifting morphism $f_X\colon X_f \longrightarrow X$ such that $F(f_X) = f$. Now we need to define the functor $P(f)$. Let $P(f)(X) = X_f$. Suppose we have $g\colon X \longrightarrow Y$ in the fibre $F^{-1}(E)$. Then we need to define $P(f)(g)$. Consider $g \circ f_X$. Note that $g \circ f_X$ maps to $1_E f = f$. In the image, $F(g \circ f_X) = f$ has a factorisation through the image of the lifting morphism, $F(f_Y) = f$ as $f = f 1_D$. By the path lifting property of f_Y, there is a unique morphism, $h\colon X_f \longrightarrow Y_f$, with $f_Y \circ h = g \circ f_X$. Hence, we let $P(f)(g) = h$.

$$\begin{tikzcd} & Y & & & E \\ X_f \arrow[ur, "gf_X"] \arrow[r, dashed, "\exists!\, h"'] & Y_f \arrow[u, "f_Y"'] \arrow[r, mapsto, "F"] & {} & D \arrow[ur, "f"] \arrow[r, "1_D"'] & D \arrow[u, "f"'] \end{tikzcd}$$

Conversely, given a pseudofunctor $P\colon \mathcal{C} \longrightarrow \mathbf{Cat}^{op}$, we can construct a fibration [3, p. 388]. We first construct the category, $\mathcal{E}$. Objects in $\mathcal{E}_0$ are pairs, (I, X), where $I \in \mathcal{C}_0$ and $X \in P(I)_0$. A morphism $(J, Y) \longrightarrow (I, X)$ in $\mathcal{E}_1$ is a pair (g, f), where $g\colon J \longrightarrow I$ and $f\colon Y \longrightarrow P(g)(X)$ are morphisms in $\mathcal{C}_1$ and $P(J)_1$, respectively. Then, the fibration F is just the first component functor, $F(I, X) = I$ and $F(g, f) = g$.

2.11.1 Substructures as Object Images

Proposition 2.10.2 yields a pseudofunctor $P\colon \mathbf{CRing} \longrightarrow \mathbf{Cat}^{op}$. A commutative ring R gets sent to all surjective ring homomorphisms with domain R onto integral domains, which forms a category (representing all prime ideals). Morphisms in $P(R)$ are a commutative triangles

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr, "g"'] & D_1 \arrow[d, "h"] \\ & D_2 \end{tikzcd}$$

and a homomorphism $R_1 \longrightarrow R_2$ gets sent to a functor: $P(R_2) \longrightarrow P(R_1)$.

Chapter 3

Towards Self-dual Axioms for Sets

In this section, we generalise the axiomatic framework for Z Janelidze's projective group theory [12] into a context which holds also for concrete sets, that is, the category **Set**. Basic lattice theoretic properties are verified to hold in the general context. This includes a short result on bounded morphisms. Finally, examples for the new axiomatic context are given, including showing how the context of projective group theory holds in the context of projective set theory.

3.1 Structure

3.1.1 The Underlying Form

The premise is the following. We start out with a category $\mathcal{C}$. To each object $C \in \mathcal{C}_0$, called an *abstract set*, we assign a poset of *$\mathcal{A}$-subobjects*, ΣC. The terminology $\mathcal{A}$-subobject instead of subset is to mean *abstract subobject*. In the concrete set theory case, the $\mathcal{A}$-subobjects are equivalence relations, not subsets, and applied to group theory, they are subgroups. An abstract set in the context of group theory will correspond to a group in the usual sense. This structure amounts to a faithful and amnestic functor, called a form:

$$\begin{tikzcd} \mathcal{E} \arrow[d, "F"] \\ \mathcal{C} \end{tikzcd}$$

After the first two axioms, this will later be seen to lead to a functor which is a bifibration:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "G"] \\ \mathbf{Gal} \end{tikzcd}$$

where we write the lattice $G(C)$ of $\mathcal{A}$-subobjects of $C \in \mathcal{C}_0$ as ΣC. For any morphism $f\colon A \longrightarrow B$ in $\mathcal{C}_1$, f induces a direct image map $f(-)\colon \Sigma A \longrightarrow \Sigma B$ and an inverse image map $(-)f\colon \Sigma B \longrightarrow \Sigma A$. As we shall see, Axiom 2 will force $f(-)$ and $(-)f$ to form a Galois connection, with left adjoint $f(-)$.

$$\begin{tikzcd} \Sigma A \arrow[r, bend left, "f(-)"] \arrow[r, phantom, "\perp"] & \Sigma B \arrow[l, bend left, "(-)f"] \end{tikzcd}$$

Definition 3.1.1. In order to differentiate between the underlying constructions, we may refer to $\mathcal{E} \xrightarrow{F} \mathcal{C}$ as the *infunctor construction* and $\mathcal{C} \xrightarrow{G} \mathbf{Gal}$ as the *outfunctor construction*.

3.1.2 Inmorphisms and Outmorphisms

Any $\mathcal{A}$-subobject $C \in \Sigma C$ induces a family of *inmorphisms* with codomain C and a family of *outmorphisms* with domain C. That is, to $C \in \Sigma C$ we assign index sets I and J, and a family of morphisms called inmorphisms

$$\{f_i\colon L_i \longrightarrow C \mid i \in I\}$$

and a family of morphisms called outmorphisms

$$\{g_j\colon C \longrightarrow R_j \mid j \in J\}$$

subject to the conditions of the axioms.

3.1.3 Null Morphisms

Together with the structure above, we have a distinguished class $\mathcal{N}$ of morphisms which will be required to be null in the sense that for all $f \in \mathcal{N}$, $f1 = 0$. We shall refer to these as $\mathcal{N}$-null morphisms.

3.1.4 The Concrete and Abstract Settings

When we need to differentiate between the two, we shall to refer to the category of sets **Set** as *concrete sets* in contrast to the category $\mathcal{C}$ of *abstract sets* in the axiomatic setting. The new version of Axiom 3 in our setting will be shown to hold for groups, but for the other axioms, the relevant resource to consult is [8], a collaboration of Janelidze and Goswami, which also gives a historical and future perspective and contains many interesting references.

When we say a concept is self-dual, usually it is meant that the concept is expressed in terms of $F\colon \mathcal{E} \longrightarrow \mathcal{C}$ and at the same time the suitably translated notion holds in $F^{op}\colon \mathcal{E}^{op} \longrightarrow \mathcal{C}^{op}$.

3.1.5 On the Term *Projective*

Grandis introduces the term projective in the sense that projectivity between two groups occurs exactly when their lattices of subgroups are isomorphic.

Axiom 2 can also be found in the formulation of Grandis's modular connections [9], however, since the lattices that we will be working with are not always modular, there will be conormality and normality conditions (see Section 3.3) for the rest of Grandis's formulation of modular connections to hold in our setting.

One of the main motivations behind this work is to formulate a theory in a self-dual context. The term *projective set theory* alludes to self-dual aspects of projective geometry. In projective geometry, points are dual to lines: every two points have a unique line going through them and every two lines intersect at a unique point.

If one wants to think of projective set theory as an analogue of projections of shapes in geometry a suggestion is as follows. In projective set theory, it will turn out that the axioms are satisfied by letting elements of ΣX for a set X be the set of equivalence relations on X, ordered by inclusion. Now we can isolate any subset $S \subseteq X$ by the equivalence relation which has only one non-trivial equivalence class S and trivial equivalence classes everywhere else. However, when $S \simeq \{\varnothing\}$, i.e., S is a singleton, then if we want to isolate S by an equivalence relation we get the smallest equivalence relation, but any other singleton $T \simeq \{\varnothing\}$ also generates the smallest equivalence relation and moreover, it isolates instead *every* singleton—there is no non-trivial equivalence class. We can't differentiate between the singleton classes to know which one S isolates.

Hence, in terms of subsets, we may think of projective set theory as a way to project the powerset of a set so that it conserves each non-singleton non-empty subset, but projects all singleton subsets and the empty subset onto the same *point*. If we let $\mathscr{P}(X)$ be the powerset of X and $\mathcal{R}$ be the set of equivalence relations on X with at most one non-trivial equivalence class, then we have the spacial projection $\rho\colon \mathscr{P}(X) \longrightarrow \mathcal{R}$. This is a surjection, which is injective on all but the singleton and empty subsets. This ρ will not be used in this thesis. Rather, it is presented here as an intuitive description. This perspective coincides with the manner in which Grandis uses the term, i.e., two sets are projective to each other when the lattices of equivalence relations on them are isomorphic.

3.2 Axioms and Basic Consequences

In this section, the axioms that form the basis of our abstract context are formulated and basic but important consequences of these axioms are exposited. Each of the axioms are functorially self-dual. They were developed from axioms that hold for groups [8]; and perhaps surprisingly, by working with equivalence relations instead of subsets, many of the properties of groups that follow from the original axioms can be emulated in **Set**.

Axiom 1. The $\mathcal{A}$-subobject posets are bounded lattices and direct and inverse images maps are monotone. Moreover, the following equalities hold for direct and inverse image maps, whenever the sides of the equalities suitably compose.

1. $(fg)(T) = f(g(T))$
2. $(R)(fg) = ((R)f)g$
3. $1_X(S) = S = (S)1_X$

Axiom 2. For any morphism $f\colon A \longrightarrow B$ and $\mathcal{A}$-subobjects $S \in \Sigma A, R \in \Sigma B$, we have:

1. $(fS)f = S \vee (0)f$
2. $f(Rf) = R \wedge f(1)$

Definition 3.2.1. A morphism $f\colon A \longrightarrow B$ is an *isomorphism* if there exists some $g\colon B \longrightarrow A$ such that $fg = 1_B$ and $gf = 1_A$.

Proposition 3.2.2. *From Axioms 1 and 2, we have that for any isomorphism, the corresponding direct and inverse image maps are isomorphisms of lattices, and moreover, they are inverse to each other.*

Proof. Let $f\colon A \longrightarrow B$ be an isomorphism with inverse g. By monotonicity, $g(1) \leqslant 1 \Rightarrow fg(1) \leqslant f(1)$ but $fg(1) = 1_B(1) = 1$ and hence $1 \leqslant f(1)$ so $f(1) = 1$. Then $f(Rf) = R \wedge f(1) = R \wedge 1 = R$. Dually, $0 \leqslant (0)g$ so $(0)f \leqslant ((0)g)f = 0(gf) = 0$. Hence, $(fS)f = S \vee 0(f) = S$. Consequently, the direct and inverse images of f are inverses to each other. That the two maps are morphisms of lattices follows from the fact that the maps are monotone bijections. Suppose $Q \leq fS_1$ and $Q \leq fS_2$. Then $Qf \leq (fS_1)f = S_1$ and $Qf \leq S_2$ and hence $Qf \leq S_1 \wedge S_2$. By monotonicity again, $f(Qf) = Q \leq f(S_1 \wedge S_2)$ and hence $f(S_1 \wedge S_2) = f(S_1) \wedge f(S_2)$ by the definition of the greatest lower bound. The dual, for least upper bounds, follows. □

Proposition 3.2.3. *A consequence of Axiom 2 is that the direct and inverse image maps form a Galois connection where the direct image is the left adjoint.*

Proof. $S \leqslant S \vee (0)f = (fS)f$ and similarly $f(Rf) = R \wedge f(1) \leqslant R$. □

Definition 3.2.4. Two inmorphisms of an $\mathcal{A}$-subobject R $f\colon A \longrightarrow B$ and $f'\colon A' \longrightarrow B$ are isomorphic if there exists an isomorphism $h\colon A' \longrightarrow A$ with $fh = f'$. Dually, two outmorphisms $g\colon B \longrightarrow C, g'\colon B \longrightarrow C'$ are isomorphic if there exists an isomorphism $h'\colon C' \longrightarrow C$ with $h'g' = g$. We write this as $f \simeq f'$ and $g \simeq g'$.

Definition 3.2.5. A *null morphism* f is a morphism such that $f1 = 0$, or equivalently, $0f = 1$.

The next axiom is introduced to handle morphisms with the null property $(f1 = 0)$ and to capture properties of how null functions behave in **Set**. For the homomorphism theorems that will follow, the theorems will not hold in specific instances where $\mathcal{A}$-subobjects that are either images or kernels of null morphisms are concerned. However, for the specialisation to **Grp**, the theorems hold for the null instances.

Axiom $\mathcal{N}$. ($\mathcal{N}$-null morphisms) For each instance of this theory, we have a class of morphisms, $\mathcal{N}$. Morphisms in the class $\mathcal{N}$ (called $\mathcal{N}$-null morphisms) have the properties described below. Now, for the full axiom, assume f is a morphism in our abstract category $\mathcal{C}$ in each of the points that follow.

1. If $f \in \mathcal{N}$, then f is null.

2. If $f \in \mathcal{N}$, then $fg \in \mathcal{N}$ and $hf \in \mathcal{N}$, for any suitable g and h that compose.

3. Suppose $f \notin \mathcal{N}$. Then for any inmorphism m, $mf \notin \mathcal{N}$ and for any outmorphism r, $fr \notin \mathcal{N}$.

4. Suppose that $g1 \leqslant f1$. If $g \notin \mathcal{N}$, then $f \notin \mathcal{N}$. Dually, suppose $0f \leqslant 0g$. If $g \notin \mathcal{N}$, then $f \notin \mathcal{N}$.

5. Suppose an inmorphism of S is $\mathcal{N}$-null. Then all null inmorphisms of S are $\mathcal{N}$-null. Dually, suppose an outmorphism of S is $\mathcal{N}$-null. Then all null outmorphisms of S are $\mathcal{N}$-null.

6. Suppose that $S = f1$ for $f \notin \mathcal{N}$. Then, for any $f' \in \mathcal{N}$, $S \neq f'1$. Dually, suppose that $S = 0f$ for $f \notin \mathcal{N}$. Then, for any $f' \in \mathcal{N}$, $S \neq 0f'$.

7. If an inmorphism $m \notin \mathcal{N}$, then $0m \neq 0f$ for any $f \in \mathcal{N}$. Dually, if an outmorphism $r \notin \mathcal{N}$, then $r1 \neq f1$ for any $f \in \mathcal{N}$.

8. For some $f \in \mathcal{N}$, $f1 = 1 \in \Sigma X$ if and only if for some $g \in \mathcal{N}$, $0g = 0 \in \Sigma X$.

Remark 3.2.6. In order to understand the intuitive significance of this axiom, the reader is referred to Section 3.4.1, where the $\mathcal{A}$-subobjects and also the inmorphisms and outmorphisms are defined for the specialisation to **Set**.

Remark 3.2.7. Note that all of Axiom $\mathcal{N}$ holds when $\mathcal{N}$ is chosen to be all null morphisms. This corresponds to the specialisation to the **Set** context. Further note that all of Axiom $\mathcal{N}$ holds when $\mathcal{N}$ is chosen to be empty, which corresponds to the **Grp** context.

Notation 3.2.8. **We shall call the class of images of $\mathcal{N}$-null morphisms $\mathcal{N}_0$ and the class of kernels of $\mathcal{N}$-null morphisms $\mathcal{N}_1$.** In other words $R \in \mathcal{N}_0$ exactly when $f1 = 0 = R$ for some f and $S \in \mathcal{N}_1$ exactly when $0f = 1 = S$ for some f.

Proposition 3.2.9. *Suppose we have $f \notin \mathcal{N}$ for $f\colon A \longrightarrow B$. Then $1 \in \Sigma A$ and $1 \in \Sigma B$ are not images of $\mathcal{N}$-null morphisms and $0 \in \Sigma A$ and $0 \in \Sigma B$ are not kernels of $\mathcal{N}$-null morphisms.*

Proof. We can prove this from Points 4, 6 and 8. Suppose that $f \notin \mathcal{N}$. Then, $f1 \leqslant 1 = i1$ for an isomorphism i, which implies that $i \notin \mathcal{N}$ and hence for any $f' \in \mathcal{N}$, $f'1 \neq i(1) = 1$. Thus, for any $f'' \in \mathcal{N}$, $0f'' \neq 0 \in \Sigma A$. By the dual argument using f, we then also have the result for $0, 1 \in \Sigma B$. □

Definition 3.2.10. We define a morphism f to be *$\mathcal{Z}$-empty* if

$$0f = 0 \quad \textit{and } f \textit{ is not an inmorphism}$$

and the class of all such morphisms is $\mathcal{Z}$.

Axiom 3 is an adaptation of the axiom for groups in [8] to hold for **Set**. It is split into two dual parts.

Axiom 3a. Let $\{f_i | i \in I\}$ be the family of inmorphisms of S. For any morphism f we have:

1. $I \neq \varnothing$.
2. Each f_i is a monomorphism.
3. $f_i 1 \leqslant S$ for all $i \in I$.
4. $(f1 \leqslant f_i 1 \text{ and } f \notin \mathcal{N}) \Rightarrow \exists u(f = f_i u)$.
5. Fix f_j. Then, $\exists f_i(f_i = f_j u) \Leftrightarrow u$ is iso.
6. For $f_i \notin \mathcal{N}$ and $f_j \notin \mathcal{N}$,
   $\exists u \exists v(f_i u = f_j v \wedge (f_i u \text{ not } \mathcal{Z}\text{-empty})) \Rightarrow f_i \simeq f_j$
7. $f1 \leqslant S \Rightarrow \exists f_i \exists u(f = f_i u)$.

Axiom 3b. Let $\{g_j | j \in J\}$ be the family of outmorphisms of S. For any morphism g we have:

1. $J \neq \varnothing$.
2. Each g_j is an epimorphism.
3. $S \leqslant 0g_j$ for all $j \in J$.
4. $(0g_j \leqslant 0g$ and $g \notin \mathcal{N}) \Rightarrow \exists u(g = ug_j)$.
5. Fix g_i. Then, $\exists g_j(g_j = ug_i) \Leftrightarrow u$ is iso.
6. For $g_j \notin \mathcal{N}$ and $g_i \notin \mathcal{N}$,
   $\exists u \exists v(ug_j = vg_i \wedge (ug_j$ not $\mathcal{Z}$-empty$)) \Rightarrow g_i \simeq g_j$.
7. $S \leqslant 0g \Rightarrow \exists g_j \exists u(g = ug_j)$.

Axiom 3. We shall refer to Axioms 3a and 3b together as Axiom 3.

Definition 3.2.11. An $\mathcal{A}$-subobject C is *conormal* if $C = f1$ for some morphism f. An $\mathcal{A}$-subobject N is *normal* if $N = 0g$ for some morphism g.

Proposition 3.2.12. $1 \in \Sigma X$ *is normal and conormal. Dually,* 0 *is normal and conormal.*

Proof. $1 = 1_X(1)$ and $1 \leqslant 0g \Rightarrow 1 = 0g$ where g is an outmorphism of 1. The dual follows. □

Proposition 3.2.13. *The inverse image of a normal $\mathcal{A}$-subobject along some f is normal and the image of a conormal $\mathcal{A}$-subobject along f is conormal.*

Proof. Suppose N is a normal $\mathcal{A}$-subobject of $S \in \Sigma X$ and $f\colon A \longrightarrow X$ is a morphism. Then, $N = 0g$ for some g and hence $Nf = (0g)f = 0(gf)$. Hence, Nf is normal. Dually, images of conormal $\mathcal{A}$-subobjects are conormal. □

Proposition 3.2.14. *If C is conormal then all non-$\mathcal{N}$ inmorphisms of C are isomorphic. Dually, if N is normal then all non-$\mathcal{N}$ outmorphisms of N are isomorphic.*

Proof. Suppose C is conormal and has a non-$\mathcal{N}$ inmorphism, let's say f_k. Then, $f_k1 \leqslant C$. Now, $C = h1$ where $h\colon A \longrightarrow B$ is thus non-$\mathcal{N}$ by Axiom $\mathcal{N}$.4. Then h factorises as $f_iq = h$, where $f_i\colon A' \longrightarrow B$ is an inmorphism of C. $f_i1 \leqslant C$ and $C = f_iq1 \leqslant f_i1$ implies that $f_i1 = C$, which also implies that f_i is not $\mathcal{N}$-null by Axiom $\mathcal{N}$.4. Suppose f_j is another non-$\mathcal{N}$ inmorphism of C. Then $f_j1 \leqslant f_i1 \Rightarrow f_j = f_iu$ by Axiom 3a.4 and u is an isomorphism by Axiom 3a.5. Hence all non-$\mathcal{N}$ inmorphisms are isomorphic. The dual follows. □

Proposition 3.2.15. *Suppose C is conormal. If $C = 0$, then it is the image of all of its null inmorphisms. Else, it is the image of its non-null inmorphisms (all of which are isomorphic). Dually, if N is normal, either $N = 1$ and it is the preimage of all of its null outmorphisms, or it is the preimage of its non-null outmorphisms, all of which are isomorphic.*

Proof. Suppose C is conormal. If $C = 0$ then for every inmorphism f_i we have $f_i 1 \leqslant 0 = C$ and hence $f_i 1 = 0 = C$. Else suppose $C \neq 0$. C is conormal, so $C = h1$ for some non-null morphism h. So we have a unique factorisation, $h = f_i q$, where f_i is an inmorphism of C. $f_i \leqslant C$ and $C = h1 = f_i q 1 \leqslant f_i 1$. Hence, $f_i 1 = C$ and every non-null inmorphism f_j of C has $f_i \simeq f_j$ by Proposition 3.2.14. The dual argument follows. □

Proposition 3.2.16. *An inmorphism f_i of S is an inmorphism of $f_i 1$. Dually, an outmorphism g_j of S is an outmorphism of $0g_j$.*

Proof. Let $f_i \colon A \longrightarrow B$ be an inmorphism of S. Since $f_i 1 \leqslant f_i 1$, f_i factors through one of the inmorphisms $f_j \colon A' \longrightarrow B$ of $f_i 1$ as $f_j q = f_i$. But, since f_j is an inmorphism of $f_i 1$, $f_j 1 \leqslant f_i 1 \leqslant S$ which by Axiom 3a.7 gives that $f_j = f_k p$, where f_k is an inmorphism of S. But $f_k pq = f_j q = f_i$, which by 3a.5 means that pq is an iso. Then, $f_k p = f_j = f_i (pq)^{-1} p = f_j q (pq)^{-1} p$ and since f_j is a mono, $q(pq)^{-1} p = 1_{cod(q)}$. Moreover, $(pq)^{-1} pq = 1_{dom(q)}$, which proves that q is an iso. Hence, f_i is also an inmorphism of $f_i 1$.

The dual argument gives that an outmorphism g if S is an outmorphism of $0g$. □

Proposition 3.2.17. *The preimage of an $\mathcal{A}$-subobject S along any inmorphism of S is the largest $\mathcal{A}$-subobject. Dually, the image along an outmorphism is the smallest $\mathcal{A}$-subobject.*

Proof. Let f be an inmorphism of S. Then $f1 \leqslant S \Rightarrow 1 \leqslant (f1)f \leqslant Sf \Rightarrow 1 = Sf$. Let g be an outmorphism of S. Then by the dual argument, $gS = 0$. □

Proposition 3.2.18. *Isomorphisms are inmorphisms of 1 and outmorphisms of 0.*

Proof. Let i be an isomorphism into X. Since $i(1) = 1 \leqslant 1$, we have that for some inmorphism f_j of 1, $i = f_j u$. Also, $i(i^{-1} f_j) = f_j$ and being monomorphisms factoring through each other, $f_j \simeq i$, and hence i is an inmorphism of 1. Dually, isomorphisms are outmorphisms of 0. □

Proposition 3.2.19. *Inmorphisms of 1 are isomorphisms and dually, outmorphisms of 0 are isomorphisms.*

Proof. Suppose $f_i \colon A \longrightarrow X$ is an inmorphism of $1 \in \Sigma X$. Let i be an isomorphism into X. Then i is an inmorphism of 1. But $f_i = ii^{-1}f_i$ and hence by Axiom 3a.5, we have that $f_i \simeq i$.

By duality, outmorphisms of 0 are isomorphisms. □

Corollary 3.2.20. *Consequently, if $1 \in \Sigma X$ has a null inmorphism f, then f is an isomorphism and hence $f1 = 0 = 1$ and ΣX is the trivial one element poset. So, all isomorphisms into X are null. Dually, if $0 \in \Sigma X$ has a null outmorphism g then $(0)g = 1 = 0$ and ΣX is again the trivial one element poset.*

Proposition 3.2.21. *Let X be an abstract set. The inmorphisms of the biggest $\mathcal{A}$-subobject of X are precisely the isomorphisms with codomain X and dually, outmorphisms of the smallest $\mathcal{A}$-subobject of X are precisely the isomorphisms with domain X.*

Proof. This follows from Proposition 3.2.19 and Proposition 3.2.18. □

Axiom 4. Any non-$\mathcal{N}$ morphism $f \colon A \longrightarrow B$ factorises as $f = me$ where m is an inmorphism of $f1$ and e is an outmorphism of $0f$.

Remark 3.2.22. Axiom 4 in the context of group theory translates to the first isomorphism theorem. For a homomorphism $f \colon G \longrightarrow H$, we have $f = me$ and since e is a surjection one can verify that $cod(e) \simeq G/0f$.

Proposition 3.2.23. *For a decomposition, $f = me$, $0e = 0f$ and $m1 = f1$.*

Proof. Since m is an inmorphism of $f1$, we have $m1 \leqslant f1$. Also, $f1 = me1 \leqslant m1$ and hence, $f1 = m1$. The dual follows. □

Proposition 3.2.24. *If there are two factorisations of a non-null morphism f as $f = me$ and $f = m'e'$, then $m \simeq m'$ and $e \simeq e'$.*

Proof. Suppose f is non-$\mathcal{N}$ and factors as $f = me$ and $f = m'e'$, where m and m' are inmorphisms of $f1$ and e and e' are outmorphisms of $0f$. Now, f factors through an inmorphism of $f1$, $f = me$. Since m' is also an inmorphism of $f1$ and $f = m'e'$ (and f is an inmorphism and hence not $\mathcal{Z}$-empty), we have that $m' \simeq m$. Dually, we have $e' \simeq e$. □

Proposition 3.2.25. *If f is a non-$\mathcal{N}$ inmorphism of some S and f decomposes as $f = me$, then $f \simeq m$. Dually, if f is a non-null outmorphism then $f \simeq e$.*

Proof. Suppose an non-$\mathcal{N}$ inmorphism $f\colon A \longrightarrow X$ of S decomposes as $f = me$. By 3.2.16, f is an inmorphism of $f1$. Since f factors through an inmorphism m of $f1$ as $f = me$ and f factors through itself as $f1_A$, we have by Axiom 3a.5 that $m \simeq f$. The dual argument also holds. □

Proposition 3.2.26. *A morphism which is both an inmorphisms of an $\mathcal{A}$-subobject S and an outmorphism of an $\mathcal{A}$-subobject T is exactly an isomorphism.*

Proof. By Proposition 3.2.18, we know isomorphisms are both inmorphisms of 1 and outmorphisms of 0. Conversely, suppose a morphism $f\colon A \longrightarrow X$ is both an inmorphism of S and an outmorphism of T. Then f is also an inmorphism of $f1$ and an outmorphism of $0f$.

Suppose f is null. Since f forms a Galois connection, we have $f0 = 0$. Then $f1 = 0 = f0 \Rightarrow 1 = 0$ in ΣA since f is an inmorphism and the direct image is an injection. So, ΣA is trivial and f is an outmorphism of $0f = 0$, and by Proposition 3.2.21, an isomorphism.

Else suppose f is not null. Decompose $f = me$. By Proposition 3.2.25, we have $f \simeq m$ and $f \simeq e$. So $mi = f = me$, for an isomorphism i, and hence $e = i$. Since $f \simeq e = i$, f is thus an isomorphism. □

Proposition 3.2.27. *A non-$\mathcal{N}$ morphism $f\colon A \longrightarrow X$ is a inmorphism if and only if $0f = 0$. Dually, f is an outmorphism if and only if $f1 = 1$. If f is an $\mathcal{N}$-null inmorphism, then $0f = 0$; dually, if f is an $\mathcal{N}$-null outmorphism, then $f1 = 1$.*

Proof. Suppose $f\colon A \longrightarrow X$ is an inmorphism of some S and hence also of $f1$.

Suppose that f is $\mathcal{N}$-null and hence is null by Axiom $\mathcal{N}$.1. Then, $1f = 1 \in \Sigma A = 0f$. But, $f1 = 0 = f0$, and since f is an inmorphism, $1 = 0 \in \Sigma A$. Hence, $0f = 1 = 0$.

Else, suppose f is not $\mathcal{N}$-null, and decompose f uniquely as $f = me$. Then, by Proposition 3.2.25, $f \simeq m$. Then $mi = me \Rightarrow e = i$. So $0f \leqslant 0e = 0$ since e is an outmorphism of $0f$. Hence, $0f = 0$. Conversely, suppose $0f = 0$ with the decomposition $f = me$. Then, since e is an outmorphism of 0, we have by Proposition 3.2.21 that e is an isomorphism. Hence, $f \simeq m$ and f is an inmorphism (of $f1$). The dual argument follows. □

Corollary 3.2.28. *If the image map and inverse image map of a non-$\mathcal{N}$ morphism f are isomorphisms of lattices, then f is an isomorphism. Hence, by Proposition 3.2.2, non-$\mathcal{N}$ isomorphisms are then exactly the non-$\mathcal{N}$ morphisms whose image map and inverse image map are isomorphisms of lattices.*

Proof. Suppose the image and preimage of a non-$\mathcal{N}$ f are isomorphisms of lattices. Then, $f1 = f(1f) = 1$. Similarly, $0f = (f0)f = 0$. Hence, since f is then both an inmorphism and outmorphism, then by Proposition 3.2.26 f is an isomorphism. □

Proposition 3.2.29. *A non-$\mathcal{N}$ morphism f is an inmorphism if and only if, for all S, $(fS)f = S$. Dually, a non-$\mathcal{N}$ f is an outmorphism if and only if, for all T, $f(Tf) = T$. Moreover, if f is a $\mathcal{N}$-null inmorphism, then $(fS)f = S$ and dually if f is a $\mathcal{N}$-null outmorphism, then $f(Tf) = T$.*

Proof. Suppose f is an inmorphism of some G. Then, by Proposition 3.2.27, $0f = 0$. Then, for all S, $(fS)f = S \vee 0f = S \vee 0 = S$. Conversely, suppose f is not $\mathcal{N}$-null and $(fS)f = S$ for all S. In particular, $(f0)f = 0$. So, $0 \vee 0f = 0$, which means that $0f \leqslant 0$ and thus $0f = 0$. Hence, f is an inmorphism. The dual argument follows. □

Corollary 3.2.30. *A non-$\mathcal{N}$ morphism f is an inmorphism if and only if the direct image map is injective and the inverse image map is surjective. Dually, a non-$\mathcal{N}$ f is an outmorphism if and only if the direct image map is surjective and the inverse image map is injective. Furthermore, an $\mathcal{N}$-null inmorphism f has an injective direct image map and a surjective inverse image map. Dually, an $\mathcal{N}$-null outmorphism f has a surjective direct image map and an injective inverse image map.*

Proof. It follows directly from Proposition 3.2.29 and since functions that are left invertible are exactly the injections and functions that are right invertible are exactly the surjections. □

Corollary 3.2.31. *The respective classes of non-$\mathcal{N}$ inmorphisms and of non-$\mathcal{N}$ outmorphisms are both closed under composition.*

Proof. For $f\colon A \longrightarrow X$ and $f'\colon X \longrightarrow B$ with $0f = 0$ and $0f' = 0$, we have $0f'f = 0f = 0$. If the composite is $\mathcal{N}$-null, then $0f = 0f'f = 1 \in \mathcal{N}_1$, i.e., f is $\mathcal{N}$-null. But by assumption it is not, hence the composite is not $\mathcal{N}$-null. Composition of outmorphisms follows dually. □

Proposition 3.2.32. *If f is not $\mathcal{N}$-null and $f = uw$ with u an inmorphism of some S and w an outmorphism of some T, then u is an inmorphism of $f1$ and w is an outmorphism of $0f$.*

Proof. Suppose for an non-$\mathcal{N}$ f we have $f = uw$ with u an inmorphism of some S and w an outmorphism of some T. Since w is an outmorphism we have $f1 = uw1 = u1$. Hence, since u is an inmorphism of $u1$, it is also of $f1 = u1$. Dually, since u is an inmorphism, we have $0f = 0uw = 0w$ and w is an outmorphism of $0w = 0f$. □

Proposition 3.2.33. *If a non-$\mathcal{N}$ composite uw is an inmorphism, then so is w. Dually, if uw is an outmorphism, then so is u.*

Proof. Suppose uw is a non-$\mathcal{N}$ inmorphism of some S. Then $0w \leqslant 0uw = 0 \Rightarrow 0w = 0$, w is not $\mathcal{N}$-null since then the composite would be $\mathcal{N}$-null and hence, w is an inmorphism (of $w1$). The dual follows. □

Axiom 5. The join of two normal $\mathcal{A}$-subobjects is normal and the meet of two conormal $\mathcal{A}$-subobjects is conormal.

Proposition 3.2.34. *Equivalently to Axiom 5, the class of normal $\mathcal{A}$-subobjects is closed under direct images along outmorphisms, and dually, the class of conormal $\mathcal{A}$-subobjects is closed under inverse images along inmorphisms.*

Proof. Suppose $N \in \Sigma X$ is normal. Suppose $g \colon X \longrightarrow B$ is an outmorphism of some S (and hence also of $0g$). If $N \vee 0g$ is normal, then by Proposition 3.2.15 for one of its outmorphisms, $g' \colon X \longrightarrow B'$, we have $N \vee 0g = 0g'$. Now, $0g \leqslant 0g'$ and hence we have a unique factorisation, $q\bar{g} = g'$, where $\bar{g}$ is an outmorphism of $0g$. If $\bar{g}$ is null, then g' is null. Then $(gN)g = N \vee 0g = 1$. By the definition of a Galois connection, we then have $g1 \leqslant gN$. If g is $\mathcal{N}$-null, then $gN = 0$ is normal (by Proposition 3.2.12). Otherwise, since g is an outmorphism, by Proposition 3.2.27 $g1 = 1$ and hence, $gN = 1$. Since 1 is normal by Proposition 3.2.12, gN is normal.

Else, suppose that $\bar{g}$ is not $\mathcal{N}$-null. Then, g is not null, since $0g \leqslant 0\bar{g}$. And thus, $\bar{g} \simeq g$ (by minimality of outmorphisms in Axiom 3b.4 and 3b.5), say $g = i\bar{g}$.

$$\begin{aligned} 0qi^{-1} &= g((0qi^{-1})g) = g(0qi^{-1}i\bar{g}) = g(0q\bar{g}) = g(0g') \\ &= g(N \vee 0g) = g(N) \vee g(0g) = g(N) \vee 0 = g(N) \end{aligned}$$

And consequently, $g(N)$ is normal.

Conversely, suppose direct images of normal $\mathcal{A}$-subobjects along outmorphisms are normal. Now suppose $N, M \in \Sigma X$ are normal, and suppose $0g = M$ where g is an outmorphism of M. By assumption, gN is normal. From Proposition 3.2.13, inverse images of normal $\mathcal{A}$-subobject are normal, and hence $(gN)g = N \vee 0g = N \vee M$ is normal. The dual characterisation of conormal $\mathcal{A}$-subobjects follows. □

3.3 Lattice Theoretic Properties

3.3.1 Conditional Modularity

Theorem 3.3.1. *For any three $\mathcal{A}$-subobjects, X, Y, Z of an abstract set A, if either Y is normal and Z is conormal, or if Y is conormal and X is normal, then we have:*

$$X \leqslant Z \Rightarrow X \vee (Y \wedge Z) = (X \vee Y) \wedge Z.$$

Proof. The second case where Y is conormal and X is normal is dual to the first case. So we show only the result when Y is normal and Z is conormal. Suppose then that $Y = 0g$ and $Z = f1$ for morphisms g and f. We must prove that

$$X \leqslant f1 \Rightarrow X \vee (0g \wedge f1) = (X \vee 0g) \wedge f1.$$

Suppose that $X \leqslant f1$. Then

$$\begin{aligned}
X \vee (0g \wedge f1) &= (X \wedge f1) \vee (0g \wedge f1) && \text{since } X \leqslant f1\\
&= f(Xf) \vee f(0gf) && \text{by Axiom 2 on } f \text{ twice}\\
&= f(Xf \vee 0gf) && \text{left adjoints preserve colimits}\\
&= f((gf(Xf))gf) && \text{using Axiom 2 inside the brackets on } gf \text{ and } Xf\\
&= f((g(X \wedge f1))gf) && \text{Axiom 2 again on } f \text{ and } X\\
&= f((g(X))gf) && \text{since } X \leqslant f1\\
&= f((X \vee 0g)f) && \text{by Axiom 2 on } g \text{ and } X\\
&= (X \vee 0g) \wedge f1 && \text{by Axiom 2 on } f \text{ and } X \vee 0g
\end{aligned}$$

which is what we wanted to prove. □

3.3.2 A Result on Bounded Morphisms

I would like to thank to Professor Rewitzky for the interesting talks that indirectly led to the result in Theorem 3.3.4, a collaboration with Professor Z Janelidze.

Theorem 3.3.2. *For any morphism $f\colon A \longrightarrow B$, $Y \in \Sigma B$ and conormal $\mathcal{A}$-subobject X of A, we have:*

$$f(X \wedge Yf) = f(X) \wedge Y.$$

Proof. Suppose X is conormal, then

$$\begin{aligned}
f(X \wedge Yf) &= f(X \wedge Yf) \wedge f1 && \text{since } f(X \wedge Yf) \leqslant f1\\
&= f((f(X \wedge Yf))f) && \text{Axiom 2 on } f \text{ and } f(X \wedge Yf)\\
&= f(0f \vee (X \wedge Yf)) && \text{Axiom 2 of } f \text{ inside the brackets on } X \wedge Yf\\
&= f((0f \vee X) \wedge Yf) && \text{by Theorem 3.3.1; } 0f \text{ normal and } X \text{ conormal}\\
&= f((fX)f \wedge Yf) && \text{Axiom 2 on } f \text{ and } X\\
&= f((fX \wedge Y)f) && \text{right adjoints preserves limits}\\
&= f1 \wedge fX \wedge Y && \text{Axiom 2 on } f \text{ and } fX \wedge Y\\
&= fX \wedge Y && \text{since } fX \leqslant f1
\end{aligned}$$

which proves the theorem. □

The following proposition is a weakening of the previous theorem which shows that in fact for any Galois connection the one sided relation holds.

Proposition 3.3.3. *Suppose that $f\colon A \longrightarrow B$ forms a Galois connection and that A and B have meets. Then for all $X \in A$ and $Y \in B$ we have*

$$f(Yf \wedge X) \leqslant Y \wedge fX.$$

Proof. Note that

$$Yf \wedge X \leqslant Yf \Rightarrow f(Yf \wedge X) \leqslant f(Yf)$$

and since we have a Galois connection,

$$f(Yf) \leqslant Y$$

furthermore,

$$Yf \wedge X \leqslant X \Rightarrow f(Yf \wedge X) \leqslant fX.$$

Then, by the definition of the greatest lower bound, we have that

$$f(Yf \wedge X) \leqslant Y \wedge fX.$$

□

Theorem 3.3.4. *Assume $f\colon A \longrightarrow B$ forms a Galois connection. Also assume that A and B have meets. Then the following are equivalent for all $X \in A$ and $Y \in B$:*

1. $\downarrow \{fX\} = f(\downarrow \{X\})$
2. $Y \wedge fX = f(Yf \wedge X)$

Proof. Assume that $\downarrow \{fX\} = f(\downarrow \{X\})$ holds.

$$\begin{aligned} &Y \wedge fX \leqslant fX \\ \Rightarrow\ &Y \wedge fX \in \downarrow \{fX\} \\ \Rightarrow\ &Y \wedge fX \in f(\downarrow \{X\}) \end{aligned}$$

So we have that there exists some C such that $Y \wedge fX = fC$ and $C \leqslant X$. Now,

$$\begin{aligned} C \leqslant (fC)f &= (fX \wedge Y)f \\ &= (fX)f \wedge Yf \quad \text{since right adjoints preserve limits} \end{aligned}$$

So, $C \leqslant Yf$ and thus,

$$\begin{aligned} C &\leqslant Yf \wedge X \\ fC &\leqslant f(Yf \wedge X) \\ Y \wedge fX &\leqslant f(Yf \wedge X) \end{aligned}$$

and hence, by Proposition 3.3.3, we have that

$$Y \wedge fX = f(Yf \wedge X).$$

Now we need to show that 2 $\Rightarrow$ 1. Firstly, $f(\downarrow \{X\}) \subseteq \downarrow \{fX\}$ holds by monotonicity. For the reverse, let $Y \in \downarrow \{fX\}$. Then, $Y \leqslant fX$.

$$f(Yf \wedge X) = Y \wedge fX = Y$$

And also, $Yf \wedge X \leqslant X$. Thus, $Y \in f(\downarrow \{X\})$ and hence

$$f(\downarrow \{X\}) = \downarrow \{fX\}$$

which proves the backwards implication. Hence, the theorem holds. □

Remark 3.3.5. A map with Property 1 is sometimes referred to in the literature as a *bounded morphism*. Property 2 is due to Frobenius.

Definition 3.3.6. A functor $F\colon \mathcal{X} \longrightarrow \mathcal{A}$ is a *discrete fibration* if for every object X in $\mathcal{X}$, and every morphism of the form $g\colon A \longrightarrow F(X)$ in $\mathcal{A}$ there is a unique morphism $h\colon Y \longrightarrow X$ in $\mathcal{X}$ such that $F(h) = g$.

Notation 3.3.7. Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be the functor corresponding to $f\colon A \longrightarrow B$ where A and B are viewed as discrete categories. Then the concrete direct image map $f(-)\colon \mathscr{P}(A) \longrightarrow \mathscr{P}(B)$ has a corresponding functor between poset categories, written as $F(-)\colon \mathcal{A}_{\mathscr{P}} \longrightarrow \mathcal{B}_{\mathscr{P}}$. Similarly, we have $(-)F\colon \mathcal{B}_{\mathscr{P}} \longrightarrow \mathcal{A}_{\mathscr{P}}$. Hence, in this notation, $\mathcal{A}_{\mathscr{P}}$ is the poset category corresponding to the poset $\mathscr{P}(A)$.

Theorem 3.3.8. *Let $f\colon A \longrightarrow B$ be a function. Then we have a functor $F(-)\colon \mathcal{A}_{\mathscr{P}} \longrightarrow \mathcal{B}_{\mathscr{P}}$. Let X be any $X \subseteq A$ which is an element in the Boolean algebra. The following are equivalent:*

1. $\forall X(\downarrow \{fX\} = f(\downarrow \{X\}))$
2. *$F(-)$ is a discrete fibration*

Proof. Let us label an element $S \in X$ and the corresponding object in the discrete category the same way, that is, as $S \in \mathcal{X}_0$.

Let $f\colon A \longrightarrow B$ be a function which corresponds to a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ between discrete categories. Suppose that for all $X \subseteq A, \downarrow \{fX\} = f(\downarrow \{X\})$. Then, for $Q \leqslant fX$, it implies that there exist Y such that $fY = Q$ and $Y \leqslant X$. Now suppose we have $g'\colon Q \longrightarrow FX$ in $\mathcal{B}_{\mathscr{P}}$. Then it translates to the following: we have that there is a $g\colon Y \longrightarrow X$ such that $F(g) = g'$, which together with uniqueness of g (since it is a poset category) is exactly the definition for a discrete fibration.

Conversely suppose that $F(-)\colon \mathcal{A}_{\mathscr{P}} \longrightarrow \mathcal{B}_{\mathscr{P}}$ is a discrete fibration corresponding to the direct image function $f(-)$. Suppose that we have $Q \leqslant fX$. Then we have $Q \longrightarrow fX$, which means that there is a unique $g\colon Y \longrightarrow X$ such that $F(g) = g'$, i.e., $fY = Q$ and $Y \leqslant X$. This is exactly the boundedness condition. □

3.3.3 Isomorphism Restriction

Theorem 3.3.9. *For any morphism $f\colon A \longrightarrow B$, the direct and inverse image maps along f restrict to an isomorphism of lattices between the lattice of $\mathcal{A}$-subobjects of A greater or equal to $0f$ and the lattice of $\mathcal{A}$-subobjects of B smaller or equal to $f1$.*

Proof. Suppose that $0f \leqslant S$. Note that $fS \leqslant f1$. Hence, $(fS)f = S \vee 0f = S$. The dual follows. That they are lattice homomorphisms follows from the fact that they are monotone bijections. □

3.4 Examples

3.4.1 The Category of Sets

Our main motivating example is the category of sets. For the context of sets, we choose $\mathcal{N}$ to consist of all null morphisms (that is, all f such that $f1 = 0$).

We shall now more explicitly define our setting of *concrete sets*. Firstly, we work in the category of sets. For a concrete set X, X is an object in **Set**. Then, we define ΣX as the lattice of equivalence relations on X ordered by set inclusion (or equivalently, the lattice of epimorphisms with the order reversed). The direct image map is formed by taking a pushout and the inverse image map by an epi-mono factorisation.

Specifically, given a morphism $f\colon A \longrightarrow B$, we define the direct image map $f(-)$ by defining it on each equivalence relation on A. Let g be a surjection of which R is the kernel relation on A. Then we define the relation $f(R)$ as follows by forming the pushout:

$$\begin{tikzcd} A' \arrow[r] & B' \\ A \arrow[u, two heads, "g"] \arrow[r, "f"'] \arrow[ur, phantom, "\lrcorner", very near end] & B \arrow[u, two heads, "q"'] \end{tikzcd}$$

where $f(R)$ is defined as the kernel relation of q.

We then define the inverse image map $(-)f$, by defining it as an equivalence relation on A determined by an equivalence relation S on B. Let h be a surjection of which S is the kernel relation. Then form the surjection-injection (epi-mono) factorisation $em = hf$ in the category of sets:

$$\begin{tikzcd} A'' \arrow[r, hook, "m"] & B'' \\ A \arrow[u, two heads, "e"] \arrow[r, "f"'] & B \arrow[u, two heads, "h"'] \end{tikzcd}$$

We define $(S)f$ to be the kernel relation of e.

When we need to refer to the direct image map and inverse image map between powersets in the usual sense, we shall resort to the terms, *concrete image map* and *concrete preimage map*. The motivation for this is that our concepts of image maps and preimage maps in the abstract setting are generalisations of some properties that we want to preserve of the usual Galois connection between powersets, albeit with some additional properties, like the properties in Axiom 2. In contrast to the abstract image and preimage maps, the concrete image and preimage maps only satisfy the second half of Axiom 2.

Proposition 3.4.1. *Given a function $f\colon A \longrightarrow B$, let g be a surjection of which R is the kernel relation on A. The direct image map applied to R can then equivalently be defined as:*

$$(b_1, b_2) \in f(R) \quad \iff \quad b_2 \in fg^{-1}gf^{-1}\dots fg^{-1}gf^{-1}\{b_1\} \quad \textit{or} \quad b_1 = b_2$$

repeating the sequence $fg^{-1}gf^{-1}$ i times for $i \in I$, where I is some index set.

Proof. Let $f\colon A \longrightarrow B$ be a function, $g\colon A \longrightarrow C$ a surjection and let $R = 0g$. The definition of the direct image is $f(R) = 0\iota_2$ where $(C +_A B, \iota_1, \iota_2)$ is a pushout.

In Proposition 2.6.8 of Chapter 2, we computed the pushout concretely. Now, $(b_1, b_2) \in f(R)$ if and only if $\iota_2(b_1) = \iota_2(b_2)$, which exactly the case (for the equivalence relation Q defined in Proposition 2.6.8) when $[f^{-1}\{b_1\}]_Q = [f^{-1}\{b_2\}]_Q$ or when $b_1 = b_2$ (to include the isolated points). Hence,

$$f^{-1}\{b_2\} \subseteq f^{-1}fg^{-1}g\dots f^{-1}fg^{-1}gf^{-1}\{b_1\}$$

which is equivalent to

$$ff^{-1}\{b_2\} \subseteq fg^{-1}g\dots f^{-1}fg^{-1}gf^{-1}\{b_1\}.$$

If b_2 is not an isolated point, then $\{b_2\} = ff^{-1}\{b_2\}$. That concludes the proof. □

Remark 3.4.2. With this notation we use f^{-1} to mean the inverse image of subsets (i.e., the concrete inverse image map) applied firstly to the singleton subset $\{b_1\}$ of B. We use f to mean the direct image of subsets (i.e., the concrete image map). We similarly use the notation g and g^{-1}.

Proposition 3.4.3. *In sets, the inverse image map of a function $f\colon A \longrightarrow B$ applied to an equivalence relation S can equivalently be defined as*

$$(a_1, a_2) \in (S)f \quad \iff \quad (fa_1, fa_2) \in S$$

Proof. The definition of the inverse image map is that $(S)f = 0e$ where $S = 0g$ for some surjection g and where $me = gf$ is a surjection-injection factorisation in **Set**. Note that $0e = 0gf$, that is, $e(x) = e(y) \Leftrightarrow gf(x) = gf(y)$, since m is an injection. Hence, $(a_1, a_2) \in (S)f$ exactly when $e(a_1) = e(a_2)$, which is exactly $gf(a_1) = gf(a_2)$, which means that $(fa_1, fa_2) \in S$. Moreover, $(fa_1, fa_2) \in S \Rightarrow gf(a_1) = gf(a_2) \Rightarrow e(a_1) = e(a_2) \Rightarrow (a_1, a_2) \in (S)f$. That concludes the proof. □

For a function f, the equivalence relation $0f$ is the inverse image of the smallest equivalence relation. Hence, $(a_1, a_2) \in 0f$ exactly when $(fa_1, fa_2) \in 0$, i.e., $fa_1 = fa_2$. So $0f$ is the kernel relation of f.

Proposition 3.4.4. *In concrete sets,* **Set**, *the abstract image of a function $f\colon A \longrightarrow B$ has at most one non-trivial (non-singleton) equivalence class, which is equal to the concrete image of f in the usual sense.*

Proof. The image $f1$ is defined by taking a pushout:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads, "h"] & B \arrow[d, two heads, "\iota_1"] \\
\{\emptyset\} \arrow[r, "\iota_2"'] \arrow[ur, phantom, "\ulcorner", very near end] & \{\emptyset\} +_A B
\end{tikzcd}$$

where $\{\emptyset\}$ can be substituted for any singleton set. The image of f is defined by $0\iota_1 = f1$. Now, ι_1 has to collapse everything in the concrete image of f to a point for the diagram to commute, i.e., the point $\iota_2(\emptyset)$. Every other element b in B is seen to not collapse (by calculation of the pushout) and injectively maps into the pushout. Hence $f1$ consists of trivial equivalence classes with at most one non-trivial equivalence class which is equal to the concrete image $im(f)$. $\square$

Proposition 3.4.5. *Axiom 1 holds for concrete sets.*

Proof. Let $f\colon B \longrightarrow C$ and $g\colon A \longrightarrow B$ be functions and h a surjection with the kernel relation T on A. Let q be a function with kernel relation gT. Consider the following diagram:

$$\begin{tikzcd}
A' \arrow[r] & B' \arrow[r] & C' \\
A \arrow[u, two heads, "h"] \arrow[r, "g"'] \arrow[ur, phantom, "\ulcorner", very near end] & B \arrow[u, two heads, "q"'] \arrow[r, "f"'] \arrow[ur, phantom, "\ulcorner", very near end] & C \arrow[u, two heads]
\end{tikzcd}$$

The left square is a pushout and the outer rectangle is a pushout. Hence, from elementary category theory, we have that the right square is a pushout. That is,

$$fg(T) = f(gT).$$

Let l be a surjection with kernel relation R on C. Let e be the surjection in an surjection-injection factorisation $me = lf$ with kernel relation Rf. Let $m'e'$ be the surjection-injection factorisation of eg. Consider the diagram:

$$\begin{tikzcd}
A'' \arrow[r, hook, "m'"] & B'' \arrow[r, hook, "m"] & C'' \\
A \arrow[u, two heads, "e'"] \arrow[r, "g"'] & B \arrow[u, two heads, "e"] \arrow[r, "f"'] & C \arrow[u, two heads, "l"']
\end{tikzcd}$$

By commutativity, we have that $mm'e' = lfg$. Since mm' is a mono, we have that $(m \circ m') \circ e'$ is an surjection-injection factorisation of $l \circ (f \circ g)$ and hence:

$$(R)fg = (Rf)g.$$

Finally, let g be a surjection with kernel relation S. Then consider the pushout:

$$\begin{tikzcd}
X' \arrow[r, "1_{X'}"] & X' \\
X \arrow[u, two heads, "g"] \arrow[r, "1_X"'] & X \arrow[u, two heads, "g"'] \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Thus we have that

$$1_X(S) = S.$$

Now consider the surjection-injection factorisation, where g' is taken to be a surjection, again with kernel relation S:

$$\begin{tikzcd}
X' \arrow[r, hook, "1_{X'}"] & X' \\
X \arrow[u, two heads, "g'"] \arrow[r, "1_X"'] & X \arrow[u, two heads, "g'"']
\end{tikzcd}$$

And therefore,

$$S = (S)1_X.$$

Finally, we need to show that the direct and inverse image maps are monotone. Suppose we have a function $f\colon A \longrightarrow B$ and that $g\colon A \longrightarrow C$ and $g'\colon A \longrightarrow C'$ are surjections with kernel relations R and R', respectively and $R \leqslant R'$, in other words, seen as subsets of the cartesian product the equivalence relations have $R \subseteq R'$. Then we have that g factors through g'. Since $C +_A B$ is a pushout, and since we have $(\iota_1' h)g = \iota_2' f$, we have by the universal property of the pushout that there exists a morphism $u\colon C +_A B \longrightarrow C' +_A B$ such that both $u\iota_1 = \iota_1' h$ $u\iota_2 = \iota_2'$.

$$\begin{tikzcd}
C \arrow[rr, "\iota_1"] \arrow[dd, "h"'] & & C +_A B \arrow[dd, dashed, bend left, "u"] \\
& A \arrow[ul, two heads, "g"] \arrow[r, "f"] \arrow[dl, two heads, "g'"] & B \arrow[u, "\iota_2"'] \arrow[d, "\iota_{2'}"] \\
C' \arrow[rr, "\iota_1'"'] & & C' +_A B
\end{tikzcd}$$

Hence, from the factorisation in last equality we conclude that $fR \leqslant fR'$. This proves monotonicity of the image map.

Consider now the case where $f' \colon B \longrightarrow A$ is a function:

$$\begin{tikzcd}
C \arrow[dd, "h"'] & & D \arrow[ll, hook', "m"'] \\
& A \arrow[ul, two heads, "g"] \arrow[dl, two heads, "g'"'] & B \arrow[l, "f'"] \arrow[u, two heads, "e"'] \arrow[d, two heads, "e'"] \\
C' \arrow[rr, bend right, "s"'] & & D' \arrow[ll, hook', "m'"']
\end{tikzcd}$$

where $me = gf'$ and $m'e' = g'f'$ are surjection-injection factorisations. We can let s be a left inverse of m' since monomorphisms in **Set** are split. Then $(shm)e = shgf' = sg'f' = sm'sg'f' = sm'sm'e' = e'$. This means that e' factors through e and hence, $Rf \leqslant R'f$.

Hence, Axiom 1 holds. □

Proposition 3.4.6. *Axiom 2 holds for concrete sets.*

Proof. We can determine $(fS)f$ from the following diagram:

$$\begin{tikzcd}
A' \arrow[r, "\iota_1"] & A' +_A B \arrow[r, bend left, "s"] & C \arrow[l, hook', "m"'] \\
A \arrow[u, two heads, "g"] \arrow[r, "f"'] & B \arrow[u, two heads, "\iota_2"] & A \arrow[l, "f"] \arrow[u, two heads, "e"']
\end{tikzcd}$$

The left square is a pushout and the right square is a surjection-injection factorisation. We take g to be a surjection with kernel relation S. Hence, ι_2 has kernel relation fS and e (and me) have kernel relation $(fS)f$. Let s be a left inverse of m.

It can be shown that the join $S \vee 0f$ is exactly determined by taking the pushout of f and g, that is, $S \vee 0f = 0\iota_1 g$. But $me = \iota_2 f = \iota_1 g$ and therefore, $(fS)f = 0e = 0me = 0\iota_1 g = S \vee 0f$.

Similarly, we can prove the dual part of Axiom 2 by determining $f(Rf)$ from the next diagram:

$$\begin{tikzcd}
B' & C \arrow[l, hook', "m"'] \arrow[r, "\iota_1"] & C +_A B \arrow[ll, dashed, bend right, "u"'] \\
B \arrow[u, two heads, "g"] & A \arrow[l, "f"'] \arrow[u, two heads, "e"] \arrow[r, "f"] & B \arrow[u, two heads, "\iota_2"']
\end{tikzcd}$$

The left square is constructed by an surjection-injection factorisation and the right square is a pushout. We take g to be a surjection with kernel relation R. Then $0e = Rf$ (i.e., e has kernel relation Rf) and $0\iota_2 = f(Rf)$. By monotonicity, we have $f(Rf) \leqslant f1$. Now we show that $f(Rf) \leqslant R$. That is, that g factors through ι_2. But

$me = gf$ and hence by the universal property of the pushout we have that there is a function $u\colon C +_A B \longrightarrow B'$ such that $u\iota_1 = m$ and $u\iota_2 = g$. The second equality then implies that $f(Rf) \leqslant R$ and hence $f(Rf) \leqslant R \wedge f1$.

Now we show the converse, i.e., that $f1 \wedge R \leqslant f(Rf)$. Note that $0gf = 0me = 0e = Rf$ and $f(0e)$ can be computed as a pushout.

The final part of the proof is not categorical and requires an argument on elements. Let us suppose that for $x \neq y$, $(x, y) \in R$, and x and y are in the concrete image of f, in other words, $(x, y) \in f1 \wedge R$. Then we have that for $x = fz$ and $y = fq$, $(fz, fq) \in R$, which means that $(z, q) \in Rf$ by Proposition 3.4.3 and hence that $e(z) = e(q)$ and consequently $z \in e^{-1}\{e(q)\}$ which implies that $y \in fe^{-1}ef^{-1}\{x\}$. Hence, by Proposition 3.4.1, $(x, y) \in f(Rf)$. Finally, $f1 \wedge R \leqslant f(Rf)$ and therefore $f1 \wedge R = f(Rf)$. For the case where $x = y$, note that automatically $(x, x) \in f(Rf)$ by reflexivity in equivalence relations. □

Proposition 3.4.7. *Axiom 3 holds for concrete sets.*

Proof. Let X be a set. Let S be an equivalence relation on X. Then, let $\{f_i \mid i \in I\}$ be the family of injections for every equivalence class of S whose concrete image chooses exactly an equivalence class of S. If $X = \emptyset$ then the only function $\emptyset \longrightarrow \emptyset$ is the inmorphism of the only equivalence relation on $\emptyset$.

1. $I \neq \emptyset$ since there is at least one injection into X selecting an equivalence class of S.

2. Each f_i is a monomorphism since it is injective.

3. For each i, $f_i1 \leqslant S_i \leqslant S$, where S_i is an equivalence class of S.

4. For $f1 \leqslant f_i1$, f not $\mathcal{N}$-null $\Rightarrow \exists u(f = f_iu)$, since a function factors through the injection into its concrete image and hence factors through a larger concrete image.

5. Fix f_j. $\exists f_i(f_i = f_ju) \Leftrightarrow u$ is iso, since there is exactly one injection up to isomorphism into each equivalence class and hence if an inmorphism factor through another, they must be isomorphic. If f_i is the map $\emptyset \longrightarrow \emptyset$, then so is f_j, forcing $f_i = f_j$.

6. For f_i and f_j not $\mathcal{N}$-null, $\exists u \exists v(f_iv = f_ju \wedge (f_ju \text{ not empty})) \Rightarrow f_i \simeq f_j$, since if $f_iv = f_ju$ and f_ju is non-empty (in the usual set theory sense), then their concrete images overlap and hence they must be isomorphic. In our setting, $\emptyset \longrightarrow \emptyset$ is also non-empty, but in that case $f_i = f_j$.

7. $f1 \leqslant S \Rightarrow \exists f_i \exists u(f = f_iu)$ holds since either f is empty and factors through all inmorphisms, or f factors through the injection f_i containing the image $im(f)$.

Now we need to show the dual statements for the outmorphisms. Let g_j be a surjection with kernel relation S.

1. $J \neq \emptyset$ since there is exactly one $g_j \in J$, plus all isomorphic surjections.
2. Since g_j is a surjection, it is an epimorphism.
3. $S = 0g_j$.
4. If $0g_j \leqslant 0g$, then $g = vg_j$, where v collapses all points in X which g collapse but g_j does not.
5. All elements of J are isomorphic by definition.
6. Again, all elements of J are isomorphic by definition.
7. For $S = 0g_j \leqslant 0g$, we again use the argument in Point 4.

□

In concrete sets, we choose the $\mathcal{N}$-null functions to be exactly the null functions. Hence, for an $\mathcal{N}$-null function f, f has the necessary and sufficient condition that $f1 = 0$. That means that the concrete image of f is either a singleton, or empty. Hence, the requirement on f to be $\mathcal{N}$-null is exactly that it is either a constant function or an empty function.

Proposition 3.4.8. *Axiom 4 holds for concrete sets.*

Proof. Let f be a non-$\mathcal{N}$ function (that is, neither empty nor constant) and $f = me$ be the epi-mono factorisation of f in the category of sets. That is, the factorisation which is a surjection onto the image followed by an injection. Then, the concrete images are equal as $im(m) = im(f)$, which implies that the abstract images are equal, $m1 = f1$. Since m is an injection (selecting the equivalence class $im(m)$), by our definition of inmorphisms for concrete sets, it is an inmorphism of $m1$. Also, $0f = 0me = 0e$, and e is a surjection and thus an outmorphism of $0e$. This is because the inverse image of 0 under an injection is always the finest equivalence relation.

Note that $m1 = f1$ does not imply $im(m) = im(f)$ in general, in particular, it does not hold exactly when m is constant and maps to a different point than f, or when either one of m or f is empty and the other is not. □

Proposition 3.4.9. *Axiom 5 holds for concrete sets.*

Proof. Any equivalence relation of sets is normal since it is the kernel relation of some function (namely, the equaliser of the kernel pair) and so the join of any two $\mathcal{A}$-subobjects is normal.

A conormal $\mathcal{A}$-subobject in concrete sets is an equivalence relation with at most one non-trivial equivalence class. Hence the meet of conormal C and D will have only at most one non-trivial equivalence class (namely, the intersection of the non-trivial equivalence classes in C and D) and is thus conormal. □

Proposition 3.4.10. *The $\mathcal{Z}$-empty morphisms in* **Set** *are exactly the morphisms of the form $\emptyset \longrightarrow A$ where $A \neq \emptyset$.*

Proof. By Proposition 3.2.27 and using Definition 3.2.10, $\mathcal{Z}$-empty morphisms are the $\mathcal{N}$-null functions which are not inmorphisms. Since the $\mathcal{N}$-null morphisms are null, an empty morphism which is not an inmorphism excludes morphisms of the form $\bullet \longrightarrow A$ where $\bullet \simeq \{\emptyset\}$ since these are inmorphisms. Secondly, since $\emptyset \longrightarrow \emptyset$ is also an inmorphism, it is not $\mathcal{Z}$-empty. Hence, the remaining functions in $\mathcal{N}$ which are at the same time $\mathcal{Z}$-empty are exactly of the form $\emptyset \longrightarrow A$ where $A \neq \emptyset$. □

Remark 3.4.11. Under the usual definition in set theory, any $\emptyset \longrightarrow A$ can be called an empty map. As stated in the previous theorem, we now however have that under Definition 3.2.10, the prototypical empty map $\emptyset \longrightarrow \emptyset$ is not $\mathcal{Z}$-empty.

3.4.2 Group-like Structures

It has been shown by Janelidze and Goswami [8] that the Axiom 1, 2 and 5 hold for groups. Axiom 4 in the current context holds since it is a weakening of the Axiom 4 employed in projective group theory. Hence, we need to show that Axiom 3 holds for groups. In fact, it can be seen that Axiom 3 in the abstract set context reduces to the exact Axiom 3 for groups in the context of projective group theory.

For the context of **Grp**, we choose $\mathcal{N}$ so that it is empty. Hence there are no $\mathcal{N}$-null morphisms.

Proposition 3.4.12. *Axiom 3 holds for groups, where the class of $\mathcal{N}$-null morphisms is chosen to be empty.*

Proof. Let G be a group. Let S be a subgroup of G. Let f_i be the subgroup inclusion of S (and all f_k for $k \in I$ are isomorphic) and let g_j be the quotient by the smallest normal subgroup containing S (and again all g_n for $n \in J$ are isomorphic).

1. $I \neq \emptyset$ and $J \neq \emptyset$ since we explicitly defined their elements.

2. We have that f_i is a monomorphism since it is injective and g_j is an epimorphism since it is surjective.

3. $f_i 1 = S$ and $S \leqslant 0g_j$ since $0g_j$ is the normal closure of S.

4. If $f1 \leqslant S = f_i 1$, then $f = f_i u$, with exactly the same factorisation as in the set case. If $S \leqslant 0g$, then $0g_j \leqslant 0g$ since it is the smallest normal subgroup containing S. The factorisation $g = vg_j$ follows from standard group theory: $0g/0g_j$ is a normal subgroup of $G/0g_j$ and v is the characteristic map $v \colon G/0g_j \longrightarrow (G/0g_j)/(0g/0g_j) = G/0g$.

5. All elements indexed by I and by J are isomorphic.

6. All elements indexed by I and by J are isomorphic.

7. This was already handled in Point 3.

$\square$

3.4.3 A Linearly Ordered Set

A linearly ordered set $(X, \leqslant)$ with intervals (a, b) such that $a, b \in X$ and $a \leqslant b$ satisfies the axioms in our abstract setting. Morphisms $(a, b) \longrightarrow (c, d)$ are unique and exist exactly when $a \leqslant c$ and $b \leqslant d$. Substructures of (a, b) are all x such that $a \leqslant x \leqslant b$. The inmorphisms and outmorphisms for each x are unique. They are $f \colon (a, x) \longrightarrow (a, b)$ and $g \colon (a, b) \longrightarrow (x, b)$.

3.4.4 Topological Spaces

Topological spaces with quotients (equivalence relations on the underlying set) do not satisfy Axiom 4, assuming outmorphisms have the induced quotient topology and inmorphism the induced subspace topology.

Consider the continuous map:

$$f \colon A = (\{0, 1\}, \tau_A) \longrightarrow B = (\{0, 1\}, \tau_B)$$

where f is the identity map, τ_A is the discrete topology and τ_B is the in indiscrete topology. Since any morphism into an indiscrete topology is a continuous map, f is continuous.

We have the surjection-injection factorisation:

$$\begin{tikzcd} & C = (\{0,1\}, \tau_C) \arrow[dr, hook, "g"] & \\ A \arrow[ur, two heads, "h"] \arrow[rr, "f"'] & & B \end{tikzcd}$$

Since h is an outmorphism, it needs to have the induced quotient topology. But τ_A is discrete, h is the identity; hence, since $h^{-1}(\{0\}) = \{0\}$ and $h^{-1}(\{1\}) = \{1\}$, both of which are open, we have that $\{0\}$ and $\{1\}$ need to be open in τ_C and hence τ_C is discrete. Now, however, since g is an inmorphism, it should have the induced subspace topology. Since the underlying map of g is an identity, it forces τ_C to be indiscrete like τ_B, which contradicts that τ_C is discrete.

Hence, Axiom 4 fails for topological spaces. If we allow the alternative for factorisation, that is, to let τ_C be indiscrete (and relax the requirements that an outmorphism's codomain needs to be the quotient topology and an inmorphisms domain be the subspace topology), then Axiom 4 will break because of uniqueness of the factorisation failing, i.e., f will factor through two topological spaces, (C, τ_C) and (C, τ_C*), one discrete and the other indiscrete. They are not homeomorphic and hence not isomorphic categorically.

Chapter 4

Homomorphism Theorems

In this chapter we explore the results of Chapter 3. The necessary language to handle chasing of subgroups and proofs of isomorphism theorems are developed here analogously to Z Janelidze's notes [12]. This leads to analogues for the isomorphism theorems for **Grp**, but now in the context of abstract sets, of which **Set** is the prototypical example. Thereafter, the isomorphism theorems in Tholen's 1974 doctoral thesis are reformulated as far as possible for them to hold in our context. The $\mathcal{N}$-null cases are given special attention since they do not hold in the general self-dual theory of abstract sets. By selectively adding properties of **Set**, one can eventually fully recover Tholen's results for the specialisation to **Set** but at the cost of losing self-duality of the theory. As explained at the end, a specific empty case of the Zassenhaus lemma can be perhaps considered an exception, unless one adds more structure. The basic reason is that an empty map has the same abstract image as a constant map with the same codomain ($\emptyset 1 = 0$ and $f1 = 0$) and the empty map factors through all such constant maps.

4.1 Pyramid Lemmas

4.1.1 Induced Morphism

The purpose of our pyramid lemmas is to provide a way to induce a universal morphism for a sequence of morphisms of which the directions may be not composable.

The strategy for such an induced morphism is to construct a pyramid from a base of morphisms, after which the outer triangle of sequences will be composable by taking inverses of isomorphisms.

Axiom 4 holds only for functions which are not constant nor empty and hence by design the universal morphism in this context can only be induced if constant or empty functions do not arise in the construction of the pyramid. In Section 4.4 an alternative strategy is briefly explained which would make the requirements on the base sequence weaker: the criteria that the composite relation of the sequence is in fact a function. If one were to follow the latter strategy, it may not be possible to show that the pyramid is constructible and furthermore the morphisms will be required to necessarily also be functions. Hence, one could do away with the empty set completely (removing the need to separately handle null morphisms); or, one could employ an alternative strategy to induce a morphism rather than using a pyramid. However, for the purposes of this thesis, we maintain the same method and approach throughout, which we will describe below.

Definition 4.1.1. Consider a sequence of morphisms:

$$A_0 \xrightarrow{f_1} A_1 \xleftarrow{f_2} A_2 \xrightarrow{f_3} A_3 \xleftarrow{f_4} A_4 \text{ - - - - - - } A_{n-1} \xleftarrow{f_n} A_n$$

We will call such a sequence a *zigzag*. Furthermore, we say that an $\mathcal{A}$-subobject T of A_n is obtained from an $\mathcal{A}$-subobject S of A_0 by chasing it along the zigzag when

$$T = (f_{n-1} \ldots f_3((f_1 S)f_2)) \ldots f_n.$$

Or, equally, by writing the direct image as $f(-)$ and the inverse image as $f^{-1}(-)$,

$$T = f_n^{-1} \ldots f_3 f_2^{-1} f_1 S.$$

Our objective is to complete such a sequence to a commutative *pyramid*. In the construction of the pyramid, there are diagrams of four different types that need to be considered that are each completed into a *diamond*.

Each diagram corresponds to one of the four possible pairs of directions that two morphisms can have by switching the domain and co-domain.

Notation 4.1.2. We number the four possible diagrams (and their associated completed diamond) as follows.

1.
$$\begin{tikzcd} \bullet \arrow[dr] & & \bullet \\ & \bullet \arrow[ur] & \end{tikzcd}$$

2.
$$\begin{tikzcd} \bullet & & \bullet \arrow[dl] \\ & \bullet \arrow[ul] & \end{tikzcd}$$

3.
$$\begin{tikzcd} \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd}$$

4.
$$\begin{tikzcd} \bullet & & \bullet \\ & \bullet \arrow[ul] \arrow[ur] & \end{tikzcd}$$

In practice, diagrams of the last two types, Type 3 and Type 4, will consist of two inmorphisms and two outmorphisms, respectively.

Definition 4.1.3. For diagrams of the first two types, the completion into a diamond is defined as taking the outmorphism-inmorphism factorisation of the bottom composite in the sense of Axiom 4. This then forms the top part of the diagram. Hence, the diamond is only constructible when the composite is not $\mathcal{N}$-null.

$$\begin{tikzcd}
& \bullet \arrow[dr, hook] & \\
\bullet \arrow[ur, two heads] \arrow[dr] & & \bullet \\
& \bullet \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \bullet \arrow[dl, hook'] & \\
\bullet & & \bullet \arrow[ul, two heads] \arrow[dl] \\
& \bullet \arrow[ul] &
\end{tikzcd}$$

Lemma 4.1.4. *For diamonds of Type 1 and 2, chasing an $\mathcal{A}$-subobject along the top of the diagram is the same as chasing it along the bottom.*

Proof. The chasing property is an immediate consequence of commutativity. □

We now turn to the completion into a diamond for diagrams of Type 3 and 4.

Definition 4.1.5. Suppose that we have two adjacent inmorphisms as follows:

$$\begin{tikzcd}
\bullet \arrow[dr, hook, "m_1"'] & & \bullet \arrow[dl, hook', "m_2"] \\
& \bullet &
\end{tikzcd}$$

We define the completion into a diamond, as indicated by the dotted arrows, by taking m_1' to be an non-$\mathcal{N}$ inmorphism of $(m_2 1)m_1$ (assuming $(m_2 1)m_1$ is not in $\mathcal{N}_0$), which is conormal by Proposition 3.2.34. By Proposition 3.2.14 all non-$\mathcal{N}$ inmorphisms are isomorphic, and hence the choice of m_1' is unique up to isomorphism. Furthermore, by Proposition 3.2.15, $m_1' 1 = (m_2 1)m_1$.

$$\begin{tikzcd}
& \bullet \arrow[dl, hook', dashed, "m_1'"'] \arrow[dr, hook, dashed, "m_2'"] & \\
\bullet \arrow[dr, hook, "m_1"'] & & \bullet \arrow[dl, hook', "m_2"] \\
& \bullet &
\end{tikzcd}$$

Remark 4.1.6. Note that if $(m_2 1)m_1 \in \mathcal{N}_0$, then we define the diagram as not having a completion (since the possible ways to complete the diagram may be not unique).

Lemma 4.1.7. *There is an induced m_2' determined by the rest of the diagram that makes the diamond commute.*

Proof. Note that since m_1' is not $\mathcal{N}$-null, then m_1m_1' is not $\mathcal{N}$-null by Axiom $\mathcal{N}$.3. Now,

$$m_1m_1'1 = m_1((m_21)m_1) = m_21 \wedge m_11 \leqslant m_21.$$

Hence, by Axiom 3a.7, we have that m_1m_1' uniquely factors through an inmorphism of m_21, say f as $m_1m_1' = fq$. But $m_1m_1'1 = fq1 \leqslant f1$ implies that f is not $\mathcal{N}$-null, and since $f1 \leqslant m_21$ we have that by maximality of inmorphisms, Axiom 3a.4 and 3a.5, that $f \simeq m_2$. (Explicitly: $f1 \leqslant m_21$ implies that for some u, $f = m_2u$; and since m_2 and f are both inmorphisms of m_21, we have that u is an isomorphism.) Hence we have a unique factorisation, $m_1m_1' = fq = m_2uq = m_2q'$. Hence any m_2' as in the diagram forces $m_2' = q'$. □

Lemma 4.1.8. *m_2' is a non-$\mathcal{N}$ inmorphism of $(m_11)m_2$ and $m_2'1 = (m_11)m_2$.*

Proof. Note that $m_2' \notin \mathcal{N}$. For if it were, then $m_2m_2' \in \mathcal{N}$ and hence $m_1m_1' \in \mathcal{N}$.

Since $m_1m_1' \notin \mathcal{N}$ and both m_1 and m_1' are non-$\mathcal{N}$ inmorphisms, then by Proposition 3.2.31, m_1m_1' is an inmorphism. Hence, $m_2m_2' = m_1m_1'$ is an inmorphism. Now, $0m_2m_2' = 0 \Rightarrow 0m_2' = 0$. Therefore, since m_2' is not $\mathcal{N}$-null, by Proposition 3.2.27 m_2' is an inmorphism (of $m_2'1$).

$$\begin{aligned}
m_2'1 &= (m_2m_2'1)m_2 && \text{since } m_2 \text{ is an inmorphism}\\
&= (m_1m_1'1)m_2 && \text{since } m_1m_1' = m_2m_21\\
&= (m_1((m_21)m_1))m_2 && \text{since } m_1'1 = (m_21)m_1\\
&= (m_21 \wedge m_11)m_2 && \text{expanding } m_1 \text{ and } m_21 \text{ with Axiom 2}\\
&= (m_21)m_2 \wedge (m_11)m_2 && \text{since right adjoints preserve limits}\\
&= 1 \wedge (m_11)m_2 && \text{since } 0m_2 \vee 1 = 1\\
&= (m_11)m_2 && \text{since } 1 \wedge X = X \text{ for any } X
\end{aligned}$$

Hence m_2' is a inmorphism of $(m_11)m_2$ and $m_2'1 = (m_11)m_2$. □

Lemma 4.1.9. *Chasing an $\mathcal{A}$-subobject along the bottom of a diamond of Type 3 is the same as chasing it along the top.*

Proof. Let $S \in \Sigma\, dom(m_1)$.

$$\begin{aligned}
(m_1S)m_2 &= (m_1S)m_2 \wedge (m_11)m_2 && \text{since } S \leqslant 1\\
&= (m_1S)m_2 \wedge m_2'1 && \text{by Lemma 4.1.8}\\
&= m_2'(((m_1S)m_2)m_2') && \text{using Axiom 2 on } (m_1S)m_2 \text{ and } m_2'\\
&= m_2'((m_1S)m_2m_2') && \text{removing brackets}\\
&= m_2'((m_1S)m_1m_1') && \text{the diamond commutes by Lemma 4.1.7}\\
&= m_2'(Sm_1') && \text{the direct image map of an inmorphism is injective}
\end{aligned}$$

Similarly, chasing backwards, for $T \in \Sigma\, dom(m_2)$, $(m_2T)m_1 = m_1'(Tm_2')$, by duality of the diamond.

Hence, chasing an $\mathcal{A}$-subobject along the top of the diagram is the same as chasing it along the bottom. $\square$

Corollary 4.1.10. *Since a diagram of Type 4 is dual to Type 3, it then follows that the diamond constructed from a diagram of Type 4 (using outmorphisms) also has the property that chasing an $\mathcal{A}$-subobject along the top of the diagram is the same as chasing it along the bottom of the diagram.*

The pyramid is constructed level by level, and we have now covered the four possible diagrams that can arise in the construction. The first level of the pyramid is the horizontal zigzag. On the second level of the pyramid, each morphism is factored in the sense of Axiom 4. On the third level of the pyramid, diagrams of Type 3 and 4 are used to construct diamonds of Type 3 and 4, respectively. Subsequent levels are constructed analogously, until an outer triangle is formed.

$$\begin{tikzcd}
 & & & & & & A_0^n \arrow[dl, dash] \arrow[dr, dash] & & & & & & \\
 & & & & & A_0^{n-1} \arrow[dl, dash, dashed] \arrow[dr, dash] & & A_1^{n-1} \arrow[dl, dash] \arrow[dr, dash] & & & & & \\
 & & & & A_0^4 \arrow[dl, dash] \arrow[dr, dash] & & A_1^{n-2} \arrow[dl, dash, dashed] \arrow[dr, dash] & & A_2^{n-2} \arrow[dl, dash] \arrow[dr, dash] & & & & \\
 & & & A_0^3 \arrow[dl, dash] \arrow[dr, dash] & & A_1^3 \arrow[dl, dash] \arrow[dr, dash] & & A_2^{n-3} \arrow[dl, dash, dashed] \arrow[dr, dash] & & A_3^{n-3} \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & A_0^2 \arrow[dl, dash] \arrow[dr, dash] & & A_1^2 \arrow[dl, dash] \arrow[dr, dash] & & A_2^2 \arrow[dl, dash] \arrow[dr, dash] & & A_3^{n-4} \arrow[dl, dash, dashed] \arrow[dr, dash] & & A_4^{n-4} \arrow[dl, dash] \arrow[dr, dash, dashed] & & \\
 & A_0^1 \arrow[dl, dash] \arrow[dr, dash] & & A_1^1 \arrow[dl, dash] \arrow[dr, dash] & & A_2^1 \arrow[dl, dash] \arrow[dr, dash] & & A_3^1 \arrow[dl, dash] \arrow[dr, dash] & & A_4^{n-5} \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & & A_{n-1}^1 \arrow[dl, dash] \arrow[dr, dash] & \\
A_0^0 \arrow[rr, dash] & & A_1^0 \arrow[rr, dash] & & A_2^0 \arrow[rr, dash] & & A_3^0 \arrow[rr, dash] & & A_4^0 \arrow[rr, dash, dashed] & & A_{n-1}^0 \arrow[rr, dash] & & A_n^0
\end{tikzcd}$$

Lemma 4.1.11. *The pyramid will have the property that on each level, chasing an $\mathcal{A}$-subobject along a zigzag will be the same as chasing the $\mathcal{A}$-subobject horizontally along the base of the triangle.*

Proof. This follows directly from the $\mathcal{A}$-subobject chase property of the 4 types of diamonds. $\square$

As mentioned in Remark 4.1.6, diagrams of Type 3 and Type 4 are not always constructible, since we made the assumption that $(m_2 1)m_1$ is not in $\mathcal{N}_0$ (or in the dual case, $e_2(0e_1))$ is not in $\mathcal{N}_1$, where e_1 and e_2 are the outmorphisms of a diagram of Type 4). Similarly, diagrams of Type 1 and Type 2 are only constructible when the composite of the two base morphisms is not $\mathcal{N}$-null.

Furthermore, we are interested in pyramids that canonically induce a morphism. The following lemmas will determine the conditions for which a pyramid is constructible and induces a morphism.

Definition 4.1.12. An $\mathcal{N}$*-null expansion* is a short zigzag of morphisms

$$A_0 \xrightarrow{f_0} A_1 \xleftarrow{f_1} A_2$$

such that $(f_0 1)f_1 \in \mathcal{N}_0$, or a short zigzag

$$A_0 \xleftarrow{g_0} A_1 \xrightarrow{g_1} A_2$$

such that $g_1(0g_0) \in \mathcal{N}_1$.

Definition 4.1.13. A pyramid is *constructible* if on each subsequent level of construction it does not lead to the formation of an $\mathcal{N}$-null expansion or an $\mathcal{N}$-null morphism.

Remark 4.1.14. One can readily see that Definition 4.1.13 corresponds exactly to the condition which allows the construction of diamonds of Type 3 and Type 4 (no $\mathcal{N}$-null expansions) and types 1 and 2 (no $\mathcal{N}$-null morphisms).

We will need the following axiom to handle some of the $\mathcal{N}$-cases when we want to induce a morphism.

Axiom $\mathcal{N}$ (Chasing). We continue from the axioms on $\mathcal{N}$-null morphisms in Section 3.2 and add a 9th point to Axiom $\mathcal{N}$.

9. If we chase an $\mathcal{A}$-subobject $S \in \mathcal{N}_0$ along a sequence of inverse image maps and direct image maps to $0 \in \Sigma X$, then $\Sigma X \ni 0 \in \mathcal{N}_0$. Dually, if we start out with an $\mathcal{A}$-subobject $T \in \mathcal{N}_1$ and chase to $1 \in \Sigma Y$, then $\Sigma Y \ni 1 \in \mathcal{N}_1$.

The constructibility of the pyramid covers all the cases for which chasing subgroups would not lead to an $\mathcal{N}$-null morphism. For, if for a base sequence the pyramid is not constructible then it means that either it forms an $\mathcal{N}$-null morphism or it forms an $\mathcal{N}$-null expansion. In both cases this will force that chasing subgroups will chase 1 forwards to $0 \in \mathcal{N}_0$, which means that an induced morphism would have been $\mathcal{N}$-null.

The next lemma handles the cases for which a constructed pyramid canonically induces a morphism. For criteria for both constructibility and induction of a morphism at the same time see Lemma 4.1.17 thereafter.

Lemma 4.1.15. *Suppose that the pyramid induced by a zigzag $A_0 \dots A_n$ is constructible (i.e., satisfies the requirements in Definition 4.1.13). Then the following are equivalent.*

i) Along the left side of the outer triangle all the inmorphisms are isomorphisms and along the right side of the outer triangle all the outmorphisms are isomorphisms.

ii) There exists a morphism $f\colon A_0 \longrightarrow A_n$ for which the direct image map and inverse image map are defined by chasing $\mathcal{A}$-subobjects forwards and backwards, respectively, along the zigzag.

iii) Chasing $\mathcal{A}$-subobjects along backwards and forwards, respectively, gives a Galois connection between ΣA_0 and ΣA_n.

iv) Chasing the smallest $\mathcal{A}$-subobject of A_0 forwards along the zigzag results in the smallest $\mathcal{A}$-subobject of A_n, and chasing the largest $\mathcal{A}$-subobject of A_n backwards along the zigzag results in the largest $\mathcal{A}$-subobject of A_0

Proof. $i) \Rightarrow ii)$: As was shown earlier, chasing $\mathcal{A}$-subobjects along the outer triangle using direct and inverse images is the same as chasing along the base sequence. Assuming the outer triangle has isomorphisms for inmorphisms on the left and for outmorphisms on the right, we can form f by composing along the sides of the outer triangle using the inverses of isomorphisms. For an isomorphism i, the direct image is equal to the inverse image of i^{-1}. To see this, note that $i(S) = (i^{-1}i(S))i^{-1} = (S)i^{-1}$ since i^{-1} is an inmorphism and $i^{-1}i = 1_{dom(i)}$. This proves that chasing $\mathcal{A}$-subobjects along f is the same as chasing it along the outside of the outer triangle.

$ii) \Rightarrow iii)$: f is a morphism and hence induces a Galois connection between its abstract image and inverse image maps and chasing $\mathcal{A}$-subobjects along f is the same as along the outer triangle (and thus the same as along the bottom zigzag).

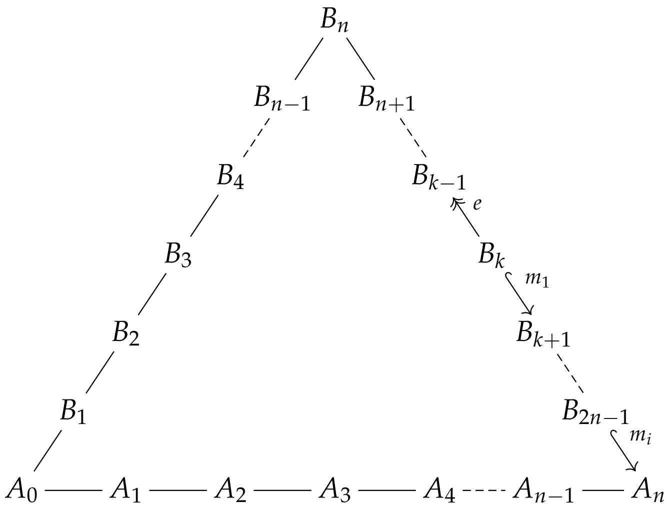
$iii) \Rightarrow iv)$: Since we have a Galois connection f by chasing $\mathcal{A}$-subobjects we have

$$[f1 \leqslant 1 \Leftrightarrow 1 \leqslant 1f] \Rightarrow 1f = 1.$$

Similarly,

$$[0 \leqslant 0f \Leftrightarrow f0 \leqslant 0] \Rightarrow f0 = 0.$$

$iv) \Rightarrow i)$: Let us suppose that statement $iv)$ holds. Let $e\colon B_k \longrightarrow B_{k-1}$ be the first outmorphism along the right side of the triangle which is not an isomorphism. By Proposition 3.2.26 it is not an inmorphism and since the pyramid in constructible, e is not $\mathcal{N}$-null. Hence, by Proposition 3.2.27, we have that $0e \neq 0$. (Even if e is $\mathcal{N}$-null we can infer that $0e \neq 0$. Since if $0e = 0$, then e is an outmorphism of $0 \in \Sigma B_k$, and hence must be an isomorphism by Proposition 3.2.18, but we assumed that e was not an isomorphism. This is important for Section 4.3.) Chasing 0 up to $cod(e) = B_{k-1}$ leads to $0 \in \Sigma B_{k-1}$ since every outmorphism e' along the left side of the triangle has $e'0 = 0$ and every inmorphism m along the left side of the triangle has $0m = 0$. Furthermore inmorphisms m' along the right side of the triangle have $m'0 = 0$ and every outmorphism e'' on the right side of the triangle leading up to e is an isomorphism and hence $0e'' = 0$. Suppose w.l.o.g. that the last morphism along the outer triangle on the right side is an inmorphism m_i. Let us label the first outmorphism after e (moving towards the right along the triangle) as e_1 and the first inmorphism after e as m_1 and so forth up to m_i. By assumption and since $0 \in \Sigma A_0$ chases to $0 \in \Sigma B_{k-1}$, we have that $m_i((m_{i-1}\ldots(m_1(0e))e_1\ldots)e_{i-1}) = 0$.



By Corollary 3.2.30, the direct image maps of inmorphisms are injective and the inverse image maps of outmorphisms are injective. One then readily shows by induction that $0e = 0$. The base case is

$$m_i((m_{i-1}\ldots(m_1(0e))e_1\ldots)e_{i-1}) = 0 \Rightarrow (m_{i-1}\ldots(m_1(0e))e_1\ldots)e_{i-1} = 0$$

The inductive step follows from, for an outmorphism r and inmorphism l, that

$$(X)r = 0 \Rightarrow X = 0$$

and that

$$l(Y) = 0 \Rightarrow Y = 0.$$

Hence, since there is no first e which is not an isomorphism, all outmorphisms along the right side of the outer triangle are isomorphisms. By the dual argument, all inmorphisms m along the left side of the triangle have $m1 = 1$ and hence are isomorphisms. $\square$

Remark 4.1.16. Inducing a morphism is subject to the conditions of 0 chasing to $0 \notin \mathcal{N}_0$ and 1 chasing backwards to $1 \notin \mathcal{N}_1$. In concrete sets, i.e., for **Set**, this will cause some examples of base zigzags to not induce a morphism even though upon inspection there is a clear morphism that is *suggested*. More specifically, this will correspond to cases where the induced relation is in fact a function. By design this route is not followed in this thesis and the interested reader may consult Section 4.4.3, Point 3, for a discussion around the topic and examples.

Lemma 4.1.17. *Suppose we have a sequence*

$$A_0 \xrightarrow{f_0} A_2 \dashrightarrow A_n$$

A pyramid of this sequence is constructible and induces a non-$\mathcal{N}$ morphism if both:

a) Chasing the smallest $\mathcal{A}$-subobject forwards results in the smallest $\mathcal{A}$-subobject and chasing the largest $\mathcal{A}$-subobject forwards does not result in an $\mathcal{A}$-subobject that is in $\mathcal{N}_0$. That is,

$$(f_{n-1}\dots(f_0 0)\dots)f_n = 0$$

and

$$(f_{n-1}\dots(f_0 1)\dots)f_n \notin \mathcal{N}_0$$

b) Chasing the largest $\mathcal{A}$-subobject backwards results in the largest $\mathcal{A}$-subobject and chasing the smallest $\mathcal{A}$-subobject backwards does not result in an $\mathcal{A}$-subobject which is in $\mathcal{N}_1$. That is,

$$f_0(\dots(1f_n)\dots f_1) = 1$$

and

$$f_0(\dots(0f_n)\dots f_1) \notin \mathcal{N}_1.$$

Proof. We need only show that conditions a) and b) lead to a constructible pyramid. By Lemma 4.1.15 we will then know that it induces a morphism f. This f will be non-$\mathcal{N}$ by the second part of condition a) which implies that $f1$ is not in $\mathcal{N}_0$.

Suppose that during the construction of the pyramid we have w.l.o.g. the first $\mathcal{N}$-null expansion on the layer most recently constructed:

$$A_k \xrightarrow{f_k} A_{k+1} \xleftarrow{f_{k+1}} A_{k+2}$$

Then $(f_k 1)f_{k+1}$ is in $\mathcal{N}_0$. Chasing any $\mathcal{A}$-subobject S up to $S' \in \Sigma A_k$ at this point then has $(f_k S')f_{k+1} \leqslant (f_k 1)f_{k+1} \in \mathcal{N}_0$. This means that $0 \in \Sigma A_0$ and $1 \in \Sigma A_0$ both chase to $\Sigma A_{k+2} \ni 0 \in \mathcal{N}_0$. Hence chasing $0 \in \Sigma A_0$ and $1 \in \Sigma A_0$ all the way will lead to the same $S'' \in \Sigma A_n$.

Condition a) of the lemma then requires that $S'' = 0 \notin \mathcal{N}_0$. By Axiom $\mathcal{N}$.9, $S'' = 0 \Rightarrow S'' \in \mathcal{N}_0$. This is a contradiction and hence the pyramid does not lead to an $\mathcal{N}$-null expansion of the kind described. A similar dual argument works for an $\mathcal{N}$-null expansion of the other type.

Suppose now that during the construction of the pyramid we have w.l.o.g. an $\mathcal{N}$-null morphism:

$$B_k \xrightarrow{g_k} B_{k+1}$$

Then similarly to the $\mathcal{N}$-null expansion case, for any $S' \in \Sigma B_k$, we have that $g_k 1 \in \mathcal{N}_0 \Rightarrow g_k S' \in \mathcal{N}_0$. Hence, again, $0 \in \Sigma A_0$ and $1 \in \Sigma A_0$ chase to $0 \in \Sigma B_{k+1}$; chasing 0 up to the end of the zigzag gives $S'' \in \Sigma A_n$ with the requirement that $0 = S'' \notin \mathcal{N}_0$, which is contradiction since $S'' = 0 \Rightarrow S'' \in \mathcal{N}_0$ by Axiom $\mathcal{N}$.9. Hence the pyramid does not lead to a $\mathcal{N}$-null morphism in the construction. A similar argument works for a null-morphism in the reverse direction.

Since the process of construction of the pyramid will not lead to a $\mathcal{N}$-null expansion or $\mathcal{N}$-null morphism, the pyramid is constructible. $\square$

Proposition 4.1.18. *Each inmorphism or outmorphism that is formed during the construction of the pyramid is unique up to isomorphism.*

Proof. This follows from the uniqueness of choice for completing diagrams of the four types. □

4.2 Isomorphism Theorems

As with the previous sections, the layout of this section (especially the order of theorems) is by analogue of a set of informal notes by Z Janelidze [12]. Those notes were published ultimately in an adapted format in [8]. The objective here is to follow the layout of the original notes in order to be able to compare the **Set** case and **Grp** case directly.

Definition 4.2.1. Let X and Y be $\mathcal{A}$-subobjects of a set G. We say that X is *normal under* Y if for every inmorphism of Y, $f\colon A \longrightarrow G$, Xf is a normal $\mathcal{A}$-subobject of A. Dually, we say that Y is *conormal over* X if for every outmorphism of X, $g\colon G \longrightarrow B$, gY is a conormal $\mathcal{A}$-subobject of B. When in addition $X \leqslant Y$, then we write $X \lhd Y$ to express that X is normal under Y and $X \subset Y$ to say that Y is conormal over X.

Proposition 4.2.2. *If X is normal under Z, then X is normal under any $\mathcal{A}$-subobject $Y \leqslant Z$. Dually, if Z is conormal over X, then Z is conormal over any Y with $X \leqslant Y$.*

Proof. Suppose $X \in \Sigma G$ is normal under some $Z \in \Sigma G$. Then for every inmorphism $f_i\colon A_i \longrightarrow G$ of Z we have that Xf_i is a normal $\mathcal{A}$-subobject of A_i. Now suppose that $Y \leqslant Z$. Any inmorphism of Y, $f'_k\colon A'_k \longrightarrow G$, has $f'_k 1 \leqslant Z$. This means that we have (for some k') $f'_k = f_{k'}q$ by Axiom 3a.7. Hence, since $Xf_{k'}$ is normal, $Xf_{k'}q \in \Sigma A'_k$ is normal since inverse images of normal $\mathcal{A}$-subobjects are normal by Proposition 3.2.13. The dual argument then follows. □

Corollary 4.2.3. *If an $\mathcal{A}$-subobject $X \in \Sigma G$ is normal under 1, then X is a normal $\mathcal{A}$-subobject. Dually, if X is conormal over 0, then X is conormal.*

Proof. Suppose X is normal under 1. By Proposition 3.2.18, 1_G is an inmorphism of 1. Then $X1_G = X$ is a normal $\mathcal{A}$-subobject of G. The dual follows. □

Definition 4.2.4. For an $\mathcal{A}$-subobject X of a set G, we will write l_X to denote an inmorphism of X (when it's clear from the context which inmorphism is being referred to). Dually, we write r_X te denote an outmorphism of X. R_X will be used to refer to the codomain of r_X and dually L_X is used to refer to the domain of l_X.

Proposition 4.2.5. *For an $\mathcal{A}$-subobject X of a set G, we have $X \lhd X$ and $X \subset X$.*

Proof. Let l_X be any inmorphism of X. Then $Xl_X = 1$ by Proposition 3.2.17, which is normal, and hence $X \lhd X$. Let r_X be any outmorphism of X. Then $r_X X = 0$, which is conormal and hence $X \subset X$. □

Proposition 4.2.6. *If $X = 0$, then ΣL_X is trivial. Dually, if $X = 1$ the ΣR_X is trivial.*

Proof. Suppose $X = 0$. Then, by Proposition 3.2.27 and Proposition 3.2.17, $0 = 0l_X = Xl_X = 1$. The dual follows. □

For the theorems that follow, one should notice that from Axiom $\mathcal{N}.8$ that if $X \notin \mathcal{N}_0$ then it is equivalent to $X \notin \mathcal{N}_1$; however, for clarity, both conditions may be stated next to each other.

Theorem 4.2.7. *Let X and Y be $\mathcal{A}$-subobjects of a set G. Fix an inmorphism l_X of X and an outmorphism r_Y of Y such that the composite $r_Y \circ l_X$ is not $\mathcal{N}$-null. Then this induces a non-$\mathcal{N}$ outmorphism r_0 of $(Y)l_X$ and a non-$\mathcal{N}$ inmorphism l_0 of $r_Y(X)$. There is a canonically induced morphism $f\colon R_{(Y)l_X} \longrightarrow L_{r_Y(X)}$. When X is conormal and Y is normal, f is an isomorphism.*

Proof. Let X and Y be $\mathcal{A}$-subobjects of a set G and suppose we have fixed an l_X and r_Y. If $r_Y l_X$ is not $\mathcal{N}$-null, then both r_Y and l_X are not $\mathcal{N}$-null.

$$Y \leqslant 0r_Y \Rightarrow (Y)l_X \leqslant 0r_Y l_X$$

This means that for some outmorphism $r_{(Y)l_X} = r_0$ of $(Y)l_X$, we have a unique factorisation $qr_0 = r_Y l_X$:

$$\begin{tikzcd} R_Y & L_X \arrow[l, "r_Y l_X"'] \arrow[dl, two heads, "r_0"] \\ R_{(Y)l_X} \arrow[u, dashed, "q"] & \end{tikzcd}$$

with $0r_0 \leqslant 0qr_0 = 0r_Y l_X$. Hence, r_0 is not $\mathcal{N}$-null by $\mathcal{N}.4$. Dually, we have a non-$\mathcal{N}$ $l_{r_Y(X)} = l_0$ such that $l_0 p = r_Y l_X$.

Therefore, we have the following zigzag:

$$R_{(Y)l_X} \xleftarrow{r_0} L_X \xrightarrow{l_X} G \xrightarrow{r_Y} R_Y \xleftarrow{l_0} L_{r_Y(X)}$$

If we want to show that this induces a constructible pyramid and canonical morphism, by Lemma 4.1.17, we need to chase $\mathcal{A}$-subobjects backwards and forwards.

$$r_Y l_X(0r_0) \leqslant r_Y l_X(0r_Y l_X) = r_Y l_X 1 \wedge 0 = 0$$

And, since l_0 is an inmorphism, we have $0 = 0l_0$ and combining the results, that $0 = (r_Y l_X(0r_0))l_0$. Hence the smallest $\mathcal{A}$-subobject chases to the smallest $\mathcal{A}$-subobject in the forwards direction. Now let us chase 1 forwards. The equality $1r_0 = 1$ holds for any Galois connection. By assumption, $r_Y l_X 1 \notin \mathcal{N}_0$. And then finally, for the factorisation of $r_Y l_X$, we have $(r_Y l_X 1)l_0 = (l_0 p1)l_0 = p1$. But $p1 \in \mathcal{N}_0$ would imply that $l_0 p1 \in \mathcal{N}_0$ by Axiom $\mathcal{N}.2$ and hence that $r_Y l_X$ is $\mathcal{N}$-null, which is not true by assumption. Hence 1 does not chase forwards to $0 \in \mathcal{N}_0$. The second part of Lemma 4.1.17 that needs to be satisfied, condition b), holds dually. Hence, the zigzag is constructible to a pyramid and induces a morphism.

Now let us suppose that X is conormal and Y is normal. The morphism will be an isomorphism exactly when chasing smallest and largest $\mathcal{A}$-subobjects backwards and forwards give smallest and largest $\mathcal{A}$-subobjects, respectively. Note that since both of l_X and r_Y are not $\mathcal{N}$-null and since X is conormal and Y is normal, this makes the choice of l_X and r_Y unique up to isomorphism.

As earlier, $1r_0 = 1$. Then, by Proposition 3.2.15, $l_X 1 = X$, since X is conormal. Chasing further along r_Y, we have $r_Y(X)$. Lastly, $(r_Y(X))l_0 = (r_Y(X))l_{r_Y(X)} = 1$. The last equality follows from Proposition 3.2.17. The argument for chasing 0 backwards is exactly dual and hence chases to 0. Therefore, since the induced non-$\mathcal{N}$ morphism f has both $0f = 0$ and $f1 = 1$, it is at the same time an inmorphism and an outmorphism and hence is an isomorphism. □

Theorem 4.2.8. *Consider a morphism $f\colon A \longrightarrow B$, and $\mathcal{A}$-subobjects $S \notin \mathcal{N}_0$ and $X \notin \mathcal{N}_0$ of A such that*

$$0f \leqslant S < X$$

where X is conormal and hence we have some inmorphism $l_X 1 = X$. Also, assume $f l_X 1 \notin \mathcal{N}_0$. Then $S \lhd X$ if and only if $fS \lhd fX$. When this is the case, then assuming that $(S)l_X \notin \mathcal{N}_1$ and $(fS)l_{fX} \notin \mathcal{N}_1$, they induce two outmorphisms, $r_0 = r_{(S)l_X}$ and $r_1 = r_{(fS)l_{fX}}$ and furthermore, there is a canonical isomorphism

$$R_{(S)l_X} \simeq R_{(fS)l_{fX}}.$$

Proof. Since $X \notin \mathcal{N}_0$ is conormal, it is the image of its non-$\mathcal{N}$ isomorphic inmorphisms. Chose such a representative $l_X \notin \mathcal{N}$. Then, $fX = f l_X 1 \notin \mathcal{N}_0$. Hence we can also choose a $l_{fX} \in \mathcal{N}$ up to isomorphism.

We then look at the following zigzag:

$$L_X \xrightarrow{l_X} A \xrightarrow{f} B \xleftarrow{l_{fX}} L_{fX}$$

The $\mathcal{A}$-subobject $0 \in \Sigma L_X$ chases forwards to $0 \in \Sigma B$, and then finally, $0l_{fX} = 0$ since it is an inmorphism. Chasing 1 backwards we have $l_{fX}1 = fX$. Then, by Proposition 3.2.17, $1 = (X)l_X \leqslant (fX)fl_X$; the last inequality holds since in a Galois connection, $X \leqslant (fX)f$. Hence 1 chases backwards to 1.

Chasing 1 forwards gives $l_X 1 = X$. Then $(fX)l_{fX} = 1$. Since $l_{fx} \notin \mathcal{N}$, we have by Axiom $\mathcal{N}$.6 that $1 \in \Sigma L_{fx}$ is not in $\mathcal{N}_1$ and hence $0 \in \Sigma L_{fx}$ is not in $\mathcal{N}_0$ by Axiom $\mathcal{N}$.8. Chasing 0 backwards, $l_{fX}0 = 0$. Then $0fl_X \notin \mathcal{N}_1$ since fl_x is not $\mathcal{N}$-null. Hence, there is a canonical morphism, $h\colon L_X \longrightarrow L_{fX}$. Since $h1 = 1$ and since h is not $\mathcal{N}$-null, h is an outmorphism. $0fl_X = (l_{fX}0)fl_X = 0h$. Hence, h is an outmorphism of the kernel of fl_X. l_{fX} is an inmorphism of the image $fl_X 1 = fX$. Therefore, $l_{fX} \circ h$ is an out-in factorisation of fl_X in the sense of Axiom 4.

We want to show that S being normal under X is equivalent to fS being normal under fX. Suppose $S \lhd X$. Then $(S)l_X$ is normal. The morphism induced above, h, is an outmorphism so its image of a normal $\mathcal{A}$-subobject is also normal. Hence, $h((S)l_X) = (fl_X((S)l_X))l_{fX} = (f(S \wedge l_X 1))l_{fX} = (f(S \wedge X))l_{fX} = (fS)l_{fX}$ is normal. Also, for any $l'_{fX} \in \mathcal{N}$, we have $fSl'_{fX} = 0$, which is normal. This thus gives that $fS \lhd fX$.

Conversely, let us assume that $fS \lhd fX$. Then $(fS)l_{fX}$ is normal. Since the preimage of a normal $\mathcal{A}$-subobject is normal, we have that

$$\begin{aligned}((fS)l_{fX})h &= (l_{fX}((fS)l_{fX}))fl_X \\ &= (fS \wedge l_{fX}1)fl_X = (fS \wedge fX)fl_X \\ &= (fS)fl_X = (S \vee 0f)l_X \\ &= (S)l_X\end{aligned}$$

So, $(S)l_X$ is normal. Again, for any $l'_X \in \mathcal{N}$, $Sl'_X = 0$ which is normal. Hence, $S \lhd X$. Let us now assume that $S \lhd X$. Since $(S)l_X \notin \mathcal{N}_1$ and since $(fS)l_{fX} \notin \mathcal{N}_1$, we have outmorphisms, $r_0 = r_{(S)l_X} \notin \mathcal{N}$ and $r_1 = r_{(fS)l_{fX}} \notin \mathcal{N}$. Both are unique up to isomorphism.

Now consider this zigzag:

$$R_{(S)l_X} \xleftarrow{r_0} L_X \xrightarrow{l_X} A \xrightarrow{f} B \xleftarrow{l_{fX}} L_{fX} \xrightarrow{r_1} R_{(fS)l_{fX}}$$

Let us now chase $\mathcal{A}$-subobjects. Chasing 0 forwards along r_0 gives $0r_0 = (S)l_X$. Chasing further gives $l_X((S)l_X) = S \wedge l_X 1 = S \wedge X = S$. Then we have fS, and then chasing again we have $(fS)l_{fX}$. Finally, $r_1((fS)l_{fX}) = 0$, by Proposition 3.2.17. Chasing 1 backwards gives $1r_1 = 1$. Then, $l_{fX}1 = fX$. Once more gives, $(fX)fl_X = (X \vee f0)l_X = (X)l_X = 1$. Finally, $r_0 1 = 1$ since r_0 is an outmorphism.

Let us chase 1 forwards. For any morphism we have $1r_0 = 1$. Then, $fl_X 1 = fX$. $(fX)l_{fX} = 1$. Finally, $r_1 1 = 1$, since r_1 is an outmorphism. Also, $1 = r_1 1 \notin \mathcal{N}_0$ since $r_1 \notin \mathcal{N}$. We then chase 0 backwards. Firstly, $0r_1 = (fS)l_{fX}$. Then, $l_{fX}((fS)l_{fX}) = fS \wedge l_{fX}1 = fS \wedge fX = fS$. Then along the next morphism we have $(fS)fl_X = (S \vee 0f)l_X = (S)l_X$. Then along the last morphism, $r_0((S)l_X) = 0$, again by Proposition 3.2.17. Since $r_0 \notin \mathcal{N}$, we have that $0 = 0r_0 \notin \mathcal{N}_1$. Thus, by chasing 0 and 1 both forwards and backwards, we have established that there is a canonical isomorphism $g\colon R_{(S)l_X} \longrightarrow R_{(fS)l_{fX}}$, and hence

$$R_{(S)l_X} \simeq R_{(fS)l_{fX}}.$$

$\square$

Corollary 4.2.9 (Isomorphism III). *Let $N \notin \mathcal{N}_1$ be a normal $\mathcal{A}$-subobject of a set G. For any $\mathcal{A}$-subobject Y such that $\mathcal{N}_0 \not\ni Y \notin \mathcal{N}_1$ of R_N there exists an $\mathcal{A}$-subobject X of G with $\mathcal{N}_0 \not\ni X \notin \mathcal{N}_1$ such that $N = 0r_N \leqslant X$, and further, Y is normal if and only if X is normal, and when this is the case, we have a canonical isomorphism*

$$R_X \simeq R_Y.$$

Proof. Consider Theorem 4.2.8 with $f = r_N$ where r_N is such that $0r_N = N$. Let Y satisfying $\mathcal{N}_1 \not\ni Y \notin \mathcal{N}_0$ be an $\mathcal{A}$-subobject of R_N. Let $X = Yr_N$. Suppose $X \in \mathcal{N}_0$. Then, $r_N 0 = 0 \leqslant Y \in \Sigma R_n$ and $\Sigma R_n \ni 0 \in \mathcal{N}_0$ by Axiom $\mathcal{N}.2$. Moreover, since $Yr_N \in \mathcal{N}_0$ by assumption, it implies that $0r_N = 0 = Yr_N \in \mathcal{N}_0$ by monotonicity.

Then, $0r_N = Yr_N \Rightarrow Y = 0 \in \Sigma R_n$ (and $Y \in \mathcal{N}_0$) by injectivity of the inverse image map for outmorphisms. By contradiction this implies $X \notin \mathcal{N}_0$. Also, $0r_N = N \leqslant Yr_N = X \leqslant 1$. Furthermore, $r_N 1_G 1 = r_N 1 = 1$ (the last equality by Proposition 3.2.27 and since r_N is an outmorphism). Hence $r_N 1_G \in \mathcal{N}$ only if ΣR_N is trivial and $1 = 0 \in \Sigma R_N$ has $\Sigma R_N \ni 0 \in \mathcal{N}_0$. But $0r_N = N \notin \mathcal{N}_1$ implies that $r_N \notin \mathcal{N}$, which implies that $0 \in \Sigma R_N$ cannot be in $\mathcal{N}_0$ (by Axiom $\mathcal{N}.1$ if it were, then $r_N 1 \in \mathcal{N}_0$). Hence, $r_N 1_G \notin \mathcal{N}$. Lastly, $Y \notin \mathcal{N}_1$, and hence $Y = r_N(Yr_N) = r_N X \notin \mathcal{N}_1$. If $X = X \vee N = (r_N X)r_N \in \mathcal{N}_1$, then $r_N X = r_N(Yr_N) = Y \in \mathcal{N}_1$ by applying Axiom $\mathcal{N}.9$ (with a sequence consisting of only one direct image, r_N, where $r_N 1 = 1$ since it is an outmorphism). Thus, $X \notin \mathcal{N}_1$. All the conditions from Theorem 4.2.8 are satisfied.

The corresponding zigzag from Theorem 4.2.8 is thus:

$$R_X \xleftarrow{r_X} G \xrightarrow{1_G} G \xrightarrow{r_N} R_N \xleftarrow{1_{R_N}} R_N \xrightarrow{r_Y} R_Y$$

Note that, without loss of generality, r_A and l_B with $A = 0$ and $B = 1$ can always be taken to be identities, since the options for r_A and l_B are exactly the isomorphisms by Proposition 3.2.21. Furthermore, the identity morphism is always an isomorphism. Then it follows that $l_{r_N 1} = l_{1 \in \Sigma R_N} = 1_{R_N}$ and $l_{1 \in \Sigma G} = 1_G$, and also, $r_{(r_N X)l_{r_N 1}} = r_{Yl_{1 \in \Sigma R_N}} = r_{Y1_{R_N}} = r_Y$.

X is normal if and only if $X \lhd 1$. This is the case, if and only if $r_N X = r_N(Yr_N) = Y \lhd 1 = r_N 1$ (i.e., that Y is normal) by Theorem 4.2.8. Hence we have the canonical isomorphism:

$$R_X = R_{Xl_1} = R_{X1_G} \simeq R_{(r_N X)l_{r_N 1}} = R_{Yl_{1 \in \Sigma R_N}} = R_Y.$$

$\square$

Lemma 4.2.10. *Let $S, X \in \Sigma G$ such that $S \lhd X$ and such that X is conormal. Then $Sf \lhd Xf$ and $gS \lhd gX$ for any morphism $f\colon A \longrightarrow G$ and any outmorphism $g\colon G \longrightarrow X$.*

Proof. Suppose $X = 0$, then $S = 0$ and hence, $Sf \lhd Xf$ and $gS \lhd gX$. Let us then assume that $X \neq 0$.

Consider the zigzag:

$$L_{Xf} \xrightarrow{l_{Xf}} A \xrightarrow{f} G \xleftarrow{l_X} L_X$$

where $l_X \notin \mathcal{N}$ is a (unique up to isomorphism) inmorphism of X, and l_{Xf} is any inmorphism of Xf. If $l_{Xf} \in \mathcal{N}$ or $fl_{Xf} \in \mathcal{N}$, then $Sfl_{Xf} = 1$, which is normal. Let us then suppose that both $l_{Xf} \notin \mathcal{N}$ and $fl_{Xf} \notin \mathcal{N}$.

We want to show that this induces a morphism. Chasing 0 forwards, we have $fl_{Xf}0 = 0$. Then along the last morphism, $0l_X = 0$, since it is an inmorphism. Since $fl_{Xf}1 \leqslant f(Xf) = X \wedge f1 \leqslant X$, we have that fl_{Xf} factors through an inmorphism of X. But since X is conormal and since $fl_{Xf} \notin \mathcal{N}$, it means that fl_{Xf} factors through $l'_X \notin \mathcal{N}$, say as $fl_{Xf} = l'_X q$ (since it cannot factor through a $\mathcal{N}$-null morphism, for if it did, it would also be $\mathcal{N}$-null). But X is conormal and its non-$\mathcal{N}$ inmorphisms are isomorphic, thus we have $fl_{Xf} = l'_X q = l_X iq$ for an isomorphism i. So $(fl_{Xf}1)l_X = (l_X iq1)l_X = iq1 \vee l_X 1$ which is not in $\mathcal{N}_0$ since $iq \notin \mathcal{N}$ (if $iq \in \mathcal{N}$ then $0iq = 0q \in \mathcal{N}_1$ so $q \in \mathcal{N}$ and hence $fl_{Xf} = l'_X q \in \mathcal{N}$). Chasing 1 backwards, we have $l_X 1 = X$. Then, X chases to Xf. Lastly, $Xfl_{Xf} = 1$ since the inverse images of an $\mathcal{A}$-subobject along its inmorphisms are the largest $\mathcal{A}$-subobjects. Finally, we chase 0 backwards. Along l_X we have $l_X 0 = 0$. Then, $0fl_{Xf} \notin \mathcal{N}_1$ since fl_{Xf} $/\!-in\mathcal{N}$. Hence 0 does not chase backwards to $1 \in \mathcal{N}_1$ and consequently we have a induced morphism $h\colon L_{Xf} \longrightarrow L_X$. Since $S \lhd X$, we have that Sl_X is a normal $\mathcal{A}$-subobject of L_X. Hence, $(Sl_X)h$ is normal. But $(Sl_X)h = (l_X(Sl_X))fl_{Xf} = (S \wedge l_X 1)fl_{Xf} = (S \wedge X)fl_{Xf} = (S)fl_{Xf} = (Sf)l_{Xf}$. That is, $(Sf)l_{Xf}$ is a normal $\mathcal{A}$-subobject of L_{Xf}. Since the choice of l_{Xf} was arbitrary, we thus have by definition 4.2.1 that $Sf \lhd Xf$.

Now we aim to prove that $gS \lhd gX$. Suppose that $g \circ l_X \in \mathcal{N}$. Then, $gX = gl_X 1 \in \mathcal{N}_0$. This means that for any inmorphism l_{gX} of gX, $l_{gX}1 \leqslant gX \in \mathcal{N}_0$ and hence $l_{gX} \in \mathcal{N}$. Then $(gS)l_{gX} \in \mathcal{N}_1$, which is normal, and hence $gS \lhd gX$. Suppose now that $g \circ l_X$ is not $\mathcal{N}$-null. Then, since X is conormal and $0g$ is normal, we have by Theorem 4.2.7 the following diagram:

$$\begin{tikzcd}
 & G \arrow[dr, "g"] & \\
X \arrow[ur, "l_X"] \arrow[dr, "r_{(0g)l_X}"'] & & B \\
 & R_{(0g)l_X} \simeq L_{r_{0g}(X)} \arrow[ur, "l_{gX}"'] &
\end{tikzcd}$$

Note that $g = r_{0g}$ and that $l_{r_{0g}X} = l_{gX}$ is uniquely determined as in Theorem 4.2.7. Moreover, since X is conormal, gX is conormal and hence any choice of l'_{gX} is isomorphic to l_{gX}.

Consider the following argument that chases along any $T \in \Sigma X$.

$$
\begin{aligned}
& l_{gX} i r_{(0g)l_X} T \quad \text{where } i\colon R_{(0g)l_X} \simeq L_{r_{0g}(X)} \text{ is an isomorphism} \\
&= l_{gX}((gl_X((r_{(0g)l_X}T)r_{(0g)l_X}))l_{gX}) \quad \text{expanding } i \text{ by chasing the } \mathcal{A}\text{-subobject } T \\
&= l_{gX}((gl_X(0r_{(0g)l_X} \vee T))l_{gX}) \quad \text{using Axiom 2 on } T \text{ and } r_{(0g)l_X} \\
&= l_{gX}((gl_X(0gl_X \vee T))l_{gX}) \quad \text{since } r_{(0g)l_X} \text{ is not } \mathcal{N}\text{-null; } 0gl_X \text{ is normal} \\
&= l_{gX}((gl_X(0gl_X) \vee gl_X T)l_{gX}) \quad \text{left adjoints preserve colimits} \\
&= l_{gX}((0 \vee gl_X T)l_{gX}) \quad \text{since } gl_X(0gl_X) = 0 \wedge gl_X 1 = 0 \\
&= l_{gX}((gl_X T)l_{gX}) \quad \text{since } 0 \vee gl_X T = gl_X T \\
&= gl_X T \wedge l_{gX} 1 \quad \text{using Axiom 2 on } l_{gX} \text{ and } gl_X T \\
&= gl_X T \wedge gX \quad \text{since } l_{gX} \text{ is non-}\mathcal{N} \text{ and } gX \text{ is conormal} \\
&= gl_X T \quad \text{since } l_X T \leqslant l_X 1 = X
\end{aligned}
$$

Hence, chasing along the top is the same as chasing along the bottom of the diagram.

Since Sl_X is normal, we have that $r_{0gl_X}(Sl_X)$ is normal and also that $ir_{0gl_X}(Sl_X)$ is normal.

But we have

$$
\begin{aligned}
ir_{0gl_X}(Sl_X) &= (l_{gX}(ir_{0gl_X}(Sl_X)))l_{gX} \quad \text{since } l_{gX} \text{ is an inmorphism} \\
&= (gl_X(Sl_X))l_{gX} \quad \text{using the argument above} \\
&= (g(S \wedge l_X 1))l_{gX} \quad \text{expanding Axiom 2 for } S \text{ and } l_X \\
&= (g(S \wedge X))l_{gX} \quad \text{since } l_X 1 = X \\
&= (gS)l_{gX} \quad \text{since } S \leqslant X
\end{aligned}
$$

Hence, $(gS)l_{gX}$ is normal. Also, for any $l_{gX} \in \mathcal{N}$, $(gS)l_{gX} \in \mathcal{N}_1$, which is normal. This proves normality for any choice of l_{gX} and hence $gS \lhd gX$. That completes the proof. $\square$

Theorem 4.2.11 (Isomorphism II). *Consider two $\mathcal{A}$-subobjects X and Y of a set G such that $X \wedge Y \notin \mathcal{N}_0$ and both $(X \wedge Y)l_Y \notin \mathcal{N}_1$ and $Xl_{X\vee Y} \notin \mathcal{N}_1$. If Y and $X \vee Y$ are conormal, and $X \lhd X \vee Y$, then $X \wedge Y \lhd Y$ and there is a canonical isomorphism*

$$R_{(X\wedge Y)l_Y} \simeq R_{Xl_{X\vee Y}}.$$

Proof. Consider the zigzag:

$$L_Y \xrightarrow{l_Y} G \xleftarrow{l_{X\vee Y}} L_{X\vee Y}$$

where l_Y and $l_{X\vee Y}$ are chosen such that $l_Y \notin \mathcal{N}$ and $l_{X\vee Y} \notin \mathcal{N}$ (the choices of which will be unique up to isomorphism). Let us show that this induces a morphism. Firstly, $(l_Y 0)l_{X\vee Y} = 0l_{X\vee Y} = 0$. Also, $(l_Y 1)l_{X\vee Y} = Yl_{X\vee Y} \in \mathcal{N}_0$ implies that $l_{X\vee Y}(Yl_{X\vee Y}) = l_{X\vee Y}0 = 0 \in \mathcal{N}_0$ by Axiom $\mathcal{N}$.2. This in turn means that $l_{X\vee Y}(Yl_{X\vee Y}) = Y \wedge l_{X\vee Y}1 = Y \wedge (X \vee Y) = Y \in \mathcal{N}_0$, which contradicts the assumption that $Y \notin \mathcal{N}_0$. Hence, 1 does not chase forwards to $0 \in \mathcal{N}_0$. Chasing 1 backwards, we have $l_{X\vee Y}1 = X \vee Y$. Chasing further, we have $(X \vee Y)l_Y$. Note that $Yl_Y = 1$ and hence $1 \leqslant (X \vee Y)l_Y$, i.e., $(X \vee Y)l_Y = 1$. Finally, chasing 0 backwards we have $l_{X\vee Y}0 = 0$, and then $0l_Y = 0 \notin \mathcal{N}_1$ since $l_Y \notin \mathcal{N}$. Hence, we have an induced morphism, $h\colon L_Y \longrightarrow L_{X\vee Y}$.

Now, $Xl_{X\vee Y}$ is normal by assumption. This means that $(Xl_{X\vee Y})h$ is normal. Furthermore

$$\begin{aligned}
(Xl_{X\vee Y})h &= (l_{X\vee Y}(Xl_{X\vee Y}))l_Y \\
&= (X \wedge l_{X\vee Y}1)l_Y \quad \text{by applying Axiom 2 to } X \text{ and } l_{X\vee Y} \\
&= (X \wedge (X \vee Y))l_Y \quad \text{since } X \vee Y \text{ is conormal} \\
&= Xl_Y \quad \text{since } X \leqslant (X \vee Y) \\
&= Xl_Y \wedge 1 \quad \text{since } Xl_Y \leqslant 1 \\
&= Xl_Y \wedge Yl_Y \quad \text{by Proposition 3.2.17} \\
&= (X \wedge Y)l_Y \quad \text{since right adjoints preserve limits}
\end{aligned}$$

Hence, $(X \wedge Y)l_Y$ is normal. Finally, since every $l'_Y \in \mathcal{N}$ will have $(X \wedge Y)l'_Y = 1$ (which is normal) and every $l''_Y \notin \mathcal{N}$ is isomorphic to l_Y, we have that $X \wedge Y \lhd Y$.

Now consider the following zigzag:

$$R_{(X\wedge Y)l_Y} \xleftarrow{r_{(X\wedge Y)l_Y}} L_Y \xrightarrow{l_Y} G \xleftarrow{l_{X\vee Y}} L_{X\vee Y} \xrightarrow{r_{Xl_{X\vee Y}}} R_{Xl_{X\vee Y}}$$

By assumption, $(X \wedge Y)l_Y \notin \mathcal{N}_1$ and $Xl_{X\vee Y} \notin \mathcal{N}_1$.

This, in addition to $(X \wedge Y)l_Y$ being normal means that there is a $r_0 \notin \mathcal{N}$ with $r_0 = r_{(X\wedge Y)l_Y}$ such that $0r_0 = (X \wedge Y)l_Y$ and which is unique up to isomorphism. The same holds for $r_1 = r_{Xl_{X\vee Y}}$.

Chasing 0 forwards gives $0r_0 = (X \wedge Y)l_Y$. Then we have $l_Y((X \wedge Y)l_Y) = (X \wedge Y) \wedge l_Y 1 = X \wedge Y \wedge Y = X \wedge Y$. Chasing further we get $(X \wedge Y)l_{X\vee Y} \leqslant (X)l_{X\vee Y}$ and $r_1((X)l_{X\vee Y}) = 0$ (once again by 3.2.17). Chasing 1 forwards gives $1r_0 = 1$. Then, $l_Y 1 = Y$, and next $r_1(Yl_{X\vee Y})$. But now note that:

$$
\begin{aligned}
r_1(Yl_{X\vee Y}) &= r_1((r_1(Yl_{X\vee Y}))r_1) \quad \text{since } r_1 \text{ is an outmorphism}\\
&= r_1((l_{X\vee Y}((r_1(Yl_{X\vee Y}))r_1))l_{X\vee Y}) \quad \text{since } l_{X\vee Y} \text{ is an inmorphism}\\
&= r_1((l_{X\vee Y}(Yl_{X\vee Y} \vee 0r_1))l_{X\vee Y}) \quad \text{by Axiom 2 on } r_1 \text{ and } Yl_{X\vee Y}\\
&= r_1((l_{X\vee Y}(Yl_{X\vee Y} \vee Xl_{X\vee Y}))l_{X\vee Y}) \quad \text{expanding } 0r_1\\
&= r_1((l_{X\vee Y}(Yl_{X\vee Y}) \vee l_{X\vee Y}(Xl_{X\vee Y}))l_{X\vee Y}) \quad \text{left adjoints on colimits}\\
&= r_1(((Y \wedge l_{X\vee Y}1) \vee (X \wedge l_{X\vee Y}1))l_{X\vee Y}) \quad \text{Axiom 2 twice: } X, Y \text{ with } l_{X\vee Y}\\
&= r_1(((Y \wedge (X \vee Y)) \vee (X \wedge (X \vee Y)))l_{X\vee Y}) \quad \text{since } l_{X\vee Y}1 = X \vee Y\\
&= r_1((Y \vee X)l_{X\vee Y}) \quad \text{since } X \leqslant X \vee Y \text{ and } Y \leqslant X \vee Y\\
&= r_1(1) \quad \text{by Proposition 3.2.17 again}\\
&= 1 \quad \text{since } r_1 \text{ is an outmorphism}
\end{aligned}
$$

Note that $r_1 \notin \mathcal{N}$ implies that $r_1 1 \notin \mathcal{N}_0$ by definition. Chasing 1 backwards we have $1r_1 = 1$. Then, $l_{X\vee Y}1 = X \vee Y$. Further again we get $(X \vee Y)l_Y$ and $1 = Yl_Y \leqslant (X \vee Y)l_Y$. Finally, $r_0 1 = 1$. Chasing 0 backwards we have $0r_1 = Xl_{X\vee Y}$. Then $l_{X\vee Y}(Xl_{X\vee Y}) = X \wedge l_{X\vee Y}1 = X \wedge (X \vee Y) = X$. Chasing all the way, we have

$$
\begin{aligned}
r_0(Xl_Y) &= r_{(X\wedge Y)l_Y}(Xl_Y) \quad \text{by the definition of } r_0\\
&= r_{(X\wedge Y)l_Y}(Xl_Y \wedge 1) \quad \text{since } Xl_Y \wedge 1 = Xl_Y\\
&= r_{(X\wedge Y)l_Y}(Xl_Y \wedge Yl_Y) \quad \text{since } Yl_Y = 1 \text{ by Proposition 3.2.17}\\
&= r_{(X\wedge Y)l_Y}((X \wedge Y)l_Y) \quad \text{since right adjoints preserve limits}\\
&= 0 \quad \text{again by Proposition 3.2.17}
\end{aligned}
$$

Since $r_0 \notin \mathcal{N}$, we have that $\mathcal{N}_1 \not\ni 0 \in R_{(X\wedge Y)l_Y}$. Hence, both a constructible pyramid is induced, and the resulting morphism is an isomorphism. □

Theorem 4.2.12 (Zassenhaus). *Let $S' \lhd S$ and $T' \lhd T$ all be conormal $\mathcal{A}$-subobjects of a set G. Suppose further that $(T' \wedge S) \vee S'$ and $(S' \wedge T) \vee T'$ are conormal. Then the following holds:*

i) $S' \vee (S \wedge T') \lhd S' \vee (S \wedge T)$

ii) $(S \wedge T') \vee (S' \wedge T) \lhd S \wedge T$

iii) $T' \vee (T \wedge S') \lhd T' \vee (T \wedge S)$

and there are cononical isomorphisms

$$R_{(S'\vee(S\wedge T'))l_{(S'\vee(S\wedge T))}} \simeq R_{((S\wedge T')\vee(S'\wedge T))l_{S\wedge T}} \simeq R_{(T'\vee(T\wedge S'))l_{(T'\vee(T\wedge S))}}.$$

Proof. We will show that we have $S' \vee (S \wedge T') \lhd S' \vee (S \vee T)$ and $(S \wedge T') \vee (S' \wedge T) \lhd S \wedge T$, with a canonical isomorphism

$$R_{(S' \vee (S \wedge T'))l_{(S' \vee (S \wedge T))}} \simeq R_{((S \wedge T') \vee (S' \wedge T))l_{S \wedge T}}.$$

The rest follows by symmetry of exchanging S with T and S' with T'.

Consider the morphisms, where l_S is chosen such that $l_S 1 = S$.

$$G \xleftarrow{l_S} L_S \xrightarrow{r_{S'l_S}} R_{S'l_S}$$

Since $S'l_S$ is normal, we can choose $r_{S'l_S}$ such that $S'l_S = 0r_{S'l_S}$.

By applying Lemma 4.2.10, we have $T'l_S \lhd Tl_S$, and applying it again, we have $r_{S'l_S}(T'l_S) \lhd r_{S'l_S}(Tl_S)$. By applying it a third time for the preimage, we have:

$$\begin{aligned}(r_{S'l_S}(T'l_S))r_{S'l_S} &= T'l_S \vee 0r_{S'l_S} \\ &= T'l_S \vee S'l_S \lhd (r_{S'l_S}(Tl_S))r_{S'l_S} = Tl_S \vee 0r_{S'l_S} = Tl_S \vee S'l_S\end{aligned}$$

Now,

$$T'l_S = T'l_S \wedge 1 = T'l_S \wedge Sl_S = (T' \wedge S)l_S$$

and similarly, $Tl_S = (T \wedge S)l_S$. Hence, we can rewrite the condition as

$$(T' \wedge S)l_S \vee S'l_S \lhd (T \wedge S)l_S \vee S'l_S.$$

Also,

$$\begin{aligned}(T' \wedge S)l_S \vee S'l_S &= (l_S((T' \wedge S)l_S \vee S'l_S))l_S && \text{since the image of } l_S \text{ is injective} \\ &= (l_S((T' \wedge S)l_S) \vee l_S(S'l_S))l_S && \text{left adjoints preserve colimits} \\ &= (((T' \wedge S) \wedge l_S 1) \vee (S' \wedge l_S 1))l_S && \text{by Axiom 2; } l_S, T' \wedge S, S' \\ &= ((T' \wedge S \wedge S) \vee (S' \wedge S))l_S && \text{since } l_S 1 = S \\ &= ((T' \wedge S) \vee S')l_S && \text{since } S \wedge S = S \text{ and } S \leqslant S'\end{aligned}$$

and similarly,

$$(T \wedge S)l_S \vee S'l_S = ((T \wedge S) \vee S')l_S$$

therefore we have:

$$((T' \wedge S) \vee S')l_S \lhd ((T \wedge S) \vee S')l_S.$$

We want to apply Theorem 4.2.8. Note that $0l_S = 0 \leqslant ((T' \wedge S) \vee S')l_S \leqslant ((T \wedge S) \vee S')l_S$. If $((T' \wedge S) \vee S')l_S = 0$ or $((T \wedge S) \vee S')l_S = 0$, we have $0 = (T' \wedge S) \vee S' \lhd (T \wedge S) \vee S'$. So, $l_S(((T' \wedge S) \vee S')l_S) = l_S 0 = 0$.

Else, suppose both are non-zero. Since $((T' \wedge S) \vee S')l_S$ is conormal and we have $l_S l_{((T' \wedge S) \vee S')l_S} 1 = l_S(((T' \wedge S) \vee S')l_S) = ((T' \wedge S) \vee S') \wedge S = (T' \wedge S) \vee S' \neq 0$, by

Theorem 4.2.8, we then have that:

$$\begin{aligned} l_S(((T' \wedge S) \vee S')l_S) &\lhd l_S(((T \wedge S) \vee S')l_S) \\ ((T' \wedge S) \vee S') \wedge l_S 1 &\lhd ((T \wedge S) \vee S') \wedge l_S 1 \quad \text{by Axiom 2 twice on } l_S \\ ((T' \wedge S) \vee S') \wedge S &\lhd ((T \wedge S) \vee S') \wedge S \quad \text{since } l_S 1 = S \end{aligned}$$

Now, $S' \leqslant S$ and $(T' \wedge S) \leqslant S$ implies that $(T' \wedge S) \vee S' \leqslant S$. Similarly, $(T \wedge S) \vee S' \leqslant S$. Hence

$$(T' \wedge S) \vee S' \lhd (T \wedge S) \vee S'. \tag{4.1}$$

By Theorem 3.3.9, we have that $((T' \wedge S) \vee S') \wedge (S \wedge T) = l_S((((T' \wedge S) \vee S') \wedge (S \wedge T))l_S)$ and that the image and preimage of l_S restricts to an isomorphism of lattices for $\mathcal{A}$-subobjects in the codomain less than S (and greater than $0l_S = 0$ in the domain). Now

$$\begin{aligned} &(((T' \wedge S) \vee S') \wedge (S \wedge T))l_S \\ &= ((T' \wedge S) \vee S'))l_S \wedge (S \wedge T)l_S \quad \text{right adjoints preserve limits} \\ &= ((T' \wedge S)l_S \vee (S')l_S) \wedge (Sl_S \wedge Tl_S) \quad \text{iso's of lattices preserve join} \\ &= (T'l_S \vee S'l_S) \wedge Tl_S \quad \text{since } Sl_S = 1 \\ &= T'l_S \vee (S'l_S \wedge Tl_S) \quad \text{by Theorem 3.3.1, since } T'l_S \leqslant Tl_S \\ &\qquad \text{and } S'l_S \text{ is normal and } Tl_S \text{ is conormal} \\ &\qquad \text{by Proposition 3.2.34} \\ &= (Sl_S \wedge T'l_S) \vee (S'l_S \wedge Tl_S) \quad \text{since } Sl_S = 1 \\ &= (S \wedge T')l_S \vee (S' \wedge T)l_S \quad \text{right adjoints preserve limits} \\ &= ((S \wedge T') \vee (S' \wedge T))l_S \quad \text{since } l_S \text{ restricted to a lattice isomorphism} \\ &\qquad \text{preserves join} \end{aligned}$$

Applying the image map of l_S both sides gives

$$\begin{aligned} l_S((((T' \wedge S) \vee S') \wedge (S \wedge T))l_S) &= ((T' \wedge S) \vee S') \wedge (S \wedge T) \\ &= (S \wedge T') \vee (S' \wedge T) \\ &= l_S(((S \wedge T') \vee (S' \wedge T))l_S) \end{aligned}$$

so we have

$$((T' \wedge S) \vee S') \wedge (S \wedge T) = (S \wedge T') \vee (S' \wedge T) \tag{4.2}$$

If $(S \wedge T') \vee (S' \wedge T) = 0$ we automatically have $(S \wedge T') \vee (S' \wedge T) \lhd S \wedge T$. Hence, let us assume $(S \wedge T') \vee (S' \wedge T) \neq 0$.

We want to complete the proof using Theorem 4.2.11 with $X = S' \vee (S \wedge T')$ and $Y = S \wedge T$. This means that $X \vee Y = (S' \vee (S \wedge T')) \vee (S \wedge T) = S' \vee (S \wedge$

$T')\vee(S\wedge T)=S'\vee(S\wedge T)$. As a requirement to apply the theorem we have that $((S'\vee(S\wedge T'))\wedge S\wedge T)l_{S\wedge T}\notin\mathcal{N}_1$ and $(S'\vee(S\wedge T'))l_{(S'\vee(S\wedge T'))\vee(S\wedge T)}\notin\mathcal{N}_1$. We also require that $(S'\vee(S\wedge T'))\wedge(S\wedge T)\notin\mathcal{N}_0$. Examining the first case in the positive we note $((S'\vee(S\wedge T'))\wedge S\wedge T)l_{S\wedge T}=((S\wedge T')\wedge(S\wedge T))l_{S\wedge T}\in\mathcal{N}_1$ and since all $l'_{S\wedge T}\notin\mathcal{N}$ are isomorphic and all $l'_{S\wedge T}\in\mathcal{N}$ have $((S\wedge T')\wedge(S\wedge T))l'_{S\wedge T}=1$, we have that $(S\wedge T')\vee(S'\wedge T)\lhd S\wedge T$. Examining the second part in the positive we have $(S'\vee(S\wedge T'))l_{(S'\vee(S\wedge T'))\vee(S\wedge T)}=(S'\vee(S\wedge T'))l_{S'\vee(S\wedge T)}\in\mathcal{N}_1$. Since $(S\wedge T)l_{S'\vee(S\wedge T')}=1$ and $((S'\vee(S\wedge T'))\wedge(S\wedge T))l_{S'\vee(S\wedge T')}=((S\wedge T')\vee(S'\wedge T))l_{S'\vee(S\wedge T')}\in\mathcal{N}_1$ and $(S\wedge T')\vee(S'\wedge T)\lhd S'\vee(S\wedge T')$, hence, by Proposition 4.2.2 $(S\wedge T')\vee(S'\wedge T)\lhd S\wedge T'$. Moreover, in both the previous cases the desired isomorphism holds. Thus we can assume the first requirement. Note that $(S'\vee(S\wedge T'))\wedge(S\wedge T)=(S\wedge T')\vee(S'\wedge T)\neq 0$, which gives the final requirement.

$Y=S\wedge T$ is conormal since it is the meet of two conormal $\mathcal{A}$-subobjects. Now, we have $X\vee Y=S'\vee(S\wedge T)$, moreover, $l_S((S'\vee(S\wedge T))l_S)=S'\vee(S\wedge T)$ proves that $X\vee Y$ is conormal. $X\lhd X\vee Y$ is proven above in Equation 4.1. Hence, by Theorem 4.2.11, we have that $X\wedge Y\lhd Y$, i.e.

$$(S'\vee(S\wedge T'))\wedge(S\wedge T)=(S\wedge T')\vee(S'\wedge T)\lhd S\wedge T$$

and there is a canonical isomorphism

$$R_{((S\wedge T')\vee(S'\wedge T))l_{S\wedge T}}\simeq R_{(S'\vee(S\wedge T'))l_{S'\vee(S\wedge T)}}.$$

This completes the proof. □

4.2.1 Each Isomorphism Theorem has a Dual Theorem

Since each of the preceding isomorphism theorems is formulated using self-dual properties, we can infer that for each theorem, there will also be a dual theorem which automatically holds. The substitution is by taking the op-functor. These dual theorems are beyond the scope of this thesis, but to write down what they are would be a relatively straightforward interchange of duals. It is interesting to know that they exist.

4.3 Comparison to Tholen's Approach

In this section the aim is to briefly investigate Tholen's doctoral thesis [22] and to explain the similarities and differences between the three isomorphism theorems in his work with the analogues in this thesis when the theorems are specialised to concrete sets.

The methodology is to look at the specialisation of each isomorphism theorem to sets in both this work and and Tholen's work. A visual sample diagram of a specific choice of sets and substructures is given. Then, the theorem is reformulated in the present context to more closely resemble Tholen's theorem and proven using the methodology at our disposal.

As will be shown, if we relax the definition of when a pyramid is constructible; firstly use the fact that in **Set** there is always only one outmorphism up to isomorphism; and secondly use factorisation of non-empty null morphisms (more specifically, morphisms that are not $\mathcal{Z}$-empty can all be out-in factorised), then the theorems can be recovered almost completely.

For convenience of comparison, we will use similar notation for objects and morphism as in Tholen's work.

4.3.1 Isomorphism Theorem II

Theorem 4.3.1. *Suppose we have a morphism $u\colon U \longrightarrow A$, and a morphism $c\colon A \longrightarrow C$, such that the composite $cu \notin \mathcal{N}$. Let m' be an inmorphism of $(cu1)c$ such that $m't = u$. Let $m''c'$ and mq be out-in factorisations of cm' and cu, respectively. Let $cod(q) = dom(m) = D$ and $cod(c') = dom(m'') = D'$. Then $D \simeq D'$.*

Proof.

$$\begin{tikzcd}
U \arrow[r, "u"] \arrow[rr, bend left, dashed, "t"] \arrow[d, two heads, "q"] & A \arrow[d, "c"] & P \arrow[l, hook', "m'"'] \arrow[d, two heads, "c'"] \\
D \arrow[r, hook, "m"] & C & D' \arrow[l, hook', "m''"']
\end{tikzcd}$$

The composite $cu \notin \mathcal{N}$ and hence can be factorised, say $cu = mq$. Now, $u1 \leqslant (cu1)c$ and hence we have a factorisation through an in-morphism m' of $(cu1)c$ as $m't = u$. Also, $cu = cm't \notin \mathcal{N}$ and hence $cm' \notin \mathcal{N}$ and factorises as say $cm' = m''c'$. Note that:

$$\begin{aligned}
0q &= 0mq \quad \text{since } m \text{ is an inmorphism} \\
&= 0cu \\
&= 0cm't \\
&= 0m''c't \\
&= 0c't \quad \text{since } m'' \text{ is an inmorphism}
\end{aligned}$$

Also, we have $m'1 \leqslant (cu1)c$. And hence,

$$(m'1)m' \leqslant (cu1)cm' = (cu1)m''c'.$$

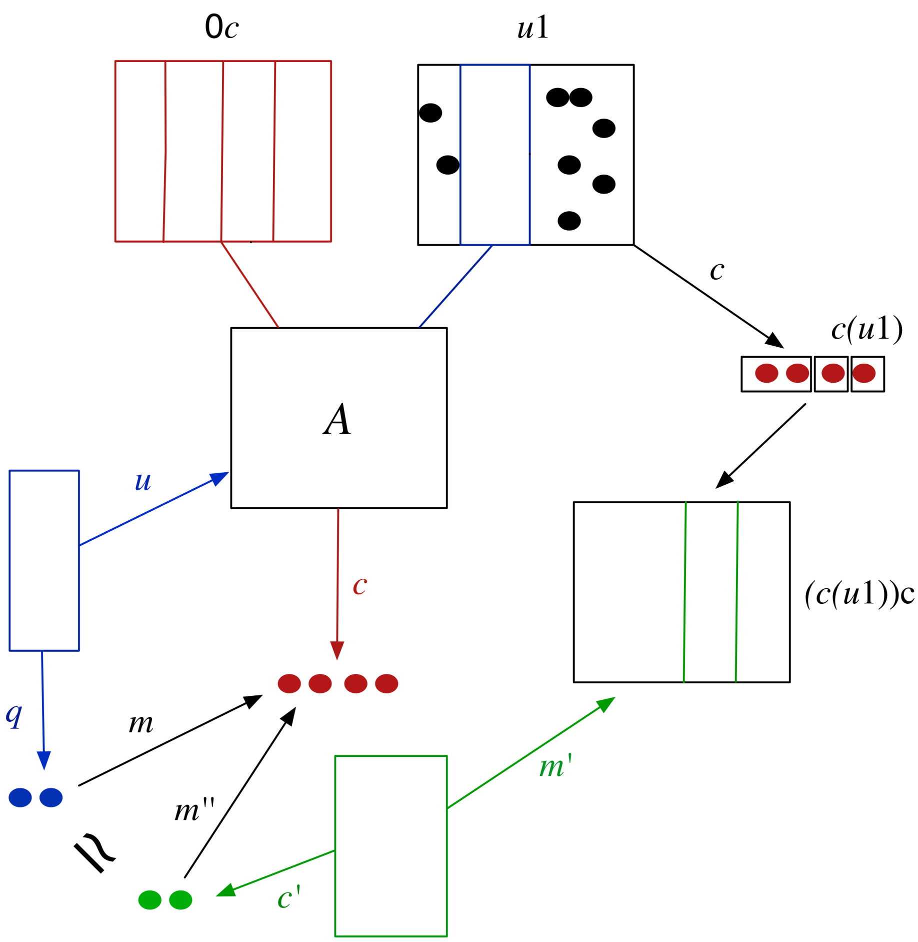


Figure 4.1: An example of the second isomorphism theorem for a given u and c. The black dots represent singleton equivalence classes in $u1 \in \Sigma A$, blocks represent sets (and so do aligned dots without borders). Blocks within blocks or around coloured dots represent equivalence classes.

Therefore,

$$\begin{aligned} c'((m'1)m') &\leqslant c'((cu1)m''c') \\ &= (cu1)m'' \quad \text{since } c' \text{ is an outmorphism} \\ &= (cm't1)m'' \\ &= (m''c't1)m'' \\ &= c't1 \quad \text{since } m'' \text{ is an inmorphism} \end{aligned}$$

But $c'((m'1)m') = c'1$ since m' is an inmorphism and $c'1 = 1$ since c' is an outmorphism. Hence, $c't1 = 1$.

Consider now the zigzag:

$$D \xleftarrow{q} U \xrightarrow{t} P \xrightarrow{c'} D'$$

Chasing 1 forwards, we have proven above that $c't(1q) = c't1 = 1$. Chasing 0 forwards, we have $c't(0q) = c't(0c't) = 0 \wedge c't1 = 0$. Then, chasing 1 backwards, we have $q(1c't) = q1 = 1$. Finally, chasing 0 backwards, we have $q(0c't) = q(0q) = 0 \wedge q1 = 0$. Hence, by chasing 0 and 1 both forwards and backwards, we have that $D \simeq D'$.

Since $cu \notin \mathcal{N}$, we have that $q \notin \mathcal{N}$, and hence that $\Sigma D \ni 0 \notin \mathcal{N}_1$. □

We have now shown that we can recover Tholen's isomorphism theorem II in the specialisation to sets, provided that the composite cu is not null.

4.3.1.1 Adding the Non-Empty Null Case

If we now use the fact that **Set** in fact has out-in factorisation of not $\mathcal{Z}$-empty null morphisms and each $\mathcal{A}$-subobject has only one outmorphism up to isomorphism, we can prove Tholen's theorem almost completely, except for the case where u is $\mathcal{Z}$-empty.

We will not use the previous convention in Definition 4.1.5. Rather, we define diamonds of Type 3 to now always be constructible, and we explicitly construct the pyramid and then we show that it induces an isomorphism.

Definition 4.3.2. We define a diagram of Type 3 to always be constructible if the setting has only one outmorphism (up to isomorphism) for each $\mathcal{A}$-subobject.

Theorem 4.3.3. *Suppose that we work in the following setting.*

1. *The family of outmorphisms for each $\mathcal{A}$-subobject consists of only one morphism (up to isomorphism).*

2. *A composite of any two outmorphisms is also an outmorphism (i.e., Corollary 3.2.31 then holds in the $\mathcal{N}$-null case as well).*

3. *Finally, if a morphism is not out-in factorisable, then its composite with another morphism is also not out-in factorisable.*

Let us now suppose we have a morphism $u\colon U \longrightarrow A$ and a morphism $c\colon A \longrightarrow C$ where cu is out-in factorisable. Then, as is Theorem 4.3.1, we have that $D \simeq D'$.

Proof. The proof is the same as in Theorem 4.3.1, except that we need to explicitly construct the pyramid.

$$\begin{tikzcd}
 & & X_3 \simeq X_3' & & \\
 & X_1 \arrow[ur, two heads, "e_3"] \arrow[dl, hook, "m_1"'] & & X_2 \arrow[ul, two heads, "e_4"'] \arrow[dr, hook, "m_2"] & \\
D & & U \arrow[ll, two heads, "q"] \arrow[ul, two heads, "e_1"'] \arrow[ur, two heads, "e_2"] \arrow[r, dashed, "t"'] & P \arrow[r, two heads, "c'"'] & D'
\end{tikzcd}$$

Firstly, m_1e_1 and m_2e_2 are out-in factorisations of q and $c't$, respectively. Now, note that unlike in our previous definition of constructibility, we now have that the diamond centred at e_1 and e_2 is constructible. We construct it as follows. Let e_3 be the outmorphism of $e_1(0e_2)$, which is normal by Proposition 3.2.34 and hence $0e_3 = e_1(0e_2)$. Then, $0e_3e_1 = (e_1(0e_2))e_1 = 0e_2 \vee 0e_1$. And since $0e_2 \leqslant 0e_2 \vee 0e_1$ we then have that e_3e_1 factors through e_2 as $e_3e_1 = qe_2$. But if we define e_4 to be the outmorphism of $e_2(0e_1)$, then we will have $0e_4e_2 = 0e_2 \vee 0e_1 = 0e_3e_1$. Since composites of outmorphisms are outmorphisms, we have that e_3e_1 and e_4e_2 are outmorphisms (of the same $\mathcal{A}$-subobject) and hence are isomorphic.

Chasing top and bottom elements along the zigzag will now be the same as in Theorem 4.3.1, which completes the proof. □

4.3.1.2 Adding the Empty Case

Now, when we assumed that cu is out-in factorisable, we essentially assumed that for the concrete set case, that if $U = \varnothing$, then $A = \varnothing$, excluding $\mathcal{Z}$-empty functions. Or put differently, we assumed that u is an inmorphism and that c is an outmorphism (if u is already a monomorphism and c already an epimorphism). This exactly excludes $\mathcal{Z}$-empty functions, which are functions of the form $f\colon \varnothing \longrightarrow A$ where $A \neq \varnothing$. However, Theorem 4.3.3 can be even further adapted to hold even when cu is not factorisable. For the purposes of this theorem, we can assume that u is a monomorphism since we can always replace u with a monomorphism with the same image, and dually that c is an epimorphism by replacing it with an epimorphism with the same kernel.

Theorem 4.3.4. *Suppose that we work in a setting with the next requirements.*

1. *Each $\mathcal{A}$-subobject has only one outmorphism up to isomorphism.*

2. *A composite of any two outmorphisms is an outmorphism.*

3. *If a morphism is not out-in factorisable, then its composite with another morphism is also not out-in factorisable.*

Let us now suppose we have a morphism $u\colon U \longrightarrow A$ and a morphism $c\colon A \longrightarrow C$. Let q be the outmorphism of $0cu$. Hence, we have a factorisation through q, $cu = qm$.

4. *Suppose we have that $(P, m', \bar{m})$ is the pullback of c and m.*

Let c' be the outmorphism of $0cm'$. Then, if $cod(c') = D'$ and $cod(q) = D$, we have that $D \simeq D'$.

Proof.

$$\begin{tikzcd}
U \arrow[r, "u"] \arrow[d, two heads, "q"] \arrow[rr, dashed, bend left, "t"] & A \arrow[d, "c"] & P \arrow[l, "m'"'] \arrow[d, two heads, "c'"] \arrow[ll, dl, "\bar{m}"] \\
D \arrow[r, dashed, "m"] & C & D' \arrow[l, dashed, "m''"']
\end{tikzcd}$$

Let q be the outmorphism of $0cu$. And since $0cu$ is normal, we have $0q = 0cu$. Similarly, let c' be the outmorphism of $0cm'$. Then $0c' = 0cm'$ and we have an out-in factorisation, $m''c' = cm'$.

Since q is an outmorphism, we have that the inverse image map is injective, that is, $0q = 0mq \Rightarrow 0 = 0m$. Similarly, $0m''c' = 0c' \Rightarrow 0m'' = 0$.

Then

$$\begin{aligned} 0q &= 0cu \\ &= 0cm't \\ &= 0c't. \end{aligned}$$

Since we have a pullback, and $mq = cu$, we have t such that $\bar{m}t = q$ and $m't = u$.

Now, $m'1 \leqslant (cm'1)c = (m\bar{m}1)c \leqslant (m1)c = (mq1)c = (cu1)c$. Again, as in Theorem 4.3.1,

$$
\begin{aligned}
c'((m'1)m') \leqslant\ & c'((cu1)m''c') \\
=\ & (cu1)m'' \quad \text{since } c' \text{ is an outmorphism} \\
=\ & (cm't1)m'' \\
=\ & (m''c't1)m'' \\
=\ & c't1 \vee 0m'' \\
=\ & c't1 \vee 0 \quad \text{since } 0m'' = 0 \text{ as shown above} \\
=\ & c't1
\end{aligned}
$$

But $c'((m'1)m') = c'(1 \vee 0m') = c'1$ and $c'1 = 1$ since c' is an outmorphism. Hence, $c't1 = 1$. □

This theorem with the stated additional requirements on our abstract setting then also holds for the specialisation to **Set** for Tholen's isomorphism theorem II. In other words, what we have done is devise a new theorem which captures Tholen's theorem in the **Set** case. All the assumptions that we have on the theorem are assumptions that are valid in **Set**.

4.3.1.3 A Note on Duality for Sets

It is interesting to note that in order for the isomorphism theorem to hold for the $\mathcal{N}$-null and empty cases, one needs to break the duality inherent to the theory. In **Set** many $\mathcal{A}$-subobjects have more than one inmorphism. Dually, in the general case we could have that there is a theory that satisfies the axioms for which an $\mathcal{A}$-subobject could have many outmorphisms. The requirement that each set has only one outmorphism cannot be replaced by for example requiring that each $\mathcal{A}$-subobject is normal. Although a normal $\mathcal{A}$-subobject S has only one outmorphism r up to isomorphism such that $0r = S$, there is a problem in the null case (i.e., $S = 1$) that there may be another non-isomorphic r_2 such that $0r_2 = S$. To see a concrete example, consider the dual situation. For an $\mathcal{A}$-subobject $S = 0$ of a concrete set, there may be many inmorphisms f_i such that $f_i1 = 0$. Let $X = \{0, 1\}$ and S be the smallest equivalence relation on X. Then we have inmorphisms $f_01 = 0$ and $f_11 = 0$ where $f_0\colon \{0\} \longrightarrow \{0, 1\}$ and $f_1\colon \{0\} \longrightarrow \{0, 1\}$ are defined on the point as $f_0(0) = 0$ and $f_1(0) = 1$, the elements of $\{0, 1\}$. They have different concrete images, but their abstract images coincide. This phenomenon only occurs in the null case. The result of this is that for some function $g\colon A \longrightarrow \{0, 1\}$ with the abstract image $g1 = 0$ (i.e., g is a constant function), we don't know a priori whether g factors through f_0 or f_1. Hence, in the abstract setting where we don't work with concrete images, we will see f_0 and f_1 as two non-isomorphic inmorphisms, but we can only observe their different properties as different ways in which functions factor through them.

Hence, as is evident in Axiom 3, one of the main insights into projective set theory is to distinguish between factorisation properties and image or kernel properties. Unlike group theory, in set theory we cannot for example equate two monomorphisms when their abstract images are equal. In this sense, projective set theory is a *point free* theory in that the smallest equivalence relation on a set isolates each point, yet it does not separate the points individually from each other.

4.3.2 Isomorphism Theorem III

The version of isomorphism theorem III in Tholen's work coincides with that of this text in Section 4.2, with the exception that we have the requirement that (in the notation of Corollary 4.2.9) $N \notin \mathcal{N}_1$, $X \notin \mathcal{N}_1$ and $X \notin \mathcal{N}_0$.

We can relax the condition that $X \notin \mathcal{N}_0$ and still prove the theorem in this paper's context.

Theorem 4.3.5. *Let π_k and π_n be outmorphisms with domain A such that $0\pi_k \leqslant 0\pi_n$. Assume that $0\pi_n \notin \mathcal{N}_1$. We have that for some outmorphism of $\pi_k(0\pi_n)$, $0r_{\pi_k(0\pi_n)} = \pi_k(0\pi_n)$ since $\pi_k(0\pi_n)$ is normal. Moreover, cod$(\pi_n) \simeq R_{\pi_k(0\pi_n)}$.*

Proof.

$$D \xtwoheadleftarrow{r_{\pi_k(0\pi_n)}} B \xtwoheadleftarrow{\pi_k} A \xtwoheadrightarrow{\pi_n} C$$

Chasing 0 forwards, we have $r_{\pi_k(0\pi_n)}(\pi_k(0\pi_n)) = 0$ by Proposition 3.2.17 since we chase the group along its outmorphism. Chasing 1 forwards all the way, we have $r_{\pi_k(0\pi_n)}\pi_k(1\pi_n) = r_{\pi_k(0\pi_n)}\pi_k(1) = r_{\pi_k(0\pi_n)}1 = 1$.

Then, chasing 0 backwards, we have

$$\begin{aligned}
\pi_n((0r_{\pi_k(0\pi_n)})\pi_k) &= \pi_n((\pi_k(0\pi_n))\pi_k) \quad \text{by normality} \\
&= \pi_n(0\pi_n \vee 0\pi_k) \quad \text{by Axiom 2} \\
&= \pi_n(0\pi_n) \quad \text{since } \pi_k \leqslant \pi_n \\
&= 0 \wedge \pi_n 1 = 0
\end{aligned}$$

Finally, chasing 1 backwards, we have $\pi_n((1r_{\pi_k(0\pi_n)})\pi_k) = \pi_n(1\pi_k) = \pi_n 1 = 1$. □

4.3.2.1 Adding the Null and Empty Cases

As with the previous isomorphism theorem, we can again relax our criteria for constructibility which will again fully recover Tholen's specialisation to **Set**, including the empty case.

Theorem 4.3.6. *Suppose again that we work in a setting with the following requirements.*

1. *Each $\mathcal{A}$-subobject has only one outmorphism up to isomorphism.*

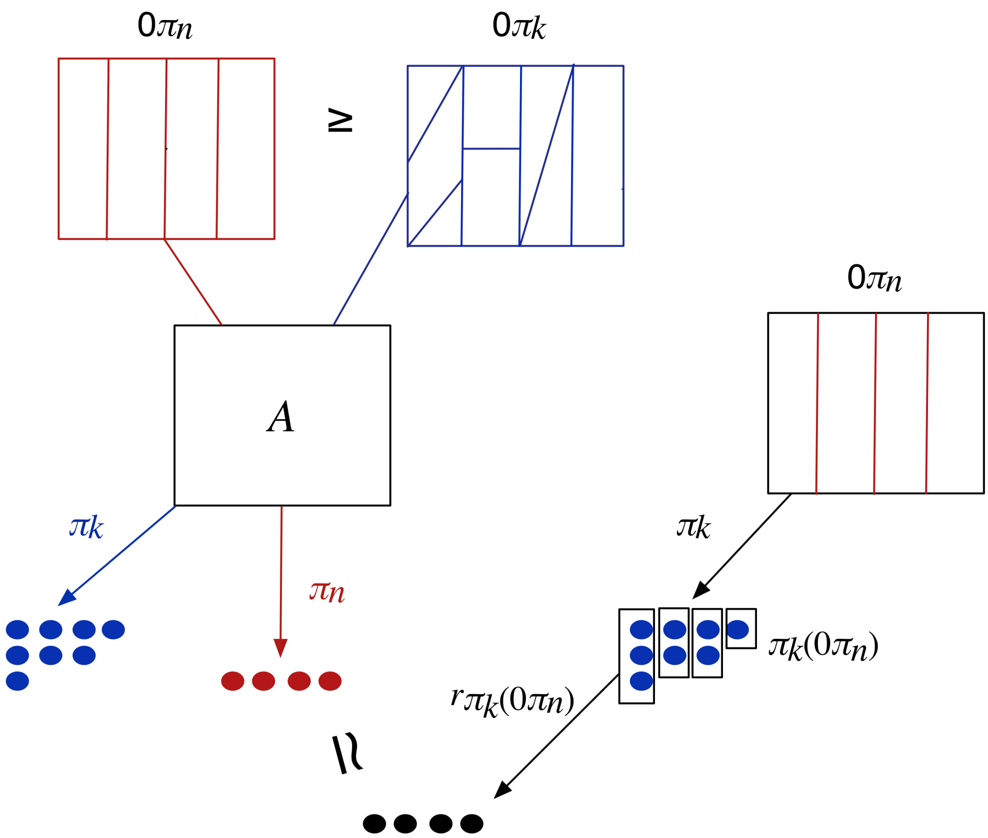


Figure 4.2: An example of the third isomorphism theorem for a given π_n and π_k. Collections of dots represent sets with elements. Borders around dots represent equivalence classes. Blocks are sets, and blocks within them are equivalence classes.

2. *Composites of outmorphisms are outmorphisms.*
3. *Outmorphisms are out-in factorisable.*

Now, let π_k and π_n be outmorphisms with domain A such that $0\pi_k \leqslant 0\pi_n$. Then, we have that $0r_{\pi_k(0\pi_n)} = \pi_k(0\pi_n)$, since $\pi_k(0\pi_n)$ is normal. Moreover, let $C = cod(\pi_n)$ and $R_{\pi_k(0\pi_n)} = D$. Then $C \simeq D$.

Proof. Again, the proof is the same as for Theorem 4.3.5, but we again need to show that the pyramid is constructible.

$$\begin{tikzcd}
 & & X_3 & & \\
 & X_1 \arrow[ur, two heads, "e_3"] \arrow[dl, hook, "m_1"'] & & X_2 \arrow[ul, two heads, "e_4"'] \arrow[dr, hook, "m_2"] & \\
D & B \arrow[l, two heads, "r_{\pi_k(0\pi_n)}"] & A \arrow[l, two heads, "\pi_k"] \arrow[ul, two heads, "e_1"'] \arrow[ur, two heads, "e_2"] \arrow[rr, two heads, "\pi_n"'] & & C
\end{tikzcd}$$

The construction of the diamond is the same method as done in Theorem 4.3.1. Then, chasing $\mathcal{A}$-subobjects is the same as in Theorem 4.3.5. □

This last theorem also covers the empty case: if either π_k or π_n are empty functions, then $A = \emptyset$, and since they are outmorphisms, $B = C = \emptyset$ and hence $D = \emptyset$. The rest of the sets in the pyramid will also be empty. The reason why Lemma 4.1.15 still holds is that for sets, $\emptyset \longrightarrow A$ with A non-empty (i.e., a $\mathcal{Z}$-empty morphism) is not out-in factorisable. Moreover, it is neither an inmorphism nor an outmorphism and hence cannot occur as a morphism on an outer layer of the pyramid during construction. On the other hand, $\emptyset \longrightarrow \emptyset$ is both an inmorphism and an outmorphism and is out-in factorisable.

4.3.3 Zassenhaus's Lemma

Firstly, we redraw Tholen's diagram to fit our setting.

$$\begin{tikzcd}
R_{0\pi_k} & & A & & R_{0\pi_n} \\
U \arrow[u, two heads, "\pi_k"] \arrow[urr, hook, "l_u"] & & & & V \arrow[u, two heads, "\pi_n"'] \arrow[ull, hook, "l_v"'] \\
 & & U \wedge V \arrow[uu, hook, "l_{u \vee v}"] \arrow[ull, dashed, hook, "v'"] \arrow[urr, dashed, hook, "u'"'] \arrow[dll, dashed, hook, "x"'] \arrow[drr, dashed, hook, "y"] \arrow[dd, two heads, "r_{Q_2}"] & & \\
L_{0\pi_k \vee v'1} \arrow[uu, hook, "i"] \arrow[d, two heads, "r_{Q_1}"] & & & & L_{0\pi_n \vee u'1} \arrow[uu, hook, "j"'] \arrow[d, two heads, "r_{Q_3}"] \\
R_{Q_1} & & R_{Q_2} & & R_{Q_3}
\end{tikzcd}$$

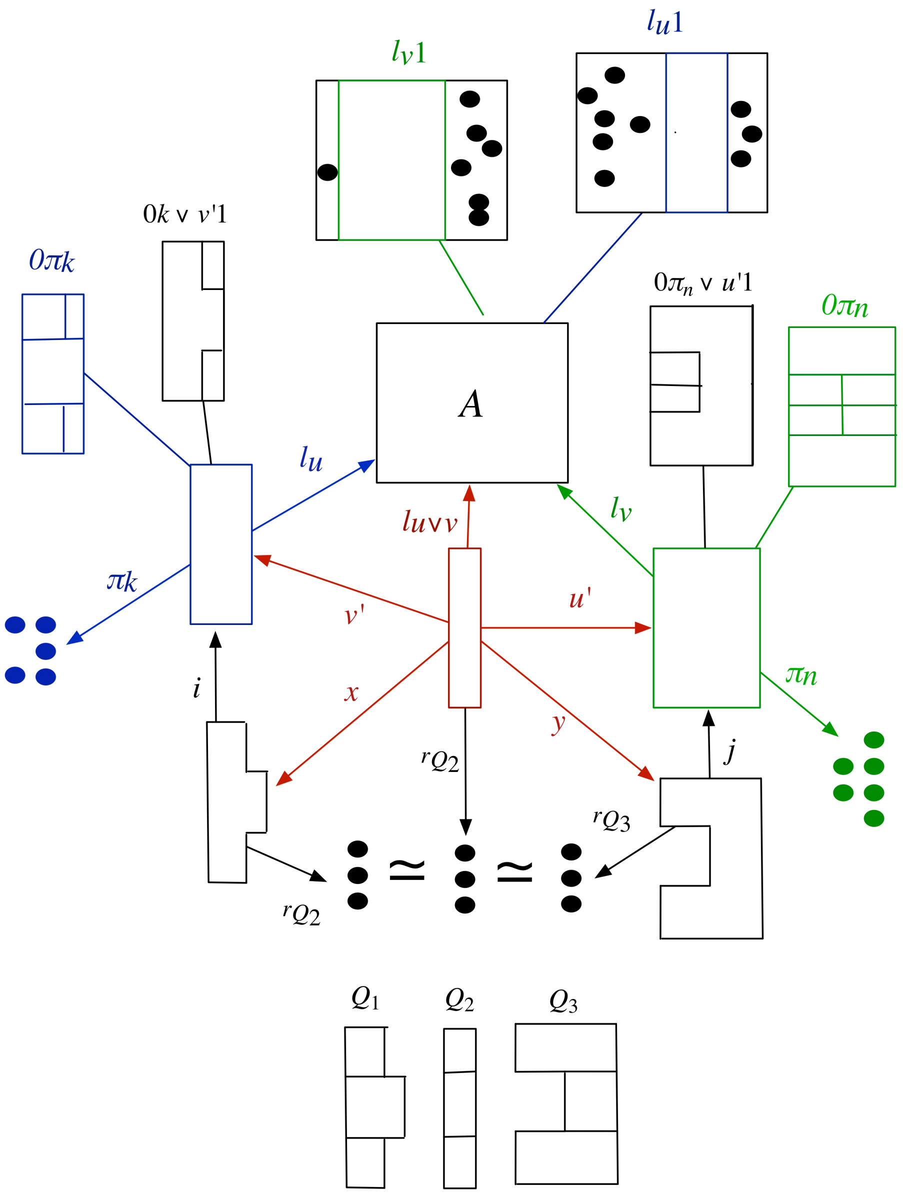


Figure 4.3: An example of the Zassenhaus lemma for a given l_u, l_v, π_n and π_k. Dots outside borders represent points within sets. Dots within blocks represent singleton equivalence classes. Blocks are sets, and the blocks within are equivalence classes.

We start out with two inmorphisms, l_u and l_v with $l_u 1 = u$ and $l_v 1 = v$, together with two outmorphisms, π_k and π_n. Since u and v are conormal, we have that $u \wedge v$ is conormal by Axiom 5, and hence has an inmorphism $l_{u\wedge v}$ with $l_{u\wedge v}1 = u \wedge v$. Assuming that $u \wedge v \neq 0$ and since $u \wedge v \leqslant v$ and $u \wedge v \leqslant v$, this induces the factorisations $l_u v' = l_{u\wedge v}$ and $l_v u' = l_{u\wedge v}$. The sets U, V and $U \vee V$ are L_u, L_v and $L_{u\vee v}$, respectively.

Now, $v'1 \leqslant 0\pi_k \vee v'1$ and $u'1 \leqslant 0\pi_n \vee u'1$. This induces factorisations through some inmorphisms $l_{0\pi_k \vee v'1} = i$ and $l_{0\pi_n \vee u'1} = j$ as $ix = v'$ and $jy = u'$.

Finally, we will assign the $\mathcal{A}$-subobjects $Q_1 \in \Sigma L_{0\pi_k \vee v'1}$, $Q_2 \in \Sigma(U \wedge V)$ and $Q_3 \in \Sigma L_{0\pi_n \vee u'1}$. They are computed as follows by a direct analogue of Tholen's case. Let $0\pi_k = K$ and $0\pi_n = N$.

$$\begin{aligned}
Q_1 &= (0\pi_k) l_{0\pi_k \vee v'1} \vee x(0\pi_n u') &&= (K)i \vee x((N)u') \\
Q_2 &= 0\pi_k v' \vee 0\pi_n u' &&= (K)v' \vee (N)u' \\
Q_3 &= (0\pi_n) l_{0\pi_n \vee u'1} \vee y(0\pi_k v') &&= (N)j \vee y((K)v')
\end{aligned}$$

In order for r_{Q_1} and r_{Q_3} to be defined as expected (i.e., unique up to isomorphism) we need to assume that Q_1 and Q_3 are normal, which can be done by assuming that $x(Nu')$ and $y(Kv')$ are normal. Note that Q_2 is normal since it is the join of two normal $\mathcal{A}$-subobjects.

Theorem 4.3.7. *Consider the zigzag:*

$$R_{Q_1} \xleftarrow{r_{Q_1}} L_{K\vee v'1} \overset{x}{\longleftarrow\!\!-\!\!\!\!\longrightarrow} U \vee V \xrightarrow{r_{Q_2}} R_{Q_2}$$

If the pyramid is constructible, then we have the isomorphisms, $R_{Q_1} \simeq R_{Q_2} \simeq R_{Q_3}$.

Proof. As before, we need to chase $\mathcal{A}$-subobjects. Firstly, we need a preliminary result.

$$\begin{aligned}
x((x(Nu') \vee Ki)x) &= (x(Nu') \vee Ki) \wedge x1 \\
&= x(Nu') \vee (Ki \wedge x1) \quad \text{by } x(Nu') \leqslant x1;\ Ki \text{ normal, } x1 \text{ conormal} \\
&= x(Nu') \vee x(Kix) \quad \text{since } x(Kix) = Ki \wedge x1
\end{aligned}$$

By applying the inverse image map under x we then have:

$$\begin{aligned}
(x(Nu') \vee x(Kix))x &= (x((x(Nu') \vee Ki)x))x \quad \text{by substituting the above result} \\
&= (x(Nu') \vee Ki)x \quad \text{since inmorphism image map is injective}
\end{aligned}$$

Now, chasing 0 forwards we have $0r_{Q_1} = Ki \vee x(Nu')$. Then

$$\begin{aligned}
(Ki \vee x(Nu'))x &= (x(Nu') \vee x(Kix))x \\
&= (x(Nu' \vee Kix))x \quad \text{since left adjoints preserve colimits} \\
&= Nu' \vee Kix \quad \text{since } x \text{ is an inmorphism} \\
&= Nu' \vee Kv' \quad \text{since } ix = Kv'
\end{aligned}$$

Then, finally, $r_{Q_2}(Kv' \vee Nu') = 0$. We need to chase 1 backwards now to ensure the induced morphism. Again, we'll first compute a preliminary result.

$$\begin{aligned}
(r_{Q_1}x1)r_{Q_1} &= x1 \vee (Ki \vee x(Nu')) \quad \text{by Axiom 2 with } x1 \text{ under } r_{Q_1} \\
&= x1 \vee Ki \quad \text{since } x1 \vee x(Nu') = x1 \\
&= (i(x1 \vee Ki))i \quad \text{since } i \text{ inmorphism implies image map injective} \\
&= (ix1 \vee i(Ki))i \quad \text{since left adjoints preserve colimits} \\
&= (ix1 \vee (K \wedge i1))i \quad \text{by Axiom 2 with } K \text{ under } i \\
&= ((ix1 \vee K) \wedge i1)i \quad \text{since } ix1 \leqslant i1; K \text{ is normal } i1 \text{ is conormal} \\
&= (i((ix1 \vee K)i))i \quad \text{by Axiom 2 with } ix1 \vee K \text{ under } i \\
&= (ix1 \vee K)i \quad \text{since the image map of } i \text{ is injective} \\
&= (v'1 \vee K)i \quad \text{since } ix = v' \\
&= 1 \quad \text{since } i \text{ is an inmorphism of } v'1 \vee K
\end{aligned}$$

Using this result, we have $1 = r_{Q_1}1 = r_{Q_1}((r_{Q_1}x1)r_{Q_1}) = r_{Q_1}x1$, the last equality since r_{Q_1} is an outmorphism and the inverse image map is injective and finally, $1 = r_{Q_1}x1 = r_{Q_1}x(1r_{Q_2})$. This shows that 1 chases back to 1. Hence (since we assumed constructibility of the pyramid), we have an induced morphism, $R_{Q_1} \longrightarrow R_{Q_2}$.

Chasing 1 forwards we have $r_{Q_2}((1)r_{Q_1}x) = r_{Q_2})1 = 1$. Chasing 0 backwards, we have $0r_{Q_2} = Kv' \vee Nu'$. Then, since $Kix = Kv'$, $x(Kix \vee Nu') = x(Kix) \vee x(Nu')$. Now, since $x(Kix) \vee x(Nu') \leqslant Ki \vee x(Nu')$, $r_{Q_1}(Ki \vee x(Nu')) = 0 \Rightarrow r_{Q_1}(x(Kix) \vee x(Nu')) = 0$. Hence, we have $R_{Q_1} \simeq R_{Q_2}$ and by symmetry of the argument we can relabel to get $R_{Q_2} \simeq R_{Q_3}$. □

4.3.3.1 Concerning the Null and Empty Cases

When we assumed that the pyramid is constructible we essentially assume that none of r_{Q_1}, r_{Q_2} or x are $\mathcal{N}$-null or forms part of an $\mathcal{N}$-null expansion. (By symmetry we also assume that for r_{Q_3} and y).

We can again attempt to modify our constructibility criteria as before to accommodate what happens in the **Set** case. If $u \wedge v = 0$, the smallest equivalence relation, then there is a possibility that $l_{u\wedge v}$ does not factorise through both l_u and l_v, that is, suppose that if $l_{u\wedge v} = l_u v'$, then a for a different $l'_{u\wedge v} = l_v u'$.

Translating to the language of concrete set theory in Tholen's setting, this will either force the rest of the diagram's objects to be all singletons, or all empty. The former occurs when the concrete images U of l_u and V l_v intersect at a single point, $U \cap V = \bullet \simeq \{\varnothing\}$. In this case, we can choose $l_{u\wedge v} \colon \bullet \longrightarrow A$ amongst all the inmorphisms such that $l_{u\wedge v}$ factors through both l_u and l_v and then selects that point. The latter occurs when $U \cap V = \varnothing$ and then the function selecting the concrete image is the empty function $\varnothing \longrightarrow A$. Since empty functions are not inmorphisms if $A \neq \emptyset$; that is, when they are $\mathcal{Z}$-empty, then we cannot choose $l_{u\wedge v}$ to be $\mathcal{Z}$-empty. However, $u \wedge v = 0$ will have inmorphisms, but in this case it will be impossible to choose one such that it factors through both l_u and l_v since their concrete images are disjoint.

Barring the case where $U \cap V = \emptyset$, one can still complete the diagram for the former (singleton) case.

Theorem 4.3.8. *Suppose that*

1. *Each $\mathcal{A}$-subobject has only one outmorphism up to isomorphism.*
2. *Outmorphisms are out-in factorisable.*
3. *The composite $r_{Q_2}x$ is out-in factorisable.*
4. *$l_{u\wedge v}$ factors through both l_u and l_v.*

Then we have that $R_{Q_1} \simeq R_{Q_2} \simeq R_{Q_3}$.

Proof.

$$\begin{tikzcd}
 & & X_3 & & \\
 & X_1 \arrow[ur, two heads, "e_3"] \arrow[dl, hook, "m_1"'] & & X_2 \arrow[ul, two heads, "e_4"'] \arrow[dr, hook, "m_2"] & \\
R_{Q_1} & & L_{K\vee v'1} \arrow[ll, two heads, "r_{Q_1}"] \arrow[ul, two heads, "e_1"'] \arrow[ur, two heads, "e_2"] \arrow[r, hook, dashed, "x"'] & U \vee V \arrow[r, two heads, "r_{Q_2}"'] & R_{Q_2}
\end{tikzcd}$$

The pyramid is directly constructible by using the additional assumptions: the out-in factorisations on the first level are due to an outmorphism and $r_{Q_2}x$ respectively being factorisable, e_3 and e_4 are chosen uniquely since any $\mathcal{A}$-subobjects has only one outmorphism; finally, since the pyramid is directly constructible, by Theorem 4.3.7 we have the desired isomorphism. $\square$

4.3.3.2 The Empty Case

If $r_{Q_2}x$ is not out-in factorisable for concrete sets, then necessarily $L_{K\vee v'1} = \emptyset$, which in turn means that $U \cap V = \emptyset$, where U is the concrete image of l_u and V the concrete image of l_v. But then $\emptyset \longrightarrow A$ is not an inmorphism, unless also $A = \emptyset$, in which case all objects in the Zassenhaus diagram are empty, and then the previous Theorem 4.3.8 holds. Hence, if $r_{Q_2}x$ is not out-in factorisable, then the function $\emptyset \longrightarrow A$ is exactly $\mathcal{Z}$-empty. Put differently, $U \cap V = \emptyset$ and $A \neq \emptyset$ is exactly what happens in Tholen's case when there is no $l_{u\wedge v}$ which factorises through both l_u and l_v.

But (arguing in concrete sets), if we let $l_{u\wedge v}$ be the $\mathcal{Z}$-empty map instead of an inmorphism, then we don't know which i and j to use, since any empty map in

Set factors through all inmorphisms. However, if we just let i and j be any of the inmorphisms of $0\pi_k \vee v'1$ and $0\pi_n \vee u'1$, respectively, we will have $Q_1 = (0\pi_k)i = (0\pi_k)l_{0\pi_k} = 1$, hence R_{Q_1} collapses everything, i.e., $R_{Q_1} = \{\emptyset\}$ or, in the case that $U = \emptyset$, $R_{Q_1} = \emptyset$. However, since $U \cap V = \emptyset$, we necessarily have that $R_{Q_2} = \emptyset$.

Hence we conclude that if $U \neq \emptyset$, where i and j are any inmorphisms, then the desired isomorphism theorem fails, since $\{\emptyset\} \neq \emptyset$; hence, choosing i and j to be arbitrary inmorphisms is not a good approach. Yet, we know from Tholen's specialisation to sets that his isomorphism theorem holds even in the empty case, by choosing suitable i and j. Upon careful examination of Tholen's Zassenhaus lemma, it is apparent how this is possible. Although in our comparison we formulate the *meaning* to be the same as for Tholen, the mechanisms to our disposal are different and in the empty case cease to be the same—we work with inmorphisms rather than monomorphisms. Since Tholen works with concrete images, he defines i as $\pi_k^{-1}\pi_k(v')$, where v' is seen as a subobject, i.e., a monomorphism. Hence we have i as the result of $i = \pi_k^{-1}\pi_k(\emptyset) = \emptyset$, the empty map, which is a monomorphism but not an inmorphism exactly when it is also $\mathcal{Z}$-empty; namely, when $U \neq \emptyset$. To remedy the discrepancy one could work with **monomorphisms rather than inmorphisms** and again relax the definition of constructibility. See the concluding Section 4.4.3, Point 3, for a note on this.

In summary, it is not clear how to formulate the Zassenhaus lemma (with inmorphisms) to handle all the empty cases for projective set theory, in particular when $U \cap V = \emptyset$, but $A \neq \emptyset$ and the only option for $l_{u\vee v}$ is $\mathcal{Z}$-empty. Specifically, **one cannot choose suitable** i **and** j **which are at the same time inmorphisms**. There are not clear alternative maps to assign to i and j in this case. Of course one may introduce and take concrete direct images, as Tholen does, and select exactly $i = \pi_k^{-1}\pi_k(v')$ and $j = \pi_n^{-1}\pi_n(u')$ but this then interferes with the idea of working with only one Galois connection. The general observation is that handling null and empty cases for sets in the three isomorphism theorems above breaks the self-dual nature of the projective set theory setting.

This is meaningful for further work and applications especially from a conceptual point of view. We have a set theory setting where we can argue about any two or more points, but not isolate a single point. At the level of $\mathcal{A}$-subobjects we cannot differentiate between two points, yet we can uniquely identify a single point as a constant function which factors through two non-null functions with a singleton common in their concrete image. Hence the distinction between factorisation properties and image or kernel properties is important.

4.4 Final Remarks

4.4.1 Duality on the Underlying Form

As we saw, the setting in which we work in this thesis can be seen to study a form $G\colon \mathcal{C} \longrightarrow \mathbf{Gal}$, where $\mathbf{Gal}$ is the category of posets with Galois connections between them. The self-dual property of each of the axioms translates into self-duality when switching between G and G^{op}. Let us give an example. Suppose that the first part of Axiom 5 holds. Then, when $X = 0f$ and $Y = 0g$ for some f and g we have $X \vee Y = 0h$ for some h. In other words, the right adjoints determined by the morphisms f and g map the smallest elements of the lattices respectively to X and Y. If we require this property in G^{op} and translate it to the context of G, then it means that the left adjoints of some f' and g' map the largest element of the lattices to some X' and Y' (i.e., the *opposite* Galois connection), that is, $X' = f'1$ and $Y' = g'1$. Since $X \vee Y = 0h$ holds in F, this implies then that $X' \wedge Y' = h'1$ holds, i.e., the same property in G^{op}, but translated back to G. Hence, the functorial dual of the first part of Axiom 5 is exactly the second part of Axiom 5. Therefore, Axiom 5 is functorially self-dual. This type of functorial self-duality has been studied by Janelidze and Weighill [14, 15] and these sources can be consulted for some historical notes on the subject.

In Chapter 2 we started to develop the idea of substructures determined by a functor. In the form setting the idea manifests in the following way: an $\mathcal{A}$-subobject of an abstract set X is an element of the lattice ΣX and furthermore, $\Sigma X = G(X)$.

Furthermore, in the case of concrete sets, which we have shown to satisfy the axioms, the functor G is the outfunctor construction of the quotient object bifibration in the category of sets, but with the order of the posets reversed. This is the same as the op-functor of the outfunctor construction of the subobject bifibration in $\mathbf{Set}^{op}$. Note that $\mathbf{Gal}^{op} \simeq \mathbf{Gal}$, by reversing the order of posets and switching right and left adjoints.

4.4.2 Chain of Adjunctions

The form structure in our context also leads to a series of generalised adjunctions:

$$\begin{tikzcd}[row sep=huge]
\mathcal{E} \arrow[d, bend right=70, "C"'] \arrow[d, leftarrow, bend right=30, "I"'] \arrow[d, ""] \arrow[d, leftarrow, bend left=30, "F"'] \arrow[d, bend left=70, "S"', "D"]
\\
\mathcal{C}
\end{tikzcd}$$

$$C \dashv I \dashv F \dashv S \dashv D$$

However, unlike in [8, pp. 2–3, p. 25] and [23, p. 10], D and C are not functors, but relations between categories. The top category $\mathcal{E}$ is the category of pairs consisting of sets with equivalence relations on them (i.e., F is the quotient object bifibration) where F is the projection of a pair $(X, R) \mapsto X$. In the abstract axiomatic setting F is thus the form sending $\mathcal{A}$-subobjects in ΣX to X. The functor I selects for a set X, the smallest equivalence relation on it, $(X, 0)$, and the functor S selects for a set X the largest equivalence relation on it, $(X, 1)$. Correspondingly for the abstract setting, I selects for X the bottom element of the poset $0 \in \Sigma X$ and S selects the top element $1 \in \Sigma X$. For a pair (X, R), C relates it to the codomains of all outmorphisms and D to the domains of all inmorphisms. For the group case as expected, C and D reduce to being functors. However, one can generalise the definition of an adjunction to extend to adjunctions between functor relations which then formalises the leftmost $C \dashv I$ and rightmost $S \dashv D$ adjunctions for **Set**.

4.4.3 Further Topics

1. Each of the isomorphism theorems in the abstract setting of the projective set theory that is laid out in this thesis has a dual version of the theorem, replacing each concept with its dual and then stating the resultant theorem, analogously to projective geometry. An initial investigation of some of these dual theorems would be interesting, especially to see for instance what they would translate for concrete sets. One could draw diagrams for specific examples and write the theorems' dual versions to see what exact properties of **Set** they describe.

2. The new generalisation of the chain of adjunctions above can be studied in more detail. In particular, instead of working with functor relations, one may work with functors into the category of families $\mathbf{Fam}(\mathcal{C})$ over a category $\mathcal{C}$.

3. In the context presented in this thesis, the strategy was for the criteria of constructibility of a pyramid to exclude all $\mathcal{N}$-null morphisms and $\mathcal{N}$-null expansions. This choice was deliberate and allows for example an elegant way of accommodating that Axiom 4 need not hold for $\mathcal{N}$-null morphisms. However, one may instead require for example that the composite relation determined by the base zigzag is a function and use that as an alternative notion for constructibility. This was not done in the present text. For example, to simplify the setting, when one removes the $\mathcal{Z}$-empty maps ($\varnothing \longrightarrow A$ where A is non-empty) from **Set** and studies such a setting, then the axioms also hold. In this case one may want to modify the conditions for the pyramid to be constructible to handle zigzags such as the singleton zigzag:

$$\bullet \longmapsto \bullet \longleftarrow\!\!\!\dashv \bullet$$

with $\bullet \in \{\varnothing\}$. Within the current context, the functions in the diagram are $\mathcal{N}$-null and hence do not induce a morphism. However, by composing the two functions as relations, the final relation is in fact a function. By composing relations, it will for example allow us to induce constant functions, which in the current context are always $\mathcal{N}$-null and hence cannot be induced through a pyramid. Moreover, one may carefully rephrase the constructibility conditions for the pyramid since the example above induces a clear pyramid

$$\begin{tikzcd}
& & \bullet \arrow[dl, mapsto] \arrow[dr, mapsto] & & \\
& \bullet \arrow[dr, mapsto] & & \bullet \arrow[dl, mapsto] & \\
\bullet \arrow[ur, mapsto] \arrow[rr, mapsto] & & \bullet & & \bullet \arrow[ul, mapsto] \arrow[ll, mapsto]
\end{tikzcd}$$

although it may not be necessary to work with pyramids in this case. Thus, one may make the condition for inducing a morphism the condition that by composing relations, the final relation is in fact a function. This condition allows inducing a morphism in broader settings than in the present thesis. For example, consider the sequence of functions and its action on elements:

$$\begin{tikzcd}
A \arrow[r, "f"] & B & C \arrow[l, "g"'] \\
\bullet \arrow[r, mapsto] & \bullet & \bullet \arrow[l, mapsto] \\
\bullet \arrow[r, mapsto] & \bullet & \bullet \arrow[l, mapsto] \\
& \bullet & \bullet \arrow[l, mapsto] \\
& & \bullet \arrow[ul, mapsto]
\end{tikzcd}$$

The induced relation $r\colon A \longrightarrow C$ is a function, but 0 does not chase forward to 0. As mentioned earlier in Remark 4.1.16 this is an example for concrete sets where the induced relation is a function, but for which in the present work the design is to not induce a function in such cases. One would need to investigate the new pyramids which become constructible to make sure that the new, broader definition does in fact hold in all of the cases that are expected. Moreover, if the morphisms in $\mathcal{C}_1$ that one works with in the context are not functions, then this definition of course would not make sense, and one would have to investigate further. Although the current setting may thus seem limiting in the light of this discussion, one should notice that the latter example did not occur in the cases that we are interested in. Furthermore, it is not clear whether this alternative definition can be used effectively in proofs to the same extent as the definition in the text.

4. Another investigation which may be interesting would be to see how well this context of projective set theory generalises the notion of a regular category.

Although there are subtle differences between the isomorphism theorems, as we discussed in Section 4.3, it may be worthwhile to see how this setting covers other regular categories (making **Set** the departure point as a regular category rather than **Grp**) and whether it could be modified to generalise regular categories in a similar way that semi-abelian categories generalise the category of groups (and the other group-like structures that hold in the axiomatic context of projective group theory).

5. One may see if it is possible to modify Axiom 4 so that it holds for **Top**.

6. Following the style of the present text, one could propose more self-dual axioms which holds for **Set** and perhaps more generally for a topos, such as a self-dual notion for a subobject classifier (which would then of course not hold for **Grp**).

4.4.4 Conclusion

In this thesis a self-dual approach to set theory is developed which allows one to study sets in a way which may feel at the same time slightly counterintuitive but rather insightful. Indeed, from this perspective, the categories of **Grp** and **Set** satisfy the same set of self-dual axioms.

In this sense, because the present work recovers projective group theory, it is also a perspective on how to compare features of **Grp** and **Set**. The conversion from familiar non-dual structures to self-dual structures itself may be a good notion of understanding a mathematical structure abstractly. In this spirit, it is interesting to see how the exact notion of duality in question is developed to adequately capture the mathematical structures of interest. It will be meaningful to see how far the notion of functorial duality can be employed for this purpose in different branches of mathematics; and specifically, when and how further structure that capture specific notions of duality would need to be introduced.

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