The Lévy-LIBOR model with default risk

by

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Declaration

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Abstract

In recent years, the use of Lévy processes as a modelling tool has come to be viewed more favourably than the use of the classical Brownian motion setup. The reason for this is that these processes provide more flexibility and also capture more of the 'real world' dynamics of the model. Hence the use of Lévy processes for financial modelling is a motivating factor behind this research presentation.

As a starting point a framework for the LIBOR market model with dynamics driven by a Lévy process instead of the classical Brownian motion setup is presented. When modelling LIBOR rates the use of a more realistic driving process is important since these rates are the most realistic interest rates used in the market of financial trading on a daily basis.

Since the financial crisis there has been an increasing demand and need for efficient modelling and management of risk within the market. This has further led to the motivation of the use of Lévy based models for the modelling of credit risky financial instruments. The motivation stems from the basic properties of stationary and independent increments of Lévy processes. With these properties, the model is able to better account for any unexpected behaviour within the market, usually referred to as "jumps".

Taking both of these factors into account, there is much motivation for the construction of a model driven by Lévy processes which is able to model credit risk and credit risky instruments. The model for LIBOR rates driven by these processes was first introduced by Eberlein and Özkan (2005) and is known as the *Lévy-LIBOR model*. In order to account for the credit risk in the market, the *Lévy-LIBOR model with default risk* was constructed. This was initially done by Kluge (2005) and then formally introduced in the paper by Eberlein *et al.* (2006). This thesis aims to present the theoretical construction of the model as done in the above mentioned references. The construction includes the consideration of recovery rates associated to the default event as well as a pricing formula for some popular credit derivatives.

Opsomming

In onlangse jare, is die gebruik van Lévy-prosesse as 'n modellerings instrument baie meer gunstig gevind as die gebruik van die klassieke Brownse bewegingsproses opstel. Die rede hiervoor is dat hierdie prosesse meer buigsaamheid verskaf en die dinamiek van die model wat die praktyk beskryf, beter hierin vervat word. Dus is die gebruik van Lévy-prosesse vir finansiële modellering 'n motiverende faktor vir hierdie navorsingsaanbieding.

As beginput word 'n raamwerk vir die LIBOR mark model met dinamika, gedryf deur 'n Lévy-proses in plaas van die klassieke Brownse bewegings opstel, aangebied. Wanneer LIBOR-koerse gemodelleer word is die gebruik van 'n meer realistiese proses belangriker aangesien hierdie koerse die mees realistiese koerse is wat in die finansiële mark op 'n daaglikse basis gebruik word.

Sedert die finansiële krisis was daar 'n toenemende aanvraag en behoefte aan doeltreffende modellering en die bestaan van risiko binne die mark. Dit het verder gelei tot die motivering van Lévy-gebaseerde modelle vir die modellering van finansiële instrumente wat in die besonder aan kridietrisiko onderhewig is. Die motivering spruit uit die basiese eienskappe van stasionêre en onafhanklike inkremente van Lévy-prosesse. Met hierdie eienskappe is die model in staat om enige onverwagte gedrag (bekend as spronge) vas te vang.

Deur hierdie faktore in ag te neem, is daar genoeg motivering vir die bou van 'n model gedryf deur Lévy-prosesse wat in staat is om kredietrisiko en instrumente onderhewig hieraan te modelleer. Die model vir LIBOR-koerse gedryf deur hierdie prosesse was oorspronklik bekendgestel deur Eberlein and Özkan (2005) en staan beken as die Lévy-LIBOR model. Om die kredietrisiko in die mark te akkommodeer word die Lévy-LIBOR model met "default risk" gekonstrueer. Dit was aanvanklik deur Kluge (2005) gedoen en formeel in die artikel bekendgestel deur Eberlein *et al.* (2006). Die doel van hierdie tesis is om die teoretiese konstruksie van die model aan te bied soos gedoen in die bogenoemde verwysings. Die konstruksie sluit ondermeer in die terugkrygingskoers wat met die wanbetaling geassosieer word, sowel as 'n prysingsformule vir 'n paar bekende krediet afgeleide instrumente.

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Lastly, to my colleagues in the AIMS research center, thank you for inspiring and encouraging me to work hard up until the end. Working with you has indeed been a pleasant experience. I wish you all the best for your future careers, be it in academia or industry.

Dedications

To my parents, Dawood and Karima,
and siblings, Abduraghmaan and Sadia.

"Education is the most powerful weapon which you can use to change the world."

Nelson Mandela

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Chapter 1

Introduction

Financial modelling refers to the construction of a sophisticated mathematical model that represents the behaviour or performance of a financial instrument, usually an asset or investment. A hypothesis about the behaviour of the asset being modelled is converted into a numerical prediction.

The aim of this thesis is to discuss the modelling of interest rates, specifically the *LIBOR (London Interbank Offered Rate)* rates. More importantly, the models considered are those driven by a time-inhomogeneous Lévy process rather than the classical Brownian motion setup. In addition to this, a credit risk factor is added to the model. So instead of only considering the default-free model for these rates, the model is extended to the case of defaultable rates and so the modelling of this default risk becomes a focal point of the study. The motivation behind the study is twofold: firstly it is motivated by the increasing use of Lévy processes as driving tools for financial modelling and secondly by the need for robust models of default risk.

In recent years, the use of Lévy processes as a modelling tool has become much more favourable than the use of the classical Brownian motion setup. Applebaum (2004) states a few reasons as to why these processes are so important:

- They are generalizations of random walks to continuous time.
- The simplest class of processes having paths of continuous motion interspersed with jump discontinuities of random size occurring at random times.
- The structure contains many features that generalize to much wider classes of processes such as semi-martingales and Feller-Markov processes.
- They are a natural model of noise used to build stochastic integrals and as driving force behind stochastic differential equations.

- The structure is mathematically robust which allows for many generalizations, from Euclidean space to Banach and Hilbert spaces, Lie groups and symmetric spaces, and algebraically to quantum groups.

Lévy processes play a central role in several fields of science, such as *physics*, in the study of turbulence, laser cooling and quantum field theory; in *engineering*, for the study of networks, queues and dams; in *economics*, for continuous time-series models; in *actuarial science*, for the calculation of insurance and re-insurance risk and of course in *mathematical finance*, for financial modelling, derivatives pricing and risk management (Papapantoleon, 2008). Interest herein of course lies in the latter. The main motivation for the use of Lévy processes for financial modelling is taken from the financial market itself. This is evident since fluctuations or 'jumps' in asset prices such as stocks and bonds can be observed within the actual market. These fluctuations in prices are usually unexpected and hence not continuous in time. So essentially the use of a discontinuous model such as those driven by Lévy processes allow practitioners to better account for the unexpected jumps within the market. In doing this, much less reality of the market will be lost as is the case when using models driven by classical diffusion setup. The work of Tankov (2007) and Applebaum (2004) give introductions to Lévy processes and their application in mathematical finance. For detailed explanations on the basics of these processes and their applications in finance, the interested reader is referred to the more subjective texts of Cont and Tankov (2004), Øksendal and Sulem (2004) and Schoutens (2003).

Interest rates, coupon bonds and their derivatives are the main financial instruments of the debt market and essentially represent the interplay of time with economic activity, money capital and real (tangible) assets.

Interest in its simplest form refers to the fee paid by the borrower of money (or an asset) to the owner thereof. The fee is a form of compensation for the use of the asset. The interest rate describes the rate at which interest is paid to the lender by the borrower. The interest rate itself cannot be traded, hence bonds and other fixed income instruments depending on the interest rate are traded. Bonds are the primary financial instruments in the market where the *time value*[1] of money is traded (Filipovic, 2009). The texts of Brigo and Mercurio (2006), Baaquie (2010) and Filipovic (2009) provide all the basic requirements for the understanding of interest rates and the modelling thereof.

In the financial market there are different classes of interest rates. Interest herein lies in the class of *interbank* interest rates. Generally, interbank rates refer to the interest rates at which fixed deposits are exchanged between banks

[1]The idea that the value of money today is worth more than the same amount of money tomorrow. The amount is still the same but the value has changed. In other words "A dollar today is worth more than a dollar tomorrow".

in the capital markets. The most important interbank rate usually used as a reference for contracts is the LIBOR rate. The LIBOR refers to the average interest rate that banks are charged for borrowing money from each other in the interbank market. Libor was officially introduced into the financial market in 1986 by the *British Bankers' Association* (BBA) and is now considered as one the most important interest rate instruments used in the debt market. The rate is calculated for ten currencies with fifteen maturities ranging from overnight to 12 months. Hence for each business day there are 150 rates quoted. The three-month LIBOR is the benchmark rate that forms the basis of the LIBOR derivatives market (Baaquie, 2010). The calculation of LIBOR is done by *Thomson Reuters* for the BBA. Every morning each LIBOR contributor bank is asked how much they would charge other banks for a short-term loan. The LIBOR rate produced by Thomson Reuters is calculated using a trimmed arithmetic mean. The rates received from the contributor banks are ranked in descending order and the highest and lowest quartiles are removed. This "trimming" ensures that all outliers are removed from the final calculation. The remaining figures are used to calculate an arithmetic average that is then published by the BBA as the daily LIBOR.

As implied, the rate is usually used by bankers, however changes in the rate can have major impacts on ordinary borrowers. Libor has been used as the basis for consumer loans in many countries around the world. These include small business and student loans as well as credit cards. The slightest change in LIBOR will lead to the interest charged on credit cards, car loans and adjustable rate mortgages either moving up or down. Hence movement in the LIBOR rate is crucial in determining the ease of borrowing not only amongst banks but also amongst companies and consumers.

The current literature boasts a wide range of work on the subject of interest modelling. Of the most popular developments are those of Alan Brace, Dariusz Gaterek, Marek Musiela, Marek Rutkowsi and Farshid Jamshidian to name but a few, dating back to the early and late 1990's. One of the earliest developments on the modelling of the term structure of interest rates is that of Heath *et al.* (1992) and Heath *et al.* (1990), in which the authors present a new methodology for the pricing of interest rate derivatives based on equivalent martingale measure techniques. The methodology also provides arbitrage-free prices that are independent of the market price of risk. Miltersen *et al.* (1997) present a unified model in line with the arbitrage-free structure of the HJM framework, however it is done under the assumption of log-normally distributed simple rates. Later in the 1990's, Brace *et al.* (1997) introduced a new model for the term structure of LIBOR rates. In their paper, Brace *et al.* (1997) aim to show that the market practice of pricing can be made consistent with an arbitrage-free term structure model. Musiela and Rutkowski (1997) similarly present a model that focuses on bond prices and LIBOR rates rather than the instantaneous rates considered in most traditional models. They present a construction of the log-normal model of forward LIBOR rates under a discrete-tenor setup. Jamshidian (1997) further presents theory for the pricing and hedging of LIBOR rates by arbitrage. His approach to modelling LIBOR rates was motivated by the work of Musiela and Rutkowski (1997). The work of the above mentioned authors form the basis of the development of the *LIBOR Market Model*. Later Jamshidian extends the already developed LIBOR market model to the class of general martingales in his paper: (Jamshidian, 1999). To this point credit risk, specifically default risk, and jump processes had not been included in any of the models. Extensions of the LIBOR market model to the inclusion of default risk can be found in Schönbucher (1999) and Grbac and Papapantoleon (2013). Term structure models extended to the case of using Lévy processes as the driving force can be found in Eberlein and Raible (1999) as well as Eberlein and Özkan (2003) amongst others. The extension of the LIBOR market model to the use of Lévy processes instead of classical Brownian motion was introduced by Eberlein and Özkan (2005), known as the *Lévy-LIBOR model*. Their work can be seen as a special case of Jamshidian's approach, however the LIBOR rates are driven by Lévy processes. They also show that in order to have non-negative rates, the LIBOR rates can be represented as an ordinary exponential of a stochastic integral drive by a Lévy process. With this, arbitrage free conditions as well as pricing formula's for interest rate derivatives are given.

Due to the unpredictability of the market there has been a steady increase in the need for efficient modelling and management of the risks faced in the market, particularly that of credit risk. The basic details of credit risk can be found in Bielecki and Rutkowski (2004). The need for efficient and effective models has lead to the use of Lévy processes as the driving force behind such models. Jumps that occur in the market when a credit event, most likely the default event, takes place can be much better accounted for with the use of Lévy processes. Texts such as that of Schoutens and Cariboni (2009), Cariboni (2007) and Kluge (2005) introduce this new approach to modelling credit risk. The modelling of credit risk using Lévy processes was then further extended to the modelling of LIBOR rates. This was in introduced by Eberlein *et al.* (2006) in their paper *The Lévy-LIBOR model with default risk*. Here the already established Lévy-LIBOR model is extended to include the case of default risk. They show how the standard model can be extended to model defaultable rates while maintaining arbitrage-free conditions. The model presented is a generalization of Schönbucher's *LIBOR Market Model with Default Risk* to a Lévy driven setting. This forms the main focus of this thesis presentation.

Thesis Structure

The main aim of this thesis is to present an overview of the theory of the Lévy-LIBOR model with defualt risk as introduced by Eberlein *et al.* (2006).

The structure of the rest of this thesis is as follows. Following the discussion of some basic mathematical concepts pertaining to stochastic analysis and the modelling of the financial market in this chapter, a general introduction to Lévy processes is given in chapter 2. The discussion includes some important features of Lévy processes such as path structure among others. Concepts regarding Ito-Calculus for Lévy processes are then mentioned. This is followed by a brief introduction to the generalisation of Lévy processes referred to as the class of *time-inhomogeneous* Lévy processes as well as an introduction to various Lévy processes used for application to financial modelling. The chapter will form the mathematical basis on Lévy processes used throughout the work to follow.

Chapter 3 gives an overview of some theory of interest rate modelling, in particular the LIBOR market model. The dynamics of the LIBOR market model is given along with a construction of the model under the terminal measure, the *LIBOR Forward Rate Model*. The chapter is concluded with the presentation of the *Lévy-LIBOR Model* as constructed in (Eberlein and Özkan, 2005). The model dynamics under the terminal measure are also specified therein.

In chapter 4, the concept of default risk is introduced and defined. This is then incorporated into the Lévy-LIBOR model, which leads to the construction of the defaultable model. The construction is done as in (Eberlein *et al.*, 2006) and (Kluge, 2005) following the canonical construction of a default time as in (Bielecki and Rutkowski, 2004). Following the construction of the default time, the drift term has to be specified so that certain conditions are satisfied. A brief mention of defaultable forward measures as well as some valuation formula is given towards the end of the chapter. The chapter is concluded with a discussion of some recovery rules and respective pricing formula for bonds. This chapter is the main part of the thesis as these concepts and constructions encapsulate the main theory behind the model.

Following the construction and presentation of the model in chapter 4, an introduction to credit derivatives and the pricing thereof is given in chapter 5. This includes the classification of credit derivatives in general and then the pricing of some of the more popular credit derivatives traded in the credit market. The conclusion of the entire document is found at the end of this chapter. This conclusion includes a brief mention of future work carried out pertaining the topic discussed herein.

1.1 Concepts and Terminologies

Throughout this thesis the filtered probability space denoted by $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$, where $\mathcal{F}$ is a σ-algebra on the non-empty set Ω, $\{\mathcal{F}_t\}_{t\geq 0}$ the filtration and $\mathbb{P}$ the real-world probability measure on $(\Omega, \mathcal{F})$, is used.

Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a stochastic process is defined as a sequence of random variables $\{X_t\}_{t\,\in\, S}$ indexed by time. With the above notation, a stochastic process is said to be an *adapted process* if for each $t \geq 0$, X_t is $\mathcal{F}_t$-measurable. In other words, for each $t \geq 0$, the value of X_t is revealed at time t.

The following definitions are crucial in understanding the financial theory, see (Shreve, 2004) for in-depth descriptions.

Definition 1.1 (Martingale)**.** *Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}\right)$ be a filtered probability space and $\mathcal{M}_t$ an adapted stochastic process such that*

$$\mathbb{E}\left[\mathcal{M}_t|\mathcal{F}_s\right] = \mathcal{M}_s \quad \forall \quad 0 \leq s \leq t \leq T,$$

then $\mathcal{M}_t$ is a martingale. (It has no tendency to rise or fall.)

This means that the expectation or best prediction of a future value of $\mathcal{M}$ (at time t), given information available today, is the observed present value of $\mathcal{M}$ today (at time s). Thus a martingale has no systemic drift (Björk, 1998).

A martingale can be further generalized into two cases where the current observation is either an upper or lower bound on the future conditional expectation. If $\mathbb{E}\left[\mathcal{M}_t|\mathcal{F}_s\right] \geq \mathcal{M}_s \quad \forall \quad 0 \leq s \leq t \leq T$, then $\mathcal{M}_t$ is said to be a *submartingale*. Conversely, if $\mathbb{E}\left[\mathcal{M}_t|\mathcal{F}_s\right] \leq \mathcal{M}_s \quad \forall \quad 0 \leq s \leq t \leq T$, then $\mathcal{M}_t$ is said to be a *supermartingale*.

Definition 1.2 (Markov Process)**.** *Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}\right)$ be a filtered probability space and $\{X_t\}_{t\geq 0}$ an adapted stochastic process. If for every non-negative Borel-measurable function f, there is another Borel-measurable function g such that*

$$\mathbb{E}\left[f(X_t)|\mathcal{F}_s\right] = g\left(X_s\right) \quad \forall \quad 0 \leq s \leq t \leq T,$$

then X_t is a Markov process.

A Markov process is a specific type of stochastic process where the present value of the variable under consideration is the only relevant information needed for predicting the future of the variable. All previous information on the behaviour of the variable is irrelevant.

Definition 1.3 (Brownian Motion)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{\mathcal{B}_t\}_{t\geq 0}$ be a family of continuous real valued measurable functions $\mathcal{B}_t$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathcal{B}_0 = 0$. If the following conditions are satisfied* *(i)* $\mathcal{B}_{s+t} - \mathcal{B}_t$ *is normally distributed with mean* 0 *and variance* s, $\forall\, 0 \leq s \leq t$

(ii) The increments $\mathcal{B}_{t_{i+1}} - \mathcal{B}_{t_i}$, $\quad i = 0, 1, \ldots, n-1$ *are independent* $\forall \quad 0 \leq t_0 < t_1 < \ldots < t_n$

then $\mathcal{B}_t$ *is a* $1-$*dimensional Brownian motion.*

Some properties of Brownian motion include that of being a martingale, a Markov process, having infinite first variation $a.s$ and the scaling property: for any $c \neq 0$, $\{\hat{\mathcal{B}}_t = c\mathcal{B}_{t/c^2}, t \geq 0\}$ is a Brownian motion. A sample path of a Brownian Motion is shown in Figure 1.1.

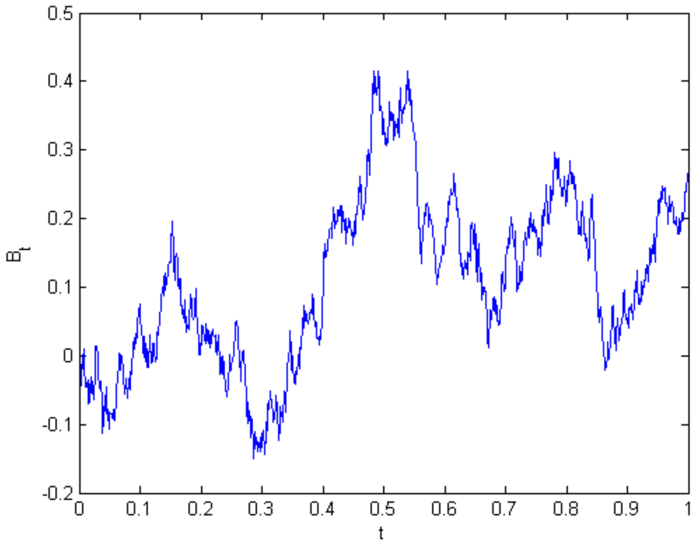

Figure 1.1: A sample path of a Brownian motion.

Geometric Brownian motion is defined as a function of the standard Brownian motion. The stochastic differential equation (linear) describing geometric Brownian motion is given by

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t d\mathcal{B}_t, \\ S_0 &= s \end{aligned} \tag{1.1.1}$$

where is a $\mathcal{B}_t$ a Brownian motion and μ (the drift) and σ (the diffusion or volatility) are constants. The unique solution of (1.1.1) is given by

$$S_t = s \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\mathcal{B}_t\right].$$

For detailed solution, the reader is referred to Øksendal (2003). It is important to note that a process following geometric Brownian motion will never take on negative values given that the initial value is greater than or equal to 0, i.e $s \geq 0$. This can be seen in Figure 1.2. For this reason geometric Brownian motion is often used to model stock prices and is particularly popular as a result of its use as a modelling tool for stock prices in the Black-Scholes model.

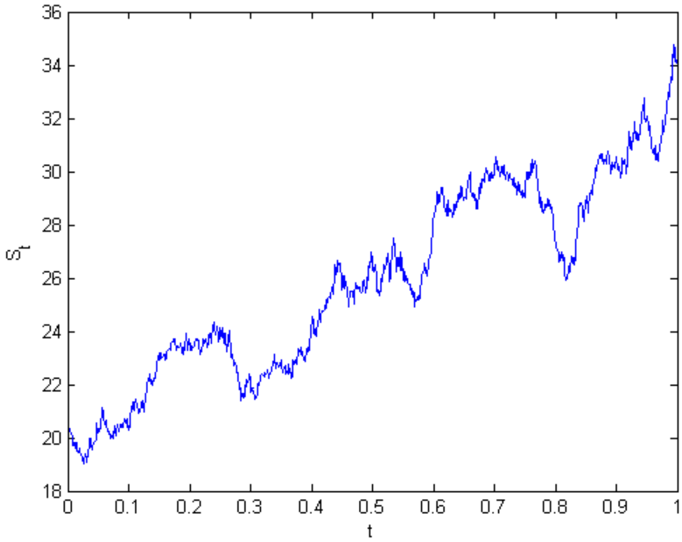

Figure 1.2: A sample path of a geometric Brownian motion with parameters $\mu = 0.5$ and $\sigma = 0.5$.

1.1.1 Girsanov Theory

In probability theory, Girsanov theory describes the change in the structure of a stochastic process when the original probability measure is changed to an equivalent probability measure.

In this section, a general introduction to the main concepts related to Girsanov theory, before discussing details thereof relating to Lévy processes, is given. In order to create a structured link between martingales and Girsanov theory, the following concepts are discussed. The discussion is with reference to Musiela and Rutkowski (2005), Øksendal (2003) and Øksendal and Sulem (2004).

Definition 1.4 (Local Martingale)**.** *An $\mathcal{F}_t$-adapted stochastic process $Z_t \in \mathbb{R}^n$ is called a local martingale with respect to the given filtration, $\mathcal{F}_t$, if there exists* *an increasing sequence of $\mathcal{F}_t$-stopping times*[2] *,τ_k, such that*

$$\tau_k \to \infty \ a.s \ as \ k \to \infty$$

and

$$Z(t \wedge \tau_k) \text{ is an } \mathcal{F}_t \text{ martingale} \ \forall \ k.$$

Every martingale is also a local martingale, however the converse is generally not true. As mentioned in Musiela and Rutkowski (2005), it is important to note that the class of local martingales form a larger class than that of general continuous martingales.

Definition 1.5 (Semimartingale)**.** *A real-valued, continuous, $\mathcal{F}_t$-adapted process X is called a (real-valued) continuous semimartingale if it admits the decomposition*

$$X_t = X_0 + M_t + A_t \qquad \forall \, t \in [0, T],$$

where X_0, M and A satisfy

(i) X_0 is an $\mathcal{F}_0$-measurable random variable,

(ii) M is a continuous local martingale with $M_0 = 0$ and

(iii) A is a continuous process whose almost all sample paths are of finite variation on the interval $[0, T]$ with $A_0 = 0$.

So semimartingales can be considered as an extension of local martingales. To strengthen the link between martingales and Girsanov theorem the concept of an *equivalent probability measure* is introduced into the discussion.

Definition 1.6 (Equivalent Probability Measure)**.** *Given the filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\big)$ and $\mathbb{Q}$ another probability measure on $\mathcal{F}_t$. The two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are said to be equivalent if*

$$\mathbb{P}(A) = 0 \quad \Longleftrightarrow \quad \mathbb{Q}(A) = 0, \qquad \forall \quad A \in \mathcal{F}_T.$$

So $\mathbb{P}$ and $\mathbb{Q}$ have the same zero sets in $\mathcal{F}_T$ and write $\mathbb{P} \sim \mathbb{Q}$.

---

[2]Given an increasing family of σ-algebras $\{\mathcal{F}_t\}$, an $\{\mathcal{F}_t\}$-*stopping time* can be defined as a function $\tau : \Omega \to [0, \infty]$ such that

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \qquad \forall \, t \geq 0.$$

This means that with the given knowledge of $\{\mathcal{F}_t\}$, one can decide whether $\tau \leq t$ has occured or not.

$\mathbb{Q}$ is said to be equivalent to $\mathbb{P}|\mathcal{F}_T$ if

$$\mathbb{P}|\mathcal{F}_T \ll \mathbb{Q} \quad \text{and} \quad \mathbb{Q} \ll \mathbb{P}|\mathcal{F}_T.$$

By the Radon-Nikodym theorem, this is true if and only if

$$d\mathbb{Q}(\omega) = Z(T)d\mathbb{P}(\omega) \quad \text{and} \quad d\mathbb{P}(\omega) = Z^{-1}(T)d\mathbb{Q}(\omega) \quad \text{on} \quad \mathcal{F}_T, \tag{1.1.2}$$

for some $\mathcal{F}_T$-measurable random variable $Z(T) > 0$ a.s $\mathbb{P}$, (Øksendal and Sulem, 2004). (1.1.2) can be rewritten as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z(T) \quad \text{and} \quad \frac{d\mathbb{P}}{d\mathbb{Q}} = Z^{-1}(T) \quad \text{on} \quad \mathcal{F}_T.$$

This concept of equivalence between two measures can now be connected to that of martingales. This is done with the following two useful observations.

Lemma 1.7. *(Øksendal and Sulem, 2004) Let $\mathbb{P}$ and $\mathbb{Q}$ be two equivalent probability measures such that $\mathbb{Q} \ll \mathbb{P}$ and $Z(T) = \frac{d\mathbb{Q}}{d\mathbb{P}}$ on $\mathcal{F}_T$. Then*

$$\mathbb{Q}|\mathcal{F}_t \ll \mathbb{P}|\mathcal{F}_t \quad \forall\, t \in [0,T].$$

Particularly, $Z(t) = \mathbb{E}_{\mathbb{P}}\big[Z(T)|\mathcal{F}_t\big]$ is a unique, positive $\mathbb{P}$-martingale such that

$$\begin{aligned} Z(t) &= \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{F}_t\right] \\ &= \frac{d\big(\mathbb{Q}|\mathcal{F}_t\big)}{d\big(\mathbb{P}|\mathcal{F}_t\big)}, \quad 0 \le t \le T. \end{aligned}$$

In this case, (lemma 1.7), $Z(t)$ is referred to as the density of $\mathbb{Q}$ with respect to the measure $\mathbb{P}$.

Lemma 1.8. *(Øksendal and Sulem, 2004) Let $\mathbb{P}$ and $\mathbb{Q}$ be two equivalent probability measures such that $\mathbb{Q} \ll \mathbb{P}$ and $Z(T) = \frac{d\mathbb{Q}}{d\mathbb{P}}$ is positive on $\mathcal{F}_T$. Also, let $X(t)$ be an adapted process such that $Z(t)X(t)$ is a martingale with respect to $\mathbb{P}$. Then $X(t)$ is a martingale with respect to $\mathbb{Q}$. Similarly, if $Z(t)X(t)$ is a local martingale with respect to $\mathbb{P}$, then $X(t)$ is a local martingale with respect to $\mathbb{Q}$.*

Girsanov theorem can now be stated as:

Theorem 1.9 (Girsanov)**.** *Let $\mathcal{B}_t$ be a Brownian motion with respect to $\mathbb{P}$ on $\mathcal{F}_t$ and $q(t)$ an adapted stochastic process such that $\int_0^t q(s)^2 ds < \infty$ a.s. Let $\mathbb{Q}$ be equivalent to $\mathbb{P}$ on $\mathcal{F}_t$ such that*

$$\begin{aligned} d\mathbb{Q} &= \exp\left(-\frac{1}{2}\int_0^t q(s)^2 ds + \int_0^t q(s) d\mathcal{B}(s)\right) d\mathbb{P} \\ &= Z(t) d\mathbb{P}. \end{aligned}$$

Then under the measure $\mathbb{Q}$, the process

$$\tilde{\mathcal{B}}_t = \mathcal{B}_t - \int_0^t q(s)ds$$

is a Brownian motion and satisfies the stochastic differential equation

$$\begin{aligned} d\tilde{\mathcal{B}} &= d\mathcal{B} - q(t)dt \\ \text{with} \quad dZ(t) &= Z(t)q(t)d\mathcal{B} \end{aligned}$$

Further details and proof of Girsanov theorem can be found in texts such as Øksendal (2003) and Shreve (2004).

1.2 Modelling the Financial Market

In this section the concept of a financial market as well as the modelling thereof will be introduced. In general, the financial market refers to the market (environment) where the trading of financial assets (instruments) take place between buyers and sellers.

Consider a market $\mathcal{M}$ consisting of n financial assets. These assets are continuously tradable over the time period $[0, T]$. The future prices of these assets add uncertainty to the market. This uncertainty in the market is captured and modelled through a Brownian motion defined on the filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}\big)$ introduced in the previous section. Usually the assets considered in the market would be risk-less bonds (money-market accounts) and risky stocks.

Now the main concepts (components) involved in the market which allow for the modelling thereof are defined as follows.

Buyers and sellers trading in the market are referred to as investors. These investors are responsible for the management and maintenance of portfolios of traded assets.

Definition 1.10 (Portfolio)**.** *A portfolio, also referred to as a trading strategy, is a predictable* [3] *stochastic process*

$$\phi = \left(\phi_t^0, \phi_t^1, \ldots, \phi_t^n\right), \quad 0 \le t \le T,$$

where ϕ_t^i, $i = 0, \ldots, n$ represent the number of shares of asset i, held between the trading period $t-1$ and t.

[3] (Cont and Tankov, 2004) The predictable σ-algebra is the σ-algebra $\mathcal{P}$ generated on $[0, T] \times \Omega$ by all adapted left-continuous processes. A mapping $X : [0, T] \times \Omega \to \mathbb{R}^d$ which is measurable w.r.t $\mathcal{P}$ is called a predictable process. Hence, all predictable processes are "generated" from left-continuous processes.

Definition 1.11 (Self-financing). *A portfolio $\phi = (\phi^1, \phi^2)$ is referred to as self-financing if it requires no external input of cash-flow. The value process of such a portfolio is given by*

$$V_t(\phi) = \phi_t^1 S_t + \phi_t^2 B_t, \quad 0 \leq t \leq T,$$

which satisfies the following condition

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u^1 dS_u + \int_0^t \phi_u^2 dB_u, \quad 0 \leq t \leq T,$$

where S and B respectively represent the stocks and bonds invested in.

So the net gain is caused by the price changes between time t and initial time t_0.

1.2.1 Arbitrage Theory

Definition 1.12 (Arbitrage Opportunity). *An arbitrage opportunity is a self-financing strategy φ satisfying the following value process properties:*

1. $V_0(\varphi) = 0$
2. $V_T(\varphi) \geq 0$
3. $\mathbb{P}\big(V_t(\varphi > 0)\big) > 0, \quad 0 \leq t \leq T.$

The investor has no initial capital at time 0 and has a net non-negative profit at time T, with positive probability of the profit being strictly positive at time T.

From now on, the market considered in this work will be assumed to have no such arbitrage opportunity. The market $\mathcal{M}$ is then said to be *arbitrage-free.* The pricing (modelling) of financial instruments in the market has to be done in a way that no arbitrage opportunities will come into existence within the market. This arbitrage-free assumption ensures that no profits can be made in the market without any initial capital invested. Without this assumption (condition), there would be no equilibrium in the market and the correct (fair) pricing and modelling of financial instruments within the market would not be possible.

Chapter 2

Lévy Processes

Lévy processes refer to a rich class of stochastic processes named after french mathematician *Paul Lévy*. His contribution to both probability theory and modern theory of stochastic processes, enables both the understanding and characterization of processes with stationary and independent increments including jump processes of this type.

Michel Loève, French-American probabilist (mathematical statistician) gives a colourful description of Paul Lévy's contributions: *"Paul Lévy was a painter in the probabilistic world. Like the very great painting geniuses, his palette was his own and his paintings transmuted forever our vision of reality. His three main, somewhat overlapping, periods were: the limit laws period, the great period of additive processes and of martingales painted in pathtime colours, and the Brownian pathfinder period."*

In this chapter, general definitions related to Lévy processes are given. Along with the definitions and examples, connections to infinitely divisible distributions (Lévy-Khintchine formula) as well as concepts relating Lévy processes to topics in Itô-calculus will be discussed. These concepts are mentioned and discussed as they will provide the mathematical basis on the subject needed throughout this study. This is followed by an introduction to the class of time-inhomogeneous Lévy processes. The chapter is concluded with a section introducing some more interesting Lévy processes, along with their construction, used for application to financial modelling. Most of this chapter is with reference to Øksendal and Sulem (2004), Cont and Tankov (2004) and Kyprianou (2006).

2.1 Basic Definitions

Basically, a Lévy process is described as a stochastic process having both stationary and independent increments. Mathematically, it is defined as follows:

Definition 2.1 (Lévy process)**.** *Let* $\left(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\right)$ *be a filtered probability space and* $\{\eta_t\}_{t\geq 0}$ *an* $\mathcal{F}_t$*-adapted stochastic process with values in* $\mathbb{R}^d$ *such that* $\eta_0 = 0$*. If the following properties hold*

(i) Independence of increments: For each increasing sequence of times, $t_0,\ t_1,\ \ldots,\ t_n$*, the random variables* $\eta_{t_0},\ \eta_{t_1} - \eta_{t_0},\ \ldots,\ \eta_{t_n} - \eta_{t_{n-1}}$ *are independent.*

(ii) Stationarity of increments: The law of $\eta_{t+h} - \eta_t$ *does not depend on* t*. Hence, for* $s < t$*,* $\eta_t - \eta_s$ *is equal in distribution to* η_{t-s} *(since* $\eta_t - \eta_s \sim \eta_{t-s} - \eta_0 \sim \eta_{t-s}$*).*

(iii) Stochastic continuity: For all $\epsilon > 0$*,* $\lim_{h\to 0} \mathbb{P}\left(|\eta_{t+h} - \eta_t| \geq \epsilon\right) = 0$*.*

then η_t *is a Lévy process.*

Using the first two properties it can be shown that a Lévy process satisfies the Markov property and is hence also a Markov process. The last property implies that the probability of a jump occurring at any given time t is 0. This means that the jumps themselves are not predictable, therefore they occur at random times.

Most texts do not include the *càdlàg*[1] property in the definition of a Lévy process, however it can be stated in the following theorem.

Theorem 2.2. *(Protter, 2004) Let* η_t *be a Lévy process. Then there exists a unique modification of* η_t *which is càdlàg and also a Lévy process.*

As stated in Cont and Tankov (2004), càdlàg functions appear to be natural models for processes with jumps and hence without loss of generality the càdlàg property of Lévy processes can be assumed.

Define the jump of the Lévy process η_t at time $t \geq 0$ by

$$\Delta\eta_t := \eta_t - \eta_{t-}.$$

With this the *Poisson random measure* (or *Jump measure*) of a Lévy process can be defined as follows:

Definition 2.3 (Poisson random measure)**.** *Let* $\mathbb{B}_0$ *be the family of Borel sets* $U \subset \mathbb{R}$ *whose closure* $\bar{U}$ *does not contain* 0*. The jump measure of* $\eta(.)$*, denoted by* $N(t, U)$*, is the number of jumps of size* $\Delta\eta_s \in U$ *which occur before or at time* t*. Mathematically*

$$N(t, U) := \sum_{s:0<s\leq t} \chi_U(\Delta\eta_s) \text{ for } U \in \mathbb{B}_0.$$

[1] A function $f : [0, T] \to \mathbb{R}^d$ is said to be càdlàg if it is right-continuous with left limits. Any continuous function is càdlàg, but càdlàg functions may have discontinuities. These discontinuities are referred to as "jumps".

From this the *Lévy measure* of η_t is defined as the set function

$$\nu(U) := \mathbb{E}[N(1, U)],$$

where $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ is the expectation w.r.t the probability measure $\mathbb{P}$. This gives the expected number of jumps per unit time, whose jump size belongs to U.

Another interesting property of Lévy processes is that of having *infinite divisibility*[2]in distribution. If η is a Lévy process, then for any $t > 0$, η_t is infinitely divisible. Furthermore, a representation of characteristic functions of all infinitely divisible distributions is given by the Lévy-Khintchine formula.

Theorem 2.4 (Lévy-Khintchine formula)**.** *(Øksendal and Sulem, 2004) Let* $\{\eta_t\}$ *be a Lévy process. Then there exists a measure* ν *and constants* α *and* σ*, with* $\int_{\mathbb{R}} min(1, z^2)\nu(dz) < \infty$*,* $R \in [0, \infty]$ *and*

$$\mathbb{E}\big[e^{iu\eta_t}\big] = e^{t\psi(u)}, \ u \ \in \ \mathbb{R} \tag{2.1.1}$$

where

$$\psi(u) = -\frac{1}{2}\sigma^2 u^2 + i\alpha u + \int_{z<R} \big\{e^{iuz} - 1 - iuz\big\}\nu(dz) + \int_{z \geq R} \big(e^{iuz} - 1\big)\nu(dz). \tag{2.1.2}$$

Conversely, given constants α*,* σ*, and a measure* ν *on* $\mathbb{B}_0$ *s.t*

$$\int_{\mathbb{R}} min(1, z^2)\nu(dz) < \infty, \tag{2.1.3}$$

there exists a Lévy process $\eta(t)$ *(unique in law) such that (2.1.1) and (2.1.2) hold.*

A detailed proof can be found in Sato (1999). From the theorem above the Lévy triplet can be defined as follows:

Definition 2.5 (Lévy triplet)**.** *Given a Lévy process* η_t *with characteristic function* $\mathbb{E}\big[e^{iu\eta_t}\big]$ *as in equations (2.1.1) and (2.1.2), the Lévy triplet of* η_t *is given by the triplet* (σ^2, α, ν)*, where* $\alpha \in \mathbb{R}$*,* $\sigma \geq 0$ *and* ν *is the Lévy measure of* η_t *satisfying (2.1.3).*

The Lévy measure ν controls the jumps of the Lévy process. In other words, it describes the expected number of jumps of a certain height in a time interval of length 1 (Papapantoleon, 2008).

It is important to note that a Lévy process could have infinitely many jumps over a finite time interval t, but only a finite number of jumps of size

[2]A probability distribution F on $\mathbb{R}^d$ is said to be infinitely divisible if for any integer $n \geq 2$, there exists n i.i.d random variables $Y_1, \ldots, Y_n$ such that $Y_1 + \ldots + Y_n$ has distribution F.

$\geq \epsilon$ for any $\epsilon > 0$. In this case the infinite jumps in the trajectories of the Lévy process must be small in size. Such processes are known as *infinite activity* Lévy processes which, due to their flexibility, are found to be very interesting in financial modelling. *Finite activity* Lévy processes, on the other hand, are processes characterized by a finite amount of jumps over any finite (bounded) interval. Such Lévy processes are referred to as *jump processes.* A more detailed discussion of these processes is given in subsection 2.5.1.

Essentially, the Lévy measure contains useful information about the structure of the Lévy process. This is summarised in the following proposition:

Proposition 2.6. *(Papapantoleon, 2008) Let η_t be a Lévy process with triplet (σ^2, α, ν).*

1. *If $\nu(\mathbb{R}) < \infty$, then almost all paths of η_t have a finite number of jumps on every compact interval. In that case, the Lévy process has finite activity.*

2. *If $\nu(\mathbb{R}) = \infty$, then almost all paths of η_t have an infinite number of jumps on every compact interval. In that case, the Lévy process has infinite activity.*

The proof of the above proposition can be found in Sato (1999), Theorem 21.3. For further details on Lévy processes and infinite divisibility the reader is referred to Sato (1999), Applebaum (2009) and Cont and Tankov (2004).

2.2 Examples of Lévy Processes

An obvious example of a Lévy process is Brownian motion. This is clear since it satisfies definition 2.1. Here two more common and simple examples of Lévy processes are briefly discussed. These processes are important since all other Lévy processes can be built from these processes.

2.2.1 Poisson Process

One of the simplest examples of a Lévy process is the Poisson process. The Poisson process is based on a random variable having Poisson distribution. A random variable N follows the Poisson distribution with parameter λ, if

$$\mathbb{P}(N = n) = e^{-\lambda}\frac{\lambda^n}{n!} \quad \forall\, n \,\in\, \mathbb{N}_0.$$

The parameter λ is referred to as the *intensity* of the process and describes the expected number of events or arrivals occurring per unit time. Two important properties of the Poisson distribution is that of *stability under convolution*[3]

[3]Stability under convolution is when given two independent Poisson distributed variables Y_1 and Y_2 with parameters λ_1 and λ_2, the sum $Y_1 + Y_2$ is also Poisson distributed with parameter $\lambda_1 + \lambda_2$.

and *infinite divisibility.* Given these two properties, the Poisson process can be defined as follows:

Definition 2.7 (Poison Process)**.** *(Schoutens and Cariboni, 2009) A stochastic process $\{N_t\}_{t\geq 0}$ taking values in $\mathbb{N}\cup\{0\}$ with intensity parameter $\lambda > 0$ is a Poisson process if it satisfies the following conditions:*

1. $N_0 = 0$
2. *Independent increments*
3. *Stationary increments*
4. *For $s < t$, the random variable $N_t - N_s$ has Poisson distribution with parameter $\lambda(t-s)$, i.e*

$$\mathbb{P}(N_t - N_s = n) = e^{-\lambda(t-s)}\frac{\lambda^n(t-s)^n}{n!}.$$

From this definition it is clear that the Poisson process is a Lévy process, with jump size always equal to 1. This can be seen by looking at a sample path of the process as shown in Figure 2.1. Although the jump sizes are predictable (always equal to 1), the jumps times are unpredictable, hence the Poisson process is stochastic. Another definition can be found in Cont and Tankov (2004), wherein the authors describe the Poisson process as a counting process: Based on an *i.i.d* sequence of exponential variables $(T_n - T_{n-1})_{n\geq 1}$, the process $N_t = \sum_{n\geq 1} 1_{t\geq T_n}$ counts the number of random times T_n occurring between 0 and t.

2.2.2 Compound Poisson Process

The compound Poisson process is considered a generalization of the Poisson process in the sense that jumps can be taken from any arbitrary distribution.

Definition 2.8 (Compound Poisson Process)**.** *(Cont and Tankov, 2004) A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f, is a stochastic process X_t defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where jump sizes Y_i, are i.i.d random variables with distribution f and N_t is a Poisson process with intensity λ independent of $(Y_i)_{i\geq 1}$.

As shown in Figure 2.2, the jump sizes of a compound Poisson process are random with Poisson distributed number of jump times occurring by time t.

The following three important properties of the compound Poisson process are to be noted (Cont and Tankov, 2004):

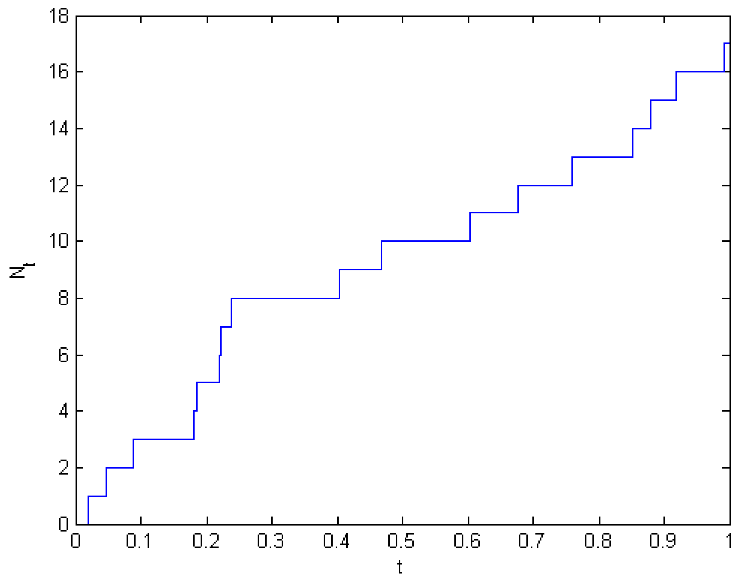

Figure 2.1: A sample path of a Poisson process with intensity $\lambda = 10$.

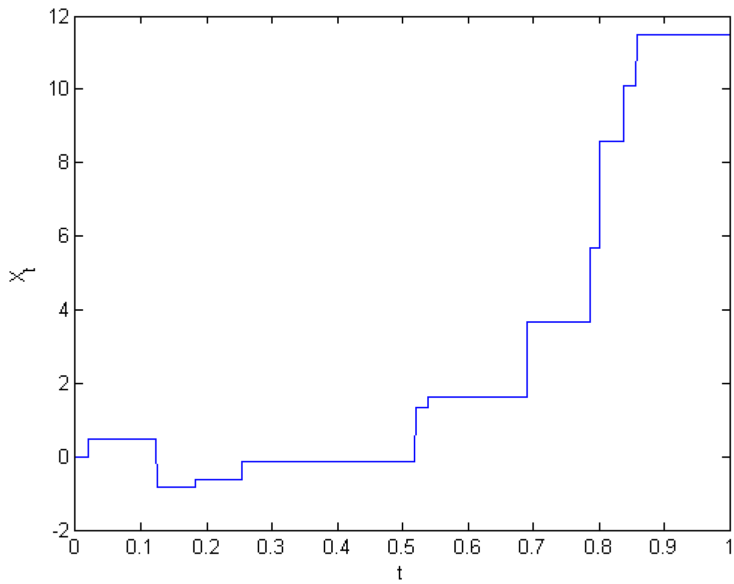

Figure 2.2: A sample path of a compound Poisson process with intensity $\lambda = 10$.

1. Sample paths of X are piecewise constant functions.
2. Jump times $(T_i)_{i\geq 1}$ follow the same law as jump times of the Poisson process N_t.

3. Jump sizes $(Y_i)_{i\geq 1}$ are i.i.d with law f.

The usual Poisson process is a special case of the compound Poisson process where $Y_i = 1,\ i \geq 1$. The compound Poisson process is unique in the sense that it is the only Lévy process characterized by piecewise constant sample paths (Cont and Tankov, 2004). This leads to the following proposition:

Proposition 2.9. $(X_t)_{t\geq 0}$ *is a compound Poisson process iff it is a Lévy process and its sample paths are piecewise constant functions.*

A detailed proof can be found in Cont and Tankov (2004).
For much more detail and examples of Lévy processes the reader is referred to Schoutens (2003).

2.3 Lévy Processes and Itô-Calculus

Here the path structure of Lévy processes is briefly discussed. This is given by the Itô-Lévy decomposition.

Proposition 2.10 (Itô-Lévy decomposition)**.** *(Øksendal and Sulem, 2004) Let* η_t *be a Lévy process. Then* η_t *has the decomposition*

$$\eta_t = \alpha t + \sigma \mathcal{B}(t) + \int_{|z|<R} z\tilde{N}(t,dz) + \int_{|z|\geq R} zN(t,dz), \tag{2.3.1}$$

for constants $\alpha,\ \sigma \in \mathbb{R}$*, and* $R \in [0,\infty]$*. Here*

$$\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$$

is the compensated Poisson[4] *random measure of* $\eta(.)$*,* $\mathcal{B}_t$ *is a Brownian motion independent of* $\tilde{N}(dt,dz)$ *and* $N(dt,dz)$ *the Poisson random measure of* $\eta(.)$*.*

The Itô-Lévy decomposition describes the structure of a general Lévy process in terms of three independent Lévy processes (Kyprianou, 2006). The first two terms form a continuous Gaussian Lévy process characterized by a linear Brownian motion with drift. The other two terms are discontinuous processes describing the jumps of η_t. The first integral being a limit of compound Poisson processes (martingale) with jumps (could be infinitely many) of size $< R$ and the other is the sum of a finite number of bigger jumps of size $> R$.

From the decomposition (2.3.1), more general stochastic integrals can be considered which lead to the following definition.

[4]The *compensated Poisson process* defined by

$$\tilde{N}_t = N_t - \lambda_t$$

is a centred version of the Poisson process N_t, where $(\lambda_t)_{t\geq 0}$ is referred to as the compensator of $(N_t)_{t\geq 0}$. This quantity allows for $\tilde{N}_t$ to be a martingale when subtracted from N_t. The martingale property is one of the important properties of the compensated Poisson process. The other is that of having independent increments.

Definition 2.11 (Itô-Lévy Process). *An Ito-Lévy process is a stochastic process described by a stochastic integral of the form*

$$X(t) = X(0) + \int_0^t \alpha(s,w)ds + \int_0^t \beta(s,w)d\mathcal{B}(s) + \int_0^t \int_{\mathbb{R}} \gamma(s,z,w)\tilde{N}(ds,dz),$$

where

$$\tilde{N}(ds,dz) = \begin{cases} N(ds,dz) - \nu(dz)ds & \text{if } |z| < R \\ N(ds,dz) & \text{if } |z| \geq R \end{cases}.$$

The shorthand differential notation for $X(t)$ given by

$$dX(t) = \alpha(t)dt + \beta(t)d\mathcal{B}(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz).$$

Next the Itô formula for such Itô-Lévy processes as mentioned in definition 2.11 is discussed. This is a fundamental result in the stochastic calculus of Lévy processes.

Theorem 2.12 (The One-Dimensional Itô Formula). *(Øksendal and Sulem, 2004) Suppose $X(t) \in \mathbb{R}$ is an Itô-Lévy process of the form*

$$dX(t) = \alpha(t,w)dt + \beta(t,w)d\mathcal{B}(t) + \int_{\mathbb{R}} \gamma(t,z,w)\tilde{N}(dt,dz),$$

where

$$\tilde{N}(ds,dz) = \begin{cases} N(ds,dz) - \nu(dz)ds & \text{if } |z| < R \\ N(ds,dz) & \text{if } |z| \geq R \end{cases},$$

for some $R \in (0,\infty)$. Let $f \in C^2(\mathbb{R}^2)$ and define $Y(t) = f\big(t, X(t)\big)$. Then $Y(t)$ is again an Itô-Lévy process and

$$\begin{aligned} dY(t) = &\frac{\partial f}{\partial t}\big(t,X(t)\big)dt + \frac{\partial f}{\partial x}\big(t,X(t)\big)\Big[\alpha(t,w)dt + \beta(t,w)d\mathcal{B}(t)\Big] \\ &+ \frac{1}{2}\beta^2(t,w)\frac{\partial^2 f}{\partial x^2}\big(t,X(t)\big)dt + \int_{|z|<R} \Big\{ f\big(t, X(t^-) + \gamma(t,z)\big) \\ &\qquad - f\big(t,X(t^-)\big) - \frac{\partial f}{\partial x}\big(t,X(t^-)\big)\gamma(t,z)\Big\}\nu(dz)dt \\ &+ \int_{\mathbb{R}} \Big\{ f\big(t,X(t^-)+\gamma(t,z)\big) - f\big(t,X(t^-)\big)\Big\}\bar{N}(dt,dz). \end{aligned}$$

Remark 2.13. *If $R = 0$ then $\bar{N} = N$ everywhere.*
If $R = \infty$ then $\bar{N} = \tilde{N}$ everywhere.

An example of the use of the one-dimensional Itô formula for solving Itô-Lévy stochastic differential equations is given as follows.

Example 2.14 (Geometric Lévy Process)**.** *Geometric Lévy processes are often used as a tool in the modelling of stock prices. The stochastic differential equation (sde) describing geometric Lévy processes is given by*

$$dS(t) = S(t^-)\Big[\alpha dt + \beta d\mathcal{B}(t) \ + \int_{\mathbb{R}} \gamma(t,z)\bar{N}(dt,dz)\Big], \tag{2.3.2}$$

where α and β are constants and $\gamma(t,z) \geq -1$.

(2.3.2) can be rewritten as

$$\frac{dS(t)}{S(t^-)} = \alpha dt + \beta d\mathcal{B}(t) \ + \int_{\mathbb{R}} \gamma(t,z)\bar{N}(dt,dz). \tag{2.3.3}$$

Note that the notation $S^c(t)$ refers to the continuous part of the sde (2.3.3), i.e $S^c(t) = \alpha dt + \beta d\mathcal{B}(t)$. Now the one-dimensional Itô formula (Theorem 2.12) is applied to $Z(t) = \ln S(t)$ so that

$$\begin{aligned} dZ(t) =& \frac{\partial \ln S(t)}{\partial s} dS^c(t) + \frac{1}{2}\left(\frac{\partial^2 \ln S(t)}{\partial s^2}\right) d[S^c]_t{}^5 + \int_{z<R} \Big\{ \ln\Big(S(t^-) \\ & + \gamma(t,z)S(t^-)\Big) - \ln\Big(S(t^-)\Big) - \frac{1}{S(t^-)} S(t^-)\gamma(t,z)\Big\}\nu(dz)dt \\ & + \int_{\mathbb{R}} \Big\{ \ln\Big(S(t^-) + \gamma(t,z)S(t^-)\Big) - \ln\Big(S(t^-)\Big)\Big\}\bar{N}(dt,dz). \end{aligned}$$

Substituting $dS^c(t)$ it follows that

$$\begin{aligned} dZ(t) =& \frac{1}{S(t)} S(t)\Big[\alpha dt + \beta d\mathcal{B}(t)\Big] - \frac{1}{2S^2(t)} S^2(t)\beta^2 dt \\ & + \int_{z<R} \left\{ \ln\left(\frac{S(t^-)\big(1+\gamma(t,z)\big)}{S(t^-)}\right) - \gamma(t,z)\right\}\nu(dz)dt \\ & + \int_{\mathbb{R}} \left\{ \ln\left(\frac{S(t^-)\big(1+\gamma(t,z)\big)}{S(t^-)}\right)\right\}\bar{N}(dt,dz) \\ =& \left(\alpha - \frac{1}{2}\beta^2 + \int_{z<R} \Big\{\ln\big(1+\gamma(t,z)\big) - \gamma(t,z)\Big\}\nu(dz)\right) dt + \beta d\mathcal{B}(t) \\ & + \int_{\mathbb{R}} \Big\{\ln\big(1+\gamma(t,z)\big)\Big\}\bar{N}(dt,dz). \end{aligned}$$

[5] The *quadratic variation* of an Itô process X_t given by the sde $dX(t) = \mu dt + \sigma d\mathcal{B}(t)$ is denoted by $d[X]_t = \big(dX(t)^2\big)$ and is computed according to the rules $(dt)(dt) = (dt)\big(d\mathcal{B}(t)\big) = \big(d\mathcal{B}(t)\big)(dt) = 0$ and $\big(d\mathcal{B}(t)\big)\big(d\mathcal{B}(t)\big) = dt$.

Recalling that $Z(t) = \ln S(t)$, *integrating from* 0 *to* t *yields*

$$\begin{aligned}\ln S(t) - \ln S(0) = &\int_0^t \left(\alpha - \frac{1}{2}\beta^2 + \int_{z<R} \Big\{ \ln\big(1+\gamma(s,z)\big) - \gamma(s,z) \Big\} \nu(dz) \right) ds \\ &+ \int_0^t \beta d\mathcal{B}(s) + \int_0^t \int_{\mathbb{R}} \Big\{ \ln\big(1+\gamma(s,z)\big) \Big\} \bar{N}(ds,dz) \\ \ln S(t) = &\ln S(0) + \Big(\alpha - \frac{1}{2}\beta^2\Big)t + \beta\mathcal{B}(t) + \int_0^t \int_{z<R} \Big\{ \ln\big(1+\gamma(s,z)\big) \\ &- \gamma(s,z) \Big\} \nu(dz) ds + \int_0^t \int_{\mathbb{R}} \Big\{ \ln\big(1+\gamma(s,z)\big) \Big\} \bar{N}(ds,dz) \\ \implies S(t) = &S(0) \exp\Bigg[\Big(\alpha - \frac{1}{2}\beta^2\Big)t + \beta\mathcal{B}(t) + \int_0^t \int_{z<R} \Big\{ \ln\big(1+\gamma(s,z)\big) \\ &- \gamma(s,z) \Big\} \nu(dz) ds + \int_0^t \int_{\mathbb{R}} \Big\{ \ln\big(1+\gamma(s,z)\big) \Big\} \bar{N}(ds,dz) \Bigg],\end{aligned}$$

which is the solution to the stochastic differential equation given by (2.3.2).

2.3.1 Girsanov Theory

This section serves as a continuation to subsection 1.1.1 in which general concepts of equivalent measures and Girsanov theory was introduced. Here a brief mention of Girsanov theorem for semimartingales and more specifically Itô-Lévy processes is made.

Theorem 2.15 (Girsanov Theorem for Semimartingales)**.** *(Øksendal and Sulem, 2004) Let* $\mathbb{Q}$ *be a probability measure on* $\mathcal{F}_T$ *and assume that* $\mathbb{Q}$ *is equivalent to* $\mathbb{P}$ *on* $\mathcal{F}_T$, *with*

$$d\mathbb{Q}(\omega) = Z(T)d\mathbb{P}(\omega) \quad on \quad \mathcal{F}_t, \quad t \in [0,T].$$

Let $M(t)$ *be a local* $\mathbb{P}$*-martingale. Then the process* $\widehat{M}(t)$ *defined by*

$$\widehat{M}(t) := M(t) - \int_0^t \frac{d\langle M, Z\rangle(s)}{Z(s)}$$

is a local $\mathbb{Q}$*-martingale.*

In the theorem above, $\langle M, Z\rangle(s)$ refers to the *quadratic covariation*[6] of $M(s)$ and $Z(s)$.

[6] The *quadratic covariation* of two semimartingales $X(t)$ and $Y(t)$ is defined as the unique semimartingale such that

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s^-)dY(s) + \int_0^t Y(s^-)dX(s) + \langle X, Y\rangle(t)$$

and is usually denoted by $\langle X, Y\rangle(t)$.

Theorem 2.16 (Girsanov Theorem for Itô-Lévy processes)**.** *(Øksendal and Sulem, 2004) Let $X(t)$ be an n-dimensional Itô-Lévy process of the form*

$$dX(t) = \alpha(t,\omega)dt + \sigma(t,\omega)d\mathcal{B}(t) + \int_{\mathbb{R}} \gamma(t,z,\omega)\tilde{N}(dt,dz), \quad 0 \leq t \leq T.$$

Assume there exists predictable processes $u(t) = u(t,\omega) \in \mathbb{R}^m$ and $\phi(t,\omega) = \phi(t,z,\omega) \in \mathbb{R}^l$ such that

$$\sigma(t)u(t) + \int_{\mathbb{R}^l} \gamma(t,z)\phi(t,z)\nu(dz) = \alpha(t) \quad \textit{for a.a } (t,\omega) \in [0,T]\times\Omega,$$

and such that the process

$$\begin{aligned} Z(t) := \exp\Bigg[&- \int_0^t u(s)d\mathcal{B}(s) - \frac{1}{2}\int_0^t u^2(s)ds \\ &+ \sum_{j=1}^{l}\int_0^t\int_{\mathbb{R}} \ln\big(1-\phi_j(s,z)\big)\tilde{N}_j(ds,dz) \\ &+ \sum_{j=1}^{l}\int_0^t\int_{\mathbb{R}} \Big\{\ln\big(1-\phi_j(s,z)\big) + \phi_j(s,z)\Big\}\nu_j(dz)ds\Bigg], \quad 0 \leq t \leq T \end{aligned}$$

is well defined and satisfies

$$\mathbb{E}[Z(T)] = 1.$$

Define the probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ by $d\mathbb{Q}(\omega) = Z(T)d\mathbb{P}(\omega)$. Then $X(t)$ is a local martingale with respect to $\mathbb{Q}$.

The measure $\mathbb{Q}$ as described in Theorem 2.16 is referred to as an *Equivalent Local Martingale Measure* (ELMM) for the process $X(t)$. If $X(t)$ is a martingale with respect to $\mathbb{Q}$, then $\mathbb{Q}$ is called an *Equivalent Martingale Measure* (EMM) for $X(t)$ (Øksendal and Sulem, 2004).

2.4 Time-inhomogeneous Lévy Processes

Time-inhomogeneous or *non-homogeneous* Lévy processes are generalizations of standard Lévy processes in that they do not have the property of stationary increments. By relaxing the property of stationary increments, more flexibility is added to the model under consideration. For this reason, time-inhomogeneous Lévy processes are preferred as the driving process behind most models used for application in mathematical finance. Further details of such Lévy processes can be found in (Cont and Tankov, 2004) and (Kluge, 2005).

Definition 2.17 (Time-inhomogeneous Lévy process)**.** *Let $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}\big)$ be a filtered probability space and $L = \{L_t\}_{t \geq 0 \geq T^*}$ an $\mathcal{F}_t$-adapted stochastic process with values in $\mathbb{R}^d$ such that $L_0 = 0$. If the following properties hold*

(i) Independence of increments: $L_t - L_s$ is independent of $\mathcal{F}_s$ for $0 \leq s < t \leq T^*$.

(ii) For each $t \in [0, T^*]$, the law of L_t has characteristic function

$$\mathbb{E}\Big[e^{i\langle u, L_t\rangle}\Big] = \exp \int_0^t \Bigg(i\langle u, b_s\rangle - \frac{1}{2}\langle u, c_s u\rangle + \int_{\mathbb{R}^d} \Big(e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle \mathbb{1}_{\{|x|\leq 1\}}\Big) F_s(dx)\Bigg) ds,$$

where $b_s \in \mathbb{R}^d$, c_s is a symmetric non-negative definite $d\times d$ matrix, and F_s a measure on $\mathbb{R}^d$ integrating $\big(|x|^2 \wedge 1\big)$ satisfying $F_s(\{0\}) = 0$. $\langle ., .\rangle$ denotes the Euclidian scalar product on $\mathbb{R}^d$ and $|.|$ the respective norm. It is further assumed that

$$\int_0^{T^*} \Bigg(|b_s| + ||c_s|| + \int_{\mathbb{R}^d} \Big(|x|^2 \wedge 1\Big) F_s(dx)\Bigg) ds < \infty,$$

where $||.||$ denotes any norm on the set of $d \times d$ matrices.

then L_t is a time-inhomogeneous Lévy process with characteristics given by $(b, c, F) := (b_s, c_s, F_s)_{0\leq s\leq T^*}$.

Properties of time-inhomogeneous Lévy processes include that of being infinitely divisible in distribution, additive in law and a semi-martingale with respect to the stochastic basis $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}\big)$. Formal statements and proofs of these properties can be found in (Kluge, 2005).

The following assumption will be required for financial application since both asset prices and interest rate processes are usually modelled as exponential processes and need to be martingales with respect to the relative risk-neutral measure. For this reason the finiteness of exponential moments of the driving process is required:

Assumption 2.18 ($\mathbb{EM}$)**.** *(Kluge, 2005) There exist constants M, $\epsilon > 0$ such that*

$$\int_0^{T^*} \int_{|x|>1} \exp\langle u, x\rangle F_t(dx) dt < \infty,$$

for every $u \in [-(1+\epsilon)M, (1+\epsilon)M]^d$. In particular, without loss of generality, assume that $\int_{|x|>1} \exp\langle u, x\rangle F_t(dx) < \infty$, for all $t \in [0, T^*]$.

There is another assumption that will be needed for application in financial modelling. The assumption is mathematically stronger than the previous one, although practically equivalent.

Assumption 2.19 ($\mathbb{SUP}$)**.** *(Kluge, 2005) The following hold:*

$$\sup_{0 \leq s \leq T^*} \left(|b_s| + ||c_s|| + \int_{\mathbb{R}^d} \left(|x|^2 \wedge |x| \right) F_s(dx) \right) < \infty,$$

and there are constants M*,* $\epsilon > 0$ *such that for every* $u \in [-(1+\epsilon)M, (1+\epsilon)M]^d$

$$\sup_{0 \leq s \leq T^*} \left(\int_{|x|>1} \exp\langle u, x\rangle F_t(dx) \right) < \infty.$$

2.5 Lévy Processes for Financial Modelling

As previously mentioned, the class of Lévy processes has gained popularity as a tool for financial modelling, due to their flexibility and versatility when compared to classical Brownian motion (diffusion) processes. This flexibility and versatility stems from the fact that there are many (different) types of Lévy processes that can be used for modelling purposes. This section aims to introduce some of these specific Lévy processes along with their construction and application to financial modelling. It should be noted that the more commonly used and less complicated processes are mentioned in this study. For further details on more complicated processes the reader is referred to texts such as Cont and Tankov (2004), Schoutens (2003) and Schoutens and Cariboni (2009).

As a starting point, the two main classes of Lévy processes are briefly discussed followed by a brief introduction on the subclass of Lévy processes called *subordinators*. This leads to and is necessary for the discussion on methods of constructing Lévy processes used for financial modelling. More detail on the approach of Brownian subordination is given as this method is used to build the more popular processes used for financial modelling purposes. These models include the Gamma, Inverse-Gaussian, Variance-Gamma and Normal Inverse Gaussian processes.

2.5.1 Jump-Diffusions and Infinite Activity Processes

Lévy processes for financial modelling can be categorised into two classes: Jump-diffusion and infinite activity processes. A brief introduction to these processes was given in section 2.1, however here more detail regarding the use of these processes for financial modelling is given.

In jump-diffusion models the evolution of prices is given by the diffusion component while the jump components represents rare or extreme events such as a market crash affecting the price movement. When using jump-diffusion processes the prices are modelled as a Lévy process with a non-zero Gaussian component and a jump part, usually a compound Poisson process with a finite number of jumps in every time interval Cont and Tankov (2004). Generally, jump-diffusion type Lévy processes take the form

$$X_t = \gamma t + \sigma \mathcal{B}_t + \sum_{i=1}^{N_t} Y_i,$$

where $N_{tt \geq 0}$ is the Poisson process counting the jumps of X and Y_i are the jump sizes . Popular examples of these models are the *Merton jump-diffusion model* with Gaussian jumps and the *Kou model* with double exponential jumps. As explained in Cont and Tankov (2004), in the Merton model jumps in the log-price X_t are assumed to have a Gaussian distribution $Y_i \sim N(\mu, \delta^2)$, whereas in the Kou model, the distribution of jump sizes is an asymmetric exponential. Details of both models can be found in the respective papers of Merton (Merton, 1976) and Kou (Kou, 2002). Key properties of both models are summarized by Cont and Tankov (2004) (see table 4.3).

The second class of Lévy processes, infinite activity models, is characterised by an infinite number of jumps in every time interval which represent the high activity of the price process. As a requirement, the frequency of bigger jumps is always less than that of the smaller jumps (Riemer, 2008). As explained in Cont and Tankov (2004), a Brownian component does not need to be included since the dynamics of the jumps is rich enough to generate non-trivial small time behaviour and so it can be argued that these models give a more realistic description of the price process at various time scales. Furthermore, most of these infinite activity Lévy processes are constructed through Brownian subordination, which gives them additional analytical tractability when compared to jump-diffusion models. A summary of these two types of Lévy processes is given in Table 2.1 taken from Cont and Tankov (2004).

2.5.2 Subordinators

Subordinators refer to a subclass of increasing Lévy processes used for the construction of other Lévy processes. These processes are characterised by non-negative increments, hence they are very useful for financial modelling. The following definition follows from Cont and Tankov (2004) (Prop. 3.10).

Definition 2.20 (Subordinator)**.** *Let* $\{S_t\}_{t\geq 0}$ *be a Lévy process on* $\mathbb{R}$*.* S_t *is said to be a subordinator iff it satisfies one of the following properties:*

(i) $S_t \geq 0$ *a.s for some* $t > 0$

Jump-diffusion Models	**Infinite activity Models**
Must contain a Brownian component.	Do not necessarily contain a Brownian component.
Jumps are rare events.	The process moves essentially by jumps.
Distribution of jump sizes is known.	"Distribution of jump sizes" does not exist: jumps arrive infinitely often.
Perform well for implied volatility smile interpolation.	Give a realistic description of the historical price process.
Densities not known in closed form.	Closed from densities available in some cases.
Easy to simulate.	In some cases can be represented via Brownian subordination, which gives additional tractability.

Table 2.1: A comparison of two approaches to modelling Lévy processes.

(ii) $S_t \geq 0$ *a.s for every* $t > 0$

(iii) Sample paths of S_t *are a.s non-decreasing, i.e,* $t \geq s \Longrightarrow S_t \geq S_s$

(iv) The characteristic triplet of S_t *satisfies* $A = 0$, $\nu\big((-\infty, 0]\big) = 0$, $\int_0^\infty (x \wedge 1)\nu(dx) < \infty$ *and* $b \geq 0$. *That is,* S_t *has no diffusion component, only positive jumps of finite variation and positive drift.*

Basically, subordinating Lévy processes can be thought of as random models of time evolution (Applebaum, 2009). As explained in Sato (1999), subordination is a transformation of a stochastic process through random time by an increasing Lévy process (subordinator) independent of the original process. The following theorem illustrates the importance of such processes as tools used for the "time changing" of other Lévy processes.

Theorem 2.21 (Subordination of Lévy processes)**.** *(Cont and Tankov, 2004) Fix a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *Let* $\{X_t\}_{t\geq 0}$ *be a Lévy process on* $\mathbb{R}^d$ *with characteristic exponent* $\Psi(u)$ *and triplet* (A, ν, γ) *and let* $\{S_t\}_{t\geq 0}$ *be a subordinator with Laplace exponent* $l(u)$ *and triplet* $(0, \rho, b)$. *Then the process* $\{Y_t\}_{t\geq 0}$ *defined for each* $\omega \in \Omega$ *by* $Y(t,\omega) = X\big(S(t,\omega),\omega\big)$ *is a Lévy process. Its characteristic function is*

$$E[e^{iuY_t}] = e^{tl\left(\Psi(u)\right)}, \tag{2.5.1}$$

i.e, the characteristic exponent of Y *is obtained by composition of the Laplace exponent of* S *with the characteristic exponent of* X*. the triplet* (A^Y, ν^Y, γ^Y)

of Y is given by

$$A^Y = bA,$$

$$\nu^Y(B) = b\nu(b) + \int_0^\infty p_s^X(B)\rho(ds),\ \forall\ B \in \mathbb{B}(\mathbb{R}^d), \tag{2.5.2}$$

$$\gamma^Y = b\gamma + \int_0^\infty \rho(ds) \int_{|x|\leq 1} x\rho_s^X(dx), \tag{2.5.3}$$

where p_t^X is the probability distribution of X_t.

$\{Y_t\}_{t\geq 0}$ is said to be subordinate to the process $\{X_t\}_{t\geq 0}$.

A detailed proof of the theorem can be found in Sato (1999) (see Theorem 30.1).

2.5.3 Construction of Lévy Processes

In order to use Lévy processes for financial modelling purposes they have to be constructed in a certain manner. Furthermore the construction should be done in such a way that the new Lévy process remains invariant. There are quite a few approaches to constructing these process so that they remain invariant, however in this section three popular methods will be discussed. These methods are Brownian subordination, specification of the Lévy measure and specification of the probability density of increments. The method of Brownian subordination is discussed in more detail as this will be used for the construction of some processes in the subsequent subsection to follow.

2.5.3.1 Brownian Subordination

In this approach a Brownian motion is subordinated by an independent increasing Lévy process in order to obtain a new Lévy process. The time variable in the Brownian motion $\{\mathcal{B}_t\}_{t\geq 0}$ is replaced by the stochastic process $\{S_t\}_{t\geq 0}$ so that $X_{S_t} = \mu S_t + \sigma\mathcal{B}(S_t)$ is a new Lévy process. If observed on a new time scale, the stochastic time scale given by S_t, the new process is a Brownian motion. the subordinator in this case is interpreted as *"business time"*, i.e, the integrated rate of information arrival Cont and Tankov (2004). Although this interpretation makes models constructed from Brownian subordination easier to understand, an explicit form of the Lévy measure might not always be available. However, characterization of such processes is given by the following theorem.

Theorem 2.22. *(Cont and Tankov, 2004) Let ν be a Lévy measure on $\mathbb{R}$ and $\mu \in \mathbb{R}$. There exists a Lévy process $\{X_t\}_{t\geq 0}$ with Lévy measure ν such that $X_t = \mathcal{B}(Z_t) + \mu Z_t$ for some subordinator $\{Z_t\}_{t\geq 0}$ and some Brownian motion $\{\mathcal{B}_t\}_{t\geq 0}$ independent from Z if and only if the following conditions are satisfied:*

1. *ν is absolutely continuous with density $\nu(x)$.*

2. $\nu(x)e^{-\mu x} = \nu(-x)e^{\mu x}$ *for all* x.

3. $\nu(\sqrt{u})e^{-\mu\sqrt{u}}$ *is a completely monotonic function on* $(0, \infty)$.

This theorem allows for the description of the jump structure of a process represented as a time-changed Brownian motion. If the simulation of a valid subordinating Lévy process is known, simulation of processes using this technique becomes tractable and easy. Popular models constructed via Brownian subordination include that of the Variance-Gamma and Normal Inverse-Gaussian processes, where the Brownian motion is time-changed by the Gamma and Inverse-Gaussian processes respectively.

2.5.3.2 Specification of the Lévy Measure

This method entails direct specification of the Lévy measure. An advantage of which is the fact that the jump structure can be modelled directly so that one has a clear description of the path-wise structure of the process. The distribution of the process at any time is also known through the Lévy-Khintchine formula. Simulation however, is more involved and complicated, although this method does provide the modeller with a wide variety of models. A popular example of models generated using this method is that of tempered stable processes.

2.5.3.3 Specification of the Probability Density

For this approach an infinitely divisible density is specified as the density of increments at a given time scale, say Δ. If date is sampled with the same period Δ, estimation of the parameters of distribution is easy. Similarly, increments at the same time scale are easy to simulate. However, in general the Lévy measure is not known (Cont and Tankov, 2004). Most common processes constructed using this method is the Generalized Hyperbolic process.

For further details on these methods for constructing Lévy processes the reader is referred to Cont and Tankov (2004).

2.5.4 Lévy based Models

In this subsection certain processes used to build Lévy based models are briefly introduced. This includes basic definitions and some important properties. The figures shown here can be simulated based on algorithms found in Cont and Tankov (2004). The Matlab code used for the actual simulations was taken from Deville (2007).

2.5.4.1 The Gamma Process

The Gamma process refers to the Lévy process associated to the exponential distribution. The density function of a Gamma(a, b) distributed random variable X is given by

$$f_{\text{Gamma}}(x, a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-xb), \qquad \text{for } x, a \text{ and } b > 0,$$

and characteristic function given by

$$\phi_{Gamma}(u; a, b) = \left(1 - \frac{iu}{b}\right)^{-a},$$

where a is the shape parameter and b the scale. The Gamma process can formally be defined as follows:

Definition 2.23 (Gamma Process). *A stochastic process $X^{Gamma} = \{X_t^{Gamma}\}_{t\geq 0}$ with parameters a and b is a Gamma process if it fulfils the following conditions:*

(i) $X_0^{Gamma} = 0$

(ii) Independent increments

(iii) Stationary increments

(iv) For $s < t$, the random variable $X_t^{Gamma} - X_s^{Gamma}$ has a $Gamma\big(a(t-s), b\big)$ distribution.

The Gamma process is an increasing Lévy process with Lévy measure

$$\nu_{\text{Gamma}} = a \exp(-bx) x^{-1} \mathbb{1}_{\{x>0\}}.$$

Two important properties of the Gamma process include that of being infinitely divisible in distribution and the scaling property: given a Gamma(a, b) random variable X, for any $c > 0$, cX is a Gamma$(a, b/c)$ distributed random variable. A sample path of a Gamma process is shown in Figure 2.3.

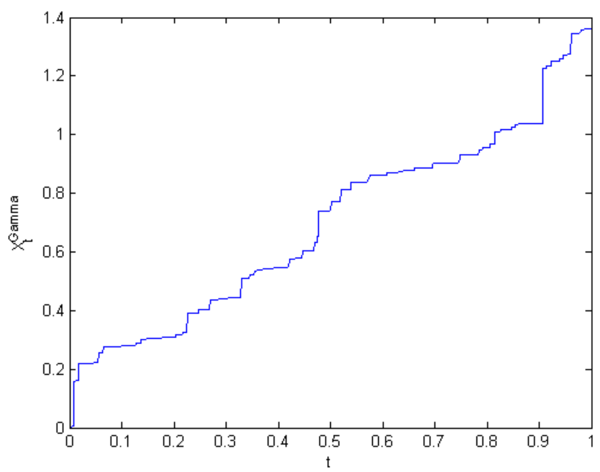

Figure 2.3: A sample path of a Gamma process with parameters $a = 30$ and $b = 18$.

2.5.4.2 The Inverse-Gaussian Process

The Inverse-Gaussian (IG) process is based on an Inverse-Gaussian distributed random variable with density function

$$f_{\mathrm{IG}}(x,a,b) = \frac{a}{\sqrt{2\pi}} \exp(ab) x^{-3/2} \exp\big(-(a^2x^{-1}+b^2x)/2\big) \qquad \text{for } x>0$$

and characteristic function

$$\phi_{IG}(u;a,b) = \exp\Big(a\big(\sqrt{-2iu+b^2}-b\big)\Big).$$

The distribution describes the distribution of time taken by a Brownian motion with positive drift to reach a fixed positive level (Riemer, 2008). The Inverse-Gaussian process can be defined as a process $X^{\mathrm{IG}} = \{X_t^{\mathrm{IG}}\}_{t\geq 0}$ with parameters $a,b>0$, initial value 0, independent and stationary increments and Lévy measure given by

$$\nu_{\mathrm{IG}}(x) = (2\pi)^{-1/2} a x^{-3/2} \exp\big(-b^2x/2\big) \mathbb{1}_{\{x>0\}}.$$

A sample path of an Inverse-Gaussian process is shown in Figure 2.4.

Similarly, the Inverse-Gaussian process is infinitely divisible in distribution with scaling property: for a given $\mathrm{IG}(a,b)$ random variable X, there exists $c>0$ such that the random variable cX is $\mathrm{IG}\big(\sqrt{c}a, b/\sqrt{c}\big)$ distributed.

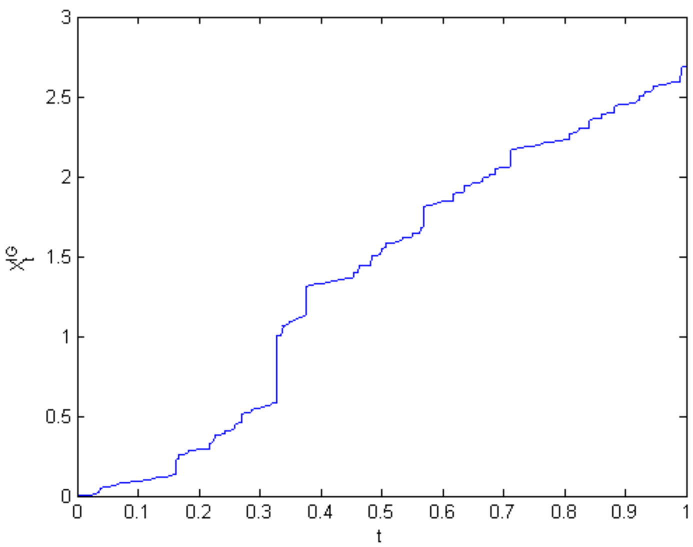

Figure 2.4: A sample path of an Inverse-Gaussian process with parameters $a=10$ and $b=2$.

2.5.4.3 The Variance-Gamma Process

In order to overcome the shortcomings of the Black-Scholes model the Variance-Gamma (VG) process was introduced as an extension of geometric Brownian motion. Motivated by the fact that Gamma distributed increments are derived from the memoryless exponential distribution, Maden and Seneta introduced the VG process as a Brownian motion time-changed by a drift-less Gamma process. Just as the Black-Scholes model is used to model the dynamics of the log return of stock prices, VG based models are used for the same purpose, although they provide more modelling flexibility. VG based models are characterised by three parameters explicitly controlling kurtosis (symmetric increase in both left and right tail probabilities), skewness (asymmetry in both left and right tails) in return distribution and volatility of the subordinated Brownian motion (Riemer, 2008).

The following discussion follows from Schoutens and Cariboni (2009). As a starting point there are two ways of generating VG random variables. The first approach is as the difference of two Gamma distributed random variables. Suppose X is a Gamma$(a = C, b = M)$ random variable and Y is an independent Gamma$(a = c, b = G)$ random variable. Then the random variable $Z = X - Y$ is VG(C, G, M) distributed.

The second approach involves the combination of a Normal distribution with a Gamma random variable. A random variable X is selected from a Gamma$(a = 1/\nu, b = 1/\nu)$ distribution and another random variable Z is sampled from a $\mathcal{N}(\theta X, \sigma^2 X)$ distribution, then Z follows a VG(σ, ν, θ) distribution, where $\nu, \sigma > 0$ and $\theta \in (\infty, \infty)$. The characteristic function of the VG(σ, ν, θ) distribution is given by

$$\phi_{\text{VG}}(u; \sigma, \nu, \theta) = \left(1 - iu\theta\nu + \sigma^2\nu u^2/2\right)^{-1/\nu}. \tag{2.5.4}$$

The parameters θ and ν respectively control the skewness and kurtosis of the distribution. The VG process is infinitely divisible in distribution and can be defined as follows:

Definition 2.24 (Variance-Gamma process)**.** *A stochastic process $X^{VG} = \{X_t^{VG}\}_{t\geq 0}$ is a Variance-Gamma process if it satisfies the following properties:*

(i) $X_0^{VG} = 0$

(ii) Independent increments

(iii) Stationary increments

(iv) The increments $X_{s+t}^{VG} - X_s^{VG}$ over the time interval $[s, t + s]$ follows a VG$(\sigma\sqrt{t}, \nu/t, t\theta)$ distribution given by equation (2.5.4).

Similar to the construction of VG random variables there are also two approaches to the construction of VG processes. In line with the (C, G, M)

parametrization, a VG process can be expressed as the difference of two independent Gamma processes $G_t^{(1)}$ and $G_t^{(2)}$. So the Lévy process generated by the VG process is given by $X_t^{\mathrm{VG}} = G_t^{(1)} = G_t^{(2)}$, where $G_t^{(1)}$ and $G_t^{(2)}$ respectively represent up and down movements of the Lévy process.

The second approach is the construction via Brownian subordination. Here the VG process is defined as a Gamma time-changed Brownian motion with drift. The process is given by

$$X_t^{\mathrm{VG}} = \theta X_t^{\mathrm{Gamma}} + \sigma \mathcal{B}_{X_t^{\mathrm{Gamma}}},$$

where X_t^{Gamma} is a Gamma process with parameters $a = 1/\nu$ and $b = 1/\nu$ and $\mathcal{B}_t$ a standard Brownian motion. A sample path of a Variance-Gamma process generated via Brownian subordination is shown in Figure 2.5.

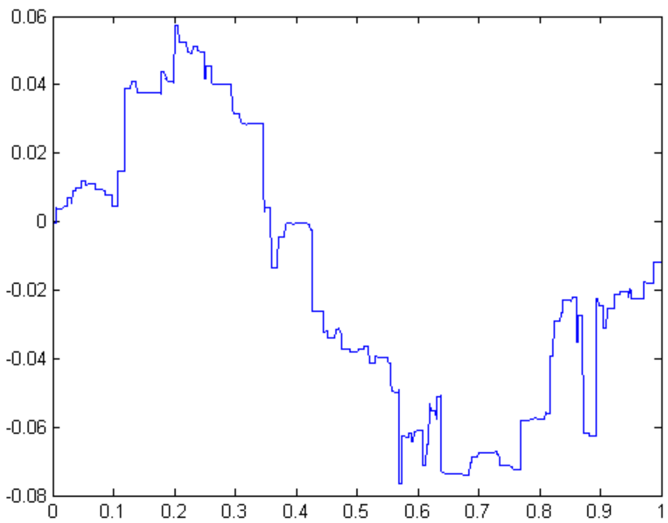

Figure 2.5: A sample path of a Variance-Gamma process with parameters $\sigma = 0.1$, $\nu = 0.05$ and $\theta = 0.15$.

Further details on the Lévy measure and parametrization of the Variance-Gamma process can be found in Schoutens and Cariboni (2009), Schoutens (2003) and Riemer (2008).

2.5.4.4 The Normal Inverse-Gaussian Process

The Normal Inverse-Gaussian (NIG) process refers to the Lévy process obtained through Inverse-Gaussian subordination of a Brownian motion. The process is also known as a *normal variance-mean mixture* where the mixing density is the IG distribution. As explained in Riemer (2008), this means that the process has normally distributed increments, conditional on an IG time-change. The process is typically characterised by three parameters; the drift and volatility of the Brownian motion and variance of the IG subordinator. The NIG process is infinitely divisible in distribution with characteristic function

$$\phi_{\mathrm{NIG}}(u;\alpha,\beta,\delta) = \exp\Big(-\delta\sqrt{\alpha^2-(\beta+iu)^2} - \sqrt{\alpha^2-\beta^2}\Big).$$

Hence, the NIG process can be defined as a process $X^{\mathrm{NIG}} = \{X_t^{\mathrm{NIG}}\}_{t\geq 0}$ with initial value 0, stationary and independent increments and Lévy measure

$$\nu_{\mathrm{NIG}} = \frac{\delta\alpha}{\pi}\frac{\exp(\beta x)K_1(\alpha|x|)}{|x|},$$

where $K_\lambda(x)$ is the modified Bessel function of the third kind with index λ[7].

As mentioned before, the NIG process can be defined as a IG time-changed Brownian motion. Given a standard Brownian motion $\mathcal{B}_t = \{\mathcal{B}_t\}_{t\geq 0}$ and IG process $I_t = \{I_t\}_{t\geq 0}$ with parmeters $a = 1$, $b = \delta\sqrt{\alpha^2-\beta^2}$ for $\alpha > 0$, $\infty < \beta < \infty$ and $\delta > 0$, the stochastic process given by

$$X_t = \beta\delta^2 I_t + \delta\mathcal{B}_{I_t},$$

is a NIG process with parameters α, β and δ.

A sample path of a Normal Inverse-Gaussian process generated via Brownian subordination is shown in Figure 2.6.

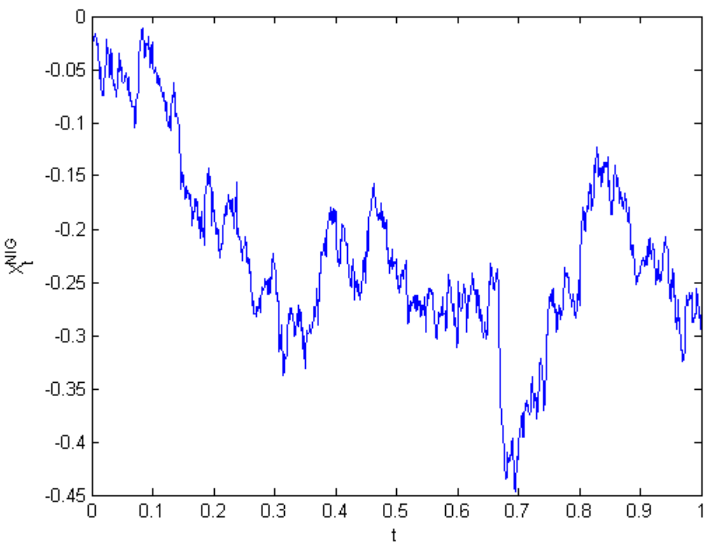

Figure 2.6: A sample path of a Normal Inverse-Gaussian process with parameters $\alpha = 85$, $\beta = 2$ and $\delta = 1$.

[7]See (Schoutens, 2003) Appendix A

Chapter 3

LIBOR Market Modelling

The LIBOR market model (LMM) can be considered as the industry standard financial model for interest rates. The model is also referred to as the *Brace-Gatarek-Musiela (BGM)* model after some of its inventors. The model provides practitioners with an alternative approach to the traditional modelling of the term structure of interest rates. Applying the LMM, the term structure of the interest rate is modelled using the simple rate instead of the instantaneous short rate. Rather than modelling the short or instantaneous forward rates as in the HJM framework, a set of discrete forward rates are modelled directly in the LMM. The advantage of modelling such variables lies in the fact that they are directly observable in the market as well as having volatilities that are naturally linked to traded contracts. As implied by its name, the LMM only uses forward LIBOR rates as its modelling instrument, and hence may also be considered as a collection of forward LIBOR dynamics for various forward rates with varying maturities.

In this chapter a brief discussion of the LIBOR market model is given. In order to do this in a systematic manner, some important information on interest rate theory has to be mentioned first. For further details on interest rate and term-structure modelling the reader is referred to texts such as Brigo and Mercurio (2006) and Baaquie (2010). Thereafter the LIBOR market model and the dynamics thereof is introduced along with a construction of the model under the terminal (forward) measure. The chapter is then concluded by presenting the dynamics of the LIBOR market model driven by Lévy processes: the Lévy-LIBOR model. The dynamics of the model under the terminal measure is also given. This presentation of the Lévy-LIBOR model will be used in the next chapter for the construction of the defaultable model.

3.1 Interest Rates

This section gives a description of how closely bonds and interest rates are connected to each other and hence form the main components of the debt market. In essence, modelling the interest rate is equivalent to the modelling of bond prices. Specifically, it can be shown that all interest rates can be defined with respect to zero-coupon bond[1] prices.

The following definitions are with reference to Brigo and Mercurio (2006), Filipovic (2009) and Baaquie (2010). First the workings of the short rate is explained by defining the bank account as follows:

Definition 3.1 (Bank Account)**.** *Define* $\mathbf{B}(t)$ *to be the value of a bank account at time* $t \geq 0$*. Assume* $\mathbf{B}(0) = 1$ *and that the bank account evolves according to the following differential equation:*

$$d\mathbf{B}(t) = r(t)\mathbf{B}(t)dt, \ \mathbf{B}(0) = 1,$$

where $r(t)$ *is a positive function of time. As a consequence,*

$$\mathbf{B}(t) = \exp\left(\int_0^t r(s)ds\right). \tag{3.1.1}$$

So an investment of one unit at time 0 will yield the value in equation (3.1.1) at time t. Here $r(t)$ is the rate at which the investment grows and is referred to as the *short rate.*

Definition 3.2 (Continuously-compounded Spot Rate)**.** *The spot rate, denoted* $R(t,T)$*, is the constant rate at which an investment (usually a bond) of* $B(t,T)$ *units of currency at time t accrues (accumulates) continuously to yield one unit of currency at time of maturity,* T*. Mathematically it is given by the formula*

$$R(t,T) := -\frac{\ln B(t,T)}{T-t}. \tag{3.1.2}$$

Equation (3.1.2) indicates that the spot rate is consistent with the zero-coupon bond price such that

$$B(t,T)e^{R(t,T)(T-t)} = 1,$$

and hence the bond price can be expressed in terms of the spot rate as

$$B(t,T) = e^{-R(t,T)(T-t)}.$$

[1](Björk, 1998) A zero-coupon bond with maturity date T, also called a T-bond, is a contract which guarantees the holder 1 unit of currency (usually 1 dollar) to be paid on the date T. The price at time t of a bond with maturity date T is denoted by $B(t,T)$.

So an investment of M units at time t will yield an amount of $Me^{R(t,T)(T-t)}$ at time T. Conversely, if a pre-fixed amount M is expected to be received at future time T, the current value at time t of that same amount would be $M/e^{R(t,T)(T-t)}$.

In contrast to the continuously-compounded spot rate the simple spot rate is defined as follows.

Definition 3.3 (Simple Spot Rate)**.** *The simple spot rate at time t, denoted $F(t,T)$, is the constant rate at which an investment (bond) of $B(t,T)$ units of currency at time t, has to be made to produce one unit of currency at maturity when accrued proportionally to the investment period. Mathematically the rate is described by the formula*

$$F(t,T) = \frac{1}{T-t}\left(\frac{1}{B(t,T)} - 1\right). \tag{3.1.3}$$

From this rate the bond price can be expresses as

$$B(t,T) = \frac{1}{1+(T-t)F(t,T)},$$

so that an investment of an amount M at time t will yield the amount $M\big(1 + (T-t)F(t,T)\big)$ at maturity time T. If one is to receive a pre-fixed amount M at a future time T, the value of that same amount at current time t would be $M/\big(1+(T-t)F(t,T)\big)$.

3.1.1 Forward Rates

Forward interest rates form a pivotal role in the study of interest rate and coupon bond markets. They are characterized by three time instants; the time at which the rate is considered t, the expiration time T and the maturity time S. The forward interest rate is locked in today for an investment in a future time period and is set in consistency with the current term structure of the discounting factors. In terms of the short rate, the forward rate is considered as an unbiased estimator thereof. In terms of bond prices, both the compound and simple forward rates are defined as follows:

Definition 3.4 (Continuously-compounded Forward Rate)**.** *The forward rate denoted $f(t,T,S)$, can be described as the continuous rate available in the debt market such that one can lock-in, at time t, the interest rate for a deposit from future time T to S, where $S > T$. Mathematically it can be formulated such* *that*

$$
\begin{aligned}
B(t,T) &= e^{-(S-T)f(t,T,S)}B(t,S) \\
\implies f(t,T,S) &= -\frac{1}{S-T}\ln\left(\frac{B(t,T)}{B(t,S)}\right) \\
&= -\frac{\ln B(t,T) - \ln B(t,S)}{S-T}.
\end{aligned}
$$

As stated in Baaquie (2010), the yields at future times T and S, of an investment made at time t are given by $e^{R(t,T)(T-t)}$ and $e^{R(t,S)(S-t)}$, respectively. The value of these investments are connected since one can take the payoff obtained at time T and lock-in the interest at time t, for the period T to S using $f(t,T,S)$. The principle of no arbitrage yields

$$e^{R(t,S)(S-t)} = e^{R(t,T)(T-t)}e^{f(t,T,S)(S-T)}.$$

In relation to the forward rate the forward bond price as are defined follows:

Definition 3.5 (Forward Bond Price)**.** *Suppose a zero-coupon bond will be issued at a future time $T > t$ with expiry date $S > T$. The forward price of the bond is the price paid at time t to ensure delivery of the bond when it is issued at time T and is given by*

$$f_b(t,T,S) = \frac{B(t,T)}{B(t,S)}. \tag{3.1.4}$$

In terms of the forward rate, $f(t,T,S)$ it follows that

$$f_b(t,T,S) = e^{-f(t,T,S)(S-T)}.$$

Equation (3.1.4) is also referred to as the *forward continuous discount.* This yields the time S value of an investment made at time T. Considering all three time instants t, T and S, an investment of 1 unit made at time T is worth $B(t,T)$ at time t and $\frac{B(t,T)}{B(t,S)}$ at time S.

Definition 3.6 (Simple Forward Rate)**.** *The simple forward rate denoted $F(t,T,S)$, is the value of the fixed rate in a prototypical FRA*[2] *with expiry T and maturity S that renders the FRA a fair contract at time t and is defined as*

$$F(t,T,S) = \frac{1}{S-T}\left(\frac{B(t,T)}{B(t,S)} - 1\right).$$

[2](Brigo and Mercurio, 2006) A prototypical Forward-Rate Agreement or FRA is a contract that gives its holder an interest rate payment for the time period between T and S. The contract involves three time instants; the current time t, the expiry time $T > t$ and the maturity time $S > T$. At maturity time S, a fixed payment based on on a fixed rate K is exchanged against a floating payment based on the spot rate $F(T,S)$ resetting in T and with maturity S. In essence, the contract allows one to lock in the interest rate between time T and S at a desired value K.

So an investment of M units made between times T and S will yield an amount of $M\big(1 + (S - T)F(t, T, S)\big)$.
The following section indicates how forward interest rates form a very fundamental part of the dynamics of the LIBOR market model.

3.2 Dynamics of the LIBOR Market Model

The LIBOR Market model is driven by *forward LIBOR interest rates* characterised by a non-linear evolution equation having both stochastic drift and volatility functions.

Consider a market with a sequence of expiry dates

$$0 \leq T_0 < T_1 < \ldots < T_n < T_{n+1}$$

referred to as the *tenor structure* of the market with each individual time period (date) being termed as a *standard tenor*. The notation $\delta_i = T_{i+1} - T_i$, refers to a constant trading (investment) period within the market and define $T_0 = 0$ as the present time. As in the previous section, the bond price $B(t, T_i)$ is used to define the relative interest rate. This leads to the following definitions.

Definition 3.7 (Spot LIBOR rate)**.** *The spot LIBOR rate, denoted $L(T_i)$, can be defined as the fixed rate set between time T_i and T_{i+1} such that the bond yields a value of* 1 *at time T_{i+1}. Mathematically*

$$B(T_i, T_{i+1})\Big(1 + \delta_i L(T_i)\Big) = 1,$$

and so the spot LIBOR rate is given by

$$L(T_i) = \frac{1}{\delta_i}\left(\frac{1}{B(T_i, T_{i+1})} - 1\right), \qquad i = 1, \ldots, n.$$

So if one is to receive 1 unit amount at future time T_{i+1}, the value of the investment at time T_i is $1/\big(1 + \delta_i L(T_i)\big)$.

In order to define the *LIBOR forward rate*, consider the LIBOR Forward rate agreement (FRA). With this there are two maturity dates T_i and T_{i+1} such that $T_i \leq T_{i+1}$, and the bond price at time t with maturity T_i is denoted by $B(t, T_i)$. The LIBOR FRA refers to the agreement to invest an amount of $B(t, T_{i+1})$ discounted by $B(t, T_i)$ at time T_i, with a yield of 1 unit amount at time T_{i+1}.

Definition 3.8 (LIBOR Forward Rate)**.** *The rate agreed upon as part of the LIBOR FRA is known as the LIBOR forward rate, denoted $L(t, T_i, T_{i+1})$. The rate is agreed upon at time t and is fixed throughout the investment (accrual) period, $\delta_i = T_{i+1} - T_i$. Mathematically this formulated as*

$$\frac{B(t, T_{i+1})}{B(t, T_i)}\Big(1 + \delta_i L(t, T_i, T_{i+1})\Big) = 1,$$

so that the LIBOR forward rate is defined as

$$\begin{aligned} L(t,T_i,T_{i+1}) &= \frac{1}{\delta_i}\left(\frac{B(t,T_i)}{B(t,T_{i+1})}-1\right) \\ &= \frac{1}{\delta_i}\Big(f_b(t,T_i,T_{i+1})-1\Big) \end{aligned} \tag{3.2.1}$$

From definition 3.8, the bond price in terms of the LIBOR forward rate is given by

$$B(t,T_{i+1}) = \frac{B(t,T_i)}{\delta_i L(t,T_i,T_{i+1})+1}. \tag{3.2.2}$$

Given a set of LIBOR forward rates $L_k(T_i)$, $k = i+1,\dots,n$, the bond price $B(T_i,T_n)$ can be determined using the forward rates between times T_i and T_n. This bond price is then given by

$$B(T_i,T_n) = \frac{1}{\prod_{k=i+1}^{n}(1+\delta_k L_k(T_i))}, \tag{3.2.3}$$

where $L_k(T_i) = L(T_i,T_k,T_{k+1})$.

3.2.1 The Model

Consider a market, $\mathcal{M}$, consisting of $n+1$ assets, usually bonds, with maturities $T_1 < \dots < T_{n+1}$. The bond price processes, defined as $B_i(t) := B(t,T_i)$, $i = 1,\dots,n+1$ are Itô processes modelled by the stochastic differential equation

$$\frac{dB_i(t)}{B_i(t)} = \mu_i(t)dt + \sigma_i(t)d\mathcal{B}(t),$$

with

$$B_i(0) = b_{0,i}^{\mathcal{M}}, \quad i = 1,\dots,n+1,$$

where $b_{0,i}^{\mathcal{M}}$ is the market bond price at time 0 and $\mu_i(t)$ and $\sigma_(t)$ represent the drift and diffusion respectively.

Also, for each maturity there are n forward LIBOR rates $L_i(t) := L(t,T_i,T_{i+1})$ with tenor δ_i. Each forward LIBOR rate process is also a stochastic Itó process and is modelled by the stochastic differential equation

$$dL_i(t) = L_i(t)\Big(\mu_i(t)dt + \sigma_i(t)d\mathcal{B}_i(t)\Big),$$

where $\mathcal{B}_i(t)$ is the Brownian motion and $\mu_i(t)$ the drift term for $L_i(t)$. The drift term $\mu_i(t)$ depends on the probability measure that $L_i(t)$ is measured under.

Theorem 3.9. *(Doux, 2004) If the market is arbitrage-free, then for each $i = 1, \dots, n$, there exists an equivalent martingale measure, denoted by $\mathbb{P}_{i+1}$, under which the LIBOR rate process $L_i(t)$ is a martingale.*

So under the equivalent martingale measure $\mathbb{P}_{i+1}$, the LIBOR rate process is a martingale and has no drift term and is hence given by

$$dL_i(t) = L_i(t)\sigma_i(t)d\mathcal{B}_{i+1}(t), \tag{3.2.4}$$

where $\sigma_i(t)$ is a deterministic function and $\mathcal{B}_{i+1}(t)$ the Brownian motion under the equivalent probability measure $\mathbb{P}_{i+1}$.

The expression in (3.2.4) can be solved by applying Itó's Lemma to the substitution $Z_t = \ln L(t)$. From this it follows that the solution to equation (3.2.4) is given by

$$L_i(t) = L_i(0) \exp\left(-\frac{1}{2}\int_0^t \sigma(s)^2 ds + \int_0^t \sigma(s) d\mathcal{B}_{i+1}(s) \right), \quad i = 1, \dots, n.$$

This concludes the discussion on the general workings of the LIBOR Market Model.

3.3 The LIBOR Forward Rate Model

In this section, a set-up of the LIBOR market model under the terminal measure is presented.

The *terminal measure*, denoted $\mathbb{P}^M$, is defined as the last equivalent martingale measure under which the last forward LIBOR rate process $L_{M-1}(t)$ is a martingale. Respectively, $L_{M-1}(t)$ is referred to as the terminal forward LIBOR rate process.

So for each $i = 1, \dots, M-1$, all LIBOR rate processes $L_i(t)$ are also martingales with respect to each corresponding $\mathbb{P}^{i+1}$. The aim is to be able to express all the LIBOR rate processes, $L_i(t)$, $i = 1, \dots, M-1$, with respect to one measure[3]: the terminal measure, $\mathbb{P}^M$. This is the main motivation behind the construction of the LIBOR forward rate model.

The model will be based on the construction of the bond market introduced in the following theorem:

Theorem 3.10. *(Becker, 2009) Given the value of the bond at time T_M, $B_M(t)$, the initial bond prices obtained in the market, $B_i^{\mathcal{M}}(0)$, and volatilities, σ_i, for $i = 1, \dots, M-1$. Then there exists LIBOR rates, $L_i(t)$, whose*

[3] Any measure, $\mathbb{P}^i$, $i = 1, \dots, M-1$ could be chosen as the single measure under which the LIBOR rates are modelled. Given this single measure, say $\mathbb{P}^M$, other measures $\mathbb{P}^i$, $i = 1, \dots, M-1$ can be determined.

logs have volatilities, σ_i, and bonds, $B_i(t)$, $i = 1, \dots, M-1$, with initial conditions $B_i(0) = B_i^{\mathcal{M}}(0)$, which form an arbitrage-free bond market in which the LIBOR rates are consistent with the bond prices. The numéraire[4] which makes all bonds numéraire-based martingales is $B_M(t)$.

Recall the LIBOR forward rate introduced in the previous section as:

$$L(t, T_i, T_{i+1}) = \frac{1}{\delta_i}\left(\frac{B(t, T_i)}{B(t, T_{i+1})} - 1\right).$$

The following notation:

$$\tilde{L}_i(t) = \delta_i L_i(t) \quad \text{with,} \quad L_i(t) = L(t, T_i, T_{i+1})$$

will be used.

Suppose there is a market consisting of bonds, $B_M, \dots, B_{i+1}$ with respective forward measures, $\mathbb{P}^M, \dots, \mathbb{P}^{i+1}$ and corresponding Brownian motions, $\mathcal{B}_M, \dots, \mathcal{B}_{i+1}$. Also that the LIBOR rates $\tilde{L}_{M-1}, \dots, \tilde{L}_i$ have been defined.

Define the LIBOR rate, $\tilde{L}_i(t)$ as the solution of

$$\frac{d\tilde{L}_i(t)}{\tilde{L}_i(t)} = \sigma_i(t) d\mathcal{B}_{i+1}, \tag{3.3.1}$$

with initial condition

$$\tilde{L}_i(0) = \frac{B_i^{\mathcal{M}}(0)}{B_{i+1}^{\mathcal{M}}(0)} - 1,$$

where

$$\begin{aligned} B_i(t) &= \big(1 + \tilde{L}_i(t)\big) B_{i+1}(t) \\ \Longrightarrow \tilde{L}_i(t) &= \frac{B_i(t)}{B_{i+1}(t)} - 1 \\ &= \frac{B_i(t) - B_{i+1}(t)}{B_{i+1}(t)}. \end{aligned}$$

The measure $\mathbb{P}^i$ is equivalent to the measure $\mathbb{P}^{i+1}$ which corresponds to $B_i(t)$. In changing between these measures, Girsanov's theorem is applied and the Brownian motion $\mathcal{B}_i(t)$ is obtained there from . So (by Girsanov) it follows that

$$d\mathbb{P}^i = Z_i(t) d\mathbb{P}^{i+1},$$

[4]The term *numéraire* refers to the basic unit of account by which value or price is computed. Hence, it can be a traded asset always taking on positive values usually modelled by a log-normal SDE.

where

$$\begin{aligned} Z_i(t) &= \frac{B_i(t)/B_(0)}{B_{i+1}(t)/B_{i+1}(0)} \\ &= \frac{B_{i+1}(0)}{B_i(0)}\big(1+\tilde{L}_i(t)\big). \end{aligned} \tag{3.3.2}$$

To this point, the bonds and LIBOR rates with indices $1,\ldots,M-2$ have been defined conditionally upon $\tilde{L}_{M-1}(t)$, $B_{M-1}(t)$ and $\mathcal{B}_M$ having been defined. So $\mathcal{B}_M$ has to be defined as the Brownian motion relative to the forward measure at time T_M, with corresponding bond $B_M(t)$. This further requires the existence of a bank account and risk-neutral measure under which the discounted bond price $B_M(t)/\mathbf{B}(t)$ is a martingale.

By definition $B_M(0) = B_M^{\mathcal{M}}(0)$ and if $B_{i+1}(0) = B_{i+1}^{\mathcal{M}}(0)$ then (by definition)

$$\begin{aligned} B_i(0) &= \big(1+\tilde{L}_i(0)\big)B_{i+1}(0) \\ &= B_{i+1}(0)\frac{B_i^{\mathcal{M}}(0)}{B_{i+1}^{\mathcal{M}}(0)} \\ &= B_i^{\mathcal{M}}(0), \end{aligned}$$

which is true for all i (by induction). Hence, the bonds defined have initial conditions which are consistent with the market. Also, for the LIBOR rates it follows that

$$\tilde{L}_i(t) = \frac{B_i(t) - B_{i+1}(t)}{B_{i+1}(t)},$$

so that both the LIBOR rates and bond prices are consistent.

From equation (3.3.2)

$$\begin{aligned} dZ_i(t) &= \frac{B_{i+1}(0)}{B_i(0)}d\tilde{L}_i(t) \\ &= \frac{B_{i+1}(0)}{B_i(0)}\sigma_i(t)\tilde{L}_i(t)d\mathcal{B}_{i+1} \\ &= \frac{\sigma_i(t)\tilde{L}_i(t)}{1+\tilde{L}_i(t)}\frac{B_{i+1}(0)}{B_i(0)}\Big(1+\tilde{L}_i(t)\Big)d\mathcal{B}_{i+1} \\ &= \frac{\sigma_i(t)\tilde{L}_i(t)}{1+\tilde{L}_i(t)}Z_i(t)d\mathcal{B}_{i+1}. \end{aligned} \tag{3.3.3}$$

By Girsanov theorem, a change of measure function

$$dZ_i(t) = q_i(t)Z_i(t)d\mathcal{B}_{i+1}(t), \tag{3.3.4}$$

with

$$d\mathcal{B}_i(t) = d\mathcal{B}_{i+1}(t) - q_i(t)dt$$

is used. Comparing equations (3.3.3) and (3.3.4) it can be deduced that

$$q_i(t) = \frac{\sigma_i(t)\tilde{L}_i(t)}{1+\tilde{L}_i(t)}. \tag{3.3.5}$$

Hence

$$\begin{aligned} d\mathcal{B}_i(t) &= d\mathcal{B}_{i+1}(t) - q_i(t)dt \\ &= d\mathcal{B}_{i+1}(t) - \frac{\sigma_i(t)\tilde{L}_i(t)}{1+\tilde{L}_i(t)}dt, \end{aligned}$$

so that

$$\begin{aligned} d\mathcal{B}_{i+1}(t) &= d\mathcal{B}_{i+2}(t) - q_{i+1}(t)dt \\ d\mathcal{B}_{i+2}(t) &= d\mathcal{B}_{i+3}(t) - q_{i+2}(t)dt \\ &\vdots \end{aligned}$$

which implies that

$$\begin{aligned} d\mathcal{B}_{i+1}(t) &= d\mathcal{B}_{i+3}(t) - q_{i+2}(t)dt - q_{i+1}(t)dt \\ &= d\mathcal{B}_{i+3}(t) - \big(q_{i+1}(t) + q_{i+2}(t)\big)dt \\ &= d\mathcal{B}_{i+p}(t) - \left(\sum_{k=i+1}^{i+p-1} q_k(t)\right)dt. \end{aligned}$$

So under the terminal measure, $\mathbb{P}^M$, the Brownian motion is given by

$$d\mathcal{B}_{i+1}(t) = d\mathcal{B}_M(t) - \left(\sum_{k=i+1}^{M-1} q_k(t)\right)dt. \tag{3.3.6}$$

Now substituting equations (3.3.6) and (3.3.5) into equation (3.3.1) it follows that

$$\begin{aligned} \frac{d\tilde{L}_i(t)}{\tilde{L}_i(t)} &= \sigma_i(t)\left[d\mathcal{B}_M(t) - \left(\sum_{k=i+1}^{M-1} q_k(t)\right)dt\right] \\ &= -\sigma_i(t)\left(\sum_{k=i+1}^{M-1} \frac{\sigma_k(t)\tilde{L}_k(t)}{1+\tilde{L}_k(t)})\right)dt + \sigma_i(t)d\mathcal{B}_M(t). \end{aligned}$$

So under the terminal (forward) measure the LIBOR rate dynamics are given by

$$d\tilde{L}_i(t) = -\left(\sum_{k=i+1}^{M-1} \frac{\sigma_k(t)\tilde{L}_k(t)}{1+\tilde{L}_k(t)})\right)\sigma_i(t)\tilde{L}_i(t)dt + \sigma_i(t)\tilde{L}_i(t)d\mathcal{B}_M(t), \qquad (3.3.7)$$

for $0 \leq i \leq M-1$.

It should be noted that when $i = M-1$ the sum is taken to be 0. Also as mentioned before, the LIBOR rate process driven by the terminal measure, $\tilde{L}_{M-1}(t)$ is the only LIBOR rate process that is a martingale. This is true since all other LIBOR rate processes will be characterised by a non-zero drift term

$$\mu_i(t) = -\left(\sum_{k=i+1}^{M-1} \frac{\sigma_k(t)\tilde{L}_k(t)}{1+\tilde{L}_k(t)})\right)\sigma_i(t)\tilde{L}_i(t). \qquad (3.3.8)$$

The market so defined can be shown to be arbitrage-free. In general, the drift term given in equation (3.3.8) is a non-linear function of $\tilde{L}_i(t)$ and is also stochastic in behaviour. Because of its non-linearity, the drift term cannot be evaluated analytically, and so numerical techniques such as Monte Carlo methods are used instead. This however falls out of the scope of this thesis as these methods are too time-consuming. Details pertaining to Monte Carlo methods can be found in Zhang (2009), Lesniewski (2008) and Doux (2004). The fact that Monte Carlo methods are so time-consuming leads to another motivation for the use of Lévy based models.

3.4 The Lévy-LIBOR Model

In this section, a presentation of a model for LIBOR rates driven by Lévy processes rather than classical Brownian motion is given. Along with this conditions necessary to maintain an arbitrage-free market is discussed. This section is with reference to Eberlein and Özkan (2005), in which the authors discuss various methods for the modelling of the LIBOR rates. They begin by modelling the instantaneous forward rates, which is very advantageous since the LIBOR rates can be embedded in an existing HJM set-up. They further show that the modelling of the instantaneous forward rates can be done in both the semi-martingale and Lévy settings. They also briefly discus "a forward price model" in which the forward processes (or LIBOR rates) are modelled directly. So it is known that LIBOR rates can be derived from either bond prices or forward prices. For the purpose of this study the framework in which the LIBOR rates are directly used for the construction of the model is considered. The discrete-tenor set-up as discussed in Eberlein and Özkan (2005) is used as the main reference.

3.4.1 The Discrete-Tenor Lévy-LIBOR Model

Consider a discrete-tenor structure similar to the one introduced in section 3.2

$$0 = T_0 < T_1 < \ldots < T_n < T_{n+1} = T^*,$$

where $\delta = T_{i+1} - T_i,\ i = 1, \ldots, n.$

The model is driven by a time-inhomogeneous Lévy process $\mathcal{L}^{T^*}$ on a complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_{T^*})$, where $\mathbb{F} = \{\mathcal{F}_s\}_{0 \leq s \leq T^*}$ and $\mathbb{P}_{T^*}$ is the terminal forward measure associated to the settlement date T^*. Hence the construction is done via backward induction.

The law of $\mathcal{L}_t$ described by the Lévy-Khintchine formula is given by

$$\mathbb{E}_{\mathbb{P}_{T^*}}[e^{iu\mathcal{L}_t}] = \exp\Big(\int_0^t \kappa_s(iu)ds\Big),$$

where κ_s is the *cumulant generating function* associated to the infinitely divisible distribution of $\mathcal{L}^{T^*}$, with Lévy triplet $(0, c, \mathbf{F}_s^{T^*})$ so that

$$\kappa_s(iu) = -\frac{c_s}{2}u^2 + \int_{\mathbb{R}} \Big(e^{iux} - 1 - iux\Big)\mathbf{F}_s^{T^*}(dx).$$

Since $\mathcal{L}^{T^*}$ satisfies assumption (2.18) it can be written in its canonical decomposition

$$\mathcal{L}_t^{T^*} = \int_0^t \sqrt{c_s}d\mathcal{B}_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x\big(\mu - \nu^{T^*}\big)(ds, dx),$$

where $\mathcal{B}_t^{T^*}$ denotes the standard Brownian motion, μ the random measure associated to the jumps of $\mathcal{L}^{T^*}$ and $\nu^{T^*}(dt, dx) = \mathbf{F}_s^{T^*}(dx)dt$ is the compensator of μ. The characteristics of $\mathcal{L}^{T^*}$ is given by $\big(0, c, \nu^{T^*}\big)$ and $\mathcal{L}^{T^*}$ is assumed to be driftless. The following assumptions are further made (Kluge, 2005):

($\mathbb{LL}$.1): For any maturity T_i, there exists a deterministic function $\lambda(\cdot, T_i) : [0, T^*] \to \mathbb{R}^d$, which represents the volatility of the forward LIBOR rates $L(\cdot, T_i)$ such that,

$$\sum_{i=1}^{n-1} |\lambda(s, T_i)| \leq M,\ \forall\ s\ \in\ [0, T^*], \tag{3.4.1}$$

where M is the constant from assumption (2.18) and $\lambda(s, T_i) = 0,\ \forall\ s > T_i$.

($\mathbb{LL}$.2): The initial term structure, $B(0, T_i),\ i\ \in\ \{1, \ldots, n\}$, is strictly positive and decreasing (in i), such that the initial condition of the LIBOR forward rates is given by

$$L(0, T_i) = \frac{1}{\delta_i}\left(\frac{B(0, T_i)}{B(0, T_{i+1})} - 1\right).$$

Under the forward measure $\mathbb{P}_{T_{k+1}}$, the dynamics of the forward LIBOR rates are given by

$$L(t,T_k) = L(0,T_k)\exp\left(\int_0^t b^L(s,T_k)ds + \int_0^t \lambda(s,T_k)d\mathcal{L}_s^{T_{k+1}}\right). \tag{3.4.2}$$

The sum of $\mathcal{L}^{T^*}$ and some drift term gives $\mathcal{L}^{T_{k+1}}$. This drift term however, has to be specified in such a way that $\mathcal{L}^{T_{k+1}}$ is driftless and becomes a martingale under the forward measure $\mathbb{P}_{T_{k+1}}$.

The specification of the drift term is given by

$$\begin{aligned} b^L(s,T_k) &= -\frac{1}{2}\lambda^2(s,T_k)c_s \\ &\quad - \int_{\mathbb{R}^d}\left(\exp^{\lambda(s,T_k)} -1 - \lambda(s,T_k)x\right)\mathbf{F}_s^{T_{k+1}}(dx). \end{aligned}$$

The relationship between the terminal forward measure $\mathbb{P}_{T^*}$ and the forward measure $\mathbb{P}_{T_{k+1}}$ is given by

$$\begin{aligned} \frac{d\mathbb{P}_{T_{k+1}}}{d\mathbb{P}_{T^*}} &= \prod_{l=k+1}^{n-1} \frac{1+\delta_l L(T_{k+1},T_l)}{1+\delta_l L(0,T_l)} \\ &= \frac{B(0,T^*)}{B(0,T_{k+1})}\prod_{l=k+1}^{n-1}\big(1+\delta_l L(T_{k+1},T_l)\big). \end{aligned}$$

The restriction to the σ-field $\mathcal{F}_t$ for $t \in [0,T_{k+1}]$ yields

$$\left.\frac{d\mathbb{P}_{T_{k+1}}}{d\mathbb{P}_{T^*}}\right|_{\mathcal{F}_t} = \frac{B(0,T^*)}{B(0,T_{k+1})}\prod_{l=k+1}^{n-1}\big(1+\delta_l L(t,T_l)\big). \tag{3.4.3}$$

Furthermore, $\mathcal{L}^{T_{k+1}}$ now has the canonical decomposition

$$\mathcal{L}_t^{T_{k+1}} = \int_0^t \sqrt{c_s}d\mathcal{B}_s^{T_{k+1}} + \int_0^t\int_{\mathbb{R}^d} x\big(\mu - \nu^{T_{k+1}}\big)(ds,dx),$$

where

$$\mathcal{B}_t^{T_{k+1}} := \mathcal{B}_t^{T^*} - \int_0^t \sqrt{c_s}\left(\sum_{l=k+1}^{n-1} l(s^-,T_l)\lambda(s,T_l)\right)ds \tag{3.4.4}$$

is the standard Brownian motion with respect to $\mathbb{P}_{T_{k+1}}$ and

$$\nu^{T_{k+1}} = \left(\prod_{l=k+1}^{n-1}\beta(s,x,T_l)\right)\nu^{T^*}(dt,dx) =: \mathbf{F}_t^{T_{k+1}}(dx)dt, \tag{3.4.5}$$

is the $\mathbb{P}_{T_{k+1}}$- compensator of μ (the random measure of jumps of L^{T^*}) with

$$\beta(s,x,T_l) := 1 + l(s^-,T_l)\left(e^{\lambda(s,T_l)x} - 1\right) \tag{3.4.6}$$

and

$$l(s^-,T_l) := \frac{\delta_l L(s^-,T_l)}{1+\delta_l L(s^-,T_l)}. \tag{3.4.7}$$

3.4.1.1 Terminal Measure Dynamics

Using the above connections between the forward measure $\mathbb{P}_{T_{k+1}}$ and the terminal measure $\mathbb{P}_{T^*}$, the dynamics of the forward LIBOR rates $L(\cdot, T_k)$ under the terminal measure $\mathbb{P}_{T^*}$ is given by

$$L(t, T_k) = L(0, T_k) \exp\left(\int_0^t b^L(s, T_k)ds + \int_0^t \lambda(s, T_k) d\mathcal{L}_s^{T^*} \right), \tag{3.4.8}$$

where $\mathcal{L}^{T^*}$ is the $\mathbb{P}_{T^*}$ time-inhomogeneous Lévy process and the drift is specified as

$$b^L(s, T_k) = -\frac{1}{2}\lambda^2(s, T_k)c_s - c_s\lambda(s, T_k) \sum_{l=k+1}^{n-1} l(s^-, T_l)\lambda(s, T_l) \tag{3.4.9}$$

$$- \int_{\mathbb{R}} \left(\left(e^{\lambda(s,T_k)x} - 1 \right) \prod_{l=k+1}^{n-1} \beta(s, x, T_l) - \lambda(s, T_k)x \right) \mathbf{F}_s^{T_*}(dx), \tag{3.4.10}$$

where $\beta(s, x, T_l)$ and $l(s^-, T_l)$ is given in equations (3.4.6) and (3.4.7).

Chapter 4

Credit Risk in Lévy-LIBOR Modelling

The term *credit risk* refers to the risk associated to any kind of credit-linked event, such as the default event, changes in credit quality and variations of credit spreads. Credit risk can further be characterised more generally in terms of the following components:

1. The obliger - also referred to as the reference entity, the party who agrees to fulfil certain contractual obligations.

2. The set of criteria defining the default - the criteria which define how the obligations agreed upon have to be fulfilled.

3. The time horizon or maturity - this defines the time period over which the risk is spread.

More specifically, *default risk* refers to the possibility of an obliger not fulfilling their contractual agreement to meet their obligations as stated in the contract. If this happens the obliger is said to have defaulted or a default event has occurred. In general, companies and bonds are dealt with instead of people and loans. In view of this, default can be defined in a variety of ways such as the complete financial bankruptcy of the obliger (reference entity) or a rating downgrade, restructuring or merger of a company with another. Regarding the default event there are three unknowns: the probability of the event occurring, the time when the even will occur and the severity of the loss once it has occurred. Associated with the default event are the two probabilities:

1. Survival Probability:
$\mathbb{P}_{Surv}(t) =$ Probability that the default event will not occur in $[0, t]$.

2. Default Probability:
$\mathbb{P}_{Def}(t) =$ Probability that the default event will occur in $[0, t]$.

From basic probability theory $\mathbb{P}_{Def}(t) = 1 - \mathbb{P}_{Surv}(t)$, $\forall\, 0 \leq t \leq T$.

This chapter starts with a discussion of concepts related to credit risk and the modelling thereof. The purpose of this is to provide a setting and motivation for the defaultable Lévy-LIBOR model to be introduced later in the chapter. The defaultable model discussed here is simply an extension of the default-free model discussed in the previous chapter. In the current literature there is a range of texts relating to the subject of credit risk. The reader is referred to Bielecki and Rutkowski (2004), Schönbucher (2000), Schoutens and Cariboni (2009), Cariboni (2007), Özkan (2002) and Kluge (2005) for further detail.

Following this discussion, a presentation of the *Lévy-LIBOR model with default risk* as introduced by Eberlein *et al.* (2006) is given. This will be the main reference, however use of other literature on the subject such as Eberlein and Özkan (2003), Huehne (2007), Kluge (2005) and Özkan (2002) is also made. Important components of the model are introduced as well as discussion of arbitrage-free conditions that need to be maintained throughout the construction of the model. Instead of modelling the forward LIBOR rates as in the default-free model, the forward default intensities are modelled instead. One important component, the default time and the construction thereof is focused on as this is the focal point behind the construction of the model. This is due to the fact that the dynamics of these intensities can only be specified once a specification of the default time is given. This leads to the canonical construction of the default time as done in Eberlein *et al.* (2006). As an introduction to this construction method, a discussion of hazard processes is given. This is significant in the construction of the default time since the construction is only possible once the hazard process is given.

Once the default time is constructed and the dynamics of the modelled intensities coincide with those derived from the real world default probabilities, the drift coefficient of the model has to be specified so that certain arbitrage-free conditions are maintained. The specification is of course done under corresponding forward measures. This specification of the drift however is mathematically intense, hence most of the section presents statements of theorems for which proofs can be found in Kluge (2005).

As with any mathematical model some form of analysis of the model is required. For this a discussion on the defaultable forward measures is given following the two versions introduced by Schönbucher (1999) and Bielecki and Rutkowski (2004). These defaultable forward measures are then used to price contingent claims.

The last section of the chapter deals with the recovery rules regarding pricing of default-contingent payoffs. As stated in Eberlein *et al.* (2006) these payoffs are important for the modelling of recovery payoffs of real-world defaultable securities such as coupon-bearing bonds or credit default swaps. To model this recovery, the recovery of par approach is used as Eberlein *et al.*

(2006) view it as the most realistic parameterization of recovery payoffs.

4.1 Credit Risk Modelling

As mentioned in Cariboni (2007), modelling credit risk requires the definition of a default event and an estimation of its associated probability of default. To this there are two main approaches: *structural models* and *intensity-based models*.

Structural models focus on the modelling and pricing of credit risk associated specifically to the obliger's economic capital (the value of the firm). The credit event is triggered by the movement of the firm's value relative to some credit barrier. Hence, the approach is also referred to as the *firm's value model*. The model is concerned only with the default credit event and so the credit event mentioned above will be referred to as default and the credit barrier the default barrier. With structural models the value process of the firm is modelled directly and the default event directly linked to this. In this case, the default event is defined as the first stopping-time at which the firm's value process falls below the specified default barrier within the agreed contractual time frame.

On the other hand, in intensity-based models, the probability of a default event occurring is modelled directly. The approach is based on the idea that there is a probability of the obliger defaulting at any time within the contract period. Here the default event is defined and modelled as the first jump of a counting process, usually a Poisson process with random intensity is used. In contrast to the structural approach, the intensity-based model does not necessarily depend on the firm's value, but rather the specification of the hazard rate. The hazard rate defines the intensity that models the rate of default of the reference entity. Hence this approach is also referred to as the *hazard-rate model*. Most credit risk models follow the intensity-based approach, since an increase in the default intensity (hazard rate) would imply an increase in the probability of default. For the construction of the *Lévy-LIBOR model with default risk* the intensity-based approach to modelling the default risk is applied. For further details on both modelling approaches the reader is referred to Bielecki and Rutkowski (2004) and to Schoutens and Cariboni (2009) and Cariboni (2007) for details of the Lévy based models thereof.

4.2 Presentation of the Model

The credit risk model presented in this section is an extension of the model presented in chapter 3. The model is extended in such a way that the dynamics of defaultable market rates can be captured. In contrast to the default-free model, the dynamics of defaultable LIBOR rates cannot be modelled freely. As stated in Kluge (2005), they follow by arbitrage-free arguments from the specification of the default time and the default-free forward LIBOR rates. This is discussed in further detail later in this section.

Consider a fixed time horizon T^* and a discrete tenor structure given by

$$0 = T_0 < T_1 < \ldots < T_n = T^*,$$

with $\delta_k := T_{k+1} - T_k$ for $k = 0, \ldots, n-1$.

Assume that both default-free and defaultable zero-coupon bonds with maturities $T_1, \ldots, T_n$ are traded in the market. Denote the price of the default-free bond at time t, with maturity T_k by $B(t, T_k)$ and the price of the defaultable bond by $B^0(t, T_k)$. The pre-default value of the defaultable bond is given by $\bar{B}(t, T_k)$ and τ the time of default. The bond prices are related to each other by

$$B^0(t, T_i) = \mathbb{1}_{\{\tau > t\}} \bar{B}(t, T_i) \text{ and } \bar{B}(T_i, T_i) = 1 \text{ for } i \in \{1, \ldots, n\}. \tag{4.2.1}$$

In constructing the model, rather than modelling the bond prices directly, the dynamics of the forward LIBOR rates will be specified instead. As in Schönbucher (1999) and Eberlein *et al.* (2006), the following notation will be used:

The default-free LIBOR forward rates are defined as

$$L(t, T_k) = \frac{1}{\delta_k} \left(\frac{B(t, T_k)}{B(t, T_{k+1})} - 1 \right) \qquad \text{for} \quad k \in \{1, \ldots, n-1\}.$$

The defaultable LIBOR forward rates are defined by

$$\bar{L}(t, T_k) = \frac{1}{\delta_k} \left(\frac{\bar{B}(t, T_k)}{\bar{B}(t, T_{k+1})} - 1 \right) \qquad \text{for} \quad k \in \{1, \ldots, n-1\}.$$

The forward LIBOR spreads are defined by

$$S(t, T_k) = \bar{L}(t, T_k) - L(t, T_k) \qquad \text{for} \quad k \in \{1, \ldots, n-1\}.$$

The default risk factors [1] are defined by

$$D(t, T_k) = \frac{\bar{B}(t, T_k)}{B(t, T_k)} \qquad \text{for} \quad k \in \{1, \ldots, n-1\}.$$

[1] The default risk factors $D(t, T_k)$ allow to separate the influence of default risk on the defaultable bond prices from the standard discounting with default-free interest rates (Schönbucher, 1999).

The forward default intensities are defined by

$$H(t, T_k) = \frac{1}{\delta_k}\left(\frac{D(t, T_k)}{D(t, T_{k+1})} - 1\right) \qquad \text{for} \quad k \in \{1, \ldots, n-1\}.$$

The default-free part of the model is discussed in subsection 3.4.1. For further details on this construction the reader is referred to Eberlein and Özkan (2005). The model is once again driven by a time-inhomogeneous Lévy process, $\mathcal{L}^{T^*}$ on a complete stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \mathbb{P}_{T^*})$, where $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_s\}_{0 \leq s \leq T^*}$ and $\tilde{\mathbb{P}}_{T^*}$ is the terminal forward measure associated to the settlement date T^*. A slightly stronger condition is put on $\mathcal{L}^{T^*}$ as Assumption (2.19) is assumed to be satisfied and the notation is changed from $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_{T^*})$, as in subsection 3.4.1, to $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \mathbb{P}_{T^*})$. However, all basic properties of the default-free model as mentioned in section 3.4 still hold. The aim is now to include the defaultable LIBOR rates into the Lévy-LIBOR model.

A seemingly obvious way to do this would be to specify the dynamics of the defaultable forward LIBOR rates in a similar way as done in equation (3.4.2). As stated in Kluge (2005), $\bar{L}(T_k, T_k) < L(T_k, T_k)$ would imply that $\bar{B}(T_k, T_{k+1}) > B(T_k, T_{k+1})$ which would create an arbitrage opportunity within the market, provided that $B^0(\cdot, T_{k+1})$ has not defaulted until time T_k. To ensure no arbitrage within the market, defaultable bonds should always be worth less than default-free bonds. So in order to maintain an arbitrage-free market the model has to be specified so that the defaultable forward LIBOR rates are always higher than the default-free rates. Instead of specifying the defaultable forward LIBOR rates directly, the forward LIBOR spreads or forward default intensities are modelled as positive processes. The defaultable forward LIBOR rates are then derived by

$$\bar{L}(t, T_k) = S(t, T_k) + L(t, T_k),$$

where by definition

$$\begin{aligned}
S(t, T_k) &= \frac{1}{\delta_k}\left(\frac{\bar{B}(t, T_k)}{\bar{B}(t, T_{k+1})} - 1\right) - \frac{1}{\delta_k}\left(\frac{B(t, T_k)}{B(t, T_{k+1})} - 1\right) \\
&= \frac{1}{\delta_k}\left(\frac{\bar{B}(t, T_k)}{\bar{B}(t, T_{k+1})} - \frac{B(t, T_k)}{B(t, T_{k+1})}\right) \\
&= \frac{1}{\delta_k}\left(\frac{\bar{B}(t, T_k)}{B(t, T_k)}\frac{B(t, T_{k+1})}{\bar{B}(t, T_{k+1})} - 1\right)\frac{B(t, T_k)}{B(t, T_{k+1})} \\
&= \frac{1}{\delta_k}\left(\frac{D(t, T_k)}{D(t, T_{k+1})} - 1\right)\frac{B(t, T_k)}{B(t, T_{k+1})} \\
&= H(t, T_k)\frac{B(t, T_k)}{B(t, T_{k+1})} \\
&= H(t, T_k)\big(1 + \delta_k L(t, T_k)\big) + L(t, T_k),
\end{aligned}$$

so that

$$\bar{L}(t,T_k) = H(t,T_k)\big(1+\delta_k L(t,T_k)\big) + L(t,T_k). \tag{4.2.2}$$

Unfortunately H and S cannot be specified directly since their dynamics depend on the specification of the default time τ.

Suppose that τ has already been constructed and is a stopping time with respect to the filtration $\tilde{\mathbb{F}}$. Then the terminal value of a defaultable bond is given by

$$B^0(T_k,T_k) = \mathbb{1}_{\{\tau>T_k\}}\bar{B}(T_k,T_k) = \mathbb{1}_{\{\tau>T_k\}}.$$

In the default-free model the price of a contingent claim[2] X at time t with a payout of $\mathbb{1}_{\{\tau>t\}}$ at time T_k is given by

$$X_t := B(t,T_k)\mathbb{E}_{\mathbb{P}_{T_k}}\Big[\mathbb{1}_{\{\tau>T_k\}}|\tilde{\mathcal{F}}_t\Big] = \mathbb{1}_{\{\tau>t\}}B(t,T_k)\mathbb{E}_{\mathbb{P}_{T_k}}\Big[\mathbb{1}_{\{\tau>T_k\}}|\tilde{\mathcal{F}}_t\Big].$$

For the deafaultable model to be consistent, the value of the defaultable bond at time t is given by

$$B^0(t,T_k) = \mathbb{1}_{\{\tau>t\}}B(t,T_k)\mathbb{E}_{\mathbb{P}_{T_k}}\Big[\mathbb{1}_{\{\tau>T_k\}}|\tilde{\mathcal{F}}_t\Big],$$

which by (4.2.1) implies the pre-default value of the bond to be

$$\bar{B}(t,T_k) = B(t,T_k)\mathbb{E}_{\mathbb{P}_{T_k}}\Big[\mathbb{1}_{\{\tau>T_k\}}|\tilde{\mathcal{F}}_t\Big],$$

and so the default risk factors becomes

$$D(t,T_k) = \mathbb{E}_{\mathbb{P}_{T_k}}\Big[\mathbb{1}_{\{\tau>T_k\}}|\tilde{\mathcal{F}}_t\Big].$$

So given this specification of $D(t,T_k)$ the formulas for both H and S can be specified. Hence, the dynamics of H and S depend on the specification of the default time τ. Since the specification of τ is not given, H can only be "specified" by giving a prespecification. The prespecification is made so that the default time τ can be constructed in such a way that the actual dynamics of H implied by this τ will match the prespecification, denoted by $\hat{H}$. Hence H would be specified based on the construction of τ. In addition to assumptions ($\mathbb{LL}$.1) and ($\mathbb{LL}$.2) made in subsection (3.4.1), the following slightly stronger assumptions are made (Kluge, 2005):

($\mathbb{DR}$.1): For any maturity T_i there is a deterministic function $\gamma(\cdot,T_i) : [0,T^*] \to \mathbb{R}^d_+$, which represents the volatility of the forward default intensity $H(\cdot,T_i)$. Suppose that $\gamma(s,T_k) = 0$ for $T_k < s \leq T^*$. Moreover, it is required that

[2]A contingent claim X represents the terminal pay-off of a contract depending on the occurrence of a specific event.

the function $\lambda(\cdot, T_i)$ from ($\mathbb{LL}$.1) map to $\mathbb{R}^d_+$ and condition (3.4.1) is tightened by assuming that

$$\sum_{i=1}^{n-1} |\lambda(s, T_i)| + |\gamma(s, T_i)| \leq M, \ \forall \, s \ \in \ [0, T^*]. \tag{4.2.3}$$

($\mathbb{DR}$.2): The initial term structure $\bar{B}(0, T_i)$, $i \ \in \ \{1, \ldots, n\}$ of defaultable zero coupon bond prices satisfies $0 < \bar{B}(0, T_i) \leq B(0, T_i)$ for all T_i as well as $\bar{L}(0, T_i) \geq L(0, T_i)$. The later implies that

$$\frac{\bar{B}(0, T_i)}{\bar{B}(0, T_{i+1})} \geq \frac{B(0, T_i)}{B(0, T_{i+1})}.$$

As in Kluge (2005) the prespecification of the forward default intensities $\hat{H}$ is postulated to be given by

$$\begin{aligned} \hat{H}(t, T_i) = H(0, T_i) \exp \Bigg(\int_0^t b^H(s, T_i, T_{i+1}) ds + \int_0^t \sqrt{c_s} \gamma(s, T_i) d\mathcal{B}_s^{T_{i+1}} \\ + \int_0^t \int_{\mathbb{R}} \langle \gamma(s, T_i), x \rangle (\mu - \nu^{T_{i+1}})(ds, dx) \Bigg), \end{aligned} \tag{4.2.4}$$

with initial condition

$$H(0, T_k) = \frac{1}{\delta_k} \left(\frac{\bar{B}(0, T_i) B(0, T_{i+1})}{B(0, T_i) \bar{B}(0, T_{i+1})} - 1 \right).$$

$\mathcal{B}^{T_{i+1}}$ is the Brownian motion defined in equation (3.4.4), $\nu^{T_{i+1}}$ the compensator defined in equation (3.4.5) and $b^H(s, T_i, T_{i+1})$ the drift term which will be specified in section 4.4. For the time being it can assumed that $b^H(s, T_i, T_{i+1}) = 0$ for $T_i < s \leq T^*$, i.e. $\hat{H}(t, T_i) = \hat{H}(T_i, T_i)$ is required for $t \in [T_i, T^*]$ (Eberlein *et al.*, 2006).

4.3 Construction of the Time of Default

As in Eberlein *et al.* (2006), the default time is constructed in a canonical way, i.e. for a given $\tilde{\mathbb{F}}$-hazard process (Γ) a stopping time (τ) has to be constructed on an enlarged probability space. The important step is selecting a specific hazard process such that the actual default intensities, H, match the pre-specified default intensities, $\hat{H}$, as given by equation (4.2.4).

Before presenting details of the canonical construction, basic definitions of the hazard process are briefly mentioned. This section is with reference to Bielecki and Rutkowski (2004) to whom the reader is referred to for further details.

4.3.1 Hazard Processes

Let τ denote a non-negative random variable on a probability space $(\sigma, \mathcal{G}, \mathbb{P})$ with filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$ such that $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$ for any $t \in \mathbb{R}_+$. H denotes a right-continuous process such that $H_t = \mathbb{1}_{\{\tau \leq t\}}$ with the filtration $\mathbb{H} : \mathcal{H}_t = \sigma(H_u : u \leq t)$. The process H_t defines the jump process associated to the random time τ. The following assumptions regarding the filtrations have to be made (Bielecki and Rutkowski, 2004):

(ℍℙ.1) Assume that an auxiliary filtration $\mathbb{F}$ such that $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ is given. Then $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$ for any $t \in \mathbb{R}_+$.

(ℍℙ.1a) For every $t \in \mathbb{R}_+$, the event $\{\tau \leq t\}$ belongs to the σ-field $\mathcal{F}_t$. So τ is an $\mathbb{F}$-stopping time.

Under condition (ℍℙ.1a), $\mathbb{G} = \mathbb{F}$ so that τ is also a $\mathbb{G}$-stopping time. Some models however, only postulate a partial observation of the random time τ. This leads to the next condition.

(ℍℙ.1b) For some dates $t \in \mathbb{R}_+$, the event $\{\tau \leq t\}$ does not belong to the σ-field $\mathcal{F}_t$.

Let $\hat{\mathbb{H}} \subset \mathbb{H}$ denote the filtration associated to the partial observations of τ. Then the enlarged filtration is given by $\mathbb{G} = \hat{\mathbb{H}} \vee \mathbb{F}$. Under condition (ℍℙ.1) the process H is $\mathbb{G}$-adapted, i.e τ is a $\mathbb{G}$-stopping time. However, under condition (ℍℙ.1b) H is not $\mathbb{G}$-adapted, i.e τ is not a $\mathbb{G}$-stopping time. In both cases the following condition is satisfied:

(ℍℙ.2) For every $t \in \mathbb{R}_+$, $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{H}_t \vee \mathcal{F}_t$.

For any $t \in \mathbb{R}_+$, G denotes the $\mathbb{F}$-survival process of τ given by

$$G_t := 1 - F_t = \mathbb{P}(\tau > t|\mathcal{F}_t) \ \forall\, t \in \mathbb{R}_+, \tag{4.3.1}$$

where $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$. The $\mathbb{F}$-hazard process of a random time is defined as follows:

Definition 4.1 ($\mathbb{F}$-hazard process)**.** *The $\mathbb{F}$-hazard process of random time τ under $\mathbb{P}$ is denoted by Γ and defined as*

$$\begin{aligned} &1 - F_t = e^{-\Gamma_t} \\ &\Longrightarrow \Gamma_t := -\ln G_t = -\ln(1 - F_t), \end{aligned}$$

$\forall\, t \in \mathbb{R}_t$ and $F_t < 1$.

From this point, Γ is referred to as the $\mathbb{F}$-hazard process of τ and $F_t < 1$ is always assumed to be true in order for Γ to be well-defined. The following lemma gives some useful results regarding the conditional expectations of τ given the filtrations $\mathcal{G}_t$ and $\mathcal{F}_t$:

Lemma 4.2. *(Bielecki and Rutkowski, 2004)*

(i) Assume that ($\mathbb{HP}$.2) holds. Then for any $\mathcal{G}$-measurable random variable Y and any $t \in \mathbb{R}_+$

$$\mathbb{E}_{\mathbb{P}}\Big[\mathbb{1}_{\{\tau>t\}}Y|\mathcal{G}_t\Big] = \mathbb{P}\big(\tau > t|\mathcal{G}_t\big)\frac{\mathbb{E}_{\mathbb{P}}\Big[\mathbb{1}_{\{\tau>t\}}Y|\mathcal{F}_t\Big]}{\mathbb{P}\big(\tau > t|\mathcal{F}_t\big)}. \tag{4.3.2}$$

(ii) If, in addition, $\mathcal{H}_t \subseteq \mathcal{G}_t$ (so that ($\mathbb{HP}$.1) holds) then

$$\mathbb{E}_{\mathbb{P}}\Big[\mathbb{1}_{\{\tau>t\}}Y|\mathcal{G}_t\Big] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}\Big[Y|\mathcal{G}_t\Big] = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}\Big[\mathbb{1}_{\{\tau>t\}}Y|\mathcal{F}_t\Big]}{\mathbb{P}\big(\tau > t|\mathcal{F}_t\big)}. \tag{4.3.3}$$

In particular, for any $t \le s$

$$\mathbb{P}\big(t < \tau \le s|\mathcal{G}_t\big) = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{P}\big(t < \tau \le s|\mathcal{F}_t\big)}{\mathbb{P}\big(\tau > t|\mathcal{F}_t\big)}. \tag{4.3.4}$$

Using definition 4.1 and equations (4.3.1) and (4.3.4), it follows that

$$\begin{aligned}
\mathbb{P}(t < \tau \le T\ |\mathcal{G}_t) &= \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}\Big[e^{-\Gamma_t} - e^{-\Gamma_T}|\mathcal{F}_t\Big]}{1 - F_t} \\
&= \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{P}}\Big[e^{-\Gamma_t} - e^{-\Gamma_T}|\mathcal{F}_t\Big]}{e^{-\Gamma_t}} \\
&= \mathbb{1}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}_{\mathbb{P}}\Big[e^{-\Gamma_t} - e^{-\Gamma_T}|\mathcal{F}_t\Big] \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}\Big[1 - e^{\Gamma_t-\Gamma_T}|\mathcal{F}_t\Big].
\end{aligned} \tag{4.3.5}$$

The hazard process, Γ, of τ is also assumed to have the following integral representation

$$\Gamma_t = \int_0^t \gamma_u du,\ \forall\, t\ \in\ \mathbb{R}_+,$$

where γ is the $\mathbb{F}$-hazard rate or $\mathbb{F}$-intensity of τ, also referred to as the *stochastic intensity* of τ. In terms of γ, equation (4.3.5) becomes

$$\begin{aligned}
\mathbb{P}(t < \tau \le T\ |\mathcal{G}_t) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}\Big[1 - e^{\int_0^t \gamma_u du - \int_0^T \gamma_u du}|\mathcal{F}_t\Big] \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}\Big[1 - e^{-\int_T^t \gamma_u du}|\mathcal{F}_t\Big].
\end{aligned} \tag{4.3.6}$$

The intensity of the time of default is referred to as the *intensity function* of τ when τ is non-random. When the trivial filtration $\mathbb{F}$ (such that $\mathbb{G} = \mathbb{H}$) is chosen as the reference filtration, the intensity function, $\gamma(t)$, is used instead of γ_t and so equation (4.3.6) becomes

$$\mathbb{P}(t < \tau \le T\ |\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{P}}\Big[1 - e^{-\int_T^t \gamma(u)du}|\mathcal{F}_t\Big].$$

4.3.2 Canonical Construction of τ

Here the construction of the default time associated to a given hazard process Γ is described. The method used is known as the *canonical construction*, for which details can be found in Bielecki and Rutkowski (2004). The discussion in this section is with reference to Bielecki and Rutkowski (2004), Kluge (2005) and Eberlein *et al.* (2006).

Let Γ be an $\tilde{\mathbb{F}}$-adapted, right-continuous, increasing process on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \mathbb{P}_{T^*})$ such that $\Gamma_0 = 0$ and $\lim_{t\to\infty} \Gamma_t = \infty$. The aim now is to construct the default time τ such that Γ is the $\mathbb{F}$-hazard process of τ. For this the probability space $\tilde{\Omega}$ has to be enlarged. By doing this, Γ will no longer be the $\mathbb{F}$-hazard process of τ under the measure $\mathbb{P}_{T^*}$, but rather the $\mathbb{F}$-hazard process under $\mathbb{Q}_{T^*}$, the extension of $\mathbb{P}_{T^*}$.

Let η is a random variable on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}_{T^*}})$ uniformly distributed over the interval $[0, 1]$. Consider the product space $(\Omega, \mathcal{G}, \mathbb{Q}_{T^*})$ defined by

$$\Omega := \tilde{\Omega} \times \hat{\Omega}, \quad \mathbb{G} := \tilde{\mathcal{F}} \otimes \hat{\mathcal{F}}, \quad \mathbb{Q}_{T^*} := \mathbb{P}_{T^*} \otimes \hat{\mathbb{P}},$$

and $\mathbb{F}$ the extension of $\tilde{\mathbb{F}}$ to the enlarged probability space $(\Omega, \mathcal{G}, \mathbb{Q}_{T^*})$. All stochastic processes from the default-free part of the model are extended to the enlarged probability space by setting $L^{T^*}(\tilde{\omega}, \hat{\omega}) := L^{T^*}(\tilde{\omega})$ (Eberlein *et al.*, 2006).

Define the random time $\tau : \Omega \to \mathbb{R}_+$ by

$$\tau := \inf\{t \in \mathbb{R}_+ : e^{-\Gamma_t} \leq \eta\}.$$

Recall $\mathcal{H}_t := \sigma(\mathbb{1}_{\tau \leq u} | 0 \leq u \leq t)$ and $\mathcal{G}_t := \mathcal{H}_t \vee \mathcal{F}_t$ for $t \in [0, T^*]$ as discussed in section 4.3.1. So τ is a stopping-time with respect to the filtration $\mathbb{G} = \{\mathcal{G}_s\}_{0 \leq s \leq T^*}$, since $\{\tau \leq t\} \in \mathcal{H}_t \subset \mathcal{G}_t$ (Eberlein *et al.*, 2006). Also, $\{\tau > t\} = \{\eta < e^{-\Gamma_t}\}$ and Γ_t is $\tilde{\mathcal{F}}$-measurable so that

$$\mathbb{Q}_{T^*}\big(\tau > t | \tilde{\mathcal{F}}\big) = \mathbb{Q}_{T^*}\big(\eta < e^{-\Gamma_t} | \tilde{\mathcal{F}}\big) = \tilde{\mathbb{P}}\big(\eta < e^{-\Gamma_t}\big) = e^{-\Gamma_t}. \tag{4.3.7}$$

Also,

$$1 - F_t = \mathbb{Q}_{T^*}\big(\tau > t | \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}_{T^*}}\Big[\mathbb{Q}_{T^*}\big(\tau > t | \tilde{\mathcal{F}}\big) | \mathcal{F}_t\Big] = \mathbb{E}_{\mathbb{Q}_{T^*}}\Big[e^{-\Gamma_t} | \mathcal{F}_t\Big] = e^{-\Gamma_t}. \tag{4.3.8}$$

So Γ is the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}_{T^*}$.

From (4.3.7) and (4.3.8) the following interesting property of the canonical construction of τ is obtained:

$$\mathbb{Q}_{T^*}\big(\tau \leq t | \tilde{\mathcal{F}}\big) = \mathbb{Q}_{T^*}\big(\tau \leq t | \mathcal{F}_t\big) \; \forall \, t \in \mathbb{R}_+. \tag{4.3.9}$$

Moreover, for $0 \leq s \leq t \leq T^*$ (Eberlein *et al.*, 2006)

$$\mathbb{Q}_{T^*}\big(\tau > s | \mathcal{F}_{T^*}\big) = \mathbb{Q}_{T^*}\big(\tau > s | \mathcal{F}_t\big) = \mathbb{Q}_{T^*}\big(\tau > s | \mathcal{F}_s\big) = e^{-\Gamma_s}. \tag{4.3.10}$$

The question now is whether or not $\mathcal{L}^{T^*}$ is a time-inhomogeneous Lévy process with respect to $\mathbb{Q}_{T^*}$ and the enlarged filtration $\mathbb{G}$. As a result of the canonical construction of the default time τ and equation (4.3.9), the following conditions are satisfied:

($\mathbb{C}$.1) For any $t \in [0, T^*]$, $\mathcal{F}_{T^*}$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$ under the measure $\mathbb{Q}_{T^*}$, i.e. for any bounded $\mathcal{F}_{T^*}$-measurable variable X and any $\mathcal{H}_t$-measurable variable Y

$$\mathbb{E}_{\mathbb{Q}_{T^*}}\big[XY|\mathcal{F}_t\big] = \mathbb{E}_{\mathbb{Q}_{T^*}}\big[X|\mathcal{F}_t\big]\mathbb{E}_{\mathbb{Q}_{T^*}}\big[Y|\mathcal{F}_t\big].$$

($\mathbb{C}$.2) For any bounded, $\mathcal{F}_{T^*}$-measurable random variable X

$$\mathbb{E}_{\mathbb{Q}_{T^*}}\big[X|\mathcal{G}_t\big] = \mathbb{E}_{\mathbb{Q}_{T^*}}\big[X|\mathcal{F}_t\big], \quad t \in [0, T^*].$$

Conditions ($\mathbb{C}$.1) and ($\mathbb{C}$.2) are equivalent. For further discussion on these conditions the reader is referred to section 6.1 (pg 166-167) in Bielecki and Rutkowski (2004) and Kluge (2005) for a detailed proof thereof.

Proposition 4.3. *(Kluge, 2005) $\mathcal{L}^{T^*}$ is a non-homogeneous Lévy process on the stochastic basis $(\Omega, \mathcal{G}_{T^*}, \mathbb{G}, \mathbb{Q}_{T^*})$ with characteristics $(0, c, F^{T^*})$.*

Proof. $\mathcal{L}^{T^*}$ is an adapted, càdlàg process satisfying $\mathcal{L}_0^{T^*} = 0$. Using the definition of both the extended probability measure $\mathbb{Q}_{T^*}$ and the process $\mathcal{L}^{T^*}(\tilde{\omega}, \hat{\omega})$, the characteristic function of $\mathcal{L}^{T^*}$ under $\mathbb{Q}_{T^*}$ is given by

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_{T^*}}\Big[\exp\big(iu\mathcal{L}_t^{T^*}\big)\Big] &= \int_{\mathbb{Q}_{T^*}} \exp\big(iu\mathcal{L}_t^{T^*}\big)d\mathbb{Q}_{T^*} \\
&= \int_{\tilde{\Omega}\times\hat{\Omega}} \exp\big(iu\mathcal{L}_t^{T^*}(\tilde{\omega}, \hat{\omega})\big)d\big(\mathbb{P}_{T^*} \otimes \hat{\mathbb{P}}\big)(\tilde{\omega}, \hat{\omega}) \\
&= \int_{\tilde{\Omega}} \exp\big(iu\mathcal{L}_t^{T^*}(\tilde{\omega})\big)d\big(\mathbb{P}_{T^*}\big)(\tilde{\omega}) \\
&= \mathbb{E}_{\mathbb{P}_{T^*}}\Big[\exp\big(iu\mathcal{L}_t^{T^*}\big)\Big].
\end{aligned}$$

So the characteristic function of $\mathcal{L}_t^{T^*}$ is preserved under $\mathbb{Q}_{T^*}$. It remains to show that the increment $\mathcal{L}_t^{T^*} - \mathcal{L}_s^{T^*}$ is independent of $\mathcal{G}_s$ for $s < t$. Let $B \in \mathfrak{B}^d$ and $A \in \mathcal{G}_s$, then using condition ($\mathbb{C}$.2) for $X := \mathbb{1}_B(\mathcal{L}_t^{T^*} - \mathcal{L}_s^{T^*})$ and the fact that $\mathcal{L}_t^{T^*} - \mathcal{L}_s^{T^*}$ is independent of $\mathcal{F}_s$ it follows that

$$\begin{aligned}
\mathbb{Q}_{T^*}\Big(A \cap \big\{\big(\mathcal{L}_t^{T^*} - \mathcal{L}_s^{T^*}\big) \in B\big\}\Big) &= \int_A \mathbb{1}_B\big(\mathcal{L}_t^{T^*} - \mathcal{L}_s^{T^*}\big)d\mathbb{Q}_{T^*} \\
&= \int_A \mathbb{E}_{\mathbb{Q}_{T^*}}\Big[\mathbb{1}_B(\mathcal{L}_t^{T^*} - \mathcal{L}_s^{T^*})|\mathcal{F}_s\Big]d\mathbb{Q}_{T^*} \\
&= \int_A \mathbb{E}_{\mathbb{Q}_{T^*}}\Big[\mathbb{1}_B(\mathcal{L}_t^{T^*} - \mathcal{L}_s^{T^*})\Big]d\mathbb{Q}_{T^*} \\
&= \mathbb{Q}_{T^*}(A)\mathbb{Q}_{T^*}\Big(\big\{\big(\mathcal{L}_t^{T^*} - \mathcal{L}_s^{T^*}\big) \in B\big\}\Big).
\end{aligned}$$

□

The following lemma further shows that Γ is not only the $\mathbb{F}$-hazard process of the default time τ with respect to the measure $\mathbb{Q}_{T^*}$, but also the $\mathbb{F}$-hazard process of τ under all other forward measures.

Lemma 4.4. *(Kluge, 2005) Γ is the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}_{T_k}$, $\forall\, k \in \{1,\ldots,n\}$.*

Proof. Fix a k and let ψ denote the Radon-Nikodym derivative of $\mathbb{Q}_{T_k}$ with respect to $\mathbb{Q}_{T^*}$. Using abstract Bayes rule and the conditional expectation given in condition ($\mathbb{C}$.1) it follows that

$$\begin{aligned}
\mathbb{Q}_{T_k}\big(\tau > s|\mathcal{F}_s\big) &= \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}\Big[\psi\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s\Big]}{\mathbb{E}_{\mathbb{Q}_{T^*}}\Big[\psi|\mathcal{F}_s\Big]} \\
&= \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}\Big[\psi|\mathcal{F}_s\Big].\mathbb{E}_{\mathbb{Q}_{T^*}}\Big[\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s\Big]}{\mathbb{E}_{\mathbb{Q}_{T^*}}\Big[\psi|\mathcal{F}_s\Big]} \\
&= \mathbb{E}_{\mathbb{Q}_{T^*}}\Big[\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s\Big] \\
&= \mathbb{Q}_{T^*}\big(\tau > s|\mathcal{F}_s\big) \\
&= 1 - F_s \\
&= e^{-\Gamma_s}.
\end{aligned}$$

□

A specific hazard process Γ has to be chosen so that the forward default intensities H can coincide its prespecification $\hat{H}$. As in section 4.2, for the defaultable model to be consistent, the value of the defaultable bond has to be

$$B^0(t,T_k) = B(t,T_k)\mathbb{Q}_{T_k}\{\tau > T_k|\mathcal{G}_t\} \tag{4.3.11}$$

Applying equation (4.3.3) from lemma 4.2, equation (4.3.11) becomes

$$\begin{aligned}
B^0(t,T_k) &= B(t,T_k)\mathbb{1}_{\{\tau>T_k\}}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}\Big[\mathbb{1}_{\{\tau>T_k\}}|\mathcal{F}_t\Big]}{\mathbb{Q}_{T_k}\big(\tau > T_k|\mathcal{F}_t\big)} \\
&= B(t,T_k)\mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}\Big[\mathbb{1}_{\{\tau>T_k\}}|\mathcal{F}_t\Big]}{\mathbb{E}_{\mathbb{Q}_{T_k}}\Big[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\Big]}.
\end{aligned} \tag{4.3.12}$$

From equation (4.3.12) the pre-default value of the bond can be given by

$$\begin{aligned}
\bar{B}(t,T_k) &= B(t,T_k)\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}\Big[\mathbb{1}_{\{\tau>T_k\}}|\mathcal{F}_t\Big]}{\mathbb{E}_{\mathbb{Q}_{T_k}}\Big[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t\Big]} \qquad (4.3.13) \\
&= B(t,T_k)\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}\Big[\mathbb{1}_{\{\tau>T_k\}}|\mathcal{F}_t\Big]}{e^{-\Gamma_t}},
\end{aligned}$$

and so the default risk factors become

$$\begin{aligned}
D(t,T_k) &= \frac{\mathbb{E}_{\mathbb{Q}_{T_k}}\left[\mathbb{1}_{\{\tau>T_k\}}|\mathcal{F}_t\right]}{e^{-\Gamma_t}} \\
&= \mathbb{E}_{\mathbb{Q}_{T_k}}\left[e^{\Gamma_t-\Gamma_{T_k}}|\mathcal{F}_t\right].
\end{aligned}$$

Hence the forward default intensities can now be given by

$$\begin{aligned}
H(t,T_k) &= \frac{1}{\delta_k}\left(\frac{D(t,T_k)}{D(t,T_{k+1})}-1\right) \\
&= \frac{1}{\delta_k}\left(\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}\left[e^{\Gamma_t-\Gamma_{T_k}}|\mathcal{F}_t\right]}{\mathbb{E}_{\mathbb{Q}_{T_k}}\left[e^{\Gamma_t-\Gamma_{T_{k+1}}}|\mathcal{F}_t\right]}-1\right). \\
&= \frac{1}{\delta_k}\left(\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}\left[e^{-\Gamma_{T_k}}|\mathcal{F}_t\right]}{\mathbb{E}_{\mathbb{Q}_{T_k}}\left[e^{-\Gamma_{T_{k+1}}}|\mathcal{F}_t\right]}-1\right) \qquad (4.3.14)
\end{aligned}$$

From equation (4.3.14), it is clear that the matching of H to the prespecification $\hat{H}$ only depends on the dynamics of the hazard process Γ_{T_k} for $k \in \{1,\ldots,n\}$. Using equation (4.3.3) in lemma 4.2 and equation (4.3.13), it follows that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_{T_k}}\left[e^{-\Gamma_{T_k}}|\mathcal{F}_{T_{k-1}}\right] &= \mathbb{E}_{\mathbb{Q}_{T_k}}\left[\mathbb{1}_{\{\tau>T_k\}}\middle|\mathcal{F}_{T_{k-1}}\right] \\
&= \mathbb{E}_{\mathbb{Q}_{T_k}}\left[\mathbb{E}_{\mathbb{Q}_{T_k}}\left[\mathbb{1}_{\{\tau>T_k\}}\middle|\mathcal{G}_{T_{k-1}}\right]\middle|\mathcal{F}_{T_{k-1}}\right] \\
&= \mathbb{E}_{\mathbb{Q}_{T_k}}\left[\mathbb{1}_{\{\tau>T_{k-1}\}}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}\left[\mathbb{1}_{\{\tau>T_k\}}\middle|\mathcal{F}_{T_{k-1}}\right]}{\mathbb{E}_{\mathbb{Q}_{T_k}}\left[\mathbb{1}_{\{\tau>T_{k-1}\}}\middle|\mathcal{F}_{T_{k-1}}\right]}\middle|\mathcal{F}_{T_{k-1}}\right] \\
&= \mathbb{E}_{\mathbb{Q}_{T_k}}\left[\mathbb{1}_{\{\tau>T_{k-1}\}}\frac{\bar{B}(T_{k-1},T_k)}{B(T_{k-1},T_k)}\middle|\mathcal{F}_{T_{k-1}}\right] \\
&= \frac{\bar{B}(T_{k-1},T_k)}{B(T_{k-1},T_k)}\mathbb{E}_{\mathbb{Q}_{T_k}}\left[\mathbb{1}_{\{\tau>T_{k-1}\}}\middle|\mathcal{F}_{T_{k-1}}\right] \\
&= \frac{\bar{B}(T_{k-1},T_k)}{B(T_{k-1},T_k)}e^{-\Gamma_{T_{k-1}}} \\
&= e^{-\Gamma_{T_{k-1}}}D(T_{k-1},T_k) \\
&= e^{-\Gamma_{T_{k-1}}}\frac{1}{1+\delta_{k-1}H(T_{k-1},T_{k-1})} \quad , \qquad (4.3.15)
\end{aligned}$$

where equation (4.3.15) follows from the definition of $D(t,T_k)$. The hazard process Γ can now be defined recursively with the initial condition $\Gamma_0 := 0$ so that

$$\begin{aligned}\Gamma_{T_k} := &\ \Gamma_{T_{k-1}} + \log\left(1+\delta_{k-1}\hat{H}(T_{k-1},T_{k-1})\right),\ \text{for } k\in\{1,\dots,n\}\\ = &\sum_{i=0}^{k-1}\log\left(1+\delta_{k-1}\hat{H}(T_{k-1},T_{k-1})\right). \end{aligned} \tag{4.3.16}$$

It now remains to check if the implied dynamics of H match that of the prespecification $\hat{H}$. Using equation (4.3.14) it follows that

$$\begin{aligned} H(t,T_1) &= \frac{1}{\delta_1}\left(\frac{\mathbb{E}_{\mathbb{Q}_{T_1}}\left[e^{-\Gamma_{T_1}}|\mathcal{F}_t\right]}{\mathbb{E}_{\mathbb{Q}_{T_2}}\left[e^{-\Gamma_{T_2}}|\mathcal{F}_t\right]}-1\right)\\ &= \frac{1}{\delta_1}\left(\frac{1}{\mathbb{E}_{\mathbb{Q}_{T_2}}\left[e^{-\Gamma_{T_2}+\Gamma_{T_1}}|\mathcal{F}_t\right]}-1\right),\end{aligned}$$

where from equation (4.3.16)

$$\begin{aligned} -\Gamma_{T_2}+\Gamma_{T_1} &= \log\left(1+\delta_1\hat{H}(T_1,T_1)\right)\\ \implies e^{-\Gamma_{T_2}+\Gamma_{T_1}} &= \frac{1}{1+\delta_1\hat{H}(T_1,T_1)},\end{aligned}$$

so that $H(t,T_1)$ now becomes

$$H(t,T_1) = \frac{1}{\delta_1}\left(\frac{1}{\mathbb{E}_{\mathbb{Q}_{T_2}}\left[\frac{1}{1+\delta_1\hat{H}(T_1,T_1)}|\mathcal{F}_t\right]}-1\right),$$

which can be written as

$$\mathbb{E}_{\mathbb{Q}_{T_2}}\left[\frac{1}{1+\delta_1\hat{H}(T_1,T_1)}\middle|\mathcal{F}_t\right] = \frac{1}{1+\delta_1 H(T_1,T_1)}. \tag{4.3.17}$$

This means that if $\left(\frac{1}{1+\delta_1\hat{H}(T_1,T_1)}\right)_{0\le t\le T_1}$ is a martingale with respect to $\mathbb{Q}_{T_2}$, then $H(T_1,T_1)$ coincides with its prespecification $\hat{H}(T_1,T_1)$. The following general result can be stated.

Lemma 4.5. *(Kluge, 2005) $H(\cdot,T_k)$ meets its prespecification $\hat{H}$ if $\left(\prod_{i=1}^{l}\frac{1}{1+\delta_i\hat{H}(t,T_i)}\right)_{0\le t\le T_l}$ is a $\mathbb{Q}_{T_{l+1}}$-martingale for all $l\in\{1,\dots,k\}$.*

Proof. The case of $k=1$ has been proven (equation (4.3.17)). Using equations (4.3.14), (4.3.16) and the requirement of $\hat{H}(t,T_i)=\hat{H}(T_i,T_i)$ for $t\in[T_i,T^*]$ as mentioned in section 4.2, it follows that

$$\begin{aligned} H(t,T_k) &= \frac{1}{\delta_k}\left(\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}\left[\prod_{i=0}^{k-1}\frac{1}{1+\delta_i\hat{H}(T_i,T_i)}|\mathcal{F}_t\right]}{\mathbb{E}_{\mathbb{Q}_{T_{k+1}}}\left[\prod_{i=0}^{k}\frac{1}{1+\delta_i\hat{H}(T_i,T_i)}|\mathcal{F}_t\right]}-1\right)\\ &= \frac{1}{\delta_k}\left(\left(1+\delta_k\hat{H}(t,T_k)\right)-1\right) = \hat{H}(t,T_k). \end{aligned}$$

□

4.4 Specification of the Drift

Recall that the drift coefficients $b^H(\cdot, T_k, T_{k+1})$ from equation (4.2.4) still have to be specified so that the requirement of lemma 4.5 is satisfied. This specification of the drift is done recursively as in Kluge (2005). First a process $b^H(\cdot, T_1, T_2)$ has to be specified so that $\left(\frac{1}{1+\delta_1 \hat{H}(t,T_1)}\right)_{0\le t\le T_1}$ becomes a $\mathbb{Q}_{T_2}$-martingale. Then $b^H(\cdot, T_2, T_3)$ must be specified so that $\left(\frac{1}{\left(1+\delta_1 \hat{H}(t,T_1)\right)\left(1+\delta_2 \hat{H}(t,T_2)\right)}\right)_{0\le t\le T_2}$ becomes a $\mathbb{Q}_{T_3}$-martingale and so on. The following two lemmata will be required:

Lemma 4.6. *(Kluge, 2005) Let X be a real-valued semimartingale with $X_0 = 0$ and $\Delta X > -1$. Then*

$$\left(\mathcal{E}(X)\right)^{-1} = \mathcal{E}\Big(-x + \langle X^c, X^c\rangle + \Big(\frac{1}{1+x} - 1 + c\Big) * \mu\Big).$$

Here $\mathcal{E}$ denotes the *stochastic exponential* [3] of X and $\langle X^c, X^c\rangle$ its *quadratic covariation* [4]

Lemma 4.7. *(Kluge, 2005) For $k \in \{2, \dots, n\}$ and ß $\in \{1, \dots, k-1\}$*

$$\begin{aligned}\hat{H}(t,T_i) =& H(0,T_i)\mathcal{E}_t\Bigg(\int_0^\bullet a(s,T_i,T_k)ds + \int_0^\bullet \sqrt{c}_s \gamma(s,T_i) d\mathcal{B}_s^{T_k} \\ &+ \int_0^\bullet \int_{\mathbb{R}^d} \Big(e^{\langle \gamma(s,T_i),x\rangle} - 1\Big)\big(\mu - \nu^{T_k}\big)(ds,dx)\Bigg),\end{aligned}$$

where

$$\begin{aligned}a(s,T_i,T_k) :=& b^H(s,T_i,T_k) + \frac{1}{2}\big\langle \gamma(s,T_i), c_s\gamma(s,T_i)\big\rangle \\ &+ \int_{\mathbb{R}^d}\Big(e^{\langle\gamma(s,T_i),x\rangle} - 1 - \big\langle\gamma(s,T_i),x\big\rangle\Big)\mathbf{F}_s^{T_k}(dx), \end{aligned} \tag{4.4.1}$$

and

$$\begin{aligned}b^H(s,T_i,T_k) :=& b^H(s,T_i,T_{i+1}) - \Bigg\langle \gamma(s,T_i), c_s\Bigg(\sum_{l=i+1}^{k-1} \alpha(s,T_l,T_{l+1})\Bigg)\Bigg\rangle \\ &- \int_{\mathbb{R}^d} \big\langle\gamma(s,T_i),x\big\rangle\Bigg(\prod_{l=i+1}^{k-1}\beta(s,x,T_l,T_{l+1}) - 1\Bigg)\mathbf{F}_s^{T_k}(dx).\end{aligned} \tag{4.4.2}$$

[3]Definition and details of the stochastic exponential can be found in Jacob and Shiryaev (2003), chapter 8.

[4]Definition of the quadratic covariation can be found on page 22.

Proofs of lemma 4.6 and 4.7 can be found in Kluge (2005). The reader is also referred to Kallsen and Shiryaev (2002), Lemma 2.6 which is used in these proofs.

The first specification of $b^h(\cdot, T_i, T_{i+1})$ is given in the following proposition:

Proposition 4.8. *(Kluge, 2005)* $\left(\frac{1}{1+\delta_1 \hat{H}(t,T_1)}\right)_{0\le t\le T_1}$ *is a $\mathbb{Q}_{T_2}$-martingale if for $s \in [0, T_1]$*

$$\begin{aligned} b^H(s,T_1,T_2) =& \left(Y^1_{s-} - \frac{1}{2}\right)\big\langle \gamma(s,T_1), c_s\gamma(s,T_1)\big\rangle \\ &+ \int_{\mathbb{R}^d} \left(\big\langle \gamma(s,T_1), x\big)\big\rangle - \frac{e^{\langle\gamma(s,T_1),x)\rangle} - 1}{1 + Y^1_{s-}\left(e^{\langle\gamma(s,T_1),x)\rangle} - 1\right)}\right)\mathbf{F}^{T_2}_s(dx), \end{aligned} \tag{4.4.3}$$

where $Y^1_s := \frac{\delta_1 \hat{H}(s,T_1)}{1+\delta_1\hat{H}(s,T_1)}$.

Proof. From Lemma 4.7 it follows that

$$\begin{aligned} \hat{H}(t,T_1) =& H(0,T_1)\mathcal{E}_t\left(\int_0^\bullet a(s,T_1,T_2)ds + \int_0^\bullet \sqrt{c_s}\gamma(s,T_1)d\mathcal{B}^{T_2}_s\right. \\ &\left. + \int_0^\bullet \int_{\mathbb{R}^d} \left(e^{\langle\gamma(s,T_1),x\rangle} - 1\right)\left(\mu - \nu^{T_2}\right)(ds,dx)\right), \end{aligned}$$

with

$$\begin{aligned} a(s,T_1,T_2) =& b^H(s,T_1,T_2) + \frac{1}{2}\big\langle\gamma(s,T_1), c_s\gamma(s,T_1)\big\rangle \\ &+ \int_{\mathbb{R}^d}\left(e^{\langle\gamma(s,T_1),x\rangle} - 1 - \big\langle\gamma(s,T_1),x\big\rangle\right)\mathbf{F}^{T_2}_s(dx). \end{aligned} \tag{4.4.4}$$

Let $X^1_t := 1 + \delta_1\hat{H}(t,T_1)$ and apply Itô-Lévy formula to $\hat{H}(t,T_1)$ from equation (4.2.4) then

$$\begin{aligned} dX^1_t &= \delta_1 d\hat{H}(t,T_1) \\ &= \delta_1\hat{H}(t,T_1)\left(a(t,T_1,T_2)dt + \sqrt{c_t}\gamma(t,T_1)d\mathcal{B}^{T_2}_t\right. \\ &\qquad\left. + \int_{\mathbb{R}^d}\left(e^{\langle\gamma(t,T_1),x\rangle} - 1\right)\left(\mu - \nu^{T_2}\right)(dt,dx)\right) \\ &= X^1_{t-}\left(Y^1_{t-}a(t,T_1,T_2)dt + Y^1_{t-}\sqrt{c_t}\gamma(t,T_1)d\mathcal{B}^{T_2}_t\right. \\ &\qquad\left. + \int_{\mathbb{R}^d} Y^1_{t-}\left(e^{\langle\gamma(t,T_1),x\rangle} - 1\right)\left(\mu - \nu^{T_2}\right)(dt,dx)\right). \end{aligned}$$

Lemma 4.6 along with the fact that $dX_t = X^1_{t-}dZ_t \Longrightarrow X_t = X_0\mathcal{E}(Z_t)$ implies that

$$\left(X^1_t\right)^{-1} = \left(X^1_0\right)^{-1}\mathcal{E}_t\Bigg(\int_0^\bullet A(s,T_2)ds - \int_0^\bullet Y^1_{s-}\sqrt{c_s}\gamma(s,T_1)d\mathcal{B}^{T_2}_s \\ + \int_0^\bullet \int_{\mathbb{R}^d} \left(\left(1 + Y^1_{s-}(e^{\langle\gamma(s,T_1),x\rangle} - 1)\right)^{-1} - 1\right)\left(\mu - \nu^{T_2}\right)(ds,dx)\Bigg),$$

where

$$A(s,T_2) := -Y^1_{s-}a(s,T_1,T_2) + (Y^1_{s-})^2\big\langle\gamma(s,T_1), c_s\gamma(s,T_1)\big\rangle \\ + \int_{\mathbb{R}^d} Y^1_{s-}\left(e^{\langle\gamma(s,T_1),x\rangle} - 1 - \frac{e^{\langle\gamma(s,T_1),x\rangle} - 1}{1 + Y^1_{s-}\left(e^{\langle\gamma(s,T_1),x\rangle} - 1\right)}\right)\mathbf{F}^{T_2}_s(dx). \tag{4.4.5}$$

If $A(s,T_2) \equiv 0$, then $\left(1 + \delta_1\hat{H}(t,T_1)\right)^{-1}$ is a $\mathbb{Q}_{T_2}$-local martingale. Since it is bounded by 0 and 1 it is also a martingale.

Setting $A(s,T_2)$ in equation (4.4.5) to 0 implies that

$$Y^1_{s-}a(s,T_1,T_2) =(Y^1_{s-})^2\big\langle\gamma(s,T_1), c_s\gamma(s,T_1)\big\rangle \\ + \int_{\mathbb{R}^d} Y^1_{s-}\left(e^{\langle\gamma(s,T_1),x\rangle} - 1 - \frac{e^{\langle\gamma(s,T_1),x\rangle} - 1}{1 + Y^1_{s-}\left(e^{\langle\gamma(s,T_1),x\rangle} - 1\right)}\right)\mathbf{F}^{T_2}_s(dx).$$

Substituting $a(s,T_1,T_2)$ from equation (4.4.4) it follows that

$$\begin{aligned} b^H(s,T_1,T_2) =& Y^1_{s-}\big\langle\gamma(s,T_1), c_s\gamma(s,T_1)\big\rangle - \frac{1}{2}\big\langle\gamma(s,T_1), c_s\gamma(s,T_1)\big\rangle \\ & - \int_{\mathbb{R}^d}\left(e^{\langle\gamma(s,T_1),x\rangle} - 1 - \big\langle\gamma(s,T_1),x\big\rangle\right)\mathbf{F}^{T_2}_s(dx) \\ & + \int_{\mathbb{R}^d}\left(e^{\langle\gamma(s,T_1),x\rangle} - 1 - \frac{e^{\langle\gamma(s,T_1),x\rangle} - 1}{1 + Y^1_{s-}\left(e^{\langle\gamma(s,T_1),x\rangle} - 1\right)}\right)\mathbf{F}^{T_2}_s(dx) \\ =& \left(Y^1_{s-} - \frac{1}{2}\right)\big\langle\gamma(s,T_1), c_s\gamma(s,T_1)\big\rangle \\ & + \int_{\mathbb{R}^d}\left(\big\langle\gamma(s,T_1),x)\big\rangle - \frac{e^{\langle\gamma(s,T_1),x\rangle} - 1}{1 + Y^1_{s-}\left(e^{\langle\gamma(s,T_1),x\rangle} - 1\right)}\right)\mathbf{F}^{T_2}_s(dx), \end{aligned}$$

which completes the proof. □

More generally, the following proposition is stated:

Proposition 4.9. *(Kluge, 2005)* $\left(\prod_{i=1}^{k-1}\frac{1}{1+\delta_i\hat{H}(t,T_i)}\right)_{0\le t\le T_{k-1}}$ *is a martingale with respect to* $\mathbb{Q}_{T_k}$ *for* $k\in\{2,\dots,n\}$ *if* $\forall\, i\in\{1,\dots,k-1\}$ *and* $s\in[0,T_1]$

$$\begin{aligned}
b^H(s,T_i,T_k) &= \sum_{j=1}^{i} Y_{s-}^j\big\langle\gamma(s,T_j),c_s\gamma(s,T_j)\big\rangle-\frac{1}{2}\big\langle\gamma(s,T_i),c_s\gamma(s,T_i)\big\rangle\\
&\quad+\sum_{j=1}^{i-1}\left(\frac{Y_{s-}^j}{Y_{s-}^i}\big\langle\gamma(s,T_j),c_s\alpha(s,T_i,T_{i+1})\big\rangle\right)\\
&\quad+\int_{\mathbb{R}^d}\left(\big\langle\gamma(s,T_i),x\big\rangle-\frac{e^{\langle\gamma(s,T_i),x\rangle}-1}{\prod_{j=1}^{i}\Big(1+Y_{s-}^j\big(e^{\langle\gamma(s,T_j),x\rangle}-1\big)\Big)}\right)\mathbf{F}_s^{T_{i+1}}(dx)\\
&\quad+\big(Y_{s-}^i\big)^{-1}\int_{\mathbb{R}^d}\big(\beta(s,x,T_i,T_{i+1})-1\big)\\
&\qquad\qquad\times\left(1-\prod_{j=1}^{i-1}\Big(1+Y_{s-}^j\big(e^{\langle\gamma(s,T_j),x\rangle}-1\big)\Big)^{-1}\right)\mathbf{F}_s^{T_{i+1}}(dx),
\end{aligned}\tag{4.4.6}$$

where $Y_s^i := \frac{\delta_i\hat{H}(s,T_i)}{1+\delta_i\hat{H}(s,T_i)}$.

The proof of the above proposition is calculationally intense. The reader is therefore referred to Appendix A in Kluge (2005) for explicit details thereof.

The drift $b^H(s,T_i,T_{i+1})$ however cannot be specified by equation (4.4.6) since it involves the Y_s^i term which depends on $\hat{H}(s,T_i)$ which in turn depends on $b^H(s,T_i,T_{i+1})$. For this reason a stochastic differential equation has to be introduced. Suppose that there exists a stochastic differential equation, say $h(t,T_i)$, having a unique solution for each $i\in\{1,\dots,k-1\}$ such that

$$\begin{aligned}
h(t,T_i) = h(0,T_i) &+ \int_0^t f^i\big(s,h(s-,T_i)\big)ds+\int_0^t\sqrt{c_s}\gamma(s,T_i)d\mathcal{B}_s^{T_{i+1}}\\
&+\int_0^t\int_{\mathbb{R}^d}\langle\gamma(s,T_j),x)\rangle\big(\mu-\nu^{T_{i+1}}\big)(ds,dx),
\end{aligned}\tag{4.4.7}$$

with

$$h(0,T_i) := \log H(0,T_i)$$

and

$$f^i(s,x) := f_1^i(s)+f_2^i(s,x)+f_3^i(s,x)+f_2^i(s,x),$$

where

$$
\begin{aligned}
f_1^i(s) &:= \sum_{j=1}^{i-1} \frac{\delta_j e^{h(s-,T_j)}}{1+\delta_j e^{h(s-,T_j)}} \langle \gamma(s,T_j), c_s \gamma(s,T_j) \rangle - \frac{1}{2} \langle \gamma(s,T_i), c_s \gamma(s,T_i) \rangle \\
&\quad - \int_{\mathbb{R}^d} \left(e^{\langle \gamma(s,T_i), y \rangle} - 1 - \langle \gamma(s,T_i), y \rangle \right) \mathbf{F}_s^{T_{i+1}}(dy) \\
f_2^i(s,x) &:= \frac{\delta_i e^x}{1+\delta_i e^x} \langle \gamma(s,T_i), c_s \gamma(s,T_i) \rangle \\
&\quad + \frac{1+\delta_i e^x}{\delta_i e^x} \sum_{j=1}^{i-1} \left(\frac{\delta_j e^{h(s-,T_j)}}{1+\delta_j e^{h(s-,T_j)}} \langle \gamma(s,T_j), c_s \alpha(s,T_i,T_{i+1}) \rangle \right) \\
f_3^i(s,x) &:= \frac{1+\delta_i e^x}{\delta_i e^x} \int_{\mathbb{R}^d} \left(\beta(s,y,T_i,T_{i+1}) - 1 \right) \qquad (4.4.8) \\
&\quad \times \left(1 - \prod_{j=1}^{i-1} \left(1 + \frac{\delta_j e^{h(s-,T_j)}}{1+\delta_j e^{h(s-,T_j)}} \left(e^{\langle \gamma(s,T_j), y \rangle} - 1 \right) \right)^{-1} \right) \mathbf{F}_s^{T_{i+1}}(dy),
\end{aligned}
$$

and

$$
\begin{aligned}
f_4^i(s,x) &:= \int_{\mathbb{R}^d} \left(e^{\langle \gamma(s,T_i), y \rangle} - 1 \right) \left(1 - \left(1 + \frac{\delta_i e^x}{1+\delta_i e^x} \left(e^{\langle \gamma(s,T_i), y \rangle} - 1 \right) \right)^{-1} \right. \\
&\quad \left. \times \prod_{j=1}^{i-1} \left(1 + \frac{\delta_j e^{h(s-,T_j)}}{1+\delta_j e^{h(s-,T_j)}} \left(e^{\langle \gamma(s,T_j), y \rangle} - 1 \right) \right)^{-1} \right) \mathbf{F}_s^{T_{i+1}}(dy).
\end{aligned}
$$

Then $\hat{H}(s,T_i) := \exp h(s,T_i)$ satisfies equation (4.2.4) with drift $b^H(s,T_i,T_{i+1})$ given by equation (4.4.6) and so proposition 4.9 holds. In order to show that the stochastic differential equation (4.4.7) has a unique solution the following theorem which is a direct consequence of Theorems V.6 and V.7 of Protter (2004) is stated:

Theorem 4.10. *(Kluge, 2005) Assume a (1 -dimensional) semimartingale Z with $Z_0 = 0$ on a complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ to be given and let $f : \mathbb{R}_+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ be such that*

1. *for fixed $x \in \mathbb{R}$, $(t,\omega) \mapsto f(t,\omega,x)$ is an adapted càdlàg process,*
2. *there exists a finite random variable K such that for all $t \in \mathbb{R}_+$*

$$|f(t,\omega,x) - f(t,\omega,y)| \le K(\omega)|x-y|.$$

Then the stochastic differential equation

$$X_t + X_0 + Z_t + \int_0^t f(s,.,X_{s-})ds,$$

where X_0 is a constant, has a unique (strong) solution. This solution is a semimartingale.

Unfortunately, f_2^i and f_3^i in equation (4.4.7) do not satisfy condition 2 of theorem 4.10, i.e they are not globally Lipschitz. However, this condition can be weakened by assuming that f is locally Lipschitz and satisfies a growth condition, as shown in the following proposition:

Proposition 4.11. *(Kluge, 2005) Assume a d-dimensional special semimartingale* $S := \int_0^\bullet \sqrt{c_s} d\mathcal{B}_s + \int_0^\bullet \int_{\mathbb{R}^d} x(\mu - \nu)(ds, dx)$ *on a complete stochastic basis* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$*, where* $\mathcal{B}$ *is a standard Brownian motion,* c *is deterministic, and* μ *is the random measure associated with the jumps of* S *with compensator* $\nu(ds, dx) = \mathbf{F}_s(dx)ds$*, is given. Suppose that* $\sigma : \mathbb{R}_+ \to \mathbb{R}^d$ *is a bounded function and let* $f : \mathbb{R}_+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ *be such that*

1. *for fixed* $x \in \mathbb{R}$, $(t, \omega) \mapsto f(t, \omega, x)$ *is an adapted càdlàg process,*
2. *for all* $r > 0$ *there is a real number* K_r *such that for all* (t, ω) *and all* $x, y \in \mathbb{R}$ *with* $|x|, |y| \leq r$
$$|f(t, \omega, x) - f(t, \omega, y)| \leq K_r |x - y| \text{ and } |f(t, \omega, x)| \leq K_r.$$
3. *there is a constant* B_1 *such that for all* (t, ω) *and all* $x \in \mathbb{R}$
$$x f(t, \omega, x) \leq B_1(1 + x^2).$$

Suppose further that there is a constant B_2 *such that for all* (t, ω)

$$\langle \sigma(t), c_t \sigma(t) \rangle + \int_{\mathbb{R}} \langle \sigma(t), y \rangle^2 \mathbf{F}_s(dy) \leq B_2. \tag{4.4.9}$$

Then the stochastic differential equation

$$X_t + X_0 + \int_0^t f(s, \cdot, X_{s-}) ds + \int_0^t \sigma(s) dS_s, \tag{4.4.10}$$

where X_0 *is a constant, has a unique (non-exploding) solution which is a semimartingale.*

Proposition 4.11 can now be used to check that for a one-dimensional driving process $\mathcal{L}$, the stochastic differential equation (4.4.7) has a unique non-exploding solution. The following proposition is stated:

Proposition 4.12. *(Kluge, 2005) Assume* $d = 1$*. Suppose that* $\gamma(\cdot, T_i)$ *is a càdlàg function for each* $i \in \{1, \ldots, n-1\}$ *and that the characteristics of* $\mathcal{L}^{T^*}$ *are chosen in such a way that* $f_1^i(\cdot, \cdot, x), \ldots, f_4^i(\cdot, \cdot, x)$ *have càdlàg paths for each* $x \in \mathbb{R}$*. Then the stochastic differential equation (4.4.7) admits a unique (non-exploding) solution for each* $i \in \{1, \ldots, n-1\}$*.*

In order to prove the existence of a solution to equation (4.4.7) for $d > 1$ further restrictions would have to be put on the characteristics of $\mathcal{L}$ to meet the growth condition (condition 3) of proposition 4.11.

From here on, the drift terms $b^H(\cdot, T_i, T_{i+1})$ are chosen as described in proposition 4.9, hence no further distinction between $\hat{H}$ and H needs to be made.

4.5 Defaultable Forward Measures

Just as using forward measures for the pricing of derivatives in default-free interest rate models, defaultable forward measures are similarly used to price derivatives in the defaultable model. The defaultable claims dealt with usually have zero recovery with settlement date T_i. The concepts discussed in this section are taken from Bielecki and Rutkowski (2004), Schönbucher (1999) and Kluge (2005). The defaultable forward measure is defined as follows:

Definition 4.13. *The defaultable forward (martingale) measure $\bar{\mathbb{Q}}_{T_i}$, for the settlement date T_i defined on $(\Omega, \mathcal{G}_{T_i})$ is given by*

$$\frac{d\bar{\mathbb{Q}}_{T_i}}{d\mathbb{Q}_{T_i}} := \frac{B(0,T_i)}{B^0(0,T_i)} B^0(T_i,T_i).$$

The above can also be referred to as the Radon-Nikodym density of $\bar{\mathbb{Q}}_{T_i}$ with respect to $\mathbb{Q}_{T_i}$. Since $B^0(0,T_i) = \bar{B}(0,T_i)$ and $B^0(T_i,T_i) = \mathbb{1}_{\{\tau > T_i\}}$ it follows that

$$\frac{d\bar{\mathbb{Q}}_{T_i}}{d\mathbb{Q}_{T_i}} = \frac{B(0,T_i)}{\bar{B}(0,T_i)} \mathbb{1}_{\{\tau > T_i\}}.$$

$\bar{B}(T_i,T_i) = B(T_i,T_i) = 1$ further implies that

$$\begin{aligned}\frac{d\bar{\mathbb{Q}}_{T_i}}{d\mathbb{Q}_{T_i}} &= \mathbb{1}_{\{\tau > T_i\}} \frac{B(0,T_i)}{\bar{B}(0,T_i)} \frac{\bar{B}(T_i,T_i)}{B(T_i,T_i)} \\ &= \mathbb{1}_{\{\tau > T_i\}} \frac{D(T_i,T_i)}{D(0,T_i)} \\ &= \frac{\mathbb{1}_{\{\tau > T_i\}}}{\mathbb{Q}_{T_i}(\tau > T_i)}.\end{aligned}$$

The deafultable forward measure $\bar{\mathbb{Q}}_{T_i}$ is absolutely continuous with respect to the forward measure $\mathbb{Q}_{T_i}$, since for any $A \in \mathcal{G}_{T_i}$ such that $\mathbb{Q}_{T_i}(A) = 0$ it is also true that $\bar{\mathbb{Q}}_{T_i}(A) = 0$. More specifically, for any $t \in (0, T_i]$ the event $A = \{\tau \geq t\}$ has probability of 0 under $\bar{\mathbb{Q}}_{T_i}$ and a strictly positive probability under $\mathbb{Q}_{T_i}$.

Furthermore, the probability measure $\bar{\mathbb{Q}}_{T_i}$ corresponds to the choice of the process $B^0(\cdot, T_i)$ as a discounting factor or numéraire. It should however be noted that under $\mathbb{Q}_{T_i}$ the value of $B^0(\cdot, T_i)$ is not strictly positive with probability 1. Because of this the two probability measures are $\mathbb{Q}_{T_i}$ and $\bar{\mathbb{Q}}_{T_i}$ are not mutually equivalent.

Schönbucher (1999) refers to $\bar{\mathbb{Q}}_{T_i}$ as the T_i-survival measure since it only assigns probabilities to survival events. Again for any $A \in \mathcal{G}_{T_i}$ the density yields

$$\bar{\mathbb{Q}}_{T_i}(A) = \frac{\mathbb{Q}_{T_i}(A \cap \tau > T_i)}{\mathbb{Q}_{T_i}(\tau > T_i)} = \mathbb{Q}_{T_i}(A | \tau > T_i).$$

So the probability of A conditional on survival up to time T_i, under the measure $\mathbb{Q}_{T_i}$, is equivalent to the probability of A under the T_i-survival measure $\bar{\mathbb{Q}}_{T_i}$, i.e the defaultable forward measure $\bar{\mathbb{Q}}_{T_i}$ can be regarded as the forward measure $\mathbb{Q}_{T_i}$ conditioned on survival up to time T_i.

The Radon-Nikodym can now be restricted to the σ-field $\mathcal{G}_t$. Since the discounted process

$$\frac{B^0(t,T_i)}{B(t,T_i)} = \mathbb{Q}_{T_i}(\tau > T_i|\mathcal{G}_t)$$

is a $\mathbb{G}$-martingale under $\mathbb{Q}_{T_i}$ it follows that

$$\begin{aligned}\left.\frac{d\bar{\mathbb{Q}}_{T_i}}{d\mathbb{Q}_{T_i}}\right|_{\mathcal{G}_t} &= \mathbb{1}_{\{\tau>t\}}\frac{B(0,T_i)}{\bar{B}(0,T_i)}\frac{\bar{B}(t,T_i)}{B(t,T_i)} \\ &= \mathbb{1}_{\{\tau>t\}}\frac{B(0,T_i)}{\bar{B}(0,T_i)}\frac{\mathbb{Q}_{T_i}(\tau > T_i|\mathcal{F}_t)}{\mathbb{Q}_{T_i}(\tau > t|\mathcal{F}_t)} \\ &= \frac{\mathbb{1}_{\{\tau>t\}}}{D(0,T_i)}\frac{\mathbb{Q}_{T_i}(\tau > T_i|\mathcal{F}_t)}{\mathbb{Q}_{T_i}(\tau > t|\mathcal{F}_t)}.\end{aligned}$$

Similarly, a restriction to the σ-field $\mathcal{F}_t$ yields

$$\left.\frac{d\bar{\mathbb{Q}}_{T_i}}{d\mathbb{Q}_{T_i}}\right|_{\mathcal{F}_t} = \frac{B(0,T_i)}{\bar{B}(0,T_i)}\mathbb{Q}_{T_i}(\tau > T_i|\mathcal{F}_t).$$

The restricted defualtable forward measure is defined as follows:

Definition 4.14. *The restricted defaultable forward (martingale) measure $\bar{\mathbb{P}}_{T_i}$, for the settlement date T_i defined on $(\Omega, \mathcal{F}_{T_i})$ is given by*

$$\frac{d\bar{\mathbb{P}}_{T_i}}{d\mathbb{P}_{T_i}} = \frac{B(0,T_i)}{\bar{B}(0,T_i)}\mathbb{Q}_{T_i}(\tau > T_i|\mathcal{F}_{T_i}),$$

where $\mathbb{P}_{T_i}$ denotes the restriction of $\mathbb{Q}_{T_i}$ to the σ-field $\mathcal{F}_{T_i}$.

More explicitly, this density can be expressed as

$$\begin{aligned}\frac{d\bar{\mathbb{P}}_{T_i}}{d\mathbb{P}_{T_i}} &= \frac{B(0,T_i)}{\bar{B}(0,T_i)}e^{-\Gamma_{T_i}} \\ &= \frac{B(0,T_i)}{\bar{B}(0,T_i)}\prod_{k=0}^{i-1}\frac{1}{1+\delta_k H(T_k,T_k)}.\end{aligned} \tag{4.5.1}$$

Since $\prod_{k=0}^{i-1}\frac{1}{1+\delta_k H(\cdot,T_k)}$ is a $\mathbb{P}_{T_i}$-martingale, the restriction to $\mathcal{F}_{T_i}$ yiels

$$\left.\frac{d\bar{\mathbb{P}}_{T_i}}{d\mathbb{P}_{T_i}}\right|_{\mathcal{F}_t} = \frac{B(0,T_i)}{\bar{B}(0,T_i)}\prod_{k=0}^{i-1}\frac{1}{1+\delta_k H(t,T_k)}. \tag{4.5.2}$$

The standard Brownian motion and compensator of μ with respect to $\bar{\mathbb{P}}_{T_i}$ are respectively given by (Kluge, 2005)

$$\bar{\mathcal{B}}_t^{T_i} := \mathcal{B}_t^{T_i} + \int_0^t \sum_{l=1}^{i-1} Y_{s-}^l \sqrt{c_s} \gamma(s, T_l) ds \tag{4.5.3}$$

and

$$\bar{\nu}^{T_i}(ds, dx) = \prod_{l=1}^{i-1} \left(1 + Y_{s-}^l \left(e^{\{\gamma(s,T_l),x\}} - 1 \right) \right)^{-1} \nu^{T_i}(ds, dx) := \bar{\mathbf{F}}_s^{T_i}(dx) ds. \tag{4.5.4}$$

Just as in the default-free model a connection between restricted defaultable forward measures for different settlement dates can be made. This is given in the following lemma:

Lemma 4.15. *(Kluge, 2005) The defaultable LIBOR rate $\big(\bar{L}(t, T_i)\big)_{0 \le t \le T_i}$ is a $\bar{\mathbb{P}}_{T_{i+1}}$-martingale and*

$$\left.\frac{d\bar{\mathbb{P}}_{T_i}}{d\bar{\mathbb{P}}_{T_{i+1}}}\right|_{\mathcal{F}_t} = \frac{\bar{B}(0, T_{i+1})}{\bar{B}(0, T_i)} \big(1 + \delta_i \bar{L}(t, T_i)\big) \quad \textit{for} \quad 0 \le t \le T_i.$$

Proof. From equation (4.2.2) it follows that

$$\begin{aligned} \big(1 + \delta_i \bar{L}(t, T_i)\big) &= \big(1 + \delta_i H(t, T_i)\big)\big(1 + \delta_i L(t, T_i)\big) \\ &= \prod_{k=0}^{i} \big(1 + \delta_k H(t, T_k)\big)\big(1 + \delta_i L(t, T_i)\big) \prod_{k=0}^{i-1} \big(1 + \delta_k H(t, T_k)\big)^{-1}. \end{aligned}$$

Applying equations (3.4.3) and (4.5.2) yields:

$$\begin{aligned} \big(1 + \delta_i \bar{L}(t, T_i)\big) &= \frac{B(0, T_{i+1})}{\bar{B}(0, T_{i+1})} \left.\frac{d\mathbb{P}_{T_{i+1}}}{d\bar{\mathbb{P}}_{T_{i+1}}}\right|_{\mathcal{F}_t} \cdot \frac{B(0, T_i)}{B(0, T_{i+1})} \left.\frac{d\mathbb{P}_{T_i}}{d\mathbb{P}_{T_{i+1}}}\right|_{\mathcal{F}_t} \cdot \frac{\bar{B}(0, T_i)}{B(0, T_i)} \left.\frac{d\bar{\mathbb{P}}_{T_i}}{d\mathbb{P}_{T_i}}\right|_{\mathcal{F}_t} \\ &= \frac{\bar{B}(0, T_i)}{\bar{B}(0, T_{i+1})} \left.\frac{d\bar{\mathbb{P}}_{T_i}}{d\bar{\mathbb{P}}_{T_{i+1}}}\right|_{\mathcal{F}_t}, \end{aligned}$$

so that

$$\left.\frac{d\bar{\mathbb{P}}_{T_i}}{d\bar{\mathbb{P}}_{T_{i+1}}}\right|_{\mathcal{F}_t} = \frac{\bar{B}(0, T_{i+1})}{\bar{B}(0, T_i)} \big(1 + \delta_i \bar{L}(t, T_i)\big)$$

as required. □

As mentioned at the start of this section, defaultable forward measures can be used for pricing of defaultable contingent claims. For $t \in [0, T_i]$, the time-t value of a defaultable claim with payoff X at settlement date T_i and zero recovery at default is given by

$$\pi_t^X := \mathbb{1}_{\{\tau>t\}} B(t,T_i) \mathbb{E}_{\mathbb{Q}_{T_i}} \left[X \mathbb{1}_{\{\tau>T_i\}} | \mathcal{G}_t \right].$$

In the following proposition, the general case when the payoff X is $\mathcal{G}_{T_i}$-measurable as well as the more common case of X being $\mathcal{F}_{T_i}$-measurable is taken into consideration.

Proposition 4.16. *(Kluge, 2005) Assume that the promised payoff X is $\mathcal{G}_{T_i}$-measurable and integrable with respect to $\bar{\mathbb{Q}}_{T_i}$. Then*

$$\pi_t^X := \mathbb{1}_{\{\tau>t\}} \bar{B}(t,T_i) \mathbb{E}_{\bar{\mathbb{Q}}_{T_i}} \left[X | \mathcal{G}_t \right] = B^0(t,T_i) \mathbb{E}_{\bar{\mathbb{Q}}_{T_i}} \left[X | \mathcal{G}_t \right].$$

If X is $\mathcal{F}_{T_i}$-measurable, then

$$\pi_t^X := \mathbb{1}_{\{\tau>t\}} \bar{B}(t,T_i) \mathbb{E}_{\bar{\mathbb{P}}_{T_i}} \left[X | \mathcal{F}_t \right] = B^0(t,T_i) \mathbb{E}_{\bar{\mathbb{P}}_{T_i}} \left[X | \mathcal{F}_t \right].$$

Additional details of proof can be found in Bielecki and Rutkowski (2004).

4.6 Recovery Rules and Bond Prices

Thus far the evolution of defaultable zero-coupon bonds with zero recovery has been specified. However, in reality all defaultable bonds have positive recovery. To maintain consistency with real markets, recovery rules must be included in the model framework. Detailed explanations of these rules can be found in Bielecki and Rutkowski (2004), Schönbucher (1999) and Duffie and Singleton (1999) to whom this section is in reference to. Typically there are three approaches (assumptions) regarding the inclusion of recovery in the modelling of default:

1. *Fractional Recovery of Par Value.* The owner (creditor) receives a fraction of the promised par (face) value immediately upon default. The advantage of this approach is that real-world recovery procedures are closely adhered to.

2. *Fractional Recovery of Treasury Value.* The creditor receives a fixed fraction of par of an equivalent default-free bond with the same maturity. The recovery is paid at the maturity date of the original defaultable bond. If the bond is coupon paying, the recovery upon default includes a fraction of the promised post-default coupon payment.

3. *Fractional Recovery of Market Value.* The creditor receives a fraction of the pre-default vaue of the defaulted bond. The expected recovery at time s, in the event of default at time $s+1$, is a fraction of the risk-neutral expected survival-contingent market value at time $s+1$.

When modelling default risk via the intensity-based approach the recovery upon default is specified in terms of recovery of defaultable zero-coupon bonds. For defaultable zero-coupon bonds the fractional recovery of treasury value approach is adopted, i.e the creditor receives a specific amount π at maturity T. The value of this bond is then given by

$$\begin{aligned} B^{\pi}(T,T) &:= \mathbb{1}_{\{\tau > T\}} + \pi \mathbb{1}_{\{\tau \le T\}} \\ &= \pi + (1-\pi)\mathbb{1}_{\{\tau > T\}}. \end{aligned}$$

Hence the bond value at time $t \le T$ is

$$B^{\pi}(t,T) = \pi B(t,T) + (1-\pi)\mathbb{1}_{\{\tau > T\}}\bar{B}(t,T).$$

The recovery of these defaultable zero-coupon bonds are specified and then used for the pricing of defaultable claims such as coupon bearing bonds. This is done by decomposing the coupon bond into a series of defaulatble zero-coupon bonds. The recovery of these defaultable coupon bonds can then be calculated as the sum of all the recoveries of the individual zero-coupon bonds. Note that the assumption that all the individual zero-coupon bonds have the same recovery cannot be made. As stated in Schönbucher (1999), these modelling approaches seem to ignore some fundamental differences between principal and coupon claims in real-world proceedings: *"The claim of a creditor on the defaulted debtor's assets is only determined by the outstanding principal and accrued interest payments of the defaulted loan or bond, any future coupon payments do not enter the consideration. The recovery rate gives the fraction of this claim that is paid off after a default, and this payoff is measured in cash and not in terms of default-free bonds or pre-default market value."*

In a realistic recovery model, the decomposition of defaultable coupon bonds should be into two classes of elementary claims: zero recovery claims $B^0(t,T)$ and positive recovery claims $B^p(t,T)$ having recovery of π times the face value at default. As in Schönbucher (1999) for coupon bearing bonds the recovery of par value approach is applied.

Assumption 4.17 (Recovery of par)**.** *The recovery of a defaultable coupon bearing bond that defaults in the time interval $(T_k, T_{k+1}]$ is composed of the recovery rate π times the sum of the notional of the bond and the accrued interest over $(T_k, T_{k+1}]$. The accrued interest can be one of two types:*

1. *A constant (c) in the case of a fixed-coupon bond with coupon c. The recovery is then given by $\pi(1+c)$.*

2. *L_k in the case of a floating rate bond. The recovery is then given by $\pi(1 + \delta_k L_k(T_k))$.*

The recovery is paid in cash at time $T_{k(\tau)}$, i.e at the next tenor date T_{k+1} if default occurred in the period $(T_k, T_{k+1}]$.

Let $e_k^X(t)$ denote the time-t value of receiving an amount of X at time T_{k+1} if and only if default has occurred in the preceding time interval $(T_k, T_{k+1}]$ (Kluge, 2005). It can further be assumed that claims of the same seniority have the same recovery rate π at time of default. The following useful lemma can now be stated:

Lemma 4.18. *(Kluge, 2005) Let X be $\mathcal{F}_{T_k}$-measurable. Then, for $t \leq T_k$*

$$e_k^X(t) = \mathbb{1}_{\{\tau > t\}} \bar{B}(t, T_{k+1}) \delta_k \mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}} \big[X H(T_k, T_k) | \mathcal{F}_t \big].$$

Using lemma 4.18, the price (at time 0) of a defaultable bond with positive recovery and maturity date T_k is

$$B^p(0, T_k) = \bar{B}(0, T_k) + \pi \sum_{i=1}^{k} e_{i-1}^1(0).$$

The time-0 price of a defaultable fixed[5] coupon bond paying out m coupons of value c at times T_i, $i, \ldots, m$ is defined by

$$\begin{aligned} B_{\text{fixed}}^{\pi}(0; c, m) : &= \bar{B}(0, T_m) + \sum_{k=0}^{m-1} c\bar{B}(0, T_{k+1}) + \sum_{k=0}^{m-1} \pi(1+c) e_k^1(0) \\ &= \bar{B}(0, T_m) + \sum_{k=0}^{m-1} \bar{B}(0, T_{k+1}) \Big(c + \pi(1+c)\delta_k \mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}} \big[H(T_k, T_k) \big] \Big). \end{aligned}$$

At the time of default there is a positive recovery π on the notional amount (taken as 1) as well as on the next outstanding coupon payment c.

On the other hand for floating[6] coupon bonds the coupons themselves have zero recovery. Instead recovery only depends on the notional amount and the outstanding coupons at default. Once again using lemma 4.18 the price of a defaultable floating coupon bond can be determined. In this case the bond is assumed to pay an interest rate composed of the default-free LIBOR rate and a constant spread x. Further assume that the bond has m coupon payments

[5]The interest payment remains the same (fixed) throughout the investment period. Provided that default does not take place, the bondholder knows with certainty how much interest will be earned over the investment period. However there is some risk involved with this type of bond. The investor faces the risk of the interest rate in market increasing at any time. If this happens the value of the bond will decrease.

[6]These are bonds having variable interest rate payments. The interest rate is usually composed of some benchmark rate, such as the LIBOR or federal funds rates, and some constant spread. The choice of the bondholder determines how frequently the interest rate will change during the investment period. This change can range from once a day, once a month or even once a year. Unlike fixed coupon bonds, these bonds face very little risk where changes in interest rates are concerned, but are rather subject to credit or default risk.

so that the holder is promised an amount of $\delta_k\big(L(T_k,T_k)+x\big)$ at the dates $T_{k+1},\ k=0,\dots,m-1$. The time-0 price of this bond is then given by

$$\begin{aligned}
B^{\pi}_{\text{floating}}(0;x,m) :&= \bar{B}(0,T_m)+\sum_{k=0}^{m-1}\delta_k\bar{B}(0,T_{k+1})\Big(x+\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[L(T_k,T_k)\big]\Big)\\
&\quad+\sum_{k=0}^{m-1}\pi\Big((1+\delta_k x)e^1_k(0)+\delta_k e_k^{L(T_k,T_k)}(0)\Big)\\
&= \bar{B}(0,T_m)+\sum_{k=0}^{m-1}\delta_k\bar{B}(0,T_{k+1})\Bigg(x+\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[L(T_k,T_k)\big]\\
&\qquad+\pi(1+\delta_k x)\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[H(T_k,T_k)\big]\\
&\qquad+\pi\delta_k\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[H(T_k,T_k)L(T_k,T_k)\big]\Bigg).
\end{aligned}$$

To explicitly price such defaultable coupon bonds expressions for the expectations,

$$\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[H(T_k,T_k)\big],\ \mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[L(T_k,T_k)\big]\text{ and }\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[H(T_k,T_k)L(T_k,T_k)\big]$$

have to be derived. The reader is referred to Kluge (2005) for the mathematical derivation thereof. The abbreviations

$$V^i_t := \frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)}\text{ and }Y^i_t := \frac{\delta_i H(t,T_i)}{1+\delta_i H(t,T_i)}$$

are used. By approximating the stochastic terms V^i_{s-} and Y^i_{s-} by their deterministic initial values V^i_0 and Y^i_0, Kluge (2005) derive expressions for $\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[H(T_k,T_k)\big]$ and $\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[L(T_k,T_k)\big]$ given by

$$\begin{aligned}
\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[H(T_k,T_k)\big] \approx H(0,T_k)\exp\Bigg(&\int_0^t\sum_{l=1}^{k-1}\frac{Y^k_0V^k_0}{Y^k_0}\langle\gamma(s,T_l),c_s\lambda(s,T_k)\rangle ds\\
&+\int_0^t\int_{\mathbb{R}^d}\frac{V^k_0}{Y^k_0}\big(e^{\langle\lambda(s,T_k),x\rangle}-1\big)\Big(1+Y^k_0\big(e^{\langle\gamma(s,T_k),x\rangle}-1\big)\Big)\\
&\times\Bigg(\prod_{l=1}^{k-1}\Big(1+Y^l_0\big(e^{\langle\gamma(s,T_l),x\rangle}-1\big)\Big)-1\Bigg)\tilde{\nu}^{T_{k+1}}(ds,dx)\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[L(T_k,T_k)\big] \approx L(0,T_k)\exp\Bigg(&-\int_0^t\sum_{l=1}^{k}Y^l_0\langle\gamma(s,T_l),c_s\lambda(s,T_k)\rangle ds\\
&-\int_0^t\int_{\mathbb{R}^d}\big(e^{\langle\lambda(s,T_k),x\rangle}-1\big)\\
&\times\Bigg(\prod_{l=1}^{k}\Big(1+Y^l_0\big(e^{\langle\gamma(s,T_l),x\rangle}-1\big)\Big)-1\Bigg)\tilde{\nu}^{T_{k+1}}(ds,dx)\Bigg),
\end{aligned}$$

where $\tilde{\nu}^{T_{k+1}}$ is an approximation for $\bar{\nu}^{T_{k+1}}$ given by

$$\tilde{\nu}^{T_{k+1}}(ds, dx) = \prod_{l=1}^{k} \Big(1 + Y_0^l\big(e^{\langle \gamma(s,T_l), x\rangle} - 1\big)\Big)^{-1} \times \prod_{l=k+1}^{n-1} \Big(1 + V_0^l\big(e^{\langle \lambda(s,T_l), x\rangle} - 1\big)\Big)\nu^{T^*}(ds, dx).$$

Kluge (2005) derive the expression for $\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[H(T_k, T_k)L(T_k, T_k)\big]$ by using the $\bar{\mathbb{P}}_{T_{k+1}}$-martingale property of the defaultable LIBOR rates $\bar{L}(\cdot, T_k)$ as well as equation (4.2.2).

From equation (4.2.2)

$$\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[1 + \delta_k \bar{L}(T_k, T_k)\big] = \mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\Big[\big(1 + \delta_k L(T_k, T_k)\big)\big(1 + \delta_k H(T_k, T_k)\big)\Big].$$

Using the martingale property

$$\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[1 + \delta_k \bar{L}(T_k, T_k)\big] = 1 + \delta_k \bar{L}(0, T_k),$$

it follows that

$$1 + \delta_k \bar{L}(0, T_k) = 1 + \delta_k \mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[L(T_k, T_k)\big] + \mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[H(T_k, T_k)\big] + (\delta_k)^2 \mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[H(T_k, T_k)L(T_k, T_k)\big],$$

which yields

$$\mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[H(T_k, T_k)L(T_k, T_k)\big] = \frac{1}{\delta_k}\Big(\bar{L}(0, T_k) - \mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[L(T_k, T_k)\big] - \mathbb{E}_{\bar{\mathbb{P}}_{T_{k+1}}}\big[H(T_k, T_k)\big]\Big).$$

Chapter 5

Credit Derivatives Pricing

Credit derivatives refer to financial securities whose price is derived from the value of an underlying (reference) asset which is subject to credit risk. Hence, the payoff of the credit derivative depends on and is affected by the default of the underlying asset. These underlying assets are usually loans or bonds and could be either privately or governmentally owned.

The most prominent feature of credit derivatives is its use for the transfer, management and hedging of credit risk. By transferring credit risk between counter-parties involved, credit derivatives have proven to be useful as tools for the control of credit exposure as well. Generally investors are concerned with two components of risk; the market risk and the credit risk associated to the specific asset. With credit derivatives investors can isolate the specific credit risk from the total market risk. As stated in Duncan (2004), *"The feature of credit derivatives which distinguishes them from that of the more traditional credit instruments, is the precision with which credit derivatives are able to isolate and transfer the component parts of credit risk (i.e. default, spread, credit migration, restructuring) from an underlying asset, as opposed to credit risk in it's entirety."*

There is wide selection of literature on the subject of credit derivatives, the market and modelling thereof. These include texts such as Bielecki and Rutkowski (2004), Schönbucher (2003), Das (1998) and O'Kane (2008) to whom sections of this chapter is in reference to as well as Duncan (2004). More specifically, Schoutens and Cariboni (2009) give an introduction to modelling credit risk and pricing credit derivatives using Lévy processes.

In the following section the three categories of credit derivatives are more formally introduced and described. This will enable the pricing of specific credit derivatives under the Lévy-LIBOR setting which follows in the subsequent sections of this chapter. The pricing of credit derivatives is done in order to demonstrate the flexibility and applicability of the modelling approach (Eberlein *et al.*, 2006). The relative instruments will be defined in later sections of the chapter wherein the pricing thereof is considered.

5.1 Classification of Credit Derivatives

There is a large degree of flexibility in the specification of credit derivatives which enable such agreements to be customized to specific needs of the investor. Financial derivatives are typically classified into three main types of agreements: forwards, swaps and options. Similarly, credit derivatives can be classified into the following three general categories:

1. Credit event instruments
2. Credit spread instruments
3. Total rate of return instruments

The payoffs of these instruments depend on the reference asset which could be associated to either a single name or to multiple names. Furthermore, the reference asset can be either a cash instrument (such as a bond or loan) or a synthetic instrument (such as another credit derivative) (Duncan, 2004).

Credit event instruments refer to the class of derivatives directly linked to a credit event. The payoff of these instruments depend on the occurrence of a specific credit event, usually the default event. Hence they allow the transfer and assumption of pure credit risk, in relation to the default event of the nominated issuer (Das, 1998). This class of credit derivatives is also referred to as *credit default instruments* since their primary purpose is the isolation of the risk associated to the default of the credit obligation. Instruments within this class of credit derivatives include *credit default swaps (CDS')*, *credit linked notes* and *credit debt obligations (CDO's)* amongst others.

Credit spread instruments refer to the class of derivatives linked to changes in the credit quality of the underlying asset. The payoffs of such instruments depend on the movement in credit spreads, regardless of the reason for the movement (Duncan, 2004). As explained in Das (1998), here credit spreads[1] represent the margin relative to the risk-free rate designed to compensate the investor for the risk of default on the underlying security. These derivatives are constructed from credit spread levels directly observed from the market. The construction is done so that the credit spread risk can be easily transferred between the parties involved. Credit derivatives within this class can further be divided into two formats: credit spreads relative to the risk-free benchmark (absolute spread) and credit spreads between two credit-sensitive assets (relative spread). Credit spread instruments enable the use of credit spreads to trade, hedge or monetize expectations on future credit spreads (Das, 1998). Instruments within this class of credit derivatives include *credit spread forwards (swaps)* and *credit spread options*.

[1] The credit spread is calculated as the difference between the yield of the security or loan and the yield of the corresponding risk-free security.

Total rate of return instruments refer to the class of instruments whose payoff depend on both the behaviour of credit spreads and credit events such as default. These derivatives enable the complete transfer of the risk associated to an asset between two parties. As stated in Das (1998), the central concept of these credit derivative structure is the replication of the total performance of the credit asset (loan or bond). Instruments within this class of credit derivatives include *total rate of return swaps* and *asset swaps*.

5.2 Pricing under the LLM

In this section valuation formula is derived for the some of the most popular credit derivatives traded in the credit market. The valuation or pricing of these derivatives is done under the modelling framework introduced in the previous chapter(s). In order to do this the parties generally involved will be referred to as parties A and B for convenience. Party A refers to the insured counterparty, i.e the party who will receive a payment in the event of default. Correspondingly, party B refers to the insurer, i.e the party who will have to make the payment if default occurs. There is a third party involved, party C; referred to as the reference entity. Party C is usually the issuer of the reference credit or asset. The section is divided into two subsections; the one dealing with credit sensitive swaps and the other with credit options. This section is with reference to Kluge (2005) and Schönbucher (1999).

5.2.1 Credit Sensitive Swaps

Here specific swap contracts that are sensitive to credit events such as default are considered for valuation under the defaultable Lévy-LIBOR framework.

Credit default swaps

Credit default swaps (CDS') are the simplest and most popular credit derivatives traded in the credit derivatives market. They are also the most liquid single-name derivatives traded in the over-the-counter market and form one of the main building blocks for other credit derivatives instruments. Hence they take up a relatively large share of the market. A CDS can simply be defined as a bilateral contract enabling the isolation and transfer of the credit risk of a reference entity between the counterparties involved. These counterparties are referred to as the protection buyer (payer) and protection seller (receiver).

As agreed to in the contract the protection buyer (A) makes predetermined periodic payments to the protection seller (B) until either the maturity of the contract period is reached or the credit event occurs (e.g the default of the reference asset). In exchange for these payments B promises a specified payment to A if the default event occurs. As explained in O'Kane (2008) the protection buyer is insuring themselves against the credit risk assumed by the protection seller. Hence CDS' can be considered as some sort of debt insurance contract.

Generally CDS agreements differ in the specification of the credit event and consequent default payment. For simplicity a standard default swap agreement is considered. The contract matures at time T_m. As in Schönbucher (1999), two specific payments are involved:

1. The *fee stream* (payment): A pays an amount s at times $T_i,\ i = 0, \ldots, m$ or until the credit event (default) occurs.
2. The *default payment*: At the time of default B pays the difference between the post-default price of the reference asset (usually a bond issued by C) and its par (face) value.

If default occurs in the period $(T_k, T_{k+1}]$ for $k \in \{0, \ldots m-1\}$, A receives (at time T_{k+1}) the amount

$$1 - \pi(1 + c)$$

for a fixed coupon bond and

$$1 - \pi\Big(1 + \delta_k\big(L(T_k, T_k) + x\big)\Big)$$

in the case of a floating coupon bond.

The initial value of the fee stream is given by

$$s \sum_{k=1}^{m} \bar{B}(0, T_{k-1}).$$

The initial value of the default payment on a fixed coupon bond and a floating coupon bond is given by

$$\sum_{k=1}^{m} \Big(1 - \pi(1 + c)\Big) e^{1}_{k-1}(0)$$

and

$$\sum_{k=1}^{m} \left(\Big(1 - \pi(1 + \delta_{k-1} x)\Big) e^{1}_{k-1}(0) - \pi \delta_{k-1} e^{L(T_{k-1}, T_{k-1})}_{k-1}(0) \right)$$

respectively.

In order to price the CDS the default swap rate has to be determined. This is the level s of the fee stream that would ensure a fair price of the swap. By equating the fee payment and the default payment, s is determined for both cases of fixed and floating bonds respectively as follows:

$$s_{\text{fixed}} \sum_{k=1}^{m} \bar{B}(0, T_{k-1}) = \sum_{k=1}^{m} \Big(1 - \pi(1 + c)\Big) e^{1}_{k-1}(0)$$

which implies that

$$s_{\text{fixed}} = \Big(1 - \pi(1+c)\Big) e^1_{k-1}(0) \frac{\sum_{k=1}^m e^1_{k-1}(0)}{\sum_{k=1}^m \bar{B}(0, T_{k-1})},$$

where from lemma 4.18

$$e^1_{k-1}(0) = \bar{B}(0, T_k)\delta_{k-1}\mathbb{E}_{\bar{\mathbb{P}}_{T_k}}\big[H(T_{k-1}, T_{k-1})\big],$$

so that

$$s_{\text{fixed}} = 1 - \pi(1+c) \frac{\sum_{k=1}^m \Big(\bar{B}(0, T_k)\delta_{k-1}\mathbb{E}_{\bar{\mathbb{P}}_{T_k}}\big[H(T_{k-1}, T_{k-1})\big]\Big)}{\sum_{k=1}^m \bar{B}(0, T_{k-1})}.$$

Similarly

$$s_{\text{floating}} \sum_{k=1}^m \bar{B}(0, T_{k-1}) = \sum_{k=1}^m \left(\Big(1 - \pi(1+\delta_{k-1}x)\Big) e^1_{k-1}(0) - \pi\delta_{k-1} e^{L(T_{k-1}, T_{k-1})}_{k-1}(0) \right)$$

which implies that

$$\begin{aligned} s_{\text{floating}} = \left(\sum_{k=1}^m \bar{B}(0, T_{k-1})\right)^{-1} \sum_{k=1}^m \Bigg(\Big(1 - \pi(1+\delta_{k-1}x)\Big) & e^1_{k-1}(0) \\ & - \pi\delta_{k-1} e^{L(T_{k-1}, T_{k-1})}_{k-1}(0) \Bigg), \end{aligned}$$

where again from lemma 4.18

$$e^{L(T_{k-1}, T_{k-1})}_{k-1}(0) = \bar{B}(0, T_k)\delta_{k-1}\mathbb{E}_{\bar{\mathbb{P}}_{T_k}}\big[H(T_{k-1}, T_{k-1})L(T_{k-1}, T_{k-1})\big],$$

so that

$$\begin{aligned} s_{\text{floating}} &= \left(\sum_{k=1}^m \bar{B}(0, T_{k-1})\right)^{-1} \sum_{k=1}^m \Bigg(\Big(1 - \pi(1+\delta_{k-1}x)\Big) \bar{B}(0, T_k)\delta_{k-1} \\ &\qquad \times \mathbb{E}_{\bar{\mathbb{P}}_{T_k}}\big[H(T_{k-1}, T_{k-1})\big] \\ &\qquad - \pi\delta_{k-1}\bar{B}(0, T_k)\delta_{k-1} \\ &\qquad \times \mathbb{E}_{\bar{\mathbb{P}}_{T_k}}\big[H(T_{k-1}, T_{k-1})L(T_{k-1}, T_{k-1})\big] \Bigg) \\ &= \left(\sum_{k=1}^m \bar{B}(0, T_{k-1})\right)^{-1} \sum_{k=1}^m \Bigg(\bar{B}(0, T_k)\delta_{k-1}\Big(\Big(1 - \pi(1+\delta_{k-1}x)\Big) \\ &\qquad \times \mathbb{E}_{\bar{\mathbb{P}}_{T_k}}\big[H(T_{k-1}, T_{k-1})\big] - \pi\delta_{k-1} \\ &\qquad \times \mathbb{E}_{\bar{\mathbb{P}}_{T_k}}\big[H(T_{k-1}, T_{k-1})L(T_{k-1}, T_{k-1})\big]\Big) \Bigg). \end{aligned}$$

The expectations, $\mathbb{E}_{\bar{\mathbb{P}}_{T_k}}\big[H(T_{k-1}, T_{k-1})\big]$ and $\mathbb{E}_{\bar{\mathbb{P}}_{T_k}}\big[H(T_{k-1}, T_{k-1})L(T_{k-1}, T_{k-1})\big]$ can be obtained as in section 4.6 of the previous chapter.

Total rate of return swaps

Total rate of return swaps are agreements in which the total return on some reference asset is exchanged for other cash flows (periodic fixed or floating payments). These contracts enable investors to obtain the cash flow benefits of owning an asset without having to actually hold the physical asset on their balance sheet (Duncan, 2004). Here the payer (A) owns the reference asset and agrees to pay the total return of the asset on a notional amount to the receiver (B) who in turn agrees to make periodic payments throughout the contract period based on an agreed interest rate. As explained in Bielecki and Rutkowski (2004), if default on the reference asset occurs during the contract period, the contract terminates immediately. The receiver is however obligated to cover the change in value of the underlying asset by paying the payer the difference in price since inception of the contract. The receiver therefore accepts the price risk, including credit risk, of the underlying reference asset. In other words, a total rate of return swap has an embedded default swap in which the payer is the protection buyer (Bielecki and Rutkowski, 2004).

Consider a total rate of return swap with maturity date T_m and a fixed coupon bond (issued by C) maturing at date T_M ($m \leq M \leq n$) as the reference asset. As in Kluge (2005) the payment streams involved are as follows:

1. If no default occurs in $(T_k, T_{k+1}]$ for $0 \leq k \leq m-1$, then B receives an amount of $(c-s)$ at time T_{k+1}, where s is the fixed periodic payment of B and c is the coupon of the underlying bond.

2. If default occurs in $(T_k, T_{k+1}]$ for $0 \leq k \leq m-1$, then B receives an amount of $\pi(1+c) - B^{\pi}_{\text{fixed}}(0, c, M)$ at time T_{k+1} and the swap contract terminates.

3. If no default occurs until maturity T_m, B receives an amount of $B^{\pi}_{\text{fixed}}(T_M, c, M) - B^{\pi}_{\text{fixed}}(0, c, M)$ at time T_M.

Let $v^{\pi}_0(m, M, c)$ denote the time-0 value of receiving an amount of $B^{\pi}_{\text{fixed}}(T_M, c, M)$ at time T_m if no default has occurred until then. From the point of view of the receiver (B), the initial value of the contract is given by

$$(c-s)\sum_{k=1}^{m} \bar{B}(0, T_k) + \Big(\pi(1+c) - B^{\pi}_{\text{fixed}}(0, c, M)\Big)\sum_{k=1}^{m} e^{1}_{k-1}(0) \\ + v^{\pi}_0(m, M, c) - \bar{B}(0, T_m)B^{\pi}_{\text{fixed}}(0, c, M).$$

The fixed periodic payment s, that renders a fair contract, i.e a contract with initial value of zero, can be determined as follows:

$$(c-s)\sum_{k=1}^{m}\bar{B}(0,T_k) = \bar{B}(0,T_m)B^{\pi}_{\text{fixed}}(0,c,M) - v^{\pi}_0(m,M,c) \\ - \Big(\pi(1+c) - B^{\pi}_{\text{fixed}}(0,c,M)\Big)\sum_{k=1}^{m} e^1_{k-1}(0),$$

so that

$$c-s = \left(\sum_{k=1}^{m}\bar{B}(0,T_k)\right)^{-1}\Bigg(\bar{B}(0,T_m)B^{\pi}_{\text{fixed}}(0,c,M) - v^{\pi}_0(m,M,c) \\ - \Big(\pi(1+c) - B^{\pi}_{\text{fixed}}(0,c,M)\Big)\sum_{k=1}^{m} e^1_{k-1}(0)\Bigg),$$

which implies that

$$s = c - \left(\sum_{k=1}^{m}\bar{B}(0,T_k)\right)^{-1}\Bigg(\bar{B}(0,T_m)B^{\pi}_{\text{fixed}}(0,c,M) - v^{\pi}_0(m,M,c) \\ + \Big(B^{\pi}_{\text{fixed}}(0,c,M) - \pi(1+c)\Big)\sum_{k=1}^{m} e^1_{k-1}(0)\Bigg).$$

For the detailed calculation of $v^{\pi}_0(m,M,c)$ the reader is referred to Kluge (2005).

5.2.2 Credit Options

Credit options refer to the class of options on specific reference credit with specified maturity. As in the case of classical option derivatives, there are options to either buy (call options) or sell (put options) protection on the underlying reference credit. The reference credit is usually a defaultable fixed or floating rate bond, although the option could be on other credit derivatives such as asset swaps and credit default swaps amongst others.

The mathematical derivation of pricing formula for these instruments are rather complicated hence only the approximated prices will be noted in the sections below. The reader is referred to Kluge (2005) for the detailed mathematical derivation thereof.

Credit spread options

Credit spread options are similar to credit spread forwards[2] except that they have the additional feature of classical options contracts, i.e the holder has the right but not the obligation to buy or sell the credit spread. Hence, as in Bielecki and Rutkowski (2004), these credit spread instruments can be described as option-like agreements having payoffs associated to the yield difference of two credit-sensitive assets. Furthermore, credit spread options enable the isolation of the firm-specific credit risk from that of the market. Regarding the parties involved, the option seller faces unlimited loss and limited profit potential whereas the opposite holds for the option buyer, i.e unlimited profit potential and limited loss. Just as in the case of classical options contracts, credit spread put and call options are dealt with. A credit spread put option gives the holder the right to sell the credit spread and hence benefit from an increase (widening) therein. Similarly, a credit spread call option gives the holder the right to buy the credit spread and hence benefit from a decrease (tightening) therein. As in Kluge (2005), the following definition is taken from Schmid (2004):

Definition 5.1. *A credit spread call (put) option with maturity T and strike spread K on a defaultable bond $B^\pi(\cdot, U)$ with maturity $U \geq T$ gives the holder the right to buy (sell) the defaultable bond at time T at a price that corresponds to a yield spread of K above the yield of an otherwise identical non-defaultable bond $B(\cdot, U)$.*

Consider a call option with maturity date T_i and strike spread K. Once the underlying defaultable bond $B^\pi(\cdot, T_m)$, $i < m \leq n$, defaults, the call option is knocked out. The time-T_i value of the option is given by

$$\begin{aligned}
\pi_{T_i}^{\mathrm{CSO}}(K, T_i, T_m) :&= \mathbb{1}_{\{\tau > T_i\}} \Big(B^\pi(T_i, T_m) - e^{-(T_m - T_i)K} B(T_i, T_m) \Big)^+ \\
&= \mathbb{1}_{\{\tau > T_i\}} \Big(\pi B(T_i, T_m) + (1-\pi)\bar{B}(T_i, T_m) \\
&\qquad\qquad - e^{-(T_m - T_i)K} B(T_i, T_m) \Big)^+ \\
&= \mathbb{1}_{\{\tau > T_i\}} \Big((1-\pi)\bar{B}(T_i, T_m) - \big(e^{-(T_m - T_i)K} - \pi \big) B(T_i, T_m) \Big)^+ .
\end{aligned}$$

The initial value of the credit spread option is then found to be

$$\pi_0^{\mathrm{CSO}}(K, T_i, T_m) = \bar{B}(0, T_m)(g * \varphi)(0),$$

with $g(x) := \big(\upsilon(x)\big)^+$ and

$$\upsilon(x) := (1-\pi) - \big(e^{-(T_m - T_i)K} - \pi\big) \prod_{l=i}^{m-1} \left(1 + \delta_l H(0, T_l) \exp\Big(-\frac{\gamma l}{\sigma_{\mathrm{sum}}} x + B_l^H \Big) \right).$$

[2]Credit spread forwards or swaps are credit-risk sensitive agreements in which one party makes payments based on the yield to maturity of a specific issuer's debt and the other party makes payments based on comparable treasury yields (Bielecki and Rutkowski, 2004).

The above pricing formula was obtained using similar arguments to that of the derivation of the price of options on defaultable bonds, details of which can be found in Kluge (2005) along with that of the pricing of credit default swaptions.

5.3 Conclusion

Following the work of Eberlein and Özkan (2005), a model for LIBOR rates driven by a time-inhomogeneous Lévy process was presented. Motivation for this lies in the fact that Lévy processes provide much more modelling flexibility than the standard Brownian motion. As shown in Eberlein and Özkan (2005), the model can be constructed in various ways, however the focus herein lies in the framework under which the LIBOR rates are modelled directly. This setup constitutes the discrete-tenor Lévy-LIBOR model. As in the case of the LIBOR forward rate model, the Lévy-LIBOR model can be constructed via backward induction and is driven by a process that is generally only a Lévy process under one forward measure. Positive rates are guaranteed since the LIBOR rates are represented as the ordinary exponential of a stochastic integral driven by a Lévy process. Under certain specification of the drift term, the forward LIBOR rate process is a martingale and so the arbitrage-free conditions of the model are satisfied. Dynamics of the model with respect to the terminal measure is also specified using the connections between the one forward measure and the terminal measure. Eberlein and Özkan (2005) and Kluge (2005) further present valuation formula for caps and floors under the Lévy-LIBOR model setting. This was not included in the current presentation as the main focus lies in the use of the model for the discussion of the defaulatble model.

The standard LIBOR market model was first extended to model defaultable rates by Schönbucher (1999). The main part of this thesis presentation is the same extension of the LIBOR market model but of course once again under a Lévy setting. This follows from the work of Eberlein *et al.* (2006) and Kluge (2005). The approach of Schönbucher (1999) was followed where instead of modelling the defaultabe rates directly, the forward default intensities are modelled. However, the specification of the dynamics of the defaultable rates as well as the forward default intensities depend directly on the specification of the time of default. For this reason the dynamics of the defaultable rates were specified by first giving a pre-specification for these dynamics and then constructing the time of default that imply the actual dynamics. It is then shown that the canonical construction of the default time ensures that the modelled dynamics of the default intensities do in fact correspond to those derived from real-world probability measures. Once again, arbitrage-free conditions within the model need to be satisfied. For this a specification of the drift term had to be given just as in the case of the default free model. Both the defaultable and restricted defaultable measures were then introduced, again following the work of Schönbucher (1999), in order to present pricing formula for contingent claims. Recovery rules are included in the modelling framework to maintain consistency with real markets. The recovery of par approach was used since Eberlein *et al.* (2006) consider it as the most realistic to pricing recovery. The price of a recovery unit payoff can then be given in terms of an expectation of the default intensities under the forward survival measure.

As with any model, the flexibility and applicability of the model has to be demonstrated. For this the pricing of credit derivatives was presented. Since the main purpose of the model was to account for the credit risk in the market it is only natural that the model be applied to the pricing of such financial instruments. Using the valuation formula obtained from the defaultable measures, pricing formula for some of the more commonly used credit derivatives such as credit sensitive swaps and options can be deduced. The pricing formula presented are approximate pricing solutions since closed form solutions do not exist.

In summary, the Lévy-LIBOR model with default risk has been constructed and introduced by Kluge (2005) as a combination of Schönbucher (1999)'s extension: the LIBOR market model with default risk and Eberlein and Özkan (2005)'s generalization: the Lévy-LIBOR model. Due to the flexibility and robustness of the model's driving Lévy process practitioners are better equipped to account for the risk found in the market, particularly that of credit risk.

Further research relating to the topic at hand include extensions of the model to rating classes. This has been introduced by Grbac (2009) and Eberlein and Grbac (2013). As stated in their paper, Eberlein and Grbac (2013) develop an arbitrage-free model for defaultable forward Libor rates related to defaultable bonds with credit ratings and call this the *Rating Based Lévy-LIBOR model*. With this they also present a modelling framework for credit migration of these bonds via a conditional Markov chain approach. Naturally, valuation formula for commonly traded credit credit derivatives is presented.

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