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An Analogue of the Andre-Oort Conjecture for products of Drinfeld Modular Surfaces

by

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ABSTRACT

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Doctoral Promoter: Prof F. Breuer

Doctoral CoPromoter: Dr. A.P. Keet

This thesis deals with a function field analog of the André-Oort conjecture. The (classical) André-Oort conjecture concerns the distribution of special points on Shimura varieties. In our case we consider the André-Oort conjecture for special points in the product of Drinfeld modular varieties. We in particular manage to prove the André-Oort conjecture for subvarieties in a product of two Drinfeld modular surfaces under a characteristic assumption.

Uittreksel

'n Analoog van die André-Oort Vermoeding vir 'n produkt van twee Drinfeldse modulvalakke

deur

Archiebold Karumbidza

Doktorale Promotor: Professor F. Breuer

Doktorale MedePromotor: Dr. A.P. Keet

Hierdie tesis handel van 'n funksieliggaam analoog van die André-Oort Vermoeding. Die (Klassieke) André-Oort Vermoeding het betrekking tot die verspreiding van spesiale punte op Shimura varietiete. Ons geval beskou ons die André-Oort Vermoeding vir spesiale punte op die produk Drinfeldse modulvarietiete. In die besonders, bewys ons die André-Oort Vermoeding vir ondervarieteite van 'n produk van twee Drinfeldse modulvarietiete, onderhewig aan 'n karakteristiek-aanname.

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I dedicate this thesis to my parents who have always supported me unconditionally and wished me success at every step.

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CHAPTER I

Introduction

This thesis deals with a function field analogue of the André-Oort conjecture. In our particular case we consider the André-Oort conjecture for subvarieties in the product of Drinfeld modular varieties. In this direction we prove some instances of this characteristic p analog of the André-Oort conjecture. The classical André-Oort conjecture concerns subvarieties of Shimura varieties and we will first review this. We however do not attempt to give all the definitions involved in formulating the classical André-Oort conjecture, referring the reader instead to the ample body of literature that deals with this. We particularly recommend Noot's Bourbaki talk as a starting point (see [37]).

1.1 Classical André-Oort conjecture

Motivated by the Manin-Mumford conjecture (see [46],[47],[51], [16],[32]) Frans Oort and Yves André (for curves) independently observed that:

If V is a Hodge subvariety in the moduli space of principally polarized g-dimensional abelian varieties $A_g \otimes \mathbb{C}$, then V contains a Zariski dense set of special points, i.e., points corresponding to abelian varieties with large endomorphism rings. It is then natural to wonder if the converse might be true. This motivates the following conjecture.

Conjecture 1.1.1 (André-Oort)**.** *(With the same notation as above). Let $V \subset A_g \otimes \mathbb{C}$ be a subvariety of $A_g \otimes \mathbb{C}$ which contains a Zariski dense set of CM points, then V is of Hodge type.*

To be precise, Yves André first stated this conjecture for curves containing infinitely many special points in a general Shimura variety as a question in his book (see [2]). Furthermore in [1] he mentions the similarity of this conjecture with the Manin-Mumford conjecture. He in fact proposes (see [1]) a version generalising both the André-Oort and the Manin-Mumford conjecture. Independently, Oort (see [38]) raised the question for general subvarieties in the moduli space of principally polarized abelian varieties. From this it is natural to leap to the following generalisation.

Conjecture 1.1.2 (André-Oort)**.** *Let V be a subvariety of the Shimura variety $\mathrm{Sh}_K(G, X)(\mathbb{C})$ such that V contains a Zariski dense set of special points. Then V is a Hodge subvariety.*

That one direction of this statement is true follows from the well known fact that starting with a special point in a Shimura variety then the Hecke orbit of this point is Zariski dense (even analytically dense in the complex topology) in an irreducible component of the Shimura variety (see for instance [22]). Thus it is the converse that has ellicited more than a decade of work.

Remark 1.1.3. *The André-Oort conjecture has been vastly generalised by Pink (see [44]) to mixed Shimura varieties to give a conjecture combining the Mordell-Lang conjecture with an important special case of the André-Oort conjecture.*

In the absence of technical definitions, the following might clarify the meaning of the conjecture. The irreducible Shimura subvarieties of Hodge type are irreducible components of sub-Shimura varieties or translates of such by a Hecke correspondence.

To be precise let (G, X) be a Shimura datum and let K be a compact open subgroup of $G(\mathbb{A}_f)$. Then $V \subset \mathrm{Sh}_K(G, X)(\mathbb{C})$ is a Hodge subvariety if there exists a Shimura subdatum (G', X'), a morphism of Shimura data $f : (G', X') \to (G, X)$ and $g \in G(\mathbb{A}_f)$, such that V is an irreducible component of the image of the map

$$\mathrm{Sh}(G', X')(\mathbb{C}) \xrightarrow{f} \mathrm{Sh}(G, X)(\mathbb{C}) \xrightarrow{\cdot g} \mathrm{Sh}(G, X)(\mathbb{C}) \to \mathrm{Sh}_K(G, X)(\mathbb{C})$$

The special points then are the Hodge subvarieties of zero dimension.

In the simplest case $A_1 \otimes \mathbb{C} \cong \mathbb{C}$ is the coarse moduli space of elliptic curves and Hodge subvarieties here correspond to elliptic curves with complex multiplication. The André-Oort conjecture is trivial in this case, but already nontrivial for then next simplest moduli. Indeed let K be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$ and let V be an irreducible subvariety of the Shimura variety $\mathrm{Sh}_K(\mathrm{GL}_2 \times \mathrm{GL}_2, X)(\mathbb{C})$. The Shimura variety $\mathrm{Sh}_K(\mathrm{GL}_2 \times \mathrm{GL}_2, X)(\mathbb{C})$ is none other than $(A_1 \otimes \mathbb{C}) \times (A_1 \otimes \mathbb{C}) \cong \mathbb{C} \times \mathbb{C}$ and the Hodge subvarieties in this case are simply

- $\mathbb{C} \times \{\text{CM point}\}$ or $\{\text{CM point}\} \times \mathbb{C}$ and
- graphs of Hecke correpondences in $\mathbb{C} \times \mathbb{C}$ i.e. images of $\mathbb{C}$ in $\mathbb{C} \times \mathbb{C}$ via the map $x \to (g_1 \cdot x, g_2 \cdot x)$ with $(g_1, g_2) \in \mathrm{GL}_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{A}_f)$.

These varieties are also alternatively called the modular varieties. We have the following theorem independently due to Bas Edixhoven (see [21]) and Yves André (see [3]). Yves André proof is unconditional and does not assume the Generalized Riemann Hypothesis (GRH).

Theorem 1.1.4. *Let $V \subset \mathbb{C} \times \mathbb{C}$ be an irreducible algebraic variety. Then V contains a Zariski-dense set of CM points if and only if V is a modular variety .*

In general if one assumes the Generalized Riemann Hypothesis (GRH) for CM fields then the André-Oort conjecture i.e. conjecture 1.1.2 above, is now a theorem thanks to the work of Klingler, Ullmo and Yafaev (see [4],[56]), building on the efforts and collaboration of many people. The chronological progress being as follows:

In his thesis [35], Ben Moonen proved that the conjecture is true for subvarieties $V \subset A_g \otimes \mathbb{C}$ for which there exists a prime number p at which a Zariski dense set of CM points of V have an ordinary reduction of which they are the canonical lift.

Edixhoven [21] and André [3], then proved that the conjecture holds for the moduli space of pairs of elliptic curves. Edixhoven has also generalised this result to arbitrary products of modular curves [23]. Yves Andre's proof is unconditional and uses the Galois action on the CM-points, and a Diophantine approximation result of Masser on the j-function. Edixhoven's proof on the other hand depends on the validity of the Generalised Riemann Hypothesis (GRH) for imaginary quadratic fields.

Yafaev then proved (generalising the result of Edixhoven [21]) the conjecture for curves in the products of two Shimura curves associated to quaternion algebras over $\mathbb{Q}$ (see [57]).

Edixhoven and Yafaev [24] proved the conjecture for curves in Shimura varieties assuming that all the Hodge structures of a Zariski dense set of CM points on the curves belong to the same isomorphism class.

Clozel and Ullmo [14], prove that the conjecture is true for subvarieties of $\mathrm{Sh}_K(G, X)(\mathbb{C})$ with G among GSp_{2n} and GL_n, such that sets of the form $T_p x$, with x in $G(\mathbb{Q})\backslash G(\mathbb{A})/K$ where T_p are certain Hecke operators with p tending to infinity are equidistributed.

Edixhoven proves the conjecture for Hilbert modular surfaces assuming GRH, (see [22]). Assuming the Generalised Riemann Hypothesis (GRH) for CM fields in [58] Yafaev extends the result of Moonen to Shimura varieties. Yafaev [59] then also proves the conjecture for C an irreducible closed algebraic curve contained in the Shimura variety $\mathrm{Sh}_K(G, X)$ such that C contains an infinite set of special points, this is the original conjecture of André.

Clozel and Ullmo [15] proved the conjecture for strongly special subvarieties of a Shimura variety using ergodic theoretic methods. This was extended to T-special subvarietes by Yafaev and Ullmo (see [56]). Combining the ergodic results of Ullmo with the Edixhoven-Yafaev strategy Klingler and Yafaev [4] finally settled the André-Oort conjecture, though assuming the Generalised Riemann Hypothesis (GRH) for CM fields. As we alluded to before, the André-Oort conjecture bears remarkable structural similarities with the Manin-Mumford conjecture, which was proved by Raynaud (see [46],[47]).

Conjecture 1.1.5 (Manin, Mumford)**.** *Let A be an abelian variety and let V be an irreducible closed subvariety of A. Then $V \subset A$ contains a Zariski dense set of torsion points if and only if $V = a + B$, with $a \in A_{\mathrm{tor}}$ and $B \subset A$ an abelian subvariety.*

More recently there has been spectacular progress on André-Oort type conjectures using methods from logic, more precisely model theory. Pila [39] using methods from his joint work with Wilkie [40] has managed to prove the André-Oort conjecture for arbitrary products of the complex plane without assuming the Generalised Riemann Hypothesis (GRH) for CM fields. His further work with Tsimmerman proves the André-Oort conjecture unconditionally for Shimura varieties of dimension up to six. These results have put model theory square in the field of Arithmetic. Here the main input from model theory is a counting theorem for the the number of rational points on non semialgebraic sets. For a recent exposition of these ideas we refer the reader to [50] .

We can make the following analogy between the André-Oort and the Manin-Mumford conjectures:

The Shimura varieties correspond to abelian varieties, while the special points correspond to torsion points and the subvarieties of Hodge type correspond to translates of abelian subvarieties by a torsion point. These structural similarities have in fact been some of the inspiring intuition leading to the final solution of the André-Oort conjecture by Klingler and Yafaev and in fact a uniform proof strategy can now be implemented for both conjectures in view of the aforementioned analogy (see [55]).

1.2 Drinfeld Analog of the André-Oort conjecture

In his seminal paper [20], Drinfeld introduced the notion of Drinfeld (elliptic) modules for the purpose of realising the Langlands correspondence over function fields. More precisely, to l-adic representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ arising from elliptic curves Deligne [19] attached automorphic representations of the adele group $\mathrm{GL}_r(\mathbb{A}_{\mathbb{Q}})$. Drinfeld modules of rank two were then introduced to serve as the characteristic p analogue of elliptic curves to transport Deligne's theory over to the function field setting. The above mentioned theory of Deligne crucially depends on the moduli of elliptic curves. Drinfeld therefore also constructs the moduli of Drinfeld modules with level structure. These moduli and their subvarieties are the subject of this thesis. Drinfeld moduli spaces are affine varieties over function fields and serve as the analogues of Shimura varietes. Drinfeld modules have turned out to be a powerful tool in the study of the arithmetic of function fields, indeed many of the concepts that arise in the theory of abelian varieties have analogues in the setting of Drinfeld modules.

In particular we have a notion of complex multiplication (CM) for Drinfeld modules. Having the notion of complex multiplication in hand we can consider Hodge subvarieties which are the locus in the moduli of Drinfeld modules of Drinfeld modules which have prescribed extra endomorphisms or Hecke translates of such. The irreducible components of zero dimensional Hodge subvarieties are precisely the CM points in the Drinfeld sense.

It is then natural to consider analogues of the André-Oort conjecture in the setting of function fields.

Conjecture 1.2.1 (André-Oort for Drinfeld Modules)**.** *Let $M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ denote the moduli scheme of rank r Drinfeld A-modules over $\mathbb{C}_\infty$ with $\mathcal{K}$-level structure. Let X be an algebraic subvariety contained in an arbitrary product $M_A^{r_1}(\mathcal{K})_{\mathbb{C}_\infty} \times \cdots \times M_A^{r_n}(\mathcal{K})_{\mathbb{C}_\infty}$ of Drinfeld moduli schemes. Denote by Σ a set of special points contained in X. Then Σ is Zariski-dense in X if and only if X is a Hodge (Modular) subvariety.*

The results in the function field case are mostly due to the efforts of Florian Breuer. In his thesis (see [7]) he proves the conjecture for subvarieties contained in the product of rank two Drinfeld moduli varieties over a rational function field of odd characteristic. This is a characteristic p analogue of the result of Edixhoven [23] (see theorem 1.1.4 above). Moreover he obtains an effective characterisation of modular subvarieties based of the height of the CM points.

The precise results of Breuer are as follows:

Theorem 1.2.2. *Assume that q is odd. Let d and m be given positive integers, and g a given non-negative integer. Then there exists an effectively computable constant $B = B(d, m, g)$ such that the following holds. Let X be an irreducible algebraic curve in $M_A^2(\mathcal{K})_{\mathbb{C}_\infty} \times M_A^2(\mathcal{K})_{\mathbb{C}_\infty} = \mathbb{A}^1(\mathbb{C}_\infty) \times \mathbb{A}^1(\mathbb{C}_\infty)$ of degree d, defined over a finite extension F of K of degree $[F : K] = m$ and genus g.*

Then X is a modular curve $Y_0(N)$ for some $N \in A$ or

$X = \{CM\ point\} \times \mathbb{A}^1(\mathbb{C}_\infty)$ or

$X = \mathbb{A}^1(\mathbb{C}_\infty) \times \{CM\ point\}$

if and only if $X(\mathbb{C}_\infty)$ contains a CM point of arithmetic height at least B.

Suppose that q is odd. Let $X \subset M_A^2(\mathcal{K})_{\mathbb{C}_\infty} \times \cdots \times M_A^2(\mathcal{K})_{\mathbb{C}_\infty}$ be an irreducible algebraic variety. Then $X(\mathbb{C}_\infty)$ contains a Zariski-dense subset of CM points if and only if X is a modular variety.

Subsequently Breuer [11] has managed to extend the above result to the case where A the ring of integers of "K" is non-rational but still of odd characteristic. In his paper [11] and in [9], he applies the above result to the study of Heegner points on elliptic curves over function fields, obtaining characteristic p analogues of results of Cornut (see [17]).

When the rank of the Drinfeld modules is arbitrary, Breuer in [8] has proved the conjecture for a curve lying in $M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$, thus completely for $M_A^3(\mathcal{K})_{\mathbb{C}_\infty}$. For an arbitrary subvariety X of $M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ such that X has a Zariski dense set of CM points that have canonical behaviour above a prime p, Breuer has proved the following result, which is an analog of Ben Moonen's theorem in the setting of Shimura varieties.

Theorem 1.2.3. *[Breuer [8]] Let $X \subset M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ be an irreducible algebraic subvariety. Let $\mathfrak{p} \subset A$ be a non-zero prime and let $m \in \mathbb{N}$. Suppose that X contains a Zariski-dense set Σ of CM points x for which $\mathfrak{p}$ is residual (see definition 3.3.3) in the quotient field of* $\mathrm{End}(\varphi^x)$ *and $\mathfrak{p}^m$ does not divide the conductor of* $\mathrm{End}(\varphi^x)$ *for all x. Then X is special. In particular, if Σ lies in one Hecke orbit then X is special.*

While this thesis was underway, Patrik Hubschmid in his recent thesis has proven the André-Oort conjecture for arbitrary subvarieites of $M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$, under a mild assumption on the field of definition of the CM points. The recent result of Hubschmid essentially completes the André-Oort conjecture for subvarieties of $M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$. His precise result is,

Theorem 1.2.4. *Let Σ be a set of special points in $M_A^r(\mathcal{K})_F$. Suppose that the reflex fields of all special points in Σ are separable over F. Then each irreducible component over $\mathbb{C}_\infty$ of the Zariski closure of Σ is a special subvariety of $M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$.*

We make the following remark about the André-Oort conjecture for Drinfeld modular varieties. The first thing to note is that the Riemann hypothesis is true for function fields. Hence results over functions fields tend to be unconditional except maybe on the characteristic. The function field setting is also relatively easier to understand that the number field case since the geometry is easier and we have to only deal with the group GL_r.

1.3 Results

In this thesis we prove the André-Oort conjecture for subvarieties in the product of two Drinfeld modular surfaces. The precise result states the following

Theorem 1.3.1. *Let X be an irreducible subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$. Suppose that X is defined over F and that the characteristic of F is not 3. Then $X(\mathbb{C}_\infty)$ contains a Zariski dense set of CM points if and only if X is a Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$.*

The strategy we adopt to prove the above results is that of Klingler, Ullmo and Yafaev. This strategy originates in Edixhoven's work on the André-Oort conjecture. In the Klingler-Ullmo-Yafaev strategy, there is a dichotomy between subvarieties with bounded Galois orbits and those with unbounded Galois orbits. Ergodic theorems apply to the former while Galois theoretic methods apply to the latter. The function field setting is not well suited for the ergodic approach. However Patrik Hubschmid has informed us that in the function field case, one realises that by using the degree of a Galois orbit of a Hodge subvariety instead of the naive cardinality one always has unbounded Galois orbits. Much of the background that goes into this thesis owes its existence to Florian Breuer's work on the subject (see [11],[7]), a collaboration with Richard Pink [12] and his paper [8] and some deep results of Pink on openness of compact subgroups of algebraic groups over local fields [42].

1.4 Organisation

The plan of the thesis is as follows:

In Chapter One we gather background material on Drinfeld modules and their moduli, this material mainly eminates from Drinfeld's seminal paper [20].

In Chapter Two we introduce the theory of complex multiplication for Drinfeld modules a la Hayes [31].

In Chapter Three we introduce the notion of a Hecke correpondence and prove the finiteness of a family of Hecke correspondences with bounded degree. We further more expose the theory of complex multiplication and Drifeld modular varieties.

In Chapter Four we state various useful results. To start, we give lower bounds for Galois orbits of Hodge subvarieties. We then determine the monodromy groups of a non-isotrivial family of Drinfeld modules on the product of two Drinfeld modular varieties. We also give results on irreducibility of Hecke correspondences and finally we state results on the degrees of Hecke correspondences. In most cases these results are adaptations of results due to Breuer [8], so that they apply to the case of a product of Drinfeld modular varieties

In chapter five we gather all the material from the previous chapters to prove instances of the André-Oort conjecture. In particular in our main result we prove the André-Oort conjecture for subvarieties in the product of two Modular surfaces under a mild characteristic assumption.

CHAPTER II

Drinfeld Modules

CHAPTER III

Hodge Subvarieties

3.1 Hecke Correspondences

Recall that for a general admissible level structure $\mathcal{K}$ the functor

$$M_A^r(\mathcal{K}) : \mathbf{Sch}\,/A \to \mathrm{Set} : S \longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of rank-}r \\ \text{Drinfeld } A\text{-modules over } S \\ \text{endowed with a } \mathcal{K}\text{-level structure} \end{array} \right\}.$$

is representable by the affine moduli scheme $M_A^r(\mathcal{K})$ over $\mathrm{Spec}(A)$. Drinfeld (see [20]) proved this only for $\mathcal{K} = \mathcal{K}(\mathfrak{n})$ with $\mathfrak{n}$ an admissible ideal of A. The extension to general admissible level structures can be found in Boeckle [5, thm 1.15]. An inclusion $\mathcal{K} \subset \mathcal{K}'$ of general level structures induces a functorial and canonical morphism $M_A^r(\mathcal{K}) \to M_A^r(\mathcal{K}')$ corresponding to the restriction of level structures. Taking the projective limit over these level structures we obtain a normal affine pro-scheme $M_A^r := \varprojlim_{\mathcal{K}} M_A^r(\mathcal{K})$ which is faithfully flat over $\mathrm{Spec}(A)$. $\mathrm{GL}_r(\mathbb{A}_f)$ naturally acts on M_A^r via multiplication by $g \in \mathrm{GL}_r(\mathbb{A}_f)$ (see [34], [28]) and this induces a correspondence T_g on $M_A^r(\mathcal{K})$. We call this the *Hecke correspondence* attached to g.

The graph of the correspondence T_g is the image of M_A^r in $M_A^r(\mathcal{K}) \times M_A^r(\mathcal{K})$ via the map $x \to (\pi(x), \pi \circ g(x))$. The diagram below summarises this situation.

$$\begin{tikzcd} M_A^r \arrow[r, "g"] \arrow[d, "\pi"'] & M_A^r \arrow[d, "\pi"] \\ M_A^r(\mathcal{K}) & T_g \arrow[l, dotted] \arrow[r, dotted] & M_A^r(\mathcal{K}) \end{tikzcd}$$

Here the map $\pi : M_A^r \to M_A^r(\mathcal{K})$ is the canonical projection. Let $\mathcal{K}_g := \mathcal{K} \cap g^{-1}\mathcal{K}g$, then the correspondence T_g factors through $M_A^r(\mathcal{K}_g)$, hence we can diagramatically represent the correspondence T_g as

$$M_A^r(\mathcal{K}) \xleftarrow{\pi} M_A^r(\mathcal{K}_g) \xrightarrow{\pi\circ g} M_A^r(\mathcal{K}).$$

From the diagram above we can see that the action of T_g on a subset $X \subset M_A^r(\mathcal{K})$ is given by

$$T_g(X) := \pi \circ g(\pi^{-1}(X)).$$

Moreover T_g is a finite algebraic correspondence of degree $\deg(T_g) = [\mathcal{K} \cdot Z : \mathcal{K}_g \cdot Z]$ where Z is the center of $\mathrm{GL}_r(\hat{A})$. Let $M_A^r(\mathcal{K}) = \mathrm{GL}_r(K)\backslash \mathrm{GL}_r(\mathbb{A}_f) \times \Omega^r/\mathcal{K}$, we note for future use that T_g may also be defined as

$$T_g : [\omega, s] \in M_A^r(\mathcal{K}) \longmapsto \{[\omega, skg^{-1}] \in M_A^r(\mathcal{K}) \mid \text{for representatives } k \in \mathcal{K}/\mathcal{K}_g\}$$

Thus T_g depends on g only up to $\mathcal{K} \cdot Z(\mathrm{GL}_r(\mathbb{A}_f))$.

The Drinfeld modules corresponding to the points of $T_g(x)$ are linked to the Drinfeld module corresponding to x by isogenies specified by the choice of g.

Definition 3.1.1. *Let T_g be a Hecke correspondence on $M_A^r(\mathcal{K})(\mathbb{C}_\infty)$. We say T_g is irreducible if* $\det(\mathcal{K}_g) = \det(\mathcal{K})$

The projection $M_A^r(\mathcal{K}_g) \longrightarrow M_A^r(\mathcal{K})$ induces via the determinant morphism a morphism $K^\times\backslash\mathbb{A}_f^\times/\det(\mathcal{K}_g) \longrightarrow K^\times\backslash\mathbb{A}_f^\times/\det(\mathcal{K})$ between the connected components of $M_A^r(\mathcal{K}_g)_{\mathbb{C}\infty}$ and $M_A^r(\mathcal{K})_{\mathbb{C}\infty}$. Thus if T_g is irreducible this morphism is a bijection.

This definition is not vacuous, for example if $\mathcal{K} \subset \mathrm{GL}(\hat{A})$ is an open subgroup which contains $\mathrm{diag}(\det(\mathcal{K}), 1, \ldots, 1)$, then T_g is irreducible on $M_A^r(\mathcal{K})$ for every diagonal element $g \in \mathrm{GL}_r(\mathbb{A}_f)$ and when $\mathcal{K} = \mathrm{GL}_r(\hat{A})$ and $g = \mathrm{diag}(\mathfrak{n}, 1, \ldots, 1) \in \mathrm{GL}_r(\mathbb{A}_f)$ for $\mathfrak{n}$ a generator of an ideal of $\hat{A}$. Then

$$\mathcal{K}_g = \{(a_{ij}) \in \mathrm{GL}_r(\hat{A}) \mid a_{2,1}, a_{3,1}, \ldots, a_{r,1} \in \mathfrak{n}\hat{A}\},$$

and $\det(\mathcal{K}_g) = \det(\mathrm{GL}_r(\hat{A})) = \hat{A}^\times$, thus T_g is irreducible on $M_A^r(1)$. Similarly, if $\tilde{g} = \mathrm{diag}(1, \mathfrak{n}, \ldots, \mathfrak{n}) \in \mathrm{GL}_r(\mathbb{A}_f)$, then $T_{\tilde{g}}$ is also irreducible on $M_A^r(1)$.

We are particularly interested in correspondences that encode cyclic isogenies, i.e. isogenies with kernel $(A/\mathfrak{n})$ for some ideal $\mathfrak{n} \subset A$. Thus we will often make the following specific choice of g: Denote by $\mathfrak{n} \in \hat{A}$ a chosen generator of the principal ideal $\mathfrak{n}\hat{A}$, set $g = \mathrm{diag}(\mathfrak{n}, 1, \ldots, 1) \in \mathrm{GL}_r(\mathbb{A}_f)$ and set $\tilde{g} = \mathrm{diag}(1, \mathfrak{n}, \ldots, \mathfrak{n})$. Then the Hecke correspondence T_g encodes cyclic isogenies of degree $\mathfrak{n}$ on $M_A^r(1)(\mathbb{C}_\infty)$ while the the Hecke correspondence $T_{\tilde{g}}$ encodes dual isogenies (each of kernel $(A/\mathfrak{n})^{r-1}$). In this special case we will denote the Hecke correspondences T_g and $T_{\tilde{g}}$ respectively by $T_\mathfrak{n}$ and $T_{\tilde{\mathfrak{n}}}$.

3.2 Finiteness of Hecke Correspondences

In this section we show that there are finitely many Hecke correspondences of bounded degree. We first set some notation. Let $U_\mathfrak{p}$ be an open subgroup of $\mathrm{GL}_r(K_\mathfrak{p})$ such that $U_\mathfrak{p} = \mathrm{GL}_r(A_\mathfrak{p})$ for almost all $\mathfrak{p}$ with finitely many exceptions. Define

$$U := \prod_\mathfrak{p} U_\mathfrak{p} \subset \mathrm{GL}_r(\mathbb{A}_f).$$

Given $\mathcal{K}$ a compact open subgroup of $\mathrm{GL}_r(\hat{A})$ then we assume that $\mathcal{K}$ has the form given by U. If not then there exists U defined as above such that U has finite index in $\mathcal{K}$. To see this we note that since $\mathcal{K}$ is compact open there is a finite set of primes

S such that $\mathcal{K} := G_S \times \prod_{\mathfrak{p} \notin S} \mathrm{GL}_r(A_\mathfrak{p})$ where G_S is a compact open subgroup of $\prod_{\mathfrak{p} \in S} \mathrm{GL}_r(K_\mathfrak{p})$. Thus shrinking $\mathcal{K}$ we obtain $U = \prod_{\mathfrak{p} \notin S} \mathrm{GL}_r(A_\mathfrak{p})$ which is of finite index in $\mathcal{K}$. Let $g = (g_\mathfrak{p}) \in \mathrm{GL}_r(\mathbb{A}_f)$, then for such $\mathcal{K}$ we obtain

$$\mathcal{K} \backslash \mathcal{K} g \mathcal{K} = \prod_{\mathfrak{p}} U_\mathfrak{p} \backslash U_\mathfrak{p} g_\mathfrak{p} U_\mathfrak{p}.$$

Define the local degree of g at $\mathfrak{p}$ to be

$$\deg_{U_\mathfrak{p}}(g) := |U_\mathfrak{p} \backslash U_\mathfrak{p} g_\mathfrak{p} U_\mathfrak{p}|$$

Since we have a bijection of sets between $g^{-1}\mathcal{K}g \cap \mathcal{K} \backslash \mathcal{K}$ and $\mathcal{K} \backslash \mathcal{K} g \mathcal{K}$, we therefore have a local decomposition of the degree

$$\deg_{\mathcal{K}}(g) = \prod_{\mathfrak{p}} \deg_{U_\mathfrak{p}}(g).$$

We note that we in fact have an equality $\deg(T_g) = \deg_{\mathcal{K}}(g)$. We define the support of g to be

$$\mathrm{supp}(g) = \{\mathfrak{p} \mid \deg_{U_\mathfrak{p}}(g) > 1\}$$

We note that $\deg_{U_\mathfrak{p}}(g) = 1$ if and only if $g_\mathfrak{p} U_\mathfrak{p} = U_\mathfrak{p} g_\mathfrak{p}$, hence if $U_\mathfrak{p} = \mathrm{GL}_r(A_\mathfrak{p})$ then $\deg_{\mathrm{GL}_r(A_\mathfrak{p})}(g) = 1$ if and only if $g_\mathfrak{p} \in \mathrm{GL}_r(A_\mathfrak{p})$.

Proposition 3.2.1. *Let $\{g_n\}$ be a infinite sequence of elements in $\mathrm{GL}_r(\mathbb{A}_f)$. Let B be a fixed constant and suppose that $\deg(g_n) < B$ for all n. Then the sequence $\{g_n\}$ belongs to finitely many cosets of $\mathcal{K} \cdot Z(\mathrm{GL}_r(\mathbb{A}_f))$*

Proof. Since there exist a compact open U of finite index in $\mathcal{K}$ which can be written as a product we may assume $\mathcal{K}$ is of this product form. By assumption $\deg(g_n) < B$ for all n. Hence $\deg_{U_\mathfrak{p}}(g_n) < B$ for all n and for all $\mathfrak{p}$. Denote by S_n the support of g_n. We clearly have $|S_n| < B$. Thus $|S_n|$ is uniformly bounded independent of n and $\deg_{U_\mathfrak{p}}(g_n) = 1$ except for the finite set for primes S_n. In particular the sequence

$\{g_n\}$ has representatives in $\prod_{\mathfrak{p}\in S_n} \mathrm{GL}_r(K_\mathfrak{p}) \times \prod_{\mathfrak{p}\notin S_n} \mathrm{GL}_r(A_\mathfrak{p})$.

We now show that S_n is contained in a finite subset that is independent of n and that we can bound the exponents of each local component $g_{n,\mathfrak{p}}$ of g_n. We have a bijection

$$g_p^{-1} U_\mathfrak{p} g_p \cap U_\mathfrak{p} \backslash U_\mathfrak{p} = U_\mathfrak{p} \backslash U_\mathfrak{p} g_\mathfrak{p} U_\mathfrak{p}$$

and since $\mathrm{GL}_r(A_\mathfrak{p}) \subseteq U_\mathfrak{p}$

$$|\,\mathrm{GL}_r(A_\mathfrak{p}) \backslash \mathrm{GL}_r(A_\mathfrak{p}) g_\mathfrak{p} \,\mathrm{GL}_r(A_\mathfrak{p})| \le |U_\mathfrak{p} \backslash U_\mathfrak{p} g_\mathfrak{p} U_\mathfrak{p}|$$

Therefore $|\,\mathrm{GL}_r(A_\mathfrak{p}) \backslash \mathrm{GL}_r(A_\mathfrak{p}) g_\mathfrak{p} \,\mathrm{GL}_r(A_\mathfrak{p})| < B$ for all $\mathfrak{p}$. The space $\mathrm{GL}_r(A_\mathfrak{p}) \backslash \mathrm{GL}_r(A_\mathfrak{p}) g_\mathfrak{p} \,\mathrm{GL}_r(A_\mathfrak{p})$ is related to the Bruhat-Tits building for $\mathrm{GL}_r(K_\mathfrak{p})$. More precisely $\mathrm{GL}_r(A_\mathfrak{p})$ acts on the space of rank r $A_\mathfrak{p}$-lattices in $K_\mathfrak{p}$. Let L be the standard lattice, this has stabiliser $\mathrm{GL}_r(A_\mathfrak{p})$ which implies the stabiliser of $g_p L$ in $\mathrm{GL}_r(A_\mathfrak{p})$ is $g_p^{-1} GL_r(A_\mathfrak{p}) g_p \cap GL_r(A_\mathfrak{p})$. Thus

$$g_p^{-1}\,\mathrm{GL}_r(A_\mathfrak{p}) g_p \cap \mathrm{GL}_r(A_\mathfrak{p}) \backslash \mathrm{GL}_r(A_\mathfrak{p}) = \mathrm{GL}_r(A_\mathfrak{p}) \backslash \mathrm{GL}_r(A_\mathfrak{p}) g_\mathfrak{p} \,\mathrm{GL}_r(A_\mathfrak{p})$$

is the orbit of $g_p L$ under $\mathrm{GL}_r(A_\mathfrak{p})$. By the Cartan decomposition

$$\mathrm{GL}_r(K_\mathfrak{p}) = \mathrm{GL}_r(A_\mathfrak{p}) \,\mathrm{diag}(\mathfrak{p}^{\omega_1}, \cdots, \mathfrak{p}^{\omega_r}) \,\mathrm{GL}_r(A_\mathfrak{p}),$$

where $\omega_1 \le \omega_2 \le \cdots \le \omega_r$. If g_p is of the form $\mathrm{diag}(\mathfrak{p}^{\omega_1}, \cdots, \mathfrak{p}^{\omega_r})$ then the orbit of $g_p L$ consists of lattices M such that $L/M \cong \oplus_i^r A/\mathfrak{p}^{\omega_i} A$. However by Cartan decomposition above the general element of $\mathrm{GL}_r(K_\mathfrak{p})$ has the form $g_p = g_1 \cdot \mathrm{diag}(\mathfrak{p}^{\omega_1}, \cdots, \mathfrak{p}^{\omega_r}) \cdot g_2$ which implies the orbit of $g_p L$ lies in the orbit of $\mathrm{diag}(\mathfrak{p}^{\omega_1}, \cdots, \mathfrak{p}^{\omega_r}) L$. Therefore we still have that the general orbit consists of lattices M such that $L/M \cong \oplus_i^r A/\mathfrak{p}^{\omega_i} A$. Let $\omega = \max\{\omega_i\}$ and let $p : (K)^r \to (K)^{r-1}$ be the projection that forgets the component of the lattice L with $\mathfrak{p}^\omega$. Denote by L' and M' the projection of L and M respectively. We have a split exact sequence

$$0 \to A/\mathfrak{p}^\omega A \to L/M \to L'/M' \to 0$$

Thus by induction to determine $|\{M \subset L | L/M \cong \oplus_i^r A/\mathfrak{p}^{\omega_i} A\}|$ we are reduced to $g_p = \mathrm{diag}(\mathfrak{p}^{\omega}, \cdots, 1)$. In this case $|\operatorname{GL}_r(A_{\mathfrak{p}}) \backslash \operatorname{GL}_r(A_{\mathfrak{p}}) g_{\mathfrak{p}} \operatorname{GL}_r(A_{\mathfrak{p}})|$ counts cyclic isogenies of degree $|A/\mathfrak{p}^{\omega} A|$. By lemma 3.2.3 the number of cyclic isogenies of degree $|A/\mathfrak{p}^{\omega} A|$ is polynomial in $\mathfrak{p}^{\omega}$. Since $|\operatorname{GL}_r(A_{\mathfrak{p}}) \backslash \operatorname{GL}_r(A_{\mathfrak{p}}) g_{\mathfrak{p}} \operatorname{GL}_r(A_{\mathfrak{p}})| < B$ this implies

(i). The primes $\mathfrak{p}$ are bounded independent of n,

(ii). As $\mathfrak{p}$ and n vary, there are only finitely many exponents $(\omega_1, \cdots, \omega_r)$ appearing in the sequence $\{g_{n,p}\}$.

The primes $\mathfrak{p}$ therefore belong to a finite set which we call S and the sequence $\{g_n\}$ has representatives in $\prod_{\mathfrak{p} \in S} \operatorname{GL}_r(K_{\mathfrak{p}}) \times \prod_{\mathfrak{p} \notin S} \operatorname{GL}_r(A_{\mathfrak{p}})$. Thus modulo $\mathcal{K} \cdot Z(\operatorname{GL}_r(\mathbb{A}_f))$ the sequence $\{g_n\}$ has representatives in $\prod_{\mathfrak{p} \in S} \operatorname{GL}_r(K_{\mathfrak{p}})$ and these representatives furthermore have bounded exponents $(\omega_1, \cdots, \omega_r)$. This proves our claim. $\square$

Corollary 3.2.2. *Let $\{g_n\}$ be a infinite sequence of elements in $\operatorname{GL}_r(\mathbb{A}_f)$. Let B be a fixed constant and suppose that $\deg(g_n) < B$ for all n. Then the set of Hecke correspondences $\{T_{g_n}\}$ is finite.*

Proof. By proposition 3.2.1 above the sequence $\{g_n\}$ belong to finitely many cosets of $U \cdot Z(\operatorname{GL}_r(\mathbb{A}_f))$. Since Hecke correspondences are well defined up to $\mathcal{K} \cdot Z(\operatorname{GL}_r(\mathbb{A}_f))$ and $U/\mathcal{K}$ is finite we obtain our claim. $\square$

Lemma 3.2.3. *Let φ be a rank r Drinfeld A-module over $\mathbb{C}_\infty$ and let N be an ideal in A. Then the number of rank r Drinfeld A-module over $\mathbb{C}_\infty$ which are linked to φ by a cyclic isogeny with kernel A/NA are bounded by a polynomial in $|N|$.*

Proof. We note that it suffices to consider the case where $N = \mathfrak{p}^{\omega}$, with $\mathfrak{p}$ a prime. Let Λ_φ be a the rank r A-lattice corresponding to φ. Choosing a basis for Λ_φ, define the following lattices $\Lambda_{\varphi,i} := \{(x_1, \cdots, x_r) \in \Lambda_\varphi \mid [x_1 : x_2 : \cdots : x_r] = i \bmod \mathfrak{p}^{\omega}\}$ for

$i \in A/\mathfrak{p}^\omega A$, with respect this basis. Each $\Lambda_{\varphi,i}$ is of index $A/\mathfrak{p}^\omega A$ in Λ_φ. Moreover each $\Lambda_{\varphi,i}$ identifies with a point in projective space $\mathbb{P}^r(A/\mathfrak{p}^\omega A)$. Thus it might seem that there are at least $|\mathbb{P}^r(A/\mathfrak{p}^\omega A)|$ such cyclic isogenies. But it is possible that two of the above lattices give rise to equivalent Drinfeld modules. Suppose this is the case, i.e. $\Lambda_{\varphi,i} = \Lambda' = \Lambda_{\varphi,j}$, then for $i = 1, 2$ let $f_i : \Lambda_\varphi \longrightarrow \Lambda'$ be these two isogenies. Then we consider the two isogenies $f_i{}' : \Lambda_\varphi / \ker(f_1) \cap \ker(f_2) \longrightarrow \Lambda'$ for $i = 1, 2$. Let M be their degree, then $f_2{}' \cdot \hat{f_1}{}'$ is an endomorphism of the Drinfeld module corresponding to $\Lambda_\varphi / \ker(f_1) \cap \ker(f_2)$ of degree $(A/MA)^r$. There are at most $r^{\pi(M)}$ such endomorphisms, where $\pi(M)$ is the number of prime divisors of the ideal M. Thus our claims follows from the fact that $|\mathbb{P}^r(A/\mathfrak{p}^\omega A)|/r^{\pi(M)}$ is polynomial in $|A/\mathfrak{p}^\omega A|$. □

3.3 Complex Multiplication

Let φ be a rank r Drinfeld A-module with endomorphism ring R. Then R is a commutative A-algebra with rank dividing r as a projective A-module (see for example [29]) and R is an order in its quotient field K', with K' a *totally imaginary* extension of K i.e. there is only one prime in K' lying above ∞ .

Definition 3.3.1. *Let φ be a rank r Drinfeld A-module. We say that φ has* complex multiplication *(CM) by R if $[K' : K] = [R : A] = r$.*

We can extend this definition to pairs of Drinfeld A-modules in a product $M_A^r(\mathcal{K})(\mathbb{C}_\infty) \times M_A^r(\mathcal{K})(\mathbb{C}_\infty)$ of Drinfeld moduli spaces. A point $x = (x_1, x_2)$ in $M_A^r(\mathcal{K})(\mathbb{C}_\infty) \times M_A^r(\mathcal{K})(\mathbb{C}_\infty)$ has complex multiplication (is a *CM point*) if each of the corresponding Drinfeld modules φ^{x_i} has complex multiplication.

If φ is a Drinfeld A-module over L with *CM* by R, then by definition R is the centraliser of the image of A in $L\{\tau\}$ (see for example [29]). We thus have an embedding $\varphi_R : R \longrightarrow L\{\tau\}$. It is tempting and fortunately correct to think that φ_R is a Drinfeld R-module over L. There are however some technicalities to overcome, for instance R might not integrally closed, hence not a Dedekind domain. In any case Hayes (see [31]) has given a satisfactory theory of rank 1 Drinfeld modules over R even when R is not integrally closed. Since φ has complex multiplication by R, by looking at the $\mathfrak{n}$ torsion of φ_R, we easily see that φ_R is a rank 1 Drinfeld R-module. Let $M_R^1(\mathbb{C}_\infty)$ be the pro-affine scheme over $\mathbb{C}_\infty$ whose closed points are in bijection with rank 1 Drinfeld R-modules over $\mathbb{C}_\infty$ with full level structure. $\mathrm{GL}_1(\mathbb{A}_{f,K'})$ acts on $M_R^1(\mathbb{C}_\infty)$ as follows. Let $\alpha \in \hat{A}^\times$ then α acts on K'/R by multiplication with kernel H. If $E = (\mathbb{G}_{a,\mathbb{C}_\infty}, \varphi)$ is a rank 1 Drinfeld R-module over $\mathbb{C}_\infty$ with full level structure then $E/H = (\mathbb{G}_{a,\mathbb{C}_\infty}/H, \varphi)$ is also a rank 1 Drinfeld R-module over $\mathbb{C}_\infty$ with full level structure [27] which is furthermore isogenous to E. We encapsulate this in the following commutative diagram.

$$\begin{tikzcd}
K'/R \arrow[r] \arrow[d, "\alpha"] & E = (\mathbb{G}_{a,\mathbb{C}_\infty}, \varphi) \arrow[d, "\alpha"] \\
K'/R \arrow[r] & E/H = (\mathbb{G}_{a,\mathbb{C}_\infty}/H, \varphi)
\end{tikzcd}$$

If $\alpha \in (R-0)$ then the action on K'/R is trivial hence $\hat{A}^\times/(R-0) \cong \mathbb{A}_{f,K'}^\times/K'^\times$ acts on M_R^1.

Let $\mathrm{Pic}(R)$ be the ideal class group of R. The group $\mathrm{Pic}(R)$ acts on the isomorphism classes of Drinfeld R-modules over $\mathbb{C}_\infty$ as follows. Let $\mathfrak{a}$ be an ideal of R. We denote by $I_{\varphi,\mathfrak{a}}$ the left ideal in $\mathbb{C}_\infty\{\tau\}$ generated by the image of $\mathfrak{a}$ in $\mathbb{C}_\infty\{\tau\}$. $\mathbb{C}_\infty\{\tau\}$ is a left principal ideal domain, so that we can find some $\varphi_\mathfrak{a} \in I_{\varphi,\mathfrak{a}}$ such that $I_{\varphi,\mathfrak{a}} = \mathbb{C}_\infty\{\tau\}\varphi_\mathfrak{a}$. For any $\mathfrak{a} \in R$, there is an element $\psi_a \in \mathbb{C}_\infty\{\tau\}$ such that $\varphi_\mathfrak{a}\varphi_a = \psi_a\varphi_\mathfrak{a}$. Then $a \to \psi_a$ defines a Drinfeld R-module $\psi : R \to \mathbb{C}_\infty\{\tau\}$.

We write $\psi = \mathfrak{a} * \varphi$. It satisfies the next conditions ([31]):

(i). If $[\mathfrak{a}] = [\mathfrak{b}]$ in $\mathrm{Pic}(R)$, then $\mathfrak{a} * \varphi = \mathfrak{b} * \varphi$;

(ii). $R * \varphi = \varphi$;

(iii). $\mathfrak{a} * (\mathfrak{b} * \varphi) = \mathfrak{a}\mathfrak{b} * \varphi$.

Hence we have the action of $\mathrm{Pic}(R)$. Hayes shows that if we put $R_1 = \mathrm{End}(\mathfrak{a})$, there is a unique Drinfeld R_1-module over $\mathbb{C}_\infty$ whose restriction to R is $\mathfrak{a} * \varphi$. For φ^Λ, the Drinfeld R-module associated to an R-lattice Λ, we have $\mathfrak{a} * \varphi^\Lambda = \varphi^{\mathfrak{a}^{-1}\Lambda}$.

Let $\sigma \in \mathrm{Aut}(\mathbb{C}_\infty/K')$ and let $\varphi : A \longrightarrow \mathbb{C}_\infty\{\tau\}$ be a Drinfeld A-module over $\mathbb{C}_\infty$. if $\varphi_a = \sum_{i=0}^m a_i\tau^i$ then we define the action of σ on φ by

$\varphi_a^\sigma := \sum_{i=0}^m a_i^\sigma \tau^i$ for all $a \in A$.

We now have two actions, the next theorem shows that these actions (i.e. of $\mathbb{A}_{f,K'}$ and σ) are compatible. Let H_R be the ring class field of R i.e. the class field associated to $K^\times\backslash\mathbb{A}_{f,K'}^\times/\hat{R}^\times$, then $\mathrm{Gal}(H_R/K') = \mathrm{Pic}(R)$. The maximal abelian extension of K' in which ∞ splits completely is then the union $K'^{\mathrm{ab}} = \bigcup_{R\subset K'} H_R$ where R runs through all the orders of K.

Let φ^Λ be a rank 1 Drinfeld R-module over $\mathbb{C}_\infty$ with associated lattice Λ. The lattice Λ has the representation $\Lambda = c^{-1}\mathfrak{a}$ where $c \in \mathbb{C}_\infty^\times$ and $\mathfrak{a} \subset A$ an ideal of R. Via the exponential function attached to Λ we have an isomorphism $e_\Lambda\colon K/\Lambda \cong \mathrm{tor}(\varphi)$. Let

$$[*, K'^{\mathrm{ab}}/K']\colon K^\times\backslash\mathbb{A}_{f,K'}^\times \longrightarrow \mathrm{Gal}(K'^{\mathrm{ab}}/K')$$

be the Artin map.

Theorem 3.3.2. *(Main Theorem of Complex Multiplication) Let $A = \mathbb{F}_q[T]$ and $\sigma \in \mathrm{Aut}(\mathbb{C}_\infty/K')$ and let $s \in \mathbb{A}_{f,K'}$ such that $\sigma|_{K'^{\mathrm{ab}}} = [s, K'^{\mathrm{ab}}/K']$, then there exists a unique isomorphism $d : \varphi^{s^{-1}\Lambda} \longrightarrow \varphi^\sigma$ such that the following diagram is* *commutative*

$$\begin{tikzcd} K/\Lambda \arrow[r, "e_\Lambda"] \arrow[d, "s^{-1}"] & \mathrm{tor}(\varphi) \arrow[d, "\sigma"] \\ K/\Lambda \arrow[r, "ds^{-1}e_\Lambda"] & \mathrm{tor}(\varphi^\sigma) \end{tikzcd}$$

Proof. (see Gekeler [27]). □

Via the Artin map we have an isomorphism $K'^{\times}\backslash\mathbb{A}^{\times}_{f,K'}/\mathcal{K} \cong \mathrm{Gal}(H_{\mathcal{K}}/K')$ where $H_{\mathcal{K}}/K'$ is the class field associated to the idèle class group $K'^{\times}\backslash\mathbb{A}^{\times}_{f,K'}/\mathcal{K}$. In the particular case that $\mathcal{K} = GL_1(\hat{R})$, then $K'^{\times}\backslash\mathbb{A}^{\times}_{f,K'}/\mathcal{K} \cong \mathrm{Pic}(R) \cong \mathrm{Gal}(H_R/K')$ and since $M^1_R(\mathcal{K}) = \mathcal{K}\backslash M^1_R$, we see that $\mathbb{A}^{\times}_{f,K'}$ acts on $M^1_R(\mathcal{K})$ via its quotient $K'^{\times}\backslash\mathbb{A}^{\times}_{f,K'}/\mathcal{K} \cong \mathrm{Pic}(R)$. Thus the $\mathbb{A}^{\times}_{f,K'}$ action is consistent with the action of $\mathrm{Pic}(R)$. Therefore $M^1_R(\mathcal{K})_{\mathbb{C}_\infty}$ consists of $\mathrm{Pic}(R)$ points, each defined over $H_{\mathcal{K}}$, and $\mathrm{Gal}(H_{\mathcal{K}}/K')$ acts on these points via isogenies determined by the Artin symbol. If $g \in \mathrm{GL}_1(\hat{R})$ and $\sigma_g \in \mathrm{Gal}(K'^{\mathrm{ab}}/K')$ is the corresponding Frobenius element, then for $x \in M^1_R(\mathcal{K})(\mathbb{C}_\infty)$ our discussion shows that $\sigma_g(x) \in T_g(x)$.

Let $\mathfrak{p}$ be a prime of K and let $\mathfrak{P}$ be an unramified prime above it in R. If $\mathfrak{P}$ has residue degree one, then $R/\mathfrak{P} \cong A/\mathfrak{p}$. Thus if a rank 1 Drinfeld R-module over $\mathbb{C}_\infty$, φ has an isogeny with kernel $R/\mathfrak{P}$ then this remains a cyclic isogeny of φ as a rank r Drinfeld A-module over $\mathbb{C}_\infty$ with kernel $A/\mathfrak{p}$. This becomes a definition.

Definition 3.3.3. *Let M/L be an algebraic field extension. A prime $\mathfrak{p}$ of L is called* residual *in M if there exists a prime $\mathfrak{P}$ of M above $\mathfrak{p}$ with residual degree $f(\mathfrak{P}|\mathfrak{p}) = 1$. If R is an order in M, then $\mathfrak{p}$ is residual in R if it is residual in M and $\mathfrak{p}$ does not divide the conductor of R.*

Let K'/K be a finite extension, let R be an order in K' containing A, and let $\mathfrak{n} \subset A$ be an ideal such every prime factor of $\mathfrak{n}$ is residual in R. We call $\mathfrak{n}$ a *residual ideal* of R. The preceding discussion can be summed up in:

Proposition 3.3.4. *Let $x \in M_A^r(1)(\mathbb{C}_\infty)$ be a CM point with $R = \mathrm{End}(\varphi^x)$ an order in the purely imaginary extension K'/K, and let $\mathfrak{n} \subset A$ be a residual ideal in R. Let $\mathfrak{N} \subset R$ be an ideal such that $R/\mathfrak{N} \cong A/\mathfrak{n}$, let $\sigma = (\mathfrak{N}, H_R/K') \in \mathrm{Gal}(H_R/K')$ denote the Frobenius element corresponding to $g = \mathrm{diag}(\mathfrak{n}, 1, \ldots, 1) \in \mathrm{GL}_r(\mathbb{A}_f)$. Then there exists a cyclic isogeny $\varphi \to \sigma(\varphi)$ of degree $\mathfrak{n}$ and furthermore $\sigma(x) \in T_g(x)$.*

3.4 Drinfeld modular subvarieties

Let C and C' be two smooth, projective and geometrically irreducible curves over $\mathbb{F}_q$. Let $\pi : C' \longrightarrow C$ be a fixed finite morphism of degree n. Let $\infty \in C$ be a closed point which does not split in C and $\{\infty'\} = \pi^{-1}(\infty)$. We set $A := \Gamma(C \setminus \infty, \mathcal{O}_C)$ and $A' := \Gamma(C' \setminus \infty', \mathcal{O}_{C'})$, these are the rings of regular functions away from ∞ and ∞' respectively. A' is a flat A-algebra via the map $\pi^{\#} : A \to A'$.

Following Hendler and Hartl [54], we say that the morphism π defines a restriction of coefficients functor from Drinfeld A'-modules over S to Drinfeld A-modules over S. We rigidify the above functor by adding level structure data, indeed let $(\mathbb{G}_{a,\mathcal{L}'/S}, \varphi', \alpha')$ be a triple, where $(\mathbb{G}_{a,\mathcal{L}'/S}, \varphi')$ is a rank r' Drinfeld A'-module over S and α' is a level-$\mathfrak{n}'$ structure. Via the composition

$$\varphi : A \xrightarrow{\pi^{\#}} A' \xrightarrow{\varphi'} \mathrm{End}(\mathbb{G}_{a,\mathcal{L}'/S})$$

we obtain a rank r Drinfeld A-module $(\mathbb{G}_{a,\mathcal{L}'/S}, \varphi \cdot \pi^{\#})$. Choosing an isomorphism $(\mathfrak{n}^{-1}/A)^r \xrightarrow{\sim} (\mathfrak{n}'^{-1}/A')^{r'}$ and letting α be the restriction of α' to A, we obtain a level-$\mathfrak{n}$ structure on φ i.e.

$$\alpha : (\mathfrak{n}^{-1}/A)^r \xrightarrow{\sim} (\mathfrak{n}'^{-1}/A')^{r'} \xrightarrow{\alpha'} \mathcal{L}(S).$$

The following proposition can be found in ([54, section 2]).

Proposition 3.4.1. *The morphism $\pi : C^{'} \longrightarrow C$ defines a restriction of coefficients functor $\mathcal{F}_{\pi^{\#}} : (\mathbb{G}_{a,\mathcal{L}'/S}, \varphi^{'}, \alpha^{'}) \longrightarrow (\mathbb{G}_{a,\mathcal{L}'/S}, \varphi = \varphi^{'} \cdot \pi^{\#}, \alpha)$ from rank $r^{'}$ Drinfeld $A^{'}$-modules over S with level-$\mathfrak{n}'$ structure $\alpha^{'}$ to rank $r^{'}n$ Drinfeld A-modules over S with level-$\mathfrak{n}$ structure α.*

Proof. The statement of the proposition is clear from the discussion above. The only thing left to check is the claim about the rank of the drinfeld module $(\mathbb{G}_{a,\mathcal{L}'/S}, \varphi = \varphi^{'} \cdot \pi^{\#}, \alpha)$. The claim follows from the fact that $\pi^{-1}(\infty) = \{\infty^{'}\}$. Therefore $\mathrm{rank}\varphi_a = \mathrm{rank}{\varphi^{'}}_a \cdot \pi^{\#} = \mathrm{ord}_{\infty'}(a) \cdot \deg(\infty^{'}) = n \cdot \mathrm{ord}_{\infty}(a) \cdot \deg(\infty)$ for all $a \in A$.

□

The restriction of coefficients functor above induces a functorial (with respect to the choice of an isomorphism above) morphism between the moduli schemes classifying Drinfeld $A^{'}$-modules respectively Drinfeld A-modules with level structure, i.e. we have a morphism $f : M_{A'}^{r'}(\mathfrak{n}') \to M_A^{r^{'}n}(\mathfrak{n})$.

Indeed since $\mathcal{M}_{A'}^{r'}(\mathfrak{n}')$ is representable, let $(\mathbb{G}_{a,\mathcal{L}'/S}, \varphi^{'*}, \alpha^{'*})$ be the universal Drinfeld module on $M_{A'}^{r'}(\mathfrak{n}')$. Since every Drinfeld A-module over S comes via pullback from the universal Drinfeld module on $M_A^r(\mathfrak{n})$ there exist a morphism $f_{A'/A} : S \to S$ such that $(\mathbb{G}_{a,\mathcal{L}'/S}, \varphi = \varphi^{'*} \cdot \pi^{\#*}, \alpha)$ is the pullback of the universal Drinfeld module on $M_A^r(\mathfrak{n})$ via $f_{A'/A}$.

The morphism $f_{A'/A}$ is separated and of finite type since $M_{A'}^{r'}(\mathfrak{n}')$ and $M_A^{r^{'}n}(\mathfrak{n})$ are spectra of A-algebras of finite type. Furthermore this morphism is proper as the following theorem due to Breuer [8] shows.

Theorem 3.4.2. *Let $r', (r = r' \cdot n), K', A', \mathfrak{n}$ and $\mathfrak{n}'$ be as above. Suppose that $\mathfrak{n}$ is admissible, so that $\mathfrak{n}'$ is admissible and $M_A^r(\mathfrak{n})$ and $M_{A'}^{r'}(\mathfrak{n}')$ are fine moduli schemes. Then the canonical morphism $M_{A'}^{r'}(\mathfrak{n}')_{K'} \to M_A^r(\mathfrak{n})_{K'}$ is proper.*

The above proposition allows us to view $f_{A'/A}(M_{A'}^{r'}(\mathfrak{n}')_{K'})$ the image of the affine scheme $M_{A'}^{r'}(\mathfrak{n}')_{K'}$ as a closed subvariety of $M_A^r(\mathfrak{n})_{K'}$. $M_{A'}^{r'}(\mathfrak{n}')_{K'}$ is the locus of those Drinfeld modules with endomorphism ring containing A'.

3.4.1 Subvarieties of Hodge Type

Following Breuer [8] we define the Hodge or special subvarieties of a Drinfeld modular variety as follows.

Definition 3.4.3. *Let $X \subset M_A^r(\mathfrak{n})_{\mathbb{C}_\infty}$ be an irreducible subvariety. Then X is a* Hodge *subvariety if X is an irreducible component of $T_g(M_{A'}^{r'}(\mathfrak{n})_{\mathbb{C}_\infty})$ for some $g \in \mathrm{GL}_r(\mathbb{A}_f)$ and $r'|r$ and $A'|A$.*

Using this we can define *Hodge* subvarieties of $M_A^r(1)_{\mathbb{C}_\infty}$ as the irreducible components of images of *Hodge* subvarieties in $M_A^r(\mathfrak{n})_{\mathbb{C}_\infty}$ via the natural projection $M_A^r(\mathfrak{n})_{\mathbb{C}_\infty} \longrightarrow M_A^r(1)_{\mathbb{C}_\infty}$.

Finally for an irreducible subvariety $X \subset M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$, we say X is a *Hodge* subvariety if its image under the canonical projection $M_A^r(\mathcal{K})_{\mathbb{C}_\infty} \to M_A^r(1)_{\mathbb{C}_\infty}$ is a *Hodge* subvariety.

We want to describe the Hodge subvarieties in the product of two Drinfeld modular varietes $M_A^r(\mathcal{K})_{\mathbb{C}_\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$. Taking a cue from the case of Shimura Varieties [21] and rank 2 Drinfeld moduli [10], we give the following ad hoc definition of Hodge subvarieties.

Definition 3.4.4. *Let $X \subset M_A^r(1)_{\mathbb{C}_\infty} \times M_A^r(1)_{\mathbb{C}_\infty}$ be an irreducible subvariety. Then X is a* Hodge *subvariety if X is an irreducible component of*

(i). the image of $T_{g_1} M_{A_1}^{r_1}(1)_{\mathbb{C}_\infty} \times T_{g_2} M_{A_2}^{r_2}(1)_{\mathbb{C}_\infty}$ in $M_A^r(1)_{\mathbb{C}_\infty} \times M_A^r(1)_{\mathbb{C}_\infty}$ for some $g_1 \in \mathrm{GL}_{r_1}(\mathbb{A}_f)$, $g_2 \in \mathrm{GL}_{r_2}(\mathbb{A}_f)$, $r_i|r$ and $A_i|A$ for $i \in \{1, 2\}$ or

(ii). the image of $T_g(M^{r'}_{A'}(1)_{\mathbb{C}_\infty})$ *in* $M^r_A(1)_{\mathbb{C}_\infty} \times M^r_A(1)_{\mathbb{C}_\infty}$ *for some* $g \in \mathrm{GL}_r(\mathbb{A}_f)$ *and* $r'|r$ *and* $A'|A$*, embedded under the map*

$$x \longrightarrow (g_1 \cdot x\,,\, g_2 \cdot x) \textit{ for some } g_1\,,\, g_2 \in GL_r(\mathbb{A}_f).$$

For an irreducible subvariety $X \subset M^r_A(\mathcal{K})_{\mathbb{C}_\infty} \times M^r_A(\mathcal{K})_{\mathbb{C}_\infty}$*, we say* X *is a* Hodge *subvariety if its image in* $M^r_A(1)_{\mathbb{C}_\infty} \times M^r_A(1)_{\mathbb{C}_\infty}$ *under the canonical projection* $M^r_A(\mathcal{K})_{\mathbb{C}_\infty} \times M^r_A(\mathcal{K})_{\mathbb{C}_\infty} \to M^r_A(1)_{\mathbb{C}_\infty} \times M^r_A(1)_{\mathbb{C}_\infty}$ *is a* Hodge *subvariety.*

Let $g_1\,,\, g_2 \in GL_r(\mathbb{A}_f)$ and let $N = g_1^{-1}g_2$ then we denote the image of the map

$$M^r_A(1)_{\mathbb{C}_\infty} \longrightarrow M^r_A(1)_{\mathbb{C}_\infty} \times M^r_A(1)_{\mathbb{C}_\infty}; x \longrightarrow (x\,,\, g_1^{-1}g_2 \cdot x) \text{ for some } g_1\,,\, g_2 \in GL_r(\mathbb{A}_f),$$

by ${Y_2}^r(N)$. The variety ${Y_2}^r(N)$ is analogous to the rank 2 Drinfeld Modular varieties defined by Breuer [10]. By abuse of notation we will use the same notation for its irreducible components as well.

Definition 3.4.5. *We say* $X \subset M^r_A(\mathcal{K})_{\mathbb{C}_\infty}$ *is Hodge generic if* X *is not contained in a proper Hodge subvariety of positive codimension of* $M^r_A(\mathcal{K})_{\mathbb{C}_\infty}$*. When* $X \subset M^r_A(\mathcal{K})_{\mathbb{C}_\infty} \times M^r_A(\mathcal{K})_{\mathbb{C}_\infty}$ *we say that* X *is Hodge generic if* X *is not contained in a proper Hodge subvariety of positive codimension of* $M^r_A(\mathcal{K})_{\mathbb{C}_\infty} \times M^r_A(\mathcal{K})_{\mathbb{C}_\infty}$*.*

This definition means that $X \subset M^r_A(\mathcal{K})_{\mathbb{C}_\infty} \times M^r_A(\mathcal{K})_{\mathbb{C}_\infty}$ has an underlying pair of non-isogenous Drinfeld modules such that the endomorphism rings are A.
From the definitions we see that a *Hodge* subvariety of $X \subset M^r_A(\mathcal{K})_{\mathbb{C}_\infty}$ is the locus of those points corresponding to Drinfeld modules with endomorphism rings containing certain orders A' which are integral extensions of A. In particular, the *Hodge* subvarieties of dimension zero are precisely the CM points with exact endomorphism rings A'.

CHAPTER IV

Monodromy and Irreducibility of Hecke Correspondences

In this chapter we give a description of the monodromy group associated to a family of Drinfeld modules. These results follow from some very deep results, namely the Tate-conjecture for Drinfeld modules due to Tamagawa [53] and a characterisation of compact subgroups of linear algebraic groups due to Pink [42]. The result of Pink allows us to easily conclude in suitable circumstances that the monodromy group is an open subgroup. We use this openness result to show the irreducibility of Hecke correspondences.

4.1 Fundamental Groups

Let $k \subset \mathbb{C}_\infty$ be a subfield which is finitely generated over K and let $X \subset M_A^r(\mathcal{K})_k$ be an irreducible and closed subvariety of the moduli space of rank r Drinfeld A-modules over k with sufficiently high level structure $\mathcal{K}$. Suppose also that X has positive dimension. Write F^{sep} for a separable closure of F, where F is the function field of X. Let $\eta : \mathrm{Spec}(F) \to X$ be a generic point of X and $\bar{\eta} : \mathrm{Spec}(F^{\mathrm{sep}}) \to X$ the geometric point above η via the inclusion $F \hookrightarrow F^{\mathrm{sep}}$. Let k^{sep} be the separable closure of k in F^{sep}. Write $X_{k^{\mathrm{sep}}} = X \times_k k^{\mathrm{sep}}$. We denote the étale fundamental group and the geometric fundamental group of X with base point $\bar{\eta}$ by $\pi_1^{\mathrm{et}}(X, \bar{\eta})$ and

$\pi_1^{\text{et}}(X_{k^{\text{sep}}}, \bar{\eta})$ respectively. There is a short exact sequence,

$$1 \longrightarrow \pi_1^{\text{et}}(X_{k^{sep}}, \bar{\eta}) \longrightarrow \pi_1^{\text{et}}(X, \bar{\eta}) \longrightarrow \text{Gal}(k^{sep}/k) \longrightarrow 1.$$

This is the analog of the fibration exact sequence in topology. The assumption on $\mathcal{K}$ ensures that $M_A^r(\mathcal{K})$ is a fine moduli scheme, hence the embedding $X_{\mathbb{C}\infty} \subset M_A^r(\mathcal{K})(\mathbb{C}_\infty)$ determines a non-isotrivial family φ of Drinfeld modules. Let φ_η be the Drinfeld module corresponding to η. Let $\hat{T}(\varphi_\eta)$ denote the adelic Tate module of φ_η i.e.

$$\hat{T}(\varphi_\eta) := \prod_{\mathfrak{p}} T_{\mathfrak{p}}(\varphi_\eta),$$

where $T_{\mathfrak{p}}(\varphi_\eta)$ is the $\mathfrak{p}$-adic Tate module. It is well known that $T(\varphi_\eta)$ is a free $\hat{A}$-module of rank r i.e. $T(\varphi_\eta) \cong \hat{A}^r$.

The étale fundamental group $\pi_1^{\text{et}}(X, \bar{\eta})$ acts on the adelic Tate module via the monodromy action (see [6]) and a choice of basis for $\hat{A}^r$ gives us the associated monodromy representation.

$$\rho : \pi_1^{\text{et}}(X, \bar{\eta}) \longrightarrow \text{GL}_r(\hat{A}) \subset \text{GL}_r(\mathbb{A}_f)$$

Let Γ^{geo} and Γ denote the images of $\pi_1^{\text{et}}(X_{k^{sep}}, \bar{\eta})$ and $\pi_1^{\text{et}}(X, \bar{\eta})$ in $\text{GL}_r(\hat{A})$ via ρ respectively. Since $\pi_1^{\text{et}}(X_{k^{sep}}, \bar{\eta})$ is a normal subgroup of $\pi_1^{\text{et}}(X, \bar{\eta})$, we have the normal subgroup inclusion $\Gamma^{\text{geo}} \triangleleft \Gamma$.

We now assume that $X_{\mathbb{C}_\infty}$ is a smooth irreducible and locally closed algebraic subvariety of $M_A^r(\mathcal{K})_{\mathbb{C}\infty}$ of positive dimension. Then $X^{\text{an}}{}_{\mathbb{C}\infty}$ is contained in an irreducible component of $M_A^r(\mathcal{K})^{\text{an}}{}_{\mathbb{C}\infty}$ and since we have a rigid analytic isomorphism (see theorem **??**)

$$M_A^r(\mathcal{K})^{\text{an}}{}_{\mathbb{C}\infty} \xrightarrow{\sim} \coprod_{s \in S} \Gamma_s \backslash \Omega^r,$$

$X_{\mathbb{C}\infty}$ lies in some connected component $\Gamma_s \backslash \Omega^r$ for some s. Let $\Delta := \Gamma_s$ then Δ is a congruence subgroup of $SL_r(K)$ commensurable with $SL_r(A)$.

Let $\pi : \Omega^r \longrightarrow \Delta\backslash\Omega^r$ be the universal covering map and let Ξ be an irreducible component of the inverse image of $X_{\mathbb{C}_\infty}^{\mathrm{an}}$ via π. Then the restriction of π to Ξ is an unramified Galois covering with Galois group $\Delta_\Xi := Stab_\Delta(\Xi)$. Recall that the embedding $X_{\mathbb{C}\infty} \subset M_A^r(\mathcal{K})(\mathbb{C}_\infty)$ determines a non-isotrivial family φ of Drinfeld modules. Let $\varphi_{\eta_{\mathbb{C}\infty}}$ be the Drinfeld module corresponding to the generic point $\eta_{\mathbb{C}\infty}$ of $X_{\mathbb{C}\infty}$. Let $\bar{\eta}_{\mathbb{C}\infty}$ be the geometric point above $\eta_{\mathbb{C}\infty}$. Denote by $\varphi_{\bar{\eta}_{\mathbb{C}\infty}}$ the pullback of $\varphi_{\eta_{\mathbb{C}\infty}}$ to $\bar{\eta}_{\mathbb{C}\infty}$. Breuer and Pink [12], prove the following statement.

Theorem 4.1.1. *With the situation as above, if X is Hodge generic and if $End_{\bar{\eta}_{\mathbb{C}\infty}}(\varphi_{\bar{\eta}_{\mathbb{C}\infty}}) = A$ then the closure of Δ_Ξ in $SL_r(\mathbb{A}_f)$ is an open subgroup of $SL_r(\mathbb{A}_f)$.*

For the application in mind we need a version of the above theorem that applies to subvarieties in the product of two Drinfeld modular varieties.

Let $X_{\mathbb{C}\infty}$ be a smooth irreducible and locally closed algebraic subvariety of positive dimension in the product of the fine moduli schemes $M_A^r(\mathcal{K})_{\mathbb{C}\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}\infty}$, futhermore assume that the projection of $X_{\mathbb{C}\infty}$ on to its coordinate factors is not constant. These assumptions will stay in force until the end of this section.

$X_{\mathbb{C}\infty}$ determines a non isotrivial family of pairs of Drinfeld modules. Let $\eta := (\eta_1, \eta_2)$ denote the generic point of X. Then $\pi_1^{\mathrm{et}}(X, \bar{\eta})$, the étale fundamental group of X acts on the adelic Tate module $\hat{T}(\varphi_{\eta_2}) \times \hat{T}(\varphi_{\eta_2})$ via the monodromy action and a choice of basis for $\hat{A}^r$ gives us the associated monodromy representation.

$$\rho : \pi_1^{\mathrm{et}}(X, \bar{\eta}) \longrightarrow \mathrm{GL}_r(\hat{A}) \times \mathrm{GL}_r(\hat{A}) \subset \mathrm{GL}_r(\mathbb{A}_f) \times \mathrm{GL}_r(\mathbb{A}_f)$$

As in the previous paragraphs we denote by Γ^{geo} and Γ the images of $\pi_1^{\mathrm{et}}(X_{k^{sep}}, \bar{\eta})$ and $\pi_1^{\mathrm{et}}(X, \bar{\eta})$ in $\mathrm{GL}_r(\hat{A}) \times \mathrm{GL}_r(\hat{A})$ via ρ respectively.

Since $X_{\mathbb{C}_\infty}^{\mathrm{an}}$ is irreducible it is contained in an irreducible component of ${M_A^r(\mathcal{K})^{\mathrm{an}}}_{\mathbb{C}\infty} \times$

$M_A^r(\mathcal{K})^{\mathrm{an}}{}_{\mathbb{C}\infty}$ of the form $\Delta\backslash\Omega^r \times \Delta'\backslash\Omega^r$ where Δ and Δ' are congruence subgroups of $SL_r(K)$ that are commensurable with $SL_r(A)$.

Let Ξ be an irreducible component of the inverse image of $X_{\mathbb{C}_\infty}^{\mathrm{an}}$ via the morphism

$$\pi \times \pi : \Omega^r \times \Omega^r \to \Delta\backslash\Omega^r \times \Delta'\backslash\Omega^r,$$

and let $\Delta_\Xi := Stab_{\Delta\times\Delta'}(\Xi)$, be the stabilizer of Ξ in $\Delta \times \Delta'$. Then

$$\pi \times \pi \mid_\Xi : \Xi \to X_{\mathbb{C}\ \infty}^{\mathrm{an}}$$

is an unramified Galois covering with Galois group Δ_Ξ.

Let I be the index set $\{1, 2\}$ denote by Δ_{Ξ_i} the projection of Δ_Ξ to the i-th coordinate factor. Let N_i denote the kernel of the two projections. By Goursat's lemma [48, lemma 5.2.1] we have that the image of Δ_Ξ in $\Delta_{\Xi_1}/N_2 \times \Delta_{\Xi_2}/N_1$ is the graph of an isomorphism $\rho : \Delta_{\Xi_1}/N_2 \cong \Delta_{\Xi_2}/N_1$. In the ensuing sections we attempt to describe more precisely Δ_Ξ under various assumptions on the family of Drinfeld modules from which it arises.

For the following proposition, we ask the reader to recall the notation introduced at the beginning of this section.

Proposition 4.1.2. Γ^{geo} *is the closure of* $g^{-1} \cdot \Delta_\Xi \cdot g$ *in* $\mathrm{SL}_r(\mathbb{A}_f) \times \mathrm{SL}_r(\mathbb{A}_f)$ *for some* $g := (g_1, g_2) \in \mathrm{GL}_r(\mathbb{A}_f) \times \mathrm{GL}_r(\mathbb{A}_f)$

Proof. Let $i : K^{\mathrm{sep}} \hookrightarrow \mathbb{C}_\infty$ be a choice of an embedding and let (ξ_1, ξ_2) be a point above (η_1, η_2), the generic point of X. We denote by Λ_1 and Λ_2 the A-lattices corresponding to ξ_1 and ξ_2 respectively. The choice of a basis for the Tate module $\hat{T}(\eta_1)$ results in a embedding

$$\hat{A}^r \cong \Lambda \otimes_A \hat{A} \hookrightarrow F \otimes_A \hat{A} \cong {\mathbb{A}_f}^r$$

which is given by multiplication by some $g_1 \in \mathrm{GL}_r(\mathbb{A}_f)$, this holds for $\hat{T}(\eta_2)$ as well. Since the group $\Delta \times \Delta'$ stabilizes Λ_1 and Λ_2 we have

$$(g_1^{-1}, g_2^{-1}) \cdot \Delta_\Xi \cdot (g_1, g_2) \in \mathrm{SL}_r(\hat{A}) \times \mathrm{SL}_r(\hat{A}).$$

We now need to show that $\pi_1^{\mathrm{et}}(X_{k^{sep}}, \bar{\eta})$ and $(g_1^{-1}, g_2^{-1}) \cdot \Delta_\Xi \cdot (g_1, g_2)$ have the same images in $\mathrm{GL}_r(A/\mathfrak{a}A) \times \mathrm{GL}_r(A/\mathfrak{a}'A)$ for all nonzero ideals $\mathfrak{a}, \mathfrak{a}' \subset A$. For this we consider the étale Galois cover

$$\pi_\mathfrak{a} : M_A^r(\mathfrak{a})_{\mathbb{C}\infty} \times M_A^r(\mathfrak{a}')_{\mathbb{C}\infty} \longrightarrow M_A^r(1)_{\mathbb{C}\infty} \times M_A^r(1)_{\mathbb{C}\infty}.$$

This has Galois group G contained in $\mathrm{GL}_r(A/\mathfrak{a}A) \times \mathrm{GL}_r(A/\mathfrak{a}'A)$. Let

$$\Delta(\mathfrak{a}) = \{h \in \mathrm{GL}(\hat{A}) \mid {g_1}^{-1} h g_1 \equiv \mathrm{id} \bmod \mathfrak{a}\hat{A}\} \bigcap \mathrm{GL}_r(K)$$

and

$$\Delta'(\mathfrak{a}') = \{h \in \mathrm{GL}(\hat{A}) \mid {g_2}^{-1} h g_2 \equiv \mathrm{id} \bmod \mathfrak{a}'\hat{A}\} \bigcap \mathrm{GL}_r(K).$$

Then $\Delta(\mathfrak{a})\backslash\Omega^r \times \Delta'(\mathfrak{a}')\backslash\Omega^r$ is a connected component of $M_A^r(\mathfrak{a})_{\mathbb{C}\infty} \times M_A^r(\mathfrak{a}')_{\mathbb{C}\infty}$ lying above $\Delta\backslash\Omega^r \times \Delta'\backslash\Omega^r$. By abuse of notation we denote again by $\pi_\mathfrak{a}$ the restriction

$$\pi_\mathfrak{a} : \Delta(\mathfrak{a})\backslash\Omega^r \times \Delta'(\mathfrak{a}')\backslash\Omega^r \longrightarrow \Delta\backslash\Omega^r \times \Delta'\backslash\Omega^r,$$

of $\pi_\mathfrak{a}$ to $\Delta(\mathfrak{a})\backslash\Omega^r \times \Delta'(\mathfrak{a}')\backslash\Omega^r$. Finally let $X_\mathfrak{a}^{\mathrm{an}}$ be any connected component of ${\pi_\mathfrak{a}}^{-1}(X^{\mathrm{an}})$, one of these is $\Delta_\Xi \cap (\Delta(\mathfrak{a}) \times \Delta'(\mathfrak{a}'))\backslash\Xi$. This has Galois group

$$\Delta_\Xi/\Delta_\Xi \cap (\Delta(\mathfrak{a}) \times \Delta'(\mathfrak{a}')),$$

over $X_{\mathbb{C}_\infty}^{\mathrm{an}} := \Delta_\Xi\backslash\Xi$. Since Δ_Ξ is a quotient of $\pi_1^{\mathrm{et}}(X_{k^{sep}}, \bar{\eta})$ the composite morphism

$$\Delta_\Xi \hookrightarrow \ \Delta \times \Delta \longrightarrow \Delta \times \Delta/(\Delta(\mathfrak{a}) \times \Delta'(\mathfrak{a}')) \hookrightarrow G \hookrightarrow \mathrm{GL}_r(A/\mathfrak{a}A) \times \mathrm{GL}_r(A/\mathfrak{a}'A),$$

shows that the image of $\pi_1^{\mathrm{et}}(X_{k^{sep}}, \bar{\eta})$ in $\mathrm{GL}_r(A/\mathfrak{a}A) \times \mathrm{GL}_r(A/\mathfrak{a}'A)$ is $\Delta_\Xi/\Delta_\Xi \cap (\Delta(\mathfrak{a}) \times \Delta'(\mathfrak{a}'))$. Now the reduction homomorphism

$$\Delta_\Xi \hookrightarrow (g_1^{-1}, g_2^{-1}) \cdot \Delta_\Xi \cdot (g_1, g_2) \hookrightarrow \mathrm{SL}_r(\hat{A}) \times \mathrm{SL}_r(\hat{A}) \longrightarrow \mathrm{SL}_r(A/\mathfrak{a}A) \times \mathrm{SL}_r(A/\mathfrak{a}'A)$$

has kernel $\Delta_\Xi \cap (\Delta(\mathfrak{a}) \times \Delta'(\mathfrak{a}'))$ while the image of $(g_1^{-1}, g_2^{-1}) \cdot \Delta_\Xi \cdot (g_1, g_2)$ via the reduction homomorphism is

$$(g_1^{-1}, g_2^{-1}) \cdot \Delta_\Xi \cdot (g_1, g_2)/((g_1^{-1}, g_2^{-1}) \cdot \Delta_\Xi \cdot (g_1, g_2)) \cap \Gamma(\mathfrak{a})$$

The isomorphism

$$\Delta_\Xi/\Delta_\Xi \cap (\Delta(\mathfrak{a}) \times \Delta'(\mathfrak{a}')) \cong (g_1^{-1}, g_2^{-1}) \cdot \Delta_\Xi \cdot (g_1, g_2)/((g_1^{-1}, g_2^{-1}) \cdot \Delta_\Xi \cdot (g_1, g_2)) \cap \Gamma(\mathfrak{a})$$

then show that the image of $(g_1^{-1}, g_2^{-1}) \cdot \Delta_\Xi \cdot (g_1, g_2)$ in $\mathrm{SL}_r(A/\mathfrak{a}A) \times \mathrm{SL}_r(A/\mathfrak{a}'A)$ is $\Delta_\Xi \cap (\Delta(\mathfrak{a}) \times \Delta'(\mathfrak{a}'))$. In particular $\pi_1^{\mathrm{et}}(X_{k^{sep}}, \bar{\eta})$ and $(g_1^{-1}, g_2^{-1}) \cdot \Delta_\Xi \cdot (g_1, g_2)$ have the same image in $\mathrm{SL}_r(A/\mathfrak{a}A) \times \mathrm{SL}_r(A/\mathfrak{a}'A)$ for all nonzero ideals $\mathfrak{a}, \mathfrak{a}' \subset A$. Taking the direct limit over all nonzero pairs of ideals $\mathfrak{a}, \mathfrak{a}'$ proves our claim.

□

We quote the following particular case of a very deep theorem of Pink (see [42, Thm 0.2],[41, Thm 2.8]). It gives us an easy criterion to decide openess of images of Galois representations.

Theorem 4.1.3. *Let L_1 and L_2 be local fields. Let $L := L_1 \oplus L_2$ and let Γ be a compact subgroup of $GL_n(L) = \mathrm{GL}_n(L_1) \times \mathrm{GL}_n(L_2)$. Let $i \in \{1, 2\}$, denote by ${\rho_i}^{\mathrm{ad}}$ the adjoint representation of Γ_i and set $\rho^{\mathrm{ad}} = {\rho_1}^{\mathrm{ad}} \oplus {\rho_2}^{\mathrm{ad}}$. Furthermore let $O_{\rho^{\mathrm{ad}}} \subset L$ be the closure of the subring generated by 1 and by $\mathrm{tr}(\rho^{\mathrm{ad}}(\Gamma))$.*
Let $E_{\rho^{\mathrm{ad}}} = \{\frac{x}{y} \mid x, y \in O_{\rho^{\mathrm{ad}}}, y \in L^\times\} \subset L$.
Suppose that Γ_i, the image of Γ in $\mathrm{PGL}_n(L_i)$ is Zariski dense. Then if $E_{\rho^{\mathrm{ad}}} = L$, the closure of the commutator subgroup of Γ is open in $\mathrm{SL}_n(L) = \mathrm{SL}_n(L_1) \times \mathrm{SL}_n(L_2)$.

Proposition 4.1.4. *Let φ_1 and φ_2 be a pair of Drinfeld modules defined over a common field of definition L and such that $End(\varphi_1) = A = End(\varphi_2)$. Let K be a finitely generated extension of L. Let $\rho^1_{\mathfrak{p}}$ and $\rho^2_{\mathfrak{p}}$ be the Galois representations attached to the rational $\mathfrak{p}$-adic Tate modules of φ_1 and φ_2 respectively. Let*

$$\rho_{\mathfrak{p}} := (\rho^1_{\mathfrak{p}}, \rho^2_{\mathfrak{p}}) : \mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow \mathrm{GL}_r(L_{\mathfrak{p}}) \times \mathrm{GL}_r(L_{\mathfrak{p}})$$

be the joint $\mathfrak{p}$-adic representation. Denote by $\Gamma_{\mathfrak{p}}$ the image of $\rho_{\mathfrak{p}}$ in $\mathrm{GL}_r(L_{\mathfrak{p}}) \times \mathrm{GL}_r(L_{\mathfrak{p}})$ and for brevity set $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$.

(i). If φ_1 and φ_2 are isogenous then $\rho^1_{\mathfrak{p}}(G_K) = \rho^2_{\mathfrak{p}}(G_K)$ and the image of G_K is open in each factor $\mathrm{GL}_r(L_{\mathfrak{p}})$. Furthermore $\Gamma_{\mathfrak{p}}$ is the graph of an inner automorphism of $\rho^1_{\mathfrak{p}}(G_K)$ in $\mathrm{GL}_r(L_{\mathfrak{p}}) \times \mathrm{GL}_r(L_{\mathfrak{p}})$.

(ii). Suppose $(r, p) \neq (2, 2)$. If φ_1 and φ_2 are non-isogenous of rank greater than two then the Zariski closure of the derived group of $\Gamma_{\mathfrak{p}}$ is open in $SL_r(L_{\mathfrak{p}}) \times SL_r(L_{\mathfrak{p}})$.

Proof. (i). If φ_1 and φ_2 are isogenous then by the Tate conjecture [53] the Galois representations $\rho^1_{\mathfrak{p}}$ and $\rho^2_{\mathfrak{p}}$ are equivalent. Thus there exists $h \in \mathrm{GL}_r(L_{\mathfrak{p}})$ such that $\rho^2_{\mathfrak{p}} = h^{-1}\rho^1_{\mathfrak{p}}h$. Hence the image of ρ in $\mathrm{GL}_r(L_{\mathfrak{p}}) \times \mathrm{GL}_r(L_{\mathfrak{p}})$ is

$$\Gamma_{\mathfrak{p}} = \{(\rho^1_{\mathfrak{p}}(g), h^{-1}\rho^1_{\mathfrak{p}}(g)h) \mid g \in \mathrm{Gal}(K^{\mathrm{sep}}/K)\},$$

i.e. $\Gamma_{\mathfrak{p}}$ is the graph in $\mathrm{GL}_r(L_{\mathfrak{p}}) \times \mathrm{GL}_r(L_{\mathfrak{p}})$ of an inner automorphism of $\rho^1_{\mathfrak{p}}(G_K)$. The second claim follows directly from a theorem of Pink (see [41, Thm 0.1]) on the openness of the image of Galois representations attached to Drinfeld modules without complex multiplication in generic characteristic.

(ii). Let G be the Zariski closure of $\Gamma_{\mathfrak{p}}$ in $\mathrm{GL}_{r,L_{\mathfrak{p}}} \times \mathrm{GL}_{r,L_{\mathfrak{p}}}$. Denote by G_i the projection of $\Gamma_{\mathfrak{p}}$ in $\mathrm{GL}_{r,L_{\mathfrak{p}}} \times \mathrm{GL}_{r,L_{\mathfrak{p}}}$ to the i-th factor. Also set $L = L_{\mathfrak{p}} \oplus L_{\mathfrak{p}}$. Denote by

$\Gamma_{\mathfrak{p},i}$ the projection of $\Gamma_{\mathfrak{p}}$ to the i-th coordinate factor of $\mathrm{GL}_r(L_{\mathfrak{p}}) \times \mathrm{GL}_r(L_{\mathfrak{p}})$. Since each $\Gamma_{\mathfrak{p},i}$ is the image of a representation coming from a non CM Drinfeld module it is open in $\mathrm{GL}_r(L_{\mathfrak{p}})$ by theorem [41, Thm 0.1]. Furthermore $(\mathfrak{p}, r) \neq (2, 2)$ hence by proposition [42, Prop 7.1] and [42, Prop 0.6] respectively the projections to each summand $(L_{\mathfrak{p}}, G_i, \Gamma_{\mathfrak{p},i})$ are minimal and $E_{\rho_i{}^{\mathrm{ad}}} = L_{\mathfrak{p}}$ respectively. Hence by theorem 4.1.3 above to prove our theorem it suffices to show that $E_{\rho^{\mathrm{ad}}} = L$ (see theorem 4.1.3 for notation). By theorem [42, Thm 2.3 9(a)], $E_{\rho^{\mathrm{ad}}}$ is semisimple, therefore if $E_{\rho^{\mathrm{ad}}} \neq L_{\mathfrak{p}} \oplus L_{\mathfrak{p}}$ we have that $E_{\rho^{\mathrm{ad}}}$ is a subfield of L. By theorem [42, Thm 0.2 (a)] there is a quasi model $(E_{\rho^{\mathrm{ad}}}, H, \varphi)$ of $(L, G, \Gamma_{\mathfrak{p}}$ where $\varphi : H \times_{E_{\rho^{\mathrm{ad}}}} L \longrightarrow G$ is an isogeny and H is a absolutely simple adjoint group over $E_{\rho^{\mathrm{ad}}}$ with $\varphi(H(E_{\rho^{\mathrm{ad}}})) \subset G(L)$. Since the projections to each summand $(L_{\mathfrak{p}}, G_i, \Gamma_{\mathfrak{p},i})$ are minimal we have by proposition [42, Prop 3.13 (b)], that there exists ρ_0 such that $\rho^{\mathrm{ad}} := \rho_1^{\mathrm{ad}} \oplus \rho_2^{\mathrm{ad}} = \rho_0 \cdot \varphi$ where ρ_0 is a representation of H and φ is an isomorphism. This implies that the adjoint representations ρ_i^{ad} are isomorphic after applying the automorphism $x \to (x^t)^{-1}$ of $L_{\mathfrak{p}}$. Hence the image of the Galois representation in $\mathrm{PGL}_{n,L_{\mathfrak{p}}} \times \mathrm{PGL}_{n,L_{\mathfrak{p}}}$ is in either the diagonal or is a graph of the automorphism of $\mathrm{PGL}_{n,L_{\mathfrak{p}}}$ induced by the automorphism $x \to (x^t)^{-1}$ of $L_{\mathfrak{p}}$. If the image lies (up to conjugation) in the graph of the automorphism $x \to (x^t)^{-1}$ then the tensor product of the two representations has weight $0 < 2/r < 1$ but possesses a quotient representation of dimension one, which is a contradiction. Thus the image is in the diagonal. Then the image(lifts) of the Galois representations in $\mathrm{GL}_{n,L_{\mathfrak{p}}}$ differ by a character. Hence the tensor product of the first representation and the dual of the second representation is pure of weight zero and possesses a quotient representation of dimension one. Let ψ be the character associated with this representation.

ψ factors through G_K^{ab} and by compactness lands in $O_{L\mathfrak{p}}^*$. By class field theory, ψ and the Artin map induces a character $\psi : K^*\backslash\mathbb{A}_{fK}^* \to O_{L\mathfrak{p}}^*$. This has component only at $\mathfrak{p}$ and factors through an open subgroup hence is a finite character. Therefore the original ψ itself is finite. Hence after extending the base field the two represention are isomorphic. This contradicts the following fact: if φ_1 and φ_2 are non-isogenous then by the analog of the Tate conjecture for Drinfeld modules we have that ρ_1 and ρ_2 are not equivalent. Therefore we have $E_\rho = L_\mathfrak{p} \oplus L_\mathfrak{p}$ and consequently that $\Gamma_\mathfrak{p}$ is open in $\mathrm{GL}_r(L_\mathfrak{p}) \times \mathrm{GL}_r(L_\mathfrak{p})$ by theorem 4.1.3. Hence $\Gamma_\mathfrak{p}$ is also open in $SL_r(L_\mathfrak{p}) \times SL_r(L_\mathfrak{p})$.

□

The second part of the following proposition will allow us to deduce an irreducibility statement for Hecke correspondences (Theorem 4.2.2) in the next section. We remark that the proof presented here is modeled on that of [12, Thm 1.1] and we sometimes quote from this reference. Although some of the statements we quote from there are stated only for SL_r and $X \subset M_A^r(\mathcal{K})_{\mathbb{C}\infty}$, many arguments go through vertabim for $SL_r \times SL_r$ or indeed for any general semisimple linear group and $X_{\mathbb{C}\infty} \subset M_A^r(\mathcal{K})_{\mathbb{C}\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}\infty}$. When this is the case we will just refer to the above mentioned reference.

Let φ be the non-isotrivial family of ordered pairs of Drinfeld modules over $\mathbb{C}_\infty$ determined by the embedding $X_{\mathbb{C}\infty} \subset M_A^r(\mathcal{K})_{\mathbb{C}\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}\infty}$. Let $\eta_{\mathbb{C}\infty} = (\eta^1_{\mathbb{C}\infty}, \eta^2_{\mathbb{C}\infty})$ be the generic point of $X_{\mathbb{C}\infty}$ and $\bar{\eta}_{\mathbb{C}\infty}$ the geometric point above it. Let $\varphi_{\bar{\eta}_{\mathbb{C}\infty}}$ denote the pullback of φ to $\bar{\eta}_{\mathbb{C}\infty}$. Let $k \subset \mathbb{C}_\infty$ be a finitely generated extension of K such that $X_{\mathbb{C}_\infty} = X \times_k \mathbb{C}_\infty$ for some $X \subset M_A^r(\mathcal{K})_k \times M_A^r(\mathcal{K})_k$. Let F be the function field of X, then η corresponds to the morphism $\mathrm{Spec}(F) \longrightarrow X$.

Lemma 4.1.5. *Let $X_{\mathbb{C}\infty} \subset M_A^r(\mathcal{K})_{\mathbb{C}\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}\infty}$ as above. Then the analytic* *fundamental group is* Δ_Ξ *is infinite.*

Proof. Suppose to the contrary that Δ_Ξ is finite. Then after increasing the level structure we may assume that $\Delta_\Xi = 1$. This implies that the Geometric fundamental group $\Gamma^{\mathrm{geo}} = 1$. Thus the monodromy representation factors through

$$\pi_1(X, \bar{\eta}) \to \mathrm{Gal}(k^{\mathrm{sep}}/k) \to \mathrm{GL}_r(F_\mathfrak{p}) \times \mathrm{GL}_r(F_\mathfrak{p})$$

After a suitable finite extension of the constant field k we may assume that X possesses a k-rational point x. Denote by φ_x the pair of Drinfeld modules over k corresponding to x. Via the embedding $k \longrightarrow K$ we may consider the pair of Drinfeld modules as defined over K and compare it with the pair $\varphi_{\bar{\eta}}$. The factorization above implies that the Galois representations on the component $\mathfrak{p}$-adic Tate modules of φ_x and $\varphi_{\bar{\eta}}$ are isomorphic. By the Tate conjecture [53] this implies that there exists an isogeny between the components of $\varphi_{\bar{\eta}}$ and φ_x over K. The kernels of each of these isogenies is finite and therefore defined over some finite extension k' of k. The quotients by these kernels are isomorphic to a Drinfeld modules defined over k'. But by assumption $\dim X > 1$ hence $\bar{\eta}$ is not a closed point of $M_A^r(\mathcal{K})_k \times M_A^r(\mathcal{K})_k$; hence $\varphi_{\bar{\eta}}$ cannot be defined over a finite extension of k. We therefore have a contradiction. $\square$

Lemma 4.1.6. *Let H denote the Zariski closure of Δ_Ξ in $\mathrm{GL}_{r,F} \times \mathrm{GL}_{r,F}$. Then H is a normal subgroup of $\mathrm{GL}_{r,F} \times \mathrm{GL}_{r,F}$.*

Proof. Let $\mathfrak{p} \neq \infty$ be a prime of F and denote by $\Gamma_\mathfrak{p}$ the projection of Γ to $\mathrm{GL}_r(F_\mathfrak{p}) \times \mathrm{GL}_r(F_\mathfrak{p})$. The base change $H_\mathfrak{p}$ is the Zariski closure of Δ_Ξ in $\mathrm{GL}_{r,F_\mathfrak{p}} \times \mathrm{GL}_{r,F_\mathfrak{p}}$. Lemma 4.1.2 implies that $g^{-1}H_\mathfrak{p}g$ is the Zariski closure of ${\Gamma_\mathfrak{p}}^{\mathrm{geo}}$ in $\mathrm{GL}_{r,F_\mathfrak{p}} \times \mathrm{GL}_{r,F_\mathfrak{p}}$. Since $\Gamma_\mathfrak{p}$ normalises ${\Gamma_\mathfrak{p}}^{\mathrm{geo}}$ it normalises $g^{-1}H_\mathfrak{p}g$. But by proposition 4.1.4(ii) $\Gamma_\mathfrak{p}$ is open in $\mathrm{GL}_r(F_\mathfrak{p}) \times \mathrm{GL}_r(F_\mathfrak{p})$ hence it is Zariski dense in $\mathrm{GL}_{r,F} \times \mathrm{GL}_{r,F}$. Therefore $\Gamma_\mathfrak{p}$ also normalises $g^{-1}H_{\mathfrak{p}}g$. This implies that $\mathrm{GL}_{r,F} \times \mathrm{GL}_{r,F}$ normalises $g^{-1}H_{\mathfrak{p}}g$ and hence also $H_{\mathfrak{p}}$. Therefore H is a normal subgroup of $\mathrm{GL}_{r,F_{\mathfrak{p}}} \times \mathrm{GL}_{r,F_{\mathfrak{p}}}$. $\square$

Proposition 4.1.7. *Let X be a subvariety of $M_A^r(\mathcal{K})_{\mathbb{C}\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}\infty}$ such that* $\mathrm{End}(\bar{\eta}_{1\mathbb{C}_\infty}) = A = \mathrm{End}(\bar{\eta}_{2\mathbb{C}_\infty})$.

(i). Suppose that the pair of Drinfeld modules underlying $\varphi_{\bar{\eta}_{\mathbb{C}\infty}}$ the generic point of X are isogenous. Then

$$\Delta_\Xi = \{(\tilde{g}, \rho(\tilde{g})) \mid \tilde{g} \in G\},$$

where G is a subgroup of $\mathrm{SL}_r(F)$ and ρ is an automorphism of G.

(ii). Suppose that the pair of Drinfeld modules underlying the point $\varphi_{\bar{\eta}_{\mathbb{C}\infty}}$ are non-isogenous. Then $\hat{\Delta}_\Xi$ is an open subgroup of $SL_r(\mathbb{A}_f) \times SL_r(\mathbb{A}_f)$.

Proof. (i). Suppose the pair of Drinfeld modules underlying the point $\varphi_{\bar{\eta}_{\mathbb{C}\infty}}$ are isogenous. We consider the representation attached to the pair of Drinfeld modules corresponding to η_1 and η_2. Now F is a common field of definition of η_1 and η_2. By lemma 4.1.4[(1)] the image of the $\mathfrak{p}$-adic representation $\rho_{\mathfrak{p}}$ is of the form

$$\rho_{\mathfrak{p}}(G_L) = \{(g, {h_{\mathfrak{p}}}^{-1}gh_{\mathfrak{p}}) \mid g \in H_{\mathfrak{p}}\},$$

where $H_{\mathfrak{p}}$ is an open subgroup of $\mathrm{GL}_r(F_{\mathfrak{p}})$ and $h \in \mathrm{GL}_r(F_{\mathfrak{p}})$. Since $H_{\mathfrak{p}}$ is open in $\mathrm{GL}_r(F_{\mathfrak{p}})$, the kernel of $\mathrm{GL}_r(F_{\mathfrak{p}})$ acting on the $H_{\mathfrak{p}}$ cosets is open and normal in $\mathrm{GL}_r(F_{\mathfrak{p}})$ and contained in $H_{\mathfrak{p}}$. Now every open normal subgroup of $\mathrm{GL}_r(F_{\mathfrak{p}})$ contains $\mathrm{SL}_r(F_{\mathfrak{p}})$ therefore $H_{\mathfrak{p}}$ contains $\mathrm{SL}_r(F_{\mathfrak{p}})$. We obtain

$$\rho(G_L) = \prod_{\mathfrak{p}} \rho_{\mathfrak{p}} = \prod_{\mathfrak{p}} \{(g_{\mathfrak{p}}, (h_{\mathfrak{p}})^{-1}g_{\mathfrak{p}}h_{\mathfrak{p}}) \mid g_{\mathfrak{p}} \in H_{\mathfrak{p}}\}.$$

Now the rational adelic Galois representation ρ is the composite homomorphism

$$\mathrm{Gal}(F^{\mathrm{sep}}/F) \cong \pi_1(\eta, \bar{\eta}) \to \pi_1(X, \bar{\eta}) \to \mathrm{GL}_r(\mathbb{A}_f) \times \mathrm{GL}_r(\mathbb{A}_f)$$

therefore Γ, the image of $\pi_1(X, \bar{\eta})$ coincides with the image of the rational adelic Galois representation $\rho(G_L)$. Furthermore $\Gamma_{\mathfrak{p}}$, the projections of Γ to $\mathrm{GL}_r(F_{\mathfrak{p}})$ contains $\mathrm{SL}_r(F_{\mathfrak{p}})$. By lemma 4.1.2

$$\overline{g^{-1} \cdot \Delta_{\Xi} \cdot g} = \Gamma^{\mathrm{geo}} \subset \Gamma \subset \mathrm{GL}_r(\mathbb{A}_f) \times \mathrm{GL}_r(\mathbb{A}_f).$$

Since Γ^{geo} is the Zariski closure of $g^{-1} \cdot \Delta_{\Xi} \cdot g \subset \mathrm{SL}_r(\hat{A}) \times \mathrm{SL}_r(\hat{A})$ and Γ^{geo} is a subgroup of Γ this implies that Δ_{Ξ} is a proper subgroup of $\mathrm{SL}_r(F) \times \mathrm{SL}_r(F)$ hence $\Delta_{\Xi} = \{(\tilde{g}, \rho(g)) \mid \tilde{g} \in G \subset \mathrm{SL}_r(F)\}$, where G is a subgroup of $\mathrm{SL}_r(F)$ and ρ is an automorphism of G.

(ii). If the pair of Drinfeld modules underlying the point $\varphi_{\bar{\eta}_{\mathbb{C}\infty}}$ are non-isogenous, then as in the previous argument the image of Galois coincides with Γ. By proposition 4.1.4(ii) it follows that $\Gamma_{\mathfrak{p}}$, the projection of Γ to $\mathrm{GL}_r(F_{\mathfrak{p}}) \times \mathrm{GL}_r(F_{\mathfrak{p}})$ is open. Let H denote the Zariski closure of Δ_{Ξ} in $\mathrm{GL}_{r,F} \times \mathrm{GL}_{r,F}$. By construction $H \subset SL_{r,F} \times \mathrm{SL}_{r,F}$ and by lemma 4.1.6 H is a normal subgroup of $\mathrm{GL}_{r,F} \times \mathrm{GL}_{r,F}$. Furthermore Δ_{Ξ} is infinite by lemma 4.1.5, hence H is not contained in the center of $SL_{r,F} \times \mathrm{SL}_{r,F}$ and has surjective projections to each factor of $SL_{r,F} \times \mathrm{SL}_{r,F}$. Let N_1 and N_2 denote the kernels of these projections. Then

$$H/N_1 \times N_2 \cong \mathrm{SL}_{r,F} / N_1 \cong \mathrm{SL}_{r,F} / N_2.$$

Since H has surjective projections to each factor of $SL_{r,F} \times \mathrm{SL}_{r,F}$ we must have that N_1 and N_2 are not central i.e. $N_1 = \mathrm{SL}_{r,F} = N_2$. Therefore $H = SL_{r,F} \times \mathrm{SL}_{r,F}$. Therefore Δ_{Ξ} is Zariski dense in $SL_{r,F} \times \mathrm{SL}_{r,F}$. Now Δ_{Ξ} is contained in a congruence subgroup commensurable with $SL_r(A) \times SL_r(A)$, hence it is integral at all places except infinity. We note that our proof of 4.1.4(ii) also shows that the triple $(F \oplus F, SL_{r,F} \times \mathrm{SL}_{r,F}, \Delta_{\Xi})$ is minimal, in the sense of [42]. Thus in summary Δ_Ξ is a Zariski dense subgroup of the connected, semisimple group $SL_r(F) \times SL_r(F)$ that is minimal, integral at all places except infinity. Therefore theorem [43, Thm 0.2] of Pink applies and we conclude that $\hat{\Delta}_\Xi$ is an open subgroup of $SL_r(\mathbb{A}_f) \times SL_r(\mathbb{A}_f)$

□

4.2 Irreducibility of Hecke Correspondences

Let S in $\mathrm{GL}_r(\mathbb{A}_f)$ be a set of representatives of the finite set $\mathrm{GL}_r(K)\backslash \mathrm{GL}_r(\mathbb{A}_f)/\mathcal{K}$. For each $s \in S$ we define the arithmetic group $\Gamma_s := s\mathcal{K}s^{-1} \cap \mathrm{GL}_r(K)$. We have a bijection

$$\mathrm{GL}_r(K)\backslash \mathrm{GL}_r(\mathbb{A}_f) \times \Omega^r/\mathcal{K} \xrightarrow{\sim} \textstyle\coprod_{s\in S} \Gamma_s\backslash\Omega^r$$

Hence

$$M_A^3(\mathcal{K})^{\mathrm{an}}_{\mathbb{C}_\infty} \xrightarrow{\sim} \coprod_{s\in S} \Gamma_s\backslash\Omega^3.$$

For $g = \mathrm{diag}(\mathfrak{p}, 1, \dots, 1) \in \mathrm{GL}_r(\mathbb{A}_f)$ and $x = (x_1, x_2) \in M_A^r(\mathcal{K})_{\mathbb{C}_\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ we define the Hecke orbit $(T_g \times T_g)^\infty(x)$ inductively as follows

$$(T_g \times T_g)(x) = (T_g \times T_g)(x_1, x_2) \cup (T_{g^{-1}} \times T_{g^{-1}})(x_1, x_2)$$

$$(T_g \times T_g)^{n+1}(x) = (T_g \times T_g)^n(x) \cup (T_g \times T_g)(x)$$

$$(T_g \times T_g)^\infty(x) = \bigcup_{n>0} (T_g \times T_g)^{n-1}(x)$$

Theorem 4.2.1. *Let $x \in M_A^r(\mathcal{K})_{\mathbb{C}_\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ where $\mathcal{K}_\mathfrak{p}$ is a principal congruence subgroup of $\mathrm{GL}_r(A_p)$, $g_\mathfrak{p} = \mathrm{diag}(\mathfrak{p}, 1, \dots, 1) \in \mathrm{GL}_r(\mathbb{A}_f)$ and $g = g_1 \cdot g_\mathfrak{p} \cdot g_2$ where $g_1, g_2 \in \mathrm{GL}_r(\mathbb{A}_f)$, then for any integer k, the Hecke orbit $(T_{g^k} \times T_{g^k})^\infty(x)$ is Zariski dense in an irreducible component of $M_A^r(\mathcal{K})_{\mathbb{C}_\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$.*

Proof. Assume $x_1 \in \Gamma_{s_0}\backslash\Omega^r$, then by [33][Theorem 5.1.2]

$$(T_{g^k})^\infty(x_1) \cap \Gamma_s\backslash\Omega^r = \{[T\omega] \in \Gamma_s\backslash\Omega^r : T \in s < \mathcal{K}g^k\mathcal{K} > {s_0}^{-1} \cap \mathrm{GL}_r(K)\},$$

where $< \mathcal{K}g^k\mathcal{K} >$ denotes the subgroup of $\mathrm{GL}_r(\mathbb{A}_f)$ generated by the double coset $\mathcal{K}g^k\mathcal{K}$. Thus again by the content of [33][Theorem 5.2.2] to prove our theorem it suffices to show that the image of subgroup of $\mathrm{GL}_r(F_p)$ generated by the $\mathfrak{p}$-component of g^k in $\mathrm{PGL}_r(F_p)$ is unbounded. Let

$$f(\lambda) = \lambda^r + a_{r-1}\lambda^{r-1} + \cdots + a_1\lambda + a_0$$

be the characteristic polynomial of $(g_{1\mathfrak{p}}g_{\mathfrak{p}}g_{1\mathfrak{p}})^k$. Since $\mathrm{Det}((\mathrm{g}_{1\mathfrak{p}}), \mathrm{Det}((\mathrm{g}_{1\mathfrak{p}}) \in {F_{\mathfrak{p}}}^*$, the $\mathfrak{p}$-valuation

$$v_{\mathfrak{p}}(a_0) = v_{\mathfrak{p}}(\mathrm{Det}(g_{1\mathfrak{p}}g_{\mathfrak{p}}g_{1\mathfrak{p}})^k) = k \cdot v_{\mathfrak{p}}(\mathrm{Det}\, g_{\mathfrak{p}}) = -k$$

Now

$$a_{r-1} = -\mathrm{tr}((g_{1\mathfrak{p}}g_{\mathfrak{p}}g_{1\mathfrak{p}})^k) = -\sum_{i,j}((g_{1\mathfrak{p}}g_{\mathfrak{p}}g_{1\mathfrak{p}})^k)_{i,j},$$

and since $g_{1\mathfrak{p}}$ and $g_{2\mathfrak{p}}$ are in the principal congruence subgroup of $\mathrm{GL}_r(A_p)$ by assumption, we have $v_{\mathfrak{p}}((g_{1\mathfrak{p}})_{i,j}), v_{\mathfrak{p}}((g_{1\mathfrak{p}})_{i,j}) \geq 0$, with equality if $i = j$. This implies $(g_{1\mathfrak{p}}g_{\mathfrak{p}}g_{1\mathfrak{p}})^k)_{i,j}$ has $\mathfrak{p}$-valuation k. Therefore

$$v_{\mathfrak{p}}(a_{r-1}) = -k,$$

in particular the newton polygon of $f(\lambda)$ has at least two lines. This implies the image of subgroup of $\mathrm{GL}_r(F_p)$ generated by the $\mathfrak{p}$-component of g^k in $\mathrm{PGL}_r(F_p)$ is unbounded. Then same as in [33][5.2.2] the Hecke orbits $(T_g \times T_g)^\infty(x_1)$ and $(T_g \times T_g)^\infty(x_2)$ are dense in an irreducible component of $M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$, since these two coordinates are independent, the Hecke orbit $(T_g \times T_g)^\infty(x)$ in $M_A^r(\mathcal{K})_{\mathbb{C}_\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ is Zariski dense as well in an irreducible component of $M_A^r(\mathcal{K})_{\mathbb{C}_\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$. $\square$

Let X be a Hodge generic and F-irreducible subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$ defined over F and assume that the underlying pair of Drinfeld modules of the generic point of X are non-isogenous, and have both endomorphism rings A.

Let X' be an irreducible component of the preimage of X in $M_A^r(\mathcal{K})_{\mathbb{C}_\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ via the canonical projection. X' is defined over a a finite extension L of F. After replacing X' with its non singular locus we may assume X' is smooth and still Hodge generic with the generic point of X' consisting of non-isogenous underlying Drinfeld modules which have both endomorphism rings A. Assume that $\mathcal{K} = \prod_{\mathfrak{p}} \mathcal{K}_{\mathfrak{p}} = \mathcal{K}_{\mathfrak{p}} \times \mathcal{K}^{\mathfrak{p}}$ where $\mathcal{K}_{\mathfrak{p}}$ is contained in $\mathrm{GL}_r(K_{\mathfrak{p}})$, the first component $\mathcal{K}_{\mathfrak{p}}$ is a principal congruence subgroup of $\mathrm{GL}_r(A_p)$ and that

$$X' \subset \sigma \cdot (T_g \times T_g)(X')$$

for any g as in 4.2.1 and σ, a Galois automorphim of K. Let

$$\pi \times \pi : M_A^3(\mathcal{K}^{\mathfrak{p}}) \times M_A^3(\mathcal{K}^{\mathfrak{p}}) \to M_A^3(\mathcal{K}) \times M_A^3(\mathcal{K}),$$

then $\pi \times \pi$ is a $\mathcal{K}_{\mathfrak{p}} \times \mathcal{K}_{\mathfrak{p}}$-cover and by proposition 4.1.7 the monodromy representation associated with $\pi \times \pi$ has open image $\mathcal{K}'_{\mathfrak{p}} \times \mathcal{K}'_{\mathfrak{p}}$ in $\mathcal{K}_{\mathfrak{p}} \times \mathcal{K}_{\mathfrak{p}}$. Now

$$\pi' \times \pi' : M_A^3(\mathcal{K}'_{\mathfrak{p}} \times \mathcal{K}^{\mathfrak{p}}) \times M_A^3(\mathcal{K}'_{\mathfrak{p}} \times \mathcal{K}^{\mathfrak{p}}) \to M_A^3(\mathcal{K}) \times M_A^3(\mathcal{K})$$

is a $\mathcal{K}'_{\mathfrak{p}} \times \mathcal{K}'_{\mathfrak{p}}$-cover of $M_A^3(\mathcal{K}) \times M_A^3(\mathcal{K})$ thus $Z' := (\pi' \times \pi')^{-1}(X')$ is irreducible. Therefore $(T_g \times T_g)(Z')$ is irreducible as well.

Since $(T_g \times T_g)(Z')$ is irreducible and $Z' \subset \sigma \cdot (T_g \times T_g)(Z')$ we have

$$Z' = \sigma \cdot (T_g \times T_g)(Z').$$

Iterating we see that

$$Z' = \sigma \cdot (T_g \times T_g)(Z') = \sigma^2 \cdot (T_g \times T_g)((T_g \times T_g)(Z')) \supset \sigma^2 \cdot (T_{g^2} \times T_{g^2})(Z'),$$

e.t.c. Since Z' is irreducible and both sides have the same dimension we have

$$Z' = \sigma^k \cdot (T_{g^k} \times T_{g^k})(Z'),$$

for all $k \geq 0$. Therefore for large enough integers k,

$$Z' = (T_{g^k} \times T_{g^k})(Z')$$

By theorem 4.2.1, the $(T_{g^k} \times T_{g^k})$-orbit of any point in Z' is Zariski-dense in an irreducible component of $M_A^3(\mathcal{K})_{\mathbb{C}_\infty} \times M_A^3(\mathcal{K})_{\mathbb{C}_\infty}$, but this orbit lies completely inside Z'. Therefore Z' is an irreducible components of $M_A^3(\mathcal{K})_{\mathbb{C}_\infty} \times M_A^3(\mathcal{K})_{\mathbb{C}_\infty}$. Thus its projection X is a Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$. We have thus proved the following.

Theorem 4.2.2. *Let $X \subset M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$ be an F-irreducible Hodge generic subvariety and assume that the underlying pair of Drinfeld modules of the generic point of X are non-isogenous, have both endomorphism rings A and the characristic of F is not 3. Let $\mathcal{K}$ be an admissible level structure of the form $\mathcal{K} = \prod_{\mathfrak{p}} \mathcal{K}_{\mathfrak{p}}$ where $\mathcal{K}_{\mathfrak{p}}$ is contained in $\mathrm{GL}_r(K_{\mathfrak{p}})$, and large enough so that $M_A^3(\mathcal{K})_{\mathbb{C}_\infty}$ is a fine moduli scheme. Let X' be the smooth locus of a Hodge generic, irreducible component of the preimage of X in $M_A^r(\mathcal{K})_{\mathbb{C}_\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ such that for $g = \mathrm{diag}(\mathfrak{p}, 1, \ldots, 1) \in \mathrm{GL}_r(\mathbb{A}_f)$ we have $X' \subset \sigma \cdot (T_g \times T_g)(X')$. Then X is a Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$.*

We remark that it is in this theorem that we need the condition that the characteristic is not 3.

4.3 Modular Polynomials

For most of this section we assume that $A = \mathbb{F}_q[T]$ until we signal to drop this restriction. Let φ and φ' be two A-Drinfeld modules of rank r with

$$\varphi_T = T\tau^0 + g_1\tau + \cdots + g_{r-1}\tau^{r-1} + g_r\tau^r, \qquad \Delta = g_r \neq 0.$$

Then φ and φ' are isomorphic if and only if there exists $\lambda \neq 0$ such that $g_i' = \lambda^{q^i-1}g_i$ for $i = 1, \ldots, r$. It follows that the tuple $(g_1, \ldots, g_r)$ representing φ is equivalent to $(\lambda^{q^1-1}g_1', \ldots, \lambda^{q^r-1}g_r)$, hence the coarse moduli scheme parameterizing Drinfeld A-modules of rank $\leq r$ is the weighted projective space

$$\mathbb{P}_A(q-1, q^2-1, \ldots, q^r-1) := \operatorname{Proj} A[g_1, \ldots, g_{r-1}, \Delta],$$

where each g_i has weight $q^i - 1$ and Δ has weight $q^r - 1$. The moduli scheme of rank r Drinfeld A-modules $M_A^r(1)$, is then the quasiprojective subvariety of $\mathbb{P}_A(q-1, q^2-1, \ldots, q^r-1)$ given by the open condition that $\Delta \neq 0$. When $A = \mathbb{F}_q[T]$, Potemine [45] has given an explicit description of the affine scheme $M_A^r(1)$ by giving expicitly the generators of its coordinate ring.

Consider a multi-index $(k_1, \ldots, k_l)$ with $1 \leq k_1 < \cdots < k_l \leq r-1$. Set $d(k_1, \ldots, k_l) := \gcd(k_1, \ldots, k_l)$ and let $\delta_1, \ldots, \delta_l, \delta_r$ be non-negative integers such that

(i). $\delta_1(q^{k_1} - 1) + \cdots + \delta_l(q^{k_l} - 1) = \delta_r(q^r - 1)$,

(ii). $0 \leq \delta_i \leq (q^r - 1)/(q^{\gcd(i,r)} - 1)$ for $\leq i \leq l$ and $d(\delta_1, \ldots, \delta_l, \delta_r) = 1$.

Definition 4.3.1. *With the above notation the* basic J-invariants *of φ are defined by*

$$J_{k_1,\ldots,k_l}^{\delta_1,\ldots,\delta_l}(\varphi) := \frac{g_{k_1}^{\delta_1} \cdots g_{k_l}^{\delta_l}}{\Delta^{\delta_r}} = u_{k_1}^{\delta_1} \cdots u_{k_l}^{\delta_l},$$

where $u_{k_i} = \frac{g_i}{\Delta^{(q^{k_i}-1)/(q^k-1)}}$.

We furthermore let

$$j_k(\varphi) = J_k^{\delta_k}(\varphi) = \frac{g_k^{(q^r-1)/(q^{d(k,r)}-1})}{g_r^{(q^k-1)/(q^{d(k,r)}-1)}}; 1 \leq k \leq r-1$$

We call $j_k(\varphi)$ the j_k-*invariant* of φ and we call the tuple $j(\varphi) = (j_1(\varphi), \ldots, j_{r-1}(\varphi))$ the j-*invariant* of φ. When $r = 2$

$$j(\varphi) = j_1^{\delta_1}(\varphi) = \frac{g_1^{q+1}}{\Delta},$$

which is the usual j-invariant of rank 2 Drinfeld modules. We have the following theorem due to Potemine (see [45]).

Theorem 4.3.2. *(i). If two Drinfeld modules φ and φ' of rank r over an A-scheme S are isomorphic then all $J_{k_1,\ldots,k_l}^{\delta_1,\ldots,\delta_l}(\varphi)$-invariants coincide.,*

(ii). Let φ and φ' be Drinfeld modules of rank r over a separably closed A-field L having the same j-invariant $j(\varphi) = j(\varphi')$. Then there exists a Drinfeld module φ'' isomorphic to φ such that

$$g_k(\varphi') = \mu_k \cdot g_k(\varphi''); \quad \mu_k^{(q^k-1)/(q^{d(k,r)}-1)} = 1; \quad 1 \le k \le r-1$$

(iii). Let φ and φ' be Drinfeld modules of rank r over a separably closed A-field L. If their basic J-invariants coincide:

$$J_{k_1,\ldots,k_l}^{\delta_1,\ldots,\delta_l}(\varphi) = J_{k_1,\ldots,k_l}^{\delta_1,\ldots,\delta_l}(\varphi')$$

for all integers $\delta_1, \ldots, \delta_l, \delta_r$ satisfying (1) and (2) then these modules are isomorphic.

As a consequence of this he also proves in the same paper

Theorem 4.3.3. *The affine toric A-variety of relative dimension $r-1$*

$$M_A^r(1) \cong \mathrm{Spec}(A[\{J_{k_1,\ldots,k_l}^{\delta_1,\ldots,\delta_l}\}])$$

is the coarse moduli scheme of Drinfeld A-modules of rank r.

Recall that analytically the moduli space $M_A^r(1)_{\mathbb{C}_\infty} = \mathrm{GL}_r(A)\backslash\Omega^r$. Two points $\tau, \tau' \in \Omega^r$ correspond to isomorphic (respectively isogenous) Drinfeld modules φ^τ and $\varphi^{\tau'}$ if and only if there exist $\sigma \in \mathrm{GL}_r(A)$ (respectively $\mathrm{GL}_r(K)$) such that $\sigma\tau = \tau'$. If f is a cyclic isogeny (see section 3.1) from φ^τ to $\varphi^{\tau'}$ and then $\sigma_f\tau = \tau'$ where

$$\sigma_f \in \mathbb{M}_n = \mathrm{GL}_r(A)\backslash \mathrm{GL}_r(A) \begin{pmatrix} \mathfrak{n} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \mathrm{GL}_r(A)$$

is a suitable representative of the above coset and $\mathfrak{n}$ is a nonzero ideal of A. Let $\{J_{k_1,\dots,k_l}^{\delta_1,\dots,\delta_l}\} = \{J_1, J_2 \dots, J_N\}$ be the list of all *basic J-invariants*. We consider as in the classic case the expression

$$P_{J_i,n} := \prod_{\varphi \xrightarrow{n} \varphi'} (J_i(\varphi) - J_i(\varphi'))$$

where the product runs over all φ' linked to φ by a cyclic n isogeny. Let $\{\sigma_i\}$ be a finite set of distinct representatives of the double coset $\mathbb{M}_n$ then this polynomial can be written as

$$P_{J_i,n} := \prod_{\sigma_i} (J_i(\varphi) - J_i(\sigma_i \cdot \varphi))$$

From this we can define the *modular polynomials*

$$P_{J_i,n}(X) := \prod_{\sigma_i} (X - J_i \cdot \sigma_i(\varphi)) \tag{4.0}$$

By [13, Theorem 1.1] the degree of X in $P_{J_i,n}(X)$ is

$$\prod_{p|n} \frac{|p|^r - 1}{|p|^r - |p|^{r-1}}.$$

It is clear that $P_{J_i,n}(X) = 0$ if and only if $X = J_i(\varphi')$ with $\varphi' \in T_n(\varphi)$. Let $I_g = \deg(T_g)$ be the degree of T_g as a correspondence then expanding expression 4.0

we obtain a polynomial $\sum_{n=0}^{I_g} a_{j,n} X^n$ in X with coefficients $a_{j,n}$. These coefficients are symmetric functions in $J_j \cdot \sigma_i$ of degree $I_g - n$ and permuted by Γ. Hence $a_{j,n} \in \mathcal{O}(\Gamma_{g_i} \backslash \Omega^r)$ i.e. the $a_{j,n}$ are holomorphic functions on $\Gamma \backslash \Omega^r$.

Definition 4.3.4. *Let $I \in A[J_1, \cdots, J_N]$, we denote by $w(I)$ the weighted degree of I, where each monomial is assigned the weight*

$$w(J_{k_1,\ldots,k_l}^{\delta_1,\ldots,\delta_l}) := \delta_r.$$

Let K be the quotient field of A, i.e. $K = \mathbb{F}_q(T)$. For a polynomial $n \in A$ we denote $|n| = q^{\deg n}$. The following theorem is due to Breuer and Rück[13].

Theorem 4.3.5. *Let $I \in A[J_1, \cdots, J_N]$ be an invariant of weighted degree $w(I)$, and $n \in A$ monic. Then*

(i). We have $P_{I,n}(X) \in A[J_1, \cdots, J_N][X]$, which has degree

$$\#J(n) = |n|^{r-1} \prod_{p|n} \frac{|p|^r - 1}{|p|^r - |p|^{r-1}}$$

in X, and is irreducible in $\mathbb{C}_\infty[J_1, \cdots, J_N][X]$.

(ii). The weighted degree of the coefficient $a_i \in A[J_1, \cdots, J_N]$ of X^i in $P_{I,n}(X)$ is bounded by:

$$w(a_i) \leq \left(|n|^{2(r-1)} \prod_{p|n} \frac{|p|^r - 1}{|p|^r - |p|^{r-1}} - i \right) w(I).$$

4.4 Degrees of Hecke Correpondences

To compute the degrees of Hecke correspondences on $M_A^r(1) \times M_A^r(1)$ we construct explicit polynomials (following Breuer and Rück [13]) that define the correspondence T_g for $g \in \mathrm{GL}_r(\mathbb{A}_f)$ then compute the degrees of these polynomials. We want to compute the degrees (as a variety) of Hecke correspondences on $M_A^r(1) \times M_A^r(1)$. We still assume that $A = \mathbb{F}_q[T]$. Then in this case the moduli of rank r Drinfeld A-modules without level structure is $M_A^r(1) = \mathrm{Spec}(A[J_1, \cdots, J_N])$, where the J_i are a list of all the *basic J-invariants*. The morphism $\theta : A[J_1, \cdots, J_N] \to A[X_1, \cdots, X_N]$ given by sending a basic J-invariant J_i to the indeterminate X_i, realizes $M_A^r(1)$ as a subvariety of the affine space $\mathbb{A}^N$ hence of the projective space $\mathbb{P}^N$.

Definition 4.4.1. *Let $X \subset M_A^r(1) \subset \mathbb{A}^N$ and let $\overline{\theta(X)}$ be the Zariski closure of X in $\mathbb{P}^N$. If Y_i are the irreducible components of this Zariski closure then $\overline{\theta(X)} = \bigcup Y_i$. We define the intersection degree of X to be the sum of the degrees of the irreducible components Y_i*

It follows from the definition that the graph of the Hecke correspondence T_g is a subvariety of $M_A^r(1) \times M_A^r(1)$. The locus of this graph is equal to the subvariety of corresponding to the ideal of modular polynomials

$$\langle P_{J_1,n}(Y_1; X_1, \ldots, X_N), \ldots, P_{J_N,n}(Y_N; X_1, \ldots, X_N)\rangle \subset \mathbb{C}_\infty[X_1, \ldots, X_N, Y_1, \ldots, Y_N]$$

The degree $\mathrm{Deg}(T_g)$ of $T_g \subset M_A^r(1) \times M_A^r(1) \subset \mathbb{A}^{2N}$ as a projective variety is given by

$\mathrm{Deg}(T_g) = \mathrm{Deg}\left(T_g \cap (M_A^r(1)_{\mathbb{C}_\infty} \times M_A^r(1)_{\mathbb{C}_\infty})\right)$ In particular if we let $g = \mathrm{diag}(\mathfrak{n}, 1, \cdots, 1)$ then by a variant of Bezout's Theorem for $\mathbb{A}^N$ [26, example 8.4.6], and by theorem 4.3.5

$$\mathrm{Deg}(T_\mathfrak{n}) = \mathrm{Deg}\left(T_\mathfrak{n} \cap (M_A^r(1)_{\mathbb{C}_\infty} \times M_A^r(1)_{\mathbb{C}_\infty}))\right) \leq \mathrm{Deg}\left(M_A^r(1)_{\mathbb{C}_\infty}\right)^2 \prod_{j=1}^{N} \left(|\mathfrak{n}|^{r-1}\psi_r(\mathfrak{n})^2 w(J_i)\right),$$

We now drop the restriction on A, so A is once again a general base ring (as defined in the introduction). For every transcendental element $T \in A$ we have $\mathbb{F}_q[T] \subset A$ and $K/\mathbb{F}_q(T)$ is an imaginary extension. Let d_T be the degree of this extension. We have a canonical morphism $\rho_T : M_A^r(1)_K \longrightarrow M_{\mathbb{F}_q[T]}^{rd_T}(1)_K$. For any subvariety

$X \subset M_A^r(1)_K$ we define the T*-degree* $\deg_T(X)$ to be the degree of $\rho_T(X) \subset \mathbb{A}_K^N$ as in definition 4.4.1. The following result is due to Breuer [8] and will be important to us later.

Proposition 4.4.2. *Let $T \in A$ be a transcendental element, then there exist computable positive constants $c = c(T)$ and $n = n(T)$ such that the following holds. Let $X \subset M_A^r(1)_K$ be an irreducible algebraic subvariety, and let $\mathfrak{P} \subset A$ be a prime which has residual degree one over $\mathfrak{p} := \mathfrak{P} \cap \mathbb{F}_q[T]$. Then*

$$\deg_T(T_{\mathfrak{P}}(X)) \leq c \deg_T(X)|\mathfrak{P}|^n.$$

In particular, if $X \cap T_{\mathfrak{P}}(X)$ is finite, then

$$|X \cap T_{\mathfrak{P}}(X)| \leq c \deg_T(X)^2|\mathfrak{P}|^n.$$

Corollary 4.4.3. *Let $T \in A$ be a transcendental element, then there exist computable positive constants $c = c(T)$ and $n = n(T)$ such that the following holds. Let $X \subset M_A^r(1)_K \times M_A^r(1)_K$ be an irreducible algebraic subvariety, and let $\mathfrak{P} \subset A$ be a prime which has residual degree one over $\mathfrak{p} := \mathfrak{P} \cap \mathbb{F}_q[T]$. Then*

$$\deg_T((T_{\mathfrak{P}} \times T_{\mathfrak{P}})(X)) \leq c^2 \deg_T(X)^2|\mathfrak{P}|^{2n}.$$

In particular, if $(X \cap T_{\mathfrak{P}} \times T_{\mathfrak{P}})(X)$ is finite, then

$$|X \cap (T_{\mathfrak{P}} \times T_{\mathfrak{P}})(X)| \leq c^2 \deg_T(X)^3|\mathfrak{P}|^{2n}.$$

Proof. Since $M_A^r(1) \subset \mathbb{A}^N$ we can consider X as a subvariety of $\mathbb{P}^N \times \mathbb{P}^N$. The Chow group of $\mathbb{P}^N$ has the particular simple form $K[\varepsilon_1, \cdots, \varepsilon_N]$ with $\varepsilon_i^2 = 0$. Hence the class of the graph of the Hecke correspondence $T_{\mathfrak{P}}$ in $(\mathbb{P}^N \times \mathbb{P}^N) := K[\varepsilon_1, \cdots, \varepsilon_N, \eta_1, \cdots, \eta_N]$ is $[T_{\mathfrak{P}}] = (\deg(T_{\mathfrak{P}})\varepsilon, \deg(T_{\mathfrak{P}})\eta)$, where $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_N)$ and $\eta = (\eta_1, \cdots, \eta_N)$. Therefore the class of $(T_{\mathfrak{P}} \times T_{\mathfrak{P}})(X)$ in $(\mathbb{P}^N \times \mathbb{P}^N \times \mathbb{P}^N \times \mathbb{P}^N)$ is

$$[T_{\mathfrak{P}}] \times [T_{\mathfrak{P}}] = \deg(T_{\mathfrak{P}})^2[H] \cdot [X],$$

where $[H]$ is the class of the generator of $(\mathbb{P}^N \times \mathbb{P}^N \times \mathbb{P}^N \times \mathbb{P}^N)$ and $[X]$ is the class of X. By proposition 4.4.2 it is now easy to see that

$$\deg_T((T_{\mathfrak{P}} \times T_{\mathfrak{P}})(X)) \leq c^2 \deg_T(X)^2 |\mathfrak{P}|^{2n}.$$

The second claim is rather evident. □

Let X be a irreducible subvariety of $M_A^r(1)_K \times M_A^r(1)_K$ and let

$$pr_i : M_A^r(1)_K \times M_A^r(1)_K \longrightarrow M_A^r(1)_K$$

denote the projection to the i-th factor of $M_A^r(1)_K \times M_A^r(1)_K$. Let Y be the image of X in $M_A^r(1)_K$ under this projection.

By abuse of notation we denote still by $pr_i : X \longrightarrow Y$ the restriction of pr_i to X. Assume that pr_i is quasi finite hence that its degree by $\deg(pr_i) := [\mathbb{C}_\infty(X) : \mathbb{C}_\infty(Y)]$ is well defined. Let $\bar{X}$ denote the Zariski closure of the image of X in $\mathbb{P}^N \times \mathbb{P}^N$ induced by ρ_T and let $\bar{Y}$ denote the Zariski closure of the image of Y in $\mathbb{P}^N$ under the embedding into projective space afforded by ρ_T. Let

$$pr_{i,T} : \bar{X} \longrightarrow \bar{Y}$$

denote the projection morphism and denote its degree by $\deg_T(pr_i)$. Since X and $\bar{X}$ are birational, similarly Y and $\bar{Y}$, we have $\deg_T(pr_i) = \deg(pr_i)$. In computing the T-degree of a subvariety X in the product $M_A^r(1)_K \times M_A^r(1)_K$ one is faced with the following question. On one hand one can compute the degree of the class of X in the Chow group of $\mathbb{P}^N \times \mathbb{P}^N$, on the other hand $\mathbb{P}^N \times \mathbb{P}^N$ is not projective. Thus $\mathbb{P}^N \times \mathbb{P}^N$ can be further embedded in a projective space $\mathbb{P}^M$ using the very ample line bundle $\mathcal{L} = \mathcal{O}(1,1)$. With this embedding one has a different degree $\deg_{\mathcal{L}}(X)$ at his disposal. The following simple lemma gives a relation between $\deg_T(pr_i)$, $\deg_T(X)$ and $\deg_{\mathcal{L}}(X)$.

Lemma 4.4.4. *With the situation as above we have*

$$\deg_{\mathcal{L}}(X) = (\dim(X))!(\deg_T(pr_1) + \deg_T(pr_2)) = (\dim(X))! \cdot \deg_T(X)$$

Proof. The class of X in the Chow group $CH(\mathbb{P}^M) = \mathbb{Z}[\varepsilon]/(\varepsilon^{M+1})$ (see [30]) under the embedding injuced by $\mathcal{L}$ is $[X] \cdot [\mathcal{L}]^d$ (see [18]). Hence

$$\deg_{\mathcal{L}}(X) = (\dim(X))!(\deg_T(pr_1(X)) + \deg_T(pr_2(X))),$$

which gives the first equality. Since $CH(\mathbb{P}^N \times \mathbb{P}^N) = \mathbb{Z}[\varepsilon]/(\varepsilon^{N+1}) \oplus \mathbb{Z}[\varepsilon]/(\varepsilon^{N+1})$, the class of X in $CH(\mathbb{P}^N \times \mathbb{P}^N)$ is $d_1 \cdot {\varepsilon_1}^{k_1} + d_2 \cdot {\varepsilon_2}^{k_2}$ where d_i is the degree of the i-th projection and k_i is the codimension of the i-th projection. Therefore

$$\deg_T(X) = \deg_T(pr_1) + \deg_T(pr_2),$$

hence the second equality.

$\square$

CHAPTER V

André-Oort Conjecture

In this chapter we prove the André-Oort conjecture for a product of two Drinfeld modular surfaces when the characteristic is not 3. We follow the approach successfully employed by Edixhoven, Klingler-Yafaev and Breuer (see for instance [21],[10],[4] , [8]).

We first recall the statement of the André-Oort Conjecture for products of Drinfeld modular varieties.

Conjecture 5.0.5 (André-Oort for Products of Drinfeld Modular Varieties)**.** *Let X be an irreducible subvariety of $M_A^{r_1}(\mathcal{K})_{\mathbb{C}_\infty} \times \cdots \times M_A^{r_n}(\mathcal{K})_{\mathbb{C}_\infty}$. Suppose Σ is a non empty set of Hodge subvarieties in X such that Σ is dense in X. Then X is a Hodge subvariety of $M_A^{r_1}(\mathcal{K})_{\mathbb{C}_\infty} \times \cdots \times M_A^{r_n}(\mathcal{K})_{\mathbb{C}_\infty}$.*

In the case that Σ is a set of CM points in X, the conjecture has the more familiar expression.

Conjecture 5.0.6 (André-Oort for products of Drinfeld Modular Varieties)**.** *Let X be an irreducible subvariety of $M_A^{r_1}(\mathcal{K})_{\mathbb{C}_\infty} \times \cdots \times M_A^{r_n}(\mathcal{K})_{\mathbb{C}_\infty}$. Then X contains a Zariski dense set of CM points Σ if and only if X is a Hodge subvariety of $M_A^{r_1}(\mathcal{K})_{\mathbb{C}_\infty} \times \cdots \times M_A^{r_n}(\mathcal{K})_{\mathbb{C}_\infty}$.*

To be precise: What constitutes the André-Oort conjecture has classically been the forward statement. In one direction the conjecture is much easier. Indeed if X is a Hodge subvariety of $M_A^{r_1}(\mathcal{K})_{\mathbb{C}_\infty} \times \cdots \times M_A^{r_n}(\mathcal{K})_{\mathbb{C}_\infty}$ then given a CM point in X we can consider the Zariski closure of a Hecke orbit (see Breuer [8, 4.2]). This is Zariski dense in an irreducible component of X.

The André-Oort conjecture is insensitive to the level structures. The inclusion $\mathcal{K} \hookrightarrow \mathrm{GL}_r(\hat{A})$ induces a morphism

$$f : M_A^{r_1}(\mathcal{K})_{\mathbb{C}_\infty} \times \cdots \times M_A^{r_n}(\mathcal{K})_{\mathbb{C}_\infty} \longrightarrow M_A^{r_1}(1)_{\mathbb{C}_\infty} \times \cdots \times M_A^{r_n}(1)_{\mathbb{C}_\infty}$$

and $X \subset M_A^{r_1}(\mathcal{K})_{\mathbb{C}_\infty} \times \cdots \times M_A^{r_n}(\mathcal{K})_{\mathbb{C}_\infty}$ is a Hodge subvariety if and only if its image $f(X) \subset M_A^{r_1}(1)_{\mathbb{C}_\infty} \times \cdots \times M_A^{r_n}(1)_{\mathbb{C}_\infty}$ is a Hodge subvariety. Therefore it suffices to state and prove the André-Oort conjecture for subvarieties of $M_A^{r_1}(1)_{\mathbb{C}_\infty} \times \cdots \times M_A^{r_n}(1)_{\mathbb{C}_\infty}$. This is the form of the conjecture we shall use for the rest of this chapter. We now state the main result of this thesis.

5.1 Main Theorem

Theorem 5.1.1 (André-Oort for a product of Drinfeld Modular Surfaces)**.** *Let X be an irreducible subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$. Suppose that X is defined over F and that the characteristic of F is not 3. Then $X(\mathbb{C}_\infty)$ contains a Zariski dense set of CM points if and only if X is a Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$.*

5.2 Existence of small primes

We first state a utility proposition: It concerns the existence of small primes with respect to the Picard groups of CM endomorphism rings. This will be needed to define suitable Hecke correspondences. That such primes exist is an application of an effective form of the Čebotarev Theorem. This is possible in the function field case since the Riemann hypothesis has been settled in this case. In the characteristic zero situation, this is precisely where the Generalized Riemann Hypothesis is used. We first recall some notation from the section on Complex Multiplication. Let φ be a rank r Drinfeld A-module with endomorphism ring R. Then R is a commutative A-algebra with rank dividing r as a projective A-module (see for example [29]) and R is an order in its quotient field K', with K' a *totally imaginary* extension of K i.e. there is only one prime in K' lying above ∞ .

Definition 5.2.1. *Let φ be a rank r Drinfeld A-module. We say that φ has* complex multiplication *(CM) by R if $[K' : K] = [R : A] = r$.*

Let φ be a rank r Drinfeld A-module with complex multiplication by an order R in K', as above. Denote by $\mathfrak{c}$ be the conductor of R in A' the integral closure of A in K', set $|\mathfrak{c}| = |A'/\mathfrak{c}|$, and finally denote the genus of K' by $g(K')$.

Definition 5.2.2. *The CM-height of φ is defined by*

$$H_{CM}(\varphi) = q^{g(K')} \cdot |\mathfrak{c}|^{1/r}.$$

Proposition 5.2.3. *(see Breuer [8])*

For every $\varepsilon > 0$ there is a computable constant $C_\varepsilon > 0$ such that the following holds. If φ is Drinfeld module with complex multiplication by R, then

$$|\operatorname{Pic}(R)| > C_\varepsilon H_{CM}(\varphi)^{1-\varepsilon}.$$

If $x = (x_1, x_2)$ is a CM point in $M_A^r(1)(\mathbb{C}_\infty) \times M_A^r(1)(\mathbb{C}_\infty)$ then we define the CM height of x as

$$H_{\mathrm{CM}}(x) = \max\{H_{\mathrm{CM}}(\varphi_1), H_{\mathrm{CM}}(\varphi_2)\}.$$

Let K_1, K_2 denote the CM fields of x_1 and x_2 and let $K_1(x_1), K_2(x_2)$ denote choices of fields of definition for x_1 and x_2. Hence a field of definition of x is the compositum $L := K_1K_2(x_1, x_2)$ over K_1K_2. We have the following simple corollary.

Corollary 5.2.4. *Let $m \geq 1$ be an integer. For every $\varepsilon > 0$ there is a computable constant $C_\varepsilon > 0$ (which depends on the degree m) such that, for every field F with $m = [F : K]$ and every CM point $x = (x_1, x_2) \in M_A^r(1)(\mathbb{C}_\infty) \times M_A^r(1)(\mathbb{C}_\infty)$ we have*

$$|\operatorname{Gal}(F^{\text{sep}}L/FK_1K_2) \cdot x| \geq C_\varepsilon H_{CM}(x)^{1-\varepsilon},$$

where K_1, K_2 and L are as above.

Proof. We have

$$\operatorname{Gal}(F^{\text{sep}}L/FK_1K_2) \cdot x = (\operatorname{Gal}(FK_1(x_1)/FK_1) \cdot x_1, \operatorname{Gal}(FF_2(x_2)/FK_2) \cdot x_2)$$

but size of the Galois orbit of x is a least the size of the Galois orbits of its coordinates hence

$$|\operatorname{Gal}(F^{\text{sep}}L/FK_1K_2) \cdot x| \geq \max\{|\operatorname{Gal}(FF_1(x_1)/FK_1) \cdot x_1|, |\operatorname{Gal}(FF_2(x_2)/FK_2) \cdot x_2|\}.$$

Now $|\operatorname{Gal}(FF_i(x_i)/FK_i) \cdot x_i| \geq |\operatorname{Pic}(R_i)|/[F_i(x_i) : F]$ where R_i is the endomorphism ring of the Drinfeld module corresponding to x_i and $F_i(x_i)$ is its field of definition. By lemma 5.2.3 if we are given x_i, then for every $\varepsilon > 0$ there is a computable constant $C_{i,\varepsilon} > 0$, depending on the CM field of x_i such that $|\operatorname{Pic}(R_i)| \geq C_{i,\varepsilon} H_{\mathrm{CM}}(x_i)^{1-\varepsilon}$, therefore

$$|\operatorname{Gal}(F^{\text{sep}}L/FK_1K_2) \cdot x| \geq \max\{C_{1,\varepsilon}/[F_1(x_1) : F] \cdot H_{\mathrm{CM}}(\varphi_1)^{1-\varepsilon}, C_{2,\varepsilon}/[F_2(x_2) : F] \cdot H_{\mathrm{CM}}(\varphi_2)^{1-\varepsilon}\}$$

Setting $C_\varepsilon = \min\{C_{1,\varepsilon}/[F_1(x_1) : F], C_{2,\varepsilon}/[F_2(x_2) : F]\}$ we obtain

$$|\operatorname{Gal}(F^{\text{sep}}L/FK_1K_2) \cdot x| \geq C_\varepsilon H_{\mathrm{CM}}(x)^{1-\varepsilon}.$$

□

Let X be a closed subvariety of $M_A^r(1)_{\mathbb{C}_\infty} \times M_A^r(1)_{\mathbb{C}_\infty}$ defined over a finite extension F of K. Let $x = (x_1, x_2)$ be a CM point of X and denote by R_1 and R_2 the endomorphism rings of the Drinfeld modules corresponding to x_1, x_2 respectively. Also denote by $\mathfrak{c}_1$ and $\mathfrak{c}_2$ the conductors of R_1 and R_2 respectively.

Let $T \in A$ be a transcendental element such that K is a finite separable geometric extension of $\mathbb{F}_q(T)$. Let F_s be the separable closure of K in F and let $K_1 = \mathrm{Quot}(R_1)$, $K_2 = \mathrm{Quot}(R_2)$ and let $L = K_1K_2$ and let L_s denote the separable closure of $\mathbb{F}_q(T)$ in L. Let M be the Galois closure of F_sL_s over $\mathbb{F}_q(T)$.

Let $t \in \mathbb{N}$ and define

$$\pi_M(t) := \{P \subset \mathbb{F}_q[T] \mid P \text{ is a prime, splits in } M \text{ and } |P| = q^t\ \}.$$

Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_q$ in M. Let $n_c = [\mathbb{F} : \mathbb{F}_q]$ be the constant extension degree of $M/\mathbb{F}_q(T)$ and $n_g = [M : \mathbb{F}(T)]$ the geometric extension degree.

The Čebotarev Theorem for function fields can be stated as follows

Theorem 5.2.5. *[25, 5.16] Let g_M denote the genus of M, then*

- *if $n_c \nmid t$, then $\pi_M(t) = \emptyset$*
- *if $n_c|t$, then* $\quad ||\pi_M(t)| - \frac{1}{n_g}q^t/t| < 4\big(g_M + 2\big)q^{t/2}$.

By the Castelnuovo inequality [52, III.10.3], we have bounds for g_M in terms of g' (the genus of L). Applying these bounds to $|\pi_M(t)|$ above we obtain a bound of the form

$$|\pi_M(t)| > C_1q^t/t - \big(C_2(g') + C_3\big)q^{t/2}, \tag{5.0}$$

where C_1, C_2 and C_3 are absolutely computable positive constants, independent of t and depending only on $[F : K]$

The following lemma is essentially due to Breuer and can be found in [8, section 7]. What we mainly do is provide a version of it suitable for our application.

Lemma 5.2.6. *(with the same notation as above) Let X be a subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$ of dimension $d \geq 1$ defined over F. Suppose X contains a Zariski dense set of CM points. Then there exists $x = (x_1, x_2) \in X$ a CM point, principal primes $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_{d-1} \subset A$ and for all $\varepsilon \geq 0$,*

$$C_\varepsilon H_{CM}(x)^{1-\varepsilon} > c^{2.(3^{d-1}+..+3+1)} \deg_T(X)^{3^{d-1}} \prod_{k=0}^{d-1} |\mathfrak{p}_k|^{2n \cdot 3^{d-k}},$$

where C_ε is the constant from Corollary 5.2.4 which is independent of x, c and n are some constants and

(1) $|\mathfrak{p}_i| \geq (d)! \cdot \deg_T(X)$,

(2) each $\mathfrak{p}_i$ is residual in $R_i = \mathrm{End}(\varphi_i^x)$,

(3) each $\mathfrak{p}_i$ has residue degree one over $\mathfrak{p}_i \cap \mathbb{F}_\mathfrak{p}[T]$.

(4) $|\mathfrak{p}_{i+1}| \geq c^{2.(3^i+..+3+1)} \deg_T(X)^{3^i} \prod_{k=0}^{i} |\mathfrak{p}_k|^{2n \cdot 3^{i+1-k}}$

(5) $|\operatorname{Pic}(R_i)/[F:K]| > c^{2.(3^{d-1}+..+3+1)} \deg_T(X)^{3^{d-1}} \prod_{k=0}^{d-1} |\mathfrak{p}_k|^{2n \cdot 3^{d-k}}$.

Proof. Let $x = (x_1, x_2)$ be an arbitrary CM point of X with conductor $(\mathfrak{c}_1, \mathfrak{c}_2)$ and choose $t > \max\{\log|\mathfrak{c}_1|, \log|\mathfrak{c}_2|\}$, then for every prime $P = \mathfrak{p} \cap \mathbb{F}_q[T]$ in $\pi_M(t)$ we have that $\mathfrak{p}$ is unramified in K_1K_2 since $\mathfrak{p}$ does not divide $|\mathfrak{c}_1||\mathfrak{c}_2|$. Furthermore P has residue degree over $\mathfrak{p}_i \cap \mathbb{F}_q[T]$ since P splits completely in $F_sL_s/\mathbb{F}_q(T)$ and is totally ramified in FL/F_sL_s. If we in addition assume $t > \log(d)! \cdot \deg_T(X)$ then $|P| \geq (d)! \cdot \deg_T(X)$ for every prime in $\pi_M(t)$. Thus taking

$$t > \max\{\log|\mathfrak{c}_1|, \log|\mathfrak{c}_2|\} + \log(d)! \cdot \deg_T(X),$$

we have that all the primes in $\pi_M(t)$ satisfy condition (1), (2), (3) and (4).

By the Castelnuovo inequality (see equation 5.0 above)

$$|\pi_M(t)| > C_1 q^t/t - \big(C_2(g') + C_3\big) q^{t/2},$$

thus the cardinality of $\pi_M(t)$ is unbounded as t goes to infinity. We are therefore guaranteed an infinite supply of primes in $\pi_M(t)$ as t goes to infinity. By taking logarithms condition (4) may be written as linear inequality

$$t_{i+1} > \log(c^{2.(3^i+..+3+1)} \deg_T(X)^{3^i}) + \Sigma_{k=0}^{i}(2n \cdot 3^{d-k})t_k$$

Therefore starting with any integer t_0, we can by an iterative process, find integers $t_{i+1}, t_i, \cdots, t_0$ such that the spacing condition (4) holds. Thus we can ensure that conditions (1),(2),(3) and (4) are satisfied by the primes $\mathfrak{p}_i$ above $P_i \in \pi_M(t_i)$. Now in light of (4), to satisfy condition (5) it suffices to show that

$$|\operatorname{Pic}(R_i)| > c^{2.(3^{d-1}+..+3+1)}[F : K] \deg_T(X)^{3^{d-1}} |\mathfrak{p}_{d-1}|^{2n \cdot 3^d}$$

Since for all $\varepsilon \geq 0$, there is a computable constant C_ε such that

$$|\operatorname{Pic}(R_i)| > C_\varepsilon H_{\mathrm{CM}}(x_i)^{1-\varepsilon},$$

it follows that condition (5) holds if

$$C_\varepsilon H_{\mathrm{CM}}(x_i)^{1-\varepsilon} > c^{2.(3^{d-1}+..+3+1)}[F' : K] \deg_T(X)^{3^{d-1}} |\mathfrak{p}_{d-1}|^{2n \cdot 3^d}$$

Since $H_{\mathrm{CM}}(x_i) = q^{g(K')} \cdot |\mathfrak{c}_i|^{1/3}$, condition (5) holds if

$$C_\varepsilon \max\{q^{g_i(K')} \cdot |\mathfrak{c}_i|^{1/3}\}^{1-\varepsilon} > c^{2.(3^{d-1}+..+3+1)}[F : K] \deg_T(X)^{3^{d-1}} |\mathfrak{p}_{d-1}|^{2n \cdot 3^d}$$

Let

$$\log|\mathfrak{c}| = \max\{\log|\mathfrak{c}_1|, \log|\mathfrak{c}_2|\} + \log(d)! \cdot \deg_T(X),$$

and let $\gamma = c^{2.(3^{d-1}+..+3+1)}[F : K] \deg_T(X)^{3^{d-1}}$. Since $|\mathfrak{c}_i|$ grows much faster than $\log|\mathfrak{c}_i|$ and we have a Zariski dense set of CM points the inverval

$$[\log|\mathfrak{c}|, C_\varepsilon \max\{q^{g_i(K')} \cdot |\mathfrak{c}_i|^{1/3}\}]$$

is unbounded as we move among the CM points. Since $C_1 q^{t_i}/t_i - \big(C_2(g') + C_3\big) q^{t_i/2}$ is dominated by $\gamma \cdot q^{t \cdot 2n \cdot 3^d}$ for large t, if we choose a CM point x such that $H_{\mathrm{CM}}(x)$ is very large then there exist integers $t_0, t_1, t_2, \cdots, t_{d-1}$ such that the intervals $[C_1 q^{t_i}/t_i - \big(C_2(g') + C_3\big) q^{t_i/2}, \gamma \cdot q^{t_i \cdot 2n \cdot 3^d}]$ are contained in $[\log|\mathfrak{c}|, C_\varepsilon \max\{q^{g_i(K')} \cdot |\mathfrak{c}_i|^{1/3}\}]$. These integers $t_0, t_1, t_2, \cdots, t_{d-1}$ (by construction) then satisfy

(i). $C_1 q^{t_i}/t_i - \big(C_2(g') + C_3\big) q^{t_i/2} > \max\{\log|\mathfrak{c}_1|, \log|\mathfrak{c}_2|\} + \log(d)! \cdot \deg_T(X)$

(ii). $t_{i+1} > \log(c^{2.(3^i + .. + 3 + 1)} \deg_T(X)^{3^i}) + \Sigma_{k=0}^{i}(2n \cdot 3^{d-k}) t_k$

(iii). $C_\varepsilon \max\{q^{g_i(K')} \cdot |\mathfrak{c}_i|^{1/3}\}^{1-\varepsilon} > \gamma \cdot q^{t_{d-1} \cdot 2n \cdot 3^d}$

Picking primes $\mathfrak{p}_i$ above primes in $\pi(t_i)$ respectively we obtain the primes we seek. $\square$

5.3 Characterisation of Hodge varieties

In this section we give a characterisation of Hodge subvarieties in the product of two Drinfeld modular surfaces. This is analogous to a result of Breuer (see [10, theorem 6]), which characterises Hodge subvarieties in the product of Drinfeld modular curves.

The following auxiliary lemmas are crucial ingredients for our characterisation of Hodge subvarieties.

Consider the functor $\mathcal{F}^r$, from A-Schemes to *Set* given by

$$\mathcal{F}^r : \mathbf{Sch}\,/A \to \mathrm{Set}\ ; S \longmapsto \left\{ \begin{array}{l} \text{isomorphism classes of rank-}r \\ \\ \text{Drinfeld } A\text{-modules over } S \text{ with } \Gamma_0(\mathfrak{p}) \text{ level structure.} \end{array} \right\}.$$

Then $\mathcal{F}^r$ is representable by a scheme Z (see [36] chapter 3 for a proof in the elliptic curve case. The Drinfeld situation works the same way). The modular variety Z parametrises pairs of rank r Drinfeld A-modules linked by a cyclic isogeny of degree $\mathfrak{p}$. The analytification of Z is a disjoint union of quotient spaces of the Drinfeld upper half plane by congruence subgroups. Let $Y_0^r(\mathfrak{p}) = \Gamma_0(\mathfrak{p})\backslash\Omega^r$ be one such irreducible component. By [36] corollary 5.3.3 there is a natural finite morphism $Z \longrightarrow M_A^r(1)$. Let

$$Y^r(1) := \mathrm{GL}_r(A)\backslash\Omega^r$$

be a connected component of $M_A^r(1)$ lying below $Y_0^r(\mathfrak{p})$, then we have a finite covering map $Y_0^r(\mathfrak{p}) \longrightarrow Y^r(1)$.

For a ring R, let $Z(R^\times)$ denote the subgroup of scalar matrices in $\mathrm{GL}_n(R)$. Let

$$\Gamma(\mathfrak{p}) = \{\gamma \in \mathrm{GL}_r(A) \mid \gamma \equiv \mathrm{id} \bmod \mathfrak{p}\}/Z(\mathbb{F}_q{}^\times),$$

$$\Gamma_2(\mathfrak{p}) = \{\gamma \in \mathrm{GL}_r(A) \mid \gamma \bmod \mathfrak{p} \in Z((A/\mathfrak{p}A)^\times)\}/Z(\mathbb{F}_q{}^\times).$$

By abuse of notation we shall also denote the projectivisation of $\Gamma_0(\mathfrak{p})$ by $\Gamma_0(\mathfrak{p})$ and still define the projective quotient as $Y_0^r(\mathfrak{p}) := \Gamma_0(\mathfrak{p})\backslash\Omega^r$. We further define $Y_2^r(\mathfrak{p}) := \Gamma_2(\mathfrak{p})\backslash\Omega^r$ and $Y^r(\mathfrak{p}) := \Gamma(\mathfrak{p})\backslash\Omega^r$. We can now state the following lemmas, which are generalisations of [10, Theorem 4].

Lemma 5.3.1. *Suppose $r \geq 2$ and $\gcd(|\mathfrak{p}|-1, r) = 1$. Then $Y_2^r(\mathfrak{p})$ is a Galois cover of $Y^r(1)$ with Galois group $\mathrm{PSL}_r(A/\mathfrak{p}A)$*

Proof. Since $\Gamma(\mathfrak{p}) \subset \Gamma_2(\mathfrak{p}) \subset \mathrm{PGL}_r(A)$, we have a tower of covers

$$Y^r(1) \subset Y_2^r(\mathfrak{p}) \subset Y^r(\mathfrak{p}).$$

The cover $Y^r(\mathfrak{p})$ is Galois over $Y^r(1)$ with Galois group $\mathrm{PGL}_r(A)/\Gamma(\mathfrak{p})$. Let $G(p)$ be the image of the reduction map

$$r_1 : \mathrm{PGL}_r(A) \longrightarrow \mathrm{PGL}_r(A/\mathfrak{p}A)$$

and denote by $\tilde{G(p)}$ the preimage of $G(p)$ in $\mathrm{GL}_r(A/\mathfrak{p}A)$. Set $Z(G(p)) := Z((A/\mathfrak{p}A)^*) \cap \tilde{G(p)}$, the scalar matrices in $\tilde{G(p)}$. Then $\mathrm{PGL}_r(A)/\Gamma(\mathfrak{p})$ is isomorphic to $G(p)$ via the reduction map r_1. The cover $Y_2^r(\mathfrak{p})$ corresponds to the normal subgroup $Z(G(p))$ hence

$$\mathrm{Gal}(Y_2^r(\mathfrak{p})/Y^r(1)) = \tilde{G(p)}/Z(G(p)) \subset \mathrm{PGL}_r(A/\mathfrak{p}A).$$

By the strong approximation theorem for $\mathrm{SL}_r(A)$, the reduction map

$$r_1 : \mathrm{SL}_r(A) \longrightarrow \mathrm{SL}_r(A/\mathfrak{p})$$

is surjective. Hence it is surjective on $\mathrm{PSL}_r(A/\mathfrak{p}A)$ as well. We therefore have $\mathrm{PSL}_r(A/\mathfrak{p}A) \subset G(p)$ and consequently $\mathrm{PSL}_r(A/\mathfrak{p}A) \subset \mathrm{Gal}(Y_2^r(\mathfrak{p})/Y^r(1))$. Let $q = |A/\mathfrak{p}A|$ then it is well known that (see [49, page 74])

$$|\,\mathrm{PGL}_r(\mathbb{F}_q)| = q^{r(r-1)/2}(q^r-1)(q^{r-1}-1)\cdots(q^2-1))$$

and

$$|\,\mathrm{PSL}_r(\mathbb{F}_q)| = q^{r(r-1)/2}(q^r-1)(q^{r-1}-1)\cdots(q^2-1))/(q-1,r)$$

Therefore if $\gcd(q-1,r)=1$ we have $\mathrm{PSL}_r(A/\mathfrak{p}A) = \mathrm{PGL}_r(A/\mathfrak{p}A)$. Consequently $\mathrm{Gal}(Y_2^r(\mathfrak{p})/Y^r(1)) = \mathrm{PSL}_r(A/\mathfrak{p}A)$. Proving our claim.

$\square$

Let $g_{\mathfrak{p}} = \mathrm{diag}(\mathfrak{p}, 1, \cdots, 1)$ and denote by S the quotient space $(\mathrm{GL}_r(A)\backslash\, \mathrm{GL}_r(A) g\, \mathrm{GL}_r(A))^2$. Let $\{t_{i,j} = (t_i\,,\ t_j) \mid (i,j) \in I \times I\}$ (for some index set I), be a set of representatives of S. Recall that $M_A^r(1)_{\mathbb{C}_\infty} \times M_A^r(1)_{\mathbb{C}_\infty}$ is the disjoint union of the quotient spaces $\Gamma_s\backslash\Omega^r \times \Gamma_{s'}\backslash\Omega^r$, where Γ_s and $\Gamma_{s'}$ are congruence subgroups of $\mathrm{GL}_r(A)$. Let

$$\pi \times \pi' : \Omega^r \times \Omega^r \longrightarrow \Gamma_s\backslash\Omega^r \times \Gamma_{s'}\backslash\Omega^r,$$

be the natural projection maps. Since the primes $\mathfrak{p}$ are prime, the Hecke correspondence is irreducible on $\Gamma_s\backslash\Omega^r \times \Gamma_{s'}\backslash\Omega^r$ and the action of the Hecke correspondence

$(T_{\mathfrak{p}} \times T_{\mathfrak{p}})$ on $x \in \Gamma_s \backslash \Omega^r \times \Gamma_{s'} \backslash \Omega^r$ is given by

$$(T_{\mathfrak{p}} \times T_{\mathfrak{p}})(x) = \bigcup_{(i,j) \in I \times I} (\pi \times \pi^{'})(t_{i,j} \cdot z),$$

where z is a preimage of x via $\pi \times \pi^{'}$.

The Drinfeld modular variety $Y_0^3(\mathfrak{p})$ parametrises pairs of rank three Drinfeld A-modules linked by a cyclic isogeny of degree $\mathfrak{p}$ and since the Hecke correspondence $T_{\mathfrak{p}} \times T_{\mathfrak{p}}$ encodes pairs of cyclic isogenies of degree $\mathfrak{p}$ we may also describe the Hecke correspondence $T_{\mathfrak{p}} \times T_{\mathfrak{p}}$ on $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$ in the following fashion. Since every point on the modular variety $Y_0^3(\mathfrak{p})$ corresponds to a pair of Drinfeld A-modules linked by a cyclic $\mathfrak{p}$ isogeny $\varphi_1 \xrightarrow{\mathfrak{p}} \varphi_1{}'$. The image of

$$T : Y_0^3(\mathfrak{p}) \times Y_0^3(\mathfrak{p}) \longrightarrow M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$$
$$((\varphi_1 \xrightarrow{\mathfrak{p}} \varphi_1{}'), (\varphi_2 \xrightarrow{\mathfrak{p}} \varphi_2{}')) \longrightarrow (\varphi_1, \varphi_2, \varphi_1{}', \varphi_2{}')$$

is then the usual definition of the Hecke correspondence $T_{\mathfrak{p}} \times T_{\mathfrak{p}}$.

Proposition 5.3.2. *Let X_F be an F-irreducible, Hodge generic subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$ and suppose that the characteristic of F is not 3. Suppose that both projections of X_F are dominant onto irreducible components of $M_A^3(1)_{\mathbb{C}_\infty}$. Suppose that X_F is the union of a finite number of* $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ *conjugates of an irreducible variety X . Suppose furthermore that $X_F \subset (T_{\mathfrak{p}} \times T_{\mathfrak{p}})X_F$. Then every* $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ *conjugate of X is a Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$.*

Proof. Since $X_F \subset (T_{\mathfrak{p}} \times T_{\mathfrak{p}})X_F$ we have $\bigcup_\sigma X^\sigma \subset \bigcup_{\sigma_i} (T_{\mathfrak{p}} \times T_{\mathfrak{p}})X^{\sigma_i}$. Hence $X^\sigma \subset (T_{\mathfrak{p}} \times T_{\mathfrak{p}})X^{\sigma_i}$ for some σ_i. Let $\mathcal{K}$ be a large enough admissible level structure, so that $M_A^3(\mathcal{K})_{\mathbb{C}_\infty}$ is a fine moduli scheme, assume that $\mathcal{K} = \prod_{\mathfrak{p}} \mathcal{K}_{\mathfrak{p}}$ where $\mathcal{K}_{\mathfrak{p}}$ is contained in $\mathrm{GL}_r(K_{\mathfrak{p}})$ and the first component $\mathcal{K}_{\mathfrak{p}}$ is a principal congruence subgroup of $\mathrm{GL}_r(A_p)$

Let

$$\pi \times \pi : M_A^r(\mathcal{K})_{\mathbb{C}_\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}_\infty} \to M_A^r(1)_{\mathbb{C}_\infty} \times M_A^r(1)_{\mathbb{C}_\infty}$$

and let X' be an F-irreducible component of the preimage of X_F in $M_A^r(\mathcal{K})_{\mathbb{C}_\infty} \times M_A^r(\mathcal{K})_{\mathbb{C}_\infty}$ via the canonical projection $\pi \times \pi$. After replacing X' with its non singular locus we may assume X' is smooth and still Hodge generic with the generic point of X' consisting of non-isogenous underlying Drinfeld modules which have both endomorphism rings A.

Since $X_F \subset (T_\mathfrak{p} \times T_\mathfrak{p})X_F$ we have $X' \subset \sigma_i \cdot (T_\mathfrak{p} \times T_\mathfrak{p}) \cdot (\pi \times \pi)^{-1}(X_F)$, consequently that $X' \subset \sigma_i \cdot (T_g \times T_g)X'$, where $g = g_1 g_\mathfrak{p} g_2$ and g_1, g_2 are representatives of the coset $\mathcal{K}\backslash \mathrm{GL}_3$. Then by theorem 4.2.2 X_F is a Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$.

□

Definition 5.3.3. *Let X be a subvariety of $M_A^r(1)_{\mathbb{C}_\infty} \times M_A^r(1)_{\mathbb{C}_\infty}$. We say X is weakly Hodge generic if for a generic point (x, y) in X, the corresponding pair of Drinfeld modules φ_x and φ_y have endomorphism ring* $\mathrm{End}(\varphi_x) = A = \mathrm{End}(\varphi_y)$.

We remark that a Hodge generic subvariety is weakly Hodge generic, but the definition of weakly Hodge generic allows points to have isogenous coordinates.

Definition 5.3.4. *Let X be a subvariety of $M_A^r(1)_{\mathbb{C}_\infty} \times M_A^r(1)_{\mathbb{C}_\infty}$. We say X is degenerate if X is an irreducible component of $\{x\} \times M_A^r(1)_{\mathbb{C}_\infty}$ or $M_A^r(1)_{\mathbb{C}_\infty} \times \{y\}$ with $x, y \in M_A^r(1)_{\mathbb{C}_\infty}$.*

Lemma 5.3.5. *Let X be an irreducible subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$ of positive dimension. Suppose that X is not degenerate. Then X is a weakly Hodge generic subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$.*

Proof. If X is not weakly Hodge generic then X is in particular not Hodge generic. Thus X is contained in a proper Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$. By definition, the non-degenerate proper Hodge subvarieties of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$ are Hecke graphs.

Since X is not weakly Hodge generic $\tilde{X}$, the preimage of X in $M_A^r(1)_{\mathbb{C}_\infty}$ via

$$f : M_A^3(1)_{\mathbb{C}_\infty} \longrightarrow M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty},$$

is not Hodge generic, here f sends x a point of $M_A^3(1)_{\mathbb{C}_\infty}$ to the point $(x, g \cdot x)$ of $M_A^3(1)_{\mathbb{C}_\infty} \longrightarrow M_A^3(1)_{\mathbb{C}_\infty}$ where $g \in \mathrm{GL}(K)$. Indeed if $\tilde{X}$ is Hodge generic then the generic point of $\tilde{X}$ would have endomorphism A, consequently its image via f would be weakly Hodge generic. The non Hodge generic subvarieties of $M_A^3(1)_{\mathbb{C}_\infty}$ are zero dimensional. This means that $\tilde{X}$ is of zero dimension in $M_A^3(1)_{\mathbb{C}_\infty}$, hence also its image X, contrary to our assumption that X has positive dimension.

□

Proposition 5.3.6. *Let X be an irreducible and weakly Hodge generic but not Hodge generic subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$ of dimension greater than one. Then X is a Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$.*

Proof. If X is weakly Hodge generic but not Hodge generic, then X is contained in the graph of a Hecke correspondence. This has dimension two. Since X has dimension bigger than one it must be of dimension two. X is thus an irreducible component of this graph of a Hecke correspondence, therefore a Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$. □

5.4 André-Oort Conjecture

In this section we prove our main theorem. We begin with the following lemma which settles the Andre-Oort conjecture for curves in $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$.

Proposition 5.4.1. *Let X be an curve in $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$. If X contains a Zariski dense set of CM points then X is a Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$.*

Proof. The images of the projections of such a curve to the factors of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$ have a Zariski dense set of CM points, therefore are Hodge subvarieties of $M_A^3(1)_{\mathbb{C}_\infty}$ by the André-Oort conjecture for $M_A^3(1)_{\mathbb{C}_\infty}$. Since there are no Hodge subvarieties of dimension one in $M_A^3(1)_{\mathbb{C}_\infty}$ and X is a curve these images must be CM points. X therefore is a CM point and our statement is vacuously true. □

Proposition 5.4.2. *Let X be an irreducible subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$ of positive dimension. Suppose that X is defined over F and that the characteristic of F is not 3. Suppose further that X contains a Zariski dense set of CM points Σ. Then X contains a Zariski dense set Hodge subvarieties of positive dimension Σ' such that almost all CM points in Σ are properly contained in some Hodge subvariety in Σ'.*

Proof. Since X contains a Zariski dense of CM points Σ, F is a finite extension of K. We may also assume (as in the proof of proposition 5.4.1) that both projections of X are dominant onto some irreducible component of $M_A^3(1)_{\mathbb{C}_\infty}$. By lemma 5.3.5 and proposition 5.4.1 we may assume that X is weakly Hodge generic. Let d be the dimension of X, which we may assume is 2 or 3. If X is weakly Hodge generic but not Hodge generic then by proposition 5.3.6 X is a Hodge subvariety. Thus we may further assume that X is Hodge generic.

By lemma 5.2.6 there exists $x = (x_1, x_2)$ a CM point of X and primes $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_{d-1}$

of A such that the $\mathfrak{p}_i$ have properties (1)-(5) of that lemma. In particular

$$C_\varepsilon H_{\mathrm{CM}}(x)^{1-\varepsilon} > c^{2.(3^{d-1}+..+3+1)} \deg_T(X)^{3^{d-1}} \prod_{k=0}^{d-1} |\mathfrak{p}_k|^{2n\cdot 3^{d-k}},$$

for some constants c, n as in theorem 4.4.2.

If $X \subset (T_{\mathfrak{p}_1} \times T_{\mathfrak{p}_1})X$ then by theorem 5.3.2 the F-irreducible components of X are Hodge subvarieties. In particular X is a Hodge subvariety. So suppose that $X \subsetneq (T_{\mathfrak{p}_1} \times T_{\mathfrak{p}_1})X$. We will show that there exists Y_i a proper Hodge subvariety of X properly containing x.

If $X \subset X \cap (T_{\mathfrak{p}_1} \times T_{\mathfrak{p}_1})X$ then by theorem 5.3.2 X is a Hodge subvariety which properly contains x. So assume $X \subsetneq X \cap (T_{\mathfrak{p}_1} \times T_{\mathfrak{p}_1})X$ and let Y_1 be a closed irreducible component of $X \cap (T_{\mathfrak{p}_1} \times T_{\mathfrak{p}_1})X$ which contains x. Let $(Y_1)_F$ be an F-irreducible component such that $(Y_1)_F \times \mathbb{C}_\infty = \mathrm{Gal}(F^{\mathrm{sep}}/F) \cdot Y_1$. If $(Y_1)_F \subset (T_{\mathfrak{p}_2} \times T_{\mathfrak{p}_2})(Y_1)_F$ then Y_1 is a Hodge subvariety otherwise we iterate the procedure getting Y_{i+1} a closed irreducible component of $Y_i \cap (T_{\mathfrak{p}_{i+1}} \times T_{\mathfrak{p}_{i+1}})Y_i$ which contains x. Let $Y_0 := X$, we observe that $(Y_i)_F$ has the following properties.

(i). $\dim(Y_i)_F < \dim Y_{i-1}$

(ii). $\mathrm{Gal}(F^{\mathrm{sep}}L/FK_1K_2) \cdot x \subsetneq (Y_i)_F \subset Y_{i-1} \cap (T_{\mathfrak{p}_1} \times T_{\mathfrak{p}_1})Y_{i-1}$.

Thus $(Y_i)_F$ has positive dimension. The second property can be explained as follows: If $x = (Y_i)_F$ then $\mathrm{Gal}(F^{\mathrm{sep}}L/FK_1K_2) \cdot x$ is the union of the irreducible components of $Y_{i-1} \cap (T_{\mathfrak{p}_{i+1}} \times T_{\mathfrak{p}_{i+1}})Y_{i-1}$, hence

$$|\,\mathrm{Gal}(F^{\mathrm{sep}}L/FK_1K_2) \cdot x)| = \deg_T(Y_{i-1} \cap (T_{\mathfrak{p}_{i+1}} \times T_{\mathfrak{p}_{i+1}})Y_{i-1}) \leq c^2 \cdot \deg_T(Y_{i-1})^3 |\mathfrak{p}_{i+1}|^{2n}.$$

This implies that

$$C_\varepsilon H_{\mathrm{CM}}(x)^{1-\varepsilon} \leq c^2 \cdot \deg_T(X)^3 |\mathfrak{p}_{i+1}|^{2n},$$

which contradicts our lower bound for $C_\varepsilon H_{\mathrm{CM}}(x)^{1-\varepsilon}$. Now

$$\cdots \dim(Y_i)_F < \dim(Y_2)_F < \dim(Y_1)_F < \dim X$$

Therefore in at most d steps either $(Y_i)_F \subset (T_{\mathfrak{p}_{i+1}} \times T_{\mathfrak{p}_{i+1}})(Y_i)_F$ or $(Y_{i+1})_F$ is finite. If $(Y_{i+1})_F$ is finite then since

$$\mathrm{Deg}_T((X) \cap (T_{\mathfrak{p}_1} \times T_{\mathfrak{p}_1})(X)) \leq c^{2\cdot(3^{d-1}+\cdots+3+1)} \cdot \deg_T(X)^{3^{d-1}} \cdot |\mathfrak{p}_1|^{2n.(3^{d-1}+\cdots+3+1)},$$

and on the other hand by proposition 5.2.4 we have

$$C_\varepsilon H_{\mathrm{CM}}(x)^{1-\varepsilon} \leq |\,\mathrm{Gal}(F^{\mathrm{sep}}L/FK_1K_2) \cdot x|.$$

We obtain the following inequalities

$$C_\varepsilon H_{\mathrm{CM}}(x)^{1-\varepsilon} \leq |\,\mathrm{Gal}(F^{\mathrm{sep}}L/FK_1K_2)\cdot x| \leq c^{2\cdot(3^{d-1}+\cdots+3+1)}\cdot\deg_T(X)^{3^{d-1}}\cdot|\mathfrak{p}_1|^{2n.(3^{d-1}+\cdots+3+1)},$$

which contradict our assumption that

$$C_\varepsilon H_{\mathrm{CM}}(x)^{1-\varepsilon} > c^{2\cdot(3^{d-1}+\cdots+3+1)} \cdot \deg_T(X)^{3^{d-1}} \cdot \prod_{k=1}^{d} |\mathfrak{p}_k|^{2n\cdot 3^{d-k}},$$

and we have a contradiction same as above. Therefore by theorem 5.3.2 we then see that the irreducible components of $(Y_i)_F$ are Hodge subvarieties. In particular Y_i is a Hodge subvariety of X which properly contains x. Now set $V_x = Y_i$ and replace x by V_x.

We can therefore replace each x that satisfies the lower bound hypothesis with a higher dimensional Hodge subvariety and since there are only finitely many x below the bound we see that this new set of Hodge subvarieties is still Zariski dense.

□

Corollary 5.4.3. *With the same notation as above, there is a Zariski dense subset of Hodge subvarieties $V' \in \Sigma'$ such that each V' is the graph of a Hecke correspondence T_g, for some g.*

Proof. By definition there are no Hodge subvarieties of dimension one and three in $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$. This implies that the dimension of each Hodge subvariety in Σ' must be two. The Hodge subvarieties of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$ of dimension two are irreducible components of $\{x\} \times M_A^3(1)_{\mathbb{C}_\infty}$, $M_A^3(1)_{\mathbb{C}_\infty} \times \{y\}$ where x, y are CM points and irreducible components of a of Hecke graph attached to $g \in \mathrm{GL}(K)$. If $V' = \{x_1\} \times M$ for an infinite subset of Σ', where M is an irreducible component of $M_A^3(1)_{\mathbb{C}_\infty}$ and x_1 is the projection of a CM point x. The Zariski closure of the CM points x_1 is special by the Andre-Oort conjecture for $M_A^3(1)_{\mathbb{C}_\infty}$ hence the Zariski closure of the union of the V's is an irreducible component of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$. This contradicts the dimension assumption on X. Therefore V' is the graph of some Hecke correspondence T_g, for almost all $V' \in \Sigma'$. After removing the above set of degenerate Hodge subvarieties we have a Zariski dense set of Hecke graphs. $\square$

We can now prove the André-Oort conjecture for subvarieties in the product of two Drinfeld modular surfaces.

Theorem 5.4.4. *Let X be an irreducible subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$. Suppose that X is defined over F and that the characteristic of F is not 3. Then X contains a Zariski dense set of CM points Σ if and only if X is a Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$.*

Proof. We proceed by induction on the dimension of X. Assume the theorem holds for all subvarieties of dimension less than $\dim X$. The theorem holds vacuously if X is of dimension one. Let p_i, with $i \in \{1, 2\}$ denote the two projections of $M_A^3(1)_{\mathbb{C}_\infty} \times M_A^3(1)_{\mathbb{C}_\infty}$ to each coordinate factor. We may further assume that both projections are dominant onto an irreducible component of $M_A^3(1)_{\mathbb{C}_\infty}$: indeed if X contains a Zariski dense set of CM points Σ, then the projections $p_i(\Sigma)$ are Zariski

dense sets of CM points in $p_i(X) \subset M_A^3(1)_{\mathbb{C}_\infty}$. Since X is irreducible the images of the projections p_i are irreducible. The André-Oort conjecture is true for subvarieties of $M_A^3(1)_{\mathbb{C}_\infty}$ (see Breuer [8]) therefore $p_i(X)$ is a Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty}$.
If the dimension of $p_i(X)$ is less than or equal to one then $p_i(X)$ is a CM point or a special curve in $M_A^3(1)_{\mathbb{C}_\infty}$ for one of the projections. However $M_A^3(1)_{\mathbb{C}_\infty}$ does not contain any special curves. Therefore $p_i(X)$ must be a CM point. In this case X is one of the following

- $X = \{\text{CM point}\} \times \{\text{CM point}\}$
- An irreducible component of $\{\text{CM point}\} \times M_A^3(1)_{\mathbb{C}_\infty}$ or $M_A^3(1)_{\mathbb{C}_\infty} \times \{\text{CM point}\}$

We may therefore assume that both projections of X to each coordinate factor are dominant and consequently that the dimension of X is greater than one. By 5.3.5 we may also assume that X is weakly hodge generic. If X is weakly Hodge generic but not Hodge generic then by theorem 5.3.6 X is a Hodge subvariety. We are thus reduced to the Hodge generic situation.
By proposition 5.4.2 there exists a Zariski dense set Σ' of strictly positive dimensional Hodge subvarieties contained in X such that each CM point in Σ is contained in a Hodge subvariety in Σ'. By lemma 5.4.3 these Hodge subvarieties are graphs of Hecke correspondences T_g. Therefore we are done if X is of dimension two.
Let $\mathfrak{p}_1$ be a prime chosen as in theorem 5.2.6. If $X \subset (T_{\mathfrak{p}_1} \times T_{\mathfrak{p}_1})X$ then by theorem 5.3.2 the F-irreducible components of X are Hodge subvarieties hence X is a Hodge subvariety. So suppose $X \subsetneq (T_{\mathfrak{p}_1} \times T_{\mathfrak{p}_1})X$.
Let y be a CM point of $M_A^3(1)_{\mathbb{C}_\infty}$ and set $X_y := X \cap (\{y\} \times M_A^3(1)_{\mathbb{C}_\infty})$. Let $\bar{X}_y$ denote the Zariski closure of the union $\bigcup_{V \in \Sigma'} V \cap X_y$.
Now as $V = T_g$ ranges through Σ' either $\deg(g) = |T_g(y)|$ is bounded or unbounded.

By corrollary 3.2.2 for a fixed bound B, there are only finitely many Hecke correspondences T_g for which $\deg(g) < B$. Since X is irreducible we have in the first case that X is the graph of some Hecke correspondence T_g and we are done. So assume that $\deg(g)$ is unbounded.

Since each V in Σ' is the graph of some Hecke correspondence T_g, the intersection $V \cap X_y$ contains a CM point and consists of a finite set of CM points linked by some isogeny determined by g. Since $\deg(g)$ is unbounded by assumption, $\bigcup_{V \in \Sigma'} V \cap X_y$ is not finite. Hence its Zariski closure $\bar{X}_y$ has positive dimension in addition to containing a Zariski dense set of CM points. Now $\dim \bar{X}_y < \dim X$ otherwise we would have $\bar{X}_y = X \subset \{y\} \times M_A^3(1)_{\mathbb{C}_\infty}$ contradicting that X has dominant projections. By induction we conclude that the geometrically irreducible components of $\bar{X}_y$ are Hodge subvarieties.

Now the geometrically irreducible components of $\bar{X}_y$ are of the form $\{y\} \times V_y$ where V_y is a Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty}$ and V_y has positive dimension. By the Andre-Oort conjecture for $M_A^3(1)_{\mathbb{C}_\infty}$, V_y is an irreducible component of $M_A^3(1)_{\mathbb{C}_\infty}$. Since there are only finitely many irreducible component of $M_A^3(1)_{\mathbb{C}_\infty}$ we may assume that as y varies V_y is fixed and equals to V.

Let Z be the Zariski closure of $\bigcup_y X'_y$, where the union runs over the Zariski dense set of CM points in $M_A^3(1)_{\mathbb{C}_\infty}$. Then $Z = \tilde{V} \times V$, where $\tilde{V}$ is a subvariety of $M_A^3(1)_{\mathbb{C}_\infty}$ of positive dimension which contains a Zariski dense set of CM points. By the Andre-Oort conjecture for $M_A^3(1)_{\mathbb{C}_\infty}$, $\tilde{V}$ is a Hodge subvariety of $M_A^3(1)_{\mathbb{C}_\infty}$. Therefore Z is a Hodge subvariety. Since $Z \subset X$ and $\dim Z \geq 3$ we have $X = Z$. Therefore X is a Hodge subvariety. $\square$

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