THE PROCESS OF MATHEMATISATION IN MATHEMATICAL MODELLING OF NUMBER PATTERNS IN SECONDARY SCHOOL MATHEMATICS

by

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DECLARATION

By submitting this dissertation electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the owner of the copyright thereof (unless to the extent explicitly otherwise stated) and that I have not previously in its entirety or in part submitted it for any qualification.

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Abstract

Research has confirmed the educational value of mathematical modelling for learners of all abilities. The development of modelling competencies is essential in the modelling approach. Little research has been done to identify and develop the *mathematising* modelling competency *for specific sections of the mathematics curriculum*. The study investigates the development of mathematising competencies during the modelling of number pattern problems. The RME theory has been selected as the theoretical framework for the study because of its focus on mathematisation. Mathematising competencies are identified from current literature and developed into models for horizontal and vertical (complete) mathematisation. The complete mathematising competencies were developed for number patterns and mapped on a continuum. They are *internalising*, *interpreting*, *structuring*, *symbolising*, *adjusting*, *organising* and *generalising*. The study investigates the formulation of a hypothetical trajectory for algebra and its associated local instruction theory to describe how effectively learning occurs when the mathematising competencies are applied in the learning process. Guided reinvention, didactical phenomenology and emergent modelling are the three RME design heuristics to form an instructional theory and were integrated throughout the study to comply with the design-based research's outcome: to develop a learning trajectory and the means to support the learning thereof. The results support research findings, that modelling competencies develop when learners partake in mathematical modelling and that a heterogeneous group of learners develop complete mathematising competencies through the learning of the modelling process. Recommendations for additional studies include investigations to measure the influence of mathematical modelling on individualised learning in secondary school mathematics.

Key words: mathematical modelling, modelling competencies, mathematisation (horizontal and vertical), hypothetical learning trajectory, local instructional theory, number patterns

Abstrak

Navorsing steun die opvoedkundige waarde van modellering vir leerders met verskillende wiskundige vermoëns. Die ontwikkeling van modelleringsbevoegdhede is noodsaaklik in 'n modelleringsraamwerk. Daar is min navorsing wat die identifikasie en ontwikkeling van die bevoegdhede vir *matematisering vir spesifieke afdelings van die wiskundekurrikulum* beskryf. Die studie ondersoek die ontwikkeling van matematiseringsbevoegdhede tydens modellering van getalpatrone. Die Realistiese Wiskundeonderwysteorie is gekies as die teoretiese raamwerk vir die studie, omdat hierdie teorie die matematiseringsproses sentraal plaas. Matematiseringsbevoegdhede vanuit die bestaande literatuur is geïdentifiseer en ontwikkel tot modelle wat horisontale en vertikale (volledige) matematisering aandui. Hierdie matematiseringsbevoegdhede is spesifiek vir getalpatrone ontwikkel en op ‘n kontinuum geplaas. Hulle is *internalisering*, *interpretasie*, *strukturering*, *simbolisering*, *aanpassing*, *organisering* en *veralgemening*. Die studie lewer die formulering van ‘n hipotetiese leertrajek vir algebra, die gepaardgaande lokale onderrigteorie en beskryf hoe effektiewe leer plaasvind wanneer die ontwikkelde matematiseringsbevoegdhede volledig in die leerproses toegepas word. Die RME ontwikkellingsheuristieke, begeleidende herontdekking, didaktiese fenomenologie en ontluikende modellering, is geïntegreer in die studie sodat dit aan die uitkoms van ‘n ontwikkelingsondersoek voldoen. Die uitkoms is ‘n leertrajek en ‘n beskrywing hoe die leerproses ondersteun kan word. Die analise het tot die formulering van ‘n lokale-onderrig-teorie vir getalpatrone gelei. Die resultate van die studie kom ooreen met navorsingsbevindings dat modelleringsbevoegdhede ontwikkel wanneer leerders deelneem aan modelleringsaktiwiteite, en bewys dat ‘n groep leerders met gemengde vermoëns volledige matematiseringsbevoegdhede ontwikkel wanneer hulle deur die modelleringsproses werk. 'n Aanbeveling vir verdere navorsing is om die uitwerking van die modelleringsperspektief op individuele leer in hoërskool klaskamers te ondersoek.

Sleutelwoorde: wiskunde modellering, modellerings bevoegdhede, matematisering (horisontaal en vertikaal), hipotetiese leertrajek, lokale onderrigteorie, getalpatrone

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CHAPTER 1
INTRODUCTION AND OVERVIEW

1.1 BACKGROUND TO THE STUDY

Mathematics education has endured many a change over the past decade. This change can be attributed to the change in nature of mathematics and what mathematics means to the average learner, his life and career choices. The Programme for International Student Assessment (PISA) document emphasises the usefulness of mathematics in the world and the importance of learners' understanding the relevance of it (Organisation for Economic Cooperation and Development [OECD], 2003). Meaningful mathematical experiences demand the teaching and learning of mathematics be more applied.

Research has established the educational value of mathematical modelling (Barbosa, 2006; Kaiser & Schwarz, 2006; Lingefjärd, 2006; Maaβ, 2006). From an educational perspective, mathematical modelling is dealt with as a means and as a goal. As a means, the mathematical modelling concept has the advantage of developing and constructing mathematical knowledge, and as a goal it has the advantage of developing mathematical skills and mathematical thinking (Sjuts, 2005). This is true for learners with different abilities at all levels of school mathematics. Average learners are capable of forming powerful mathematical models and constructs: these conceptual systems that form the basis of these models, are often are more sophisticated than those they are currently taught in schools (Lesh, & English, 2005; Kaiser & Schwarz, 2006). Mathematical modelling ensures a richer learning experience as it embraces the aspect of doing mathematics (Burkhardt, 2006). To ensure the outcome of the purpose of mathematics education, to produce critical, reflective thinkers who are prepared to solve problems in their lives, attention must be given to important and relevant mathematics (Brenner, 1998). When a teacher adopts a mathematical modelling perspective and consequently a problem-centred approach to teaching, it has the possibility to change the perspective of teaching and learning mathematics from the teacher's perspective and the learners' perspective.

The learning of mathematics involves constructing concepts based on an existing reference frame. It is a socio-constructive experience. Mathematics education employs constructivism from a cognitive position as well as a methodological position (Hanley, 1994; Noddings, 1990). The constructivist learning theory is the process whereby individuals construct their own knowledge and understanding for themselves. Thus the learning process is a social phenomenon where communication is vital (Cobb & Yackel, 1998; Ward, 2005). The ability to solve mathematical problems is more valuable than having the knowledge but lacking the competencies to apply that knowledge. “Knowledge should be intelligent and applicable, not an inactive and isolated one, because knowledge should be developed into ability” (Sjuts, 2005, p. 424). The constructivist theory requires a learner-centred, problem-centred and collaborative approach to teaching, where the learner has the opportunity to interact with their awareness, as well as the opportunity to construct their own knowledge. Cobb (1999) regards the prime responsibility of the teacher to generate a collaborative, problem-solving environment. Solving mathematical problems adequately initially requires inductive skills but later also deductive skills in the development of their reasoning, which forms an important basis for higher mathematical thinking. Through mathematical modelling, learners develop the necessary competencies and skills to solve mathematical modelling problems. Maaβ (2006) discusses the importance of using problem solving skills and divergent thinking when dealing with a mathematical modelling problem.

Mathematical models are representations constructed from real life problems to aid the problem-solving process. Mathematical modelling can be described as the process where an authentic problem is solved by forming a model of the real situation, constructing a mathematical model from the real situation, constructing a mathematical model from the real model, finding a mathematical solution, and interpreting and validating the solution with regards to the original problem (Borromeo Ferri, 2006; Lingefjärd, 2006; Maaβ, 2006). Various problem-solving skills and competencies are enhanced while working through the modelling cycle on unseen, non-routine problems. The modelling cycle explains the modelling process from a real world problem to a validated solution. The competencies noted as a learner moves through the modelling cycle are understanding the task, simplifying the task, mathematising, working mathematically, interpreting and validating the solution. During this study various competencies are explored but the focus of this study is on the competency *mathematising*. The following section provides an explanation of the mathematisation process and its relevance to the study.

1.2 RATIONALE OF THE STUDY

Realistic Mathematics Education (RME) is a teaching and learning theory that was first introduced and developed by the Freudenthal Institute in the Netherlands. RME has a certain view on mathematics education. Freudenthal (see Gravemeijer, 1994) envisioned that mathematics education needs to be connected to reality and mathematics as a human activity. If mathematics is connected to reality it will allow learners to reinvent mathematics so that they experience a similar invention-process. This will allow for mathematics to be meaningful and relevant in their lives. They have developed six principles that depicts the essence of RME: the activity principle, reality principle, level principle, intertwinement principle, interaction principle and the guidance principle (Van den Heuvel-Panhuizen, 2000, pp. 5-9). If the RME principles are integrated within mathematics education learners will work with a series of progressive realistic problems during guided instruction and will have the opportunity to reinvent mathematics by doing it.

Mathematics as human activity involves an explorative type of modelling which occurs at the "level of concept formation" (Andresen, 2007, p. 2042). This is the activity of mathematising. Mathematising involves the sense making, quantifying and coordinating of experiences using different mathematical methods (Lesh & Sriraman, 2005). Treffers signifies eight characteristics that need to occur during the learning process so that opportunities for mathematising are best possible. These include activity, differentiation, vertical planning, structural character, applicability, language, dynamics and a specific approach (Treffers, 1987, p.58-71). Treffers (1987) distinguished between horizontal and vertical mathematisation. In horizontal mathematisation learners organise and solve a problem that is real to them and in vertical mathematisation learners reorganise the mathematical system itself (Van den Heuvel-Panhuizen, 2003). According to Wessels (2009), learners find the mathematising component the principle problem area within the mathematical modelling process. The study will integrate the RME principles and characteristics for mathematising when planning and executing the teaching and learning activities.

During mathematical modelling, learners construct models in different parts of the modelling process and at different activity levels. Figure 1.1 shows the relationship between the four activity-levels that result in the merging of models. Horizontal mathematising is the first step in the modelling process. When a learner translates a contextual problem into a mathematical model, the learner's own experience is being reinvented. As the problem is being analysed, activity at situational level occurs. Horizontal mathematising occurs at the situational level. As the learner further engages in the problem, a model is formed at referential level. This model is constructed to be a *model of* a specific situation. Models are further used to model other situations: this is known as the activity at general level which occurs as the *model of* now becomes a *model for*. Vertical mathematisation occurs as a learner moves between the referential and the formal activity levels.

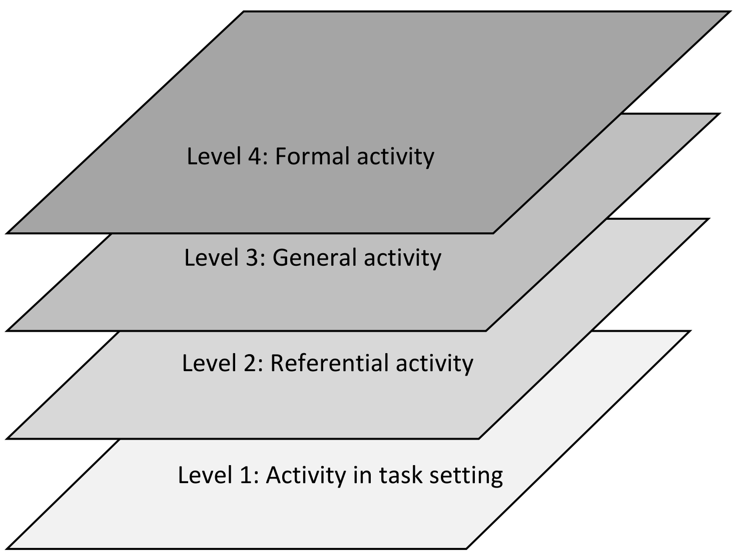


Figure 1.1: Levels of activity (Gravemeijer, Cobb, Bowers & Whitenack, 2000, p. 243)

The objective of the RME theory is to deliver a local instructional theory (LIT) that includes learning activities and rationales to provide the necessary support for teachers to adapt and implement the instructional theory for their specific classes (Gravemeijer, 1999, 2004). The RME theory's design heuristics provide the necessary guide to develop a local instructional theory (LIT). The design heuristics are guided reinvention, didactical phenomenology and emergent modelling (Gravemeijer, 1999). The (LIT) develops through the formulation of an actual observed learning trajectory (LT) by planning a hypothetical learning trajectory (HLT). A HLT provides a teacher with a path of development which is directed towards a specific goal and changes continually to support teaching and learning. Simon (1995, p. 133) notes the importance of a HLT. When constructing a HLT, it needs to incorporate learning goals, the learning activities and the thinking and learning of learners. Instructional tasks are designed to match the levels of thinking to progressively work towards the goal of that specific trajectory. A different number of problems constitute the HLT. As a learner moves through the trajectory, strategic thinking is developed, and he is able to construct a higher or further level of formal mathematics beyond informal mathematics.

In the South African curriculum, a fraction of the time is allocated to modelling problems.
If the focus is shifted to a problem-solving mathematics curriculum, the possibility exists that learners can achieve a higher level of thinking. Higher order thinking involves and results in the connections a learner makes between different mathematical knowledge and constructs. Van den Heuvel-Panhuizen (2003) notes that models can be utilised to bridge the gap between informal and formal mathematics; resulting in formal understanding. Through the learner's construction of emerging models and the process of progressive mathematisation, it leads to building strategic knowledge which will assist him moving up to a higher level of understanding mathematics and constructing mathematical knowledge.

1.3 PROBLEM STATEMENT

The contributing value of mathematical modelling to the learning process of the individual learner and the complementing role in mathematics education cannot be ignored. The transmission approach to mathematics subject didactics involves an over-reliance on a series of procedures of algebraic manipulation and solving routine problems from a theoretical point of view. This type of mathematics has done little to enhance learners' ideas of mathematics in real life or the understanding of real life problems (Brown, 2008). The process of mathematisation is also part of the traditional teaching approach but *the sense making, quantifying and coordinating* *of experiences using different mathematical methods* (see Lesh & Sriraman above) which is the heart of mathematics understanding, is missing in many mathematics classrooms. Mathematising is directly related to the underlying processes when a learner grapples with unseen, non-routine problems. The emphasis in this study will be placed on the mathematisation process, by formulating a teaching-learning trajectory which will focus on the understanding of number patterns that is being learned. As the learner moves through the different activity levels in no specific order, based on Gravemeijer's model (see Figure 1.1), the different elements that suggest horizontal and vertical mathematisation processes can be noted. It is possible to make a distinction between horizontal and vertical mathematisation and to explore the competencies of each mathematisation process. The following research questions will guide the study.

Research question:

How does the development of a local instructional theory influence learners' development of mathematising competencies when modelling number pattern problems?

The study will aim to:

Aim 1: describe a mathematical modelling perspective towards the teaching and learning of mathematics

Aim 2: analyse the process of mathematisation, and mathematising competencies

Aim 3: analyse number patterns in terms of the processes of mathematisation

Aim 4: design of hypothetical learning trajectory that will form a learning trajectory

Aim 5: design a learning trajectory that will form a framework for a local instructional theory

Sub research questions:

1.1 What is the nature and scope of the didactics and curriculum theory that should be formulated to address the research question?

1.2 What is a modelling perspective towards the teaching and learning of mathematics?

2.1 What constitutes the process of mathematisation?

2.2 What are the differences between horizontal and vertical mathematisation based on a mathematical modelling framework?

2.3 What are horizontal and vertical mathematising competencies?

3.1 How can number patterns be explained in terms of the processes of mathematisation?

3.2 What are mathematising competencies for number pattern problems?

4.1 What constitutes a hypothetical learning trajectory?

4.2 How does the type of activity contribute to the aim/objective of the activity?

4.3 What are the roles of the activities which make up the hypothetical learning trajectory?

4.4 How does a hypothetical learning trajectory contribute to and influence a learner's mathematising?

5.1 When does a hypothetical learning trajectory become a learning trajectory?

5.2 Does a local instructional theory assist the application of modelling in mathematics to lead to a better understanding of mathematisation by the learners?

The intention for this study is to contribute to the current research on learning and teaching of mathematical modelling.

1.4 AIM OF THE INVESTIGATION

The aim of the study focuses on the foundation: learning for meaningful understanding. An improved adaptation would be: *understanding for meaningful learning*. When combining a mathematical modelling approach with a problem-centred perspective towards the teaching and learning of mathematics, the learner experiences mathematics as a process created by the learner himself. For meaningful mathematical experiences to occur, it is important for a learner to build his own understanding. Section 1.2 explained that a RME theory will result in meaningful learning when mathematics is reinvented through guided instruction. The RME theory will be incorporated in all aspects of the study.

In Chapter 2 the various perspectives of mathematical modelling will be explored to establish the characteristics of the mathematical modelling process. The mathematical modelling competencies will be identified and explained to focus the study on the activities that identify the competency *mathematising*. The process of mathematisation will be investigated to distinguish between horizontal and vertical mathematising competencies and to develop models for mathematising competencies. The goals for the study will be selected by means of a phenomenological analysis in Chapter 3 so that the HLT in Chapter 4 is in line with the RME theory's design heuristics for an instructional theory. Mathematical modelling learning activities will be selected for the HLT based on the researcher's predicted learning goals. The learning activities in the HLT will guide the learning process as learners complete the different activities. The learning activites will support learners' reasoning and mathematical development. Investigating the dimensions of development of the models and adapting the LT accordingly will accommodate the facilitation of further understanding. The learners will learn the modelling process during the teaching experiment while working through the modelling problems. As the learners work towards emergent modelling, a comprehensive description will be given to analyse the mathematisation process. It is then possible to investigate the learners' horizontal and vertical mathematising competencies. Chapter 5 will analyse the horizontal and vertical mathematising competencies and provide a rationale for each learning activity. A local instructional theory will evolve through the actual observed learning trajectory which will form a theory for number patterns.

1.5 RESEARCH METHODOLOGY

1.5.1 Research design

The project will be carried out in the context of a typical design research framework within a qualitative research process (Gravemeijer et al., 2000). This research design is flexible and evolves as theories develop throughout the research process. It is characterised by planning and creating educational settings for investigating the teaching and learning process. The design-based research methodology consists of the preparation phase for a teaching experiment, the teaching experiment to support learning and a retrospective analysis for the collected data (Cobb & Gravemeijer, 2008). A feasibility study will be conducted to develop a local instructional theory for a specific topic in the mathematics curriculum, namely Number Patterns. A hypothetical learning trajectory will be used as guidance for the study and data-analysis. The data-analysis will be aimed at understanding the horizontal and vertical mathematisation processes of the learners.

1.5.2 Sample

The investigation will be conducted in a multi-cultural, English medium school. The classroom will follow a typical mathematical modelling perspective constituting a problem-centred approach. During their Grade 10 year, the learners were introduced to the problem-centred approach to teaching and learning mathematics. The learners will therefore be familiar with a collaborative culture of learning. A focus group will be randomly selected consisting of six learners from the Grade 10 mathematics class.

1.5.3 Method

1.5.3.1 Research instruments

Research instruments to assist in collecting data will be designed and used during the teaching experiment. A baseline assessment will be developed to assess the learners' pre-knowledge of number patterns. It will help formulate a clear goal for a trajectory. The baseline assessment will provide the starting points of the initial HLT. The number pattern competencies will be documented throughout the teaching experiment. An interview questionnaire will be used to gain insight into learners' opinions about the modelling process. The researcher observation guide will be used to collect valuable information during the teaching experiment.

1.5.3.2 Development of the hypothetical learning trajectory (HLT)

High quality modelling problems will ensure that learners move through all the steps of the modelling process. The design of the problems will be guided according to the RME principles. Learning activities will be selected by the researcher to support the learners' learning. The activities need to be authentic, applicable and appropriate (Busse, 2006; Kaiser & Schwarz, 2006; Maaβ, 2006). The development of a HLT will be described. Each learning activity which constitutes the LT will be analysed and explained. The data from the learners' written work and audio recordings will be used to compare the researcher's predicted learning goals with the actual observed learning. The teaching experiment of the study (macro cycle) consists of micro cycles (teaching experiments) that form a local instructional theory (LIT). A LIT will be developed for number patterns.

1.5.3.3 Collecting data

(a) The focus group will be audio recorded and transcribed
(b) Class discussions will be video recorded and transcribed
(c) Field reports will be collated on a daily basis
(d) Portfolios of the instructional activities and the focus group's written work will be collected
(e) The elements indicating the different activity levels and mathematisation processes will be noted
(f) Learners' written transcripts will be analysed
(g) Horizontal mathematising will be identified and analysed
(h) Vertical mathematising will be identified and analysed
(i) The role of the activities will be explored
(j) Progress during the modelling process will be noted

1.5.3.4 Research criteria

Cobb, Stephan, McClain and Gravemeijer (2001) explained three criteria that an investigation should fulfil for the study to contribute to an improvement in mathematics education. The first criterion requires the documentation of the development of the collaborative classroom during the course of the teaching experiment. During the retrospective analysis, the learning activities will be analysed so that the development of the class as a community of mathematical learners can be appreciated. The second criterion focuses on the documentation of the mathematical reasoning of individual learners as they participate in the classroom community. The retrospective analysis will include a task-based analysis which focuses specifically on the individual's attainment of learning goals. The longitudinal analysis also references specific learners and records their contributions to the study. The third criterion recommends that the outcome of the investigation should result in an improvement of the instructional design. This is an outcome of the DBR methodology, the RME theory and therefore an outcome of the study. Through the development of a LIT the learners' learning is supported and the improvement of the individual and collective group is shown throughout the teaching experiment (macro cycle). This also contributes to the reliability of the study that will be discussed in Chapter 4.

1.5.3.5 Ethical considerations

Permission was given by the ethical committee of Stellenbosch University (Appendix D1) and the KwaZulu-Natal Department of Education to conduct the research. All measures were taken to minimise risks and maximise benefits during the study. Informed consent was given by the principal, head of department and the participants. The researcher used a coding system to protect the privacy of the participants. The participants were coded alphabetically and in no specific order from *A* to *Q*.

1.6 LAYOUT OF THE DISSERTATION

Chapter 1 gives a brief overview of the theoretical framework. The attention is directed at the motivation, aims and objectives of the investigation. The problem statement is described to focus the investigation on the information subject to the importance of the research. The research design is shortly described as direction for the study and lists important elements for investigation within the literature review.

Chapter 2 provides a review of past and current research focuses on a mathematical modelling perspective in mathematics education. This serves as a framework towards the teaching and learning of mathematics. Literature is compared and clarified to serve as a basis of the study. It is important to note that limited research has been done in relation to the actual mathematisation processes: horizontal and vertical mathematising. In this chapter, these processes will be explained, analysed and incorporated within mathematical modelling.

Chapter 3 is the first phase of the design study. A phenomenological analysis will result in the goals for the study. This will assist with the development of adequate mathematical modelling problems to lead the empirical study. A baseline assessment will be designed, explained and analysed to identify the baseline knowledge of the learners. This chapter will deliver a specific situation analysis on which the HLT will be based on.

Chapter 4 describes the second phase of the design study. The specific research problems will focus the study on the aims and objectives of the study. Research design methods and data collection methods will be explained. This chapter will show how the HLT is developed throughout the investigation by continuously predicting and assessing the learning tract of the learners.

Chapter 5 is the retrospective analysis of the study. A complete didactical framework will be presented which involves the reasons for the chosen activities and an analysis of the activities including information on horizontal and vertical mathematising. A goal description will be given of each activity. The function of the mathematical material will be explained in detail.

Chapter 6 draws conclusions from the data and the retrospective analysis. It also includes the limitations of the study and provides recommendations for further areas of study.

CHAPTER 2

MATHEMATICAL MODELLING PERSPECTIVES TOWARD THE TEACHING AND LEARNING OF MATHEMATICS

2.1 INTRODUCTION

To adequately understand the full implication of a mathematical modelling perspective towards the teaching and learning of mathematics, it is beneficial to investigate the current perspectives supporting the movement towards modelling. The different perspectives of mathematical modelling and the classification of each modelling perspective is tabulated by Kaiser and Sriraman (2006, p. 304). It summarises the central aims of each perspective and also relates it to earlier perspectives. This chapter investigates mathematical modelling under the educational modelling perspective, the socio-critical perspective, the cognitive perspective, the contextual perspective and the epistemological perspective. The investigation will help the reader to establish the connectedness of these perspectives. It will also establish a suitable focal point to provide an essential basis for the empirical study. The socio-critical perspective, discussed in Section 2.1, explains the value of mathematical modelling and describes the educational goals of a modelling approach to mathematics education. A contextual perspective towards mathematical modelling in Section 2.2 elaborates on how flexible mathematics can benefit from the understanding of real life applications. The educational perspective, discussed in Section 2.3, describes how mathematical modelling can be used as a vehicle for learning. It revisits the modelling cycle, the modelling process, modelling competencies and focuses on the different types of educational modelling: didactical and conceptual. The cognitive perspective in Section 2.4 looks at the different processes involved in the mathematical modelling process. It also forms a conceptual framework for teaching and learning mathematics. A realistic perspective in Section 2.5 explains how real problems serve as motivation for learning mathematics. A socio-cultural perspective with the emphasis on semiotics in Section 2.6 will discuss how learners' representations form the basis of learning when they are working collaboratively through a modelling problem. Modelling under an epistemological perspective is based on Freudenthal's explanation of modelling as an activity of mathematising. This perspective is investigated in Section 2.7, and forms the basic structure of the study.

2.2 MATHEMATICAL MODELLING: A SOCIO-CRITICAL PERSPECTIVE

2.2.1 The value of mathematical modelling

The mathematical modelling approach to teaching allows for the kind of valuable exploration in mathematics that has been absent to date. Zbiek and Conner (2006) describe the aims of mathematical modelling as being: to provide a learner with the opportunity to design powerful models; to provide them with an alternative and engaging setting in which learners' learn mathematics; to motivate learners by showing them real world applicability and to provide them with the opportunities to integrate mathematics with other areas of the curriculum. Zbiek and Conner (2006) investigate how mathematical modelling can act as a vehicle to construct new concepts. Their study is regarded as an important guide, as this investigation is aimed at providing evidence that a modelling perspective to teaching can form a basis of conceptual understanding, create deeper meaning, form new constructs and therefore improve the learning of mathematics. Mathematics education will be able to provide students with knowledge and abilities of mathematics, and knowledge and abilities concerning other subjects (Blum & Niss, 1991).

2.2.2 Educational goals

Mathematical modelling involves the various processes an individual needs to work through to acquire the mathematical modelling competencies needed for successfully solving future modelling problems. Learners can use and observe the mathematics they learn at school, in their real lives. When mathematics is connected to reality, it provides experiences which are relevant to learners' experiences and relevant to society (Van den Heuvel-Panhuizen & Wijers, 2005). Zbiek and Conner (2006) argue the need to design rich learning experiences to make the learning of mathematics more meaningful. Mathematical modelling contributes towards giving more meaning to the teaching and learning of mathematics (Blum, 1993) thus learners will feel more motivated and positive in the mathematics classroom. This form of modelling develops mathematics which seems useful to the learners as it enables them to have an increased sense of control over their experiences. The purpose of mathematical modelling is to teach learners that the mathematics they learn can be related to their real life experiences (Mukhopadahyay & Greer, 2001). Learners must be able to tackle any kind of problem when dealing with real life issues and making important decisions. Life, after all, is about making choices. An important point is made by Burkhardt (2006) when noting that modelling is a means to "guide understanding and sensible decisions" (p. 182). When modelling with mathematics, one of the aims is to produce critical, politically engaged citizens (Barbosa, 2006). Mathematical modelling is perhaps a rudiment of this very important educational goal.

2.3 MATHEMATICAL MODELLING: A CONTEXTUAL PERSPECTIVE

Learning mathematics through real world situations is appealing to learners. Real world problems motivate learners to study mathematics and provide insight into the real world. Teachers can develop the opportunity for learners to learn through valuable applications to solve important problems and make invaluable decisions in their future endeavours. Doerr and English (2003, p. 110) are of the opinion that a modelling approach to teaching and learning "shifts the focus of the learning activity from finding a solution to a particular problem to creating a system of relationships that is generalisable". In all areas of life, we are confronted with new challenges. An inward looking mathematics curriculum is not the solution and is inadequate for extra-mathematical areas. Extra-mathematical areas refer to those sciences or contexts in which mathematics can be applied. The learning of mathematics must be concerned with flexible and not problem-specific mathematics (Brown, 2008; Kaiser & Schwarz, 2006). The product formed when problem-specific mathematics is taught would be *mathematical theory imitators* instead of the much-needed *mathematical thinkers*.

2.4 MATHEMATICAL MODELLING: AN EDUCATIONAL PERSPECTIVE

2.4.1 Refocusing mathematics education

Mathematics education at school level must be focused on a holistic design of teaching and learning: a design based on different perspectives to meet the standards of teaching learners who have had different mathematical experiences and have different mathematical abilities. When a learner is given the opportunity to grapple with a mathematical problem, a learner uses his previous experiences, mathematical and non-mathematical; to find a solution to the problem (Treffers, 1987). The task of a teacher is to encourage mathematical understanding by thinking mathematically. Teachers often assume that learners are thinking mathematically when they can do a certain computation or master a certain skill. Doing a computation or mastering a skill does not prove that deeper conceptualising of mathematics has taken place. Lesh and English (2005) note the trend of mathematics education moving from "mathematics as computation towards mathematics as conceptualization, description and explanation" (p. 487). This shift could possibly support a deeper understanding of mathematics.

2.4.2 The influence of affect

Various influences determine the input and outcome of the teaching and learning of mathematics. Affect has an effect on cognitive abilities (Hannula, 2006). Affect includes attitudes, beliefs, emotions, values, motivation, feeling, mood, conception, interest, anxiety and view. Various psychological needs will influence the goals of a learner which is influenced by certain beliefs about accomplishing these goals (Hannula, 2006). The most prominent factor is the teacher's beliefs. A teacher's beliefs about the nature of mathematics and mathematical knowledge are spectacles through which we look at teaching and learning (Presmeg, 1998). The teacher's ideas and perspectives of the nature and role of mathematics are inevitably moulded in his teaching and mirrored in the learner, regardless of the true meaning of it.

Mathematical modelling can ensure the development of a mathematics curriculum where a learner can do mathematics because he truly understands the deeper connections and not because of a mere procedure. Learners will have the opportunity to feel motivated and positive in the mathematics classroom. Teaching mathematical modelling can perhaps enable teachers to change their conception of teaching practice (Bassanezi, 1994). The models and modelling approach provides the learners opportunities to develop general competencies and problem solving competencies (Borromeo Ferri & Blum, 2008). The importance of the applicability of school mathematics is a central factor for the planning of a curriculum, and, more importantly, the approach to mathematics education. Mathematics education needs to be suitable for the learners attempting further studies at tertiary level. It also needs to be relevant to the learners seeking other career opportunities.

2.4.3 Didactical modelling

Mathematical modelling as a means to education involves the various processes and competencies a learner goes through and develops when working with models. The development of mathematical modelling competencies is considered to be a goal of mathematics education (Jensen, 2007; Kaiser & Schwarz, 2006). Modelling competencies are the knowledge and skills required to move through the modelling process with a positive attitude. The pedagogical idea behind mathematical modelling competency is to emphasize the holistic aspect of modelling (Blomhøj & Kjeldsen, 2006). The focus needs to be shifted to the mathematical ability of a learner. "What does the learner know?" needs to be modified into: "What can the learner do?" Freudenthal (see Gravemeijer & Terwel, 2000) believed that doing mathematics was more important than working on a ready-made product. He further noted that mathematics as a human activity is an activity of solving problems whether from reality or mathematical matter. The ability to solve mathematical problems is more valuable than having the knowledge but lacking the competencies to apply that knowledge. Knowledge should be intelligent and applicable, not an inactive and isolated one, because knowledge should be developed into ability (Sjuts, 2005). Mathematics in the classroom situation needs to be a multi-dimensional study of a doing-mathematics within a real life context (Burkhardt, 2006).

Mathematical modelling can be considered as an advanced form of competency-based learning as the requirement of competencies will ensure successful modelling experiences (Blomhøj & Kjeldsen, 2006; Maaβ, 2006; Kaiser & Schwarz; 2006). If a learner is introduced to a new concept, his conceptual development is based on similar previous mathematical experiences. Modelling activities motivate the learning process and help learners to establish a basis for the construction of mathematical concepts (Blomhøj & Kjeldsen, 2006). Lesh and Sriraman (2005) argue that the focus of mathematics education is based on the development of conceptual systems rather than the development of different tools or thinking patterns to express and to operate within these conceptual systems. Ernest (Almedia & Ernest, 1996) views the aim of mathematics education as ideally being to:

> …foster critical mathematical literacy and thus empower students to become critical citizens in modern society. This involves having a sound knowledge of significant subset of school mathematics and the confident possession of the process skills of applying mathematical knowledge independently to solve and

> pose problems and evaluating the situations critically. However, it also necessitates the ability to interpret and critically evaluate the mathematics embedded in social and political claims and systems, from advertisements to government pronouncements (Introduction, para. 11).

Galbraith (Haines & Crouch, 2007) describes modelling as the means of collecting and constructing mathematical knowledge. Competencies can be regarded as critical aims for modelling to be successful. These include: understanding the task, simplifying and structuring, mathematising, working mathematically, interpreting, validating, presenting and reflecting. Biccard's (2010) investigation into the above mentioned competencies provided a good insight on how these competencies develop as learners regularly work through mathematical modelling problems.

Mathematical modelling can be defined as solving problems based on real life .We can therefore characterise the mathematics in our daily living. The modelling process requires learners to form a mathematical model and to use mathematics to find a solution. Mathematical modelling provides learners with knowledge and skills to deal with life outside the classroom (Haines & Crouch, 2007). A higher level of thinking, as well as combinational thinking, can be developed. Transmission methods fail to deliver this very important facet of mathematics education: to be able to engage in higher order mathematical thinking.

2.4.4 Conceptual modelling

The mathematical modelling concept has the advantage not only of developing and constructing mathematical knowledge, but also of developing mathematical skills and mathematical thinking (Sjuts, 2005). Modelling at a level of concept formation involves the actual learning and understanding of mathematics by acquiring knowledge, skills, attitudes and values. The different levels of constructing mathematical understanding that lead to mathematical knowledge can be related directly to mathematical modelling and the outcomes thereof. Modelling has been assigned as a possible answer to address conceptual difficulties (Lesh & Lehrer, 2003). Lesh and Harel (2003) explain that the conceptual development of a learner engaged in a modelling session of 60-90 minutes is similar to that of learners' conceptual development of several years. Blomhøj and Kjeldsen (2006) emphasizes that mathematical modelling is an educational goal in its own right and can be used as a tool for motivating and supporting the learning of mathematics. The skills, competencies and knowledge we want learners to acquire (the objectives for learning) is directly proportional to the reason for applying a specific teaching approach. Modelling activities motivate the learning process and help learners to establish cognitive roots for the construction of mathematical concepts (Blomhøj and Kjeldsen, 2006). Lesh and Sriraman (2005) argue that the focus of mathematics education is based on the development of conceptual systems rather than the development of different tools or thinking patterns to express and to operate within these conceptual systems. Mathematical modelling is a tool to develop knowledge and skills and enhance learners' confidence and thinking abilities and build a positive attitude towards mathematics (Gellert, Jablonka & Keitel, 2001). Learners must be able to apply the various steps in the modelling cycle to various open-ended problems (Haines & Crouch, 2007). This approach can be applied to other problems to solve them successfully and to develop a critical, reflective individual. Mathematical literacy can be fostered through the mathematical modelling process, where a learner needs to build a mathematical model. The learner needs to be able to verbalise a real problem and translate it into a mathematical problem by using mathematical language. A mathematical solution is found and that needs to be translated to a real answer. Another aim alluded to above is the confident solving of problems regarding real issues. Mathematical modelling links mathematics to real life. The learner must also be able to evaluate critically the problems and solutions in terms of the real situation.

2.4.5 The process of mathematical modelling

The modelling cycle from a cognitive perspective (Figure 2.1) is used to further explain the modelling process from a real world problem to an interpreted solution. It also gives a clear layout of all the necessary competencies and sub-competencies during the modelling cycle. The *real situation* represents the situation or the given problem. The learner shows some understanding when moving from the real problem to a *mental representation*. Any representation shows a degree of understanding. These mental representations differ as mathematical thinking, experiences and extra-mathematical knowledge vary from individual to individual. From the mental representation a *real model* is formed by identification and idealisation.

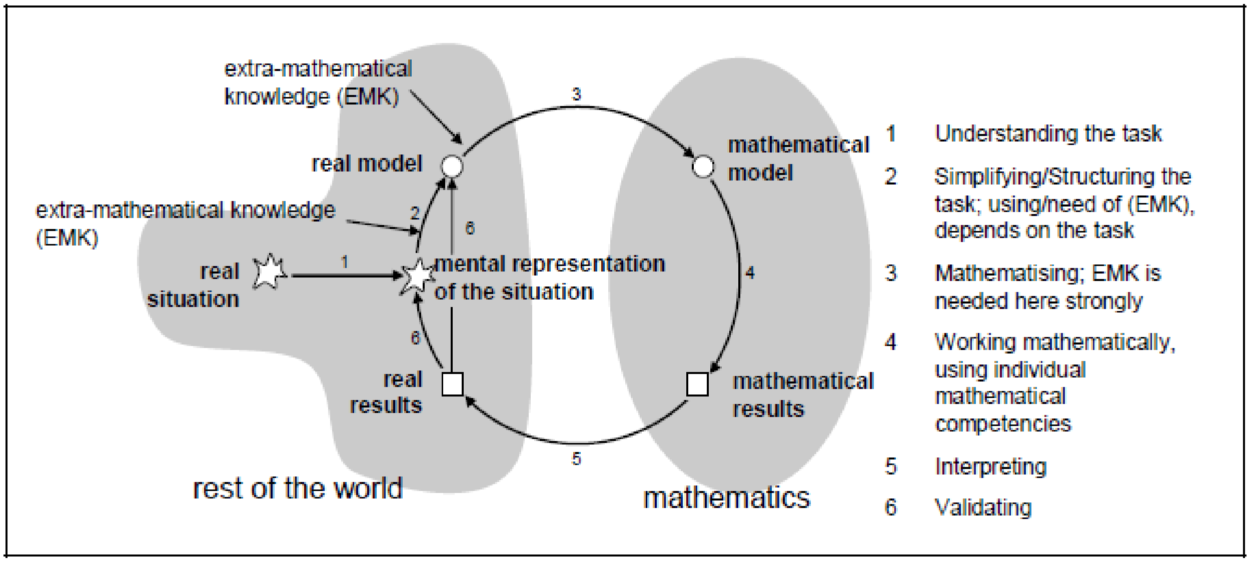


Figure 2.1: Blum and Leiβ's modelling cycle from a cognitive perspective (Borromeo Ferri, 2006, p. 92; Haines & Crouch, 2007, p. 2)

Setting up the model includes the following competencies: identifying the relevant mathematics within the realistically-set problem; representing the problem in a different way; organising the problem according to mathematical concepts and assumptions; understanding the relationships between the language of the problem and the symbolic and formal knowledge needed to understand it mathematically; finding regularities, relations and patterns; recognising aspects that are isomorphic with known problems; and translating the problem into mathematical model. When moving from the real model to a *mathematical model*, mathematising takes place. Mathematisation is the process where something obviously not mathematical is converted into something that is mathematical (Wheeler, 1982; 2001). When a learner is engaged in the process of mathematisation, he is required continuously to make and build on, assumptions, conditions, limitations, and constraints (Zbiek &Conner, 2006). Extra-mathematical knowledge is used to build this mathematical model. Verbal statements are now on a mathematical level and the transition into mathematics is completed at this stage. Working within the mathematical world to obtain *mathematical results* includes competencies such as: using and switching between different operations and representations, refining and adjusting mathematical models, combining and integrating models, argumentation, and generalisation. The learners write down their results based on the model. When learners interpret their results, they are transitioning mathematical results to *real results*. These results are discussed and validated. *Validation* needs to be made in relation to the real results of the mental representation. Interpreting, validating and reflecting competencies include interpreting mathematical solutions in a real context, understanding the extent and limits of mathematical concepts, reflecting on mathematical arguments, explaining and justifying results, and critiquing the model and its limits. When a learner is reporting his modelling process, he is involved in communicating the processes verbally and through written work, on matters dealing with mathematical content, and understanding other learners and their explanations.

2.4.6 Mathematical modelling competency

Mathematical modelling competency includes those skills and knowledge considered necessary when working through all the steps in the modelling process (Blomhøj & Kjeldsen, 2006). Niss, Blum, and Galbraith, (2007) give a much more comprehensible definition as:

> …the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation, as well as the ability to analyze or compare given models by investigating the assumptions being made, checking properties and scope of a given mode (p. 12).

A learner must develop the competency to understand the real problem and set up a real model based on the mental representation obtained as a result of his understanding (Maaβ, 2006). This means that the learner must acquire the necessary skills to be able to simplify the problem, construct relations or patterns, and look for important, available and relevant information. According to Haines and Crouch (2007), learners find it difficult to move from the real model to the mathematical model. Biccard (2010) also commented that the learners in her modelling groups found it challenging to mathematise the real problems. This may be due to a learner's weak knowledge base and lack of abstract thinking (Haines & Crouch, 2007). These aspects can be improved by working with mathematical models on a regular basis. Tanner and Jones (1995; Blomhøj & Kjeldsen, 2006) warn that knowledge alone is not enough for successful modelling: the learner should also monitor his own process and progress throughout the modelling process.

An important metacognitive-competency is when the learner is able to plan, monitor and validate his own actions (Maaβ, 2006). The learner must now set up a mathematical model from the information gathered and the real model they constructed (Maaβ, 2006). Setting up the mathematical model requires the representation of the real model in mathematical form. Once the mathematical model is constructed, the learner must use his mathematical knowledge to solve mathematical problems adequately within this model (Maaβ, 2006). Mathematical results must be interpreted in accordance with the real situation (Maaβ, 2006). Learners interpret mathematical results in extra-mathematical contexts. Once the solution is obtained, a learner must validate his results to consider the appropriateness in relation to the original problem (Maaβ, 2006). Validating a solution entails critically checking and reflecting on found solutions, reviewing and going through the modelling process again if the solution does not fit. As discussed earlier, the reflection through the entire learning process is a vital component of successful modelling as a learner is constantly reviewing his own work and thinking, and is fully in control of his own learning and the modelling process. Tanner and Jones (Maaβ, 2006) indicate that knowledge alone is not enough when dealing with the modelling process: a learner must be able to use this knowledge and monitor his process and progress. A learner can now judge the effectiveness, adequacy and value of his own model as well as the process in its entirety.

2.4.7 A closer look at mathematical modelling competencies and sub-competencies

As many attributing factors influence mathematical performance, we need to focus on those factors which influence modelling competencies and hence the overall performance, progress and development in the mathematical-modelling process. According to the COM²-project: progress regarding the attainment of mathematical modelling competencies can be described according to three aspects, which can thus also be assessed regarding the three competencies (see Blomhøj & Kjeldsen, 2006, p. 167; Haines & Crouch, 2007, pp. 5-6):

i. Technical level: this is measured according to the level of mathematics and the flexibility of the mathematics the learners are using.
ii. Radius of action: the radius of action is measured according to the domain of the situations in which learners can perform modelling activities.
iii. Degree of coverage: this is measured according to which part of the modelling process the learners are working with as well as the level of the reflections by the learner.

As we acquire different metacognitive strategies to develop metacognition, thus further expanding our cognitive abilities, we need sub-competencies to utilise competencies in order to deal with the modelling process efficiently (Maaβ, 2006). There is a definite connection between modelling competencies and mathematical knowledge. When carrying out single steps of the modelling process, sub-competencies are the skills and processes that need to be carried out so that competencies can be attained. Sub-competencies are those competencies needed to achieve various modelling competencies when modelling a problem. The following table differentiates between competencies and sub-competencies by listing the activities of sub-competencies activities and the corresponding competencies of mathematical modelling.

Competencies	**Sub-competencies**
Understanding the real problem	Make assumptions, recognise variables, construct relations between variables, distinguish between relevant and irrelevant information
Setting up a mathematical model	Mathematising, simplifying, choosing appropriate mathematical notations, representing the situation
Solve the mathematical problem	Heuristics, mathematical knowledge
Interpreting the results	Interpreting mathematical results in extra-mathematical contexts, generalising solutions, using appropriate mathematical language, communicating about the solution
Validating the solution	Critically checking and reflecting on solutions, reviewing parts of the process, reflecting on other ways to solve the problem, generally questioning the model

Table 2.1: Differentiating between competencies and sub-competencies (Maaß, 2006, pp.116-117)

To make use of the modelling process successfully sub-competencies need to be developed. Opinions as to which part of the modelling process is more important vary. Biccard (2010) noted that some competencies in the cycle were more significant than others. Without these competencies the pure mathematics gets lost in the process. It is known that learners are able to reconstruct their knowledge by constructing concepts to rectify misconceptions (Cobb, 1999; Borromeo Ferri, 2006). Metacognitive modelling competencies could be reconstructed for many learners (Maaß, 2006). Weaknesses within metacognitive modelling competencies matched general misconceptions on the modelling process based on weaknesses in general problem-solving skills (Lucangeli, Tressoldi & Cendron, 1998; Maaβ, 2006). "Knowledge about the modelling process influences the acquirement of modelling competencies positively" (Galbraith & Clatworthy, 1990, p. 158). Learners need to know what is expected of them. Learners' attention must be brought to the important aspects when setting up a real model as it is the starting point of the modelling process. The validation process must be discussed in detail. There is a relationship between the quality of meta-knowledge and the competencies of modelling a problem (Yimer & Ellerton, 2006).

When working with the modelling process it is important to have a sense of direction (Maaß, 2006). It is a common problem for learners to 'get stuck'. They lose motivation and give up their efforts. When a learner is familiar with a specific process and a framework is set in place, it seems less difficult to tackle a problem, a learner feels more motivated and comfortable in his doing, and shows ownership regarding the problem, especially when facing an unknown task (Jonassen & Land, 2000). Trelibs (1979) recognises a sense of direction as important for modelling. Learners need to be sure of the direction in which they are going. They need to focus and work towards an end result. Competencies in arguing seem to be an important variable concerning the modelling process (Duffy & Cunningham, 1996; Hein, 1991; Matthews, 2003). It is important for a teacher to create a suitable social environment for the construction of knowledge within the learners to occur effectively (Hein, 1991; Janvier, 1996). Communication is an important factor of learning as it goes hand-in-hand with explaining, reasoning and justifying decisions and ideas especially in the sense-making procedure (Lesh & Yoon, 2007). When a learner can communicate his thought processes, findings and conclusion in a formal mathematical language, we are given an indication of what the learner knows, if the learner has attained certain competencies, metacognitive abilities and if he has mastered a skill or even satisfactory constructed knowledge (Borromeo Ferri, 2006). Communication is a tool whereby a learner heightens his awareness of his own learning. Emphasis must be placed on the modelling process when arguing, rather than arguing in relation to learners' own experiences (Maaß, 2006). Learners must focus on writing down all arguments and reasons for decisions they make and routes they follow. Argumentative mathematical skills form part of communicating mathematics appropriately. Brown and Redmond (2007) place emphasis on collective argumentation as this bases argumentation in a social context which is applied to establish deeper meanings.

2.5 MATHEMATICAL MODELLING: A COGNITIVE PERSEPCTIVE

2.5.1 Cognitive processes during mathematical modelling

The modelling approach enhances sense making (Mousoulides, 2009). Modelling problems promote understanding thus growth in understanding, as there is no memorised answer and no 'correct' solution. The ability to solve algorithmic procedures does not indicate depth of understanding. Students capable of solving these types of problems often cannot connect these manipulations to the real world (Schoenfeld, 1988). Problem solving based on a mathematical modelling perspective enables the learner to develop a deeper understanding of mathematics as it allows the learner to construct a stable understanding about situations (Izsák, 2004). The knowledge and conceptual tools developed via the modelling process is situated cognition (Lamberts & Goldstone, 2005; Lesh and English, 2005). Situated cognition refers to the knowledge emerging from authentic, context-bound activities as well as from social constructions (Olivier, 1999). Anderson, Reder and Simon (1997) argue that learning is not bound to the specific situation of its application; knowledge can transfer between different tasks. The idea is that instruction does not have to take place in a complex social setting only; there is also value to individual learning. A negative aspect of the transmission approach is the lack of preparation for unseen problems. The transmission approach results in a learner reconstructing existing objective knowledge (Murray, Olivier & Human, 1998). Learning needs to be a personal experience where a learner creates his own subjective knowledge rather than reconstructing objective knowledge. The active reasoning and organising of information promote meaningful learning (Chamberlin, n.d.).

Encoding and decoding define the cognitive elements of the modelling process (Occelli, 2001). Encoding involves the formulation of abstract pictures from the observable reality, thus building models representing these pictures, while decoding involves referring back to the observed reality (Occelli, 2001). Occelli (2001) differentiates between two types of "fundamental loops"

(para. 10) in the modelling activity: the internal loop describes the abstraction during the modelling process and the external loop represents the general modelling context of the modelling problem. Cognitive processes integrate the abstraction and context to initiate the modelling activity. Lesh and Sriraman (2005) argue that concept development transforms into understanding and forms mathematical modelling perspectives. The different levels of constructing mathematical understanding, which lead to mathematical knowledge, can be compared directly to mathematical modelling and the outcomes thereof.
Borromeo Ferri (2006) explains that if a teacher looks at a learner's modelling process from a cognitive viewpoint, he can only refer to the verbal descriptions and representations, and external illustrations and representations to identify if the learner is successful at beginning and understanding the first steps of the modelling cycle and working through the process effectively. The mathematical-modelling concept has the advantage, not only of developing and constructing mathematical knowledge but also of developing mathematical skills and mathematical thinking (Sjuts, 2005).

Mathematical understanding and continuous focus and activation of the learner's metacognitive processes are requirements for mathematical modelling. Metacognition refers to: "The active monitoring and consequent regulation and orchestration of processes in relation to the cognitive aspects on which they bear, usually in the sense of some concrete goal or objective" (Stillman & Galbraith, 1998, p. 162). Metacognitive knowledge consists of knowledge and beliefs about factors that influence the outcomes of cognitive knowledge and understanding. Metacognition is divided into the following processes: monitoring, control, orienting and reflecting. Metacognition entails the knowledge of monitoring a process and involves verifying and acknowledgement of one's own diagnosis. Metacognitive teaching in a cooperative setting has shown the potential to enhance problem-solving skills and mathematical-modelling construction (Mevarech, Zion & Michalsky, 2005). If teachers are knowledgeable concerning the mental processes and the learners' thinking habits, and misconceptions they will adjust and adapt their type of instruction to enhance thinking, understanding, knowledge construction, the development of models, and meaningful learning. Knowledge is a development and procession of thinking, learning and making sense of complex situations (Burkhardt, 2006; Mickelson, 2006). Thus the knowledge of metacognitive strategies enables the management of thinking and learning.

Jagals' (2013) study on the role of reflection and confidence when doing mathematics and the various components of metacognition were explored. Two of Jagals' concluding remarks emphasises the importance of the activity of reflection: "The act of reflection stimulates the level of confidence relating to planning and monitoring of tasks" (p. 179), and "the act of reflecting possibly manipulate and vary the knowledge and feelings associated with person, strategy and task characteristics during problem solving" (p. 180). Sjuts (2005) identifies metacognitive thinkers as self-regulated learners, who plan, organise, self-instruct, self-monitor, and self-evaluate during the process of learning. Research has shown that the higher the metacognitive ability of a learner, the higher the learner's thinking abilities and academic levels (Garofolo & Lester, 1985; Mevarech et al., 2005). Self-regulated learners are in control of their affective processes (Sjuts, 2005). Affect determines the rate of the development of learning and especially learners' mathematical development (Hannula, 2006; Janvier, 1996). Affect controls cognitive processes (Owens, 2008). The importance of conscious self-regulation and extreme control over a learner's own learning process is essential in all areas of the problem-solving process. Self-regulated learning describes the ability of a learner to set his own goals, to use appropriate methods and techniques regarding the content and the goal, and to review, as well as judge, his own processes (Maaß, 2006). It forces a learner to be in charge of his metacognitive processes. When learners take responsibility for their own learning experiences, metacognitive abilities become an important factor of learning. It is important for a learner to be aware of his own thinking and understanding as they form the foundation for the construction of knowledge.

2.5.2 Conceptual development

The cognitive structure in the human mind is associated with the concept image which is all those mental representations and processes regarding the concept (Tall, 1988). Tall (1988) also states when a learner meets an old concept in a new context, it is the concept image with all its assumptions and not the concept definition that reacts to the new task. The roles of emotions, consciousness and physical environments have effects on mathematical thinking, therefore influencing the forming and development of concepts (Thagard, 2008). Negative emotions, consciousness and physical environments have a negative effect and positive emotions, consciousness and physical environments have a positive effect on learning. This implies that the learning milieu determines the cognisance of a learner. Mayer and Hagarty (1996, p. 34) summarize the cognitive processes during the problem-solving process:

i. Translating: a learner constructs a mental representation.
ii. Integrating: the learner makes assumptions and recognitions during the construction of mental representations.
iii. Planning: devising a plan for solving the problem
iv. Executing: carrying out the plan and computations

Polya's (1973) four steps to the problem-solving process involve: understanding the problem, developing a plan based on connections made, carrying out the plan, and evaluating that solution by looking back. The process above is very similar to that of Polya's, yet Polya's shows more reflective characteristics.

Building blocks of mental representations are perceptions, past actions and the result of reflecting on, and abstracting from, already existing representations. These will often involve identifying and retrieving a sample or prototype (Lutz & Huitt, 2003). Tall (1988) refers to the sample type or prototype as a concept image. Abstract objects emerge at intersections in development of mathematical knowledge when some new process is introduced and applied by another already existing and known process. Abstraction is the activity when a learner becomes aware of similarities amongst objects, symbols and concepts. The stages in abstraction are listed as follows: internalisation, condensation, reification, generalisation, synthesis and abstraction (Mitchelmore, 2002). Abstract objects mediate between the product of lower process and higher level manipulation (Sfard & Linchevski, 1994). Structural development is considered to be a difficult operation as it is more abstract and advanced stage of conceptual development.

Mathematical awareness and assumptions influence the extent of seeing mathematical structure in situations during model development (Zbiek & Conner, 2006). A conceptual theory is closely associated with model building. The conceptual theory forms as a learner revises, refines and extends his ways of thinking. During this revising, refining and extending conceptual tools such as constructing, describing and explaining are developed. As models are continually revised and revisited, new ways of thinking are developed. This local conceptual development is similar to Piaget and Van Hiele's theories of conceptual development (see Lesh & Harel, 2003). Their theories emphasise mathematical thinking being the interpretation and observation of situations.

The product of an individual's conceptual operations and development is mathematical knowledge (Voigt, 1996).

The development of conceptual systems involves more than accumulating relations, operations and principles. They occur along a variety of dimensions. These dimensions can be specific to general, concrete to abstract, simple to complex, intuitive to formal and situated to de-contextualised (Lesh & Lehrer, 2003). When a learner reflects on and abstracts from past experiences and perceptions he builds on existing representations to form new conceptualisations. These cognitive structures are connected by conceptual schema and mediate interactions between conceptual understanding of situations and symbolic representations (Izsák, 2004). These processes are reversible and are organised by logic. The stages of logical modelling are composed by conceptualisation, verbalisation, and formalisation (French & Finlay, 2000). Learners' representations of situations are based on their understanding of physical patterns, existing conceptual representations and symbolic patterns. Learners use these patterns to recognise situations which can be modelled and connect these to conceptual representations through symbolic representations. Models are therefore focused on describing patterns and other mathematical representations so that learners can build an understanding of the system that is modelled. "Mathematical models are conceptual systems that are: (a) expressed for some specific purpose and (b) using some representational media" (Lesh & Lehrer, 2003, pp. 111-112).

Lesh and Lehrer's (2003) developed a modelling cycle to show the process a learner goes through when they develop models. Figure 2.2 shows how the model is represented by some representational media and it is described or explained. Both the describing and representing is based on the purpose of the model. This purpose of the model is the "end-in-view" (p. 112). Lesh and Lehrer (2003) state that the modelling cycle can be revisited and revised more than once, and through this revisiting and revising they predict the learner to show different ways of thinking regarding the symbols the learner chose to use, the process he followed, and the way he got to the solution. Learners make sense of new information by relating it to previous understanding and experience. Olivier (1992) explains that learning depends on the schema formed in a previous learning experience. Learners make sense of new information by relating it to earlier learning. It is therefore easier for a learner to add to his existing schema, than to change it.

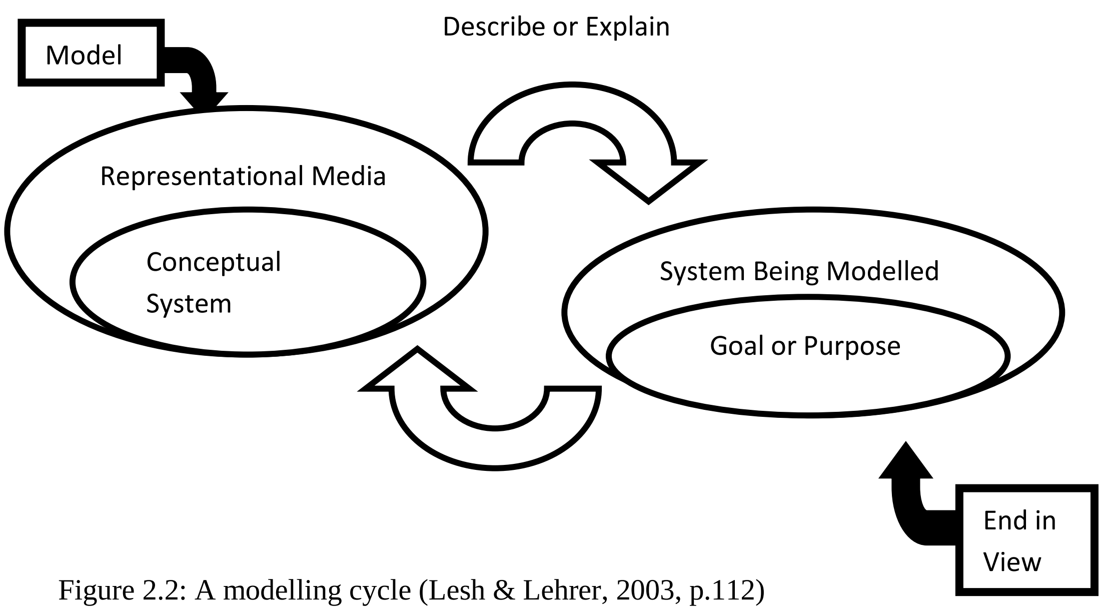


Figure 2.2: A modelling cycle (Lesh & Lehrer, 2003, p.112)

Olivier (1992, p. 200) further elaborates on “overgeneralization”; when a learner applies the same operation to a new situation, because he was successful in previous instance. A state of equilibrium occurs when an individual experiences mismatches between internal conceptual understanding and the understanding of the modelling situation (Izsák, 2004). As the informal understanding of models evolves through multiple conceptual systems, levels of understanding become mature and mathematical thinking can now be organised around abstractions rather than experiences (Lesh, Doerr, Carmona & Hjalmarson, 2003).

2.5.3 Conceptual systems

Educational requirements needed for the development of mathematics as a pure science need to be more than just a functional numeracy (Lesh & English. 2005). Sfard and Linchevski (1994) note the difficulties a learner experiences when he is introduced to a new mathematical concept. This is known as the ability to reify. The possibility of performing higher-level processes on processes by representing compact expressions, spurs structural thinking. Through reification learning becomes more meaningful and retrieval processes are faster. Sfard (1991) defines reification as “an ontological shift - a sudden ability to see something familiar in a totally new light” (p. 19). If a learner does not understand, it means that he is not reifying. Reification is when a learner regards an abstract object to be a concrete one. This process of reification evolves through reflective abstraction to develop new and modified constructs (Simon, 1995). Mathematics can be viewed operationally and structurally. The development of mathematical ideas into understanding naturally occurs from the operational stage to the structural stage. The operational way of thinking is regarded as the verb, the process or work that has to be done, whereas structural approach is regarded as working concretely with the objects. In context of the mathematical modelling approach to teaching and learning, the operational stage can be compared with horizontal mathematisation and the structural stage can be compared with vertical mathematisation. A learner uses reflecting activities to move from the operational (ideas) to structural (understanding) stage, whereas in the transition between horizontal (organising information according to a learner's understanding) and vertical mathematising (working with symbols), the term objectification is used. Objectification is explained in Section 2.6.

The transition between the operational to structural conceptions leads to insight and an increase in confidence during the problem solving process (Sfard, 1991). When a learner cannot bridge the gap between the operational and the structural stage, the understanding of operating within the structural stage is lost. It becomes instrumental not the desired relational, and the learner cannot donate meaning to the structural conceptions. The learners are able to do the complex structural operations, yet the operations are meaningless. This is known as "pseudostructural conceptions" (Sfard & Linchevski, 1994, p. 220). In our current curriculum, learners are not given the opportunity to move form operational conceptions to structural conceptions. The operational-structural duality is missing. The reification concept is a major influencing factor in this study. It will be a lens through which the researcher will look at the learners' constructions, by analysing their operational and structural conceptions.

Kaiser and Sriraman (2006), Carlsen (2010) and French and Finlay (2000) also note that conceptual systems are human constructs which are developed in a social setting through establishing shared meanings. The modelling approach shifts the focus of mathematical learning to a system of relationships which are developed to be reusable and sharable in a community of collaboration (Doerr & English, 2003).

The constructivist approach is a philosophy of learning which deals with the concept that knowledge is constructed internally. The learning process is a social happening where communication is vital (Cobb & Yackel, 1998, Ward, 2005). Vygotsky (1978) believed that community contributes to making meaning. Students make progress though Vygotky's zone of proximal development (ZPD). He defined the learner's zone of proximal development as the distance between the actual developmental level and the potential level of development (Allal & Ducrey, 2000; Vygotsky, 1978). Vygotsky sees the ZPD as the sensitive area where guidance and encouragement are given to a learner to allow the learner to develop knowledge. "ZPD is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). Vygotsky's theory also suggests collaborative learning in order to move through the development and understanding of mathematical structures. Voigt (1994) shows how learners do not necessarily share meanings but negotiate and collaborate to form a consensus on an emergent meaning. When working collaboratively, learners work in groups: to seek shared understanding, meaning and solutions. This explains the advantages of learners working collaboratively; a learner who is able to help another can do so while they learn something too. This gives rise to the group composition in the current study. They need to be heterogeneous for optimal learning to occur.

The collaborative approach classifies learning as an active and constructive process dependent on rich learning environments, diverse environments, and a social experience (Smith & MacGregor, 1992). Ideas and actions are integrated in a reflective inquiry situation (Hiebert et al., 1997). This leads to organising which results in conceptual development (Human, 2009). On evaluating the various elements of the collaborative teaching approach to the mathematical modelling perspective, the complementary nature of the two when dealing with the mathematical modelling process is clearly noted. The traditional teaching approach seems to be inadequate when dealing with a dynamic process such as problem solving through modelling (Blum, Galbraith, Henn & Niss, 2007; Lingefjärd, 2006).

2.5.4 Advancing mathematical thinking

Emergent properties from higher level systems evolve from systems of interaction and primitive, concrete, enactive and intuitive levels (Lesh & Lehrer, 2003). This implies that a learner will build on his informal mathematical thinking to develop a formal mathematical thinking: the level of mathematics will be based on how he reflects on his experiences. This leads to remarkable achievements by learners judged to be too young and lacking in ability for such sophisticated and powerful forms of mathematical thinking (Blum & Niss, 1991). Sjuts (2005) notes that higher cognitive achievements are possible at a much younger age. Mathematics education needs to focus on higher level skills and deeper flexibility (Burkhardt, 2006). Modelling activities motivate the learning process and help learners to establish cognitive roots for the construction of mathematical concepts (Blomhøj & Kjeldsen, 2006).

Thinking mathematically involves making sense of, computing, mathematising, quantifying, and coordinating through the use of other mathematical systems (Lesh & Sriraman, 2005). The way in which learners invent, understand and work with mathematical facts and connections using internal imaginations and external representations is linked to their mathematical thinking styles (Borromeo Ferri & Blum, 2008). The different types of thinking styles can be categorised as visual, analytical and integrated. Mathematical thinking is overcoming and not avoiding mathematical complex mathematical situations. Lesh and Harel (2003, p. 187) suggest mature levels of thinking develop when:

i. learners are challenged to develop models and conceptual tools to be sharable and reusable,
ii. learners are introduced to powerful representational systems for expressing constructs, and
iii. learners are encouraged to go beyond thinking with and about constructs.

When mature levels of mathematical thinking develop, significant expectations for generalisations exist. Three types of generalisations occur progressively: expansive generalisation entails the applicability of existing schema expanded without reconstruction; reconstructive generalisation involves existing schema which are reconstructed to a wider range of applicability; and disjunctive generalisation occurs when new schema is constructed which is applicable to new contexts (Zazkis & Applebaum, 2007). Within the development of powerful constructs and conceptual systems, the likelihood of a small number of big ideas leads to thinking beyond what is expected of the learner (Lesh & Lehrer, 2003). The spiral concept of learning is where concepts are introduced in an intuitive, early stage and later revisited and connected to other knowledge. In this phase a higher level of abstraction is reached. Visualising makes abstract ideas more tangible. Reflective abstraction is introduced by Piaget to describe the individual's development of abstractions (see Dubinski, 1999). When learners organise their experiences and then compare it with current abstractions to construct new combinations reflective abstraction occurs. Guided reflection contributes to advancing mathematical thinking (Zazkis & Applebaum, 2007). This involves rigorous deductive reasoning. The requirements of advancing mathematical thinking are the effective use of outside tools and reconstructive generalisation (Zazkis & Applebaum, 2007). The knowledge to exercise control over various thinking processes enables a learner to plan and adjust strategies, ideas and even his thinking. It is essential to reflect on strategies and attempts to advance mathematical thinking. Knowledge is a development and procession of thinking, learning and making sense of complex situations (Burkhardt, 2006; Mickelson, 2006).

2.6 MATHEMATICAL MODELLING: A SOCIO-CULTURAL PERSPECTIVE

A socio-cultural perspective focuses on the external and internal representations that are influenced by culture when learners do mathematics. The internal representations include mental and computational representations (Hesselbart, 2007, p. 13). Goldin (2002, p. 210) includes natural language, personal symbols, visual and spatial images, problem solving heuristics and affect as internal representations of a learner that solves mathematical problems. Semiotic representations are external representations used to access mathematical objects (Duval, 1999, 2006). External representations can have many purposes to symbolise an internal representation (see Goldin, 2002, p. 208). Radford (2008a, 2008b) adopts a cultural-semiotic approach to mathematics education. In a cultural-semiotic approach, culture is the tool whereby mathematical objects acquire meaning. Radford (2008a) notes that artifacts support thinking; "neither merely aids to thinking or simple amplifiers, but rather constitutive and consubstantial parts of thinking" (p. 218). Semiotic representations originate at the level of task-setting where a learner builds a real model from the real problem (see Chapter 1, Section 1.2) using various representations. Radford's theory of knowledge objectification is based on the way a learner makes sense of his world and the social aspect of learning (Radford, 2008b). Through social interaction (discussion and discourse) objectification may take place. In mathematical modelling, when a learner moves from semiotic representations (representing information as objects/symbols using cultural knowledge) to working with symbols (making these symbols subjective in terms of the mathematical problem using reflective strategies) objectification takes place.

Each semiotic system needs specific registers of representations for different types of mathematical thinking to take place (Duval, 1999). Registers of representation can include natural language, symbolic language, graphs and geometrical figures. There are three main cognitive activities of semiotics: *formation, treatment and conversion* (Hesselbart, 2007, p.11). Formation is the term used when representations in a specific register is constructed. When these representations are transformed into another transformation within that register, it has undergone treatment. New mathematical knowledge is constructed through the treatment activity (Hesselbart, 2007, p.13). When a representation in a register is transformed into a representation belonging to a different register, conversion takes place. Hesselbart (2007) notes that one of the characteristics of mathematical activity is that learners often work with representations that belong to more than one semiotic register in a single problem. It is also possible to move from representations from one register to representations of another register. Although learners find this shifting between representation of different registers challenging, it is the interrelated feature of mathematics that offer such valuable connections for advanced mathematical thinking and understanding. Goldin (2002) notes that "effective teachers continuously make inferences about students' internal representations, their mathematical conceptions and misconceptions, based on their interaction with or production of external representations" (p. 211).

2.7 MATHEMATICAL MODELLING: A REALISTIC PERSPECTIVE

The National Council of Teachers of Mathematics (NCTM) argues that problem solving needs to develop into a more prominent focus in the school curriculum (NTCM, 1989, p. 6). Learning becomes a meaningful experience when learners grapple with unfamiliar and non-limiting problems. Problem solving is considered to be a form of conceptual understanding as these problems promote understanding (Chamberlin, n.d.). In a didactical setting learners work collaboratively and learners take responsibility for their own learning (Blomhøj & Kjeldsen, 2006). The learners are viewed as problem solvers and the learners need to seek alternative and more productive ways to think about a situation. Cognitive abilities and metacognitive abilities are used during the enquiry stages and through the steps in the problem solving process (Panitz, 1997). The importance of mathematical modelling in a mathematics curriculum is to develop problem-solving skills and attitudes (Danusso, Testa & Vicentini, 2009). There is significance in a model and modelling experience when a learner constructs his own personal knowledge. The models and modelling perspective acknowledges this fact (Carlsen, 2010). Conceptual tools in problem solving include constructing, describing and explaining. These are consistent with the conceptual tools used in mathematical modelling (Lesh & Harel, 2003).

The Dutch approach to mathematics education acknowledges the value of mathematics as a human activity and provides learners the opportunity to experience this directly. Realistic Mathematics Education (RME) is portrayed through six principles (Van den Heuvel-Panhuizen, 2000, pp. 5-9).

i. *Activity* principle: A learner is actively involved in their own learning by developing mathematical constructs and understanding.
ii. *Reality* principle: Learners grapple with problems that are context-rich and real to them. They can link mathematics to reality and understand the usefulness of mathematics in everyday life.
iii. *Level* principle: Learners use reflecting skills to pass through levels of increasing understanding. Tasks are constructed so that they have progressive and coherent elements.
iv. *Intertwinement* principle: Mathematical content is interrelated. This principle allows learners to find connections between various concepts within mathematics.
v. *Interaction* principle: RME supports the constructivist approach to mathematics education. Knowledge as a shared activity can lead to higher levels of understanding through reflection when learners share their ideas and strategies.
vi. *Guidance* principle: Teachers facilitate the learning process by guiding them; they provide the learners with the opportunities to reinvent mathematics in order to construct mathematical knowledge.

To summarise the RME approach: Learners engage in a series of realistic problems which provide them with the opportunity to create mathematical constructs by reinventing them (Kwon, 2005; Van den Heuvel-Panhuizen, 2000; Van den Heuvel-Panhuizen & Wijers, 2005). The construction of models in RME reflects the important and relevant aspects of the mathematical concepts of a problem situation (Grigoraș & Hoede, 2008). When a problem-solver revisits, revises and tests cycles of model construction, new ways of thinking are developed and the possibility exists that new constructs are ready to be developed (Lesh & Harel, 2003). These concepts are much more refined than constructs developed around traditional methods. Model-eliciting activities (MEAs) are simulations of contextual situations which involve several modelling cycles (Lesh & Lehrer, 2003). Learners need to construct symbolic representations of meaningful situations. Average-ability students are capable of developing powerful mathematical models and constructs. Conceptual systems that support these models often are more sophisticated that anybody has tried to teach the relevant students at school (Lesh & English, 2005). In a problem-solving session local conceptual development occurs and continues to develop and change when a learner moves through similar activities (Lesh & Harel, 2003).

2.8 MATHEMATICAL MODELLING: AN EPISTEMOLOGICAL PERSPECTIVE

2.8.1 Model building

From the above explanation it is evident that the purpose of models is to bridge the gap between different levels of understanding. It also gives a learner access to constructing formal knowledge. Van den Heuvel-Panhuizen (2003) shows that models can accommodate level-raising: from informal level of mathematics to a formal level of mathematics. They also allow a change in perspective when broader applicability can be achieved through the progression towards the stage of reflective generalisation. These new perspectives and possibilities allow for higher levels of understanding during a problem-solving situation. Models also provide flexibility so that they are useful in higher level activities (Van den Heuvel-Panhuizen, 2000). The driving force of RME is mathematisation. Üzel and Mert Uyangör (2006) note the RME reinvention principle uses mathematisation as a guide. When a learner attempts a mathematical problem, he observes extra and relevant information; he contextualises the problem and mathematises the situation (Carlsen, 2010). Wheeler (2001) also argues that mathematisation consist of those mental processes that produce mathematics through modelling and not the activity of making a model. However, Steiner (1968) notes that mathematisation forms part of the activity of model building. When a chain of models is developed, it enables discourse which results in deeper understanding of mathematical principles (Presmeg, 2003). RME offers a complete and effective pedagogy which enhances understanding of the processes producing mathematics (Wheeler, 2001). The level principle of RME provides a learner with growth in understanding mathematics and gives the curriculum a longitudinal coherency of progressive growth (Van den Heuvel-Panhuizen, 2000).

2.8.2 Emergent modelling

Emergent modelling is a dynamic process in which models emerge to support the emergence of formal mathematical knowledge (Gravemeijer, 2002; Doorman & Gravemeijer, 2009). As discussed previously, models are fundamental in the RME theory. Models are entities in their own right and a means of mathematical reasoning (Doorman & Gravemeijer, 2009). The activity of modelling is an organising activity from which models emerge. In RME, formal mathematics grows from the learner's activities. Expressive models can serve as a model for emergent modelling because models emerge from learners' activities where problem contexts are experientially real for the learner (Doorman & Gravemeijer, 2009). Emergent models represent a bottom-up approach to learning as informal symbolisations derive their meaning from contextual problems and, over time, develop an independent meaning of symbolisations to be classified as formal mathematics (Gravemeijer & Stephan, 2002). Emergent modelling constitutes two processes (Gravemeijer, 2002, pp. 1-2):

i. Translating, which involves the translation of problem situations in mathematical expression which functions as a model
ii. Organising, which involves structuring the problem situation so that a model emerges

A model derives its meaning from an emerging framework and becomes more important as it becomes a base for reasoning. A *model of* can be described as the situation-specific solution of a problem. It is also the process of executing and describing the solution methods and mathematical meaning from the basis of conceptual development. A *model for* can be described as the development of formal mathematical reasoning. This model is generalisable so that it is no longer solution specific (Gravemeijer & Stephan, 2002). Learners use various symbolisations and each activity experienced is a natural expansion of an activity with previously-constructed symbolisations. A *model of* can be described as symbolising and a *model for* becomes symbolising to reason mathematically. The transfer from *models of* to the *model for* is a dynamic shift. It is also a holistic, metaphorical concept. Three interrelated processes encompass emergent models (Gravemeijer, 2002). Firstly, an overarching model illustrates the informal activities of the learner. Secondly a *model of* to the *model for* shift demonstrates the formal activities of the learner. Thirdly, the emergent model represents the series of symbolisations which are now connected to generalisable arguments. Formal mathematical symbols will eventually be rooted in concrete activities of students (Gravemeijer, 2002). Learners develop formal mathematics by mathematising informal mathematics (Gravemeijer & Stephan, 2002). The four types of activity are important factors when dealing with emergent modelling:

i. Task setting: situation-specific solutions related to a specific setting
ii. Referential activity: the situation is now described by symbolisations specific to the instructional task (model of)
iii. General activity: reasoning and acting independently on specific situation (model for)
iv. Formal mathematical reasoning: reasoning is no longer dependent on the support of ‘model for’ mathematical reasoning

Freudenthal’s domain-specific instructional theory gives learners the opportunity to reinvent mathematics by mathematising real matter and real mathematics. In the following sections, the four activity levels will be integrated with the mathematisation process and, more specifically, with horizontal and vertical mathematisation.

2.8.3 The nature of mathematisation

The instructional design of RME is organised so that learners can organise subject matter at one level to produce a new understanding at a higher level (Kwon, 2005). Van Hiele’s levels prove that activities of mathematising on a lower level can be the subject of enquiry on a higher level (Van den Heuvel-Panhuizen, 2003). Well-chosen subject matter enhances the opportunity to develop informal mathematical thinking and context-specific solutions (Doorman & Gravemeijer, 2009). Through organising, formalising and structuring, a higher level of mathematical thinking can be achieved. Organising, formalising and structuring are facets of mathematising (Wheeler, 1982). If pedagogy is focused on the process of mathematisation and all its constituent elements, the learner has to accept responsibility for his own learning process (Wheeler, 2001). This is in accordance with the fact that RME principles are rooted in, and developed from, a constructivist approach to learning. Wheeler (1982) further notes that mathematisation is closer to a phenomenology of awareness and convictions we experience, when doing mathematics. This process powers and empowers mathematical thought. The nature of mathematisation includes the following: to perceive relationships, to idealise relationships, to operate on relationships, to internalise, to visualise actions, perceptions and transformations, to alter frames of reference, to refocus neglected attributes, to coordinate and contrast the real and ideal, to recast problems and to synthesise perception (Wheeler, 1982). The presence of mathematisation can be noted through the following activities (Wheeler, 1982, p. 47):

i. Structuring: searching for patterns, putting structure into a structure (Wheeler, 2001)
ii. Dependence: putting ideas into relation and coordinating them
iii. Infinity: all mathematics implicitly or explicitly linked to infinity, generalisability or universability of ideas and structures

A typical, generalised learning process of a mathematisation pedagogy starts with an authentic problem. Informal and intuitive knowledge is developed and a mathematical model is constructed by the process of horizontal mathematisation. Vertical mathematisation occurs when a learner solves, compares and discusses the model to form a solution. The model is then further developed into a reusable and sharable system which can be used to interpret other contextual problems (Üzel & Mert Uyangör, 2006). Informal mathematics using outside tools and reflective generalisation support the development of abstract thinking which ultimately results in more formal mathematics. Different levels of understanding can be stimulated according to the chosen mathematical problem. Üzel & Mert Uyangör (2006) categorise the four types of mathematics education: in a mechanistic approach, no vertical or horizontal mathematisation is evident; an empiricist approach only satisfies horizontal mathematisation; and a structuralist approach caters for horizontal mathematisation only; a realistic approach however, will serve for horizontal and vertical mathematisation. It is important for mathematical problems to be rooted in a realistic, imaginable and flexible context so that the created systems can be developed and applied on a general level to serve the ultimate purpose of reinventing mathematics (Van den Heuvel-Panhuizen, 2003). The following two sections will define, elaborate on, and identify the modelling competencies and the process levels of mathematics involved within horizontal and vertical mathematisation.

2.8.4 Defining horizontal and vertical mathematisation

Horizontal and vertical mathematisation cannot be separated; they are two different processes, separately definable, yet they are complementary in nature and in purpose. Horizontal mathematising is the crossing between the real world situation and a mathematical model of the situation. Van den Heuvel-Panhuizen (2000) classifies horizontal mathematising as the activity occurring when a learner comes up with tools to organise and solve activities in a real life situation. Treffers describes horizontal mathematising as “going from the world of life to the world of symbols” (Van den Heuvel-Panhuizen, 2003, p. 12). Horizontal mathematisation occurs when any of the following activities can be identified: identifying or describing specific mathematics in a general context; schematising; formalising and visualising a problem in different ways; recognising relations and regularities; recognising isomorphic aspects in different problems; and transferring a real world problem into a mathematical problem (Üzel & Mert Uyangör, 2006).

Vertical mathematising concerns finding a mathematical solution to the mathematical model which was constructed during the horizontal mathematisation phase. It is defined as the process of organising within the mathematical system itself (Van den Heuvel-Panhuizen, 2000). Treffers described vertical mathematisation as “moving within the world of symbols” (Van den Heuvel-Panhuizen, 2003, p. 12). This implies identifying shortcuts when discovering connections and patterns between different systems. Vertical mathematising occurs when the following activities can be identified: reorganising within a mathematical system, representing a relation in a formula; proving regularities; refining and adjusting models; using different models; combining and integrating models; formulating a mathematical model; and generalising a mathematical model (Üzel & Mert Uyangör, 2006). These activities are not hierarchical. During mathematical modelling at the stage of vertical mathematisation, a learner combines mathematical entities by creating a mathematical object from other mathematical objects. Analysing refers to manipulating and interpreting the mathematical entity mathematically. The learner then analyses the mathematical entity to derive new parameters and properties of mathematical entity. This involves solving the mathematical problem. After the learner has obtained a mathematical solution, he associates the solution with the real world problem. Highlighting occurs when the conclusion is being revised according to the context. A learner then aligns his ideas when reflecting on the appropriateness of the solution, and reconciling inconsistent results. Vertical mathematisation is considered to be the vehicle to advance mathematical thinking. The advancing of mathematical thinking involves abstraction and deductive reasoning, and skills, which can be acquired and developed within the process of vertical mathematisation (Zbiek & Conner, 2006).

2.8.5 Developing mathematising competencies

The first step in the modelling process is to explore the real situation and to identify conditions and assumptions in the real world contexts, so that various constraints and restrictions are acknowledged. The competencies involved and developed when working through the mathematical modelling process are described in Section 2.4.7. The first mathematical-modelling competency a learner needs to master is the competency to understand the real problem (Maaß, 2006, p. 116). The sub-competencies relating to horizontal mathematising are: making assumptions regarding the problem and simplifying the situation; recognising quantities that influence the situation; naming them and identifying key variables; constructing relations between variables; looking for available information; and differentiating between relevant and irrelevant mathematics. The second competency is related to setting up a model based on the real problem. The sub-competencies relating to horizontal mathematising are: mathematising relevant quantities and their relations, simplifying relevant quantities and their relations, reducing their number and complexity, choosing appropriate mathematical notations and representing situations graphically.

The competencies and sub-competencies developed during vertical mathematising deal with the competencies to solve mathematical questions within the mathematical model. The third competency in the mathematical modelling process is focused on the activities when the learner: uses heuristic strategies such as division of the problem into part problems, establishing relations with similar or analog problems, rephrasing the problem, viewing the problem in a different form, varying the quantities or the available data, and using mathematical knowledge to solve the problem. The fourth competency requires interpreting mathematical results in a real situation. These involve: interpreting mathematical results in extra-mathematical contexts; generalising solutions that were developed for a specific situation; and viewing solutions to a problem by using appropriate mathematical language and/or communicating about the solutions. The fifth competency is validating the solution. The sub-competencies include: critically checking and reflecting on found solutions; reviewing some parts of the model or the whole model; going through the modelling process again if solutions do not fit the situation; reflecting on other ways of solving the problem, or reflecting and adjusting models if solutions can be developed differently; and generally questioning the model. It must be noted that throughout the process of mathematisation, during horizontal or vertical mathematisation, a learner needs to develop and use metacognitive competencies. These involve competencies such as: meta-cognitive questioning, practicing, reviewing, obtaining mastery on higher and lower cognitive processes, verification and, most importantly, reflection (Mevarech & Kramarski, 1997, p. 386).

2.8.6 Models for horizontal and vertical mathematising

As noted above, knowledge development occurs during the transition from a ‘model of’ a situation to the more general ‘model for’ a situation. The progression of knowledge evolves from a level of pre-informal, to informal, to pre-formal, and finally to a more formal mathematics. Figure 2.3 represents a horizontal mathematising model, and includes the central components, steps and modelling competencies required, to reach a level of adequate constructing of pre-informal knowledge. This is also the first step in emergent modelling. At this stage, the problem situation needs to be internalised. The model begins at the task-setting level where a learner must attempt to understand the realistic problem. This activity level is dependent on the learner’s intra- and extra-mathematical experiences. The next component involves the interpreting of the real problem so that the learner can identify the problem, and the different problem areas within the problem, thereby advancing to the actual model of importance. The learner then needs to build an understanding of the relationships within the information, to emerge at the next component dealing with structuring the problem. Structuring involves finding various patterns, regularities, relations and isomorphic elements. This is normally based on the existing reference frame of the learner. A learner will obtain a more meaningful experience if the patterns, relations and regularities are new to him, compared to a situation where he has worked with similar ideas and constructs before. The construction of informal knowledge occurs between the transitions from horizontal to vertical mathematisation. As noted in Section 2.6, the process that describes moving from symbolising to working with symbols is termed objectification. This signifies the transition between horizontal and vertical mathematisation. Objectification is an essential component when assessing modelling problems.

Figure 2.4 attempts to organise the activity of vertical mathematising by focusing on the relevant activities and competencies involved during the process. The first stage in the vertical mathematisation process is the referential activity, which was the final stage of the horizontal mathematisation process. This involves symbolisations which are context specific. The next level introduces the refining and manipulating of objects, in order to adjust adequately the mathematical model. Organising the mathematical model concerns combining, structuring and integrating symbols, when simplifying the mathematical model. The symbolisations are still situation specific, and through the activity of formalising they become more general. The last stage is the most important activity in the emergent modelling process. It depicts that moment when the model is not situation specific any more, but can be developed for more than one situation. A learner acquires a stage of formal knowledge when the level of generalisation is reached. At this level of generalisation new concepts can be discovered.

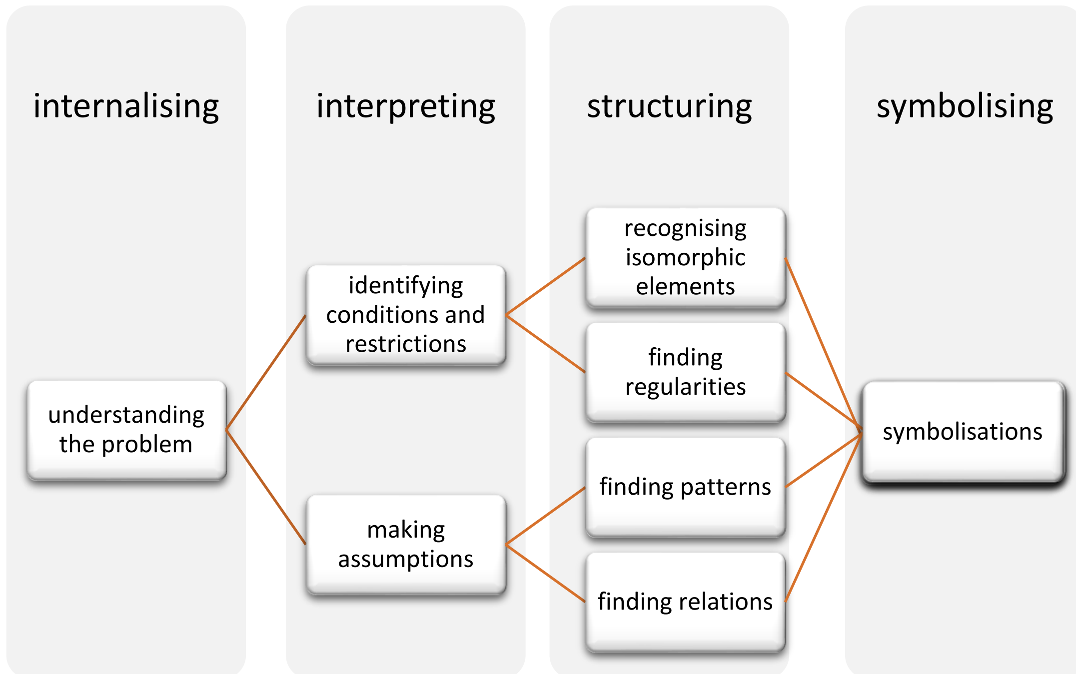


Figure 2.3: A model for horizontal mathematising (own representation)

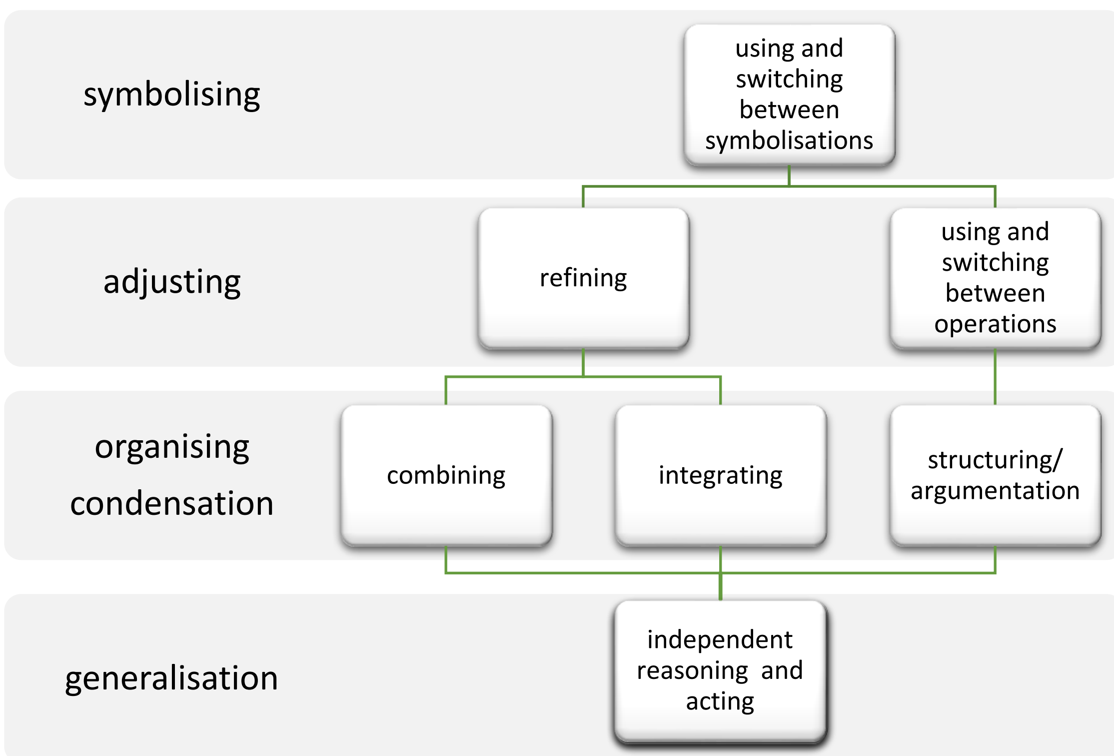


Figure 2.4: A model for vertical mathematising (own representation)

2.9 PROGRESSING THE PROCESS OF HORIZONTAL AND VERTICAL MATHEMATISATION

Figure 2.5 shows progressive mathematisation when a reified model is constructed to build new or adapted knowledge and constructs. These constructs are reusable and sharable and lead to the construction of theories by means of a deductive approach. From the process of vertical mathematisation, models are re-organised. New knowledge is constructed based on reified material.

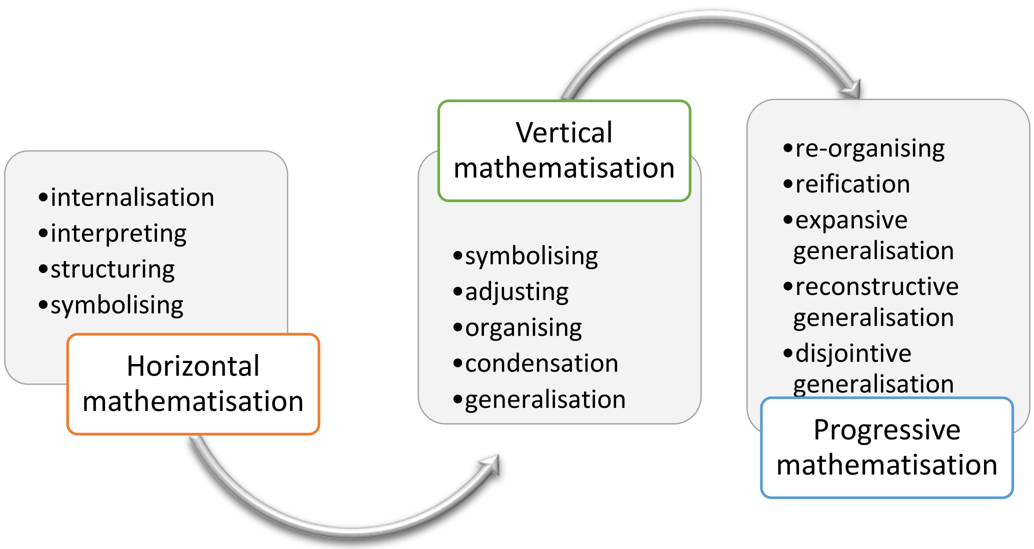


Figure 2.5: A model for progressive mathematisation (own representation)

Generalisation occurs at different levels resulting in the emergence of abstraction and advanced mathematical thinking. A repeated process of horizontal and vertical mathematisation, can lead to the process of progressive mathematisation where models can be used over an array of situations.

2.10 SUMMARY

The importance of mathematical modelling as a curriculum theory cannot be questioned. Based on the previous sections, it is evident that mathematical modelling offers the learner a chance to learn mathematics in a way which can be utilised in real-life. The purpose of mathematical modelling is to teach learners that the mathematics they learn can be related to their real-life experiences (Mukhopadahyay & Greer, 2001). To overcome the irrelevance of learning imitative mathematics, and for mathematics to become a noticeable variable in our society, a solution could be to implement mathematical modelling. Learners will feel more motivated and positive in the mathematics classroom. The mathematical modelling approach is an important one to consider when looking at enhancing the teaching and learning of mathematics. All learners are able to engage in modelling activities, although the complexities and the level of modelling may differ (Kaiser & Schwarz, 2006). There is a high demand for higher-level thinking. Owing to the fact that mathematical modelling requires, and then develops metacognitive abilities, higher-order thinking will be the resultant product. Research has shown that some average ability learners are able to construct complex models that are incomparable with any mathematics they have learnt at school (Kaiser & Schwarz, 2006). The mathematical modelling approach to instruction allows for the kind of valuable exploration in mathematics that has been absent to date. This can be rectified when a learner works with mathematical modelling as a problem-solving activity. Problemising mathematics is a way of extending the inquiry component of mathematics. Problemising mathematics is a dynamic occurrence which ultimately leads to conceptual development. It is a valuable component in the learning of mathematics, because it adds a collaborative element to the learning of mathematics. It also guides the learners’ attention on more productive ideas and mathematical thoughts, which would have been absent in other approaches of teaching and learning. The most important aspect of mathematics through problem solving is the opportunity for learning to become a meaningful experience. Learning becomes the responsibility of the learner.

The process of mathematisation occurs during the mathematical modelling process when the emphasis is on the activities of model building and emergent modelling. This is evident when a learner needs to construct a mathematical model from a real situation, solve the mathematical model to find a mathematical solution and then interpret the solution with respect to the original real contextual problem. Mathematisation involves conceptualising, verbalising, and formalising mathematical systems. The process of mathematisation can be divided into two sub-processes: horizontal mathematising and vertical mathematising. Figures 2.3 and 2.4 (Section 2.8.6) clearly show the difference between the different components when engaging in the process of horizontal and vertical mathematisation. However, the modelling process does not end there. The aim is to develop emergent models to serve the purpose of reinventing mathematics when models are created which are sharable and reusable. Mathematical modelling as a means to education enhances the possibility of creating formal mathematical knowledge and mathematical thinking. Therefore, the mathematisation model will now be constructed to include horizontal and vertical mathematising competencies for number patterns. In order for emergent modelling to occur, the focus is on progressive mathematising. Progressive mathematising is when a model is reorganised and generalised to become a universal model. It can also be defined when a learner rises through the four levels of activity, which is identical to moving through the process of horizontal and vertical mathematisation (Andresen, 2007). Zazkis and Applebaum (2007) also state that advancing mathematical thinking is when a learner engages in continuously and reflectively looking back, when working through horizontal and vertical mathematising. Doorman and Gravemeijer (2009) verify formalisation and generalisation as the main components of progressive mathematisation. A higher level of thinking is developed and evolves into increasingly abstract mathematical reasoning. This follows after sequences of activities are modelled and the learner is now able to construct a new theory. This theory is created through meaningful mathematics experience. It provides a referential base for formal reasoning (Gravemeijer, 2002). Reflective abstraction leads to new or modified concepts (Simon, 1995).

CHAPTER 3

TOWARDS PROGRESSIVE MATHEMATISATION

PHASE 1: Preliminary design of an instructional sequence

3.1 INTRODUCTION

This chapter provides a specific requirement for the study. It offers a description and explanation for the preparation phase of the developmental research design. The preparation and design stage of the triangulation sequence of a Design Based Research (DBR) study will introduce and guide the development of the instructional activities (Bakker, 2004, p. 40). These activities will formulate a hypothetical learning trajectory (HLT) which will form the basis of the classroom experiment. In Chapter 4, the second phase of the DBR will use the HLT to guide the teaching experiment. The HLT becomes a learning trajectory (LT). A day-to-day process of reflecting on the learners' understanding and the strategies they apply to solve the problems might lead to refining and adjusting the trajectory (Bakker, 2004). An outcome for this study is to design a LT which provides a teacher with a framework for a local instructional theory (LIT) for number patterns. This LIT can then be adjusted and refined for a specific classroom according to the needs for the given situation.

Various factors need to be discussed before the HLT is designed. The first decision to be made is the goals and outcomes of the subject content. Studies using DBR have emphasised the value of the goals and subject content being selected from a phenomenological analysis (Bakker, 2004; Bakker & Van Eerde, in press; Gravemeijer & Bakker, 2006). The RME instructional theory is the result of a continual DBR (Gravemeijer & Bakker, 2006). This instructional theory can be applied by looking at the three major heuristics: guided reinvention, didactical phenomenology and emergent modelling (Gravemeijer, 1999). The three instructional design heuristics will be discussed in the following sections: historical phenomenology and didactical phenomenology. A phenomenological analysis will focus the goals and outcomes on a selected few that will be discussed throughout the chapter. The aim of this chapter is to develop a curriculum for the HLT consisting of learning material that will be used in the teaching experiment. The learning material selected must fulfil the requirements of generating meaning through guided reinvention and constructing mathematical knowledge during the learning process.

3.2 PHENOMENOLOGICAL ANALYSIS

A historical phenomenology and a didactical analysis will be used to identify goals and successive outcomes for the subject content.

3.2.1 Historical phenomenology

Bakker (2004, p. 51) acknowledges the importance of a historical investigation of a topic because it serves as a basis for preparing the teaching of the content. The information that is collected through a historical phenomenology can also assist finding the starting points for instruction. In Chapters 2.7 and 2.8 the guidance principal of the RME theory was discussed. The guidance principal is based on the idea that learners should have the opportunity to construct knowledge by reinventing mathematics. A look into a topic's history can accommodate this reinvention principle by understanding the development of the learning process as technology and new ways of thinking developed.

3.2.2 Didactical phenomenology

According to Bakker (2004) a didactical phenomenology can identify problem situations that might occur during teaching and learning a specific concept. Problem areas that occurred in history might give researchers' ideas for future problem areas and suggest possible ways to overcome these problems. "A didactical phenomenological analysis, is closely connected with the idea of guided reinvention: it informs the researcher/designer about a possible reinvention route" (Gravemeijer & Bakker, 2006, p. 2). The reinvention principal can have its roots by considering the historical phenomenology and the didactical phenomenology. A didactical phenomenology originates from three analyses: the literature study, historical phenomenology and learners' prior knowledge (Bakker, 2004, p. 91). The following section is a phenomenological analysis to identify the goals of the content and suggested outcomes for each goal.

3.2.3 Phenomenological analysis: Selecting the goals and outcomes for the subject content

The goals and outcomes will be selected from the following sources: Research on the teaching and learning of number patterns in the past, the literature study done in Chapter 2 and the baseline assessment. The baseline assessment is a tool that will be used to establish the learners' prior knowledge.

The first goal of the subject content emphasises the value of representations. In Section 3.3 the different ways patterns can be represented will be discussed. From Chapter 2 it is evident that learners use representations to move from the real image of the real problem to the mathematical solution in the modelling process. The internal and external representation a learner uses and creates gives us a glimpse of the reasoning that accompanies the representations. The roles of these representations will be investigated when learners generalise number patterns.

The second goal for determining the content for the instructional activities addresses two separate matters. The South African mathematics teaching guidelines for number patterns are focused on investigating patterns. The exploration of patterns may provide learners the opportunity to generalise number patterns. Research has shown that generalising algebraic expressions for situations that are represented arithmetically is an activity learners find difficult (see Ellis, 2007a, 2007b; see Warren & Cooper, 2008). Section 3.4 will investigate why learners find moving from the real world to symbols challenging. The different ways learners can generalise number patterns using functional and recursive relationships will also be described. Ellis (2007a, 2007b) has developed a taxonomy to categorise generalising actions and reflections. This taxonomy will be useful when mathematising competencies are developed for number patterns in Section 3.5.

Section 3.5 identifies the third goal, to develop mathematising competencies for number patterns. Mathematising competencies will be developed exclusively for number pattern problems so that they can be easily recognised and identified in Chapter 5 when data is analysed.

The construction of the baseline assessment is a component of the phenomenological analysis and the fourth goal in the analysis. The outcome of the baseline assessment is to establish the learners' prior knowledge. In Section 3.6 the construction of a baseline assessment will be analysed. The results of the baseline assessment will be discussed per question. The aim of the baseline assessment is to establish the learners' prior knowledge. The prior knowledge of the learners will serve as a starting point for the HLT.

In Section 3.7 the elements of a HLT will be described. Lesh' principles for designing MEA's and the RME theory's design principles will be used to explore selecting quality learning material for the LT.

3.3 GOAL 1: REPRESENTATIONS

The importance of representations in mathematical modelling is evident from the literature study in Chapter 2. The modelling cycle in Section 2.4.5 shows the mathematical modelling competencies a learner develops when working through the modelling process. Setting up the mathematical model is the second competency in Maaß's competency list and involves the sub-competencies: mathematising, simplifying, choosing appropriate mathematical notations and representing the situation (2006, pp. 116-117). By setting up a mathematical model, a mental representation needs to be formed from the real situation and a learner needs to produce an external representation by internalising the representation. The purpose of representations is to help formulate and communicate ideas and information (Zarkis & Liljedahl, 2004). The following section describes the role of representations within the modelling process.

3.3.1 The role of representations

Section 2.5.1 explains that encoding and decoding are elements of the modelling process when a learner builds mental images of a real problem (encoding) while referring back to the problem (decoding). The process of encoding and decoding can be compared with the two-sided process of internalisation and externalisation as explained by Pape and Tchoshanov (2001). Their view that representations are built from a cultural perspective is similar to Radford's (2008b) view that social interaction using culture as a vehicle gives meaning to representations. Pape and Tchoshanov (2001) suggest that the main role of representations is to aid the understanding of abstract ideas. These abstract ideas can then be externally represented so that relationships can be identified and models can be constructed. Bruner's learning model (see Pape & Tchoshanov, 2001, p. 123) identifies three levels of engagement with representations: the enactive, iconic and symbolic. The enactive engagement involves manipulating concrete material, the iconic pictures and graphs while the symbolic represents working with numerals. Lesh, Landau and Hamilton (1983) list five types of representations: real life experiences, manipulative models, pictures and diagrams, spoken words and written symbols. Bruner's engagement model can be used to identify the representations. Real life experiences and manipulative models are enactive engagement, pictures and diagrams that form part of the iconic engagement and spoken words and written symbols are symbolic engagement. The five representations mentioned involve both internal and external representations. The following two sections discuss internal and external representations. This will provide valuable information for the study when mathematising competencies are developed for number patterns in Section 3.5.

3.3.2 Representing representations

In Section 2.5.2 internal representations and the different elements of concept formation were discussed. Mathematics is dependent on visualisation of concrete and abstract ideas (Arcavi, 2003). Visualisation is a mental representation. Internal representations include mental representations and computational representations (Hesselbart, 2007). Mental representations will develop as a learner reflects on existing representations based on past experiences and social interaction. Pape and Tchoshanov (2001) show that a combination of visual, concrete and abstract representations will result in increased understanding. Internal abstractions can only be adequately represented as a learner's experience representing his abstractions increases (Pape & Tchoshanov, 2001). By analysing a learner's external representations his internal representations can be noted. Cognitive representation is the interrelated functioning of internalising and externalising processes (Pape & Tchoshanov, 2001). As the skills of internalising his abstract ideas develop, cognitive representations will increase. Chapter 2.6 investigates a socio-cultural perspective towards mathematical modelling which is focused around external representations. Duval (1999, 2006) noted that semiotic representations are external representations. External representations are those representations that can be observed. From a modelling perspective the representations from Lesh, Post and Behr (1987) can be related to the representations that a learner could possibly use or construct during the modelling process. Lesh et al. (1987) list five systems of representations: experience based scripts that are organised around real world problems, manipulative models, pictures or diagrams, spoken language and written symbols. In the following section, representations will be used to generalise solutions.

3.4 GOAL 2: GENERALISATIONS

In Section 2.4.7 it is noted that *generalising* is a sub-competency when a learner is working within the mathematical world of symbols. In Sections 2.8.4 to 2.8.6 the competency *generalising* is described and represented as a vertical mathematising competency. Ellis (2007a, p. 225) provides a few definitions of generalisation: the rule about relations, extension of reasoning and identification of commonalities. Generalisation can also be described as expressing a general rule for common elements (Ellis, 2007b, Samson, 2012). The value of generalising cannot be ignored.

The South African Department of Education (DoE) strongly encourages the notion that learners are exposed to mathematical experiences that will give them the opportunity to develop mathematical reasoning to prepare them for more abstract mathematics in tertiary education. The Curriculum and Assessment Policy Statement (CAPS) provides the following teaching guidelines for number patterns at Grade 10 level: “Investigate number patterns leading to those where there is a constant difference between consecutive terms, and the general term is therefore linear” (Department of Education (DoE, 2011, p. 12). The Grade 11 teaching guidelines show the progression from linear to quadratic: “Investigate number patterns leading to those where there is a constant second difference between consecutive terms, and the general term is therefore quadratic” (DoE, 2011, p. 12).

De Villiers (2007) notes that the modelling perspective is not suited for all topics in the South African mathematics curriculum. The teaching guidelines for number patterns provide opportunities for exploration and can be provided when a modelling approach to the teaching and learning is implemented. The chance for exploration can also be limited if the transmission approach is applied and learners solve imitative, routine problems. Learners should be given the opportunity to explore patterns so that they can form conjectures and ultimately generalise number patterns. Carraher, Schliemann, Brizuela and Earnest (2006, p. 3) regard generalisation as: "the heart of algebraic reasoning". Generalisation is the link between arithmetic expressions and functions and allows us to engage with concepts and their meanings. It is therefore important for a learner to be able to switch between different representations. In Section 2.4.7 is it explained that as a learner works though the modelling cycle, he needs to be able to switch between different operations and representations. This will lead to more general solutions. According to Arcavi (2003), learners find it difficult to switch between representations. The next section will discuss the difficulties learners experience when generalising and suggest some of the reasons for these difficulties.

3.4.1 Difficulties in generalisation

Van den Heuvel-Panhuizen (2003) notes that the ultimate purpose of reinventing mathematics is for a learner to create models by using the process of generalising. Generalisation was the second goal selected for the subject content because of its value for mathematical reasoning and because of the difficulties learners encounter. Warren and Cooper (2008, p. 172) summarise the following reasons for learners' generalisation difficulties:

i. The transition of patterns to functions
ii. Finding a functional relationship and exploring a concept as a variable
iii. Lack of appropriate language needed to describe relationships
iv. Inability to visualise spatially
v. Older students battle with the process of generalisation

Warren and Cooper (2008) explain the meaningless and monotonous activities learners complete in class to practise generalising patterns. Learners are given a simple pictorial or numeral pattern and asked to continue the pattern, to identify the repeating part and then generalise a rule for the pattern. Ellis (2007b) notes that learners can easily recognise the pattern but cannot generalise the pattern successfully. Traditional algebra lessons are focused on manipulating symbols which negatively impacts a learner's algebraic understanding and reasoning (Ellis, 2007b). Brenner et al. (1997) have based their study on multiple representations. The aim of the study was to examine if pre-algebra students can represent symbols, words and graphics more effectively if the instruction was more focused on understanding word problems rather than symbol manipulation skills. Instruction moved away from teaching manipulation skills and focused on guided discovery using meaningful problem solving contexts. Learners showed a better understanding in the topic but also showed an improvement of representational skills (Brenner et al., 1997). Samson (2012) suggests that a more visual approach to introduce algebra can support the important processes when a learner generalises solutions. Arcavi (2003) similarly notes that visualisation is a contributing factor when a learner engages in reasoning and problem-solving.

3.4.2 Generalising patterns

Mathematics is dependent on visualising concrete and abstract ideas (Arcavi, 2003). Abstraction is when a learner recognises a pattern within a representation and generalisation is when he extends that pattern to a general solution. When a learner uses the processes of recognising and generalising, he uses inductive reasoning to formulate these conjectures.

Andrews (1990, p. 9) gives his thoughts on how a learner would generalise the following problem which he refers to as a "standard approach to a linear relation":

n	1	2	3	4	5	6
$f(n)$	3	7	11	15	19	23

The first difference is constant: 4

A recursive relationship can be represented by the general solution: $t(n+1) = t(n) + 4, n \in N$

The functional relationship can be represented by the general solution: $f(n) = 4n - 1$

A more structural way of generalising the pattern is by considering the structure of the values. A structural analysis requires deductive reasoning. During vertical mathematisation learners generalise patterns using abstraction and deductive skills (Zbiek & Connor, 2006)

$$
\begin{aligned}
f(1) &= 3 \\
f(2) &= 3 + 4 &&= 3 + 1 \times 4 \\
f(3) &= 3 + 4 + 4 &&= 3 + 2 \times 4 \\
f(4) &= 3 + 4 + 4 + 4 &&= 3 + 3 \times 4 \\
f(n) &= 3 + (n-1) \times 4
\end{aligned}
$$

Samson (2012) differentiates between local and global visualisation. Local visualisation is identified by a recurring addition of a unit that results in the next term. Local visualisation is the visualisation strategies applied to establish a recursive relationship or a functional relationship. Global visualisation is a broader visualisation that focuses on the relationship between the term and its position. Global visualisation results in a generalised functional relationship. The following pictorial sequence is used to explain local and global visualisation:

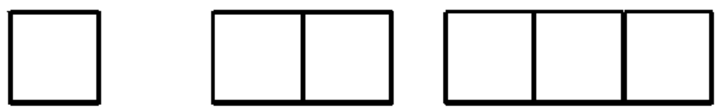


Figure 3.1: Pictorial sequence

The fifth shape will be used to explain the local visualisation. In Figure 3.2, both generalisations were obtained by using an additive unit in the pattern. In the first rule, $T_n = 1 + 3n$ was generalised by adding three matches to the original starting match each time to get the next term. In the second rule, a four-match constant was added to three matches of $n - 1$ to get to the next term.

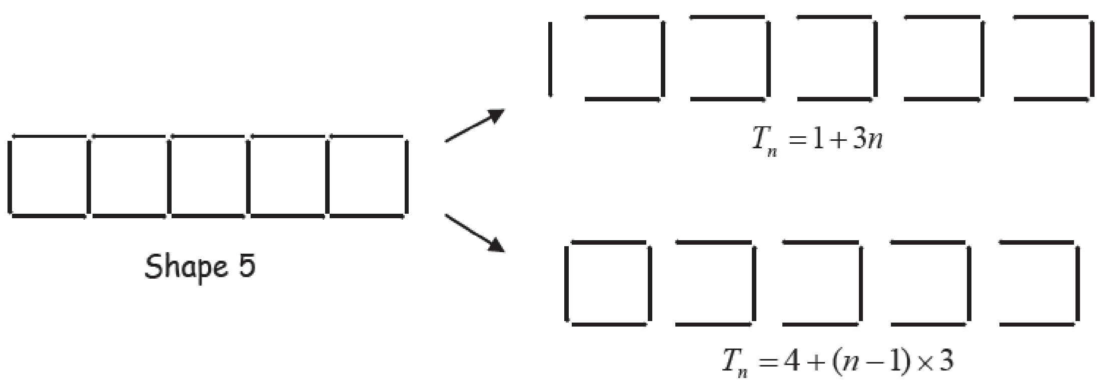


Figure 3.2: Local visualisation of Term 5 (Samson, 2012, p. 3)

Global visualisation does not make use of the recurring feature of the pattern. In Figure 3.3, each term is divided into a top row and a bottom row of n matches and a middle row of $n - 1$ matches which results in the rule: $T_n = 2n + (n + 1)$. The second generalisation was obtained by correcting the over count groups of four matches $(4n)$ because there are $n - 1$ overlapping matches. The rule is $T_n = 4n - (n - 1)$.

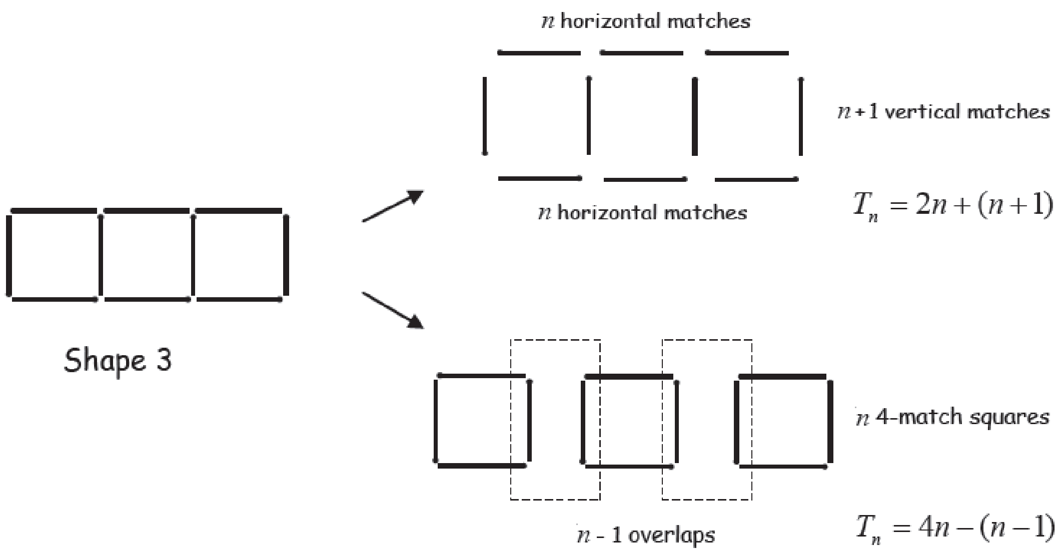


Figure 3.3: Global visualisation of Term 3 (Samson, 2012, p. 3)

The construction of triangular numbers follows the rule that each new polygon is formed by adding another row of dots to the existing polygon.

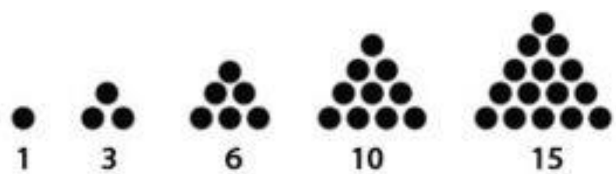


Figure 3.4: A pictorial quadratic sequence

The structure of the sequence results in a functional relationship:

Term 1: $1 = 1 \times (1 + 1) \div 2$

Term 2: $3 = 2 \times (2 + 1) \div 2$

Term 3: $6 = 3 \times (3 + 1) \div 2$

Term 4: $10 = 4 \times (4 + 1) \div 2$

Term n: $T_n = n \times (n + 1) \div 2$

The recursive relationship shows the important characteristic of a quadratic pattern: it has a constant second difference.

The recursive relationship is generalised:

$t(n + 1) = t(n) + [rule\ of\ the\ first\ difference's\ pattern]$

$t(n + 1) = t(n) + [n + 1]$, $t(1) = 1$ and $n \epsilon N$

Warren and Cooper (2008) suggest that a better understanding of algebraic reasoning skills will deepen a learner's understanding of the structure in mathematics. Ellis (2007a, 2007b) emphasises the value of justification and generalisation. Although a learner can easily generalise a recursive relationship it leaves them with little opportunity for further exploration. If a learner adapts a functional approach to generalisation, he has the opportunity to really engage and explore functions. As learners' generalising skills increase, their reasoning with patterns, quantities and real life problems will increase (Ellis, 2007b). This will not only link algebra to real life problems but give learners the opportunities to find meaning in algebraic reasoning and manipulation.

3.4.3 Categorising generalisations

In Section 2.5.4 the three levels of generalisation were described according to Zazkis and Applebaum (2007): expansive generalisation involves the relevance of existing schema expanded without reconstruction; reconstructive generalisation involves existing schema which are reconstructed to a wider range of applicability; and disjunctive generalisation occurs when new schema is constructed which is relevant to new contexts. Ellis (2007a, 2007b) has developed taxonomy to categorise generalisations. The taxonomy will be explored and compared with Zazkis and Applebaum's three levels of generalisation. Ellis' taxonomy categorises different levels of generalising. It differentiates between learners' activity as they generalise (generalising actions) and their final statements of generalisation (reflection generalisations). Generalising actions can be divided into three categories: relating (Type I), searching (Type II) and extending (Type III) (Ellis, 2007a, p. 235). These activities can be viewed in a hierarchical manner. When a learner is relating, he forms associations between two or more problems, situations, ideas or mathematical objects. This is noted when a learner recalls a similar previous situation or problem. When a learner is searching, he attempts to locate a similar element or idea by looking for relationships, procedures, patterns or solutions. Extending involves the expansion of a pattern, relationship or rule into a more general format. At this point in the generalisation process, their reasoning is extended past the problem, situation or case in which it started. Reflection generalisations categorise the activities that will result in a general solution that are articulated verbally or in written symbols. The reflection activities are identification or statement (Type IV), definition (Type V) and influence (Type VI) (Ellis, 2007a, p. 245). The activities for the reflection generalisations can take place in any order. In the Type IV reflection activity the learner's generalisation can be identified according to general patterns, properties, rules or common elements. In the definition activity, a rule is produced that can be applied to all the elements in the relationship or pattern. The influence activity in a previously generalisation is implemented or adapted so that it can be applied to a new situation.

The levels of Zazkis and Applebaum's (2007) generalisation are not as clearly defined as Ellis' taxonomy. Expansive generalisation can be grouped into Type III, the extending action. During the extending action of generalisation a learner expands the range of a case from which it originated. Reconstructive generalisation forms part of Type IV, the identification or statement. During the identification or statement stage, the strategy or idea can be implemented beyond a specific case. Disjunctive generalisation can be paired with Type VI, influence. At this stage, the idea or strategy can be modified to a new problem or situation. Ellis' taxonomy provides a framework to effectively be able to pinpoint the exact nature of a learner's generalising activities. The taxonomy will be incorporated when mathematising competencies are developed for number patterns. It will also be a useful tool when competencies are identified so that data can be effectively analysed in Chapter 5.

3.5 GOAL 3: DEVELOPING MATHEMATISING COMPETENCIES FOR NUMBER PATTERNS

In Section 1.2 and Section 2.8.6, the process of mathematisation and model building was discussed in terms of Gravemeijer's activity levels. In Section 2.4.7 mathematising competencies have been elaborated and defined. In Section 2.8.6 models for horizontal and vertical mathematising competencies were constructed by linking Üzel and Mert Uyangör's activities for mathematising to the horizontal and vertical competencies. Figures 2.3 and 2.4 model these activities for horizontal and vertical mathematisation. The next step involves adjusting and interpreting mathematising competencies specifically for number patterns. These mathematising competencies need to be developed so that horizontal and vertical mathematising competencies can be identified and described as a learner works through the modelling process. Various frameworks will be considered when describing horizontal and vertical mathematising competencies so that a valid and reliable tool can be developed to assess mathematising competencies during Phase 2 in the teaching experiment (Chapter 4). The model that will be developed in this section will be a guideline for recognising mathematising competencies in the modelling cycle. It can be readjusted and refined at any stage. In the process of understanding the various competencies for mathematising for number patterns, various frameworks will be compared and aligned. Table 3.1 maps competencies within key frameworks to provide a reliable development for number pattern competencies. Gravemeijer's activity levels (Gravemeijer, Cobb, Bowers & Whitenack, 2002), Van den Heuvel-Panhuizen (2003) classification of mathematising competencies, Üzel and Mert Uyangör's activities for horizontal and vertical mathematising (2006) and Ellis' generalising taxonomy (2007a, 2007b) will be compared and scrutinised.

3.5.1 Mapping mathematising competencies with current frameworks

As explained in Chapter 1.2, Gravemeijer's activity levels show the relationship between the activity and the development of emergent models through the motion of progressive schematising. Üzel and Mert Uyangör (2006) noted that horizontal mathematisation occurs when any of the following activities can be identified: identifying or describing specific mathematics in a general context, schematising, formalising and visualising a problem in different ways, recognising relations and regularities, recognising isomorphic aspects in different problems, and transferring a real world problem into a mathematical problem. In the activity of task setting the following horizontal mathematising competencies can be noted, internalising and structuring. In Ellis' generalising taxonomy, the generalising actions can be compared with the horizontal mathematising competencies of internalising, interpreting and structuring. *Internalising* focuses on the competency of understanding the real problem. When a learner understands a problem, he uses previous experiences to make sense of a new one. Relating (Type I) can refer to relating situations or relating objects (Ellis, 2007a). A learner relates situations or objects when he can connect to previous situations or objects based on similar elements or when he creates new situations based on those similar properties. Searching (Type II) can be compared with *structuring* (Table 3.1). When a learner is searching, he attempts to locate a similar element or idea by looking for relationships, procedures, patterns or solutions. Ellis (2007a) notes that a learner can search for a similar relationship, procedure, pattern or solution or result using a previous situation as a frame of reference. When *structuring* during horizontal mathematising, a learner recognises isomorphic elements, regularities, patterns or relations.

Objectification takes place when a learner moves from horizontal mathematisation to vertical mathematisation. On Gravemeijer's activity levels this movement is known as the referential activity. During the referential activity a learner moves from horizontal to vertical mathematisation and *symbolising* takes place. The referential activity can be compared with Ellis' extending (Type III). Extending involves the expansion of a pattern, relationship or rule into a more general format. The constructed model is the 'model of' a specific situation. Reflection generalisations in Ellis' taxonomy categorises the activities that will result in a general solution that are articulated verbally or in written symbols. This categorisation reflects the activities in Üzel and Uyangör's (2006) activities for vertical mathematising. These activities are reorganising within a mathematical system, representing a relation in a formula; proving regularities; refining and adjusting models; using different models; combining and integrating models; formulating a mathematical model; and generalising a mathematical model. During Gravemeijer's general activity and formal activity, vertical mathematising competencies can be noted. The competency of *adjusting* can be compared with Ellis' identification or statement (Type IV). In the Type IV reflection activity the learner's generalisation can be identified according to general patterns, properties, rules or common elements. The identification or statement level can include: identifying situations or objects that are similar, constructing a general rule or identifying a pattern, extending strategies or procedures beyond a specific case. The *organising* competency can be compared with the generalisation activity of definition (Type V). In the definition activity, a rule is produced that can be applied to all the elements in the relationship or pattern. In the level of formal activity, the vertical mathematical competency of *generalising* is aligned. Models can now be used to model other situations. This is known as the activity at general level (see Table 3.1) which occurs as the 'model of' now becomes a 'model for'.

	Gravemeijer's activity levels (Gravemeijer, Cobb, Bowers & Whitenack, 2000),	Van den Heuvel-Panhuizen's competencies (2003)	Üzel and Mert Uyangör's activities for horizontal and vertical mathematising (2006)	Ellis' taxonomy (2007a, 2007b)
Horizontal mathematisation	Level 1 Activity of task setting	Internalising	Understanding the problem	Type I Relating
		Interpreting	Making assumptions, identifying conditions, constraints and restrictions, quantities that influence the situation	
		Structuring	Recognising isomorphic elements, finding regularities, finding patterns, finding relations	Type II Searching
	Level 2 Referential activity	Symbolising	Symbolising	Type III Extending
Vertical mathematisation			Using and switching between symbolisations	
	Level 3 General activity	Adjusting	Refining, using and switching between operations	Type IV Identification/Statement
		Organising	Combining, integrating, structuring, argumentation	Type V Definition
	Level 4 Formal activity	Generalising	Independent reasoning and acting	Type VI Influence

Table 3.1: Mapping the mathematising competencies with Gravemeijer's activity levels and Ellis' generalising taxonomy

In Ellis' taxonomy the reflection of influence (Type VI) is when a generalisation is implemented or adapted so that it can be applied to a new situation.

The models for mathematisation in Section 2.8.6 need to be refined for mathematising competencies when a learner models number pattern problems. The next section will use Table 3.1 as a basis to develop competencies so that the learners' external representations can be used as a competency indicator as he moves through the modelling process.

3.5.2 Number pattern competencies for mathematising

In Section 3.5.1 number pattern competencies have been mapped (Table 3.1) so that mathematising competencies for number patterns can be developed. Table 3.2 separates horizontal and vertical mathematising competencies and focuses each competency on one or more sub-competency. Ellis' taxonomy (2007a, 2007b) uses various activities to characterise a learner's generalisation activities into specific levels. Using these levels and explanations, competencies can now be identified by looking at a learner's external representations, i.e. what the learner does, says, makes and writes. These activities serve as indicators to acknowledge whether horizontal and vertical competencies are utilised when working through a number pattern modelling problem.

Internalising is the horizontal mathematising competency when a learner explores the real problems so that he understands the problem and is able to simplify the problem. The competency *internalising* can be identified when a learner rephrases the problem into his own language, when he explains or notes important information. According to the generalisation taxonomy (Ellis, 2007a, 2007b) a learner relates back to previous problems when attempting to make sense of a current problem. During the competency *interpreting*, the learner recognises quantities that influence the situation and make assumptions to note conditions that will work or not work for a problem. When a learner is *interpreting* he recognises quantities that influence the situation. In Ellis' taxonomy relating objects can be compared with *interpreting* while searching can be compared with the competency *structuring*. *Structuring* involves setting up a real model based on relationships and patterns. The external representations of a learner when he is *structuring* will demonstrate finding and stating patterns or relationship within the problem.

	Competencies	Sub-competencies	What the learner does, says, makes or writes
Horizontal	Internalising	• Understanding the problem • Distinguishing between relevant and irrelevant information • Simplifying the situation	• Learner states the problem in language he understands • Learner notes/explains important information • Learner notes/explains/relates a previous problem that is similar to the current one
	Interpreting	• Making assumptions • Identifying conditions • Identifying constraints • Recognising quantities that influence situation	• Learner makes assumptions • Learner notes conditions that will work/not work for a problem • Learner recognises quantities that influence the situation
	Structuring	• Setting up a real model • Naming quantities • Identifying key variables • Recognise patterns • Recognise relationships	• Learner looks for a pattern/relationship • Learner notes a recurring value or situation in the problem • Learner recognises a pattern/relationship • Learner states the relationship or pattern
	Symbolising	• Choosing appropriate mathematical symbols • Using symbols • Setting up a mathematical model • Switching between symbolisations	• Learner draws pictures to represent the problem • Learner draws pictures to show the relationship/pattern • Learner uses objects to build the pattern
Vertical			• Learner uses symbols to represent his pictures/patterns • Learner forms a pattern using symbols • Learner extends his pattern • Learner formulates a rule using symbols • Learner creates a model *of*
	Adjusting	• Rephrasing the problem • Refining • Using and switching between operations	• Learner adapts his pattern so that it makes sense for the situation • Learner tests his pattern • Learner refines his pattern after testing it • Learners reflects back to the pattern/symbols • Learner reflects back to the real problem • Learner creates a model *for*
	Organising	• Viewing problem in a different form • Use mathematical knowledge to solve problem • Using heuristics • Combining • Integrating	• Learner constructs a rule that works for all elements • Learner reflects back to the real problem • Learner uses the rule to solve a problem • Learner validates his solution • Learner creates a model *for*
	Generalising	• Establishing similar relationships in different problems • Independent reasoning and acting	• Learner uses deductive reasoning to prove his rule • Learner uses/adapts the rule for another situation

Table 3.2: Number pattern competencies for mathematising

The competency *symbolising* can be a horizontal or a vertical competency. Activities that identify horizontal competencies can be: the learner draws pictures to represent the problem, he shows the relationship or pattern in the problem, he uses objects to build the pattern. Vertical mathematising can be identified when learners use symbols to represent the problem. When a learner is *adjusting*, the emphasis is on refining the symbolisations. *Adjusting* can be noted when a learner refines his symbolisations. Ellis' taxonomy (2007a, 2007b) summarises the identification or statement to be focused around a continuing phenomenon, a statement of commonality or similarity or a general principle. This means the act of finding a pattern or rule that symbolises a common element. The competency *organising* is identifiable when a learner constructs a rule that works for all elements. *Generalising* involves independent reasoning and acting and a learner can now use or adapt the rule for another situation.

The third goal in the phenomenological analysis was to develop mathematising competencies specifically for number patterns. The number pattern competency continuum will be used in Chapter 5 as a tool to identify competencies during the teaching experiment.

3.6 GOAL 4: ESTABLISHING THE LEARNERS' PRE-KNOWLEDGE

The starting point of the HLT is dependent on the learners' prior knowledge. A baseline assessment is an effective tool to establish a learner's mathematical skills and knowledge. The baseline assessment provides a teacher the opportunity to effectively plan the learning process. Kyriakides (2002) notes that the baseline assessment is used to identify what the learner can and cannot do so that differentiated learning needs can be targeted. The baseline assessment also serves as a basis for measuring future progress. The baseline assessment will provide important information about groups and individuals in the mathematics classroom and how the groups in a class can be structured. It will focus the HLT.

3.6.1 Analysis of the baseline assessment

Question 1 focuses on recognising, describing, extending, explaining patterns in different settings.

1.1 The given sequence consists of regular polygons staring with a 3-sided regular polygon (triangle), then a 4-sided regular polygon (square) and then a 5-sided regular polygon (a pentagon). Different choices are given, but the learner is not limited to one choice. If a learner selects:

A: The learner is not extending the given pattern, but he recognises that the next shape needs to have 6 sides.

B: The learner recognises the pattern, he can extend the sequence correctly, and he understands that the shapes are all regular.

C: The learner knows that the next shape in the sequence must have six sides, but he does not understand regular shapes.

D: The learner cannot extend the sequence; he does not recognise the pattern.

1.2 The given sequence consists of an unfamiliar combination sequence:

1, 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, 1, ...

Different choices are given, but the learner is not limited to one choice. If a learner chooses:

A: He recognises the linear or arithmetic part of the sequence only.

B: He recognises the linear or arithmetic part of the sequence and realises there must be a one in the sequence.

C: He recognises the combination sequence and extends the pattern correctly.

D: He does not recognise the pattern, he cannot extend it.

1.3 In the open sequence: 1, 2, 4, ... the learner has the opportunity to:

A: He recognises a quadratic sequence, even though he might not know that it is called a quadratic pattern, he still recognises that there is a pattern that emerges with the first differences.

B: Recognise an exponential pattern, even though he might not know that it is called an exponential/geometric pattern, he still recognises that there is a pattern that emerges when you multiply by a constant value.

C: No pattern. The learner cannot extend this sequence, which means he cannot relate to forming any next term.

D: No pattern. The learner cannot extend this sequence, which means he cannot relate to forming the next term.

1.4 The following sequence bids quite a challenge. $1, 3, 5, 7, 5, 3, 1, 3, \ldots$ The learner has four options to try and describe the pattern. If he chooses:

A: He recognises the first part of the sequence that is adding three each time. He does not apply this recursive rule to the second part of the sequence.

B: He recognises the second part of the sequence that is subtracting three each time. He does not apply this recursive rule to the first part of the sequence.

C: He cannot see a constant pattern, so the learner decides not to attempt his own description.

D: The learner attempts his own description for the pattern he sees. He might even attempt to extend the pattern.

1.5 In this question, a sequence is given, and a learner needs to select the correct rule which will work for each term. This question attempts to assess whether a learner can relate a rule to a sequence. If he chooses:

A: He only applied the rule for the first term.

B: Could apply the rule for the first two terms only. This means that the learner did not test the rule for all the terms.

C: He could test the rule for each term successfully.

D: He does not understand how to relate the rule to the sequence.

1.6 This question does not involve extending or recognising or even describing a pattern. During the modelling problems it is important for a learner to apply the correct operation rules, this question will assess if learners can use the BODMAS (brackets of division, multiplication, addition and subtraction) from left to right.

Question 2 focuses on recognising, describing, extending and explaining patterns with different properties. This question is an open question with the aim to assess the knowledge of patterns that learners have. Learners need to construct any linear number pattern, describe it by giving the first few terms and explaining how they would find more terms. This will clearly show if a learner knows that there is a constant first difference between two consecutive terms.

In the second part of this question, the learner needs to construct any quadratic number pattern, describe it by giving the first few terms and explaining how they would find more terms. Even though they might not know what a quadratic number pattern looks like, the question hints that there is a constant second difference. A learner might use a simple sequence such as $1, 4, 9, 16, \ldots$ and can at least explore the possibility of getting a second difference to be constant.

In the third part of the question, learners have the opportunity to construct any sequence other than a linear or quadratic. This question will give the teacher a clear understanding of the learner when he describes different sequences. He might describe and extend a cubic, a geometric sequence or a Fibonacci sequence. This accounts for the learner thinking of patterns and relationships between numbers.

Question 3 assesses the use of the *In-Out* table or diagram. The *In-Out* table shows the functional relationship of a pattern clearly (see Section 3.4.2). During the modelling problems, learners can choose to use the tables as a model building tool to represent the real problem as a mathematical problem. Once again, the learner needs to recognise patterns that are represented in a different way. This question is constructed so that the level of pattern recognition can be established. Question 3.1 involves a straight-forward sequence. Terms are given in order, so the learner might look at the *Out* column to find a relationship between the patterns. Learners need to fill in the missing terms in the sequence. In 3.2 the terms are not given in any order, and learners are forced to find a general rule to represent the *Out* values in terms of the *In* values in order to fill in the missing values. In the last question (3.3), information is given in no specific order, and is presented in a table. Learners need to find a general rule to represent the *Out* values in terms of the *In* values in order to fill in the missing values.

3.6.2 Results of the baseline assessment

The results of the baseline assessment will be discussed per question.

1.1. All the learners got this question correct. All learners recognised the pattern, they could extend the sequence correctly, and they understood that the shapes were regular.

1.2. All but one learner answered this question correctly. Sixteen learners recognised the combination sequence and extended the pattern correctly. One learner recognised the linear part of the sequence but added in a one because he realised there had to be an extra one in the sequence.

1.3. Six learners answered A. Answer A suggests that the learners recognised that to get to the second term one was added, to get to the third term two was added so to get to the fourth term three was to be added (quadratic pattern). Nine learners answered B. B suggests that the learner multiplied each term by two to get to the next (exponential number pattern). Two learners answered C which means they ignored the first term and added two to get to the next term.

1.4. All learners answered D. The majority attempted to explain a pattern:

F: Add two to the first three numbers, subtract two from the next three numbers, add two to the next three numbers and so forth.

D: 5 will come next because the pattern first begins with #1-7 then #5-1, so #3,… can be followed. This is a reverse pattern.

1.5. The six learners that answered B did not test the rule for all the terms or they might have guessed the answer. The learners correctly answered C and they could apply the rule for all the terms in the sequence. One learner did not answer the question.

1.6. Twelve out of seventeen learners correctly inserted brackets to make the equation true. Three learners incorrectly inserted brackets and two learners left it blank.

2.1. Fourteen learners could construct a linear number pattern correctly. Some examples were:

I: 2, 4, 6, 8, 10…

The learner described the pattern: *You add two every time.*

The learner wrote down the incorrect rule: $T_n = n + 1$

M:	*5*	*10*	*15*	*20*	*25*	*30*
	1	*2*	*3*	*4*	*5*	*6*

The learner's incorrectly explained: *Add 5*

2.2. When constructing a quadratic number pattern, seven learners correctly noted a quadratic pattern. Nine learners answered incorrectly. Of the learners that answered incorrectly, the majority constructed a linear pattern. One learner did not provide an answer. An example of a learner that gave a correct answer was:

A: 1, 2, 4, 7, 11, 16 …

The learner described the pattern: *To get to a new term, consecutive natural numbers must be added.*

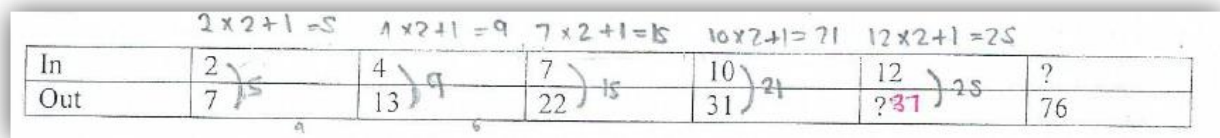
2.3. Two learners constructed quadratic number patterns, fourteen learners either wrote down a linear pattern or one that had no relationship between the terms and one learner did not answer the question.

3.1. Eleven learners explained the rule in words. Three learners did not explain the rule correctly and three learners did not answer the question. The following response was very well articulated:

F: The out is one less than three times the in.

3.2. All seventeen learners explained the rule in words.

3.3. Only four out of the seventeen learners gave a clear explanation of the relationship in the pattern.



2×2+1 =5 | 4×2+1 =9 | 7×2+1=15 | 10×2+1=21 | 12×2+1 =25

In	2	4	7	10	12	?
Out	7	13	22	31	?37	76

Figure 3.5: Learner *A* searches for patterns

Although the learner *A* did not complete his rule as a sentence, she searched for patterns by working with the differences between the *In* and the *Out* values (see Figure 3.5).

3.6.3 Using the baseline assessment to group learners in the modelling classroom

In the previous section, the results of the baseline assessment were explained. From the results, the learners need to be grouped within the heterogeneous classroom. Linchevski and Kutscher’s (1998) reports on a study where the aim was to investigate the effect of mixed ability teaching on achievement. This study was carried out over two years involving twelve schools. It was concluded that it is possible for all ability levels in mathematics to effectively increase in heterogeneous classes (Linchevski & Kutscher, 1998). Baer (2003, p. 170) notes:

> What little research that has been done in the elementary and secondary levels suggest a pattern similar to that found in non-cooperative learning settings: high achievers do much better in homogeneous groups; among average and low achievers there is little difference between students in heterogeneous and homogeneous groups.

Slavin (1987) suggests positive and negative factors to both ability and mixed ability grouping. Pigford (1990) notes three positive heterogeneous effects: increased student achievement, positive race relations and increased self-esteem. In Chapter 2.7 it was explained that low or average achievers can develop powerful models to scaffold their learning in the modelling classroom. When learners work on modelling problems, they work collaboratively towards the same result. Appendix B3 is a table used to summarise the results of the baseline assessment and shows the random, heterogeneous grouping of the learners. In Section 4.4.2 the selecting of the groups will be discussed. Section 4.4.6 will focus on selecting the learning activities for the teaching experiment.

3.7 HLT IN THE PRELIMINARY PHASE

3.7.1 Getting the HLT ready

In Chapter 1, a HLT was explained as the hypothesised path of development of a student's thinking and learning. Simon's (1995) study explains the challenges of developing an instructional design that would allow learners to reinvent mathematics. This study is based on a constructivist perspective. Section 2.5.3 explains the value of community in the learning process. The constructivist learning theory constitutes learning as a process whereby individuals construct their own knowledge and understanding for themselves in a social setting. The learner is in charge and takes complete responsibility for his own learning. Cobb (1999) emphasises the educator's facilitating role in a constructivist classroom, which involves guiding the learner to be an active participant in their learning. An educator needs to create and maintain a rich environment which stimulates active involvement of the learners and their own learning (Janvier, 1996). Progressive education involves the social aspect as an integral element of learning (Hein, 1991). The constructivist theory requires a learner-centred, cooperative and problem solving approach to teaching, where the learner has the opportunity to interact with their awareness as well as the opportunity to construct their experiences (Hein, 1991; Ward, 2005). The important role of the mathematics educator is emphasised by Confrey (1990):

> An educator should promote and encourage the development for each individual within his class of a repertoire for powerful mathematical constructions and should seek to develop in students the capacity to reflect on, and evaluate, the quality of this construction (p. 108).

The constructivist approach provides educators with important information on the understanding of the learning of mathematics, but neglects to establish a model for teaching (Simon, 1995; Matthews, 2003). Hein (1991) notes that learners construct knowledge based on previous knowledge constructions. If the learners' preconceived knowledge seems to be incorrect, incomplete or invalid a learner world reformulate existing constructs only if the knowledge is connected to their existing knowledge base (Hanley, 1994). Reflecting upon the learning process during the learning process is considered to be a principle of learning (Hein, 1991). The learning and understanding of mathematics is a progressive developmental process. This progressive process needs to be understood by the teacher. He needs to be able to set goals for each lesson and predict what the learner should be learning next. These goals will establish the direction of a HTL. Simon (1995) notes that a LT needs to incorporate learning goals, the learning activity and the thinking and learning of learners. The teacher then needs to provide the learners with instructional tasks that are designed to match the levels of thinking so that learners can progressively work towards the goal of that specific trajectory.

Figure 3.6 shows a mathematical teaching cycle which explains how a teacher uses a HLT to plan for a learning activity. Initially the baseline assessment will direct the HLT because it assesses the learners' existing knowledge. During the course of the teaching experiment in Chapter 4, the iterative process of setting learning goals, planning and setting the learning activity and adjusting the HLT will lead to an actual learning trajectory. The final learning trajectory will include the activities and the order of the activities that the learners complete to achieve the learning goals set by the teacher. It is evident that the learning activities in the LT have essential roles. The learning activities must be a means for the learner to have the opportunity to reinvent mathematics, progress their mathematical thinking and construct knowledge through building models.

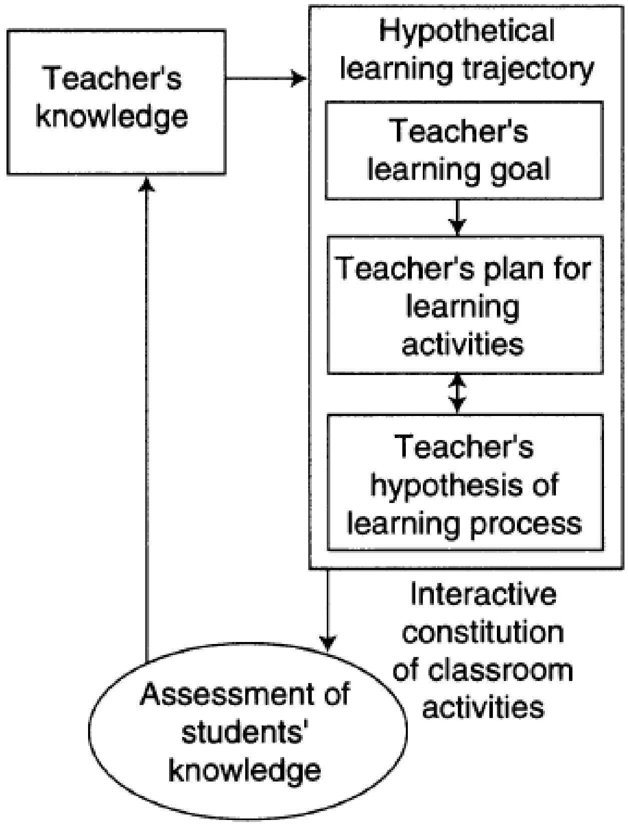


Figure 3.6: Mathematics teaching cycle (Simon, 1995, p. 136)

In the next section the properties of a modelling activity that will adhere to the before mentioned roles will be investigated.

3.7.2 Model-eliciting activities

In the previous section the function of a HLT was explored. The activities that form the LT need to promote the learners' mathematical thinking. Due to the nature of mathematical modelling and the aims of the study, it is appropriate to use model-eliciting activities (MEAs) for the learning activities. MEAs are used to investigate learners' thinking (Chamberlin & Coxbill, n.d.). Lesh and Lehrer (2003) note that the mathematical thinking learners' need to solve MEAs is different to the understanding and mathematical thinking needed when solving textbook questions. MEAs involve constructing symbolisations from real situations when working through the modelling cycle (Chamberlin & Coxbill, n.d.; Lesh & Harel, 2003; Lesh & Lehrer, 2003). MEAs are solved in an express, test and revise approach. Lesh's six design principles form the basis to create formal MEAs and capture the important properties modelling problems must encompass for progressive mathematisation can occur. The design principles to create MEAs are the model-construction principle, reality principle, self-assessment principle, model documentation principle, model share-ability and reusability principle, and the effective prototype principle (Lesh, Hoover, Hole, Kelly & Post, 2000).

i. *Model construction* principle. The model construction principle articulates that a model needs to be devised to solve the problem.
ii. *Reality* principle. The reality principle refers to a meaningful activity that is real and relevant to the learners.
iii. *Self-assessment* principle. The activity needs to be such that the learners are able to validate their solutions and decide whether they have done enough.
iv. *Model documentation* principle. Learners need to document and represent their solutions in a way that explains their mathematical thought processes and thinking.
v. *Model share-ability and reusability* principle. This principle emphasises the shift from a model-of a specific situation to a model-for a general situation. The latter is a model that can be reused or modified for a new situation.
vi. *Effective prototype* principle. The effective prototype principle refers to a model that is simple yet effective for the situation so that others can interpret and use it successfully.

In the next section, a checklist will be designed to effectively note the correlation of the learning activities with the RME principles and MEA design principles.

3.7.3 Selecting criteria to develop a checklist for mathematical modelling problems

In Section 2.7 the RME principles were discussed. Together with Lesh's six design principles noted in Section 3.7.2 a checklist will be developed to ensure that the learning activities in the learning trajectory adhere to these principles. Doerr (2006) investigated a MEA and selected four of the six principles that she found relevant for the task. These questions will be used as a guideline to develop a tool to evaluate modelling problems. Table 3.3 shows the criteria for each RME principle and the converted question for the checklist. Five principles are selected from Lesh's MEA principles and the RME principles to form the criteria for the checklist. These principles are relevant to the types of learning activities that will foster mathematical modelling perspectives. The first principle selected from the MEA principles is the reality principle. The reality principle is also a RME principle. The questions on the checklist will be focused on the meaningfulness of the problem. Problems and their solutions need to be real and meaningful to the learners.

Principle	Criteria	Questions
Model construction principle	• Learners are able to make meaningful connections within the problem • Problems need to allow for creating models for relationships, patterns and rules for these relationships	• Does the activity involve constructing, describing or explaining a structurally significant system? (Doerr, 2006, p. 7)
Reality principle	• Problems and solutions are real and meaningful to the learners	• Is the context of the activity realistic and useful? • Will students be encouraged to make sense of the situation based on extensions of their own personal knowledge and experiences? (Doerr, 2006, p. 7)
Self-assessment principle	• Learners can judge the usefulness of their solution based on the problem • The learning activity has sufficient information for the learner to know when he has accomplished his goal and when his solution complete	• Does this activity provide enough information for a learner to establish if he has done enough?
Level principle	• The learning activity forms part of the vertical planning component of the RME principles • The learning activity is progressive or forms part of progressive activities	• Is this task progressive or form part of a progressive sequence of activities? • Can this task be used in a higher level of activity?
Language	• The language used in the learning activity is appropriate for the learners	• Is the language of the activity appropriate for the learners?

Table 3.3: Checklist for mathematical modelling problems

The second principle is the model principle selected from Lesh's design principles. The criteria for this perspective will be based on whether the learner is able to make meaningful connections with the problem. Problems need to allow for creating models for relationships, patterns and rules for these relationships. The third selected principle is the self-assessment principle. Learners must be able to judge the usefulness of their solutions. The learning activity needs to have sufficient information for a learner to know when he has accomplished his goal and when his solution is complete. The level principle is selected from the RME principles and will establish whether the learning activity forms part of the vertical planning component of the RME principles. This will ensure that the problems are progressive. The fifth principle is the language principle. The language principle is a RME principle and the criteria for the checklist will be based on the appropriateness of the language for the learners. Treffers (1987) explains the starting points of mathematics education through: “activity, differentiation, vertical planning, structural character, the language aspect, the applicability, the dynamics and the specific approach” (p. 59). By developing a checklist of specific criteria, the learning activities are designed so that these starting points are achievable. By building models to represent realistic situations learners relate mathematics to their previous experiences and learn mathematics by doing. When the learning activities form a LT that shows progression, mathematics is learnt according to the needs of the learner’s mathematical thinking and logic. Each activity that will be considered for the HLT will adhere to the principles that were selected in Table 3.3.

3.8 SUMMARY

The purpose of this chapter was to select the goals and outcomes of the subject content for the study. Goals were selected by means of a phenomenological analysis so that the HLT was focused in terms of Gravemeijer’s guidelines for an instructional theory: guided reinvention, didactical phenomenology and emergent modelling. The first goal was focused on the external representation of learners. The outcome for investigating representations was to explore the different representations a learner might use to build a real model and a mathematical model. The way in which learners represent their thinking and understanding will assist identifying and describing the revealed mathematising competencies during the analysis of the teaching experiment. The second goal was to investigate the process of generalisation so that the different levels of generalisation could be identified and characterised. Ellis’ taxonomy provided a framework to effectively be able to locate the learner’s generalisation activity. Mathematising competencies were then developed specifically for number patterns modelling problems. The third goal was developing mathematising number pattern competencies. These mathematising competencies will be used in the next chapter when the learners’ activities are analysed. The fourth goal was to construct and use a baseline assessment to establish learners' prior knowledge to focus the HLT and therefore the LT. The RME principles and Lesh's MEA design principles were used to develop criteria for a checklist to easily review the learning activities. This will ensure that the learning activities are true thought-provoking mathematical modelling problems. In the next chapter the HLT becomes a LT. Chapter 4 will explain the teaching experiment as guided reinvention takes place by implementing the HLT.

CHAPTER 4

EMPIRICAL INVESTIGATION

PHASE 2: Educational experiment and adjusted, elaborated, refined sequence

4.1 INTRODUCTION

The main purpose of this section is to explain the methodology and the hypothetical learning trajectory (HLT) that becomes the learning trajectory (LT) in the experimental phase. Chapter 2 has served as the theoretical background for the study. The various perspectives of mathematical modelling were investigated to explore the purpose of mathematical modelling and the aims for each perspective. The RME approach to the teaching and learning of mathematics has been investigated in Chapter 2 and 3 and implemented in this empirical design of the study. Chapter 3 started the preparation stage of the study. The instructional goals were identified through a phenomenological analysis considering goals and outcomes that were evident in the history of number patterns and the overarching literature study in Chapter 2.

Four main goals were selected to focus and prepare the study for the experimental phase. The goals were: investigating the role of presentations to relate learners' presentations with their reasoning, investigating generalisation because this was classified as a difficult area in the phenomenological analysis and is an outcome in the mathematics curriculum's guidelines, developing mathematising competencies for number pattern problems so that mathematising competencies could be identified, and developing and discussing the baseline assessment to investigate learners' prior knowledge to focus the HLT and to start the LT.

The purpose of Section 4.2 is to address the research problem and aims of the study. Referring back to the research question and aims in Chapter 1 will amplify the problem statement and aims for Chapter 4. These aims will be reviewed in Section 4.2. The design-based research (DBR) is the frame of the study. Section 4.3 will explain the purpose of DBR and how it's been interwoven in the study. The research instruments in Section 4.4 will be explained as they were developed and used in the experimental phase. Section 4.5 will explain the HLT in the experimental phase. The triangulation component of DBR explains the merging of the data collection methods and coding procedure that was followed to ensure validity and reliability. The phases of the cycles in the experimental phase will be explained and the retrospective analysis will be linked to the local instructional theory (LIT).

4.2 SPECIFYING THE RESEARCH PROBLEM AND AIMS

The research question of the study is: *How does the development of a local instructional theory influence learners' development of mathematising competencies when modelling mathematical number pattern problems*? Five aims were constructed to direct the study so that the research question can be explored. Chapter 2 addressed the *first* aim which was to describe a mathematical modelling perspective towards the teaching and learning of mathematics. All the modelling perspectives were investigated. A socio-critical perspective introduced the value of mathematical modelling and the educational goals depicted when a learner models mathematics. The contextual perspective related mathematics to real life and explained that mathematics can be meaningful to a learner if they were subjected to flexible, real life problems. An educational perspective involved mathematical modelling as a concept-developing aid where mathematical modelling competencies are developed through the modelling process. The cognitive perspective explored the cognitive and meta-cognitive processes required when a learner grapples with a modelling problem. The roles of representations were the focus of the socio-critical perspective and explored the process of objectification which is the transition between the horizontal and vertical mathematisation processes.

The RME theory was discussed in the realistic modelling perspective and the process of horizontal and vertical mathematisation was investigated in the epistemological perspective. The RME framework and the development of mathematising competencies form the basis of the study. The epistemological perspective of modelling (Section 2.8) explored mathematising competencies and the development of these competencies through model building. Exploring the horizontal and vertical mathematisation processes was the *second* aim of the study. The horizontal and vertical mathematising competencies were modelled in Figures 2.3 and 2.4 (Section 2.8.6). As mentioned in Section 4.1 one of the goals as directed by the phenomenological analysis was to develop mathematising competencies specifically for number patterns. The *third* aim of the study was directed towards the analysis of number patterns in terms of the processes of mathematisation. Table 3.2 explains the competencies and sub-competencies for number patterns. It introduces indicators described by the learners' representations so that competencies can be identified and coded. In Chapter 6 the generalisability of this model will be explored. Section 3.7 addressed HLT in the preliminary phase. The *fourth* aim in the study was to explore the design of a HLT. In Section 4.4.3 the selection of the tasks will explain how the HLT started from the baseline assessment and how the HLT developed throughout the experimental phase. Section 4.5 will initiate the *fifth* and final aim of the study and will focus on the retrospective analysis that will result in a LIT for number patterns.

4.3 RESEARCH DESIGN

As mentioned in the previous section, the main outcome of the study is to produce a domain-specific, local instructional theory. In a design experiment that concerns the development of a domain-specific, instructional theory, the goal is to develop an empirically grounded theory including the processes of students' learning in that domain and the means by which this learning can be supported (Cobb & Gravemeijer, 2008, p. 86). DBR is characterised as a flexible research methodology used by researchers to directly improve educational practices (Bakker, 2004; Brown, 1992; Burkhardt & Schoenfeld, 2003; Edelson, 2002; Gravemeijer & Bakker, 2006; Wang & Hannafin, 2005). Section 3.1 provided a brief summary of the main aspects of a DBR, the three phases and the HLT. The three phases of the DBR methodology are: the planning phase, the experimental phase and the retrospective analysis (Cobb & Gravemeijer, 2008).

Learning situations are planned by hypothesising the path of a learner's reasoning and understanding. The HLT in the planning phase is aligned with a learner's pre-knowledge determined by a baseline assessment. Section 4.4.4 will explain how the baseline assessment contributes to the starting points and thus the initial HLT. Phase 2 of the DBR is the teaching experiment. The experiment takes place in a classroom setting. The researcher sets learning goals for the learners so that an activity can be selected to support the learner's learning. Section 4.4.3 explains how activities are selected based on shifts in the learner's reasoning. Gravemeijer and Bakker (2006, p. 1) explain the DBR as "design cycles of preparing, designing, testing and revising". These activities that result from the cyclic processes of preparing, designing, testing and revising form the actual learning trajectory. During a DBR study, educational practices can directly be improved because theories are developed as instructional sequences are implemented and learning is supported throughout the sequence. A DBR framework results in useful results in the form of an instructional theory (Edelson, 2002). It provides teachers and researchers the opportunity to refine the instructional sequence to their situation (Cobb & Gravemeijer, 2008). Through the development of a LT an associated LIT for number patterns will emerge. This will capture the mathematisation processes that learners developed throughout the teaching experiment.

4.4 EMPIRICAL DESIGN

The empirical design of the study involves the design and administration of the teaching experiment. In the following sections the different components of the study will be explained. The pilot study will introduce the first trail of a baseline assessment and learning activities in Section 4.4.1. A detailed explanation of the selection of the learners will be given in Section 4.4.2. In Section 4.4.3 the different research instruments that were developed and the purpose of each instrument will be explained. The data collecting method in Section 4.4.4 outlines the means of data collection and the role of the researcher for each activity. Section 4.4.5 will explain how validity and reliability of the study were enhanced. The aspect of triangulation is pertinent in the validity of the data and its interpretations. The selecting of the learning activities is a focal component of the chapter. Section 4.4.6 describes the HLT before and the LT during the teaching experiment.

4.4.1 Pilot study

The pilot study took place during August 2012. Six learners participated in the pilot study. These learners would not be involved in the teaching experiment. The purpose of the pilot study was to assess the adequacy of the baseline assessment (see Appendix B1) and the activities (Appendices A13-A17). Learners did not understand the questions. No previous research existed on the development of a baseline assessment for modelling problems. Once the learning goals were identified using the phenomenological analysis, a more adequate baseline assessment could be designed. The development of the baseline assessment was discussed in Section 3.6.1. The baseline assessment in the pilot provided some starting points for the initial activity in the pilot study. As the six learners worked through the activities, the learners showed interesting ways of pattern generalisation. The activities also had a definite vertical progression. Although the activities were real to the students, e.g. building matches and working with chords in a circle, the need to ascertain the quality of a modelling problem arose. The design of a checklist with criteria from RME principles and Lesh's principles to design MEAs led to the development of Table 3.3 (see Section 3.7.3). Although the activities adhered to the checklist of the principles for mathematical modelling problems, it lacked the chance for learners to construct real models from a real problem. The real model was given (representations of the patterns) in most of the activities. The activities were retained as possible activities for the LT in the teaching experiment. The pilot study contributed to planning time frames, constructing and adjusting research instruments and selecting quality modelling problems for the LT.

4.4.2 Selecting the learners

The design experiment's participants were seventeen Grade 10 learners. The teaching experiments took place on a Monday afternoon for 90 minutes and on Saturdays for three hours. The study could not take place during the school day. The topic number patterns was completed in adherence to the curriculum documents early in February. Due to the rigid and full Grade 10 syllabus, the only time that the design experiment could be conducted was outside of an ordinary school day. After explaining the aims and objectives for the study to a class of twenty five, seventeen learners were interested to see how mathematics was reinvented from modelling problems. The learners were ardent in their intent to improve their understanding of number patterns. The seventeen learners' abilities were heterogeneous. Mixed-ability and same-ability grouping was discussed in Section 3.6.3. Research suggests that heterogeneous ability grouping for low-ability and average-ability allow for the best improvement while the high-ability learners show the same improvement in either a heterogeneous or homogeneous group (Baer, 2003;

Linchevski & Kutscher, 1998; Pigford, 1990). Modelling problems have the advantage that models can be constructed at different levels. In a heterogeneous group, although the learners work collaboratively, learners may construct models at different levels. In Section 2.10 it was stated that all learners are able to engage in modelling activities, although the complexities and the level of modelling may differ (Kaiser & Schwarz, 2006). The learners were randomly grouped into three heterogeneous groups. A focus group was selected at random to follow the development of their mathematising competencies throughout the design experiment. In the next section the research instruments will be discussed.

4.4.3 Developing the research instruments

Research instruments were developed to document the development of number pattern competencies (NPCs). The research instruments and their contribution towards the study will be explained in the following section. Edelson (2002) notes that research instruments provide helpful data to support the last phase of the DBR, the retrospective analysis. All “conjectures about shifts in reasoning” (Cobb & Gravemeijer, 2008, p. 69) and “contextual influences” (Wang and Hannafin, 2005, p. 18) need to be recorded. The research instruments were designed to support these conjectures and influences.

4.4.3.1 Baseline assessment and table (see Appendix B2 & B3)

The baseline assessment was analysed in Section 3.6. The objective of the baseline assessment was to establish the learners’ current modes of reasoning and their ZPD, in other words what they are ready to learn with support. A table (Appendix B2) was used to summarise the data. It provided the researcher with information for the first learning activity in the LT. The baseline assessment had no effect on the selection of the learners in the groups, but it did contribute to the selection of the initial learning activity. The selection of the learning activities will be discussed in Section 4.4.6.

4.4.3.2 Interview questionnaire (see Appendix B4)

The interview questionnaire was designed to understand the learners’ views about the types of questions in the teaching experiment. The questions were aimed to investigate if the learners felt accomplishment and motivation before, during and after the learning activities. It also investigated the participants' feeling with working in a group and their confidence when grappling with unknown problems. The questionnaire was conducted before the last Saturday session.

4.4.3.3 Researcher observation guide (see Appendix B5)

The researcher observation guide was designed for multiple purposes: Group interaction was documented, shifts in understanding were noted and meta-cognitive strategies were noted. A component for the teacher/researcher's reflection was also constructed. In the current study, the researcher is also the teacher. The researcher use the reflection part on the guide to reflect on the facilitating role, the learning activities, the questions asked or answered, and what she could do to improve her practice. The researcher observation guide was used during and after the modelling sessions.

4.4.3.4 Number pattern competency continuum (see Appendix B6)

A research instrument was needed to adequately assess mathematical competencies. In Section 2.4.7, the COM²-project rated the progress of mathematical modelling competencies according to three aspects: the technical level, the radius of action and the degree of coverage (see Blomhøj & Kjeldsen, 2006, p. 167; Haines & Crouch, 2007, pp. 5-6). The research observation guide needed to encompass these. Biccard (2010) used the three aspects as an assessment tool. She skilfully indexed competencies to measure the levels of attained competencies. In the current study the aim is to teach the modelling process and competencies while a learner works through the learning competencies. The number pattern competency (NPC) continuum will be used to identify the competency revealed during the modelling sessions. Section 3.5.2 explains how the NPCs for mathematising were developed. Table 3.2 in Section 3.5.2 explain the indicators to identify mathematising competencies when a learner models a number pattern problem.

4.4.4 Collecting the data

There were six means of data collection. The data collection's function was to collect information so that inferences could be made about the learners' mathematising competencies. Each learning activity started with a whole group discussion to share ideas, individual or small group activity and then a whole group discussion of the analyses (Cobb and Gravemeijer, 2008).

Audio recordings were made of the focus group while the learners worked through the learning activities. Video recordings of the class discussions and presentation sessions were made. The focus groups' written work was collated at the end of each modelling session. All interviews, video and audio recordings were transcribed by the researcher. The audio transcriptions, researcher observations and written work were assessed by the researcher on a daily basis to determine the HLT in the experimental phase, in other words the LT. The learners' mathematising competencies were identified using the NPC continuum. This enabled a visual performance of the learners' Zone of Proximal Development (ZPD). The written work and transcripts of the recordings were coded per NPC. The researcher used informal mini-interviews to better understand what the learners were doing, saying and writing during the modelling process. The questions posed by the researcher did not interfere with the groups' direction of thinking and reasoning but rather helped their reflective processes and guided the modelling process. The researcher posed questions like: What have you got? Can you explain? How do you think this will help you? These mini-interviews were noted during the transcribing of the audio recordings.

A data collection plan in Table 4.1 shows the data collecting method, the means by which the data was collected and the role of the researcher for the different methods.

Data collection method	Means of data collection	Role of the researcher	Appendix
Baseline assessment	Function and results discussed in Section 3.6 to form groups and starts the LT in Section 4.4.6	Developing and analysing the baseline assessment	B2, B3
Learning activities in LT	Written work of the learners	Developing and/or sourcing modelling problems, checking the quality of the modelling problems, making verbatim transcripts, coding the data per competency, analysing data and interpreting data	
	Audio recordings of focus group and video recording of group discussions		
Interview questionnaire	Audio recorded activities	Scheduling appointments, preparing and conducting interviews, making verbatim transcripts, coding and interpreting data	B4
Mini interviews	Audio recorded activities	Asking thought-provoking questions, making verbatim transcripts, coding and interpreting data	
Researcher observation guide	Written notes	Coding and interpreting data	B5

Table 4.1: Data collecting method

4.4.5 Validity and reliability

4.4.5.1 Internal validity

The validity of a study is based on the credibility of the data and the analysis of the data. To improve the validity of a study two methods can be employed: data triangulation and using counter examples to test conjectures during the retrospective analysis (Bakker, 2004; Bakker & Van Eerde, in press; Denscombe, 2007). Triangulation refers to “the practice of viewing things from more than one perspective” (Denscombe, 2007, p. 134). Section 4.4.4 explored data collection from six sources, a baseline assessment, learners’ written work, interviews (including mini interviews), and verbatim transcripts from mechanical audio and video recordings. The confidence in the internal validity is enhanced through the multiple methods of data collection. The series of sequential teaching cycles (see Section 4.5.1) makes it possible to test and compare the conjectures developed in experiments. This serves as a confirmation of conjectures and interpretations during the retrospective analysis. Direct quotations used in the retrospective analysis also improves the credibility and hence the validity of the study.

4.4.5.2 External validity

Bakker and Van Eerde (in press, p.25) note that external validity is based on the “generalisability” and “transferability” of the results. The application of a design research is specific to the starting points of a HLT and the development of learners’ reasoning during the micro cycles (see Section 4.5.1). Conjectures, expectations and interpretations throughout the teaching experiment have been documented in such a way that other researchers can adjust, refine and use them according to their contexts.

4.4.5.3 Internal reliability

The internal reliability concerns the level of dependence of the data and analysis of the data by the researcher (Bakker and Van Eerde, in press). The internal reliability of the study was improved by the following methods:

- audio and video recordings were collected from mechanical devices,
- Section 4.4.3 explained the development of the research instruments,
- Section 4.4.4 discussed the method of data collection,
- Section 4.4.4 discussed the data coding, and

- inferences in the retrospective analysis were supported by data triangulation and the literature review.

4.4.5.4 External reliability

External reliability refers to whether the same results will be obtained by a different researcher. The three phases of DBR that has framed the study has been clearly explained and linked to the aims and outcomes of the study. Every step of the preparation phase has been clearly documented. The various elements of the experimental phase were discussed. Table 4.2 shows how an *initial* HLT was constructed based on the researchers conjectures, expectations and interpretations. Table 4.3 shows how the HLT changed *during* the teaching experiment to support the learners' thinking and reasoning. The retrospective analysis will give a detailed explanation of the results. The reader can therefore track all the researcher's decisions and conclusions. Due to these documentations it is clear that the research is dependent on the participants and the conditions.

4.4.6 Selecting the learning activities

The fourth aim of the study (discussed in Section 4.1) is to design a HLT. Section 3.7.1 explained the typical cycle of a HLT. The learning activities in the HLT will be used to scaffold learners' ZPD so that models of increasing level can be constructed throughout the actual learning trajectory. Edelson (2002) notes that the learning activity is used to "teach mathematising competencies, create a record of progress, monitoring progress, communicating progress" (pp. 113-114). Although the learners are used to a culture of problemising mathematics, they had to learn the modelling process and modelling competencies through the teaching experiment. This section will summarise the learning activities in the HLT and the LT. Some of the activities might not be used in the LT. The HLT develops in the preparatory phase but is flexible and can change (Bakker, 2004; Edelson, 2002; Wang & Hannafin, 2005). The HLT guides the researcher. If he knows the current point of a learner's reasoning he can predict the path of a reasoning and predict a new learning goal. The learning activity will help achieve that goal. The goal might not be reached through that activity and he needs to reformulates a HLT. Table 4.2 gives an overview of the predicted HLT *before* the teaching experiment.

Learning activity	Instruments used to collect data	Conjectured starting points	Researcher's conjectured goals and expectations	Conjectured outcomes
Baseline assessment	Baseline assessment, table of baseline assessment results	Unknown, the baseline assessment revealed starting points	Learners should be able to write the rule for a linear number pattern (generalising)	Learners could identify and extend linear and some quadratic patterns (generalising)
Modelling session 1 Broken eggs (Appendix A1)		Known: Learners could identify and extend linear and some quadratic patterns	Learners should be able to identify the unknown quantity by generalising a rule for a linear pattern (generalising)	Learners could identify the unknown quantity by generalising a rule for a linear pattern (generalising)
Modelling session 2 Marcella's doughnuts (Appendix A3)		Learners can identify the unknown quantity by generalising a rule	Learners should be able to create a model for Marcella's doughnuts and work backwards to find the unknown quantity (adjusting)	Learners created a model for Marcella's doughnuts and work backwards to find the unknown quantity (adjusting)
Modelling session 3 Extended doughnuts (Appendix A4)		Learners created a model for Marcella's doughnuts and work backwards to find the unknown quantity	Learners should find the rule for Marcella's doughnuts and explain how the values are dependent on each other (generalising)	Learners could use and adapt the rule for another situation (generalising)
Modelling session 4 Consecutive sums (Appendix A7)		Learners can generalise and use and adapt a rule for another situation	Learners should be able to investigate patterns and generalise linear patterns (generalising)	Learners can generalise linear patterns (generalising) and they can form quadratic patterns using symbols (symbolising)
Modelling session 5 Pulling out roots (Appendix A10)		Learners investigate patterns and can generalise linear patterns, they can form quadratic patterns using symbols	Learners should be able to identify and generalise quadratic patterns (generalising)	Learners can identify and form quadratic patterns using symbols (symbolising)
Modelling session 6 Squares (Appendix A6)		Learners can identify and form quadratic patterns using symbols	Learners should be able to identify and generalise quadratic patterns (generalising)	Learners are able to identify and generalise quadratic patterns (generalising)
Modelling session 7 Cutting through the layers		Learners can investigate patterns and can generalise linear and quadratic patterns	Learners should be able to investigate patterns and can generalise linear, quadratic	Learners investigate patterns and can generalise linear and quadratic patterns

(Appendix A12)			and other patterns (generalising)	(generalising) and can extend cubic and exponential patterns (symbolising)
Modelling session 8 The garden border (Appendix A8)		Learners can investigate patterns and can generalise linear and quadratic patterns	Learners should be able to identify linear patterns (symbolising)	Learners were able to identify and generalise linear patterns (generalising)

Table 4.2: HLT of the design experiment

Learning activity	**Instruments used**	**Starting points**	**Researcher's goals and expectations**	**Actual outcomes**
Baseline assessment	Baseline assessment	Unknown, the baseline assessment revealed starting points	To identify the learner's current points of reasoning (generalising)	Learners could identify and extend and find the general rule for linear and for some quadratic patterns (generalising)
Modelling session 1 Broken eggs (Appendix A1)	Baseline assessment, activity	Learners can identify and extend linear and some quadratic patterns	Learners should be able to identify the unknown quantity by grouping the eggs and then use multiples of 7 to find the number of eggs (symbolising)	Learners could identify the unknown quantity by extending the linear pattern (symbolising)
Modelling session 2 More broken eggs (Appendix A2)	Researcher observation guide, NPC continuum to assess learner's written work and audio recordings, video recordings of the group discussions	Learners can identify the unknown quantity by extending the linear pattern	Learners should be able to generalise a rule for the linear pattern (adjusting)	Learners could generalise the pattern by using $T_n = a + (n-1)d$
Modelling session 3 Marcella's doughnuts (Appendix A3)	Researcher observation guide, NPC continuum to assess learner's written work and audio recordings, video recordings of the group discussions	Learners can identify the unknown quantity by generalising a rule using $Tn = a + (n-1)d$	Learners should be able to create a model for Marcella's doughnuts and work backwards to find the unknown quantity (adjusting)	Learners created a model for Marcella's doughnuts and worked backwards to find the unknown quantity (adjusting)
Modelling session 4 Extended doughnuts (Appendix A4)	Researcher observation guide, NPC continuum to assess learner's written work and	Learners created a model for Marcella's doughnuts and work backwards to find the	Learners should be able to formulate a pattern and generalise a rule for the linear	Learners can generalise linear patterns (generalising)

	audio recordings, video recordings of the group discussions	unknown quantity Learners can generalise and use and adapt a rule for another situation	pattern (generalising)	
Modelling session 5 Thinking diagonally (Appendix A5)	Researcher observation guide, NPC continuum to assess learner's written work and audio recordings, video recordings of the group discussions	Learners investigate patterns and can generalise linear patterns	Learners should be able to identify and generalise quadratic patterns (generalising)	Learners can identify and form quadratic patterns using symbols (symbolising)
Modelling session 6 Squares (Appendix A6)	Researcher observation guide, NPC continuum to assess learner's written work and audio recordings, video recordings of the group discussions	Learners can identify and form quadratic patterns using symbols	Learners should be able to identify and generalise quadratic patterns (generalising)	Learners are able to identify and generalise quadratic patterns (generalising)
Modelling session 7 Consecutive sums (Appendix A7)	Researcher observation guide, NPC continuum to assess learner's written work and audio recordings, video recordings of the group discussions	Learners can investigate patterns and can generalise linear and quadratic patterns	Learners should be able to investigate patterns and can generalise linear, quadratic and other patterns (generalising)	Learners investigate patterns and can generalise linear and quadratic patterns (generalising)
Modelling session 8 The garden border (Appendix A8)	Researcher observation guide, NPC continuum to assess learner's written work and audio recordings, video recordings of the group discussions	Learners can investigate patterns and can generalise linear and quadratic patterns	Learners should be able to identify and extend linear and quadratic patterns (symbolising)	Learners were able to identify and generalise linear patterns (generalising)
Modelling session 9 Folding paper (Appendix A9)	Researcher observation guide, NPC continuum to assess learner's written work and audio recordings, video recordings of the group discussions	Learners can investigate patterns and can generalise linear and quadratic patterns	Learners should be able to identify and generalise simple exponential patterns (generalising)	Learners were able to identify and generalise simple exponential patterns (generalising)

Table 4.3: LT of the design experiment

The starting points in the HLT for the first learning activity is based on the baseline assessment. The results of the baseline assessment (Appendix B3) indicated that the learners could identify linear patterns and extend them. The first activity's purpose was to introduce the learners to the modelling process by giving them a realistic problem which resulted in a linear pattern. This is also in line with the curriculum guidelines for number patterns (see Section 3.4). The table gives the order of the activities based on the researcher's conjectured starting points, expectations and conjectured outcomes. Table 4.3 is the LT as it developed *during* the course of the teaching experiment. The table shows the starting points, the researcher's expectations and goals, and the actual observed learning goals that were attained during the learning activities. The instruments used during the modelling session are also listed, and the actual observed learning during the modelling sessions provided the starting points for the successive sessions. The first activity's purpose was to introduce the learners to the modelling process using a realistic problem which resulted in a linear pattern. The HLT in Table 4.2 and the LT in Table 4.3 are summative representations. The retrospective analysis in Chapter 6 will give a comparison of the LT and the actual learning outcomes, and a description of the associated LIT for number patterns.

4.5 HLT IN THE EXPERIMENTAL PHASE

4.5.1 A conjectured local instructional theory

Local instructional theories are developed during DBR. A feature of the DBR is its cyclic nature. Figure 4.1 represents several micro cycles that forms one macro cycle. A micro cycle represents a learning activity in the modelling session. The macro cycle represents the teaching experiment of the study. Each instructional experiment is preceded by a thought experiment. If the actual observed learning goal at the end of the instructional experiment does not confirm the conjectured learning goal, the conjectured LT needs to be adjusted or reassigned for the next instructional experiment. The new micro cycle commences after the reflective analysis where the new learning goals are conjectured. During the HLT each learning activity was selected to support the learner's reasoning at that specific point. The thought experiment was referred to as starting points in Section 4.4.6. The micro cycles form the macro cycles which develops into a LIT.

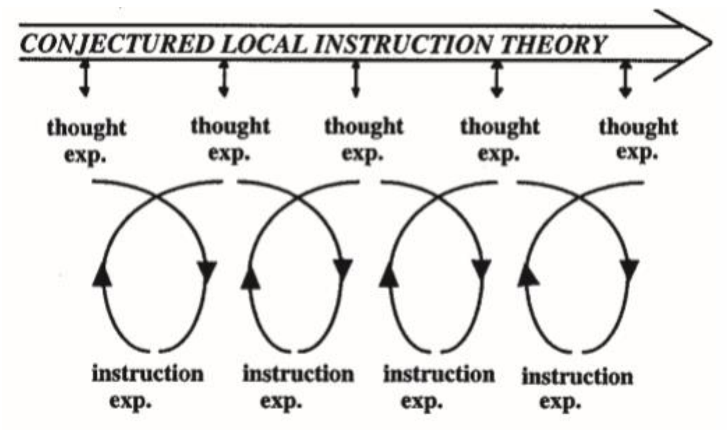


Figure 4.1: Local instructional theory and thought and instruction experiments (Gravemeijer, 1994, p.9)

Figure 4.2 is a diagrammatic representation of the first four micro cycles of the LT (from Table 4.3). It also incorporates the three phases of DBR within each micro cycle and the macro cycle (red ink). The macro cycle is referred to as the "encompassing learning trajectory" by Cobb and Gravemeijer (2008, p. 77).

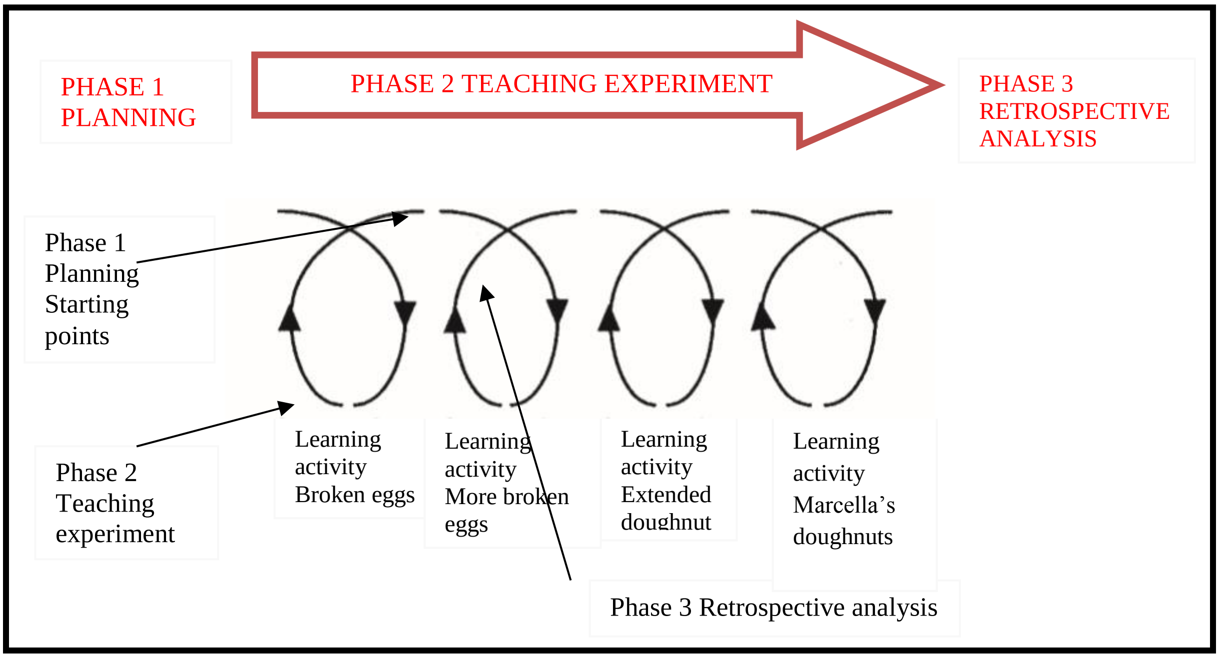


Figure 4.2: Macro cycle and micro cycles of the conjectured LIT (adapted from Gravemeijer, 1994, p.9)

The diagram (Figure 4.2) represents the conjectured LIT for number patterns.

4.5.2 Towards a local instructional theory

The aim of the study is to use a DBR framework and develop a LIT that is domain specific. Cobb and Gravemeijer (2008) discuss the issues that have an impact on the analysis and the findings of the research. Argumentative grammar refers to the logical reasoning used to analyse data. In Chapter 3 the goals for the study were developed using a phenomenological analysis. Through these goals mathematising competencies were developed. When learners collaboratively engage in mathematical modelling problems they develop mathematical modelling competencies. These competencies will be identified and explicitly described in the retrospective analysis. “The Hawthorne Effect refers to the fact that any intervention tends to have positive effects merely because of the attention of experimental teams to the subjects’ welfare” (Brown, 1992, p. 163). Brown rejects this argument because there is a relationship between the activities practiced during a design experiment and the improvements shown by participants’. The Hawthorne effect will be proven incorrect in the following ways: When the researcher plans and implements a teaching experiment, all the shifts in learners’ understanding will be documented as well as the means of support for the reasoning. Wang and Hannafin (2005) note that claims need to be evidence based. It is clear from Section 4.4.5 that the researcher is going to follow all the mentioned procedures to ensure the validity and reliability of the study.

4.6 SUMMARY

The aim of the chapter was to discuss the HLT in the experimental phase. The problem statement and aims of the study were reviewed and aligned with the research design. The review of the problem statement and aims of the study led to the development of the research instruments. The research instruments were designed to document the development of NPCs during the modelling sessions of the teaching experiment. The six methods of data collection were summarised in Table 4.1 and the role of the researcher for each method was explained. Section 4.4.6 explained the selection of the learning activities and how the researcher’s conjectured expectations and goals for the learners’ would be supported by a learning activity. Table 4.2 shows the first HLT and the conjectured learning path the researcher predicted before the design experiment. Table 4.3 shows the LT during the design experiment and how the actual outcomes for each learning activity served as the starting point for the learning activity in the next modelling session. The trustworthiness and repeatability of the study were explained by exploring ways in which the validity and reliability of the data analysis would be improved. Section 4.5 focused the chapter towards the development of a domain-specific, LIT. The retrospective analysis in Chapter 5 will produce a comparative analysis of the LT and the actual observed learning outcomes. The goals for each activity will be clarified and the function of the learning material will be described. This will form a domain-specific instructional theory based on a holistic three-dimensional goal description (Treffers, 1987). Through the development of a LT and hence a LIT learners will have the opportunity to learn the modelling process and develop modelling competencies to construct new concepts by reinventing mathematics. It is evident that a local instructional theory is suitable for a mathematical modelling perspective to teaching and learning.

CHAPTER 5
DEVELOPING A LOCAL INSTRUCTIONAL THEORY
PHASE 3: Retrospective analysis

5.1 INTRODUCTION

The purpose of this chapter is not only to produce a retrospective analysis of the collected data but also to provide a local instructional theory (LIT) for the topic number patterns. Chapter 2 gave an overview of the different modelling perspectives and provided a theoretical framework for the study. The RME framework together with a DBR paradigm provided the foundation necessary to effectively plan the teaching experiment in the first phase (Chapter 3) and guide the teaching experiment in the second phase (Chapter 4). According to Bakker and Van Eerde (in press, p. 21) two types of analyses are valuable in DBR, the comparison of the learners' actual learning during the different tasks in the learning trajectory (LT) and an overall more longitudinal approach. These two analyses will be explained in Section 5.2. In Sections 5.3 to 5.11 each learning activity will be analysed and explained in a rationale. The analyses and rationales form the basis of the local instructional theory which will be highlighted in Section 5.12.

5.2 DESIGN-BASED RESEARCH ANALYSES

5.2.1 Comparing the HLT and LT

The problem statement and aims of the study were explained in Chapter 1 and revisited in Section 4.2. The fifth aim of the study focuses on the retrospective analysis that will result in the LIT. Table 4.2 showed the conjectured path of learning based on the baseline assessment results before the design experiment and Table 4.3 gave a summary of the LT of the learners for the duration of the design experiment. Dierdorp et al. (2011, p. 139) note that it is not possible to identify learners' actual learning but it is possible to describe their observed learning. To coincide with the aims and objectives of the study, the retrospective analysis will deliver a detailed comparison of the LT and the actual observed learning targeting the conjectures and the researcher's final decisions based on the learners' reasoning. A data matrix analysis will be used for the comparison so that data collected from the transcripts, field notes and interviews can be used as evidence or counter-evidence to clarify whether the conjectured goals in the LT (see Table 4.3) is on par with the learners actual observed learning. When the conjectures of the learners' learning and correlated learning activities are compared with the learners' actual observed learning a conclusion can be reached as to how well the learners' learning was supported.

Table 5.1 is a data matrix analysis that shows the following information (Bakker & Van Eerde, in press, p. 22; Dierdorp et al, 2011, p. 139):

i. The first column is the learning activity the researcher used. It is noted that the activities in the LT (Table 4.3) will be used because these are the learning activities the learners worked on during the design experiment.
ii. The second column describes the task.
iii. The third column is the researcher's conjectured learning goals based on the baseline assessment in the initial activity and then based on the actual learning perceived in the preceding lessons.
iv. The fourth column provides excerpts from verbatim transcripts to deliver evidence which will support or contradict the prediction made in the third column. The coding during the data analysis was labelled: conjecture, transcript, clarification.
v. In the fifth column the transcripts will be clarified and explained based on the field notes and transcripts
vi. The sixth and final column is the result column. This column shows how well the HLT and the LT match. Three categories will be selected $(+, \pm, -)$. Dierdorp et al. (2011, p. 139) explain that two categories would be "too coarse for an evaluation and more than three categories would suggest more precision that can be justified". If the conjecture is confirmed for at least two thirds of the learners, a '$+$' sign will be allocated. If the conjecture is confirmed for a third or less of the learners a '$-$'sign will be allocated. For the confirmation of the intermediate group, a '$\pm$' will be allocated. If there wasn't enough information to assign one of the three categories, it would be left blank.

In the analyses the letters *A* to *Q* represent the learners in the class. The class is grouped into three groups. Group 1 is the focus group. Learners *A*, *B*, *D*, *F*, *I* and *M* are the learners in the focus group. *R* is the code denoted for the researcher which is also the teacher in the teaching experiment. Table 5.2 is a table that summarises the results of the data matrix analysis.

Learning activity in the LT	Task	Conjecture	Transcript excerpt	Clarification	Result
Modelling session 1 Broken eggs (Appendix A1)	Learners need to determine how many eggs the farmer had originally.	Learners should be able to identify the unknown quantity by grouping the eggs and then using multiples of 7 to find the number of eggs.	*D: Group 1 to group 7, then I grouped them in two's and three's and so on and leave the left ones over* *B: Wait, so won't it be on group 2? Wait, are they saying that in each group there is one left over every time?* *A: Group two, two eggs, group three, three eggs, four, four eggs. Can't you see? If you get groups of seven, loose eggs add them up, divide them and see that you get groups of seven.* *B: I thought of a number that can go into all of these numbers, I think it will be 301* *M: When you minus, can't we try numbers that go into 7?* *I: I figured out if it ends, takes a while but I am adding 7 the whole time*	5 out of the 6 learners draw groups with one left over in the groups of 2, 3, 4, 5 and 6. The group looks at numbers that could be divided by 7 evenly and tests them. They use decimals to check if their answers work: *A: For a 2, you need a comma five, for a 3 you need a comma three recurring* The group found that 301 was not the only number that worked: *I: Try 721* *B: It does work*	+
Modelling session 2 More broken eggs (Appendix A2)	Learners need to find more solutions to the problem. They need to generalise a rule to find more solutions.	Learners should be able to identify the unknown quantity by generalising a rule for a linear pattern (generalising).	*M: It's* $T_n = a + (n-1)d$ *A: Times the* d*?* *D: Minus* d*?* *A: It works* *M: It does* *A: What is this called again? General formula?* *D: Explain it to me* *I and M: It's* $T_n = a + (n-1)d$, d *is the difference*	The learners used multiples of 7 to find a number that works for the rules in the first learning activity. Each answer had to end in a one, e.g. 301, 721, 1141, 1561 etc. The learners find a constant difference of 420 when they subtract the numbers and *M* remembers the general formula	$\pm$

			A: Which is 420 *D: What is a?* *A: a is your first number* *D: This is 7* *A: No, n is the number you are trying to find* *D: Oh* *I: $T_n = a + (n-1)d$, so the difference is 420, 420 times n, 420times -1 which is -420. You try move them around so, so it's gonna be 420n-420 and you carry on from there* *R: Why are you using that specific formula?* *I: Because we are trying to find T_n* *R: Are you saying that this formula will work for any pattern?* *A: No, we are just trying to see something*	for linear patterns. Three of the six learners confidently use the general formula ($T_n = a + (n-1)d$) for linear number patterns.	
Modelling session 3 Marcella's doughnuts (Appendix A3)	Learners need to figure out how many doughnuts Marcella originally had by working backwards.	Learners should be able to create a model for Marcella's doughnuts and work backwards to find the unknown quantity (adjusting).	*A: Can we make an equation of this, 'cause we don't know what the number is* *B: So she says half so it's x over 2, for each of you so then plus two* *A: You guys, I just thought of something! Why can't you have like a pie thing?* *D: Pie chart* *A: How many parts are there?* *I: Can't you like do percentages?* *D: How?* *I: Like 100% and then you broke it in half and then you have 50% and broke it in half and then you are left with something* *I: Guys, I started working backwards, because the only way to figure this out is 8 divided by 2 minus 2, that's the only way that I think there*	*A* and *B* try to construct an equation to represent that half of the doughnuts are taken and then two more. Learner *A* discuss an idea to use a pie chart with the group. *A*'s pie chart idea gives learner *I* the idea of working with percentages. Although this doesn't work it gives them the sense of the values they need to work with and the process of Marcella's doughnuts. Learner *I* realises that they need to work backwards to find the answer.	+

			is 92 thingies, because I divided by 2 minus 2, give us 44, divided by 2 minus two, 20 divided by 2 minus 2, divided by 2 minus 2 *D: Please explain?* *B: Write it down* *I: I can't write it down. I started working backwards, I tried to figure out the number, if you would actually divided by two and minus 2 would be equal to two*	The evidence from the transcripts and written work show that the conjecture is confirmed for 5 out of the 6 learners in the group.	
Modelling session 4 Extended doughnuts (Appendix A4)	The learners need to construct a mathematical model and then use the model to generalise a formula to show how the number of doughnuts at the end depend on the number of doughnuts Marcella originally had.	Learners should be able to formulate a pattern and generalise a rule for the linear pattern (generalising).	*A: We chose number four, right. We decided that Marcella came back with 4 doughnuts instead of two* *A: If we find the original amount, it has to be 44, it only works with the 44 because it's a two. I'm saying two won't be a sixty, it won't work. So with the two it has to be a 44* *I: I think for two it's only 44 and for 4 it's only 60, so there should be a pattern somewhere there that is making them all cooperate together so I think we should start working with different numbers: 1, 2, 3, 4, 5* *M: Try every number* *A: Different original amounts, and that's how it's dependent on it 'cause it can't be any other number* *B: I think I've got it, the formula is 28 plus* *I: 28 is zero* *A: 28 plus n times 8* *R: Okay, how does this relate to the original question?* *D: Exactly it doesn't*	4 out of the 6 learners confirmed the conjecture that learners are able to formulate a pattern. The learners realise that each number at the end had a specific number at the beginning. They decide to use the rule for Marcella's doughnuts in the previous activity to find the corresponding values for the numbers at the end. 2 out of the 6 learners generalise a formula and show the group why they use 28 and 8 in the general term. After their explanation, there is evidence to show that 4 out of the 6 learners	+

			I: The difference of all of them is 8, the first one, the 0 is 28 so we got that *D: Why are we adding 28 again?*	understand the relationship with the 28 and the 8. From the transcript, *D* still can't relate the mathematical model to the real situation.	
Modelling session 5 Thinking diagonally (Appendix A5)	Learners need to determine the number of diagonals a 100-gon.	Learners should be able to identify and generalise quadratic patterns so that they can use the generalisation to solve a problem (generalising).	*B: I see a pattern* *I: What pattern do you see?* *Four sides then there is two diagonals, so multiply it by a half. There is five sides and five diagonals multiply by 1. There is six sides and 9 diagonals, so you multiply by three over two,* *I: Say that, multiply 9 by 3 over 2* *A: I am lost* *M: How did you figure this out?* *B: I just noticed that each time the difference and I wanted to find out how you multiply* *R: So what could you do with that in terms of finding the 100-gon?* *I: I'm guessing there is going to have to be a divide by two somewhere* *B: You'll say n minus 3 over two. So you will say 100 minus three over two and you get 57 over two* *M: 57 or 97* *D: That's 97* *B: Times 97 over two and that's how many diagonals it is*	The conjecture that learners will be able to identify and generalise a quadratic pattern is confirmed for 3 of the 6 learners. The learners use the general term to calculate how many diagonals a 100-gon has.	±
Modelling session 6 Squares (Appendix A6)	Learners had to determine how many squares a 40-stack square	Learners should be able to identify and generalise quadratic patterns (generalising).	*I: I've got a formula* *B: What is the formula?* *I: n times n plus one over two*	The learners recognise that there is a second constant difference similar to the previous question. They generalise a formula to calculate the number of blocks in a stack.	+

	has.		*D: You add a block each time to the set of blocks* *R: I see* *A: What is happening to be able to get to here* *I: I double them and I add one more, this is three, you double three and you add one more* *A: Yah, you add one more to each stack* *M: The median of 40, one and 40, which is 20 and 21, right* *A: Guys, it's the median times the number in the stack* *B: That's what I said* *A: The median, that's 20* *F: That's 20 and a half*	The learners relate the rule back to the blocks in the stacks and find that the $n + 1$ can be explained by the block that is being added to the stack each time. The group works with the numbers and find that if they multiply the median of the stack number by the stack number it gives them the number of blocks in the stack. The conjecture is confirmed for all the students.	
Modelling session 7 Consecutive sums (Appendix A7)	Learners search and identify patterns (linear or quadratic or other) and generalise these patterns.	Learners should be able to investigate patterns and generalise linear, quadratic or other patterns (generalising).	*B: Did anyone notice that if you take the median of the numbers that are in it and you multiply it by the number of digits in the sequence, it gives the final answer* *R: So does that mean I can take a number, let's take 135, can I find the five consecutive numbers?* *A: You can't* *D: You can, you divide it by five, and then how many numbers is there in succession?* *R: How many numbers do you want there to be?* *M: Six* *B: You'd divide it by 6*	Learner *B* recognises a pattern and generalises a formula by using mathematical knowledge gained in the previous modelling sessions. Using this rule, the consecutive numbers can be calculated by dividing the value by the number of consecutive numbers in the series. This gives them the median number and they can add the preceding and succeeding numbers in the series. In the post-discussion (see Section 5.9.1.6), the learners find that not all numbers can be written as consecutive sums.	+

<table>
<tr><td></td><td></td><td></td><td>R: Let's do it
M: 135 divided by 6, 22 comma 5
D: Oh I get it
F: Oh I get it now

I: If there is five numbers here, five times five plus one over two, 10. Three numbers there, 3 times 3 plus one over two, 6.

I: If you started at one, what if there are 7 numbers and you started at 2? That's the problem. Let me try though.
M: What are you trying to do now? If you're starting with two, and the rule is to start with one
I: Just wait, we're on to something, oh I got it, seven plus seven times seven plus three over two
A: It's 35
M: So three has to starts with n+5?

A: So the plus thingy changes all the time to an odd number?
I: Yes</td><td>The previous activity's generalisation was used to find the sum of n numbers when starting at 1.

The group members discuss the possibility of starting at another position and reach the following conclusion: they will use $n+1$ when starting a series of consecutive numbers at one, $n+3$ when starting the consecutive numbers at two, $n+5$ when starting the consecutive series at 3. The translation will always increase by two.

The conjecture is confirmed for all the learners in the group.</td><td></td></tr>
<tr><td>Modelling session 8
The garden border
Task 1
(Appendix A8)</td><td>Learners determine how many tiles a 10m by 10 m a square garden border has.</td><td>Learners should be able to count the number of tiles in the garden border.</td><td>M: 36
D: Yah, somewhere around there
M: Is 36 the square?
B: 6?
I: Six times plus the ten

D: So you just counted to get the answer
B: Task 1 is 36</td><td>The conjecture is confirmed for all the learners in the group. The learners counts 36 tiles in the 10m by 10m garden border.</td><td>+</td></tr>
<tr><td>Modelling session 8
The garden</td><td>The learners change the dimensions of</td><td>Learners should be able to create a mathematical</td><td>B: I think it's 4n minus 4

B: Well we found T_n so that we can find any</td><td>The conjecture is confirmed for 4 out of the six learners. The learners find the rule $4n-4$ and learns how the other</td><td>+</td></tr>
</table>

border Task 2 (Appendix A8)	the square garden border and then calculate the number of tiles needed.	model and generalise linear patterns (generalising).	*number* *A: What is the* T_n*?* *B: 4n minus 4* *D: What is the* n*?* *B: The term*	group found the formula structurally (see Section 5.10.2)	
Modelling session 8 The garden border Task 3 (Appendix A8)	Learners change the garden into a rectangle and then determine the amount of tiles they will need for the garden border.	Learners should be able to create a mathematical model and generalise a linear pattern (generalising).	*A: We need to find the median of the two and then use the formula 4n-4, so 7 is the median of that and the median of 9 and 7 is 8* *A: Since the formula is 4n-4,* *M: Substitute the median* *A: Minus four, which is 36*	The conjecture is confirmed for 5 of the 6 learners. Thee learners explain how to find the number of tiles for a 9m by 11m rectangle. By finding the median, they essentially turn the rectangle into a 10m by 10m square and use the previous activity's generalisation to solve for the number of tiles.	+
Modelling session 9 Folding paper (Appendix A9)	Learners search for patterns by folding paper and need to figure out how many folds they have when they fold it infinite times.	Learners should be able to identify and generalise an exponential pattern (generalising).	*B: It's two to the power n and it works for all of them* *D: How did you get the power thing? Oh, I knew that* *F: Wait, what's n?* *B: It's the number of folds*	The conjecture was confirmed for 5 of the 6 learners.	+

Table 5.1: A data matrix analysis to compare the conjectures in the LT with the actual learning outcomes

+	×		×	×		×	×	×	×	×	×
±		×			×						
−											
Task	1	2	3	4	5	6	7	8.1	8.2	8.3	9

Table 5.2: Actual results compared with the conjectures for the learning activities in the LT

5.2.2 A three-dimensional goal description

Bakker and Van Eerde (in press, p. 4) note that DBR has an "explanatory and advisory aim, namely to give theoretical insights into how particular ways of teaching and learning can be promoted". In the Wiskobas project, Treffers (1987) developed a holistic three-dimensional goal description. The three dimensional goal description is formed by incorporating the instructional activties, learning activities and the instructional aids (Treffers, 1987, p. 187). The instructional activities are formed around leading questions, suggestions and working methods while the learning activities explore difficulties in the task and indicate slolution methods. Bakker and Van Eerde's explanatory and advisory aims will be met by developing a three-dimensional goal description approach for each learning activity.

Treffers (1987, p. 188) notes that the components of a holistic goal description are not isolated, which means that the aims and objectives are included in a description of the teaching and learning process. He adds that this description can be used with parts of transcripts of lessons to indicate important learning instants. Section 4.4.4 explained that video and audio recordings were transcribed. Transcripts were coded in terms of mathematising competencies that were developed in Section 3.5.2 and are located on the NPC continuum (Table 3.2). Evidence that confirms and does not confirm the development of mathematising competencies will ensure a more cyclic and longitudinal method of data analysis. It is important to determine the development of the learners' mathematising competencies as they work through the modelling problems. Colour coding was used to identify the mathematising competencies.

For each activity in Sections 5.3 to 5.11, the learning activity will be analysed. The analysis will be holistic and will outline the working procedures, specific difficulties in the activities and solution approaches. Each competency will be described using excerpts from the transcripts, learners' written work and explained using the NPC continuum. The retrospective analysis is therefore directly linked to the research question and aims of the study. A rationale for each activity will include the starting points for the activities, notes and reflections from the researcher observation sheet, and overall goals for the modelling sessions. The goals that will be addressed may include learning goals, teaching goals and activity goals. The task oriented analysis in Section 5.2.1 does not include the role of the teacher and learners but will be discussed in the following sections.

5.3 ACTIVITY 1

Broken eggs (Appendix A1)

5.3.1 Analysis of the learning activity

It is apparent from the transcripts in Table 5.1 that the conjecture that learners should be able to identify the unknown quantity by using multiples of seven was confirmed. Learners were able to identify the unknown quantity by grouping the eggs and then using multiples of seven to find the number of eggs. The task-based analysis provided an individual assessment of the focus group. The following analysis will use the NPC continuum (Appendix B6) to identify and explain the development of the group's number pattern competencies.

5.3.1.1 Internalising

The competency *internalising* is noted when the learner states the problem in language he understands. The researcher asked the group how they were going to go about the task.
D: We are going to put them in groups

Learner *A* stated the important information out loud: *Wait, so won't it be in group 2? Wait, are they saying that in each group there is one left over every time?*

The above question provides evidence of how the learner tries to understand the problem by using language she understands.

5.3.1.2 Interpreting

Interpreting occurs when learners make assumptions and specify important information. The learners talk about the different groups. *M* tries to see if they know how many eggs there were while *D* immediately starts drawing a diagram to show that there is one left over for the groups two through six.

M: Do we know how many eggs there were?

M: In group two there's two

D: There is two in a group, three in a group

B: Is there any order?

D: O, wait, see here. So there is two here, and then there is one left over, all of those three, add to group three, one left over, take it to group 5, and so on

The rules that are formulated in the above excerpt will be used in the next learning activity too.

Figure 5.1 shows the learner's real model. The learners correctly made the assumption that there is one left over for each of the groups of two, three, four, five and six.

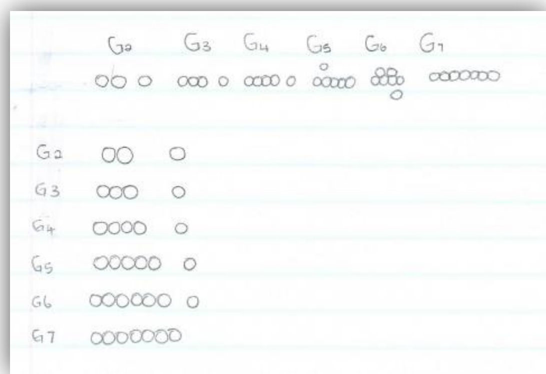


Figure 5.1: *D*'s interpretation of the real problem by setting up a real model

5.3.1.3 Structuring

The following excerpt shows that the learners notice a value in the problem that is important, in this case using groups of seven.

A: It works for two, works with three, only works with two

D: I've got 27 but it only works for four groups

B: You should use the information we have, should see what we get

A: We should use like multiples of seven, with each one

D: Okay

5.3.1.4 Symbolising

As mentioned in the data matrix analysis (Table 5.1), five out of the six learners in the groups illustrated actual groups of eggs in their real models, while *B* constructed a table to show the remainder of eggs after grouping:

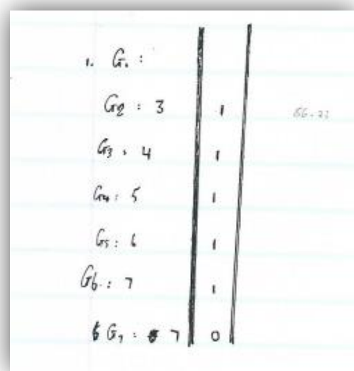


Figure 5.2: The learners' real models (*A* and *B*'s written work)

Four of the six learners wrote down all the multiples of seven, knowing that one of the multiples had to work for the rules given in the question.

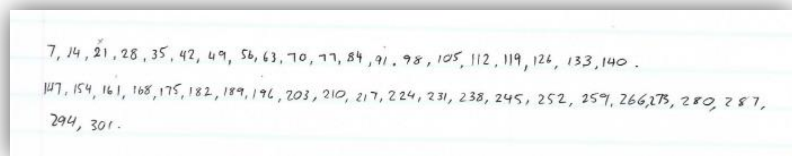


Figure 5.3: Learners write the multiples of seven (*B*'s written work)

5.3.1.5 Adjusting

The competency *adjusting* is noted when a learner tests his pattern. The learners notice that the number 301 works for all the rules. As the learners test more numbers, they notice decimal values they needed to get when dividing the number of eggs by 2, 3, 4, etc.

A: For a five, we need something comma 45, for a three, something comma three recurring, for a two, comma 5

R: How are you going

I: Getting there. There is a pattern, for a 2, you need to get a comma five, for a three you need a comma three recurring

The learners use the decimals as a rule to test for more numbers. Figure 5.4 shows how the learners formulated the decimal-rules.

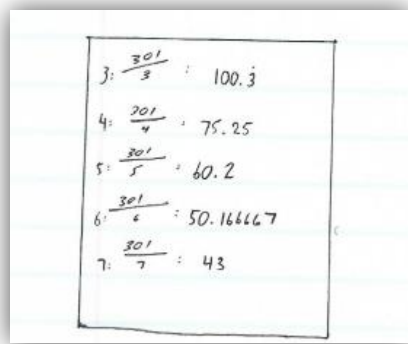


Figure 5.4: *B* writes the decimal values to help them to test solutions

Although the group does not generalise a single rule to calculate the number of eggs the farmer originally had, the learners still devise a set of rules that will enable them to work more efficiently.

I: (To *B*) *Try 721*

A: It does work

B: It works

The excerpt above indicates that 721 also works for their decimal-rules.

5.3.1.6 Organising

Organising can be noted when a learner constructs a rule that works for all the elements. The rule is validated by reflecting it back to the real problem. The next learning activity is designed to reveal evidence for this competency.

5.3.1.7 Generalising

The competency *generalising* occurs when the general terms are found deductively, and/or a rule is used for another situation. The next learning activity is designed to reveal evidence for this competency.

5.3.2 Rationale for the activity

The baseline assessment provides valuable information to indicate the starting points for the HLT. Learners were able to extend linear and some quadratic number patterns in the baseline assessment. With support they would be able to generalise linear number patterns. Keeping in mind that the learners need to learn the modelling process, the first problem had to be an authentic problem where the terms in the pattern was not evident. The first learning activity formed the foundation for the second activity. The learning activity was checked with the checklist for modelling problems based on RME principles and Lesh's principles for designing MEAs (Lesh et al., 2000). To give the reader an idea of the reasoning behind the principles, the first learning activity will be explained according to the principles listed in the checklist (Appendix C1). The learning activity involves constructing a structurally significant system because learners had to group the farmer's eggs following certain rules: When she puts the eggs in groups of two, there was one egg left over, when she put the eggs in groups of three, there was one egg over, the same thing happened when she puts them in groups of four, five and six but when she groups them in groups of seven, she ended up with complete groups. The second principle was the reality principle. The context was real and useful because the learners could imagine this happening; it was in their own personal knowledge and experiences. The self-assessment principle ensures that the learners are given enough information to judge if they have done enough. The learning activity was quite explicit in its instruction. The level principle questions whether the learning activity forms part of a progressive activity. The second learning activity is the extension of the first learning activity and forms part of a progressive activity. Although the learners found one solution, they tested more numbers. This proves that the task can be used in a higher level of activity. The language of the task was appropriate, this is apparent from questions the learners answered successfully in the pilot study and the baseline assessment.

The lesson started with a class discussion of possible ideas for starting points. In a modelling classroom the teacher needs to create and maintain a rich environment which stimulates active involvement of the learners and their own learning (Janvier, 1996, p. 453). The teacher which is also the researcher needs to therefore guard against sabotaging the learners' reasoning but rather give them the opportunity to formulate their own ideas and reasoning. Hein (1991) acknowledges teachers' difficulties concerning their need to construct meaning for their learners and their concern whether their learners' would be able to construct meaning on their own. This implies a radical change in the teacher's belief system. This goal was realised by using a reflection section on the researcher observation guide to keep track of the researcher's involvement during the modelling session, the questions the researcher asked and the answers the researcher provided.

During the first modelling session, the researcher noted not giving the learners enough time to work with new ideas.

Throughout the modelling sessions mini-interviews helped the researcher to gain insight about the learners' ideas and reasoning. The mini-interviews also provided the learner with the opportunity to reflect on his own ideas and reasoning. The following example is taken from the first modelling session:

R: So how are you going to go about this?

D: We are going to put them into groups

R: So what have you got here?

D: Group 1 to group 7, then I grouped them in two's, three's and so on and leave the left ones over

A: Wait, so won't it be on group 2? Are they saying that in each group there is one left over in each time?

D: Are they saying that if you have two here and two here, makes sense what happens to the eggs. Shall we try it?

A: Let's try it

The researcher gained insight into learners *A* and *D*'s internalising processes. Learner *D*'s initial idea that she shared with the researcher became a discussion between *D* and *A*. Learner *A* stated the question in her own words. After the reflection of their idea the learners formed a real model by grouping the eggs.

The first learning activity's purpose was to introduce the modelling process to the learners. Blomhøj and Kjeldsen (2006, p. 167) note that the modelling task will determine the success of the modelling experience. The modelling task needs to be valuable, authentic and real. The intention of giving the learners a real problem that didn't have an obvious pattern was to reveal and develop the learners' horizontal competencies in the NPC continuum.

In the learning activity, learners were given the opportunity to group values according to rules so that they can find a number that works for all the rules in the real problem. The researcher allowed the learners to test and revise until they found a solution that they were satisfied with. In the first learning activity the learners were able to identify the unknown quantity by grouping the eggs to show that the groups two through six had one egg left over each time except for the seven groups which had no eggs left over. They found 301 to be the number of eggs the farmer originally had. The duration of the first modelling session was 90 minutes.

5.4 ACTIVITY 2

More broken eggs

5.4.1 Analysis of the learning activity

5.4.1.1 Internalising

The competency *internalising* is noted when the learner states the problem in language he understands and notes a similar problem. The group realises that the second activity is a continuation of the first. The learners understand that they need to find more numbers that will work for the grouping rules.

D: We have already found 301

I: We need to find more, that is the problem

5.4.1.2 Interpreting

Interpreting takes place when learners make assumptions, specify important information or note conditions that influence the situation. Learner *A* and *D* discuss the numbers that can and cannot work.

D: So are you saying that if you add 103… So you're saying the number of what?

I: We don't have a pattern

D: It doesn't work

A: It does

D: It doesn't work

A: It does

D: No, it doesn't work, 320 plus 420 is 103

A: No it's 721

With the help of *A* and *I*, learner *D* realises that every time they added 420, the number worked for the rules of the farmer's eggs. Learner *D* asked a good question:

D: What's the lowest number that works?

I: It's 301

The lowest number that worked for the farmer's rules was indeed 301, but it could have been a good learning moment if learner *D* had questioned it further.

5.4.1.3 Structuring

The competency *structuring* occurs when the learner looks for a relationship or a pattern. Learner *I*'s objective was to look for a pattern. The group decides that they are going to start with the number 301. *B* notices that if he adds 420 every time, the answers work for the rules in the first activity.

D: We have already found 301

I: We need to find more, that is the problem

B: That there is a 420 difference every single time, the ones that work perfectly

After *B* notes the relationship, the researcher questions learner *A*:

R: How many solutions have you got?

A: Only one

R: The question said there is more than one. What do you think? At the moment you have only one number

A: Well we have four, mam

R: You have four? What were the other numbers?

A: First solution is we got 7 time 43 and then we added 420 each time

A stated the recursive relationship in words when she said they added 420 each time. The researcher suggested that the learners read the problem again to get more focused. This directive was to guide the modelling process so that the learners get used to working through the modelling cycle (see Section 2.4.5).

5.4.1.4 Symbolising

Two out of the six learners note down the pattern and show the recurring nature of the 420. In the previous activity, the learners found that 721 also worked for the rules for the farmer's broken eggs. The learners then added the difference of the two numbers to the 721 and saw that the next number, 1141 also worked (see Figure 5.5).

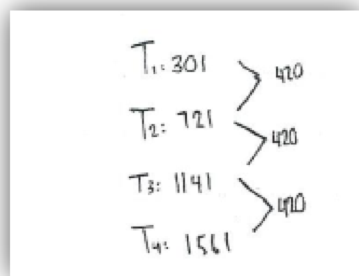


Figure 5.5: Learners show the relationship of the terms (*B*'s written work)

5.4.1.5 Adjusting

The competency *adjusting* occurs when the learner tests his pattern and refines it after testing it. It is interesting to note that learner *I* has expanded the term values in Figure 5.6. If she expanded further, she might have ended up with a rule to represent the functional relationship of the terms. In the above excerpt of the written work, it is evident that the learner has established a recursive rule to represent the relationship of the term values.

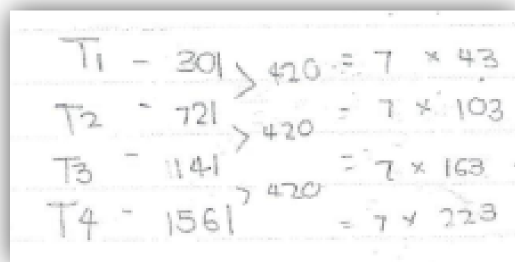


Figure 5.6: Learner *I* expands the terms

5.4.1.6 Organising

Organising involves formulating a rule that works for all the terms. The learners then remember a general formula that they were introduced to earlier in the year. They substituted the values and found that it worked for all the elements in their number pattern,

D: Explain it to me

I and *M: It's* $T_n = a + (n - 1)d$, *d is the difference*

A: Which is 420

D: What is a?

A: a is your first number

D: This is 7

A: No, n is the number you are trying to find

D: Oh

I: $T_n = a + (n-1)d$, so the difference is 420, 420 times n, 420times -1 which is -420. You try move them around so, so it's gonna be 420n-420 and you carry on from there

T: Why are you using that specific formula?

I: Because we are trying to find T_n

T: Are you saying that this formula will work for any pattern?

A: No, we are just trying to see something

5.4.1.7 Generalising

The learners formulate the general term $420n - 420$ by using integrating, a pre-internalised construct for this situation.

5.4.2 Rationale for the activity

The learning activity was checked and was in line with the checklist for modelling problems based on RME principles and Lesh's principles for designing MEAs (Appendix C1). The groups discuss what the task entail and possible ideas to help them towards a solution. The researcher then approaches each group to discover their ideas. The teacher should guide discussions in a non-directive but supportive manner (Burkhardt, 2006, p. 188). Teachers need to give students sufficient time to complete the activities as well as support during the modelling process. Guidance must be given strategically, remembering that learners need to develop competencies and sub-competencies on their own. Supplementary questions will enhance the learners' progress and lead them to go further. The questions that the researcher asked in this modelling session were: How many solutions have you got? What do you think? What are you doing? How are you guys going? Are you making progress? The questions the teacher asks can consciously steer the modelling process and motivate and support learners. The modelling session was 80 minutes in length, this included a representation of each group's solutions at the end where the class made valuable comments and suggestions.

It is important for the teacher to be knowledgeable about the modelling process. Before the modelling sessions started, the researcher explained the modelling process to the learners. Learners need to be aware of their expectations with respect to their different roles and actions during the modelling process. Learner *I* referred back to the problem throughout the activity, questioning why a certain method would work. Learner *G* has not yet made a contribution even though she was engaged in the activity.

During the learning activity learners had the opportunity to find more solutions that would yield the same result for the farmer's broken eggs. The conjecture was based on the fact that learners would be able to generalise a linear number pattern so that they could find more solutions. The learners found two solutions.

A: The second solution is the T_n

After investigating the transcripts and researcher's observation sheet, the learners incorrectly used the term solution rather than ways to find the different solutions. All six learners in the group confidently worked with a recursive relationship. Three out of the six learners confidently used the previously-internalised general term $T_n = a + (n-1)d$.

5.5 ACTIVITY 3

Marcella's doughnuts

5.5.1 Analysis of the learning activity

5.5.1.1 Internalising

From the NPC continuum, the learner is *internalising* when he states the problem in his own words, he notes or explains important information, he relates a previous similar problem to the current one. The group discusses the problem:

I: On the side of the road she sees two people, so that means minus two and then, um

M: Where is that?

B: And then it says some surfers, it doesn't tell you how many

A: But the two doesn't really help 'cause they took half the bag

I: No, for the people that were there, take one for each of them

B: So you might as well, and they took two more, minus two

I: After that they took another half, so that's another x over 2

B: And then Susan took half the bag again

A: So she gave the people from the homeless half the bag and then two more

For *A*, the people who collected food for the needy in her words became "homeless people". From the excerpt above, it is obvious that the group recreates the story as it happened.

5.5.1.2 Interpreting

For the *interpreting* competency the learner recognises quantities that influence the situation. The learners recognise that Marcella gives away half the bag and two more every time she stops. They calculate that Marcella gave away six doughnuts because she stopped three times and had two left over in the end:

I: For Susan, she had two more

M: She also took half and two more, then there were two left

D: So the twos are like 8 together, but like the halves are

A: (To *B*) *Is that why you've got x over two here?*

D: Because it's half and we don't know how many there is

The learners get a sum of eight because she gave away six on the way home and had two left over.

5.5.1.3 Structuring

Structuring is when the learner notes a pattern or a relationship.

R: What do you know?

A: She had more than 8

R: What else do you know?

A: She gave away half of her bags three times

R: What else do you know?

M: She's got two left

A: If you have half the bags three times and 8 okay, half of the bags three times. Can you say half of the bags three times?

M: I did that

I: No, you broke it half so you got another number, so you broke that in half and you got another number, and you broke that on half, and you get another number
M: It's like the same half every time

The researcher questions the group and they communicate that she gives half the bag away each time. At the end, *M* notes that is the same half every time.
A has an idea: *You guys! I thought of something. Why can't we have a pie thing?*

After *A* shares her idea of using a pie chart, learner *I* suggests that they should use percentages.
I: Like 100% and then you broke it in half and then you have 50% and broke it in half and then you are left with something
A: Okay so we start with 100%, so she gave away half, which is 50

The learners seem to forget that Marcella also gives away two more the three times she stops. After investigating, the researcher discovers that they did not forget about the two that Marcella gives away each time.
*R: (*To *B) What have you got written down here? Looks quite interesting, problem is there is not 8 doughnuts left*
B: In the story 8 doughnuts are given away

And later *A* noted:
A: Can't we add it in the end?

5.5.1.4 Symbolising

Symbolising is when learners use object or symbols to show their relationships or patterns.

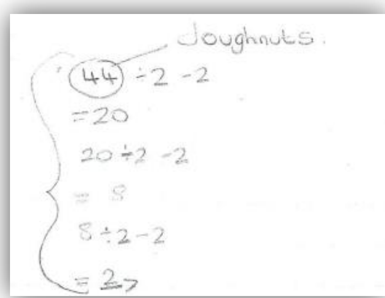


Figure 5.7: Learners show the relationship of the terms

In Figure 5.7, learner *D* shows how half of the doughnuts were given away and two more. This process happens three times because Marcella stopped three times.

5.5.1.5 Adjusting

In the competency *adjusting*, patterns are adjusted so that they make sense for the situation.

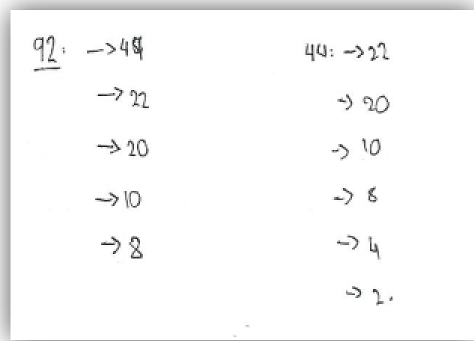


Figure 5.8: Learners use the rules from the real problem

In Figure 5.8 learner *B* tests his model with ninety two and calculates there is eight doughnuts left after Marcella gives away half of the doughnuts each time and then two more. He then tests forty four and finds an answer of two. The learners established a set of rules for the real problem and used these rules to find values that relate to the real problem.

5.5.1.6 Organising

The learners did not construct a single rule that worked for all elements.

5.5.1.7 Generalising

The learners did not construct a rule deductively.

5.5.2 Rationale for the activity

The conjecture of the previous learning activity was not confirmed with the observed based on the observed outcomes in the transcripts and written work. This provided the researcher with a challenge for the next activity. The options were to give the learners another linear problem or a quadratic problem. The baseline assessment's results indicated that the learners were not ready to generalise a quadratic number pattern, especially not this early in the learning trajectory. The next modelling task had to be a problem where the learners could associate with the activity to find deeper connections and not rely on $T_n = a + (n-1) + d$ which they had used in the previous activity. Although learners were given the opportunity to generalise the linear number pattern in the second learning activity, they were not able to. The learners had to be given the opportunity to generalise a rule for a linear number pattern.

Marcella's doughnuts problem was the first part of a progressive activity. The activity was checked against the checklist for modelling problems based on RME principles and Lesh's principles for designing MEAs (Appendix C1). A sign of the authenticity of the problem was when *A* mentioned: *You guys this is crazy, already two people took half of her bag and two more!*

The self-assessment principle was evident in the following excerpt:

D: How do we know if we are right or not?

A: I don't know

B: It will be a whole number

The modelling session was 90 minutes in length. The learners had the opportunity to discuss their initial understanding in their groups. The question was very wordy and the learners had to really take their time to read it a couple of times. The researcher's self-reflection was positive. The questions asked by the researcher were non-directive and the questions the researcher answered were thought-provoking rather than thought-directing. The class discussion at the end was positive. The learners were able to successfully create a model for Marcella's doughnuts and work backwards to find the unknown quantity.

5.6 ACTIVITY 4

Extended doughnuts

5.6.1 Analysis of the learning activity

5.6.1.1 Internalising

The competency *internalising* is noted when the learner states the problem in language he understands. The learners have a chance to communicate some ideas of what the question might mean in the class discussion. The researcher asks the learners what the questions might ask. Learner *C* (Group 3) says that the question is asking for a formula.

The teacher leads the discussion:

R: Okay, what do you think, do you also read formula when you read this question?

E: I think that the question is saying we cannot work out how much she had in the beginning if we don't know what she has in the end

R: Okay, so what you just said is that we know what is happening with one number to know what is happening with the other. So what is happening with the one number to get to the other number?

I: From the solution downwards it's solution divide by two minus 2, but from the thingy, the number of when she gets home it's times and then you add 2

The class then discuss what would happen if the end number would change.

Q says: *Mam, would it be good or bad to change the answers? If you are changing the answers, then the original ones would have to change, I don't see a problem with it.*

R: If she changes the amount that she comes home with?

A: It just means the original numbers would be changing

T: How would they be changing?

I: You are still using the same formula, but different numbers

The learners internalised the activity of how the solution to Marcella's doughnuts depend on the number of doughnuts she has when she gets home. They understand that there is a formula that will stay the same even though the numbers change.

5.6.1.2 Interpreting

The competency *interpreting* focuses on smaller details that impacts the situation. The conversation in the group is focused around the set of rules they used in the previous learning activity.

D: Plus two times two, minus two, divide by two. So two is part of the main to get the answer. So that's how it will be dependent on, 'cause if you say for times three you are not going to get the same thing

M: And that will be 8

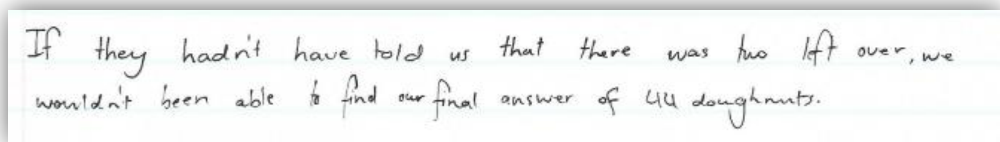

If they hadn't have told us that there was two left over, we wouldn't been able to find our final answer of 44 doughnuts.

Figure 5.9: Interpreting the real problem

From the above excerpts it is noted that the two is an important quantity that influences the situation. Figure 5.9 is evidence that the learner finds a connection between the 2 and the 44.

5.6.1.3 Structuring

Structuring involves looking for a relationship or a pattern. The group's focus changes to finding a pattern.

A: If we find the original amount, it has to be 44, it only works with the 44 because it's a two. I'm saying two won't be a sixty, it won't work. So with the two it has to be a 44

I: I think for two it's only 44 and for 4 it's only 60, so there should be a pattern somewhere there that is making them all cooperate together so I think we should start working with different numbers: 1, 2, 3, 4, 5

M: Try every number

A: Different original amounts, and that's how it's dependent on it 'cause it can't be any other number

The learners find that the difference between the original numbers, when they changed the end values has a constant difference of eight.

5.6.1.4 Symbolising

The learners use the set of rules from the previous activity when Marcella only had two doughnuts left when she arrived home. They calculate the values if she had different numbers of eggs left.

B's working (left in Figure 5.10) shows the original values if Marcella had three, five, six and seven doughnuts at the end. *B* writes out how many eggs there were when Marcella got home with the corresponding number of eggs she originally had (right in Figure 5.10). Learner *M*'s questions lead the group into the right direction:

M: What happens to zero to get to 28? What happens to one to get a 36? What happens to 2 to get 44? What happens to 3 to get 53?

D: We are adding two and timesing the answer by two. That's what we are doing

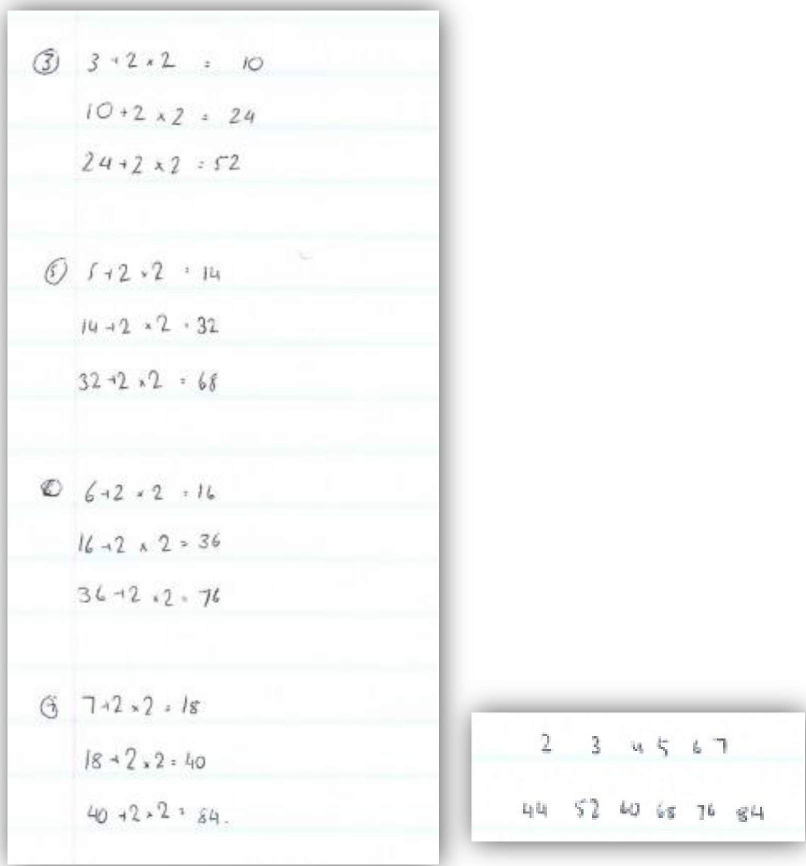


Figure 5.10: Symbolising (left) to find the terms of the pattern (right)

5.6.1.5 Adjusting

In the following excerpt, the learner tries to formulate a rule by relating numbers from the real problem:

A: Since she stopped three times we can make it like a rule, you'd be like two plus two, like two squared so we can shorten the thing

M: I put the three in brackets but it still didn't work

M tested her rule but it didn't work for all the elements.

5.6.1.6 Organising

The group constructs a rule for the real problem that works for all the elements.

B: I think I've got it, the formula is 28 plus
I: 28 is zero

They then relate it back to the real problem

A: 28 plus n times 8

R: Okay, how does this relate to the original question?

D: Exactly it doesn't

I: The difference of all of them is 8, the first one, the 0 is 28 so we got that

The excerpt above shows that learner *D* cannot relate the computation to the real problem. There is also no observed evidence that learner *G* shows an understanding of the generalisation. Based on these results the conjecture is only confirmed for four out of the six learners (see Table 5.1).

5.6.1.7 Generalising

The group has generalised a rule for the formula using deductive reasoning (see Section 5.6.1.6). The generalisation is a structural analysis. They focus on the original number that had no eggs left over at the end and the eight eggs (six from stopping three times and giving two away, and two because Marcella had two left over). The question they had to answer was: How does the solution of Marcella's doughnuts depend on the number of doughnuts she has when she gets home?

The group's generalisation for the computational procedure reads:

B: Number she has left multiplied by 8 and then you add 28 to it

5.6.2 Rationale for the activity

The duration of the modelling session was 90 minutes and included a class discussion before, during and at the end of the session. The time allocated for class discussion during the session was because the groups struggled to formulate ideas. Although the learners looked for other values by changing the number at the end, the concept of dependence was new to them. Towards the end of the modelling session, learner *A* summarised the task:

Guys, we were just saying how you get to the original number.

The group interaction was particularly good during this modelling session. All the group members made valuable contributions. The researcher noted metacognitive strategies during the modelling session: Learners reflected often and related their symbolisations back to the real problem. It was noted in the transcripts of the audio recordings that the group referred back to the learning activity four times.

The learners worked through the modelling cycle three times (refer to Figure 2.1). The first cycle was noted in the beginning of the modelling session when learners tried to understand the task by relating the dependence question to changing the number of doughnuts at the end. They set up a mathematical model and found different answers for the different eggs at the end. They related their model back to the original problem and noticed amounts that would influence the situation. The second modelling cycle was on a higher level than the first, learners changed the number of doughnuts that were given away. They set up a mathematical model by using a third and giving away three doughnuts every time. The learners were trying to investigate other situations similar to Marcella's. In the third cycle learners used their model from the first cycle and focus on the importance of the value two. They wanted to set up a simpler mathematical model and used the constant difference of the values they calculated in the first cycle. The group noticed that the difference depends on the number of times she stopped. The learners set up a model that represented a shortcut for the doughnuts at the beginning from the number they had at the end. The learners validated their results and were satisfied with their solution. The task focused on the relationship of the values which led them to look for connections instead of relying on the general formula or T_n that they used in the third learning activity. The learners were given the opportunity to generalise a linear number pattern. Through the scaffolding nature of the learning activity, the learners were able to formulate a rule for the number pattern because they realised that they needed a shortcut to get to the original number of eggs.

5.7 ACTIVITY 5

Thinking diagonally

5.7.1 Analysis of the learning activity

5.7.1.1 Internalising

The competency *internalising* is identified when the learner notes important information. The learners decide to draw more polygons to see how many diagonals they have. They agree that to find the diagonals in the 100-gon they will need to find a pattern or a formula to help them.

M: (To *B*) *Isn't there a formula for this?*

B: We will have to find a pattern, here there is a two and here there is a 5

D: Okay, but there is 4 sides and 2 diagonals and then 5 sides and 5 diagonals. So what's that going to say

D: There is no link here, you think you should draw it?

The link is discussed in the next section.

5.7.1.2 Interpreting

D, *F* and *A* want to see how many pentagons they need to make a hundredgon,

F: 20 of these will make a 100

D: Wait, so just five times what then

A: Five times twenty, you guys

They discard the idea and decide to link the question to the interior angles in a polygon.

After they discuss the purpose of the formula $180(n-2)$, the group decides that calculating the angles in the triangle will not get them closer to calculating the diagonals for a polygon. The above explanation shows how the learners communicate different ideas that could work or not work for the situation.

5.7.1.3 Structuring

When a learner is *structuring* he is searching for a pattern or a relationship. The next conjecture the learners make involves the number of sides and the diagonals of a polygon. If the sides of the polygon is an odd number then the diagonals will equal the number of sides, and if the polygon has an even number of sides, then the number of diagonals is half the number of sides. The learners find the number of diagonals for a hundredgon:

A: A hundred divided by two has fifty diagonals

R: How did you find this?

A: Even number diagonals are half of it, for six it is 3, 8 it is 4, and therefore a hundred is 50

D asks a valid question: *How do we prove this?*

The learners try to formulate a pattern. They refer back to Marcella's doughnuts but note the conditions that will not work for this problem:

A: Remember the pattern we had, there was the Marcella-thing B came up with, do you think this will give us something like that pattern?

I: There is no constant difference, remember that time the difference was eight

The learners notice that the rule found in the previous activity would be related only to a pattern with a constant difference. They sketched more polygons to see how many diagonals each of them has in order to find a pattern.

5.7.1.4 Symbolising

Learner *D* makes the following suggestion: *I think we need to write the number of sides and diagonals and find the difference*

The learner's suggestion proved to be a useful one. The learners discovered that the first difference increased by one every time.

The group use their pattern to predict how many diagonals the next polygon should have and then tested it by drawing the polygon and its diagonals

A: Wait, why should 8 be 20?

B: Well then our ten must be wrong because the difference is always one extra, like there the difference is three, there the difference is four, there the difference is five, there we haven't done yet

I: Nine should be 27, let's try that out

B: Then draw a big octagon and a bigger nonagon, like do the size of a page then we can clearly see

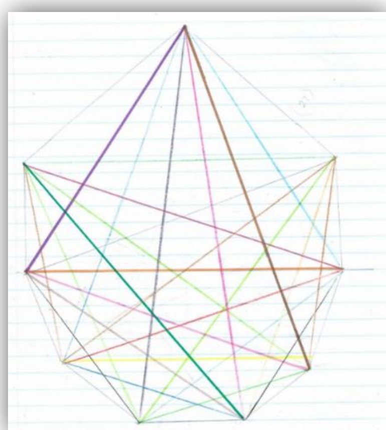


Figure 5.11: Counting the diagonals in a nonagon

Figure 5.11 shows the nonagon the group constructed to show that it has 27 diagonals.

5.7.1.5 Adjusting

Learner *D* notices a pattern by looking at the diagonals and the sides.

B: I see a pattern

I: What pattern do you see?

B: Four sides then there is two diagonals, so multiply it by a half. There is five sides and five diagonals multiply by 1. There is six sides and 9 diagonals, so you multiply by three over two,

I: Say that, multiply 9 by 3 over 2

Learner *A* is obviously confused and *M* asks learner *D* how he figured it out,

A: I am lost

M: How did you figure this out?

B: I just noticed that each time the difference and I wanted to find out how you multiply

The researcher asks the group if they can use their rule to find the number of diagonals in a hundredgon,

R: So what could you do with that in terms of finding the 100-gon?

I: I'm guessing there is going to have to be a divide by two somewhere

B: You'll say n minus 3 over two. So you will say 100 minus three over two and you get 57 over two

M: 57 or 97

D: That's 97

B: Times 97 over two and that's how many diagonals it is

In the discussion above, the competency *adjusting* is identified when the learner reflects back to the problem and uses the diagonals and sides to relate to the mathematical model.

5.7.1.6 Organising

In the competency *organising* the learner tests his pattern/rule and reflects back to the real solution. Figure 5.12 is taken from *D*'s written work, she uses the rule to show that a hexagon has 9 diagonals and nonagon has 27 diagonals.

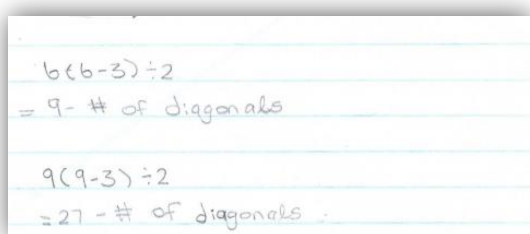


Figure 5.12: Rule to calculate the number of diagonals in a hexagon and nonagon

5.7.1.7 Generalising

The learners deductively generalise a rule for the quadratic pattern:

$n(n-3) \div 2$

$: 100(100-3) \div 2$

$= 4850$ diagonals

$\therefore$ The 100-gon will have 4850 diagonals.

Figure 5.13: The diagonals in a hundredgon (A’s written work)

The group used their generalisation to find the number of diagonals for a hundredgon.

5.7.2 Rationale for the activity

The modelling session was broken into two parts. The first part’s duration was 60 minutes where the learners had the opportunity to use their pre-knowledge to connect with the problem. The learners formed some conjectures based on incorrect assumptions. The learners only connected opposite angles in the polygon. They did not include all the diagonals in the polygons. This led them to the conjecture that if the number of sides of the diagonals is an odd number then the diagonals will equal the number of sides, if the polygon has an even amount of sides, then the number of diagonals is half the number of sides (see Section 5.7.1.3). The researcher noticed that some groups were not familiar with the notion of diagonals and used a discussion during the activity to let the knowledgeable groups help the other learners. The second part of the modelling session was 90 minutes in length. The researcher expected that the learners would take longer to generalise a quadratic number pattern than a linear number pattern. The use of technology could be valuable for learners to discover unfamiliar concepts. The learners spent some time drawing the diagonals searching for patterns. Section 5.7.1.5 explains how learner *B* found a rule by looking for a structural relationship. The learners in the group were interested to learn how the learner used the diagonals and sides to derive the generalisation.

The researcher noted metacognitive strategies that were observed during the modelling session: learners sketched polygons to count diagonals, they worked with different mathematical ideas (working with degrees) but discarded the ideas when the learners realised that it would not bring them closer to the relationship between the sides and the diagonals of the polygons. The researcher was impressed with the use of mathematical terms the learners used fluently (difference, nonagon). It is evident that the learners' mathematical proficiency is improving as they work through the modelling problems.

5.8 ACTIVITY 6

Squares

5.8.1 Analysis of the learning activity

5.8.1.1 Internalising

The learners relates the learning activity to the previous one. In the pre-activity discussion, *E* notes that: *it's the same question*.

5.8.1.2 Interpreting

Interpreting can be noted when learners make assumptions and specify important information. There is no evidence in the data to depict the competency *interpreting*.

5.8.1.3 Structuring

The learners look for a pattern by drawing in more stacks. Learner *D* draws a five-stack and discovers that it has ten blocks.

5.8.1.4 Symbolising

D starts drawing the 40-high stack.

M: This is going to be so huge

I: Are you actually drawing it?

Figure 5.14 shows *D*'s real model. It indicates the competency *symbolising* which is part of horizontal mathematisation.

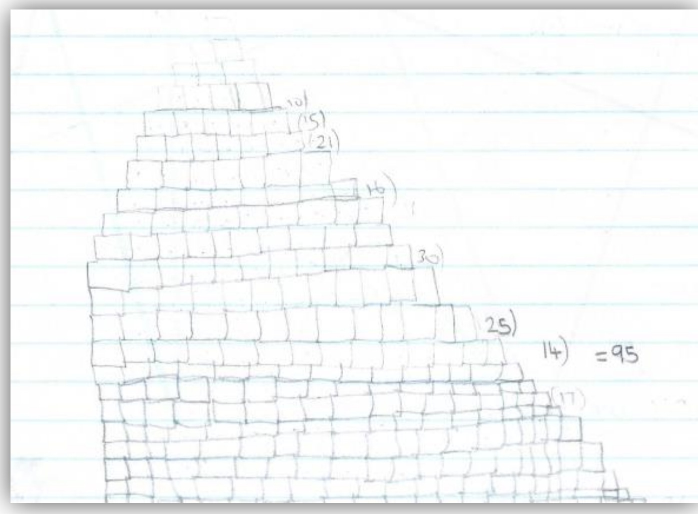


Figure 5.14: *D*'s symbolisation

Learners *M* and *D* draw the next stack to see that fifteen blocks will form a five-stack. In Figure 5.15 learner *B* predicts the amount of squares in the next terms using the recurring value.

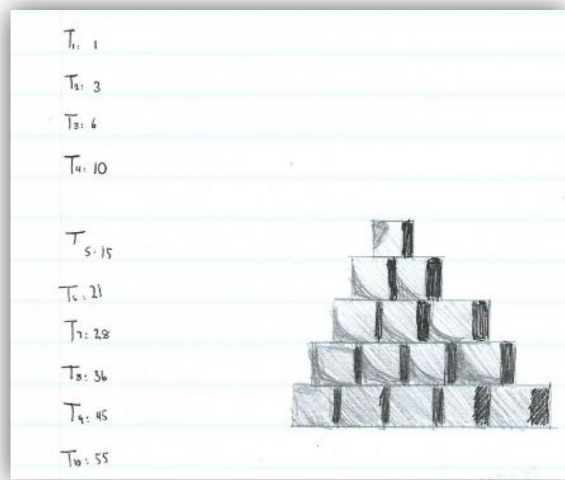


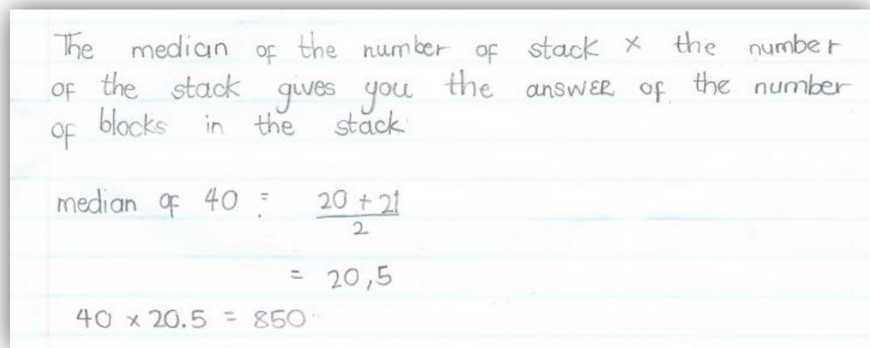
Figure 5.15: Predicting the stacks and squares (*B*'s written work)

5.8.1.5 Adjusting

The learners look at the numbers to find a rule that is a simplified version of the generalised rule $\frac{n(n+1)}{2}$. The learners relate the rule for the quadratic pattern to the real problem. Learner *D* explains it in terms of blocks in the set of blocks:

D: You add a block each time to the set of blocks

In the second generalisation, the learners invented the generalisation in Figure 5.16 which is the same as $\frac{n(n+1)}{2}$


The median of the number of stack × the number of the stack gives you the answer of the number of blocks in the stack

median of 40 = $\frac{20+21}{2}$

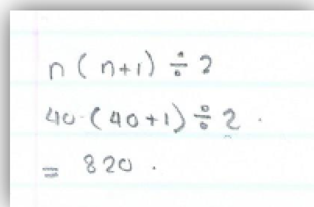
= 20,5

40 × 20.5 = 850

Figure 5.16: The number of blocks in a 40-high stack (*I*'s written work)

5.8.1.6 Organising

The learners generalise the a rule which worked for all the elements in their pattern: $\frac{n(n+1)}{2}$. They use the rule to find how many squares a forty-stack would have.


n (n+1) ÷ 2

40 (40+1) ÷ 2

= 820

Figure 5.17: The nth term to find the squares in a 40-stack (*A*'s written work)

The learners used the 40-stack values as an example and discussed ways of simplifying the process of multiplying 40 by 20 and a half.

D said: *n times the median of the,* and got interrupted by his group. The group later returned to this idea:

M: The median of 40, one and 40, which is 20 and 21, right

In the previous excerpt the learner explains that the midpoint between one and forty is between the values twenty and twenty one. The midpoint of twenty and twenty one is twenty and a half.

A: Guys, it's the median times the number in the stack

B: That's what I said

A: The median, that's 20

F: That's 20 and a half

5.8.1.7 Generalising

The learners use deductive reasoning to find the rule $\frac{n(n+1)}{2}$.

The rule they find afterwards, $median \times number\ of\ stacks$ is an astute way to incorporate the median value of stacks. In the next learning activity the learners use this rule for another problem.

5.8.2 Rationale for the activity

The modelling session's duration was 60 minutes. The learning activity aimed to develop the learner's internalising and structuring competencies by relating the question to the previous one and by recognising a quadratic number pattern. The observed mathematising competencies were more vertical than horizontal. Section 2.8.4 explored horizontal and vertical mathematisation. It was noted that the activities of vertical mathematisation do not necessarily occur in a specific order. When the NCP continuum was designed using Ellis' generalisation taxonomy in Section 3.4.3, the reflection activities which were compared with vertical mathematisation also did not take place in a specific order. The above explanation corresponds with the analysis in Section 5.8.1 because the vertical mathematising competencies (symbolising, adjusting, organising and generalising) were observed in no specific order and overlapped at times.

The learners related the pattern with the previous one within the first ten minutes of the session. Although the conjecture in the previous learning activity was only confirmed for three out of the six learners, the data shows that all the learners in the group learned from the previous experience. They successfully generalised the quadratic pattern. During the activity, the researcher noted the possibility to swap Activities 6 and 7 in a future LT. In retrospect, the learning activities proved useful in their current positions. During the discussion at the end of the session the groups shared their generalisations. The researcher used the fourth activity's wording for the current problem: *How does the stack*, and the class completed the sentence like a recitation, *depend on the number of boxes we have*.

5.9 ACTIVITY 7

Consecutive sums

5.9.1 Analysis of the learning activity

5.9.1.1 Internalising

The group attempts to understand the problem by discussing ideas to start the problem.

M: What do they want? I'm trying to get an idea

A: I have no idea

B: Well they want you to like pick any random number and then like start a pattern with consecutive numbers and then they want to see if we can distinguish a pattern as in n+1, n+2, n+3

M: Hm, any random number

B: It can be any number, then we can like find a pattern

M: So, find any number and find its consecutive numbers

B: So basically you have to find the nth term

The learners form consecutive series using their favourite numbers to start with. The learners allocate group members to construct series with specific starting values and specific number of terms.

5.9.1.2 Interpreting

The competency *interpreting* involves recognising quantities that would influence the situation. Learner *M* and *I* add five consecutive numbers in the following excerpt and notice a pattern.

M: I think it depends on the number you add, how many numbers we add up

I: It depends on how many numbers you add because this time the difference is five

5.9.1.3 Structuring

The learners begin to work more methodically by starting their series at different numbers but keeping the number of terms in the sequence the same. In *M*'s written work (Figure 5.18) she firstly adds five consecutive numbers starting at three and repeats this for four, five, six and then seven consecutive terms in each series. The group investigates series starting with the next number of the last series. *M* finds a constant difference of twenty five when adding five consecutive numbers starting at three, eight and thirteen. She does the same with four consecutive terms and finds a constant difference of sixteen between the sums.

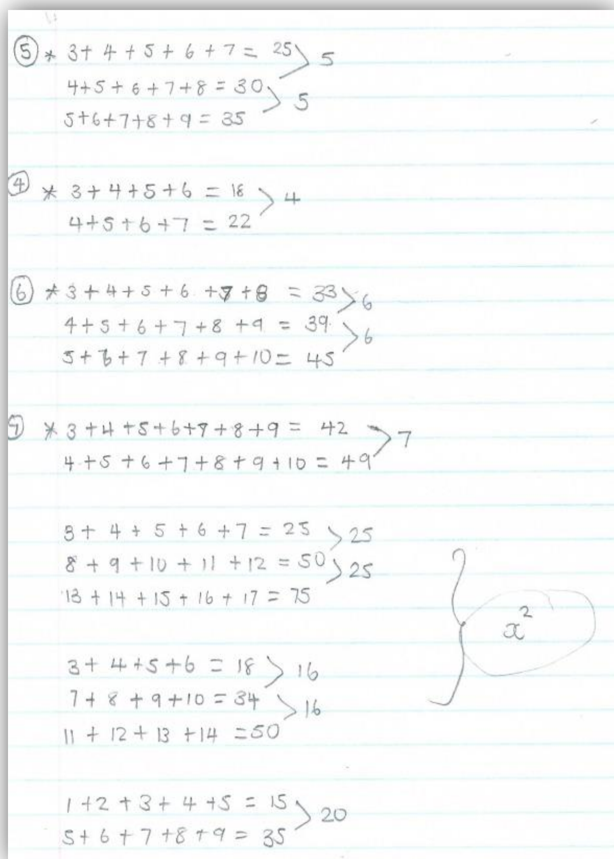


Figure 5.18: Setting up series to find patterns (*M*'s written work)

5.9.1.4 Symbolising

In Figure 5.18 the learner uses symbolisations to add consecutive numbers with the aim to search for patterns. The group use x^2 to describe the relationship between the differences of the sum of five consecutive numbers starting at three. The amount of terms squared equals the difference between successive sums. This also works when the learners add four consecutive numbers and find that the constant difference between the sums is four squared.

The learners formulate a rule using the median to work out the consecutive numbers,

D: If you know your median, you can work out the consecutive

A: No, if you like add them together

B: If you add the consecutive sums you will get 60

I: Consecutive numbers, numbers going this way and that way, is it consecutive?

B: Well if they grow after each other then they are consecutive

I: So you can work down and you can work the other way as well

The symbolisation in Figure 5.19 is a rule to find the sum of n consecutive numbers with the first term starting at one.

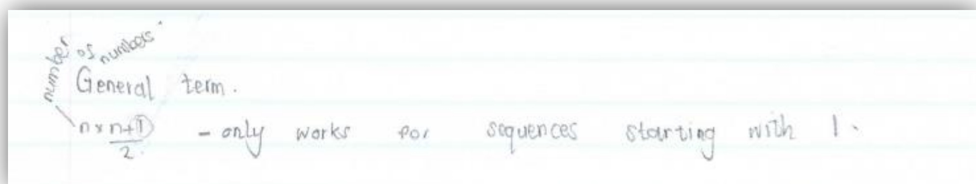


Figure 5.19: The nth term for finding the sum of n consecutive numbers (*I*'s written work)

Section 5.9.1.6 explains how learner *I* adapts the rule to determine the consecutive sum of a series starting from numbers other than one.

5.9.1.5 Adjusting

The learners test their generalisations. In Figure 5.20 the learners test their median-formula that the sum of a consecutive series can be calculated when the median number of a series is multiplied by the number of terms in the series.

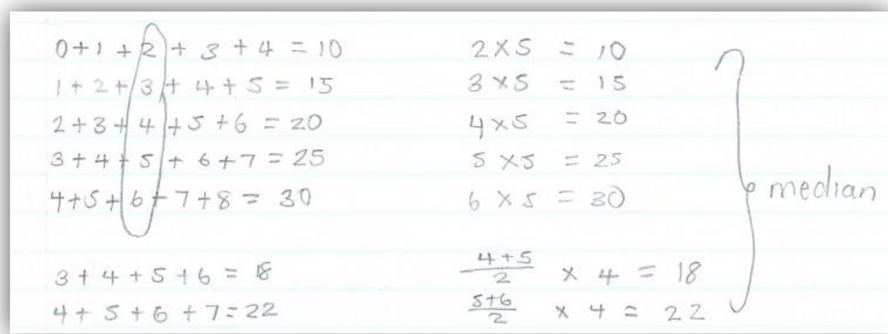


Figure 5.20: Testing the median-formula (*M*'s written work)

Figure 5.21 is an excerpt from learner *I*'s written work which shows that the sum is calculated in a series consisting of seven and five consecutive terms using the general term.

Figure 5.21: Testing the nth term (I's written work)

5.9.1.6 Organising

B notes: *Did anyone notice that if you take the median of the numbers that are in it and you multiply it by the number of digits in the sequence, it gives the final answer?*

The learners discuss how this rule would work when a sequence consists of an odd amount of numbers. They decide that they would find the midpoint of the two numbers by adding them together and then dividing the sum by two.

The group develops a rule to determine the consecutive terms of a sum that has a known amount of consecutives.

R: So does that mean I can take a number, let's take 135, can I find the five consecutive numbers?

A: You can't

D: You can, you divide it by five, and then how many numbers is there in succession?

R: How many numbers do you want there to be?

M: Six

B: You'd divide it by 6

R: Let's do it

M: 135 divided by 6, 22 comma 5

D: Oh I get it

F: Oh I get it now

The formula explained in the excerpt above will not work for all numbers because all numbers cannot be written as consecutive sums. During the post-activity discussion the group explain their formula to the class and confidently say that this formula would work for any number. They use 135 to explain their formula,

A: If we divide this by 6 we get 22 and a half. So our median is 22 and 23. So 23 and 22 will be

our middle numbers, and there has to be six numbers in the sequence, so it's, 24, 25, 21 and 20, is you add all of this together you will get 135
D: That's if you had to work with consecutive numbers and you were given the answer
E: I don't understand how you get to the 6 numbers, how would you know?
A: Cos you divide by six, the six tells you how many numbers there has to be
D: Yah
A: Even if you divided this by 4, you get the median thing, okay, 135 divided by 3, is equal to 45, so our median number is 45, and there has to be, so the number in our sequence has to be 3, so it's 46 and 44, if you plus all of these numbers together, you will get 135

The class asks the group to do one more example, to write fifty five as the sum of three consecutive numbers. *B* logically explains why it cannot work:
Your statement was that if you gave us 55, you had to find three consecutive numbers, and our group said that any consecutive numbers, what was it the median would, the number of numbers in the sequence multiplied by the median that would be your final answer, but 55 is one of those numbers that will never find a, what's it, constant pattern
R: So we can safely say some numbers you cannot write as consecutive sums. So it's important to always check your findings and to see if it fits for all numbers.

In the above excerpt, the learners constructed a rule that works for elements that have consecutive sums. They used their rule to solve a problem and provided the class with useful feedback when the rule did not work for all the numbers.

5.9.1.7 Generalising

The learners develop a formula to write the consecutive sums for values (if they exist). This rule was the same as the one they constructed in the previous activity. Learner *I*'s generalisation (Figure 5.21) is the same as the rule used in the previous activity to determine the number of blocks in a 40-high stack. In the previous learning activity it was used to find the sum of n number of consecutive terms starting from one. She generalises further:
I: If you started at one, what if there are 7 numbers and you started at 2? That's the problem. Let me try though.
M: What are you trying to do now? If you're starting with two, and the rule is to start with one
I: Just wait, we're on to something, oh I got it, seven plus seven times seven plus three over two

A: It's 35

M: So three has to starts with n+5?

Learner *A* summarises what needs to happen to the n:

A: So the plus thingy changes all the time to an odd number?

I: Yes

The task-based analysis in Section 5.1 and the explanation in the above sections demonstrate that the learners used the generalisations that were constructed in the previous learning activity and adapted it to for the current learning activity. Modelling activities provide learners with the opportunity to make meaningful connections. This does not happen in traditional classrooms and the learners forget what they learn.

5.9.2 Rationale for the activity

The learners had the opportunity to explore the concept of consecutive numbers in this learning activity. The analysis in Section 5.9.1 provided a detailed description of the 90-minute modelling session. They used the knowledge they constructed in the previous learning activities to generalise patterns successfully. Metacognitive strategies that were noted during the modelling session included: learners kept track of their progress, learners predicted an unsuccessful outcome and then followed with another idea. The group members allocated or volunteered specific tasks and used each other's series and the sums when they searched for patterns. The learners were able to share their patterns and generalisations at the end of the session. The post-discussion delivered important learning monents discussed in Section 5.9.1.6.

5.10 ACTIVITY 8

Garden border

5.10.1 Analysis of the learning activity

5.10.1.1Internalising

The learners use their own words to state the problem. In the first part of the modelling session, the learners try to establish the meaning of dimension and discuss whether the tiles will change or the border will change. *A* asks the researcher how the dimensions can change. The researcher formulates her responses so that it helps the learner's reflection without giving her an answer.

A: Mam, how does she change the dimensions of the garden?

R: How do you think you change the dimensions?

A: Make it bigger?

R: What do you think?

A: Well is it like add more tiles to the border?

R: Don't you think when we enlarge an area, you have to add more tiles to the border?

The group discuss the meaning of dimension. They use a dictionary to look up the meaning,

A: It's a measurement in length, breadth or thickness. An example is, the dimensions of the box are 20cm by 10 cm by 4cm

M: Is it like the length, breadth and height?

Once the learners are confident with the new term, the consider the dimensions 23 by 20:

B: You know it says 23 by 20 it doesn't mean that you multiply it, it's like 23 by 20

A: The tile wouldn't be square

B: I know

The above discussion provides a summary of what the learners regard important for second task: the meaning of dimension, increasing the border by adding more tiles and making the garden square.

In the third task the learners discuss what the question means,

A: I don't understand task 3, it's not really asking you a question

M: So it's like 8

D: Wouldn't you just double it?

I: It's not a square anymore

After the discussion the learners understand that the next task requires them to find the number of tiles in a rectangular garden.

5.10.1.2 Interpreting

Interpreting involves making assumptions, noting conditions that will work or not work for a problem, and recognising conditions that influence the situation. In the second task the learners discuss the possibility of a garden around a border with only one tile:

A: Guys, how can 1m by 1m be 0 tiles?

I: That's what we said at first but it doesn't

B: Tile one is 1m, one metre is a single line

A: That's what I'm trying to say, so it can't be 0 metres

B: It's zero tiles

The learners discuss how to change the dimension of the border. They can either change the tile size or change the amount of tiles in the border. They refer to the real problem and come to the same conclusion, they need to change the number of tiles and not the tile size.

B: It is only the amount of tiles that changed, you can't change the tiles

In the second task the square garden changes to a rectangular garden.

D suggests: *Okay there's 24 tiles here. If it was a rectangle, wouldn't you just double it? 'Cause it's metre by metre and it'll turn into two metres, so wouldn't it be 48 then? Or not?*

The learner makes an incorrect assumption, she does not take into consideration that the longer side of the border will be two tiles wider than the breadth of the border. The group refers back to the problem and creates a garden border that is two metres wide.

5.10.1.3 Structuring

The learners draw square borders in search for a pattern (Figure 5.22). They write down each dimension and the total number of tiles.

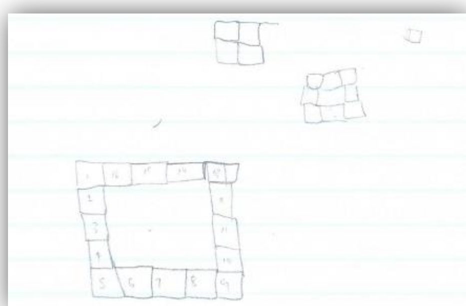


Figure 5.22: Constructing the real model of the square garden (*I*'s written work)

The learners write down the dimensions of the square borders and the sum of the tiles for each border. They note a constant difference of four between the number of tiles in the square garden starting with a one by one border.

Learner *I* finds a relationship between the square garden border in the second task and the rectangular garden border in the third task,
I: If it's a square garden, wouldn't the dim, the thingies be the same? Wouldn't this be a 6m by 6m wide and if it's a rectangle? And it's two metres wider, so that would be 8m

They use this idea to establish a rule that works for all the rectangular garden borders that are two metres longer than they are wide (see 5.10.1.6).

5.10.1.4 Symbolising

The NPC continuum shows that learners are *symbolising* when they use pictures or symbols to show the relationship of the problem. In Figure 5.23, the learners search for a relationship between the terms by using a real model. The diagrams indicate the length of each side without the blocks. They use mathematical symbols that would lead them to working independently from the real problem.

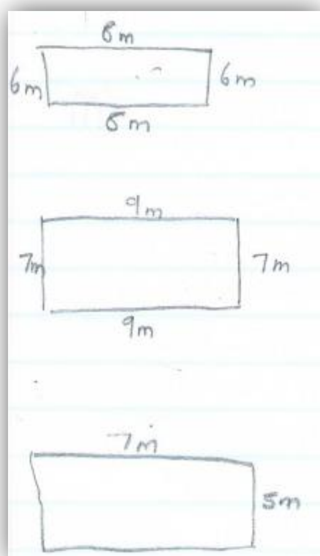


Figure 5.23: Searching for a constant difference (*M*'s written work)

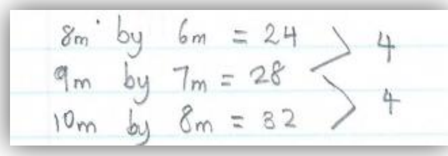


Figure 5.24: Stating the constant difference using symbols (*M*'s written work)

The learners find the sum of the tiles for each border and discover a constant difference of four (Figure 5.24). Learner *A* relates the constant difference to the previous task's constant difference.

A: If we only change the length of the garden and don't change the tile, we are pretty much going to go with this four difference thing all the time

5.10.1.5 Adjusting

During the post-discussion, Group 3 explain how they calculate the answer of the first task.

L (Group 3): We were given the garden with 10 metres, so we took 10 multiplied it by 4, subtracted 4 and found the answer

R: Why did you subtract four?

The learners explain that they multiply the ten tiles from each side by four because there are four sides in a square. Four of the blocks were shared so they subtracted four from the total. Samson explained this as local visualisation (see Section 3.4.2). In Figure 5.25 the learners write the general rule referring to the real problem in the second task. Group 1 does not explain the general term as clearly as Group 3.

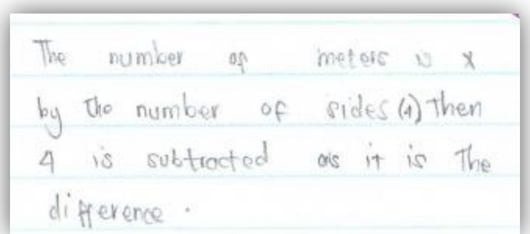


Figure 5.25: Relating the rule to the real problem (*D*'s written work)

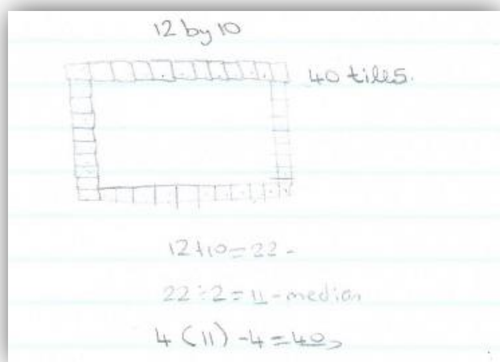


Figure 5.26: Testing the rule (*D*'s written work)

The competency *adjusting* is described as testing the pattern or rule. In Figure 5.26 the learner tests the rule by substituting eleven which is the median of ten and twelve into the rule $4n - 4$.

5.10.1.6 Organising

The researcher asks the group to explain their generalised rule for the second task,

B: We saw that it was four sides, so we had to use something like four, because a square has four sides

R: So are you saying that if you had a hexagon, that you would times it by six?

B: You could. But would you?

R: I don't know, that's the question I am asking you

R: Did your rule work for each and every

B and M: Yes it did

A: If we don't change the tile

R: What do you mean if you don't change the tile?

B: The dimension of the tile

The learners turn the rectangles into squares by finding the median of the length and breadth of the rectangular border's dimension:

I: I think it's the same formula but since it's 6 and 4, it's 8 and 6, you find the median. Really you do, because it's 7, so it's got to be 4 times 7 minus 4

After the group tests this idea, they formulate a rule for the rectangular garden border. The learners explain that the dimensions of the rectangles can be written as dimension of squares by finding the median of the sides of the rectangles,

I: It's more or less the same thing, it's just that you are working with two numbers. We find a median of them and we work with the same formula

The rule for finding the number of tiles in a rectangular square garden border will be $4n - 4$, where n is the median of the length and the breadth of the rectangular garden border.

5.10.1.7 Generalising

The learners derived a general term for the number of tiles in a square border. The rule to find the number of tiles in a n by n square garden border is $4n - 4$, n representing the number of tiles in one of the side borders of a square garden. In task three, the learners used the rule derived in task two for the rectangular garden in task three. The median of the rectangular dimension of a garden border results in the number of tiles in the dimension for a square garden border.

5.10.2 Rationale for the activity

The learning activity consists of three tasks that progressively build on one another. In the first task learners had to count the number of 1m square tiles in a 10 m by 10 m square garden. In the second activity the group had to generalise a rule to determine how many tiles the gardener would need if the square garden's dimensions changed. In the third activity the learners had to determine the number of tiles the gardener would need if the garden was rectangular and not square. The learners could imagine the garden and its borders. The use of building blocks or square counters may be useful for the learners to represent the real model. The use of technology (a drawing tool) may be useful. The learning activity's local and global visualisation possibilities (see Section 3.4.2) makes it a good learning aid to support learners if they have difficultiies to connect the real problem with a general term.

The learners found the use of a dictionary useful to define unfamiliar words (see Section 5.10.1.1). The learners went through the modelling cycle four times in the second task. The learner that directed the modelling cycle back to the real problem asked the following questions:

A: Do you think that if the tile isn't a square, would it make that border a square?

A: You guys, do we change even the length of the border? Only the tiles?

A: Do we only change the size of the tile?

In the data matrix analysis (Table 5.1) the conjecture was unconfirmed for learner A.

The analysis in Section 5.10.1 discussed the observed mathematising competencies throughout the 80 minute modelling session. As the researcher predicted, the learning activity provided the learners the opportunity to develop and reveal horizontal mathematising competencies. This modelling problem gives learners the opportunity to explore linear number patterns and is suitable in a LT's earlier learning activities. In retrospect, the researcher would replace this activity with a modelling problem that would result in a quadratic generalisation or a simple cubic generalisation. The reason for this change would be to develop and reveal vertical mathematising competencies.

5.11 ACTIVITY 9

Folding paper

5.11.1 Analysis of the learning activity

5.11.1.1 Internalising

The competency *internalising* is noted when the learner states the problem in language he understands. The learners decide to fold paper to see what happens to the regions. Learner *I* is not convinced when she asks: *Mam, what are we really supposed to do?* The group explain to *I* that they need to fold the paper to investigate how the regions change. If they fold the paper once, the paper is divided into two regions. If the paper is folded twice, they count four regions.

5.11.1.2 Interpreting

Learner *I* makes the following prediction:

I: Five will probably be 20

A: No, wouldn't it be 16 times 2? 'Cause 2 times 2 is 4, 4 times 2 is 8, 8 times 2 is 16, 16 times 2 is what?

I: 32

F: Yah

A: The number just keeps doubling

Learner *A* helps learner *I* by explaining why five folds will not have twenty regions. Learner *A* makes an accurate assumption: *So if it's two, there is no limit, it will be double folds infinity*

5.11.1.3 Structuring

The competency *structuring* occurs when a learner looks for patterns or relationships. Figure 5.27 lists the number of regions for the number of folds. The learners try to find a pattern by searching for a constant difference. They calculate the first differences of the values in Figure 5.27.

A: 2, 6

F: 8

A: 2, 4, 8, 16

F: It's not constant, that's the thing

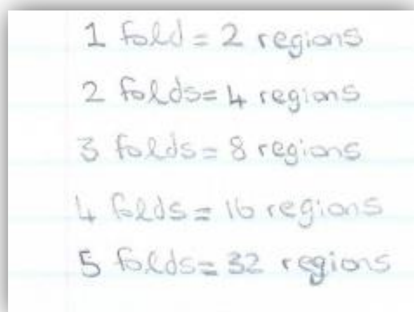


Figure 5.27: The learner notes the regions as she is folding the paper (*D*'s written work)

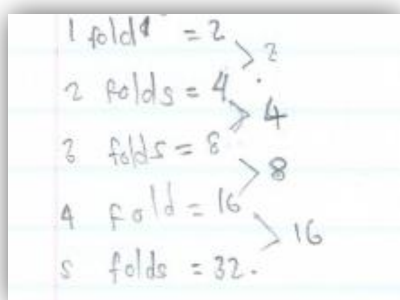


Figure 5.28: The learners search for a pattern (*A*'s written work)

The learners notice the that each value doubles to get to the next:

A: For every fold the previous number doubles

M: Doubles!

5.11.1.4 Symbolising

In Figure 5.28 the learners use mathematical symbols to show the relationship of the regions. Figure 5.29 shows the rule for the folding paper learning activity.

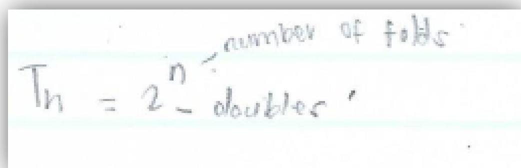


Figure 5.29: The nth term (*D*'s written work)

5.11.1.5 Adjusting

When the learners adapt their pattern to make sense for the situation and test his pattern, the competency *adjusting* can be noted. In the following excerpt, the learners relates the rule to the real problem:

F: Wait, what is n*?*

I: It's the thingy

B: The number of folds

In Figure 5.29 the learners indicate that the n in the nth term represents the folds and the two is used because the region is doubled with every fold.

5.11.1.6 Organising

The *organising* competency can be identified when learners test patterns or rules and reflects back to the real problem.

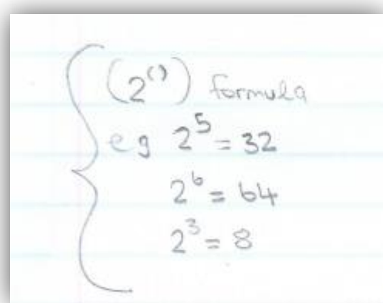


Figure 5.30: The learners test the nth term (D's written work)

In Figure 5.30, the learner tests the formula to see if two to the power of the number of folds gives them the regions they counted when they worked with the real problem.

5.11.1.7 Generalising

Learner *B* generalised an exponential pattern and tested it. It worked for all the terms.

B: It is two to the power n and it works for all of them

D: How do you get the power thing? Oh, I knew that!

5.11.2 Rationale for the activity

The learners had to fold a piece of paper. The first fold resulted in two regions. They had to investigate how the number of regions depend on the number of folds. The learning activity stated: Imagine there is no limit to the number of folds possible. The learners excitedly folded the paper. The analysis in Section 5.11.2 explained the mathematising competencies revealed during this 60 minute modelling session.

Learner *D* shared an experiment that he saw on a television program. The team of scientists folded a football-size piece of paper eleven times. They demonstrated that the myth of not being able to fold a piece of paper more than seven times was false.

D: They used a tape measure and everything to be precise and they used a piece of paper the size of a giant football field, it was insane!

http://www.youtube.com/watch?v=kRAEBbotuIE

The learners tried to find a constant difference (see Section 5.11.1.3) but once they discovered a constant ratio, they were able to generalise the rule. The conjecture was confirmed for five out of the six learners because no evidence in the data was found that learner *F* showed an understanding of the exponential generalisation.

5.12 THE PROGRESSION OF A LOCAL INSTRUCTIONAL THEORY

This section will explain the first steps of the progression from the RME theory to an attempt to a number pattern theory. Bakker (2004, p. 243) uses a reflective component and a prospective component to explain an instructional theory. Section 5.12.1 will present a reflective component from a RME theory that will address the research question and general results relevant to the instructional theory. Section 5.12.2 will discuss a prospective component that will form a theory for number patterns and will include suggestions for a number pattern theory at secondary school level (Year 8 to Year 11).

5.12.1 RME theory

The research question of the present study is:

How does the development of a local instructional theory influence learners' development of mathematising competencies when modelling number pattern problems?

Cobb and Gravemeijer (2008, p. 86) note:

> In a design experiment that concerns the development of a domain-specific, instructional theory, the goal is to develop an empirically grounded theory about both the processes of students' learning in that domain and the means by which this learning can be supported.

The above mentioned empirically grounded theory starts with an all-encompassing theory. The RME theory was selected for this study. The following section explains how the goal of developing an empirically grounded theory for number patterns has been developed in the study: This study has been formulated around a DBR in the wider context on RME and the Netherlanders' conception of it. A literature study on a mathematical modelling perspective to teaching and learning was used to explore the mathematisation process and the nature of horizontal and vertical mathematisation. A NPC continuum (see Section 3.5.2) was formulated so that the horizontal and vertical mathematising competencies could be identified during the modelling sessions. A HLT was designed and implemented in the design experiment to predict the learners learning goals and selecting learning activities to scaffold their learning. "The objective of the RME approach is that students experience formal mathematics no differently from informal mathematics" (Gravemeijer, 1999, p. 160). During the teaching experiment learners worked with the modelling process and constructed models. The learners' construction of models provided them the opportunity to reinvent formal mathematics. Their informal *models* *of* emerged to *models for* at a formal level when learners adapted and used their generalisations in other situations.

Chapter 5 provided a retrospective analysis which incorporated the two types of analyses that are valuable in DBR. The first analysis was a task-based data matrix analysis that was used to compare the HLT with the learners' actual observed learning (see Section 5.2.1). The second analysis was a longitudinal analysis to identify and explain the revealed mathematising competencies during the design experiment. Sections 5.3 to 5.11 presented a holistic three-dimensional goal description which formed the basis of the LIT.

Table 5.3 summarises the learning activities in the LT and the horizontal and vertical competencies that the learners revealed during the modelling sessions. The symbol '✓' indicates that the mathematising competency was revealed and the '×' indicates that the mathematising competency was not revealed during the learning activity. The results in Figure 6.1 are based on the analysis in Section 5.3 to 5.11. For the first and the third learning activity, the symbol '-' indicates that the task did not give the learners the opportunity to reveal the competencies *organising* and *generalising*. No evidence was found in the transcripts, written work or interviews to suggest that the competency was revealed. Table 5.3 indicates that the researcher's prediction of the learning goals were consistent with the actual observed learning outcomes.

	Mathematising competencies						
	Horizontal mathematising				Vertical mathematising		
Learning activities in the LT	Internalising	Interpreting	Structuring	Symbolising	Adjusting	Organising	Generalising
Learning activity 1 Broken eggs	✓	✓	✓	✓	✓	-	-
Learning activity 2 More broken eggs	✓	✓	✓	✓	✓	✓	✓
Learning activity 3 Marcella's doughnuts	✓	✓	✓	✓	✓	-	-
Learning activity 4 Extended doughnuts	✓	✓	✓	✓	✓	✓	✓
Learning activity 5 Thinking diagonally	✓	✓	✓	✓	✓	✓	✓
Learning activity 6 Squares	✓	×	✓	✓	✓	✓	✓
Learning activity 7 Consecutive sums	✓	✓	✓	✓	✓	✓	✓
Learning activity 8 The garden border	✓	✓	✓	✓	✓	✓	✓
Learning activity 9 Folding paper	✓	✓	✓	✓	✓	✓	✓

Table 5.3: Mathematising competencies revealed in the LT

The learning activities were used to scaffold the learners reasoning so that they could generalise more challenging number patterns in the next modelling sessions. The powerful modelling problems allowed for the development of mathematising competencies. The RME theory used in a DBR framework has resulted in the following outcomes for the study:

1. The learners learned the modelling process
2. The learners developed mathematical modelling competencies
3. The learners developed horizontal and vertical mathematising competencies
4. The learners successfully generalised linear, quadratic and exponential number patterns

5.12.2 A number pattern theory

Gravemeijer (1999) notes that the final outcome of a design study is to formulate an instructional theory based on "theoretical deliberations and empirical assessments" (p. 157). The instructional theory explains how the learning of a specific topic can be supported. The key features of a first level number pattern generalisation theory provides guidance for this support. The goal of the RME theory is to establish means by which learners can reinvent mathematics (Gravemeijer, 1999). This reinvention principle occurs when learners construct models through the process of mathematising. Table 5.4 outlines the key features of the instructional design for a number pattern theory by referring to the RME principles that will ensure emergent modelling.

Key features of the instructional design	**RME principle**
Learners are active participants in their learning	Activity principle
The curriculum is directed towards pattern generalisation: Learners give meaning to patterns in various kind of situations that are real to them	Reality principle
The curriculum contains a teaching strategy in progressive stages in which pattern generalisation is developed through model building: searching for and extending patterns (linear, quadratic and exponential), generalising linear patterns, generalising quadratic number patterns, generalising exponential number patterns	Level principle
Learners are given the opportunity to use pre-established knowledge and tools to solve problems	Intertwinement principle
Group and class discussions are used to share ideas and evoke reflection to construct meaning	Interaction principle
Learners are offered the opportunity to reinvent mathematics through guidance in the form of well-structured and selected learning activities	Guidance principle

in a LT	

Table 5.4: Key features of a number pattern theory

These key features can be noted throughout the learning activities' analyses and rationales. The planning phase of the teaching experiment, the development of the teaching experiment and the retrospective analysis have provided the reader with the basis of an instructional theory and has incorporated the RME design heuristics throughout.

During the phenomenological analysis in Section 3.2 the goals for the content of the study were formulated. The second goal of the study (see Section 3.4) was the notion of generalisation. The historical and didactical phenomenological analyses indicated that the process of generalisation is a problem area for learners. The South African mathematics curriculum provided teaching guidelines that focused on investigating and generalising linear number patterns in Grade 10 and quadratic number patterns in Grade 11. The LT curriculum therefore focused on a generalisation goal. The results from the study show (see Tables 5.2 & 5.3) that a mixed ability group of Grade 10 learners are able to generalise linear, quadratic and exponential number patterns when their learning is predicted and adequately supported throughout the teaching experiment.

The mathematical modelling approach to the teaching and learning of number patterns can successfully be integrated into a mathematics curriculum. It will ensure the development of sophisticated models, the opportunity to reinvent formal mathematics and create meaningful mathematical experiences. Table 5.5 gives an overview of the role of the suggested learning activities in an instructional sequence by referring to the progression stages (see Table 5.4) and topics of mathematical discourse noted from the teaching experiment.

Gravemeijer (1999, 2004) summarises that the local instructional theory is the whole instructional sequence, the general theory, and the framework against which teachers can develop a HLT to fit their classrooms. The suggested learning activities produced the first attempt to such an instructional sequence. The general theory was explained in the literature study, applied in the preparation phase and implemented in the teaching experiment.
The developmental nature of the study provides the teacher with a framework to develop a LT and guidance to successfully support learners' learning by following a mathematical modelling perspective. The requirements of LIT are evident in the theory that has been developed for

number patterns.

Suggested learning activities	Progression stages	Potential activity and mathematical discourse topics
Learning activity 1 Broken eggs	Searching for patterns	Grouping Using multiples of seven
Learning activity 2 More broken eggs	Extending linear number pattern Generalising linear number pattern	Many solutions follow the same rules Finding a shorter way to get more solutions
Learning activity 3 Marcella's doughnuts	Searching for patterns	Trying different mathematical strategies to solve the problem Backtracking to find the solution
Learning activity 4 Extended doughnuts	Extending linear number pattern Generalising linear number pattern	Dependence of one value on another The same process was repeated three times
Learning activity 5 Thinking diagonally	Extending quadratic number pattern Generalising quadratic number pattern	What is a diagonal? Absence of a constant difference Relating the diagonals to the shape's sides
Learning activity 6 Squares	Extending quadratic number pattern Generalising quadratic number pattern	Absence of a constant difference Relating the squares to the stacks
Learning activity 7 Consecutive sums	Extending quadratic number pattern Generalising quadratic number pattern	What is consecutive? Noticing relationships Different consecutives to be tested
Learning activity 8	Extending exponential number patterns Generalising exponential number patterns	Folding paper Using differences to search for a pattern

Table 5.5: Role of the suggested learning activities in the instructional sequence

5.13 SUMMARY

The retrospective analysis featured the two analyses that would result in important evidence for a

DBR study. A task oriented comparison was essential to compare the conjectured learning with the actual learning that was observed by the researcher based on the data. A data matrix analysis presented these comparisons and matched the accuracy of the predictions in Table 5.1. Table 5.2 showed a results summary. A longitudinal analysis was implemented in Sections 5.3 to 5.11. The analysis provided evidence for the learners' mathematising competencies as they worked through the learning activities. This analysis also contributed to the rationale for the modelling sessions. This analysis was in line with Treffers' (1987) aim of a holistic three-dimensional goal description: it provides informed guidelines to support teachers by giving a constructive analysis of the learning materials and didactics. Each selected activity in the LT has played a significant role in the learners' learning of the modelling process and the development of number pattern competencies.

The confirmation of the conjectures in Table 5.2, the development of the mathematising competencies for number patterns and the explanation of the activities in the analysis have formed the basis of the LIT. Section 5.12.1 explained how the RME theory was used throughout the study to form a theory for number patterns. The LIT for number patterns was discussed in Section 5.12.2. Chapter 6 will provide the conclusions for this study and will include the limitations of the study and recommendations for further areas of study.

CHAPTER 6
CONCLUDING REMARKS AND RECOMMENDATIONS

6.1 CONCLUSIONS

The study investigated the process of mathematisation in secondary school mathematics. The research question focused on developing a local instructional theory (LIT) for number patterns and its influence on the development of mathematising competencies. This investigation proved to be progressive in nature, building on elements to move from one step to the next.

A literature study to the mathematical modelling approach to teaching and learning mathematics showed that mathematical modelling offers the learners the opportunity to learn mathematics in a way that is meaningful. Mathematical modelling from **different** perspectives explored the **various** social, cognitive and emotional benefits that learners can experience when they are exposed to a mathematical modelling environment. When learners engage constructively in the mathematical modelling process they develop modelling competencies.

The process of mathematisation occurs through the activity of model building during the mathematical modelling process. Learners construct models so that formal knowledge and mathematical thinking emerge through sharable and reusable models. Mathematisation can be viewed as two identifiable but interrelated processes, horizontal mathematising and vertical mathematising. Horizontal mathematising can be explained as converting a real problem into a model of symbols. Vertical mathematising occurs when a learner organises mathematical symbols into a mathematical model to produce a mathematical solution. The study produced models for horizontal and vertical mathematising which inlcuded sub-competencies for each.

The RME theory was selected as an all-encompassing theory for the study, guiding the different steps towards the LIT. The goals for the study were selected by means of a phenomenological analysis so that the hypothetical learning trajectory (HLT) was in line with the RME theory's guidelines for an instructional theory: guided reinvention, didactical phenomenology and emergent modelling. Mathematising competencies were developed specifically for number patterns. The RME principles and principles to construct model-eliciting activities were used to develop criteria for a checklist to easily review the learning activities.

Seventeen learners participated in the teaching experiment. The learners were randomly grouped. The learners worked on nine mathematical modelling learning activities that were selected for the HLT based on the researcher's predicted learning goals. The aim of the activities was to support learners' reasoning and mathematical development. Mathematising competencies of the focus group were recorded and analysed and the classroom discussions provided valuable reflections and contributions.

The retrospective analysis had a three-fold function: it featured an analysis of the mathematising competencies, it compared the conjectured learning in the learning trajectoy (LT) with the actual learning that was observed by the researcher based on the data, and it provided the basis of the LIT. The learners' number pattern competencies were identified and explained. The horizontal mathematising competencies *internalising*, *interpreting*, *structuring* and *symbolising* were revealed hierarchically. The vertical mathematsing comeptencies *symbolysing*, *adjusting*, *organising* and *generalising* were revealed in no specific order and showed overlapping elements. The learning activities in the HLT supported the learners' learning and the learning of the modelling process. The learning activities in the HLT encouraged the development of number pattern competencies.

The RME's objective is to design instructional trajectories that provide teachers the support to form their own HLT (Gravemeijer, 2004, p. 107). The RME's principles of teaching and learning has been incorporated throughout the study; the planning of the teaching experiment, the implementation of the teaching experiment, the retrospective analysis and the formation of a number pattern theory. During the teaching experiment, the learners learned the modelling process while working through the modelling problems. The learners developed competencies because of their participation in the design experiment. When learners are given the opportunity to reinvent mathematics through a mathematical modelling perspective, mathematising competencies are developed that may have never developed in a traditional classroom.

6.2 LIMITATIONS OF THE STUDY

In mathematics education there has been a focus on mathematical modelling as a teaching and learning framework. In mathematics teaching methodologies a special focus has developed for mathematical modelling as a teaching and learning approach. The focus of mathematics education has moved towards a modelling perspective within education. The limited research done on mathematising competencies must be stated as a limitation of the study, although it has given the researcher the opportunity to develop horizontal and vertical mathematising competencies that are generalisable for different topics.

Only one pilot study was implemented before the teaching experiment. Biccard (2012) used two pilot studies to test and refine the research instruments and trial the learning activities. The pilot study undertaken in the planning phase of the teaching experiment fulfilled questions regarding the baseline assessment, learning activities, time frames and research instruments. The pilot's baseline assessment was discarded. The baseline assessment needs to provide the researcher with the learners' current mathematical reasoning. This information is essential for the LT. The teaching experiment's baseline assessment was an improvement on the first. The researcher often questioned: What does a baseline assessment look like for a mathematical modelling perspective to teaching and learning? Although the baseline assessment fulfilled its purpose of establishing the learners pre-knowledge for the starting points of the HLT, further experience with the modelling process might produce a better version.

Bakker and Van Eerde (in press, p. 13) note that pre-tests and post-tests are typically implemented before and at the end of the study so that the results can be compared. The purpose of the study was to investigate the development of mathematising competencies while learners were working through mathematical modelling problems. The retrospective analysis provided evidence that competencies developed. The researcher is yet to explain how a criterion referenced test (post-test) would be employed, perhaps to measure goals directed by curriculum standards. This leads to the question of assessment which will be discussed in Section 6.4.

This teaching experiment in the study was the researcher's second experience a mathematical modelling classroom. The researcher relied on past and current research to aid the planning and executing of the design study. The researcher selected learning activities based on conjectures and intuition. An outcome of this study is that the researcher can better judge which competencies would be revealed in modelling problems. This will be a valuable skill to take into the classroom and further studies in mathematical modelling.

The teaching experiment was completed twice a week over a period of four weeks (see Section 4.4.2). At least two learning activities were completed on the Saturday classes. This required the researcher to have a range of modelling problems prepared if the HLT had changed. A HLT changes when the researcher's conjectured learning goals do not match with the actual observed learning. The researcher was also left with a short reflection period in-between the modelling sessions.

The researcher was the teacher and single coder during the study. The researcher guarded against directing the learners' ideas and reasoning but let them instigate their own ideas and reasoning. This was enhanced by consciously reflecting on the researcher's practice and the aims for the study. Bakker and Van Eerde (in press, p. 23) note that "peer examination" of interpretations ensures the validity of the results. Strategies were implemented (see Section 4.4.5) to improve the validity and reliability of the study to eliminate this shortcoming.

6.3 SUMMARY OF CONTRIBUTIONS

This study has made multi-faceted contributions to the research on the teaching and learning of mathematical modelling focusing on mathematising competencies. Chapter 2 is a comprehensive literature study exploring the different modelling perspectives and the functions of each perspective. The educational and social benefits when adopting a mathematical modelling perspective in the classroom were highlighted. The modelling process and the development of mathematical modelling competencies have been pronounced in mathematics education research (see Section 2.4). The study shows that modelling competencies develop when learners engage in mathematical modelling problems.

The focus was narrowed to the mathematisation process. Models for horizontal and vertical mathematisation (Section 2.8.6) were produced. The models represent the competencies for horizontal and vertical mathematising. The horizontal mathematising competencies include *internalising*, *interpreting*, *structuring* and *symbolising*. The vertical mathematising competencies were *symbolising*, *adjusting*, *organising* and *generalising*.

The researcher developed a number pattern competency (NPC) continuum (see Section 3.5.2). The NPC continuum was used to identify the horizontal and vertical mathematising competencies revealed during the modelling sessions. Corresponding with Ellis' generalisation taxonomy, the horizontal mathematising competencies were revealed in a specific order while the vertical mathematising competencies were revealed in no particular order (see Section 5.8.2). The vertical mathematising competencies also displayed overlapping elements.

Research shows that average-ability learners can develop sophisticated models. This study shows that a heterogeneous group of learners can develop mathematising competencies during the development of models. The prerequisite of developing powerful models are: the teacher supports the learners' learning by predicting learning goals, the teacher uses quality modelling problems to support their learning, the teacher gives learners time to discover new ideas, and the teacher facilitates but *does not direct* the learners' thinking and reasoning by allowing them to reinvent mathematics.

The study delivered a LIT that can the incorporated into different year levels and class groups. Chapter 2 provides the teacher with background information relevant to the modelling classroom. Chapter 3 guides the teacher to plan a teaching experiment by setting goals, developing a baseline assessment and selecting modelling tasks. A checklist to judge the quality of a modelling problem was devised in Section 3.7.3. Chapter 4 introduces the development of a LT and shows how it can be adapted and changed to suit a specific class at a specific moment. Chapter 5's three-dimensional goal description provides the teacher with a guide to identify learning moments and provide support in a modelling classroom.

6.4 RECOMMENDATIONS FOR FURTHER STUDY

In a homogeneous or heterogeneous classroom there are learners with multi-ability levels and different past-experiences that influence their mathematical reasoning and abilities. The teacher needs to support every student's learning by locating their zone of proximal development (ZPD) to ensure that learning is progressive. The initial ZDP can be located by using a baseline assessment. A further study which includes the development of a baseline assessment for a mathematical modelling perspective is required. The development of a baseline assessment specific to mathematical modelling might influence the development of formal and summative assessment for mathematical modelling integrating the competency-assessment aspects described in Section 2.4.7. This study showed that learners revealed horizontal and vertical competencies while working with mathematical modelling problems. The researcher suggests an investigation to measure the influence of a mathematical modelling perspective to teaching and learning for *individualised learning* in a secondary school mathematics classroom.

A challenge is to introduce the mathematical modelling perspective into everyday classrooms. In Section 2.4.2 it was noted that a teacher's belief is inevitably moulded in his teaching practice. To change a teacher's practice, he needs to be an active participant in his own development. Teacher development and ongoing support are necessary components for the implementation of a mathematical modelling curriculum. If the modelling perspective is introduced during the teacher training programs at tertiary institutions it could possibly initiate the beginning of a mathematical modelling trend.

In the study mathematising competencies were developed for number patterns. The development of mathematising competencies need to be developed for different topics. The development of mathematising competencies for different topics would lead to the development of local instructional theories for different domains. The result would be a coherent mathematical modelling curriculum for secondary school mathematics.

The study's introductory statement is that mathematics education has suffered many changes attributable to the change in nature of mathematics and what mathematics means to the average learner, his life and career choices. The study has shown that a mathematical modelling perspective to the teaching and learning of mathematics will not only develop mathematising competencies but expose learners to the discovery of meaningful mathematics.

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APPENDICES

Appendix A

Appendix A1

The broken eggs

(Fendel, Resek & Interactive Mathematics Program [IMP], 1997, p. 12)

The situation

A farmer is taking her eggs to market in her cart, but she hits a pothole, which knocks over all the containers of eggs.

Though she herself is unhurt, every egg is broken.

So she goes to her insurance agent, who asks her how many eggs she had. She says she doesn't know, but she remembers some things from various ways she tried picking the eggs.

She knows that when she put the eggs in groups of two, there was one egg left over. When she put them in groups of three, there was also one egg left over. The same thing happened when she put them in groups of four, groups of five and groups of six.

But when she put them in groups of seven, she ended up with complete groups of seven with no eggs left over.

Your task

Your task is to answer the insurance agent's question.

Appendix A2

More broken eggs

(Fendel, Resek & IMP, 1997, p. 70)

In Activity 1, you found a possible number of eggs that the farmer might have had when her cart was knocked over.

You may have found only one solution to the problem, but there are actually many solutions.

Your task

Your task is to look for other solutions to the problem. Find as many as you can. If possible, find and describe a pattern for getting all the solutions and explain why all solutions fit your pattern.

Here are the facts you need to know:

- When the farmer put the eggs in groups of two, there was one egg left over.
- When she put them in groups of three, there was also one egg left over. The same thing happened when she put them in groups of four, groups of five or groups of six.
- When she put them in groups of seven, she ended up with complete groups of seven with no eggs left over.

Appendix A3

Marcella's Doughnuts

(Fendel, Resek & IMP, 1997, p. 19)

Have you ever been really in the mood to eat a doughnut? There are some pretty amazing things that can get in the way of this pursuit. Marcella was walking home from the beach one day. She had just bought a big bag of doughnuts and was going to share them with her daughter, Sonya. (Sonya loves doughnuts.)

On the side of the road she saw two people collecting food for needy families. Well, Marcella decided that she had quite a few doughnuts in her bag. Sonya didn't need that many doughnuts.

"Here," said Marcella, "you can have half of my doughnuts for the needy." The people were very happy to get the doughnuts. Marcella thought for a moment and then said, "Aw, take one for each of you."

As Marcella walked along the beach, some surfers came out of the water. They saw, and even smelled, the fresh doughnuts she had. "Could you by any chance spare a few doughnuts?" they pleaded. "We are so-o-o hungry after riding all of those gnarly waves."

As you might imagine, Marcella was not thrilled. But she had a good heart and recognised hunger after physical exertion, so she handed her bag to the surfers. They took half of her doughnuts and then, just as they were about to hand the bag back, they took two more.

Now Marcella was a very reasonable person who liked to help others. She thought she still had enough doughnuts left to make Sonya happy. She walked on.

As many of you may already have guessed, Marcella didn't get far before she had another encounter. Just before she reached home, her friend Susan approached. After exchanging greetings, Susan explained that she was on her way to get some doughnuts for her family. Susan seemed in a bit of a rush.

Generosity overtook Marcella and she found herself saying, "Why don't you save yourself the trip and take some of my doughnuts? As you can see, I've got several." So Susan took half of what Marcella had in the bag and two more.

Marcella finished her walk home without further interruption. When she opened her once bulging bag of doughnuts, she discovered that there were only two left! She had a doughnut for lunch with her daughter Sonya, and then there were none!

After lunch, Sonya asked her mother how many doughnuts had been in the bag to begin with. Marcella told her story of her walk and then said that if Sonya could figure it out herself, Marcella would take her rollerblading the next day. Sonya took a while, but then she figured it out and got her rollerblade outing.

What was Sonya's answer?

Appendix A4

Extended Doughnuts

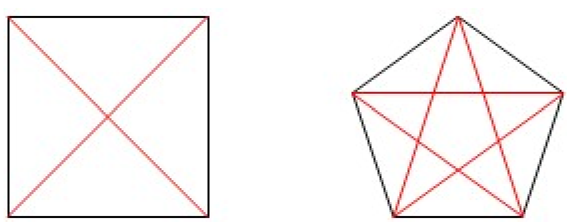
(Fendel, Resek & IMP, 1997, p. 21)

This extended section gives you the opportunity to examine what the critical elements of the problem are and how they could be changed.

How does the solution to Marcella's doughnuts depend on the number of doughnuts she has when she gets home?

Appendix A5

Thinking diagonally



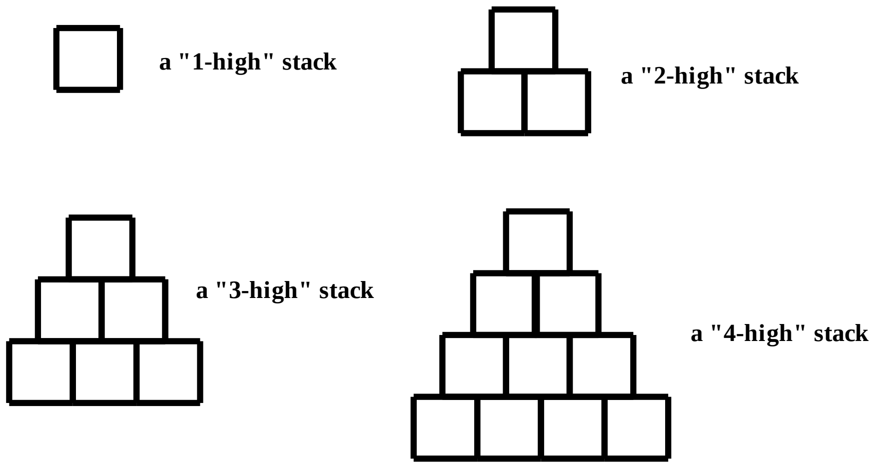
A square has two diagonals and a pentagon has five diagonals. How many diagonals does a $100-$gon have?

Appendix A6

Squares

(Fendel, Resek & IMP, 1997, p. 56)

Suppose some squares are stacked in piles of different heights as shown in the pictures below.



Find the number of squares in the stacks below. Generalise a description for the number of squares in a “40-high” stack.

Appendix A7

Consecutive sums

(Fendel, Resek & IMP, 1997, p. 28)

A consecutive sum is a sum of a sequence of consecutive numbers. So each expression below is a consecutive sum.

$2 + 3 + 4$

$8 + 9 + 10 + 11$

$23 + 24$

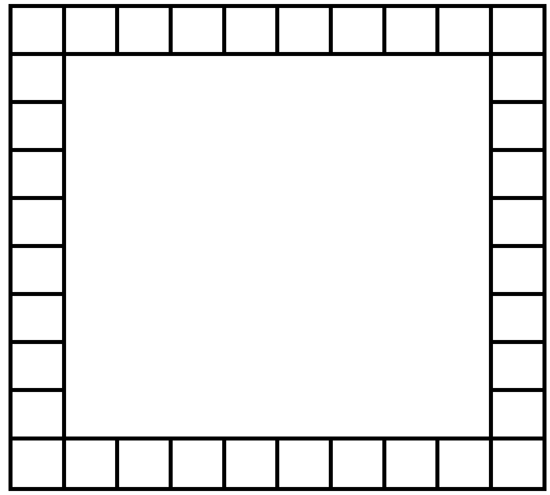
For this activity, you should only consider consecutive sums involving positive whole numbers. These are also called the natural numbers or counting numbers.

Your task

Explore the idea of consecutive sums. Try to find patterns and make generalisations.

Appendix A8

The garden border



(Fendel, Resek & IMP, 1997, p. 57)

Leslie was planning an oriental garden.

Task 1

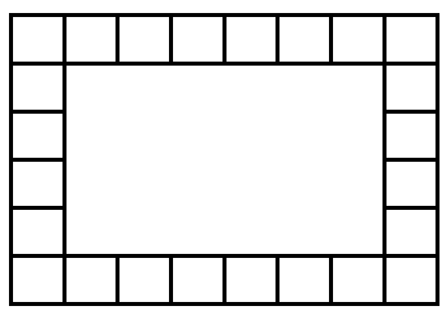
She wanted the garden to be square, 10 meters on each side, and she wanted part of this area to be used for a border of tiles. The tiles she wanted were each 1 meter by 1 meter square.

How many tiles does Leslie need?

Task 2

She decides to change the dimensions of the square garden. How many tiles does she need?

Task 3



Suppose Leslie's garden is not square.

Leslie thinks of an alternative shape, a rectangular garden. If she has a garden that is 6m by 8 m, the tiles would look like the diagram on the right. How many tiles would Leslie need to build any rectangular border?

Appendix A9

Folding paper

Take a piece of paper, and fold it as many times as you can. After one fold there will be two regions. Imagine there is no limit to the number of folds possible.

Appendix A10

Pulling out rules

(Fendel, Resek & IMP, 1997, p. 14)

The supervisor of a community garden project organises volunteers to help dig out weeds. The supervisor has found that the more people they have, the more weeds get pulled out. That is not surprising, but the results get even better than one might think. Although one person will only pull out one bag a day, two people will pull out about three bags a day and three people will pull out about seven bags per day. It is the beginning of spring, and the garden must be cleared of huge amount of winter weeds. The supervisor estimates that there are about 100 bags worth of weeds to be pulled.

How many volunteers would the supervisor needs in order to get the job done in one day?

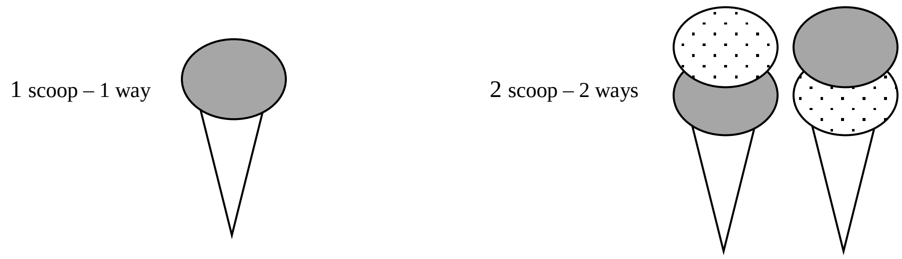
Appendix A11

Scoops

(Fendel, Resek & IMP, 1997, p. 56)

Suppose you have some scoops of ice cream, and each scoop is a different flavour. How many different flavours can you arrange the scoops in a stack?

The pictures show the cases of one scoop and two scoops.



Find the number of ways to arrange the scoops if there were 100 scoops.

Appendix A12

Cutting through the layers

(Fendel, Resek & IMP, 1997, p. 61)

Imagine a single piece of string, which can be bent back and forth. In the picture the string is bent so that it has three “layers”. But it is still one piece of string.

Imagine now you take scissors and cut across the bent string, as indicated by the dotted line. The result will be four separate pieces of string, as shown in the diagram on the right.

You could have made more than one cut across the bent string, creating more pieces. In the next picture, two cuts have been made, creating a total of seven pieces.

Your task

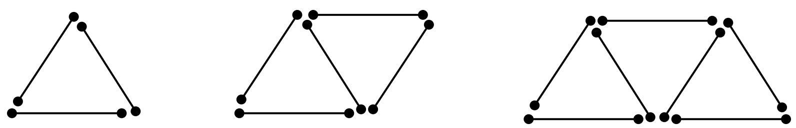
Suppose the number of layers is L and the number of cuts is C. Find a rule for the formula expressing the number of pieces as a function of both L and C.

Appendix A13

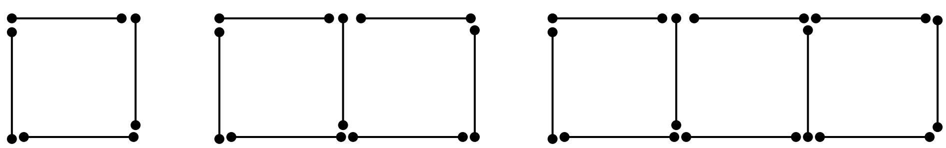
The researcher would like to acknowledge MALATI and the Open Society Foundation for South Africa for the following problem.

Activity 1

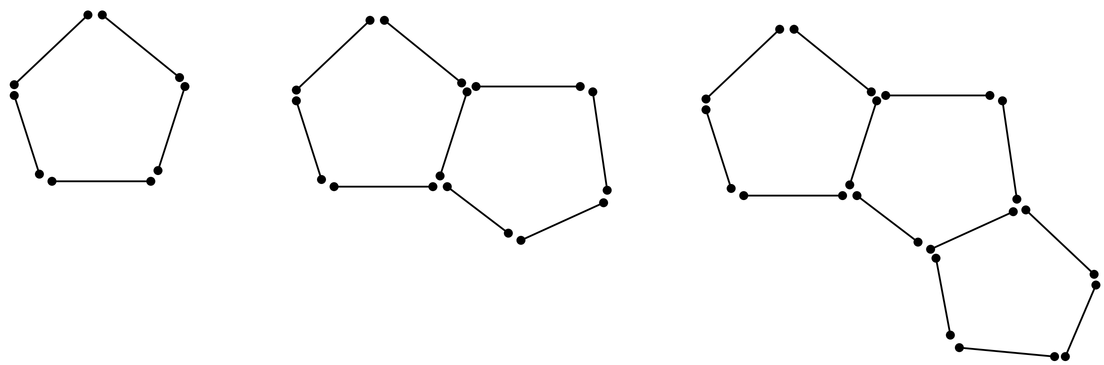
1. Lynn forms triangle patterns with matches. How many matches will she need to build 100 triangle patterns? And n triangle patterns?



2. On the next day Lynn forms square patterns with matches. How many matches does she need to build 100 square patterns? And n square patterns?



3. On the third day Lynn forms pentagon patterns. How many matches does she need to build 100 pentagon patterns? And n pentagon patterns?



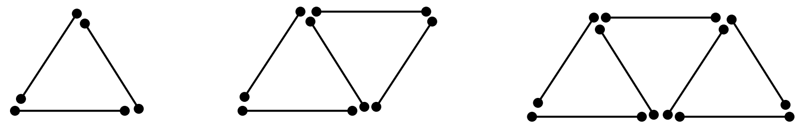
4. On the fourth day Lynn forms hexagon patterns. How many matches does she need to build 100 hexagon patterns? And n hexagon patterns?

5. On the fifth day Lynn forms decagon patterns. How many matches does she need to build 100 decagon patterns? And n decagon patterns?

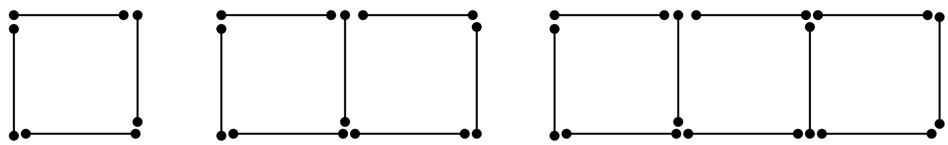
Appendix A14

Activity 2

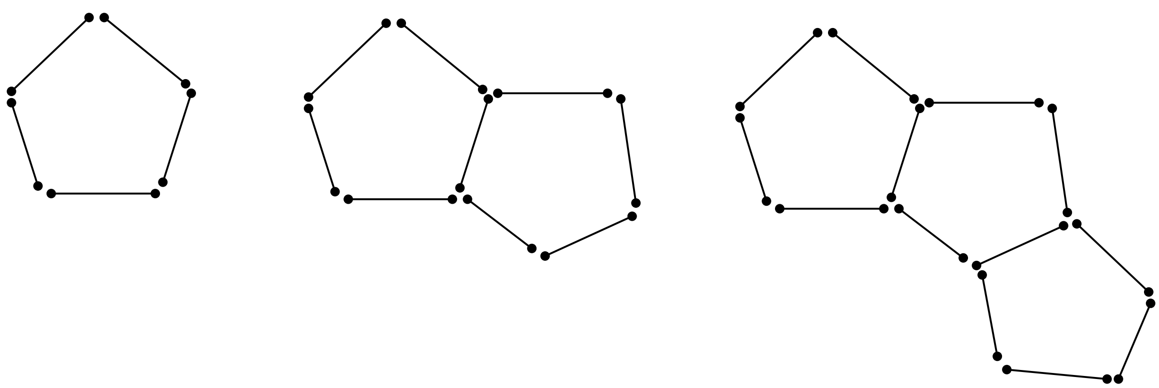
1. Nokwanda forms triangle patterns with matches. How many matches will she need in total to build the whole triangle sequence up to 100 triangle patterns? And the nth pattern?



2. On the next day Nokwanda forms square patterns with matches. How many matches will she need in total to build the whole sequence up to 100 square patterns? And the nth pattern?



3. On the third day Nokwanda forms pentagon patterns. How many matches will she need in total to build the whole sequence up to 100 pentagon patterns? And the nth pattern?



4. On the fourth day Nokwanda forms hexagon patterns. How many matches will she need in total to build the whole sequence up to 100 hexagon patterns? And the nth pattern?

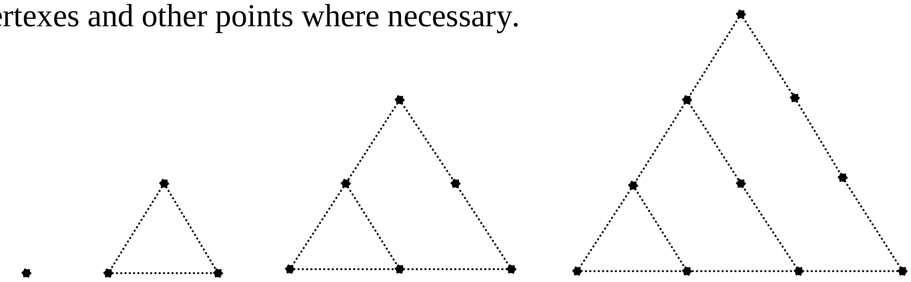
5. On the fifth day Nokwanda forms decagon patterns. How many matches will she need in total to build the whole sequence up to 100 decagon patterns? And the nth pattern?

Appendix A15

Activity 3

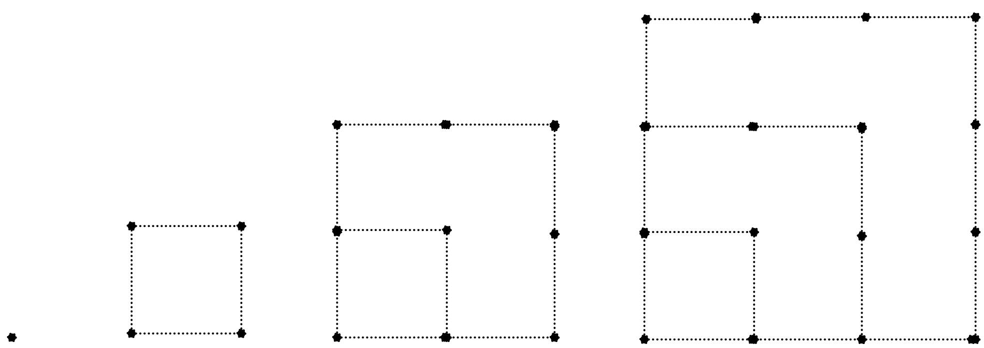
Poly forms patterns by using the number of vertexes in a figure formed by polygons. The first number in any group of polygonal numbers is always 1, or a point. The second number is equal to the number of vertexes of the polygon. The third polygonal number is made by extending two of the sides of the polygon from the second polygonal number, completing the larger polygon and placing vertexes and other points where necessary.

1.



How many points will Poly need to form the nth pattern in the triangular patterns?

2.



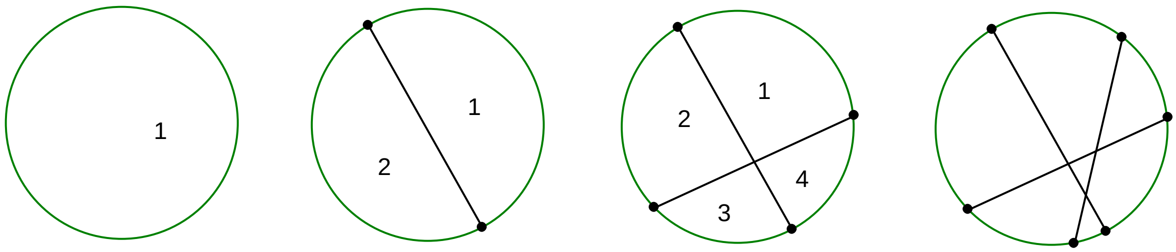
How many points will Poly need to form the nth pattern in the square patterns?

3. How many points will Poly need to form the nth pattern using pentagonal patterns?

4. How many points will Poly need to form the nth pattern using hexagonal patterns?

Appendix A16

Activity 4



2 chords divide a circle into 4 regions. What is the maximum number of regions 6 chords will divide a circle into? And 20 chords? And n chords?

# chords (n)	0	1	2	3	4	5	6	20	n
# regions (R)									

Appendix A17

Activity 5

1. Two teams have the same number of players. Each player on one team shakes hand with each player on the other team. How many handshakes will take place between two 5-player teams? Between two 10-player teams?

Write an equation for the number of handshakes h between two n player teams.

2. One team has one more player than the other. Each player on the one team shakes hands with each player on the other team. How many handshakes will take place between a 6-player team and a 7-player team? Between an 8 player team and a 9-player team?

Write an equation for the number of handshakes h between an n-player team and an $(n-1)$ player team.

3. Each member of a team gives a high five to each teammate. How many high fives will take place among a team with 4 members? Among a team with 8 members?

Write an equation for the number of high fives h among a team with n members.

Appendix B Research instruments

Appendix B1 Baseline assessment for pilot study

Baseline assessment

Name: ____ Date: ____________

Instructions: Answer the following questions in the spaces provided. The additional paper for rough work must be handed in together with the assessment task

1. Extend the following patterns by completing the following tables.

1.1

n	1	2	3	4	5	6	10		n
$T(n)$	2	4	6	8				44	

1.2

n	1	2	3	4	5	6	10	32	n
$T(n)$	5	6	7	8					

1.3

n	1	2	3	4	5	6	10		n
$T(n)$	5	8	11	14				77	

1.4

n	1	2	3	4	5	6	10		n
$T(n)$	1	4	9	16				324	

1.5

n	1	2	3	4	5	6	7
$T(n)$	6	9	14	21			

1.6

n	1	2	3	4	5	6	7
$T(n)$		7	17	31			

1.7

n	1	2	3	4	5	6	10		n
$T(n)$	1	8	27	64				4096	

1.8

n	1	2	3	4	5	6	7
$T(n)$	2	16		128			

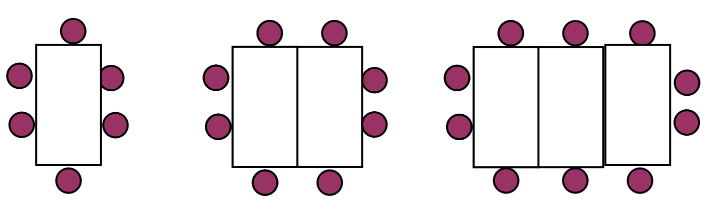
1.9

n	1	2	3	4	5	6	10		n
$T(n)$	1	3	9	27				4782969	

1.10

n	1	2	3	4	5	6	10		n
$T(n)$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$				$\frac{19}{20}$	

2. The following diagrams represent customers seated around different tables in a restaurant



2.1 Draw the next two diagrams in the sequence.

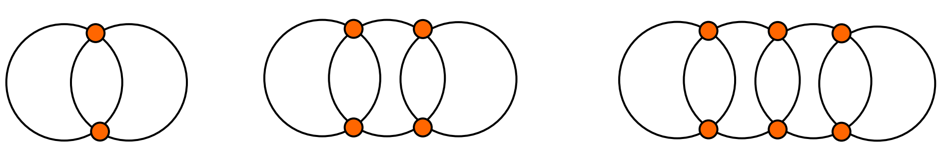
2.2 Write down the sequence generated by the number of dots in each diagram.

2.3 Write down the 6th and 7th terms in this sequence.

2.4 Write down in words how the pattern continues in relation to the diagrams.

2.5 Try do describe in words or symbols the *n*th term (general term) for this sequence in the form $T(n) =$

4. To join circles together they need a fastener as indicated in the representation below.



4.1 Complete the table from the diagrams.

Circles	x	2	3	4
Fasteners	y		4	

4.2 Represent the pattern using your table as a graph by plotting the number of circles on the x-axis against the number of fasteners on the y-axis.

4.4 Use your graph to predict:

(i) How many fasteners are needed for 7 circles? ______________________________

(ii) How many circles use 18 fasteners exactly? ______________________________

Appendix B2 Baseline assessment for design experiment

Baseline assessment

Number patterns

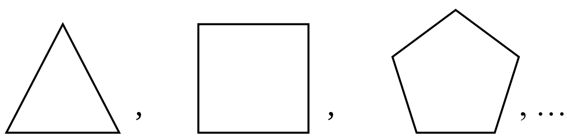
Mathematics often involves looking for patterns in various situations. A collection of shapes or numbers is called sequences. Each shape or number is called a term of the sequence, and the terms are separated by commas.

Instructions: Answer the following questions in the spaces provided. Do all your rough work on the question paper. **SHOW ALL YOUR WORKING**, even if you think it is not important.

Question 1

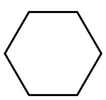
Four options are given in each question. Circle the letter(s) of the correct answer.

1.1 The following regular polygons make up a sequence:

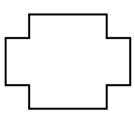


The next polygon in the pattern is:

A. six

B. 

C. 

D. 

1.2 Consider the following sequence: 1, 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, 1, ...

The next three terms in the following sequence is:

A. 7, 8, 9

B. 7, 1, 1

C. 7, 1, 8

D. 6, 1, 5

1.3 The following sequence is given: $1, 2, 4, \ldots$

The next term is:

A. 7

B. 8

C. 6

D. none of the above

1.4 $1, 3, 5, 7, 5, 3, 1, 3, \ldots$

The description of the above sequence can be:

A. Add three to each term

B. Subtract three from each term

C. I can't see a pattern

D. Explain your own pattern: ____

____

1.5 Look at the following sequence: $2,\ 6, 12,\ 20,\ 30, \ldots$

A possible rule for this sequence could be:

A. $n \times n + 1$

B. $4n - 2$

C. $n \times n + n$

D. I don't understand what this means

1.6 Insert brackets so that the resulting statement forms a correct equation.

$12 - 8 . 1 + 7 = 32$

Question 2

Make three different number sequences of your own:

2.1 LINEAR: A sequence with a constant first difference.

2.2 QUADRATIC: A sequence with a constant second difference.

2.3 ANY: And any **other** sequence (cannot be linear or quadratic).

Describe it by giving the first few terms and explaining how you would find more terms.

2.1 LINEAR: A sequence with a constant first difference.

2.2 QUADRATIC: A sequence with a constant second difference.

2.3 ANY: And any **other** sequence (cannot be linear or quadratic).

Question 3

Supply the missing entries in the following In-Out tables/diagrams.

Then write a rule for each table that tells what to do with the *In* to get to the *Out*. **Express each rule as a sentence,** such as: The *Out* is one more than four times the *In*. Be as clear as you can!

3.1

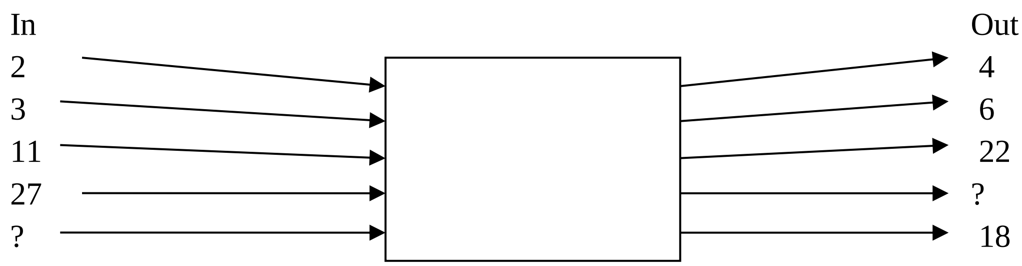
In	Out
1	2
2	5
3	8
4	?
5	?

Rule as a sentence: ____

____

____

3.2



Rule as a sentence: ____

____

____

3.3

In	2	4	7	10	12	?
Out	7	13	22	31	?	76

Rule as a sentence: ____

____

____

Appendix B3 Baseline assessment table

Group	Learner	1.1	1.2	1.3	1.4	1.5	1.6	2.1	2.2	2.3	3.1	3.2	3.3
1	*A*	B	C	A	D	B	✓	✓	✓	×	✓	✓	×
2	*B*	B	C	B	D	C	✓	✓	✓	×	✓	✓	×
1	*C*	B	C	A	D	C	✓	✓	×	×	✓	✓	×
2	*D*	B	C	A	D	C	✓	✓	×	×	✓	✓	✓
3	*E*	B	C	B	C	C	✓	✓	×	×	✓	✓	✓
1	*F*	B	C	B	D	C	✓	✓	✓	×	✓	✓	✓
3	*G*	B	C	B	D	B	×	✓	✓	quad	✓	✓	×
2	*H*	B	C	B	D	C	✓	×	✓	×	×	×	×
1	*I*	B	C	A	D	C	✓	✓	✓	×	✓	✓	×
2	*J*	B	B	A	C	C	×	✓	×	quad	✓	×	×
3	*K*	B	C	B	D	C	✓	×	×	×	-	✓	×
3	*L*	B	C	B	D	C	✓	✓	✓	×	✓	✓	✓
1	*M*	B	C	A	C	B	×	✓	×	×	✓	✓	×
2	*N*	B	C	C	D	B	✓	×	×	×	×	✓	×
2	*O*	B	C	B	D	B	×	✓	×	×	×	×	×
3	*P*	B	C	C	D	B	-	✓	×	×	-	✓	×
3	*Q*	B	C	B	D	-		-	-	-	-	✓	×

✓Correct × Incorrect - Not done

Appendix B4 Interview questionnaire

Name: ___________________________ Date: _________________

1. What do you believe mathematics is all about?

   ____

2. Are you very good at mathematics? Why?

   ____

3. What do you enjoy the most during our sessions?

   ____

4. What do you enjoy the least during our sessions?

   ____

5. Are you enjoying the activities you are doing? Why?

   ____

6. Was there a time(s) when you have impressed yourself?

   ____

7. Have you learnt something new?

   ____

8. Have you learnt something that you can apply in your classes at school?

   ____

9. What did you learn about yourself during this program?

   ____

10. Do you feel more confident when tackling problems/group work?

    ____

11. Would you be able to assist other learners better?

    ____

Appendix B5 Researcher's observation guide

Activity: ____

Date: ________________________

	Positives	Negatives	Other
Watching/Listening			
Group interaction			
Meta-cognitive strategies			
Questions asked			
Sense of direction			
Other			
Other			
Reflection			

Appendix B6 Number pattern competency (NPC) continuum

	Competencies	**Sub-competencies**	**What the learner does, says, makes or writes**
Horizontal	Internalising	• Understanding the problem • Distinguishing between relevant and irrelevant information • Simplifying the situation	• Learner states the problem in language he understands • Learner notes/explains important information • Learner notes/explains/relates a previous problem that is similar to the current one
	Interpreting	• Making assumptions • Identifying conditions • Identifying constraints • Recognising quantities that influence situation	• Learner makes assumptions • Learner notes conditions that will work/not work for a problem • Learner recognises quantities that influence the situation
	Structuring	• Setting up a real model • Naming quantities • Identifying key variables • Recognise patterns • Recognise relationships	• Learner looks for a pattern/relationship • Learner notes a recurring value or situation in the problem • Learner recognises a pattern/relationship • Learner states the relationship or pattern
	Symbolising	• Choosing appropriate mathematical symbols • Using symbols • Setting up a mathematical model • Switching between symbolisations	• Learner draws pictures to represent the problem • Learner draws pictures to show the relationship/pattern • Learner uses objects to build the pattern
Vertical			• Learner uses symbols to represent his pictures/patterns • Learner forms a pattern using symbols • Learner formulates a rule using symbols • Learner creates a model *of*
	Adjusting	• Rephrasing the problem • Refining • Using and switching between operations	• Learner adapts his pattern so that it makes sense for the situation • Learner tests his pattern • Learner refines his pattern after testing it • Learners reflects back to the pattern/symbols • Learner reflects back to the real problem • Learner creates a model *for*
	Organising	• Viewing problem in a different form • Use mathematical knowledge to solve problem • Using heuristics • Combining • Integrating	• Learner constructs a rule that works for all elements • Learner reflects back to the real problem • Learner uses the rule to solve a problem • Learner validates his solution • Learner creates a model *for*
	Generalising	• Establishing similar relationships in different problems • Independent reasoning and acting	• Learner uses deductive reasoning to prove his rule • Learner uses/adapts the rule for another situation

Appendix C Checklists

Appendix C1 Checklist for modelling problems

Principle	**Questions**	**YES**	**NO**
Model construction principle	Does the task involve constructing, describing or explaining a structurally significant system?		
Reality principle	Is the context real and useful?		
	Will students be encouraged to make sense of the situation based on extensions of their own personal knowledge and experiences?		
Self-assessment principle	Does this task provide enough information for a learner to establish if he has done enough?		
Level principle	Is this task progressive or form part of a progressive activity?		
	Can this task be used in a higher level of activity?		
Language	Is the language of the task appropriate for the learners?		

Appendix D Permission documents

Appendix D1 Ethical clearance from the Stellenbosch University

UNIVERSITEIT·STELLENBOSCH·UNIVERSITY
jou kennisvennoot • your knowledge partner

31 October 2012

Tel.: 021 - 808-9003
Enquiries: Mrs S. Oberholzer
Email: oberholzer@sun.ac.za

Reference No. 456/2010

Ms A Knott
Faculty of Education

Dear A Knott,

LETTER OF ETHICS CLEARANCE

With regard to your application, **No. 456/2010** I would like to inform you that the project," An Analysis of the Process of Mathematisation in Mathematical Modelling in Secondary School Mathematics", has been approved.

1. The researcher will remain within the procedures and protocols indicated in the proposal, particularly in terms of any undertakings made in terms of the confidentiality of the information gathered.
2. The research will again be submitted for ethical clearance if there is any substantial departure from the existing proposal.
3. The researcher will remain within the parameters of any applicable national legislation, institutional guidelines and scientific standards relevant to the specific field of research.
4. The researcher will consider and implement the foregoing suggestions to lower the ethical risk associated with the research.
5. This ethics clearance is valid for one year from **31 October 2012 – 30 October 2013**

We wish you success with your research activities.

Best regards

S. Oberholzer

MRS S. OBERHOLZER
REC Coordinator: Research Ethics Committee: Human Research (Humaniora)
Registered with the National Health Research Ethics Council (NHREC): REC-050411-032

Afdeling Navorsingsontwikkeling • Division for Research Development

Privaatsak/Private Bag X1 • Matieland 7602 • Suid-Afrika/South Africa
Tel: +27 21 808 9184 • Faks/Fax: +27 21 808 4537
www.sun.ac.za/research

Appendix D2 Permission from the KwaZulu-Natal Education Department

4. Sep. 2012 16:03

kzn education
Department:
Education
KWAZULU-NATAL

Enquiries: Sibusiso Alwar | Tel: 033 341 8610 | Ref.: 0072/2010

Axanthé Knott

EXTENTION OF RESEARCH PERIOD:
PERMISSION TO CONDUCT RESEARCH IN THE KZN DoE INSTITUTIONS

1. Your application for the extension of the period for collecting data in the KwaZulu-Natal Department of Education Institutions has been approved.
2. The research period has been extended to the 31 December 2013.
3. Please note that only the period has been extended, the other conditions stipulated in the original approval still hold. Therefore this letter must always be presented with the original approval.
4. Hope you find this in order.

PROVINCE OF KWAZULU-NATAL
UMNYANGO WEMFUNDO
DEPARTMENT OF EDUCATION
DEPARTEMENT VAN ONDERWYS
2012 -08- 24
PRIVATE BAG X9137
PIETERMARITZBURG 3200
RESOURCE PLANNING

...dedicated to service and performance
beyond the call of duty.

KWAZULU-NATAL DEPARTMENT OF EDUCATION
POSTAL: Private Bag X9137, Pietermaritzburg, 3200, KwaZulu-Natal, Republic of South Africa
PHYSICAL: Office G.25, 188 Pietermaritz Street, Metropolitan Building, Pietermaritzburg 3201
TEL: Tel. +27 33 341 8610 | Fax: +27 33 341 8612 | E-mail: [illegible]@kzndoe.gov.za
Web: www.kzneducation.gov.za

kzn education

Department:
Education
KWAZULU-NATAL

AXANTHE KNOTT
PO BOX 46
UMTENTWENI
4235

Enquiries: Sibusiso Alwar

Date: 31/08/2010

Reference: 0072/2010

RESEARCH PROPOSAL: AN ANALYSIS OF THE PROCESS OF MATHEMATISATION IN MATHEMATICAL MODELLING IN SECONDARY SCHOOL MATHEMATICS.

Your application to conduct the above-mentioned research in schools in the attached list has been approved subject to the following conditions:

1. Principals, educators and learners are under no obligation to assist you in your investigation.
2. Principals, educators, learners and schools should not be identifiable in any way from the results of the investigation.
3. You make all the arrangements concerning your investigation.
4. Educator programmes are not to be interrupted.
5. The investigation is to be conducted from 31 August 2010 to 31 August 2011.
6. Should you wish to extend the period of your survey at the school(s) please contact Mr Sibusiso Alwar at the contact numbers above.
7. A photocopy of this letter is submitted to the principal of the school where the intended research is to be conducted.
8. Your research will be limited to the schools submitted.
9. A brief summary of the content, findings and recommendations is provided to the Director: Resource Planning.

...dedicated to service and performance
beyond the call of duty.

KWAZULU-NATAL DEPARTMENT OF EDUCATION
POSTAL: Private Bag X9137, Pietermaritzburg, 3200, KwaZulu-Natal, Republic of South Africa
PHYSICAL: Office G25; 188 Pietermaritz Street; Metropolitan Building; PIETERMARITZBURG 3201
TEL: Tel: +27 33 341 8610/8611 | Fax: +27 33 341 8612 | E-mail: [illegible] / [illegible]

kzn education

Department:
Education
KWAZULU-NATAL

10. The Department receives a copy of the completed report/dissertation/thesis addressed to:

The Director: Resource Planning
Private Bag X9137
Pietermaritzburg
3200

We wish you success in your research.

Kind regards

R. Cassius Lubisi (PhD)
Superintendent-General

...dedicated to service and performance
beyond the call of duty.

KWAZULU-NATAL DEPARTMENT OF EDUCATION

POSTAL : Private Bag X9137, Pietermaritzburg, 3200, KwaZulu-Natal, Republic of South Africa

PHYSICAL: Office G25; 188 Pietermaritz Street: Metropolitan Building; PIETERMARITZBURG 3201

TEL: Tel: +27 33 341 8610/8611 | Fax: +27 33 341 8612 | E-mail:

kzn education
Department:
Education
KWAZULU-NATAL

AXANTHE KNOTT
PO BOX 46
UMTENTWENI
4235

Enquiries: Sibusiso Alwar
Date: 31/08/2010
Reference: 0072/2010

PERMISSION TO INTERVIEW LEARNERS AND EDUCATORS

The above matter refers.

Permission is hereby granted to interview Departmental Officials, learners and educators in selected schools of the Province of KwaZulu-Natal subject to the following conditions:

1. **You make all the arrangements concerning your interviews.**
2. **Educators' programmes are not interrupted.**
3. **Interviews are not conducted during the time of writing examinations in schools.**
4. **Learners, educators and schools are not identifiable in any way from the results of the interviews.**
5. **Your interviews are limited only to targeted schools.**
6. **A brief summary of the interview content, findings and recommendations is provided to my office.**
7. **A copy of this letter is submitted to District Managers and principals of schools where the intended interviews are to be conducted.**

The KZN Department of education fully supports your commitment to research: **An analysis of the process of mathematisation in mathematical modelling in secondary school mathematics.**

It is hoped that you will find the above in order.

Best Wishes

R Cassius Lubisi, (PhD)
Superintendent-General

...dedicated to service and performance
beyond the call of duty.

KWAZULU-NATAL DEPARTMENT OF EDUCATION
POSTAL : Private Bag X9137, Pietermaritzburg, 3200, KwaZulu-Natal, Republic of South Africa
PHYSICAL: Office G25; 188 Pietermaritz Street; Metropolitan Building; PIETERMARITZBURG 3201
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kzn education

Department:
Education
KWAZULU-NATAL

AXANTHE KNOTT
PO BOX 46
UMTENTWENI
4235

Enquiries:	Sibusiso Alwar
Date:	31/08/2010
Reference:	0072/2010

LIST OF SCHOOLS

1. Port Shepstone High School

Kind regards

R Cassius Lubisi, (PhD)
Superintendent-General

...dedicated to service and performance
beyond the call of duty.

KWAZULU-NATAL DEPARTMENT OF EDUCATION

POSTAL: Private Bag X9137, Pietermaritzburg, 3200, KwaZulu-Natal, Republic of South Africa

PHYSICAL: Office G25; 188 Pietermaritz Street; Metropolitan Building; PIETERMARITZBURG 3201

TEL: Tel: +27 33 341 8610/8611 | Fax: +27 33 341 8612 | E-mail: [illegible] / [illegible]

Appendix D3 Permission from the principal of Port Shepstone High School

UNIVERSITEIT·STELLENBOSCH·UNIVERSITY
jou kennisvennoot • your knowledge partner

STELLENBOSCH UNIVERSITY
PERMISSION TO CONDUCT RESEARCH

An analysis of the process of mathematisation in mathematical modelling in secondary school mathematics.

Principal of Port Shepstone High School

I am conducting a research study for my master's degree in mathematics education at the University of Stellenbosch. This letter is to obtain permission to conduct the research at Port Shepstone High school. The KwaZulu-Natal Department of Education granted permission for the study to take place.

The participants in this study are selected because they are Grade 10 learners at Port Shepstone High School. This study will take place on **Monday** afternoons (from 14:15 – 16:00) and **Saturdays** (from 08:00 to 13:00) starting on **15 October 2012** until **5 November 2012**.

1. PURPOSE OF THE STUDY

The study seeks to understand the mathematical processes when modelling mathematical problems in number patterns.

2. PROCEDURES

If a participant chooses to participate in this study, he will:

- form part of a study group with other Grade 10 learners,
- complete a sequence of selected mathematical problems,
- present your group's findings and solutions to the other groups involved in the study,
- be interviewed during the study to share experiences related to the activities,
- be audio taped during each lesson,
- be videotaped during each lesson,
- complete a criterion referenced problem.

3. POTENTIAL RISKS AND DISCOMFORTS

There are no foreseeable risks or discomforts involved by taking part of this study.

4. POTENTIAL BENEFITS TO SUBJECTS AND/OR TO SOCIETY

The potential benefits of the study to participants may be:

- An increased awareness of the usefulness of mathematics in solving everyday problems.
- An increased enjoyment of solving real life mathematical problems.
- The benefit of collaborating with peers in the group.
- Better understanding and better achievement in the content domain.

5. PAYMENT FOR PARTICIPATION

No payment will be made for the participation in the study.

6. CONFIDENTIALITY

Any information that is obtained in connection with this study and that can be identified with the learner will remain confidential and will be disclosed only with the parent/guardian's permission or as required by law. Confidentiality will be maintained by means of only using the participants' first name initial in the coding procedures; the name of the school will not be disclosed. In the research itself, and in any papers following the research, pseudonyms will be used and not the learners' real names. Data will be kept by the researcher. Data will be locked up and kept for a period of three years after the dissertation is submitted. Thereafter the tapes will be erased. No persons other than the student and supervisor will have access to the data.

The working sessions will be audio recorded and video recorded. You have the right to access and view the tapes. The tapes will be available to the research student and are for educational purposes only.

Confidentiality resulting from any publications from the study will be treated with the same confidentiality as the study itself. If exact transcripts are necessary then only the participant's first name will be used.

7. PARTICIPATION AND WITHDRAWAL

The participant can choose to be in the study or not. A participant may withdraw at any time without consequences of any kind. A participant may also refuse to answer any questions he does not want to answer and still remain in the study. The investigator may withdraw a participant from this research if circumstances arise which warrant doing so.

8. IDENTIFICATION OF INVESTIGATORS

If you have any questions or concerns about the research, please feel free to contact Mrs A. Knott at school.

9. RIGHTS OF RESEARCH SUBJECTS

A participant may withdraw your consent at any time and discontinue participation without penalty. A participant is not waiving any legal claims, rights or remedies because of your participation in this research study. If a participant or his parent/guardian have questions regarding their rights as a research subject, contact Ms Marléne Fouché at the Research Develop Division, University of Stellenbosch at mfouche@sun.ac.za; 021 808 4622.

SIGNATURE OF PRINCIPAL OF PORT SHEPSTONE HIGH SCHOOL

The information above was described to me by Mrs. A. Knott in English and I am in command of this language. I was given the opportunity to ask questions and these questions were answered to my satisfaction.

I hereby give my permission voluntarily for the researchers to do the research in my school for this study. I have been given a copy of this form.

PORT SHEPSTONE HIGH SCHOOL

Name of School

MR P.F. MCKILLEN

Name of Principal

Signature of Principal

PORT SHEPSTONE HIGH SCHOOL
08 OCT 2012
PRINCIPAL P.F. MC KILLEN

Date

SIGNATURE OF INVESTIGATOR

I declare that I explained the information given in this document to Mr McKillen. He was encouraged and given ample time to ask me any questions. This conversation was conducted in English.

Knott.

Signature of Investigator

08-10-2012

Date

Appendix D4 Permission from the Mathematics Head of Subject

UNIVERSITEIT-STELLENBOSCH-UNIVERSITY
jou kennisvennoot • your knowledge partner

STELLENBOSCH UNIVERSITY
PERMISSION TO CONDUCT RESEARCH

An analysis of the process of mathematisation in mathematical modelling in secondary school mathematics.
Mathematics, Head of Subject of Port Shepstone High School

I am conducting a research study for my master's degree in mathematics education at the University of Stellenbosch. This letter is to obtain permission to conduct the research at Port Shepstone High school. The KwaZulu-Natal Department of Education granted permission for the study to take place.

The participants in this study are selected because they are Grade 10 learners at Port Shepstone High School. This study will take place on **Monday** afternoons (from 14:15 – 16:00) and **Saturdays** (from 08:00 to 13:00) starting on **15 October 2012** until **5 November 2012**.

1. PURPOSE OF THE STUDY

The study seeks to understand the mathematical processes when modelling mathematical problems in number patterns.

2. PROCEDURES

If a participant chooses to participate in this study, he will:

- form part of a study group with other Grade 10 learners,
- complete a sequence of selected mathematical problems,
- present your group's findings and solutions to the other groups involved in the study,
- be interviewed during the study to share experiences related to the activities,
- be audio taped during each lesson,
- be videotaped during each lesson,
- complete a criterion referenced problem (post assignment).

3. POTENTIAL RISKS AND DISCOMFORTS

There are no foreseeable risks or discomforts involved by taking part of this study.

4. POTENTIAL BENEFITS TO SUBJECTS AND/OR TO SOCIETY

The potential benefits of the study to participants may be:

- An increased awareness of the usefulness of mathematics in solving everyday problems.
- An increased enjoyment of solving real life mathematical problems.
- The benefit of collaborating with peers in the group.
- Better understanding and better achievement in the content domain.

5. PAYMENT FOR PARTICIPATION

No payment will be made for the participation in the study.

6. CONFIDENTIALITY

Any information that is obtained in connection with this study and that can be identified with the learner will remain confidential and will be disclosed only with the parent/guardian's permission or as required by law. Confidentiality will be maintained by means of only using the participants' first name initial in the coding procedures; the name of the school will not be disclosed. In the research itself, and in any papers following the research, pseudonyms will be used and not the learners' real names. Data will be kept by the researcher. Data will be locked up and kept for a period of three years after the dissertation is submitted. Thereafter the tapes will be erased. No persons other than the student and supervisor will have access to the data.

The working sessions will be audio recorded and video recorded. You have the right to access and view the tapes. The tapes will be available to the research student and are for educational purposes only.

Confidentiality resulting from any publications from the study will be treated with the same confidentiality as the study itself. If exact transcripts are necessary then only the participant's first name will be used.

7. PARTICIPATION AND WITHDRAWAL

The participant can choose to be in the study or not. A participant may withdraw at any time without consequences of any kind. A participant may also refuse to answer any questions he does not want to answer and still remain in the study. The investigator may withdraw a participant from this research if circumstances arise which warrant doing so.

8. IDENTIFICATION OF INVESTIGATORS

If you have any questions or concerns about the research, please feel free to contact Mrs A. Knott at school.

9. RIGHTS OF RESEARCH SUBJECTS

A participant may withdraw your consent at any time and discontinue participation without penalty. A participant is not waiving any legal claims, rights or remedies because of your participation in this research study. If a participant or his parent/guardian have questions regarding their rights as a research subject, contact Ms Marléne Fouché at the Research Develop Division, University of Stellenbosch at mfouche@sun.ac.za; 021 808 4622.

SIGNATURE OF RESEARCH SUBJECT OR LEGAL REPRESENTATIVE

The information above was described to me by Mrs. A. Knott in English and I am in command of this language. I was given the opportunity to ask questions and these questions were answered to my satisfaction.

I hereby give my permission voluntarily for the researcher to do the research. I have been given a copy of this form.

Port Shepstone High
Name of School

Catherine Foster
Name of Head of Subject

CA Foster
Signature of Head of Subject

8/10/2012
Date

SIGNATURE OF INVESTIGATOR

I declare that I explained the information given in this document to Mrs. Foster. He was encouraged and given ample time to ask me any questions. This conversation was conducted in English.

Signature of Investigator

08-10-2012
Date

Appendix D5 Consent to participate in research

UNIVERSITEIT·STELLENBOSCH·UNIVERSITY
jou kennisvennoot • your knowledge partner

STELLENBOSCH UNIVERSITY
CONSENT TO PARTICIPATE IN RESEARCH

An analysis of the process of mathematisation in mathematical modelling in secondary school mathematics.

Parent/guardian of Grade 10 learners of Port Shepstone High School

I am conducting a research study for my master's degree in mathematics education at the University of Stellenbosch. This letter is written to invite your son/daughter to be a participant in this study. The KwaZulu-Natal Department of Education granted permission for the study to take place.

This study will take place on Monday, Tuesday, Wednesday and Thursday afternoons from 14:45 – 16:00 starting on 15 October 2012 until 15 November 2012.

1. PURPOSE OF THE STUDY

The study seeks to understand the mathematical processes when modelling mathematical problems in number patterns.

2. PROCEDURES

If a you chooses to let your son/daughter participate in this study, he/she will:

- form part of a study group with other Grade 10 learners,
- complete a sequence of selected mathematical problems,
- present your group's findings and solutions to the other groups involved in the study,
- be interviewed during the study to share experiences related to the activities,
- be audio taped during each lesson,
- be videotaped during each lesson,
- complete a criterion referenced problem.

3. POTENTIAL RISKS AND DISCOMFORTS

There are no foreseeable risks or discomforts involved by taking part of this study.

4. POTENTIAL BENEFITS TO SUBJECTS AND/OR TO SOCIETY

The potential benefits of the study to participants may be:

- An increased awareness of the usefulness of mathematics in solving everyday problems.
- An increased enjoyment of solving real life mathematical problems.
- The benefit of collaborating with peers in the group.
- Better understanding and better achievement in the content domain.

Fakulteit Opvoedkunde • Faculty of Education

Departement Kurrikulumstudies • Department of Curriculum Studies

Privaat Sak/Private Bag X1 • Matieland 7602 • Suid-Afrika/South Africa

5. PAYMENT FOR PARTICIPATION

No payment will be made for the participation in the study.

6. CONFIDENTIALITY

Should you declare your son/daughter to participate in this study, confidentiality will be guaranteed. Any information that is obtained in connection with this study and that can be identified with you will remain confidential and will be disclosed only with your permission or as required by law. Confidentiality will be maintained by means of only using the participants' first name initial in the coding procedures; the name of the school will not be disclosed. In the research itself, and in any papers following the research, pseudonyms will be used and not the learners' real names. Data will be kept by the researcher. Data will be locked up and kept for a period of three years after the dissertation is submitted. Thereafter the tapes will be erased. No persons other than the student and supervisor will have access to the data.

The working sessions will be audio recorded and video recorded. You have the right to access and view the tapes. The tapes will be available to the research student and are for educational purposes only.

Confidentiality resulting from any publications from the study will be treated with the same confidentiality as the study itself. If exact transcripts are necessary then only the participant's first name will be used.

7. PARTICIPATION AND WITHDRAWAL

If you give permission for your son/daughter to be in this study, they may withdraw at any time without consequences of any kind. Your son/daughter may also refuse to answer any questions they don't want to answer and still remain in the study. The investigator may withdraw your son/daughter from this research if circumstances arise which warrant doing so.

8. IDENTIFICATION OF INVESTIGATORS

If you have any questions or concerns about the research, please feel free to contact Mrs. A. Knott at school.

9. RIGHTS OF RESEARCH SUBJECTS

You may withdraw your consent at any time and discontinue participation without penalty. You are not waiving any legal claims, rights or remedies because of your participation in this research study. If you have questions regarding your son/daughter's rights as a research subject, contact Ms Marléne Fouché at the Research Develop Division, University of Stellenbosch at mfouche@sun.ac.za; 021 808 4622.

If you are willing to let your son/daughter participate in this study, please sign this letter as a declaration of your consent.

______________________ ______________

Parent or guardian's signature **Date**

E-mail address ______________________ Contact number ______________

Knott.

______________________________ 08-10-2012

Signature of Investigator **Date**

Appendix D6 Informed assent to participate in research

UNIVERSITEIT·STELLENBOSCH·UNIVERSITY
jou kennisvennoot • your knowledge partner

STELLENBOSCH UNIVERSITY
INFORMED ASSENT TO PARTICIPATE IN RESEARCH

An analysis of the process of mathematisation in mathematical modelling in secondary school mathematics.
Grade 10 learners of Port Shepstone High School

I am conducting a research study for my master's degree in mathematics education at the University of Stellenbosch. This letter is written to invite you to be a participant in this study. The KwaZulu-Natal Department of Education granted permission for this study to take place.

This study will take place on **Monday** afternoons (from 14:15 – 16:00) and **Saturdays** (from 08:00 to 13:00) starting on **15 October 2012** until **15 November 2012**.

1. PURPOSE OF THE STUDY

The study seeks to understand the mathematical processes when modelling mathematical problems in number patterns.

1. PROCEDURES

If you choose to participate in this study, you will:

- form part of a study group with other Grade 10 learners,
- complete a sequence of selected mathematical problems,
- present your group's findings and solutions to the other groups involved in the study,
- be interviewed during the study to share experiences related to the activities,
- be audio taped during each lesson,
- be videotaped during each lesson,
- complete a criterion referenced problem.

2. POTENTIAL RISKS AND DISCOMFORTS

There are no foreseeable risks or discomforts involved by taking part in this study.

3. POTENTIAL BENEFITS TO SUBJECTS AND/OR TO SOCIETY

The potential benefits of the study to participants may be:

- An increased awareness of the usefulness of mathematics in solving everyday problems.
- An increased enjoyment of solving real life mathematical problems.
- The benefit of collaborating with peers in the group.
- Better understanding and better achievement in the content domain.

4. PAYMENT FOR PARTICIPATION

No payment will be made for the participation in the study.

5. CONFIDENTIALITY

Should you declare yourself willing to participate in this study, confidentiality will be guaranteed. Any information that is obtained in connection with this study and that can be identified with you will remain confidential and will be disclosed only with your permission or as required by law. Confidentiality will be maintained by means of only using the participants' first name initial in the coding procedures; the name of the school will not be disclosed. In the research itself, and in any papers following the research, pseudonyms will be used and not the learners' real names. Data will be kept by the researcher. Data will be locked up and kept for a period of three years after the dissertation is submitted. Thereafter the tapes will be erased. No persons other than the student and supervisor will have access to the data.

The working sessions will be audio recorded and video recorded. You have the right to access and view the tapes. The tapes will be available to the research student and are for educational purposes only.

Confidentiality resulting from any publications from the study will be treated with the same confidentiality as the study itself. If exact transcripts are necessary then only the participant's first name will be used.

6. PARTICIPATION AND WITHDRAWAL

You can choose to be in the study or not. If you volunteer to be in this study, you may withdraw at any time without consequences of any kind. You may also refuse to answer any questions you don't want to answer and still remain in the study. The investigator may withdraw you from this research if circumstances arise which warrant doing so.

7. IDENTIFICATION OF INVESTIGATORS

If you have any questions or concerns about the research, please feel free to contact Mrs A. Knott at school.

8. RIGHTS OF RESEARCH SUBJECTS

You may withdraw your consent at any time and discontinue participation without penalty. You are not waiving any legal claims, rights or remedies because of your participation in this research study. If you have questions regarding your rights as a research subject, contact Ms Marléne Fouché at the Reseach Develop Division, University of Stellenbosch at mfouche@sun.ac.za; 021 808 4622.

If you are willing participate in this study, please sign this letter as a declaration of your consent.

____________________ ____________

Participant's signature **Date**

E-mail address ____________________ Contact number ____________

____________________ 08-10-2012

Signature of Investigator **Date**