From ‘proofs without words’ to ‘proofs that explain’ in secondary mathematics

M Faaiz Gierdien

Stellenbosch University
Email: faaiz@sun.ac.za

Introduction

From the mid 1970s onwards in almost every issue of the undergraduate mathematics journals *Mathematics Magazine* and *College Mathematics Journal* there is at least one ‘proof without words’ (Nelsen, 1993). A proof without words can be thought of as a ‘proof’ that makes use of visual representations, that is, pictures or other visual means to show a mathematical idea, equation or theorem (Casselman, 2000). It does not contain any *words* other than literal or numerical symbols and geometrical drawings, for example. There is debate around whether a proof without words really qualifies as a proof. It helps the observer see *why* a particular mathematical statement may be true, and also to see *how* one might begin to go about proving it true. It may also have an equation or two, arrows or shading in order to guide the reader in this process. In it there is a clear emphasis on providing visual clues to the reader in order to stimulate thinking with the eventual goal of writing a proof. Many proofs without words in the referred journals are directly related to the secondary Mathematics curriculum in South African schools although not exclusively so.

Interpreting a proof without words requires explanations that draw on various mathematical ideas not necessarily evident in the proof without words. When the reader starts to unpack and explain the diagrams or pictures in the proof without words, it can become a ‘proof that explains’ as opposed to a ‘proof that proves.’ More needs to be said about the last two notions. Writing explanations for and discussing a suitable proof without words can present opportunities to develop insights about and connections between different mathematical ideas. These are also ways to popularise proof in general in the secondary mathematics curriculum (De Villiers, 1990; Volmink, 1990).

References to proof appear in current South African policy documents on school mathematics reform at the secondary level. For example, “competence descriptions” for learners by the end of grade 12 include “being able to critically analyse and compare mathematical arguments and proofs” and being able to “demonstrate an understanding of proof in local axiomatic systems” (Department of Education, 2003: 83). The question becomes, what means are available to align learners and teachers with these competence descriptions? Elsewhere the document mentions “mathematical process skills” which include generalising, explaining, describing, observing, inferring, specialising, justifying, representing, refuting and predicting (*ibid.*: 19). As possible visual processes these mathematical skills can stimulate thinking about proofs and also proofs without words. In this regard visualisation is a key construct which will be explored in this paper. A reference to proof also appears in the study of series and sequences (*ibid.*).

The purpose of this paper is to explore an epistemic role for visualisation with respect to proofs without words in secondary mathematics in the current South African education policy context. Visualisation as process and product can be a means to examining proofs without words by turning them into proofs that explain. In this way students can develop insights and explanations for the mathematics they encounter in the secondary curriculum. The proofs without words chosen are those that show analytic and visual representations of series and sequences. In the secondary curriculum series and sequences are mainly represented analytically. It will be shown that a thoughtful interpretation and explanation through visualisation of such proofs without words connects different strands in the bureaucratically stated secondary curriculum found in the policy document (Department of Education, 2003). There is more mathematics embedded and ‘unseen’ in these proofs without words.

Visualisation as process and product

It is difficult to conceptualise a neat division between visualisation as process and product when we interpret a proof without words. Visualisation as product can be thought of as the proof without words or the final *picture*. On the other hand visualisation as process involves employing various techniques to understand and to interpret the proof without words. Visualisation has a special attraction in the case of a proof without words because the reader is drawn 'to fill in the words' in order to make the theorem or statement in the proof without words true. Literature on visualisation sometimes refers to visualising (Giaquinto, 1993; 1994), visual reasoning (Hershkowitz, Arcavi, & Bruckheimer, 2001) or simply visualisation (Arcavi, 2003). Many proofs without words rely on visual means to communicate a mathematical statement. Visualisation – as both the product and the process of creation, interpretation and reflection upon pictures and images – is gaining increased popularity in mathematics and mathematics education (Arcavi, 2003). What we do upon seeing a proof without words is process a product. Depending on the proof without words, the reader can be drawn into "seeing the unseen and perhaps also proving," according to Arcavi (2003). One can think of interpreting, creating and reflecting as examples of visualisation as process, which can also include scribbling notes or diagrams on paper, or making gestures and utterances. Interestingly, visualisation as product can include explanatory notes that result from interpretation of and reflection on a proof without words, in addition to the final *picture* or proof without words.

In a proof without words of the infinite geometric series

$$\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^6+\ldots=\frac{1}{3} \text{ (see Figure 1),}$$

Arcavi (2003) argues that a proof without words is (a) neither "without words" nor (b) "a proof." The reader is most likely to decode the picture through words (a) – either mentally or aloud – and according to Hilbert's standard for a proof, it must be "arithmetisable" (b), otherwise it is non-existent (Hadamard, 1954, in Arcavi, 2003). This explains the cautious use of "visual proof" in the case of the infinite geometric series. What is clear is the controversy around what constitutes a proof. From the former, we infer that what is seen – or visualisation as process or product – might actually be complemented by verbalisation. Hence the notion of "without words" in proof without words should not be understood literally.

There is continuum between process and product interpretations of visualisation which is illustrated using the proof without words of the same infinite geometric series. The proof without words presented in Figure 1 is a product of the proof creator's visualisation which the reader has to process. It provides us with cues that make our process of visualisation easier. We may not be instantly convinced of the result. Also, we potentially see how a proof for the geometric series is done. There are non-trivial bits of numerical manipulations that the reader has to process, for example, interpreting

$$\left(\frac{1}{2}\right)\times\left(\frac{1}{2}\right) \text{ and } \left(\frac{1}{2}\right)^2\times\left(\frac{1}{2}\right)^2 \text{ as } \textit{areas}.$$

The use of *areas* is an example of Arcavi's (2003) notion of how the reader is attracted to "seeing the unseen" or "filling in the words." To make the statement true we may be attracted to look at the final *picture* or product.

Visualisation as process in the infinite geometric series has its attendant problems, namely, a particularity objection and unintended exclusions (Giaquinto, 1993). What is the particularity objection in visualisation as process in the case of this infinite geometric series? We cannot do a visualisation process of the geometric series that goes up to infinity. We can only do a visualisation process up to a particular number of *areas*. Visualisation as process thus cannot include every *area* in the infinite geometric series but it can specify some *areas*. Also, in the process of visualising the infinite geometric series there are some *areas* which will be excluded from the content of visualising. This is not to say that a precise number of *areas* is visualised. There will be numerical vagueness in the visualisation process, but not so much vagueness that no number of *areas* is excluded. For instance, we cannot visualise 41 specific *areas*, meaning that there will be "unintended exclusions" (*ibid.*). From the way Figure 1 is shown we are typically unable to carry out a visualisation process that includes exactly 41

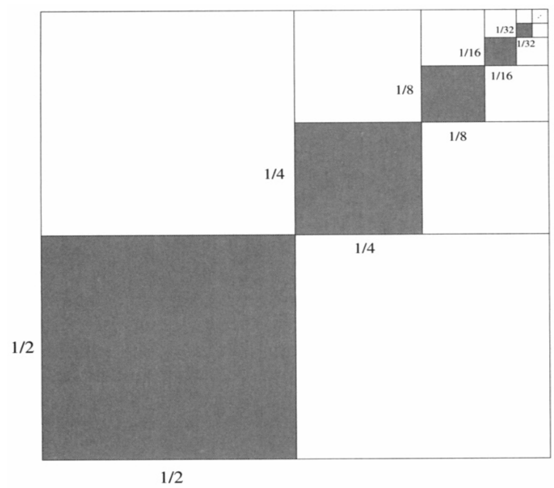


Figure 1. A proof without words of the infinite geometric series

areas. The best we can do is to visualise an arrangement of roughly 6 such *areas*. There will thus be a problem of unintended exclusions the more we specify the number of *areas*. This problem does not negate the use of visualisation as process in this infinite geometric series. It does, however, pull us in the direction of the final *picture* or visualisation as product so that we can hopefully conclude that the sum to infinity equals $\frac{1}{3}$.

Visualisation as process and product in the case of the infinite geometric series can take us in the direction of analysis. The unseen mathematics in Figure 1 is far more than meets the eye. Seeing the unseen mathematics depends on the reader's insights. By visualising the first few steps in the process the reader gets an idea of the common nature of each step: we divide the large unmarked square into quarters, marks the lower left of these quarters and leaves the other three unmarked (to be divided into quarters in the next step). A crucial thought becomes apparent: at each stage there is a shading of one of the four squares. The reader has to come to believe the theorem that the limit of the series

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \ldots = \frac{1}{3}.$$

Also, it is clear that at no particular stage of the division of the squares do the areas of the shaded parts of the figure add up to give $\frac{1}{3}$. It also seems clear that no area at the top right-hand corner is so small that it will not eventually "fill up" the open space. There will be unintended exclusions. One can think of $\frac{1}{3}$ as the least upper bound of the sequence. The truth of the theorem can be inferred from this, taking it as known that a monotonic increasing function sequence bounded above converges to its least upper bound. The arguments presented here take us into the realm of elementary real analysis, involving the limit of an infinite process (Giaquinto, 1994). To get to see why the series has a limit of $\frac{1}{3}$, the reader's eye has to digest several pieces of numerical information that are in the picture. This would entail seeing and eventually proving that the limit of the series is $\frac{1}{3}$ through real analysis. On the other hand the reader can simply *see* or trust that $\frac{1}{3}$ of the area of the outer square is being shaded.

According to Giaquinto (1994) there is insight garnered from the picture of such an infinite series. Looking at the picture or proof without words we understand *why* the series has the sums it does. This picture is not a proof of what the limit of the infinite series is. Implicit in the above are characteristic properties (Steiner, 1978, in Hanna, 1990), which will be discussed, in more detail later on. This, however, brings us to an important distinction between proofs that demonstrate *that* a theorem or statement is true and proofs that show *why* a theorem or statement is true. Visualisation as both process and product plays a key role in turning a proof without words into a proof that explains. How do we distinguish a proof that proves from a proof that explains?

Distinguishing between proofs that prove and proofs that explain

One of Hanna's (1983; 1990; 1998a; 1998b) major contributions to literature on the nature of proof in mathematics and mathematics education is a distinction between proofs that prove and proofs that explain. This distinction has a long and interesting history and is stated slightly differently at times. "Verifying" is used when proofs demonstrate *that* a theorem or statement is true and "clarifying" is used when proofs show *why* a theorem or statement is true (De Villiers, 1990). The former has to do with "convincing" or "making certain," while the latter has to do with "explaining." This distinction is quite important. For example, the mathematician Bolzano (in Hanna, 1990) makes a similar distinction "making certain" (gewissmachung) and "building a foundation" (begründung). "Making certain" and "building a foundation" are synonymous with a proof that proves or verifies and a proof that explains or clarifies, respectively. Hanna (1990) uses "explain" when a proof reveals, and makes use of the mathematical ideas that motivate it and hence refers to an "explanatory proof." Such a proof focuses on "building a foundation" or clarifying, and is consonant with Volmink's (1990) notion of proof as a means of communication. For example, in classrooms, teaching and explaining a proof becomes a form of discourse in which visualisation as process and product can lead to insight and connections among mathematical ideas. On the other hand, a proof that proves does not illuminate the appearance of particular symbols, whether literal or numerical, in a proof.

Hanna (1998a) cites mathematical induction as the example of a proof that proves or verifies. We have to unpack mathematical induction by firstly examining *induction* and then *mathematical* induction. Induction is the process of discovering general laws by the observation of and combination of particular instances. It aims at finding regularity and coherence behind observations. How do we insert a mathematical aspect to observations done via induction? According to Pólya (1945/1988) there is in mathematics a higher authority than observation and induction: rigorous proof. This is where *mathematical* induction comes in. It is to the mathematical aspects of mathematical induction that we turn to next.

Pólya (1954) lists several steps in mathematical induction before its actual technique. These are the inductive phase, the demonstrative phase, examining transitions and, finally, the technique of mathematical induction. During the inductive phase we suspect that a particular mathematical relationship, theorem or statement is true. Driven by what we suspect, we formulate a conjecture about the mathematical statement which we test for particular cases to see if it is true. We check to see if the conjecture is true for several cases and we ask how we can test the conjecture more efficiently. During the demonstrative phase we examine whether the conjecture passes a severe test. This is done by taking what is supposedly true to what is incontestably true and finally consequently true.

The statement $1+2+3+...+n=\frac{n^2}{2}+\frac{n}{2}$ which appears in secondary mathematics will be considered. During the inductive phase there is an examination of several numerical values, where we can tabulate the results for n = 1, 2, 3, …. For example, we would end up with a conjecture that

$$1+2+3+...+n=\frac{n^2}{2}+\frac{n}{2}$$

is probably true.

Proving this truth would involve testing whether the conjecture is true. In the demonstrative phase we increase our doubts by first, assuming that it is *supposedly* true. The conjecture is then shown to be *incontestably* true and then *consequently* true. Examining the transition from n to $n + 1$, is the last reasoning to conclude that $1+2+3+...+n=\frac{n^2}{2}+\frac{n}{2}$ is true for all integers. To summarise, see Box 1.

Going through this exercise we end up being certain *that* the statement is true. It is not difficult to see how all these steps are about convincing and making certain (gewissmachung).

A curious student or learner following the steps in a proof via mathematical induction will certainly have questions, such as why is there a $\frac{1}{2}$ in the statement, $1+2+3+...+n=\frac{n^2}{2}+\frac{n}{2}$?

This question calls for a ‘proof that explains.’ A visual representation or proof without words of this statement is shown in Figure 2.

To answer the question about the $\frac{1}{2}$ the student will have to use visualisation processes such as describing and observing a triangle and *blocks* or square units in order to interpret the proof constructor’s product or proof without words. Describing and observing can include ‘filling in the words’ or verbalising and ‘seeing the unseen.’ It becomes clear that processing a visualisation of the arithmetic (Giaquinto, 1993) in the statement is supported geometrically. Some of the seen and unseen mathematics is the area of a triangle with a height of length n units and a base of length n units. This area turns out to be $\frac{n^2}{2}$.

The missing area to be added is $\frac{1}{2}$ multiplied by n, the number of n square units. These are the shaded half squares, $\frac{n}{2}$. It should be noted that the proof without words is about a general theorem in arithmetic: for all positive integers n, the sum of the first n positive integers is a half of $n^2 + n$. The proof without words has a particular number of squares, meaning that in the visualisation processes there will be unintended exclusions. A similar

$$
\begin{aligned}
1+2+3+\ldots+n &= \frac{n^2}{2}+\frac{n}{2} \\
\therefore 1+2+3+\ldots+n+(n+1) &= \frac{n^2}{2}+\frac{n}{2}+(\mathrm{n}+1) \\
&= \frac{n^2}{2}+\frac{n}{2}+\frac{2n}{2}+\frac{2}{2} \\
&= \frac{n^2+2n+1}{2}+\frac{n+1}{2} \\
&= \frac{(n+1)^2}{2}+\frac{(n+1)}{2}
\end{aligned}
$$

Box 1. The transition from *supposedly* true to *incontestably* true, to *consequently* true, for all positive integers

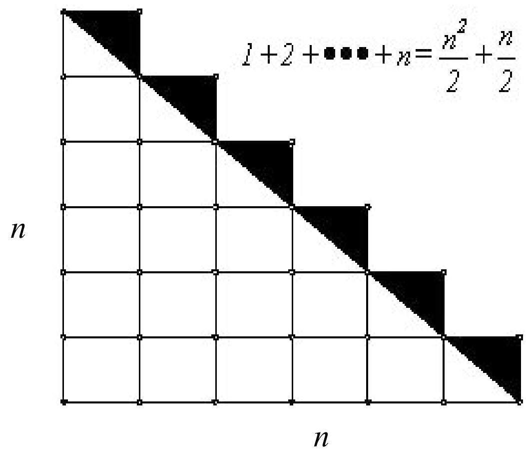


Figure 2. Visual representation or proof without words

point was raised in the case of the infinite geometric series. Alternately, we can find the area of a square of side length *n*, halving this area (n^2) and then adding $\frac{1}{2}$ of *n* blocks to yield $\frac{n^2}{2}+\frac{n}{2}$, to find 1 + 2 + 3 + …+ *n*. There is thus a geometrical justification in terms of the area of a triangle that explains the statement, $1+2+3+...+n=\frac{n^2}{2}+\frac{n}{2}$ which includes an explanation for the appearance of $\frac{1}{2}$.

There are contrasts between proofs that prove and proofs that explain in the case of the said mathematical statement. In the entire proof that proves via mathematical induction there is no translation back and forth between different representations. Only a numerical or analytic representation is used. In contrast, the proof that explains uses far more mathematics with the hope of bringing about understanding. Here there is the possibility that the student will develop insights depending on how his or her visualisation as process and product interacts and unfolds. The mathematical statement is about a general arithmetic theorem which is proved via mathematical induction, for all positive integers *n*. In contrast the proof that explains with its geometric justification makes use of the area of a particular triangle, although the height and base of the triangle is stated as general, namely, '*n*'. More needs to be said about proving and explaining with respect to proofs without words.

Prove and explain

So far it is evident that visualisation as process and product plays an important role in turning suitable proofs without words into explanatory proofs or proofs that explain. Central to this is seeking characterising properties in the proofs without words. Steiner (1978: 143) and Hanna (1990: 10) – who cites Steiner – characterise an explanatory proof as follows:

> *…an explanatory proof makes reference to a characterising property on an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depends on the property. It must be evident, that is, that if we substitute in the proof a different object of the same domain, the theorem collapses; more, we should be able to see as we vary the object, how the theorem changes in response.*

For example, what characteristic properties are entailed in proofs without words of the following two statements?

$$1+3+6+...+\frac{n(n+1)}{2}=\frac{n(n+1)(n+2)}{6} \quad \text{and}$$

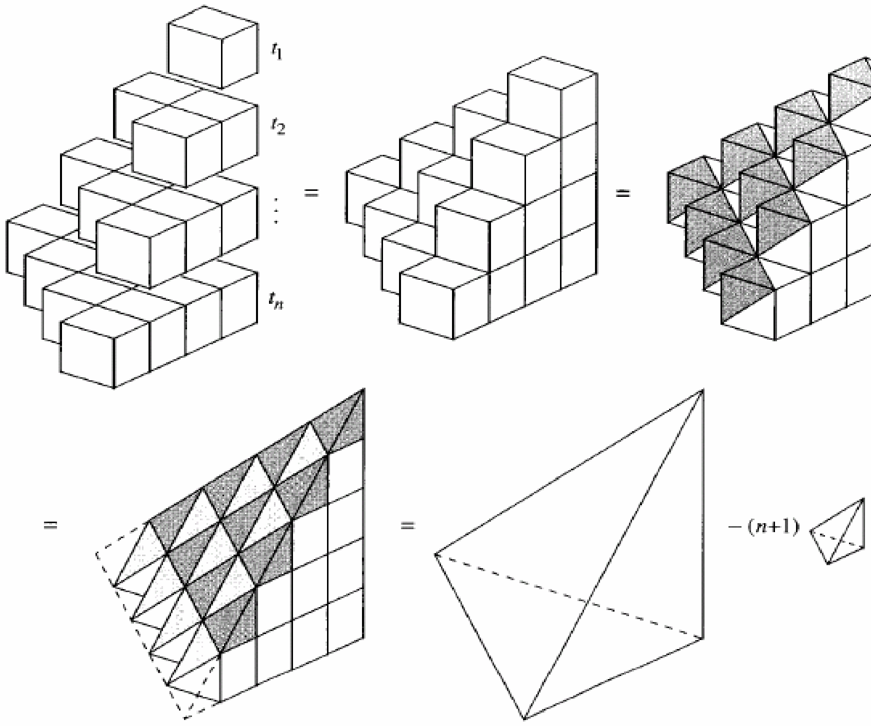
$$1^3+2^3+3^3+...+n^3=\left(\frac{n(n+1)}{2}\right)^2 ?$$

Also, how do we go about finding the characterising properties in proofs without words that motivate, explain and compel the truths of the statements? Much of the answer lies with Chinese mathematicians for whom a proof consisted of "any explanatory note, which served to convince or to enlighten" (Siu, 1993: 346). They practiced "proof as explanation" in ways that were very different from Greek mathematicians' axiomatics and deductive proofs, because these had not reached them (Hanna, 1998b). Explanatory notes as instances of visualisation as process and product thus play an important role in searching for characteristic properties.

How can we explain each of the symbols in Figure 3, $\frac{1}{6}(n+1)^3-\frac{1}{6}(n+1)=\frac{n(n+1)(n+2)}{6}$, the mathematical statement for the sum of *n* triangular numbers?

Visualisation as product in the form of a proof without words showing the sum of *n* triangular numbers is shown in Figure 3. The triangular numbers – 1, 3, 6, … , $\frac{n(n+1)}{2}$ – are represented geometrically as the cubes in the layers t_1, t_2, t_3…, t_n respectively. The cubes forming the triangular numbers at each stage are arranged in a way where they form three-dimensional objects, which suggests that volume will come into play. In uncovering the characteristic properties of this

$$t_n = 1 + 2 + \cdots + n \Rightarrow t_1 + t_2 + \cdots + t_n = \frac{n(n+1)(n+2)}{6}$$



$$t_1 + t_2 + \cdots + t_n = \frac{1}{6}(n+1)^3 - (n+1) \cdot \frac{1}{6} = \frac{n(n+1)(n+2)}{6}$$

Figure 3. Proof without words: sum of triangular numbers (from Nelsen, 2005)

proof without words, the reader's eye is guided by the visualisation in the arrangement of the triangular numbers as cubes and the equal signs ending with the generalised pyramid of height (n+1) units and base of area $\frac{1}{2}(n+1)^2$ square units. This forms a carefully assembled chain of reasoning and qualifies as a "good mathematical illustration" (Casselman, 2000) that entices the reader to visualise the processes that make the mathematical statement true.

Where does $\frac{1}{6}$ come from? In the third arrangement of the triangular number as cubes, in Figure 3, one sixth of the volume of the top small cube is shaded. A small cube on its own will consist of 3 small pyramids having the same height. This is what Calculus tells us, namely, the volume of a pyramid having the same height

$$V = \frac{1}{3}(\text{area of base} \times \text{height}).$$

The base, however, is halved, meaning that the volume of the shaded part of the cube becomes

$$V = \frac{1}{6}(\text{area of base} \times \text{height}).$$

This is indicated by the shaded part in the top, small cube, which forms the first triangular number. In the second triangular number two such slices are shaded. The pattern continues where three such slices are shaded for the third triangular number, and so on.

In the second row of the arrangement in Figure 3, the shaded pyramids are turned upwards to a generalised pyramid of height (n + 1), with a halved base which is explained by the $\frac{1}{6}$. The full volume of a cubic arrangement of triangular numbers of side length (n+1) is therefore $(n+1)^3$. In the case of summing the triangular numbers, we are only interested in $\frac{1}{6}$ of the volume. The *extra* volumes of $\frac{1}{6}$ of 1 cubic unit of which there will be (n+1) have to be subtracted. This illuminates the line $\frac{1}{6}(n+1)^3 - \frac{1}{6}(n+1)$.

Obtaining the right-hand side of the statement $\frac{n(n+1)(n+2)}{6}$ is a matter of factoring. As in the previous cases the proof without words contains unintended exclusions because it aims at drawing the reader into seeing a generalised arithmetic theorem. Proofs without words for the statement

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

entail characteristic properties that are combinatorial and geometric in nature (see Figures 4 and 5).

Suggestion 1: combinatorial proof without words

The combinatorial proof takes its name from the combinations of the sum of positive integers starting with 1 (see Figure 4). By examining the combinatorial proof the reader can be encouraged into visualisation processes such as 'seeing the unseen' or 'filling in the words.' On the left hand side (Figure 4), there is the sum of the positive integers in the first row beginning with 1 up to *n*. In the second row, each of these integers is multiplied by 2. In the third row, each of the integers in the first row is multiplied by 3. This pattern continues. The last two rows on the left-hand side can be factorised and reduced to the following:

$(1 + 2 + 3 + \ldots + n) \times (\sum_{i=1}^{n} i)$ which becomes $(\sum_{i=1}^{n} i) \times (\sum_{i=1}^{n} i)$. On the right hand side of Figure 4 the combinations of numbers as indicated are added in the following way:

1 + (2 + 4 + 2) + (3 + 6 + 9 + 6 + 3) + $\ldots n + 2n + 3n + \ldots + n^2 + \ldots + 3n + 2n + n$.

Non-trivial bits of algebraic manipulation will have to be done to show that any L shape analytically represented as

$$n + 2n + 3n + \ldots + n^2 + \ldots + 3n + 2n + n$$

sums to $n(n^2)$.

Interestingly, these bits are not visualised at all. The L shapes sums are as follows:

1 = 1(1)

2 + 4 + 2 = 2(1 + 2 + 1)

3 + 6 + 9 + 6 + 3 = 3(1 + 2 + 3 + 2 + 1).

This generalises as set out in Box 2.

$$n + 2n + 3n + \ldots + n^2 + \ldots + 3n + 2n + n = n(1 + 2 + 3 + \ldots + n) + n(1 + 2 + 3 + \ldots + n - 1)$$
$$= n\left(\frac{n(n+1)}{2} + \frac{n(n-1)}{2}\right)$$

Box 2. Generalising for a combinatorial proof

Manipulating $n\left(\frac{n(n+1)}{2} + \frac{n(n-1)}{2}\right)$ yields

$$n\left(\frac{n(n+1) + n(n-1)}{2}\right)$$
$$= n\left(\frac{n^2 + n + n^2 - n}{2}\right)$$
$$= n(n^2)$$

	1	**2**	**3**	.	.	.	***n***
+	**2**	**4**	**6**	.	.	.	**2*n***
+	**3**	**6**	**9**	.	.	.	**3*n***
+	.	.	.	.	.	.	.
	.	.	.	.	.		.
	.	.	.	.	.	.	.
+	***n***	**2*n***	**3*n***	.	.	.	***n*²**

$$= \sum_{i=1}^{n} i + 2\sum_{i=1}^{n} i + 3\sum_{i=1}^{n} i + \ldots + n\sum_{i=1}^{n} i$$
$$= (\sum_{i=1}^{n} i)^2$$
$$= \left(\frac{n(n+1)}{2}\right)^2$$

	1	**2**	**3**	.	.	.	***n***
+	**2**	**4**	**6**	.	.	.	**2*n***
+	**3**	**6**	**9**	.	.	.	**3*n***
+	.	.	.	.	.	.	.
	.	.	.	.	.	.	.
	.	.	.	.	.	.	.
+	***n***	**2*n***	**3*n***	.	.	.	***n*²**

$$= 1(1^2) + 2(2)^2 + \ldots + n(n)^2$$
$$= \sum_{i=1}^{n} i^3$$

Figure 4. Suggestion 1 – combinatorial proof (adapted from Pouryoussefi, 1989)

The previous manipulation is a deductive proof that shows that

$$n + 2n + 3n + ... + n^2 + ... + 3n + 2n + n = n(n^2)$$

The combinatorial proof without words, in fact, uses a result from a previous proof without words about consecutive integers that was discussed earlier, namely: $1 + 2 + 3 + ... + n = \frac{n^2}{2} + \frac{n}{2}$

By focusing on these combinations of the numbers, we gain a sense of the truth of the original statement, namely:

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Suggestion 2: geometric proof without words

In the geometric proof without words, a focus on the area of a square of side length compels the truth in the original statement with some qualification (see Figure 5). The reader's attention can be directed to visualising the area of a square with a side length (1 + 2 + 3 + 4 + 5), or $\frac{5(5+1)}{2}$. This is a particular side length meaning that the particularity objection mentioned earlier may be applicable. The area of the square is $\left(\frac{5(5+1)}{2}\right)^2$ and not $\left(\frac{n(n+1)}{2}\right)^2$. This means that there are unintended exclusions in the geometric proof without words. This does not mean that our visualising experience cannot be in the direction of a general arithmetical or mathematical statement as in this case. The area inferred in the geometric proof is not stated as an arithmetical theorem about all positive integers (Giaquinto, 1994) as compared to the combinatorial proof in suggestion 1. The problem of the unintended exclusions does not support a negative view of the utility of visualisation processes which are geometric in this instance. The connection between 'series and sequences' and the geometry of the areas of squares are not surprising because we have such a connection for summing the integers 1 + 2 + 3 + …+ n as we saw earlier on.

Implications for teaching

The ideas discussed in this paper have implications for what might happen in teaching. Each of the proofs without words became proofs that explain via visualisation as process and product. The latter is therefore a means to align policy statements about learners' competence descriptions with respect to proof. Recall that learners have to be able to critically analyse and compare mathematical arguments and proofs.

So what might the teacher do? He or she should encourage learners to do the explaining when poring over a proof without words. They should be encouraged to 'fill in the words' and to try to 'see the unseen' mathematics through visualisation. They could do so collectively or individually. Applicable here are visualisation processes such as generalising, observing, inferring, representing, predicting, describing through writing down what is observed and verbalising collectively and individually. Note that that these processes are in concert with the Department of Education's "mathematical process skills" according to the South African policy document for secondary mathematics (2003: 19). Learners could record their utterances or verbalisation on the sheet containing the proof without words. This would be the product of their visualisation in addition to the proof constructor's proof without words which they will be interpreting and explaining. The teacher must explicitly tell learners that any proof without words is a proof constructor's final product that they have to process. The teacher would have the challenging task of orchestrating a discussion that has the goal of linking learners' visualisation process and product with the proof without words that they are examining.

What is gained by learners explaining what they see in a proof without words? They might see how mathematical ideas in the secondary curriculum are related

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25

Figure 5. Suggestion 2 – geometric proof (adapted from Pouryoussefi,1989)

through different representations. Who would have thought that a proof without words of a compact statement such as

$$1+2+3+...+n=\frac{n^2}{2}+\frac{n}{2}$$

– on series and sequences – can be explained via the area of a triangle, or that an analytic representation such as

$$1+3+6+...+\frac{n(n+1)}{2}=\frac{n(n+1)(n+2)}{6}$$

can be explained using the volume of a pyramid? They could learn from the insights that fellow learners present during explaining. Here the teacher plays a critical role because he or she will have to figure out what learners are saying in relation to what they 'see" and 'don't see.' A broad base of knowledge which is a prerequisite for mathematical insight (Hanna, 1983) could be gained by explaining through visualisation as process and product. In a proof that proves, learners would not be able to come up with explanations for the appearance of $\frac{1}{2}$ or a $\frac{1}{6}$ as in Figure 3.

In a proof without words what could be gained by the explanation itself? The explanation can certainly help in terms of Bolzano's "building a foundation" (begründung) (in Hanna, 1990). Any explanation itself, however, will have to contend with unintended exclusions and particularity objections as shown earlier. In the first proof without words in Figure 1,

$$\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^6+...=\frac{1}{3}$$

it is not possible to exercise visualisation as process that goes to infinity. Furthermore, the last three proofs without words (Figure 3) are about general arithmetic theorems or series, namely, the sum of consecutive positive integers starting with 1, the sum of consecutive triangular numbers starting with 1 and the sum of consecutive cubes starting with 1. As 'informal proofs,' the proofs without words discussed highlight the slippage from dealing with specific numbers to dealing with infinity and general arithmetic theorems. Explanations must take this slippage into account. Learners might want to know whether there is another method to deal with the problems of unintended exclusions and particularity objections. Would this pave the way for proofs that prove? What could done in the case of the learner who cannot 'see' the deductive proof for

$n+2n+3n+...+n^2+...+3n+2n+n = n(n^2)$?

This deductive proof does not 'explain'.

If we are to align learners' competence descriptions with respect to proof then we must in our teaching aim for a level of proof that explains. The deductive mechanisms of mathematical induction and deductive proof do not have the goal of mathematical understanding (Hanna, 1983; 1990).

Concluding remarks

This paper has shown that visualisation as both process and product can play an epistemic role in changing selected proofs without words into proofs that explain. It can be a means to help learners to critically analyse and compare mathematical arguments and proofs at the secondary level. What has to be mentioned is the debate around the role of visualisation itself in the learning of mathematics. Sfard (1998) cites a prominent member of the mathematics community who states that visualisation is not mathematics. The possibility of the "devaluation of visualisation" (Presmeg, 1997) is therefore likely to permeate right to the classroom, curriculum materials and teacher education, according to Arcavi (2003). Also, there are cognitive difficulties around visualisation. In simplistic terms the issue raised reads as follows: is 'visual' easier or more difficult? We saw the cognitive demand was certainly high in turning the combinatorial proof without words of

$$1^3+2^3+3^3+...+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

into a proof that explains. In fact, it depends on a previous proof without words. Learners would need to attain flexible and competent translation back and forth between visual and analytic representations. Learners working on their competences would thus have to be ready for long-winded, non-linear and even tortuous processes (Schoenfeld, Smith & Arcavi, 1993). Last but not least, a difficulty arises from the fact that the proofs without words in this paper were taken from mathematics journals associated with tertiary or higher education. In teaching proofs that explain words via visualisation in secondary schools there will be the inevitable "didactical transposition" (Chevallard, 1985). There will be a transformation of the knowledge associated with changing proofs without words into proofs that explain. By its very nature this process linearises, compartmentalises and possibly also algorithmetises knowledge, thereby stripping it of any rich interconnections (Arcavi, 2003).

Acknowledgement
I would like to thank Dr Helen Chick for providing insightful feedback to an earlier draft of this paper.

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"In order to translate a sentence from English into French two things are necessary. First, we must understand thoroughly the English sentence. Second, we must be familiar with the forms of expression peculiar to the French language. The situation is very similar when we attempt to express in mathematical symbols a condition proposed in words. First, we must understand thoroughly the condition. Second, we must be familiar with the forms of mathematical expression."

George Pólya