



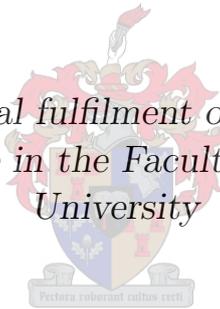
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Contributions to the analysis of approximate counting

by

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*Thesis presented in partial fulfilment of the requirements for the
degree of Doctor of Science in the Faculty of Science at Stellenbosch
University*



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March 2016

Declaration

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Date: December 1, 2014

Abstract

Approximate Counting is a classical technique with very challenging questions related to its performance analysis. It is also somewhat similar to parameters around Digital Search trees. Surprising links to q -analysis and the theory of partitions exist. The author has contributed to the analysis during the last decades; the relevant papers have been collected in this thesis. Some emphasis is on a recent development, namely, to introduce a parameter m (m counters instead of one).

Opsomming

Benaderde Aftelling is 'n klassieke tegniek met baie uitdagende vrae in verband met sy prestasie-analise. Dit is ook verwant aan parameters van digitale soekbome. Daar bestaan 'n verrassende verband met q -analise en die teorie van partisies. Die outeur het in die afgelope dekades tot hierdie analise bygedra; die relevante artikels is in hierdie tesis versamel. Klem word getoon op 'n onlangse ontwikkeling, naamlik om 'n parameter m (m tellers in plaas van een) by te voeg.

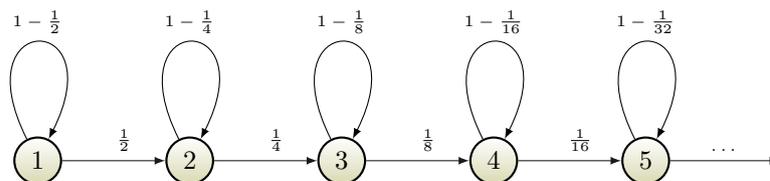
Acknowledgments. Thanks are due to my coauthors who shared the enthusiasm for *approximate counting* with me during 30 years. Stephan Wagner thankfully agreed to be the coordinator of this project. Elmar Teufl helped me with the subtleties of high level L^AT_EX packages and is responsible for the appearance of the thesis.

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1 Introduction

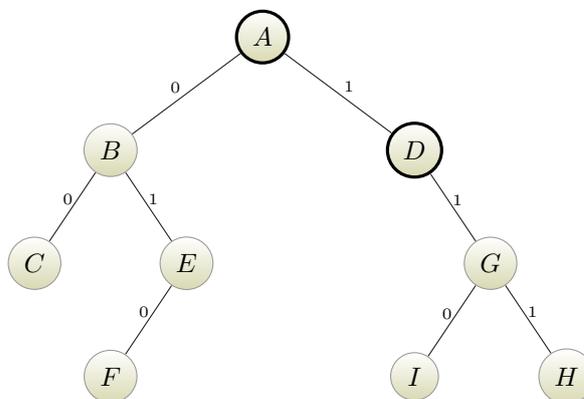
Approximate counting in its most basic form can be described as follows: There is a counter C , initially set to 1, and when the counter has the value k and a random element arrives (to be counted), the counter advances to value $k + 1$ with probability 2^{-k} , and with the remaining probability $1 - 2^{-k}$, it stays the same. After n random increments, the counter has a value between 1 and n , but is typically around $\log_2 n$. So, instead of exactly counting n , one counts approximately $\log_2 n$. One needs only about $\log \log n$ bits to keep that count. The evolution of the counter value is depicted in the following picture, which is somewhat self-explanatory.



The first analysis is due to Flajolet [Fla85]. The probabilities $p_{n,l}$ that the counter has value l after n random steps were exactly computed and approximated. An application of the Mellin transform method gives asymptotic expressions for the first two moments, and they involve (tiny) periodic oscillations. To deal with them is very interesting and challenging in itself.

There is another structure that appears frequently in this thesis: Digital Search Trees. Here is an example.

A : 1001
 B : 0110
 C : 0000
 D : 1111
 E : 0100
 F : 0101
 G : 1101
 H : 1110
 I : 1100



Data A, B, \dots, H arrive and will be inserted into the first available slot. If a slot is not available, a probabilistic choice is made to go left or right and try again. Here, 0 means go left, and 1 means go right. The (insertion) depth of a node is its distance from the root, and the path length of the digital search tree is the sum of the distances of all nodes in the tree.

I contributed to the approximate counting complex over the last 30 years. The relevant papers are collected in this thesis.

2 Brief description of the papers collected in this thesis

Approximate counting: An alternative approach, P. Kirschenhofer and H. Prodinger, RAIRO Theoretical Informatics and Applications 25 (1991), 43–48.

This is my first paper on the subject. It offers an alternative analysis to the one presented by Flajolet [Fla85]. Using Euler's partition identities from the theory of partitions, the probabilities $p_{n,l}$ that the counter C has values l after n random increments, can be written in a different way. The advantage is that the expectation (and higher moments as well) may be explicitly expressed as alternating sums, amenable to Rice's method to perform the asymptotic analysis when $n \rightarrow \infty$.

There is a further observation in it, namely that a constant that appears in the variance may be replaced by a much simpler one – the reason is a modular identity due to Dedekind. These identities (partially due to Ramanujan) appear in several other papers of mine, also unrelated to approximate counting.

Hypothetical analyses: approximate counting in the style of Knuth, path length in the style of Flajolet, Helmut Prodinger, Theoretical Computer Science 100 (1992), 243–251.

It was noticed that approximate counting and the parameter path length in digital search trees (formed by n random nodes) are somewhat similar. The latter analysis was first performed by Knuth in his celebrated series of books *The Art of Computer Programming*. I exchanged the two eminent scientists Knuth and Flajolet and analyzed approximate counting as Knuth would have done it, and digital search as Flajolet would. In a final section, a model of n players was introduced to explain the evolution in the approximate counting model. This became important also in later publications.

How to advance on a stairway by coin flippings, H. Prodinger, Proceedings "Applications of Fibonacci Numbers", Kluwer 5 (1993), 473–479.

Tries are a data structure similar to digital search trees. It is shown that certain aspects of them and of approximate counting can be analyzed together, by introducing a parameter p . The paper was presented at a conference on Fibonacci numbers. Thus, an attempt was made to describe the subject in popular terms, with a demon that at every step in (discrete) time may take out a player of a party of initially n people. This concept of a demon became popular and occurred also in later writings.

Asymptotic analysis of a class of functional equations and applications, P. Grabner, H. Prodinger, and R.F. Tichy, Journal de Theorie des Nombres de Bordeaux 5 (1993), 365–381.

The idea of a demon was further developed, by allowing a finite number of them. The recursions become much more tricky, but the techniques of Flajolet and Richmond [FR92] that appeared around the time, allowed us to proceed.

A coin tossing algorithm for counting large numbers of events, Peter Kirschenhofer and Helmut Prodinger, Mathematica Slovaca 42 (1992), 531–545.

The original algorithm is analyzed again, but this time details that were previously ignored are taken care of. There is a random generator involved in the algorithm, and is realized here using the simple procedure of flipping a coin. This changes the behaviour, but only in lower order terms. It requires to use Knuth fundamental analysis of carry propagations [Knu78] – something that was ever present in my research, also in many papers not cited here.

Approximate counting via Euler transform, Helmut Prodinger, Mathematica Slovaca 44 (1994), 569–574.

The first idea to link approximate counting to the theory of partitions was picked up again, but this time on a higher level, namely on the level of q -(basic)-generating functions. A transformation due to Heine was used to bring the double generating function marking n , the number of random increments and l , the value of the counter, into a convenient shape, so that the moments come out in a neat way. One need Rice’s method again to get to asymptotics.

Generalized approximate counting revisited, Guy Louchard and Helmut Prodinger, Theoretical Computer Science 391 (2008), 109–125.

Charalambides and a team around K. Simon had rediscovered the approximate counting scheme, probably without knowledge of the earlier writings. We undertook to write a historic account, linked all the approaches, but added a fair amount of our own results.

Approximate counting with m counters: A detailed analysis, Helmut Prodinger, Theoretical Computer Science 439 (2012), 58–68.

Cichon and Machyna [CM11] came up with the idea of using m counters instead of just one, and distribute the incoming random items fairly (at random) to one of the counters. The result of the procedure is the sum of the m counters. Since the paper did not contain any analysis, I jumped in and worked it out. It was a real tour de force – an old analysis of digital search trees [KHS94] had to be revitalised. The final results about expectation and variance are not surprising and intuitively clear, but the analysis was non-trivial and beautiful.

Approximate Counting via the Poisson-Laplace-Mellin Method, Michael Fuchs, Chung-Kuei Lee, and Helmut Prodinger, DMTCS proc. AQ (2012), 13–28.

My paper on m -approximate counting became popular quite quickly: Michael Fuchs and his student Chung-Kuei Lee used a different approach, following the footsteps of the deep paper [HFZ10]. My contribution was about the constant in the asymptotic expansion: it came out in a generic, but quite different form from the one that one is used to see. To show that the two forms are indeed the same was surprisingly difficult, and required some knowledge in the manipulation of q -series.

This technique simplified another constant from the study of the path length in digital search trees as well, making its numerical calculation much easier.

Approximate counting with m counters: a probabilistic analysis, Guy Louchard and Helmut Prodinger.

Guy Louchard was the other person who liked the m -approximate counting version immediately. He used a machinery that he developed and perfected over the years, and invited me to contribute a few smaller observations.

Words coding set partitions, Kamilla Oliver and Helmut Prodinger, Appl. Anal. Discrete Math. 5 (2011), 55–59.

The words in question are sometimes called words with a restricted growth property. Scanning a word from left to right, a number that is larger than all the previous ones, can only be one larger than the previous left-to-right maximum. It is not difficult to see that they code set partitions. Using the geometric weights that occurred in many of my publications, the natural question is how many such words of length n exist, or, rather, what is the probability that such a word enjoys the restricted growth property. Approximate counting does not directly appear here, but in a follow-up publication.

Set partitions, words, and approximate counting with black holes, Helmut Prodinger, Australasian Journal of Combinatorics 54 (2012), 303–310.

It was noted that the words with the restricted growth property may be modeled using an automaton akin to the one that is traditional in the approximate counting algorithm. The transition probabilities are different, and, since words may violate the restricted growth property, there are transitions leading to a “sink”; such words are rejected. Consequently, an effort was made to use the machinery known from approximate counting also in the present instance.

Words with a generalized restricted growth property, Michael Fuchs and Helmut Prodinger, Indagationes Mathematicae 24 (2013), 1024–1033.

In a further step of this saga, the restricted growth property has been relaxed, now depending on a parameter d . The analysis is more difficult, but was manageable.

This paper was an invited paper for a memorial issue dedicated to N. G. de Bruijn. Among many other things, de Bruijn was a pioneer in asymptotics, as related to questions typically arising in Number Theory and Combinatorics.

The subject became quite popular, and M. Fuchs has continued to write about it.

Digital search trees with m trees: Level polynomials and insertion costs, H. Prodinger, Discrete Mathematics and Theoretical Computer Science 13 (3), 2011, 1–8.

As we have seen, approximate counting and digital search trees are related. The idea of introducing the m parameter to approximate counting, as seen before, is now

transferred. We keep m digital search trees, and choose at random, into which one to insert a new element. The parameter “insertion depth” will be analyzed using so-called level polynomials; their coefficients are the expected number of nodes on a given level. The machinery of q -series, in particular, Heine’s transformation formula, is used here to provide an elegant analysis.

Advancing in the presence of a demon, G. Louchard and H. Prodinger, Mathematica Slovaca 58 (2008), 263–276.

I gave a talk in Brussels and mentioned the old papers about the demon and advancing on a staircase. Guy Louchard liked this and suggested to write another paper about the subject. Naturally, using his methods about the Gumbel (extreme-value) distribution, we obtained much stronger results now, including results about all moments.

Number of survivors in the presence of a demon, Guy Louchard, Helmut Prodinger, and Mark Daniel Ward, Periodica Mathematica Hungarica 64 (2012), 101–117.

And, after the previous paper appeared, another fan of the demon saga was created: Mark Daniel Ward. He suggested to investigate another parameter (as mentioned in the title). This led to somewhat heavy computations, where again Rice’s method was employed.

Digital search trees again revisited: The internal path length perspective. P. Kirschenhofer, H. Prodinger and W. Szpankowski, SIAM Journal on Computing 23 (1994), 598–616.

This paper about digital search trees was written around 1987/88, but appeared only much later. A true tour de force – it was necessary to compute many terms in order to analyze the variance of the path length, due to the heavy cancellations. It is included in this collection because it played a major role in the analysis of the m -model of approximate counting. There was a previous version, that appeared in some conference proceedings. It is interesting, because the analytic continuation of a critical function was done in a quite different way.

Periodic oscillations in the analysis of algorithms and their cancellations, H. Prodinger, JIRSS 3 (2004), 251–270.

The subject are the periodic fluctuations that occur so frequently in the analysis of algorithms, in particular, in approximate counting. There are non-trivial identities satisfied by them, and hence cancellations in higher order moments. There are different methods to derive such identities, which are akin to classical identities for modular functions (Dedekind, Ramanujan). One is the Mellin transform, and another one the residue calculus.

There is a whole complex of papers of mine which are somewhat related, about loser (leader) election by rounds of coin flippings, also with “swedish stopping” etc.

To keep this thesis within reasonable length, it was decided not to include this material here.

There is another complex called probabilistic counting, which is different from approximate counting but shares several characteristics. These papers are not included here either.

At the end of the thesis, there is a (hopefully) complete list of my papers, so that interested readers can orientate themselves for further reading.

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- [CM11] J. Cichoń and W. Macyna. Approximate counters for flash memory. *17th IEEE International Conference on Embedded and Real-Time Computing Systems and Applications (RTCSA), Toyama, Japan*, 2011.
 - [Fla85] P. Flajolet. Approximate counting: A detailed analysis. *BIT*, 25:113–134, 1985.
 - [FR92] P. Flajolet and B. Richmond. Generalized digital trees and their difference–differential equations. *Random Structures and Algorithms*, 3:305–320, 1992.
 - [HFZ10] H.-K. Hwang, M. Fuchs, and V. Zacharovas. Asymptotic variance of random symmetric digital search trees. *Discrete Mathematics and Theoretical Computer Science*, 12(2):103–166, 2010.
 - [KHS94] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. Digital search trees again revisited: The internal path length perspective. *SIAM Journal on Computing*, 23:598–616, 1994.
 - [Knu78] D. E. Knuth. The average time for carry propagation. *Indagationes Mathematicae*, 40:238–242, 1978.

Informatique théorique et Applications/Theoretical Informatics and Applications
(vol. 25, n° 1, 1991, p. 43 à 48)

APPROXIMATE COUNTING: AN ALTERNATIVE APPROACH (*)

by Peter KIRSCHENHOFER ⁽¹⁾ and Helmut PRODINGER ⁽¹⁾

Abstract. – *In this note an alternative analysis of approximate counting is presented by using a lemma from the calculus of finite differences instead of the Mellin integral transform.*

Résumé. – *Cette note présente une analyse de l'algorithme de comptage approximatif qui repose sur un lemme du calcul des différences finies au lieu d'une utilisation de la transformée de Mellin.*

R. Morris [6] has proposed a probabilistic algorithm that maintains an approximate count in the interval 1 to n using only about $\log_2 \log_2 n$ bits. Approximate counting (with a binary base) starts with counter $C=1$. After n increments C should contain a good approximation to $\lfloor \log_2 n \rfloor$; thus C should be increased by 1 after another n increments approximately. Since only C is known the algorithm has to base its decision on the content of C alone. The following principle is used to increment the counter:

$$C := C + \begin{cases} 0 & \text{with probability } 1 - 2^{-C} \\ 1 & \text{with probability } 2^{-C} \end{cases}$$

Compare Flajolet [2] for a more detailed description. In the same paper a detailed analysis of this algorithm is presented and the following results are shown:

THEOREM 1: *After n successive increments the average content \bar{C}_n of the counter satisfies:*

$$\bar{C}_n \sim \log_2 n + \frac{\gamma}{\log 2} - \alpha + \frac{1}{2} + \delta_1(\log_2 n);$$

(*) Received January 1989, accepted in January 1990.

We thank an anonymous referee for his/her guidance concerning a better presentation of this paper.

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the variance σ_n^2 satisfies

$$\sigma_n^2 \sim \sigma_\infty^2 + \delta_2(\log_2 n)$$

where

$$\sigma_\infty^2 = \frac{\pi^2}{6 \log^2 2} - \alpha - \beta + \frac{1}{12} - \frac{1}{\log 2} \sum_{k \geq 1} \frac{1}{k \sinh(\theta k)}$$

with

$$\alpha = \sum_{k \geq 1} \frac{1}{2^k - 1}, \quad \beta = \sum_{k \geq 1} \frac{1}{(2^k - 1)^2}, \quad \theta = \frac{2\pi^2}{\log 2},$$

and $\delta_1(x)$ as well as $\delta_2(x)$ are periodic functions with period 1, mean value 0, and amplitude less than 10^{-4} .

Flajolet achieves this result by computing the probability $p_{n,l}$ of having counter value l after n increments:

$$p_{n,l} = \sum_{j=0}^{l-1} \frac{(-1)^j 2^{-\binom{j}{2}} (1 - 2^{-(l-j)})^n}{Q_j Q_{l-1-j}}, \quad (1)$$

where

$$Q_n = \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{4}\right) \cdots \left(1 - \frac{1}{2^n}\right). \quad (2)$$

The quantities

$$\bar{C}_n = \sum_{l=1}^n l p_{n,l} \quad \text{and} \quad \sigma_n^2 = \sum_{l=1}^n l^2 p_{n,l} - (\bar{C}_n)^2 \quad (3)$$

are then analyzed using real analysis and certain properties of the Mellin integral transform.

In the sequel we analyze the quantities in (3) by an alternative approach which avoids completely the unpleasant real analysis by making use of a classical lemma from the calculus of finite differences. This method has been attributed by Knuth [5] to S. O. Rice.

LEMMA 2: Let C be a curve surrounding the points $1, 2, \dots, n$ in the complex plane and let $f(z)$ be analytic inside C . Then

$$\sum_{k=1}^n \binom{n}{k} (-1)^k f(k) = -\frac{1}{2\pi i} \int_C \langle n; z \rangle f(z) dz,$$

where

$$\langle n; z \rangle = \frac{(-1)^{n-1} n!}{z(z-1)\dots(z-n)}.$$

Moving the contour of integration it turns out that the asymptotic expansion of the alternating sum is obtained via

$$\sum \text{Res}(\langle n; z \rangle f(z)) \tag{4}$$

where the sum is taken over all poles different from $1, \dots, n$. (Flajolet and Sedgewick have used this technique to simplify the analysis of digital search trees in [3].)

In order to apply Lemma 2 we write (1) as

$$p_{n,t} = \sum_{k=0}^n \binom{n}{k} (-1)^k \sum_{j=0}^{t-1} \frac{(-1)^j 2^{-\binom{j}{2}}}{Q_j Q_{t-1-j}} 2^{-(t-j)k}. \tag{5}$$

By Euler's *partition identities* [1]

$$\sum_{j \geq 0} \frac{(-1)^j 2^{-\binom{j}{2}} x^j}{Q_j} = \prod_{m \geq 0} \left(1 - \frac{x}{2^m}\right)$$

and

$$\sum_{j \geq 0} \frac{x^j}{Q_j} = \prod_{m \geq 0} \left(1 - \frac{x}{2^m}\right)^{-1},$$

the inner sum in (5) is the coefficient

$$[x^{t-1}] \sum_{j \geq 0} \frac{(-1)^j 2^{-\binom{j}{2}} x^j}{Q_j} \sum_{s \geq 0} 2^{-(s+1)k} \frac{x^s}{Q_s} = 2^{-k} [x^{t-1}] \prod_{m=0}^{k-1} \left(1 - \frac{x}{2^m}\right).$$

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Thus

$$\bar{C}_n = \sum_{l=1}^n l p_{n,l} = \sum_{k=0}^n \binom{n}{k} (-1)^k 2^{-k} \sum_{l=1}^n l [x^{l-1}] \prod_{m=0}^{k-1} \left(1 - \frac{x}{2^m}\right),$$

where the summation on l can be extended to infinity without changing the value. Since

$$\sum_{l \geq 1} l [x^{l-1}] g(x) = g'(1) + g(1)$$

we get immediately the following identity for \bar{C}_n which we state as a Proposition, since it is interesting in its own right.

PROPOSITION 3

$$\bar{C}_n = 1 - \sum_{k=0}^n \binom{n}{k} (-1)^k 2^{-k} Q_{k-1}. \quad (6)$$

With

$$Q_z = \frac{\prod_{j \geq 1} (1 - (1/2^j))}{\prod_{j \geq 1} (1 - (1/2^{z+j}))}$$

we may write (6) as

$$\bar{C}_n = 1 - \sum_{k=1}^n \binom{n}{k} (-1)^k f(k) \quad \text{with } f(z) = 2^{-z} Q_{z-1}. \quad (7)$$

The most significant pole of $\langle n; z \rangle f(z)$ different from $1, \dots, n$ is $z=0$, and the residue is

$$-\frac{H_n}{L} + \frac{1}{2} + \alpha \sim -\log_2 n - \frac{\gamma}{L} + \frac{1}{2} + \alpha,$$

with $L = \log 2$ and H_n the n -th Harmonic number. Further poles in $z = 2k\pi i/L$ with residue equal to

$$\frac{1}{L} \Gamma\left(\frac{2k\pi i}{L}\right)$$

give rise to the periodic fluctuation $\delta_1(x)$.

The next significant poles have real part $z = -1$ (simple!), so that the error term is of order n^{-1} ,

A similar approach allows us to evaluate the variance σ_n^2 in a few lines. Since

$$\sum_{l \geq 1} l^2 [x^{l-1}] g(x) = g^n(1) + 3g'(1) + g(1)$$

we have

$$\sum_{k=1}^n l^2 p_{n,l} = 1 + \sum_{k=1}^n \binom{n}{k} (-1)^k f(k)$$

with

$$f(k) = 2^{-k} Q_{k-1} (2T_{k-1} - 3), \quad T_{k-1} = \sum_{i=1}^{k-1} \frac{1}{2^i - 1}.$$

The continuation of Q_k has been referred above; T_k can be continued via

$$T_z = \alpha - \sum_{i \geq 1} \frac{1}{2^{z+i} - 1}.$$

The most important pole is again $z=0$ (triple) with residue

$$\begin{aligned} & \frac{2}{L^2} \left(\frac{H_n^2 + H_n^{(2)}}{2} + H_n L \left(\frac{1}{2} - \alpha \right) \right) + \alpha^2 - 4\alpha - \beta - \frac{2}{3} \\ & \sim \log_2^2 n + (\log_2 n) \left(\frac{2\gamma}{L} - 2\alpha + 1 \right) + \alpha^2 - 2\alpha - \beta + \frac{\gamma}{L} + \frac{\gamma^2}{L^2} - \frac{2\alpha\gamma}{L} + \frac{\pi^2}{6L^2} - \frac{2}{3}. \end{aligned}$$

Subtracting \bar{C}_n^2 yields for the variance (apart from smaller order terms)

$$\frac{\pi^2}{6L^2} - \alpha - \beta + \frac{1}{12} - \frac{2}{L^2} \sum_{k \geq 1} \left| \Gamma \left(\frac{2k\pi i}{L} \right) \right|^2 + \delta_2(\log_2 n),$$

where the series originates from the mean of $\delta_1^2(x)$. Observe that

$$|\Gamma(iy)|^2 = \frac{\pi}{y \sinh \pi y}, \quad y \in \mathbf{R},$$

to obtain the form of σ_∞^2 , whence Theorem 1.

Finally, it is striking to note that the constant σ_∞^2 , whose numerical value is

$$\sigma_\infty^2 = 0.7630141871107195182 \dots$$

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differs from the simpler

$$\frac{1}{2 \log 2} + \frac{1}{24} = 0.76301418711114837034 \dots$$

only in the twelfth digit after the decimal point.

Indeed, using a transformation result for Dedekind's η -function

$$\eta(\tau) = e^{ni\tau/12} \prod_{n \geq 1} (1 - e^{2\pi in\tau}), \quad \Im \tau > 0,$$

the constant σ_{∞}^2 appearing in (8) turns out to be

$$\frac{1}{2L} + \frac{1}{24} - h_1\left(\frac{2\pi^2}{L}\right) + \frac{2\pi^2}{L^2} h_2\left(\frac{4\pi^2}{L}\right),$$

where

$$h_1(x) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k(e^{kx} - 1)} \quad \text{and} \quad h_2(x) = \sum_{k \geq 1} \frac{e^{kx}}{(e^{kx} - 1)^2},$$

so that the numerical coincidence follows by the smallness of the involved functions $h_1(x)$ and $h_2(x)$.

Compare [4], where a closely related constant with relevance to digital search trees was evaluated by the authors and J. Schoisengeier.

Finally it should be noted that the expansion to arbitrary bases of the counter can be analyzed in the same style.

REFERENCES

1. G. E. ANDREWS, The Theory of Partitions, *Addison Wesley*, 1976.
2. P. FLAJOLET, Approximate Counting: A detailed Analysis, *BIT*, 1985, 25, pp. 113-134.
3. P. FLAJOLET and R. SEDGEWICK, Digital Search Trees Revisited, *S.I.A.M. J. Comput.*, 1986, 15, pp. 748-767.
4. P. KIRSCHENHOFER, H. PRODINGER and J. SCHOISSENGEIER, Zur Auswertung gewisser numerischer Reihen mit Hilfe modularer Funktionen, in *Zahlentheoretische Analysis II*, E. HLAWKA ed., Springer, Berlin, 1987, pp. 108-110.
5. D. E. KNUTH, The Art of Computer Programming, 3, *Addison Wesley*, 1973.
6. R. MORRIS, Counting Large Numbers of Events in Small Registers, *Comm. A.C.M.*, 1978, 21, pp. 840-842.

Theoretical Computer Science 100 (1992) 243–251
Elsevier

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Note

Hypothetical analyses: approximate counting in the style of Knuth, path length in the style of Flajolet

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Communicated by J. Díaz

Received September 1990

Revised October 1991

Abstract

Prodinger, H., Hypothetical analyses: approximate counting in the style of Knuth, path length in the style of Flajolet, Theoretical Computer Science 100 (1992) 243–251.

The first analysis of *approximate counting* is due to Flajolet (1985), whereas the first satisfactory analysis of the *average path length in digital search trees* has been performed by Knuth (1973). Both authors have used the *Mellin integral transform*, but in rather different ways. It was shown by Kirschenhofer and Prodinger (1991) that both problems are very similar. (This note contains also an “explanation” of this phenomenon.)

It is amusing to figure out what Flajolet and Knuth would have done by considering the exchanged problems. The aim of this note is to perform these analyses.

1. Introduction

The first analysis of *approximate counting* is due to Flajolet [1], whereas the first satisfactory analysis of the *average path length in digital search trees* (DSTs) has been performed by Knuth [4, p. 497ff]. Both authors have used the *Mellin integral transform* (see [2] for a brilliant introduction), but in rather different ways. It was shown by Kirschenhofer and Prodinger [3] that both problems are very similar (not to say “almost identical”).

It is now amusing to figure out what Flajolet and Knuth would have done by considering the exchanged problems. The aim of this note is to perform these analyses.

It is the author's opinion that this is entertaining and amusing both from the methodological as well as the *historical* point of view.

2. Flajolet's analysis of approximate counting

Flajolet considered $C_N^{[F]}$, the average value of a counter after N increments in a probabilistic counting algorithm. He computed $C_N^{[F]}$ via

$$C_N^{[F]} = \sum_{l \geq 1} l p_{N,l}^{[F]},$$

with

$$p_{N,l}^{[F]} = \sum_{j=0}^{l-1} \frac{(-1)^j 2^{-\binom{j}{2}} (1-2^{j-l})^N}{Q_j Q_{l-1-j}}$$

and

$$Q_m = \prod_{i=1}^m \left(1 - \frac{1}{2^i}\right).$$

Then he approximated $p_{N,l}^{[F]}$ by $\Phi(N/2^l)$, with

$$\Phi(x) = \frac{1}{Q_\infty} \sum_{j \geq 0} \frac{(-1)^j 2^{-\binom{j}{2}}}{Q_j} e^{-x2^j}.$$

Here,

$$Q_\infty = \lim_{m \rightarrow \infty} Q_m = \prod_{i \geq 1} \left(1 - \frac{1}{2^i}\right) = 0.288788 \dots$$

Then the Mellin transform was computed:

$$\Phi^*(s) = \Gamma(s) \zeta(s),$$

with

$$\zeta(s) = \frac{1}{Q_\infty} \sum_{j \geq 0} \frac{(-1)^j 2^{-\binom{j}{2}}}{Q_j} 2^{-js}.$$

Finally,

$$C_N^{[F]} \sim F(N) = \sum_{l \geq 1} l \Phi\left(\frac{N}{2^l}\right),$$

and

$$F^*(s) = \frac{2^s}{(2^s - 1)^2} \Phi^*(s).$$

By considering the *negative residues* of $F^*(s)x^{-s}$ for $\Re s = 0$, he obtained Theorem 2.1.

Theorem 2.1 (Flajolet [1]).

$$C_N^{[F]} \sim \log_2 N + \frac{\gamma}{\log 2} - \alpha + \frac{1}{2} - \delta_0(\log_2 N),$$

where

$$\alpha = \sum_{i \geq 1} \frac{1}{2^i - 1} = 1.6066\dots,$$

$$\gamma = 0.57721\dots \text{ (Euler's constant),}$$

$$\delta_0(x) = \frac{1}{\log 2} \sum_{k \neq 0} \Gamma\left(-\frac{2k\pi i}{\log 2}\right) e^{2k\pi i x}.$$

3. Knuth's analysis of the average path length in DSTs

Knuth considered

$$L_N^{[K]} = \sum_{k=2}^N \binom{N}{k} (-1)^k Q_{k-2}. \tag{K}$$

He changed it to

$$L_N^{[K]} \sim \sum_{n \geq 0} a_{n+1} \sum_{m \geq 0} 2^{m(1-n)} \left(e^{-N/2^m} - 1 + \frac{N}{2^m} \right),$$

with

$$a_i = \frac{(-1)^{i+1} 2^{-\binom{i}{2}}}{Q_{i-1}}.$$

By ‘‘Mellin’’¹ he obtained

$$L_N^{[K]} \sim \sum_{n \geq 0} a_{n+1} \frac{1}{2\pi i} \int_{-3/2-i\infty}^{-3/2+i\infty} \Gamma(s) N^{-s} \frac{1}{1-2^{1-n+s}} ds.$$

The negative residues of the integrand for $\Re s = -1$ give the following theorem.

Theorem 3.1 (Knuth [4]).

$$L_N^{[K]} \sim N \left(\log_2 N + \frac{\gamma}{\log 2} - \alpha + \frac{1}{2} - \frac{1}{\log 2} + \delta_{-1}(\log_2 N) \right),$$

¹ Actually, Knuth never mentioned the word ‘‘Mellin transform’’. He basically used the formula $e^{-x} = (1/2\pi i) \sum_{1/2-i\infty}^{1/2+i\infty} \Gamma(s) x^{-s} ds$ and some variants that can be proved directly by residue calculus. Because of the ubiquitous appearance of the Gamma function he christened the approach ‘‘Gamma function method’’. Also, he gives the credits for this procedure to N.G. de Bruijn.

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with

$$\delta_{-1}(x) = \frac{1}{\log 2} \sum_{k \neq 0} \Gamma\left(-1 - \frac{2k\pi i}{\log 2}\right) e^{2k\pi i x}.$$

Now we could show [3] that

$$C_N^{[F]} = 1 - \sum_{k=1}^N \binom{N}{k} (-1)^k 2^{-k} Q_{k-1}, \quad (\text{F})$$

and this similarity with (K) motivated this note. Section 6 contains an attempt to “explain” this phenomenon.

4. Knuth’s analysis of approximate counting

We need some “Q-formulae” (see, e.g. [1; 4, Exercise 5.1.1–16]). Let

$$Q(x) = \prod_{i \geq 1} \left(1 - \frac{x}{2^i}\right),$$

then

$$Q_k = \frac{Q_\infty}{Q(2^{-k})}. \quad (\text{Q1})$$

Euler’s formulae:

$$\frac{1}{1-t} \frac{1}{Q(t)} = \sum_{n \geq 0} \frac{t^n}{Q_n} \quad (\text{E1})$$

$$Q(t) = \sum_{n \geq 0} a_{n+1} t^n \quad (\text{E2})$$

Finally we need

$$\sum_{i \geq 1} \frac{a_{i+1}}{1-2^{-i}} = -\alpha;$$

this follows from

$$\frac{Q_\infty}{Q(x)} = \sum_{i \geq 1} \frac{a_i}{1-x2^{-i}}.$$

Now

$$C_N^{[F]} - 1 = -Q_\infty \sum_{k=1}^N \binom{N}{k} (-1)^k 2^{-k} \prod_{l \geq 0} \frac{1}{1-2^{-l-k}} \quad (\text{by Q1})$$

$$= -Q_\infty \sum_{k=1}^N \binom{N}{k} (-1)^k 2^{-k} \sum_{m \geq 0} 2^{-km} \frac{1}{Q_m} \quad (\text{by E1})$$

$$\begin{aligned}
 &= - \sum_{m \geq 0} \left(\sum_{k=0}^N \binom{N}{k} (-1)^k 2^{-k(m+1)} - 1 \right) Q(2^{-m}) \quad (\text{by Q1}) \\
 &= - \sum_{m \geq 0} ((1 - 2^{-m-1})^N - 1) \sum_{n \geq 0} a_{n+1} 2^{-mn} \quad (\text{by E2}) \\
 &\sim - \sum_{n \geq 0} a_{n+1} \sum_{m \geq 0} 2^{-mn} (e^{-N/2^{m+1}} - 1) \\
 &= - \sum_{n \geq 0} a_{n+1} \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \Gamma(s) N^{-s} \frac{2^s}{1-2^{-n+s}} ds.
 \end{aligned}$$

The negative residues at $s=0$ and $s=2k\pi i/\log 2$ ($k \neq 0$) give Flajolet's result: From $n=0$ we obtain

$$\log_2 N + \frac{\gamma}{\log 2} - \frac{1}{2} - \delta_0(\log_2 N);$$

from $n \geq 1$ we obtain

$$\sum_{n \geq 1} a_{n+1} \frac{1}{1-2^{-n}} = -\alpha.$$

5. Flajolet's analysis of the average path length in DSTs

We want to go back from (K) to

$$L_N^{[K]} = \sum_{l \geq 1} lp_{N,l}^{[K]};$$

this representation is of course not unique. Let

$$g(x) = \prod_{m=0}^{k-2} \left(1 - \frac{x}{2^m} \right).$$

Then

$$g'(1) = \begin{cases} 0 & \text{for } k=0, \\ -Q_{k-2} & \text{for } k \geq 1. \end{cases}$$

Furthermore, we have in general

$$\sum_{l \geq 1} l[x^l]g(x) = g'(1),$$

where $[x^l]g(x)$ denotes the coefficient of x^l in the Taylor series expansion of $g(x)$.

Thus,

$$L_N^{[K]} = - \sum_{k=1}^N \binom{N}{k} (-1)^k \sum_{l \geq 1} l[x^l] \prod_{m=0}^{k-2} \left(1 - \frac{x}{2^m} \right).$$

By (E1) and (E2) we have

$$\begin{aligned} \prod_{m=0}^{k-2} \left(1 - \frac{x}{2^m}\right) &= \prod_{m \geq 0} \left(1 - \frac{x}{2^m}\right) \prod_{m \geq 0} \left(1 - \frac{x}{2^{m+k-1}}\right)^{-1} \\ &= \sum_{j \geq 0} \frac{(-1)^j 2^{-\binom{j}{2}} x^j}{Q_j} \sum_{s \geq 0} \frac{1}{Q_s} (2^{1-k})^s x^s \end{aligned}$$

and, thus,

$$[x^l] \prod_{m=0}^{k-2} \left(1 - \frac{x}{2^m}\right) = \sum_{j=0}^l \frac{(-1)^j 2^{-\binom{j}{2}}}{Q_j Q_{l-j}} 2^{-(l-j)(k-1)}.$$

So, we can write

$$L_N^{[K]} = \sum_{l \geq 1} l p_{N,l}^{[K]},$$

with

$$\begin{aligned} p_{N,l}^{[K]} &= - \sum_{k=1}^N \binom{N}{k} (-1)^k \sum_{j=0}^l \frac{(-1)^j 2^{-\binom{j}{2}}}{Q_j Q_{l-j}} 2^{-(l-j)(k-1)} \\ &= \sum_{j=0}^l \frac{(-1)^j 2^{-\binom{j}{2}}}{Q_j Q_{l-j}} 2^{l-j} [1 - (1 - 2^{-(l-j)})^N]. \end{aligned}$$

Flajolet would approximate $p_{N,l}^{[K]}/N$ by $\Psi(N/2^l)$, with

$$\Psi(x) = \frac{1}{Q_\infty} \sum_{j \geq 0} \frac{(-1)^j 2^{-\binom{j}{2}}}{Q_j} \frac{1}{x 2^j} (1 - e^{-x 2^j}).$$

Now we perform the Mellin transform and find, for $-1 < \Re s < 0$,

$$\left(\frac{1 - e^{-x}}{x}\right)^* (s) = \int_0^\infty (1 - e^{-x}) x^{s-2} dx = -\Gamma(s-1).$$

Therefore,

$$\Psi^*(s) = -\xi(s) \Gamma(s-1),$$

where the function $\xi(s)$ was defined in Section 2. Now

$$\frac{L_N^{[K]}}{N} \sim K(N)$$

with

$$K(x) = \sum_{l \geq 1} l \Psi\left(\frac{x}{2^l}\right).$$

As before,

$$K^*(s) = \frac{2^s}{(2^s - 1)^2} \Psi^*(s).$$

By Mellin's inversion theorem we have

$$K(x) = -\frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{2^s}{(2^s-1)^2} \zeta(s) \Gamma(s-1) x^{-s} ds.$$

The negative residue of the integrand at $s=0$ gives the contribution

$$\log_2 N - \frac{1}{\log 2} + \frac{\gamma}{\log 2} - \alpha + \frac{1}{2};$$

the residues at $s=2k\pi i/\log 2$ produce the contribution $\delta_{-1}(\log_2 N)$.

6. An “explanation” of the similarities of the two problems

After a first reading of this paper (1991), an anonymous referee asked for a combinatorial explanation of the surprising similarity of the two seemingly different problems. This is of course a natural question and I failed in 1990 to give a good answer. However, now I can offer something.

The *probability generating functions* for the path length in digital search trees obey the following recursion ($N \geq 1$, $F_1(z) = F_0(z) = 1$):

$$F_{N+1}(z) = z^N \sum_{k=0}^N 2^{-N} \binom{N}{k} F_k(z) F_{N-k}(z).$$

(The averages L_N studied by Knuth are obtained via $L_N = F'_N(1)$.)

Now I will give a combinatorial explanation of a similar recursion for the approximate counting problem. Let, as in Section 1, $p_{N,l}^{[F]}$ denote the probability that the counter has the value l (“level l ”) after N “inputs”, and

$$F_N(z) = \sum_{l \geq 0} p_{N,l}^{[F]} z^l.$$

It should be noted that the counter is initially set to 1. To have a good idea about what is going on, let us say that there are N “players”; whenever a player has his turn, he tries to increase the counter by 1. If the counter has the value l , he will succeed with probability 2^{-l} ; otherwise his attempt has no effect. For instance, we can ask him to throw l consecutive “heads” (or 1's) with a fair coin. But we can think about it in another way: He starts at level 1 and flips a fair coin and can proceed if he has a 1. If he reaches the level $l+1$, it is exactly what we need. Now we no longer have to think about N consecutive players; each of them can throw the dice as long as he has 1's; throwing a 0 means to stop.

How is the value of counter computed from these data? We start at level 1 and, being at level j , we can go on if there are at least j players who made it to level $j+1$. The maximal level that we reach in that way is the value of the counter. Indeed, to proceed to a higher level means to kick out a successful player. From this interpretation we can set up a recursion for the probability generating functions. If $k \geq 1$ players have been

successful at level 1, $k-1$ are going to continue. But if $k=0$, the recursion stops. Hence,

$$F_N(z) = z \sum_{k=1}^N 2^{-N} \binom{N}{k} F_{k-1}(z) + z 2^{-N}, \quad N \geq 1, \quad F_0(z) = z.$$

Now set, as usual, $C_N = F'_N(1)$. Then the recursion turns into

$$C_N = 1 + 2^{-N} \sum_{k=1}^N \binom{N}{k} C_{k-1}, \quad N \geq 1, \quad C_0 = 1.$$

If we subtract the analogous recursion with N replaced by $N-1$, it turns into the more feasible recursion

$$2C_N = C_{N-1} + 1 + 2^{1-N} \sum_{k=0}^{N-1} \binom{N-1}{k} C_k, \quad N \geq 1, \quad C_0 = 1.$$

The standard method to solve this recursion is by the use of the exponential-generating function $C(z) = \sum_{N \geq 0} C_N z^N / N!$. Then

$$2C'(z) = C(z) + e^z + e^{z/2} C\left(\frac{z}{2}\right).$$

Now one sets $D(z) = e^{-z} C(z)$ to obtain the easier equation,

$$2D'(z) = -D(z) + D\left(\frac{z}{2}\right) + 1.$$

Then we consider the coefficients:

$$2D_{N+1} = -(1 - 2^{-N}) D_N, \quad N \geq 1, \quad D_0 = 1, \quad D_1 = 1/2.$$

Iterating this we find

$$D_N = 2^{-N} (-1)^{N-1} Q_{N-1}, \quad N \geq 1, \quad D_0 = 1.$$

Since $C(z) = e^z D(z)$, we have

$$C_N = \sum_{k=0}^N \binom{N}{k} D_k = 1 - \sum_{k=1}^N \binom{N}{k} (-1)^k 2^{-k} Q_{k-1};$$

this is formula (F).

Similar ideas are used in a forthcoming paper on probabilistic counting algorithms by Peter Kirschenhofer, Wojciech Szpankowski and the author.

7. Conclusion

Now it is worthwhile to discuss the two ‘‘Mellin’’ styles. Knuth and Flajolet attacked their problems from different angles (both of them very natural in their respective contexts). Since in ‘‘approximate counting’’ it is natural to compute the

probabilities directly, it was a good idea to approximate them immediately and use “Mellin” after that. In the tree context the recursions (as in Section 6) are most natural, so that Knuth *had* to use Euler’s formulae in order to avoid the cancellations in (K). After that “Mellin” can be used. Flajolet used first “Mellin” and then “Euler” (in order to analyze the ζ function); in Knuth’s case it is the other way around.

It should be stated here explicitly that this note contains no *judgement* about the relative qualities of the approaches. Both of them are most valuable and have influenced many people.

References

- [1] P. Flajolet, Approximate counting: a detailed analysis, *BIT* **25** (1985) 113–134.
- [2] P. Flajolet and J. Vitter, Average-case analysis of algorithms and data structures, in: *Handbook of Theoretical Computer Science, Vol. A: Algorithms and Complexity* (North-Holland, Amsterdam, 1990) 431–524.
- [3] P. Kirschenhofer and H. Prodinger, Approximate counting: an alternative approach, *RAIRO Theoretical Informatics* **25** (1991) 43–48.
- [4] D.E. Knuth, *The Art of Computer Programming, Vol. 3* (Addison-Wesley, Reading, MA, 1973).

HOW TO ADVANCE ON A STAIRWAY BY COIN FLIPPINGS

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ABSTRACT. Assume that N players start at stair 1 and flip a fair coin to advance one step. From the surviving people, a demon additionally kills one with probability $1-p$, i.e. with probability p he does not interfere. The average level that such a party will reach is of interest. In this paper we show that this quantity behaves like $\log_2 N$ and give a tutorial survey how to attack such problems. This problem is related to themes in Theoretical Computer Sciences, namely *Approximate Counting* and *Tries*.

Assume that N persons want to advance on a stairway. The rule is as follows: They start at level 1. The $k \geq 1$ persons who advanced to the level j flip a (fair) coin; the persons with the “1” advance; the others (with a “0”) die. However, there is a demon supervising the game. Usually, he “consumes” one of the survivors. But, with a probability p , he resigns and does not interfere. (The demon can only interfere at levels 2 or larger.) The question is, how far can the party advance, on the average?

In this example the party gets to level 4; at one stage the demon resigns.

This recreational description has of course a serious background. If $p = 0$, the model is exactly equivalent to an algorithmic procedure called *Approximate counting* [1], as was shown in [7]. If $p = 1$ the resulting recurrences (see the discussion below) resembles recurrences that are customary in the context of *tries*; see for instance the brandnew book of H. Mahmoud [5]. The idea of introducing a probability p of escaping the demon is borrowed from [8]; in this thesis U. Schmid studied the collision of N transmitted data. Usually they are all destroyed, but with a probability p one package survives.

In this paper we want to solve this problem. The main interest is not so much in the actual answer that we will finally obtain; its main aim is to give a rather tutorial introduction to the machinery that is usually used within the context of such problems.

To simplify the notation, let, as usual, $q = 1 - p$. We introduce probability generating functions $F_N(z)$. The coefficient of z^k is the probability that N persons

came to level k (and not farther). There is a recursion relating these functions:

$$F_N(z) = z \sum_{k=1}^N 2^{-N} \binom{N}{k} \left[qF_{k-1}(z) + pF_k(z) \right] + z2^{-N}, \quad N \geq 1, \quad F_0(z) = z$$

To understand it, note that $2^{-N} \binom{N}{k}$ is the probability that k of N have flipped a “1”. With z we mark a level that is already passed. With probability p , k people go on, with probability q , only $k-1$. (If $k=1$, and the demon takes him, we nevertheless say that the level was reached!) If $k=0$, the recursion stops.

The desired average C_N may be obtained via $F'_N(1)$. By differentiation, we get the recursion

$$C_N = 1 + q \sum_{k=1}^N 2^{-N} \binom{N}{k} C_{k-1} + p \sum_{k=1}^N 2^{-N} \binom{N}{k} C_k, \quad N \geq 1, \quad C_0 = 1.$$

Now it is customary to study the exponential generating function

$$C(z) = \sum_{N \geq 0} C_N \frac{z^N}{N!}.$$

Multiplying the recursion by $(2z)^N/N!$ and summing up we get

$$C(2z) = e^{2z} + pe^z C(z) - pe^z + q \sum_{N \geq 1} \frac{z^N}{N!} \sum_{k=1}^N \binom{N}{k} C_{k-1}.$$

To get rid of the unpleasant sum, we differentiate this equation and consider the difference of the new and the old equation. This gives us

$$2C'(2z) = 2e^{2z} + pe^z C(z) + pe^z C'(z) - pe^z + q \sum_{N \geq 1} \frac{z^{N-1}}{(N-1)!} \sum_{k=1}^N \binom{N}{k} C_{k-1},$$

or

$$2C'(2z) - C(2z) = e^{2z} + pe^z C'(z) + qe^z C(z).$$

Here we have used that $\binom{N+1}{k} - \binom{N}{k} = \binom{N}{k-1}$. The next step is the introduction of the *Poisson transformed* function $D(z) = e^{-z} C(z)$. Then $D'(z) + D(z) = e^{-z} C'(z)$, and we obtain the easier equation

$$2D'(2z) + D(2z) = 1 + pD'(z) + D(z).$$

Now we consider the coefficients D_N of this (exponential) generating function. Equating the coefficients of $z^N/N!$, we find

$$2^{N+1} D_{N+1} + 2^N D_N = \delta_{N,0} + pD_{N+1} + D_N, \quad N \geq 1, \quad D_0 = 1.$$

It means $D_0 = 1$, $D_1 = \frac{1}{2-p}$ and for $N \geq 1$

$$\left(1 - \frac{p}{2^{N+1}}\right) D_{N+1} = -\frac{1}{2} \left(1 - \frac{p}{2^N}\right) D_N.$$

We rewrite it as

$$D_N = -\frac{1}{2} D_{N-1} \frac{1 - \frac{1}{2^{N-1}}}{1 - \frac{p}{2^N}}$$

and iterate it to get

$$D_N = \frac{1}{2-p} (-1)^{N-1} 2^{1-N} \frac{Q_{N-1}}{Q_{N-1}\left(\frac{p}{2}\right)}, \quad N \geq 1.$$

Here,

$$Q_m(a) = \left(1 - \frac{a}{2}\right) \left(1 - \frac{a}{2^2}\right) \dots \left(1 - \frac{a}{2^m}\right)$$

and $Q_m = Q_m(1)$. Since $C(z) = e^z D(z)$, we have

$$\begin{aligned} C_N &= \sum_{k=0}^N \binom{N}{k} D_k \\ &= 1 + \frac{1}{2-p} \sum_{k=1}^N \binom{N}{k} (-1)^{k-1} 2^{1-k} \frac{Q_{k-1}}{Q_{k-1}\left(\frac{p}{2}\right)} \\ &= 1 - \frac{1}{2-p} \sum_{k=1}^N \binom{N}{k} (-1)^k f(k) \end{aligned}$$

with

$$f(k) = 2^{1-k} \frac{Q_{k-1}}{Q_{k-1}\left(\frac{p}{2}\right)}.$$

The method to evaluate such an alternating sum is called *Rice's method*, [2], [4], [5], [6].

Lemma. *Let \mathcal{C} be a curve surrounding the points $1, 2, \dots, N$ in the complex plane and let $f(z)$ be analytic inside \mathcal{C} . Then*

$$\sum_{k=1}^N \binom{N}{k} (-1)^k f(k) = -\frac{1}{2\pi i} \int_{\mathcal{C}} [N; z] f(z) dz,$$

where

$$[N; z] = \frac{(-1)^{N-1} N!}{z(z-1)\dots(z-N)} = \frac{\Gamma(N+1)\Gamma(-z)}{\Gamma(N+1-z)}. \quad \square$$

Extending the contour of integration it turns out that under suitable growth conditions on $f(z)$ (compare [8]) the asymptotic expansion of the alternating sum is given by

$$\sum \text{Res}([N; z]f(z)) + \text{smaller order terms}$$

where the sum is taken over all poles z_0 different from $1, \dots, N$.

Thus we must find a complex function $f(z)$, such that $f(k)$ are the numbers $2^{1-k} \frac{Q_{k-1}}{Q_{k-1}(\frac{p}{2})}$. For this task it is convenient to introduce the infinite product

$$Q(x) = \left(1 - \frac{x}{2}\right) \left(1 - \frac{x}{4}\right) \left(1 - \frac{x}{8}\right) \cdots.$$

Then it is easy to see that

$$Q_m = \frac{Q(1)}{Q(2^{-m})},$$

and, more generally,

$$Q_m(a) = \frac{Q(a)}{Q(a2^{-m})}.$$

It makes sense to replace m in the last equality by a complex variable z . Now we have to consider the residues of

$$[N; z] 2^{1-z} \frac{Q_{z-1}}{Q_{z-1}(\frac{p}{2})}$$

at $z = 0$ and then also at $z = 2k\pi i/L$, were we set here and later $L = \log 2$. Also, the notion $\chi_k = 2k\pi i/L$ is very useful. The meaning of Q_{z-1} etc. is as it was just described. Let's start as follows:

$$Q_{z-1} = \frac{Q(1)}{Q(2^{1-z})} = \frac{1}{1 - 2^{-z}} \cdot \frac{Q(1)}{Q(2^{-z})}.$$

Now

$$1 - 2^{-z} = 1 - e^{-Lz} \sim 1 - \left(1 - Lz + \frac{L^2 z^2}{2} \mp \dots\right) \sim Lz \left(1 - \frac{Lz}{2}\right)$$

and thus

$$\frac{1}{1 - 2^{-z}} \sim \frac{1}{Lz} \left(1 + \frac{Lz}{2}\right).$$

Furthermore,

$$Q(2^{-z}) \sim Q(1) + Q'(1) \cdot (-L) \cdot z,$$

and

$$Q'(x) = -Q(x) \cdot \sum_{k \geq 1} \frac{1}{2^k - x},$$

so that

$$Q(2^{-z}) \sim Q(1) \cdot (1 + Lz\alpha),$$

where we use the abbreviation

$$\alpha = \sum_{k \geq 1} \frac{1}{2^k - 1}.$$

Alltogether we find

$$Q_{z-1} \sim \frac{1}{Lz} \left(1 + \frac{Lz}{2}\right) (1 - Lz\alpha).$$

The computations for $Q_{z-1}(\frac{p}{2})$ are entirely similar, so that we just mention the result:

$$Q_{z-1}(\frac{p}{2}) \sim \frac{1}{1 - \frac{p}{2}} \cdot (1 - p\alpha(p)Lz)$$

with

$$\alpha(p) = \sum_{k \geq 1} \frac{1}{2^k - p}.$$

We need one more local expansion:

$$[N; z] \sim -\frac{1}{z} (1 + zH_N),$$

with the N -th harmonic number

$$H_N = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \sim \log N + \gamma + \dots$$

Plugging things together we have

$$\begin{aligned} f(z) &\sim 2(1 - Lz) \frac{1}{Lz} \left(1 + \frac{Lz}{2}\right) (1 - Lz\alpha) \left(1 - \frac{p}{2}\right) (1 + p\alpha(p)Lz) \\ &\sim \frac{2-p}{Lz} \left(1 + z \left(-\frac{L}{2} - \alpha L + p\alpha(p)L\right)\right). \end{aligned}$$

Therefore the residue of $[N; z]f(z)$ at $z = 0$ is

$$-\frac{2-p}{L} \left(H_N - \frac{L}{2} - \alpha L + p\alpha(p)L\right).$$

We can use the asymptotics for H_N and then go back to the formula for the average C_N and get the main contribution

$$1 - \frac{1}{2-p} \cdot \left(-\frac{2-p}{L} \left(\log N + \gamma - \frac{L}{2} - \alpha L + p\alpha(p)L\right)\right).$$

This readily simplifies to

$$\log_2 N + \frac{\gamma}{L} + \frac{1}{2} - \alpha + p\alpha(p).$$

From this we can easily see the dependency of p as p varies between 0 and 1. As was to be expected, this quantity increases monotonically and for $p = 1$ the α -terms disappear.

Now we have to turn to the residues $z = \chi_k$. This time it is easier because there is only a single pole, originating from Q_{z-1} . We list the local expansions of the factors of

$$f(z) = 2^{1-z} \frac{Q_{z-1}}{Q_{z-1}(\frac{p}{2})} :$$

$$\begin{aligned}
2^{1-z} &\sim 2 \\
Q_{z-1} &\sim \frac{1}{Lz} \\
\frac{1}{Q_{z-1}\left(\frac{p}{2}\right)} &\sim \frac{Q(p)}{Q\left(\frac{p}{2}\right)} = 1 - \frac{p}{2} \\
[N; z] &\sim N^{\chi_k} \cdot \Gamma(-\chi_k)
\end{aligned}$$

Here, we used the well-known fact that

$$\frac{\Gamma(N+a)}{\Gamma(N+b)} \rightarrow N^{a-b}.$$

The residue of $[N; z]f(z)$ at $z = \chi_k$ is thus

$$2 \cdot \frac{1}{L} \cdot \left(1 - \frac{p}{2}\right) \cdot N^{\chi_k} \Gamma(-\chi_k) = (2-p) \frac{1}{L} \Gamma(-\chi_k) N^{\chi_k}.$$

It is now customary to collect all these contributions for $k \in \mathbb{Z}$, $k \neq 0$ in one function $\delta(x)$. Set

$$\delta(x) = \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) e^{2k\pi i \cdot x},$$

then the periodic quantity of our desired quantity C_N is (don't forget the factor $-\frac{1}{2-p}$)

$$-\delta(\log_2 N).$$

Observe that this quantity is independent of p . However, in the smaller order terms, the parameter p appears.

We summarize our results as follows:

Theorem. *Let C_N be the average level that a random party of N people can reach in presence of a demon (as described in the Introduction). Then we have the following asymptotic formula*

$$C_N \sim \log_2 N + \frac{\gamma}{L} + \frac{1}{2} - \alpha + p\alpha(p) - \delta(\log_2 N)$$

where

$$\alpha(p) = \sum_{k \geq 1} \frac{1}{2^k - p}, \quad \alpha = \alpha(1)$$

and

$$\delta(x) = \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) e^{2k\pi i \cdot x}$$

is a periodic function of period 1, mean 0 and small amplitude ($\approx 10^{-6}$). It is given as a Fourier series.

REFERENCES

1. P. Flajolet, *Approximate Counting: A Detailed Analysis*, BIT **25** (1985), 113–134.
2. P. Flajolet and R. Sedgewick, *Digital Search Trees Revisited*, SIAM J. Comput. **15** (1986), 748–767.
3. P. Kirschenhofer and H. Prodinger, *Approximate Counting: An Alternative Approach*, RAIRO Informatique Théorique et Applications **25** (1991), 43–48.
4. D.E. Knuth, *The Art of Computer Science Vol.3*, Addison-Wesley, Reading, MA, 1973.
5. H. M. Mahmoud, *Evolution of Random Search Trees*, Wiley-Interscience Series in Discrete Mathematics and Optimization, New York, 1992.
6. N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Chelsea, New York, 1954.
7. H. Prodinger, *Hypothetic Analyses: Approximate Counting in the Style of Knuth, Path length in the Style of Flajolet*, Theoret. Comput. Sci. (to appear).
8. U. Schmid, *Analyse von Collision-Resolution Algorithmen in Random-Access Systemen mit dominanten Übertragungskanälen*, Dissertation, TU Wien, 1986.

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*Journal de Théorie des Nombres
de Bordeaux* 5 (1993), 365–381

Asymptotic analysis of a class of functional equations and applications

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ABSTRACT. Flajolet and Richmond have invented a method to solve a large class of divide-and-conquer recursions. The essential part of it is the asymptotic analysis of a certain generating function for $z \rightarrow \infty$ by means of the Mellin transform. In this paper this type of analysis is performed for a reasonably large class of generating functions fulfilling a functional equation with polynomial coefficients. As an application, the average life time of a party of N people is computed, where each person advances one step or dies with equal probabilities, and an additional “killer” can kill at any level up to d survivors, according to his probability distribution.

1. Introduction

In [3] Flajolet and Richmond presented an ingenious method to deal with a class of recursions where a typical example looks like

$$(1.1) \quad f_{n+d} = 1 + \sum_{k=0}^n 2^{-n} \binom{n}{k} (f_k + f_{n-k}).$$

Here, $d \geq 0$ is a fixed integer. For $d = 0$ and $d = 1$ such recursions (and their solutions!) occur frequently as *divide-and-conquer* recursions in the *Analysis of Algorithms*, (see [6] as a general reference on the subject. However, the new approach allows it for the first time to deal with the cases $d = 2, 3, \dots$; it consists of several stages. First, the recursion is translated into the language of exponential generating functions. Then, a related function (sometimes called the *Poisson transform*) is considered. Then, from its coefficients, an *ordinary generating function* $G(z)$ is built up. The analytic behaviour of the latter for $z \rightarrow \infty$ is then to be analyzed by *Mellin transform* methods, (this is the heart of the method). Then, by a substitution, the asymptotic behaviour as $z \rightarrow 1$ of the ordinary generating

Manuscrit reçu le 19 mars 1993.

†These authors are supported by the Austrian Science Foundation Project P8274-PHY. All authors are supported by the Austrian–Hungarian Science Cooperation Program, project 10U3.

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function of the desired numbers f_n is found. As the last step, this local information is translated to the asymptotic behaviour of the coefficients by *transfer theorems*.

The relevant function $G(z)$ fulfills a functional equation

$$(1.2) \quad G(z)(1+z)^d = 2z^d G\left(\frac{z}{2}\right) + P_0(z),$$

where the polynomial $P_0(z)$ depends on the starting values.

We devote this paper to the general study of recursions as in (1.2), with general polynomials as factors and “ $\frac{1}{2}$ ” replaced by an arbitrary parameter $0 < \lambda < 1$.

This is not only interesting by itself, but can be applied in the last section to a stochastic process which is a generalization of one presented in [10]. Although formulated in terms of “recreational mathematics” the recursions describe either average “trie” parameters or the behaviour of an “approximate counter”. In order to keep this paper short, we refrain from describing those algorithms and data structures and refer for all computer science algorithms to [6] and [7]. For the Mellin transform, which has proven to be very useful especially in number theory and in theoretical computer science, we refer to [1, chapter 13], [6], [8] and to the survey [4]. We note here that the classical applications of the Mellin transform occur in prime number theory and, more recently, in the analysis of digital problems (cf. [9] and for a detailed survey see [2]).

Let us recall the fundamental properties of the Mellin transform

$$(1.3) \quad f^*(s) = \int_0^\infty f(x)x^{s-1} dx.$$

If $f(x)$ is a piece-wise continuous function with

$$f(x) = O(x^\alpha) \quad \text{for } x \rightarrow 0 \quad \text{and} \quad f(x) = O(x^\beta) \quad \text{for } x \rightarrow \infty,$$

and $\alpha > \beta$, then the Mellin transform exists as a holomorphic function in the vertical strip $-\alpha < \Re s < -\beta$ (“fundamental strip”). Our basic tool is Mellin’s inversion formula

$$(1.4) \quad f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s} ds$$

for any $-\alpha < c < -\beta$. This formula yields a correspondence between the singularities of $f^*(s)$ and the asymptotics of $f(x)$ (cf. [8, chapter 1.5]). This technique is especially useful in the asymptotic analysis of *harmonic sums* $h(x) := \sum_k \lambda_k f(a_k x)$ since

$$(1.5) \quad h^*(s) = f^*(s) \sum_k \frac{\lambda_k}{a_k^s}.$$

Thus the Mellin transform of a harmonic sum is the product of a generalized Dirichlet series and the transform of the base function.

In Section 2 we introduce a class of functional equations with polynomial coefficients generalizing (1.2) and perform the asymptotic analysis by the Mellin transform approach. Two cases have to be distinguished: in the first one the main term does not contain an oscillating part (Section 2) and in the second one there are oscillations in the main term (Section 3). Section 4 is devoted to a direct extension of [3]. In the final section this method is applied to analyze a special evolution process where N persons are climbing up an infinite staircase as follows: at each step either a person goes to the next step or the person dies. We compute the average of the maximum lifetime.

2. A class of functional equations

We consider the functional equation

$$(2.1) \quad G(z)P_1(z) = G(\lambda z)P_2(z) + P_0(z),$$

where $P_0(z) = a_0 + a_1 z + \dots + a_{d_0} z^{d_0}$, $P_1(z) = b_0 + b_1 z + \dots + b_{d_1} z^{d_1}$ and $P_2(z) = c_0 + c_1 z + \dots + c_{d_2} z^{d_2}$ are polynomials of degrees d_0, d_1, d_2 (with $d_1 \leq d_2, d_0 \geq 0$), respectively, assume that P_1 and P_2 have no non-negative real roots, $P_1(0) \neq P_2(0)$ and $0 < \lambda < 1$. Our main result gives an asymptotic expansion for the unique analytic solution of (2.1). Formally iterating the functional equation we get

$$G(z) = \sum_{n=0}^{\infty} P_0(\lambda^n z) \frac{P_2(z) \cdots P_2(\lambda^{n-1} z)}{P_1(z) \cdots P_1(\lambda^n z)}.$$

Let now

$$Q_1(z) = \prod_{n=0}^{\infty} \frac{1}{b_0} P_1(\lambda^n z) \quad \text{and} \quad Q_2(z) = \prod_{n=0}^{\infty} \frac{1}{c_0} P_2(\lambda^n z).$$

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Thus we can write

$$(2.2) \quad G(z) = \frac{Q_2(z)}{b_0 Q_1(z)} \sum_{n=0}^{\infty} P_0(\lambda^n z) \left(\frac{c_0}{b_0}\right)^n \frac{Q_1(\lambda^{n+1} z)}{Q_2(\lambda^n z)}.$$

Our next step is to establish an asymptotic expansion of $G(z)$ for $z \rightarrow \infty$. This will be done by a Mellin-transform technique. Let for the following

$$(2.3) \quad \frac{1}{b_0} P_1(z) = \prod_{k=1}^{r_1} (1 + \alpha_k z)^{\mu_k} \quad \text{and} \quad \frac{1}{c_0} P_2(z) = \prod_{k=1}^{r_2} (1 + \beta_k z)^{\nu_k}$$

(μ_k and ν_k denoting the multiplicities of the roots) and take the Mellin-transforms of the logarithms of these equations

$$\begin{aligned} \left(\log \frac{1}{b_0} P_1\right)^*(s) &= \frac{\pi}{s \sin \pi s} \sum_{k=1}^{r_1} \mu_k \alpha_k^{-s} \quad \text{and} \\ \left(\log \frac{1}{c_0} P_2\right)^*(s) &= \frac{\pi}{s \sin \pi s} \sum_{k=1}^{r_2} \nu_k \beta_k^{-s} \end{aligned}$$

for $-1 < \Re s < 0$. Observing that $\log Q_1$ and $\log Q_2$ are harmonic sums yields

$$(2.4) \quad \begin{aligned} (\log Q_1)^*(s) &= \frac{1}{1 - \lambda^{-s}} \frac{\pi}{s \sin \pi s} \sum_{k=1}^{r_1} \mu_k \alpha_k^{-s} \quad \text{and} \\ (\log Q_2)^*(s) &= \frac{1}{1 - \lambda^{-s}} \frac{\pi}{s \sin \pi s} \sum_{k=1}^{r_2} \nu_k \beta_k^{-s}. \end{aligned}$$

In order to establish an asymptotic expansion for Q_1 and Q_2 we apply Mellin’s inversion formula

$$(2.5) \quad \log Q_1(z) = \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} (\log Q_1)^*(s) z^{-s} ds$$

(and similarly for Q_2). Inserting (2.4) and shifting the line of integration to $\Re s = \frac{1}{2}$ (cf. [8, chapter 1.5]) yields

$$(2.6) \log Q_1(z) =$$

$$\sum_{m \in \mathbb{Z}} \operatorname{Res}_{s=\chi_m} \frac{1}{\lambda^{-s}-1} \frac{\pi}{s \sin \pi s} \sum_{k=1}^{r_1} \mu_k \alpha_k^{-s} z^{-s} + \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} (\log Q_1)^*(s) z^{-s} ds,$$

where $\chi_m = \frac{2m\pi i}{\log \frac{1}{\lambda}}$ are the roots of $1 - \lambda^{-s} = 0$. Note that for $s = 0$ we have a triple pole and in the other cases simple poles. Computing residues (and applying Vieta's rule) we get

$$\begin{aligned} \operatorname{Res}_{s=0} \frac{1}{1-\lambda^{-s}} \frac{\pi}{s \sin \pi s} \sum_{k=1}^{r_1} \mu_k \alpha_k^{-s} z^{-s} = \\ - \frac{d_1}{2 \log \frac{1}{\lambda}} \log^2 z - \left(\frac{\log \frac{b_0}{b_{d_1}}}{\log \frac{1}{\lambda}} - \frac{d_1}{2} \right) \log z - B, \end{aligned}$$

where

$$B = \frac{1}{2 \log \frac{1}{\lambda}} \sum_{k=1}^{r_1} \mu_k \log^2 \alpha_k - \frac{1}{2} \log \frac{b_0}{b_{d_1}} - \frac{\log \frac{1}{\lambda}}{12} - \frac{\pi^2}{6 \log \frac{1}{\lambda}}.$$

For the residues at $\chi_m \neq 0$ we obtain

$$\begin{aligned} \operatorname{Res}_{s=\chi_m} \frac{1}{1-\lambda^{-s}} \frac{\pi}{s \sin \pi s} \sum_{k=1}^{r_1} \mu_k \alpha_k^{-s} z^{-s} = \\ \frac{\pi}{\chi_m (\log \lambda) \sin \pi \chi_m} \sum_{k=1}^{r_1} \mu_k \alpha_k^{-\chi_m} z^{-\chi_m}. \end{aligned}$$

Thus we have

$$\begin{aligned} \log Q_1(z) = \\ \frac{d_1}{2 \log \frac{1}{\lambda}} \log^2 z + \left(\frac{\log \frac{b_0}{b_{d_1}}}{\log \frac{1}{\lambda}} - \frac{d_1}{2} \right) \log z + B + \varphi_1 \left(\frac{\log z}{\log \frac{1}{\lambda}} \right) + O \left(z^{-\frac{1}{2}} \right), \end{aligned}$$

where $\varphi_1(u)$ is a continuous periodic function of period 1, whose Fourier expansion is given by

$$(2.7) \quad \varphi_1(u) = - \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{\pi}{\chi_m (\log \lambda) \sin \pi \chi_m} \sum_{k=1}^{r_1} \mu_k \alpha_k^{\chi_m} e^{2\pi i m u}.$$

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Similar calculations yield the asymptotic expansion for Q_2 :

$$\log Q_2(z) = \frac{d_2}{2 \log \frac{1}{\lambda}} \log^2 z + \left(\frac{\log \frac{c_0}{c_{d_2}}}{\log \frac{1}{\lambda}} + \frac{d_2}{2} \right) \log z + C + \varphi_2 \left(\frac{\log z}{\log \frac{1}{\lambda}} \right) + O \left(z^{-\frac{1}{2}} \right),$$

where

$$C = \frac{1}{2 \log \frac{1}{\lambda}} \sum_{k=1}^{r_2} \nu_k \log^2 \beta_k - \frac{1}{2} \log \frac{c_0}{c_{d_2}} - \frac{\log \frac{1}{\lambda}}{12} - \frac{\pi^2}{6 \log \frac{1}{\lambda}}$$

and $\varphi_2(u)$ is a continuous periodic function of period 1, whose Fourier expansion is given by

$$(2.8) \quad \varphi_2(u) = - \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{\pi}{\chi_m(\log \lambda) \sin \pi \chi_m} \sum_{k=1}^{r_2} \nu_k \beta_k^{\chi_m} e^{2\pi i m u}.$$

Setting $\Phi(u) = \exp(C - B + \varphi_2(u) - \varphi_1(u))$, $A = \frac{d_2 - d_1}{2 \log \frac{1}{\lambda}}$, $L = \frac{\log \frac{c_0 b_{d_1}}{c_{d_2} b_0}}{\log \frac{1}{\lambda}}$ and observing the rate of growth of the transforms in (2.4), we derive

$$(2.9) \quad \frac{Q_2(z)}{Q_1(z)} = \begin{cases} z^L \Phi \left(\frac{\log z}{\log \frac{1}{\lambda}} \right) \left(1 + O(z^{-\frac{1}{2}}) \right) & \text{if } d_1 = d_2 \\ e^{A \log^2 z} z^{L + \frac{d_2 - d_1}{2}} \Phi \left(\frac{\log z}{\log \frac{1}{\lambda}} \right) \left(1 + O(z^{-\frac{1}{2}}) \right) & \text{if } d_1 < d_2 \end{cases}$$

for $z \rightarrow \infty$ inside an angular region

$$|\arg z| \leq \delta < \delta_0 = \pi - \min_k (|\arg \alpha_k|, |\arg \beta_k|).$$

As above we obtain

$$(2.10) \quad \frac{Q_1(\lambda z)}{Q_2(z)} = \begin{cases} \frac{b_{d_1}}{b_0 \lambda^{d_1}} z^{-L - d_1} \frac{1}{\Phi \left(\frac{\log z}{\log \frac{1}{\lambda}} \right)} \left(1 + O(z^{-\frac{1}{2}}) \right) & \text{if } d_1 = d_2 \\ e^{-A \log^2 z + O(\log z)} & \text{if } d_1 < d_2 \end{cases}$$

for $|\arg z| \leq \delta < \delta_0$.

Let us first consider the case $d_1 = d_2 = d$ and $a_0 \neq 0$. Then we need $|c_0| < |b_0|$ to ensure convergence in (2.2). Setting

$$(2.11) \quad H(z) = \sum_{n=0}^{\infty} P_0(\lambda^n z) \left(\frac{c_0}{b_0}\right)^n \frac{Q_1(\lambda^{n+1} z)}{Q_2(\lambda^n z)},$$

and observing that this is a harmonic sum yields for its Mellin transform

$$(2.12) \quad H^*(s) = \frac{b_0}{b_0 - c_0 \lambda^{-s}} \left(P_0(z) \frac{Q_1(\lambda z)}{Q_2(z)} \right)^* (s).$$

Notice that this Mellin transform only exists if $-L + d_0 - d < 0$, since “the order at ∞ ” has to be smaller than “the order at 0”. From our asymptotic expansion (2.10) we want to extract some information on the singularities of the transformed harmonic sum $H^*(s)$. The fundamental strip of this transform is

$$0 < \Re s < -\max \left(-L + d_0 - d, \frac{\log \frac{c_0}{b_0}}{\log \frac{1}{\lambda}} \right).$$

Thus we have to distinguish two cases. First we consider the case occurring in our applications in Section 3, i.e.

$$(2.13) \quad b_d \lambda^{d_0} \leq c_d \lambda^d.$$

This is the case where the singularities originate directly from the individual summands (in the other case the singularities are caused by the harmonic summation and a special example of this type was extensively studied by Flajolet and Richmond [3]). For $b_d \lambda^{d_0} < c_d \lambda^d$, by (2.10) we have

$$(2.14) \quad H(z) = \frac{b_d^2}{(b_d - c_d \lambda^{d-d_0}) b_0 \lambda^d} z^{-L-d+d_0} \frac{1}{\Phi \left(\frac{\log z}{\log \frac{1}{\lambda}} \right)} (1 + O(z^{-\varepsilon})),$$

where ε is a suitable positive number. Combining (2.14) and (2.9) the fluctuation Φ cancels out in the asymptotics of $G(z)$ and we obtain

$$(2.15) \quad G(z) = \frac{b_d^2 a_{d_0}}{b_0^2 (b_d - c_d \lambda^{d-d_0}) \lambda^d} z^{d_0-d} (1 + O(z^{-\varepsilon})).$$

For $b_d \lambda^{d_0} = c_d \lambda^d$ the function $H^*(s)$ has double poles at $-L-d+d_0+\chi_m$. Thus the asymptotic formula (2.14) has to be replaced by

$$(2.16) \quad H(z) = \left(\frac{\log z}{\log \frac{1}{\lambda}} \frac{1}{\Phi \left(\frac{\log z}{\log \frac{1}{\lambda}} \right)} + \Psi \left(\frac{\log z}{\log \frac{1}{\lambda}} \right) \right) z^{-L-d+d_0} (1 + O(z^{-\varepsilon})).$$

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Combining (2.16) and (2.9) we derive

$$(2.17) \quad G(z) = \left(\frac{1}{b_0} \frac{\log z}{\log \frac{1}{\lambda}} + \frac{1}{b_0} \Psi \left(\frac{\log z}{\log \frac{1}{\lambda}} \right) \right) z^{d_0-d} (1 + O(z^{-\varepsilon})),$$

where Ψ is an unspecified periodic fluctuation.

Until now we have restricted ourselves to the case $a_0 \neq 0$. In general let ℓ be defined by $P_0(z) = z^\ell \tilde{P}_0(z)$ with a polynomial $\tilde{P}_0(z)$ such that $\tilde{P}_0(0) \neq 0$. Then the equation corresponding to (2.11) is

$$(2.18) \quad H(z) = z^\ell \sum_{n=0}^{\infty} \tilde{P}_0(\lambda^n z) \left(\frac{\lambda^\ell c_0}{b_0} \right)^n \frac{Q_1(\lambda^{n+1} z)}{Q_2(\lambda^n z)}.$$

The sum in this expression is exactly of the type considered above and therefore we need the condition $\lambda^\ell |c_0| < |b_0|$ to ensure convergence. Thus we obtain the same asymptotic expansions also in the general case $\ell \geq 0$.

Summing up what we have proved until now yields

THEOREM 1. *Let $G(z)$ be the unique analytic solution of a functional equation of type (2.1) and let ℓ be the smallest number such that $a_\ell \neq 0$. Furthermore assume the additional properties $d_1 = d_2 = d$, $\lambda^\ell |c_0| < |b_0|$ and $b_d \lambda^{d_0} \leq c_d \lambda^d$. Then $G(z)$ has an asymptotic expansion for $z \rightarrow \infty$, ($|\arg z| \leq \delta < \pi - \min_k (|\arg \alpha_k|, |\arg \beta_k|)$), of the form*

$$G(z) = \begin{cases} \frac{b_d^2 a_{d_0}}{b_0^2 (b_d - c_d \lambda^{d-d_0}) \lambda^d} z^{d_0-d} (1 + O(z^{-\varepsilon})) & \text{if } b_d \lambda^{d_0} < c_d \lambda^d \\ \left(\frac{1}{b_0} \frac{\log z}{\log \frac{1}{\lambda}} + \frac{1}{b_0} \Psi \left(\frac{\log z}{\log \frac{1}{\lambda}} \right) \right) z^{d_0-d} (1 + O(z^{-\varepsilon})) & \text{if } b_d \lambda^{d_0} = c_d \lambda^d, \end{cases}$$

where ε is a suitable positive number and Ψ denotes a continuous periodic function of period 1.

Remark 1: We note that in the case covered by Theorem 1 the periodic fluctuations do not occur in the main term.

3. Fluctuations in the main term

First we consider functional equations of type (2.1) with $d_1 = d_2 = d$ satisfying the condition $b_d \lambda^{d_0} > c_d \lambda^d$ converse to (2.13). In the previous section the convergence of the Fourier series for φ_1 and φ_2 was immediate. In the case treated now we establish a preparatory lemma generalizing Lemma 5 in [3], from which the convergence of the corresponding Fourier series can be deduced. In the following let v denote the maximal order of a zero of the entire function Q_2 . Note that v is finite, but may be larger than $\max_k \nu_k$.

LEMMA 1. Let $F(z) = P_0(z) \frac{Q_1(\lambda z)}{Q_2(z)}$. Then the Mellin transform $F^*(s)$ admits the representation

$$\frac{\pi}{\sin \pi s} (E_0(\lambda^{-s}) + (s-1)E_1(\lambda^{-s}) + \cdots + (s-1) \cdots (s-v+1)E_{v-1}(\lambda^{-s}))$$

where the E_i 's are entire functions. Thus for $\sigma > 0$

$$(3.1) \quad |F^*(\sigma + it)| = O(|t|^{v-1} e^{-\pi|t|}) \quad \text{as } |t| \rightarrow \infty.$$

Proof. Proceeding as in [3] let

$$J(s) = \int_{\mathcal{H}} F(z)(-z)^{s-1} dz,$$

where \mathcal{H} denotes a Hankel contour that goes from $+\infty - i0$, circles around 0 clockwise and returns to $+\infty + i0$. Then a standard argument (cf. [11]) yields $J(s) = 2i \sin \pi s F^*(s)$. Now we want to evaluate the loop integral by residues. For this purpose we have to know polynomial upper bounds for $F(z)$, which follow from

$$(3.2) \quad \left| \prod_{n=0}^{\infty} \left(\frac{1 + \alpha \lambda^n z}{1 + \beta \lambda^n z} \right) \right| \leq K_1 |z|^{K_2} \quad \text{for } |1 + \beta \lambda^n z| \geq \varepsilon,$$

where α, β and $\frac{\lambda}{4} > \varepsilon > 0$ are given numbers and K_1 and K_2 are positive constants depending on those numbers and on λ .

For proving this assertion let $\gamma = \max\{1, 2|\alpha|, 2|\beta|\}$ and $n_0 = \left\lfloor \frac{\log 2\gamma z}{\log \frac{1}{\lambda}} \right\rfloor + 1$. Thus we have $|\lambda^{n_0} z \gamma| \leq \frac{1}{2}$. We split the product (3.2) into two parts $n < n_0$ and $n \geq n_0$. The second product can be estimated by taking logarithms

$$\begin{aligned} & \left| \sum_{n=n_0}^{\infty} \log \left(\frac{1 + \alpha \lambda^n z}{1 + \beta \lambda^n z} \right) \right| \\ &= \left| \sum_{n=n_0}^{\infty} \left((\alpha - \beta) z \lambda^n - (\alpha^2 - \beta^2) \frac{z^2}{2} \lambda^{2n} + (\alpha^3 - \beta^3) \frac{z^3}{3} \lambda^{3n} \pm \cdots \right) \right| \\ &\leq \gamma |z| \lambda^{n_0} \frac{1}{1 - \lambda} + \gamma^2 |z|^2 \lambda^{2n_0} \frac{1}{1 - \lambda^2} + \cdots \leq \frac{1}{1 - \lambda}. \end{aligned}$$

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For estimating the first part of the product in the region $|1 + \beta\lambda^n z| \geq \varepsilon$ for $n = 0, \dots, n_0 - 1$ we note that the function $f(w) = \frac{1+\alpha w}{1+\beta w}$ is continuous on the compact subregion $|1 + \beta w| \geq \varepsilon$ of the Riemannian sphere, and hence attains its maximum $M = M(\varepsilon, \alpha, \beta, \lambda) \geq 1$. Thus we obtain

$$\left| \prod_{n=0}^{\infty} \left(\frac{1 + \alpha\lambda^n z}{1 + \beta\lambda^n z} \right) \right| \leq M^{n_0} e^{\frac{1}{1-\lambda}} \leq K_1 |z|^{K_2},$$

and (3.2) is proved.

Observing that $\frac{Q_1}{Q_2}$ can be decomposed into finitely many factors of the type (3.2) we get a polynomial upper bound for $F(z)$ in the region

$$|1 + \beta_k \lambda^n z| \geq \varepsilon, \quad (k = 1, \dots, r_2 \text{ and } n = 0, 1, \dots).$$

Notice that ε is chosen small enough that arbitrarily large circles centered at the origin are contained in this region. Again by a standard argument we derive

$$J(s) = 2\pi i \sum_{k=1}^{r_2} \sum_{n=0}^{\infty} \operatorname{Res}_{z=-\frac{\lambda^{-n}}{\beta_k}} F(z)(-z)^{s-1}$$

for $\Re s < \sigma_0 < 0$. Next we calculate the residues and observe that the maximum order of the poles is v . Let us consider a pole $-\frac{\lambda^{-n}}{\beta_k}$ of multiplicity b and set

$$X_{k,n}(z) = F(z)(1 + \beta_k \lambda^n z)^b.$$

First we have for $w = z + \frac{\lambda^{-n}}{\beta_k}$

$$(3.3) \quad (-z)^{s-1} = \lambda^{-n(s-1)} \beta_k^{-s+1} \left(1 + \frac{(1-s)}{1!} (\beta_k \lambda^n w) + \frac{(1-s)(2-s)}{2!} (\beta_k \lambda^n w)^2 + \dots \right).$$

Expanding $X_{k,n}(-\frac{\lambda^{-n}}{\beta_k} + w)$ around $w = 0$ we get

$$\begin{aligned} X_{k,n} \left(-\frac{\lambda^{-n}}{\beta_k} + w \right) &= P_0(z) Q_1(\lambda z) \prod \left(1 - \frac{\beta_l}{\beta_k} \lambda^{m-n} \right)^{-1} \prod \left(1 + \frac{w}{\frac{\lambda^{-m}}{\beta_l} - \frac{\lambda^{-n}}{\beta_k}} \right)^{-1} \\ &= \sum_{j=0}^{\infty} \Lambda_j(k, n) w^j, \end{aligned}$$

where the products range over all (l, m) such that $\beta_l \lambda^m \neq \beta_k \lambda^n$. We observe

$$(3.4) \quad \Lambda_j(k, n) = O\left(\lambda^{\frac{n(n+1)}{2} + jn} \eta_k^{-j}\right)$$

with an absolute O -constant, where η_k is the minimal distance of $\frac{1}{\beta_k}$ to the other roots of Q_2 .

Multiplying (3.3) and the expansion of $X_{k,n}$ yields for the coefficient of w^{b-1}

$$\lambda^{-n(s-1)} \beta_k^{-s+1} \sum_{j=0}^{b-1} \Lambda_{b-j-1}(k, n) \frac{(1-s)(2-s)\cdots(j-s)}{j!}.$$

Each residue provides one term in the power series expansion of the $E_i(\lambda^{-s})$. Because of the upper bound (3.4) the power series expansion of E_i is hyperexponentially convergent and E_i is an entire function. \square

Now we are able to treat functional equations of type (2.1) with equal degrees and fluctuations in the main term.

THEOREM 2. *Let $G(z)$ be the unique analytic solution of a functional equation of type (2.1) and let ℓ be the smallest number such that $a_\ell \neq 0$. Furthermore assume the additional properties $d_1 = d_2 = d$, $\lambda^\ell |c_0| < |b_0|$ and $b_d \lambda^{d_0} > c_d \lambda^d$. Then $G(z)$ has an asymptotic expansion for $z \rightarrow \infty$, ($|\arg z| \leq \delta < \pi - \min_k(|\arg \alpha_k|, |\arg \beta_k|)$), of the form*

$$G(z) = z^M \Psi_1\left(\frac{\log z}{\log \frac{1}{\lambda}}\right) (1 + O(z^{-\varepsilon})),$$

where ε is a suitable positive number, $M = \frac{\log \frac{b_d}{c_d}}{\log \frac{1}{\lambda}}$ and Ψ_1 some continuous periodic fluctuation (the Fourier expansion of which is discussed in the proof).

Proof. We apply Mellin's inversion formula to (2.12) and observe that the first singularities encountered when shifting the line of integration to the right originate from the factor $\frac{b_0}{b_0 - c_0 \lambda^{-s}}$. Notice that evaluations of the E_i 's of Lemma 1 at the poles are constants and that the Fourier expansion of the arising fluctuation is exponentially convergent because of the estimate (3.1). The fluctuation Ψ_1 is obtained by multiplying with the periodic

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function $\frac{1}{b_0}\Phi$ as in (2.9). The proof is completed by multiplying with the first case in the asymptotic formula (2.9). \square

Up to now we have only considered polynomials P_1 and P_2 of equal degree. What remains for a complete asymptotic analysis of functional equations of type (2.1) is the case $d_1 < d_2$, (we note here that the converse case $d_1 > d_2$ cannot be treated by the Mellin transform approach).

Remark 2: Lemma 1 remains true in the case $d_1 < d_2$. The only problem is to find a polynomial upper bound for $\frac{Q_1(\lambda z)}{Q_2(z)}$. This product can be split into products of type (3.2) and $d_2 - d_1$ additional linear factors in the denominator. For these additional factors we easily find the polynomial lower bound $K\varepsilon^{(d_2-d_1)\log z}$.

By the same reasoning as above we obtain

THEOREM 3. *Let $G(z)$ be the unique analytic solution of a functional equation of type (2.1) with $d_1 < d_2$. Then $G(z)$ has an asymptotic expansion for $z \rightarrow \infty$, ($|\arg z| \leq \delta < \pi - \min_k(|\arg \alpha_k|, |\arg \beta_k|)$), of the form*

$$G(z) = e^{A \log^2 z} z^{M + \frac{d_2 - d_1}{2}} \Psi_2 \left(\frac{\log z}{\log \frac{1}{\lambda}} \right) (1 + O(z^{-\varepsilon})),$$

where ε is a suitable positive number, $A = \frac{d_2 - d_1}{2 \log \frac{1}{\lambda}}$, $M = \frac{\log \frac{b_d}{c_d}}{\log \frac{1}{\lambda}}$ and Ψ_2 some continuous periodic fluctuation.

4. The Flajolet–Richmond case

In [3] Flajolet and Richmond analyze a tree-partitioning process in which n elements split into d at the root of a tree (d is a design parameter), the rest going recursively into two subtrees with a binomial probability distribution. They prove an asymptotic formula for the expected number f_n of non-empty nodes in such a random tree, by studying the generating function $G(z) = \sum_{n \geq 0} g_n z^n$ where $f_n = \sum_{k=0}^n \binom{n}{k} g_k$. $G(z)$ satisfies the functional equation

$$(4.1) \quad G(z)(1+z)^d = 2z^d G\left(\frac{z}{2}\right) + P_0(z),$$

where P_0 is a polynomial of degree d . This is not a functional equation of type (2.1) since $c_0 = 0$. Thus the above theorems cannot be applied. In the following we extend our general approach to the case $c_0 = 0$, $d_1 = d_2 = d$ which covers the special case (4.1) (in the case $b_0 = 0$ there is no analytic solution, for $d_1 \neq d_2$ the Mellin transform approach cannot be applied).

THEOREM 4. *Let $G(z)$ be the unique analytic solution of the functional equation*

$$G(z)P_1(z) = G(\lambda z)P_2(z) + P_0(z) \quad (0 < \lambda < 1),$$

where $P_1(z) = b_0 + b_1z + \dots + b_dz^d$, $P_2(z) = c_\kappa z^\kappa + \dots + c_dz^d$ (with $b_0 \neq 0$, $c_\kappa \neq 0$, $\kappa > 0$) are polynomials of equal degree d with no positive real roots and let $P_0(z) = a_0 + a_1z + \dots + a_{d_0}z^{d_0} \neq 0$. Then $G(z)$ has an asymptotic expansion for $z \rightarrow \infty$, ($|\arg z| \leq \delta < \pi - \min_k(|\arg \alpha_k|, |\arg \beta_k|)$), of the form

$$G(z) = \begin{cases} z^{M+d_0-d} \Psi_3 \left(\frac{\log z}{\log \frac{1}{\lambda}} \right) (1 + O(z^{-\varepsilon})) & \text{if } M < d - d_0 \\ \left(\frac{a_d \log z}{c_d \log \frac{1}{\lambda}} + \Psi_4 \left(\frac{\log z}{\log \frac{1}{\lambda}} \right) \right) z^{d_0-d} (1 + O(z^{-\varepsilon})) & \text{if } M = d - d_0 \\ \frac{a_{d_0}}{c_d} z^{d_0-d} (1 + O(z^{-\varepsilon})) & \text{if } M > d - d_0, \end{cases}$$

where ε is a suitable positive number, $M = \frac{\log \frac{b_d}{c_d}}{\log \frac{1}{\lambda}}$ and Ψ_3, Ψ_4 some continuous periodic fluctuations.

Proof. Using the substitution $t = \frac{1}{z}$ we can rewrite the explicit formula for $G(z)$

$$G\left(\frac{1}{t}\right) = t^{d-d_0} \sum_{n=0}^{\infty} \bar{P}_0(\lambda^{-n}t) \lambda^{n(d_0-d)} \frac{\bar{P}_2(t) \dots \bar{P}_2(\lambda^{-(n-1)}t)}{\bar{P}_1(t) \dots \bar{P}_1(\lambda^{-n}t)},$$

where $\bar{P}_0(z) = z^{d_0}P_0(\frac{1}{z})$, $\bar{P}_1(z) = z^dP_1(\frac{1}{z})$ and $\bar{P}_2(z) = z^dP_2(\frac{1}{z})$ are polynomials with $\deg \bar{P}_1 = d > d - \kappa = \deg \bar{P}_2$. Proceeding as in Section 2 we set

$$\bar{Q}_1(t) = \prod_{n=0}^{\infty} \frac{1}{b_d} \bar{P}_1(\lambda^n t)$$

$$\bar{Q}_2(t) = \prod_{n=0}^{\infty} \frac{1}{c_d} \bar{P}_2(\lambda^n t)$$

and obtain

$$(4.2) \quad G\left(\frac{1}{t}\right) = t^{d-d_0} \frac{\bar{Q}_1(\lambda t)}{c_d \bar{Q}_2(\lambda t)} \sum_{n=0}^{\infty} \bar{P}_0(\lambda^{-n}t) \lambda^{n(d_0-d)} \frac{\bar{Q}_2(\lambda^{-(n-1)}t)}{\bar{Q}_1(\lambda^{-n}t)}.$$

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Let $F(t) = \bar{P}_0(t) \frac{\bar{Q}_2(\lambda t)}{\bar{Q}_1(t)}$. Notice that Lemma 1 can be applied to this function. Furthermore we have, for $t \rightarrow \infty$, $F(t) \leq \exp(-\eta \log^2 t + O(\log z))$ with some constant $\eta > 0$ and $F(0) = a_{d_0}$. Thus the fundamental strip of $F^*(s)$ is $0 < \Re s < \infty$. Denoting the sum in (4.2) by $H(t)$ we obtain

$$(4.3) \quad H^*(s) = F^*(s) \frac{b_d}{b_d - c_d \lambda^{d_0 - d + s}}$$

with the fundamental strip $\max(0, -M - d + d_0) < \Re s < \infty$. This yields the desired asymptotic expansion for $H(t)$, $t \rightarrow 0$ which implies the expansion for $G(z)$, $z \rightarrow \infty$. \square

Remark 3: The computer science problem originally considered by Flajolet and Richmond is covered by the first alternative of Theorem 4.

5. An evolution process with killer

We consider the following stochastic process. N persons (starting at level 1) are climbing up an infinite staircase. At every step for each person there are two possibilities of equal probability: (i) go to the next step (ii) the person dies. After this at any level an outside killer kills j persons with given probability p_j , ($0 \leq j \leq d$, d is fixed; $p_0 \neq 0$), (if there are only $< j$ persons available, all of them are killed). This situation is a generalization of [10] where the case $d = 1$ was discussed (emerging from some computer science problems).

We ask for the expectation C_N of the maximum lifetime. To this end, we introduce the probability generating functions $F_N(z)$ where the coefficient of z^k is the probability that the maximum level of N persons is k . We have the recursion ($N \geq 1$)

$$(5.1) \quad F_N(z) = z \sum_{k=1}^N 2^{-N} \binom{N}{k} [p_0 F_k(z) + \cdots + p_{k-d} F_{k-d}] + z 2^{-N},$$

(observe that $2^{-N} \binom{N}{k}$ is the probability that k out of N people have made it to the next level. The variable z marks the level, and the effect of the killer is described by $[p_0 F_k(z) + \cdots + p_{k-d} F_{k-d}]$). The cumbersome special cases when there are less persons than the killer wants to kill can be avoided by setting $F_i(z) = z$ for $i \leq 0$.

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Now the expectations C_N are obtained by differentiating the $F_N(z)$ and then evaluating at $z = 1$. Doing this we obtain the recursion ($N \geq 1$)

$$(5.2) \quad C_N = 1 + \sum_{j=0}^d p_j \sum_{k=1}^N 2^{-N} \binom{N}{k} C_{k-j},$$

and the starting values $C_i = 1$ for $i \leq 0$.

Setting up the exponential generating function $C(z) = \sum_{N \geq 0} C_N z^N / N!$ we find

$$(5.3) \quad C(z) = e^z - e^{z/2} + \sum_{j=0}^d p_j e^{z/2} \sum_{k \geq 0} C_{k-j} \frac{(z/2)^k}{k!}.$$

Now it is customary (in order to get an easier equation) to set $D(z) = e^{-z} C(z)$ (the ‘‘Poisson transform’’). For simplicity, we replace z by $2z$, multiply the equation by e^{-z} and differentiate it d times, yielding

$$(5.4) \quad e^z \sum_{k=0}^d \binom{d}{k} 2^k D^{(k)}(2z) = e^z + \sum_{j=0}^d p_{d-j} C^{(j)}(z).$$

Furthermore, we can express (by Leibniz’ rule)

$$C^{(j)}(z) = e^z \sum_{i=0}^j \binom{j}{i} D^{(i)}(z),$$

so that we obtain after another multiplication by e^{-z}

$$(5.5) \quad \sum_{j=0}^d \binom{d}{j} 2^j D^{(j)}(2z) = 1 + \sum_{j=0}^d p_{d-j} \sum_{i=0}^j \binom{j}{i} D^{(i)}(z).$$

Now set $D(z) = \sum_{N \geq 0} D_N z^N / N!$ and $G(z) = \sum_{N \geq 0} D_N z^N$. Comparing coefficients we find

$$(5.6) \quad \sum_{j=0}^d \binom{d}{j} 2^{N+j} D_{N+j} = \delta_{N,0} + \sum_{j=0}^d p_{d-j} \sum_{i=0}^j \binom{j}{i} D_{N+i}.$$

Finally, we multiply by $(\frac{z}{2})^{N+d}$ and sum up for all $N \geq 0$ to obtain

$$(5.7) \quad G(z) \left(1 + \frac{z}{2}\right)^d = G\left(\frac{z}{2}\right) P_2(z) + P_0(z)$$

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with

$$(5.8) \quad P_2(z) = \sum_{j=0}^d p_{d-j} \sum_{i=0}^j \binom{j}{i} \left(\frac{z}{2}\right)^{d-i} = \sum_{j=0}^d p_j \left(\frac{z}{2}\right)^j \left(1 + \frac{z}{2}\right)^{d-j}$$

and

$$(5.9) \quad P_0(z) = \left(\frac{z}{2}\right)^d + \sum_{j=0}^d \binom{d}{j} \left(\frac{z}{2}\right)^{d-j} \sum_{i=0}^{j-1} D_i z^i - \sum_{j=0}^d p_{d-j} \sum_{i=0}^j \binom{j}{i} \left(\frac{z}{2}\right)^{d-i} \sum_{h=0}^{i-1} D_h \left(\frac{z}{2}\right)^h.$$

We want to apply Theorem 4 and find $d_0 = d$, $a_d = b_d = c_d = 2^{-d}$ and thus $M = 0$. Hence the second alternative of Theorem 4 gives

$$(5.10) \quad G(z) = \log_2 z + \Psi_4(\log_2 z) + O(z^{-\epsilon}).$$

This leads to

$$(5.11) \quad F(z) = \sum_{N \geq 0} C_N z^N = \frac{1}{1-z} G\left(\frac{z}{1-z}\right) \\ = \frac{1}{1-z} \log_2 \frac{1}{1-z} + \frac{1}{1-z} \Psi_4\left(\log_2 \frac{1}{1-z}\right) + O((1-z)^{-1+\epsilon}),$$

for $z \rightarrow 1$. Now, since

$$[z^N] \frac{1}{1-z} \log \frac{1}{1-z} = 1 + \frac{1}{2} + \dots + \frac{1}{N} = \log N + \gamma + O\left(\frac{1}{N}\right),$$

singularity analysis (cf. [5]) leads to Theorem 5.

THEOREM 5. *The average level C_N that a party of N people with an additional killer will reach is asymptotically given by*

$$C_N = \log_2 N + \varphi(\log_2 N) + O(N^{-\epsilon})$$

where $\varphi(x)$ is a periodic function with period 1. Its Fourier coefficients could be given in principle by evaluating an appropriate Mellin integral at the points $2k\pi i / \log 2$, $k \in \mathbb{Z}$. $\varphi(x)$ depends on the probability distribution (p_0, \dots, p_d) , (with $p_d \neq 0$) of the killer.

REFERENCES

- [1] T. M. APOSTOL, *Introduction to Analytic Number Theory*, Springer, New York, 1984.

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- [2] P. FLAJOLET, P. GRABNER, P. KIRSCHENHOFER, H. PRODINGER and R. F. TICHY, *Mellin Transforms and Asymptotics: Digital Sums*, Theoret. Comput. Sci. (to appear).
- [3] P. FLAJOLET and L. B. RICHMOND, *Generalized Digital Trees and Their Difference-Differential Equations*, Random Structures and Algorithms 3 (1992), 305–320.
- [4] P. FLAJOLET, M. RÉGNIER and R. SEDGEWICK, *Some Uses of the Mellin Integral Transform in the Analysis of Algorithms*, Combinatorial Algorithms on Words (A. Apostolico and Z. Galil, eds.), Springer, New York, 1985.
- [5] P. FLAJOLET and A. ODLYZKO, *Singularity Analysis of Generating Functions*, SIAM J. Disc. Math. 3 (1990), 216–240.
- [6] P. FLAJOLET and J. VITTER, *Average-Case Analysis of Algorithms and Data Structures*, Handbook of Theoretical Computer Science Vol. A “Algorithms and Complexity” North-Holland, 1990, 431–524.
- [7] D. E. KNUTH, *The Art of Computer Science Vol. 3*, Addison-Wesley, Reading, MA, 1973.
- [8] H. M. MAHMOUD, *Evolution of Random Search Trees*, Wiley-Interscience Series in Discrete Mathematics and Optimization, New York, 1992.
- [9] J.-L. MAUCLAIRE and L. MURATA, *On q -Additive Functions, II*, Proc. Japan Acad. 59 (1983), 441–444.
- [10] H. PRODINGER, *How to Advance on a Staircase by Coin Flippings*, Proceedings “Fibonacci Numbers and Applications 5” (1992) (to appear).
- [11] E. T. WHITTAKER and G. N. WATSON, *A Course in Modern Analysis*, Cambridge University Press, 1927.

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A COIN TOSSING ALGORITHM FOR COUNTING LARGE NUMBERS OF EVENTS

PETER KIRSCHENHOFER — HELMUT PRODINGER

ABSTRACT. R. Morris has proposed a probabilistic algorithm to count up to n using only about $\log_2 \log_2 n$ bits. In this paper a slightly more general concept is introduced that allows to obtain a smoother average case behaviour. This concept is general enough to cover the analysis of an algorithm where the randomness is simulated by coin tossings.

1. Introduction

“Approximate counters” are realized by probabilistic algorithms that maintain an approximate count in the interval 1 to n using only about $\log_2 \log_2 n$ bits. The algorithmic principle was proposed by R. Morris [7]:

Starting with counter $C = 1$, after n increments C should contain a good approximation to $\log_2 n$. Thus C should be increased by 1 after other n increments approximately. Since only C is known the algorithm has to base its decision on the content of C alone.

The principle to increment the counter is now

$$C := C + \begin{cases} 0 & \text{with probability } 1 - 2^{-C} \\ 1 & \text{with probability } 2^{-C} \end{cases} \quad (*)$$

Flajolet [2] has analysed this algorithm in detail; another method of analysis has been proposed by the authors [4]. In [2; p. 127ff] Flajolet also discusses variants of the incremental procedure (*), essentially replacing base 2 by base a . For $a < 2$ a smoother behaviour of the counter is obtained.

The aim of this paper is twofold.

(i) On the one hand we substitute (*) by an incremental process that adds $\frac{1}{b}$ to the counter with suitable probability. For $a = 2^{1/b}$ the resulting “automaton” (compare Fig. 1) is closely related to Flajolet’s smoothing procedure just

AMS Subject Classification (1991): Primary 68Q25.

Key words: Probabilistic algorithms, Analysis of algorithms, Asymptotic expansions.

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described. Nevertheless we point out that our algorithm is slightly more general and flexible enough to deal with a related problem described below.

(ii) On the other hand we describe a simple coin tossing algorithm that simulates the probabilities in (*). We compare the results on this algorithm with another one whose analysis is closely related to a problem studied by Knuth in the context of binary addition [6]. It turns out that the variance in our instance is significantly smaller.

In Section 2 of this paper we present the analysis of the general incremental principle mentioned in (i) above.

Section 3 refers to the coin tossing problems mentioned in (ii). It turns out that the analysis of the behaviour of these algorithms is essentially (i.e. with neglectable error terms) covered by the analysis of Section 2.

2. A general incremental procedure

We consider the following incremental procedure. Starting with $C' = 1$ we increment as follows (b is a fixed natural number, $0 < d < 2^{1/b}$).

$$C' := C' + \begin{cases} 0 & \text{with probability } 1 - d \cdot 2^{-C'/b} \\ 1 & \text{with probability } d \cdot 2^{-C'/b} \end{cases}$$

(The constant d will be chosen appropriately later on.) To get a good approximation for $\log_2 n$ it is meaningful to rescale the countervalues of counter C' by considering

$$C = 1 + \frac{C' - 1}{b} .$$

A reformulation of the incremental procedure in terms of C reads

$$C := C + \begin{cases} 0 & \text{with probability } 1 - d \cdot 2^{-C - \frac{1}{b} + 1} \\ \frac{1}{b} & \text{with probability } d \cdot 2^{-C - \frac{1}{b} + 1} \end{cases} \quad (**)$$

We denote for abbreviation the possible counter values of C by

$$c_j = 1 + \frac{j - 1}{b} , \quad j = 1, 2, 3, \dots .$$

Then we have for the *transition* $c_j \rightarrow c_{j+1}$ the *probability* $d \cdot 2^{-j/b}$. In the following we set

$$a = 2^{1/b} .$$

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In order to illustrate the above concept we may use the following “automaton”:

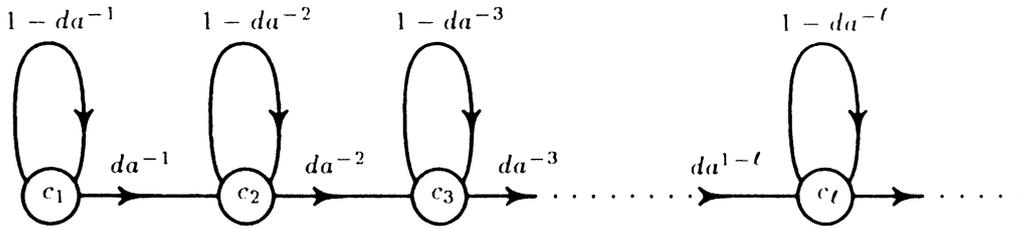


Fig. 1

The reader might observe that the instance $d = 1$ is the instance studied by Flajolet in [2; p. 127ff]. In the following derivation we digress from Flajolet’s analysis after having established the probabilities as in formula (6). Instead of Flajolet’s Mellin-type analysis we use a contour integral representation for the alternating sums coming up (compare Lemma 1).

Let $p_{n,l}$ denote the probability to reach “state” c_l in n steps and $H_l(z)$ the corresponding probability generating function, i.e.,

$$H_l(z) = \sum_{n \geq 0} p_{n,l} z^n. \tag{1}$$

With $\alpha_i = 1 - da^{-i}$ it follows immediately from Figure 1 that

$$\begin{aligned} H_l(z) &= \frac{1}{1 - \alpha_1 z} \cdot da^{-1} z \cdot \frac{1}{1 - \alpha_2 z} \cdot da^{-2} z \dots \frac{1}{1 - \alpha_l z} \\ &= d^{l-1} a^{-\binom{l}{2}} z^{l-1} \prod_{i=1}^l \frac{1}{1 - \alpha_i z}. \end{aligned} \tag{2}$$

Let

$$H_l(z) = \sum_{j=1}^l \frac{A_{l,j}}{1 - \alpha_j z} \tag{3}$$

denote the partial fraction decomposition of $H_l(z)$. It is a straightforward computation to derive

$$A_{l,j} = \frac{(-1)^{l-j} a^{-\binom{l-j}{2}}}{Q_{j-1}(a)Q_{l-j}(a)}, \tag{4}$$

where

$$Q_i(a) = \left(1 - \frac{1}{a}\right) \left(1 - \frac{1}{a^2}\right) \dots \left(1 - \frac{1}{a^i}\right). \tag{5}$$

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From (3) and (4) we find (interchanging j and $l - j$)

$$p_{n,l} = \sum_{j=0}^{l-1} \frac{(-1)^j a^{-\binom{j}{2}}}{Q_j(a) Q_{l-1-j}(a)} (1 - da^{j-l})^n. \quad (6)$$

By the binomial theorem we may express $p_{n,l}$ by an *alternating sum*:

$$p_{n,l} = \sum_{k=0}^n \binom{n}{k} (-1)^k d^k \sum_{j=0}^{l-1} \frac{(-1)^j}{Q_j(a) Q_{l-1-j}(a)} \cdot a^{-\binom{j}{2} + (j-l)k}.$$

Using Euler's famous partition identities [1]

$$\sum_{j \geq 0} \frac{(-1)^j a^{-\binom{j}{2}} t^j}{Q_j(a)} = \prod_{m \geq 0} \left(1 - \frac{t}{a^m}\right) \quad (7)$$

and

$$\sum_{j \geq 0} \frac{t^j}{Q_j(a)} = \prod_{m \geq 0} \left(1 - \frac{t}{a^m}\right)^{-1}, \quad (8)$$

the second sum in $p_{n,l}$ may be associated to a product.

$$p_{n,l} = \sum_{k=0}^n \binom{n}{k} (-1)^k \left(\frac{d}{a}\right)^k [t^{l-1}] \prod_{m=0}^{k-1} \left(1 - \frac{t}{a^m}\right). \quad (9)$$

For the expectation E_n of the value of the counter C after n steps we find

$$E_n = \sum_{l \geq 1} p_{n,l} \cdot cl = 1 + \frac{1}{b} \sum_{k=0}^n \binom{n}{k} (-1)^k \left(\frac{d}{a}\right)^k \sum_{l \geq 0} l [t^l] \prod_{m=0}^{k-1} \left(1 - \frac{t}{a^m}\right).$$

Now $\sum_{l \geq 0} l [t^l] f(t) = f'(1)$, so that

$$E_n = 1 - \frac{1}{b} \sum_{k=1}^n \binom{n}{k} (-1)^k \left(\frac{d}{a}\right)^k Q_{k-1}(a) = 1 - \frac{1}{b} \Sigma_n. \quad (10)$$

An asymptotic evaluation of an alternating sum as in (10) may be performed using the following

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LEMMA 1. [8] *Let C be a curve surrounding the points $s, s + 1, \dots, n$ ($s \in \mathbb{N}$) in the complex plane and let $f(z)$ be analytic inside C . Then*

$$\sum_{k=s}^n \binom{n}{k} (-1)^k f(k) = -\frac{1}{2\pi i} \int_C [n; z] f(z) dz,$$

where

$$[n; z] = \frac{(-1)^{n-1} n!}{z(z-1)\dots(z-n)}. \quad \square$$

Extending the contour of integration it turns out that under suitable growth conditions on $f(z)$ (compare [3]) the asymptotic expansion of the alternating sum is given by

$$\sum \text{Res}([n; z]f(z)) + O(n^{\sigma+\epsilon}), \quad \text{any } \epsilon > 0,$$

where the sum is taken over all poles z_0 different from s, \dots, n with $\Re z_0 > \sigma$.

In our sum we have $s = 1$ and $f(k) = \left(\frac{d}{a}\right)^k Q_{k-1}(a)$, which may be continued analytically by

$$f(z) = \left(\frac{d}{a}\right)^z Q_{z-1}(a), \quad \text{where} \quad Q_z(a) = \frac{Q_\infty(a)}{\prod_{j \geq 1} \left(1 - \frac{1}{a^{z+j}}\right)}.$$

In order to find the residues of $[n; z]f(z)$ at the double pole $z = 0$ we use the local expansions

$$\begin{aligned} [n; z] &\sim -\frac{1}{z}(1 + zH_n) \\ \left(\frac{d}{a}\right)^z &\sim 1 + z \log\left(\frac{d}{a}\right) \\ Q_{z-1}(a) &\sim \frac{1}{z \log a} \left(1 + z(\log a) \left(\frac{1}{2} - \alpha_a\right)\right) \quad \text{where} \quad \alpha_a = \sum_{k \geq 1} \frac{1}{a^k - 1}. \end{aligned}$$

With $\log a = \frac{1}{b} \log 2$ we have the following contribution $\Sigma_{n,0}$ to Σ_n :

$$-\Sigma_{n,0} = \log_a n + \frac{\gamma}{\log a} + \log_a d - \frac{1}{2} - \alpha_a. \tag{11}$$

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Near the simple poles $z_k = \frac{2k\pi i}{\log a} = \lambda_k(a)$, $k \in \mathbb{Z}$, $k \neq 0$, we have the expansions

$$\begin{aligned} [n; z] &\sim n^{\lambda_k(a)} \cdot \Gamma(-\lambda_k(a)) = e^{2k\pi i \cdot \log_a n} \cdot \Gamma(-\lambda_k(a)) \\ \left(\frac{d}{a}\right)^z &\sim e^{\lambda_k(a) \cdot \log d} \\ Q_{z-1}(a) &\sim \frac{1}{\log a} \cdot \frac{1}{z - \lambda_k(a)}, \end{aligned}$$

so that the poles z_k yield the following fluctuating contribution $\Sigma_{n,k}$ to Σ_n :

$$\Sigma_{n,k} = \frac{1}{\log a} \Gamma(-\lambda_k(a)) e^{2k\pi i \cdot \log_a(nd)}.$$

In total we have

THEOREM 2. *The expected value E_n of the counter C after n random increments with the generalized incremental procedure (**) fulfils*

$$E_n = \log_2 n + \frac{\gamma}{\log 2} + \log_2 d + 1 - \frac{1}{2b} - \frac{\alpha_a}{b} + \delta_1(\log_a n + \Delta) + O(n^{-1}),$$

where

$$\alpha_a = \sum_{k \geq 1} \frac{1}{a^k - 1}, \quad a = 2^{1/b}$$

and the periodic function

$$\delta_1(x) = -\frac{1}{\log 2} \sum_{k \neq 0} \Gamma(-\lambda_k(a)) e^{2k\pi i x},$$

where

$$\lambda_k(a) = \frac{2k\pi i}{\log a} \quad \text{and} \quad \Delta = \log_a d. \quad \square$$

Since

$$|\Gamma(iy)|^2 = \frac{\pi}{y \cdot \sinh(\pi y)},$$

we have $|\Gamma(-\lambda_k(a))|^2 = O\left(b e^{-\frac{2\pi^2 b}{\log^2}}\right)$ for $b \rightarrow \infty$, so that the fluctuating term, which is already very small for the classical case $b = 1$, becomes even smaller for b getting large.

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In order to find the appropriate value for the constant d we consider the expectation S_n of 2^C after n random increments:

$$S_n = \sum_l p_{n,l} 2^{C_l}. \quad (12)$$

Since

$$p_{n,l} = p_{n-1,l} (1 - da^{-l}) + p_{n-1,l-1} da^{1-l} \quad (13)$$

we have

$$S_n = S_{n-1} - \frac{2d}{a} + 2d, \quad n \geq 1; \quad S_0 = 2;$$

and thus

$$S_n = 2d \left(1 - \frac{1}{a}\right) n + 2. \quad (14)$$

A good choice for d is therefore

$$d = \frac{a}{2(a-1)}, \quad (15)$$

so that $S_n = n + 2$.

Nevertheless we can analyse the *variance* V_n of the content of C after n steps for arbitrary d in an analogous manner as the expectation E_n . Omitting the technical details we obtain, neglecting periodic fluctuations of mean 0,

$$V_n \sim \frac{\pi^2}{6 \log^2 2} - \frac{\beta_a}{b^2} - \frac{\alpha_a}{b^2} + \frac{1}{12b^2} - [\delta_1^2]_0, \quad (16)$$

where $[\delta_1^2]_0$ is the mean of the square of the periodic function $\delta_1(x)$ defined in Theorem 2.

In order to study the influence of b on V_n , it is helpful to make use of the following remarkable transformation of the constant in (16): We have

$$\alpha_a + \beta_a = \sum_{k \geq 1} \frac{a^k}{(a^k - 1)^2} = h_1(\log a),$$

where

$$h_1(x) = \sum_{k \geq 1} \frac{e^{kx}}{(e^{kx} - 1)^2}. \quad (17)$$

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Now it follows from Dedekind’s functional equation for the η -function that

$$h_1(x) = \frac{\pi^2}{6x^2} - \frac{1}{2x} + \frac{1}{24} - \frac{4\pi^2}{x^2} h_1\left(\frac{4\pi^2}{x}\right)$$

(compare [5]; observe the typo therein). Thus we have

$$\alpha_a + \beta_a = \frac{\pi^2}{6 \log^2 a} - \frac{1}{2 \log a} + \frac{1}{24} - \frac{4\pi^2}{\log^2 a} h_1\left(\frac{4\pi^2}{\log a}\right). \tag{18}$$

On the other hand, again using $|\Gamma(iy)|^2 = \frac{\pi}{y \cdot \sinh(\pi y)}$, we get

$$[\delta_1^2]_0 = \frac{\log a}{\log^2 2} \sum_{k \geq 1} \frac{1}{k \cdot \sinh\left(\frac{2k\pi^2}{\log a}\right)} = \frac{2 \log a}{\log^2 2} h_2\left(\frac{2\pi^2}{\log a}\right) \tag{19}$$

with

$$h_2(x) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k (e^{kx} - 1)}. \tag{20}$$

Combining (18) and (19) and regarding $\log a = \frac{\log 2}{b}$, we finally have

THEOREM 3. *The variance V_n of the content of the counter C after n steps of the generalized incremental procedure (**) fulfils*

$$V_n = \frac{1}{2b \log 2} + \frac{1}{24b^2} + \frac{4\pi^2}{\log^2 2} h_1\left(\frac{4\pi^2 b}{\log 2}\right) - \frac{2}{b \log 2} h_2\left(\frac{2\pi^2 b}{\log 2}\right) + \delta_3(\log_a n + \Delta) + O\left(\frac{\log n}{n}\right),$$

where $h_1(x)$ and $h_2(x)$ are defined in (17) and (20), and where $\delta_3(x)$ is a periodic function of mean zero, and $\Delta = \log_a d$. □

$\delta_3(x)$ is a combination of $\delta_1(x)$ and $\delta_2(x)$, whose Fourier coefficients could be computed in principle.

The reader should note that the main term is of order b^{-1} in b ; the values $h_1\left(\frac{4\pi^2 b}{\log 2}\right)$ resp. $h_2\left(\frac{2\pi^2 b}{\log 2}\right)$ tend to 0 exponentially fast in b . The result quantifies the smoother behaviour of the approximate counting procedure for increasing b . For $d = 1$ Theorem 3 should be compared with Flajolet’s result [2; eq. (50)], where only the leading term $\frac{1}{2b \log 2}$ appears.

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3. Approximate counting by coin tossings

In this section we apply the previous results to the analysis of an approximate counting algorithm where the incremental procedure (*) is simulated by *coin tossings*. We flip a fair coin and increase the counter C by 1 after having seen C consecutive heads occurring since the previous incremental step. More formally the set of sequences yielding a counter $C = l$ is described by the regular expression

$$0^*1\{0, 10\}^*11\{0, 10, 110\}^* \dots 1^{l-1}\{0, 10, \dots, 1^{l-1}0\}^*\{\varepsilon, 1, \dots, 1^{l-1}\}.$$

The corresponding generating function equals

$$F_l(z) = \frac{1 - z^l}{(1 - z)^2} z^{\binom{l}{2}} \prod_{j=1}^{l-1} \frac{1 - z}{1 - 2z + z^{j+2}}, \quad l \geq 1, \tag{21}$$

so that the probability $p_{n,l}$ that counter $C = l$ after n tossings is

$$p_{n,l} = 2^{-n} [z^n] F_l(z). \tag{22}$$

It is our aim to show that the analysis of this coin tossing algorithm may be reduced to the analysis of Section 2 by approximating the probabilities $p_{n,l}$ step by step.

For $l \geq 2$, $F_l(z)$ is a rational function having $l - 1$ first order poles $\rho_1, \dots, \rho_{l-1}$ of absolute value $\leq \frac{3}{4}$: For z traversing the circle $|z| = \frac{3}{4}$ the value of $1 - 2z + z^{j+2}$ winds around the origin exactly once, so that $1 - 2z + z^{j+2}$ has exactly one root ρ_j in $|z| \leq \frac{3}{4}$. For later purposes we note that

$$|1 - 2z + z^{j+2}| \geq \left| -\frac{1}{2} + \left(\frac{3}{4}\right)^3 \right| = \frac{5}{64} \quad \text{for } j \geq 1 \quad \text{and} \quad |z| = \frac{3}{4}.$$

Therefore

$$|F_l(z)| \leq C_1 \left(\frac{3}{4}\right)^{\binom{l}{2}} \left(\frac{7}{4}\right)^{l-1} \left(\frac{64}{5}\right)^{l-1} = O\left(\eta^{\binom{l}{2}}\right) \quad \text{for any } \eta > \frac{3}{4}. \tag{23}$$

The reader should observe that because of

$$2\rho_j = 1 + \rho_j^{j+2} \tag{24}$$

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ρ_j will be close to $1/2$ for j getting large. More precisely, using *Lagrange's inversion formula*, we have

$$\rho_j = \frac{1}{2} \left(1 + \sum_{k \geq 1} \frac{1}{k} \binom{k(j+2)}{k-1} \frac{1}{2^{k(j+2)}} \right) = \frac{1}{2} \left(1 + \frac{1}{2^{j+2}} + \dots \right). \tag{25}$$

Now it follows that

$$2^n p_{n,l} = - \sum_{j=1}^{l-1} \text{Res}(F_l(z); z = \rho_j) \rho_j^{-n-1} + \frac{1}{2\pi i} \int_{|z|=\frac{3}{4}} F_l(z) \frac{dz}{z^{n+1}},$$

where the integral is $O\left(\eta \binom{l}{2} \left(\frac{4}{3}\right)^n\right)$ for any $\eta > \frac{3}{4}$, with an absolute O -constant. From this observation it is immediate that $p_{n,l}$ may be substituted by

$$q_{n,l} = - \sum_{j=1}^{l-1} \text{Res}(F_l(z); z = \rho_j) \rho_j^{-1} (2\rho_j)^{-n} \tag{26}$$

with exponentially small errors in expectation and variance. The computation of the residues gives

$$\text{Res}(F_l(z); z = \rho_j) = \frac{(1 - \rho_j^l) \rho_j \binom{l}{2} (1 - \rho_j)^{l-1}}{(1 - \rho_j)^2 (-2 + (j+2)\rho_j^{j+1})} A_j B_{j,l},$$

where (compare (24))

$$\begin{aligned} A_j (1 - \rho_j)^{-2} &= \prod_{i=0}^{j-1} (1 - 2\rho_j + \rho_j^{i+2})^{-1} = \prod_{i=0}^{j-1} (\rho_j^{i+2} - \rho_j^{j+2})^{-1} \\ &= \rho_j^{1 - \binom{j+2}{2}} / Q_j(\rho_j^{-1}), \\ B_{j,l} &= \rho_j^{-(j+2)(l-1-j)} (-1)^{l-1-j} / Q_{l-1-j}(\rho_j^{-1}). \end{aligned}$$

Using formula (24) it can be shown by some straightforward algebra that

$$\begin{aligned} q_{n,l} &= \sum_{j=1}^{l-1} \frac{(1 - \rho_j^l) (1 - \rho_j^{j+1})^{l-1}}{(1 - (j+1)\rho_j^{j+2})} \cdot \frac{(-1)^{l-1-j} \rho_j^{\binom{l-1-j}{2}}}{Q_j(\rho_j^{-1}) Q_{l-1-j}(\rho_j^{-1})} (2\rho_j)^{-n} \\ &= \sum_{j=1}^{l-1} a_{l,j} (2\rho_j)^{-n}, \quad \text{say.} \end{aligned} \tag{27}$$

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This formula suggests the quantity

$$r_{n,l} = \sum_{j=1}^{l-1} \frac{(-1)^{l-1-j} 2^{-\binom{l-1-j}{2}}}{Q_j Q_{l-1-j}} (2\rho_j)^{-n} = \sum_{j=1}^{l-1} b_{l,j} (2\rho_j)^{-n}, \quad (28)$$

where $Q_k = Q_k(2)$, as a good approximation of $q_{n,l}$. Indeed, we have

$$\sum_{l \geq j+1} l a_{l,j} = \frac{(1 - \rho_j^{j+1})^j}{(1 - (j+1)\rho_j^{j+2}) Q_j (\rho_j^{-1})} \sum_{l \geq 0} (l+j+1) s_{l,j}$$

with

$$s_{l,j} = (1 - \rho_j^{j+1} \rho_j^l) (1 - \rho_j^{j+1})^l (-1)^l \rho_j^{\binom{l}{2}} / Q_l (\rho_j^{-1}). \quad (29)$$

Using Euler's identity (7)

$$\sum_{l \geq 0} s_{l,j} = \prod_{m \geq 0} (1 - (1 - \rho_j^{j+1}) \rho_j^m) - \rho_j^{j+1} \prod_{m \geq 0} (1 - (1 - \rho_j^{j+1}) \rho_j^{m+1}) = 0. \quad (30)$$

Differentiating formula (7) we have

$$\sum_{l \geq 0} l \frac{(-1)^l a^{-\binom{l}{2}} t^l}{Q_l(a)} = -t \left(\sum_{m \geq 0} \frac{1}{a^m - t} \right) \prod_{m \geq 0} \left(1 - \frac{t}{a^m} \right), \quad (31)$$

so that, after a short computation,

$$\sum_{l \geq 0} l s_{l,j} = - (1 - \rho_j^{j+1}) \prod_{m \geq 1} (1 - (1 - \rho_j^{j+1}) \rho_j^m). \quad (32)$$

Altogether

$$\sum_{l \geq j+1} l a_{l,j} = - \frac{(1 - \rho_j^{j+1})^{j+1}}{(1 - (j+1)\rho_j^{j+2}) Q_j (\rho_j^{-1})} \prod_{m \geq 1} (1 - (1 - \rho_j^{j+1}) \rho_j^m). \quad (33)$$

In the same way we derive

$$\sum_{l \geq j+1} l b_{l,j} = - \frac{Q_\infty}{Q_j}, \quad \text{where } Q_\infty = Q_\infty(2), \quad Q_j = Q_j(2). \quad (34)$$

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The sum is up to a factor $1 + O(j2^{-j})$ the right hand side of (33). Therefore the series

$$\sum_{j \geq 1} (2\rho_j)^{-n} \sum_{l \geq j+1} l(a_{l,j} - b_{l,j})$$

converges absolutely and may be rearranged as $\sum_{l \geq 1} l(q_{n,l} - r_{n,l})$.

It is easily seen that $\sum_{j \geq 1} (2\rho_j)^{-n} O(2^{-j})$ is $o(1)$ for $n \rightarrow \infty$. In order to get a sharper estimate we may apply Lemma 1: We have $(2\rho_j)^{-1} = 1 - \delta_j$, where $\delta_j = O(2^{-j})$ so that

$$\sum_{j \geq 1} (2\rho_j)^{-n} O(2^{-j}) = \sum_{k=0}^n \binom{n}{k} (-1)^k f(k)$$

with $f(k) = \sum_{j \geq 1} \delta_j^k O(2^{-j})$. Since $\sum_{j \geq 1} \delta_j^z O(2^{-j})$ defines an analytic function for $\Re z > -1$ we find from Lemma 1 that the whole sum is $O(n^{-1+\epsilon})$ for any $\epsilon > 0$.

Thus the expectation E_n of the content of the counter C after n tossings fulfils

$$\begin{aligned} E_n &= \sum_{l \geq 1} lq_{n,l} + O\left(\left(\frac{2}{3}\right)^n\right) = \sum_{l \geq 1} lr_{n,l} + \sum_{l \geq 1} l(q_{n,l} - r_{n,l}) + O\left(\left(\frac{2}{3}\right)^n\right) \\ &= \sum_{l \geq 1} lr_{n,l} + O(n^{-1+\epsilon}), \quad \text{any } \epsilon > 0. \end{aligned}$$

In order to evaluate $\sum_{l \geq 1} lr_{n,l}$ we approximate $(2\rho_j)^{-1}$ by $1 - 2^{-j-2}$ (compare (25)):

$$\begin{aligned} \Delta_n &= \sum_{j \geq 1} \left((2\rho_j)^{-n} - (1 - 2^{-j-2})^n \right) \frac{Q_\infty}{Q_j} \\ &= \sum_{k=1}^n \binom{n}{k} (-1)^k \sum_{j \geq 1} \left(\delta_j^k - 2^{-(j+2)k} \right) \frac{Q_\infty}{Q_j}, \end{aligned}$$

where $(2\rho_j)^{-1} = 1 - \delta_j$. From (31) we have $\delta_j^k - 2^{-(j+2)k} = O\left(j2^{-2j}k2^{-j(k-1)}\right) = O\left(kj2^{-j(k+1)}\right)$ so that $f(z) = \sum_{j \geq 1} \left(\delta_j^z - 2^{-(j+2)z} \right) \frac{Q_\infty}{Q_j}$ is again an analytic function for $\Re z > -1$. Since $f(0) = 0$, $[n; z]f(z)$ has a removable singularity at $z = 0$ and $\Delta_n = O(n^{-1+\epsilon})$, any $\epsilon > 0$.

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We finally have

$$\begin{aligned}
E_n &= \sum_{l \geq 1} l r_{n,l} + O(n^{-1+\epsilon}) = \sum_{l \geq 1} l \sum_{j=1}^{l-1} b_{l,j} (2\rho_j)^{-n} + O(n^{-1+\epsilon}) \\
&= \sum_{l \geq 1} l \sum_{j=1}^{l-1} b_{l,j} (1 - 2^{-j-2})^n + O(n^{-1+\epsilon}) \\
&= \sum_{l \geq 1} l t_{n,l} - \sum_{l \geq 1} l b_{l,0} \left(\frac{3}{4}\right)^n + O(n^{-1+\epsilon}) \\
&= \sum_{l \geq 1} l t_{n,l} + O(n^{-1+\epsilon}),
\end{aligned} \tag{35}$$

where $t_{n,l}$ coincides with $p_{n,l}$ from (6) for $a = 2$, $d = \frac{1}{2}$. Therefore E_n from this section coincides with E_n from Section 2 with an error of order $O(n^{-1+\epsilon})$. The same result may be proved for the variance V_n by an analogous reasoning.

THEOREM 4. *The expected value E_n of counter C after n coin tossings fulfils*

$$E_n = \log_2 n + \frac{\gamma}{\log 2} - \frac{1}{2} - \alpha + \delta_1(\log_2 n) + O(n^{-1+\epsilon}), \quad \text{any } \epsilon > 0,$$

with

$$\alpha = \sum_{k \geq 1} \frac{1}{2^k - 1} \approx 1.6066 \dots$$

The variance V_n fulfils for any $\epsilon > 0$

$$\begin{aligned}
V_n &= \frac{1}{2 \log 2} + \frac{1}{24} + \frac{4\pi^2}{\log^2 2} h_1\left(\frac{4\pi^2}{\log 2}\right) - \frac{2}{\log 2} h_2\left(\frac{2\pi^2}{\log 2}\right) + \delta_3(\log_2 n) \\
&\quad + O(n^{-1+\epsilon}) \approx 0.7630 \dots
\end{aligned}$$

with $\delta_1, \delta_3, h_1, h_2$ from Theorems 2 and 3.

Theorem 4 shows that our coin tossing algorithm is a very good simulation of the general incremental procedure mentioned in the first two sections with parameters $b = 1$, $d = 1/2$.

We finally compare these results with the following alternative variant of a coin tossing algorithm: The Counter C is incremented by 1 if after another flipping of the coin the sequence ends up with a run of C ones. With other words, C maintains 1 plus the length of the longest run of ones. Results on

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this problem may be found e.g. in [9], [10]; the analysis is closely related to Knuth's analysis of the average time for carry propagation in binary addition [6]. Intuitively it is clear that the latter proposal leads to a less smooth behaviour than our algorithm from above.

The regular expression corresponding to the set of sequences yielding a counter $C \leq l$ is

$$\{0, 10, \dots, 1^{l-1}0\}^* \{\varepsilon, 1, \dots, 1^{l-1}\}$$

with corresponding generating function

$$F_l(z) = \frac{1 - z^l}{1 - 2z + z^{l+1}}.$$

The dominating singularity is ρ_{l-1} , where ρ_j fulfils (24). In Knuth's analysis [6] the corresponding generating function is

$$G_l(z) = \frac{1}{1 - 2z + \frac{1}{2}z^l}.$$

The dominating singularity τ_{l-1} of $G_l(z)$ fulfils $\tau_{l-1} = \rho_{l-1} + O(l2^{-2l})$, so that Knuth's asymptotic result on the expectation covers the coin tossing problem, too. For comparison we cite the results from [6] and [10]:

Expectation E_n and variance V_n of the counter C in the modified algorithm are asymptotic to (neglecting the small periodic fluctuations of mean zero)

$$\begin{aligned} E_n &\sim \log_2 n + \frac{\gamma}{\log 2} - \frac{1}{2}, \\ V_n &\sim \frac{\pi^2}{6 \log^2 2} + \frac{1}{12} - [\delta_1^2]_0 \approx 3.5070 \dots \end{aligned} \tag{36}$$

with δ_1 from Theorem 2. Comparing with (16) we find that the variance in our algorithm is smaller by $\alpha + \beta = \sum_{k \geq 1} \frac{2^k}{(2^k - 1)^2} \approx 2.7440 \dots$

REFERENCES

- [1] ANDREWS, G. E.: *The Theory of Partitions*, Addison Wesley, Reading-Mass, 1976.
- [2] FLAJOLET, P.: *Approximate counting: A detailed analysis*, BIT **25** (1985), 113–134.
- [3] FLAJOLET, P.—SEGEWICK, R.: *Digital search trees revisited*, SIAM J. Comput. **15** (1986), 748–767.
- [4] KIRSCHENHOFER, P.—PRODINGER, H.: *Approximate counting: An alternative approach*, RAIRO Inform. Théor. Appl. **25** (1991), 43–48.

A COIN TOSSING ALGORITHM FOR COUNTING LARGE NUMBERS OF EVENTS

- [5] KIRSCHENHOFER, P.- PRODINGER, H.—SCHOISSENGEIER, J.: *Zur Auswertung gewisser numerischer Reihen mit Hilfe modularer Funktionen*. In: *Zahlentheoretische Analysis II. Lecture Notes in Math. 1262* (E. Hlawka, ed.), Springer, Berlin, 1987, pp. 108-110.
- [6] KNUTH, D. E.: *The average time for carry propagation*, *Indag. Math.* **40** (1978), 238-242.
- [7] MORRIS, R.: *Counting large numbers of Events in small registers*, *Comm. ACM* **21** (1978), 840-842.
- [8] NÖRLUND, N. E.: *Vorlesungen über Differenzenrechnung*, Chelsea, New York, 1954.
- [9] PRODINGER, H.: *Über längste 0-1-Folgen*. In: *Zahlentheoretische Analysis II. Lecture Notes in Math. 1262* (E. Hlawka, ed.), Springer, Berlin, 1987, pp. 124-133.
- [10] SCHMID, U.: *Abzählprobleme der theoretischen Informatik*. Diplomarbeit, TU, Wien, 1985.

Received December 13, 1991

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APPROXIMATE COUNTING VIA EULER TRANSFORM

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(Communicated by Stanislav Jakubec)

ABSTRACT. In this short note we would like to emphasize how some elements of “ q -analysis” (basic hypergeometric functions) allow some shortcuts in the enumerative part.

Approximate Counting might be described as follows. There is a counter C which is initially set to 1 and incremented randomly depending on the counter value. If this value is k , the probability that the counter will be increased by 1, is 2^{-k} . Otherwise, the counter value will stay the same. The idea is that after n random increments the counter should have a value close to $\log_2 n$. It is convenient to replace $\frac{1}{2}$ by q .

There is another useful way to imagine this procedure: There are the states $1, 2, \dots$, and in one step one may either advance from state i to state $i+1$ with probability q^i , or stay in the state i with probability $1 - q^i$. The interesting parameter is the state that one reaches after n random steps, starting in state 1. This is clearly the value of the counter C .

The original analysis was performed by Flajolet [2] and consists of an enumerative (or algebraic) part and an asymptotic (or analytic) part. The latter was done by the Mellin transform. In [5], some additional manipulations allowed to rewrite the sought quantities in such a way that an alternative asymptotic technique (Rice’s method) could be used. See also the related papers [6], [7].

In this short note we would like to emphasize how some elements of “ q -analysis” (basic hypergeometric functions) allow some shortcuts in the enumerative part. Let $H_l(x)$ be the generating function, where the coefficient of x^n is the probability that n random steps have led to state l . We find a rather explicit form for the bivariate generating function $\sum_{l \geq 1} H_l(x)y^l$, which is interesting in

AMS Subject Classification (1991): Primary 11B68.

Key words: Generating function, Euler transform.

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itself and might lead to additional insight. We only use it here to find representations for the expectation and the second factorial moment as alternating sums, with binomial coefficients and some simple quantities, which is essential for the use of Rice's method. One could avoid to use such explicit formulae and, using the (new) generating functions, derive the asymptotics directly by the ingenious method in [4]. However, here, we only deal with the enumerative part.

We need a few concepts from q -analysis which are taken from [1].

q-Pochhammer symbol:

$$(a)_n := (1-a)(1-aq)\dots(1-aq^{n-1}), \quad (a)_0 = 1, \quad (a)_\infty = \lim_{n \rightarrow \infty} (a)_n.$$

Cauchy's formula:

$$\sum_{n \geq 0} \frac{(a)_n t^n}{(q)_n} = \frac{(at)_\infty}{(t)_\infty}.$$

Heine's transformation:

$$\sum_{n \geq 0} \frac{(a)_n (b)_n t^n}{(q)_n (c)_n} = \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{n \geq 0} \frac{(c/b)_n (t)_n b^n}{(q)_n (at)_n}.$$

Euler's transformation: If

$$f(x) = \sum_{n \geq 0} a_n x^n,$$

then

$$\frac{1}{1-x} f\left(\frac{x}{x-1}\right) = \sum_{n \geq 0} \left(\sum_{k=0}^n \binom{n}{k} (-1)^k a_k \right) x^n.$$

This is very easily computed directly and was used with great success in [3] and [4].

The generating function $H_l(x)$ was computed in [2]. Using a decomposition of a path from 1 to l into stages, it is not hard to see that

$$H_l(x) = \frac{x^{l-1} q^{\binom{l}{2}}}{\prod_{i=1}^l (1-x(1-q^i))} = \frac{\frac{1}{x} \left(\frac{x}{1-x}\right)^l q^{\binom{l}{2}}}{\left(\frac{xq}{x-1}\right)_l}.$$

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The coefficient of x^n in $\sum_{l \geq 1} H_l(x)$ must be 1 for all n , since each path of length n must simply lead somewhere. Let us see how we find the formula

$$\sum_{l \geq 1} H_l(x) = \frac{1}{1-x}$$

by some properties of “ q -analysis”. It is equivalent to showing

$$\sum_{l \geq 1} \frac{(-z)^l q^{\binom{l}{2}}}{(qz)_l} = -z,$$

with $z = \frac{x}{x-1}$. Let us show that

$$\sum_{l \geq 0} \frac{(-z)^l q^{\binom{l}{2}}}{(qz)_l} = 1 - z.$$

First, write

$$(-1)^l q^{\binom{l}{2}} = (0-1)(0-q)\dots(0-q^{l-1}) = \lim_{\varepsilon \rightarrow 0} \varepsilon^l (1/\varepsilon)_l.$$

Then we can use the transformation of Heine and compute

$$\sum_{l \geq 0} \frac{(1/\varepsilon)_l (\varepsilon z)^l}{(qz)_l} \cdot \frac{(q)_l}{(q)_l} = \frac{(q)_\infty (z)_\infty}{(qz)_\infty (\varepsilon z)_\infty} \sum_{n \geq 0} \frac{(z)_n (\varepsilon z)_n q^n}{(q)_n (z)_n}.$$

Now we can perform the limit $\varepsilon \rightarrow \infty$ on the right hand side without problems and obtain

$$\frac{(q)_\infty (z)_\infty}{(qz)_\infty} \sum_{n \geq 0} \frac{q^n}{(q)_n}.$$

We use the trivial fact $(z)_\infty = (1-z)(qz)_\infty$ for the first factor and Cauchy’s identity (the special case which is attributed to Euler), which evaluates the sum to $1/(q)_\infty$, which gives us the desired $1 - z$.

Now let us attack the generating function of the expectations,

$$\sum_{l \geq 1} l H_l(x).$$

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For that, we consider the bivariate generating function

$$H(x, y) = \sum_{l \geq 0} H_l(x) y^l,$$

differentiate it w.r.t. y and evaluate at $y = 1$. We obtain

$$H(x, y) = \frac{1}{x} \sum_{l \geq 0} \frac{(-yz)^l q^{\binom{l}{2}}}{(qz)_l}, \quad z = \frac{x}{x-1}.$$

Using Heine’s transformation as before, we get

$$\sum_{l \geq 0} \frac{(-yz)^l q^{\binom{l}{2}}}{(qz)_l} = \frac{(q)_\infty (yz)_\infty}{(qz)_\infty} \sum_{n \geq 0} \frac{(z)_n q^n}{(q)_n (yz)_n}.$$

For $y = 1$ we could evaluate the sum immediately. Now we transform the sum again by “Heine”, with $a = 0$, $b = z$, $c = yz$ and $t = q$ and find (for the sum only)

$$\frac{(z)_\infty}{(yz)_\infty (q)_\infty} \sum_{n \geq 0} \frac{(y)_n (q)_n z^n}{(q)_n},$$

which gives

$$H(x, y) = \frac{1}{x} (1-z) \sum_{n \geq 0} (y)_n z^n, \quad z = \frac{x}{x-1}.$$

Now observe that

$$\frac{d}{dy} (y)_n \Big|_{y=1} = -(q)_{n-1}, \quad n \geq 1,$$

so that the generating function of the expectations $E(x)$ is

$$E(x) = -\frac{1}{x} (1-z) \sum_{n \geq 0} (q)_n z^{n+1} = (1-z)^2 \sum_{n \geq 0} (q)_n z^n.$$

According to Euler’s transformation we have to extract the coefficient of z^n in

$$\frac{1}{1-z} \cdot (1-z)^2 \sum_{n \geq 0} (q)_n z^n = (1-z) \sum_{n \geq 0} (q)_n z^n,$$

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which is

$$(q)_n - (q)_{n-1} = -q^n (q)_{n-1}$$

for $n \geq 1$ and 1 for $n = 0$. Hence the expected value E_n is

$$E_n = 1 - \sum_{k=1}^n \binom{n}{k} (-1)^k q^k (q)_{k-1},$$

a formula already reported in [5].

To obtain the generating function $E_2(x)$ for the second factorial moments we have to differentiate twice w.r.t. y and evaluate at $y = 1$. Observe that for $n \geq 2$

$$\frac{d^2}{dy^2} (y)_n \Big|_{y=1} = 2(q)_{n-1} \sum_{k=1}^{n-1} \frac{q^k}{1 - q^k}.$$

Let us abbreviate

$$T_n = \sum_{k=1}^n \frac{q^k}{1 - q^k}.$$

Then

$$E_2(x) = \frac{1}{x} (1 - z) \cdot 2 \sum_{n \geq 1} (q)_n T_n z^{n+1}.$$

As before, we have to extract the coefficient of z^n in

$$\frac{1}{1 - z} E_2(x) = -2(1 - z) \sum_{n \geq 1} (q)_n T_n z^n,$$

which is

$$-2((q)_n T_n - (q)_{n-1} T_{n-1}) = 2q^n (q)_{n-1} (T_{n-1} - 1).$$

Hence the second factorial moment $E_n^{(2)}$ is given by

$$E_n^{(2)} = \sum_{k=1}^n \binom{n}{k} (-1)^k \cdot 2q^k (q)_{k-1} (T_{k-1} - 1).$$

This is equivalent to the formula given in [5].

As stated before, asymptotics follow, as demonstrated in [5], by Rice’s method. For the sake of completeness we cite the expectation E_n (‘the average value of the counter C after n random increments’) and refer for the variance to the literature.

$$E_n = \log_2 n + \frac{\gamma}{\log 2} + \frac{1}{2} - \alpha + \delta(\log_2 n) + \mathcal{O}\left(\frac{1}{n}\right),$$

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where $\gamma = 0.577\dots$ is Euler's constant,

$$\alpha = \sum_{k \geq 1} \frac{1}{2^k - 1} = 1.606695\dots$$

and $\delta(x)$ is a periodic function of period 1, mean 0, small amplitude and known Fourier expansion.

We would also like to mention that, since we have the explicit forms of the generating functions of the moments, an approach like the one in [4] would give the asymptotics even a little bit more directly. We do not work it out, because the computations are somehow similar to those which occur with Rice's method.

Acknowledgement

I thank Christian Krattenthaler for his help with q -hypergeometric functions.

REFERENCES

- [1] ANDREWS, G. E.: *The Theory of Partitions*, Addison Wesley, 1976.
- [2] FLAJOLET, P.: *Approximate counting: A detailed analysis*, BIT **25** (1985), 113–134.
- [3] FLAJOLET, P.—LABELLE, G.—LAFOREST, L.—SALVY, B.: *Hypergeometrics and the cost structure of quadrees*, Random Structures and Algorithms (1995) (To appear).
- [4] FLAJOLET, P.—RICHMOND, B.: *Generalized digital trees and their difference-differential equations*, Random Structures and Algorithms **5** (1992), 305–320.
- [5] KIRSCHENHOFER, P.—PRODINGER, H.: *Approximate counting: An alternative approach*, RAIRO Informatique Théorique et Applications **25** (1991), 43–48.
- [6] KIRSCHENHOFER, P.—PRODINGER, H.: *A coin tossing algorithm for counting large numbers of events*, Math. Slovaca **42** (1992), 531–545.
- [7] PRODINGER, H.: *Hypothetic analyses: Approximate counting in the style of Knuth, path length in the style of Flajolet*, Theoretical Computer Science **100** (1992), 243–251.

Received November 25, 1994

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Theoretical Computer Science 391 (2008) 109–125

Theoretical
Computer Science

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Generalized approximate counting revisited

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Abstract

A large class of q -distributions is defined on the stochastic model of Bernoulli trials in which the probability of success (= advancing to the next level) depends geometrically on the number of trials and the level already reached. If the dependency is only on the level already reached, this is an algorithm called *approximate counting*.

Two random variables, X_n (level reached after n trials) and Y_k (number of trials to reach level k) are of interest. We rederive known results and obtain new ones in a consistent way, based on generating functions.

We also discuss asymptotics. The classical instance of *approximate counting* is more interesting from a mathematical point of view. On the other hand, if the number of trials also decreases the probability of success (advancing to the next level), then the limits are constants which are straightforward to compute.

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Keywords: Approximate counting; q -analysis; q -distribution; Markov chain; Transition probability

1. Introduction

The Markov chain

$$\mathbb{P}(X_{n+1} = k + 1 \mid X_n = k) = q^{an+bk+c}, \quad \mathbb{P}(X_{n+1} = k \mid X_n = k) = 1 - q^{an+bk+c} \quad (1)$$

with the initial condition $\mathbb{P}(X_0 = 0) = 1$ was recently revisited by Charalambides [3], also based on some earlier work [2]. We will adopt the notation $p_C(n, k) = \mathbb{P}(X_n = k)$. So, this process starts at time 0 in state 0, and the likelihood to advance to the next state decreases both with time and level already reached.

Sometimes it is more convenient to start in state 1. This amounts to relabel the states from $0, 1, \dots$ to $1, 2, \dots$. Then parameters (a, b, c) must be changed to $(a, b, c - b)$, to have an equivalent model.

Crippa, Simon and Trunz [5] considered a special case, defined by

$$p_{CST}(n, k) = \lambda_{n,k-1} p_{CST}(n-1, k-1) + (1 - \lambda_{n,k}) p_{CST}(n-1, k) \quad (2)$$

where $\lambda_{n,k} = q^{a(n-1)+bk}$ and either $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$. The starting condition is here $p_{CST}(0, 1) = 1$.

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This recursion, with $a = 0, b = 1, c = -1$, i.e., $\lambda_{n,k} = q^{k-1}$, is known as *approximate counting*. This algorithm starts with a counter C initialized to 1. At each increment, we add 1 to C with probability 2^{-C} . After n increments, the counter C contains a good approximation of $\lfloor \log_2 n \rfloor$, with high accuracy and concentration. This was originally analysed by Flajolet [6].

As a motivation, Crippa, Simon, and Trunz mention, apart from approximate counting, an algorithm by Simon to compute the transitive closure on acyclic digraphs. In [4], applications in biology, particle physics and queue theory are mentioned.

We will reconsider in this paper the recursion (reformulation of the Markov chain (1))

$$p(n, k) = q^{a(n-1)+b(k-1)+c} p(n-1, k-1) + (1 - q^{a(n-1)+bk+c}) p(n-1, k)$$

with one of the initial conditions $p(0, 0) = 1$ or $p(0, 1) = 1$, depending on the context.

The aim in this paper is to derive old and new results with a general approach that is based on generating functions. In this way, we will recover as particular cases many results from the literature.

We also discuss asymptotics. This is more interesting for $a = 0$, which is essentially the approximate counting case, with an expectation of order $\log n$. There are several ways to derive these results: Mellin transform (Flajolet [6]), Rice’s method (Kirschenhofer and Prodinger [7]), analysis of extreme-value distributions (Louchard and Prodinger [8]), just to name a few. If $a > 0$, then each failed attempt to advance results in an additional punishment, and the expected level that will be reached is just a constant, which is given in a straightforward way by an infinite series involving the limits of the (explicit forms of the) probabilities. The paper is organized as follows. In Section 2, we collect a few results from q -analysis. Flajolet’s explicit formula for approximate counting is analysed in Section 3. Section 4 considers general formulæ for X_n distribution, with $a, b, c \geq 0$. The moments of X_n are analysed in Section 5. Section 6 is devoted to the asymptotics of the moments of X_n for $n \rightarrow \infty$. Analysis of Y_k (time to reach k) is given in Section 7. Section 8 concludes the paper. Old results will be given as theorems and new results as propositions.

2. Preliminaries

Before we start, we need to collect a few results from q -analysis. They can be found in many textbooks, e.g. [1]. For our probabilistic interpretation, we always assume $0 < q < 1$. We will use the notation

$$(x; q)_n := (1 - x)(1 - xq) \dots (1 - xq^{n-1});$$

for $(q; q)_n$ we sometimes write $(q)_n$ if no misunderstanding is possible. Furthermore we need the Gaussian coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q)_n}{(q)_k (q)_{n-k}};$$

they are polynomials in q and approach the binomial coefficients $\binom{n}{k}$ as $q \rightarrow 1$.

We also need $[n]_q := \frac{1-q^n}{1-q}$ and $[n]_q! := [1]_q [2]_q \dots [n]_q = (q)_n / (1 - q)^n$.

We recall Euler’s two partition identities:

$$\prod_{i=0}^{\infty} (1 - tq^i) = \sum_{n=0}^{\infty} \frac{(-1)^n t^n q^{\binom{n}{2}}}{(q)_n}, \tag{3}$$

$$\prod_{i=1}^{\infty} (1 - tq^{i-1})^{-1} = \sum_{n=0}^{\infty} \frac{1}{(q)_n} t^n \tag{4}$$

and the q -binomial formulæ

$$\prod_{i=1}^n (u + tq^{i-1}) = \sum_{k=0}^n q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q t^k u^{n-k}, \tag{5}$$

$$\frac{1}{(t; q)_n} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q t^k. \tag{6}$$

We use the (now standard) notation $[z^n]f(z)$ to extract the coefficient of z^n in the series expansion of $f(z)$, as well as Iverson’s notation $\llbracket P \rrbracket$, which is one if P is true, and zero otherwise.

3. Flajolet's explicit formula

Let us first rederive this formula [6, (46)] in the simplest way: Flajolet proves

Theorem 1.

$$p(n, k) = \sum_{t=0}^{k-1} \frac{(-1)^t q^{\binom{t}{2}}}{(q)_t (q)_{k-1-t}} (1 - q^{k-t})^n.$$

Proof. We have

$$p(n, k) = q^{k-1} p(n-1, k-1) + (1 - q^k) p(n-1, k), \quad p(0, 1) = 1.$$

We will use bivariate *generating functions*.

If we set

$$F(z, u) := \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} z^n u^k p(n, k),$$

we derive

$$F(z, u) - u = zuF(z, qu) + zF(z, u) - zF(z, qu),$$

or

$$F(z, u) = \frac{u}{1-z} + \frac{z(u-1)}{1-z} F(z, qu).$$

Iterating, this gives

$$\begin{aligned} F(z, u) &= \frac{u}{1-z} + \frac{z(u-1)}{1-z} \frac{uq}{1-z} + \frac{z(u-1)}{1-z} \frac{z(qu-1)}{(1-z)^2} uq^2 \\ &\quad + \frac{z(u-1)}{1-z} \frac{z(qu-1)}{1-z} \frac{z(q^2u-1)}{(1-z)^2} uq^3 + \dots \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j z^j (u; q)_j u q^j}{(1-z)^{j+1}}. \end{aligned} \quad (7)$$

This expression is already in [5, eq. (17)] but only for the moments. It was independently derived in [9], using a transformation formula due to Heine.

Now we have several ways of computing $[z^n u^k] F(z, u)$.

We write

$$(u; q)_j = \frac{(u; q)_{\infty}}{(uq^j; q)_{\infty}}, \quad (8)$$

and with Euler's partition identity, we have

$$\begin{aligned} p(n, k) &= \sum_{t=0}^{k-1} \frac{(-1)^t q^{\binom{t}{2}}}{(q)_t (q)_{k-1-t}} \sum_{j=0}^n (-1)^j q^{(k-1-t)j} q^j [z^{n-j}] (1-z)^{-(j+1)} \\ &= \sum_{t=0}^{k-1} \frac{(-1)^t q^{\binom{t}{2}}}{(q)_t (q)_{k-1-t}} (1 - q^{k-t})^n, \end{aligned} \quad (9)$$

which is exactly Flajolet's formula. ■

There are (at least) 3 other representations, given below by (10), Proposition 2, (11).

Second representation

Letting $a = 0$, $c = 0$, $b = 1$ in the formula [3, (3.2)], we obtain a second expression,

$$p_C(n, k) = \frac{q^{\binom{k}{2}}}{(q)_k} \sum_{j=k}^n (-1)^{j-k} \frac{(q)_j}{(q)_{j-k}} \binom{n}{j}, \quad p_C(0, 0) = 1. \tag{10}$$

This must be equivalent to Flajolet’s formula (with n in (9) replaced by $n - 1$, as $p_C(0, 1) = 1$). We will give an independent proof of this fact.

$$p(n - 1, k) = \sum_{j=0}^{k-1} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j (q)_{k-1-j}} (1 - q^{j-k})^{n-1} = \sum_{j=0}^k \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j (q)_{k-j}} (1 - q^{k-j})^n.$$

Let us consider the generating function

$$\begin{aligned} S &= \sum_{k \geq 0} x^k p(n - 1, k) \\ &= \sum_{k \geq 0} x^k \sum_{j=0}^k \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j (q)_{k-j}} (1 - q^{k-j})^n \\ &= \sum_{l=0}^n \binom{n}{l} (-1)^l \sum_{k \geq 0} x^k \sum_{j=0}^k \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j (q)_{k-j}} q^{(k-j)l} \\ &= \sum_{l=0}^n \binom{n}{l} (-1)^l \sum_{j \geq 0} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j} \sum_{k \geq j} x^k \frac{1}{(q)_{k-j}} q^{(k-j)l} \\ &= \sum_{l=0}^n \binom{n}{l} (-1)^l \sum_{j \geq 0} \frac{(-1)^j x^j q^{\binom{j}{2}}}{(q)_j} \sum_{k \geq 0} (xq^l)^k \frac{1}{(q)_k} \\ &= \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{(xq^l; q)_\infty} \sum_{j \geq 0} \frac{(-1)^j x^j q^{\binom{j}{2}}}{(q)_j} \\ &= \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{(xq^l; q)_\infty} (x; q)_\infty \\ &= \sum_{l=0}^n \binom{n}{l} (-1)^l (x; q)_l. \end{aligned}$$

On the other hand, let us start from the formula [3, eq. (3.2)]

$$p_C(n, k) = q^{\binom{k}{2}} \sum_{j=k}^n (-1)^{j-k} \begin{bmatrix} j \\ k \end{bmatrix}_q \binom{n}{j}$$

and consider the generating function

$$\begin{aligned} T &= \sum_{k \geq 0} x^k p_C(n, k) = \sum_{k \geq 0} x^k q^{\binom{k}{2}} \sum_{j=k}^n (-1)^{j-k} \begin{bmatrix} j \\ k \end{bmatrix}_q \binom{n}{j} \\ &= \sum_{j=0}^n \binom{n}{j} (-1)^j \sum_{k=0}^j x^k q^{\binom{k}{2}} (-1)^k \begin{bmatrix} j \\ k \end{bmatrix}_q \\ &= \sum_{j=0}^n \binom{n}{j} (-1)^j \prod_{l=0}^{j-1} (1 - q^l x) \\ &= \sum_{j=0}^n \binom{n}{j} (-1)^j (x; q)_j. \end{aligned}$$

So $S = T$, which ends the proof.

Third representation

A third expression for Flajolet’s formula consists in using a q -binomial in (7) to extract $[u^{k-1}]$.

First,

$$[u^k]F(z, u) = \sum_{j \geq 0} \frac{(-1)^j z^j q^j}{(1-z)^{j+1}} [u^{k-1}](u; q)_j = \sum_{j \geq 0} \frac{(-1)^j z^j q^j}{(1-z)^{j+1}} q^{\binom{k-1}{2}} \begin{bmatrix} j \\ k-1 \end{bmatrix}_q (-1)^{k-1},$$

and consequently:

Proposition 2.

$$\begin{aligned} p(n, k) &= [z^n] \sum_{j \geq 0} \frac{(-1)^j z^j q^j}{(1-z)^{j+1}} q^{\binom{k-1}{2}} \begin{bmatrix} j \\ k-1 \end{bmatrix}_q (-1)^{k-1} \\ &= q^{\binom{k-1}{2}} (-1)^{k-1} \sum_{j=k-1}^n (-1)^j \begin{bmatrix} j \\ k-1 \end{bmatrix}_q \binom{n}{j} q^j. \end{aligned}$$

Fourth representation

A fourth expression involving q -Stirling numbers is proved in [5, (14)] by induction. This can be directly done as follows:

First, we compute

$$\begin{aligned} \sum_n p(n, k) z^n &= q^{\binom{k-1}{2}} (-1)^{k-1} \sum_{j=k-1}^{\infty} (-1)^j z^j \begin{bmatrix} j \\ k-1 \end{bmatrix}_q q^j \frac{1}{(1-z)^{j+1}} \\ &= q^{\binom{k-1}{2}} \frac{q^{k-1} z^{k-1}}{(1-z)^k} \sum_{j=0}^{\infty} \begin{bmatrix} j+k-1 \\ k-1 \end{bmatrix}_q \left(\frac{-qz}{1-z} \right)^j \\ &= q^{\binom{k}{2}} \frac{z^{k-1}}{(1-z)^k} \frac{1}{\left(\frac{-qz}{1-z}; q \right)_k} \\ &= \frac{q^{\binom{k}{2}} z^{k-1}}{(1-z(1-q)) \dots (1-z(1-q^k))}. \end{aligned}$$

This formula was derived by Flajolet, using a direct combinatorial reasoning.

Now we want to link this to q -Stirling numbers of the second kind (subset Stirling numbers), defined by the recursion

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_q = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}_q + [k]_q \begin{Bmatrix} n-1 \\ k \end{Bmatrix}_q.$$

Let

$$b_k(z) := \sum_n \begin{Bmatrix} n \\ k \end{Bmatrix}_q z^n,$$

then

$$b_k(z) = z b_{k-1}(z) + [k]_q z b_k(z) = \frac{z b_{k-1}(z)}{1 - [k]_q z} = \frac{z^k}{(1 - [1]_q z) \dots (1 - [k]_q z)}.$$

Consequently

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_q = [z^n] \frac{z^k}{(1 - [1]_q z) \dots (1 - [k]_q z)} = (1-q)^{-n+k} [z^n] \frac{z^k}{(1-z(1-q)) \dots (1-z(1-q^k))}.$$

Comparing this with Flajolet’s generating function

$$\sum_n p(n, k) z^n = \frac{q^{\binom{k}{2}} z^{k-1}}{(1-z(1-q)) \dots (1-z(1-q^k))}$$

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we find that

$$p(n, k) = q^{\binom{k}{2}} (1 - q)^{n+1-k} \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\}_q. \tag{11}$$

The moments of (9) will be discussed in Section 5.

4. Analysis of X_n . General formulæ, with $a, b, c \geq 0$

4.1. General case

Assume as always $0 < q < 1$, which implies $0 \leq q^{an+bk+c} \leq 1$. Again, we will rederive the formula [3, (3.2)] in the simplest way

Theorem 3.

$$p_C(n, k) = p(n, k + 1) = q^{b\binom{k}{2}} \sum_{j=k}^n (-1)^{j-k} q^{a\binom{j}{2}+cj} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} \begin{bmatrix} j \\ k \end{bmatrix}_{q^b}. \tag{12}$$

We will also obtain another expression

Proposition 4.

$$p(n, k) = \sum_{t=0}^{k-1} \frac{(-1)^t q^{b\binom{k-t}{2}}}{(q^b; q^b)_t (q^b; q^b)_{k-1-t}} (q^{b(k-t)+c}; q^a)_n. \tag{13}$$

Proof of (12). We have

$$p(n, k) = q^{a(n-1)+b(k-1)+c} p(n - 1, k - 1) + (1 - q^{a(n-1)+bk+c}) p(n - 1, k), \tag{14}$$

and if we set

$$F(z, u) := \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} z^n u^k p(n, k),$$

we derive

$$\begin{aligned} F(z, u) - F(z, 0) &= \sum_{j=1}^{\infty} p(0, j) u^j \\ &= q^c z u F(q^a z, q^b u) + z[F(z, u) - F(z, 0)] - z q^c [F(q^a z, q^b u) - F(q^a z, 0)], \end{aligned}$$

or

$$F(z, u) = \frac{G(z, u)}{1 - z} + \frac{z(u - 1)q^c}{1 - z} F(q^a z, q^b u),$$

with

$$G(z, u) := (1 - z)F(z, 0) + z q^c F(q^a z, 0) + \sum_{j=1}^{\infty} p(0, j) u^j.$$

Iterating, this gives

$$\begin{aligned} F(z, u) &= \frac{G(z, u)}{1 - z} + \frac{z(u - 1)q^c}{1 - z} \frac{G(q^a z, q^b u)}{1 - q^a z} + \frac{z(u - 1)q^c}{1 - z} \frac{q^a z (q^b u - 1) q^c}{1 - q^a z} \frac{G(q^{2a} z, q^{2b} u)}{1 - q^{2a} z} \\ &\quad + \frac{z(u - 1)q^c}{1 - z} \frac{q^a z (q^b u - 1) q^c}{1 - q^a z} \frac{q^{2a} z (q^{2b} u - 1) q^c}{1 - q^{2a} z} \frac{G(q^{3a} z, q^{3b} u)}{1 - q^{3a} z} + \dots \\ &= \sum_{j=0}^{\infty} \frac{z^j q^{cj} (-1)^j (u; q^b)_j q^{a\binom{j}{2}}}{(z; q^a)_{j+1}} G(z q^{ja}, u q^{jb}). \end{aligned} \tag{15}$$

It is convenient to start with 1 at 0, $p(0, 1) = 1$, so $G(z, u) = u$ and $F(z, 0) = 0$. Then

$$F(z, u) = \sum_{j=0}^{\infty} \frac{z^j q^{cj} (-1)^j (u; q^b)_j q^{a \binom{j}{2}}}{(z; q^a)_{j+1}} u q^{jb}.$$

Therefore,

$$\begin{aligned} [u^k]F(z, u) &= \sum_{j=0}^{\infty} \frac{z^j q^{cj} (-1)^j q^{jb} q^{a \binom{j}{2}}}{(z; q^a)_{j+1}} [u^{k-1}](u; q^b)_j \\ &= \sum_{j=0}^{\infty} \frac{z^j q^{cj} (-1)^j q^{jb} q^{a \binom{j}{2}}}{(z; q^a)_{j+1}} q^{b \binom{k-1}{2}} (-1)^{k-1} \begin{bmatrix} j \\ k-1 \end{bmatrix}_{q^b} \end{aligned}$$

and so

$$\begin{aligned} p(n, k) &= [z^n u^k]F(z, u) \\ &= q^{b \binom{k-1}{2}} (-1)^{k-1} \sum_{j=0}^{\infty} q^{cj} (-1)^j q^{jb} q^{a \binom{j}{2}} \begin{bmatrix} j \\ k-1 \end{bmatrix}_{q^b} [z^{n-j}] \frac{1}{(z; q^a)_{j+1}} \\ &= q^{b \binom{k-1}{2}} (-1)^{k-1} \sum_{j=k-1}^n (-1)^j \begin{bmatrix} j \\ k-1 \end{bmatrix}_{q^b} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} q^{a \binom{j}{2}} q^{cj} q^{bj}. \end{aligned} \tag{16}$$

The quantity $p_C(n, k-1)$ from [3] corresponds to $p(n, k)$ as given by (14), with c replaced by $c-b$. Consequently

$$p_C(n, k) = p(n, k+1) = q^{b \binom{k}{2}} \sum_{j=k}^n (-1)^{j-k} q^{a \binom{j}{2} + cj} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} \begin{bmatrix} j \\ k \end{bmatrix}_{q^b}.$$

This proves the formula (12) in a simpler way. ■

We can derive a simple new expression from (8) and (15): Here is the proof of (13).

Proof.

$$\begin{aligned} p(n, k) &= \sum_{t=0}^{k-1} \frac{(-1)^t q^{b \binom{t}{2}}}{(q^b; q^b)_t (q^b; q^b)_{k-1-t}} \sum_{j=0}^n (-1)^j q^{b(k-1-t)j} q^{cj} q^{bj} q^{a \binom{j}{2}} [z^{n-j}] \frac{1}{(z; q^a)_{j+1}} \\ &= \sum_{t=0}^{k-1} \frac{(-1)^t q^{b \binom{t}{2}}}{(q^b; q^b)_t (q^b; q^b)_{k-1-t}} \sum_{j=0}^n (-1)^j q^{b(k-t)j} q^{cj} q^{a \binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} \\ &= \sum_{t=0}^{k-1} \frac{(-1)^t q^{b \binom{t}{2}}}{(q^b; q^b)_t (q^b; q^b)_{k-1-t}} \sum_{j=0}^n (-1)^j q^{b(k-t)j} q^{cj} [t^j] \prod_{i=1}^n (1 + tq^{a(i-1)}) \\ &= \sum_{t=0}^{k-1} \frac{(-1)^t q^{b \binom{t}{2}}}{(q^b; q^b)_t (q^b; q^b)_{k-1-t}} \prod_{i=1}^n (1 - q^{b(k-t)} q^c q^{a(i-1)}) \\ &= \sum_{t=0}^{k-1} \frac{(-1)^t q^{b \binom{t}{2}}}{(q^b; q^b)_t (q^b; q^b)_{k-1-t}} (q^{b(k-t)+c}; q^a)_n. \quad \blacksquare \end{aligned}$$

Remark. We also obtain $p(n, k)$ from [4, (24)], with the changes $\alpha = b, \beta = a, \gamma = c - a$. Crippa and Simon start with 1 at 1, so we must change their n into $n - 1$ and our c into $c + a$.

4.2. Case $a = 1, b = 0, c = 0$

For that instance, Crippa et al. also establish (by induction) a connection to q -Stirling numbers:

Theorem 5.

$$p(n, k) = q^{\binom{n}{2}} \left(\frac{1-q}{q}\right)^{n-1+k} s(n, k, 1/q).$$

Proof. We rederive how this can be done. However, here, we adopt the initial condition $p(0, 0) = 1$.

Here is the recursion again for the special case

$$p(n, k) = q^{n-1} p(n-1, k-1) + (1 - q^{n-1}) p(n-1, k). \tag{17}$$

Now define

$$a_n(u) = \sum_k p(n, k) u^k,$$

then

$$a_n(u) = u q^{n-1} a_{n-1}(u) + (1 - q^{n-1}) a_{n-1}(u) = (1 + q^{n-1}(u-1)) a_{n-1}(u)$$

and thus

$$a_n(u) = \prod_{i=0}^{n-1} (1 + q^i(u-1)).$$

Consider q -Stirling numbers recursively defined by

$$s(n, k) = s(n-1, k-1) + [n-1]_q s(n-1, k).$$

Let

$$b_n(u) = \sum_k s(n, k) u^k,$$

then

$$b_n(u) = u b_{n-1}(u) + [n-1]_q b_{n-1}(u) = \prod_{i=1}^{n-1} (u + [i]_q).$$

Therefore (we introduce the q -dependency explicitly)

$$\begin{aligned} s(n, k, 1/q) &= [u^k] b_n(u) = [u^k] \prod_{i=1}^{n-1} \left(u + \frac{1-1/q^i}{1-1/q}\right) = q^{-\binom{n}{2}} [u^k] \prod_{i=1}^{n-1} \left(u q^i + \frac{q^i-1}{q-1} q\right) \\ &= q^{-\binom{n}{2}} \left(\frac{q}{1-q}\right)^{n-1} [u^k] \prod_{i=1}^{n-1} (u(1-q)q^{i-1} + 1 - q^i) \\ &= q^{-\binom{n}{2}} \left(\frac{q}{1-q}\right)^{n-1+k} [u^k] \prod_{i=1}^{n-1} (u q^i + 1 - q^i) = q^{-\binom{n}{2}} \left(\frac{q}{1-q}\right)^{n-1+k} p(n, k), \end{aligned}$$

hence

$$p(n, k) = q^{\binom{n}{2}} \left(\frac{1-q}{q}\right)^{n-1+k} s(n, k, 1/q). \blacksquare$$

Remark. For $a = b$, simplification is possible:

$$\begin{aligned}
 p(n, k + 1) &= q^{b\binom{k}{2}} \sum_{j=k}^n (-1)^{j-k} q^{b\binom{j}{2}+cj} \begin{bmatrix} n \\ j \end{bmatrix}_{q^b} \begin{bmatrix} j \\ k \end{bmatrix}_{q^b} \\
 &= q^{b\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} \sum_{j=k}^n (-1)^{j-k} q^{b\binom{j}{2}+cj} \begin{bmatrix} n-k \\ j-k \end{bmatrix}_{q^b} \\
 &= q^{b\binom{k}{2}+ck} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} \sum_{j=0}^{n-k} (-1)^j q^{b\binom{j+k}{2}+cj} \begin{bmatrix} n-k \\ j \end{bmatrix}_{q^b} \\
 &= q^{2b\binom{k}{2}+ck} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} \sum_{j=0}^{n-k} (-1)^j q^{b\binom{j}{2}+bjk+cj} \begin{bmatrix} n-k \\ j \end{bmatrix}_{q^b} \\
 &= q^{2b\binom{k}{2}+ck} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} \prod_{i=k}^{n-1} (1 - q^{bi+c}) \\
 &= q^{2b\binom{k}{2}+ck} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} (q^{bk+c}; q^b)_{n-k}.
 \end{aligned}$$

Specializing further, for $b = 0$ and $c = 1$, this becomes $\binom{n}{k} q^k (1 - q)^{n-k}$, which is of course evident.

If we let $b = c = 1$, we get $p(n, k + 1) = q^{k^2} \frac{(q)_n^2}{(q)_k^2 (q)_{n-k}}$. Letting n tend to infinity and noticing that the probabilities sum to 1, we get

$$\sum_{k \geq 0} \frac{q^{k^2}}{(q)_k^2} = \frac{1}{(q)_\infty}.$$

This is due to Euler and occurs when enumerating partitions according to Durfee squares [1].

5. The moments

5.1. General case

The moments are derived from $F(z, u)$. We have, starting with 1 at 0 and $b > 0$,

$$\mathbb{E}[X_n^i] = \sum_{j=i-1}^n q^{cj} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} q^{a\binom{j}{2}} q^{bj} \left. \frac{\partial^i (u(u; q^b)_j)}{\partial u^i} \right|_{u=1}.$$

This leads easily to the following result. Note that $x^i = x(x - 1) \dots (x - i + 1)$, and the formula is about the factorial moments:

Proposition 6.

$$\begin{aligned}
 \mathbb{E}[X_n^i] &= i! \sum_{j=i-1}^n q^{cj} (-1)^{j-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} q^{a\binom{j}{2}} q^{bj} (q^b; q^b)_{j-1} \\
 &\quad \times \left[\sum_{1 \leq s_1 < \dots < s_{i-2} < j} \frac{q^{bs_1} \dots q^{bs_{i-2}}}{(q^{bs_1} - 1) \dots (q^{bs_{i-2}} - 1)} + \sum_{1 \leq s_1 < \dots < s_{i-1} < j} \frac{q^{bs_1} \dots q^{bs_{i-1}}}{(q^{bs_1} - 1) \dots (q^{bs_{i-1}} - 1)} \right]. \quad (18)
 \end{aligned}$$

For instance

$$\mathbb{E}(X_n) = 1 + \sum_{j=1}^n q^{cj} (-1)^{j-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} q^{a\binom{j}{2}} q^{bj} (q^b; q^b)_{j-1}, \quad (19)$$

$$\mathbb{E}[X_n^2] = 2 \sum_{j=1}^n q^{cj} (-1)^{j-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} q^{a \binom{j}{2}} q^{bj} (q^b; q^b)_{j-1} \left[1 + \sum_{l=1}^{j-1} \frac{q^{bl}}{q^{bl} - 1} \right], \tag{20}$$

$$\mathbb{E}[X_n^3] = 3! \sum_{j=2}^n q^{cj} (-1)^{j-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} q^{a \binom{j}{2}} q^{bj} (q^b; q^b)_{j-1} \left[\sum_{l=1}^{j-1} \frac{q^{bl}}{q^{bl} - 1} + \sum_{1 \leq s < l < j} \frac{q^{bl} q^{bs}}{(q^{bl} - 1)(q^{bs} - 1)} \right]. \tag{21}$$

Remark. Using a univariate generating function, Crippa and Simon get the first two moments in [4, (27), (28)]. For $a = 0$, the first two moments appear in many papers that were written about *approximate counting*.

Remark. Charalambides [3, (3.4) and (3.5)] also computes the first two moments, by a lengthy derivation.

5.2. Particular cases

Two particular cases are interesting.

First case: $b = 1, a = 0, c = 0$.

For $b = 1, a = 0, c = 0$, we get immediately the moments of *approximate counting*. Since we feel that this result is somehow important, we put it as a proposition:

Proposition 7.

$$\mathbb{E}[X_n^i] = i! \sum_{j=i-1}^n (-1)^{j-1} \binom{n}{j} q^j (q)_{j-1} \times \left[\sum_{1 \leq s_1 < \dots < s_{i-2} < j} \frac{q^{s_1} \dots q^{s_{i-2}}}{(q^{s_1} - 1) \dots (q^{s_{i-2}} - 1)} + \sum_{1 \leq s_1 < \dots < s_{i-1} < j} \frac{q^{s_1} \dots q^{s_{i-1}}}{(q^{s_1} - 1) \dots (q^{s_{i-1}} - 1)} \right].$$

Remark. [5, Theorem 5] gives the first two moments.

Let us briefly review how one can get dominant and periodic parts of the moments of *approximate counting*. As already mentioned, Flajolet derives results for the first two moments using the Mellin transform; Kirschenhofer and Prodinger did the same with Rice’s method. Using the methods in our recent paper [8, Sections 4.5 and 5.5], all moments can be almost automatically derived. Setting $Q := 1/q$, we obtain the following forms for the mean, the second and third centred moments:

$$\begin{aligned} \mathbb{E}(X_n - \log_Q n) &\sim \tilde{m}_1 + w_1, \\ \mathbb{E}(X_n - \log_Q n - \tilde{m}_1 - w_1)^2 &\sim \tilde{\mu}_2 + \kappa_2, \\ \mathbb{E}(X_n - \log_Q n - \tilde{m}_1 - w_1)^3 &\sim \tilde{\mu}_3 + \kappa_3. \end{aligned}$$

The detailed expressions are extracted from [8]. Since the formulæ for the third moments are new, we put them as a proposition.

Proposition 8. *With the notations*

$$\begin{aligned} L &:= \ln(Q), \\ \chi_l &:= \frac{2\pi i l}{L}, \\ C_k &:= \sum_{j=1}^{\infty} 1/(Q^j - 1)^k, \end{aligned}$$

we have

$$\begin{aligned} \tilde{m}_1 &= 1/2 - C_1 + \gamma/L, \\ \tilde{\mu}_2 &= \pi^2/(6L^2) + 1/12 - C_1 - C_2, \\ \tilde{\mu}_3 &= 2\zeta(3)/L^3 - 2C_3 - 3C_2 - C_1, \\ w_1 &= - \sum_{l \neq 0} \Gamma(\chi_l) e^{-2l\pi i \log_Q n / L}, \end{aligned}$$

$$\begin{aligned} \kappa_2 &= -w_1^2 - 2\gamma w_1/L + 2 \sum_{l \neq 0} \Gamma(\chi_l) \psi(\chi_l) e^{-2l\pi i \log_Q n / L^2}, \\ \kappa_3 &= w_1(4L^2 w_1^2 + 12w_1 L\gamma + 6\gamma^2 - \pi^2)/(2L^2) - 6(\gamma + w_1 L) \sum_{l \neq 0} \Gamma(\chi_l) \psi(\chi_l) e^{-2l\pi i \log_Q n / L^3} \\ &\quad - 3 \sum_{l \neq 0} \Gamma(\chi_l) \psi^2(\chi_l) e^{-2l\pi i \log_Q n / L^3} - 3 \sum_{l \neq 0} \Gamma(\chi_l) \psi(1, \chi_l) e^{-2l\pi i \log_Q n / L^3}. \end{aligned}$$

Remark. The (not surprising) fact that $\mathbb{E}[X_n^i] \sim (\log_Q n)^i$ in general, can also be deduced from Rice’s method. We do not give a full proof of this but rather sketch a few key steps. It is of course equivalent to consider the factorial moments instead.

$$\begin{aligned} \mathbb{E}[X_n^i] &= i! \sum_{j=i-1}^n (-1)^{j-1} \binom{n}{j} q^j (q)_{j-1} \\ &\quad \times \left[\sum_{1 \leq s_1 < \dots < s_{i-2} < j} \frac{q^{s_1} \dots q^{s_{i-2}}}{(q^{s_1} - 1) \dots (q^{s_{i-2}} - 1)} + \sum_{1 \leq s_1 < \dots < s_{i-1} < j} \frac{q^{s_1} \dots q^{s_{i-1}}}{(q^{s_1} - 1) \dots (q^{s_{i-1}} - 1)} \right]. \end{aligned}$$

Now,

$$\sum_{1 \leq s_1 < \dots < s_{i-1} < j} \frac{q^{s_1} \dots q^{s_{i-1}}}{(q^{s_1} - 1) \dots (q^{s_{i-1}} - 1)} = \frac{1}{(i-1)!} \sum_{1 \leq s < j} \left(\frac{q^s}{(q^s - 1)} \right)^{i-1} + \text{less important terms}$$

and thus we study

$$i(-1)^{i-1} \sum_{j=i-1}^n (-1)^{j-1} \binom{n}{j} q^j (q)_{j-1} \sum_{1 \leq s < j} \left(\frac{q^s}{1 - q^s} \right)^{i-1}.$$

The method (which is described in many textbooks, e.g. [10]) consists in continuing the function

$$q^j (q)_{j-1} \sum_{1 \leq s < j} \left(\frac{q^s}{1 - q^s} \right)^{i-1}$$

to the complex plane, to $\psi(z)$, say, and then computing the residue of

$$i(-1)^{i-1} \frac{\Gamma(n+1)}{\Gamma(n+1-z)\Gamma(-z)} \psi(z)$$

at $s = 0$. Observe that

$$\sum_{1 \leq s < j} \left(\frac{q^s}{1 - q^s} \right)^{i-1} = \sum_{s \geq 1} \left(\frac{1}{Q^s - 1} \right)^{i-1} - \sum_{s \geq 1} \left(\frac{1}{Q^{s+j-1} - 1} \right)^{i-1}$$

so, we use the function

$$\psi(z) = \frac{(q)_\infty}{(q^z)_\infty (Q^z - 1)} \left[\sum_{s \geq 1} \left(\frac{1}{Q^s - 1} \right)^{i-1} - \sum_{s \geq 1} \left(\frac{1}{Q^{s+z-1} - 1} \right)^{i-1} \right].$$

The computation of this residue leads to several terms, since the pole is of order $i + 1$. However, the dominant term that comes out is $(\log_Q n)^i$.

Second case: $b = 0$.

Another interesting case is $b = 0$. If we set $q^b = 1 - \varepsilon$, this leads to $(q^b; q^b)_j \sim \varepsilon^j j!$ and $\frac{q^{bs_1}}{(q^{bs_1} - 1)} \sim \frac{1}{-\varepsilon s_1}$. After a little algebra, we obtain the following result.

Proposition 9. *If $b = 0$, we have*

$$\mathbb{E}[X_n^i] = i! q^{c(i-1)} \left[\begin{matrix} n \\ i-1 \end{matrix} \right]_{q^a} q^{a \binom{i-1}{2}}.$$

Remarks. Our result include the following special cases:

For $a = 1, b = 0, c = 0$, [5, (22), (23)], derived there by induction.

The formulæ [4, (25), (26)], with $b = 0$ are immediate.

[3, Theorem 3.2] is also immediate.

5.3. q -factorial moments

The q -factorial moments in the general case are given in [3, (3.3)]: The formula is

$$\mathbb{E}[(X_n)_{m,q^b}] = \frac{1 - q^{bm}}{(1 - q^b)^m} q^{b\binom{m}{2}} \sum_{j=m}^n (-1)^{j-m} q^{a\binom{j}{2} + cj} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} \begin{bmatrix} j \\ m \end{bmatrix}_{q^b} (q^b; q^b)_{j-1}.$$

If $a > 0$, this quantity converges to the constant, as $n \rightarrow \infty$:

$$\frac{1 - q^{bm}}{(1 - q^b)^m} q^{b\binom{m}{2}} \sum_{j=m}^{\infty} (-1)^{j-m} q^{a\binom{j}{2} + cj} \frac{1}{(q^a; q^a)_j} \begin{bmatrix} j \\ m \end{bmatrix}_{q^b} (q^b; q^b)_{j-1}.$$

Notice that $\mathbb{E}[(X_n)_{m,q^b}]$ simplifies for $b = 0$, as in the sum only the term with $j = m$ survives, with the result $m!q^{a\binom{m}{2} + cm} \begin{bmatrix} n \\ m \end{bmatrix}_{q^a}$. This was derived in [3] in a separate theorem, but it follows readily from the general case (see the remark above).

6. Asymptotics of the moments of X_n for $n \rightarrow \infty$

While the q^b -moments of X_n are quite easy to deal with, as shown above, the proper ones are a bit harder. The results are again constants, but they don't look as pretty as the previous ones.

6.1. General case

We obtain the following result:

Proposition 10. With $a > 0$,

$$\begin{aligned} \mathbb{E}[X_{\infty}^i] &= i! \sum_{j=i-1}^{\infty} q^{cj} (-1)^{j-1} \frac{1}{(q^a; q^a)_j} q^{a\binom{j}{2}} q^{bj} (q^b; q^b)_{j-1} \\ &\times \left[\sum_{s_1=1}^{j-1} \dots \sum_{s_{i-2}=1}^{j-1} \mathbb{I}[s_1 < s_2 < \dots < s_{i-2}] \frac{q^{bs_1} \dots q^{bs_{i-2}}}{(q^{bs_1} - 1) \dots (q^{bs_{i-2}} - 1)} \right. \\ &\left. + \sum_{s_1=1}^{j-1} \dots \sum_{s_{i-1}=1}^{j-1} \mathbb{I}[s_1 < s_2 < \dots < s_{i-1}] \frac{q^{bs_1} \dots q^{bs_{i-1}}}{(q^{bs_1} - 1) \dots (q^{bs_{i-1}} - 1)} \right]. \end{aligned}$$

Proof. Letting $n \rightarrow \infty$, we obtain from (12)

$$p_C(\infty, k) = \frac{q^{(a+b)\binom{k}{2} + ck}}{(q^b; q^b)_k} \sum_{v=0}^{\infty} (-1)^v q^{a[v^2 + v(2k-1)]/2 + cv} \frac{(q^b; q^b)_{k+v}}{(q^b; q^b)_v (q^a; q^a)_{k+v}}.$$

From (16), we have

$$p(\infty, k + 1) = \frac{q^{(a+b)\binom{k}{2} + bk + ck}}{(q^b; q^b)_k} \sum_{v=0}^{\infty} (-1)^v q^{a[v^2 + v(2k-1)]/2 + bv + cv} \frac{(q^b; q^b)_{k+v}}{(q^b; q^b)_v (q^a; q^a)_{k+v}}.$$

We study the behaviour of the factorial moments

$$\mathbb{E}[X_{\infty}^i] = \sum_{k=0}^{\infty} k(k-1) \dots (k-i+1) p(\infty, k). \tag{22}$$

From (18) we derive, with $a > 0$, the proposition. ■

For instance,

$$\mathbb{E}(X_\infty) = 1 + \sum_{j=1}^{\infty} q^{cj} (-1)^{j-1} \frac{1}{(q^a; q^a)_j} q^{a\binom{j}{2}} q^{bj} (q^b; q^b)_{j-1},$$

$$\mathbb{E}[X_\infty^2] = 2 \sum_{j=1}^{\infty} q^{cj} (-1)^{j-1} \frac{1}{(q^a; q^a)_j} q^{a\binom{j}{2}} q^{bj} (q^b; q^b)_{j-1} \left[1 + \sum_{l=1}^{j-1} \frac{q^{bl}}{q^{bl} - 1} \right].$$

Remark. For $b = 0$, mean and $\mathbb{E}[(X_\infty^2)]$ are derived in [4].

6.2. Case $a = 1, b = 0, c = 0$

From (16), we derive in this case ($a = 1, b = 0, c = 0$) the following result.

Proposition 11. *With $a = 1, b = 0, c = 0$, we have*

$$p_C(\infty, k) = p(\infty, k + 1) = q^{\binom{k}{2}} \sum_{v=0}^{\infty} (-1)^v q^{[v^2+v(2k-1)]/2} \frac{\binom{k+v}{v}}{(q)_{k+v}}.$$

The limit when $n \rightarrow \infty$ is independent of n and *not* Gaussian (as was suggested in [5]).

In the following, we give an independent proof that $p(\infty, 1) = 0$ and that the $p(\infty, k)$ sum to 1.

$$p(\infty, 1) = \sum_{v=0}^{\infty} (-1)^v q^{\binom{v}{2}} \frac{1}{(q)_v} = (1; q)_\infty = 0,$$

by applying one of Euler’s partition identities. And now

$$\begin{aligned} S &= \sum_{k \geq 0} q^{\binom{k}{2}} \sum_{v=0}^{\infty} (-1)^v q^{[v^2+v(2k-1)]/2} \frac{\binom{k+v}{v}}{(q)_{k+v}} \\ &= \sum_{k, v \geq 0} q^{\binom{k+v}{2}} (-1)^v \frac{\binom{k+v}{v}}{(q)_{k+v}} \\ &= \sum_{n \geq 0} \frac{q^{\binom{n}{2}}}{(q)_n} \sum_{v=0}^n (-1)^v \binom{n}{v} \\ &= \sum_{n \geq 0} \frac{q^{\binom{n}{2}}}{(q)_n} \mathbb{I}[n = 0] = 1. \end{aligned}$$

6.3. Other expression

Charalambides [3, (2.11)] expresses the moments in the terms of q -Stirling numbers:

$$\mathbb{E} \left[X_\infty^i \right] = i! \sum_{r=i}^{\infty} (-1)^{r-i} (1 - q^b)^{r-i} \frac{(1 - q^b)^r}{(q^b; q^b)_r} \mathbb{E} \left[(X_\infty)_{r, q^b} \right] s_{q^b}(r, i).$$

6.4. Tail

When $k \rightarrow \infty$, $p(\infty, k)$ leads to the asymptotic equivalent for the tail

$$p_C(\infty, k) \sim \frac{q^{(a+b)\binom{k}{2}+ck}}{(q^a; q^a)_\infty}.$$

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If $(a, b) \neq (0, 0)$, this can be simplified (with less precision) to

$$p_C(\infty, k) \sim \frac{q^{(a+b)k^2/2}}{(q^a; q^a)_\infty}.$$

Also

$$P_C(\infty, k) := \sum_{i=k}^{\infty} p(\infty, i) \sim \frac{q^{(a+b)\binom{k}{2}+ck}}{(q^a; q^a)_\infty}.$$

These results about tails are new.

7. Analysis of Y_k (time to reach k)

7.1. General case and asymptotics

We obtain the following results:

Proposition 12.

$$\begin{aligned} \mathbb{E}([Y_k - 1]_{m,q^b}) &= \frac{1}{(1 - q^b)^m} q^{a(n-1)+b\binom{m+1}{2}} \\ &\times \sum_{m+1 \leq j \leq n} q^{a\binom{j-1}{2}+cj} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{q^a} (-1)^{j-m-1} (q^b; q^b)_{j-1} \begin{bmatrix} j-1 \\ m \end{bmatrix}_{q^b}. \end{aligned} \tag{23}$$

Proposition 13. *The normalized hitting time Y_k (normalized by $P_C(\infty, k)$) is asymptotically ($k \rightarrow \infty$) given by $Y_k = k + U$, where U is a random variable with probability generating function*

$$\frac{q^{au}}{(q^a; q^a)_u} (q^a; q^a)_\infty, \tag{24}$$

independent of b .

Proposition 14. *The normalized moments are given by*

$$\mathbb{E}(Y_k^i) \sim \sum_{u=0}^{\infty} \frac{q^{au}}{(q^a; q^a)_u} (q^a; q^a)_\infty (k+u)^i = \sum_{v=0}^i \sum_{u=0}^{\infty} \frac{q^{au}}{(q^a; q^a)_u} (q^a; q^a)_\infty \binom{i}{v} k^{i-v} u^v, \quad k \rightarrow \infty. \tag{25}$$

Proof of (23). The probability $q_C(n, k)$ is given in [3, (4.1), (4.2)] by a rather complicated expression, which can be simplified as

$$\begin{aligned} &q^{a/2(2n-2+k^2)+ck+b\binom{k}{2}} \sum_{v=0}^{n-k} q^{a/2[k(2v-3)+(v-1)(v-2)]+cv} (-1)^v \\ &\times \frac{(q^a; q^a)_{n-1}}{(q^a; q^a)_{n-v-k} (q^a; q^a)_{v+k-1}} \frac{(q^b; q^b)_{v+k-1}}{(q^b; q^b)_v (q^b; q^b)_{k-1}}. \end{aligned} \tag{26}$$

Actually, it is simply given, with (12), by

$$q_C(n, k) = p_C(n-1, k-1) q^{a(n-1)+b(k-1)+c}.$$

Indeed, (12) leads to

$$\begin{aligned}
 p_C(n, k) &= q^{b\binom{k}{2}} \sum_{j=k}^n (-1)^{j-k} q^{a\binom{j}{2}+cj} \begin{bmatrix} n \\ j \end{bmatrix}_{q^a} \begin{bmatrix} j \\ k \end{bmatrix}_{q^b} \\
 &= q^{b\binom{k}{2}} \sum_{j=k}^n (-1)^{j-k} q^{a\binom{j}{2}+cj} \frac{(q^a; q^a)_n}{(q^a; q^a)_j (q^a; q^a)_{n-j}} \frac{(q^b; q^b)_j}{(q^b; q^b)_k (q^b; q^b)_{j-k}} \\
 &= q^{ck} q^{(b+a)\binom{k}{2}} \sum_{v=0}^{n-k} (-1)^v q^{a/2[v^2+v(2k-1)]} q^{cv} \frac{(q^a; q^a)_n}{(q^a; q^a)_{v+k} (q^a; q^a)_{n-v-k}} \frac{(q^b; q^b)_{v+k}}{(q^b; q^b)_k (q^b; q^b)_v}.
 \end{aligned}$$

Now we compute a suitable type of moment to get nice results. As was discussed by Charalambides [3], it does not really matter *which* type of moments one computes, as one can always convert.

As already mentioned,

$$q_C(n, k) = q^{a\binom{n}{2}+b\binom{k}{2}+cn-a\binom{n-1}{2}-c(n-1)} \sum_{j=k}^n (-1)^{j-k} q^{a\binom{j-1}{2}+c(j-1)} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{q^a} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_{q^b}$$

or in simplified form:

$$q_C(n, k) = q^{a(n-1)+b\binom{k}{2}} \sum_{j=k}^n (-1)^{j-k} q^{a\binom{j-1}{2}+cj} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{q^a} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_{q^b}.$$

Which type of moments shall we choose in order to get an appealing result?

$$\sum_{1 \leq j \leq n} q^{a(n-1)+a\binom{j-1}{2}+cj} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{q^a} (-1)^j \sum_{1 \leq k \leq j} (-1)^k q^{b\binom{k}{2}} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_{q^b} \Theta_m(k)$$

$\Theta_m(k)$ must be a suitable “polynomial” of k of degree m , so that we can sum the inner sum in a nice way.

$$S = \sum_{1 \leq k \leq j} (-1)^k q^{b\binom{k}{2}} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}_{q^b} \Theta_m(k).$$

We may choose $\Theta_m(k) = (q^b; q^b)_{k-1} / (q^b; q^b)_{k-1-m} = \prod_{i=k-m}^{k-1} (1 - q^{bi})$, which is a q^b -factorial. Then,

$$S = \prod_{i=j-m}^{j-1} (1 - q^{bi}) \sum_{m+1 \leq k \leq j} (-1)^k q^{b\binom{k}{2}} \begin{bmatrix} j-m-1 \\ k-m-1 \end{bmatrix}_{q^b}.$$

So we are left with

$$\begin{aligned}
 S &= \frac{(q^b; q^b)_{j-1}}{(q^b; q^b)_{j-m-1}} (-1)^{m+1} \sum_{0 \leq k \leq j-m-1} (-1)^k q^{b\binom{k+m+1}{2}} \begin{bmatrix} j-m-1 \\ k \end{bmatrix}_{q^b} \\
 &= \frac{(q^b; q^b)_{j-1}}{(q^b; q^b)_{j-m-1}} (-1)^{m+1} q^{b\binom{m+1}{2}} \sum_{0 \leq k \leq j-m-1} (-1)^k q^{b\binom{k}{2}} \begin{bmatrix} j-m-1 \\ k \end{bmatrix}_{q^b} q^{bk(m+1)}.
 \end{aligned}$$

But we have the formula

$$\sum_{k=0}^n q^{b\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^b} t^k u^{n-k} = \prod_{i=0}^{n-1} (u + tq^{bi}).$$

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This applies for $n = j - m - 1, u = 1, t = -q^{b(m+1)}$:

$$\begin{aligned} S &= \frac{(q^b; q^b)_{j-1}}{(q^b; q^b)_{j-m-1}} (-1)^{m+1} q^{b\binom{m+1}{2}} \prod_{i=0}^{j-m-2} (1 - q^{b(i+m+1)}) \\ &= \prod_{i=j-m}^{j-1} (1 - q^{bi}) (-1)^{m+1} q^{b\binom{m+1}{2}} \prod_{i=m+1}^{j-1} (1 - q^{bi}) \\ &= (-1)^{m+1} q^{b\binom{m+1}{2}} (q^b; q^b)_{j-1} \left[\begin{matrix} j-1 \\ m \end{matrix} \right]_{q^b}. \end{aligned}$$

Then the moment becomes

$$q^{a(n-1)+b\binom{m+1}{2}} \sum_{m+1 \leq j \leq n} q^{a\binom{j-1}{2}+cj} \left[\begin{matrix} n-1 \\ j-1 \end{matrix} \right]_{q^a} (-1)^{j-m-1} (q^b; q^b)_{j-1} \left[\begin{matrix} j-1 \\ m \end{matrix} \right]_{q^b},$$

which proves (23).

Remark. In the notation of [3], the moment just computed is

$$(1 - q^b)^m \sum_k [k - 1]_{m, q^b} q_C(n, k). \blacksquare$$

If $b = 0$, only one term in the sum survives, and this yields

$$m! q^{a(n-1)+a\binom{m}{2}+cm} \left[\begin{matrix} n-1 \\ m \end{matrix} \right]_{q^a}.$$

Proof of (24). When $k \rightarrow \infty$, we have the asymptotic equivalent

$$q_C(n, k) \sim q^{a/2(2n-2+k^2)+ck+b\binom{k}{2}} \sum_{v=0}^{n-k} q^{a/2[k(2v-3)+(v-1)(v-2)]+cv} (-1)^v \frac{1}{(q^a; q^a)_{n-v-k} (q^b; q^b)_v},$$

and setting $n = k + u$, this gives

$$\begin{aligned} q_C(k + u, k) &\sim q^{a/2(-k-2+k^2)+ck+b\binom{k}{2}+au} \sum_{v=0}^u q^{a/2[2vk+(v-1)(v-2)]+cv} (-1)^v \frac{1}{(q^a; q^a)_{u-v} (q^b; q^b)_v} \\ &\sim q^{(a+b)\binom{k}{2}+ck+au} \frac{1}{(q^a; q^a)_u}. \end{aligned}$$

But we must normalize by $P_C(\infty, k)$; this gives the conditional probability of (24).

Eq. (24) is a decent function of u . Indeed by (4), with $t = q^a$,

$$\sum_{u=0}^{\infty} \frac{q^{au}}{(q^a; q^a)_u} = \prod_{i=1}^{\infty} (1 - q^a q^{a(i-1)})^{-1} = \prod_{i=1}^{\infty} (1 - q^{ai})^{-1} = \frac{1}{(q^a; q^a)_{\infty}},$$

as it should. \blacksquare

The normalized moments of (25) follow immediately.

Remark.

$$\sum_{v=0}^i \sum_{u \geq 0} \frac{q^{au}}{(q^a; q^a)_u} (q^a; q^a)_{\infty} \binom{i}{v} k^{i-v} u^v = (q^a; q^a)_{\infty} \sum_{v=0}^i \binom{i}{v} k^{i-v} \sum_{u \geq 0} \frac{q^{au}}{(q^a; q^a)_u} u^v.$$

Let us write $r = q^a$. What we need is

$$\sum_{n \geq 0} \frac{t^n}{(r; r)_n}$$

then the inner sum can be obtained via a few differentiations, and $t := r$. The sum can be written as a product, by Euler's partition identity. However, multiple differentiations lead to iterated sums, and that is all we can do with it.

7.2. Other expression

After normalization, [3, (4.3)] leads, for $k \rightarrow \infty$, to

$$\mathbb{E} \left[\frac{(q^a; q^a)_{Y_k+m-1}}{(q^a; q^a)_{Y_k-1} (1-q^a)^m} \right] \sim (1-q^a)^{m+2k}. \quad (27)$$

8. Conclusion

Using generating functions, we have rederived known results and obtained new ones on q -distributions, in a unified and consistent way.

Other forms for the transition probabilities are possible: for instance, in [4], the transition is related to $1 - q^{an+bk+c}$ (as opposed to $q^{an+bk+c}$, as in this paper). These generalizations will be the object of future work. It would be interesting to explore how general the recursion might be in order to get some meaningful results.

Acknowledgments

The insightful comments of two referees are gratefully acknowledged.

References

- [1] G. Andrews, The Theory of Partitions, in: Encyclopedia of Mathematics and its Applications, vol. 2, Addison-Wesley, 1976.
- [2] Ch.A. Charalambides, Non-central generalized q -factorial coefficients and q -Stirling numbers, *Discrete Mathematics* 275 (2004) 67–85.
- [3] Ch.A. Charalambides, Moments of a class of discrete q -distributions, *Journal of Statistical Planning and Inference* 135 (2005) 64–76.
- [4] D. Crippa, K. Simon, q -distributions and Markov processes, *Discrete Mathematics* 170 (1997) 81–98.
- [5] D. Crippa, K. Simon, P. Trunz, Markov processes involving q -Stirling numbers, *Combinatorics, Probability and Computing* 6 (1997) 165–178.
- [6] P. Flajolet, Approximate counting: A detailed analysis, *BIT* 25 (1985) 113–134.
- [7] P. Kirschenhofer, H. Prodinger, Approximate counting: An alternative approach, *RAIRO Theoretical Informatics and Applications* 25 (1991) 43–48.
- [8] G. Louchard, H. Prodinger, Asymptotics of the moments of extreme-value related distribution functions, *Algorithmica* 46 (2006) 431–467. Long version: <http://www.ulb.ac.be/di/mcs/louchard/moml.ps>.
- [9] H. Prodinger, Approximate counting via Euler transform, *Mathematica Slovaca* 44 (1994) 569–574.
- [10] W. Szpankowski, Average case analysis of algorithms on sequences, in: Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2001.

Contents lists available at [SciVerse ScienceDirect](#)

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcsApproximate counting with m counters: A detailed analysis

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ARTICLE INFO

Article history:

Received 9 June 2011

Received in revised form 25 January 2012

Accepted 10 March 2012

Communicated by G. Ausiello

Keywords:

Approximate counting

Rice's method

Cancellations

 q -analysis

ABSTRACT

The classical algorithm *approximate counting* has been recently modified by Cichoń and Macyna: instead of one counter, m counters are used, and the assignment of an incoming item to one of the counters is random. The parameter of interest is the sum of the values of all the counters. We analyse expectation and variance, getting explicit and asymptotic formulae.

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1. Introduction

Approximate counting is a technique that was first analysed by Flajolet [1]; some subsequent papers [2–5] added to the analysis.

A counter C is kept, and each time an item arrives and needs to be counted, a random experiment is performed; if the current value of the counter is i , then with probability 2^{-i} the counter is increased by 1, otherwise it keeps its value; at the beginning, the counter value is $C = 1$. After n random increments, the value of the counter is typically close to $\log_2 n$, and the cited papers contain exact and asymptotic values for average and variance.

Very recently, Cichoń and Macyna [6] have used this idea as follows: instead of 1 counter, they keep m counters, where $m \geq 1$ is an integer. For our subsequent analysis we will assume that m is fixed. When a new element arrives (and needs to be counted), it is randomly (with probability $\frac{1}{m}$) assigned to one of the m counters, and then the random experiment is performed as usual. The parameter that Cichoń and Macyna are interested in is the total number of changes of any counter. In other words, if we (for convenience) assume that the initial setting of a counter to the value 1 counts as a change, Cichoń is interested in the sum of the values of the m counters. This is the parameter that we will study in this paper.

We find an expression for $P_{N,l}$, the probability that the combined counter values are l , after N random experiments, in terms of the classical values $p_{n,i}$ for just one counter. Furthermore, we find exact and asymptotic expressions for expectation and variance. As in the classical case, the expectation is of the form $A_1 \log N + A_2 + \delta(\log_2 N) + o(1)$, with a periodic function $\delta(x)$ of period 1 and mean value 0. (The mean value is the zeroth term in its Fourier series expansion.) The variance is small: $A_3 + \delta_V(\log_2 N) + o(1)$. We will not compute the Fourier expansion of this periodic function explicitly, although it can be done.

Here are a few words about what to expect from our analysis: intuitively, we expect that roughly N/m elements will go to each counter, so we expect a result like $\mathbb{E}(C_N^{(m)}) \sim m\mathbb{E}(C_{N/m}^{(m)})$, where $C_N^{(m)}$ is the random variable describing the combined values of m counters after N random increments, and a similar formula for the variance. Indeed, our asymptotic analysis confirms these claims; differences would only occur for lower order terms that we do not compute here.

The paper clearly consists of two parts: the *explicit* computation of the first two moments (Section 2) and the asymptotic evaluation of them (Section 3). While it is quite an achievement to obtain these explicit evaluations, there is a prize to be

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paid: the computations to achieve all this are not simple. We also borrow from the paper [7] in terms of methods. Where this paper was particularly brief (because of length restrictions) we try here to re-explain things and provide more help for the interested reader. The path that we follow here is not surprising for the specialists. However, the present case offers several challenges, whence we decided to provide the calculations in full, since we feel that even reasonably experienced readers might not be able to fill them in, would they not been given.

Our approach is to use generating functions throughout. The explicit expressions that we obtain are alternating functions for which we use the convenient technique called *Rice's method* [8].

We will need a few abbreviations that we collect here: $L = \log 2$, $Q_n := (1 - \frac{1}{2}) \dots (1 - \frac{1}{2^n})$, $Q(x) := \prod_{k \geq 1} (1 - \frac{x}{2^k})$, so that $Q_n = Q(1)/Q(2^{-n})$; the latter expression is used to define Q_n for arbitrary n , not just nonnegative integers.

We will prove the following asymptotic result, where $\delta(x)$ and $\delta_V(x)$ are periodic functions of period 1 and mean zero.

Theorem 1. *The average and the variance of Cichoń–Macyna's m -approximate counter scheme admit the following asymptotic expansions as $N \rightarrow \infty$:*

$$E_N \sim m \left[\log_2 N - \log_2 m + \frac{1}{2} - \alpha + \frac{\gamma}{L} + \delta(\log_2 N) \right],$$

$$V_N \sim m \left[1 - \alpha - \beta + \frac{2\tau}{L} + \delta_V(\log_2 N) \right].$$

The constants (also independent of m) are

$$\alpha = \sum_{j \geq 1} \frac{1}{2^j - 1}, \quad \beta = \sum_{j \geq 1} \frac{1}{(2^j - 1)^2} \quad \text{and} \quad \tau = \sum_{j \geq 1} \frac{(-1)^{j-1}}{j(2^j - 1)}.$$

2. Generating functions

If we think what has happened after N random experiments, then we notice that $N = n_1 + \dots + n_m$ and n_i elements were assigned to counter C_i . The probability for such a split is a multinomial;

$$\binom{N}{n_1, \dots, n_m} \frac{1}{m^N} = \frac{N!}{n_1! \dots n_m! m^N}.$$

Then we get the probability that after N experiments the sum of the m counters is l :

$$P_{N,l} = m^{-N} \sum_{n_1 + \dots + n_m = N} \binom{N}{n_1, \dots, n_m} \sum_{l_1 + \dots + l_m = l} p_{n_1, l_1} \dots p_{n_m, l_m}.$$

Let

$$G_n(u) := \sum_{l \geq 0} p_{n,l} u^l.$$

This generating function (studied in the classical papers) is not very nice itself, but the first and second (factorial) moments $G'_n(1)$ and $G''_n(1)$ are explicitly known:

$$G'_n(1) = 1 - \sum_{k=1}^n \binom{n}{k} (-1)^k 2^{-k} Q_{k-1},$$

$$G''_n(1) = \sum_{k=1}^n \binom{n}{k} (-1)^k 2^{1-k} Q_{k-1} (T_{k-1} - 1), \quad \text{with } T_k = \sum_{j=1}^k \frac{1}{2^j - 1}.$$

We can now form a generating function for the case of m counters:

$$\sum_l P_{N,l} u^l = m^{-N} \sum_{n_1 + \dots + n_m = N} \binom{N}{n_1, \dots, n_m} G_{n_1}(u) \dots G_{n_m}(u)$$

or

$$\sum_l P_{N,l} u^l \frac{m^N}{N!} = \sum_{n_1 + \dots + n_m = N} \frac{1}{n_1! \dots n_m!} G_{n_1}(u) \dots G_{n_m}(u).$$

Multiplying this by z^N and summing, we find the fundamental relationship

$$\sum_{l,N} P_{N,l} u^l \frac{(mz)^N}{N!} = \sum_N z^N \sum_{n_1 + \dots + n_m = N} \frac{1}{n_1! \dots n_m!} G_{n_1}(u) \dots G_{n_m}(u) = \left(\sum_n \frac{z^n}{n!} G_n(u) \right)^m. \tag{1}$$

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Differentiating this with respect to u once or twice, followed by setting $u := 1$, produces generating functions for the first and second factorial moments.

Taking one derivative we get:

$$\sum_{l,N} P_{N,l} l \frac{(mz)^N}{N!} = m \left(\sum_n \frac{z^n}{n!} G_n(1) \right)^{m-1} \sum_n \frac{z^n}{n!} G'_n(1) = m e^{z(m-1)} \sum_n \frac{z^n}{n!} G'_n(1).$$

Reading off the coefficient of z^N and multiplying it by $N!/m^N$ leads to an expression for the expected value:

$$\begin{aligned} E_N = \mathbb{E}(C_N^{(m)}) &= m^{1-N} N! [z^N] e^{z(m-1)} \sum_n \frac{z^n}{n!} G'_n(1) = m^{1-N} \sum_{n=0}^N \binom{N}{n} (m-1)^{N-n} G'_n(1) \\ &= m^{1-N} \sum_{n=0}^N \binom{N}{n} (m-1)^{N-n} \left[1 - \sum_{k=1}^n \binom{n}{k} (-1)^k 2^{-k} Q_{k-1} \right] \\ &= m - m^{1-N} \sum_{k=1}^N \sum_{n=k}^N \binom{N}{n} \binom{n}{k} (m-1)^{N-n} (-1)^k 2^{-k} Q_{k-1} \\ &= m - m^{1-N} \sum_{k=1}^N \sum_{n=k}^N \binom{N}{k} \binom{N-k}{n-k} (m-1)^{N-n} (-1)^k 2^{-k} Q_{k-1} \\ &= m - m^{1-N} \sum_{k=1}^N \binom{N}{k} m^{N-k} (-1)^k 2^{-k} Q_{k-1} = m - m \sum_{k=1}^N \binom{N}{k} (-1)^k (2m)^{-k} Q_{k-1}. \end{aligned}$$

Note that these simplifications have resulted in a form that does not look too different from the classical instance $m = 1$. The corresponding simplifications for the second (factorial) moment, which are going to follow, are significantly harder. We decided to present them in full since it would cost an interested reader a few hours to fill them in herself.

Performing two differentiations, we get

$$\sum_{l,N} P_{N,l} l(l-1) \frac{(mz)^N}{N!} = m(m-1) e^{z(m-2)} \left(\sum_n \frac{z^n}{n!} G'_n(1) \right)^2 + m e^{z(m-1)} \left(\sum_n \frac{z^n}{n!} G''_n(1) \right).$$

Now we need to perform some auxiliary computations: In these, we will use

$$\sum_n \binom{N}{n} \binom{n}{k} \binom{N-n}{j} = \sum_n \binom{N}{k+j} \binom{k+j}{k} \binom{N-k-j}{n-k} = 2^{N-k-j} \binom{N}{k+j} \binom{k+j}{k}.$$

Firstly,

$$\begin{aligned} N! [z^N] \left(\sum_n \frac{z^n}{n!} G'_n(1) \right)^2 &= \sum_n \binom{N}{n} G'_n(1) G'_{N-n}(1) \\ &= \sum_n \binom{N}{n} \left[1 - \sum_{j=1}^n \binom{n}{j} (-1)^j 2^{-j} Q_{j-1} \right] \left[1 - \sum_{k=1}^{N-n} \binom{N-n}{k} (-1)^k 2^{-k} Q_{k-1} \right] \\ &= 2^N - 2 \sum_n \binom{N}{n} \sum_{j=1}^n \binom{n}{j} (-1)^j 2^{-j} Q_{j-1} \\ &\quad + \sum_n \binom{N}{n} \sum_{j=1}^n \binom{n}{j} (-1)^j 2^{-j} Q_{j-1} \sum_{k=1}^{N-n} \binom{N-n}{k} (-1)^k 2^{-k} Q_{k-1} \\ &= 2^N - 2 \sum_n \sum_{j=1}^n \binom{N}{j} \binom{N-j}{n-j} (-1)^j 2^{-j} Q_{j-1} \\ &\quad + \sum_{j=1}^N \sum_{k=1}^N (-1)^j 2^{-j} Q_{j-1} 2^{N-k-j} \binom{N}{k+j} \binom{k+j}{k} (-1)^k 2^{-k} Q_{k-1} \\ &= 2^N - 2^{N+1} \sum_j \binom{N}{j} (-1)^j 4^{-j} Q_{j-1} + \sum_{j=1}^N \sum_{k=1}^N (-1)^j 2^{-j} Q_{j-1} 2^{N-k-j} \binom{N}{k+j} \binom{k+j}{k} (-1)^k 2^{-k} Q_{k-1}. \end{aligned}$$

Then

$$\begin{aligned}
 N![z^N]e^{z(m-2)}\left(\sum_n \frac{z^n}{n!}G'_n(1)\right)^2 &= \sum_s \binom{N}{s}(m-2)^{N-s}2^s - \sum_s \binom{N}{s}(m-2)^{N-s}2^{s+1} \sum_j \binom{s}{j}(-1)^j4^{-j}Q_{j-1} \\
 &+ \sum_s \binom{N}{s}(m-2)^{N-s} \sum_{j=1}^s \sum_{k=1}^s (-1)^j2^{-j}Q_{j-1}2^{s-k-j} \binom{s}{k+j} \binom{k+j}{k} (-1)^k2^{-k}Q_{k-1} \\
 &= m^N - \sum_s (m-2)^{N-s}2^{s+1} \sum_j \binom{N}{j} \binom{N-j}{s-j} (-1)^j4^{-j}Q_{j-1} \\
 &+ \sum_s (m-2)^{N-s} \sum_{j=1}^s \sum_{k=1}^s (-1)^j2^{-j}Q_{j-1}2^{s-k-j} \binom{N}{k+j} \binom{N-k-j}{s-k-j} \binom{k+j}{k} (-1)^k2^{-k}Q_{k-1} \\
 &= m^N + \sum_{j=1}^N \binom{N}{j}2^{j+1}m^{N-j}(-1)^{j-1}4^{-j}Q_{j-1} \\
 &+ \sum_{j=1}^N \sum_{k=1}^N (-1)^j2^{-j}Q_{j-1} \binom{N}{k+j} m^{N-k-j} \binom{k+j}{k} (-1)^k2^{-k}Q_{k-1} \\
 &= m^N - 2m^N \sum_{j=1}^N \binom{N}{j} (-1)^j(2m)^{-j}Q_{j-1} + m^N \sum_{j=1}^N \sum_{k=1}^N \binom{N}{k+j} \binom{k+j}{k} (-1)^{k+j}(2m)^{-k-j}Q_{j-1}Q_{k-1}.
 \end{aligned}$$

Let us summarize what we just found:

$$\begin{aligned}
 N![z^N]e^{z(m-2)}\left(\sum_n \frac{z^n}{n!}G'_n(1)\right)^2 &= m^N - 2m^N \sum_{j=1}^N \binom{N}{j} (-1)^j(2m)^{-j}Q_{j-1} \\
 &+ m^N \sum_{j=1}^N \sum_{k=1}^N \binom{N}{k+j} \binom{k+j}{k} (-1)^{k+j}(2m)^{-k-j}Q_{j-1}Q_{k-1}.
 \end{aligned}$$

This must be multiplied by $m(m-1)$ and divided by m^N , for the first component of the second factorial moment:

$$m(m-1) - 2m(m-1) \sum_{j=1}^N \binom{N}{j} (-1)^j(2m)^{-j}Q_{j-1} + m(m-1) \sum_{k=1}^N \binom{N}{k} (-1)^k(2m)^{-k} \sum_{j=1}^{k-1} \binom{k}{j} Q_{j-1}Q_{k-1-j}.$$

The second contribution comes from here:

$$me^{z(m-1)}\left(\sum_n \frac{z^n}{n!}G''_n(1)\right).$$

We read off coefficients:

$$\begin{aligned}
 N![z^N]e^{z(m-1)}\left(\sum_n \frac{z^n}{n!}G''_n(1)\right) &= \sum_n \binom{N}{n}(m-1)^{N-n} \sum_{k=1}^n \binom{n}{k} (-1)^k2^{1-k}Q_{k-1}(T_{k-1} - 1) \\
 &= \sum_n (m-1)^{N-n} \sum_{k=1}^n \binom{N}{k} \binom{N-k}{n-k} (-1)^k2^{1-k}Q_{k-1}(T_{k-1} - 1) \\
 &= \sum_{k=1}^N \binom{N}{k} m^{N-k} (-1)^k2^{1-k}Q_{k-1}(T_{k-1} - 1) \\
 &= 2m^N \sum_{k=1}^N \binom{N}{k} (-1)^k(2m)^{-k}Q_{k-1}(T_{k-1} - 1).
 \end{aligned}$$

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Combining everything, we find the second factorial moment:

$$m(m-1) - 2m(m-1) \sum_{j=1}^N \binom{N}{j} (-1)^j (2m)^{-j} Q_{j-1} \\ + m(m-1) \sum_{k=1}^N \binom{N}{k} (-1)^k (2m)^{-k} \sum_{j=1}^{k-1} \binom{k}{j} Q_{j-1} Q_{k-1-j} + 2m \sum_{k=1}^N \binom{N}{k} (-1)^k (2m)^{-k} Q_{k-1} (T_{k-1} - 1).$$

Theorem 2. *The first and second factorial moments have the following explicit expressions:*

$$E_N = m - m \sum_{k=1}^N \binom{N}{k} (-1)^k (2m)^{-k} Q_{k-1}, \\ E_N^{(2)} = m(m-1) - 2m^2 \sum_{j=1}^N \binom{N}{j} (-1)^j (2m)^{-j} Q_{j-1} \\ + m(m-1) \sum_{k=1}^N \binom{N}{k} (-1)^k (2m)^{-k} \sum_{j=1}^{k-1} \binom{k}{j} Q_{j-1} Q_{k-1-j} + 2m \sum_{k=1}^N \binom{N}{k} (-1)^k (2m)^{-k} Q_{k-1} T_{k-1}.$$

3. Asymptotics

Our goal (explained in more detail a bit later) is to rewrite an alternating sum as a contour integral:

$$\sum_{k=1}^N \binom{N}{k} (-1)^k f(k) = \frac{1}{2\pi i} \int_e \frac{(-1)^N N!}{z(z-1)\dots(z-N)} f(z) dz.$$

The asymptotic evaluation of the integral can then be achieved via residues. The challenge is here (and also in similar problems of the past) to extend the (discrete) sequence $f(k)$ to an analytic function $f(z)$. This will be done first.

We must first rewrite a function that appears in the second factorial moment. The strategy for that was laid down in [7]. The main challenge is to define

$$\sum_{j=1}^{k-1} \binom{k}{j} Q_{j-1} Q_{k-1-j}$$

when k is allowed to be a complex number.

The computations will use the abbreviation

$$a_{n+1} = (-1)^n 2^{-\binom{n+1}{2}} / Q_n,$$

and the following relations due to Euler [9]:

$$\frac{1}{Q(t)} = \sum_{n \geq 0} \frac{t^n}{2^n Q_n}, \quad Q(t) = \sum_{n \geq 0} a_{n+1} t^n.$$

Using these (partition) identities, we will be able to apply the binomial theorem

$$\sum_{j=1}^{k-1} \binom{k}{j} X^j = (1+X)^k - 1 - X^k,$$

where X is complicated but explicit. And in this form, the extension $(1+X)^z - 1 - X^z$ is feasible.

Following this strategy, we get

$$\psi(N) := \sum_{j=1}^{N-1} \binom{N}{j} Q_{j-1} Q_{N-j-1} \\ = \sum_{j=1}^{N-1} \binom{N}{j} \frac{Q_\infty}{Q(2^{1-j})} \frac{Q_\infty}{Q(2^{1-(N-j)})} \\ = Q_\infty^2 \sum_{j=1}^{N-1} \binom{N}{j} \sum_{s,t \geq 0} \frac{2^{(1-j)s}}{2^s Q_s} \frac{2^{(1-(N-j))t}}{2^t Q_t}$$

$$\begin{aligned}
 &= Q_\infty^2 \sum_{j=1}^{N-1} \binom{N}{j} \sum_{s,t \geq 0} \frac{2^{-js} 2^{-(N-j)t}}{Q_s Q_t} \\
 &= Q_\infty^2 \sum_{s,t \geq 0} \frac{1}{Q_s Q_t} \left((2^{-s} + 2^{-t})^N - 2^{-Ns} - 2^{-Nt} \right) \\
 &= 2Q_\infty^2 \sum_{0 \leq s \leq t} \frac{1}{Q_s Q_t} \left((2^{-s} + 2^{-t})^N - 2^{-Ns} - 2^{-Nt} \right) - Q_\infty^2 \sum_{0 \leq s} \frac{1}{Q_s^2} \left((2^{1-s})^N - 2^{1-Ns} \right) \\
 &= 2Q_\infty^2 \sum_{0 \leq s \leq t} \frac{2^{-sN}}{Q_s Q_t} \left((1 + 2^{-(t-s)})^N - 1 - 2^{-N(t-s)} \right) - (2^N - 2) Q_\infty^2 \sum_{0 \leq s} \frac{1}{Q_s^2} (2^{-s})^N \\
 &= 2Q_\infty^2 \sum_{s,h \geq 0} \frac{2^{-sN}}{Q_s Q_{s+h}} \left((1 + 2^{-h})^N - 1 - 2^{-hN} \right) - (2^N - 2) Q_\infty^2 \sum_{s \geq 0} \frac{1}{Q_s^2} 2^{-Ns}.
 \end{aligned}$$

Here, we can replace N by an arbitrary value:

$$\psi(z) = 2Q_\infty^2 \sum_{s,h \geq 0} \frac{2^{-sz}}{Q_s Q_{s+h}} \left((1 + 2^{-h})^z - 1 - 2^{-hz} \right) - (2^z - 2) Q_\infty^2 \sum_{s \geq 0} \frac{1}{Q_s^2} 2^{-zs}.$$

However, we will rework this expression, so that it fits our needs. In particular, we need the expansion of $\psi(z)$ around $z = 0$, which is not obvious from the present representation, as a naive approach to get it would result in divergent series. The following computations belong to the realm of q -series (basic hypergeometric series) manipulations. The goal is to make the series in question converge very fast. Again, we decided to present the computations in full since even the present author needed quite some time to do them, and not all steps are completely straight-forward.

$$\begin{aligned}
 Q_\infty^2 \sum_{s \geq 0} \frac{1}{Q_s^2} 2^{-zs} &= Q_\infty \sum_{s \geq 0} \frac{Q(2^{-s})}{Q_s} 2^{-zs} = Q_\infty \sum_{s \geq 0} \frac{1}{Q_s} 2^{-zs} \sum_{n \geq 0} a_{n+1} 2^{-sn} \\
 &= Q_\infty \sum_{n \geq 0} a_{n+1} \sum_{s \geq 0} \frac{2^{-(z+n)s}}{Q_s} = Q_\infty \sum_{n \geq 0} a_{n+1} \frac{1}{Q(2^{1-z-n})} = \sum_{n \geq 0} a_{n+1} Q_{z+n-1}.
 \end{aligned}$$

Further,

$$\begin{aligned}
 Q_\infty^2 \sum_{s,h \geq 0} \frac{2^{-sz}}{Q_s Q_{s+h}} \left((1 + 2^{-h})^z - 1 \right) &= Q_\infty^2 \sum_{s,h \geq 0} \frac{2^{-sz}}{Q_s Q_{s+h}} \sum_{\lambda \geq 1} \binom{z}{\lambda} 2^{-h\lambda} \\
 &= Q_\infty \sum_{s,h \geq 0} \frac{2^{-sz}}{Q_s} \sum_{\lambda \geq 1} \binom{z}{\lambda} 2^{-h\lambda} Q(2^{-s-h}) \\
 &= Q_\infty \sum_{s,h \geq 0} \frac{2^{-sz}}{Q_s} \sum_{\lambda \geq 1} \binom{z}{\lambda} 2^{-h\lambda} \sum_{n \geq 0} a_{n+1} 2^{-n(s+h)} \\
 &= Q_\infty \sum_{s \geq 0} \frac{2^{-sz}}{Q_s} \sum_{\lambda \geq 1} \binom{z}{\lambda} \sum_{n \geq 0} a_{n+1} 2^{-ns} \sum_{h \geq 0} 2^{-h\lambda} 2^{-nh} \\
 &= Q_\infty \sum_{\lambda \geq 1} \binom{z}{\lambda} \sum_{n \geq 0} a_{n+1} \sum_{s \geq 0} \frac{2^{-s(n+z)}}{Q_s} \frac{1}{1 - 2^{-\lambda-n}} \\
 &= Q_\infty \sum_{\lambda \geq 1} \binom{z}{\lambda} \sum_{n \geq 0} a_{n+1} \frac{1}{Q(2^{1-n-z})} \frac{1}{1 - 2^{-\lambda-n}} \\
 &= \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \sum_{\lambda \geq 1} \binom{z}{\lambda} \frac{1}{1 - 2^{-\lambda-n}}.
 \end{aligned}$$

Lastly,

$$\begin{aligned}
 -Q_\infty^2 \sum_{s,h \geq 0} \frac{2^{-sz}}{Q_s Q_{s+h}} 2^{-hz} &= -Q_\infty^2 \sum_{s,h \geq 0} \frac{2^{-(s+h)z}}{Q_s Q_{s+h}} = -Q_\infty \sum_{s,h \geq 0} \frac{2^{-(s+h)z}}{Q_s} Q(2^{-s-h}) \\
 &= -Q_\infty \sum_{s,h \geq 0} \frac{2^{-(s+h)z}}{Q_s} \sum_{n \geq 0} a_{n+1} 2^{-(s+h)n}
 \end{aligned}$$

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$$\begin{aligned}
 &= -Q_\infty \sum_{s \geq 0} \frac{2^{-sz}}{Q_s} \sum_{n \geq 0} a_{n+1} 2^{-sn} \sum_{h \geq 0} 2^{-h(z+n)} \\
 &= -Q_\infty \sum_{s \geq 0} \sum_{n \geq 0} \frac{2^{-s(z+n)}}{Q_s} a_{n+1} \frac{1}{1 - 2^{-z-n}} \\
 &= -Q_\infty \sum_{n \geq 0} \frac{1}{Q(2^{1-z-n})} a_{n+1} \frac{1}{1 - 2^{-z-n}} \\
 &= - \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \frac{1}{1 - 2^{-z-n}}.
 \end{aligned}$$

Putting everything together, we find

$$\begin{aligned}
 \psi(z) &= 2 \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \sum_{\lambda \geq 1} \binom{z}{\lambda} \frac{1}{1 - 2^{-\lambda-n}} \\
 &\quad - 2 \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \frac{1}{1 - 2^{-z-n}} - (2^z - 2) \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \\
 &= 2 \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \sum_{\lambda \geq 1} \binom{z}{\lambda} \left[1 + \frac{1}{2^{\lambda+n} - 1} \right] - 2 \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \frac{1}{1 - 2^{-z-n}} - (2^z - 2) \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \\
 &= 2(2^z - 1) \sum_{n \geq 0} a_{n+1} Q_{z+n-1} + 2 \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \sum_{\lambda \geq 1} \binom{z}{\lambda} \frac{1}{2^{\lambda+n} - 1} \\
 &\quad - 2 \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \frac{1}{1 - 2^{-z-n}} - (2^z - 2) \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \\
 &= 2 \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \sum_{\lambda \geq 1} \binom{z}{\lambda} \frac{1}{2^{\lambda+n} - 1} - 2 \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \frac{1}{1 - 2^{-z-n}} + 2^z \sum_{n \geq 0} a_{n+1} Q_{z+n-1}.
 \end{aligned}$$

It was already computed in [7] that

$$\sum_{n \geq 1} a_{n+1} Q_{n-1} - 2 \sum_{n \geq 1} a_{n+1} Q_{n-1} \frac{1}{1 - 2^{-n}} = \alpha + \beta = \sum_{j \geq 1} \frac{2^j}{(2^j - 1)^2},$$

with

$$\alpha = \sum_{j \geq 1} \frac{1}{2^j - 1} \quad \text{and} \quad \beta = \sum_{j \geq 1} \frac{1}{(2^j - 1)^2}.$$

However, for the convenience of the reader, we will reproduce this here, also, because the presentation in [7] is very brief. We start with

$$\begin{aligned}
 \sum_{n \geq 1} a_{n+1} Q_{n-1} &= Q(1) \sum_{n \geq 1} a_{n+1} \frac{1}{Q(2^{1-n})} = Q(1) \sum_{n \geq 1} a_{n+1} \sum_{m \geq 0} \frac{2^{-nm}}{Q_m} \\
 &= \sum_{m \geq 0} Q(2^{-m}) \sum_{n \geq 1} a_{n+1} 2^{-nm} = \sum_{m \geq 0} Q(2^{-m})(Q(2^{-m}) - 1).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \sum_{n \geq 1} a_{n+1} Q_{n-1} \frac{1}{1 - 2^{-n}} &= Q(1) \sum_{n \geq 1} a_{n+1} \frac{1}{Q(2^{1-n})} \frac{1}{1 - 2^{-n}} = Q(1) \sum_{n \geq 1} a_{n+1} \sum_{m \geq 0} \frac{2^{-nm}}{Q_m} \sum_{j \geq 0} 2^{-nj} \\
 &= \sum_{j \geq 0} \sum_{m \geq 0} Q(2^{-m}) \sum_{n \geq 1} a_{n+1} 2^{-n(m+j)} = \sum_{j \geq 0} \sum_{m \geq 0} Q(2^{-m})(Q(2^{-m-j}) - 1).
 \end{aligned}$$

Thus

$$\begin{aligned}
 \sum_{n \geq 1} a_{n+1} Q_{n-1} - 2 \sum_{n \geq 1} a_{n+1} Q_{n-1} \frac{1}{1 - 2^{-n}} \\
 = \sum_{m \geq 0} Q(2^{-m})(Q(2^{-m}) - 1) - 2 \sum_{j \geq 0} \sum_{m \geq 0} Q(2^{-m})(Q(2^{-m-j}) - 1)
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{m \geq 0} (Q(2^{-m}) - 1)(Q(2^{-m}) - 1) - 2 \sum_{j \geq 0} \sum_{m \geq 0} (Q(2^{-m}) - 1)(Q(2^{-m-j}) - 1) \\
 &\quad + \sum_{m \geq 0} (Q(2^{-m}) - 1) - 2 \sum_{j \geq 0} \sum_{m \geq 0} (Q(2^{-m-j}) - 1) \\
 &= - \left(\sum_{m \geq 0} (Q(2^{-m}) - 1) \right)^2 - \sum_{m \geq 0} (2m + 1)(Q(2^{-m}) - 1) \\
 &= - \left(\sum_{n \geq 1} a_{n+1} \sum_{m \geq 0} 2^{-mn} \right)^2 - \sum_{n \geq 1} a_{n+1} \sum_{m \geq 0} 2^{-mn} - 2 \sum_{n \geq 1} a_{n+1} \sum_{m \geq 0} m 2^{-mn} \\
 &= - \left(\sum_{n \geq 1} a_{n+1} \frac{1}{1 - 2^{-n}} \right)^2 - \sum_{n \geq 1} a_{n+1} \frac{1}{1 - 2^{-n}} - 2 \sum_{n \geq 1} a_{n+1} \frac{2^{-n}}{(1 - 2^{-n})^2}.
 \end{aligned}$$

The following formula is obtained by partial fraction decomposition:

$$\frac{Q(1)}{Q(t)} = \sum_{n \geq 1} a_n \frac{1}{1 - \frac{t}{2^n}} = \sum_{n \geq 1} a_{n+1} \frac{1}{1 - \frac{t}{2^{n+1}}} + \frac{1}{1 - \frac{t}{2}}.$$

Therefore

$$\begin{aligned}
 \sum_{n \geq 1} a_{n+1} \frac{1}{1 - 2^{-n}} &= \lim_{t \rightarrow 2} \left(\frac{Q(1)}{Q(t)} - \frac{1}{1 - \frac{t}{2}} \right) = \lim_{t \rightarrow 2} \frac{1}{1 - \frac{t}{2}} \left(\frac{Q(1)}{Q(\frac{t}{2})} - 1 \right) \\
 &= \lim_{t \rightarrow 1} \frac{1}{1 - t} \left(\frac{Q(1)}{Q(t)} - 1 \right) = - \left(\frac{Q(1)}{Q(t)} \right)' \Big|_{t=1} = -\alpha.
 \end{aligned}$$

By differentiation,

$$\left(\frac{Q(1)}{Q(t)} \right)' = \sum_{n \geq 1} a_n \frac{\frac{1}{2^n}}{\left(1 - \frac{t}{2^n}\right)^2} = \sum_{n \geq 1} a_{n+1} \frac{\frac{1}{2^{n+1}}}{\left(1 - \frac{t}{2^{n+1}}\right)^2} + \frac{\frac{1}{2}}{\left(1 - \frac{t}{2}\right)^2}.$$

Therefore

$$2 \left(\frac{Q(1)}{Q(t)} \right)' - \frac{1}{\left(1 - \frac{t}{2}\right)^2} = \sum_{n \geq 1} a_{n+1} \frac{\frac{1}{2^n}}{\left(1 - \frac{t}{2^{n+1}}\right)^2}.$$

So

$$\begin{aligned}
 \sum_{n \geq 1} a_{n+1} \frac{2^{-n}}{(1 - 2^{-n})^2} &= \lim_{t \rightarrow 2} \left[2 \left(\frac{Q(1)}{Q(t)} \right)' - \frac{1}{\left(1 - \frac{t}{2}\right)^2} \right] \\
 &= 2 \lim_{t \rightarrow 2} \left[\frac{Q(1)}{Q(t)} - \frac{1}{1 - \frac{t}{2}} \right]' = - \frac{1}{2} \left(\frac{Q(1)}{Q(t)} \right)'' \Big|_{t=1} = - \frac{\alpha^2 + \beta}{2}.
 \end{aligned}$$

Putting things together,

$$\begin{aligned}
 \sum_{n \geq 1} a_{n+1} Q_{n-1} - 2 \sum_{n \geq 1} a_{n+1} Q_{n-1} \frac{1}{1 - 2^{-n}} &= - \left(\sum_{n \geq 1} a_{n+1} \frac{1}{1 - 2^{-n}} \right)^2 - \sum_{n \geq 1} a_{n+1} \frac{1}{1 - 2^{-n}} - 2 \sum_{n \geq 1} a_{n+1} \frac{2^{-n}}{(1 - 2^{-n})^2} \\
 &= -\alpha^2 + \alpha + \alpha^2 + \beta = \alpha + \beta.
 \end{aligned}$$

Now we need a few expansions that are elementary:

$$Q(x) \sim Q(1) \left[1 - \alpha(x - 1) + \frac{\alpha^2 - \beta}{2} (x - 1)^2 \right], \quad x \rightarrow 1,$$

and thus

$$Q(2^{-z}) \sim Q(1) \left[1 - \alpha(2^{-z} - 1) + \frac{\alpha^2 - \beta}{2} (2^{-z} - 1)^2 \right] \sim Q(1) \left[1 + L\alpha z + \frac{L^2}{2} (\alpha^2 - \alpha - \beta) z^2 \right].$$

Therefore

$$Q_z \sim 1 - L\alpha z + \frac{L^2}{2} (\alpha^2 + \alpha + \beta) z^2$$

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and

$$Q_{z-1} = \frac{Q_z}{1-2^{-z}} \sim \left[1 - L\alpha z + \frac{L^2}{2}(\alpha^2 + \alpha + \beta)z^2 \right] \frac{1}{1-2^{-z}} \sim \frac{1}{Lz} + \frac{1}{2} - \alpha + \frac{L}{2} \left(\alpha^2 + \beta + \frac{1}{6} \right) z.$$

Now we can expand the function $\psi(z)$ around $z = 0$:

$$\begin{aligned} \psi(z) &= 2 \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \sum_{\lambda \geq 1} \binom{z}{\lambda} \frac{1}{2^{\lambda+n} - 1} - 2 \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \frac{1}{1-2^{-z-n}} + 2^z \sum_{n \geq 0} a_{n+1} Q_{z+n-1} \\ &= 2Q_{z-1} \sum_{\lambda \geq 1} \frac{z}{\lambda} \binom{z-1}{\lambda-1} \frac{1}{2^{\lambda+n} - 1} + 2 \sum_{n \geq 1} a_{n+1} Q_{z+n-1} \sum_{\lambda \geq 1} \binom{z}{\lambda} \frac{1}{2^{\lambda+n} - 1} \\ &\quad - 2Q_{z-1} \frac{1}{1-2^{-z}} - 2 \sum_{n \geq 1} a_{n+1} Q_{z+n-1} \frac{1}{1-2^{-z-n}} + 2^z Q_{z-1} + 2^z \sum_{n \geq 1} a_{n+1} Q_{z+n-1} \\ &\sim 2 \frac{1}{Lz} \sum_{\lambda \geq 1} \frac{z}{\lambda} \binom{z-1}{\lambda-1} \frac{1}{2^\lambda - 1} + O(z) - 2 \left[\frac{1}{Lz} + \frac{1}{2} - \alpha + \frac{L}{2} \left(\alpha^2 + \beta + \frac{1}{6} \right) z \right] \left[\frac{1}{Lz} + \frac{1}{2} + \frac{Lz}{12} \right] \\ &\quad - 2 \sum_{n \geq 1} a_{n+1} Q_{n-1} \frac{1}{1-2^{-n}} + (1+Lz) \left[\frac{1}{Lz} + \frac{1}{2} - \alpha \right] + \sum_{n \geq 1} a_{n+1} Q_{n-1} \\ &\sim \frac{2}{L} \sum_{\lambda \geq 1} \frac{1}{\lambda} (-1)^{\lambda-1} \frac{1}{2^\lambda - 1} - 2 \left[\frac{1}{Lz} + \frac{1}{2} - \alpha + \frac{L}{2} \left(\alpha^2 + \beta + \frac{1}{6} \right) z \right] \left[\frac{1}{Lz} + \frac{1}{2} + \frac{Lz}{12} \right] \\ &\quad + (1+Lz) \left[\frac{1}{Lz} + \frac{1}{2} - \alpha \right] + \alpha + \beta \\ &\sim -\frac{2}{L^2 z^2} + \frac{2\alpha - 1}{Lz} - \alpha^2 + \alpha + \frac{2}{3} + \frac{2\tau}{L}, \end{aligned}$$

where we use

$$\tau := \sum_{j \geq 1} \frac{(-1)^{j-1}}{j(2^j - 1)}.$$

After these preparations, we can engage into the asymptotics, using Rice’s method. This method has been described in [8]. We briefly summarize what we need:

An alternating sum can be written as a contour integral:

$$\sum_{k=1}^N \binom{N}{k} (-1)^k f(k) = \frac{1}{2\pi i} \int_{\mathfrak{e}} \frac{(-1)^N N!}{z(z-1)\dots(z-N)} f(z) dz.$$

Here, the positively oriented curve \mathfrak{e} enclosed the poles $1, 2, \dots, N$, and no others. This formula follows from simple residue calculations. Note also that

$$\frac{(-1)^N N!}{z(z-1)\dots(z-N)} = -\frac{\Gamma(N+1)\Gamma(-z)}{\Gamma(N+1-z)}.$$

Extending the curve of integration, we encounter extra residues; in order to keep the formula correct, these residues must be subtracted. They give us the terms of the asymptotic expansion of interest. There is in all our examples a pole at $z = 0$, and it will give us the dominant contribution. From a term $2^z - 1$ in the denominator, we will also have poles at $\chi_k := \frac{2\pi i k}{L}$, and the contributions establish the (Fourier series of the) periodic oscillation in the asymptotic expansion.

Neglecting these (usually tiny) oscillations, we can write in a suggestive way:

$$\sum_{k=1}^N \binom{N}{k} (-1)^k f(k) \sim \text{Res}_{z=0} \frac{\Gamma(N+1)\Gamma(-z)}{\Gamma(N+1-z)} f(z).$$

Now we want to apply this to the expected value

$$m - m \sum_{k=1}^N \binom{N}{k} (-1)^k (2m)^{-k} Q_{k-1}.$$

Here, we need this function:

$$f(z) = -(2m)^{-z} Q_{z-1}.$$

The residue calculations can, after our preliminaries, be done by a computer. In papers written in the eighties, such computations would have been done by humans, but once all the relevant functions are expanded to a sufficient number of terms, there isn't anything that we can learn. These expansion we provided, and the other ones Maple "knows". This is in sharp contrast to the (long) computations in earlier parts of this paper; they cannot be done by a computer, at least not to the knowledge of the present author.

The result is:

$$\log_2 N - \log_2 m - \frac{1}{2} - \alpha + \frac{\gamma}{L}.$$

Altogether:

$$E_N \sim m \left[\log_2 N - \log_2 m + \frac{1}{2} - \alpha + \frac{\gamma}{L} - \frac{1}{L} \sum_{k \neq 0} \Gamma(\chi_k) e^{-2\pi i k \log_2 N} \right].$$

We do not show the residue calculation at $z = \chi_k$ but it is much easier, since there is only a simple pole.

Now we turn to the second factorial moment.

We need one more preparation:

$$T_{z-1} = \alpha - \frac{1}{2^z - 1} - \sum_{j \geq 1} \frac{1}{2^{j+z} - 1} \sim -\frac{1}{Lz} + \frac{1}{2} - \frac{Lz}{12} + L(\alpha + \beta)z.$$

Now the residue calculation at $z = 0$ can be done by a computer. There are 3 sums, that can be done individually. We do not display them, since they are quite long, each of type $A_1 \log^2 n + A_2 \log n + A_3$, and eventually there are many cancellations when one computes the variance. Recall that the variance is computed as second factorial moment plus expectation minus expectation squared. We find:

$$m - m\alpha - \frac{11}{12}m^2 - \frac{2m(m-1)\tau}{L} - m\beta + \frac{m^2\pi^2}{6L^2}.$$

This is the asymptotic main term of the variance, apart from the periodic component that we did not compute. Note carefully that the square of the expectation contains the square of a periodic function, which is again periodic, but does *not* have mean value zero. This mean value of $\delta^2(x)$ is a tiny quantity, but it is often very important, as explained for instance in [10]. Taking it into account, we find for the variance

$$m - m\alpha - \frac{11}{12}m^2 - \frac{2m(m-1)\tau}{L} - m\beta + \frac{m^2\pi^2}{6L^2} - m^2 \frac{1}{L^2} \sum_{k \neq 0} \Gamma(-\chi_k) \Gamma(\chi_k) + \delta_V(\log_2 n).$$

The coefficient of m^2 is

$$\frac{\pi^2}{6L^2} - \frac{2\tau}{L} - \frac{11}{12} - \frac{1}{L^2} \sum_{k \neq 0} \Gamma(-\chi_k) \Gamma(\chi_k).$$

But this quantity is exactly equal to zero, as explained in the survey article [11]. To wet the reader's appetite, this surprising identity is related to the identity for *Dedekind's η -function*.

This reduces the variance to

$$m - m\alpha - m\beta + \frac{2m\tau}{L} + \delta_V(\log_2 n).$$

In [12] we find another handy formula: set

$$h(x) = \sum_{k \geq 1} \frac{e^{kx}}{(e^{kx} - 1)^2},$$

then

$$h(x) = \frac{\pi^2}{6x^2} - \frac{1}{2x} + \frac{1}{24} - \frac{2\pi^2}{x^2} h\left(\frac{4\pi^2}{x}\right).$$

For $x = L$ this gives

$$\alpha + \beta = \frac{\pi^2}{6L^2} - \frac{1}{2L} + \frac{1}{24} - \frac{2\pi^2}{L^2} h\left(\frac{4\pi^2}{L}\right).$$

Using this and the previous formula, we find that the coefficient of m is

$$\frac{1}{2L} + \frac{1}{24} + \frac{2\pi^2}{L^2} h\left(\frac{4\pi^2}{L}\right) - \frac{1}{L^2} \sum_{k \neq 0} \Gamma(-\chi_k) \Gamma(\chi_k),$$

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which is numerically very close to

$$\frac{1}{2L} + \frac{1}{24}.$$

4. Conclusion

This was a first step towards a precise analysis of the model of m counters. It is possible that methods developed by Guy Louchard, as described in [13] might be suitable to derive *all* moments asymptotically.

Also, it might be possible that methods developed recently by Hwang and coauthors [14] provide a faster access to the variance.

The answer to these assumptions lies in the (near) future.

Added during the revision, January 2012. Such computations have indeed been started already and will hopefully appear soon. As expected, they lead to a faster asymptotic evaluation, but do not lead to explicit forms as obtained in this paper.

Acknowledgements

The insightful comments of two referees are gratefully acknowledged.

References

- [1] P. Flajolet, Approximate counting: a detailed analysis, *BIT* 25 (1985) 113–134.
- [2] P. Kirschenhofer, H. Prodinger, Approximate counting: an alternative approach, *RAIRO Theoretical Informatics and Applications* 25 (1991) 43–48.
- [3] H. Prodinger, Approximate counting via Euler transform, *Mathematica Slovaca* 44 (1994) 569–574.
- [4] H. Prodinger, Hypothetic analyses: approximate counting in the style of Knuth, path length in the style of Flajolet, *Theoretical Computer Science* 100 (1992) 243–251.
- [5] G. Louchard, H. Prodinger, Generalized approximate counting revisited, *Theoretical Computer Science* 391 (2008) 109–125.
- [6] J. Cichoń, W. Macyna, Approximate counters for flash memory, in: 17th IEEE International Conference on Embedded and Real-Time Computing Systems and Applications (RTCSA), Toyama, Japan.
- [7] P. Kirschenhofer, H. Prodinger, W. Szpankowski, Digital search trees again revisited: the internal path length perspective, *SIAM Journal on Computing* 23 (1994) 598–616.
- [8] P. Flajolet, R. Sedgewick, Mellin transforms and asymptotics: Finite differences and Rice's integrals, *Theoretical Computer Science* 144 (1995) 101–124.
- [9] G. Andrews, The theory of partitions, in: *Encyclopedia of Mathematics and its Applications*, vol. 2, Addison-Wesley, 1976.
- [10] H. Prodinger, Compositions and Patricia tries: no fluctuations in the variance!, in: L. Arge, G. Italiano, R. Sedgewick (Eds.), *Proceedings of the 6th Workshop on ALENEX and the 1st Workshop on ANALCO, 2004*, pp. 211–215.
- [11] H. Prodinger, Periodic oscillations in the analysis of algorithms—and their cancellations, *Journal of the Iranian Statistical Society* 3 (2004) 251–270.
- [12] P. Kirschenhofer, H. Prodinger, J. Schoissengeier, Zur Auswertung gewisser numerischer Reihen mit Hilfe modularer Funktionen, in: *Lecture Notes in Mathematics*, vol. 1262, 1987, pp. 108–110.
- [13] G. Louchard, H. Prodinger, Asymptotics of the moments of extreme-value related distribution functions, *Algorithmica* 46 (2006) 431–467. long version: <http://www.ulb.ac.be/di/mcs/louchard/moml.ps>.
- [14] H.-K. Hwang, M. Fuchs, V. Zacharovas, Asymptotic variance of random symmetric digital search trees, *Discrete Mathematics and Theoretical Computer Science* 12 (2010) 103–166.

AofA'12

DMTCS proc. **AQ**, 2012, 13–28

Approximate Counting via the Poisson-Laplace-Mellin Method

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Approximate counting is an algorithm that provides a count of a huge number of objects within an error tolerance. The first detailed analysis of this algorithm was given by Flajolet. In this paper, we propose a new analysis via the Poisson-Laplace-Mellin approach, a method devised for analyzing shape parameters of digital search trees in a recent paper of Hwang et al. Our approach yields a different and more compact expression for the periodic function from the asymptotic expansion of the variance. We show directly that our expression coincides with the one obtained by Flajolet. Moreover, we apply our method to variations of approximate counting, too.

Keywords: approximate counting, digital search tree, JS-admissibility, Laplace transform, Mellin transform

1 Introduction and Results

Approximate counting, an algorithm proposed by Morris [22] in 1978, is used for counting within a certain error tolerance a huge amount of objects with very limited space. The algorithm has found many applications such as in the analysis of the Webgraph, monitoring network traffic, finding patterns in protein and DNA sequencing, computing frequency moments of data streams, data storage in flash memory, and many variants and improvements have been proposed; see Csűrös [7], Mitchell and Day [21], Gronemeier and Sauerhoff [12], Aspnes and Censor [2], Cichoń and Macyna [4] and references therein.

Here, we are going to revisit the analysis of the classical algorithm which is described as follows: a counter C_n is maintained with initial value $C_0 = 0$. After “counting n objects”, a random decision based only on the current content of the counter determines whether or not the counter should be increased when “counting the $(n + 1)$ -st object”. More precisely, the counter obeys the following rule

$$C_{n+1} = \begin{cases} C_n + 1, & \text{with probability } q^{C_n}; \\ C_n, & \text{with probability } 1 - q^{C_n}, \end{cases} \quad (1)$$

where $0 < q < 1$ is fixed. Hence, $(C_n)_{n \geq 0}$ is a Markov chain describing a pure birth process. The same chain was also encountered in a couple of other problems: width of greedy decomposition of random

[†]Partially supported by National Science Council of ROC under the grant NSC-99-2115-M-009-007-MY2.

[‡]Partially supported by an incentive grant from the NRF (South Africa).

acyclic digraphs into node-disjoint paths (see Simon [30]), size of greedy independent set and greedy clique in random graphs (see Simon [30]) and length of the leftmost path in digital search trees (see below).

We mention in passing that many variants of the above Markov chain have been investigated as well; e.g. see Crippa and Simon [6], Louchard and Prodinger [20], Bertoin, Biane and Yor [3] and Guillemin, Robert and Zwart [13]. Applications range from Computer Science over Particle Physics to Molecular Biology; see the detailed discussion in [6].

As for the classical chain C_n , the first detailed analysis was given by Flajolet in [8] who used Mellin transform (see also Prodinger [24] for a similar analysis). Other approaches have been given by Kirschenhofer and Prodinger [17] via Rice method, Prodinger [25] via Euler transform, Louchard and Prodinger [19] via analysis of extreme value distributions, Rosenkrantz [29] via martingale theory and Robert [28] via probabilistic tools. In this paper, we are going to propose a new approach which will be based on the connection with digital search trees and the new method (nicknamed Poisson-Laplace-Mellin method) for analyzing shape parameters in digital search trees proposed in Hwang, Fuchs and Zacharovas [14].

We next explain the connection between approximate counting and digital search trees (DSTs). Therefore, we start with the definition of DSTs which are a fundamental data structure in computer science and were proposed by Coffman and Eve in [5]. Consider a set of n keys which are infinite 0-1 strings. The digital search tree is built from these n keys as follows: the first key is placed in the root; all other keys are directed to the left or right subtree according to whether the first bit is 0 or 1, respectively; finally, the first bits are removed and the subtrees are built recursively according to the same principle; see Figure 1 for an example.

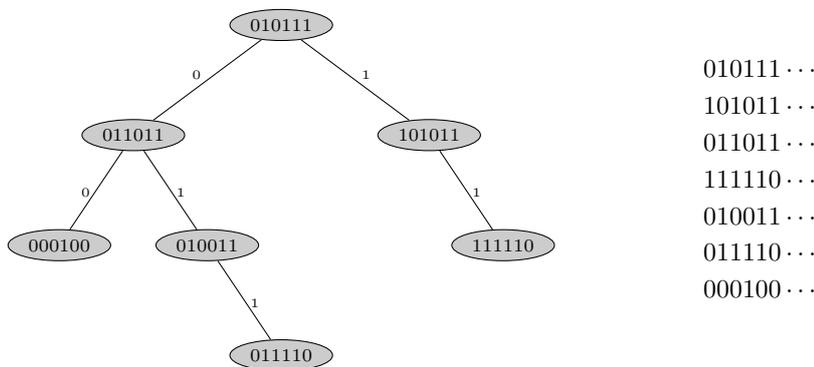


Fig. 1: A DST built from 7 keys with length of leftmost path equal to 3.

Shape parameters in random DSTs have been analyzed in many papers; see [14] and references therein. The standard random model used in such an analysis is the *Bernoulli model*. Here, the bits of the keys are assumed to be i.i.d. Bernoulli random variables with the probability of 0 being q . Shape parameters of random DSTs of size n then become random variables. One example of such a shape parameter is the length of the leftmost path which is defined as the number of vertices on the leftmost path from the root to the leftmost leaf. We denote this length in a random digital search tree of size n by X_n . Obviously, X_n

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satisfies the following distributional recurrence

$$X_{n+1} \stackrel{d}{=} X_{B_n} + 1, \quad (n \geq 0) \quad (2)$$

with $X_0 = 0$ and $B_n \stackrel{d}{=} \text{Binom}(n, q)$. This recurrence just reflects the trivial fact that X_n can be computed by starting from the root (which counts as 1) and then moving on to the left subtree (which has size B_n) where the same procedure is repeated. Now, a moment's reflection reveals that C_n is related to X_n as

$$C_n \stackrel{d}{=} X_n.$$

This relation will be the starting point of our analysis. We will use it to derive asymptotic expansions for mean and variance of C_n .

Before stating our result, we explain what is known about C_n . Flajolet in [8] showed that, as $n \rightarrow \infty$,

$$\mathbb{E}(C_n) \sim \log_{1/q} n + F_C(\log_{1/q} n),$$

where $F_C(z) = \sum_k f_k e^{2k\pi iz}$ is a 1-periodic function with Fourier coefficients

$$f_0 = \frac{\gamma}{L} + \frac{1}{2} - \alpha, \quad f_k = -\frac{\Gamma(-\chi_k)}{L} \quad (k \neq 0).$$

Here, γ is Euler's constant, $\alpha = \sum_{l \geq 1} q^l / (1 - q^l)$, $L = \log(1/q)$ and $\chi_k = 2k\pi i / L$. As for the variance, he showed that, as $n \rightarrow \infty$,

$$\text{Var}(C_n) \sim G_C(\log_{1/q} n),$$

where $G_C(z) = \sum_k g_k e^{2k\pi iz}$ is again a 1-periodic function with computable Fourier coefficients. Moreover, he gave the following expression for the average value of $G_C(z)$

$$g_0 = \frac{\pi^2}{6L^2} - \alpha - \beta + \frac{1}{12} - \frac{1}{L} \sum_{l \geq 1} \frac{1}{l \sinh(2l\pi^2/L)},$$

where $\beta = \sum_{l \geq 1} q^{2l} / (1 - q^l)^2$.

In the next section, we will use the above connection to DSTs to re-derive these results. Our approach will in particular yield a different and more compact expression for all Fourier coefficients of $G_C(z)$.

Theorem 1 *For the variance of approximate counting, we have, as $n \rightarrow \infty$,*

$$\text{Var}(C_n) \sim G_C(\log_{1/q} n),$$

where $G_C(z) = \sum_k g_k e^{2k\pi iz}$ is a 1-periodic function with

$$g_k = \frac{Q_\infty}{L\Gamma(1 + \chi_k)} \sum_{h,l,j \geq 0} \frac{(-1)^j q^{h+l+\binom{j+1}{2}}}{Q_h Q_l Q_j} \varphi(\chi_k, q^{h+j} + q^{l+j}).$$

Here, $Q_j = \prod_{l=1}^j (1 - q^l)$, $Q_\infty = \lim_{j \rightarrow \infty} Q_j$ and

$$\varphi(\chi; x) := \begin{cases} \pi(x^\chi - 1) / (\sin(\pi\chi)(x - 1)), & \text{if } x \neq 1, \\ \pi\chi / \sin(\pi\chi), & \text{if } x = 1. \end{cases}$$

Comparing with the above result, we obtain the following identity for which we will provide a direct proof in Section 3.

Corollary 1 *We have,*

$$\frac{Q_\infty}{L} \sum_{h,l,j \geq 0} \frac{(-1)^j q^{h+l+\binom{j+1}{2}}}{Q_h Q_l Q_j} \psi(q^{h+j} + q^{l+j}) = \frac{\pi^2}{6L^2} - \alpha - \beta + \frac{1}{12} - \frac{1}{L} \sum_{l \geq 1} \frac{1}{l \sinh(2l\pi^2/L)},$$

where

$$\psi(x) := \begin{cases} \log x/(x-1), & \text{if } x \neq 1; \\ 1, & \text{if } x = 1. \end{cases}$$

Finally, in Section 4, we will discuss extensions and variations of approximate counting. One such extension was proposed in [4] where instead of one counter m counters $C_n^{(1)}, \dots, C_n^{(m)}$ were used (m fixed). Then, when “counting the $(n+1)$ -st object”, one of the counters is chosen uniformly at random and increased according to the stochastic rule (1). In Prodinger [26], mean and variance of $D_n := C_n^{(1)} + \dots + C_n^{(m)}$ were derived. We will show that our approach greatly simplifies the analysis since the case of m counters can be reduced to the case of one counter.

Theorem 2 *For approximate counting with m counters, we have, as $n \rightarrow \infty$,*

$$\begin{aligned} \mathbb{E}(D_n) &\sim m \log_{1/q}(n/m) + m F_C(\log_{1/q}(n/m)), \\ \text{Var}(D_n) &\sim m G_C(\log_{1/q}(n/m)), \end{aligned}$$

where $F_C(z)$ and $G_C(z)$ are the periodic functions above.

Moreover, again in Section 4, we will show that similar simplifications can also be achieved for shape parameters in m -DSTs trees recently introduced by Prodinger in [27].

2 Analysis of Approximate Counting

Here, we are going to prove Theorem 1. Therefore, we will apply the Poisson-Laplace-Mellin method from [14]. We will first summarize the main steps of this method; for a more detailed discussion together with comparisons with other approaches, the reader is referred to [14] (in particular, see Figure 7 on page 131 in [14] which gives a comparison of the Poisson-Laplace-Mellin approach with a closely related approach of Flajolet and Richmond [10]). The main steps of our method are as follows.

- We first introduce poissonized mean and variance of X_n (or equivalently C_n) and show that they satisfy differential-functional equations of the same type;
- we use Jacquet and Szpankowski’s theory of depoissonization [16] to show that it suffices to prove our claimed result for poissonized mean and variance;
- in order to get rid of the differential operator, we use Laplace transform which then only satisfies a functional equation;
- we use a normalization factor to simplify the functional equation for the Laplace transform;

- applying Mellin transform will allow us to solve the normalized functional equation for the Laplace transform (for an excellent survey on the Mellin transform see Flajolet, Gourdon and Dumas [9]);
- finally, we first use inverse Mellin transform and then inverse Laplace transform to obtain asymptotic expansions for poissonized mean and variance.

After completing these steps, which will work in a similar fashion for both mean and variance, the compact form of the Fourier coefficients for the variance are obtained by some straightforward simplifications.

Poissonization and depoissonization. Our starting point is (2). First, denote the *Poisson generating function* of $\mathbb{E}(e^{X_n y})$ by

$$\tilde{P}(y, z) = e^{-z} \sum_{n \geq 0} \mathbb{E}(e^{X_n y}) \frac{z^n}{n!}.$$

Then, from (2), we obtain

$$\tilde{P}(y, z) + \frac{\partial}{\partial z} \tilde{P}(y, z) = e^y \tilde{P}(y, qz)$$

with $\tilde{P}(y, 0) = \exp(-z)$.

From this, by differentiation with respect to y and setting $y = 0$, we obtain for the Poisson generating functions of the first and second moment of X_n (denoted by $\tilde{f}_1(z)$ and $\tilde{f}_2(z)$, respectively)

$$\begin{aligned} \tilde{f}_1(z) + \tilde{f}_1'(z) &= \tilde{f}_1(qz) + 1, \\ \tilde{f}_2(z) + \tilde{f}_2'(z) &= \tilde{f}_2(qz) + 2\tilde{f}_1(qz) + 1, \end{aligned} \quad (3)$$

with $\tilde{f}_1(0) = \tilde{f}_2(0) = 0$. Moreover, define the *poissonized variance* as $\tilde{V}(z) := \tilde{f}_2(z) - \tilde{f}_1^2(z)$. Then, the above two relations in turn yield

$$\tilde{V}(z) + \tilde{V}'(z) = \tilde{V}(qz) + \tilde{f}_1'(z)^2 \quad (4)$$

with $\tilde{V}(0) = 0$.

We first show that in order to derive asymptotics of $\mathbb{E}(X_n)$ and $\text{Var}(X_n)$, it suffices to analyze $\tilde{f}_1(z)$ and $\tilde{V}(z)$ as $z \rightarrow \infty$, respectively. Therefore, we use the theory of analytic depoissonization due to Jacquet and Szpankowski. Recall that a function $\tilde{f}(z)$ is called JS-admissible if:

- (I) There exist $\alpha, \beta \in \mathbb{R}$ such that uniformly for $|\arg(z)| \leq \epsilon$

$$\tilde{f}(z) = \mathcal{O}(|z|^\alpha (\log_+ |z|)^\beta),$$

where $\log_+ x = \log(1 + x)$.

- (O) Uniformly for $\epsilon \leq \arg(z) \leq \pi$,

$$f(z) := e^z \tilde{f}(z) = \mathcal{O}\left(e^{(1-\epsilon)|z|}\right).$$

(Here and throughout the work, ϵ denotes a small constant whose value might be different from one occurrence to another).

JS-admissibility of given functions is easily checked due to closure properties; see Lemma 2.3 in [14]. Moreover, JS-admissibility of functions which are given by differential-functional equations of the type above is also easily checked due to the following result.

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Proposition 1 Let $\tilde{f}(z)$ and $\tilde{g}(z)$ be entire functions with

$$\tilde{f}(z) + \tilde{f}'(z) = \tilde{f}(qz) + \tilde{g}(z),$$

where $\tilde{f}(0) = 0$. Then,

$$\tilde{f}(z) \text{ is JS-admissible} \iff \tilde{g}(z) \text{ is JS-admissible.}$$

Proof: Similar to the proof of Proposition 2.4 in [14]. \square

Consequently, $\tilde{f}_1(z)$ and $\tilde{f}_2(z)$ are both JS-admissible. Depoissonization (see Proposition 2.2 in [14] and the discussion in the introduction) then yields, as $n \rightarrow \infty$,

$$\mathbb{E}(X_n) \sim \tilde{f}_1(n) \quad \text{and} \quad \text{Var}(X_n) \sim \tilde{V}(n).$$

Thus, we only have to find asymptotics of $\tilde{f}_1(z)$ and $\tilde{V}(z)$.

Analysis of the Mean. Here, we analyze the mean, where we start from (3). Since $\tilde{f}_1(z)$ is JS-admissible, we may apply Laplace transform to get rid of the differential operator. This yields

$$(s+1)\mathcal{L}[\tilde{f}_1; s] = \frac{1}{q}\mathcal{L}[\tilde{f}_1; s/q] + 1/s. \quad (5)$$

Next, we derive an exact expression for the mean. Therefore, we iterate the above functional equation and obtain

$$\mathcal{L}[\tilde{f}_1; s] = \frac{1}{s} \sum_{j \geq 0} \frac{1}{(s+1)(q^{-1}s+1) \cdots (q^{-j}s+1)}.$$

Here, we have used that $\mathcal{L}[\tilde{f}_1; s] = \mathcal{O}(1/s^2)$ as $s \rightarrow \infty$. Next, by partial fraction expansion

$$\frac{1}{(s+1)(q^{-1}s+1) \cdots (q^{-j}s+1)} = \sum_{0 \leq l \leq j} \frac{(-1)^{j-l} q^{\binom{j-l+1}{2}}}{(q^{-l}s+1) Q_l Q_{j-l}}.$$

Plugging this into the expression above yields

$$\begin{aligned} \mathcal{L}[\tilde{f}_1; s] &= \frac{1}{s} \sum_{j \geq 0} \sum_{0 \leq l \leq j} \frac{(-1)^{j-l} q^{\binom{j-l+1}{2}}}{(q^{-l}s+1) Q_l Q_{j-l}} \\ &= \frac{1}{s} \sum_{l \geq 0} \frac{1}{Q_l (q^{-l}s+1)} \sum_{j \geq 0} \frac{(-1)^j q^{\binom{j+1}{2}}}{Q_j} \\ &= \frac{Q_\infty}{s} \sum_{l \geq 0} \frac{1}{Q_l (q^{-l}s+1)}, \end{aligned}$$

where Q_j and Q_∞ have been defined in the introduction and we used the well-known identity

$$\sum_{j \geq 0} \frac{(-1)^j q^{\binom{j+1}{2}}}{Q_j} = Q_\infty.$$

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Now, by inverse Laplace transform,

$$\tilde{f}_1(z) = Q_\infty \sum_{l \geq 0} \frac{1}{Q_l} (1 - e^{-q^l z})$$

and hence

$$\mathbb{E}(X_n) = Q_\infty \sum_{l \geq 0} \frac{1}{Q_l} (1 - (1 - q^l)^n).$$

We record this result for future reference.

Proposition 2 *We have*

$$\mathbb{E}(X_n) = Q_\infty \sum_{l \geq 0} \frac{1}{Q_l} (1 - (1 - q^l)^n).$$

Next, we derive an asymptotic expansion. This will be done by using the Mellin transform. Therefore, set $\bar{\mathcal{L}}[\tilde{f}_1; s] = \mathcal{L}[\tilde{f}_1; s]/Q(-s)$, where

$$Q(-s) = \prod_{i=1}^{\infty} (1 + q^i s).$$

Then, by dividing (5) by $Q(-s/q)$,

$$\bar{\mathcal{L}}[\tilde{f}_1; s] = \frac{1}{q} \bar{\mathcal{L}}[\tilde{f}_1; s/q] + \frac{1}{sQ(-s/q)}.$$

Now, from the fact that $\tilde{f}_1(z)$ is JS-admissible and well-known growth properties of $Q(-s/q)$ (see page 127 in [14]), we obtain suitable polynomial bounds for $\mathcal{L}[\tilde{f}_1; s]$ as s tends both to zero and ∞ . This ensures the existence of the Mellin transform of $\mathcal{L}[\tilde{f}_1; s]$ in a non-trivial strip. Thus, we may apply Mellin transform and obtain

$$\mathcal{M}[\bar{\mathcal{L}}; \omega] = \frac{M_1(\omega)}{1 - q^{\omega-1}}, \quad (\Re(\omega) > 1),$$

where

$$M_1(\omega) = \int_0^\infty \frac{s^{\omega-2}}{Q(-s/q)} ds = \frac{Q(q^{1-\omega})}{Q_\infty} \Gamma(\omega+1) \Gamma(-\omega).$$

Note that the latter function is meromorphic for $\Re(\omega) > 0$ with a simple pole at $\omega = 1$. Moreover, due to rapid decay of the Γ function along vertical lines, we have $M_1(c+it) = O(e^{-\pi|t|})$ for $c > 0$ and $|t|$ large. Hence, inverse Mellin transform implies for $|\arg(s)| \leq \pi - \epsilon$ and $|s| \rightarrow 0$,

$$\bar{\mathcal{L}}[\tilde{f}_1; s] \sim \frac{1}{s} \log_{1/q} \frac{1}{s} + \frac{1}{s} \left(\frac{1}{2} - \alpha + \frac{1}{L} \sum_{k \neq 0} M_1(1 + \chi_k) s^{-\chi_k} \right),$$

where notations are as in the introduction. Since $Q(-s/q) = 1 + \mathcal{O}(s)$ for $|\arg(s)| \leq \pi - \epsilon$ and $|s| \rightarrow 0$, the above in turn yields

$$\mathcal{L}[\tilde{f}_1; s] \sim \frac{1}{s} \log_{1/q} \frac{1}{s} + \frac{1}{s} \left(\frac{1}{2} - \alpha + \frac{1}{L} \sum_{k \neq 0} M_1(1 + \chi_k) s^{-\chi_k} \right).$$

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By Proposition 2.6 of [14], we may apply inverse Laplace transform and obtain for $|\arg(z)| \leq \frac{\pi}{2} - \epsilon$ and $|z| \rightarrow \infty$,

$$\begin{aligned}\tilde{f}_1(z) &\sim \log_{1/q} z + \frac{\gamma}{L} + \frac{1}{2} - \alpha + \frac{1}{L} \sum_{k \neq 0} \frac{M_1(1 + \chi_k)}{\Gamma(1 + \chi_k)} z^{\chi_k} \\ &= \log_{1/q} z + \frac{\gamma}{L} + \frac{1}{2} - \alpha - \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) z^{\chi_k}.\end{aligned}$$

The same asymptotic expansion also holds for $\mathbb{E}(X_n)$ by dePoissonization.

Analysis of the Variance. For an asymptotic expansion of the variance, we start from (4) and proceed by the same method as above. First note that due to the above analysis and Ritt's Theorem (Theorem 4.2 of [23]), we have uniformly for $|\arg(z)| \leq \frac{\pi}{2} - \epsilon$

$$\tilde{f}'_1(z)^2 = \begin{cases} \mathcal{O}(1), & \text{if } |z| \rightarrow 0; \\ \mathcal{O}(|z|^{-2}), & \text{if } |z| \rightarrow \infty. \end{cases} \quad (6)$$

This in turn yields the following rough bounds for $\tilde{V}(z)$

$$\tilde{V}(z) = \begin{cases} \mathcal{O}(z), & \text{if } z \rightarrow 0+; \\ \mathcal{O}(z^\epsilon), & \text{if } z \rightarrow \infty. \end{cases} \quad (7)$$

Therefore, we may apply Laplace transform. Hence,

$$(s+1)\mathcal{L}[\tilde{V}; s] = \frac{1}{q}\mathcal{L}[\tilde{V}; s/q] + \tilde{g}(s),$$

where $\tilde{g}(s) = \mathcal{L}[\tilde{f}'_1{}^2; s]$. Next, set $\bar{\mathcal{L}}[\tilde{V}; s] = \mathcal{L}[\tilde{V}; s]/Q(-s)$. Dividing by $Q(-s/q)$ yields

$$\bar{\mathcal{L}}[\tilde{V}; s] = \frac{1}{q}\bar{\mathcal{L}}[\tilde{V}; s/q] + \tilde{g}(s)/Q(-s/q).$$

Now, from (7) and growth properties of $Q(-s)$, we have

$$\bar{\mathcal{L}}[\tilde{V}; s] = \begin{cases} \mathcal{O}(1/s^{1+\epsilon}), & \text{if } s \rightarrow 0+; \\ \mathcal{O}(1/s^b), & \text{if } s \rightarrow \infty, \end{cases}$$

where $b > 0$ is an arbitrary large constant. Hence, the Mellin transform of $\bar{\mathcal{L}}[\tilde{V}; s]$ exists for $\Re(\omega) > 1$. Consequently,

$$\mathcal{M}[\bar{\mathcal{L}}; \omega] = \frac{M_2(\omega)}{1 - q^{\omega-1}}, \quad (\Re(\omega) > 1),$$

where

$$M_2(\omega) = \int_0^\infty \frac{s^{\omega-1}}{Q(-s/q)} \int_0^\infty e^{-zs} \tilde{f}'_1(z)^2 dz ds.$$

Next, we have to study properties of $M_2(\omega)$. Therefore, observe that from (6) and growth properties of $Q(-s)$, we have uniformly for $|\arg(s)| \leq \pi - \epsilon$

$$\frac{\tilde{g}(s)}{Q(-s/q)} = \begin{cases} \mathcal{O}(s), & \text{if } s \rightarrow 0+; \\ \mathcal{O}(1/s^b), & \text{if } s \rightarrow \infty, \end{cases}$$

where $b > 0$ is again an arbitrary large constant. Hence, $M_2(\omega)$ is analytic for $\Re(\omega) > -1$. Moreover, from Proposition 5 in [9], we have $M_2(c + it) = O(e^{-(\pi-\epsilon)|t|})$ for $c > -1$ and $|t|$ large. Consequently, we can proceed as for the mean and obtain as $z \rightarrow \infty$,

$$\tilde{V}(z) \sim \frac{1}{L} \sum_{k \in \mathbb{Z}} \frac{M_2(1 + \chi_k)}{\Gamma(1 + \chi_k)} z^{\chi_k}.$$

The same then holds for $\text{Var}(X_n)$ as well by depoissonization.

We conclude by simplifying $M_2(1 + \chi_k)$. For that, we use

$$\tilde{f}'_1(z) = Q_\infty \sum_{l \geq 0} \frac{1}{Q_l q^{-l}} e^{-zq^l}$$

and

$$\frac{1}{Q(-s/q)} = \frac{1}{Q_\infty} \sum_{j \geq 0} \frac{(-1)^j q^{\binom{j}{2}}}{Q_j (s + q^{-j})}.$$

Plugging this into the above integral yields

$$M_2(1 + \chi_k) = Q_\infty \sum_{h, l, j \geq 0} \frac{(-1)^j q^{\binom{j}{2}}}{Q_h Q_l Q_j q^{-(l+h)}} \int_0^\infty \frac{s^{\chi_k}}{(s + q^{-j})(s + q^h + q^l)} ds.$$

Denote by

$$\varphi(\chi; x) := \begin{cases} \pi(x^\chi - 1)/(\sin(\pi\chi)(x - 1)), & \text{if } x \neq 1, \\ \pi\chi/\sin(\pi\chi), & \text{if } x = 1. \end{cases}$$

Then,

$$M_2(1 + \chi_k) = Q_\infty \sum_{h, l, j \geq 0} \frac{(-1)^j q^{\binom{j+1}{2}}}{Q_h Q_l Q_j q^{-(l+h)}} \varphi(\chi_k, q^{h+j} + q^{l+j}).$$

3 Average Value of $G_C(z)$

Here, we are going to prove Corollary 1. We will use the abbreviation $Q = 1/q$. Furthermore, in order to be closer to the q -hypergeometric world and the identities of relevance (see the book of Andrews-Askey-Roy [1]), we use the classical notation $(q)_n$ instead of Q_n .

In [17], the alternative expression

$$\mathcal{P} := \frac{\log 2}{L} - \alpha - \beta + \frac{2}{L}\tau \quad \text{with} \quad \tau := \sum_{k \geq 1} \frac{(-1)^{k-1}}{k(Q^k - 1)}$$

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was given for the constant in the variance, and we will show now the equality of this and

$$\mathcal{F} := \frac{(q)_\infty}{L} \sum_{j,l,h \geq 0} \frac{(-1)^j q^{\binom{j+1}{2} + l + h}}{(q)_j (q)_l (q)_h} \frac{\log(q^{h+j} + q^{l+j})}{q^{h+j} + q^{l+j} - 1}.$$

In this expression, we have replaced the ψ function by what it is; in some exceptional cases a limit has to be taken.

We use the symmetry in l and h and set $l = h + d$ with $d \geq 0$; then we have to take the sum over $h, d \geq 0$ twice, and subtract the sum for $h \geq 0$ and $d = 0$. Therefore

$$\mathcal{F} = 2 \sum_{j,h,d \geq 0} \dots - \sum_{j,h \geq 0, d=0} \dots.$$

We think about d as being fixed, set $h = N - j$ and fix N as well: This leads to

$$\frac{(q)_\infty [-LN + \log(1 + q^d)]}{L} \sum_{j=0}^N \frac{(-1)^j q^{\binom{j+1}{2} + 2(N-j) + d}}{(q)_j (q)_{N-j+d} (q)_{N-j}} \frac{1}{q^N + q^{N+d} - 1}.$$

By automatic summation (q -Zeilberger's algorithm) we have the simplification

$$\sum_{j=0}^N \frac{(-1)^j q^{\binom{j+1}{2} + 2(N-j) + d}}{(q)_j (q)_{N-j} (q)_{N+d-j}} \frac{1}{q^N + q^{N+d} - 1} = \frac{q^{N^2+dN}}{(q)_N (q)_{N+d}}.$$

Consequently,

$$\mathcal{F} = 2(q)_\infty \sum_{N,d \geq 0} \frac{-NL + \log(1 + q^d)}{L} \frac{q^{N^2+dN}}{(q)_N (q)_{N+d}} + (q)_\infty \sum_{N \geq 0} \frac{NL - \log 2}{L} \frac{q^{N^2}}{(q)_N (q)_N}.$$

We will soon show that

$$(q)_\infty \sum_{N,d \geq 0} \frac{\log(1 + q^d)}{L} \frac{q^{N^2+dN}}{(q)_N (q)_{N+d}} = \frac{\tau}{L} + \frac{\log 2}{L}, \quad (8)$$

which leaves us to prove that

$$2(q)_\infty \sum_{N,d \geq 0} \frac{Nq^{N^2+dN}}{(q)_N (q)_{N+d}} - (q)_\infty \sum_{N \geq 0} \frac{(N - \frac{\log 2}{L})q^{N^2}}{(q)_N (q)_N} = \frac{\log 2}{L} + \alpha + \beta.$$

Because of the identity [1, p. 567]

$$\sum_{N \geq 0} \frac{q^{N^2}}{(q)_N^2} = \frac{1}{(q)_\infty},$$

this leaves us with

$$2(q)_\infty \sum_{N,d \geq 0} \frac{Nq^{N^2+dN}}{(q)_N (q)_{N+d}} - (q)_\infty \sum_{N \geq 0} \frac{Nq^{N^2}}{(q)_N (q)_N} = \alpha + \beta. \quad (9)$$

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Now, expanding $\log(1 + q^d)$, (8) is proved once we can prove that

$$(q)_\infty \sum_{N \geq 0, d \geq 1} \frac{q^{N^2 + dN + dk}}{(q)_N (q)_{N+d}} = \frac{1}{Q^k - 1}.$$

But this follows from

$$\sum_{N \geq 0, d \geq 1} \frac{1}{(q)_d} \frac{q^{N^2 + dN + dk}}{(q)_N (q^{d+1})_N} = \sum_{d \geq 1} \frac{q^{dk}}{(q)_d} \frac{1}{(q^{d+1})_\infty} = \frac{1}{(q)_\infty} \frac{1}{Q^k - 1}.$$

We have used here the classical identity (Cauchy's identity) [1, p. 568]

$$\sum_{n \geq 0} \frac{x^n q^{n^2}}{(q)_n (xq)_n} = \frac{1}{(xq)_\infty}.$$

In order to prove (9), we will show that

$$-(q)_\infty \sum_{N \geq 0} \frac{Nq^{N^2}}{(q)_N (q)_N} = \sum_{r \geq 1} \frac{(-1)^r q^{\binom{r+1}{2}}}{1 - q^r}, \quad (10)$$

$$(q)_\infty \sum_{N, d \geq 0} \frac{Nq^{N^2 + dN}}{(q)_N (q)_{N+d}} = - \sum_{r \geq 1} \frac{(-1)^r q^{\binom{r+1}{2}}}{(1 - q^r)^2}. \quad (11)$$

Since in [18, (3.16)], it was proved that

$$\sum_{r \geq 1} \frac{(-1)^r q^{\binom{r+1}{2}}}{1 - q^r} - 2 \sum_{r \geq 1} \frac{(-1)^r q^{\binom{r+1}{2}}}{(1 - q^r)^2} = \alpha + \beta,$$

that would finish the proof. We start from

$$\sum_{n \geq 0} \frac{x^n q^{n^2}}{(q)_n (xq)_n} = \sum_{n \geq 0} \frac{x^n q^{n^2}}{(q)_n (xq)_\infty} (xq^{n+1})_\infty = \frac{1}{(xq)_\infty},$$

which is equivalent to

$$\sum_{n \geq 0} \frac{x^n q^{n^2}}{(q)_n} \sum_{k \geq 0} \frac{(-1)^k q^{\binom{k}{2}} x^k q^{(n+1)k}}{(q)_k} = 1.$$

Now differentiate this, and then set $x = 1$:

$$\sum_{n \geq 0} \frac{nq^{n^2}}{(q)_n^2} + \frac{1}{(q)_\infty} \sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} \sum_{k \geq 0} \frac{(-1)^k q^{\binom{k}{2}} k q^{(n+1)k}}{(q)_k} = 0.$$

Rearranging,

$$\sum_{n \geq 0} \frac{nq^{n^2}}{(q)_n^2} + \frac{1}{(q)_\infty} \sum_{N \geq 1} \sum_{n=0}^N \frac{q^{n^2}}{(q)_n} \frac{(-1)^{N-n} q^{\binom{N-n}{2}} (N-n) q^{(n+1)(N-n)}}{(q)_{N-n}} = 0,$$

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and again by a mechanical proof,

$$\sum_{n \geq 0} \frac{nq^{n^2}}{(q)_n^2} + \frac{1}{(q)_\infty} \sum_{N \geq 1} \frac{(-1)^N q^{\binom{N+1}{2}}}{1 - q^N} = 0.$$

This is (10). Now let us plug in $x = q^d$ after differentiation (instead of $x = 1$, as before):

$$\sum_{n \geq 0} \frac{nq^{d(n-1)}q^{n^2}}{(q)_n} \sum_{k \geq 0} \frac{(-1)^k q^{\binom{k}{2}} q^{kd} q^{(n+1)k}}{(q)_k} + \sum_{n \geq 0} \frac{q^{dn} q^{n^2}}{(q)_n} \sum_{k \geq 0} \frac{(-1)^k q^{\binom{k}{2}} k q^{(k-1)d} q^{(n+1)k}}{(q)_k} = 0.$$

After some simplifications (using Rothe's identity [1, p. 490]), this leads to

$$(q)_\infty \sum_{n \geq 0} \frac{nq^{dn} q^{n^2}}{(q)_n (q)_{n+d}} + \sum_{N \geq 1} \frac{(-1)^N q^{\binom{N+1}{2} + dN}}{1 - q^N} = 0.$$

Now sum this on d :

$$(q)_\infty \sum_{n, d \geq 0} \frac{nq^{dn} q^{n^2}}{(q)_n (q)_{n+d}} + \sum_{N \geq 1} \frac{(-1)^N q^{\binom{N+1}{2}}}{(1 - q^N)^2} = 0,$$

which is (11).

Remark. A direct proof that the Fourier coefficients, as computed here, agree with the ones given in [8], can be done in the same style.

4 Approximate Counting with m Counters and m -DSTs

Approximate Counting with m Counters. I. Here, we consider approximate counting with m counters as discussed in the introduction. Recall that D_n denoted the sum of the counters after “counting n objects”. Then, we have

$$D_n \stackrel{d}{=} C_{I_1}^{(1)} + \cdots + C_{I_m}^{(m)},$$

where $C_n^{(1)}, \dots, C_n^{(m)}$ are independent copies of C_n and I_1, \dots, I_m are random variables with joint distribution

$$P(I_1 = n_1, \dots, I_m = n_m) = \binom{n}{n_1, \dots, n_m} \frac{1}{m^n}$$

with $n_1 + \cdots + n_m = n$. Now, set

$$\tilde{Q}(y, z) = e^{-z} \sum_{n \geq 0} \mathbb{E}(e^{D_n y}) \frac{z^n}{n!}, \quad \tilde{P}(y, z) = e^{-z} \sum_{n \geq 0} \mathbb{E}(e^{C_n y}) \frac{z^n}{n!}.$$

Then, by a straight-forward computation

$$\tilde{Q}(y, z) = \tilde{P}(y, z/m)^m.$$

From this, we can derive the following relations for the Poisson generating functions of the first and second moment of D_n and C_n (denoted by $\tilde{g}_1(z), \tilde{g}_2(z)$ for the former and as above for the latter)

$$\begin{aligned}\tilde{g}_1(z) &= m\tilde{f}_1(z/m), \\ \tilde{g}_2(z) &= m(m-1)\tilde{f}_1(z/m)^2 + m\tilde{f}_2(z/m).\end{aligned}$$

Moreover, again consider the poissonized variance $\tilde{W}(z) := \tilde{g}_2(z) - \tilde{g}_1(z)^2$. Then,

$$\tilde{W}(z) = m\tilde{V}(z/m).$$

Now, it follows from the closure properties of JS-admissibility (see Lemma 2.3 in [14]) that both $\tilde{g}_1(z)$ and $\tilde{g}_2(z)$ are JS-admissible. Hence, we only have to concentrate on the $\tilde{g}_1(z)$ and $\tilde{W}(z)$ whose asymptotic expansions, due to the above formulas, follow from the case $m = 1$.

m -DSTs. m -DSTs have been introduced in [27]. They are defined as follows: again we start with n keys, but they are now stored in m DSTs. For every key, one of the m DSTs is chosen uniformly and at random and the key is then stored in the chosen tree.

Clearly, the previous analysis also gives the sum of the lengths of the leftmost paths in m -DSTs. Similarly, one can consider other shape parameters in DST and extend them linearly to m -DSTs. Our method above can then be applied to such parameters as well and again the analysis will be reduced to the case $m = 1$.

We give two examples. The first example is the depth of a random node which was discussed in [27]. As a second example, consider the total path length T_n in a random digital search tree of size n which is the sum over all distances of nodes to the root. For this quantity, it was proved for $q = 1/2$ (see Kirschenhofer, Prodinger and Szpankowski [18] and [14]) that, as $n \rightarrow \infty$,

$$\mathbb{E}(T_n) \sim n \log_2 n + nF_T(\log_2 n)$$

and

$$\text{Var}(T_n) \sim nG_T(\log_2 n),$$

where $F_T(z)$ and $G_T(z)$ are 1-periodic functions with computable Fourier coefficients (see below for a remark concerning the average value of $G_T(z)$). Similar results are known for the case $q \neq 1/2$ as well; see Jacquet and Szpankowski [15]. Now, denote by U_n the sum of all total path lengths in an m -DST. Then, with the same approach as above, we have the following result.

Theorem 3 *For the total path length in m -DSTs, we have, as $n \rightarrow \infty$,*

$$\begin{aligned}\mathbb{E}(U_n) &\sim (n/m) \log_2(n/m) + (n/m)F_T(\log_2(n/m)), \\ \text{Var}(U_n) &\sim (n/m)G_T(\log_2(n/m)),\end{aligned}$$

where $F_T(z)$ and $G_T(z)$ are the periodic functions above.

The variance of the path-length. The constant in

$$\text{Var}(T_n) \sim nG_T(\log_2 n)$$

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was given in [14] as

$$\frac{(q)_\infty}{L} \sum_{j,h,l \geq 0} \frac{(-1)^j q^{\binom{j+1}{2} + h + l}}{(q)_j (q)_h (q)_l} \varphi(q^{j+h} + q^{j+l}) \quad \text{with} \quad \varphi(x) = \frac{x-1 - \log x}{(x-1)^2}.$$

(In some cases, limits have to be taken, and the notation $(q)_n$ is again used for Q_n .) This form is a huge improvement over the form provided in [18]. However, with the methods used earlier, even this form can be further improved, in the sense that no triple sums occur anymore. This is beneficial for the numerical evaluation of this constant. The result is

$$\begin{aligned} & \frac{2\alpha}{L} + \frac{(q)_\infty}{L} \sum_{N \geq 1, d \geq 1} \frac{q^{N^2 + dN}}{(q)_N (q)_{N+d}} \frac{NL - \log(1 + q^d)}{q^N + q^{N+d} - 1} \\ & + 2(q)_\infty \sum_{N \geq 2} \frac{q^{N^2}}{(q)_N^2} \frac{N-1}{q^{N-1} - 1} - \frac{1}{L} - (q)_\infty + \frac{2}{L} \sum_{n \geq 0} \frac{(-1)^n q^{\binom{n+1}{2}}}{(q)_n} \sum_{k \geq 2} \frac{(-1)^k}{k} \frac{1}{2^{k+n-1} - 1}. \end{aligned}$$

Note that $q = \frac{1}{2}$ here. Details might appear elsewhere.

Approximate Counting with m Counters. II. Here, we again consider approximate counting with m counters, but this time we label them from 1 to m . Now, we proceed as follows: first, we use the first counter until it will be increased, then we use the second one until it will be increased, etc. until the last counter is increased then we return to the first one and repeat this procedure.

Let again D_n denote the sum of the m counters after “counting n objects”. This clearly corresponds to the length of the leftmost path in random digital search trees, where every node can hold up to m keys (here, the length is the sum of all nodes on the leftmost path weighted by the number of keys contained in the nodes). Consequently, $D_n \stackrel{d}{=} X_n$, where X_n satisfies

$$X_{n+m} \stackrel{d}{=} X_{B_n} + m, \quad (n \geq 0)$$

with $X_i = i, 0 \leq i \leq m-1$. The Poisson-Laplace-Mellin approach can be applied to this sequence as well. We only sketch some details.

First, for the Poisson generating functions of the mean and the poissonized variance (again denoted by $\tilde{f}_1(z)$ and $\tilde{V}(z)$, respectively), we have

$$\sum_{i=0}^m \binom{m}{i} \tilde{f}_1^{(i)}(z) = \tilde{f}_1(qz) + m$$

and

$$\sum_{i=0}^m \binom{m}{i} \tilde{V}^{(i)}(z) = \tilde{V}(qz) + \tilde{g}(z),$$

where $\tilde{g}(z)$ is of the form

$$\tilde{g}(z) = \left(\sum_{i=0}^m \binom{m}{i} \tilde{f}_1^{(i)}(z) \right)^2 - \sum_{i=0}^m \binom{m}{i} \left(\tilde{f}_1(z)^2 \right)^{(i)}.$$

Applying the Poisson-Laplace-Mellin method then yields asymptotic expansion of mean and variance. We content ourselves with stating the result for the variance.

Theorem 4 For approximate counting with m -counters, where counters are chosen cyclically, we have, as $n \rightarrow \infty$,

$$\text{Var}(D_n) \sim G_D(\log_{1/q} n),$$

where $G_D(z) = \sum_k g_k e^{2k\pi iz}$ is a 1-periodic function with Fourier coefficients

$$g_k = \frac{1}{L\Gamma(1 + \chi_k)} \int_0^\infty \frac{s^{\chi_k}}{Q(-s/q)^m} \left(p(s) + \int_0^\infty e^{-zs} \tilde{g}(z) dz \right) ds$$

and

$$p(s) = \frac{(s+1)^m - 1 - ms}{s^2}.$$

Acknowledgements

We thank the anonymous reviewers for helpful comments.

References

- [1] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*. Encyclopedia of Mathematics and Its Applications, The University Press, Cambridge, 1999.
- [2] J. Aspnes and K. Censor (2010). Approximate shared-memory counting despite a strong adversary, *ACM Trans. Algorithms*, **6**, 23 pages.
- [3] J. Bertoin, P. Biane, and M. Yor (2002). Poissonian exponential functionals, q -series, q -integrals, and the moment problem for log-normal distributions, Tech. Rep. PMA-705, Laboratoire de Probabilités et Modèles Aléatoires, Université Paris VI.
- [4] J. Chichoń and W. Macyna (2011). Approximate counters for flash memory, In proceedings of the seventeenth IEEE international conference on embedded and real-time computing systems and applications, 185–189.
- [5] E. G. Coffman Jr. and J. Eve (1970). File structures using hashing functions, *Commun. ACM*, **13**, 427–432.
- [6] D. Crippa and K. Simon (1997). q -distributions and Markov processes, *Discrete Math.*, **170**, 81–98.
- [7] M. Csűrös (2010). Approximate counting with a floating-point counter, Computing and Combinatorics, 16th Annual International Conference, Cocom 2010, Lecture Notes in Computer Science, **6196**, 358–367.
- [8] P. Flajolet (1985). Approximate counting: a detailed analysis, *BIT*, **25**, 113–134.
- [9] P. Flajolet, X. Gourdon, and P. Dumas (1995). Mellin transforms and asymptotics: harmonic sums, *Theoret. Comput. Sci.*, **144**, 3–58.
- [10] P. Flajolet and B. Richmond (1992). Generalized digital trees and their difference-differential equations, *Random Structures Algorithms*, **3**, 305–320.
- [11] P. Flajolet and R. Sedgewick (1986). Digital search trees revisited, *SIAM J. Comput.*, **15**, 748–767.
- [12] A. Gronemeier and M. Sauerhoff (2009). Applying approximate counting for computing the frequency moments of long data streams, *Theory Comput. Syst.*, **44**, 332–348.
- [13] F. Guillemin, P. Robert, and B. Zwart (2004). AIMD algorithms and exponential functionals, *Ann. Appl. Probab.*, **14**, 90–117.

- [14] H.-K. Hwang, M. Fuchs, and V. Zacharovas (2010). Asymptotic variance of random symmetric digital search trees, *Discrete Math. Theor. Comput. Sci.*, **12**, 103–166.
- [15] P. Jacquet and W. Szpankowski (1995). Asymptotic behavior of the Lempel-Ziv parsing scheme and digital search trees, *Theoret. Comput. Sci.*, **144**, 161–197.
- [16] P. Jacquet and W. Szpankowski (1998). Analytical de-Poissonization and its applications, *Theoret. Comput. Sci.*, **201**, 1–62.
- [17] P. Kirschenhofer and H. Prodinger (1991). Approximate counting: an alternative approach, *RAIRO Inform. Théor. Appl.*, **25**, 43–48.
- [18] P. Kirschenhofer, H. Prodinger, and W. Szpankowski (1994). Digital search trees again revisited: the internal path length perspective, *SIAM J. Comput.*, **23**, 598–616.
- [19] G. Louchard and H. Prodinger (2006). Asymptotics of the moments of extreme-value related distribution functions, *Algorithmica*, **46**, 431–467.
- [20] G. Louchard and H. Prodinger (2008). Generalized approximate counting revisited, *Theoret. Comput. Sci.*, **391**, 109–125.
- [21] S. A. Mitchell and D. M. Day (2011). Flexible approximate counting, In 15th International Database Engineering & Applications Symposium, IDEAS 2011, 233–239.
- [22] R. Morris (1978). Counting large numbers of events in small registers, *Comm. ACM*, **21**, 840–842.
- [23] F. W. J. Olver. *Asymptotics and Special Functions*. Academic Press, 1974.
- [24] H. Prodinger (1992). Hypothetic analyses: approximate counting in the style of Knuth, path length in the style of Flajolet, *Theoret. Comput. Sci.*, **100**, 243–251.
- [25] H. Prodinger (1994). Approximate counting via Euler transform, *Math. Slovaca*, **44**, 569–574.
- [26] H. Prodinger (2011). Digital search trees with m trees: level polynomials and insertion costs, *Discrete Math. Theor. Comput. Sci.*, **13**, 1–8.
- [27] H. Prodinger (2012). Approximate counting with m counters: a detailed analysis, *Theoret. Comput. Sci.*, in press.
- [28] P. Robert (2005). On the asymptotic behavior of some algorithms, *Random Structures Algorithms*, **27**, 235–250.
- [29] W. A. Rosenkrantz (1987). Approximate counting: a martingale approach, *Stochastics*, **20**, 111–120.
- [30] K. Simon (1988). Improved algorithm for transitive closure on acyclic digraphs, *Theoret. Comput. Sci.*, **58**, 325–346.

Approximate counting with m counters: a probabilistic analysis

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May 21, 2012

Abstract

Motivated by a recent paper by Cichoń and Macyna [1], who introduced m counters (instead of just one) in the approximate counting scheme first analysed by Flajolet [2], we analyse the moments of the sum of the m counters, using techniques that proved to be successful already in several other contexts [11].

Keywords: Approximate counting, Moments, dominant and fluctuating components, Complex analysis, Product of Fourier series.

1 Introduction

Approximate counting is a technique that was first analysed by Flajolet [2]; some subsequent papers [6, 12, 13, 14] added to the analysis.

A counter C is kept, and each time an item arrives and needs to be counted, a random experiment is performed; if the current value of the counter is i , then with probability 2^{-i} the counter is increased by 1, otherwise it keeps its value; at the beginning, the counter value is $C = 1$. After n random increments, the value of the counter is typically close to $\log_2 n$, and the cited papers contain exact and asymptotic values for average and variance. For instance, Flajolet [2] gives the dominant constant part of mean and variance and the periodic part of the mean.

Very recently, Cichoń and Macyna [1] used this idea as follows: Instead of one counter, they keep m counters, where $m \geq 1$ is an integer. For our subsequent analysis we will assume that m is fixed. When a new element arrives (and needs to be counted), it is randomly (with probability $\frac{1}{m}$) assigned to one of the m counters, and then the random experiment is performed as usual. The parameter that Cichoń and Macyna are interested in is the total number of changes of any counter. In other words, if we (for convenience) assume that the initial setting of a counter to the value 1 counts as a change, Cichoń and Macyna are interested in the sum of the values of the m counters.

The paper [16] provided the first analysis of Cichoń and Macyna's scheme: Based on *exact expressions*, asymptotics for expectation and variance are derived with Rice's method. There is a price to be paid for dealing with these exact expressions, as there are computational hardships to be dealt with. Let us also mention Fuchs, Lee and Prodinger [4] who analyze this algorithm via the Poisson-Laplace-Mellin method. In the present paper, the approach is different: Going to approximations immediately, one loses exact expressions, but on the other hand the computations become much more manageable, so that one can go to higher moments, which we do here, mostly, to show the power of the method.

The new interest in approximate counting that Cichoń and Macyna's scheme initiated, motivated us to provide this paper: We had a long report [10] on asymptotics of the moments of extreme-value related distribution functions, but had to shorten it in order to get it published [11]; and the analysis of classical approximate counting had to be left out. We present it here, together with additional

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material that deals with the m counters (instead of just one, as in the classical case). We also present new simplifications of products of some Fourier series.

Let $J(m, n)$ be the random variable (RV): “total value of the counter after n items have arrived”; we can write $J(m, n) = \sum_1^m J_i(n)$ where $J_i(n)$ is related to the i th counter. When we have only one counter, we can just write $J(n)$.

The most important motivation of the paper is to compute the asymptotic distribution and the moments of $J(m, n)$. The asymptotic distribution is related to the extreme-value Gumbel distribution function (DF): $\exp(-\exp(-x))$. The moments are usually given by a dominant part and a small fluctuating part. There we use Laplace and Mellin transforms and singularity analysis.

Our aim is to derive an (almost) purely mechanical computation of dominant and fluctuating components, with the help of computer algebra systems (we use Maple here). As an example, we provide the first four moments, (even the third moment is very rarely computed in the literature) but the treatment is completely automatic (with some human guidance of course). The fourth moment is particularly interesting: it presents a wide variety of combinatorial and mathematical constants as well as several types of Fourier series (including products of them).

A last but not least motivation is to simplify the analytic treatment by using only easy complex analysis: only simple poles are needed, and we do not use alternating series (so we do not need Rice’s formula). A small number of analytic functions are the only tools we need. This should be compared with the complicated techniques sometimes used in previous papers.

We have uniform integrability for the moments of our RV’s. To show that the limiting moments are equivalent to the moments of the limiting distributions, we need a suitable rate of convergence. This is related to a uniform integrability condition (see Loève [8, Section 11.4]).

The total error term related to our asymptotics of moments is detailed in [11]; it is given by $\mathcal{O}(n^{-C})$, where C is some constant.

Another technical point of interest are the periodic oscillations that always occur in approximate counting and related questions: When one goes for higher moments, there are many extra terms coming in, making the Fourier coefficients very complicated. In particular, there are high powers of some (simple) periodic functions. We present a technique to bring the Fourier coefficients of these into some standard form, using residue calculus. With a suitable function, the residues on the imaginary axis correspond to the convolution of two Fourier series, and, instead of them, one can collect the residues on the real axis. While we are convinced that this always works in our instances, we refrained from rewriting *all* Fourier coefficients occurring in the fourth moment. This can be done with some patience, but the interest in it is limited, as it would just fill many pages.

To summarize, we had several motivations to write this paper:

- show the power of the method we introduced in [11]
- present, with great details, the analysis of classical Approximate counting
- give new simplifications of product of some Fourier series
- analyze, with precision, Approximate counting with m counters

The paper is organized as follows: Definitions, notations and known properties are recalled in Section 2. Classical approximate counter (1 counter) is analyzed in Section 3. The case of m counters is considered in Section 4. The asymptotic moments of $J(n)$ are given in Section 5. Section 6 concludes the paper.

2 Definitions, notations and known properties.

Let us first give the notations we will use throughout the paper.

$$L := \ln 2,$$

$$\begin{aligned}
\log &:= \log_2, \\
\varepsilon &:= \text{small real } > 0, \\
\tilde{\alpha} &:= \alpha/L, \\
Q(j) &:= \prod_{k=1}^j (1 - 2^{-k}), \quad Q(0) = 1, \\
Q &:= Q(\infty) = 0.288788095087\dots, \\
R(j) &:= \frac{(-1)^{j+1}}{2^{j(j+1)/2} Q(j)}, \quad R(0) = -1, \\
\chi_l &:= \frac{2\pi i l}{L}, \\
\rho_1 &:= \sum_{l \neq 0} \Gamma(\chi_l) \psi(\chi_l) e^{-2l\pi i \log n}, \\
\rho_2 &:= \sum_{l \neq 0} \Gamma(\chi_l) \psi(\chi_l)^2 e^{-2l\pi i \log n}, \\
\rho_3 &:= \sum_{l \neq 0} \Gamma(\chi_l) \psi(\chi_l)^3 e^{-2l\pi i \log n}, \\
\rho_4 &:= \sum_{l \neq 0} \Gamma(\chi_l) \psi(1, \chi_l) e^{-2l\pi i \log n}.
\end{aligned}$$

These functions appear in many analyses of algorithms (see, for instance, Flajolet [2], Flajolet and Sedgewick [3], Hwang et al. [5], Louchard [9], Louchard and Prodinger [11]).

The following facts will be frequently used:

$$\begin{aligned}
Q(i) &\geq Q, \\
(1-u)^n &= e^{-nu} [1 - nu^2/2 + \mathcal{O}(nu^3)], \quad u \in]0, 1[.
\end{aligned}$$

For the integer-valued RV J (from now on, we drop i and n from $J_i(n)$ in order to ease the notation; J is now related to one counter, with n items), we set

$$p(j) := \mathbb{P}(J = j), \quad P(j) := \mathbb{P}(J \leq j).$$

Setting $\eta = j - \log n$, we will first compute f and F such that

$$p(j) \sim f(\eta), \quad P(j) \sim F(\eta), \quad n \rightarrow \infty,$$

and, of course,

$$f(\eta) = F(\eta) - F(\eta - 1).$$

Asymptotically, the distribution will be a periodic function of the fractional part of $\log n$. The distribution $P(j)$ does not converge in the weak sense, it does however converge along subsequences n_m for which the fractional part of $\log n_m$ is constant. This type of convergence is not uncommon in the Analysis of Algorithms. Many examples are given in [11].

Next, we must check that

$$\mathbb{E}(J^k) = \sum_j j^k p(j) \sim \sum_j (\eta + \log n)^k f(\eta), \quad (1)$$

by computing a suitable rate of convergence. This is related to a uniform integrability condition (see Loève [8, Section 11.4].)

3 Classical approximate counting (one counter). Analysis of $J(n)$.

In this section, we provide the asymptotic distribution and the first four asymptotic moments of the RV $J(n)$ based on n items. From Flajolet [2, Proposition 1], we have

$$p(j) = \sum_{k=0}^{j-1} 2^k \frac{-R(k)}{Q(j-1-k)} (1 - 1/2^{j-k})^n. \tag{2}$$

Letting $\eta = j - \log n$, this leads in a natural way to

$$f(\eta) = \sum_{k=0}^{\infty} 2^k \frac{-R(k)}{Q} \exp(-2^{-\eta+k}).$$

(Compare also [6, 13, 14].) It has been pointed out in [9] that it has some similarities with the Digital Search Tree distribution. Actually, as noticed by S. Janson (private communication), the distribution for approximate counting is the same as for unsuccessful search in Digital Search Trees (not only asymptotically).

The rate of the convergence problem is completely solved in Flajolet [2]. Also, we obtain by summing

$$\begin{aligned} P(j) &= - \sum_{k=0}^{j-1} \sum_{u=k+1}^j 2^k \frac{R(k)}{Q(u-1-k)} (1 - 1/2^{u-k})^n, \\ F(\eta) &= - \sum_{k=0}^{\infty} 2^k \frac{R(k)}{Q} \sum_{i=k}^{\infty} \exp(-2^{-\eta+i}). \end{aligned} \tag{3}$$

We note that the algorithm can be generalized by changing the base. The analysis is quite similar, and we won't provide details here.

The first three asymptotic moments are given in our unpublished report [10]. We take the opportunity to present them here, with some complements. In particular we analyze the product of Fourier series, which leads to convolutions of the coefficients. In order to show the power of the methods we use, we also give the fourth moment. Using the techniques we described in our published paper [11], we proceed as follows.

3.1 Some preliminary identities

Some preliminary identities are necessary.

Using a classical Euler identity, we will derive several summation formulae. This identity is

$$\prod_{k=0}^{\infty} (1 + q^k z) = \sum_{i=0}^{\infty} z^i q^{i(i-1)/2} / Q(i).$$

(for a simple proof, see Knuth [7, Ex 5.1.1–16]). Set

$$\begin{aligned} \Pi^* &:= \prod_{k=1}^{\infty} (1 + q^k z), \\ \Pi &= (1 + z)\Pi^*, \\ \Sigma_k &:= (k-1)! \sum_{i=1}^{\infty} q^{ki} / (1 + zq^i)^k. \end{aligned}$$

It is not hard to see that, with $z = -1$, $q = 1/2$, we get the following expansions

$$\Pi^* \rightarrow Q, \quad \Sigma_1 \rightarrow C_1, \quad \Sigma_2 \rightarrow C_2, \quad \Sigma_3 \rightarrow 2!C_3, \quad \Sigma_4 \rightarrow 3!C_4,$$

with the abbreviations

$$C_k := \sum_{j=1}^{\infty} \frac{1}{(2j-1)^k}.$$

Now we want to compute sums of type

$$U_k := \sum_{i=0}^{\infty} i^k 2^i R(i).$$

We obtain

$$\Pi^{*'} = \Sigma_1 \Pi^*, \quad \Pi' = \Pi^* + (1+z)\Sigma_1 \Pi^*,$$

and setting $z = -1$, $q = 1/2$, we derive $U_1 = Q$. Similarly, set $T_1 := z\Pi'$, compute T_1' , etc. With the same procedure, we obtain:

$$\begin{aligned} U_2 &= -Q(-1 + 2C_1), \\ U_3 &= Q(1 - 6C_1 - 3C_2 + 3C_1^2), \\ U_4 &= -Q(-1 + 14C_1 + 18C_2 - 18C_1^2 + 8C_3 - 12C_1C_2 + 4C_1^3), \\ U_5 &= Q(1 - 30C_1 - 75C_2 + 75C_1^2 - 80C_3 + 120C_1C_2 - 40C_1^3 - 30C_4 + 40C_1C_3 \\ &\quad + 15C_2^2 - 30C_1^2C_2 + 5C_1^4). \end{aligned} \quad (4)$$

All these values are actually necessary here. Also

$$U_0 = 0; \quad (5)$$

this is Equation (21) in Flajolet [2]. More generally, setting $z = -2^k$, $q = 1/2$ in Π , we derive

$$\sum_{i=0}^{\infty} 2^{(k+1)i} R(i) = 0. \quad (6)$$

3.2 “Slow increase property”

It will appear that all functions we use here are analytic (in some domain), depending on classical functions such as Γ , ζ , $\psi(k, s)$ (the $(k+1)$ -gamma function).

But we know that $\Gamma(s)$ decreases exponentially in the direction $i\infty$:

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2} e^{-\pi|t|/2}.$$

Also, we have a “slow increase property” for all functions we encounter: let $s = \sigma + it$,

$$|\zeta(s)| = \mathcal{O}(|t|^{1-\sigma}), \quad \sigma < 1.$$

We will also use the function $H_2(-Ls)$, with

$$H_2(\alpha) := \sum_{k=0}^{\infty} e^{\alpha k} 2^k R(k).$$

To analyze this function, we use the “sum splitting technique” as described in Knuth [7, p. 131], and used in Flajolet [2]: let $\sigma > -1$ and $\rho(x)$ be an increasing function.

For $k < \rho(|s|)$, the contribution is bounded by

$$\rho(|s|) \sup_{0 \leq k \leq \rho(|s|)} \left(\frac{1}{2^{\sigma k + k^2}} \right) = \mathcal{O} \left(1 + \rho(|s|) 2^{\rho(|s|)} \right);$$

for $k \geq \rho(|s|)$, the contribution is bounded by

$$\sum_{k=\rho(|s|)}^{\infty} \frac{2^k}{2^{k^2}} = \mathcal{O} \left(\frac{2^{\rho(|s|)}}{2^{2\rho(|s|)}} \right).$$

Choosing $\rho(x) = \log x$ insures the slow increase property.

3.3 The asymptotic moments

The detailed proofs of relations (7) to (15) are given in [11]. Let an (integer-valued) RV K be such that $\mathbb{P}(K - \log n \leq \eta) \sim F(\eta)$, where $F(\eta)$ is the DF of a continuous RV Z with mean m_1 , second moment m_2 , variance σ^2 and centered moments μ_k . Assume that $F(\eta)$ is either an extreme-value DF or a convergent series of such and that (1) is satisfied. Let

$$\varphi(\alpha) = \mathbb{E}(e^{\alpha Z}) = 1 + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} m_k = e^{\alpha m_1} \lambda(\alpha), \quad (7)$$

say, with

$$\lambda(\alpha) = 1 + \frac{\alpha^2}{2} \sigma^2 + \sum_{k=3}^{\infty} \frac{\alpha^k}{k!} \mu_k. \quad (8)$$

Also

$$\varphi(\alpha) = \int_{-\infty}^{+\infty} e^{\alpha \eta} F'(\eta) d\eta = -\alpha \int_{-\infty}^{+\infty} e^{\alpha \eta} F(\eta) d\eta, \quad (9)$$

with suitable boundary conditions (which are satisfied in all examples we present). This gives here, for $K = J(n)$,

$$\varphi(\alpha) = -\frac{H_2(\alpha) \Gamma(1 - \tilde{\alpha})}{Q(1 - e^\alpha)}, \quad \Re(\alpha) < L,$$

and

$$H_2(\alpha) := \sum_{k=0}^{\infty} e^{\alpha k} 2^k R(k).$$

This leads to (we give only the first two expressions for m_i)

$$\begin{aligned} m_1 &= \frac{\gamma}{L} - C_1, \\ m_2 &= \frac{\pi^2 + 6\gamma^2}{6L^2} - \frac{2\gamma C_1}{L} - C_2 + C_1^2 - C_1, \\ \mu_2 &= \sigma^2 = \frac{\pi^2}{6L^2} - C_1 - C_2, \\ \mu_3 &= \frac{2\zeta(3)}{L^3} - 2C_3 - 3C_2 - C_1, \\ \mu_4 &= 3C_1^2 - C_1 - 7C_2 - 12C_3 + 3C_2^2 - \frac{\pi^2 C_1}{L^2} + 6C_1 C_2 - \frac{\pi^2 C_2}{L^2} + \frac{3\pi^4}{20L^4} - 6C_4. \end{aligned}$$

Let w, κ 's (with or without subscripts) denote periodic functions of $\log n$, with period 1 and with usually small mean and amplitude. Actually, these functions depend on the fractional part of $\log n$: $\{\log n\}$.

The moments of $J(n) - \log n$ are asymptotically given by $\tilde{m}_i + w_i$, where the generating function of \tilde{m}_i is given by

$$\phi(\alpha) := \int_{-\infty}^{\infty} e^{\alpha \eta} f(\eta) d\eta = 1 + \sum_{i=1}^{\infty} \frac{\alpha^i}{i!} \tilde{m}_i = \varphi(\alpha) \frac{e^\alpha - 1}{\alpha}. \quad (10)$$

This leads to

$$\begin{aligned} \tilde{m}_1 &= \frac{1}{2} - C_1 + \frac{\gamma}{L}, \\ \tilde{m}_2 &= \frac{\pi^2 + 6\gamma^2}{6L^2} + \frac{\gamma - 2\gamma C_1}{L} + \frac{1}{3} - 2C_1 - C_2 + C_1^2. \end{aligned}$$

More generally, the centered moments of $J(n)$ are asymptotically given by $\mu_i = \tilde{\mu}_i + \kappa_i$, with the asymptotic dominant centered moment generating function given by

$$\Theta(\alpha) := 1 + \sum_{k=2}^{\infty} \frac{\alpha^k}{k!} \tilde{\mu}_k = \frac{2}{\alpha} \sinh(\alpha/2) \lambda(\alpha). \quad (11)$$

The neglected part is of order $1/n^\beta$ with $0 < \beta < 1$. We derive, with (4), the following result about these centered moments.

Theorem 3.1 *The asymptotic dominant parts of the centered moments of $J(n)$ are given by*

$$\begin{aligned} \tilde{\mu}_2 &= \frac{\pi^2}{6L^2} + \frac{1}{12} - C_1 - C_2, \\ \tilde{\mu}_3 &= \frac{2\zeta(3)}{L^3} - 2C_3 - 3C_2 - C_1, \\ \tilde{\mu}_4 &= \frac{1}{80} + 3C_1^2 - \frac{3C_1}{2} - \frac{15C_2}{2} - 12C_3 - 6C_4 + \frac{3\pi^4}{20L^4} \\ &\quad + \frac{\pi^2}{12L^2} + 6C_1C_2 - \frac{\pi^2C_1}{L^2} - \frac{\pi^2C_2}{L^2} + 3C_2^2. \end{aligned}$$

Note that

$$\Theta(\alpha) = \phi(\alpha) e^{-\alpha \tilde{m}_1}.$$

Now

$$\mathbb{E}(J(n) - \log n)^k \sim \tilde{m}_k + w_k,$$

with

$$w_k = \frac{1}{L} \sum_{l \neq 0} \Upsilon_k^*(\chi_l) e^{-2l\pi i \log n}, \quad (12)$$

and

$$\Upsilon_k^*(s) = L \phi^{(k)}(\alpha) \Big|_{\alpha=-Ls}. \quad (13)$$

With (5), we check that we have no singularity of $\phi^{(k)}$, $k > 0$, at $\alpha = 0$. The fundamental strip for (13) is $\Re(s) \in \langle -1, 0 \rangle$. We first obtain

$$w_1 = -\frac{1}{L} \sum_{l \neq 0} \Gamma(\chi_l) e^{-2l\pi i \log n}.$$

\tilde{m}_1 , $\tilde{\mu}_2$ and w_1 are identical to the expressions given in Flajolet [2].

To compute the periodic components κ_i to be added to the centered moments $\tilde{\mu}_i$, we first set

$$\mathbf{m}_1 := \tilde{m}_1 + w_1.$$

Now, we start from

$$\phi(\alpha) := 1 + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \tilde{m}_k = \varphi(\alpha) \frac{e^\alpha - 1}{\alpha}.$$

We replace \tilde{m}_k by $\tilde{m}_k + w_k$, leading to

$$\phi_p(\alpha) = \phi(\alpha) + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} w_k.$$

But it is easy to check that

$$\sum_{l \neq 0} \phi(-L\chi_l) e^{-2l\pi i \log n} = 0,$$

so we obtain

$$\phi_p(\alpha) = \phi(\alpha) + \sum_{k=0}^{\infty} \sum_{l \neq 0} \phi^{(k)}(\alpha) \Big|_{\alpha=-L\chi_l} e^{-2l\pi i \log n} \frac{\alpha^k}{k!}$$

$$= \phi(\alpha) + \sum_{l \neq 0} \phi(\alpha - L\chi_l) e^{-2l\pi i \log n}. \quad (14)$$

Finally, we compute

$$\Theta_p(\alpha) = \phi_p(\alpha) e^{-\alpha m_1} = 1 + \sum_{k=2}^{\infty} \frac{\alpha^k}{k!} (\tilde{\mu}_k + \kappa_k) = \Theta(\alpha) + \sum_{k=2}^{\infty} \frac{\alpha^k}{k!} \kappa_k, \quad (15)$$

leading to the (exponential) generating function of κ_k . By expansion and taking differences, we have a result about the oscillating parts.

Theorem 3.2 *The asymptotic oscillating parts of the centered moments of $J(n)$ are given by*

$$\begin{aligned} \kappa_2 &= -w_1^2 - \frac{2\gamma w_1}{L} + \frac{2}{L^2} \rho_1, \\ \kappa_2^2 &= w_1^4 + \frac{4\gamma^2 w_1^2}{L^2} + \frac{4}{L^4} \rho_1^2, \\ &\quad + \frac{4\gamma w_1^3}{L} - \frac{4w_1^2}{L^2} \rho_1 - \frac{8w_1}{L^3} \rho_1, \\ \kappa_3 &= \frac{4L^2 w_1^2 + 12w_1 L \gamma + 6\gamma^2 - \pi^2}{2L^2} w_1 - \frac{6(\gamma + w_1 L)}{L^3} \rho_1 \\ &\quad - \frac{3}{L^3} \rho_2 - \frac{3}{L^3} \rho_4, \\ \kappa_4 &= w_1 \left[-w_1/2 + \frac{12\gamma C_2}{L} + \frac{12\gamma C_1}{L} - 3w_1^3 + \frac{\pi^2 w_1}{L^2} - \frac{8\zeta(3)}{L^3} \right. \\ &\quad \left. + 6C_1 w_1 + 6w_1 C_2 - \frac{12\gamma^2 w_1}{L^2} - \frac{4\gamma^3}{L^3} - \frac{12w_1^2 \gamma}{L} - \frac{\gamma}{L} \right] \\ &\quad + \frac{L^2 - 12C_2 L^2 - 12C_1 L^2 + 24\gamma w_1 L + 12w_1^2 L^2 + 12\gamma^2}{L^4} \rho_1 \\ &\quad + \frac{12}{L^4} \sum_{l \neq 0} e^{-2l\pi i \log n} \psi(\chi_l) \psi(1, \chi_l) \Gamma(\chi_l) + \frac{12(w_1 L + \gamma)}{L^4} \rho_4 \\ &\quad + \frac{4}{L^4} \sum_{l \neq 0} e^{-2l\pi i \log n} \psi(2, \chi_l) \Gamma(\chi_l) + \frac{4}{L^4} \rho_3 \\ &\quad + \frac{12(w_1 L + \gamma)}{L^4} \rho_2. \end{aligned}$$

All algebraic manipulations of this paper are mechanically performed by Maple.¹

3.4 The corrections

Products of Fourier series *do* have a constant term, even if the factors do not. This term must be included in the *dominant* part of our moments. This is the object of the present subsection.

We denote by $[f]_k$ the coefficient of $e^{-2k\pi i \log n}$ in the Fourier expansion of f .

In [11], we have proved the following relations

$$c_1[0] := [w_1^2]_0 = \frac{1}{L^2} \sum_{k \neq 0} \Gamma(-\chi_k) \Gamma(\chi_k) = -\frac{2}{L} D - \frac{11}{12} + \frac{\pi^2}{6L^2}$$

with

$$D := \sum_{l \geq 1} \frac{(-1)^l}{l(2^l - 1)}.$$

¹ $\Gamma'(x) = \Gamma(x)\psi(x)$, $\Gamma''(x) = \Gamma(x)\psi(1, x) + \Gamma(x)\psi^2(x)$, $\Gamma'''(x) = \Gamma(x)\psi(2, x) + 3\Gamma(x)\psi(1, x)\psi^2(x) + \Gamma(x)\psi^3(x)$ etc.

The coefficient $c_1[k]$ of $e^{-2k\pi i \log n}$ in the Fourier expansion of w_1^2 is given by

$$\begin{aligned} c_1[k] &= \frac{1}{L^2} \sum_{j \neq 0, \neq k} \Gamma(-\chi_j) \Gamma(\chi_k + \chi_j) \\ &= \frac{2}{L} \sum_{l \geq 1} \frac{(-1)^l \Gamma(\chi_k + l)}{l!(2^l - 1)} + \frac{2}{L^2} \Gamma(\chi_k) (\psi(\chi_k) + \gamma), \end{aligned}$$

$$c_2[0] := [w_1^3]_0 = 1 + \frac{2\zeta(3)}{L^3} + \frac{1}{L} D - \frac{6}{L^2} D_1 - \frac{2 \log 3}{L} - \frac{2}{L} D_2,$$

with

$$\begin{aligned} D_1 &= \sum_{l \geq 1} \frac{(-1)^l H_{l-1}}{l(2^l - 1)}, \\ D_2 &= \sum_{l, j \geq 1} \frac{(-1)^{l+j}}{(l+j)(2^l - 1)} \left[\frac{1}{2^j - 1} + \frac{1}{2^{j+l} - 1} \right] \binom{l+j}{j}. \end{aligned}$$

This has been checked numerically and gives the tiny value $-9.428177 \times 10^{-25}$.

The coefficient $c_2[k]$ of $e^{-2k\pi i \log n}$ in the Fourier expansion of w_1^3 is given by

$$\begin{aligned} c_2[k] &= - \left\{ \frac{2}{L^3} \left[2L \sum_{l \geq 1} \frac{(-1)^l}{l!(2^l - 1)} \Gamma(l + \chi_k) (\psi(l + \chi_k) + \gamma) - L^2 \sum_{l \geq 1} \frac{(-1)^l \Gamma(l + \chi_k)}{l!} \frac{2^l}{(2^l - 1)^2} \right. \right. \\ &\quad \left. \left. + \frac{1}{12} \left(18\Psi(1, \chi_k) + 18(\psi(\chi_k) + \gamma)^2 - 11L^2 - \pi^2 \right) \Gamma(\chi_k) \right] \right. \\ &\quad \left. + \frac{2}{L^2} \sum_{l \geq 1} \frac{(-1)^l}{l!(2^l - 1)} \left[L \sum_{h \geq 0} \frac{(-1)^h}{h!(2^{h+l} - 1)} \Gamma(h + l + \chi_k) + L \sum_{h \geq 1} \Gamma(l + h + \chi_k) \frac{(-1)^h}{h!(2^h - 1)} \right. \right. \\ &\quad \left. \left. + L\Gamma(l + \chi_k) 2^{-l} - L\Gamma(l + \chi_k) - (l-1)! \Gamma(\chi_k) + \Gamma(\chi_k) \left(\psi(\chi_k) + \gamma + \frac{L}{2} \right) \right] \right. \\ &\quad \left. + \left[\frac{\pi^2}{6L^3} + \frac{1}{12L} - \frac{1}{L} - \frac{2}{L^2} \sum_{l \geq 1} \frac{(-1)^{l-1}}{l(2^l - 1)} \right] \Gamma(\chi_k) \right\}. \end{aligned}$$

For a complete description of the Fourier coefficients of the oscillations occurring in the third and fourth moment, we need the following expressions (note that Maple splits the higher derivatives of the Gamma function; if one could rework that, one could reduce the number of necessary expressions):

$$\begin{aligned} c_3[0] &:= [w_1 \rho_1]_0, & c_4[0] &:= [w_1^4]_0, & c_5[0] &:= [(\rho_1)^2]_0, & c_6[0] &:= [w_1^2 \rho_1]_0, \\ c_7[0] &:= [w_1 \rho_4]_0, & c_8[0] &:= [w_1 \rho_2]_0, & c_3[k] &:= [w_1 \rho_1]_k, & c_4[k] &:= [w_1^4]_k, \\ c_5[k] &:= [(\rho_1)^2]_k, & c_6[k] &:= [w_1^2 \rho_1]_k, & c_7[k] &:= [w_1 \rho_4]_k, & c_8[k] &:= [w_1 \rho_2]_k. \end{aligned}$$

We will show now how to “compute” some Fourier coefficients we need. “Computing” is perhaps a very ambitious word, it might be better replaced by “rewriting”. We have

$$w_1 = -\frac{1}{L} \sum_{k \neq 0} \Gamma(\chi_k) e^{-2\pi i k x},$$

and higher powers have convolutions as coefficients:

$$w_1^4 = \frac{1}{L^4} \sum_{k \neq 0} \sum_{k_1 + k_2 + k_3 + k_4 = k} \Gamma(\chi_{k_1}) \Gamma(\chi_{k_2}) \Gamma(\chi_{k_3}) \Gamma(\chi_{k_4}) e^{-2\pi i k x},$$

and none of the k_1, \dots, k_4 is allowed to be zero. The only thing that we are able to achieve is to have only one Gamma-term in the (multiple) sum, where a typical term might look like

$$\sum_{j_1 + \dots + j_t = j} C_{j_1}^{(1)} \dots C_{j_t}^{(t)} \Gamma^{(s)}(\chi_k + j) e^{-2\pi i k x}.$$

Such a representation is not a priori *better* than the straight-forward convolution, but we will sketch now how to achieve them. One (small) advantage is that the zeroth term can be explicitly determined, and extracted, and what is left is then oscillating around zero.

As an example, we consider

$$W := w_1^2 - [w_1^2]_0 = \sum_{k \neq 0} c_1[k] e^{-2\pi i k x},$$

with

$$c_1[k] = \frac{2}{L} \sum_{l \geq 1} \frac{(-1)^l \Gamma(\chi_k + l)}{l!(2^l - 1)} + \frac{2}{L^2} \left(\Gamma'(\chi_k) + \gamma \Gamma(\chi_k) \right).$$

Let us discuss W^2 . It is clear that the convolution of W with itself contains already several terms. For instance,

$$\begin{aligned} [W^2]_0 &= \frac{4}{L^2} \sum_{j, l \geq 1} \frac{(-1)^{j+l}}{j!(2^j - 1)l!(2^l - 1)} \sum_{k \neq 0} \Gamma(-\chi_k + j) \Gamma(\chi_k + l) \\ &+ \frac{8}{L^3} \sum_{l \geq 1} \frac{(-1)^l}{l!(2^l - 1)} \sum_{k \neq 0} \Gamma(-\chi_k + l) \left(\Gamma'(\chi_k) + \gamma \Gamma(\chi_k) \right) \\ &+ \frac{4}{L^4} \sum_{k \neq 0} \left(\Gamma'(-\chi_k) + \gamma \Gamma(-\chi_k) \right) \left(\Gamma'(\chi_k) + \gamma \Gamma(\chi_k) \right). \end{aligned}$$

We would like to demonstrate how to rewrite the k -sums in these expressions. The survey paper [15] has many similar examples. The approach we found most versatile is via residue calculus. One writes a suitable function, computes all the residues in the complex plane, and the sum of them is zero. There are of course some technical subtleties, like showing that integral tends to zero for larger and larger radii, and also there are usually some series that do not converge absolutely. The suitable limit of them is the *Abel* limit, i.e., consider a power series in x , and let x tend to a point at the boundary of convergence. Here, we want to concentrate on the computational part only.

The function that is suitable for the first part is

$$\frac{L}{2^z - 1} \Gamma(j - z) \Gamma(l + z).$$

A first contribution comes from the poles at $z = \chi_k$:

$$S_1 = \sum_{k \in \mathbb{Z}} \Gamma(j - \chi_k) \Gamma(l + \chi_k).$$

The next contribution stems from the poles at $z = j, j + 1, \dots$:

$$S_2 = (l + j - 1)! L \sum_{h \geq 0} \frac{(-1)^{h-1}}{2^{j+h} - 1} \binom{l + j + h - 1}{h}.$$

Now we look at the poles at $z = -l, -l - 1, \dots$:

$$\sum_{h \geq 0} \frac{L}{2^{-l-h} - 1} \Gamma(j + l + h) \frac{(-1)^h}{h!}.$$

This series does not converge absolutely. It is best to pull out the “bad” part, which leads to two contributions:

$$S_3 = (l+j-1)!L \sum_{h \geq 0} \frac{(-1)^{h-1}}{2^{l+h}-1} \binom{l+j+h-1}{h},$$

$$S_4 = (l+j-1)!L \sum_{h \geq 0} (-1)^{h-1} \binom{l+j+h-1}{h}.$$

As announced, we must interpret S_4 as a limit:

$$S_4 = -(l+j-1)!L \lim_{x \rightarrow 1} \sum_{h \geq 0} (-x)^h \binom{l+j+h-1}{h} = -(l+j-1)!L2^{-l-j}.$$

Altogether we found

$$\begin{aligned} \sum_{k \neq 0} \Gamma(j - \chi_k) \Gamma(l + \chi_k) &= -(j-1)!(l-1)! - (l+j-1)!L \sum_{h \geq 0} \frac{(-1)^{h-1}}{2^{j+h}-1} \binom{l+j+h-1}{h} \\ &\quad - (l+j-1)!L \sum_{h \geq 0} \frac{(-1)^{h-1}}{2^{l+h}-1} \binom{l+j+h-1}{h} + (l+j-1)!L2^{-l-j}. \end{aligned}$$

Now, let us look at

$$\sum_{k \neq 0} \Gamma(-\chi_k + l) \left(\Gamma'(\chi_k) + \gamma \Gamma(\chi_k) \right).$$

The proper function is

$$\frac{L}{2^z - 1} \Gamma(-z + l) \left(\Gamma'(z) + \gamma \Gamma(z) \right).$$

The poles at $z = \chi_k$, $k \neq 0$, lead to

$$S_1 = \sum_{k \neq 0} \Gamma(-\chi_k + l) \left(\Gamma'(\chi_k) + \gamma \Gamma(\chi_k) \right).$$

The pole at $z = 0$ leads to

$$S_2 = \Gamma(l) \left(\frac{\pi^2 - L^2}{12} - \frac{\gamma^2}{2} - \frac{L\gamma}{2} \right) - \Gamma'(l) \left(\gamma + \frac{L}{2} \right) - \frac{1}{2} \Gamma''(l).$$

The poles at $z = l, l+1, \dots$ lead to

$$\begin{aligned} S_3 &= L \sum_{h \geq 0} \frac{1}{2^{l+h}-1} \frac{(-1)^h}{h!} \left(\Gamma'(l+h) + \gamma \Gamma(l+h) \right) \\ &= L \sum_{h \geq 0} \frac{1}{2^{l+h}-1} \frac{(-1)^h (l+h-1)!}{h!} H_{l+h}. \end{aligned}$$

The poles at $z = -1, -2, \dots$ lead to

$$\begin{aligned} & - \sum_{h \geq 1} \frac{L}{1-2^{-h}} \Gamma(h+l) \frac{(-1)^h \gamma}{h!} \\ &= -L\gamma \sum_{h \geq 1} \frac{1}{2^h-1} \frac{(-1)^h (l+h-1)!}{h!} + L\gamma(1-2^{-l}). \end{aligned}$$

Therefore

$$\sum_{k \neq 0} \Gamma(-\chi_k + l) \left(\Gamma'(\chi_k) + \gamma \Gamma(\chi_k) \right) = -\Gamma(l) \left(\frac{\pi^2 - L^2}{12} - \frac{\gamma^2}{2} - \frac{L\gamma}{2} \right) + \Gamma'(l) \left(\gamma + \frac{L}{2} \right) + \frac{1}{2} \Gamma''(l)$$

$$-L \sum_{h \geq 0} \frac{1}{2^{l+h} - 1} \frac{(-1)^h (l+h-1)!}{h!} H_{l+h} + L\gamma \sum_{h \geq 1} \frac{1}{2^h - 1} \frac{(-1)^h (l+h-1)!}{h!} - L\gamma(1 - 2^{-l}).$$

Finally, let us consider

$$\sum_{k \neq 0} \left(\Gamma'(-\chi_k) + \gamma \Gamma(-\chi_k) \right) \left(\Gamma'(\chi_k) + \gamma \Gamma(\chi_k) \right).$$

The function of interest is

$$\frac{L}{2^z - 1} \left(\Gamma'(-z) + \gamma \Gamma(-z) \right) \left(\Gamma'(z) + \gamma \Gamma(z) \right).$$

The poles at $z = \chi_k$, $k \neq 0$, lead to

$$S_1 = \sum_{n \neq 0} \left(\Gamma'(-\chi_k) + \gamma \Gamma(-\chi_k) \right) \left(\Gamma'(\chi_k) + \gamma \Gamma(\chi_k) \right).$$

The pole at $z = 0$ leads to

$$S_2 = -\frac{11\pi^4}{360} - \frac{L^2\pi^2}{72} - \frac{L^4}{720}.$$

The poles at $z = 1, 2, \dots$ lead to

$$S_3 = L\gamma \sum_{h \geq 1} \frac{(-1)^{h-1} H_h}{h(2^h - 1)}.$$

The poles at $z = -1, -2, \dots$ lead to

$$\begin{aligned} S_3 &= -L\gamma \sum_{h \geq 1} \frac{(-1)^h H_h}{h(1 - 2^{-h})} \\ &= L\gamma \sum_{h \geq 1} \frac{(-1)^{h-1} H_h}{h(2^h - 1)} - L\gamma \sum_{h \geq 1} \frac{(-1)^{h-1} H_h}{h}. \end{aligned}$$

Altogether

$$\begin{aligned} \sum_{k \neq 0} \left(\Gamma'(-\chi_k) + \gamma \Gamma(-\chi_k) \right) \left(\Gamma'(\chi_k) + \gamma \Gamma(\chi_k) \right) &= \frac{11\pi^4}{360} + \frac{L^2\pi^2}{72} + \frac{L^4}{720} \\ -2L\gamma \sum_{h \geq 1} \frac{(-1)^{h-1} H_h}{h(2^h - 1)} + L\gamma \sum_{h \geq 1} \frac{(-1)^{h-1} H_h}{h}. \end{aligned}$$

The last alternating sum has a closed form evaluation:

$$\sum_{h \geq 1} \frac{(-1)^h H_h}{h} = \frac{L^2}{2} - \frac{\pi^2}{12}.$$

This gives us the constant term in w_1^4 :

$$\begin{aligned} c_4[0] &= [w_1^4]_0 = \left(\frac{2}{L} \sum_{h \geq 1} \frac{(-1)^{h-1}}{h(2^h - 1)} - \frac{11}{12} + \frac{\pi^2}{6L^2} \right)^2 \\ &+ \frac{4}{L^2} \sum_{j, l \geq 1} \frac{(-1)^{j+l}}{j!(2^j - 1)l!(2^l - 1)} \\ &\times \left[-(j-1)!(l-1)! - (l+j-1)!L \sum_{h \geq 0} \frac{(-1)^{h-1}}{2^{j+h} - 1} \binom{l+j+h-1}{h} \right] \end{aligned}$$

$$\begin{aligned}
& - (l+j-1)!L \sum_{h \geq 0} \frac{(-1)^{h-1}}{2^{l+h}-1} \binom{l+j+h-1}{h} + (l+j-1)!L2^{-l-j} \Big] \\
& + \frac{8}{L^3} \sum_{l \geq 1} \frac{(-1)^l}{l!(2^l-1)} \\
& \times \left[-\Gamma(l) \left(\frac{\pi^2 - L^2}{12} - \frac{\gamma^2}{2} - \frac{L\gamma}{2} \right) + \Gamma'(l) \left(\gamma + \frac{L}{2} \right) + \frac{1}{2} \Gamma''(l) - L\gamma(1-2^{-l}) \right. \\
& \left. - L \sum_{h \geq 0} \frac{1}{2^{l+h}-1} \frac{(-1)^h (l+h-1)!}{h!} H_{l+h} + L\gamma \sum_{h \geq 1} \frac{1}{2^h-1} \frac{(-1)^h (l+h-1)!}{h!} \right] \\
& + \frac{4}{L^4} \left[\frac{11\pi^4}{360} + \frac{L^2\pi^2}{72} + \frac{L^4}{720} - 2L\gamma \sum_{h \geq 1} \frac{(-1)^{h-1} H_h}{h(2^h-1)} - L\gamma \left(\frac{L^2}{2} - \frac{\pi^2}{12} \right) \right].
\end{aligned}$$

Although a few simplifications in this expression are still possible, it is clear that the complexity of the expressions does not make it attractive enough to write more similar evaluations.

We can now compute the corrected values.

Theorem 3.3 *Taking the contribution of products of Fourier series into account, the asymptotic dominant and oscillating parts of the corrected centered moments of $J(n)$ are given by*

$$\begin{aligned}
\tilde{\mu}_{2,c} &= \tilde{\mu}_2 - c_1[0] = 1 - C_1 - C_2 + 2\frac{D}{L}, \\
\tilde{\mu}_{3,c} &= \tilde{\mu}_3 + 2c_2[0] + \frac{6}{L}\gamma c_1[0] - \frac{6}{L^2}c_3[0] \\
&= -C_1 + \frac{6\zeta(3)}{L^3} - 2C_3 + 2 + 2\frac{D}{L} - \frac{12}{L^2}D_1 - \frac{12}{L^2}\gamma D - \frac{11}{2L}\gamma + \frac{\gamma\pi^2}{L^3} - 3C_2 - \frac{4}{L}D_2 - \frac{4}{L}\log(3) \\
&\quad - \frac{6}{L^2}c_3[0], \\
\tilde{\mu}_{4,c} &= \tilde{\mu}_4 - \frac{1}{2}c_1[0] - 3c_4[0] + \frac{\pi^2}{L^2}c_1[0] + 6C_1c_1[0] + 6C_2c_1[0] \\
&\quad - \frac{12}{L^2}\gamma^2c_1[0] - \frac{12}{L}\gamma c_2[0] + \frac{24}{L^3}\gamma c_3[0] + \frac{12}{L^2}c_6[0] + \frac{12}{L^3}c_7[0] + \frac{12}{L^3}c_8[0] \\
\kappa_{2,c} &= -\sum_{l \neq 0} c_1[l]e^{-2l\pi i \log n} - \frac{2\gamma w_1}{L} + \frac{2}{L^2}\rho_1, \\
\kappa_{3,c} &= 2 \sum_{l \neq 0} c_2[l]e^{-2l\pi i \log n} + \frac{6}{L}\gamma \sum_{l \neq 0} c_1[l]e^{-2l\pi i \log n} + \frac{(6\gamma^2 - \pi^2)w_1}{2L^2} \\
&\quad - \frac{6\gamma}{L^3}\rho_1 - \frac{6}{L^2} \sum_{l \neq 0} c_3[l]e^{-2l\pi i \log n} \\
&\quad - \frac{3}{L^3}\rho_2 - \frac{3}{L^3}\rho_4, \\
\kappa_{4,c} &= \left[\frac{12\gamma C_2}{L} + \frac{12\gamma C_1}{L} - \frac{8\zeta(3)}{L^3} - \frac{4\gamma^3}{L^3} - \frac{\gamma}{L} \right] w_1 \\
&\quad + \frac{L^2 - 12C_2L^2 - 12C_1L^2 + 12\gamma^2}{L^4}\rho_1 \\
&\quad + \frac{12}{L^4} \sum_{l \neq 0} e^{-2l\pi i \log n} \psi(\chi_l) \psi(1, \chi_l) \Gamma(\chi_l) + \frac{12\gamma}{L^4}\rho_4 \\
&\quad + \frac{4}{L^4} \sum_{l \neq 0} e^{-2l\pi i \log n} \psi(2, \chi_l) \Gamma(\chi_l) + \frac{4}{L^4} \sum_{l \neq 0} \rho_3 \\
&\quad + \frac{12\gamma}{L^4}\rho_2
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \sum_{l \neq 0} c_1[l] e^{-2l\pi i \log n} - 3 \sum_{l \neq 0} c_4[l] e^{-2l\pi i \log n} + \frac{\pi^2}{L^2} \sum_{l \neq 0} c_1[l] e^{-2l\pi i \log n} + 6C_1 \sum_{l \neq 0} c_1[l] e^{-2l\pi i \log n} \\
& + 6C_2 \sum_{l \neq 0} c_1[l] e^{-2l\pi i \log n} - \frac{12}{L^2} \gamma^2 \sum_{l \neq 0} c_1[l] e^{-2l\pi i \log n} - \frac{12}{L} \gamma \sum_{l \neq 0} c_2[l] e^{-2l\pi i \log n} + \frac{24}{L^3} \gamma \sum_{l \neq 0} c_3[l] e^{-2l\pi i \log n} \\
& + \frac{12}{L^2} \sum_{l \neq 0} c_6[l] e^{-2l\pi i \log n} + \frac{12}{L^3} \sum_{l \neq 0} c_7[l] e^{-2l\pi i \log n} + \frac{12}{L^3} \sum_{l \neq 0} c_8[l] e^{-2l\pi i \log n}.
\end{aligned}$$

Note that $\tilde{\mu}_{2,c}$ fits with the result given in [16].

4 m counters: asymptotic independence of the m counters

In this section, we analyze the asymptotic properties of the RV $J(m, n)$.

We will prove that, asymptotically, the counters are independent with n/m items each. We must analyze the random variable $J(m, n) = \sum_{i=1}^m J_i(n)$ where $J_i(n)$ has the distribution $p_\eta(j)$ with η is now given by ν_i : the number of items arriving in counter i . The quantity ν_i is $\text{bin}[\tilde{n}, \tilde{n}(1 - 1/m)]$ with $\tilde{n} := n/m$. Actually, $\{\nu_1, \dots, \nu_m\}$ is given by a multinomial distribution. We know that we can construct a “box”

$$[\tilde{n} - \tilde{n}^\theta, \tilde{n} + \tilde{n}^\theta]^m, \quad \frac{1}{2} < \theta < 1, \quad (16)$$

such that, by large deviation analysis, the probability that $\{\nu_1, \dots, \nu_m\}$ is outside this box is bounded by $\exp(-C\tilde{n}^{2\theta-1})$. We will analyze

$$J(m, n) - m \log(\tilde{n}) = \sum_{i=1}^m X_i, \quad \text{with } X_i := J_i - \log(\tilde{n}).$$

The rate of convergence is analyzed as follows.

4.1

Let us first assume that ν_i is exactly given by its mean \tilde{n} . As mentioned in the previous section, the rate of convergence problem is solved in Flajolet [2].

4.2

Now we assume that we are inside the box (16). We drop the \sim sign from \tilde{n} for convenience. Let $0 < \varepsilon < 1$. We must bound

$$S_2 := \sum_j j^k |p_n(j) - p_{n+\xi n^\theta}(j)|,$$

with $|\xi| < 1$. Note that $n + \xi n^\theta = n[1 + \xi n^{\theta-1}]$. Set $1 > \beta > \theta$.

- For $j < \beta \log n$, we have

$$\begin{aligned}
& 2^{-\eta} > n^{1-\beta}, \\
& |p_n(j)| \leq \frac{1}{Q^2} \sum_{k=0}^{\infty} \frac{1}{2^{k(k-1)/2}} \exp(-n^{1-\beta} 2^k) = \mathcal{O}(\exp(-n^{1-\beta})), \\
& |p_{n+\xi n^\theta}(j)| \leq \frac{1}{Q^2} \sum_{k=0}^{\infty} \frac{1}{2^{k(k-1)/2}} \exp(-n^{1-\beta} (1 + \xi n^{\theta-1}) 2^k) = \mathcal{O}(\exp(-n^{(1-\beta)(1-\varepsilon)})), \\
& |p_n(j) - p_{n+\xi n^\theta}(j)| = \mathcal{O}(\exp(-n^{(1-\beta)(1-\varepsilon)})).
\end{aligned}$$

- For $\beta \log n \leq j < 2 \log n$, we have

$$2^{-n} > \frac{1}{n},$$

$$(1 - 1/2^j)^n - (1 - 1/2^j)^{n+\xi n^\theta} = (1 - 1/2^j)^n (1 - (1 - 1/2^j)^{\xi n^\theta}) = \mathcal{O}(n^\theta / 2^j).$$

We use again the ‘‘sum splitting technique.’’ Set $r = \sqrt{2 \log n}$.

1. Truncating the sum in (2) to $k \geq r$ leads to an error E_1 :

$$E_1 \leq \frac{1}{Q^2} \sum_{k=r}^{\infty} \frac{1}{2^{k(k-1)/2}} \left[\exp\left(-\frac{2^k}{n}\right) + \exp\left(-\frac{1 + \xi n^{\theta-1}}{n} 2^k\right) \right] = \mathcal{O}\left(\frac{1}{n}\right)$$

2. The remaining sum $k \leq r$ leads to

$$E_2 \leq \frac{1}{Q^2} \sum_{k=0}^r \frac{1}{2^{k(k-1)/2}} \left[(1 - 1/2^{j-k})^n - (1 - 1/2^{j-k})^{n+\xi n^\theta} \right]$$

$$= \sum_{k=0}^r \mathcal{O}\left(\frac{n^\theta}{2^{j-k}}\right) = \mathcal{O}\left(\frac{1}{n^{(\beta-\theta)(1-\varepsilon)}}\right).$$

- For $j = 2 \log n + x$, $x \geq 0$, we set $r = \sqrt{2 \log n} + \sqrt{2x}$. So $1/2^{r^2/2} \leq 2^{-x}/n$. We proceed now as in the second range

- 1.

$$E_1 = \mathcal{O}\left(\frac{2^{-x}}{n}\right).$$

- 2.

$$E_2 \leq \frac{1}{Q^2} \sum_{k=0}^r \frac{1}{2^{k(k-1)/2}} \frac{n^\theta}{2^{j-k}} = \mathcal{O}\left(\frac{n^\theta}{n^{2(1-\varepsilon)} 2^{x(1-\varepsilon)}}\right).$$

Now we come to S_2 . We get

$$S_2 = \mathcal{O}\left((\beta \log n)^{k+1} \exp(-n^{(1-\beta)(1-\varepsilon)})\right) + (2 \log n)^{k+1} \mathcal{O}\left(\frac{1}{n^{(\beta-\theta)(1-\varepsilon)}}\right)$$

$$+ \mathcal{O}\left(\sum_{x \geq 0} (2 \log n + x)^k \mathcal{O}\left(\frac{2^{-x}}{n}\right)\right) = \mathcal{O}\left(\frac{1}{n^{(\beta-\theta)(1-\varepsilon)}}\right);$$

(not with the same ε , of course.)

4.3

Now we consider the case $\nu > n + n^\theta$. We will show that

$$S_3 := \sum_j p_\nu(j) j^k \exp(-Cn^{2\theta-1})$$

is small. We notice that $1 < \nu/n < m$. But, by the rate of convergence proved in [2], $\sum_j p_\nu(j) j^k$ is asymptotically bounded by $\mathcal{O}((\log \nu)^k) = \mathcal{O}((\log(nm))^k)$ and S_3 is asymptotically small.

4.4

The last case to consider is $0 < \nu < n - n^\theta$. We have here $0 < \nu/n < 1$. The analysis proceeds like in the previous subsection. We therefore omit the details.

In conclusion, as S_2 and S_3 are asymptotically small, we can assume that ν_i can be deterministically chosen as \tilde{n} for all i , and that the counters are asymptotically independent.

5 m counters: asymptotic moments

If J_i are iid RV, with asymptotic centered moments $\mu_k = \tilde{\mu}_k + \kappa_k$, then $J = \sum_{i=1}^m J_i$ has asymptotic distribution given by the convolution $f(\eta)^{(m)}$, mean

$$m \log(\tilde{n}) + m\mu_1$$

and asymptotic centered moments $\mu_k(m)$ given by

$$\begin{aligned}\mu_2(m) &= m\mu_2, \\ \mu_3(m) &= m\mu_3, \\ \mu_4(m) &= m[\mu_4 + 3(m-1)\mu_2^2].\end{aligned}$$

For $\mu_2(m)$ and $\mu_3(m)$, we immediately use μ_2 and μ_3 . For $\mu_4(m)$, we have $\mu_2 = \tilde{\mu}_2 + \kappa_2$, so $\mu_2^2 = \tilde{\mu}_2^2 + \kappa_2^2 + 2\tilde{\mu}_2\kappa_2$. Hence

$$\begin{aligned}\tilde{\mu}_4(m) &= m[\tilde{\mu}_4 + 3(m-1)\tilde{\mu}_2^2], \\ \kappa_4(m) &= m[\kappa_4 + 3(m-1)(\kappa_2^2 + 2\tilde{\mu}_2\kappa_2)].\end{aligned}$$

Also, we have

$$\begin{aligned}[\kappa_2^2]_0 &= c_4[0] + \frac{4\gamma^2 c_1[0]}{L^2} + \frac{4c_5[0]}{L^4} + \frac{4\gamma c_2[0]}{L} - \frac{4c_6[0]}{L^2} - \frac{8c_3[0]}{L^3}, \\ [\kappa_2^2]_k &= c_4[k] + \frac{4\gamma^2 c_1[k]}{L^2} + \frac{4c_5[k]}{L^4} + \frac{4\gamma c_2[k]}{L} - \frac{4c_6[k]}{L^2} - \frac{8c_3[k]}{L^3}.\end{aligned}$$

The corrected moments must now be computed. The interesting case is the fourth moment, since here the dependency on m is more involved: we obtain our last theorem.

Theorem 5.1 *Taking the contribution of products of Fourier series into account, the asymptotic dominant and oscillating parts of the corrected fourth centered moment of J are given by*

$$\begin{aligned}\tilde{\mu}_{4,c}(m) &= m[\tilde{\mu}_{4,c} + 3(m-1)\tilde{\mu}_2^2] + 3m(m-1)[[\kappa_2^2]_0 - 2\tilde{\mu}_2 c_1[0]], \\ \kappa_{4,c}(m) &= m\left[\kappa_{4,c} + 3(m-1)\sum_{l \neq 0} [\kappa_2^2]_l e^{-2l\pi i \log n} + 3(m-1)2\tilde{\mu}_2 \kappa_{2,c}\right].\end{aligned}$$

6 Conclusion

If we compare the approach in this paper with other ones that appeared previously, then we can notice the following. Traditionally, one would stay with exact enumerations as long as possible, and only at a late stage move to asymptotics. Doing this, one would, in terms of asymptotics, carry many unimportant contributions around, which makes the computations quite heavy, especially when it comes to higher moments. Here, however, approximations are carried out as early as possible, and this allows for streamlined (and often automatic) computations of asymptotic distributions and higher moments.

References

- [1] J. Cichoń and W. Macyna. Approximate counters for flash memory. *17th IEEE International Conference on Embedded and Real-Time Computing Systems and Applications (RTCSA), Toyama, Japan*, 2011.
- [2] P. Flajolet. Approximate counting: a detailed analysis. *BIT*, 25:113–134, 1985.

-
- [3] P. Flajolet and R. Sedgewick. Digital search trees revisited. *SIAM Journal on Computing*, 3:748–767, 1986.
- [4] M. Fuchs, C.K. Lee, and H. Prodinger. Approximate counting via the Poisson-Laplace-Mellin method. Technical report, 2011.
- [5] H.-K. Hwang, M. Fuchs, and V. Zacharovas. Asymptotic variance of random symmetric digital search trees. *Discrete Mathematics and Theoretical Computer Science*, 12:103–166, 2010.
- [6] P. Kirschenhofer and H. Prodinger. Approximate counting: An alternative approach. *RAIRO Theoretical Informatics and Applications*, 25:43–48, 1991.
- [7] D. E. Knuth. *The Art of Computer Programming*, volume 3: Sorting and Searching. Addison-Wesley, 1973. Second edition, 1998.
- [8] M. Loève. *Probability Theory, 3rd ed.* D. Van Nostrand, 1963.
- [9] G. Louchard. Exact and asymptotic distributions in digital and binary search trees. *RAIRO Theoretical Informatics and Applications*, 21(4):479–496, 1987.
- [10] G. Louchard and H. Prodinger. Asymptotics of the moments of extreme-value related distribution functions. Technical report, 2005. Long version: <http://www.ulb.ac.be/di/mcs/louchard/moml.ps>.
- [11] G. Louchard and H. Prodinger. Asymptotics of the moments of extreme-value related distribution functions. *Algorithmica*, 46:431–467, 2006.
- [12] G. Louchard and H. Prodinger. Generalized approximate counting revisited. *Theoretical Computer Science*, 391:109–125, 2008.
- [13] H. Prodinger. Hypothetic analyses: Approximate counting in the style of Knuth, path length in the style of Flajolet. *Theoretical Computer Science*, 100:243–251, 1992.
- [14] H. Prodinger. Approximate counting via Euler transform. *Mathematica Slovaca*, 44:569–574, 1994.
- [15] H. Prodinger. Periodic oscillations in the analysis of algorithms—and their cancellations. *Journal of the Iranian Statistical Society*, 3:251–270, 2004.
- [16] H. Prodinger. Approximate counting with m counters: a detailed analysis. *Theoretical Computer Science*, to appear, 2012.

APPLICABLE ANALYSIS AND DISCRETE MATHEMATICS

available online at <http://pefmath.etf.rs>APPL. ANAL. DISCRETE MATH. **5** (2011), 55–59.

doi:10.2298/AADM110205005O

WORDS CODING SET PARTITIONS

Kamilla Oliver, Helmut Prodinger

The words in the title are characterized by the fact that a smaller number must (first) appear earlier than a larger number, and that all numbers $1, \dots, k$ are present (for some k). Under the assumption that the letters are drawn from a geometric distribution, the probability that a word of length n enjoys these properties is determined, both exactly and asymptotically.

1. INTRODUCTION

For a set partition of $\{1, 2, \dots, n\}$ into k blocks, a natural coding is as follows: Element 1 is in block 1, and the smallest number not in block 1 is in block 2, and the smallest number not in blocks 1 or 2, is in block 3, etc. In this way, to every element i a number a_i is attached, namely the block in which it lies. Writing these numbers as a word $a_1 \dots a_n$, the set partition is coded in a natural way. One particular reference for this is [3].

Forgetting now about set partitions, we are talking about words where the letters are the positive integers, and, assuming that k is the largest letter that appears in the word, then the letters $1, \dots, k-1$ must also appear, and the word has exactly k (strict) left-to-right maxima, which is the same as saying that, if $i < j$, the first appearance of i is earlier than the first appearance of j . As one referee has kindly pointed out, such words are known as *restricted growth strings* in the literature [6].

Now we assign the (geometric) probability pq^{i-1} (where $p+q = 1$) to the letter i and consider P_n , the probability that a random word of length n has the *restricted growth* property. We are thus in the context of *combinatorics of geometrically distributed words*, a series of papers started with [4] and continued by the second writer as well as many others; a recent contribution is the paper [5].

2000 Mathematics Subject Classification. 00A05.

Keywords and Phrases. q -series, enumeration, asymptotics, periodic oscillation.

The present question is not only appealing from a combinatorial point of view (easy to formulate but not trivial to solve) but the approach used here (with the parameter q) leads to “richer” results, and often the instance $q = 1$ corresponds to the classical combinatorial instance, especially, when the parameter is of the *order statistics* type.

We will prove the following theorems.

Theorem 1. *The probability P_n that a random word of length n has the restricted growth property is (exactly) given by*

$$P_n = p \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} q^j (p; q)_j.$$

Here we use the (standard) notation $(x; q)_m = (1-x)(1-xq) \dots (1-xq^{m-1})$. We will also need the limit of it as $m \rightarrow \infty$, denoted by $(x; q)_\infty$, as well as the Gaussian q -binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

We need the following standard formulae:

$$\sum_{k=0}^N \begin{bmatrix} N \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} x^k = (x; q)_N, \quad \frac{1}{(w; q)_\infty} = \sum_{n \geq 0} \frac{w^n}{(q; q)_n}.$$

All this can be found in [1].

The asymptotic evaluation leads to our second theorem.

Theorem 2. *The probability that a random word of length n has the restricted growth property is asymptotically given by*

$$P_n \sim \frac{(p; q)_\infty}{L(q; q)_\infty} \Gamma\left(\frac{\log p}{\log q}\right) n^{-\frac{\log p}{\log q}} + n^{-\frac{\log p}{\log q}} \Phi(\log_Q n),$$

where $\Phi(x)$ is a 1-periodic function with mean zero. The abbreviations $Q = 1/q$ and $L = \log Q$ are used. The function is given by its Fourier series

$$\Phi(x) = \frac{(p; q)_\infty}{L(q; q)_\infty} \sum_{k \neq 0} \Gamma\left(\frac{\log p}{\log q} + \frac{2\pi i k}{L}\right) e^{-2\pi i k x}.$$

In the symmetric case $p = q$, this looks better:

$$\frac{1}{L} n^{-1} + n^{-1} \Phi(\log_2 n).$$

2. ANALYSIS

We use the natural decomposition

$$1\{\leq 1\}^*2\{\leq 2\}^*3\{\leq 3\}^* \dots k\{\leq k\}^*,$$

which translates into

$$\frac{zp}{1-(1-q)z} \frac{zpq}{1-(1-q^2)z} \dots \frac{zpq^{k-1}}{1-(1-q^k)z} = z^k p^k q^{\binom{k}{2}} \prod_{j=1}^k \frac{1}{1-(1-q^j)z}.$$

This has to be summed over all k , to get the generating function of the sought probabilities (P_n is the coefficient of z^n in this series):

$$\sum_{k \geq 1} z^k p^k q^{\binom{k}{2}} \prod_{j=1}^k \frac{1}{1-(1-q^j)z}.$$

Substituting $z = w/(w-1)$, this becomes

$$\sum_{k \geq 1} w^k (-1)^k p^k q^{\binom{k}{2}} \prod_{j=1}^k \frac{1}{1-wq^j} = \sum_{k \geq 1} \frac{w^k (-1)^k p^k q^{\binom{k}{2}}}{(wq; q)_k}.$$

Reading off coefficients:

$$\begin{aligned} P_n &= [z^n] \sum_{k \geq 1} \frac{w^k (-1)^k p^k q^{\binom{k}{2}}}{(wq; q)_k} \\ &= \frac{1}{2\pi i} \oint \sum_{k \geq 1} \frac{dz}{z^{n+1}} \frac{w^k (-1)^k p^k q^{\binom{k}{2}}}{(wq; q)_k} \quad \text{by Cauchy's integral formula} \\ &= \frac{1}{2\pi i} \oint \sum_{k \geq 1} \frac{dw(1-w)^{n-1}}{w^{n+1}} \frac{w^k (-1)^{n-k} p^k q^{\binom{k}{2}}}{(wq; q)_k} \\ &= \sum_{k=1}^n [w^{n-k}] (1-w)^{n-1} \frac{(-1)^{n-k} p^k q^{\binom{k}{2}}}{(wq; q)_k} \\ &= \sum_{k=1}^n \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j [w^{n-k-j}] \frac{(-1)^{n-k} p^k q^{\binom{k}{2}}}{(wq; q)_k} \\ &= \sum_{k=1}^n \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{n-k-j} p^k q^{\binom{k}{2}} \left[\begin{matrix} n-j-1 \\ k-1 \end{matrix} \right]_q q^{n-k-j} \quad \begin{array}{l} \text{the known expansion} \\ \text{of the denominator} \end{array} \\ &= p \sum_{j=0}^{n-1} \binom{n-1}{j} q^{n-j-1} (-1)^{n-j-1} \sum_{k=0}^{n-j-1} (-1)^k p^k q^{\binom{k}{2}} \left[\begin{matrix} n-j-1 \\ k \end{matrix} \right]_q \end{aligned}$$

$$\begin{aligned}
 &= p \sum_{j=0}^{n-1} \binom{n-1}{j} q^{n-j-1} (-1)^{n-j-1} (p; q)_{n-j-1} && \text{the sum is known} \\
 & && \text{as Rothe's sum} \\
 &= p \sum_{j=0}^{n-1} \binom{n-1}{j} q^j (-1)^j (p; q)_j.
 \end{aligned}$$

Is there a more direct way to prove this formula?

Here is an example for $n = 3$; the words enjoying the restricted growth property are 111, 112, 121, 122, 123, and they appear with probabilities $p^3, p^3q, p^3q, p^3q^2, p^3q^3$. And

$$\begin{aligned}
 p^3 + p^3q + p^3q + p^3q^2 + p^3q^3 &= p \sum_{j=0}^2 \binom{2}{j} q^j (-1)^j (p; q)_j \\
 &= p(1 - 2q(1 - p) + q^2(1 - p)(1 - pq)).
 \end{aligned}$$

For the asymptotic evaluation, we use the following integral representation as in [2]:

$$p \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} q^j (p; q)_j = \frac{-p}{2\pi i} \int_{\mathcal{C}} q^z (p; q)_z \frac{\Gamma(n)\Gamma(-z)}{\Gamma(n-z)} dz.$$

Here, \mathcal{C} enclosed the poles $0, 1, \dots, n - 1$ and no others, and the interpretation of $(p; q)_z$ is

$$(p; q)_z = \frac{(p; q)_{\infty}}{(pq^z; q)_{\infty}}.$$

For the readers' convenience we note that $n! = \Gamma(n + 1)$, and thus

$$\frac{\Gamma(n)\Gamma(-z)}{\Gamma(n-z)} = \frac{\Gamma(n)}{(n-z-1)(n-z-2)\cdots(-z)} = \frac{(-1)^n(n-1)!}{z(z-1)\cdots(z+1-n)}.$$

Furthermore, the residue of this expression at $z = k$ is

$$\frac{(-1)^n(n-1)!}{k(k-1)\cdots 1 \cdot (-1)\cdots(k+1-n)} = \frac{(-1)^{k-1}(n-1)!}{k!(n-1-k)!}.$$

To get asymptotics, we extend the contour of integration and have to consider the residues at the extra poles of

$$\frac{pq^z(p; q)_{\infty}}{(1-pq^z)(pq^{z+1}; q)_{\infty}} \frac{\Gamma(n)\Gamma(-z)}{\Gamma(n-z)}.$$

The poles with largest real part leading to the dominant contribution are at

$$z = -\frac{\log p}{\log q} + \frac{2\pi ik}{\log q}, \quad \text{for } k \in \mathbb{Z}.$$

For $k = 0$ we get the interesting term, and the others define a small fluctuation around this value. We find:

$$\begin{aligned} & \frac{pq^{-\frac{\log p}{\log q} + \frac{2\pi ik}{\log q}} (p; q)_\infty}{L(pq^{1 - \frac{\log p}{\log q} - \frac{2\pi ik}{\log q}}; q)_\infty} \frac{\Gamma(n)\Gamma\left(\frac{\log p}{\log q} + \frac{2\pi ik}{L}\right)}{\Gamma\left(n + \frac{\log p}{\log q} + \frac{2\pi ik}{L}\right)} \\ &= \frac{(p; q)_\infty}{L(q^{1 - \frac{2\pi ik}{L}}; q)_\infty} \frac{\Gamma(n)\Gamma\left(\frac{\log p}{\log q} + \frac{2\pi ik}{L}\right)}{\Gamma\left(n + \frac{\log p}{\log q} + \frac{2\pi ik}{L}\right)} \sim \frac{(p; q)_\infty \Gamma\left(\frac{\log p}{\log q} + \frac{2\pi ik}{L}\right)}{L(q; q)_\infty} n^{-\frac{\log p}{\log q} - \frac{2\pi ik}{L}}. \end{aligned}$$

The term $k = 0$ leads to

$$\frac{(p; q)_\infty \Gamma\left(\frac{\log p}{\log q}\right)}{L(q; q)_\infty} n^{-\frac{\log p}{\log q}}$$

and the other ones to $n^{-\frac{\log p}{\log q}} \Phi(\log_Q n)$, where $\Phi(x)$ is a 1-periodic function with mean zero. Note that $pq^{-t\frac{\log p}{\log q} + \frac{2\pi ik}{\log q}} = 1$, which was used in these computations.

Acknowledgments. Thanks are due to a referee who spotted a fundamental error in an earlier version. This material is based upon work supported by the National Research Foundation of South Africa under grant number 2053748.

REFERENCES

1. G. ANDREWS, R. ASKEY, R. ROY: *Special Functions*. Volume **71** of *Encyclopedia Math. Appl.*, Cambridge University Press, 1999.
2. P. FLAJOLET, R. SEDGEWICK: *Mellin transforms and asymptotics: Finite differences and Rice's integrals*. Theoret. Comput. Sci., **144** (1995), 101–124.
3. A. KNOPFMACHER, T. MANSOUR, S. WAGNER: *Records in set partitions*. Electron. J. Combin., **17** (2010), R109.
4. H. PRODINGER: *Combinatorics of geometrically distributed random variables: Left-to-right maxima*. Discrete Math., **153** (1996), 253–270.
5. H. PRODINGER: *Records in geometrically distributed words: Sum of positions*. Appl. Anal. Discrete Math., **2** (2008), 234–240.
6. D. STANTON, D. WHITE: *Constructive Combinatorics*. Springer, 1986.

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(Received August 16, 2010)
(Revised February 3, 2011)

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Set partitions, words, and approximate counting with black holes

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Abstract

Words satisfying the restricted growth property

$$w_k \leq 1 + \max\{w_0, \dots, w_{k-1}\}$$

are in correspondence with set partitions. Underlying the geometric distribution to the letters, these words are enumerated with respect to the largest letter occurring, which corresponds to the number of blocks in the set partition. It turns out that on average, this parameter behaves like $\log_{1/q} n$, where q is the parameter of the geometric distribution.

1 Introduction

Set partitions of $\{1, \dots, n\}$ can be coded by words $w_1 \dots w_n$, where the letters are positive integers, and (with $w_0 = 0$) the *restricted growth property* $w_k \leq 1 + \max\{w_0, \dots, w_{k-1}\}$ holds for $1 \leq k \leq n$; compare [8].

In [9], a study of such words was started where the letters have been equipped with geometric probabilities pq^{k-1} for $k = 1, 2, \dots$ and $p + q = 1$. If letters are drawn independently, then for $P(n)$, the probability that a word of length n has the restricted growth property, an explicit and an asymptotic formula were found [9, 8]:

$$P(n) = p \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} q^j (p; q)_j = \sum_{j=0}^n (-1)^j \binom{n}{j} (p; q)_j,$$

* This author is supported by an incentive grant from the NRF of South Africa. This note was prepared while he was a guest at the Academia Sinica in Taipei, Taiwan. The hospitality is gratefully acknowledged.

with $(x; q)_n = (1 - x)(1 - xq) \cdots (1 - xq^{n-1})$. This notation is used also for $(x; q)_\infty$ when the product is extended to infinity; see [1]. Further, with $b = \log p / \log q$, $Q = \frac{1}{q}$, $L := \log Q$, $\chi_k = \frac{2\pi ik}{L}$,

$$P(n) = \frac{(p; q)_\infty}{L(q; q)_\infty} \Gamma(b) n^{-b} + n^{-b} \frac{(p; q)_\infty}{L(q; q)_\infty} \sum_{k \neq 0} \Gamma(b + \chi_k) e^{-2\pi ik \cdot \log_Q n} + O(n^{-b-1}).$$

Note that the series is a Fourier series and represents a (small) periodic function.

A convenient way to imagine the evolution of these probabilities is by a state diagram as in Figure 1:

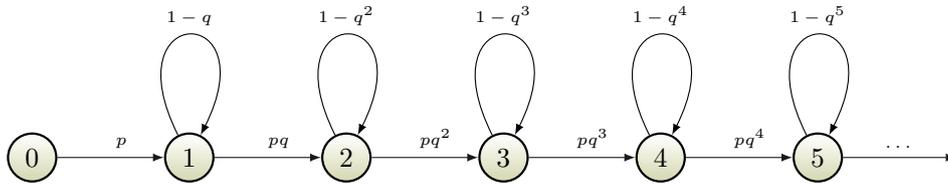


Figure 1: State diagram of the evolution of the probabilities of geometrically distributed words satisfying the restricted growth property.

The states (labelled $0, 1, \dots$) represent the largest letter seen so far, and the restricted growth property only allows staying in such a state or advancing to the state with label one higher. The edges are labelled with the probabilities to either remain in a state or advance to the next one. Note that the sum of them, $1 - q^k + pq^k = 1 - q^{k+1} < 1$, which means that in each state there is a chance to violate the restricted growth property. We call this *falling into a black hole*. Reading further letters does not help; there is no escape from a black hole.

In the recent paper [8], the analysis was extended, by computing, *inter alia*, the probability $P(n, k)$ to end up in state k after n random letters. Naturally, $P(n) = \sum_k P(n, k)$. Although it was not mentioned explicitly, the method to derive $P(n, k)$ is one that is very common when one analyzes *digital search trees*; see [3]. It is worthwhile to mention that k represents the number of blocks of the set partition corresponding to the word with the restricted growth property.

The state diagram as mentioned is, however, not uncommon in the literature. It resembles the one appearing in *approximate counting*:

In approximate counting, one is interested in the state one is in after n random steps. This state number is interpreted as the value of a certain counter. This model was first analyzed by Flajolet in [2] and remains popular to this day; [6] is a recent example, and it contains many backward pointers to the literature. Flajolet derived explicit expressions to reach state k after n random steps, and computed expectation and variance of this parameter.

Since we know so much about approximate counting and the situation is so similar, it should be possible to gain some additional insight to the analysis of the restricted growth model. This is indeed the motivation for the present note: while

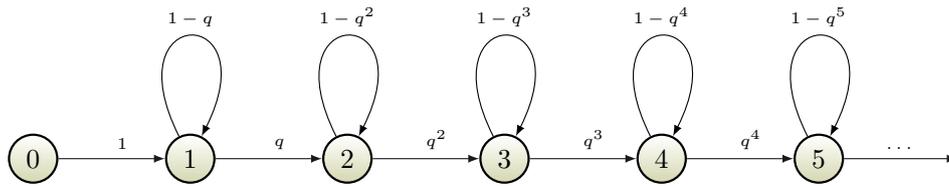


Figure 2: State diagram of the evolution of the counter in *approximate counting*. Normally, the (deterministic) initial step from state 0 to state 1 is not shown.

approximate counting might be known to some people in theoretical computer science, it is most likely new to the combinatorics community who reads this journal. As a general reference for the methods we are using, we cite the book [5] which represents the state of the art in *analytic combinatorics* and its applications to the *analysis of algorithms*.

The discussion given thus far motivates the catchy nickname *approximate counting with black holes*; for completeness, one might think about an extra state (the black hole) in which one falls from each other state k with the appropriate probability q^{k+1} , when the restricted growth property is violated for the first time. And, naturally, one cannot escape from the black hole.

The quantities

$$\frac{P(n, k)}{P(n)}$$

sum to 1 and define a probability distribution. We will, in the next section, (re)derive an explicit formula for $P(n, k)$ and then study the average value of the probability distribution just described. This answers the question “Which state does one reach, on average, after n random steps, subject to the condition that the random word satisfies the restricted growth property?” Higher moments could also be studied, but we refrain from doing that for the sake of brevity.

We cite here the average value C_n in approximate counting; Flajolet only gave it for $p = q = \frac{1}{2}$, but the analysis is easily extended:

$$C_n = \log_Q n - \alpha + \frac{\gamma}{L} + \frac{1}{2} - \frac{1}{L} \sum_{k \neq 0} \Gamma(\chi_k) e^{-2\pi i k \cdot \log_Q n} + O\left(\frac{1}{n}\right),$$

where γ is Euler’s constant and

$$\alpha = \sum_{k \geq 1} \frac{q^k}{1 - q^k}.$$

Within this range of accuracy, it does not matter whether one has one extra step in the beginning or not.

2 Analysis of approximate counting with black holes

In [8] it was shown that

$$P(n, k) = pq^{k-1}P(n-1, k-1) + (1-q^k)P(n-1, k), \quad P(0, 0) = 1.$$

This recursion is immediately understood when looking at the state diagram and how one can reach state k using the n th letter.

In the relatively recent paper [7], we collected many old and new results and techniques. We will rederive a formula for $P(n, k)$ following a technique presented there. Set

$$F(z, u) = \sum_{n, k} z^n u^k P(n, k);$$

then the recursion translates into the functional equation

$$F(z, u) - 1 = pzuF(z, qu) + zF(z, u) - zF(z, qu).$$

We want to solve this equation by *iteration*. However, since $F(z, u) = 1 + \dots$, this does not work. Thus we define $G(z, u) := F(z, u) - 1$ and rewrite the equation:

$$G(z, u) = pzu + pzuG(z, qu) + zG(z, u) - zG(z, qu),$$

which can now be iterated:

$$\begin{aligned} G(z, u) &= \frac{pzu}{1-z} + \frac{z(pu-1)}{1-z}G(z, qu) \\ &= \frac{pzu}{1-z} + \frac{z(pu-1)}{1-z} \left[\frac{pqzu}{1-z} + \frac{z(pqu-1)}{1-z}G(z, q^2u) \right] \\ &= \frac{pzu}{1-z} + \frac{z(pu-1)}{1-z} \left[\frac{pqzu}{1-z} + \frac{z(pqu-1)}{1-z} \left[\frac{pq^2zu}{1-z} + \frac{z(pq^2u-1)}{1-z}G(z, q^3u) \right] \right] \\ &= \dots = pu \sum_{j \geq 0} \frac{z^{j+1}(-1)^j (pu; q)_j q^j}{(1-z)^{j+1}}. \end{aligned}$$

From this we find, for $n \geq 1$,

$$[z^n]G(z, u) = pu \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j (pu; q)_j q^j = \sum_{j=0}^n \binom{n}{j} (-1)^j (pu; q)_j.$$

Apart from the normalization by dividing this by $P(n)$, this is a probability generating function.

Furthermore, we have for $k \geq 1$

$$\begin{aligned} [z^n u^k]G(z, u) &= P(n, k) \\ &= p^k \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j [u^{k-1}](u; q)_j q^j \\ &= p^k \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j q^j \sum_{t=0}^{k-1} \frac{(-1)^t q^{\binom{t}{2}} q^{(k-1-t)j}}{(q; q)_t (q; q)_{k-1-t}} \\ &= p^k \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \sum_{t=0}^{k-1} \frac{(-1)^t q^{\binom{t}{2}} q^{(k-t)j}}{(q; q)_t (q; q)_{k-1-t}} \\ &= p^k \sum_{t=0}^{k-1} \frac{(-1)^t q^{\binom{t}{2}}}{(q; q)_t (q; q)_{k-1-t}} (1 - q^{k-t})^{n-1}; \end{aligned}$$

in the second line a formula due to Rothe [1] has been used.

This formula is equivalent to Theorem 3.1 from [8] and provides an explicit formula for $P(n, k)$.

To find the expected value as promised, one starts from

$$[z^n]G(z, u) = \sum_{j=0}^n \binom{n}{j} (-1)^j (pu; q)_j,$$

differentiates this with respect to u , and then plugs in $u = 1$, with the result

$$g(n) := \sum_{j=0}^n \binom{n}{j} (-1)^j (p; q)_j \left[-\alpha_p + \sum_{l \geq j} \frac{pq^l}{1 - pq^l} \right].$$

Here, we use the abbreviation

$$\alpha_p = \sum_{l \geq 0} \frac{pq^l}{1 - pq^l}.$$

We rewrite this as

$$\begin{aligned} g(n) &= \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{(q^b; q)_\infty}{(q^{b+j}; q)_\infty} \left[\frac{q^{b+j}}{1 - q^{b+j}} - \alpha_p + \sum_{l \geq 1} \frac{q^{l+b+j}}{1 - q^{l+b+j}} \right] \\ &= \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{(q^b; q)_\infty}{(q^{b+j}; q)_\infty} \frac{q^{b+j}}{1 - q^{b+j}} \\ &\quad + \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{(q^b; q)_\infty}{(q^{b+j}; q)_\infty} \left[-\alpha_p + \sum_{l \geq 1} \frac{q^{l+b+j}}{1 - q^{l+b+j}} \right] =: E_1 + E_2. \end{aligned}$$

There is a convenient technique to handle such alternating sums; it is called *Rice's method*, and is described at length in [4]:

$$E_1 = -\frac{1}{2\pi i} \int_c \frac{\Gamma(n+1)\Gamma(-z)}{\Gamma(n+1-z)} \frac{(q^b; q)_\infty}{(q^{b+1+z}; q)_\infty} \frac{q^{b+z}}{(1 - q^{b+z})^2} dz,$$

where the curve \mathcal{C} encloses the poles at $0, 1, \dots, n$ and no others. To find asymptotics, one extends the curve and has, as a compensation, to subtract the extra residues that one encounters. In our case the relevant ones are at $z = -b - \chi_k, k \in \mathbb{Z}$, and we must find the residues of

$$\frac{\Gamma(n+1)\Gamma(-z)}{\Gamma(n+1-z)} \frac{(q^b; q)_\infty}{(q^{b+1+z}; q)_\infty} \frac{q^{b+z}}{(1-q^{b+z})^2}$$

at these poles. With $w = z + b + \chi_k$, we must expand

$$\frac{\Gamma(n+1)\Gamma(-w+b)}{\Gamma(n+1-w+b)} \frac{(q^b; q)_\infty}{(q^{w+1}; q)_\infty} \frac{q^w}{(1-q^w)^2}$$

to two terms around $w = 0$. So we compute

$$\begin{aligned} & [w^{-1}] \frac{\Gamma(n+1)\Gamma(-w+c)}{\Gamma(n+1-w+c)} \frac{(p; q)_\infty}{(q; q)_\infty} [1 - L\alpha w] \frac{1}{L^2 w^2} \\ &= -L\alpha \frac{\Gamma(n+1)\Gamma(c)}{\Gamma(n+1+c)} \frac{(p; q)_\infty}{(q; q)_\infty} \frac{1}{L^2} + [w^1] \frac{\Gamma(n+1)\Gamma(-w+c)}{\Gamma(n+1-w+c)} \frac{(p; q)_\infty}{(q; q)_\infty} \frac{1}{L^2} \\ &= -\alpha \frac{\Gamma(n+1)\Gamma(c)}{\Gamma(n+1+c)} \frac{(p; q)_\infty}{(q; q)_\infty} \frac{1}{L} - \frac{\Gamma(n+1)\Gamma'(c)}{\Gamma(n+1+c)} \frac{(p; q)_\infty}{(q; q)_\infty} \frac{1}{L^2} \\ & \quad + \frac{\Gamma(n+1)\Gamma(c)\psi(n+1+c)}{\Gamma(n+1+c)} \frac{(p; q)_\infty}{(q; q)_\infty} \frac{1}{L^2}, \end{aligned}$$

with $\psi = \Gamma'/\Gamma$. We have chosen the letter c to represent $b + \chi_k$.

Asymptotically, as $n \rightarrow \infty$:

$$-\alpha n^{-c} \Gamma(c) \frac{(p; q)_\infty}{(q; q)_\infty} \frac{1}{L} - n^{-c} \Gamma'(c) \frac{(p; q)_\infty}{(q; q)_\infty} \frac{1}{L^2} + n^{-c} \Gamma(c) \log_Q n \frac{(p; q)_\infty}{(q; q)_\infty} \frac{1}{L}.$$

These terms must be summed over $k \in \mathbb{Z}$, and we see that in the main term the same periodic oscillation appears that was already present in the study of $P(n)$ itself.

After normalization (division by the asymptotic equivalent of $P(n)$, mentioned earlier):

$$\log_Q n - \alpha + \text{oscillation}.$$

We also have to consider the term

$$E_2 = \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{(q^b; q)_\infty}{(q^{b+j}; q)_\infty} \left[-\alpha_p + \sum_{l \geq 1} \frac{q^{l+b+j}}{1 - q^{l+b+j}} \right],$$

which means that we have to compute the residues of

$$\frac{\Gamma(n+1)\Gamma(-z)}{\Gamma(n+1-z)} \frac{(q^b; q)_\infty}{(q^{b+1+z}; q)_\infty} \frac{1}{1 - q^{b+z}} \left[-\alpha_p + \sum_{l \geq 1} \frac{q^{l+b+z}}{1 - q^{l+b+z}} \right]$$

at $z = -b$ and also at $z = -b - \chi_k$. However, the expression at the square bracket has zeros there, and thus there are no residues originating from E_2 .

The oscillatory function is given by

$$-\frac{1}{L} \frac{\sum_{k \in \mathbb{Z}} \Gamma'(b + \chi_k) e^{-2\pi i k \cdot \log_Q n}}{\sum_{k \in \mathbb{Z}} \Gamma(b + \chi_k) e^{-2\pi i k \cdot \log_Q n}}.$$

As it stands, it is not small, but one can pull out the main term which originates from the terms for $k = 0$ in the numerator and denominator and thus has represented it as

$$-\frac{1}{L} \psi(b) + \omega(\log_Q n),$$

where $\omega(x)$ is now a tiny oscillation that occurs so frequently in the *analysis of algorithms*; see [5].

Summarizing, we have our main result.

Theorem 1 *The average state reached in approximate counting with black holes is after n random steps given by*

$$\log_Q n - \alpha - \frac{1}{L} \psi(b) + \omega(\log_Q n) + O\left(\frac{1}{n}\right);$$

the error term originates from the neglected poles at $b - 1 + \chi_k$. The periodic function $\omega(x)$ (of period 1) has small amplitude, due to the rapid decay of the Gamma function and its derivatives along vertical lines.

Note that in the symmetric case $p = q$, we have $b = 1$, and $\psi(1) = -\gamma$.

3 Conclusion

One could do many other things, following the numerous papers on approximate counting as role models. However, we are not going to do that, as the motivation to write this note was to link two areas (set partitions and approximate counting) that *a priori* do not seem to have much in common.

Further research will concentrate on the modified growth property

$$w_k \leq d + \max\{w_0, \dots, w_{k-1}\}.$$

For $d \geq 2$, it seems unlikely that explicit enumerations will work. But an asymptotic analysis should be still within reach, although with more advanced methods.

References

- [1] G. Andrews, R. Askey and R. Roy, *Special Functions*, vol. 71 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, 1999.
- [2] P. Flajolet, Approximate counting: a detailed analysis, *BIT* 25 (1985), 113–134.
- [3] P. Flajolet and R. Sedgewick, Digital search trees revisited, *SIAM J. Comp.* 15 (1986), 748–767.

- [4] P. Flajolet and R. Sedgewick, Mellin transforms and asymptotics: Finite differences and Rice's integrals, *Theoretical Comp. Sci.* 144 (1995), 101–124.
- [5] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.
- [6] M. Fuchs, C.-K. Lee, and H. Prodinger, Approximate counting via the Poisson-Laplace-Mellin Method, *DMTCS Proc. AQ* (2012), 13–28.
- [7] G. Louchard and H. Prodinger, Generalized approximate counting revisited, *Theoretical Comp. Sci.* 391 (2008), 109–125.
- [8] T. Mansour and M. Shattuck, Set partitions as geometric words, *Australas. J. Combin.* 53 (2012), 31–39.
- [9] K. Oliver and H. Prodinger, Words coding set partitions, *Applicable Analysis and Discrete Math.* 5 (2011), 55–59.

(Received 17 Mar 2012; revised 26 June 2012)

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Indagationes Mathematicae 24 (2013) 1024–1033

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mathematicae**www.elsevier.com/locate/indag

Words with a generalized restricted growth property

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Dedicated to the memory of N. G. de Bruijn

Abstract

Words where each new letter (natural number) can never be too large, compared to the ones that were seen already, are enumerated. The letters follow the geometric distribution. Also, the maximal letter in such words is studied. The asymptotic answers involve small periodic oscillations. The methods include a chain of techniques: exponential generating function, Poisson generating function, Mellin transform, depoissonization.

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Keywords: Random words; Restricted growth property; Depoissonization; Mellin transform

1. Introduction

When Knuth started his fundamental series of books *The Art of Computer Programming* [8], de Bruijn was (one of) his asymptotic advisor(s). In particular, he suggested how to evaluate sums like

$$\sum_{k \geq 1} (1 - e^{-n/2^k}) \quad \text{and} \quad \sum_{k \geq 1} d(k) e^{-k^2/n},$$

where $d(k)$ is the number of divisors of k . Although the word was not mentioned in the first editions, in essence it was the *Mellin transform* that found its way into [9]. Around the same time, the paper [2] appeared, which has 166 citations by google scholar.¹ This paper has a third

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¹ As of November 5th, 2012.

coauthor, Rice, who also suggested asymptotic methods to Knuth; there is an innocent exercise in [9], which lead later to developments called *Rice's method*; see [4].

We briefly review the Mellin transform method in asymptotic enumeration, compare with [3,5].

$$\mathcal{M}[f(x); s] = f^*(s) = \int_0^\infty f(x)x^{s-1}dx.$$

There is the *harmonic sum property*

$$\mathcal{M}\left[\sum_{k \geq 1} a_k f(b_k x); s\right] = \sum_{k \geq 1} a_k b_k^{-s} \cdot \mathcal{M}[f(x); s]. \quad (1)$$

This is particularly useful if the series has a convergent closed form evaluation (often in terms of the zeta function etc.).

Typically the Mellin transform exists in a *vertical strip* of the complex plane. There is an *inversion formula*

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s}ds,$$

where c must be in the vertical strip. Shifting the line of integration to the left/right and collecting residues provides the asymptotic expansion. The choice of left/right depends on whether one needs the expansion for $x \rightarrow \infty$ or $x \rightarrow 0$; see the converse mapping theorem in [3] for a precise statement of this fact.

The most prominent example is $f(x) = e^{-x}$, so that $f^*(s) = \Gamma(s)$, whence the term *Gamma function method* was originally coined. During the last 40 years, de Bruijn's suggestion led to numerous further developments and applications.

In the technical part of this paper, we will indeed use the Mellin transform to deal with a combinatorial (discrete probability) problem. As often in combinatorics, the problem is not difficult to describe, although the solution requires some technical machinery. We consider words $w_1 w_2 \cdots w_n$ where the letters are positive integers, and integer k appears with (geometric) probability pq^{k-1} , and $p + q = 1$. The letters are independent from each other. The *restricted growth property* is satisfied when

$$w_k \leq 1 + \max\{w_1, \dots, w_{k-1}\} \quad \text{for all } k \text{ and } w_0 = 0.$$

The words that satisfy the restricted growth property are related to *set partitions* and *approximate counting* [10,12]. The asymptotic enumeration of restricted words of length n and the asymptotic study of $\max\{w_1, \dots, w_n\}$ was done in [11,12] using the above mentioned Rice method.

Now it is a natural extension to introduce a parameter:

$$w_k \leq d + \max\{w_1, \dots, w_{k-1}\} \quad \text{for all } k \text{ and } w_0 = 0. \quad (2)$$

For $d \geq 2$, the asymptotic problems are of a more delicate nature, and that is what we will do here. Rice's method is based on explicit enumerations represented as *alternating sums*. We use here a combination of techniques that is more flexible: poissonization/depoissonization and Mellin transform. Poissonization is the process of replacing the fixed n by a random variable which is Poisson distributed with parameter z ; depoisonization is the reversed process that allows to go back from z to n . Typically, if $f(z)$ is an exponential generating function of a

sequence a_n , then, with $\tilde{f}(z) := e^{-z} f(z)$, $a_n \sim \tilde{f}(n)$, provided certain conditions are satisfied. The asymptotic study of the behaviour of the Poissonized version when $z \rightarrow \infty$ is achieved using the Mellin transform. This is the rough plan; the details are in the following sections.

Notation. We collect here some notation which we are going to use throughout this work. First,

$$Q := 1/q, \quad L := \log Q, \quad \chi_k := 2k\pi i/L, \quad k \in \mathbb{Z}.$$

Moreover, if $f(z)$ is a meromorphic function with singularity at $z = \rho$ and singularity expansion $E(z)$, then we will write $f(z) \asymp E(z)$.

2. Asymptotic enumeration of words satisfying the restricted growth property

Let p_n be the probability that a random word $w_1 \cdots w_n$ satisfies the restricted growth property (2).

We first condition on whether w_1 is l with $1 \leq l \leq d$. Note that the probability for this is pq^{l-1} and the probability that any other letter is $\leq l$ is

$$pq + pq^2 + \cdots + pq^{l-1} = p \frac{1 - q^l}{1 - q} = 1 - q^l.$$

Hence, by further conditioning on the number of letters $\leq l$ in $w_2 \cdots w_{n+1}$, we obtain

$$p_{n+1} = \sum_{l=1}^d pq^{l-1} \sum_{j=0}^n \binom{n}{j} (1 - q^l)^{n-j} q^{lj} p_j \quad (n \geq 0)$$

with initial condition $p_0 = 1$.

The binomial convolution on the right-hand side of the recurrence above suggests the use of *exponential generating functions*. Therefore, set

$$f(z) := \sum_{n \geq 0} p_n \frac{z^n}{n!}.$$

Then

$$f'(z) = \sum_{l=1}^d pq^{l-1} e^{(1-q^l)z} f(q^l z).$$

Next, consider the *Poisson generating function*

$$\tilde{f}(z) := e^{-z} \sum_{n \geq 0} p_n \frac{z^n}{n!}.$$

Then, the above differential–functional equation becomes

$$\tilde{f}(z) + \tilde{f}'(z) = \sum_{l=1}^d pq^{l-1} \tilde{f}(q^l z).$$

The goal is now to find the behaviour of $\tilde{f}(z)$ as $z \rightarrow \infty$, since $p_n \sim \tilde{f}(n)$ by general principles. This goal will be achieved using the Mellin transform. Recall that the Mellin transform of a derivative is given by [3]

$$\mathcal{M}[\tilde{f}'(z); \omega] = -(\omega - 1)\mathcal{M}[\tilde{f}(z); \omega - 1].$$

Using this and (1) from the introduction yields

$$\mathcal{M}[\tilde{f}(z); \omega] - (\omega - 1)\mathcal{M}[\tilde{f}(z); \omega - 1] = \sum_{l=1}^d pq^{l-1-l\omega} \cdot \mathcal{M}[\tilde{f}(z); \omega].$$

This functional equation can be simplified by using the Gamma function as normalization factor. Therefore, define

$$\bar{\mathcal{M}}[\tilde{f}(z); \omega] = \frac{\mathcal{M}[\tilde{f}(z); \omega]}{\Gamma(\omega)}.$$

Then

$$\bar{\mathcal{M}}[\tilde{f}(z); \omega] = \frac{\bar{\mathcal{M}}[\tilde{f}(z); \omega - 1]}{P(q^{-\omega})},$$

where $P(z) = 1 - p \sum_{l=1}^d q^{l-1} z^l$.

Next, observe that the above functional equation has the general solution

$$\bar{F}(\omega) := \bar{\mathcal{M}}[\tilde{f}(z); \omega] = \frac{c}{P(q^{-\omega})\Omega(q^{-\omega})}, \quad (3)$$

where $\Omega(s) = \prod_{j \geq 1} P(sq^j)$. Note that for $d = 1$, we have

$$\Omega(s) = \prod_{j \geq 1} (1 - pq^j s) = Q(ps) = (pqs; q)_{\infty}.$$

The notation $Q(s)$ is often used in Computer Science contexts [6], whereas $(pqs; q)_{\infty}$ is used in q -hypergeometric functions [1].

Next, we need to find c in (3). Therefore, observe that from $\tilde{f}(0) = 1$ and the direct mapping theorem from [3], we have, as $z \rightarrow 0$,

$$\mathcal{M}[\tilde{f}(z); \omega] \asymp \frac{1}{\omega}.$$

Consequently,

$$\lim_{\omega \rightarrow 0} \bar{\mathcal{M}}[\tilde{f}(z); \omega] = \lim_{\omega \rightarrow 0} \frac{\mathcal{M}[\tilde{f}(z); \omega]}{\Gamma(\omega)} = \lim_{\omega \rightarrow 0} \frac{1/\omega + \dots}{1/\omega + \dots} = 1.$$

Hence, by taking the limit as $\omega \rightarrow 0$ in (3)

$$c = P(1)\Omega(1) = q^d \Omega(1).$$

Summarizing,

$$\mathcal{M}[\tilde{f}(z); \omega] = \frac{q^d \Omega(1)\Gamma(\omega)}{P(q^{-\omega})\Omega(q^{-\omega})}. \quad (4)$$

Now that this function is known, we continue our program and (applying the inversion formula) collect residues. In order to identify them, we need the following technical lemma.

Lemma 1. *Let ρ denote the unique positive root of $P(z)$. Then, ρ is simple, $\rho > 1$ and ρ is the only root with $|z| \leq \rho$.*

Proof. Obviously, there exists a unique positive root ρ which is simple. Moreover, since

$$P(1) = 1 - p \sum_{l=1}^d q^{l-1} = q^d > 0,$$

we have that $\rho > 1$. Next, observe that for $|z| \leq \rho$

$$\left| p \sum_{l=1}^d q^{l-1} z^l \right| \leq p \sum_{l=1}^d q^{l-1} |z|^l \leq p \sum_{l=1}^d q^{l-1} \rho^l = 1.$$

If $|z| < \rho$, then the last inequality is strict; if $|z| = \rho$, then in order that z is a zero of $P(z)$, we must have equality in the triangle inequality which is only possible if z is on the positive real line. \square

From this lemma, we know that (4) has poles at $\log_Q \rho + \chi_k$ with singularity expansion

$$\mathcal{M}[\tilde{f}(z); \omega] \asymp \frac{q^d \Omega(1) \Gamma(\log_Q \rho + \chi_k)}{L \rho P'(\rho) \Omega(\rho) (\omega - \log_Q \rho - \chi_k)}.$$

Inverse Mellin transform then yields, as $z \rightarrow \infty$,

$$\tilde{f}(z) \sim - \frac{q^d \Omega(1)}{L \rho P'(\rho) \Omega(\rho)} z^{-\log_Q \rho} \sum_k \Gamma(\log_Q \rho + \chi_k) z^{-\chi_k}.$$

The last step is depoissonization. Here, we use the notation of JS-admissibility defined in Definition 1 of Section 2.3 in [6]. (The letters J and S are used to honour the authors of the early effort [7].) The following lemma is sufficient for our purpose.

Lemma 2. Let $\tilde{f}(z)$ and $\tilde{g}(z)$ be entire functions with

$$\tilde{f}(z) + \tilde{f}'(z) = \sum_{l=1}^d p q^{l-1} \tilde{f}(q^l z) + \tilde{g}(z).$$

Then

$$\tilde{f}(z) \text{ is JS-admissible} \iff \tilde{g}(z) \text{ is JS-admissible}.$$

Proof. Similar as Proposition 2.4 in [6] (only minor modifications are necessary). \square

From this result, we obtain that $\tilde{f}(z)$ is JS-admissible (since $\tilde{g}(z) = 0$ which is clearly JS-admissible). Consequently, by Proposition 2.2 in [6],

$$p_n \sim - \frac{q^d \Omega(1)}{L \rho P'(\rho) \Omega(\rho)} n^{-\log_Q \rho} \sum_k \Gamma(\log_Q \rho + \chi_k) n^{-\chi_k}.$$

We summarize this result in the following theorem.

Theorem 1. The probability p_n that a random word of length n satisfies the restricted growth property is asymptotically given by

$$p_n \sim - \frac{q^d \Omega(1)}{L \rho P'(\rho) \Omega(\rho)} \Gamma(\log_Q \rho) n^{-\log_Q \rho} + n^{-\log_Q \rho} \Psi(\log_Q n),$$

where $\Psi(z)$ is a 1-periodic function with average value equal to zero and Fourier series

$$\Psi(z) = -\frac{q^d \Omega(1)}{L\rho P'(\rho)\Omega(\rho)} \sum_{k \neq 0} \Gamma(\log_Q \rho + \chi_k) e^{-2k\pi iz}.$$

Remark 1. Note that we have small periodic oscillations if p is not too close to 1 (smallness comes from the exponential decay of the Gamma function along vertical lines). This is a phenomena often observed in the analysis of algorithms.

For $d = 1$, we have $\rho = 1/p$, and, ignoring the oscillating part, we have

$$p_n \sim \frac{qQ(p)}{LQ(1)} \Gamma(-\log_Q p) n^{\log_Q p}.$$

This matches a formula given earlier in [11,12].

3. The maximal letter X_n in restricted words

The random variable X_n as indicated in the title of this section is reminiscent of the *height* of planar (= planted plane) trees, as studied by de Bruijn, Knuth, and Rice [2].

Our goal here is to find the expected value of X_n which is given by

$$\mathbb{E}(X_n) = \frac{\sum_k k p_{n,k}}{p_n}, \tag{5}$$

where $p_{n,k}$ is the probability that a random word $w_1 \cdots w_n$ satisfying (2) has largest letter equal to k .

We will first derive a recurrence for $p_{n,k}$. Therefore, we use the same argument as in the previous section. This yields

$$p_{n+1,k} = \sum_{l=1}^d p q^{l-1} \sum_{j=0}^n \binom{n}{j} (1 - q^l)^{n-j} q^{lj} p_{j,k-l} \quad (n \geq 0; k \geq 1)$$

with initial conditions $p_{n,0} = \llbracket n = 0 \rrbracket$, $p_{0,k} = \llbracket k = 0 \rrbracket$ and $p_{j,k} = 0$ for $k < 0$. Now, again consider the Poisson generating function

$$\tilde{H}(z, u) := e^{-z} \sum_{n,k \geq 0} p_{n,k} u^k \frac{z^n}{n!}.$$

Then

$$\tilde{H}(z, u) + \frac{\partial}{\partial z} \tilde{H}(z, u) = \sum_{l=1}^d p q^{l-1} u^l \tilde{H}(q^l z, u). \tag{6}$$

Next, in order to compute the expectation of X_n , set

$$\tilde{h}(z) = \left. \frac{\partial}{\partial u} \tilde{H}(z, u) \right|_{u=1}.$$

Note that this is the Poisson generating function of the numerator of (5). Differentiating (6) with respect to u and setting $u = 1$ gives

$$\tilde{h}(z) + \tilde{h}'(z) = \sum_{l=1}^d pq^{l-1} \tilde{h}(q^l z) + \tilde{g}(z)$$

with

$$\tilde{g}(z) = \sum_{l=1}^d lpq^{l-1} \tilde{f}(q^l z).$$

We again apply Mellin transform to this differential–functional equation. This yields

$$\mathcal{M}[\tilde{h}(z); \omega] - (\omega - 1)\mathcal{M}[\tilde{h}(z); \omega - 1] = (1 - P(q^{-\omega})) \mathcal{M}[\tilde{h}(z); \omega] + \mathcal{M}[\tilde{g}(z); \omega].$$

Next, set

$$\bar{\mathcal{M}}[\tilde{h}(z); \omega] = \frac{\mathcal{M}[\tilde{h}(z); \omega]}{\Gamma(\omega)}$$

and

$$\frac{\mathcal{M}[\tilde{g}(z); \omega]}{\Gamma(\omega)} = -q^{-\omega} P'(q^{-\omega}) \bar{F}(\omega).$$

Then

$$\bar{\mathcal{M}}[\tilde{h}(z); \omega] = \frac{\bar{\mathcal{M}}[\tilde{h}(z); \omega - 1]}{P(q^{-\omega})} - \frac{q^{-\omega} P'(q^{-\omega}) \bar{F}(\omega)}{P(q^{-\omega})}.$$

The solution of this recurrence is given by

$$\bar{\mathcal{M}}[\tilde{h}(z); \omega] = - \sum_{l \geq 0} \frac{\Omega(q^{-\omega+l}) q^{-\omega+l} P'(q^{-\omega+l}) \bar{F}(\omega - l)}{P(q^{-\omega}) \Omega(q^{-\omega})} + \frac{c}{P(q^{-\omega}) \Omega(q^{-\omega})}$$

where, from $\tilde{h}(0) = 0$ and letting $w \rightarrow 0$ as in the previous section gives

$$c = -q^d \Omega(1) \alpha_p, \quad \alpha_p := - \sum_{l \geq 0} \frac{q^l P'(q^l)}{P(q^l)}.$$

Overall,

$$\begin{aligned} \mathcal{M}[\tilde{h}(z); \omega] = & - \frac{q^d \Omega(1) q^{-\omega} P'(q^{-\omega}) \Gamma(\omega)}{P(q^{-\omega})^2 \Omega(q^{-\omega})} \\ & - \Gamma(\omega) \left(\sum_{l \geq 1} \frac{\Omega(q^{-\omega+l}) q^{-\omega+l} P'(q^{-\omega+l}) \bar{F}(\omega - l)}{P(q^{-\omega}) \Omega(q^{-\omega})} + \frac{q^d \Omega(1) \alpha_p}{P(q^{-\omega}) \Omega(q^{-\omega})} \right). \end{aligned} \tag{7}$$

As before, we will use inverse Mellin transform and collect residues. Therefore, we treat the two terms on the right-hand side of (7) separately. First, for the first one, we have double poles at $c := \log_Q \rho + \chi_k$. For the singularity expansion at c note that

$$P(q^{-\omega}) = L\rho P'(\rho)(\omega - c) + \frac{L^2 \rho P'(\rho) + L^2 \rho^2 P''(\rho)}{2} (\omega - c)^2 + \dots$$

and

$$q^{-\omega} P'(q^{-\omega}) = \rho P'(\rho) + (L\rho P'(\rho) + L\rho^2 P''(\rho))(\omega - c) + \dots$$

From this,

$$\frac{q^{-\omega} P'(q^{-\omega})}{P(q^{-\omega})^2} = \frac{1}{L^2 \rho P'(\rho)(\omega - c)^2} + d^* + \dots,$$

where d^* is a constant. Next, we need

$$\frac{1}{\Omega(q^{-\omega})} = \frac{1}{\Omega(\rho)} (1 + L\alpha(\omega - c) + \dots),$$

where

$$\alpha = - \sum_{l \geq 1} \frac{\rho q^l P'(\rho q^l)}{P(\rho q^l)}.$$

Plugging this into the first term and using $\Gamma(\omega) = \Gamma(c) + \Gamma'(c)(\omega - c) + \dots$, we obtain

$$\begin{aligned} - \frac{q^d \Omega(1) q^{-\omega} P'(q^{-\omega}) \Gamma(\omega)}{P(q^{-\omega})^2 \Omega(q^{-\omega})} &\asymp - \frac{q^d \Omega(1) \Gamma(c)}{L^2 \rho P'(\rho) \Omega(\rho) (\omega - c)^2} \\ &- \frac{\alpha q^d \Omega(1) \Gamma(c)}{L \rho P'(\rho) \Omega(\rho) (\omega - c)} - \frac{q^d \Omega(1) \Gamma'(c)}{L^2 \rho P'(\rho) \Omega(\rho) (\omega - c)}. \end{aligned}$$

This gives the following contribution to the asymptotic expansion of $\tilde{h}(z)$:

$$\begin{aligned} &- \frac{q^d \Omega(1)}{L \rho P'(\rho) \Omega(\rho)} (\log_Q z) z^{-\log_Q \rho} \sum_k \Gamma(\log_Q \rho + \chi_k) z^{-\chi_k} \\ &+ \frac{q^d \Omega(1)}{L \rho P'(\rho) \Omega(\rho)} z^{-\log_Q \rho} \sum_k \left(\alpha \Gamma(\log_Q \rho + \chi_k) + \frac{\Gamma'(\log_Q \rho + \chi_k)}{L} \right) z^{-\chi_k}. \end{aligned}$$

The second term has simple poles at c with singularity expansion

$$\begin{aligned} \Gamma(\omega) &\left(- \sum_{l \geq 1} \frac{\Omega(q^{-\omega+l}) q^{-\omega+l} P'(q^{-\omega+l}) \bar{F}(\omega - l)}{P(q^{-\omega}) \Omega(q^{-\omega})} - \frac{q^d \Omega(1) \alpha_p}{P(q^{-\omega}) \Omega(q^{-\omega})} \right) \\ &\asymp \frac{q^d \Omega(1) \Gamma(c)}{L \rho P'(\rho) \Omega(\rho) (\omega - c)} (\alpha - \alpha_p). \end{aligned}$$

(In the corresponding computation in [12], there occurred a small mistake, since $\alpha - \alpha_p$ was taken to be zero. The present version corrects this.) Hence, the contribution of this term to the asymptotic expansion of $\tilde{h}(z)$ is

$$- \frac{q^d \Omega(1)}{L \rho P'(\rho) \Omega(\rho)} (\alpha - \alpha_p) z^{-\log_Q \rho} \sum_k \Gamma(\log_Q \rho + \chi_k) z^{-\chi_k}.$$

Adding up the two contributions gives

$$\begin{aligned} \tilde{h}(z) \sim & -\frac{q^d \Omega(1)}{L\rho P'(\rho)\Omega(\rho)} (\log_Q z) z^{-\log_Q \rho} \sum_k \Gamma(\log_Q \rho + \chi_k) z^{-\chi_k} \\ & + \frac{q^d \Omega(1)}{L\rho P'(\rho)\Omega(\rho)} z^{-\log_Q \rho} \sum_k \left(\alpha_p \Gamma(\log_Q \rho + \chi_k) + \frac{\Gamma'(\log_Q \rho + \chi_k)}{L} \right) z^{-\chi_k}. \end{aligned}$$

Next, note that due to [Lemma 2](#) and the closure properties of [Lemma 2.3](#) in [\[6\]](#), $\tilde{h}(z)$ is JS-admissible. Consequently, by [Proposition 2.2](#) in [\[6\]](#),

$$\begin{aligned} \sum_k k p_{n,k} \sim & -\frac{q^d \Omega(1)}{L\rho P'(\rho)\Omega(\rho)} (\log_Q n) n^{-\log_Q \rho} \sum_k \Gamma(\log_Q \rho + \chi_k) n^{-\chi_k} \\ & + \frac{q^d \Omega(1)}{L\rho P'(\rho)\Omega(\rho)} n^{-\log_Q \rho} \\ & \times \sum_k \left(\alpha_p \Gamma(\log_Q \rho + \chi_k) + \frac{\Gamma'(\log_Q \rho + \chi_k)}{L} \right) n^{-\chi_k}. \end{aligned}$$

Dividing by p_n and using [Theorem 1](#) yields

$$\frac{\sum_k k p_{n,k}}{p_n} \sim \log_Q n - \alpha_p - \frac{1}{L} \frac{\sum_k \Gamma'(\log_Q \rho + \chi_k) n^{-\chi_k}}{\sum_k \Gamma(\log_Q \rho + \chi_k) n^{-\chi_k}}.$$

Finally, by pulling out the average value in the last term on the right-hand side, we obtain our second main result.

Theorem 2. *The expected value of the largest letter in a random word of length n satisfying the restricted growth property is asymptotically given by*

$$\mathbb{E}(X_n) \sim \log_Q n - \alpha_p - \frac{\psi(\log_Q \rho)}{L} + \Phi(\log_Q n),$$

where $\Phi(z)$ is a 1-periodic function with average value equal to zero and $\psi = \Gamma'/\Gamma$ is the logarithmic derivative of the Gamma function.

Remark 2. Again $\Phi(z)$ is a periodic function with very small amplitude (if p is not close to 1).

4. Conclusion

Let us go back to

$$\tilde{f}(z) + \tilde{f}'(z) = \sum_{l=1}^d p q^{l-1} \tilde{f}(q^l z).$$

We set

$$\tilde{f}(z) = \sum_{n \geq 0} b_n \frac{z^n}{n!};$$

then

$$b_{n+1} + b_n = \sum_{l=1}^d pq^{l-1} q^{ln} b_n,$$

and

$$b_n = - \left(1 - \frac{p}{q} \sum_{l=1}^d q^{ln} \right) b_{n-1} = (-1)^n \prod_{j=1}^n \left(1 - \frac{p}{q} \sum_{l=1}^d q^{lj} \right).$$

So an explicit expression *is* available even here. However, it would be unpleasant to work with it, especially when considering the parameter maximal letter, whereas the approach, as discussed in this paper, is still quite manageable.

Acknowledgments

This work was done while the first author visited the Department of Mathematical Sciences, Stellenbosch University. He thanks the department for hospitality and support. He also acknowledges financial support of the Center of Mathematical Modeling and Scientific Computing (CMMSC). The second author is supported by the National Research Foundation of South Africa under grant number 2053748.

References

- [1] G. Andrews, R. Askey, R. Roy, Special Functions, in: Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, 1999.
- [2] N.G. De Bruijn, D.E. Knuth, S.O. Rice, The average height of planted plane trees, in: R.C. Read (Ed.), Graph Theory and Computing, Academic Press, 1972, pp. 15–22.
- [3] P. Flajolet, X. Gourdon, P. Dumas, Mellin transforms and asymptotics: harmonic sums, Theoretical Computer Science 144 (1995) 3–58.
- [4] P. Flajolet, R. Sedgewick, Mellin transforms and asymptotics: finite differences and Rice’s integrals, Theoretical Computer Science 144 (1995) 101–124.
- [5] P. Flajolet, R. Sedgewick, Analytic Combinatorics, Cambridge University Press, Cambridge, 2009.
- [6] H.-K. Hwang, M. Fuchs, V. Zacharovas, Asymptotic variance of random symmetric digital search trees, Discrete Mathematics and Theoretical Computer Science 12 (2010) 103–166.
- [7] P. Jacquet, W. Szpankowski, Analytical de-Poissonization and its applications, Theoretical Computer Science 201 (1–2) (1998) 1–62.
- [8] D.E. Knuth, The Art of Computer Programming, Volume 1: Fundamental Algorithms, third ed., Addison-Wesley, 1997, 1973.
- [9] D.E. Knuth, The Art of Computer Programming, Volume 3: Sorting and Searching, second ed., Addison-Wesley, 1998, 1973.
- [10] T. Mansour, M. Shattuck, Set partitions as geometric words, Australasian Journal of Combinatorics 53 (2012) 31–40.
- [11] K. Oliver, H. Prodinger, Words representing set partitions, Applicable Analysis and Discrete Mathematics 5 (2011) 55–59.
- [12] H. Prodinger, Set partitions, words, and approximate counting with black holes, Australasian Journal of Combinatorics 54 (2012) 303–310.

Digital search trees with m trees: Level polynomials and insertion costs

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received 23rd June 2011, accepted 23rd June 2011.

To Philippe Flajolet, who revisited Digital Search Trees and understood the power of Rice's integrals.

We adapt a novel idea of Cichon's related to Approximate Counting to the present instance of Digital Search Trees, by using m (instead of one) such trees. We investigate the level polynomials, which have as coefficients the expected numbers of data on a given level, and the insertion costs. The level polynomials can be precisely described, thanks to formulæ from q -analysis. The asymptotics of expectation and variance of the insertion cost are fairly standard these days and done with Rice's method.

Keywords: Digital search tree, level polynomial, Rice's method, cancellations, q -analysis.

1 Introduction

Digital search trees (=DSTs), level polynomials and insertion costs are certainly classical subjects: Knuth (1973); Flajolet and Sedgewick (2009); Mahmoud (1992); Flajolet and Sedgewick (1986); Louchard (1987); Kirschenhofer and Prodinger (1988); Szpankowski (1991); Prodinger (1995).

The following sentences from our own Prodinger (1995), which never appeared in a proper journal, can be reproduced here almost verbatim:

A DST is constructed like a binary search tree, but the decision to go down to the left or right is done according to the representation of the key as a binary string of bits. If the first bit is 0, the item goes to the left, otherwise to the right. Then the second bit is responsible for left or right, etc., until there is an empty node where the item can be stored. Decisions are made independently.

The classic book Knuth (1973) contains a more elaborate description.

Now, in order to study the average search costs in a DST built from n random data (i.e., in every decision, 0 and 1 is equally likely), the polynomial $H_n(u)$, which has as the coefficient of u^k the expected number of nodes on this level, is studied. By convention, the root is at level 0.

The formula for the *level polynomials*

$$H_n(u) = \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} (u)_{k-1}, \quad n \geq 1, \quad H_0(u) = 0,$$

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appears already in Knuth's book (Knuth, 1973, p. 504), compare also (Mahmoud, 1992, Section 6).

Here, we use the (classic) notation $(u)_k := (1-u)(1-uv)\dots(1-uv^{k-1})$; for applications to DSTs, we take $q = \frac{1}{2}$. We also need the infinite product $(x)_\infty := (1-x)(1-xq)\dots$.

Guy Louchard (1987) gave explicit expressions for the coefficients of $H_n(u)$; see also (Mahmoud, 1992, Section 6). We write $[u^k]H_n(u)$ for the coefficient of u^k in $H_n(u)$, and similarly for generating functions in 2 variables.

DSTs and Approximate Counting are very similar when it comes to analysis, see for instance Prodinger (1992). In 2011, Cichon (Cichoń and Macyna (2011)) had a novel idea about Approximate Counting, by introducing an additional parameter m , see also Prodinger (2011).

Translating this idea to the world of DSTs goes as follows: Instead of keeping one DST, we keep m DSTs, and an incoming data is attached to one of them. For algorithmic purposes (insertion and searching), this must be deterministic, but for the analysis we assume that a random DST is chosen with probability $\frac{1}{m}$. The meaning of the level polynomial is then obvious: the coefficient of u^k is the expected number of data on level k in all m DSTs combined.

Considering m -DSTs is equivalent (by adding an extra root) to the investigation of a tree structure with a prescribed root degree. This is not uncommon in Combinatorics and Computer Science; we just give one citation as an illustration: Kemp (1980).

2 An explicit formula for the level polynomials

We start from the classic formula (derived e. g., in Prodinger (1995)) for the level polynomials for classical DSTs $h_n(u)$

$$h_n(u) = \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} (u)_{k-1}$$

and note that $h_n(1) = n$. Then the level polynomials $H_N(u)$ for the m -version satisfy

$$H_N(u) = m^{-N} \sum_{n_1+\dots+n_m=N} \binom{N}{n_1, \dots, n_m} (h_{n_1}(u) + \dots + h_{n_m}(u)),$$

which is easy to see since a total of N data splits into n_1, \dots, n_m data each, building the m DSTs, and the individual level polynomials have to be added. The final explicit formula for the coefficients of $H_N(u)$ appears at the end of the following computations in Theorem 2.1. We can simplify:

$$\begin{aligned} H_N(u) &= m^{1-N} \sum_{n_1+\dots+n_m=N} \binom{N}{n_1, \dots, n_m} h_{n_1}(u) \\ &= m^{1-N} \sum_{n=0}^N \binom{N}{n} h_n(u) (m-1)^{N-n} \\ &= m^{1-N} \sum_{n=1}^N \binom{N}{n} \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} (u)_{k-1} (m-1)^{N-n} \\ &= m^{1-N} \sum_{k=1}^N \binom{N}{k} (-1)^{k-1} (u)_{k-1} \sum_{n=k}^N \binom{N-k}{n-k} (m-1)^{N-n} \end{aligned}$$

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$$= \sum_{k=1}^N \binom{N}{k} (-1)^{k-1} (u)_{k-1} m^{1-k}.$$

So the difference is just this extra factor m^{1-k} . Note again that for our DST application we have $q = \frac{1}{2}$, although the computations hold for general q .

Now we form a bivariate generating function:

$$\begin{aligned} H(u, x) &= \sum_{N \geq 1} H_N(u) x^N = \sum_{N \geq 1} \sum_{k=1}^N \binom{N}{k} (-1)^{k-1} (u)_{k-1} m^{1-k} x^N \\ &= \sum_{k \geq 1} \sum_{N \geq k} \binom{N}{k} (-1)^{k-1} (u)_{k-1} m^{1-k} x^N \\ &= \sum_{k \geq 1} (-1)^{k-1} (u)_{k-1} m^{1-k} \frac{x^k}{(1-x)^{k+1}} \\ &= \frac{x}{(1-x)^2} \sum_{k \geq 0} (-1)^k (u)_k m^{-k} \frac{x^k}{(1-x)^k} \\ &= \frac{x}{(1-x)^2} \sum_{k \geq 0} (u)_k z^k \quad \text{with} \quad z = \frac{x}{m(x-1)}. \end{aligned}$$

We need the classic transformation of Heine Andrews (1976):

$$\sum_{n \geq 0} \frac{(a)_n (b)_n t^n}{(q)_n (c)_n} = \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{n \geq 0} \frac{(c/b)_n (t)_n b^n}{(q)_n (at)_n}$$

and the corollary

$$\sum_{n \geq 0} (y)_n z^n = (y)_\infty \sum_{n \geq 0} \frac{y^n}{(q)_n (1-zq^n)},$$

which is obtained by setting $a = q$, $b = y$, $c = 0$, and $t = z$ and noticing that $\frac{(qz)_\infty}{(z)_\infty} = \frac{1}{1-z}$ and

$\frac{(z)_n}{(qz)_n} = \frac{1-z}{1-zq^n}$. Now we can continue:

$$\begin{aligned} H(u, x) &= \frac{x}{(1-x)^2} (u)_\infty \sum_{n \geq 0} \frac{u^n}{(q)_n (1-zq^n)} \\ &= \frac{x}{(1-x)^2} (u)_\infty \sum_{n \geq 0} \frac{u^n}{(q)_n \left(1 - \frac{x}{m(x-1)} q^n\right)} \\ &= \frac{x}{(1-x)} (u)_\infty \sum_{n \geq 0} \frac{u^n}{(q)_n \left(1 - x + \frac{x}{m} q^n\right)} \\ &= m(u)_\infty \sum_{n \geq 0} \frac{(u/q)^n}{(q)_n} \left[\frac{1}{1-x} - \frac{1}{1-x\left(1 - \frac{q^n}{m}\right)} \right] \end{aligned}$$

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$$= m \frac{1}{(1-x)(1-u/q)} - m(u)_\infty \sum_{n \geq 0} \frac{(u/q)^n}{(q)_n} \frac{1}{1-x(1-\frac{q^n}{m})}.$$

We can read off coefficients:

$$\begin{aligned} [x^N]H(u, x) &= m[x^N] \frac{1}{(1-x)(1-u/q)} - m(u)_\infty [x^N] \sum_{n \geq 0} \frac{(u/q)^n}{(q)_n} \frac{1}{1-x(1-\frac{q^n}{m})} \\ &= m \frac{1}{1-u/q} - m(u)_\infty \sum_{n \geq 0} \frac{(u/q)^n}{(q)_n} \left(1 - \frac{q^n}{m}\right)^N \end{aligned}$$

and further

$$\begin{aligned} [x^N u^l]H(u, x) &= m q^{-l} - m[u^l](u)_\infty \sum_{n \geq 0} \frac{(u/q)^n}{(q)_n} \left(1 - \frac{q^n}{m}\right)^N \\ &= m q^{-l} - m \sum_{k=0}^l \frac{(1/q)^k}{(q)_k} \left(1 - \frac{q^k}{m}\right)^N [u^{l-k}](u)_\infty \\ &= m q^{-l} - m \sum_{k=0}^l \frac{q^{-k}}{(q)_k} \left(1 - \frac{q^k}{m}\right)^N \frac{(-1)^{l-k} q^{\binom{l-k}{2}}}{(q)_{l-k}}. \end{aligned}$$

This is the explicit formula that we wanted to derive. In it, we used an expansion due to Euler:

$$(u)_\infty = \sum_{n \geq 0} \frac{(-1)^n u^n q^{\binom{n}{2}}}{(q)_n}.$$

Theorem 2.1 *The expected number of data on level l in an m -DST built from N random data is explicitly given by*

$$m q^{-l} - m \sum_{k=0}^l \frac{q^{-k}}{(q)_k} \left(1 - \frac{q^k}{m}\right)^N \frac{(-1)^{l-k} q^{\binom{l-k}{2}}}{(q)_{l-k}}.$$

The instance $m = 1$ has been known before, see Louchard (1987); Prodinger (1995); Mahmoud (1992).

3 Insertion cost

The quantity $\frac{H_N(u)}{N}$ is the probability generating function of a random variable called *insertion cost*. While it is well studied in the classical instance, we will provide here average and variance for general (fixed) m , both, explicitly and asymptotically. (The impatient reader can already jump forward to Theorem 3.1 for the results.) We need

$$H'_N(1) = \sum_{k=2}^N \binom{N}{k} (-1)^k Q_{k-2} m^{1-k},$$

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$$H_N''(1) = 2 \sum_{k=2}^N \binom{n}{k} (-1)^{k-1} Q_{k-2} T_{k-2} m^{1-k}$$

with $T(k) := \sum_{j=1}^k \frac{1}{2^j - 1}$. We use the notation $Q_n = \prod_{k=1}^n (1 - 2^{-k})$, common in Computer Science, as well as $(q)_n$.

Now we can engage into the asymptotics, using Rice's method. This method has been described in Flajolet and Sedgewick (1995):

An alternating sum can be written as a contour integral:

$$\sum_{k=2}^N \binom{N}{k} (-1)^k f(k) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{(-1)^N N!}{z(z-1)\dots(z-N)} f(z) dz.$$

Here, the positively oriented curve \mathcal{C} enclosed the poles $2, 3, \dots, N$, and no others. This formula follows from simple residue calculations. Note also that

$$\frac{(-1)^N N!}{z(z-1)\dots(z-N)} = -\frac{\Gamma(N+1)\Gamma(-z)}{\Gamma(N+1-z)}.$$

Extending the curve of integration, we encounter extra residues; in order to keep the formula correct, these residues must be subtracted. They give us the terms of the asymptotic expansion of interest. There is in all our examples a pole at $z = 1$, and it will give us the dominant contribution.

Neglecting tiny oscillations, we can write in a suggestive way:

$$\sum_{k=2}^N \binom{N}{k} (-1)^k f(k) \sim \text{Res}_{z=1} \frac{\Gamma(N+1)\Gamma(-z)}{\Gamma(N+1-z)} f(z).$$

For the expected value, we use

$$f(z) = (q)_{z-2} m^{1-z},$$

with

$$(q)_z := \frac{(q)_\infty}{(q^{z+1})_\infty}.$$

For convenience, we use $w = z - 1$, so that the expansions are around $w = 0$.

Just recently in our paper Prodinger (2011) we used (with $Q = \frac{1}{q} = 2$ and $L = \log Q = \log 2$)

$$\begin{aligned} Q_{w-1} &= \frac{Q_w}{1 - 2^{-w}} \sim \left[1 - L\alpha w + \frac{L^2}{2}(\alpha^2 + \alpha + \beta)w^2 \right] \frac{1}{1 - 2^{-w}} \\ &\sim \frac{1}{Lw} + \frac{1}{2} - \alpha + \frac{L}{2} \left(\alpha^2 + \beta + \frac{1}{6} \right) w. \end{aligned}$$

Here,

$$\alpha = \sum_{j \geq 1} \frac{1}{2^j - 1} \quad \text{and} \quad \beta = \sum_{j \geq 1} \frac{1}{(2^j - 1)^2}.$$

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So we must consider

$$(q)_{w-1} m^{-w} \frac{\Gamma(N+1)\Gamma(-w-1)}{\Gamma(N-w)};$$

the residue can be computed by a computer:

$$N \left(\log_2 N - \log_2 m + \frac{1}{2} - \alpha + \frac{\gamma-1}{L} \right) + O(1).$$

Now, we have traditionally

$$T_{z-2} = T_{w-1} = \alpha - \frac{1}{2^w - 1} - \sum_{j \geq 1} \frac{1}{2^{j+w} - 1} \sim -\frac{1}{Lw} + \frac{1}{2} - \frac{Lw}{12} + L(\alpha + \beta)w.$$

For the second factorial moment we use

$$f(z) = -2Q_{z-2} T_{z-2} m^{1-z},$$

so that we have to expand

$$-2(q)_{w-1} T_{w-1} m^{-w} \frac{\Gamma(N+1)\Gamma(-w-1)}{\Gamma(N-w)}$$

around $w = 0$ and compute the residue, which we don't display in full:

$$N \left[\log_2^2 N + 2 \log_2 N \cdot \left(\frac{\gamma-1}{L} - \alpha - \log_2 m \right) + \text{further terms} \right].$$

When we compute the variance via

$$\frac{H_N''(1)}{N} + \frac{H_N'(1)}{N} - \left(\frac{H_N'(1)}{N} \right)^2,$$

there are many cancellations. Altogether we summarize our results.

Theorem 3.1 *Expectation and variance of the parameter insertion cost in m -DSTs admit the asymptotic expansions*

$$\begin{aligned} \text{expectation} &\sim \log_2 N - \log_2 m + \frac{1}{2} - \alpha + \frac{\gamma-1}{L} + \delta_1(\log_2 N), \\ \text{variance} &\sim \frac{\pi^2}{6L^2} - \alpha - \beta + \frac{1}{12} + \frac{1}{L^2} + \delta_2(\log_2 N), \end{aligned}$$

where $\delta_1(x)$ and $\delta_2(x)$ are tiny fluctuating functions that we did not compute explicitly here.

The computation of the fluctuating functions is not difficult, and very similar computations have appeared in the relevant literature many times; here is an incomplete list of references: Kirschenhofer and Prodinger (1988); Kirschenhofer and Prodinger (1991); Flajolet and Sedgewick (1995). Note that the variance is (asymptotically) a constant (plus a tiny fluctuation) which does *not* depend on m ; $\delta_1(x)$ also does not depend on m . For $m = 1$, this result appeared already in Kirschenhofer and Prodinger (1988); Szpankowski (1991).

So, as far as the main terms in the asymptotics are concerned, the dependency on m of average and variance is very minor, and m -DSTs don't show any improved behaviour. But such a statement can only be made after some thorough analysis, which is, why we provided it here.

References

- G. Andrews. *The Theory of Partitions*, volume 2 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley, 1976.
- J. Cichoń and W. Macyna. Approximate counters for flash memory. *17th IEEE International Conference on Embedded and Real-Time Computing Systems and Applications (RTCSA)*, Toyama, Japan, 2011.
- P. Flajolet and R. Sedgewick. Digital search trees revisited. *SIAM Journal on Computing*, 15:748–767, 1986.
- P. Flajolet and R. Sedgewick. Mellin transforms and asymptotics: Finite differences and Rice’s integrals. *tcs*, 144:101–124, 1995.
- P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, Cambridge, 2009.
- R. Kemp. The average height of r -tuply rooted planted plane trees. *Computing*, 25(3):209–232, 1980.
- P. Kirschenhofer and H. Prodinger. Further results on digital search trees. *Theoretical Computer Science*, 58:143–154, 1988.
- P. Kirschenhofer and H. Prodinger. Approximate counting: An alternative approach. *RAIRO Theoretical Informatics and Applications*, 25:43–48, 1991.
- D. E. Knuth. *The Art of Computer Programming*, volume 3: Sorting and Searching. Addison-Wesley, 1973. Second edition, 1998.
- G. Louchard. Exact and asymptotic distributions in digital and binarysearch trees. *RAIRO Theoretical Informatics and Applications*, 21:479–495, 1987.
- H. Mahmoud. *Evolution of Random Search Trees*. John Wiley, 1992.
- H. Prodinger. Hypothetic analyses: Approximate counting in the style of Knuth, path length in the style of Flajolet. *Theoretical Computer Science*, 100:243–251, 1992.
- H. Prodinger. Digital search trees and basic hypergeometric functions. *EATCS Bulletin*, 56:112–115, 1995.
- H. Prodinger. Approximate counting with m counters: a detailed analysis. *submitted*, 2011.
- W. Szpankowski. A characterization of digital search trees from the successful search viewpoint. *Theoretical Computer Science*, 85:117–134, 1991.



ADVANCING IN THE PRESENCE OF A DEMON

GUY LOUCHARD* — HELMUT PRODINGER**

(Communicated by Stanislav Jakubec)

ABSTRACT. We study a parameter that contains *approximate counting*, i.e., the level reached after n random increments, driven by geometric probabilities, and *insertion costs* for *tries* as special cases. We are able to compute all moments of this parameter in a semi-automatic fashion. This is another showcase of the machinery developed in an earlier paper of these authors. Roughly speaking, it works when the underlying distributions are distributed according to the *Gumbel* distribution, or something similar.

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1. Introduction

Assume that n persons want to advance on a staircase. The rules are as follows: The party starts at level 1. The m persons who advanced to level k flip a coin. Those who flip ‘1’ (with probability q) advance to the next level; the others, who flipped ‘0’ (with probability $p = 1 - q$) die. Additionally, there is a demon, who kills one of the survivors with probability ν , but lets them alone with probability $\mu = 1 - \nu$. The demon interferes only at a level 2 or higher. If one single person is advancing to level k and is eaten, we *do not* say that this level was reached. Only people who survive the coin flipping *and* the demon count!

2000 Mathematics Subject Classification: Primary 05A16, 68P05, 68R05.

Keywords: Gumbel distribution, approximate counting, demon, trie, search cost, moment.
H. Prodinger’s research was supported by the NRF grant 2053748 and by the Center of Experimental Mathematics of the University of Stellenbosch.

G. Louchard visited this center in June 2006 and thanks for its hospitality.

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As was worked out in [10], the instance $\mu = 0$ corresponds to *approximate counting*. Let us recall what it is, to keep this paper independent. There is a counter (the state the process is in at the moment), starting at 1, and random increments, they increase the counter from i to $i+1$ with probability q^i , otherwise it stays at i . One is interested in the value of the counter after n random increments.

The other extremal case $\mu = 1$ (no demon interfering) is related to a digital data structure called *tries* ([5], [9]). Although in the previous paper [10], only the symmetrical case $p = q = \frac{1}{2}$ was considered, the arguments carry over. Let p be the probability to go left (corresponding to bit 0) and q the probability to go right (corresponding to bit 1) in a trie, we think about those who go right as the *survivors*, who repeat the experiment. In this way, we always move to the right. And we are searching for an element $.11111\dots$ (sufficiently many 1's), which is not present in the data structure, in other words we consider the *unsuccessful search cost, followed by an insertion* (which is the cost of inserting this element), provided that we have n random data in the trie. For the symmetric case, this makes perhaps more sense, as we are just interested in the parameter *unsuccessful search cost*, as we are no longer considering the path that always goes right, but rather a random path.

Of course, these two special cases are not necessary to understand the paper, but they serve as a *motivation*.

The idea of introducing a probability μ of escaping the demon is borrowed from [11]; in this thesis U. Schmid studied the collision resolution schemes, related to n transmitted data, using simple tree-algorithms (Capetanakis, Hayes, Tsybakov, Mikhailov). Unlike earlier approaches, Schmid assumes that with a positive probability μ , one of the colliding packages survives and is successfully submitted; compare also [12], [13].

In the following we are interested in the random variable (RV) $K(n)$: highest level reached by (at least one member of) a party of n players. We are able to compute *all moments* (asymptotically) in an almost automatic fashion. This will be done with the techniques worked out in [7]. Note that the expected value for the symmetric case $p = q = \frac{1}{2}$ was computed using *Rice's method* in [10].

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2. Notations

We list for convenience the notations used in this paper.

- n := number of persons,
- $\pi(n, k) := \mathbb{P}[K(n) = k], \quad \pi(n, 0) = 0, \quad \pi(0, 1) = 1,$
- $\Pi(n, k) := \sum_{i=1}^k \pi(n, i),$
- ν := probability that the demon kills a survivor, $\mu = 1 - \nu,$
- q := probability of flipping ‘1’ and advancing, $p = 1 - q,$
- $F_n(u) := \sum_{k=1}^{\infty} \pi(n, k)u^k, \quad F_0(u) = u,$
 the generating function (GF) where the coefficient of u^k gives
 the probability that the party made it exactly to level $k,$
- $G(z, u) := \sum_{n=0}^{\infty} F_n(u) \frac{z^n}{n!}, \quad G(0, u) = u,$
- $D(z, u) := e^{-z}G(z, u) = \sum_{n=0}^{\infty} \frac{z^n}{n!}D_n(u), \quad D(0, u) = u,$
 this is a classical Poissonization trick,
- $L := \ln 1/q,$
- $\log x := \log_{1/q} x,$
- $\tilde{\alpha} := \alpha/L, \quad \alpha \in \mathbb{C}$
- $\chi_l := 2l\pi i/L, \quad l \in \mathbb{Z},$
- $\{x\} :=$ fractional part of $x.$

Furthermore, we need a few concepts from q -analysis:

$$(x)_n := (1 - x)(1 - xq) \dots (1 - xq^{n-1});$$

often, one writes $(x; q)_n$ to emphasize the parameter $q,$ but that is not necessary here. $(x)_\infty := \lim_{n \rightarrow \infty} (x)_n.$

Euler’s two partition identities:

$$\prod_{i=0}^{\infty} (1 - tq^i) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q)_n}, \tag{1}$$

$$\prod_{i=0}^{\infty} (1 - tq^i)^{-1} = \sum_{n=0}^{\infty} \frac{t^n}{(q)_n}. \tag{2}$$

They are special cases of Cauchy’s formula (q -binomial theorem)

$$\frac{(at)_\infty}{(t)_\infty} = \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(q)_n},$$

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which we will use later. These concepts can be found in [1].

The following abbreviations will be useful:

$$\begin{aligned} Q_1 &:= (q)_\infty, \\ Q_2 &:= (\mu q)_\infty, \\ H_1(\alpha) &:= (e^\alpha)_\infty, \\ H_2(\alpha) &:= (\mu q e^\alpha)_\infty. \end{aligned}$$

We use the (now standard) notation $[z^n]f(z)$ to extract the coefficient of z^n in the series expansion of $f(z)$.

3. Recurrences

Among the n persons, assume that j survive, with probability $\binom{n}{j}q^j p^{n-j}$. Among the j survivors, $j - 1$ stay alive if the demon kills one of them (with probability ν) or j stay alive (with probability μ). If all of them die (with probability p^n), the highest level reached is 1.

Summing over all possible cases, we thus get the recursion

$$\pi(n, k) := \sum_{j=1}^n \binom{n}{j} q^j p^{n-j} [\nu \pi(j - 1, k - 1) + \mu \pi(j, k - 1)] + p^n \llbracket k = 1 \rrbracket.$$

The ordinary GF is given by

$$F_n(u) = u \sum_{j=1}^n \binom{n}{j} q^j p^{n-j} [\nu F_{j-1}(u) + \mu F_j(u)] + u p^n, \quad n \geq 1, \quad F_0(u) = u.$$

The exponential GF is given by

$$G\left(\frac{z}{p}, u\right) = u \mu e^z G\left(\frac{zq}{p}, u\right) - u^2 \mu e^z + \nu \sum_{n=1}^{\infty} \frac{z^n}{n!} u \sum_{j=1}^n \binom{n}{j} \left(\frac{q}{p}\right)^j F_{j-1}(u) + u e^z.$$

Now we differentiate w.r.t. z . (The prime notation refers to this.)

$$\begin{aligned} \frac{1}{p} G'\left(\frac{z}{p}, u\right) &= u \mu e^z G\left(\frac{zq}{p}, u\right) + \frac{q}{p} u \mu e^z G'\left(\frac{zq}{p}, u\right) - u^2 \mu e^z \\ &\quad + \nu \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} u \sum_{j=1}^n \binom{n}{j} \left(\frac{q}{p}\right)^j F_{j-1}(u) + u e^z. \end{aligned}$$

Now we *poissonize*; this translates into

$$\frac{1}{p} D'\left(\frac{z}{p}, u\right) + \frac{q}{p} D\left(\frac{z}{p}, u\right) = u \left[\mu \frac{q}{p} D'\left(\frac{zq}{p}, u\right) + \frac{q}{p} D\left(\frac{zq}{p}, u\right) \right].$$

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As $D(z, u) = \sum_{n=0}^{\infty} \frac{z^n}{n!} D_n(u)$, comparing coefficients, we find

$$D_n(u) = D_{n-1}(u)(uq^n - q)/(1 - u\mu q^n),$$

from which we get, upon iteration, the explicit form

$$D_n(u) = u(-1)^n q^n \frac{(u)_n}{(u\mu q)_n}.$$

Since, relating D with G ,

$$F_n(u) = \sum_{j=0}^n \binom{n}{j} D_j(u),$$

we can continue:

$$\begin{aligned} F_n(u) &= u \sum_{j=0}^n \binom{n}{j} (-1)^j q^j \frac{(u)_j}{(u\mu q)_j} \\ &= u \sum_{j=0}^n \binom{n}{j} (-1)^j q^j \frac{(u)_\infty}{(u\mu q)_\infty} \frac{(u\mu q^{j+1})_\infty}{(uq^j)_\infty} \\ &= u \frac{(u)_\infty}{(u\mu q)_\infty} \sum_{j=0}^n \binom{n}{j} (-1)^j q^j \sum_{k=0}^{\infty} \frac{(uq^j)^k (\mu q)_k}{(q)_k} \\ &= u \frac{(u)_\infty}{(u\mu q)_\infty} \sum_{k=0}^{\infty} \frac{u^k}{(q)_k} (\mu q)_k \sum_{j=0}^n \binom{n}{j} (-1)^j q^{j(k+1)} \\ &= u \frac{(u)_\infty}{(u\mu q)_\infty} \sum_{k=0}^{\infty} \frac{u^k}{(q)_k} (\mu q)_k (1 - q^{k+1})^n. \end{aligned}$$

Reading off the coefficient $[u^l]F_n(u)$, we get the following explicit result.

PROPOSITION 1. *We have*

$$\pi(n, l) = \sum_{i+j+h=l-1} \frac{(\mu q)^i}{(q)_i} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j} \frac{(\mu q)_h}{(q)_h} (1 - q^{h+1})^n.$$

Note that the special case $\mu = 0$, which restricts the summation to $i = 0$, leads to

$$\sum_{j=0}^{l-1} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j (q)^{l-1-j}} (1 - q^{l-j})^n$$

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which is exactly Flajolet’s formula ([2, (46)]). We can even derive a formula with only one summation, again by invoking the q -binomial theorem:

$$\begin{aligned} \pi(n, l) &= [u^{l-1}] \frac{(u)_\infty}{(u\mu q)_\infty} \sum_{k=0}^\infty \frac{u^k}{(q)_k} (\mu q)_k (1 - q^{k+1})^n \\ &= \sum_{k=0}^{l-1} \frac{(\mu q)_k}{(q)_k} (1 - q^{k+1})^n [u^{l-1-k}] \frac{(u)_\infty}{(u\mu q)_\infty} \\ &= \sum_{k=0}^{l-1} \frac{(\mu q)_k}{(q)_k} (1 - q^{k+1})^n \frac{(1/(\mu q))_{l-1-k}}{(q)_{l-1-k}} (\mu q)^{l-1-k}. \end{aligned}$$

However, we will not use this form; one disadvantage is that for $\mu = 0$, one must consider a limit.

4. Asymptotics

Now we set $\eta = l - \log n$ and let $n \rightarrow \infty$. This gives, in the range $\eta = \mathcal{O}(1)$, the limiting distribution

$$\begin{aligned} \pi(n, l) &\sim f(\eta) = \frac{Q_2}{Q_1} \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(\mu q)^i}{(q)_i} \frac{(-1)^j q^{\binom{j}{2}}}{(q)_j} \exp(-e^{-L\eta + L(i+j)}), \\ \Pi(n, l) &\sim H(\eta), \end{aligned}$$

with

$$f(\eta) = H(\eta) - H(\eta - 1),$$

where we recognize the Gumbel distribution function $\exp(-e^{-x})$. To show that the limiting moments are equivalent to the moments of the limiting distribution, we need a suitable rate of convergence (in particular for large and small values of η). This is related to a uniform integrability condition (see L o è v e [6, Section 11.4]). For the kind of limiting distribution we consider here, the rate of convergence is analyzed in detail in [7] and [8], we will not repeat the arguments. Asymptotically, the distribution will be a periodic function of the fractional part of $\log n$. The distribution $\Pi(n, l)$ does not converge in the weak sense, it does however converge in distribution along subsequences n_m for which the fractional part of $\log n_m$ is constant.

We will use the following result from Hitczenko and Louchard [4] related to the dominant part of the moments (the \sim sign is related to the moments of the discrete RV Y_n).

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LEMMA 2. *Let a (discrete) RV Y_n be such that $\mathbb{P}(Y_n - \log n \leq \eta) \sim F(\eta)$, where $F(\eta)$ is the distribution function of a continuous RV Z with mean m_1 , second moment m_2 . Assume that $F(\eta)$ is either an extreme-value distribution function or a convergent series of such and that we have a suitable rate of convergence. Let*

$$\varphi(\alpha) = \mathbb{E}(e^{\alpha Z}) = 1 + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} m_k.$$

Let w (with or without subscripts) denote periodic functions of $\log n$, with period 1 and with small (usually of order no more than 10^{-5}) mean and amplitude. Actually, these functions depend on the fractional part of $\log n$: $\{\log n\}$.

Then the mean of Y_n is given by

$$\begin{aligned} \mathbb{E}(Y_n - \log n) &\sim \int_{-\infty}^{+\infty} x[F(x) - F(x - 1)] dx + w_1 \\ &= \tilde{m}_1 + w_1, \quad \text{with } \tilde{m}_1 = m_1 + \frac{1}{2}. \end{aligned}$$

The neglected part is of order $1/n^\beta$ with $0 < \beta < 1$.

For the reader’s convenience, we collect some information from [7] that we use to compute moments:

The moments of $Y_n - \log n$ are asymptotically given by $\tilde{m}_i + w_i$, where the generating function of \tilde{m}_i is given by

$$\phi(\alpha) := \int_{-\infty}^{\infty} e^{\alpha \eta} f(\eta) d\eta = 1 + \sum_{i=1}^{\infty} \frac{\alpha^i}{i!} \tilde{m}_i = \varphi(\alpha) \frac{e^\alpha - 1}{\alpha}. \tag{3}$$

This leads to

$$\begin{aligned} \tilde{m}_1 &= m_1 + \frac{1}{2}, \\ \tilde{m}_2 &= m_2 + m_1 + \frac{1}{3}, \\ \tilde{m}_3 &= m_3 + \frac{3}{2}m_2 + m_1 + \frac{1}{4}. \end{aligned}$$

To analyze the periodic component w_i to be added to the moments \tilde{m}_i we proceed as in Louchard and Prodinger [7]. For instance,

$$\mathbb{E}(Y_n - \log n) \sim E^{(1)}(n) = \sum_{j=1}^{\infty} [F(j - \log n) - F(j - \log n - 1)][j - \log n]. \tag{4}$$

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Set $y = Q^{-x}$ and $G(y) = F(x)$. Equation (4) becomes

$$E^{(1)}(n) := \sum_{j=1}^{\infty} [G(n/Q^j) - G(n/Q^{j+1})] [-\log(n/Q^j)],$$

the Mellin transform of which is (for a good reference on Mellin transforms, see Flajolet et al. [3] or Szpankowski [14])

$$\frac{Q^s}{1 - Q^s} \Upsilon_1^*(s), \tag{5}$$

and

$$\begin{aligned} \Upsilon_1^*(s) &= \int_0^{\infty} y^{s-1} [G(y) - G(y/Q)] [-\log y] dy \\ &= \int_{-\infty}^{\infty} Q^{-sx} [F(x) - F(x-1)] x L dx. \end{aligned}$$

Then

$$\Upsilon_1^*(s) = L \phi'(\alpha)|_{\alpha=-Ls}. \tag{6}$$

The fundamental strip of (5) is usually of the form $s \in \langle -C_1, 0 \rangle$, $C_1 > 0$. Set also

$$\Upsilon_0^*(s) = L \phi(\alpha)|_{\alpha=-Ls}, \quad \Upsilon_0^*(0) = L.$$

We assume now that all poles of $\frac{Q^s}{1-Q^s} \Upsilon_1^*(s)$ are simple poles, which will be the case here, and given by $s = 0$, $s = \chi_l$, with $\chi_l := 2l\pi i/L$, $l \in \mathbb{Z} \setminus \{0\}$. Using

$$E^{(1)}(n) = \frac{1}{2\pi i} \int_{C_2-i\infty}^{C_2+i\infty} \frac{Q^s}{1-Q^s} \Upsilon_1^*(s) n^{-s} ds, \quad -C_1 < C_2 < 0,$$

the asymptotic expression of $E^{(1)}(n)$ is obtained by moving the line of integration to the right, for instance to the line $\Re s = C_4 > 0$, taking residues into account (with a negative sign). This gives

$$\begin{aligned} E^{(1)}(n) &= - \operatorname{Res} \left[\frac{Q^s}{1-Q^s} \Upsilon_1^*(s) n^{-s} \right] \Big|_{s=0} - \sum_{l \neq 0} \operatorname{Res} \left[\frac{Q^s}{1-Q^s} \Upsilon_1^*(s) n^{-s} \right] \Big|_{s=\chi_l} \\ &\quad + \mathcal{O}(n^{-C_4}). \end{aligned}$$

The residue at $s = 0$ gives of course

$$\tilde{m}_1 = \frac{\Upsilon_1^*(0)}{L} = \phi'(0).$$

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The other residues lead to

$$w_1 = \frac{1}{L} \sum_{l \neq 0} \Upsilon_1^*(\chi_l) e^{-2l\pi i \log n}. \tag{7}$$

More generally,

$$\mathbb{E}(Y_n - \log n)^k \sim \tilde{m}_k + w_k,$$

with

$$w_k = \frac{1}{L} \sum_{l \neq 0} \Upsilon_k^*(\chi_l) e^{-2l\pi i \log n},$$

and

$$\Upsilon_k^*(s) = L \phi^{(k)}(\alpha) \Big|_{\alpha = -Ls}.$$

It will appear that $\Upsilon_k^*(s)$ are analytic functions (in some domain), depending on classical functions such as the Γ function. But we know that $\Gamma(s)$ decreases exponentially towards $\pm i\infty$:

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi} |t|^{\sigma-1/2} e^{-\pi|t|/2}, \tag{8}$$

and all our functions will also decrease exponentially towards $\pm i\infty$.

Set

$$\begin{aligned} \phi(\alpha) &= \int_{-\infty}^{\infty} e^{\alpha\eta} f(\eta) d\eta \\ &= \frac{Q_2}{Q_1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\mu q)^i (-1)^j q^{\binom{j}{2}}}{(q)_i (q)_j} e^{\alpha(i+j)} \Gamma(-\tilde{\alpha})/L \\ &= \frac{Q_2}{Q_1} \frac{H_1(\alpha)}{H_2(\alpha)} \Gamma(-\tilde{\alpha})/L. \end{aligned}$$

This function will be the main tool we need to derive all asymptotic moments.

5. Moments

We have

$$\mathbb{E} [(K(n) - \log n)^i] \sim \tilde{m}_i + w_i + \mathcal{O}(n^{-\beta_i}), \quad \beta_i > 0,$$

where \tilde{m}_i are constants and w_i are periodic functions of $\log n$, with small $< 10^{-5}$ amplitude. All these expressions *only depend on $\phi(\alpha)$ and its derivatives*. For instance,

$$\begin{aligned} \phi(0) &= 1, \\ \tilde{m}_1 &= \phi'(0), \end{aligned}$$

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$$\begin{aligned} \tilde{m}_2 &= \phi''(0), \\ w_1 &= \sum_{l \neq 0} \varphi_1(\chi_l) e^{-2l\pi i \log n}, \\ \varphi_1(\chi_l) &= \phi'(\alpha)|_{\alpha=-L\chi_l}, \\ w_2 &= \sum_{l \neq 0} \varphi_2(\chi_l) e^{-2l\pi i \log n}, \\ \varphi_2(\chi_l) &= \phi''(\alpha)|_{\alpha=-L\chi_l}. \end{aligned}$$

Also note the following local expansions for $\tilde{\alpha}$ close to 0 resp. $-\chi_l$; recall that $\alpha = \tilde{\alpha}L$:

$$\begin{aligned} \Gamma(-\tilde{\alpha}) &= -\frac{L}{\alpha} - \gamma - \frac{\pi^2 + 6\gamma^2}{12L} \alpha + \dots, \\ \Gamma(-\tilde{\alpha}) &= \Gamma(\chi_l) - \frac{\psi(\chi_l)\Gamma(\chi_l)}{L}(\alpha + L\chi_l) \\ &\quad + \frac{\Gamma(\chi_l)(\psi(1, \chi_l) + \psi^2(\chi_l))}{2L^2}(\alpha + L\chi_l)^2 + \dots. \end{aligned}$$

With the identities presented in the appendix, this leads to our main result:

THEOREM 1. *The moments of the random variable $K(n) =$ highest level reached by (at least one member of) a party of n players satisfy the following asymptotic relation:*

$$\begin{aligned} \mathbb{E} [(K(n) - \log n)^i] &\sim \tilde{m}_i + w_i + \mathcal{O}(n^{-\beta_i}), \quad \beta_i > 0, \\ \tilde{m}_1 &= \frac{2\gamma + L - 2LC_{1,1} + 2L\mu q C_{2,1}}{L}, \\ w_1 &= \sum_{l \neq 0} \varphi_1(\chi_l) e^{-2l\pi i \log n}, \\ \varphi_1(\chi_l) &= -\frac{\Gamma(\chi_l)}{L}, \\ \tilde{m}_2 &= [\pi^2 + 6\gamma^2 + 6\gamma L - 12\gamma LC_{1,1} + 12\gamma L\mu q C_{2,1} + 2L^2 \\ &\quad - 12L^2 C_{1,1} - 6L^2 C_{1,2} + 6L^2 C_{1,1}^2 + 12L^2 \mu q C_{2,1} \\ &\quad + 6L^2 \mu^2 q^2 C_{2,2} + 6L^2 \mu^2 q^2 C_{2,1}^2 - 12L^2 \mu q C_{2,1} C_{1,1}] / (6L^2), \\ w_2 &= \sum_{l \neq 0} \varphi_2(\chi_l) e^{-2l\pi i \log n}, \end{aligned}$$

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$$\varphi_2(\chi_l) = -\frac{(-2\psi(\chi_l) + L - 2LC_{1,1} + 2L\mu q C_{2,1})\Gamma(\chi_l)}{L^2}.$$

The meaning of the various constants and functions can be found in the text and the appendix.

The first two expressions are identical to Proding er [10]. All moments can be automatically obtained by the same method.

For the reader’s convenience, we explicitly write the expected value of the maximum level that a party of (initially) n people reaches:

$$\mathbb{E}(K(n)) \sim \log_{1/q}(n) + \frac{2\gamma}{L} + 1 - 2 \sum_{i=1}^{\infty} \frac{q^i}{1 - q^i} + 2 \sum_{i=1}^{\infty} \frac{\mu q^i}{1 - \mu q^i} + \delta(\log_{1/q}(n))$$

with

$$\delta(x) = -\frac{1}{L} \sum_{l \in \mathbb{Z} \setminus \{0\}} \Gamma(\chi_l) e^{-2\pi i l x}.$$

6. Conclusion

This note is another showcase of the machinery developed in [7]. Once the underlying distribution is Gumbel distributed (extreme value distribution), moments can be computed in a semi-automatic way.

We hope to extend this series of applications in the near future.

Acknowledgement. The insightful comments of the referee are gratefully acknowledged.

Appendix A. Identities related to $H_1(\alpha)$

We find it useful to introduce the functions

$$\Sigma_{1,k}(z) := (k - 1)! \sum_{i=1}^{\infty} q^{ki} / (1 + zq^i)^k.$$

It is easily noticed that

$$\Sigma'_{1,k}(z) = -\Sigma_{1,k+1}(z).$$

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The special values

$$C_{1,k} := \sum_{i=1}^{\infty} q^{ki} / (1 - q^i)^k = \frac{1}{(k-1)!} \Sigma_{1,k}(-1)$$

are also of interest.

Logarithmic differentiation produces the following formulæ.

$$\begin{aligned} (-qz)'_{\infty} &= (-qz)_{\infty} \Sigma_{1,1}, \\ (-qz)''_{\infty} &= (-qz)_{\infty} [\Sigma_{1,1}^2 - \Sigma_{1,2}], \\ (-qz)'''_{\infty} &= (-qz)_{\infty} [-3\Sigma_{1,1}\Sigma_{1,2} + \Sigma_{1,1}^3 + \Sigma_{1,3}], \\ (-z)'_{\infty} &= (-qz)_{\infty} [1 + (1+z)\Sigma_{1,1}], \\ (-z)''_{\infty} &= (-qz)_{\infty} [2\Sigma_{1,1} + (1+z)[- \Sigma_{1,2} + \Sigma_{1,1}^2]], \\ (-z)'''_{\infty} &= (-qz)_{\infty} [-3\Sigma_{1,2} + 3\Sigma_{1,1}^2 + (1+z)[\Sigma_{1,3} - 3\Sigma_{1,2}\Sigma_{1,1} + \Sigma_{1,1}^3]]; \end{aligned}$$

we wrote here $\Sigma_{1,k}$ for $\Sigma_{1,k}(z)$.

Let ∂_{α} and ∂_z be the operators that differentiate w.r.t. α resp. z . Then we get by the chain rule for any $K(z)$, with $z = -e^{\alpha}$ or $z = -\mu q e^{\alpha}$:

$$\begin{aligned} \partial_{\alpha} K &= z \partial_z K, \\ \partial_{\alpha}^2 K &= z [z \partial_z^2 K + \partial_z K], \\ \partial_{\alpha}^3 K &= z [\partial_z K + 3z \partial_z^2 K + z^2 \partial_z^3 K]. \end{aligned}$$

This leads to (recall that $H_1(\alpha) = (e^{\alpha})_{\infty}$)

$$\begin{aligned} H_{1,0} &:= H_1(0) = 0, \\ H_{1,1} &:= \partial_{\alpha} H_1(\alpha)|_{\alpha=0} = -Q_1, \\ H_{1,2} &:= \partial_{\alpha}^2 H_1(\alpha)|_{\alpha=0} = Q_1[-1 + 2C_{1,1}], \\ H_{1,3} &:= \partial_{\alpha}^3 H_1(\alpha)|_{\alpha=0} = Q_1[-1 + 6C_{1,1} + 3C_{1,2} - 3C_{1,1}^2]. \end{aligned}$$

Note that we obtain the same expressions for $\alpha = -L\chi_l$, as $e^{-L\chi_l} = 1$.

Appendix B. Identities related to $H_2(\alpha)$

Now we deal with $H_2(\alpha) = (\mu q e^{\alpha})_{\infty}$.

We need

$$\Sigma_{2,k}(z) := (k-1)! \sum_{i=0}^{\infty} q^{ki} / (1 + zq^i)^k = \frac{(k-1)!}{(1+z)^k} + \Sigma_{1,k}(z)$$

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and

$$C_{2,k} = \sum_{i=0}^{\infty} q^{ki} / (1 - \mu q q^i)^k = \frac{1}{(k-1)!} \Sigma_{2,k}(-\mu q).$$

Since

$$\begin{aligned} (-z)'_{\infty} &= (-z)_{\infty} \Sigma_{2,1}(z), \\ (-z)''_{\infty} &= (-z)_{\infty} [\Sigma_{2,1}^2(z) - \Sigma_{2,2}(z)], \end{aligned}$$

we get

$$\begin{aligned} H_{2,0} &:= H_2(0) = Q_2, \\ H_{2,1} &:= \partial_{\alpha} H_2(\alpha)|_{\alpha=0} = -\mu q C_{2,1} Q_2, \\ H_{2,2} &:= \partial_{\alpha}^2 H_2(\alpha)|_{\alpha=0} = -\mu q Q_2 [C_{2,1} - \mu q (-C_{2,2} + C_{2,1}^2)]. \end{aligned}$$

Again we obtain the same expressions for $\alpha = -L\chi_l$, as $e^{-L\chi_l} = 1$.

REFERENCES

- [1] ANDREWS, G. E.: *The Theory of Partitions. Encyclopedia of Mathematics and its Applications, Vol. 2*, Addison-Wesley, Reading, Mass-London-Amsterdam, 1976.
- [2] FLAJOLET, P.: *Approximate counting: a detailed analysis*. BIT **25** (1985), 113–134.
- [3] FLAJOLET, P.—GOURDON, X.—DUMAS, P.: *Mellin transforms and asymptotics: Harmonic sums*, Theoret. Comput. Sci. **144** (1995), 3–58.
- [4] HITCZENKO, P.—LOUCHARD, G.: *Distinctness of compositions of an integer: a probabilistic analysis*, Random Structures Algorithms **19** (2001), 407–437.
- [5] KNUTH, D. E.: *The Art of Computer Programming. Vol. 3*, Addison-Wesley, Reading, Mass, 1998.
- [6] LOÈVE, M.: *Probability Theory* (3rd ed.), D. Van Nostrand Co., Inc., Princeton, NJ-Toronto-London, 1963.
- [7] LOUCHARD, G.—PRODINGER, H.: *Asymptotics of the moments of extreme-value related distribution functions*, Algorithmica **46** (2006), 431–467. Long version: <http://www.ulb.ac.be/di/mcs/louchard/moml.ps>.
- [8] LOUCHARD, G.—PRODINGER, H.—WARD, M. D.: *The number of distinct values of some multiplicity in sequences of geometrically distributed random variables*. In: Discrete Mathematics and Theoretical Computer Science Proceedings AD (C. Martínez, ed.). 2005 International Conference on Analysis of Algorithms. Papers from the Conference, Barcelona, Spain, June 6–10, 2005, The Association Discrete Mathematics & Theoretical Science (DMTCS), Nancy, 2005, pp. 231–256.
- [9] MAHMOUD, H. M.: *Evolution of Random Search Trees*. John Wiley, New York, 1992.

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- [10] PRODINGER, H.: *How to advance on a stairway by coin flippings*. In: Applications of Fibonacci Numbers, Vol. 5 (G. E. Bergum et al., ed.). Proceedings of the fifth international conference on Fibonacci numbers and their applications, University of St. Andrews, Scotland, July 20–24, 1992, Kluwer Academic Publishers, Dordrecht, 1993, pp. 473–479.
- [11] SCHMID, U.: *Analyse von Collision-Resolution Algorithmen in Random-Access Systemen mit dominanten Übertragungskanälen*. Ph.D. Thesis, University of Technology, Vienna, 1986.
- [12] SCHMID, U.: *The average CRI-length of a tree collision resolution algorithm in presence of multiplicity-dependent capture effects*. In: Automata, Languages and Programming (W. Kuich, ed.). 19th International Colloquium, ICALP92, Vienna, Austria, July 13–17, 1992. Proceedings. Lecture Notes in Comput. Sci. 623, Springer, Berlin, 1992, pp. 223–234.
- [13] SCHMID, U.: *On a tree collision resolution algorithm in presence of capture*, RAIRO Inform. Théor. Appl. **26** (1992), 163–197.
- [14] SZPANKOWSKI, W.: *Average Case Analysis of Algorithms on Sequences*, Wiley, New York, 2001.

Received 15. 6. 2006

Revised 16. 11 2006

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Periodica Mathematica Hungarica Vol. 64 (1), 2012, pp. 101–117
DOI: 10.1007/s10998-012-9101-9

NUMBER OF SURVIVORS IN THE PRESENCE OF A DEMON

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(Received May 3, 2010; Accepted June 10, 2010)

[Communicated by Bálint Tóth]

Abstract

This paper complements the analysis of Louchard and Prodinger [LP08] on the number of *rounds* in a coin-flipping selection algorithm that occurs in the presence of a demon. We precisely analyze a very different aspect of the selection algorithm, using different methods of analysis. Specifically, we obtain precise descriptions of the distribution and all moments of the number of *participants* ultimately selected during the execution of the algorithm. The selection algorithm is robust in at least two significant ways. The presence of a demon allows for the precise analysis even when errors may occur between the rounds of the selection process. (The analysis also handles the more traditional case, in which no demon is involved.) The selection algorithm can also use either biased or unbiased coins.

1. Introduction

We precisely analyze the number of *survivors* in a selection process that occurs in the presence of a demon. Louchard and Prodinger [LP08] recently utilized a different methodology (for extreme value distributions, often referred to as “Gumbel

Mathematics subject classification numbers: 05A16, 05A30, 60C05, 68W20, 68W40.

Key words and phrases: analysis of algorithms, leader election, coin flip, survivor, demon, asymptotic, approximation, trie, q -Pochhammer symbol, factorial moment, distribution, generating function, recurrence, poissonization.

0031-5303/2012/\$20.00
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distributions”) to analyze the number of *rounds* required to perform the selection algorithm.

The inclusion of a “demon” can be viewed as a generalization of traditional selection algorithms. The demon represents errors which might occur between rounds of the process. Another interpretation is that participants might be likely to drop out of the selection process for reasons unrelated to the coin flips in the selection process itself. In each round, exactly one participant is removed by the “demon” with probability ν ; otherwise, the demon does not affect any participants in that round. Thus, a traditional selection algorithm (with no demon involved) is just a special case (using $\nu = 0$) of our very general analysis. The special case $\nu = 0$ (i.e., with no demon involved) is a selection process using a traversal of binary retrieval trees (tries), where a coin flip of “heads” is analogous to descending one direction in the trie, and a flip of “tails” corresponds to descending in the other direction. We use p and q (respectively) for the probabilities of heads and tails on coins in the selection algorithm. The analysis is sufficiently general to handle both the unbiased ($p = q = 1/2$) and biased ($p \neq q$) processes. The involvement of a demon makes the present algorithm more complicated and realistic than the traditional trie algorithm.

We are able to give the complete distribution and all moments of the number of survivors in a selection algorithm that occurs in the presence of a demon.

2. Selection algorithm

At the start of the selection algorithm, n people are present. Each person flips a coin with probability q of heads and p of tails. If all n people flip tails, then they are all selected by the algorithm and the selection process is finished. If $j > 0$ people flip heads, then these j people remain in play, and the other $n - j$ (who flipped tails) are eliminated from further play.

Then a demon arrives and, with probability ν , removes exactly one of the survivors, so $j - 1$ remain; he leaves the j survivors alone with probability $\mu = 1 - \nu$. If he leaves the j survivors alone, then these j survivors begin another round of coin flipping. If he removes a survivor and $j - 1 = 0$ (i.e., no survivors remain) then the selection process is finished and nobody is selected. If he removes a survivor and $j - 1 > 0$ (i.e., some survivors remain) then these $j - 1$ survivors begin another round of coin flipping.

The end of the algorithm can occur in two possible ways:

(1) During a round of coin flipping, one or more people remain. All of the remaining people simultaneously flip tails at this stage, and the algorithm ends. All of the people at this last stage are selected by the algorithm.

(2) During a visit by the demon, only one person is present, and this person is removed by the demon. In this case, zero people are selected by the algorithm.

All of the coin flips (among the people and among the rounds) are conducted independently.

EXAMPLE 2.1. Suppose that the probability of heads is $q = 1/3$ and the probability of tails is $p = 2/3$. Suppose that the demon appears with probability $\nu = 1/5$. Then the selection algorithm might proceed as follows: Initially 100 people are present. Exactly 31 of them flip heads (this happens with probability $\binom{100}{31}q^{31}p^{69}$), and then the demon arrives (this happens with probability $1/5$) and removes one of the 31 survivors. So the next round begins with 30 people. Exactly 12 of the remaining 30 people flip heads (this happens with probability $\binom{30}{12}q^{12}p^{18}$), and then the demon leaves the survivors alone (this happens with probability $4/5$). So the next round begins with 12 people. Exactly 3 of the remaining 12 people flip heads (this happens with probability $\binom{12}{3}q^3p^9$), and then the demon leaves the survivors alone (this happens with probability $4/5$). So the next round begins with 3 people. Exactly 3 remaining people flip tails (this happens with probability $\binom{3}{0}q^0p^3$), and all three are selected by the algorithm.

3. Notation table

Most of the following definitions are already embedded at the appropriate places in the analysis. For the reader's convenience, we also summarize many of the terminologies used, in one succinct location. For the convenience of someone who already read [LP08], we preserve some of Louchard and Prodinger's earlier notation:

- n := number of people present at the start of the selection algorithm,
- q := probability that a coin flip shows heads,
- p := $1 - q$, probability that a coin flip shows tails,
- ν := probability that, during a visit by the demon, one survivor is removed,
- μ := $1 - \nu$, probability that the demon does not remove a survivor during a visit,
- X_n := number of people selected by the algorithm, with n initial participants,
- $\pi(n, m, j)$:= probability the algorithm selects m of the initial n people and requires j rounds; by convention, $\pi(0, 0, 1) = 1$, and otherwise $\pi(0, m, j) = 0$,
- $\pi(n, m)$:= $\mathbf{P}(X_n = m)$ = probability that the algorithm selects m of the initial n people; by convention, $\pi(0, 0) = 1$, and $\pi(0, m) = 0$ for $m \neq 0$,

$$\begin{aligned}
 F_n(u, v) &:= \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \pi(n, m, j) u^m v^j, \\
 F_n(u) &:= F_n(u, 1) = \sum_{m=0}^{\infty} \pi(n, m) u^m = \sum_{m=0}^{\infty} \mathbf{P}(X_n = m) u^m, \\
 Q &:= 1/q, \\
 L &:= \ln Q, \\
 \chi_\ell &:= 2\ell\pi i/L, \\
 H_j &:= \sum_{k=1}^j \frac{1}{k} \text{ is the } j\text{th harmonic number,} \\
 x^{\underline{j}} &:= \prod_{\ell=0}^{j-1} (x - \ell) = (x)(x-1)(x-2) \cdots (x-j+1) \text{ is the } j\text{th falling} \\
 &\quad \text{power of } x, \\
 \mathbb{E}[X_n^{\underline{j}}] &:= \mathbb{E}\left[\prod_{\ell=0}^{j-1} (X_n - \ell)\right] \text{ is the } j\text{th factorial moment of the random} \\
 &\quad \text{variable } X_n, \\
 F_n^{(s)}(u) &:= \frac{d^s}{du^s} F_n(u).
 \end{aligned}$$

We utilize some concepts from the theory of q -analysis. Since the value of q is fixed, we suppress the dependence on q . For positive integers n , we use the q -Pochhammer symbol

$$(x)_n := \prod_{j=0}^{n-1} (1 - xq^j) = (1-x)(1-xq)(1-xq^2) \cdots (1-xq^{n-1}).$$

We also define $(x)_\infty := \prod_{j=0}^{\infty} (1 - xq^j) = \lim_{n \rightarrow \infty} (x)_n$. For complex-valued z , we define

$$(x)_z := \frac{(x)_\infty}{(xq^z)_\infty}.$$

4. Results

The following two theorems precisely characterize the s th factorial moment $\mathbb{E}[X_n^{\underline{s}}]$ and the distribution $\mathbf{P}(X_n > r)$ of X_n . Each has the form $\text{const} + \delta(\log_Q n) + o(1)$. The constant and the function δ both depend on s or r , respectively. In both cases, the δ is fluctuating, because $e^{2\ell\pi i \log_Q n}$ is fluctuating, with $|e^{2\ell\pi i \log_Q n}| = 1$.

THEOREM 4.1. *The s th factorial moment of the number X_n of people selected by the algorithm, when beginning with n participants, is*

$$\begin{aligned} \mathbb{E}[X_n^s] &= \frac{(Qp)^s}{L} \left[\frac{(\mu q)_\infty}{(q)_\infty} (s-1)! + \right. \\ &\quad \left. + s(-1)^{s-1} \sum_{j \geq s-1} j^{s-1} \left[L \frac{(\mu q)_j}{(q)_j} \left(\sum_{\ell \geq 1} \frac{\mu q^{j+\ell}}{1 - \mu q^{j+\ell}} - \sum_{\ell \geq 1} \frac{q^{j+\ell}}{1 - q^{j+\ell}} \right) + \right. \right. \\ &\quad \left. \left. + \left(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j} \right) (H_j - H_{j-s+1}) \right] \right] + \\ &\quad + \sum_{\ell \neq 0} \phi_{s,\ell}(n) + O(n^{-1}), \end{aligned}$$

where

$$\begin{aligned} \phi_{s,\ell}(n) &= \frac{(Qp)^s (-1)^s}{L} \left(\frac{(\mu q)_\infty}{(q)_\infty} \chi_\ell^s + s \sum_{j \geq 0} \left(\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right) (j^{s-1} - (j + \chi_\ell)^{s-1}) \right) \times \\ &\quad \times \Gamma(-\chi_\ell) e^{2\ell\pi i \log_Q n}. \end{aligned}$$

THEOREM 4.2. *The distribution of the number X_n of people selected by the algorithm, when beginning with n participants, is*

$$\begin{aligned} \mathbf{P}(X_n > r) &= \frac{(\mu q)_\infty}{L(q)_\infty} \left[L - \sum_{s=1}^r \frac{p^s}{s} - \frac{\nu p^r}{r+1} - Q(-p)^{r+1} \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \times \right. \\ &\quad \left. \times \left(\sum_{s=1}^r \frac{(-1)^s}{s} \frac{q^{(m-1)(r-s)}}{(1 - q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r(m-1)L}}{(1 - q^{m-1})^{r+1}} \right) \right] + \\ &\quad + \sum_{\ell \neq 0} \Phi_{r,\ell}(n) + O(n^{-1}), \end{aligned}$$

where

$$\begin{aligned} \Phi_{r,\ell}(n) &= \frac{Q}{L} \frac{(\mu q)_\infty}{(q)_\infty} \left(-q \sum_{s=1}^r \frac{\chi_\ell^s}{s!} (-p)^s + (-p)^r \nu q \frac{\chi_\ell^{r+1}}{(r+1)!} + \right. \\ &\quad \left. + (-p)^{r+1} \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \left(\frac{q^{(m-1)r}}{(1 - q^{m-1})^{r+1}} - \sum_{s=0}^r \frac{\chi_\ell^s}{s!} \frac{q^{(m-1)(r-s)}}{(1 - q^{m-1})^{r-s+1}} \right) \right) \times \\ &\quad \times \Gamma(-\chi_\ell) e^{2\ell\pi i \log_Q n}. \end{aligned}$$

5. Asymptotic moments of the number of survivors

5.1. Derivation of generating functions

We next establish an exact formula for the bivariate generating function $F_n(u, v)$ that describes the probabilities associated with the number of survivors and the number of rounds in the entire algorithm.

LEMMA 5.1. *Let $F_n(u, v)$ be a bivariate generating function such that the coefficient of $u^m v^j$ is the probability that, in the algorithm, exactly m people are ultimately selected and exactly j rounds are used to complete the election. Then*

$$F_n(u, v) = \sum_{k=0}^n \binom{n}{k} \left(v(-q)^k \frac{(v)_k}{(\mu v q)_k} + (-q)^{k-1} \frac{(vq)_{k-1}}{(\mu q v)_k} \sum_{j=0}^{k-1} \frac{(1-pu)^j p v (u-1) (\mu q v)_j}{q^j (vq)_j} \right). \tag{1}$$

The proof of Lemma 5.1 utilizes some recurrences associated with $F_n(u, v)$. The proof is given in Section 7.

COROLLARY 5.2. *Setting $u = 1$ in Lemma 5.1, we obtain*

$$F_n(1, v) = \sum_{k=0}^n \binom{n}{k} v(-q)^k \frac{(v)_k}{(\mu v q)_k}. \tag{2}$$

This verifies that our results about the number of rounds agrees with the results from our previous paper.

During the remainder of the paper, we no longer pay attention to the number of rounds. We focus exclusively on the number of survivors.

LEMMA 5.3. *The s th factorial moment of X_n is*

$$\mathbb{E}[X_n^s] = \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} \varphi_s(k), \tag{3}$$

with

$$\varphi_s(z) = q^z \frac{(q)_{z-1}}{(\mu q)_z} s(Qp)^s (-1)^{s-1} \psi_s(z), \tag{4}$$

and

$$\psi_s(z) = \frac{(\mu q)_\infty}{(q)_\infty} \frac{z^s}{s} + \sum_{j \geq 0} \left[\left(\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right) j^{s-1} - \left(\frac{(\mu q)_{j+z}}{(q)_{j+z}} - \frac{(\mu q)_\infty}{(q)_\infty} \right) (j+z)^{s-1} \right].$$

5.2. Asymptotics

Now we turn our attention to the asymptotic moments of the number of survivors in the algorithm as the number n of initial participants grows large. We note that $(q)_{z-1} = \frac{(q)_\infty}{(1-q^z)(q^{z+1})_\infty}$, so $\varphi_s(z)$ has a simple pole at each of the locations of the form $z = m + \frac{2\ell\pi i}{L}$ for $\ell, m \in \mathbb{Z}$ with $m \leq 0$. By [FS95, Theorem 2], we can restrict attention to the poles, where $m = 0$, i.e., where $z = \chi_\ell$ for $\ell \in \mathbb{Z}$. Thus

$$\mathbb{E}[X_n^s] = \sum_{\ell \in \mathbb{Z}} \operatorname{Res}_{z=\chi_\ell} \left[\varphi_s(z) \frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)} \right] + O(n^{-1}).$$

We need the local expansion of $\varphi_s(z)$ and thus $\psi_s(z)$ around $z = 0$ to two terms, since $\varphi_s(z) \frac{n!}{(z)(z-1)\cdots(z-n)}$ has a double pole at $z = 0$, but only a simple pole at $z = \chi_\ell$ for $\ell \neq 0$. As $z \rightarrow 0$,

$$\frac{(\mu q)_{j+z}}{(q)_{j+z}} \sim \frac{(\mu q)_j}{(q)_j} \left[1 - zL \sum_{\ell \geq 1} \frac{\mu q^{j+\ell}}{1 - \mu q^{j+\ell}} + zL \sum_{\ell \geq 1} \frac{q^{j+\ell}}{1 - q^{j+\ell}} \right],$$

and

$$(j+z)^{s-1} \sim j^{s-1} [1 + z(H_j - H_{j-s+1})].$$

Thus

$$\begin{aligned} \psi_s(z) \sim & \frac{(\mu q)_\infty}{(q)_\infty} \frac{(-1)^{s-1}(s-1)!}{s} z + \\ & + \sum_{j \geq s-1} \left[\left(\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right) j^{s-1} - \right. \\ & \left. - \left(\frac{(\mu q)_j}{(q)_j} \left[1 - zL \sum_{\ell \geq 1} \frac{\mu q^{j+\ell}}{1 - \mu q^{j+\ell}} + zL \sum_{\ell \geq 1} \frac{q^{j+\ell}}{1 - q^{j+\ell}} \right] - \frac{(\mu q)_\infty}{(q)_\infty} \right) \times \right. \\ & \left. \times j^{s-1} [1 + z(H_j - H_{j-s+1})] \right]. \end{aligned}$$

More simply, as $z \rightarrow 0$,

$$\begin{aligned} \psi_s(z) \sim & z \left[\frac{(\mu q)_\infty}{(q)_\infty} \frac{(-1)^{s-1}(s-1)!}{s} + \right. \\ & + \sum_{j \geq s-1} j^{s-1} \left[L \frac{(\mu q)_j}{(q)_j} \left(\sum_{\ell \geq 1} \frac{\mu q^{j+\ell}}{1 - \mu q^{j+\ell}} - \sum_{\ell \geq 1} \frac{q^{j+\ell}}{1 - q^{j+\ell}} \right) + \right. \\ & \left. \left. + \left(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j} \right) (H_j - H_{j-s+1}) \right] \right]. \end{aligned}$$

Notice the absence of the constant term! Substituting into the definition of $\varphi_s(z)$ in (4), it follows that

$$\begin{aligned} \varphi_s(z) &\sim q^z \frac{(q)_{z-1}}{(\mu q)_z} s(Qp)^s (-1)^{s-1} \times \\ &\quad \times z \left[\frac{(\mu q)_\infty}{(q)_\infty} \frac{(-1)^{s-1} (s-1)!}{s} + \right. \\ &\quad \left. + \sum_{j \geq s-1} j^{s-1} \left[L \frac{(\mu q)_j}{(q)_j} \left(\sum_{\ell \geq 1} \frac{\mu q^{j+\ell}}{1 - \mu q^{j+\ell}} - \sum_{\ell \geq 1} \frac{q^{j+\ell}}{1 - q^{j+\ell}} \right) + \right. \right. \\ &\quad \left. \left. + \left(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j} \right) (H_j - H_{j-s+1}) \right] \right] \end{aligned}$$

as $z \rightarrow 0$. Also $z(q)_{z-1} \sim 1/L$ and $(\mu q)_z \sim 1$, so

$$\begin{aligned} \varphi_s(z) &\sim \frac{(Qp)^s}{L} \left[\frac{(\mu q)_\infty}{(q)_\infty} (s-1)! + \right. \\ &\quad \left. + s(-1)^{s-1} \sum_{j \geq s-1} j^{s-1} \left[L \frac{(\mu q)_j}{(q)_j} \left(\sum_{\ell \geq 1} \frac{\mu q^{j+\ell}}{1 - \mu q^{j+\ell}} - \sum_{\ell \geq 1} \frac{q^{j+\ell}}{1 - q^{j+\ell}} \right) + \right. \right. \\ &\quad \left. \left. + \left(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j} \right) (H_j - H_{j-s+1}) \right] \right] \end{aligned}$$

as $z \rightarrow 0$. Also $\frac{n!(-1)^n}{(z-1)\cdots(z-n)} \sim 1$ as $z \rightarrow 0$. Therefore

$$\begin{aligned} \operatorname{Res}_{z=0} \left[\varphi_s(z) \frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)} \right] &= \lim_{z \rightarrow 0} \varphi_s(z) \\ &= \frac{(Qp)^s}{L} \left[\frac{(\mu q)_\infty}{(q)_\infty} (s-1)! + \right. \\ &\quad \left. + s(-1)^{s-1} \sum_{j \geq s-1} j^{s-1} \left[L \frac{(\mu q)_j}{(q)_j} \left(\sum_{\ell \geq 1} \frac{\mu q^{j+\ell}}{1 - \mu q^{j+\ell}} - \sum_{\ell \geq 1} \frac{q^{j+\ell}}{1 - q^{j+\ell}} \right) + \right. \right. \\ &\quad \left. \left. + \left(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j} \right) (H_j - H_{j-s+1}) \right] \right]. \end{aligned}$$

So the s th factorial moment $\mathbb{E}[X_n^{\underline{s}}]$ of X_n is

$$\begin{aligned} \mathbb{E}[X_n^{\underline{s}}] &= \frac{(Qp)^s}{L} \left[\frac{(\mu q)_\infty}{(q)_\infty} (s-1)! + \right. \\ &\quad \left. + s(-1)^{s-1} \sum_{j \geq s-1} j^{s-1} \left[L \frac{(\mu q)_j}{(q)_j} \left(\sum_{\ell \geq 1} \frac{\mu q^{j+\ell}}{1 - \mu q^{j+\ell}} - \sum_{\ell \geq 1} \frac{q^{j+\ell}}{1 - q^{j+\ell}} \right) + \right. \right. \\ &\quad \left. \left. + \left(\frac{(\mu q)_\infty}{(q)_\infty} - \frac{(\mu q)_j}{(q)_j} \right) (H_j - H_{j-s+1}) \right] \right] + \\ &\quad + \sum_{\ell \neq 0} \tilde{\phi}_{s,\ell}(n) + O(n^{-1}), \end{aligned}$$

where

$$\begin{aligned} \tilde{\phi}_{s,\ell}(n) &= \operatorname{Res}_{z=\chi_\ell} \left[\varphi_s(z) \frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)} \right] \\ &= \operatorname{Res}_{z=\chi_\ell} [(q)_{z-1}] \frac{q^{\chi_\ell} s p^s}{(\mu q)_{\chi_\ell} q^s} (-1)^{s-1} \psi_s(\chi_\ell) \frac{n!(-1)^n}{(\chi_\ell)(\chi_\ell-1)\cdots(\chi_\ell-n)} \\ &= \frac{(Qp)^s (-1)^s}{L} \left(\frac{(\mu q)_\infty}{(q)_\infty} \chi_\ell^s + s \sum_{j \geq 0} \left(\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right) (j^{s-1} - (j + \chi_\ell)^{s-1}) \right) \times \\ &\quad \times \Gamma(-\chi_\ell) e^{2\ell\pi i \log_Q n} (1 + O(n^{-1})). \end{aligned}$$

Note that $e^{2\ell\pi i \log_Q n}$ is fluctuating, with $|e^{2\ell\pi i \log_Q n}| = 1$. This completes the proof of Theorem 4.1.

6. Asymptotic distribution of the number of survivors

6.1. Derivation of the distribution of the number of survivors

Now we derive an exact formula for the distribution of the number of survivors selected at the end of the algorithm.

LEMMA 6.1. *Let $r \geq 0$. The probability that strictly more than r out of n initial participants are selected at the end of the algorithm is*

$$\begin{aligned} \mathbf{P}(X_n > r) &= \sum_{k=1}^n \binom{n}{k} (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} p^{r+1} (-1)^r \times \\ &\quad \times \frac{(\mu q)_\infty}{(q)_\infty} \sum_{m \geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1}. \end{aligned} \tag{5}$$

The proof of Lemma 6.1 utilizes the q -binomial theorem; see Section 7.

LEMMA 6.2. *The distribution of X_n has the form*

$$\mathbf{P}(X_n > r) = \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} \varrho_r(k), \tag{6}$$

with

$$\varrho_r(z) = q^{z-1} \frac{(q)_{z-1}}{(\mu q)_z} p^{r+1} (-1)^r \Psi_r(z), \tag{7}$$

and

$$\begin{aligned} \Psi_r(z) &= \frac{(\mu q)_\infty}{(q)_\infty} \left(\frac{q}{(-p)^{r+1}} \left(1 - \sum_{s=0}^r \frac{z^s}{s!} q^{-z} (-p)^s \right) + \frac{(\mu-1)q}{p} \frac{z^{r+1}}{(r+1)!} + \right. \\ &\quad \left. + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \left(\frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} - \sum_{s=0}^r \frac{z^s}{s!} \frac{q^{(m-1)(z+r-s)}}{(1-q^{m-1})^{r-s+1}} \right) \right). \end{aligned}$$

6.2. Asymptotics

Now we turn our attention to the asymptotic distribution of the number of survivors in the algorithm as the number n of initial participants grows large. We follow the derivation for the Rice Method discussed in Section 5. As before, $(q)_{z-1} = \frac{(q)_\infty}{(1-q^z)(q^{z+1})_\infty}$, so $\varrho_r(z)$ has a simple pole at each of the locations of the form $z = m + \frac{2\ell\pi i}{L}$ for $\ell, m \in \mathbb{Z}$ with $m \leq 0$. Again, by [FS95], we focus on the poles $z = \chi_\ell$ for $\ell \in \mathbb{Z}$. Thus

$$\mathbf{P}(X_n > r) = \sum_{\ell \in \mathbb{Z}} \operatorname{Res}_{z=\chi_\ell} \left[\varrho_r(z) \frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)} \right] + O(n^{-1}).$$

Similarly to the derivation in Section 5, we need the local expansion of $\varrho_r(z)$ and thus $\Psi_r(z)$ around $z = 0$ to two terms, since $\varrho_r(z) \frac{n!}{(z)(z-1)\cdots(z-n)}$ has a double pole at $z = 0$, but only a simple pole at $z = \chi_\ell$ for $\ell \neq 0$. As $z \rightarrow 0$,

$$\sum_{s=0}^r \frac{z^s}{s!} q^{-z} (-p)^s \sim 1 - z \sum_{s=1}^r \frac{p^s}{s} + zL,$$

$$\frac{z^{r+1}}{(r+1)!} \sim z \frac{(-1)^r}{(r+1)},$$

and

$$\sum_{s=0}^r \frac{z^s}{s!} \frac{q^{(m-1)(z+r-s)}}{(1-q^{m-1})^{r-s+1}}$$

$$\sim \frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} + z \sum_{s=1}^r \frac{(-1)^{s-1}}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} - z \frac{q^{(m-1)r} L(m-1)}{(1-q^{m-1})^{r+1}}.$$

Thus

$$\Psi_r(z) \sim \frac{(\mu q)_\infty}{(q)_\infty} \left(\frac{q}{(-p)^{r+1}} \left(z \sum_{s=1}^r \frac{p^s}{s} - zL \right) + \frac{(\mu-1)q}{p} z \frac{(-1)^r}{(r+1)} + \right.$$

$$\left. + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \times \right.$$

$$\left. \times \left(-z \sum_{s=1}^r \frac{(-1)^{s-1}}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + z \frac{q^{(m-1)r} L(m-1)}{(1-q^{m-1})^{r+1}} \right) \right).$$

More simply, as $z \rightarrow 0$,

$$\Psi_r(z) \sim z \frac{(\mu q)_\infty}{(q)_\infty} \left[\frac{q(-1)^r}{p^{r+1}} \left(L - \sum_{s=1}^r \frac{p^s}{s} \right) - \frac{(1-\mu)q}{p} \frac{(-1)^r}{(r+1)} + \right.$$

$$\left. + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \times \right.$$

$$\left. \times \left(\sum_{s=1}^r \frac{(-1)^s}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r} (m-1)L}{(1-q^{m-1})^{r+1}} \right) \right].$$

As before, notice the absence of the constant term. Substitution into (7) yields

$$\begin{aligned} \varrho_r(z) &\sim q^{z-1} \frac{(q)_{z-1}}{(\mu q)_z} p^{r+1} (-1)^r \times \\ &\times z \frac{(\mu q)_\infty}{(q)_\infty} \left[\frac{q(-1)^r}{p^{r+1}} \left(L - \sum_{s=1}^r \frac{p^s}{s} \right) - \frac{(1-\mu)q}{p} \frac{(-1)^r}{(r+1)} + \right. \\ &\quad \left. + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \times \right. \\ &\quad \left. \times \left(\sum_{s=1}^r \frac{(-1)^s}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r} (m-1)L}{(1-q^{m-1})^{r+1}} \right) \right] \end{aligned}$$

as $z \rightarrow 0$. Also $z(q)_{z-1} \sim 1/L$ and $(\mu q)_z \sim 1$, so

$$\begin{aligned} \varrho_r(z) &\sim \frac{(\mu q)_\infty}{L(q)_\infty} \left[L - \sum_{s=1}^r \frac{p^s}{s} - \frac{\nu p^r}{r+1} - \right. \\ &\quad \left. - Q(-p)^{r+1} \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \times \right. \\ &\quad \left. \times \left(\sum_{s=1}^r \frac{(-1)^s}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r} (m-1)L}{(1-q^{m-1})^{r+1}} \right) \right] \end{aligned}$$

as $z \rightarrow 0$. Also $\frac{n!(-1)^n}{(z-1)\cdots(z-n)} \sim 1$ as $z \rightarrow 0$. Therefore

$$\begin{aligned} \operatorname{Res}_{z=0} \left[\varrho_r(z) \frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)} \right] &= \lim_{z \rightarrow 0} \varrho_r(z) \\ &= \frac{(\mu q)_\infty}{L(q)_\infty} \left[L - \sum_{s=1}^r \frac{p^s}{s} - \frac{\nu p^r}{r+1} - Q(-p)^{r+1} \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \times \right. \\ &\quad \left. \times \left(\sum_{s=1}^r \frac{(-1)^s}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r} (m-1)L}{(1-q^{m-1})^{r+1}} \right) \right]. \end{aligned}$$

So

$$\begin{aligned} \mathbf{P}(X_n > r) &= \frac{(\mu q)_\infty}{L(q)_\infty} \left[L - \sum_{s=1}^r \frac{p^s}{s} - \frac{\nu p^r}{r+1} - Q(-p)^{r+1} \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \times \right. \\ &\quad \left. \times \left(\sum_{s=1}^r \frac{(-1)^s}{s} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} + \frac{q^{(m-1)r} (m-1)L}{(1-q^{m-1})^{r+1}} \right) \right] + \\ &\quad + \sum_{\ell \neq 0} \tilde{\Phi}_{r,\ell}(n) + O(n^{-1}), \end{aligned}$$

where

$$\begin{aligned} \tilde{\Phi}_{r,\ell}(n) &= \operatorname{Res}_{z=\chi_\ell} \left[\varrho_r(z) \frac{n!(-1)^n}{(z)(z-1)\cdots(z-n)} \right] \\ &= \operatorname{Res}_{z=\chi_\ell} [(q)_{z-1}] \frac{q^{\chi_\ell-1} p^{r+1}}{(\mu q)_{\chi_\ell}} (-1)^r \Psi_r(\chi_\ell) \frac{n!(-1)^n}{(\chi_\ell)(\chi_\ell-1)\cdots(\chi_\ell-n)} \\ &= \frac{Q}{L} \frac{(\mu q)_\infty}{(q)_\infty} \left(-q \sum_{s=1}^r \frac{\chi_\ell^s}{s!} (-p)^s + (-p)^r \nu q \frac{\chi_\ell^{r+1}}{(r+1)!} + \right. \\ &\quad \left. + (-p)^{r+1} \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \left(\frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} - \sum_{s=0}^r \frac{\chi_\ell^s}{s!} \frac{q^{(m-1)(r-s)}}{(1-q^{m-1})^{r-s+1}} \right) \right) \times \\ &\quad \times \Gamma(-\chi_\ell) e^{2\ell\pi i \log_Q n} (1 + O(n^{-1})). \end{aligned}$$

Note that $e^{2\ell\pi i \log_Q n}$ is fluctuating, with $|e^{2\ell\pi i \log_Q n}| = 1$. This completes the proof of Theorem 4.2

7. Proofs

PROOF OF LEMMA 5.1. When starting with n participants, if all n participants are simultaneously eliminated by coin flipping, then these n participants are selected by the algorithm; this corresponds to the term $\binom{n}{0} q^0 p^n u^n v$ in recurrence (8) below. If exactly j participants obtain heads, with $1 \leq j \leq n$, then the demon arrives and removes one additional participant with probability ν , or leaves the j remaining participants alone with probability μ . This phenomenon corresponds to $v \sum_{j=1}^n \binom{n}{j} q^j p^{n-j} (\nu F_{j-1}(u, v) + \mu F_j(u, v))$ in formula (8). (Note that, since $F_0(u, v) = v$, the recurrence below also holds when $n = 0$.) So the recurrence

$$F_n(u, v) = \binom{n}{0} q^0 p^n u^n v + v \sum_{j=1}^n \binom{n}{j} q^j p^{n-j} (\nu F_{j-1}(u, v) + \mu F_j(u, v)) \quad (8)$$

holds for all integers $n \geq 0$. More simply,

$$F_n(u, v) = v \left(p^n u^n + \sum_{j=1}^n \binom{n}{j} q^j p^{n-j} (\nu F_{j-1}(u, v) + \mu F_j(u, v)) \right). \quad (9)$$

Next we define the exponential generating function

$$G(z, u, v) := \sum_{n=0}^{\infty} F_n(u, v) \frac{z^n}{n!}.$$

From the recurrence in (9), it follows that

$$\begin{aligned}
G(z, u, v) &= v \sum_{n=0}^{\infty} \left(p^n u^n + \sum_{j=1}^n \binom{n}{j} q^j p^{n-j} (\nu F_{j-1}(u, v) + \mu F_j(u, v)) \right) \frac{z^n}{n!} \\
&= v \left(e^{puz} + \sum_{j=1}^{\infty} q^j z^j (\nu F_{j-1}(u, v) + \mu F_j(u, v)) \sum_{n=j}^{\infty} \binom{n}{j} p^{n-j} \frac{z^{n-j}}{n!} \right) \\
&= v \left(e^{puz} + e^{pz} \sum_{j=1}^{\infty} \frac{(qz)^j}{j!} (\nu F_{j-1}(u, v) + \mu F_j(u, v)) \right) \\
&= v \left(e^{puz} + e^{pz} \left(\nu q \int G(qz, u, v) dz + \mu G(qz, u, v) - \mu v \right) \right).
\end{aligned} \tag{10}$$

The generating function $G(z, u, v)$ becomes simpler if we replace the fixed number n of people present at the start of the algorithm by a Poisson number of participants with mean z . For this reason, we replace $G(z, u, v)$ by the Poissonized exponential generating function

$$D(z, u, v) := G(z, u, v) e^{-z} = \sum_{n=0}^{\infty} D_n(u, v) \frac{z^n}{n!}.$$

From (10), it follows that

$$D(z, u, v) = v e^{(pu-1)z} + v e^{-qz} \left(\nu q \int G(qz, u, v) dz + \mu G(qz, u, v) - \mu v \right). \tag{11}$$

We use a succinct notation for differentiation with respect to the first of three variables:

$$D'(z, u, v) := \frac{d}{dz} D(z, u, v)$$

and

$$G'(z, u, v) := \frac{d}{dz} G(z, u, v).$$

Differentiating both sides of (11) with respect to z yields

$$\begin{aligned}
D'(z, u, v) &= (pu - 1) v e^{(pu-1)z} - v q e^{-qz} \left(\nu q \int G(qz, u, v) dz + \mu G(qz, u, v) - \mu v \right) + \\
&\quad + v e^{-qz} (\nu q G(qz, u, v) + \mu q G'(qz, u, v)).
\end{aligned}$$

It follows that

$$D'(z, u, v) + qD(z, u, v) = \mu q v D'(qz, u, v) + q v D(qz, u, v) + e^{(pu-1)z} p v (u - 1).$$

For $n \geq 1$, extracting the coefficient of $\frac{z^{n-1}}{(n-1)!}$ from $D(z, u, v) = \sum_{n=0}^{\infty} D_n(u, v) \frac{z^n}{n!}$ yields

$$D_n(u, v) + q D_{n-1}(u, v) = \mu q v D_n(u, v) q^{n-1} + q v D_{n-1}(u, v) q^{n-1} + (pu-1)^{n-1} p v (u-1),$$

or equivalently,

$$D_n(u, v) = D_{n-1}(u, v) \frac{vq^n - q}{1 - \mu v q^n} + \frac{(pu - 1)^{n-1} p v (u - 1)}{1 - \mu v q^n}.$$

Iterating this recurrence yields

$$\begin{aligned} D_n(u, v) &= v(-q)^n \frac{(v)_n}{(\mu v q)_n} + \sum_{j=0}^{n-1} \frac{(pu - 1)^j p v (u - 1) \prod_{k=j+2}^n (vq^k - q)}{\prod_{\ell=j+1}^n (1 - \mu v q^\ell)} \\ &= v(-q)^n \frac{(v)_n}{(\mu v q)_n} + (-q)^{n-1} \frac{(vq)_{n-1}}{(\mu q v)_n} \sum_{j=0}^{n-1} \frac{(1 - pu)^j p v (u - 1) (\mu q v)_j}{q^j (vq)_j}. \end{aligned}$$

Note that $F_n(u, v) = \sum_{k=0}^n \binom{n}{k} D_k(u, v)$, so Lemma 5.1 follows. □

PROOF OF LEMMA 5.3. Setting $v = 1$ in Lemma 5.1, it follows that

$$\begin{aligned} F_n(u) &= F_n(u, 1) \\ &= \sum_{k=0}^n \binom{n}{k} \left((-q)^k \frac{(1)_k}{(\mu q)_k} + (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} \sum_{j=0}^{k-1} \frac{(1 - pu)^j p (u - 1) (\mu q)_j}{q^j (q)_j} \right) \\ &= 1 + \sum_{k=1}^n \binom{n}{k} (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} \sum_{j=0}^{k-1} \frac{(1 - pu)^j p (u - 1) (\mu q)_j}{q^j (q)_j}. \end{aligned} \tag{12}$$

It follows that, for $s \geq 1$,

$$F_n^{(s)}(1) = \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} q^k \frac{(q)_{k-1}}{(\mu q)_k} s(Qp)^s (-1)^{s-1} \sum_{j=0}^{k-1} \frac{(\mu q)_j}{(q)_j} j^{s-1}. \tag{13}$$

Dissecting the summation over j in (13), we obtain

$$\begin{aligned} &\sum_{j=0}^{k-1} \frac{(\mu q)_j}{(q)_j} j^{s-1} \\ &= \frac{(\mu q)_\infty}{(q)_\infty} \sum_{j=0}^{k-1} j^{s-1} + \sum_{j=0}^{k-1} \left[\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right] j^{s-1} \\ &= \frac{(\mu q)_\infty}{(q)_\infty} \frac{k^s}{s} + \sum_{j \geq 0} \left[\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right] j^{s-1} - \sum_{j \geq k} \left[\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right] j^{s-1} \\ &= \frac{(\mu q)_\infty}{(q)_\infty} \frac{k^s}{s} + \sum_{j \geq 0} \left[\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right] j^{s-1} - \sum_{j \geq 0} \left[\frac{(\mu q)_{j+k}}{(q)_{j+k}} - \frac{(\mu q)_\infty}{(q)_\infty} \right] (j+k)^{s-1} \\ &= \frac{(\mu q)_\infty}{(q)_\infty} \frac{k^s}{s} + \sum_{j \geq 0} \left[\left(\frac{(\mu q)_j}{(q)_j} - \frac{(\mu q)_\infty}{(q)_\infty} \right) j^{s-1} - \left(\frac{(\mu q)_{j+k}}{(q)_{j+k}} - \frac{(\mu q)_\infty}{(q)_\infty} \right) (j+k)^{s-1} \right]. \end{aligned}$$

Finally, we observe that, since $F_n(u) = \sum_{m=0}^{\infty} \pi(n, m)u^m$, then $F_n^{(s)}(1) = \sum_{m=0}^{\infty} m^s \pi(n, m) = \mathbb{E}[X_n^s]$. Thus $\mathbb{E}[X_n^s]$ has the representation given in the statement of Lemma 5.3. □

PROOF OF LEMMA 6.1. First of all,

$$\mathbf{P}(X_n > r) = \sum_{m>r} \pi(n, m) = 1 - \sum_{m=0}^r \pi(n, m).$$

Note that $\sum_{m=0}^r \pi(n, m) = [u^r] \frac{F_n(u)}{1-u}$, and of course $1 = [u^r] \frac{1}{1-u}$, so

$$\begin{aligned} \mathbf{P}(X_n > r) &= [u^r] \frac{1 - F_n(u)}{1 - u} \\ &= [u^r] \frac{F_n(u) - 1}{u - 1} \\ &= [u^r] \sum_{k=1}^n \binom{n}{k} (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} \sum_{j=0}^{k-1} \frac{(1 - pu)^j p(\mu q)_j}{q^j (q)_j} \quad (\text{by equation (12)}) \\ &= \sum_{k=1}^n \binom{n}{k} (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} p^{r+1} (-1)^r \sum_{j=0}^{k-1} \binom{j}{r} \frac{(\mu q)_j}{q^j (q)_j}. \end{aligned} \tag{14}$$

We focus on the second summation in (14). Recall that $(x)_z := (x)_\infty / (xq^z)_\infty$, so

$$\frac{(\mu q)_j}{(q)_j} = \frac{(\mu q)_\infty}{(q)_\infty} \frac{(q^{j+1})_\infty}{(\mu q^{j+1})_\infty}. \tag{15}$$

Also, the q -binomial theorem states $\frac{(az)_\infty}{(z)_\infty} = \sum_{m \geq 0} \frac{(a)_m}{(q)_m} z^m$. Specifying $z = \mu q^{j+1}$ and $a = 1/\mu$,

$$\frac{(q^{j+1})_\infty}{(\mu q^{j+1})_\infty} = \sum_{m \geq 0} \frac{(1/\mu)_m}{(q)_m} (\mu q^{j+1})^m. \tag{16}$$

Combining (15) and (16) yields

$$\begin{aligned} \sum_{j=0}^{k-1} \binom{j}{r} \frac{(\mu q)_j}{q^j (q)_j} &= \sum_{j=0}^{k-1} \binom{j}{r} q^{-j} \frac{(\mu q)_\infty}{(q)_\infty} \sum_{m \geq 0} \frac{(1/\mu)_m}{(q)_m} (\mu q^{j+1})^m \\ &= \frac{(\mu q)_\infty}{(q)_\infty} \sum_{m \geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1}. \end{aligned} \tag{17}$$

Thus

$$\begin{aligned} \mathbf{P}(X_n > r) &= \sum_{k=1}^n \binom{n}{k} (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} p^{r+1} (-1)^r \frac{(\mu q)_\infty}{(q)_\infty} \sum_{m \geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1}, \end{aligned}$$

as claimed in the lemma. □

PROOF OF LEMMA 6.2. Let $r \geq 0$. The probability that strictly more than r out of n initial participants are selected at the end of the algorithm is

$$\begin{aligned} & \mathbf{P}(X_n > r) \\ &= \sum_{k=1}^n \binom{n}{k} (-q)^{k-1} \frac{(q)_{k-1}}{(\mu q)_k} p^{r+1} (-1)^r \frac{(\mu q)_\infty}{(q)_\infty} \sum_{m \geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1}. \end{aligned} \tag{18}$$

We handle the sum over m in Lemma 6.1 in three parts, $m = 0$, $m = 1$, and $m \geq 2$, as follows:

$$\begin{aligned} & \sum_{m \geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1} \\ &= \sum_{j=0}^{k-1} \binom{j}{r} q^{-j} + \frac{(1-1/\mu)(\mu q)}{(1-q)} \sum_{j=0}^{k-1} \binom{j}{r} + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1} \\ &= \sum_{j=0}^{k-1} \binom{j}{r} q^{-j} + \frac{(\mu-1)q}{p} \frac{k^{r+1}}{(r+1)!} + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1}. \end{aligned} \tag{19}$$

Now we focus our attention on the sums of the form $\sum_{j=0}^{k-1} \binom{j}{r} x^j$. Writing $D = \frac{d}{dx}$, we note

$$\sum_{j=0}^{k-1} \binom{j}{r} x^j = \frac{x^r}{r!} \sum_{j=0}^{k-1} j^r x^{j-r} = \frac{x^r}{r!} \sum_{j=0}^{k-1} D^r x^j = \frac{x^r}{r!} D^r \sum_{j=0}^{k-1} x^j = \frac{x^r}{r!} D^r \frac{1-x^k}{1-x}. \tag{20}$$

The remainder of the analysis does not depend on k being an integer. We have

$$\begin{aligned} \frac{x^r}{r!} D^r \frac{1-x^k}{1-x} &= \frac{x^r}{r!} \sum_{s=0}^r \binom{r}{s} D^s (1-x^k) \cdot D^{r-s} \left(\frac{1}{1-x} \right) \\ &= \frac{x^r}{r!} (1-x^k) \cdot \frac{r!}{(1-x)^{r+1}} - \frac{x^r}{r!} \sum_{s=1}^r \binom{r}{s} k^s x^{k-s} \cdot \frac{(r-s)!}{(1-x)^{r-s+1}} \\ &= \frac{x^r}{(1-x)^{r+1}} - \frac{x^r x^k}{(1-x)^{r+1}} - \sum_{s=1}^r \frac{k^s}{s!} \frac{x^{k+r-s}}{(1-x)^{r-s+1}} \\ &= \frac{x^r}{(1-x)^{r+1}} - \sum_{s=0}^r \frac{k^s}{s!} \frac{x^{k+r-s}}{(1-x)^{r-s+1}}. \end{aligned} \tag{21}$$

Thus, combining (20) and (21) with $x = q^{m-1}$, we can simplify the “ $m \geq 2$ ” term of (19) as follows:

$$\sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1} = \frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} - \sum_{s=0}^r \frac{k^s}{s!} \frac{q^{(m-1)(k+r-s)}}{(1-q^{m-1})^{r-s+1}}. \tag{22}$$

For $m = 0$, the analogous equation is

$$\begin{aligned} \sum_{j=0}^{k-1} \binom{j}{r} q^{-1} &= \frac{q^{-r}}{(1-q^{-1})^{r+1}} - \sum_{s=0}^r \frac{k^s}{s!} \frac{q^{-(k+r-s)}}{(1-q^{-1})^{r-s+1}} \\ &= \frac{q}{(-p)^{r+1}} \left(1 - \sum_{s=0}^r \frac{k^s}{s!} q^{-k} (-p)^s \right). \end{aligned} \tag{23}$$

Plugging the results from (22) and (23) into (19), we get

$$\begin{aligned} \sum_{m \geq 0} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \sum_{j=0}^{k-1} \binom{j}{r} (q^j)^{m-1} \\ = \frac{q}{(-p)^{r+1}} \left(1 - \sum_{s=0}^r \frac{k^s}{s!} q^{-k} (-p)^s \right) + \frac{(\mu-1)q}{p} \frac{k^{r+1}}{(r+1)!} + \\ + \sum_{m \geq 2} \frac{(1/\mu)_m (\mu q)^m}{(q)_m} \frac{q^{(m-1)r}}{(1-q^{m-1})^{r+1}} - \sum_{s=0}^r \frac{k^s}{s!} \frac{q^{(m-1)(k+r-s)}}{(1-q^{m-1})^{r-s+1}}. \end{aligned}$$

Finally, a substitution into the form of $\mathbf{P}(X_n > r)$ in Lemma 6.1 yields Lemma 6.2. □

8. Future problems

A key problem for future analysis involves a more robust demon, who might be able to remove more than one participant at a time. Another problem to be studied in the future might involve replacing the 2-outcome coins (heads versus tails) with a coin that itself involves some uncertainty. Another interpretation of this extension is that the parameters p and q are unknown before the coin is flipped. Many other possibilities exist for generalizing the present algorithm.

References

- [FS95] P. FLAJOLET and R. SEDGEWICK, Mellin transforms and asymptotics: Finite differences and Rice’s integrals, *Theoret. Comput. Sci.*, **144** (1995), 101–124.
- [FS09] P. FLAJOLET and R. SEDGEWICK, *Analytic Combinatorics*, Cambridge, 2009.
- [Loè78] M. LOÈVE, *Probability Theory* (2 volumes), 4th edition, Springer, New York, 1977, 1978.
- [LP08] G. LOUCHARD and H. PRODINGER, Advancing in the presence of a demon, *Math. Slovaca*, **58** (2008), 263–276.
- [Szp01] W. SZPANKOWSKI, *Average Case Analysis of Algorithms on Sequences*, Wiley, New York, 2001.

**DIGITAL SEARCH TREES AGAIN REVISITED:
THE INTERNAL PATH LENGTH PERSPECTIVE***PETER KIRSCHENHOFER[†], HELMUT PRODINGER,[†] AND WOJCIECH SZPANKOWSKI[‡]

Abstract. This paper studies the asymptotics of the variance for the internal path length in a symmetric digital search tree under the Bernoulli model. This problem has been open until now. It is proved that the variance is asymptotically equal to $N \cdot 0.26600 + N \cdot \delta(\log_2 N)$, where N is the number of stored records and $\delta(x)$ is a periodic function of mean zero and a very small amplitude. This result completes a series of studies devoted to the asymptotic analysis of the variances of digital tree parameters in the symmetric case. In order to prove the previous result a number of nontrivial problems concerning analytic continuations and some others of a numerical nature had to be solved. In fact, some of these techniques are motivated by the methodology introduced in an influential paper by Flajolet and Sedgewick.

Key words. digital search trees, algorithm analysis

AMS subject classifications. 68Q25, 05C80

1. Introduction. Digital trees [2], [9], [17] are experiencing a new wave of interest due to a number of novel applications in computer science and telecommunications. For example, recent developments in the context of large external files and ideas derived from the dynamic hashing (virtual hashing, dynamic hashing, extendible hashing) led to the analysis of digital trees [8], [10], [12], [14], [15], [25], [29], [30], [31]. In telecommunications, recent developments in conflict resolution algorithms [22] and data compression [19] have also brought a new interest in digital trees. Some other applications are radix exchange sort, polynomial factorizations, simulation, Huffman's algorithm, and so on [2], [9], [17].

The three primary digital tree search methods are *digital search trees (DST)*, *radix search tries* (shortly, tries), and *Patricia tries* [2], [7], [9], [17], [28], [31]. In the context of search costs one is led to investigate the *depth* of a node (search time) and the (external or internal) *path length* in digital trees. The average depth of a node for digital trees has been studied in [8], [17], [29], [30], [31], the variance in [12], [29], [30], [31], and limiting distributions in [10], [24], [25], [20]. The average value of the (external or internal) path length is closely related to the average depth of a node but *not* the variance. In [14], [15] the authors obtained the asymptotics of the variance for symmetric regular tries and Patricia tries, respectively (for asymmetric extensions of these results see [10]).

In this paper, we propose to evaluate the variance for the digital search trees, which has been an open problem until now. It has to be stressed that the variance of the internal path length in a digital search tree is the most difficult to estimate. This was already seen in the paper by Flajolet and Sedgewick [8], who establish an analytical methodology to analyze digital search trees (e.g., the average depth of a node). In this paper in the process of establishing the asymptotics of the internal path length we had to obtain some new analytic continuations of functions, which are mainly based on the famous Euler product identities. As in [12] and [13],

*Received by the editors June 25, 1990; accepted for publication (in revised form) March 10, 1993.

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to derive the final results, namely, to show the cancellation of the higher order asymptotics, we had to appeal to the theory of modular functions (cf. §3). In addition, this problem possesses nontrivial numerical challenge. A very preliminary version of our results was presented at the 1989 IFIP Congress [16].

This paper is organized as follows. In the next section, we define our model, establish the general methodology to attack the problem, and present our main results. In particular, we show that the variance of the internal path length for the *binary symmetric* digital search tree under the Bernoulli model is asymptotic to $N \cdot 0.26600 + N \cdot \delta(\log_2 N)$, where N is the number of records and $\delta(x)$ is a periodic function with a very small amplitude. Section 3 contains proofs of our main results, followed by a few concluding remarks in §4.

2. Main results. Let \mathcal{D}_N be the family of digital search trees built from N records with keys from a random stream of bits. Under the Bernoulli model, a key consists of 0's and 1's with independent and equal probability of appearance. Let L_N denote the random variable "internal path length" of trees in \mathcal{D}_N and $F_N(z)$ the corresponding probability generating functions, i.e., the coefficient $[z^k]F_N(z)$ of z^k in $F_N(z)$ is the probability that a tree in \mathcal{D}_N has internal path length equal to k . Then the following recursion, which is a direct consequence of the definition, holds:

$$(2.1) \quad F_{N+1} = z^N \sum_{k=0}^N 2^{-N} \binom{N}{k} F_k(z) F_{N-k}(z), \quad F_0(z) = 1.$$

The *expectation* l_N is given by $l_N = F'_N(1)$ and fulfills for $N \geq 0$

$$(2.2) \quad l_{N+1} = N + 2^{1-N} \sum_{k=0}^N \binom{N}{k} l_k, \quad l_0 = 0.$$

This recursion may be solved explicitly by using *exponential generating functions*. With $L(z) = \sum_{N \geq 0} l_N z^N / N!$, (2.2) translates into the functional differential equation

$$L'(z) = ze^z + 2e^{z/2}L(z/2).$$

By the substitution $\hat{L}(z) = e^z L(-z)$ we have the easier equation

$$\hat{L}(z) - \hat{L}'(z) = -z + 2\hat{L}(z/2).$$

With $\hat{L}(z) = \sum_{N \geq 0} \hat{l}_N z^N / N!$ we find for $N \geq 2$

$$\hat{l}_N = Q_{N-2}, \quad \hat{l}_0 = \hat{l}_1 = 0$$

with the finite product

$$(2.3) \quad Q_N = \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{4}\right) \cdots \left(1 - \frac{1}{2^N}\right)$$

so that finally

$$(2.4) \quad l_N = \sum_{k=2}^N \binom{N}{k} (-1)^k Q_{k-2}.$$

The reader should note that an asymptotic evaluation of (2.4) is nonelementary due to the fact that terms of almost-equal magnitude occur with alternating signs. For this reason sophisticated

methods from complex analysis are needed to find the correct order of growth. An essential step is the application of the following lemma from the calculus of finite differences.

LEMMA 1 (cf. [17, p. 38], [23]). *Let C be a path surrounding the points $j, j + 1, \dots, N$ and $f(z)$ be analytic inside C and continuous on C . Then*

$$(2.5) \quad \sum_{k \geq j} \binom{N}{k} (-1)^k f(k) = -\frac{1}{2\pi i} \int_C [N; z] f(z) dz$$

with

$$[N; z] = \frac{(-1)^{N-1} N!}{z(z-1) \cdots (z-N)}.$$

In our application $f(z)$ is a meromorphic function that continues a sequence $f(k)$, e.g., $j = 2$ and $f(k) = Q_{k-2}$ in (2.4). Moving the contour of integration, one can obtain the asymptotic expansion of the alternating sum by Cauchy’s residue theorem, that is, for any real c (2.5) becomes $\sum_{k \geq j} \binom{N}{k} (-1)^k f(k) = \sum_{z_i \in \mathcal{P}_c} \text{Res}([N; z_i] f(z_i)) + \mathcal{O}(N^c)$, where the sum is taken over the set of poles \mathcal{P}_c different from $j, j + 1, \dots, N$ with real part larger than c .

We note that the function $f(k) = Q_{k-2}$ possesses the analytic continuation $Q_z = Q_\infty / Q(2^{-z})$ where $Q(t) = \prod_{i \geq 1} (1 - t/2^i)$ [8]. Then, applying a refinement of the technique of Flajolet and Sedgewick, we can easily prove the following theorem (cf. §3).

THEOREM 2. *The expectation l_N of the internal path length of digital search trees built from N records fulfills*

$$(2.6) \quad \begin{aligned} l_N = N \log_2 N + N \left[\frac{\gamma - 1}{\log 2} + \frac{1}{2} - \alpha + \delta_1(\log_2 N) \right] + \log_2 N \\ + \frac{2\gamma - 1}{2 \log 2} + \frac{5}{2} - \alpha + \delta_2(\log_2 N) + \mathcal{O}(\log N / N) \end{aligned}$$

with $\gamma = 0.57721 \dots$ (Euler’s constant) and $\alpha = \sum_{n \geq 1} 1/(2^n - 1) = 1.60669 \dots$, where $\delta_1(x)$ and $\delta_2(x)$ are continuous periodic functions of period 1, mean 0, and very small amplitude ($< 10^{-6}$). For later use we mention the Fourier expansion of $\delta_1(x)$

$$(2.7) \quad \delta_1(x) = \frac{1}{\log 2} \sum_{k \neq 0} \Gamma \left(-1 - \frac{2k\pi i}{\log 2} \right) e^{2k\pi i x},$$

where $\Gamma(x)$ is the gamma function [1].

We mention in passing that the $\mathcal{O}(1)$ -term in (2.6) is slightly incorrect in [17].

Now we turn to the *analysis of the variance*, which is given by $\text{Var } L_N = s_N + l_N - l_N^2$ with $s_N = F_N''(1)$. From (2.1) we get the recurrence relation (for $N \geq 0; s_0 = 0$)

$$(2.8) \quad \begin{aligned} s_{N+1} = N2^{2-N} \sum_{k=0}^N \binom{N}{k} l_k + N(N-1) \\ + 2^{1-N} \sum_{k=0}^N \binom{N}{k} l_k l_{N-k} + 2^{1-N} \sum_{k=0}^N \binom{N}{k} s_k. \end{aligned}$$

In order to find an explicit solution to this recurrence, we split it into three parts: $s_N = u_N + v_N + w_N$, where

$$(2.9a) \quad u_{N+1} = 2N(l_{N+1} - N) + 2^{1-N} \sum_{k=0}^N \binom{N}{k} u_k, \quad N \geq 0, u_0 = 0,$$

$$(2.9b) \quad v_{N+1} = N(N - 1) + 2^{1-N} \sum_{k=0}^N \binom{N}{k} v_k, \quad N \geq 0, v_0 = 0,$$

$$(2.9c) \quad w_{N+1} = 2^{1-N} \sum_{k=0}^N \binom{N}{k} l_k l_{N-k} + 2^{1-N} \sum_{k=0}^N \binom{N}{k} w_k, \quad N \geq 0, w_0 = 0.$$

All of the above recurrences, as well as that for the average internal path length (2.2), fall into the following general recurrence studied in [30]. Let (x_n) be a sequence of numbers satisfying

$$(2.10) \quad x_{n+1} = a_{n+1} + 2^{1-n} \sum_{k=0}^n \binom{n}{k} x_k, \quad n \geq 2,$$

where (a_n) is any sequence of numbers. The solution of (2.10) depends on the so-called *binomial inverse relations* that are defined as

$$\hat{a}_n = \sum_{k=0}^n (-1)^k \binom{n}{k} a_k \quad \text{and} \quad a_n = \sum_{k=0}^n (-1)^k \binom{n}{k} \hat{a}_k.$$

The second equation justifies the name binomial inverse relations. For more details, see Riordan [26]. A similar treatment as in the case of (2.2) leads to the following explicit solution (for details see [30]).

LEMMA 3. *Let $x_0 = x_1 = 0$. Then the recurrence (2.10) possesses the solution*

$$(2.11a) \quad x_n = x_0 + n(x_1 - x_0) - \sum_{k=2}^n (-1)^k \binom{n}{k} \hat{x}_{k-2}$$

where

$$(2.11b) \quad \hat{x}_n = Q_n \sum_{i=1}^{n+1} [\hat{a}_i - \hat{a}_{i+1} - a_1] / Q_{i-1}$$

and Q_n is defined in (2.3).

Using Lemma 3 we immediately solve our recurrences (2.9a)—(2.9c). In particular, one proves

$$\hat{u}_k = 2Q_{k-2} \left\{ 4 + \sum_{j=1}^{k-2} \frac{1}{2^j - 1} - \sum_{j=1}^{k-2} \frac{j}{2^j - 1} - \frac{2k}{2^{k-2} - 1} \right\}$$

$$(2.12a) \quad \text{for } k \geq 3, \hat{u}_0 = \hat{u}_1 = \hat{u}_2 = 0;$$

$$(2.12b) \quad \hat{v}_k = -4Q_{k-2} \quad \text{for } k \geq 3, \hat{v}_0 = \hat{v}_1 = \hat{v}_2 = 0;$$

and

$$\hat{w}_k = -Q_{k-2} \sum_{j=4}^{k-1} \frac{2^{1-j}}{Q_{j-1}} \sum_{i=2}^{j-2} \binom{j}{i} Q_{i-2} Q_{j-i-2}$$

$$(2.12c) \quad \text{for } k \geq 5, \hat{w}_0 = \dots = \hat{w}_4 = 0.$$

Of course, the “unhatted” solutions u_N , v_N , and w_N follow from the binomial relations, as shown in (2.11a). It is also worth mentioning that the recurrence for v_N is easy, and after simple algebra one proves

$$(2.12d) \quad v_N = 4 \binom{N}{2} - 4l_N,$$

so the treatment of u_N and w_N remains to be done.

In principle, u_N and w_N may be analyzed by making use of Lemmas 1 and 3. However, it turns out to be a highly nontrivial problem to find an analytical continuation of \hat{w}_k . After lengthy and difficult computations the residue calculus leads us to the following main result of this paper, which is proved in the next section.

THEOREM 4. *The variance of the internal path length of digital search trees built from N records becomes*

$$\text{Var } L_N = N \cdot \{C + \delta(\log_2 N)\} + \mathcal{O}(\log^2 N/N)$$

where C is a constant that can be expressed as

$$(2.13) \quad C = -\frac{28}{3L} - \frac{39}{4} - 2\beta_1 + \frac{2\alpha}{L} + \frac{\pi^2}{2L^2} + \frac{2}{L^2} - \frac{2}{L} \sum_{k \geq 3} \frac{(-1)^{k+1}(k-5)}{(k+1)k(k-1)(2^k-1)} \\ + \frac{2}{L} \sum_{r \geq 1} b_{r+1} \left(\frac{L(1-2^{-r+1})/2-1}{1-2^{-r}} - \sum_{k \geq 2} \frac{(-1)^{k+1}}{k(k-1)(2^{r+k}-1)} \right) \\ + \frac{2}{L} \hat{w}'(3) - 2[\delta_1 \delta_2]_0 - [\delta_1^2]_0$$

with $L = \log 2$, $\alpha = \sum_{n \geq 1} 1/(2^n - 1)$, $\beta_1 = \sum_{n \geq 1} n2^n/(2^n - 1)^2$, and $b_{r+1} = (-1)^r 2^{-\binom{r+1}{2}}$. The fluctuating function $\delta(x)$ is continuous with period 1, mean zero, and $|\delta(x)| \leq 10^{-6}$, and $|\delta_1^2]_0| \leq 10^{-10}$, and $|\delta_1 \delta_2]_0| \leq 10^{-10}$. Finally, $\hat{w}(z)$ is a function defined as

$$(2.14) \quad \frac{\hat{w}(z+1)}{Q_{z-1}} = -2Q_\infty z + \frac{\xi(z+2)}{2^z Q_z} + \frac{\xi(z+3)}{2^{z+1} Q_{z+1}} + \sum_{j \geq 2} \left(\frac{\xi(z+j+2)}{2^{z+j} Q_{z+j}} - \frac{\xi(j+2)}{2^j Q_j} \right)$$

with $Q_z = Q_\infty/Q(2^{-z})$, where $Q(t) = \prod_{i \geq 1} (1 - t/2^i)$, $Q_\infty = Q(1)$, and

$$(2.15) \quad \xi(z+1) = \sum_{r \geq 0} \frac{b_{r+1}}{Q_r} \cdot \frac{Q_\infty}{Q(2^{3-z-r})} \\ \cdot \left\{ 2^z - \frac{2}{1-2^{1-z-r}} - \frac{2z}{1-2^{2-z-r}} + 2 \sum_{k \geq 2} \binom{z}{k} \frac{1}{2^{r+k-1}-1} \right\}.$$

Numerical evaluation of the constant C reveals that $C = 0.26600\dots$ and all five digits after the decimal point are significant.

We should point out that in order to achieve the same accuracy in C one needs to run the recurrence equations (2.9a)–(2.9c) for $N \sim 10^6$.

In the following lemma we present an explicit formula for $\hat{w}'(3)$ that is convenient for numerical evaluations.

LEMMA 5. *The following identity holds:*

$$\begin{aligned}
 \frac{2\hat{w}'(3)}{L} &= -\frac{2Q_\infty}{L} + \sum_{j \geq 2} \frac{1}{2^j Q_j} \sum_{r \geq 0} a_{r+1} Q_{r+j-2} \\
 &\cdot \left\{ -\sum_{n \geq 1} \frac{1}{2^{n+r+j+2} - 1} \cdot \left(2^{j+1} - 2j - 4 + 2 \sum_{k=2}^{j-1} \binom{j+1}{k} \frac{1}{2^{r+k-1} - 1} \right) \right. \\
 &\quad + \frac{2}{(1 - 2^{-j-r})^2} + \frac{2j+2}{(1 - 2^{1-j-r})^2} - \frac{2}{L} \frac{1}{1 - 2^{1-j-r}} \\
 &\quad - 2 \sum_{k=2}^{j+1} \binom{j+1}{k} \frac{1}{2^{r+k-1} - 1} + \frac{2}{L} \sum_{k=1}^{j+1} \binom{j+1}{k} \frac{1}{2^{r+k} - 1} \\
 &\quad \left. + \frac{2}{L} \sum_{k=0}^{j+1} \binom{j+1}{k} \sum_{i \geq 1} \frac{(-1)^i}{(i+1)(2^{r+k+i} - 1)} \right\} \\
 &\quad + \sum_{j \geq 3} \frac{\xi(j+2)}{2^j Q_j} \sum_{k \geq j+1} \frac{1}{2^k - 1}
 \end{aligned}
 \tag{2.16}$$

where

$$a_{r+1} = \frac{b_{r+1}}{Q_r} = (-1)^r 2^{-\binom{r+1}{2}} / Q_r
 \tag{2.17}$$

and

$$\xi(j+2) = \sum_{k=2}^{j-1} \binom{j+1}{k} Q_{k-2} Q_{j-k-1}
 \tag{2.18}$$

with Q_k defined in Theorem 4.

Before we proceed to the proof of our results, we first offer some remarks and extensions.

Theorem 4 and our previous results in [12] and [30] provide asymptotics for the average covariance between two different nodes in a digital search tree (DST). Let $D_N^{(i)}$ be the length of the path from the root to the i th node, i.e., the node corresponding to the i th key of the N keys. Observe that the distribution of the random variable defined by $D_N^{(i)}$ depends on i for DST, since, in contrast to tries and Patricia tries, the order of insertion of the keys is relevant in this instance. Note that the internal path length L_N is expressed in terms of $D_N^{(i)}$ as

$$L_N = \sum_{i=1}^N D_N^{(i)}.
 \tag{2.19}$$

In [12] it has been shown that, besides a small fluctuation,

$$A_N = \frac{1}{N} \sum_{i=1}^N E \left(D_N^{(i)2} \right) - \left(\frac{1}{N} \sum_{i=1}^N E D_N^{(i)} \right)^2 \sim 2.844 \dots
 \tag{2.20}$$

(The quantity NA_N was erroneously identified with the variance of the path length.) It is easily checked that

$$\sum_{i=1}^N \text{Var } D_N^{(i)} - NA_N = \frac{1}{N} \left(\sum_{i=1}^N E D_N^{(i)} \right)^2 - \sum_{i=1}^N \left(E D_N^{(i)} \right)^2.
 \tag{2.21}$$

The expectations $ED_N^{(i)} = ED_i^{(i)}$ (for $N \geq i$) may now be computed in a straightforward manner. Indeed,

$$(2.22) \quad ED_N^{(N)} = E(L_N - L_{N-1}) = -\sum_{k=1}^{N-1} \binom{N-1}{k} (-1)^k Q_{k-1},$$

and we find

$$(2.23) \quad ED_N^{(N)} = \log_2 N + \frac{\gamma}{L} + \frac{1}{2} - \alpha + \tau(\log_2 N) + \mathcal{O}\left(\frac{1}{N}\right)$$

with a small fluctuation $\tau(x)$. Altogether we get, besides the fluctuating term,

$$(2.24) \quad \sum_{i=1}^N \text{Var } D_N^{(i)} - NA_N \sim -\frac{N}{L^2},$$

and therefore, the average variance of the depth becomes

$$(2.25) \quad \frac{1}{N} \sum_{i=1}^N \text{Var } D_N^{(i)} \sim 0.763 \dots$$

(Compare also [27].) Now

$$\text{Var } L_N = \sum_{i=1}^N \text{Var } D_N^{(i)} + 2 \sum_{i \neq j} \text{Cov} \{D_N^{(i)}, D_N^{(j)}\}.$$

Using Theorem 4 and the above we find that the average value of

$$\text{Cov}\{D_N^{(i)}, D_N^{(j)}\} \text{ is } \sim -0.50 \dots /N.$$

The last statement says that, on the average, $D_N^{(i)}$ and $D_N^{(j)}$ are negatively correlated. Note that the equivalent quantity for regular tries is approximately equal to $+0.84 \dots /N$ [14] and for Patricia = $-0.63 \dots /N$ [15].

In order to select the best digital tree one needs to compare different characteristics of digital trees, namely regular tries, Patricia tries, and Digital Search Trees (DST). Table 1 contains four important parameters that are often used to predict a random shape of these trees (cf. [8], [12], [14], [15], [17], [18], [29], [30], [31]).

TABLE 1

	EL_n	$\text{Var}L_N$	“average” $\text{Var}D_N^{(i)}$	“average” $\text{Cov}(D_N^{(i)}, D_N^{(j)})$
DST	$N(\log_2 N - 1.71)$	$N \cdot 0.26$	0.763	$-0.50/N$
TRIES	$N(\log_2 N + 1.33)$	$N \cdot 4.35$	3.507	$+0.84/N$
PATRICIA	$N(\log_2 N + 0.33)$	$N \cdot 0.37$	1.000	$-0.63/N$

It can be seen from the table that the average external (internal) path length is approximately the same for all three digital trees. However, the variance of the depths and internal (external) path lengths differ significantly. We also notice that the variance of the internal path

length for DST as well as the variance of the depth are smaller than the respective quantities for Patricia.

We point out that from Theorem 4 it follows that L_N/EL_N tends to one *almost surely* (i.e., with probability one) as $N \rightarrow \infty$. This immediately follows from Theorem 4 and the Borel-Cantelli lemma. (Compare, e.g., [21] for this standard argument.)

Finally, Theorem 4 provides the missing coefficient in the variance of the numbers of phrases in the Lempel-Ziv parsing algorithm [3], [19] (cf. also [11]). More precisely, the Lempel-Ziv parsing algorithm partitions a single string into variable phrases (blocks) such that a new block is the shortest substring not seen in the past as a phrase. For example, the string 110010100010001000 is parsed into (1)(10)(0)(101)(00)(01)(000)(100). Let M_N be the number of phrases produced by the algorithm for a string of length N . Aldous and Shields [3] proved that for the symmetric alphabet $(M_N - EM_N)/\text{Var } M_N$ converges weakly to the standard normal distribution, where $EM_N \sim N/\log_2 N$ and $\text{Var } M_N = O(N/\log_2^3 N)$, however, the authors of [3] were not able to provide the coefficient at $N/\log_2^3 N$. It turns out that this coefficient is the same as the coefficient at N in the variance of the internal path length L_N , that is, $\text{Var } M_N \sim (C + \delta(\log_2 N))N/\log_2^3 N$. The details of the proof can be found in [11]. The main idea is that the number of phrases M_N can be expressed as

$$M_N = \max \left\{ M : L_M = \sum_{k=1}^M D_M^{(k)} \leq N \right\}.$$

But this equation is known in the literature as the *renewal equation*. Billingsley (compare [6], Chapter 17, Theorem 17.3) proved that (for any dependent positive random variables) if $(L - EL_N)/\sqrt{\text{Var } L_N}$ converges weakly to the standard normal distribution $\mathcal{N}(0, 1)$, then

$$\frac{M_N - N/(EL_N/N)}{\sqrt{\text{Var } L_N}(EL_N/N)^{-3/2}} \rightarrow \mathcal{N}(0, 1).$$

Hence, by the standard uniform integrability arguments and by Theorem 4, we conclude that the variance of M_N becomes

$$\text{Var } M_N = \text{Var } L_N/\log_2^3 N \sim (C + \delta(\log_2 N))N/\log_2^3 N.$$

3. Analysis. As we already pointed out in §2, it is a nontrivial problem to find appropriate analytic continuations for the sequences of values $f(k)$ that occur in alternating sums (2.5). In order to illustrate our approach, we start with the easiest case, namely, the evaluation of the expectation l_N . From (2.4) we know

$$l_N = \sum_{k=2}^N \binom{N}{k} (-1)^k Q_{k-2}.$$

As in [8] we may rewrite $f(k) = Q_{k-2}$ as

$$Q_{k-2} = \frac{Q_\infty}{Q(2^{2-k})},$$

where

$$(3.1) \quad Q(x) = \prod_{i \geq 1} \left(1 - \frac{x}{2^i}\right) \quad \text{and} \quad Q_\infty = Q(1).$$

Therefore, we have the analytic continuation

$$(3.2) \quad f(z) = \frac{Q_\infty}{Q(2^{2-z})}.$$

The main contribution to l_N is given by $\text{Res}([N; z]f(z); z = 1)$. We have with $u = z - 1 \rightarrow 0$

$$[N; z] \sim \frac{N}{u} (1 + u(H_{N-1} - 1))$$

and

$$\frac{Q_\infty}{Q(2^{2-z})} \sim \frac{1}{Lu} \left(1 + Lu \left(\frac{1}{2} - \alpha \right) \right)$$

(remember $L = \log 2$), since

$$(3.3) \quad \alpha = \left(\frac{Q_\infty}{Q(x)} \right)' \Big|_{x=1}.$$

Therefore,

$$\text{Res}([N; z]f(z); z = 1) = \frac{N}{L} \left(H_{N-1} - 1 + L \left(\frac{1}{2} - \alpha \right) \right).$$

Using the well-known asymptotics for the *harmonic numbers* H_{N-1} we get the contribution (from $z = 1$)

$$(3.4) \quad N \log_2 N + N \left(\frac{\gamma}{L} - \frac{1}{L} + \frac{1}{2} - \alpha \right) - \frac{1}{2L} + \mathcal{O} \left(\frac{1}{N} \right).$$

Besides $z = 1$ we have with the same real part 1 the simple poles $z_k = 1 + \frac{2k\pi i}{L}$, $k \in \mathbf{Z}$, $k \neq 0$, with

$$\text{Res}([N; z]f(z); z = z_k) = [N; z_k] \cdot \frac{1}{L},$$

so we get the contribution

$$(3.5) \quad \frac{N^{z_k}}{L} \Gamma(-z_k) - \frac{(z_k - 1)z_k N^{z_k - 1} \Gamma(-z_k)}{2L} + \mathcal{O}(N^{z_k - 2}).$$

The reader should take notice of the fact that the first term in (3.5) gives the Fourier coefficients of $\delta_1(x)$ in Theorem 1.

The next relevant pole is $z = 0$ and yields a contribution of

$$(3.6) \quad \log_2 N + \frac{\gamma}{L} + \frac{5}{2} - \alpha.$$

The poles $z = z_k - 1$ yield a periodic contribution of order N^0 and so on.

Collecting all contributions gives the expansion (2.6) in Theorem 2.

Next we focus our attention on the *asymptotics of* u_N . In order to find an appropriate analytic continuation of \hat{u}_k we rewrite the sums appearing in (2.12a) as follows:

$$\begin{aligned} \sum_{j=1}^{k-2} \frac{1}{2^j - 1} &= \alpha - \sum_{j \geq 1} \frac{1}{2^{k-2+j} - 1}, \\ \sum_{j=1}^{k-2} \frac{j}{2^j - 1} &= \sum_{j \geq 1} \frac{j}{2^j - 1} - \sum_{j \geq 1} \frac{k-2+j}{2^{k-2+j} - 1}. \end{aligned}$$

Thus we may continue \hat{u}_k via the function

$$(3.7) \quad \hat{u}(z) = \frac{2Q_\infty}{Q(2^{2-z})} \left[4 + \alpha - \sum_{j \geq 1} \frac{1}{2^{z-2+j} - 1} - \sum_{j \geq 1} \frac{j}{2^j - 1} + \sum_{j \geq 1} \frac{z - 2 + j}{2^{z-2+j} - 1} - \frac{2z}{2^{z-2} - 1} \right].$$

Now the main contribution to u_N in Lemma 1 originates from a second-order pole of $[N; z]\hat{u}(z)$ in $z = 2$. Further contributions that are necessary for the evaluation of the variance come from first-order poles in $z = 2 + \frac{2k\pi i}{L}$, $k \neq 0$, a third-order pole in $z = 1$, as well as second-order poles in $z = 1 + \frac{2k\pi i}{L}$, $k \neq 0$. Collecting all the above-mentioned contributions we get the following expansion of u_N (in all the following formulas $\delta_i(x)$ stands for a continuous periodic function of period 1 and mean zero).

LEMMA 6.

$$\begin{aligned} u_N \sim & 4N^2 \log_2 N + N^2 \left(\frac{4(\gamma - 1)}{L} - 6 - 4\alpha + \delta_3(\log_2 N) \right) \\ & - N \log_2^2 N + 2N \log_2 N \cdot \left(\frac{2 - \gamma}{L} + 8 + \alpha + \delta_4(\log_2 N) \right) \\ & + N \left(-\frac{\gamma^2}{L^2} + \frac{4\gamma}{L^2} + \frac{12\gamma}{L} + \frac{2\alpha\gamma}{L} - \frac{\pi^2}{6L^2} - \frac{4}{L^2} - \frac{10}{L} - \frac{2\alpha}{L} \right. \\ & \left. - \alpha^2 + \beta - 11\alpha - 2\beta_1 + \frac{133}{6} + \delta_5(\log_2 N) \right) + \mathcal{O}(\log^2 N) \end{aligned}$$

with $\beta = \sum_{k \geq 1} 1/(2^k - 1)^2$, L , α , and β_1 as in Theorem 4.

As already mentioned in §2,

$$v_N = 4 \binom{N}{2} - 4I_N,$$

so the asymptotics of v_N are given by

$$(3.8) \quad \begin{aligned} v_N \sim & 2N^2 - 4N \log_2 N \\ & + 4N \left(-1 + \alpha - \frac{\gamma}{L} + \frac{1}{L} - \delta_1(\log_2 N) \right) + \mathcal{O}(\log N). \end{aligned}$$

The most challenging task is to find an appropriate analytic continuation of $\hat{w}(z)$.

From (2.12c) we have

$$(3.9) \quad \hat{w}_{k+1} = -Q_{k-1} \sum_{j=4}^k \frac{\xi(j+1)}{2^{j-1} Q_{j-1}}$$

with

$$(3.10) \quad \xi(j+1) = \sum_{i=2}^{j-2} \binom{j}{i} Q_{i-2} Q_{j-2-i}.$$

For the following the reader should note that $\xi(j+1) \sim Q_\infty^2 2^j$. We start by rewriting (3.9) in the following manner:

With $\eta(j + 1) = \xi(j + 1) - Q_\infty^2 2^j$ we have

$$\begin{aligned}
 (3.11) \quad -\frac{\hat{w}_{k+1}}{Q_{k-1}} &= \sum_{j=4}^k \frac{\eta(j + 1) + Q_\infty^2 2^j}{2^{j-1} Q_{j-1}} \\
 &= \sum_{j \geq 4} \frac{\eta(j + 1)}{2^{j-1} Q_{j-1}} - \sum_{j \geq k+1} \frac{\eta(j + 1)}{2^{j-1} Q_{j-1}} + 2Q_\infty(k - 3) \\
 &\quad + 2Q_\infty^2 \left(\sum_{j \geq 4} \left(\frac{1}{Q_{j-1}} - \frac{1}{Q_\infty} \right) - \sum_{j \geq k+1} \left(\frac{1}{Q_{j-1}} - \frac{1}{Q_\infty} \right) \right).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \hat{w}_{k+1} &= Q_{k-1} \left[-2Q_\infty(k - 3) + \sum_{j \geq 0} \frac{\eta(j + k + 2)}{2^{j+k} Q_{j+k}} - \sum_{j \geq 3} \frac{\eta(j + 2)}{2^j Q_j} \right. \\
 &\quad \left. + 2Q_\infty^2 \left(\sum_{j \geq 0} \left(\frac{1}{Q_{j+k}} - \frac{1}{Q_\infty} \right) - \sum_{j \geq 3} \left(\frac{1}{Q_j} - \frac{1}{Q_\infty} \right) \right) \right].
 \end{aligned}$$

Since all involved series are now absolutely convergent, we may add them term-by-term and get

$$\hat{w}_{k+1} = Q_{k-1} \left[-2Q_\infty k + \frac{\xi(k + 2)}{2^k Q_k} + \frac{\xi(k + 3)}{2^{k+1} Q_{k+1}} + \sum_{j \geq 2} \left(\frac{\xi(k + j + 2)}{2^{k+j} Q_{k+j}} - \frac{\xi(j + 2)}{2^j Q_j} \right) \right].$$

From this, the representation for $\hat{w}(z + 1)$ as in (2.14) is immediate, provided we have an appropriate interpretation for $\xi(z + 1)$. This will be our next goal. The following well-known partition identities of Euler are our basic tool:

$$(3.12) \quad \frac{1}{Q(t)} = \prod_{n \geq 1} \frac{1}{(1 - t2^{-n})} = \sum_{n \geq 0} \frac{t^n}{2^n Q_n}$$

and

$$(3.13) \quad Q(t) = \prod_{n \geq 1} \left(1 - \frac{t}{2^n} \right) = \sum_{n \geq 0} a_{n+1} t^n$$

with

$$a_{n+1} = (-1)^n 2^{-\binom{n+1}{2}} / Q_n.$$

Using (3.2) and (3.12) we have

$$\begin{aligned}
 \xi(N + 1) &= \sum_{k=2}^N \binom{N}{k} \frac{Q_\infty}{Q(2^{2-k})} \frac{Q_\infty}{Q(2^{2+k-N})} \\
 &= Q_\infty^2 \sum_{i,j \geq 0} \frac{2^{i+j}}{Q_i Q_j} \sum_{k=2}^{N-2} \binom{N}{k} (2^{-i})^k (2^{-j})^{N-k},
 \end{aligned}$$

where the innermost sum is now

$$(2^{-i} + 2^{-j})^N - 2^{-iN} - 2^{-jN} - N2^{-i(N-1)-j} - N2^{-i-j(N-1)}.$$

The last expression for $\xi(N + 1)$ is symmetric in i and j . However, it turns out that for the purpose of finding an analytic continuation $\sum_{i,j \geq 0}$ should be rewritten as $-\sum_{j=i} + 2 \sum_{j \geq i \geq 0}$. Writing $j = i + h$ in the second sum we get

$$\begin{aligned}
 \xi(N + 1) = & -(2^N - 2 - 2N) \sum_{i \geq 0} \frac{Q_\infty^2}{Q_i^2} 2^{i(2-N)} \\
 & + 2 \sum_{i,h \geq 0} \frac{Q_\infty^2}{Q_i Q_{i+h}} 2^{i(2-N)} 2^h \left[(1 + 2^{-h})^N - 1 - N2^{-h} \right] \\
 & - 2 \sum_{i,h \geq 0} \frac{Q_\infty^2}{Q_i Q_{i+h}} 2^{i(2-N)} 2^{h(1-N)} \\
 & - 2N \sum_{i,h \geq 0} \frac{Q_\infty^2}{Q_i Q_{i+h}} 2^{i(2-N)} 2^{h(2-N)}.
 \end{aligned}
 \tag{3.14}$$

In expression (3.14) N can be replaced by z , yielding a meromorphic function, since all series converge uniformly. However, we are able to simplify $\xi(z + 1)$ in the following way. Consider, for example, the last term in (3.14):

$$2z \sum_{i,h \geq 0} \frac{Q_\infty^2}{Q_i Q_{i+h}} 2^{(i+h)(2-z)} = 2z \sum_{i,h \geq 0} \frac{Q_\infty Q(2^{-i-h})}{Q_i} 2^{(i+h)(2-z)},$$

which is by Euler’s identity (3.13)

$$= 2z \sum_{r \geq 0} a_{r+1} \sum_{i \geq 0} \frac{Q_\infty}{Q_i} (2^{-r+2-z})^i \sum_{h \geq 0} (2^{-r+2-z})^h$$

and by Euler’s identity (3.12)

$$= 2z \sum_{r \geq 0} a_{r+1} \frac{Q_\infty}{Q(2^{3-z-r})} \cdot \frac{1}{1 - 2^{2-z-r}}.$$

Rewriting the other terms from (3.14) in a similar way, especially using

$$(1 + 2^{-h})^z - 1 - z2^{-h} = \sum_{k \geq 2} \binom{z}{k} 2^{-hk}$$

for the second term, we finally get (2.15).

Our next task is to investigate the poles of $[N; z]\hat{w}(z)$ different from $z = 5, 6, \dots, N$.

From (3.11) we see that $\hat{w}(4) = \hat{w}(3) = 0$ (observe that $\xi(4) = 0$), so that the first poles occur with real part 2. In order to determine the residues of $[N; z]\hat{w}(z)$ in $z = 2$ (respectively, $z = 1$) we need the local behavior of $\hat{w}(z)$. Because of (2.14) this behavior will depend on the behavior of $\sigma(z) := \xi(z)/2^{z-2} Q_{z-2}$ near $z = 2, 3, \dots$. From (2.15) we see that $\xi(z)$ has second-order poles for $z = 2$ and $z = 3$ and is analytic for $z = 4, 5, \dots$. Since $[N; z]Q_{z-2}$ has already a second-order pole for $z = 1$, it will be necessary to expand $\sigma(z)$ near $z = 2, 3, \dots$ up to the linear terms. In particular, the reader should note that *all* the derivatives $\sigma'(z)$ for $z = 4, 5, \dots$ will occur in $\text{Res}([N; z]\hat{w}(z); z = 1)$. This is the main reason that the constant C in the final result is rather a complicated one.

We start with the expansion of $\sigma(z)$ about $z = 3$. Let $u = z - 3$; then we find from (2.15) after laborious computations:

$$\begin{aligned}
 \sigma(3+u) \sim & -\frac{4}{L^2 u^2} + \frac{1}{u} \left(\frac{6}{L} - \frac{2}{L^2} \right) \\
 & + u^0 \left(\frac{16}{3} + \frac{5}{L} + \frac{2\beta_2}{L} + 2 \sum_{r \geq 2} \frac{b_{r+1}}{1-2^{-r}} \left(1 - \frac{2}{1-2^{-r}} \right) \right) \\
 (3.15) \quad & + u^1 \left(-\frac{223}{18}L - \frac{23}{6} - 3C_1 + \frac{C_2}{L} - 2C_3 + 2 \sum_{r \geq 2} \frac{b_{r+1}}{1-2^{-r}} \right. \\
 & \cdot \left. \left\{ 1 - 3L + L \sum_{i=2}^{r-1} \frac{1}{2^i - 1} + \frac{2^{-1-r}L}{(1-2^{-1-r})^2} + \frac{2L-1}{1-2^{-r}} \right. \right. \\
 & \left. \left. - \frac{2L}{1-2^{-r}} \sum_{i=2}^{r-1} \frac{1}{2^i - 1} + \frac{2^{1-r}L}{(1-2^{-r})^2} + D_{r,1} \right\} \right)
 \end{aligned}$$

where

$$\begin{aligned}
 \beta_2 &= 2 \sum_{k \geq 2} \frac{(-1)^k}{(k+1)k(k-1)(2^k-1)}, \\
 C_1 &= \sum_{h \geq 0} 2^h \left[(1+2^{-h})^2 \log(1+2^{-h}) - 2^{-h} \right], \\
 C_2 &= \sum_{h \geq 0} 2^h (1+2^{-h})^2 \log^2(1+2^{-h}), \\
 C_3 &= \sum_{h \geq 0} \left[(1+2^{-h})^2 \log(1+2^{-h}) - 2^{-h} \right], \\
 D_{r,1} &= \sum_{h \geq 0} 2^{h(1-r)} \left[(1+2^{-h})^2 \log(1+2^{-h}) - 2^{-h} \right].
 \end{aligned}$$

The constant in the u^0 -term can be simplified according to the remarkable identity

$$(3.16) \quad \sum_{r \geq 1} \frac{b_{r+1}}{1-2^{-r}} \left(1 - \frac{2}{1-2^{-r}} \right) = \sum_{j \geq 1} \frac{2^j}{(2^j-1)^2} = \alpha + \beta.$$

For the proof of (3.16) we observe that the left-hand side equals

$$\sum_{r \geq 1} a_{r+1} Q_{r-1} - 2 \sum_{r \geq 1} a_{r+1} \frac{Q_{r-1}}{1-2^{-r}}.$$

Using (3.12) and (3.13) this expression becomes

$$\begin{aligned}
 & \sum_{i \geq 0} Q(2^{-i}) (Q(2^{-i}) - 1) - 2 \sum_{i,k \geq 0} Q(2^{-i}) (Q(2^{-i-k}) - 1) \\
 &= - \left(\sum_{i \geq 0} (Q(2^{-i}) - 1) \right)^2 - \sum_{j \geq 0} (2j+1) (Q(2^{-j}) - 1) \\
 &= -E_1^2 - 2E_2 - E_1,
 \end{aligned}$$

where

$$E_1 = \sum_{r \geq 1} \frac{a_{r+1}}{1 - 2^{-r}} \quad \text{and} \quad E_2 = \sum_{r \geq 1} a_{r+1} \frac{2^{-r}}{(1 - 2^{-r})^2}.$$

Now we observe

$$E_1 = \lim_{t \rightarrow 2} \left[\frac{Q_\infty}{Q(t)} - \frac{1}{1 - t/2} \right] = - \left(\frac{Q_\infty}{Q(t)} \right)' \Big|_{t=1} = -\alpha$$

and

$$E_2 = 2 \lim_{t \rightarrow 2} \left[\left(\frac{Q_\infty}{Q(t)} \right)' - \frac{1/2}{(1 - t/2)^2} \right] = -\frac{1}{2} \left(\frac{Q_\infty}{Q(t)} \right)'' \Big|_{t=1} = -\frac{\alpha^2 + \beta}{2}$$

and get the right-hand side of (3.16).

The expansion of $\sigma(z)$ about $z = 2$ reads with $u = z - 2$:

$$\begin{aligned} \sigma(2 + u) &\sim \frac{4}{L^2 u^2} + \frac{1}{u} \left(\frac{2}{L} + \frac{2}{L^2} \right) \\ &+ u^0 \left(-\frac{16}{3} - \frac{6}{L} - \frac{2}{L} (C_4 + C_6) + 2E_3 \right) \\ &+ u^1 \left(\frac{L}{6} - \frac{55}{6} - 3C_4 - \frac{C_5}{L} + C_6 - \frac{C_7}{L} \right. \\ (3.17) \quad &+ 2 \sum_{r \geq 2} \frac{b_{r+1}}{(1 - 2^{1-r})(1 - 2^{-r})} \\ &\cdot \left\{ D_{r,2} + 1 - 2L + L \sum_{i=1}^{r-2} \frac{1}{2^i - 1} + \frac{L - 1}{1 - 2^{1-r}} \right. \\ &+ L \left(\frac{1}{(1 - 2^{-r})^2} - \frac{1}{1 - 2^{-r}} \sum_{i=1}^{r-2} \frac{1}{2^i - 1} + \frac{2^{1-r}}{(1 - 2^{1-r})^2} \right. \\ &\left. \left. \left. - \frac{1}{1 - 2^{1-r}} \sum_{i=1}^{r-2} \frac{1}{2^i - 1} \right) \right\} \right) \end{aligned}$$

where

$$\begin{aligned} C_4 &= \sum_{h \geq 0} 2^h [(1 + 2^{-h}) \log(1 + 2^{-h}) - 2^{-h}], \\ C_5 &= \sum_{h \geq 0} 2^h (1 + 2^{-h}) \log^2(1 + 2^{-h}), \\ C_6 &= \sum_{h \geq 0} [(1 + 2^{-h}) \log(1 + 2^{-h}) - 2^{-h}], \\ C_7 &= \sum_{h \geq 0} (1 + 2^{-h}) \log^2(1 + 2^{-h}), \\ D_{r,2} &= \sum_{h \geq 0} 2^{h(1-r)} [(1 + 2^{-h}) \log(1 + 2^{-h}) - 2^{-h}], \\ E_3 &= \sum_{r \geq 2} \frac{b_{r+1}}{(1 - 2^{1-r})(1 - 2^{-r})} \left(1 - \frac{1}{1 - 2^{-r}} - \frac{1}{1 - 2^{1-r}} \right). \end{aligned}$$

For later simplifications we note that

$$(3.18) \quad -\frac{2}{L}(C_4 + C_6) = -8 + \frac{3}{L} - \frac{2\beta_2}{L}$$

and

$$(3.19) \quad E_3 = 2 - \alpha - \beta,$$

where (3.18) follows from the expansion of the logarithm and (3.19) by partial fraction decomposition and rearrangements of the sums.

We finally note that

$$(3.20) \quad C_5 + C_7 = C_2.$$

Next we discuss $\sigma(z)$ for z close to $j = 4, 5, \dots$

$$(3.21) \quad \sigma(j + u) \sim \sigma(j) + \frac{u}{2^{j-2}Q_{j-2}} \left(\xi'(j) - L\xi(j) + L\xi(j) \sum_{k \geq 1} \frac{1}{2^{k+j-2} - 1} \right).$$

From (3.21)

$$(3.22a) \quad \sum_{j \geq 4} (\sigma(j + u) - \sigma(j)) \sim u \left(\sum_{j \geq 4} \frac{\xi'(j) - L\xi(j)}{2^{j-2}Q_{j-2}} + L \sum_{j \geq 4} \frac{\xi(j)}{2^{j-2}Q_{j-2}} \sum_{k \geq 1} \frac{1}{2^{k+j-2} - 1} \right).$$

From (2.14) we find that the last expression equals

$$(3.22b) \quad u(2\hat{w}'(3) + 2Q_\infty),$$

where $\hat{w}'(3)$ may be computed from (2.15) to get the constant from Lemma 5.

Regarding (3.15), (3.17), and (3.22a) we find that $[N; z]\hat{w}(z)$ has third-order poles in $z = 2$ and $z = 1$ and second-order poles in $z_k = 2 + \frac{2k\pi i}{L}$, $k \in \mathbb{Z}$, $z \neq 0$, as well as in $z_k - 1$.

Our local expansions allow (after some lengthy but straightforward computations) to find the following asymptotic behavior of w_N :

LEMMA 7.

$$\begin{aligned} w_N = N^2 \log_2^2 N + N^2 \log_2 N \cdot & \left(-3 - \frac{2}{L} + \frac{2\gamma}{L} - 2\alpha + \delta_6(\log_2 N) \right) \\ & + N^2 \left(\frac{1}{3} + \alpha^2 + 3\alpha + \frac{2\alpha}{L} - \frac{3\gamma}{L} - \frac{2\alpha\gamma}{L} - \frac{\beta_2}{L} \right. \\ & \left. + \frac{2}{L} + \frac{2}{L^2} + \frac{\gamma^2}{L^2} + \frac{\pi^2}{6L^2} - \frac{2\gamma}{L^2} + \delta_7(\log_2 N) \right) \end{aligned}$$

$$\begin{aligned}
 &+3N \log_2^2 N + N \log_2 N \cdot \left(-\frac{7}{L} - 3 - 6\alpha - \frac{10\gamma}{L} + \delta_8(\log_2 N) \right) \\
 &+N \left(-22 - \frac{41}{6L} + \frac{\beta_2}{L} - \frac{3\gamma}{L} - \frac{7\gamma}{L^2} + 2\alpha - \beta + \frac{7\alpha}{L} + 3\alpha^2 - \frac{6\alpha\gamma}{L} + \frac{3\gamma^2}{L^2} \right. \\
 &\quad \left. + \frac{\pi^2}{2L^2} + \frac{6}{L^2} - \frac{2}{L} \sum_{k \geq 3} \frac{(-1)^{k+1}(k-5)}{(k+1)k(k-1)(2^k-1)} + \frac{2}{L} \sum_{r \geq 1} b_{r+1} \right. \\
 &\quad \cdot \left(\frac{L(1-2^{-r+1})/2-1}{1-2^{-r}} - \sum_{k \geq 2} \frac{(-1)^{k+1}}{k(k-1)(2^{r+k}-1)} \right) \\
 &\quad \left. + \frac{2}{L} \hat{w}'(3) + \delta_9(\log_2 N) \right) + \mathcal{O} \left(\frac{\log^2 N}{N} \right)
 \end{aligned}$$

with $L, \alpha, \beta, b_{r+1}$ as in Theorem 4 (respectively, Lemma 6) and β_2 from (3.15).

It remains to combine the previous results to get an asymptotic expansion for

$$(3.23) \quad \text{Var } L_N = u_N + v_N + w_N + l_N - l_N^2.$$

We start with an important observation concerning leading terms formed by periodic fluctuations of mean zero.

Let us assume that, at any stage, we are able to prove

$$(3.24) \quad \text{Var } L_N = \delta_{10}(\log_2 N) \cdot N^\mu \log^\nu N + R_N,$$

where $\delta_{10}(x)$ is continuous and periodic with period 1 and mean zero and $R_N = o(N^\mu \log^\nu N)$. We claim that $\delta_{10}(x)$ must vanish identically under these conditions:

Let us assume $\delta_{10}(x) \not\equiv 0$. Then, since $\delta_{10}(x)$ is continuous with mean 0, there exists an $\epsilon > 0$ and an interval, say $[a, b] \subseteq [0, 1]$, such that $\delta_{10}(x) < -\epsilon$ for $x \in [a, b]$. Since $\log_2 N$ is dense modulo 1, $\text{Var } L_N$ would be negative for an infinity of values, an obvious contradiction.

In other words: From (3.24) we may deduce that

$$(3.25) \quad \text{Var } L_N \sim R_N, \quad N \rightarrow \infty,$$

so, in order to prove that $\text{Var } L_N = \mathcal{O}(N)$ we need not collect explicitly the fluctuating contributions of mean zero.

Observing these comments we easily find that all terms of order $N^2 \log^2 N, N^2 \log N, N \log^2 N$, and $N \log N$ in $\text{Var } L_N$ cancel. The coefficient of N^2 is of a more delicate nature. The reader should note that the coefficient of N^2 in l_N^2 will contain the square δ_1^2 of the periodic fluctuation δ_1 from Theorem 2 and that the mean $[\delta_1^2]_0$ of δ_1^2 will *not* be zero. Therefore, we have to extract this term to end up with a fluctuation of mean zero and get for the coefficient of N^2 in $\text{Var } L_N$ the expression

$$(3.26) \quad \frac{1}{L^2} + \frac{\pi^2}{6L^2} - \frac{1}{L} - \frac{\beta_2}{L} - \frac{47}{12} - [\delta_1^2]_0 + \delta_{11}(\log_2 N).$$

The following lemma is crucial now.

LEMMA 8.

$$[\delta_1^2]_0 = \frac{1}{L^2} \sum_{k \neq 0} \left| \Gamma \left(-1 - \frac{2k\pi i}{\log 2} \right) \right|^2 = \frac{1}{L^2} + \frac{\pi^2}{6L^2} - \frac{1}{L} - \frac{\beta_2}{L} - \frac{47}{12}.$$

*Sketch of Proof*¹. The proof heavily relies on the following two series transformation results due to Ramanujan (compare [5]). The first is

$$\begin{aligned} (3.27) \quad & \alpha^{-N} \left(\frac{1}{2} \zeta(2N+1) + \sum_{k \geq 1} \frac{k^{-2N-1}}{e^{2\alpha k} - 1} \right) \\ & = (-\beta)^{-N} \left(\frac{1}{2} \zeta(2N+1) + \sum_{k \geq 1} \frac{k^{-2N-1}}{e^{2\beta k} - 1} \right) \\ & \quad - 2^{2N} \sum_{k=0}^{N+1} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{2N+2-2k}}{(2N+2-2k)!} \alpha^{N+1-k} \beta^k. \end{aligned}$$

Here and in the next identity, α and β have to be positive numbers with $\alpha\beta = \pi^2$, $\zeta(s)$ is the Riemann ζ -function; N has to be a positive integer, and B_n indicates the n th Bernoulli number defined by

$$\frac{z}{e^z - 1} = \sum_{n \geq 0} B_n \frac{z^n}{n!}.$$

The second identity used in the proof is

$$(3.28) \quad \sum_{k \geq 1} \frac{1}{k(e^{2\alpha k} - 1)} - \frac{1}{4} \log \alpha + \frac{\alpha}{12} = \sum_{k \geq 1} \frac{1}{k(e^{2\beta k} - 1)} - \frac{1}{4} \log \beta + \frac{\beta}{12}.$$

In fact, (3.28) is equivalent to a transformation result on Dedekind’s η -function (compare [4])

$$(3.29) \quad \eta(\tau) = e^{\pi i \tau / 12} \prod_{n \geq 1} (1 - e^{2\pi i n \tau}), \quad \Im(\tau) > 0,$$

namely,

$$(3.30) \quad \eta \left(-\frac{1}{\tau} \right) = (-i\tau)^{1/2} \cdot \eta(\tau), \quad \Im(\tau) > 0.$$

(This is a special instance of Dedekind’s famous result on the behavior of η under a transformation of the modular group.) \square

The consequences of Lemma 8 are twofold. On the one hand, we find from (3.26) that the N^2 -term in $\text{Var } L_N$ cancels, so that $\text{Var } L_N = \mathcal{O}(N)$. On the other hand we may use the identity to express β_2 by the other terms occurring in Lemma 8, including $[\delta_1^2]_0$, which yields the final form of the constant C in Theorem 2. $[\delta_1 \delta_2]_0$ is the mean of $\delta_1(x) \delta_2(x)$, originating from l_N^2 , which has to be extracted to get a fluctuation $\delta(x)$ of mean zero.

4. Concluding remarks. We would like to point out some final remarks concerning our analysis.

(i) The occurrence of the finite products Q_k gives rise to use results from the *theory of partitions*, especially the Euler product identities (3.12) and (3.13).

¹A full proof of Lemma 8 is long and difficult and included in [13].

(ii) A periodic fluctuation $\delta_1(x)$ that has mean zero and very small amplitude may be safely neglected for practical purposes as long as we are only interested in the *expectation*. In order to establish the correct order of the *variance* it is of vital importance to study the behavior of $\delta_1(x)$, especially the mean of $\delta_1^2(x)$.

(iii) The predicted value $0.26600 \dots \cdot N$ matches perfectly with the values obtained by *computer simulations*.

(iv) As we mentioned in the introduction, with this paper we achieved our goal of obtaining second-order properties for the three digital tree search structures, namely, *Tries*, *Patricia Tries*, and *Digital Search Trees* in the symmetric case. In particular, we are now able to compare the variances of the path lengths in such trees. Furthermore, we note that the analysis of the variances is a key step toward getting the respective limiting distributions (compare [3] or [10]).

(v) Our methodology does not easily extend to the asymmetric Bernoulli model. The difficulty lies in the analytical continuation of w_N . We conjecture that in the asymmetric case the variance is of order $n \log n$, but it might be difficult to obtain the coefficient at $n \log n$.

Acknowledgment. The authors would like to thank R. F. Tichy for some helpful remarks.

REFERENCES

- [1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions*, Dover, New York, 1970.
- [2] A. AHO, J. HOPCROFT, AND J. ULLMAN, *Data Structures and Algorithms*, Addison Wesley, Reading, MA, 1983.
- [3] D. ALDOUS AND P. SHIELDS, *A diffusion limit for a class of random-growing binary trees*, Probab. Theory Related Fields, 79 (1988), pp. 509–542.
- [4] T. M. APOSTOL, *Modular Functions and Dirichlet Series in Number Theory*, Springer, New York, 1976.
- [5] B. C. BERNDT, *Ramanujan's Notebooks Part II*, Springer, New York, 1989.
- [6] P. BILLINGSLEY, *Convergence of Probability Measures*, John Wiley and Sons, New York, 1968.
- [7] E. G. COFFMAN, JR. AND J. EVE, *File structures using hashing functions*, Comm. Appl. Comput. Mach., 13 (1970), pp. 427–436.
- [8] P. FLAJOLET AND R. SEDGEWICK, *Digital search trees revisited*, SIAM J. Comput., 15 (1986), pp. 748–767.
- [9] G. GONNET, *Handbook of Algorithms and Data Structures*, Addison-Wesley, Reading, MA, 1983.
- [10] P. JACQUET AND M. REGNIER, *Normal Limiting Distributions for the Size and the External Path Length of Tries*, INRIA Tech. report, 827, 1988.
- [11] P. JACQUET AND W. SZPANKOWSKI, *On the Lempel-Ziv Parsing Algorithm and Its Digital Tree Representation*, INRIA Tech. report, 1833, 1992.
- [12] P. KIRSCHENHOFER AND H. PRODINGER, *Further results on digital search trees*, Theoret. Comput. Sci., 58 (1988), pp. 143–154.
- [13] ———, *On some applications of formulae of Ramanujan in the analysis of algorithms*, Mathematika, 38 (1991), pp. 14–33.
- [14] P. KIRSCHENHOFER, H. PRODINGER, AND W. SZPANKOWSKI, *On the variance of the external path length in a symmetric digital tree*, Discrete Appl. Math., 25 (1989), pp. 129–143.
- [15] ———, *On the balance property of Patricia trees: External path length viewpoint*, Theoret. Comput. Sci., 68 (1989), pp. 1–17.
- [16] ———, *Digital search trees—further results on a fundamental data structure*, Proceedings of IFIP 89, G. X. Ritter, ed. (1989), pp. 443–447.
- [17] D. E. KNUTH, *The Art of Computer Programming, Vol. 3*, Addison-Wesley, Reading, MA, 1973.
- [18] A. G. KONHEIM AND D. J. NEWMAN, *A note on growing binary trees*, Discrete Mathematics, 4 (1973), pp. 57–63.
- [19] A. LEMPEL AND J. ZIV, *On the complexity of finite sequences*, IEEE Trans. Inform. Theory, 22 (1976), pp. 75–81.
- [20] G. LOUCHARD, *Exact and asymptotic distributions in digital and binary search trees*, RAIRO Theoret. Inform. Appl., 21 (1987), pp. 479–495.
- [21] H. MAHMOUD, *Evolution of Random Search Trees*, John Wiley and Sons, New York, 1992.
- [22] P. MATHYS AND P. FLAJOLET, *Q-ary collision resolution algorithms in random-access systems with free and blocked channel access*, IEEE Trans. Inform. Theory, 31 (1985), pp. 217–243.
- [23] N. E. NÖRLUND, *Vorlesungen über Differenzenrechnung*, Chelsea, New York, 1954.
- [24] B. PITTEL, *Paths in a random digital tree: limiting distributions*, Adv. Appl. Probab., 18 (1986), pp. 139–155.

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- [25] M. RÉGNIER AND P. JACQUET, *New results on the size of tries*, IEEE Trans. Inform. Theory, 35 (1989), pp. 203–205.
- [26] J. RIORDAN, *Combinatorial Identities*, John Wiley and Sons, New York, 1968.
- [27] W. SCHACHINGER, *Modulfunktionen und Analyse von Datenstrukturen*, Diplomarbeit, TU Vienna, 1990.
- [28] R. SEDGEWICK, *Algorithms*, Second ed., Addison-Wesley, Reading, MA, 1988.
- [29] W. SZPANKOWSKI, *Some results on V-ary asymmetric trees*, J. Algorithms, 9 (1988), pp. 224–244.
- [30] ———, *A characterization of digital search trees from the successful search viewpoint*, Theor. Comput. Sci., 85 (1991), pp. 117–134.
- [31] ———, *Patricia trees again revisited*, J. Assoc. Comput. Mach., 37 (1990), pp. 691–711.

JIRSS (2004)

Vol. 3, No. 2, pp 251-270

Periodic Oscillations in the Analysis of Algorithms and Their Cancellations

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Abstract. A large number of results in analysis of algorithms contain fluctuations. A typical result might read “The expected number of ... for large n behaves like $\log_2 n + \text{constant} + \delta(\log_2 n)$, where $\delta(x)$ is a periodic function of period one and mean zero.” Examples include various trie parameters, approximate counting, probabilistic counting, radix exchange sort, leader election, skip lists, adaptive sampling. Often, there are huge cancellations to be noted, especially if one wants to compute variances. In order to see this, one needs identities for the Fourier coefficients of the periodic functions involved. There are several methods to derive such identities, which belong to the realm of modular functions. The most flexible method seems to be the calculus of residues. In some situations, Mellin transforms help. Often, known identities can be employed. This survey shows the various techniques by elaborating on the most important examples from the literature.

*Supported by NRF Grant 2053748.

Received: October 2003, Revised: March 2004

Key words and phrases: Analysis of algorithms, approximate counting, Dedekind’s eta function, geometric random variables, Mellin transform, modular functions, periodic oscillations, residues, tries.

1 Introduction

A surprisingly large number of results in *analysis of algorithms* contain *fluctuations*. A typical result might read “The expected number of ... for large n behaves like $\log_2 n + \text{constant} + \delta(\log_2 n)$, where $\delta(x)$ is a periodic function of period one and mean zero.” Examples include various trie parameters, approximate counting, probabilistic counting, radix exchange sort, leader election, skip lists, adaptive sampling; see the classic books by Flajolet, Knuth, Mahmoud, Sedgewick, Szpankowski [23, 16, 17, 18, 25] for background.

We use the name $\delta(x)$ in a generic sense; in concrete situations we call them $\delta_0(x)$, $\delta_1(x)$, etc. An important set of such functions is

$$\delta_j(x) := \frac{1}{L} \sum_{k \neq 0} \frac{\Gamma(j - \chi_k)}{j!} e^{2\pi i k x},$$

where we use the standard abbreviations $L = \log 2$ and $\chi_k = \frac{2\pi i k}{L}$.

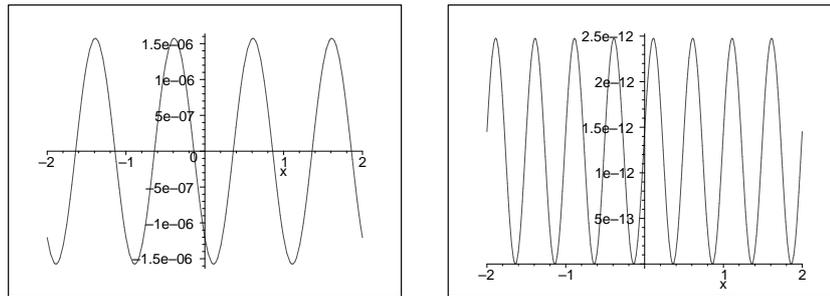


Figure 1: $\delta_0(x)$ and $\delta_0^2(x)$

As one can see from the picture, $\delta_0(x)$ has mean zero (the zeroth Fourier coefficient is not there). On the other hand, $\delta_0^2(x)$ is still periodic with period 1, but its mean is *not* zero. Why should we worry about a quantity apparently as small as $\approx 10^{-12}$?

The reason is the *variance* of such parameters, as it naturally contains the term “–expectation²,” and as such also $-\delta^2(x)$. That might not be a sufficient motivation for a casual reader if it were not the case that often *substantial cancellations* occur. In order to identify them, one has to know more about $\delta^2(x)$. If one ignores these terms, one gets wrong results, and the results are not wrong by $\approx 10^{-12}$, but *by an order of growth!* Path length in tries, Patricia

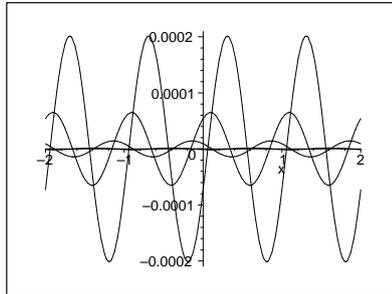


Figure 2: The functions $\delta_j(x) = \frac{1}{L} \sum_{k \neq 0} \frac{\Gamma(j - \chi_k)}{j!} e^{2\pi i k x}$ grow in amplitude

tries, and digital search trees [8, 15, 10] are such cases: the variance is in reality of order n only, but ignoring the fluctuations would lead to a (wrong) $\approx n^2$ result.

Questions like that occurred in several writings of this author (together with various coauthors), as can be seen from the references. The techniques are extremely interesting, as one has to dig deep into classical analysis. So far, it seems that the *calculus of residues* is the most versatile approach in this context. Another approach is to use (modular) identities due to Dedekind, Ramanujan, Jacobi and others (which can often be proved by Mellin transform techniques); however, often they do not quite *fit*. The residue calculus approach directly addresses the formula that is ultimately needed.

In this survey paper, we discuss all these methods by looking at various examples. The paper has also a tutorial concern, as we want to encourage the interested reader to prove his/her own identities with the methods that are provided.

Oscillating functions are usually given as *Fourier* series $f = \sum_{k \neq 0} a_k e^{2\pi i k x}$, thus representing a periodic function of period 1, and since the term a_0 is missing, oscillating around zero. We often refer to the coefficient a_k by writing $[f]_k$.

Other cancellation phenomena concerning oscillations related to Patricia tries and compositions (resp. words) were only discovered recently and presented in [22], at ANALCO04 (dedicated to Hosam Mahmoud).

Here are some examples from the literature.

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After n successive increments the average content \bar{C}_n of the counter satisfies:

$$\bar{C}_n \sim \log_2 n + \frac{\gamma}{L} - \alpha + \frac{1}{2} - \delta_0(\log_2 n),$$

with

$$\alpha = \sum_{k \geq 1} \frac{1}{2^k - 1} \quad \text{and} \quad \delta_0(x) = \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) e^{2\pi i k x},$$

with $L = \log 2$ and $\chi_k = \frac{2\pi i k}{L}$. The identity that one needs is

$$[\delta_0^2]_0 = \frac{1}{L^2} \sum_{k \neq 0} \Gamma(\chi_k) \Gamma(-\chi_k) = \frac{\pi^2}{6L^2} - \frac{11}{12} - \frac{2}{L} \sum_{h \geq 1} \frac{(-1)^{h-1}}{h(2^h - 1)}. \quad (1)$$

We will present various proofs of this identity, which will be our running example, in the next sections. The methods are residue calculus (Section 2), Mellin transform (Section 3), and identities of Ramanujan (Section 4).

Maximum of a sample of n geometric random variables [26, 13]

Assume that X is a geometric random variable such that $\mathbb{P}\{X = k\} = 2^{-k}$ (for simplicity, we only discuss this case, not the slightly more general $\mathbb{P}\{X = k\} = (1 - q)q^{k-1}$). We consider n independent trials and look for their maximum. This is a natural parameter which is also useful in the analysis of various algorithms (e. g., sliplists [14]).

The expected value is given by

$$E_n \sim \log_2 n + \frac{\gamma}{L} + \frac{1}{2} - \delta_0(\log_2 n)$$

with the same periodic function as before.

Tries [12, 8, 7]

The expected number of internal nodes in a trie built from n random data is

$$l_n = \frac{n}{L} + n\sigma(\log_2 n) + O(1),$$

with

$$\sigma(x) = \frac{1}{L} \sum_{k \neq 0} \chi_k \Gamma(1 - \chi_k) e^{2\pi i k x}.$$

The formula that one needs is

$$[\sigma^2]_0 = 3 - \frac{1}{L} - \frac{1}{L^2} + \frac{2}{L} \sum_{j \geq 2} \frac{(-1)^j j}{(j+1)(j-1)(2^j-1)}. \tag{2}$$

Partial match queries in tries [9]

The average cost (defined in the paper [9]), for random tries constructed from n random data, is

$$l_n = \sqrt{n} \left(\sqrt{\pi} \frac{1 + \sqrt{2}}{2L} + \tau(\log_2 \sqrt{n}) \right) + O(1),$$

where the fluctuating function $\tau(x) = \sum_{k \neq 0} \tau_k e^{2k\pi i x}$ has the Fourier coefficients

$$\tau_k = \frac{1}{2L} \left(1 + \sqrt{2}(-1)^k \right) \Gamma\left(\frac{-1 - \chi_k}{2}\right) \left(\frac{-1 + \chi_k}{2}\right).$$

The formula one needs is

$$[\tau^2]_0 = \frac{3}{4L} - \frac{\pi}{4L^2} (3 + 2\sqrt{2}) + \frac{3 - 2\sqrt{2}}{L} F(L) + \frac{2\sqrt{2}}{L} F\left(\frac{L}{2}\right) \tag{3}$$

with

$$F(x) = \sum_{k \geq 1} \frac{e^{-kx}}{1 + e^{-2kx}}.$$

2 Proofs by residue calculus

In this section we will show how to use residue calculus in order to prove the relevant identities. As examples of the technique, we concentrate on the identities (1), (2), and (4). However, after going through these representative examples, the reader will surely be able to prove his/her own identities, following the technique.

The following approach (“residue calculus”) to evaluate $[\delta^2]_0$ seems to be the easiest and most flexible. We start with the following example:

$$\delta_0(x) = \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) e^{2\pi i k x}.$$

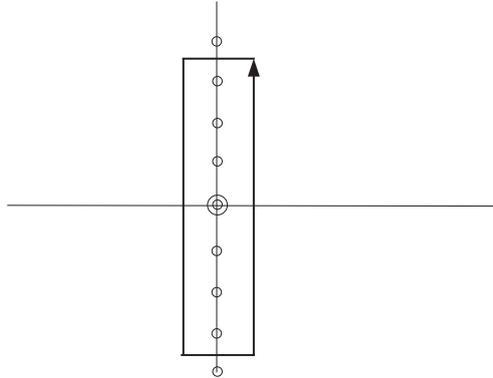


Figure 3: Path of integration; poles at χ_k are indicated, the double pole at 0 by a double circle

Find a function $F(z)$ so that $[\delta_0^2]_0$ is (apart from a few extra terms) the sum of the residues along the imaginary axis. Here, take

$$F(z) = \frac{L}{e^{Lz} - 1} \Gamma(-z) \Gamma(z).$$

If we set

$$I_1 = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(z) dz,$$

then by shifting and collecting residues,

$$I_1 = \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} F(z) dz + \sum_{k \neq 0} \Gamma(-\chi_k) \Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

What happens here is often called *closing the box*, compare Figure 3, see e. g. [25, 18]. One integrates along a rectangle with corners $\pm \frac{1}{2} \pm iM$. One can evaluate it by collecting the residues inside the rectangle. And one can let the parameter M go to infinity. In this type of problems, the integrals along the horizontal lines disappear, and we can express one integral along a vertical line by an integral along another vertical line, plus a few residues. The justification that these integrals along the horizontal lines disappear comes from the fact that the Gamma function (which is always present in our examples) becomes small extremely fast for large imaginary parts, see [27]. Since all our examples are of that nature, we will perform the relevant operations without further comments.

The emphasis of this survey is to prove identities, and this is to some extent a more algebraic than analytic endeavour.—

Now one writes

$$\frac{1}{e^z - 1} = -1 - \frac{1}{e^{-z} - 1}$$

and gets, by a simple change of variable $z := -z$,

$$I_1 = -\frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(-z)\Gamma(z)dz - I_1 + \sum_{k \neq 0} \Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

The integral

$$I_2 = -\frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(-z)\Gamma(z)dz$$

can be computed by collecting the negative residues right to the line $\Re z = -\frac{1}{2}$, viz.

$$I_2 = -\frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(-z)\Gamma(z)dz = \sum_{l \geq 1} \frac{(-1)^l}{l!} (l-1)! = -L.$$

Altogether we have

$$2I_1 = -L + \sum_{k \neq 0} \Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

On the other hand, integral I_1 is also the sum of the negative residues right of the line $\Re z = \frac{1}{2}$, i. e.,

$$I_1 = -L \sum_{l \geq 1} \frac{(-1)^l}{l!(2^l - 1)} (l-1)! = -L \sum_{l \geq 1} \frac{(-1)^l}{l(2^l - 1)}.$$

Combining these results, we get

$$-2L \sum_{l \geq 1} \frac{(-1)^l}{l(2^l - 1)} = -L + \sum_{k \neq 0} \Gamma(-\chi_k)\Gamma(\chi_k) - \frac{\pi^2}{6} - \frac{L^2}{12}.$$

This is the identity we wanted.

With not much more effort one can also compute the coefficients $[\delta_0^2]_k$, for $k \neq 0$. For this, one works with the function

$$F(z) = \frac{L}{e^{Lz} - 1} \Gamma(-z - \chi_k)\Gamma(z).$$

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One obtains

$$\begin{aligned}
 [\delta_0^2]_k &= \frac{1}{L^2} \sum_{j \neq 0, \neq k} \Gamma(-\chi_j) \Gamma(-\chi_k + \chi_j) \\
 &= \frac{2}{L} \sum_{l \geq 1} \frac{(-1)^l \Gamma(-\chi_k + l)}{l!(2^l - 1)} + \frac{2}{L^2} \Gamma(-\chi_k) (\psi(-\chi_k) + \gamma).
 \end{aligned}$$

We omit the details.

Guy Louchard, who is interested in higher moments, asked to compute the coefficients $[\delta_0^3]_k$. Here is the instance $k = 0$, the general case is very involved and not too attractive:

$$\begin{aligned}
 [\delta_0^3]_0 &= -1 - \frac{2\zeta(3)}{L^3} - \frac{1}{L} \sum_{l \geq 1} \frac{(-1)^l}{l(2^l - 1)} + \frac{6}{L^2} \sum_{l \geq 1} \frac{(-1)^l H_{l-1}}{l(2^l - 1)} + \frac{2 \log 3}{L} \\
 &\quad + \frac{2}{L} \sum_{l, j \geq 1} \frac{(-1)^{l+j}}{(l+j)(2^l - 1)} \left[\frac{1}{2^j - 1} + \frac{1}{2^{j+l} - 1} \right] \binom{l+j}{j}.
 \end{aligned}$$

(In this formula, the harmonic numbers $H_n := \sum_{1 \leq k \leq n} \frac{1}{k}$ appear.)

This has been tested numerically as well and gives $9.42817763095796606421903 \times 10^{-25}$.

Let us straight ahead do another example (identity (2)), which also occurs often:

$$\sigma(x) = \frac{1}{L} \sum_{k \neq 0} \chi_k \Gamma(-1 - \chi_k) e^{2\pi i k x}.$$

Here, we take

$$F(z) = -\frac{L}{e^{Lz} - 1} z^2 \Gamma(-1 - z) \Gamma(-1 + z).$$

Then

$$I_1 = \frac{1}{2\pi i} \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} F(z) dz + \sum_{k \neq 0} \chi_k (-\chi_k) \Gamma(-1 - \chi_k) \Gamma(-1 + \chi_k) + 1$$

and

$$2I_1 = LI_2 + \sum_{k \neq 0} \chi_k (-\chi_k) \Gamma(-1 - \chi_k) \Gamma(-1 + \chi_k) + 1$$

with

$$I_2 = \frac{1}{2\pi i} \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} z^2 \Gamma(-1 - z) \Gamma(-1 + z) dz$$

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$$\begin{aligned} &= \sum_{l \geq 2} l^2 \frac{(-1)^{l+1}}{(l+1)!} (l-2)! + \frac{L}{4} \\ &= \sum_{l \geq 2} \frac{(-1)^{l+1} l}{(l+1)(l-1)} = -L + \frac{1}{4} + \frac{1}{4} = -L + \frac{1}{2}. \end{aligned}$$

Therefore

$$2I_1 = -L^2 + \frac{L}{2} + \sum_{k \neq 0} \chi_k (-\chi_k) \Gamma(-1 - \chi_k) \Gamma(-1 + \chi_k) + 1.$$

But I_1 is also

$$\begin{aligned} I_1 &= -\frac{L}{4} + L^2 + L \sum_{l \geq 2} \frac{l^2}{2^l - 1} \frac{(-1)^{l+1}}{(l+1)!} (l-2)! \\ &= -\frac{L}{4} + L^2 + L \sum_{l \geq 2} \frac{(-1)^{l+1} l}{(2^l - 1)(l+1)(l-1)}. \end{aligned}$$

Putting things together, we find

$$\begin{aligned} 2I_1 &= -L^2 + \frac{L}{2} + \sum_{k \neq 0} \chi_k (-\chi_k) \Gamma(-1 - \chi_k) \Gamma(-1 + \chi_k) \\ &= -\frac{L}{2} + 2L^2 + 2L \sum_{l \geq 2} \frac{(-1)^{l+1} l}{(2^l - 1)(l+1)(l-1)} + 1, \end{aligned}$$

or

$$\begin{aligned} &\sum_{k \neq 0} \chi_k (-\chi_k) \Gamma(-1 - \chi_k) \Gamma(-1 + \chi_k) \\ &= -1 - L + 3L^2 + 2L \sum_{l \geq 2} \frac{(-1)^{l+1} l}{(2^l - 1)(l+1)(l-1)}, \end{aligned}$$

which is the identity in question, as it expresses the quantity $L^2[\sigma^2]_0$ in two different ways.

Here is a third example, dealing with the function

$$\frac{1}{L} \sum_{k \neq 0} \Gamma(j - \chi_k) e^{2\pi i k x},$$

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for $j \geq 1$, and the computation of the constant term of its square. The technique should be familiar by now. Consider the function

$$L \frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1}.$$

Therefore we have

$$\begin{aligned} \sum_{k \neq 0} \Gamma(j+\chi_k)\Gamma(j-\chi_k) &= \frac{L}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1} dz \\ &\quad - \frac{L}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1} dz - \Gamma(j)^2. \end{aligned}$$

($\Gamma(j)^2$ is the residue at $z=0$.)

Now we use again the decomposition

$$\frac{1}{e^{Lz}-1} = -1 - \frac{1}{e^{-Lz}-1}$$

for the second integral and get

$$\begin{aligned} & - \frac{L}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1} dz \\ &= \frac{L}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(j+z)\Gamma(j-z) dz \\ &\quad + \frac{L}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(j+z)\Gamma(j-z)}{e^{-Lz}-1} dz \\ &= \frac{L}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(j+z)\Gamma(j-z) dz \\ &\quad + \frac{L}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(j-z)\Gamma(j+z)}{e^{Lz}-1} dz. \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{k \neq 0} |\Gamma(j+\chi_k)|^2 \\ &= \frac{2L}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\Gamma(j+z)\Gamma(j-z)}{e^{Lz}-1} dz \\ &\quad + \frac{L}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(j+z)\Gamma(j-z) dz - \Gamma(j)^2 \end{aligned}$$

$$= I_1 + I_2 - \Gamma(j)^2.$$

Integral I_1 is evaluated by shifting the contour to the *right* and collecting the *negative* residues, which gives

$$I_1 = -2L \sum_{m \geq j} \frac{\Gamma(j+m) (-1)^{j-m+1}}{e^{Lm} - 1 (m-j)!}$$

and with $m = h + j$

$$\begin{aligned} &= 2L \sum_{h \geq 0} \frac{(h+2j-1)! (-1)^h}{h!} \frac{1}{2^{h+j} - 1} \\ &= 2L(2j-1)! \sum_{h \geq 0} \binom{-2j}{h} \frac{1}{2^{h+j} - 1}. \end{aligned}$$

Integral I_2 is of interest for itself and appears already in early references to the *Mellin transform technique* as by Nielsen [19, p. 224]. (It could, however, be computed as in the previous examples.)

We start with the function

$$f(x) = \frac{x^j}{(1+x)^{2j}}$$

and perform its Mellin transform (see, e.g., [6] for definitions)

$$f^*(s) = \int_0^\infty f(x)x^{s-1}dx = B(j+s, j-s) = \frac{\Gamma(j+s)\Gamma(j-s)}{\Gamma(2j)}$$

with the Beta function $B(z, w)$ (compare [1]). The *fundamental strip* is $\langle -j, j \rangle$. Therefore the inversion formula for the Mellin transform gives

$$f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(j+s)\Gamma(j-s)}{\Gamma(2j)} x^{-s} ds.$$

Now we may evaluate at $x = 1$ and get the formula

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(j+s)\Gamma(j-s) ds = \Gamma(2j)2^{-2j}.$$

This produces the formula

$$\sum_{k \neq 0} \Gamma(j + \chi_k) \Gamma(j - \chi_k)$$

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$$= 2L(2j - 1)! \sum_{h \geq 0} \binom{-2j}{h} \frac{1}{2^{h+j} - 1} + L(2j - 1)! 2^{-2j} - (j - 1)!^2. \tag{4}$$

This formula was essential in the paper [13].

Remark. The computation of the integral I_2 (as in the examples above) sometimes leads to series like

$$\sum_{l \geq 1} (-1)^l l.$$

There is nothing wrong here. The correct interpretation is as an *Abel limit*

$$\lim_{t \rightarrow 1^-} \sum_{l \geq 1} (-1)^l l t^l = \lim_{t \rightarrow 1^-} \frac{-t}{(1+t)^2} = -\frac{1}{4}.$$

3 Using the Mellin transform to prove identities

Let us start with our running example (1) and show how this can be proved using the Mellin transform. The Mellin transform is very prominent in the analysis of algorithms, and we refer to [6] for a nice survey.

We will treat again our identity (1) and might for instance start with the series

$$\sum_{h \geq 1} \frac{(-1)^{h-1}}{h(2^h - 1)}$$

and interpret it as $g(\log 2)$ with

$$g(x) := \sum_{h \geq 1} \frac{(-1)^{h-1}}{h(e^{hx} - 1)} = \sum_{h, k \geq 1} \frac{(-1)^{h-1}}{h} e^{-h k x}.$$

Now one computes the Mellin transform $g^*(s)$:

$$\begin{aligned} g^*(s) &= \sum_{h, k \geq 1} \frac{(-1)^{h-1}}{h} e^{-h k x} = \sum_{h, k \geq 1} \frac{(-1)^{h-1}}{h} h^{-s} k^{-s} \Gamma(s) \\ &= (1 - 2^{-s}) \zeta(s + 1) \zeta(s) \Gamma(s). \end{aligned}$$

The Mellin transform exists in the fundamental strip $(1, \infty)$; whence we can invoke the inversion formula for the Mellin transform. We may choose e.g. the line $\Re z = \frac{3}{2}$ since $\frac{3}{2}$ lies in the fundamental strip. So we get

$$\begin{aligned} g(x) &= \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} (1 - 2^{-s})\zeta(s+1)\zeta(s)\Gamma(s)x^{-s} ds \\ &= \frac{\pi^2}{12x} - \frac{L}{2} + \frac{x}{24} + \frac{1}{2\pi i} \int_{-\frac{3}{2}-i\infty}^{-\frac{3}{2}+i\infty} (1 - 2^{-s})\zeta(s+1)\zeta(s)\Gamma(s)x^{-s} ds \\ &= \frac{\pi^2}{12x} - \frac{L}{2} + \frac{x}{24} \\ &\quad + \frac{1}{2\pi i} \int_{-\frac{3}{2}-i\infty}^{-\frac{3}{2}+i\infty} (2^s - 1)\zeta(s+1)\zeta(s) \frac{1}{2\sqrt{\pi}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) x^{-s} ds. \end{aligned}$$

This form was obtained by taking 3 residues out and invoking the duplication formula of the Γ -function. (Observe that the exponential smallness of the Γ -function along vertical lines justifies the shifting of the line integral.) We now use the functional equation for $\zeta(s)$, namely

$$\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{5}$$

and continue:

$$\begin{aligned} g(x) &= \frac{\pi^2}{12x} - \frac{L}{2} + \frac{x}{24} \\ &\quad + \frac{1}{2\pi i} \int_{-\frac{3}{2}-i\infty}^{-\frac{3}{2}+i\infty} (2^s - 1) \frac{1}{2} \pi^{2s-\frac{1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \Gamma\left(\frac{-s}{2}\right) \zeta(-s) x^{-s} ds \\ &= \frac{\pi^2}{12x} - \frac{L}{2} + \frac{x}{24} \\ &\quad + \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} (2^{-s} - 1) \frac{1}{2} \pi^{-2s-\frac{1}{2}} \Gamma\left(\frac{1+s}{2}\right) \zeta(1+s) \Gamma\left(\frac{s}{2}\right) \zeta(s) x^s ds \\ &= \frac{\pi^2}{12x} - \frac{L}{2} + \frac{x}{24} \\ &\quad - \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} (1 - 2^{-s}) \pi^{-2s} \zeta(1+s) \zeta(s) \Gamma(s) x^s 2^{-s} ds, \end{aligned}$$

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and so

$$g(x) = \frac{\pi^2}{12x} - \frac{L}{2} + \frac{x}{24} - g\left(\frac{2\pi^2}{x}\right). \tag{6}$$

This is the formula we need, since we can also rewrite the left side of (1) in terms of this $g(x)$ function:

$$\begin{aligned} [\delta_0^2]_0 &= \frac{1}{L^2} \sum_{k \neq 0} \Gamma(\chi_k) \Gamma(-\chi_k) \\ &= \frac{1}{L} \sum_{k \geq 1} \frac{1}{k \sinh(2k\pi^2/L)} = \frac{2}{L} \sum_{k \geq 1} \frac{e^{kz}}{k(e^{2kz} - 1)}, \end{aligned}$$

with $z = 2\pi^2/L$. But

$$\begin{aligned} \sum_{k \geq 1} \frac{e^{kz}}{k(e^{2kz} - 1)} &= \sum_{k \geq 1, j \geq 0} \frac{1}{k} e^{-k(2j+1)z} \\ &= \sum_{k \geq 1, j \geq 1} \frac{1}{k} e^{-k j z} - 2 \sum_{k \geq 1, j \geq 1} \frac{1}{2k} e^{-2k j z} \\ &= \sum_{k \geq 1, j \geq 1} \frac{(-1)^{k-1}}{k} e^{-k j z} = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k(e^{kz} - 1)} = g(z), \end{aligned}$$

and so

$$[\delta_0^2]_0 = \frac{2}{L} g\left(\frac{2\pi^2}{L}\right).$$

Let us do a more complicated example in the same style: We want to rewrite $[\tau^2]_0$, to get identity (3). Note that

$$[\tau^2]_0 = 2 \sum_{k \geq 1} \tau_k \tau_{-k} = \frac{2}{4L^2} \sum_{k \geq 1} \left(3 + 2\sqrt{2}(-1)^k\right) \Gamma\left(\frac{1 - \chi_k}{2}\right) \Gamma\left(\frac{1 + \chi_k}{2}\right).$$

Now we use the formula (equivalent to the reflection formula for the Gamma function, cf. [1]) $\Gamma(z)\Gamma(1 - z) = \pi/\sin \pi z$ and obtain

$$\begin{aligned} \Gamma\left(\frac{1 - \chi_k}{2}\right) \Gamma\left(\frac{1 + \chi_k}{2}\right) &= \frac{\pi}{\sin(\pi/2 + ik\pi^2/L)} = \frac{\pi}{\cos(ik\pi^2/L)} \\ &= \frac{\pi}{\cosh(k\pi^2/L)} = 2\pi \frac{e^{-k\pi^2/L}}{1 + e^{-2k\pi^2/L}}, \end{aligned}$$

so that

$$[\tau^2]_0 = \frac{\pi}{L^2} \sum_{k \geq 1} \left(3 + 2\sqrt{2}(-1)^k\right) \frac{e^{-k\pi^2/L}}{1 + e^{-2k\pi^2/L}}. \tag{7}$$

Let us define two new functions

$$F(x) = \sum_{k \geq 1} \frac{e^{-kx}}{1 + e^{-2kx}} \quad \text{and} \quad G(x) = \sum_{k \geq 1} \frac{(-1)^{k-1} e^{-kx}}{1 + e^{-2kx}}.$$

Then, (7) in terms of $F(x)$ and $G(x)$ becomes

$$[\tau^2]_0 = \frac{3\pi}{L^2} F\left(\frac{\pi^2}{L}\right) - \frac{2\sqrt{2}\pi}{L^2} G\left(\frac{\pi^2}{L}\right). \quad (8)$$

We use a series transformation for $F(x)$ and $G(x)$. We start with

$$F(x) = \sum_{j \geq 0} (-1)^j \sum_{k \geq 1} e^{-k(2j+1)x} = \sum_{j \geq 0} \chi(j) \frac{1}{e^{jx} - 1}$$

where

$$\chi(j) = \begin{cases} 0, & \text{for } j \text{ even;} \\ 1, & \text{for } j \equiv 1 \pmod{4}; \\ -1, & \text{for } j \equiv 3 \pmod{4}. \end{cases}$$

Once we know that

$$F(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{\pi}{x} F\left(\frac{\pi^2}{x}\right), \quad (9)$$

for $x > 0$, as we shall show soon, then $G(x) = F(x) - 2F(2x)$, hence

$$G(x) = \frac{1}{4} + \frac{\pi}{x} F\left(\frac{\pi^2}{x}\right) - \frac{\pi}{x} F\left(\frac{\pi^2}{2x}\right).$$

Applying the above to (8) we finally obtain

$$[\tau^2]_0 = \frac{3}{4L} - \frac{\pi}{4L^2} (3 + 2\sqrt{2}) + \frac{3 - 2\sqrt{2}}{L} F(L) + \frac{2\sqrt{2}}{L} F\left(\frac{L}{2}\right).$$

To prove (9) we proceed as follows. Let

$$\beta(s) = \sum_{j \geq 0} (-1)^j \frac{1}{(2j+1)^s}.$$

We have

$$F(x) = \sum_{k \geq 1} \frac{e^{-kx}}{1 + e^{-2kx}} = \sum_{j \geq 0} (-1)^j \sum_{k \geq 1} e^{-k(2j+1)x},$$

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so that the Mellin transform $F^*(s) = \int_0^\infty F(x)x^{s-1}dx$ of $F(x)$ becomes $F^*(s) = \Gamma(s)\zeta(s)\beta(s)$. By the Mellin inversion formula this yields

$$F(x) = \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \Gamma(s)\zeta(s)\beta(s)x^{-s} ds.$$

Now we take the two residues $s = 1$ and $s = 0$ out from the above integral (observe that $\beta(0) = 1/2$ and $\beta(1) = \pi/4$, cf. [1]) and apply the duplication formula for $\Gamma(s)$ to obtain

$$F(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{1}{\sqrt{\pi}} 2^{s-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) x^{-s} \zeta(s) \beta(s) ds.$$

We now use the functional equations for $\zeta(s)$ and $\beta(s)$, namely

$$\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

and

$$\beta(1-s)\Gamma\left(1-\frac{s}{2}\right) = 2^{2s-1}\pi^{-s+\frac{1}{2}}\Gamma\left(\frac{s+1}{2}\right)\beta(s).$$

The first identity is Riemann's functional equation for $\zeta(s)$, and the second is an immediate consequence of the functional equation for Hurwitz's ζ -function $\zeta(s, a)$ (cf. [2]), and the fact that

$$\beta(s) = 4^{-s} \left[\zeta\left(s, \frac{1}{4}\right) - \zeta\left(s, \frac{3}{4}\right) \right].$$

Substituting $1-s = u$, we get

$$F(x) = \frac{\pi}{4x} - \frac{1}{4} + \frac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \pi^{1-2u}\Gamma(u)x^{u-1}\zeta(u)\beta(u)du,$$

which proves (9).

Using the above scheme, several other identities which one needs in the analysis of algorithms can be proved. We refer to Szpankowski's book [25].

4 Modular identities

Formulae like (6) belong to the realm of modular functions. Many of them can be found in the literature, and are due to Jacobi, Dedekind,

Ramanujan and others. Berndt’s book [4] contains a wealth of information about the subject, compare also [3].

Here is a little bit of background: Let H be the upper complex halfplane $\{z \in \mathbb{C} \mid \Im z > 0\}$. Then the Dedekind η function is defined by

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{n \geq 1} (1 - e^{2\pi i n \tau}), \quad \tau \in H;$$

there is a transformation formula:

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2} \eta(\tau).$$

C. L. Siegel [24] gave an elegant proof of this transformation formula using residue calculus.

Ramanujan considered series

$$f(z) := \sum_{k \geq 1} \frac{k^m}{e^{2kz} - 1}, \quad m \text{ an odd integer,}$$

and could relate them to $f(\pi^2/z)$. For the reader’s convenience, we give these formulæ here:

Set $m = 2N + 1$ and $N \in \mathbb{N}$, $\alpha, \beta > 0$, and $\alpha\beta = \pi^2$, then

$$\begin{aligned} & \alpha^{-N} \left\{ \frac{1}{2} \zeta(2N + 1) + \sum_{k \geq 1} \frac{k^{-2N-1}}{e^{2\alpha k} - 1} \right\} \\ &= (-\beta)^{-N} \left\{ \frac{1}{2} \zeta(2N + 1) + \sum_{k \geq 1} \frac{k^{-2N-1}}{e^{2\beta k} - 1} \right\} \\ & \quad - 2^{2N} \sum_{k=0}^{N+1} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{2N+2-2k}}{(2N+2-2k)!} \alpha^{N+1-k} \beta^k \end{aligned}$$

(this covers the exponents $-3, -5, \dots$); the B_k ’s are the Bernoulli numbers. Then

$$\sum_{k \geq 1} \frac{1}{k(e^{2\alpha k} - 1)} - \frac{1}{4} \log \alpha + \frac{\alpha}{12} = \sum_{k \geq 1} \frac{1}{k(e^{2\beta k} - 1)} - \frac{1}{4} \log \beta + \frac{\beta}{12},$$

which covers the exponent -1 . Furthermore,

$$\alpha \sum_{k \geq 1} \frac{k}{e^{2\alpha k} - 1} + \beta \sum_{k \geq 1} \frac{k}{e^{2\beta k} - 1} = \frac{\alpha + \beta}{24} - \frac{1}{4}$$

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which covers the exponent 1, and finally for $N \geq 2$,

$$\alpha^N \sum_{k \geq 1} \frac{k^{2N-1}}{e^{2\alpha k} - 1} - (-\beta)^N \sum_{k \geq 1} \frac{k^{2N-1}}{e^{2\beta k} - 1} = (\alpha^N - (-\beta)^N) \frac{B_{2N}}{4N},$$

which covers the exponents 3, 5,

The instance $m = -1$ is equivalent to the functional equation for Dedekind's eta function.

References

- [1] Abramowitz, M. and Stegun, I. A. (1964), Handbook of Mathematical Functions. Dover, 1973 A reprint of the tenth National Bureau of Standards edition.
- [2] Apostol, T. (1976), Introduction to Analytical Number Theory. Springer-Verlag.
- [3] Apostol, T. (1978), Modular Functions and Dirichlet Series in Number Theory. Springer-Verlag.
- [4] Berndt, B. (1989), Ramanujan's Notebooks. Part II, Springer-Verlag.
- [5] Flajolet, P. (1985), Approximate counting: A detailed analysis. BIT, **25**, 113–134.
- [6] Flajolet, P., Gourdon, X., and Dumas, P. (1995), Mellin transforms and asymptotics: Harmonic sums. Theoretical Computer Science, **144**, 3–58.
- [7] Jacquet, P. and Régnier, M. (1989), New results on the size of tries. IEEE Transactions on Information Theory, **35**, 203–205.
- [8] Kirschenhofer, P., Prodinger, H., and Szpankowski, W. (1989), On the variance of the external path length in a symmetric digital trie. Discrete Applied Mathematics, **25**, 129–143.
- [9] Kirschenhofer, P., Prodinger, H., and Szpankowski, W. (1993), Multidimensional digital searching and some new parameters in tries. International Journal of Foundations of Computer Science, **4**, 69–84.

- [10] Kirschenhofer, P., Prodinger, H., and Szpankowski, W. (1994), Digital search trees again revisited: The internal path length perspective. *SIAM Journal on Computing*, **23**, 598–616.
- [11] Kirschenhofer, P. and Prodinger, H. (1991), Approximate counting: An alternative approach. *RAIRO Theoretical Informatics and Applications*, **25**, 43–48.
- [12] Kirschenhofer, P. and Prodinger, H. (1991), On some applications of formulæ of Ramanujan in the analysis of algorithms. *Mathematika*, **38**, 14–33.
- [13] Kirschenhofer, P. and Prodinger, H. (1993), A result in order statistics related to probabilistic counting. *Computing*, **51**, 15–27.
- [14] Kirschenhofer, P. and Prodinger, H. (1994), The path length of random skip lists. *Acta Informatica*, **31**, 775–792.
- [15] Kirschenhofer, P., Prodinger, H., and Szpankowski, W. (1989), On the balance property of Patricia tries: External path length viewpoint. *Theoretical Computer Science*, **68**, 1–17.
- [16] Knuth, D. E. (1997), *The Art of Computer Programming. Vol. 1: Fundamental Algorithms*, Third ed. Reading, Massachusetts: Addison-Wesley.
- [17] Knuth, D. E. (1998), *The Art of Computer Programming. Vol. 3: Sorting and Searching*, 2nd ed. Reading, Massachusetts: Addison-Wesley.
- [18] Mahmoud, H. M. (1992), *Evolution of Random Search Trees*. New York: John Wiley & Sons.
- [19] Nielsen, N. (1906), *Handbuch der Theorie der Gammafunktion*. Teubner.
- [20] Prodinger, H. (1992), Hypothetic analyses: Approximate counting in the style of Knuth, path length in the style of Flajolet. *Theoretical Computer Science*, **100**, 243–251.
- [21] Prodinger, H. (1994), Approximate counting via Euler transform. *Mathematica Slovaca*, **44**, 569–574.

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- [22] Prodinger, H. (2004), Compositions and Patricia tries: No fluctuations in the variance! Proceedings of the 6th Workshop on ALENEX and the 1st workshop on ANALCO, L. Arge, G. Italiano, R. Sedgewick (Ed.), SIAM. 211–215.
- [23] Sedgewick, R. and Flajolet, P. (1996), An Introduction to the Analysis of Algorithms. Reading, Massachusetts: Addison-Wesley.
- [24] Siegel, C. L. (1954), A simple proof of $\eta(-1/\tau) = \eta(\tau)\sqrt{\tau/i}$. *Mathematika*, **1**, 4.
- [25] Szpankowski, W. (2001), Average case analysis of algorithms on sequences. New York: Wiley-Interscience.
- [26] Szpankowski, W. and Rego, V. (1990), Yet another application of a binomial recurrence: Order statistics. *Computing*, **43**, 401–410.
- [27] Whittaker, E. T. and Watson, G. N. (1927), A Course of Modern Analysis. Fourth edition, Cambridge University Press, Reprinted 1973.

21 Scientific publications of Helmut Prodinger (November 2014)

- [1] H. Prodinger and F.J. Urbanek. Language operators related to init. *Theoretical Computer Science*, 8:161–175, 1979.
- [2] H. Prodinger. On a generalization of the Dyck–language over a two letter alphabet. *Discrete Mathematics*, 28:269–276, 1979.
- [3] H. Prodinger and F.J. Urbanek. Infinite 0–1–sequences without long adjacent identical blocks. *Discrete Mathematics*, 28:277–289, 1979.
- [4] W. Kuich, H. Prodinger, and F.J. Urbanek. On the height of derivation trees. In H. Maurer, editor, *Automata, Languages and Programming*, volume 71, pages 370–384, 1979.
- [5] H. Prodinger. On the interpolation of D0L–sequences. *Theoretical Computer Science*, 10:107–108, 1980.
- [6] H. Prodinger. Congruences defined by languages and filters. *Information and Control*, 44:36–46, 1980.
- [7] H. Prodinger. Topologies on free monoids defined by closure operators of a special type. *RAIRO Theoretical Informatics and Applications*, 14:225–237, 1980.
- [8] H. Prodinger. The average height of a stack where three operations are allowed and some related problems. *Journal of Combinatorics, Information and System Sciences*, 5:287–304, 1980.
- [9] H. Prodinger. On the number of Fibonacci partitions of a set. *The Fibonacci Quarterly*, 19:463–466, 1981.
- [10] P. Kirschenhofer and H. Prodinger. On the average height of monotonically labelled binary trees. In *Finite and Infinite Sets*, volume 37, pages 509–526. North-Holland, 1981. Proceedings of Janos Bolyai Conference on Combinatorics Eger (Hungary).
- [11] H. Prodinger. Generalizing the sum of digits function. *SIAM Journal on Algebraic and Discrete Methods*, 3:35–42, 1982.
- [12] H. Prodinger and R.F. Tichy. Fibonacci numbers of graphs. *The Fibonacci Quarterly*, 19:16–21, 1982.
- [13] H. Prodinger. A note on a result of R. Kemp on r –tuply rooted planted plane trees. *Computing*, 28:363–366, 1982.
- [14] P. Kirschenhofer and H. Prodinger. On the average oscillation of the contour of monotonically labelled ordered trees. In *Proceedings of the Second World Conference on Mathematics at the Service of Man (Las Palmas)*, pages 394–399, 1982.
- [15] P. Kirschenhofer and H. Prodinger. On the average hyperoscillations of planted plane trees. *Combinatorica*, 2:177–186, 1982.
- [16] H. Prodinger. On the number of partitions of $\{1, \dots, n\}$ into two sets of equal cardinalities and equal sums. *Canadian Mathematical Bulletin*, 25:238–241, 1982.
- [17] H. Prodinger. Non-repetitive sequences and Gray code. *Discrete Mathematics*, 43:113–116, 1983.
- [18] H. Prodinger. Ordered Fibonacci partitions. *Canadian Mathematical Bulletin*, 26:312–316, 1983.
- [19] P. Kirschenhofer, H. Prodinger, and R.F. Tichy. Fibonacci numbers of graphs II. *The Fibonacci Quarterly*, 21:219–229, 1983.
- [20] H. Prodinger. Topologies over free monoids induced by families of languages. *RAIRO Theoretical Informatics and Applications*, 17:285–290, 1983.
- [21] H. Prodinger and G. Thierrin. Towards a general concept of hypercodes. *Journal of Information and Optimization Sciences*, 4:255–268, 1983.
- [22] H. Prodinger. A correspondence between ordered trees and noncrossing partitions. *Discrete Mathematics*, 46:205–206, 1983.
- [23] H. Prodinger. On the number of combinations without a fixed distance. *Journal of Combinatorial Theory, Series A*, 35:362–365, 1983.
- [24] H. Prodinger. A subword version of d’Ocagne’s formula. *Utilitas Mathematica*, 24:125–129, 1983.
- [25] H. Prodinger. An elementary approach to the stack size of regularly distributed binary trees. *Soochow Journal of Mathematics*, 9:193–196, 1983.
- [26] H. Prodinger. The height of planted plane trees revisited. *Ars Combinatoria*, 16–B:51–55, 1983.
- [27] H. Prodinger. Abzählprobleme bei bäumen. In G. Baron and P. Kirschenhofer, editors, *Publications de l’IRMA (Straßbourg)*, pages 167–173, 1983.
- [28] H. Prodinger and R.F. Tichy. Über ein zahlentheoretisches Problem aus der Informatik. *Sitzungsberichte der österreichischen Akademie der Wissenschaften, Abteilung II*, 192:385–396, 1983.
- [29] H. Prodinger. Representations of 0 as a weighted sum of the numbers $-n, \dots, n$. *Journal of Combinatorics, Information and System Sciences*, 8:271–276, 1983.
- [30] H. Prodinger. Quelques techniques analytiques pour l’étude asymptotique des paramètres dans les arbres. In *Sommerschule Ile de Ré, (Vortragsausarbeitung), english version in EATCS Bulletin 47 (1992), 180–199*, 1983.
- [31] H. Prodinger and F.J. Urbanek. On monotone functions of tree structures. *Discrete Applied Mathematics*, 5:223–239, 1983.

-
- [32] P. Kirschenhofer and H. Prodinger. Recursion depth analysis for special tree traversal algorithms. In J. Paradaens, editor, *Automata, Languages and Programming*, volume 172 of *Lecture Notes in Computer Science*, pages 303–311, 1984.
- [33] H. Prodinger. Analysis of an algorithm to construct Fibonacci partitions. *RAIRO Theoretical Informatics and Applications*, 18:387–394, 1984.
- [34] H. Prodinger. On the altitude of specified nodes in random trees. *Journal of Graph Theory*, 8:481–485, 1984.
- [35] H. Prodinger. The average height of the second highest leaf of a planted plane tree. *European Journal of Combinatorics*, 5:351–357, 1984.
- [36] P. Kirschenhofer and H. Prodinger. A short proof for a partition identity of Hwang and Wei. *Discrete Mathematics*, 52:305–306, 1984.
- [37] H. Prodinger. On the number of partitions of $\{1, \dots, n\}$ into r sets of equal cardinalities and sums. *Tamkang Journal of Mathematics*, 15:161–164, 1984.
- [38] H. Prodinger. Eine Bemerkung zur asymptotischen Bestimmung der höheren Momente der Anzahl der Inversionen. *Anzeiger der österreichischen Akademie der Wissenschaften*, 121:141–143, 1984.
- [39] P. Kirschenhofer and H. Prodinger. Subblock occurrences in positional number systems and Gray code representation. *Journal of Information and Optimization Sciences*, 5:29–42, 1984.
- [40] H. Prodinger. Die Bestimmung gewisser Parameter von binären Bäumen mit Hilfe analytischer Methoden. In E. Hlawka, editor, *Zahlentheoretische Analysis*, volume 1114 of *Lecture Notes in Mathematics*, pages 118–133, 1985.
- [41] P. Kirschenhofer, H. Prodinger, and R.F. Tichy. über mittelwerte gewichteter ziffernsummen und verwandte probleme. In E. Hlawka, editor, *Zahlentheoretische Analysis*, volume 1114 of *Lecture Notes in Mathematics*, pages 55–65, 1985.
- [42] H. Prodinger. Einige Bemerkungen zu einer Arbeit von W. Knödel. *EIK*, 21:3–7, 1985.
- [43] G. Baron, F.T. Boesch, H. Prodinger, R.F. Tichy, and J.F. Wang. The number of spanning trees in the square of a cycle. *The Fibonacci Quarterly*, 23:258–264, 1985.
- [44] H. Prodinger. The average height of the d -th highest leaf of a planted plane tree. *Networks*, 16:67–75, 1985.
- [45] W. Panny and H. Prodinger. The expected height of paths for several notions of height. *Studia Scientiarum Mathematicarum Hungarica*, 20:119–132, 1985.
- [46] P. Flajolet and H. Prodinger. Register allocation for unary–binary trees. *SIAM Journal on Computing*, 15:629–640, 1986.
- [47] P. Kirschenhofer, H. Prodinger, and R.F. Tichy. Fibonacci numbers of graphs III. In *Proceedings of the First International Conference on Fibonacci Numbers and Applications*, pages 105–120. D. Reidel, 1986.
- [48] P. Kirschenhofer and H. Prodinger. Two selection problems revisited. *Journal of Combinatorial Theory Series, A* 42:310–316, 1986.
- [49] P. Kirschenhofer and H. Prodinger. Some further results on digital search trees. In *Automata, Languages and Programming*, volume 229 of *Lecture Notes in Computer Science*, pages 177–185, 1986.
- [50] F.T. Boesch and H. Prodinger. Spanning tree formulas and Chebyshev polynomials. *Graphs and Combinatorics*, 2:191–200, 1986.
- [51] H. Prodinger. Some recent results on the register function of a binary tree. *Annals of Discrete Mathematics*, 33:241–260, 1987.
- [52] P. Kirschenhofer, H. Prodinger, and R.F. Tichy. A contribution to the analysis of in situ permutation. *Glasnik Matematički*, 22(42):269–278, 1987.
- [53] P. Flajolet and H. Prodinger. Level number sequences for trees. *Discrete Mathematics*, 65:149–156, 1987.
- [54] P. Kirschenhofer and H. Prodinger. On the recursion depth of special tree traversal algorithms. *Information and Computation*, 74:15–32, 1987.
- [55] H. Prodinger. über längste 1-teilfolgen in 0-1-folgen. In E. Hlawka, editor, *Zahlentheoretische Analysis II*, volume 1262 of *Lecture Notes in Mathematics*, pages 124–133, 1987.
- [56] P. Kirschenhofer and H. Prodinger. Asymptotische Untersuchungen über charakteristische Parameter von Suchbäumen. In E. Hlawka, editor, *Zahlentheoretische Analysis II*, volume 1262 of *Lecture Notes in Mathematics*, pages 93–107, 1987.
- [57] P. Kirschenhofer and H. Prodinger und J. Schoißengeier. Zur Auswertung gewisser numerischer Reihen mit Hilfe modularer Funktionen. In E. Hlawka, editor, *Zahlentheoretische Analysis II*, volume 1262 of *Lecture Notes in Mathematics*, pages 108–110, 1987.
- [58] P. Kirschenhofer, H. Prodinger, and R.F. Tichy. über einige funktional-differentialgleichungen in der analyse von algorithmen. In E. Hlawka, editor, *Zahlentheoretische Analysis II*, volume 1262 of *Lecture Notes in Mathematics*, pages 111–123, 1987.
- [59] P. Kirschenhofer and H. Prodinger. Further results on digital search trees. *Theoretical Computer Science*, 58:143–154, 1988.
- [60] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. Do we really need to balance patricia tries? In *Lecture Notes in Computer Science*, volume 317 of *Automata, Languages and Programming*, pages 302–316, 1988.
- [61] P. Kirschenhofer and H. Prodinger. b-tries: a paradigm for the application of numbertheoretic methods in the analysis of algorithms. In *Contributions to General Algebra*, volume 6, pages 141–153. Hölder-Pichler-Tempsky-Teubner, 1988.

-
- [62] P. Kirschenhofer and H. Prodinger. Eine Anwendung der Theorie der Modulfunktionen in der Informatik. *Sitzungsberichte der Österreichischen Akademie der Wissenschaften*, 197:339–366, 1988.
- [63] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. On the balance property of patricia tries: External path length view. *Theoretical Computer Science*, 68:1–17, 1989.
- [64] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. Digital search trees – further results on a fundamental data structure. In G.X. Ritter, editor, *Information Processing 89*, pages 443–447, 1989.
- [65] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. On the variance of the external path length in a symmetric digital trie. *Discrete Applied Mathematics*, 25:129–143, 1989.
- [66] P. Kirschenhofer and H. Prodinger. On the analysis of probabilistic counting. In E. Hlawka and R.F. Tichy, editors, *Number-theoretic Analysis*, volume 1452 of *Lecture Notes in Mathematics*, pages 117–120, 1990.
- [67] H. Prodinger. Further results on a problem of Knödel. In E. Hlawka and R.F. Tichy, editors, *Number-theoretic Analysis*, volume 1452 of *Lecture Notes in Mathematics*, pages 193–198, 1990.
- [68] P. Kirschenhofer and H. Prodinger. On some applications of formulæ of Ramanujan in the analysis of algorithms. *Mathematika*, 38:14–33, 1991.
- [69] P. Kirschenhofer and H. Prodinger. Approximate counting: An alternative approach. *RAIRO Theoretical Informatics and Applications*, 25:43–48, 1991.
- [70] H. Prodinger. Hypothetic analyses: Approximate counting in the style of Knuth, path length in the style of Flajolet. *Theoretical Computer Science*, 100:243–251, 1992.
- [71] H. Prodinger. Einige Bemerkungen zu einer Bin-Packing Aufgabe von W. Knödel. *Computing*, 47:247–254, 1992.
- [72] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. How to count quickly and accurately: A unified analysis of probabilistic counting and other related problems. In W. Kuich, editor, *Automata, Languages and Programming*, volume 623 of *Lecture Notes in Computer Science*, pages 211–222, 1992.
- [73] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. Probabilistic modeling of data structures on words. *Theoretical Computer Science*, 106:395–400, 1992.
- [74] P. Kirschenhofer and H. Prodinger. A coin tossing algorithm for counting large numbers of events. *Mathematica Slovaca*, 42:531–545, 1992.
- [75] H. Prodinger. External internal nodes in digital search trees via Mellin transforms. *SIAM Journal on Computing*, 21:1180–1183, 1992.
- [76] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. Multidimensional digital searching and some new parameters in tries. *International Journal of Foundations of Computer Science*, 4:69–84, 1993.
- [77] H. Prodinger. How to select a loser. *Discrete Mathematics*, 120:149–159, 1993.
- [78] P. Kirschenhofer and H. Prodinger. A result in order statistics related to probabilistic counting. *Computing*, 51:15–27, 1993.
- [79] H. Prodinger. How to advance on a stairway by coin flippings. In *Proceedings “Applications of Fibonacci Numbers, Volume 5”*, pages 473–479. Kluwer, 1993.
- [80] P. Grabner, P. Kirschenhofer, H. Prodinger, and R. Tichy. On the moments of the sum of digits function. In *Proceedings “Applications of Fibonacci Numbers, Volume 5”*, pages 263–271. Kluwer, 1993.
- [81] H. Prodinger and W. Szpankowski. A note on binomial recurrences arising in the analysis of algorithms (letter to the editor). *Information Processing Letters*, 46:309–311, 1993.
- [82] H. Prodinger. Combinatorial problems of geometrically distributed random variables and applications in computer science. In V. Strehl and R. König, editors, *Publications de l’IRMA (Straßbourg)*, volume 30, pages 87–95, 1993.
- [83] P. Grabner, H. Prodinger, and R.F. Tichy. Asymptotic analysis of a class of functional equations and applications. *Journal de Théorie des Nombres de Bordeaux*, 5:365–381, 1993.
- [84] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. Digital search trees again revisited: The internal path length perspective. *SIAM Journal on Computing*, 23:598–616, 1994.
- [85] P. Kirschenhofer and H. Prodinger. The path length of random skip lists. *Acta Informatica*, 31:775–792, 1994.
- [86] P. Grabner and H. Prodinger. The Fibonacci killer. *The Fibonacci Quarterly*, 32:389–396, 1994.
- [87] H. Prodinger. Some information about the binomial transform. *The Fibonacci Quarterly*, 32:412–415, 1994.
- [88] P. Kirschenhofer and H. Prodinger. The higher moments of the number of returns of a simple random walk. *Advances in Applied Probability*, 26:561–563, 1994.
- [89] H. Prodinger. An asymptotic comment on a paper by Analyti and Pramanik. *Information Processing Letters*, 49:327–328, 1994.
- [90] P. Flajolet, P. Grabner, P. Kirschenhofer, H. Prodinger, and R.F. Tichy. Mellin transforms and asymptotics: Digital sums. *Theoretical Computer Science*, 123:291–314, 1994.
- [91] P. Kirschenhofer and H. Prodinger. Multidimensional searching – alternative data structures. *Random Structures and Algorithms*, 5:123–134, 1994.
- [92] H. Prodinger. Some polynomials related to the Fibonacci polynomials. *EATCS Bulletin*, 53:248–250, 1994.
- [93] H. Prodinger. Knuth’s old sum – a survey. *EATCS Bulletin*, 54:232–245, 1994.
- [94] H. Prodinger. Approximate counting via Euler transform. *Mathematica Slovaca*, 44:569–574, 1994.
- [95] H. Prodinger. Geometric distributions and forbidden subwords. *The Fibonacci Quarterly*, 33:139–141, 1995.

- [96] H. Prodinger. Comments on the analysis of parameters in a random graph model. *RAIRO Theoretical Informatics and Applications*, 29:245–253, 1995.
- [97] X. Gourdon and H. Prodinger. A generating function approach to random subgraphs of the n -cycle. *Discrete Mathematics*, 169:227–232, 1997.
- [98] P. Kirschenhofer and H. Prodinger. Return statistics of simple random walks. *Journal of Statistical Planning and Inference*, 54:67–74, 1996.
- [99] P. Flajolet, P. Grabner, P. Kirschenhofer, and H. Prodinger. On Ramanujan’s $Q(n)$ -function. *Journal of Computational and Applied Mathematics*, 58:103–116, 1995.
- [100] W. Panny and H. Prodinger. Bottom-up mergesort: A detailed analysis. *Algorithmica*, 14:340–354, 1995.
- [101] P. Kirschenhofer, C. Martínez, and H. Prodinger. Analysis of an optimized search algorithm for skip lists. *Theoretical Computer Science*, 144:199–220, 1995.
- [102] H. Prodinger and W. Szpankowski (eds.). Special volume on the mathematical analysis of algorithms. *Theoretical Computer Science*, 144, 1995.
- [103] H. Prodinger. Solution of a problem of Yekutieli and Mandelbrot. In R. Baeza, E. Goles, and P. V. Poblete, editors, *LATIN 95*, volume 911 of *Lecture Notes in Computer Science*, pages 461–468, 1995.
- [104] H. Prodinger. The asymptotic behavior of the golden numbers. *The Fibonacci Quarterly*, 34:224–225, 1996.
- [105] A. Knopfmacher and H. Prodinger. Explicit and asymptotic formulæ for average values of order statistics of the Cantor distribution. *Statistics and Probability Letters*, 27:189–194, 1996.
- [106] P. Grabner, P. Kirschenhofer, and H. Prodinger. The sum-of-digits function for complex bases. *J. London Math. Soc.*, 57:20–40, 1998.
- [107] P. Kirschenhofer and H. Prodinger. The number of winners a in discrete geometrically distributed sample. *Annals in Applied Probability*, 6:687–694, 1996.
- [108] P. Grabner and H. Prodinger. Asymptotic analysis of the moments of the Cantor distribution. *Statistics and Probability Letters*, 26:243–248, 1996.
- [109] H. Prodinger. Combinatorics of geometrically distributed random variables: Left-to-right maxima. *Discrete Mathematics*, 153:253–270, 1996.
- [110] H. Prodinger. Digital search trees and basic hypergeometric functions. *EATCS Bulletin*, 56:112–115, 1995.
- [111] P. Kirschenhofer, C. Martínez, and H. Prodinger. Analysis of Hoare’s find algorithm with median-of-three partition. *Random Structures and Algorithms*, 10:143–156, 1997.
- [112] H. Prodinger. Asymptotics of the Luria–Delbrück distribution via singularity analysis. *Journal of Applied Probability*, 33:282–283, 1996.
- [113] H. Prodinger. Multiple quickselect — Hoare’s find algorithm for several elements. *ipl*, 56:123–129, 1995.
- [114] P. Kirschenhofer, H. Prodinger, and W. Szpankowski. Analysis of a splitting process arising in probabilistic counting and other related algorithms. *RSA*, 9:379–401, 1996.
- [115] H. Prodinger. On a problem of Yekutieli and Mandelbrot about the bifurcation ratio of binary trees. *Theoretical Computer Science*, 181:181–194, 1997.
- [116] F. Hubalek, H. K. Hwang, W. Lew, H. Mahmoud, and H. Prodinger. A multivariate view of random bucket digitalsearch trees. *Journal of Algorithms*, 44:121–158, 2002.
- [117] H. Prodinger. Depth and path length of heap ordered trees. *International Journal of Foundations of Computer Science*, 7:293–299, 1996.
- [118] H. Prodinger. Combinatorial sums in the analysis of a generalized van der Pol equation. *Maple Technical Newsletter*, 5:33–36, 1998.
- [119] H. Prodinger. On a question of Cooper and Kennedy. *The Fibonacci Quarterly*, 35:135–136, 1997.
- [120] P. Kirschenhofer and H. Prodinger. Comparisons in Hoare’s Find algorithm. *cpc*, 7:111–120, 1998.
- [121] H. Prodinger. The Cantor–Fibonacci distribution. In *Proceedings “Applications of Fibonacci Numbers, Volume 7”*, pages 311–318. Kluwer, 1998.
- [122] C. Martínez, A. Panholzer, and H. Prodinger. On the number of descendants and ascendants in random search trees. *Electronic Journal on Combinatorics*, 5:R20, 29 pages + Appendix (10 pages), 1998.
- [123] P. Grabner and H. Prodinger. The optimality of an algorithm of Reingold and Supowit. *Journal of Automata, Languages, and Combinatorics*, 1:5–11, 1996.
- [124] P. Grabner and H. Prodinger. Maximum statistics of n random variables distributed by the negative binomial distribution. *Combinatorics, Probability, and Computing*, 6:179–184, 1997.
- [125] H. Prodinger. Simplification of two papers of Ferri, Faccio, and D’Amico. <http://info.tuwien.ac.at/theoinf/proding.htm>, 1996.
- [126] H. Prodinger. The level of nodes in heap ordered trees. <http://info.tuwien.ac.at/theoinf/proding.htm>, 1996.
- [127] H. Prodinger. Explicit formulæ for some numerical triangles of Z. W. Trzaska. <http://info.tuwien.ac.at/theoinf/proding.htm>, 1996.
- [128] H. Prodinger. Descendants in heap ordered trees – or – a triumph of computer algebra. *Electronic Journal of Combinatorics*, 3:R29 (9 pages), 1996.
- [129] H. Prodinger. A note on the distribution of the three types of nodes in uniform binary trees. *Séminaire Lotharingien de Combinatoire*, 38:P2 (5 pages), 1996.
- [130] H. Prodinger. A bijection between coloured trees and set partitions. <http://info.tuwien.ac.at/theoinf/proding.htm>, 1996.

- [131] H. Prodinger. *Words, Dyck Paths, Trees, and Bijections*, pages 369–379. World Scientific, 2001.
- [132] P. Grabner and H. Prodinger. An asymptotic study of a recursion occurring in the analysis of an algorithm on broadcast communication. *Information Processing Letters*, 65:89–93, 1998.
- [133] J. W. Moon and H. Prodinger. A bijective proof of an identity concerning nodes of fixed degree in planted plane trees. *Ars Combinatoria*, 55:91–92, 2000.
- [134] A. Panholzer and H. Prodinger. Descendants and ascendants in binary trees. *Discrete Mathematics and Theoretical Computer Science*, 1:247–266, 1997.
- [135] A. Panholzer and H. Prodinger. A generating functions approach for the analysis of grand averages for multiple quickselect. *Random Structures and Algorithms*, 1999.
- [136] A. Knopfmacher and H. Prodinger. On Carlitz compositions. *European Journal on Combinatorics*, 19:579–589, 1998.
- [137] P. Flajolet and H. Prodinger. On Stirling numbers for complex arguments and Hankel contours. *SIAM J. on Discrete Mathematics*, 12:155–159, 1999.
- [138] A. Panholzer and H. Prodinger. Average case-analysis of priority trees: A structure for priority queue administration. *Algorithmica*, 22:600–630, 1998.
- [139] A. Panholzer and H. Prodinger. Towards a more precise analysis of an algorithm to generate binary trees : A tutorial. *The Computer Journal*, 41:201–204, 1998.
- [140] K. Driver, H. Prodinger, C. Schneider, and André Weideman. Padé approximations to the logarithm II: Identities, recurrences, and symbolic computation. *Ramanujan Journal*, 11:139–158, 2006.
- [141] A. Panholzer and H. Prodinger. Moments of level numbers of leaves in binary trees. *Journal of Statistical Inference*, 101:267–279, 2002.
- [142] A. Panholzer and H. Prodinger. An analytic approach for the analysis of rotations in fringe-balanced binary search trees. *Annals of Combinatorics*, 2:173–184, 1998.
- [143] A. Panholzer and H. Prodinger. Two proofs of Filipponi’s formula for Lucas numbers of odd index. *The Fibonacci Quarterly*, 38:165–166, 2000.
- [144] H. Prodinger. Epidemics with two levels of mixing: The exact moments. *SADIO*, 2:1–4, 1999.
- [145] C. Martínez, A. Panholzer, and H. Prodinger. Partial match queries in relaxed multidimensional search trees. *Algorithmica*, 29:181–204, 2001.
- [146] H. Prodinger and W. Szpankowski (eds.). Special issue on average case analysis of algorithms. *Algorithmica*, 22.4, 1998.
- [147] H. Prodinger and W. Szpankowski. Philippe Flajolet’s research in analysis of algorithms and combinatorics. *Algorithmica*, 22:366–387, 1998.
- [148] A. Knopfmacher and H. Prodinger. Combinatorics of geometrically distributed random variables: Value and position of the r th left-to-right maximum. *Discrete Mathematics*, 226:255–267, 2001.
- [149] P. Grabner, A. Knopfmacher, and H. Prodinger. Combinatorics of geometrically distributed random variables: Run statistics. *LATIN 2000, Lecture Notes in Computer Science*, 1776:457–462, 2000.
- [150] H. Prodinger. Combinatorics of geometrically distributed random variables: Inversions and a parameter of Knuth. *Annals of Combinatorics*, 5:241–250, 2001.
- [151] H. Prodinger. On Cantor’s singular moments. *Southwest J. Pure Appl. Math.*, 2000–1:27–29 (electronic), 2000.
- [152] H. Prodinger. Combinatorics of geometrically distributed random variables: Lengths of ascending runs. *LATIN2000, Lecture Notes in Computer Science 1776*, pages 473–482, 2000.
- [153] G. Louchard and H. Prodinger. Probabilistic analysis of Carlitz compositions. *Discrete Mathematics and Theoretical Computer Science*, 5:71–96, 2002.
- [154] H. Prodinger. A q -analogue of a formula of Hernandez obtained by inverting a result of Dilcher. *Australasian Journal of Combinatorics*, 21:271–274, 2000.
- [155] H. Prodinger. Combinatorics of geometrically distributed random variables: New q -tangent and q -secant numbers. *International Journal of Mathematical Sciences*, 24:825–838, 2000.
- [156] H. Prodinger. A q -analogue of the path length in binary search trees. *Algorithmica*, 31:433–441, 2001.
- [157] A. Panholzer and H. Prodinger. Bijections for non-crossing trees on a circle. *Discrete Mathematics*, 250:181–195, 2002.
- [158] H. Prodinger and W. Szpankowski (eds.). Special issue on average case analysis of algorithms. *Algorithmica*, 29, 369 pages; preface 1–2, 2001.
- [159] A. Knopfmacher and H. Prodinger. A simple card guessing game revisited. *Electron. J. Combin.*, 8(2):Research Paper 13, 9 pp. (electronic), 2001. In honor of Aviezri Fraenkel on the occasion of his 70th birthday.
- [160] H. Prodinger. On the moments of a distribution defined by the Gaussian polynomials. *Journal of Statistical Planning and Inference*, pages 237–239, 2004.
- [161] H. Prodinger. On the analysis of an algorithm to generate a random cyclic permutation. *Ars Combinatoria*, 65:75–78, 2002.
- [162] H. Prodinger. On binary representations of integers with digits $-1, 0, 1$. *Integers*, pages A8, 14 pp. (electronic), 2000.
- [163] M. E. H. Ismail, H. Prodinger, and D. Stanton. Schur’s determinants and partition theorems. *Sém. Lothar. Combin.*, 44:Art. B44a, 10 pp. (electronic), 2000.

- [164] P. Grabner and H. Prodinger. Sorting algorithms for broadcast communications: Mathematical analysis. *Theoretical Computer Science*, 289:51–67, 2002.
- [165] A. Panholzer and H. Prodinger. Binary search tree recursions with harmonic toll functions. *Journal of computational and applied mathematics*, 142:211–225, 2002.
- [166] H. Prodinger. Some applications of the q -Rice formula. *Random Structures and Algorithms*, 19:552–557, 2001.
- [167] C. Heuberger and H. Prodinger. On minimal expansions in redundant number systems: Algorithms and quantitative analysis. *Computing*, 66:377–393, 2001.
- [168] H. Prodinger. Combinatorics of geometrically distributed random variables: Value and position of large left-to-right maxima. *Discrete Mathematics*, 254:459–471, 2002.
- [169] A. Panholzer and H. Prodinger. Kirkman’s hypothesis revisited. *Integers*, 1:A05, 3 pages, 2001.
- [170] G. Andrews, A. Knopfmacher, P. Paule, and H. Prodinger. q -Engel series expansions and Slater’s identities. *Quaestiones Mathematicae*, 24:403–416, 2001.
- [171] D. Callan and H. Prodinger. An involutory matrix of eigenvectors. *The Fibonacci Quarterly*, 38:105–107, 2003.
- [172] P. Grabner and H. Prodinger. Some identities for Chebyshev polynomials. *Portugaliae Mathematicae*, 59:311–314, 2002.
- [173] G. Louchard and H. Prodinger. Ascending runs of sequences of geometrically distributed random variables: a probabilistic analysis. *Theoretical Computer Science*, 304:59–86, 2003.
- [174] R. Kemp and H. Prodinger (editors). Special issue in algorithmica, mathematical analysis of algorithms. *Algorithmica*, pages 237–457, 2001.
- [175] H. Prodinger. On a functional-difference equation of Runyon, Morrison, Carlitz, and Riordan. *Sém. Lothar. Combin.*, 46:Art. B 46a, 4 pp. (electronic), 2001.
- [176] Helmut Prodinger. Words, permutations, and representations of numbers. *Lecture Notes in Computer Science* (Ed. W. Kuich, G. Rozenberg, A. Salomaa), pages 81–99, 2002.
- [177] H. Prodinger and W. Szpankowski. Optimal versus randomized search of fixed length binary words. *IEEEIT*, 48:2614–2621, 2002.
- [178] E. Deutsch and H. Prodinger. A bijection between directed column-convex polyominoes and ordered trees of height at most four. *Theoretical Computer Science*, 307:319–325, 2003.
- [179] H. Prodinger and T.A. Tshifhumulo. On q -Olivier functions. *Annals of Combinatorics*, 6:81–94, 2002.
- [180] M. Drmota and H. Prodinger. The height of q -binary search trees. *Discrete Mathematics and Theoretical Computer Science*, 5:97–108, 2002.
- [181] H. Prodinger and W. Szpankowski (editors). Special issue in random structures and algorithms, analysis of algorithms. *Random Structures and Algorithms*, pages 149–557, 2001.
- [182] A. Panholzer, H. Prodinger, and M. Riedel. Measuring post-quickselect disorder. *Journal of the Iranian statistical society*, 3:219–249, 2004.
- [183] H. Prodinger. A note on a paper of G. H. Weiss and M. Dishon. *The Fibonacci Quarterly*, 42:290–291, 2004.
- [184] C. Heuberger and H. Prodinger. Carry propagation in signed digit representations. *European Journal of Combinatorics*, 24:293–320, 2003.
- [185] G. Louchard and H. Prodinger. Random 0-1 rectangular matrices: a probabilistic analysis. *Publicationes Mathematicae (Debrecen)*, pages 169–193, 2003.
- [186] A. Panholzer and H. Prodinger. Spanning tree size in random binary search trees. *Annals of applied probability*, 14:718–733, 2004.
- [187] P. Grabner, C. Heuberger, and H. Prodinger. Subblock occurrences in signed digit representations. *Glasgow Journal of Mathematics*, 45:427–440, 2003.
- [188] A. Panholzer and H. Prodinger. A generating functions proof of a curious identity. *Integers*, 2:A06, 3 pages, 2002.
- [189] H. Prodinger. Digits and beyond. In B. Chauvin, P. Flajolet, D. Gardy, and A. Mokkaedem, editors, *Mathematics and Computer Science II*, pages 355–377. Birkhaeuser Verlag, 2002.
- [190] H. Prodinger and B. Vallee (eds.). Special issue on analysis of algorithms (preface 1–3). *Journal of Algorithms*, 44:1–285, 2002.
- [191] F. Bassino and H. Prodinger. (q, δ) -numeration systems with missing digits. *Monatshefte für Mathematik*, 141:89–99, 2004.
- [192] A. Panholzer, H. Prodinger, and M. Riedel. Permuting in place: analysis of two stopping rules. *Journal of Algorithms*, 51:170–184, 2004.
- [193] G. Louchard and H. Prodinger. Inversions in permutations: A saddle point analysis. *Journal of integer sequences*, 6:article 3.2.8, 2002.
- [194] K. Morris, A. Panholzer, and H. Prodinger. Some parameters in heap ordered trees. *Combinatorics, Probability, and Computing*, 13:677–696, 2004.
- [195] H. Prodinger. Returns to origin for random walks on \mathbb{Z} revisited. *Missouri Journal of Mathematical Sciences*, 17:140–142, 2005.
- [196] P. Paule and H. Prodinger. Fountains, histograms, and q -identities. *Discrete Mathematics and Theoretical Computer Science*, 6:101–106, 2003.

- [197] K. Driver, H. Prodinger, and C. Schneider and André Weideman. Padé approximations to the logarithm III: Alternative methods and additional results. *Ramanujan Journal*, 12:299–314, 2006.
- [198] P. Grabner, C. Heuberger, and H. Prodinger. Distribution results for low-weight binary representations for pairs of integers. *Theoretical Computer Science*, 319:307–331, 2004.
- [199] C. Brennan and H. Prodinger. The pills problem revisited. *Quaestiones Mathematicæ*, 26:427–439, 2003.
- [200] H. Prodinger. A curious identity proved by Cauchy’s integral formula. *Mathematical Gazette*, 89:266–267, 2005.
- [201] H. Prodinger. The kernel method: a collection of examples. *Séminaire Lotharingien de Combinatoire*, B50f:19 pages, 2004.
- [202] P. Grabner, C. Heuberger, H. Prodinger, and J. Thuswaldner. Analysis of linear combination algorithms in cryptography. *Transactions on Algorithms*, 1:123–142, 2005.
- [203] H. Prodinger. Compositions and Patricia tries: no fluctuations in the variance! *Proceedings of the 6th Workshop on ALENEX and the 1st workshop on ANALCO*, L. Arge, G. Italiano, R. Sedgewick (editors), pages 211–215, 2004.
- [204] A. Panholzer and H. Prodinger. Analysis of some statistics for increasing tree families. *Discrete Mathematics and Theoretical Computer Science*, 6:437–460, 2004.
- [205] M. Archibald, A. Knopfmacher, and H. Prodinger. The number of values in a geometrically distributed sample. *European Journal of Combinatorics*, 27:1059–1081, 2006.
- [206] H. Prodinger. Periodic oscillations in the analysis of algorithms—and their cancellations. *Journal of the Iranian Statistical Society*, 3:251–270, 2004.
- [207] C. Heuberger and H. Prodinger. Analysis of alternative digit sets for nonadjacent representations. *Monatshefte für Mathematik*, 147:219–248, 2006.
- [208] A. Knopfmacher and H. Prodinger. The number of descents in samples of geometric random variables. In M. Drmota, P. Flajolet, D. Gardy, and B. Gittenberger, editors, *Mathematics and Computer Science III*, pages 339–350. Birkhaeuser Verlag, 2004.
- [209] A. Knopfmacher and H. Prodinger. The first descent in samples of geometric random variables and permutations. *Discrete Mathematics and Theoretical Computer Science*, 8:214–234, 2006.
- [210] H. Prodinger. q -identities of Fu and Lascoux proved by the q -Rice formula. *Quaestiones Mathematicæ*, 27:391–395, 2004.
- [211] L.L.Cristea and H. Prodinger. Moments of distributions related to digital expansions. *Journal of Mathematical Analysis and Applications*, 315:606–625, 2006.
- [212] C. Heuberger, R. Katti, H. Prodinger, and X. Ruan. The alternating greedy expansion and applications to left-to-right algorithms in cryptography. *Theoretical Computer Science*, 341:55–72, 2005.
- [213] A. Knopfmacher and H. Prodinger. The last descent in samples of geometric random variables and permutations. *Ars Combinatoria*, 83:365–379, 2007.
- [214] H. Prodinger. More terms of the asymptotic solution of a difference equation considered by Ramanujan. *The Australian Mathematical Gazette*, 31:340–341, 2004.
- [215] G. Louchard and H. Prodinger. Asymptotics of the moments of extreme-value related distribution functions. *Algorithmica*, 46:431–467, 2006.
- [216] H. Prodinger. Two q -identities from the theory of fountains and histograms proved a tri-diagonal determinant. *INTEGERS*, 5:A03 (3 pages), 2005.
- [217] R. Avanzi, C. Heuberger, and H. Prodinger. Scalar multiplication on Koblitz curves using the Frobenius endomorphism and its combination with point halving: Extensions and mathematical analysis. *Algorithmica*, 46:249–270, 2005.
- [218] A. Panholzer and H. Prodinger.
- [219] P. Grabner, C. Heuberger, and H. Prodinger. Counting optimal joint expansions. *INTEGERS*, 5(3):19 pages, 2005.
- [220] M. Drmota and H. Prodinger. The register function for t -ary trees. *Transactions on Algorithms*, 2:318–334, 2006.
- [221] P. Flajolet, M. Nebel, and H. Prodinger. The scientific works of Rainer Kemp (1949–2004). *Theoretical Computer Science*, 355:371–381, 2006.
- [222] G. Louchard, H. Prodinger, and M. Ward. The number of distinct values of some multiplicity in sequences of geometrically distributed random variables. *Discrete Mathematics and Theoretical Computer Science*, AD:231–256, 2005.
- [223] L.L.Cristea and H. Prodinger. Order statistics for the Cantor-Fibonacci distribution. *Aequationes mathematicæ*, 37:78–91, 2007.
- [224] A. Panholzer and H. Prodinger. The level of nodes in increasing trees revisited. *Random Structures and Algorithms*, 31:203–226, 2007.
- [225] G. Louchard and H. Prodinger. The number of gaps in sequences of geometrically distributed random variables. *dm*, 308:1538–1562, 2008.
- [226] R. Avanzi, C. Heuberger, and H. Prodinger. Minimality of the Hamming weight of the τ -NAF for Koblitz curves and improved combination with point halving. *SAC conference, Lecture Notes in Computer Science*, 3897:332–344, 2006.
- [227] H. Prodinger. q -enumeration of Salié permutations. *Annals of Combinatorics*, 11:213–225, 2007.

-
- [228] G. Louchard and H. Prodinger. The number of elements close to near-records in geometric samples. *Quaestiones Mathematicae*, 29:447–470, 2006.
- [229] C. Heuberger and H. Prodinger. On α -greedy expansions of numbers. *Advances in Applied Mathematics*, 38:505–525, 2007.
- [230] H. Prodinger. d -records in geometrically distributed random variables. *Discrete Mathematics and Theoretical Computer Science*, 8:273–284, 2006.
- [231] M. Kuba, H. Prodinger, and C. Schneider. Generalized reciprocity laws for sums of harmonic numbers. *INTEGERS*, 8:A17, 2008.
- [232] P. Grabner and H. Prodinger. Secant and cosecant sums and Bernoulli-Nörlund polynomials. *Quaestiones Mathematicae*, 30:159–165, 2007.
- [233] G. Louchard and H. Prodinger. Generalized approximate counting revisited. *Theoretical Computer Science*, 391:109–125, 2008.
- [234] G. Louchard and H. Prodinger. Analysis of a new skip list variant. *Discrete Mathematics and Theoretical Computer Science*, proc. AG:365–374, 2006.
- [235] R. Avanzi, C. Heuberger, and H. Prodinger. On redundant τ -adic expansions and non-adjacent digit sets. *SAC conference, Lecture Notes in Computer Science*, 4356:285–301, 2007.
- [236] G. Louchard and H. Prodinger. The asymmetric leader election algorithm: another approach. 12:449–478, 2009.
- [237] G. Louchard and H. Prodinger. Advancing in the presence of a demon. *Mathematica Slovaca*, 58:263–276, 2008.
- [238] C. Heuberger and H. Prodinger. Hamming weight of the non-adjacent form under various input statistics. *Period. Math. Hungar.*, 55:81–96, 2007.
- [239] G. Louchard and H. Prodinger. A combinatorial and probabilistic study of initial and end heights of descents in samples of geometrically distributed random variables and in permutations. *Discrete Mathematics and Theoretical Computer Science*, 9:137–170, 2007.
- [240] H. Prodinger and S. Wagner. Minimal and maximal plateau lengths in Motzkin paths. *Discrete Mathematics and Theoretical Computer Science*, proc. AH:353–362, 2007.
- [241] H. Prodinger. Identities involving rational sums by inversion and partial fraction decomposition. *Applicable Analysis and Discrete Mathematics*, 2:65–68, 2008.
- [242] H. Prodinger. Human proofs of identities by Osburn and Schneider. *INTEGERS*, 8:A10, 2008.
- [243] H. Prodinger. On the order of the recursion relation of Motzkin numbers of higher rank. *The open applied mathematical journal*, 2:14–15, 2008.
- [244] H. Prodinger. A continued fraction expansion for a q -tangent function: An elementary proof. *Seminaire Lotharingien de Combinatoire*, 60:B60b, 2008.
- [245] H. Prodinger. Generating functions related to partition formulae for Fibonacci numbers. *Journal of Integer Sequences*, 11:Article 08.1.8, 2008.
- [246] H. Prodinger. Records in geometrically distributed words: sum of positions. *Applicable Analysis and Discrete Mathematics*, 2:234–240, 2008.
- [247] G. Louchard and H. Prodinger. The register function for lattice paths. *Discrete Mathematics and Theoretical Computer Science*, AG:139–152, 2008.
- [248] C. Martinez, A. Panholzer, and H. Prodinger. Generating random derangements. *5th ACM-SIAM Workshop on Analytic Algorithmics and Combinatorics (ANALCO)*, AG:234–240, 2008.
- [249] G. Louchard, H. Prodinger, and S. Wagner. Joint distributions for movements of elements in Sattolo’s and Fisher-Yates’ algorithm. *Quaestiones Mathematicae*, 31:307–344, 2008.
- [250] P. Grabner and H. Prodinger. On a constant arising in the analysis of bit comparisons in quickselect. *Quaestiones Mathematicae*, 31:303–306, 2008.
- [251] C. Heuberger, H. Prodinger, and S. Wagner. Positional number systems with digits forming an arithmetic progression. *Monatshefte für Mathematik*, 155:349–375, 2008.
- [252] H. Prodinger. The location of the first maximum in the first sojourn of a dyck path. *Discrete Mathematics and Theoretical Computer Science*, 10:125–134, 2008.
- [253] H. Prodinger. On the expansion of Fibonacci and Lucas polynomials. *Journal of Integer Sequences*, 12:Article 09.1.6, 2009.
- [254] H. Prodinger. A simple bijection between a subclass of 2-binary trees and ternary trees. *Discrete Mathematics*, 309:959–961, 2009.
- [255] A. Panholzer and H. Prodinger. Bijections between certain families of labelled and unlabelled d -ary trees. *Appl. Anal. Discrete Math.*, 309:123–136, 2009.
- [256] N. Gu and H. Prodinger. Bijections for 2-plane trees and ternary trees. *European J. of Combinatorics*, 30:969–985, 2009.
- [257] A. Panholzer and H. Prodinger. Foreword for special issue in annals of combinatorics. *Annals of Combinatorics*, 12:353–355, 2009.
- [258] G. Louchard and H. Prodinger. The asymmetric leader election algorithm: another approach. *Annals of Combinatorics*, 12:449–478, 2009.
- [259] H. Prodinger and S. Wagner. The digit generating function of a polynomial. *Journal of Number Theory*, 129:1325–1337, 2009.

-
- [260] C. Martinez and H. Prodinger. Moves and displacements of particular elements in quicksort. *Theoretical Computer Science*, 410:2279–2284, 2009.
- [261] L.L.Cristea and H. Prodinger. q -enumeration of up-down words by number of rises. *Fibonacci Quarterly*, 46/47:126–134, 2008/2009.
- [262] M. Dromota, B. Gittenberger, A. Panholzer, H. Prodinger, and M.D.Ward. On the shape of the fringe of various types of random trees. *Mathematical Methods in the Applied Sciences*, 32:1207–1245, 2009.
- [263] G. Louchard and H. Prodinger. Representations of numbers as $\sum_{k=-n}^n \varepsilon_k k$: A saddle point approach. *Lecture Notes in Computer Science*, 5489:87–96, 2010.
- [264] H. Prodinger. Identities involving harmonic numbers that are of interest for physicists. *Utilitas Mathematica*, 83:291–299, 2010.
- [265] H. Prodinger. Sums of Choi, Zoernig, and Rathie – an elementary approach. *Quaestiones Mathematicae*, 32:265–267, 2009.
- [266] C. Heuberger and H. Prodinger. Analysis of complements in multi-exponentiation algorithms using signed digit representations. *Internat. J. Found. Comput. Sci.*, 20:443–453, 2009.
- [267] Markus Kuba, A. Panholzer, and H. Prodinger. Lattice paths, sampling without replacement, and limiting distributions. *Electronic Journal of Combinatorics*, R67:12 pages, 2009.
- [268] A. Panholzer and H. Prodinger. A short proof of a series evaluation in terms of harmonic numbers. *INTEGERS*, 9:491–496, 2009.
- [269] H. Prodinger. On a sum of Melham and its variants. *Fibonacci Quarterly*, 46/47:207–215, 2008/2009.
- [270] M. Kuba and H. Prodinger. A note on Stirling sums. *INTEGERS*, 2009.
- [271] M. Kuba and H. Prodinger. On a reciprocity law for finite multiple zeta values. *International Journal of Combinatorics*, 2010:Article ID 153621, 13 pages, 2010.
- [272] G. Louchard and H. Prodinger. Asymptotic results for silent elimination. *Discrete Mathematics and Theoretical Computer Science*, 12:185–196, 2010.
- [273] N.S.S. Gu, H. Prodinger, and S. Wagner. Bijections for a class of labeled plane trees. *European Journal of Combinatorics*, 31:720–732, 2010.
- [274] A. Knopfmacher, A. Munagi, H. Prodinger, and Toufik Mansour. Staircase words and Chebyshev polynomials. *Appl. Anal. Discrete Math.*, 4:81–95, 2010.
- [275] N.S.S. Gu and H. Prodinger. One-parameter generalizations of Rogers-Ramanujan type identities. *Advances in Applied Mathematics*, 45:149–196, 2010.
- [276] Emrah Kilic, Ilker Akkus, and H. Prodinger. A proof of Melham’s conjecture. *Fibonacci Quarterly*, 48:241–248, 2010.
- [277] Emrah Kilic and H. Prodinger. A generalized Filbert matrix. *Fibonacci Quarterly*, 48:29–33, 2010.
- [278] C. Heuberger and H. Prodinger. A precise description of the p -adic valuation of the number of alternating sign matrices. *Int. J. Number Theory (IJNT)*, 7:57–69, 2011.
- [279] H. Prodinger. Infinitely many identities for sums of Jacobsthal-Lucas numbers. *Journal of Algebra, Number Theory: Advances and Applications*, 4:49–55, 2010.
- [280] R. Avanzi, C. Heuberger, and H. Prodinger. Redundant τ -adic expansions I: non-adjacent digit sets and their applications to scalar multiplication. *Des. Codes Cryptogr.*, 58:173–202, 2011.
- [281] E. Kilic, I. Akkus, and H. Prodinger. A proof of Melham’s conjecture. *Fibonacci Quarterly*, 48:241–248, 2010.
- [282] K. Oliver and H. Prodinger. Consecutive records in geometrically distributed words. *Afrika Matematika*, 23:163–172, 2012.
- [283] C. Martinez, A. Panholzer, and H. Prodinger. The analysis of approximate quickselect and related problems. *Theoretical Computer Science*, 412:6537–6555, 2011.
- [284] L. L. Cristea and H. Prodinger. q -enumeration of words by their total variation. *Discrete Mathematics and Theoretical Computer Science*, pages 139–150, 2010.
- [285] G. Louchard, H. Prodinger, and M. Ward. Number of survivors in the presence of a demon. *Publ. Math. Hung.*, 64:101–117, 2012.
- [286] H. Prodinger. A generalization of Filbert matrix with 3 additional parameters. *Transactions of the Royal Society of South Africa*, 65:169–172, 2010.
- [287] E. Kilic, I. Akkus, and H. Prodinger. A generalization of a conjecture of Melham. *Utilitas Mathematica*, 94:225232, 2014.
- [288] E. Kilic and H. Prodinger. Some double binomial sums related to Fibonacci, Pell and generalized order- k Fibonacci numbers. *Rocky Mountains Journal of Mathematics*, 43:975–987, 2013.
- [289] N. S. S. Gu and H. Prodinger. On some continued fraction expansions of the Rogers-Ramanujan type. *Ramanujan Journal*, 26:323–367, 2011.
- [290] H. Prodinger. The intrinsic periodic behaviour of sequences related to a rational integral. *INTEGERS*, pages A14, 6 pages, 2010.
- [291] H. Prodinger, C. Schneider, and S. Wagner. Unfair permutations. *European Journal of Combinatorics*, 32:1282–1298, 2011.
- [292] H. Prodinger. Sums of powers of Fibonacci polynomials. *Proceedings of the Indian Academy of Sciences Mathematical Sciences*, 119:567–570, 2009.

-
- [293] H. Prodinger. Continued fraction expansions for q -tangent and q -cotangent functions. *Discrete Mathematics and Theoretical Computer Science*, 12:47–64, 2010.
- [294] R. Avanzi, C. Heuberger, and H. Prodinger. Arithmetic of supersingular Koblitz curves in characteristic three. 2014.
- [295] G. Louchard, C. Martinez, and H. Prodinger. The swedish leader election protocol: Analysis and variations. *Analco11*, 2011 Proceedings of the Eighth Workshop on Analytic Algorithmics and Combinatorics (ANALCO), SIAM Proceedings:127–134, 2011.
- [296] K. Oliver and H. Prodinger. Words representing set partitions. *Applicable Analysis and Discrete Mathematics*, 5:55–59, 2011.
- [297] H. Prodinger. Dyck paths with parity restrictions for the final runs to the origin: a study of the height. *Fundamenta Informaticae*, 117:279–285, 2012.
- [298] E. Kilic and H. Prodinger. The q -Pilbert matrix. *International Journal of Computer Mathematics*, 89:1370–1377, 2012 (10).
- [299] K. Oliver and H. Prodinger. How often do we reject a superior value? extended abstract. *Discrete Mathematics and Theoretical Computer Science*, proc. AO:741–752, 2011.
- [300] K. Oliver and H. Prodinger. How often do we reject a superior value. *Transactions of the Royal Society of South Africa*, 66:79–86, 2011.
- [301] G. Louchard and H. Prodinger. The Catalan distribution: A saddle point approach. *Annals of Combinatorics*, 15:313–329, 2011.
- [302] H. Prodinger. Continued fractions related to (t, q) -tangents and variants. *Electronic Journal of Combinatorics*, 18 (2):P18 (7 pages), 2011.
- [303] H. Prodinger. Digital search trees with m trees: Level polynomials and insertion costs. *Discrete Mathematics and Theoretical Computer Science*, 13 (3):1–8, 2011.
- [304] E. Kilic, H. Prodinger, I. Akkus, and H. Ohtsuka. Formulas for Fibonomial sums with generalized Fibonacci and Lucas coefficients. *The Fibonacci Quarterly*, 49:32–39, 2011.
- [305] R. Du and H. Prodinger. Notes on protected nodes in digital search trees. *Applied Mathematics Letters*, 25 (6):1025–1028, 2012.
- [306] H. Prodinger. Pseudo q -Engel expansions and Rogers–Ramanujan type identities. *Quaestiones Mathematicae*, 35:23–33, 2012.
- [307] H. Prodinger. Approximate counting with m counters: a detailed analysis. *Theoretical Computer Science*, 439:58–68, 2012.
- [308] A. Panholzer and H. Prodinger. Asymptotic results for the number of paths in a grid. *Bulletin of the Australian Mathematical Society*, 85:446–455, 2012.
- [309] P. Grabner and H. Prodinger. Additively irreducibles in alpha-expansions. *Publicationes Mathematicae Debrecen*, 80:405–415, 2012.
- [310] M. Fuchs, C.-K. Lee, and H. Prodinger. Approximate counting via the Poisson-Laplace-Mellin method. *Discrete Mathematics and Theoretical Computer Science*, proc. AQ:13–28, 2012.
- [311] H. Prodinger. Set partitions, words, and approximate counting with black holes. *Australasian Journal of combinatorics*, 54:303–310, 2012.
- [312] G. Louchard and H. Prodinger. The asymmetric leader election algorithm with swedish stopping: A probabilistic analysis. *Discrete Mathematics and Theoretical Computer Science*, 14 (2):91–128, 2012.
- [313] H. Prodinger. The number of restricted lattice paths revisited. *Filomat*, 26:1130–1131, 2012.
- [314] H. Prodinger. On touchard’s continued fraction and extensions: combinatorics-free, self-contained proofs. *Quaestiones Mathematicae*, 35:431–445, 2012.
- [315] K. Oliver and H. Prodinger. The continued fraction expansion of Gauss’ hypergeometric function and a new application to the tangent function. *Transactions of the Royal Society of South Africa*, 67:151–154, 2012.
- [316] H. Prodinger. On the m -enumeration of maximum and left-to-right maxima in geometrically distributed words. *P.U. M. A.*, 23:143–154, 2012.
- [317] L.-L. Cristea and H. Prodinger. The visibility parameter for words and permutations. *Central European Journal of Mathematics*, 11:283–295, 2013.
- [318] C. Elsholtz, C. Heuberger, and H. Prodinger. The number of huffman codes, compact trees, and sums of unit fractions. *IEEE Transactions on Information Theory*, 59 (2):1065–1075, 2013.
- [319] G. Louchard and H. Prodinger. The largest missing value in a composition of an integer and some Allouche-Shallit-type identities. *Journal of Integer Sequences*, Article 13.2.2:16 pages, 2013.
- [320] H. Prodinger. The m -version of binary search trees: an average-case analysis. *ISRN Journal of Combinatorics*, Article ID 450627:8 pages, 2013.
- [321] E. Kilic and H. Prodinger. Variants of the Filbert matrix. *Fibonacci Quarterly*, 51:153–162, 2013.
- [322] H. Prodinger. An identity conjectured by Lacasse via the tree function. *Electronic Journal of Combinatorics*, 20 (3):P7 (3 pages), 2013.
- [323] K. Oliver and H. Prodinger. Continued fractions related to Goellnitz’ little partition theorem. *Afrika Matematica*, 24:665–670, 2013.
- [324] M. Fuchs and H. Prodinger. Words with a generalized restricted growth property. *Indagationes Mathematicae*, 24:1024–1033, 2013.

-
- [325] H. Prodinger. The box parameter for words and permutations. *Central European Journal of Mathematics*, 12(1):167–174, 2014.
- [326] E. Kilic and H. Prodinger. Some gaussian binomial sum formulae with applications. 2014.
- [327] H. Prodinger. A short proof of Carlitz’s identity about Bernoulli numbers. *Journal of Integer Sequences*, 17:14.4.1, 2014.
- [328] K. Hare, H. Prodinger, and Shallit. Three series for the generalized golden mean. 2014.
- [329] L. L. Cristea and H. Prodinger. Order statistics of the generalised multinomial measure. *Monatshefte für Mathematik*, 175:333–346, 2014.
- [330] H. Prodinger. Mortenson’s identities and partial fraction decomposition. *Utilitas Mathematica*, 2014.
- [331] K. Oliver and H. Prodinger. The inverse Filbert matrix. *Afrika Mathematica*, 2014.
- [332] K. Oliver and H. Prodinger. Depth and path length of m -plane oriented recursive trees. 2014.
- [333] E. Kilic and H. Prodinger. Formulae related to the q -Dixon formula with applications to Fibonomial sums. *Periodica Mathematica Hungarica*, 2014.
- [334] E. Kilic and H. Prodinger. Closed form evaluation of sums containing squares of Fibonomial coefficients. *Mathematica Slovaca*, 2014.
- [335] E. Kilic and H. Prodinger. Asymmetric generalizations of the Filbert matrix and variants. *Publications de l’Institut Mathématique*, 95:267–280, 2014.
- [336] E. Kilic and H. Prodinger. The generalized q -Pilbert matrix. *Math. Slovaca*, 2014.
- [337] G. Louchard and H. Prodinger. Approximate counting with m counters: a probabilistic analysis. 2014.
- [338] H. Prodinger. Introduction to Philippe Flajolet’s work on the register function and related topics. 2014.
- [339] E. Kilic and H. Prodinger. Automatic evaluation of sums of products of generalized Fibonacci and Lucas numbers and new identities. *Periodica Mathematica Hungarica*, 2014.
- [340] N. S. S. Gu and H. Prodinger. Partial fraction decomposition proofs of some q -series identities. *JANTAA*, 10:1–27, 2013.
- [341] S. Wagner and H. Prodinger. Bootstrapping and Gumbel limit laws. 2014.
- [342] H. Prodinger. Identities of Gonzalez proved by a power series approach. *Pioneer Journal of Advances in Applied Mathematics*, 9:29–36, 2013.
- [343] E. Kilic and H. Prodinger. A note on the conjecture of Ramirez and Sirvent. *Journal of Integer Sequences*, 14:14.5.8, 2014.
- [344] H. Prodinger. Counting palindromes according to r -runs of ones using generating functions. *Journal of Integer Sequences*, 14:14.6.2, 2014.
- [345] R. Tauraso and H. Prodinger. New multiple harmonic sum identities. *Electronic Journal of Combinatorics*, 21(2):P2.43, 2014.
- [346] K. Oliver and H. Prodinger. Horizontal runs in domino tilings. *Journal of Algebra Combinatorics Discrete Structures and Applications*, 1:19–27, 2014.