On the coefficients of Drinfeld modular forms of higher rank

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Dissertation presented for the degree of Doctor of Philosophy
in the Faculty of Science at Stellenbosch University

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March 2014

Declaration

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Date: January 30, 2014

Abstract

Rank 2 Drinfeld modular forms have been studied for more than 30 years, and while it is known that a higher rank theory could be possible, higher rank Drinfeld modular forms have only recently been defined. In 1988 Gekeler published [Ge2] in which he studies the coefficients of rank 2 Drinfeld modular forms. The goal of this thesis is to perform a similar study of the coefficients of higher rank Drinfeld modular forms.

The main results are that the coefficients themselves are (weak) Drinfeld modular forms, a product formula for the discriminant function, the rationality of certain naturally defined modular forms, and the computation of some Hecke eigenforms and their eigenvalues.

Opsomming

Drinfeld modulêre vorme van rang 2 word al vir meer as 30 jaar bestudeer en alhoewel dit lankal bekend is dat daar Drinfeld modulêre vorme van hoër rang moet bestaan, is die definisie eers onlangs vasgepen. In 1988 het Gekeler die artikel [Ge2] gepubliseer waarin hy die koeffisiënte van Fourier reekse van rang 2 Drinfeld modulêre vorme bestudeer. Die doel van hierdie proefskrif is om dieselfde studie vir Drinfeld modulêre vorme van hoër rang uit te voer.

Die hoofresultate is dat die koeffisiënte self (swak) Drinfeld modulêre vorme is, 'n produk formule vir die diskriminant funksie, die feit dat sekere natuurlik gedefiniëerde modulêre vorme rasionaal is, en die vasstelling van Hecke eievorme en hul eiewaardes.

Acknowledgements

Firstly, I wish to thank the Wilhelm Frank Scholarship Fund for financial support during the last three years, and also the previous two years while I completed my Master's degree. It was an indispensable part of my studies.

Then I would like to thank the members of the staff of the Mathematics division of Stellenbosch University for all the useful discussions I had and for all the opportunities that were provided to learn. I would like, especially, to thank my advisor, Prof. Breuer, for our many discussions, his suggestions and support, and also for suggesting the topic of this thesis.

I would also like to thank Prof. Lenny Taelman for inviting me to visit Leiden University during 2012 and for the time he spent discussing mathematics with me, and Prof. Richard Pink for his time and patience during our discussions when he visited Stellenbosch University.

Thanks also to the examiners with suggestions of improvements in the clarity of certain sections and for some suggestions of further investigation.

Lastly, and most importantly, I wish to thank my wife Remerta, for her patience and understanding in stressful times, and her love and support throughout.

Contents

Introduction

In 1974, Drinfeld's paper [Dr] appeared introducing what is now called a Drinfeld module. His motivation was their use in explicit class field theory in the rank 1 case and some results in rank 2 that may fall under the Langlands programme for function fields. Since then many results have appeared which are strikingly similar to results known about elliptic curves. One such topic is that of Drinfeld modular forms. The existence of such a theory may have been implicit in [Dr], but the first to define them explicitly and study their properties was Goss in his thesis, of which a version is published as [Go1].

Even though Goss's definition is stated in arbitrary rank, a useful form of this definition is only given in rank 2. Many people have developed the theory of one-dimensional (rank 2) Drinfeld modular forms. However, it has been difficult to get a handle on higher dimensional Drinfeld modular forms. The point is that a modular form can be interpreted as a global section of a sheaf on a moduli space and that it should extend to some compactification. In the rank 2 case, the moduli space is an algebraic curve which can essentially only be compactified in one way. A major breakthrough to obtain a higher dimensional theory came when Pink (in [Pi]) constructed a *Satake-compactification* of moduli varieties that behaves well under the natural morphisms. This allowed him to define Drinfeld modular forms of higher rank algebraically. Breuer and Pink then interpreted this algebraic definition analytically to say what "holomorphic at infinity" should be in this case.

Since a holomorphic function is uniquely determined on an admissible open, we may identify a Drinfeld modular form with its Fourier expansion at infinity and hence it makes sense to study its coefficients. Since these are higher dimensional functions, the coefficients are no longer constant, but are themselves functions of one fewer parameter. The main theme of this thesis is to study these coefficients in a similar way to the way Gekeler studied the coefficients in the one-dimensional case in [Ge2]. For example, the coefficients turn out to be (weak) modular forms themselves (Proposition 3.2.7); however not satisfying the "holomorphic at the cusps" conditions.

Here follows an outline of this work. In Chapter 1 we give a short overview of the classical analogues (elliptic curves and elliptic modular forms) of the objects we shall work with. Chapter 2 gives a basic introduction to Drinfeld modules and a quick overview of their moduli space and its associated rigid analytic structure.

Chapter 3 starts our discussion of Drinfeld modular forms. Here we give the definitions of weak modular forms, modular forms, Fourier expansions at infinity and calculate some of these expansions (though often only up to the first term). At the end there is a discussion as to when such a modular form should be considered a "rational Drinfeld modular form," which might lead to questions about the behaviour of modular forms under reduction modulo ideals in A. Lastly, in Chapter 4 we define Hecke operators on the space of modular forms of weight k for the full modular group and calculate some eigenvectors and their eigenvalues, as well as proving that the Hecke algebra is completely multiplicative.

Since this is a thesis I give an outline of what my own contributions are and what I learned elsewhere. This is especially necessary in this case, since the article [BP] is not yet available and some of their results have to be reproduced in order to have a self-contained treatment. I also give such indications in the text, but here everything is together.

In Chapter 2, almost everything has been known for many years. The only results that do not appear in the literature are Lemma 2.6.15 and Proposition 2.6.16. Breuer and Pink suspected that every function on $\mathcal{U}$ must have a Laurent series expansion, but the current statement and proof of Proposition 2.6.16 are novel — the same goes for Lemma 2.6.15 on which it relies.

The material in Chapter 3 builds on the work by Breuer and Pink [BP]. Since that work is not yet available, it was necessary to reproduce their results here for the sake of completeness. Almost everything up to the end of section 3.3, with some minor modifications, are due to [BP]. I made the following modifications:

- In Definition 3.2.1 I changed their definition

$$u_{\tilde{\omega}}(\omega_1) = e_{\Lambda_U}(\omega_1)^{-1}$$

  to the way it appears in Definition 3.2.1. This is similar to the way the parameter changed for rank 2 Drinfeld modular forms. The reason for this change is that this allows us to study the rationality of Drinfeld modular forms in Section 3.6.

- By including Lemma 3.2.2 and Corollary 3.2.3, I managed to refine their argument in Proposition 3.2.4 to include the last line: "*Moreover, for every* $n \in \mathbb{N}$ *there exists* $r > 0$ *such that* $B(0,r) \times \Omega_n^{r-1} \subset \mathcal{U}$." This allows us to apply Proposition 2.6.16.

- I learned the statement of Proposition 3.2.9 from [BP], but supplied my own proof.

- The statements of Proposition 3.3.2 (a) and (c) were implicit in [BP], but I made them explicit, added (b) and supplied the proof.

- Proposition 3.2.7 is completely new.

The examples from section 3.4 appear in [BP]. However, I also made slight modifications to these examples, for example replacing the lattice A^r by a more general lattice of the form $\Lambda = A \times \tilde{\Lambda}$. This provides examples of Drinfeld modular forms on components that do not correspond to free A-modules. Another change was changing the definition of Eisenstein series for $\Gamma(N)$ (and by extension the coefficient forms) to its current form using cosets in $N^{-1}\Lambda/\Lambda$ instead of cosets in $\Lambda/N\Lambda$. This makes the presentation more natural and ensures that the consequent definitions of coefficient forms work for arbitrary ideals, and not only principal ideals. The only really new idea that was needed for this translation was the argument that $N \subseteq (a_1+v_1)^{-1}A$ during the proof of Proposition 3.4.2.

The computations of u-expansions of Drinfeld modular forms in Section 3.5 are mostly my own work. The expansion for Eisenstein series for $GL_r(A)$ are very similar to the rank 2 expansions in the original work of Goss, and use essentially the same techniques. The calculation of the expansion for Eisenstein series for $\Gamma(N)$ depends on what was obtained by Breuer and Pink up to equation (3.4) (with some modifications due to a change in the definition). However, the rest of the calculation in section 3.5 is new. The product formula for the discriminant function is also new. This provides a generalization of the formula by Gekeler in [Ge1], which is different from the generalization by Hamahata in [Ha].

In 3.6, subsection 3.6.1 was known and appears in [Ge2], while everything in 3.6.2 is new.

In Chapter 4, section 4.1 appears in [Sh] and section 4.2 is an (almost) direct translation of [Sh] Chapter 3.2 from $\mathrm{SL}_r(\mathbb{Z})$ to $\mathrm{GL}_r(A)$. Section 4.3 is new.

Chapter 1

The classical case

In this chapter, we quickly review the basic definitions of elliptic modular forms, in order to give some perspective for the analogous theory of Drinfeld modular forms that we shall discuss later. We shall restrict ourselves to discussing only elliptic modular forms. Even though our goal is to define Drinfeld modular forms of higher rank, i.e. of more than one variable, the results obtained are more closely related to those in the theory of elliptic modular forms than in modular forms in more variables, like Hilbert or Siegel modular forms. The reason for this is that the Drinfeld modular forms will have expansions in one variable, similar to the elliptic modular forms case.

1.1 Modular forms

Let $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ be the upper half-plane. There is a natural action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathcal{H}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Now, for every $n \in \mathbb{Z}$, define $\Gamma(n)$ as the set of matrices with integer entries that are congruent to the identity matrix modulo n. The group $\Gamma(n)$ is called a *principal congruence subgroup* of $\mathrm{SL}_2(\mathbb{Z})$. Any group Γ satisfying $\Gamma(n) \subseteq \Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ for some $n \in \mathbb{N}$ is called a *congruence subgroup* of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 1.1.1. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and k be an integer. A* weak modular form *of weight k with respect to Γ is a function $f : \mathcal{H} \to \mathbb{C}$ such that*

(a) *f is holomorphic on $\mathcal{H}$;*

(b) *for any* $\gamma \in \Gamma$ *of the form* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and any* $z \in \mathcal{H}$, *we have* $f(\gamma(z)) = (cz+d)^k f(z)$.

We may rewrite (b) slightly by rephrasing it as invariance by a certain action. So, for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}),$$

define

$$f[\gamma]_k(z) = (\det\gamma)^{k-1}(cz+d)^{-k} f(\gamma(z)).$$

A simple computation shows that this indeed defines an action of $\mathrm{GL}_2(\mathbb{R})$ on the set of holomorphic functions $f : \mathcal{H} \to \mathbb{C}$. Then (b) may be rephrased as

(b′) for any $\gamma \in \Gamma$ we have $f[\gamma]_k = f$ as functions on $\mathcal{H}$.

Note that the set of weak modular forms has the structure of a $\mathbb{C}$-vector space. In general it is infinite dimensional. We need another condition called "holomorphic at infinity" to find a useful finite dimensional subspace.

By definition, a congruence subgroup Γ contains a principal congruence subgroup $\Gamma(n)$, and hence contains a translation element

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

This might not be the smallest translation element, but its existence implies the existence of a smallest one. Define h to be the smallest positive integer such that

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma.$$

If we let γ be the matrix above, then $f[\gamma]_k(z) = f(z+h)$. Hence, if f is a weak modular form, then f is h-periodic. Now, any h-periodic function g on $\mathcal{H}$ factors through $q_h : \mathcal{H} \to D'$, $z \mapsto e^{2\pi z/h}$, where $D' = \{z \in \mathbb{C} \,|\, 0 < |z| < 1\}$ is the punctured unit disc (i.e. $g = \tilde{g} \circ q_h$, where $\tilde{g} : D' \to \mathbb{C}$). Moreover, if g is holomorphic on $\mathcal{H}$, then $\tilde{g}$ is holomorphic on D'. If $\tilde{g}$ turns out to be holomorphic on the whole unit disc, we say that g is *holomorphic at infinity*. This is equivalent to saying that g has an expansion of the form

$$g(z) = \sum_{n \geq 0} a_n q_h^n.$$

In order for the vector space of modular forms to be finite, we need the condition holomorphic at infinity, but we shall also need this condition at other "limit points." We define the set of cusps for Γ as the set of orbits of $\mathbb{P}^1(\mathbb{Q})$ under the action of $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\frac{m}{n}\right) = \frac{am+bn}{cm+dn} \quad \text{for } \frac{m}{n} \neq \frac{-d}{c},$$

$\gamma\left(\frac{-d}{c}\right) = \infty$ and $\gamma(\infty) = \frac{a}{c}$ (if $c = 0$, then $\gamma(\infty) = \infty$). The set of cusps is finite, since $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$ and $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ is finite.

Definition 1.1.2. *A weak modular form of weight k with respect to Γ is said to be a* modular form *of weight k for Γ if also*

(c) for every $\delta \in \mathrm{SL}_2(\mathbb{Z})$, the function $f[\delta]_k$ is holomorphic at infinity.

Moreover, if the Fourier expansion of $f[\delta]_k$ at infinity has 0 constant term for every δ, then we call f a cusp form.

We shall denote the $\mathbb{C}$-vector space of weight k modular forms for Γ by $\mathcal{M}_k(\Gamma)$.

In practice it is not necessary to check (c) for all $\delta \in \mathrm{SL}_2(\mathbb{Z})$, but only for a finite set of coset representatives of $\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})$.

Example. The Eisenstein series of weight $k \in \mathbb{Z}$ is the function $G_k : \mathcal{H} \to \mathbb{C}$ defined by

$$G_k(\tau) := \sum_{(c,d)\in\mathbb{Z}^2\setminus(0,0)} (c\tau+d)^{-k}.$$

It turns out that if $k \geq 4$, then this sum is convergent and that if k is even, then it is a non-zero modular form for $\mathrm{SL}_2(\mathbb{Z})$. Moreover, each modular form for $\mathrm{SL}_2(\mathbb{Z})$ is a polynomial in G_4 and G_6.

The discriminant function

$$\Delta(\tau) := (60G_4(\tau))^3 - 27\,(140G_6(\tau))^2$$

turns out to be the non-zero cusp form of lowest possible weight.

The weight k ($k \geq 2$, even) Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ has Fourier expansion ([DS] §1.1)

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n,$$

where ζ is the Riemann zeta function, and $\sigma_i(n)$ is the sum of the i-th powers of the divisors of n.

The q-expansion of the discriminant function Δ can be computed from a product formula ([DS] following Proposition 1.2.5):

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1-q^n)^{24}.$$

We shall prove a generalization of this formula in section 3.5.4.

1.2 Hecke operators

Hecke operators are linear operators between vector spaces $\mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$ for congruence subgroups Γ_1, Γ_2. In the case when $\Gamma_1 = \Gamma_2$ we have an endomorphism of $\mathbb{C}$-vector spaces, and there are many results on the structure of such operators. For example, the Spectral Theorem tells us that there exists a basis of cusp forms that form a system of simultaneous eigenforms for a certain infinite set of Hecke operators. In this section we simply give an indication of available results and omit details. For certain details, the reader can refer to Chapter 4, where the function field analogue is treated in more detail, and for other details we refer the reader to [DS] Chapter 5.

Let Γ_1 and Γ_2 be congruence subgroups, let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, and consider the double coset $\Gamma_1 \alpha \Gamma_2$. It can be written as the disjoint union of right cosets $\bigcup_i \Gamma_1 \beta_i$. We then define the Hecke operator associated to the double coset $\Gamma_1 \alpha \Gamma_2 : \mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$ by

$$f[\Gamma_1 \alpha \Gamma_2]_k = \sum_i f[\beta_i]_k.$$

It is not hard to check that it is well-defined and that it sends modular forms for Γ_1 to modular forms for Γ_2 and cusp forms for Γ_1 to cusp forms for Γ_2. In the special case when $\Gamma_1 = \Gamma_2$ and $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ $(a \in \mathbb{Z}_+)$, we denote the operator T_a. It turns out that these operators are multiplicative in the sense that if $\gcd(a,b) = 1$, then $T_{ab} = T_a T_b$.

It turns out that the eigenvectors for these operators (called eigenforms) have Fourier expansions that are of arithmetic interest. For example, the coefficients are multiplicative — i.e. if m, n and N (the level) are relatively prime, then $a_{mn} = a_m a_n$.

Example. Each Eisenstein series G_k is an eigenform for each operator T_n on the space of weight k modular forms for $\mathrm{SL}_2(\mathbb{Z})$. The discriminant function Δ is also an eigenform, since the space of cusp forms that contains it is one-dimensional.

Chapter 2

Drinfeld modules

From now on we let $A = \mathcal{O}_{\mathcal{X}}(\mathcal{X} \smallsetminus \infty)$ be the coordinate ring of a smooth, projective curve $\mathcal{X}$ over the finite field $\mathbb{F}_q$, minus one point denoted by ∞. The basic example is the polynomial ring $\mathbb{F}_q[t]$, where ∞ is the point for which its associated absolute value is $\left|\frac{f(t)}{g(t)}\right| = q^{\deg g - \deg f}$. In fact, we shall often make the simplifying assumption that $A = \mathbb{F}_q[t]$. A ring as described above is called a *Drinfeld ring*.

Let F be the fraction field of A, let F_∞ be the completion of F with respect to the valuation associated to the point ∞, let π be a uniformizing parameter in F_∞, let $A_\infty = \mathbb{F}_{q^{\deg \infty}}[\![\pi]\!]$ be the ring of elements in F_∞ that are regular at ∞, and let $\mathbb{C}_\infty$ be the completion of an algebraic closure of F_∞. By Krasner's Lemma, $\mathbb{C}_\infty$ remains algebraically closed. When speaking of an absolute value on $\mathbb{C}_\infty$, we shall always mean the unique extension of the absolute value on A associated to ∞, and we shall denote the valuation by $v(z)$. When $a \in A$, we shall often write $\deg a$ in stead of $v(a)$. One should think of A, F, F_∞ and $\mathbb{C}_\infty$ as analogues of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$, respectively. One of the reasons for this construction is that now one is able to develop a function theory and a geometric theory over $\mathbb{C}_\infty$.

Later on we shall also need the rings $\hat{A} = \varprojlim(A/aA)$, the profinite completion of A, and $\mathbb{A}_F^f = \hat{A} \otimes_A F$, the ring of finite adeles of A.

In this chapter, and later, we shall encounter many sums or products over expressions involving the non-zero elements of a set. We shall denote a sum or product over the non-zero elements of a set S by

$$\sum_{x \in S}{}' f(x) \qquad \text{or} \qquad \prod_{x \in S}{}' f(x).$$

2.1 Analysis on $\mathbb{C}_\infty$

Since the absolute value on $\mathbb{C}_\infty$ is non-archimedean, an infinite sum $\sum_{n\geq 0} a_n$, where $a_n \in \mathbb{C}_\infty$ for $n \geq 1$, converges if and only if $\lim_{n\to\infty} a_n \to 0$. Hence one may determine for which X a power series $\sum_{n\geq 0} a_n X^n$ converges.

Proposition 2.1.1. *Let $f(X) = \sum_{n\geq 0} a_n X^n \in \mathbb{C}_\infty[\![X]\!]$ be a power series in X with coefficients in $\mathbb{C}_\infty$. Then f defines a function on the open ball $|X| < R(f)$ (taking values in $\mathbb{C}_\infty$), where $R(f) = \liminf_{n\to\infty} |a_n|^{-1/n}$ is the* radius of convergence.

Proof. Whenever $|X| < R(f)$, we have $\lim_{n\to\infty} |a_n X^n| = 0$, and thus the series converges to a value in $\mathbb{C}_\infty$. □

Proposition 2.1.2. *For any $r < R(f)$, the function f has only finitely many zeros in the closed disc $|X| \leq r$.*

Proof. [Go4] Proposition 2.11. □

Definition 2.1.3. *The function $f(X) = \sum_{n\geq 0} a_n X^n$ is* entire *if $R(f) = \infty$, or equivalently, if the series converges for all $X \in \mathbb{C}_\infty$.*

Proposition 2.1.4. (a) *Every non-constant entire function $f(X)$ has a zero.*

(b) *Every non-constant entire function $f(X)$ is surjective.*

Proof. (a) is a direct consequence of the study of Newton polygons preceding Proposition 2.13 in [Go4], while (b) is simply (a) applied to $f(X) - c$ for an arbitrary $c \in \mathbb{C}_\infty$. □

In the following theorem we encounter an infinite product of linear terms. Under the conditions of the theorem it will define a function. We should mention explicitly that the function we are defining is the uniform limit of the polynomials defined by taking only finitely many factors at a time.

Theorem 2.1.5 (Weierstrass Factorization Theorem)**.** *Suppose that $f(X)$ is an entire function with non-zero roots $(\lambda_1, \lambda_2, \dots)$ listed with multiplicity. Also suppose that $f(X)$ has 0 as a root with multiplicity m (possibly 0). Then, for some constant $c \in \mathbb{C}_\infty$ we have*

$$\lim_{n\to\infty} |\lambda_n| = 0 \quad \textit{and} \quad f(X) = cX^m \prod_{i\geq 1} \left(1 - \frac{X}{\lambda_i}\right).$$

Conversely, given $c \in \mathbb{C}_\infty$, $m \in \mathbb{Z}_{\geq 0}$ and a sequence $(\lambda_1, \lambda_2, \dots)$ for which $\lim_{n\to\infty} |\lambda_n| = 0$, the above product defines an entire function.

Proof. If we let $g(X) = X^m \prod_{i\geq 1}\left(1 - \frac{X}{\lambda_i}\right)$, then $(f/g)(X)$ is an entire function with no zeros, and hence, by Proposition 2.1.4 (a), constant.

Conversely, let $N \in \mathbb{R}$, and suppose that there are k of the λ_i which are less than nor equal to $N/|\lambda_1|$. Then

$$\left|\prod_{i=1}^{2k}\lambda_i\right| = \left|\prod_{i=1}^{k}(\lambda_i\lambda_{2k+1-i})\right| \geq \prod_{i=1}^{k}\frac{N}{|\lambda_1|\,|\lambda_{k+1}|} \geq N^k.$$

This implies that coefficient c_{2k} in the product expansion satisfies $c_{2k}^{1/2k} \leq 1/\sqrt{N}$. Since N was chosen arbitrarily, this means that the resulting function is entire. $\square$

2.2 Exponential functions

Let L be an $\mathbb{F}_q$-linear subspace of $\mathbb{C}_\infty$ (not necessarily finite dimensional). We define the *exponential function* associated to L by

$$e_L(X) = X \prod_{\lambda\in L}{}' \left(1 - \frac{X}{\lambda}\right).$$

By Theorem 2.1.5, the product only converges to an entire function if any ball of finite radius contains only finitely many elements of L. If L satisfies this property, we call L *strongly discrete*. It turns out that $e_L(X)$ is an $\mathbb{F}_q$-linear function in the sense of the following proposition.

Proposition 2.2.1. *Let L be a strongly discrete $\mathbb{F}_q$-subspace of $\mathbb{C}_\infty$. Then the function $e_L(X)$ is $\mathbb{F}_q$-linear, i.e. satisfies the following properties:*

(a) *$e_L(cX) = ce_L(X)$ for all $X \in \mathbb{C}_\infty$ and all $c \in \mathbb{F}_q$;*

(b) *$e_L(X + Y) = e_L(X) + e_L(Y)$ for all $X, Y \in \mathbb{C}_\infty$.*

Proof. (a) If $c = 0$, it is clear, since both sides are 0. Otherwise the zero set of $e_L(X)$ is L, while the zero set of $e_L(cX)$ is $\{c^{-1}\lambda \,|\, \lambda \in L\} = L$, since L is an $\mathbb{F}_q$ vector space. The equality follows by comparing the coefficients of X.

(b) Firstly, suppose that L is finite, and hence that $e_L(X)$ is a polynomial in X. For some $Y \in \mathbb{C}_\infty$, consider the polynomial $h(X) = e_L(X+Y) - e_L(X) - e_L(Y)$. Its degree is clearly less than that of $e_L(X)$. However, every $X = z \in L$ is a root of h. Indeed, if $z \in L$, then $e_L(z) = 0$ and the roots of $e_L(z+Y)$ (as a polynomial in Y) is the set $\{\lambda - z \mid \lambda \in L\} = L$. This means that h has more roots than its degree, hence is identically 0 as a function.

Now, viewing Y as a variable, the polynomial $e_L(X+Y) - e_L(X) - e_L(Y) \in \mathbb{C}_\infty[X][Y]$ is 0 for every $Y \in \mathbb{C}_\infty$. Since $\mathbb{C}_\infty$ is an infinite field, this means that it is the zero polynomial.

The results follows by writing $L = \bigcup L_i$ as a union of finite $\mathbb{F}_q$-subspaces of $\mathbb{C}_\infty$ and noting that $e_L(X) = \lim\limits_{i\to\infty} e_{L_i}(X)$.

□

In fact, all separable entire $\mathbb{F}_q$-linear functions are constant multiples of exponential functions.

Proposition 2.2.2. *Let $f(X)$ be an entire function for which $f'(X)$, the formal derivative of f, has no common zeros with $f(X)$. Also suppose that $f(X)$ is $\mathbb{F}_q$-linear. Then its set of zeros, form a sub-$\mathbb{F}_q$-vector space of $\mathbb{C}_\infty$.*

Proof. Note that if z_1 and z_2 are zeros of f, then $f(cz_1) = cf(z_1) = 0$ and $f(z_1 + z_2) = f(z_1) + f(z_2) = 0$ as well. By the asumption on f', the roots are all simple. □

By Proposition 2.2.1 we now know that $e_L(X)$ is $\mathbb{F}_q$-linear, and thus that its power series expansion in X has non-zero coefficients only for those powers of X whose exponent is a power of q. We write

$$e_L(X) = \sum_{n\geq 0} e_n(L) X^{q^n} \tag{2.1}$$

making explicit the dependence of the coefficients on the set L. Furthermore, we know that f is entire, and hence that it is surjective.

Proposition 2.2.3. *The function $e_L : \mathbb{C}_\infty \to \mathbb{C}_\infty$ induces an isomorphism of additive groups $L\backslash\mathbb{C}_\infty \xrightarrow{\sim} \mathbb{C}_\infty$.*

Proof. We already know that it is a well-defined group homomorphism and that it is surjective. Since the kernel of e_L is L, the map is an isomorphism.

□

Lemma 2.2.4. *Let L be a strongly discrete $\mathbb{F}_q$-subspace of $\mathbb{C}_\infty$ and $z \in \mathbb{C}_\infty$. We have the estimate*

$$|e_L(z)| \geq \min\{|z - \lambda| : \lambda \in L\}.$$

Proof. By replacing z by a suitable $z + h$ $(h \in L)$, we may assume that $|z| = \min\{z - \lambda : \lambda \in L\}$. We have

$$\begin{aligned} |e_L(z)| &= |z| \prod_{\lambda \in L}{}' \left|1 - \frac{z}{\lambda}\right| \\ &= |z| \prod_{\lambda \in L}{}' \left|\frac{\lambda - z}{\lambda}\right|. \end{aligned}$$

Now we split up the product into those factors where $|z| < |\lambda|$ (in which case $\left|1 - \frac{z}{\lambda}\right| = 1$), where $|z| > |\lambda|$ (in which case $\left|1 - \frac{z}{\lambda}\right| = \left|\frac{z}{\lambda}\right| > 1$) and where $|z| = |\lambda|$. In the latter case, our choice of z implies $|z - \lambda| \geq |z| = |\lambda|$, and we conclude that $|e_\Lambda(z)| \geq |z|$. □

Proposition 2.2.5. (a) *Let L be a strongly discrete $\mathbb{F}_q$-linear subspace of $\mathbb{C}_\infty$, and $c \in \mathbb{C}_\infty$. Then*

$$e_{cL}(cX) = c e_L(X).$$

(b) *Suppose that L and M are strongly discrete $\mathbb{F}_q$-linear subspaces of $\mathbb{C}_\infty$ such that $L \subset M$. Then $e_L(M)$ is a strongly discrete $\mathbb{F}_q$-linear subspace of $\mathbb{C}_\infty$ and*

$$e_M(X) = e_{e_L(M)}(e_L(X)) \quad \textit{as power series in } X.$$

Proof. For (a) note that $e_{cL}(cX) = cX \prod_{\lambda \in L}{}' \left(1 - \frac{cX}{c\lambda}\right) = c e_L(X)$.

For (b), let S be a set of coset representatives for M/L such that each representative has minimum absolute value in the coset (this is possible since M is strongly discrete), and let $r > 0$. Pick $s \in S$ such that $|e_L(s)| < r$. By Lemma 2.2.4, this means that $\min\{|s - \lambda| : \lambda \in L\} \leq |e_L(s)| < r$. The ball around 0 with radius r has only finitely many elements, and hence s must be in a coset to which one of these elements belong. Thus, there are only finitely many choices of s for which $|e_L(s)| < r$. This means that $e_L(M)$ is strongly discrete.

The function $e_M(X)$ has simple zeros at exactly the elements of M. The function $e_{e_L(M)}(e_L(X))$ has zeros exactly when $e_L(X) \in e_L(M)$, which happens exactly when $X \in M$. Thus, the functions on the left and right hand sides have the same zero sets. The equality follows from the fact that the derivative of both sides is equal to 1. □

Like in the classical case there is also an inverse to the exponential function — the logarithmic function associated to L. We define it as the power series inverse of $e_L(X)$, i.e. as the unique power series $\log_L(X)$ such that $e_L(\log_L(X)) = \log_L(e_L(X)) = X$. The coefficients of $\log_L(X)$ may be computed by calculating these compositions of power series and comparing coefficients. It turns out that $\log_L(X)$ also has non-zero coefficients if and only if the corresponding power of X is a power of q. So, we let

$$\log_L(X) = \sum_{n\geq 0} \beta_n X^{q^n}. \tag{2.2}$$

This function is not entire, but it has a positive radius of convergence.

In Lemma 3.4.13 we shall give another interpretation of these coefficients as certain modular forms.

2.3 Drinfeld modules

To define Drinfeld modules, we shall pay special attention to those exponential functions associated to sets with even more structure — that of A-submodules of $\mathbb{C}_\infty$. In our analogy, this corresponds to $\mathbb{Z}$-submodules of $\mathbb{C}$ or lattices, which are important in the theory of elliptic curves and elliptic modular forms.

Definition 2.3.1. *A* lattice Λ *of rank r is a projective A-submodule of $\mathbb{C}_\infty$ of rank r which is strongly discrete in $\mathbb{C}_\infty$.*

The last property is necessary, since this ensures that the associated exponential function is defined. Also note that since the modules we are considering are submodules of the field $\mathbb{C}_\infty$, projective is the same as finitely generated.

Proposition 2.3.2. *A projective module $\Lambda \subset \mathbb{C}_\infty$ of rank r is a lattice if and only if $F_\infty\Lambda$ is an F_∞-vector space of dimension r.*[1]

Proof. [Go4] Propositions 4.6.2 and 4.6.3. □

Theorem 2.3.3. *Let Λ be a lattice of rank r, and $a \in A$. Then*

$$e_\Lambda(aX) = \varphi_a^\Lambda(e_\Lambda(X)), \tag{2.3}$$

[1] Note the difference between $\Lambda \otimes_A F_\infty$, an abstract r dimensional vector space, and $F_\infty\Lambda$, an F_∞-sub-vector space of $\mathbb{C}_\infty$ which may have dimension less than r.

where $\varphi_a^\Lambda(X)$ is the polynomial of degree $|a|^r = q^{r\deg a}$ given by

$$aX \prod_{\lambda\in\frac{1}{a}\Lambda/\Lambda}{}' \left(1 - \frac{X}{e_\Lambda(\lambda)}\right). \tag{2.4}$$

Proof. By Proposition 2.2.5 with $M = \frac{1}{a}\Lambda$ and $L = \Lambda$ we get $e_\Lambda(aX) = ae_{\frac{1}{a}\Lambda}(X) = \varphi_a^\Lambda(e_\Lambda(X))$. □

It is worth noting that the degree of the polynomial φ_a^Λ is $|a|^r = q^{r\deg a}$. From equation (2.3) we deduce that $\varphi_a^\Lambda(\varphi_b^\Lambda(X)) = \varphi_{ab}^\Lambda(X) = \varphi_b^\Lambda(\varphi_a^\Lambda(X))$. So, in fact, the map $a \to \varphi_a^\Lambda$ defines a ring homomorphism $\varphi^\Lambda : A \to \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a(\mathbb{C}_\infty))$, where the latter is the ring of $\mathbb{F}_q$-linear group endomorphisms of $\mathbb{C}_\infty$.

Given any such ring homomorphism, the map $A \times \mathbb{C}_\infty \to \mathbb{C}_\infty$, $(a, z) \mapsto \varphi_a(z)$ defines an A-module structure on $\mathbb{C}_\infty$, which is quite different from the usual structure. This is the *Drinfeld module* structure.

Definition 2.3.4. *Let k be a field for which there exists a morphism $\iota : A \to k$. A* Drinfeld A-module *over k is a ring homomorphism $\varphi : A \to End_{\mathbb{F}_q}(\mathbb{G}_a(k))$ such that its derivative $d\varphi = \iota$, and $\varphi \neq \iota$.*

Proposition 2.3.5. (a) *Let φ be a Drinfeld A-module over a field k. There exists an integer r such that for every $a \in A$, $\varphi_a(X)$ is a polynomial of degree $q^{r\deg(a)}$. This integer is called the* rank *of φ.*

(b) *If Λ is a lattice of rank r, then φ^Λ is a Drinfeld module of rank r.*

Proof. The proof of (a) can be found in [Go4] Proposition 4.5.3., while for (b) simply note that for any non-zero $a \in A$, the index $[\Lambda : a\Lambda] = q^{r\deg a}$ and hence, by equation (2.4) the polynomial φ_a^Λ has degree $q^{r\deg a}$. □

For a lattice Λ this means that φ^Λ is a Drinfeld module. We call it the Drinfeld module associated to Λ. It turns out that every Drinfeld module over $\mathbb{C}_\infty$ is a Drinfeld module associated to some lattice (Theorem 2.4.4).

Example. As a special example of a Drinfeld module we mention the *Carlitz module*. Assume that $A = \mathbb{F}_q[t]$. Then the Carlitz module is the unique Drinfeld module φ for which $\varphi_t(X) = tX + X^q$. By the Uniformization Theorem for Drinfeld modules (Theorem 2.4.4) there exists a lattice L of rank 1 such that $\varphi_L = \varphi$. Define $\bar{\pi} \in \mathbb{C}_\infty$ such that $L = \bar{\pi}A$. We call $\bar{\pi}$ the *Carlitz period*. It is a number which is transcendental over F (just like $\pi \in \mathbb{R}$ is transcendental over $\mathbb{Q}$) and various formulas can be given for it, e.g.

$$\bar{\pi} = \zeta \prod_{i\geq 1} \left(1 - \frac{t^{q^j} - t}{t^{q^{j+1}} - t}\right)$$

where ζ is a $(q-1)^{\text{st}}$ root of -1, which is defined up to a multiple of $\mathbb{F}_q$ (see [Go4] §3.2., where it is denoted by ξ).

2.4 Morphisms of Drinfeld modules

Definition 2.4.1. *If Λ_1 and Λ_2 are two lattices of the same rank, we define a morphism of lattices $c : \Lambda_1 \to \Lambda_2$ as an element $c \in \mathbb{C}_\infty$ such that $c\Lambda_1 \subseteq \Lambda_2$. We shall not consider any morphisms between lattices of different rank.*

Proposition 2.4.2. *Let $c : \Lambda_1 \to \Lambda_2$ be a morphism of lattices. Then*

$$P_c(X) = cX \prod_{\lambda \in c^{-1}\Lambda_2/\Lambda_1}{}' \left(1 - \frac{X}{e_{\Lambda_1}(\lambda)}\right) \tag{2.5}$$

is an $\mathbb{F}_q$-linear polynomial for which $P_c(\varphi_a^{\Lambda_1}(X)) = \varphi_a^{\Lambda_2}(P_c(X))$ for all $a \in A$.

Proof. Note that $c^{-1}\Lambda_2/\Lambda_1$ is an $\mathbb{F}_q$-vector space, which is finite, since Λ_1 and Λ_2 have the same rank. Hence equation (2.5) defines an $\mathbb{F}_q$-linear polynomial.

Now note that $P_c(e_{\Lambda_1}(X))$ is an entire function with simple zeros at the points of $c^{-1}\Lambda_2$ and with derivative c. Thus $P_c(e_{\Lambda_1}(X)) = ce_{c^{-1}\Lambda_2}(X) = e_{\Lambda_2}(cX)$ by Proposition 2.2.5 (a).

Replacing X by aX this becomes

$$P_c(\varphi_a^{\Lambda_1}(e_{\Lambda_1}(X))) = P_c(e_{\Lambda_1}(aX)) = e_{\Lambda_2}(aX) = \varphi_a^{\Lambda_2}(e_{\Lambda_2}(cX)) = \varphi_a^{\Lambda_2}(P_c(e_{\Lambda_1}(X))),$$

the last following from the final equation in the previous paragraph. Since e_{Λ_1} is surjective, it follows that $P_c(\varphi_a^{\Lambda_1}(X)) = \varphi_a^{\Lambda_2}(P_c(X))$. □

Definition 2.4.3. *Let φ and ψ be two Drinfeld modules of the same rank. A morphism $f : \varphi \to \psi$ is a polynomial $p(X)$ such that $p(\varphi_a(X)) = \psi_a(p(X))$ for all $a \in A$.*

Theorem 2.4.4 (Uniformization Theorem for Drinfeld modules)**.** *The association $\Lambda \mapsto \varphi^\Lambda$, $(c : \Lambda_1 \to \Lambda_2) \mapsto (P_c : \varphi^{\Lambda_1} \to \varphi^{\Lambda_2})$ defines an equivalence between the category of A-lattices of rank r in $\mathbb{C}_\infty$ and the category of Drinfeld A-modules of rank r over $\mathbb{C}_\infty$.*

Proof. [Go4] Theorem 4.6.9. □

We say that two lattices Λ_1 and Λ_2 are *similar* or *homothetic* if $c\Lambda_1 = \Lambda_2$ for some $c \in \mathbb{C}_\infty$. This defines an equivalence relation on the set of lattices.

It is easy to see that in this case the associated morphism P_c is linear. Any linear polynomial aX has an inverse (under composition) $\frac{X}{a}$. This means that c induces an isomorphism of Drinfeld modules. On the other hand, if $P : \varphi^{\Lambda_1} \to \varphi^{\Lambda_2}$ is an isomorphism, then it must be linear, since no other polynomial has an inverse. Then this morphism could only come from a $c \in \mathbb{C}_\infty$ for which $c\Lambda_1 = \Lambda_2$. Let us finish this section by comparing the coefficients of isomorphic Drinfeld modules.

Proposition 2.4.5. *Let $c : \varphi \to \psi$ be an isomorphism of Drinfeld modules, where $c(X) = cX$. Let $a \in A$. If $\varphi_a(X) = aX + g_1X^q + \cdots + g_nX^{q^n}$ and $\psi_a(X) = aX + h_1X^q + \cdots + h_nX^{q^n}$, then $h_i = c^{1-q^i}g_i$ for every $i = 1, \ldots, n$.*

Proof. By definition of a morphism of Drinfeld modules, we have $c\varphi_a(X) = \psi_a(cX)$, and by comparing the coefficients of X^{q^i} we obtain $cg_i = c^{q^i}h_i$, yielding the result. □

2.5 Goss polynomials

Later we shall study many expressions of the form

$$S_{k,\Lambda}(z) := \sum_{\lambda\in\Lambda}(z+\lambda)^{-k}.$$

It turns out that for any $k \geq 1$, $S_{k,\Lambda}$ is a polynomial in $S_{1,\Lambda}$, so in some respects, it will be sufficient to study $S_{1,\Lambda}$. We also give the following lemma for later use.

Lemma 2.5.1. *In the notation above, we have $S_{1,\Lambda}(X) = \dfrac{1}{e_\Lambda(X)}$.*

Proof. Note that the derivative of $e_\Lambda(X)$ is 1. Therefore taking the logarithmic derivative on both sides of

$$e_\Lambda(X) = X\prod_{\lambda\in\Lambda}{}' \left(1 - \frac{X}{\lambda}\right)$$

yields the result. □

Proposition 2.5.2. *Let $\Lambda \subset \mathbb{C}_\infty$ be a strongly discrete $\mathbb{F}_q$-linear set. There exist polynomials $P_{k,\Lambda}$ such that $S_{k,\Lambda} = P_{k,\Lambda}(S_{1,\Lambda})$. These polynomials also have the following properties:*

(a) $P_{k,\Lambda}(X) = X\left(P_{k-1,\Lambda}(X) + e_1(\Lambda)P_{k-q,\Lambda}(X) + e_2(\Lambda)P_{k-q^2,\Lambda}(X) + \cdots\right)$, *where we make the convention that $P_{k,\Lambda}(X) = 0$ if $k < 0$;*

(b) $P_{k,\Lambda}$ *is monic of degree* k*;*

(c) $P_{k,\Lambda}(0) = 0$ *and* $X^2 \mid P_{k,\Lambda}(X)$ *if* $k \geq 2$*;*

(d) $P_{pk,\Lambda}(X) = P_{k,\Lambda}(X)^p$*;*

(e) *if* $k \leq q$*, then* $P_{k,\Lambda}(X) = X^k$*;*

(f) *for* $k = q^j - 1$ *we have the formula* $P_{q^j-1}(X) = \sum_{0 \leq i < j} \beta_i X^{q^j - q^i}$*, where the* β_i *are the coefficients of the logarithm function from equation* (2.2)*;*

(g) *The indices of the non-zero coefficients of* $P_{k,\Lambda}$ *are all congruent to* k *modulo* $q-1$*;*

(h) *The coefficients of* $P_{k,\Lambda}(X)$ *lie in the ring* $\mathbb{F}_q[e_1(\Lambda), \ldots, e_m(\Lambda)]$*,*[2] *where* m *is chosen such that* $q^m \leq k < q^{m+1}$*;*

(i) $X^2 P'_{k,\Lambda}(X) = k P_{k+1,\Lambda}(X)$*.*

Proof. The proofs of (a)–(f) and (i) appear in [Ge3] and is reproduced almost exactly since we believe that it is not readily available.

The statement is clearly true for $k = 1$, with $P_{1,\Lambda}(X) = X$ for any Λ. The rest of the proof relies on the Newton relations for a polynomial which we state here without proof.

Lemma 2.5.3 (Newton relations)**.** *Let* $f(X) = \prod_{i=1}^{n}(X - \rho_i) = \sum_{i=0}^{n} a_i X^i$ *be a polynomial, and for* $k \geq 0$ *define* $S_k := \sum_{i=0}^{n} \rho_k^n$*. Then*

$$\sum_{i=0}^{k-1} a_i S_{k-i} + k a_k = 0 \quad \textit{for } n \geq k\textit{; and}$$

$$\sum_{i=0}^{n} a_i S_{k-i} = 0 \quad \textit{for } n \leq k.$$

□

First make the assumption that Λ is finite, and that $\dim_{\mathbb{F}_q} \Lambda = m$. Then e_Λ is a polynomial of degree q^m and simple roots at elements of Λ. Let f be the polynomial

$$f(X) := \frac{e_\Lambda\left(X^{-1} - z\right) X^{q^m}}{e_\Lambda(z)}.$$

[2] Recall our notation in equation (2.1) that $e_L(X) = \sum_{n \geq 0} e_n(L) X^{q^n}$.

We have

$$e_\Lambda\left(X^{-1}-z\right)=e_\Lambda(X^{-1})-e_\Lambda(z)=\sum_{i=0}^{m}e_n(\Lambda)\left(X^{-q^n}-z^{q^n}\right),$$

and conclude that $f(X)$ is a polynomial of degree q^m with roots $\left\{\frac{1}{z-\lambda}\middle|\,\lambda\in\Lambda\right\}$ and expansion

$$f(X)=X^{q^m}-\sum_{i=0}^{m}\frac{e_i(\Lambda)}{e_\Lambda(z)}X^{q^m-q^i}.$$

The Newton relations now give

$$S_{k,\Lambda}=\sum_{1\le q^i\le k-1}S_{k-q^i,\Lambda}=S_{1,\Lambda}\left(S_{k-1,\Lambda}+e_1(\Lambda)S_{k-q,\Lambda}+e_2(\Lambda)S_{k-q^2,\Lambda}+\cdots\right).$$

This defines a recurrence from which we may calculate $S_{k,\Lambda}$ in terms of $S_{1,\Lambda}$. By definition, this will give us exactly the Goss polynomials. Note that they are uniquely determined since, by Proposition 2.1.4 (b) and Lemma 2.5.1, $S_{1,\Lambda}$ takes infinitely many values.

In fact this recurrence gives us exactly (a), which implies (b), (c), (d) and (e), while (g) also follows by a simple induction. To prove (f) we show that the two polynomials $Q(X)=\sum_{0\le i<j}\beta_i(\Lambda)X^{q^i}$ and $R(X)=X^{q^j}P_{q^j-1,\Lambda}(X^{-1})$ are the same. Since $Q(X)$ is the truncation of $\log_\Lambda(X)$, we have $Q(e_\Lambda(X))=X+O(X^{q^j})$. We also have

$$\begin{aligned}R(e_\Lambda(X)) &= e_\Lambda(X)^{q^j}P_{q^j-1}\left(e_\Lambda(X)^{-1}\right)\\ &= e_\Lambda(X)^{q^j}S_{q^j-1,\Lambda},\end{aligned}$$

where the second term is $X^{1-q^i}+\sum'_{\lambda\in\Lambda}(X-\lambda)^{1-q^i}$ and the first is $X^{q^i}+O(X^{q+1})$. Noting that $(X-\lambda)$ is an invertible function in $\mathbb{C}_\infty[\![X]\!]$, we conclude that also $R(e_\Lambda(X))=X+O(X^{q^j})$. Since Q and R have degree less than q^j and are equal modulo X^{q^j}, this implies that $Q=R$.

The proof of (i) is also by induction, noting that it is true for $k=1$, and that if it is true for $j < k$, then

$$\begin{aligned} X^2 P'_{k,\Lambda}(X) &= X P_{k,\Lambda}(X) + X \left(\sum_{1 \le q^i \le k-1} e_i(\Lambda) X^2 P'_{k-q^i,\Lambda}(X) \right) \\ &= X \left(P_{k,\Lambda}(X) + \sum_{1 \le q^i \le k-1} e_i(\Lambda)(k - q^i) P_{k+1-q^i,\Lambda}(X) \right) \\ &= X \left(k P_{k,\Lambda}(X) + \sum_{q \le q^i \le k-1} e_i(\Lambda) k P_{k+1-q^i,\Lambda}(X) \right) \\ &= k P_{k+1,\Lambda}(X). \end{aligned}$$

Now, let Λ be general. Set $\Lambda_r = \Lambda \cap B(0, r)$. Then $S_{k,\Lambda} = \lim_{r\to\infty} S_{k,\Lambda_r}$ and $e_\Lambda(X) = \lim_{r\to\infty} e_{\Lambda_r}(X)$ locally uniformly. We define the polynomials $P_{k,\Lambda}(X) := \lim_{r\to\infty} P_{k,\Lambda_r}(X)$, where we take the limit coefficientwise. We have

$$P_{k,\Lambda}(S_{1,\Lambda}) = \lim_{r\to\infty} P_{k,\Lambda_r}(z)(S_{1,\Lambda_r}) = \lim_{r\to\infty} S_{k,\Lambda_r} = S_{k,\Lambda}.$$

The properties for $P_{k,\Lambda}$ follow immediately from the finite case.

Lastly, note that (h) also follows from the recursion formula (a), which we now know to be valid for arbitrary lattices. □

The polynomials $P_{k,\Lambda}$ are called the *Goss polynomials* after David Goss who first introduced them in [Go3] §6(c).

2.6 The Drinfeld Period Domain Ω^r

2.6.1 Rigid Analytic Spaces

Before defining Ω^r, we make a quick digression to define rigid analytic varieties and some related objects. This is an overview, merely stating the definitions and most important results. For a more detailed introduction, the reader may consult [Bo] or [FvdP].

Definition 2.6.1. *The* ring of strictly convergent power series *in n variables over $\mathbb{C}_\infty$ is the ring $\mathbb{C}_\infty\langle x_1, \ldots, x_n\rangle$ defined as*

$$\left\{ \sum_{i_1,\ldots,i_n \ge 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{C}_\infty[\![x_1, \ldots, x_n]\!] \,\middle|\, \lim_{i_1+\cdots+i_n\to\infty} a_{i_1,\ldots,i_n} = 0 \right\}$$

consisting of all power series that converge on the closed unit ball $\bar{B}(0,1) = \{(z_1, \ldots, z_n) \in \mathbb{C}_\infty^n \mid \max |z_i| \leq 1\}$.

Definition 2.6.2. *A* Tate algebra *is a quotient of a ring of strictly convergent power series by a finitely generated ideal. It turns out ([Bo] §1.2 Corollary 10) that every Tate algebra is also a finite extension of a strictly convergent power series ring.*

Remark. In fact, strictly convergent power series rings are Noetherian. Thus the condition that the ideal be finitely generated is superfluous.

There is a bijection between the unit ball from Definition 2.6.1 and the set of maximal ideals of $\mathbb{C}_\infty\langle x_1, \ldots, x_n\rangle$ which allows a correspondence similar to that in algebraic geometry. Hence, for any Tate algebra A, we denote the set of its maximal ideals by $\mathrm{Spm}(A)$. We also consider A to be its ring of functions. Such a space is called an *affinoid space*.

This space will be endowed with a sheaf with respect to a Grothendieck topology. To describe the sheaf we first say what an admissible open subset is.

Definition 2.6.3. *Let* $X = \mathrm{Spm}(A)$ *be an affinoid space. A subset* $U \subset X$ *is said to be an* affinoid subset *if there exists a morphism* $\varphi : \mathrm{Spm}(B) \to U$ *for some affinoid algebra* B *such that for every morphism* $\psi : Spm(C) \to U$ *(with* C *an affinoid algebra), there exists a unique morphism of affinoid algebras* $\rho : B \to C$ *such that* $\psi = \varphi \circ \mathrm{Spm}(\rho)$.

A consequence of the definition ([Bo] §1.6 Lemma 10) is that φ defines an isomorphism $\mathrm{Spm}(A) \xrightarrow{\sim} U$. As special kinds of affinoid subdomains we mention Weierstraß domains and Laurent domains. They will make an appearance in the next section when defining the Drinfeld period domain.

Definition 2.6.4. *Let* $X = \mathrm{Spm}(A)$ *be an affinoid space and let* $f_1, \ldots, f_r \in A$*, and* $g_1, \ldots, g_s \in A$ *be functions. A* Weierstraß domain *is a subset of* X *of the form*

$$X(f_1, \ldots, f_r) = \{x \in X : |f_i(x)| \leq 1\}$$

and a Laurent domain *is a subset of* X *of the form*

$$X(f_1, \ldots, f_r, g_1^{-1}, \ldots, g_s^{-1}) = \{x \in X : |f_i(x)| \leq 1, |g_j(x)| \geq 1\}.$$

By taking affinoid sets and affinoid subsets to be admissible opens, we obtain a Grothendieck topology. For completeness we include a definition of a Grothendieck topology that is suitable for us.

Definition 2.6.5. *Let X be a set. A* Grothendieck topology *on X consists of*

- *a set $S \subset \mathcal{P}(X)$ of subsets of X, called* admissible open subsets*; and*
- *a family* $(\operatorname{Cov} U)$ *for admissible opens U, where each* $\operatorname{Cov} U$ *is a set of* admissible coverings of U *whose elements are sets $\{U_i\}_{i\in I}$ of admissible opens for which $U = \bigcup_{i\in I} U_i$;*

with the properties

(a) *if $U, V \in S$, then $U \cap V \in S$;*

(b) *for each $U \in S$, $\{U\} \in \operatorname{Cov} U$;*

(c) *if $\{U_i\}_{i\in I} \in \operatorname{Cov} U$ and for each $i \in I$, $\{V_{ij}\}_{j\in J_i} \in \operatorname{Cov} U_i$, then $\{V_{ij}\}_{i\in I, j\in U_i} \in \operatorname{Cov} U$; and*

(d) *if $U, V \in S$ and $V \subset U$ and $\{U_i\}_{i\in I} \in \operatorname{Cov} U$, then $\{U_i \cap V\}_{i\in I} \in \operatorname{Cov} U \cap V$.*

The Grothendieck topology just defined is called the *weak Grothendieck topology*. However, it does not behave well under morphisms, so we need to extend it (by adding more admissible opens) to the *strong Grothendieck topology*. The following definition gives the admissible opens and admissible coverings in this case:

Definition 2.6.6. *Let X be an affinoid space. We define the strong Grothendieck topology on X as follows:*

- *The* admissible open sets *are the sets $U \subset X$ for which there exists a covering (not necessarily finite) $U = \bigcup_{i\in I} U_i$ by affinoid subdomains $U_i \subset X$ with the property that for any morphism $\varphi : Z \to X$ of affinoid spaces with $\varphi(Z) \subset U$, the covering $(\varphi^{-1}(U_i))_{i\in I}$ of Z admits a refinement which is a finite covering of Z by affinoid subspaces.*
- *The* admissible coverings *of an admissible open set V are the coverings $V = \bigcup_{j\in J} V_j$ of V by admissible opens V_j with the property that for any morphism $\varphi : Z \to X$ of affinoid spaces with $\varphi(Z) \subset V$, the covering $(\varphi^{-1}(V_j))_{j\in J}$ of X admits a refinement which is a finite covering of Z by affinoid subspaces.*

Definition 2.6.7. *A G-ringed space is a set X endowed with a Grothendieck topology $\mathfrak{T}$ and a sheaf $\mathcal{O}_X$ of rings with respect to this Grothendieck topology. A locally G-ringed space is a G-ringed space for which the stalks $\mathcal{O}_{X,x}$ are all local rings.*

A morphism of ringed spaces $(\varphi, \varphi^{\#})$ is a function $\varphi : X \to Y$ which is continuous[3] *with respect to the respective Grothendieck topologies and a morphism of sheaves $\varphi^{\#} : \mathcal{O}_Y \to \varphi_*(\mathcal{O}_X)$. A morphism of locally ringed spaces must moreover induce local homomorphisms on the stalks.*

A rigid analytic space *is locally ringed space $(X, \mathfrak{T}, \mathcal{O}_X)$ that admits an admissible covering $X = \bigcup U_i$ by affinoid sets with the Grothendieck topology from Definition 2.6.6.*

We now present an example which will be relevant later on. Let $0 < r < 1$. For any k define the annulus $B_k := \mathbb{C}_\infty\langle X, r^k X^{-1}\rangle \cong \{z \in \mathbb{C}_\infty : \left|r^k\right| \leq |z| \leq 1\}$. It is an affinoid subspace of $\mathbb{P}^1(\mathbb{C}_\infty)$. The punctured unit disc $B' = \{z \in \mathbb{C}_\infty \,|\, 0 < |z| \leq 1\}$ is the union $\bigcup_{k\geq 1} B_k$. It is an admissible open subset of $\mathbb{P}^1(\mathbb{C}_\infty)$ and the covering by annuli is an admissible covering.

The functions that are holomorphic on B_k are the Laurent series $\sum_{n\in\mathbb{Z}} a_n X^n$ where $\lim_{n\to\infty} |a_n| = 0$ and $\lim_{n\to-\infty} r^{kn} |a_n| = 0$. Thus, the functions holomorphic on B' must be the Laurent series

$$\left\{ \sum_{n\in\mathbb{Z}} a_n X^n \,\middle|\, \lim_{n\to\infty} |a_n| = 0,\, \forall R > 0 \lim_{n\to-\infty} R^n |a_n| = 0 \right\}. \tag{2.6}$$

Proposition 2.6.8. *Let $f : B' \to \mathbb{C}_\infty$ be a bounded holomorphic function on the punctured unit disc. Then it extends to a holomorphic function on the unit disc $\{z \in \mathbb{C}_\infty : |z| \leq 1\}$.*

Proof. This is a special case of [FvdP] Proposition 2.7.13. □

Lastly we consider quotients of rigid spaces. Suppose that a group Γ acts on a rigid space X. (By this we mean that for any $\gamma \in \Gamma$, the map $x \mapsto \gamma x$ is a morphism of rigid spaces.) Suppose further that the action of Γ on X is *discontinuous*. By this we mean that there exists an admissible covering $X = \bigcup_{i\in I} U_i$ of X such that for every U_i, the set $\{\gamma \in \Gamma \,|\, \gamma(U_i) \cap U_i \neq \emptyset\}$ is finite.

Proposition 2.6.9. *Let X be a rigid space, and let Γ be a group which acts discontinuously on X. Then there exists a morphism of rigid spaces $p : X \to Y$ with the universal property:*

[3] A morphism of sets with Grothendieck topologies is continuous if the inverse image of an admissible open set is admissible, and if the inverse image of an admissible covering is an admissible covering.

- *p is Γ-invariant.*
- *Let $U \subset X$ be an admissible Γ-invariant set, and let $q : U \to Z$ be a Γ-invariant morphism. Then $p(U) \subset Y$ is admissible and there is a unique morphism $r : p(U) \to Z$ such that $q = r \circ p : U \to p(U) \to Z$.*

We call Y the quotient space *of X by Γ and denote it by $\Gamma\backslash X$.*

The admissible open sets of $\Gamma\backslash X$ are the sets of the form $p^{-1}(U)$ where U is an admissible subset of X and the admissible coverings are the coverings $p^{-1}(U) = \bigcup_i p^{-1}(U_i)$ where $\bigcup_i U_i$ is an admissible covering of U. We may also describe the structure sheaf by $\mathcal{O}_{\Gamma\backslash X}(U) := \mathcal{O}_X(p^{-1}U)^\Gamma$, the subring of Γ-invariant elements of $\mathcal{O}_X(p^{-1}U)$.

More details about quotient spaces as well as an example can be found in [FvdP] Chapter 6.4.

2.6.2 The Drinfeld Period Domain Ω^r

Eventually, we shall define modular forms as holomorphic functions on the *Drinfeld period domain* Ω^r satisfying a modular functional equation. One of the goals of this section is to define the rigid analytic structure on Ω^r. Then we can say what it means for a function to be holomorphic on Ω^r. The rigid structure on Ω^r was given in Drinfeld's original paper [Dr] (Propositions 6.1 and 6.2), but is also explicitly mentioned in [SS]. We follow the approach from the latter quite closely.

Definition 2.6.10. *The Drinfeld period domain Ω^r is the complement in $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$ of the union of all F_∞-rational hyperplanes.*

This space turns out to be an admissible open subset of $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$. To prove this, we define neighbourhoods of each hyperplane, the complements of which are affinoid subsets. Unless stated otherwise, in this section we shall choose elements $\omega \in \mathbb{P}^{r-1}(\mathbb{C}_\infty)$ to be *unimodular*. This means that we pick $\omega = (\omega_1 : \omega_2 : \cdots : \omega_r)$ in such a way that $\max_{1\le i\le r}\{|\omega_i|\} = 1$.

Let $H \subset \mathbb{P}^{r-1}(\mathbb{C}_\infty)$ be an F_∞-hyperplane. It is defined by a linear form $\ell_H(\omega) = h_1\omega_1 + \cdots + h_r\omega_r$ which we may choose such that $H = \{\omega \in \mathbb{P}^{r-1}(\mathbb{C}_\infty) \,|\, \ell_H(\omega) = 0\}$ and $h_i \in A_\infty$ for every $i = 1, \ldots, r$, but at least one $h_i \notin \pi A_\infty$. Such a form is defined up to multiplication by a unit in A_∞. In particular, $|\ell_H(\omega)|$ is well-defined for any $\omega \in \mathbb{P}$. We now define neighbourhoods of such a hyperplane.

Definition 2.6.11. *Let $\varepsilon \in \mathbb{Q}_+$ and let H be an F_∞-hyperplane. The set $H(\varepsilon) = \{\omega \in \mathbb{P}^{r-1}(\mathbb{C}_\infty) : |\ell_H(\omega)| \le \varepsilon\}$, is called an* ε-neighbourhood *of the hyperplane H.*

Theorem 2.6.12. *Each set*

$$\Omega_n := \Omega(|\pi|^n) := \mathbb{P}^{r-1}(\mathbb{C}_\infty) \smallsetminus \bigcup_H H(|\pi|^n)$$

(where H ranges over all F_∞-hyperplanes) is an affinoid subspace of Ω^r. Moreover, the set $\{\Omega_n \mid n \in \mathbb{N}\}$ forms an admissible covering of Ω^r.

The key to this theorem is that each Ω_n is defined by only finitely may hyperplanes. Since the hyperplanes are defined by linear forms with coefficients in A_∞, we may define congruences between them. We say that $H_1 \equiv H_2 \pmod{\pi^n}$ if we can choose ℓ_{H_1} and ℓ_{H_2} such that $\ell_{H_1} \equiv \ell_{H_2} \pmod{\pi^n}$, where this congruence is coefficientwise. Let also $\mathcal{H}_n$ denote the set of equivalence classes of hyperplanes modulo π^n and let $\mathcal{H} = \varprojlim \mathcal{H}_n$. We also endow $\mathcal{H}$ with the profinite topology in this construction. In particular, it is compact.

Lemma 2.6.13. *Two hyperplanes H_1 and H_2 are congruent $\ell_{H_1} \equiv \ell_{H_2} \pmod{\pi^n}$ if and only if $H_1(|\pi^n|) = H_2(|\pi^n|)$.*

Proof. If $\ell_{H_1} \equiv \ell_{H_2} \pmod{\pi^n}$, then for any ω we have $|\ell_{H_1}(\omega) - \ell_{H_2}(\omega)| \leq |\pi^n|$, since ω is chosen to be unimodular. Then $\omega \in H_1(|\pi^n|)$ if and only if $\omega \in H_2(|\pi^n|)$.

Conversely, suppose that $H_1(|\pi^n|) = H_2(|\pi^n|)$. For a given unimodular ω, whether $\ell_{H_j}(\omega) \leq |\pi|^n$ $(j = 1, 2)$ depends only on $(\omega_1, \ldots, \omega_r)$ modulo π^n. Indeed, assume that $\omega \equiv \bar{\omega} \pmod{\pi^n}$ in the sense that $\omega_i - \bar{\omega}_i \in \pi^n A_\infty^r$ for $i = 1, \ldots, r$. Since the linear forms ℓ_{H_j} have coefficients in A_∞, this implies that $\ell_{H_j}(\omega) - \ell_{H_j}(\bar{\omega}) \in \pi^n A_\infty$ as well.

Thus for $j = 1, 2$, the linear forms ℓ_{H_j} associated to H_j induce linear functions $\bar{\ell}_{H_j} : (A/\pi^n A)^r \to (A/\pi^n A)$ which are easily seen to be surjective. Now, if $H_1(|\pi^n|) = H_2(|\pi^n|)$, then these maps have the same kernel, implying that they differ by a scalar which is invertible in A_∞, i.e. there exists an $\alpha \in (A_\infty/\pi^n A_\infty)^\times$ such that $\bar{\ell}_{H_1} = \alpha \bar{\ell}_{H_2}$ or equivalently we may choose ℓ_{H_2} so that $\ell_{H_1} \equiv \ell_{H_2} \pmod{\pi^n}$. $\square$

Since there are only finitely many elements in $A_\infty/\pi^n A_\infty$, there are only finitely many equivalence classes of hyperplanes modulo π^n. Hence Ω_n is the Laurent domain $\mathbb{P}^{r-1}(\mathbb{C}_\infty)\left((\pi^{-n}\ell_H^{-1})_{H \in \mathcal{H}_n}\right)$.

Proof of Theorem 2.6.12. Note that Ω_n is an affinoid subspace, since it is a Laurent domain. It is also a finite intersection of sets of the form $\mathbb{P}^{r-1}(\mathbb{C}_\infty) \smallsetminus H(|\pi|^n)$, where H is a hyperplane. But such a set is isomorphic to an open polydisc in the affine space $\mathbb{P}^{r-1}(\mathbb{C}_\infty) \smallsetminus H$. It is also known that such polydiscs form an admissible covering of the affine space. Therefore if $f : Y \to$

$\mathbb{P}^{r-1}(\mathbb{C}_\infty)$ is a rigid analytic morphism such that $f(Y) \subset \mathbb{P}^{r-1}(\mathbb{C}_\infty) \smallsetminus H$, then there is an $n(H) \in \mathbb{N}$ such that $f(Y) \subset \mathbb{P}^{r-1}(\mathbb{C}_\infty) \smallsetminus H\left(\left|\pi^{n(H)}\right|\right)$.

Now let $f : Y \to \mathbb{P}^{r-1}(\mathbb{C}_\infty)$ be a morphism such that $f(Y) \subset \Omega^r$. Then, in fact, $f(Y) \subset \mathbb{P}^{r-1}(\mathbb{C}_\infty) \smallsetminus \bigcup_{H\in\mathcal{H}} H\left(\left|\pi^{n(H)}\right|\right)$. However, by Lemma 2.6.13 the sets $\{H' \in \mathcal{H} \,|\, H' \subset H(|\pi^n|)\}$ are open in the topology on $\mathcal{H}$. Since $\mathcal{H}$ is compact, there are finitely many hyperplanes $H_1, \ldots, H_r$ and positive integers $n_1, \ldots, n_r$ such that

$$\bigcup_{H\in\mathcal{H}} H \subset H_1(|\pi^{n_1}|) \cup \cdots \cup H_r(|\pi^{n_r}|).$$

Setting $n := \max n_r$ we see that

$$\bigcup_{H\in\mathcal{H}} H \subset \bigcup_{H\in\mathcal{H}_n} H(|\pi^n|) \subset H_1(|\pi^{n_1}|) \cup \cdots \cup H_r(|\pi^{n_r}|)$$

and ultimately $f(Y) \subset \Omega_n$. Therefore the Ω_n form an admissible covering of Ω^r. □

In [Dr], Drinfeld also gives a finer admissible covering by using the Bruhat-Tits building for GL_r. We omit its discussion since we do not use it except to mention its use in Lemma 3.1.6.

Since the sets Ω_n are affinoid, there exist Tate algebras A_n such that $\mathrm{Spm}(A_n) \cong \Omega_n$. We refrain from writing them down explicitly. The holomorphic functions on Ω_n are exactly the elements of A_n. The functions that are holomorphic on Ω^r are the functions that are holomorphic on each Ω_n — hence the intersection of all the A_n. Alternatively, one may use the following equivalent definition:

Definition 2.6.14. *A function $f : \Omega_n \to \mathbb{C}_\infty$ is* holomorphic on Ω_n *if it is the uniform limit of rational functions on Ω_n with no poles in Ω_n. A function $f : \Omega^r \to \mathbb{C}_\infty$ is* holomorphic on Ω^r *if its restriction to each Ω_n is holomorphic on Ω_n.*

Later on we shall need the fact that holomorphic functions on a certain open set have a power series expansion. This is a natural place to prove this.

Lemma 2.6.15. *Let $B'(R) = \{z \in \mathrm{Spm}(\mathbb{C}_\infty) \,|\, 0 < |z| \leq R\}$ be the punctured disc of radius R, and Ω_n be the affinoid subspace of Ω^r as before. Then a function holomorphic on the product $B'(R) \times \Omega_n$ has a Laurent series expansion of the form*

$$\sum_{n\in\mathbb{Z}} f_n X^n,$$

where each f_n is a uniquely determined holomorphic function on Ω_n.

Proof. This follows from the characterization of products of affinoid domains $(\mathrm{Spm}(A) \times_{\mathrm{Spm}(C)} \mathrm{Spm}(B) = \mathrm{Spm}(A \hat{\otimes}_C B)$, the completed tensor product) and an argument similar to that leading up to equation (2.6). The f_n are uniquely determined, since otherwise there is a non-zero expansion corresponding to the zero function. This would mean that evaluating the functions f_n for some $\tilde{\omega} \in \Omega_n$, there results a non-zero Laurent series with coefficients in $\mathbb{C}_\infty$ which is zero for all $X \in B'(0, R)$. This clearly cannot be. □

Remark. The radius R in the proof plays some role as to what "size" the functions f_n can have, but we shall not need this.

Proposition 2.6.16. *Let $\mathcal{U}$ be a neighbourhood of $\{0\} \times \Omega^{r-1} \subset \mathbb{C}_\infty \times \Omega^{r-1}$ of the form $\bigcup_{n \geq 1} B(0, r_n) \times \Omega_n^{r-1}$, where for each n, $B(0, r_n)$ is the disc of radius r_n centred at 0. Also let $\mathcal{U}' = \mathcal{U} \cap (\mathbb{C}_\infty \smallsetminus \{0\}) \times \Omega^{r-1}$.*

Then any function holomorphic on $\mathcal{U}'$ has a Laurent expansion of the form

$$\sum_{k \in \mathbb{Z}} f_k X^k,$$

where each f_n is a uniquely determined holomorphic function on Ω^r.

Remark. Again, the r_n have an effect on what "size" the functions f_k may have on each Ω_n, but we are not concerned with this here.

Proof. By an extension of the argument in Lemma 2.6.15 showing that any punctured disc is an admissible open, any set of the form $B'(0, R) \times \Omega_n$ is an admissible open (since Ω_n is an affinoid domain). Then, since the intersections $B'(0, R_n) \times \Omega_n \cap B'(0, R_m) \times \Omega_m$ are admissible open sets, $\mathcal{U}'$ is the rigid space given by the admissible covering $\mathcal{U}' = \bigcup_{n \geq 1} B'(0, r_n) \times \Omega_n$.

Therefore the functions on $\mathcal{U}'$ are exactly those with Laurent series expansions $\sum_{k \in \mathbb{Z}} f_k X^k$, where for every $n \geq 1$, f_n is holomorphic on Ω_n. This is the same as saying that each f_k is holomorphic on Ω^r. The fact that they are uniquely determined follows from Lemma 2.6.15 and the sheaf property. □

We defined Ω^r by giving conditions on unimodular coordinates. However, in Chapter 3, we shall make the convention that the last coordinate $\omega_r = 1$. If we define $|\omega| := \max\{|\omega_i| : 1 \leq i \leq r\}$, then the unimodular representative and the representative where $\omega_r = 1$ differ by some factor with absolute value $|\omega|$. Note that $|\omega| \geq |\omega_r| = 1$. Also define

(2.7) $|\omega|_i := \inf\{|\ell_H(\omega)| : H \subset \mathbb{P}^{r-1}(\mathbb{C}_\infty) \text{ an } F_\infty\text{-rational hyperplane}\}.$

(The hyperplanes are still assumed to have unimodular coefficients.) In fact, this is a minimum, since any $\omega \in \Omega^r$ is in some Ω_n and on Ω_n there are only finitely many hyperplanes that define different functions $|\ell_H(\omega)|$. This serves as an analogue of the imaginary part of $z \in \mathcal{H}$ in the classical case. We may now rewrite

$$\Omega_n = \{\omega \in \Omega^r : |\omega|_i \geq |\pi|^n |\omega|\}.$$

2.7 Moduli of Drinfeld modules

It is known that rank 2 modular forms can be interpreted as the sections of a certain sheaf on the algebraic curve whose points correspond to isomorphism classes of Drinfeld modules. Such a curve is called a moduli curve. This is analogous to the classical case. The same thing can be done for Drinfeld modules of arbitrary rank, but in this case the resulting moduli variety has dimension greater than 1. Here we give a quick overview of the moduli space in general so that later we may relate analytic Drinfeld modular forms and algebraic Drinfeld modular forms. A more complete overview can be found in [Pi] §1, and even more details can be found in [DeHu].

Let S be a scheme over F. Then a *Drinfeld module over S* of rank r is a pair (E, φ), where E is a line bundle over S and φ a ring homomorphism

$$\varphi : A \to \mathrm{End}(E), \quad a \mapsto \varphi_a = \sum_{i\geq 0} \varphi_{a,i}\tau^i$$

(where τ represents the Frobenius endomorphism and $\varphi_{a,i} \in \Gamma(S, E^{1-q^i})$) such that the derivative $d\varphi : a \mapsto \varphi_{a,0}$ is the structure homomorphism and in the fibre over any $s \in S$, the sum becomes a (twisted) polynomial in τ of degree $r \deg a$.

Next, for an ideal $N \subset A$, a *level N structure* is an isomorphism of group schemes over S

$$\lambda : (N^{-1}/A)^r \xrightarrow{\sim} \varphi[N] \cong \bigcap_{a\in N} \ker(\varphi_a),$$

where $\varphi[N]$ is the group scheme of N-torsion points of φ (i.e. the elements $x \in E$ for which $\varphi_a(x) = 0$ for every $a \in N$).

Drinfeld [Dr] showed that the fine moduli space of rank r Drinfeld modules with level N structure exists, and that it is an $r-1$ dimensional irreducible smooth affine variety of finite type over F. We shall denote this variety by $M^r_{K(N)}$, where we define $K(N) := \ker(\mathrm{GL}_r(\hat{A}) \to \mathrm{GL}_r(A/N))$.

There is an isomorphism of rigid analytic spaces

$$M^r_{K(N)}(\mathbb{C}_\infty) \xrightarrow{\sim} \mathrm{GL}_r(F)\backslash \left(\Omega^r \times \mathrm{GL}_r(\mathbb{A}^f_F)/K(N)\right).$$

This isomorphism suggests that one may define the moduli space for open compact subgroups $K \subset \mathrm{GL}_r(\mathbb{A}^f_F)$. However, the quotient only has nice properties when K is also *fine*, i.e. if there exists a prime ideal $\mathfrak{p}$ such that the image of K in $\mathrm{GL}_r(A/\mathfrak{p})$ is nilpotent. (In fact, this can be done independently of this isomorphism.)

Proposition 2.7.1. *The components of M^r_K correspond to the double cosets $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}^f_F)/K$. Let G be a set of double coset representatives for it and set $\Gamma_g := gKg^{-1} \cap \mathrm{GL}_r(F)$ for each $g \in S$. Then, moreover, there is a rigid analytic isomorphism*

$$M^r_K \xrightarrow{\sim} \coprod_{g\in S} \Gamma_g\backslash\Omega^r.$$

Proof. [Hu] Proposition 2.1.3. □

Since $M^r_{K(N)}$ is a fine moduli space, there exists a *universal Drinfeld module* over $M^r_{K(N)}$ whose fiber at each point is the Drinfeld module and level structure that corresponds to that point.

2.8 The Pink-Satake compactification

Pink's observation was that since all isomorphism classes of rank r Drinfeld modules appear as points on the affine moduli space, the points on the boundary of a compactification must necessarily have a different rank. Thus, he introduced the concept of a *generalized Drinfeld module* over a scheme ([Pi] Definition 3.1). This essentially differs from the normal definition only in that the rank may vary across the scheme. For the more subtle differences we encourage the reader to read [Pi].

Definition 2.8.1. *A* generalized Drinfeld A-module over S *is a pair (E, φ) consisting of a line bundle E over S and a ring homomorphism*

$$\varphi : A \to End(E), \quad a \mapsto \varphi_a = \sum \varphi_{a,i}\tau^i$$

with $\varphi_{a,i} \in \Gamma(S, E^{1-q^i})$ satisfying the conditions:

(a) *The derivative $d\varphi : a \mapsto \varphi_{a,0}$ is the structure homomorphism $A \to \Gamma(S, \mathcal{O}_S)$.*

(b) *In the fiber over any point $s \in S$, the map φ defines a Drinfeld module of some rank $r_s \geq 1$.*

It turns out that the following definitions ([Pi] Definitions 3.9 and 4.1) define a compactification which behaves well under various natural morphisms and allows one to define Drinfeld modular forms.

Definition 2.8.2. *A generalized Drinfeld A-module (E, φ) over S is called* weakly separating *if for any Drinfeld module (E', φ') over any field L containing F, at most finitely many fibers of (E, φ) over L-valued points of S are isomorphic to (E', φ').*

Definition 2.8.3. *For any fine open compact subgroup $K \subset \mathrm{GL}_r(\hat{A})$, an open embedding $M_K^r \hookrightarrow \bar{M}_K^r$ with the properties*

(a) *$\bar{M}_K^r$ is a normal integral proper variety over F, and*

(b) *the universal family (E, φ) on M_K^r extends to a weakly separating generalized Drinfeld module $(\bar{E}, \bar{\varphi})$ over M_K^r,*

is called a Satake-Pink compactification *of M_K^r. We shall call $(\bar{E}, \bar{\varphi})$ the universal family on $\bar{M}_K^r$.*

Theorem 2.8.4. *For every fine $K \subset \mathrm{GL}_r(\mathbb{A}_F^f)$, the variety M_K^r has a Satake-Pink compactification. Moreover, this compactification and the extension of the universal family are unique up to unique isomorphism.*

Proof. [Pi] Theorem 4.2. $\square$

Chapter 3

Drinfeld modular forms

We now arrive at the main objects of this thesis, Drinfeld modular forms. We start by defining an action of $\mathrm{GL}_r(F_\infty)$ on Ω^r and an induced action on the set of holomorphic functions $\Omega^r \to \mathbb{C}_\infty$. This allows us to define weak modular forms as those functions that satisfy a certain automorphic functional equation. Like in the case of elliptic modular forms it is necessary to define holomorphy at the cusps and for this we introduce Fourier expansions at infinity. Then we introduce the main examples of Drinfeld modular forms and ultimately study their Fourier expansions at infinity. At the end we give a product formula for the Drinfeld discriminant function and study the rationality of some forms.

For the rest of this thesis we adopt the following notation: if X is some rank r object, we shall write $\tilde{X}$ for the rank $r-1$ object obtained by "forgetting the first entry."

3.1 Group Actions

From now on we shall always represent an element $\omega \in \Omega^r$ as a row matrix $\omega = (\omega_1, \tilde{\omega}) = (\omega_1, \ldots, \omega_r)$ and make the convention that $\omega_r = 1$. We would like to define an action of $\mathrm{GL}_r(F_\infty)$ on Ω^r, by $\omega \cdot \gamma^{-1}$, where the latter should be viewed as matrix multiplication.[1] To do this properly we need to make the last entry of $\gamma\omega$ equal to 1. So let the last entry of $\omega \cdot \gamma^{-1}$ be $j(\gamma, \omega)$, and define $\gamma\omega := j(\gamma, \omega)^{-1}\omega \cdot \gamma^{-1}$. Note that $j(\gamma, \omega)$ will necessarily be non-zero, because the ω_i are F_∞-linearly independent.

[1]The reason we choose this action instead of left multiplication, is that Ω^r can be identified with the set of linear functions $F^r \to \mathbb{C}_\infty$ which are injective when tensored with F_∞. The action described is the one induced from the natural action of $\mathrm{GL}_r(A)$ on the set of linear functions $F^r \to \mathbb{C}_\infty$.

It will be useful to compare the sizes of ω and $\gamma\omega$. The following lemma relates the size (absolute value of the maximum element) as well as the "imaginary absolute value" $|\cdot|_i$ defined in equation (2.7).

Lemma 3.1.1. *There exist constants c_1, c_2, c_3 depending only on γ such that*

(a) $|\omega|_i \leq |j(\gamma,\omega)| \leq c_1 |\omega|$;

(b) $1 \leq |\gamma\omega| \leq c_2 \dfrac{|\omega|}{|\omega|_i}$; *and*

(c) $c_3 \dfrac{|\omega|_i}{|\omega|} \leq |\gamma\omega|_i \leq 1$.

Proof. Since γ is fixed, there exists $c \in \mathbb{R}$ such that all the entries of γ^{-1} have absolute value less than c.

(a) Every term in $j(\gamma,\omega)$ has absolute value at most $c|\omega|$, implying $|j(\omega,\gamma)| \leq c|\omega|$. Moreover, $j(\gamma,\omega)$ is an A-linear combination of the ω_i, and hence also an F_∞-linear combination. It might not be unimodular, but making it unimodular will only decrease it. Therefore $|j(\gamma,\omega)| \geq |\omega|_i$, which is the smallest any F_∞-linear combination can be.

(b) Clearly $|\gamma\omega| \geq 1$, since $\gamma\omega$ is normalized so that its r-th entry is 1, hence the maximum of its entries is at least 1. We have $|\omega\gamma^{-1}| \leq c|\omega|$ (here $\omega\gamma^{-1}$ is the matrix product) and hence $|\gamma\omega| = |j(\gamma,\omega)|^{-1} |\omega\gamma^{-1}| \leq c|\omega| / |\omega|_i$, using (a).

(c) Since ω_r is a unimodular F_∞ hyperplane, and we always normalize so that $\omega_r = 1$, the upper bound is immediate. The lower bound is trickier. We may express a linear form ℓ as a column matrix $(\ell_1,\ldots,\ell_r)^T$, when the value of $|\ell(\omega)|$ is simply the absolute value of the element $\omega\ell \in \mathbb{C}_\infty$. The action of γ on Ω^r affects this as follows: $\ell(\gamma\omega) = j(\gamma,\omega)^{-1}\omega\gamma^{-1}\ell$.

Note that $j(\gamma,\omega)$ is independent of ℓ, thus we may focus on minimizing $\omega\gamma^{-1}\ell$. We interpret $\gamma^{-1}\ell$ as a linear form. Clearly ℓ is F_∞-linear if and only if $\gamma^{-1}\ell$ is. However, it might not be unimodular. Denote its entry with maximum absolute (choose one if there are more than one) value by $m(\gamma,\ell)$. Then $\ell_\gamma := m(\gamma,\ell)^{-1}\gamma^{-1}\ell$ is a unimodular linear form.

Now

$$\begin{aligned} |\ell(\gamma\omega)| &= |j(\gamma,\omega)|^{-1} |m(\gamma,\ell)|^{-1} |\omega\cdot\ell_\gamma| \geq \\ &\geq \frac{|\omega|_i}{c|\omega|} |m(\gamma,\ell)|^{-1}, \end{aligned}$$

so we only need an upper bound for $|m(\gamma,\ell)|$. But the largest entry of ℓ has absolute value 1, while the entries of γ^{-1} all have entries at most c. Hence $|m(\gamma,\ell)| \leq c$. In summary

$$|\ell(\gamma\omega)| \geq \frac{|\omega|_i}{c^2\,|\omega|}.$$

Since ℓ was arbitrary, this holds for all F_∞-linear forms, and hence

$$|\gamma\omega|_i \geq \frac{|\omega|_i}{c^2\,|\omega|}.$$

□

Definition 3.1.2. *The weight k, type m* factor of automorphy *is the function $\alpha_{k,m} : \mathrm{GL}_r(F_\infty) \times \Omega^r \to \mathbb{C}_\infty$ defined by $\alpha_{k,m}(\gamma,\omega) = (\det\gamma)^{-m} j(\gamma,\omega)^{-k}$.*

Lemma 3.1.3. *The factor of automorphy $\alpha_{k,m}$ satisfies the following properties:*

(a) $\alpha_{k_1,m_1}(\gamma,\omega)\alpha_{k_2,m_2}(\gamma,\omega) = \alpha_{k_1+k_2,m_1+m_2}(\gamma,\omega)$;

(b) $\alpha_{k,m}(\gamma_1\gamma_2,\omega) = \alpha_{k,m}(\gamma_1,\gamma_2\omega)\alpha_{k,m}(\gamma_2,\omega)$.

Proof. (a) is trivial from the definition of α and for (b), the right hand side is

$$j(\gamma_1,\gamma_2\omega)^{-k}(\det\gamma_1)^{-m} j(\gamma_2,\omega)^{-k}(\det\gamma_2)^{-m} = (\det\gamma_1\gamma_1)^{-m}\left(j(\gamma_1,\gamma_2\omega)j(\gamma_2,\omega)\right)^{-k},$$

so it follows from the fact that $j(\gamma_1,\gamma_2\omega)$ is the right-most entry of $\left(j(\gamma_2,\omega)^{-1}(\omega\gamma_2^{-1})\right)\gamma_1^{-1}$, so $j(\gamma_1,\gamma_2\omega)j(\gamma_2,\omega)$ is the right-most entry of $\omega\gamma_2^{-1}\gamma_1^{-1}$. □

The factor of automorphy can be used to define an operator on the set of holomorphic functions $f : \Omega^r \to \mathbb{C}_\infty$. This operator will then be used to define which functions are modular.

Definition 3.1.4. *For any $\gamma \in \mathrm{GL}_r(F_\infty)$, define the operator $[\gamma]_{k,m}$ as the operator that assigns to the function $f : \Omega^r \to \mathbb{C}_\infty$, the function $f[\gamma]_{k,m}(\omega) := \alpha_{k,m}(\gamma,\omega) f(\gamma\omega)$.*

Lemma 3.1.5. (a) *If $f : \Omega^r \to \mathbb{C}_\infty$ is holomorphic on Ω^r, then so is $f[\gamma]_{k,m}$.*

(b) *We have the equality $f[\gamma_1\gamma_2]_{k,m}(\omega) = (f[\gamma_1]_{k,m})\,[\gamma_2]_{k,m}(\omega)$, and hence the operators $[\gamma]_{k,m}$ define a right action of $\mathrm{GL}_r(F_\infty)$ on the set of holomorphic functions on $\mathbb{C}_\infty$.*

Proof. (a) This follows immediately from the fact that $\gamma : \Omega^r \to \Omega^r$ is an isomorphism of rigid analytic spaces.

(b)

$$\begin{aligned} f[\gamma_1]_{k,m}[\gamma_2]_{k,m}(\omega) &= (f[\gamma_1]_{k,m})(\gamma_2\omega)\alpha_{k,m}(\gamma_2,\omega) \\ &= f(\gamma_1\gamma_2\omega)\alpha_{k,m}(\gamma_1,\gamma_2\omega)\alpha_{k,m}(\gamma_2,\omega) \\ &= f(\gamma_1\gamma_2\omega)\alpha_{k,m}(\gamma_1\gamma_2,\omega) = f[\gamma_1\gamma_2]_{k,m}(\omega) \end{aligned}$$

□

We say that two subgroups G_1, G_2 of some group G are *commensurable* if $G_1 \cap G_2$ has finite index in both G_1 and G_2. It can be verified that commensurability is an equivalence relation. Then $\Gamma \subset \mathrm{GL}_r(F)$ is said to be an *arithmetic subgroup* of $\mathrm{GL}_r(F)$ if it is a subgroup of $\mathrm{GL}_r(F)$ which is commensurable with $\mathrm{GL}_r(A)$. Let us fix Γ as an arithmetic subgroup of $\mathrm{GL}_r(F)$.

Lemma 3.1.6. *The space Ω^r has an admissible covering by admissible open sets (U_i) such that the sets $\{\gamma \in \Gamma \mid \gamma U_i \cap U_i \neq \emptyset\}$ are finite for each i, i.e. Γ acts* discontinuously *on Ω^r.*

Proof. A covering satisfying these conditions is given in [Dr] Proposition 6.2. A discussion of this Lemma (where the term *discrete action* instead of discontinuous is used) is contained in [Dr] §6 (B), which comes shortly after the stated Proposition. □

Remark. The reason the proof of Lemma 3.1.6 is omitted is that it requires an interpretation of Ω^r through the Bruhat-Tits building. Though this is important in the theory of Drinfeld modular forms in rank 2, and is worth pursuing in higher rank, it is not needed for the rest of this work.

Definition 3.1.7. *A holomorphic function $f : \Omega^r \to \mathbb{C}_\infty$ is called a* weak modular form *of weight k and type m for Γ if $f[\gamma]_{k,m}(\omega) = f(\omega)$ for all $\gamma \in \Gamma$.*

Remark. Note that if γ is a scalar matrix cI ($c \in \mathbb{F}_q^\times$), then $f[\gamma](\omega)$ is $c^{-k+rm}f(\omega)$. Hence a weak modular form can be non-zero only if $k \equiv rm$ modulo the size of $\{cI \mid c \in \mathbb{F}_q\} \cap \Gamma$. In particular, if $\Gamma = \mathrm{GL}_r(A)$, then a weak modular form can be non-zero only if $k \equiv rm \pmod{q-1}$.

3.2 Fourier expansion at the cusps

In general, the space of weak modular forms of weight k will be infinite dimensional. However, if we impose a condition that such a function must stay bounded when extended to a certain compactification of Ω^r, then the space will become finite dimensional. This boundedness condition is easier to explain when studying a Fourier expansion of such a function at a "cusp" of Ω^r.

Let U be the algebraic subgroup of $\mathrm{GL}_r(F)$ consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ v_2 & & & \\ \vdots & & \mathrm{id}_{r-1} & \\ v_r & & & \end{pmatrix}$$

where we take the identity matrix and allow further non-zero entries only in the first column. We may consider the first column as an $(r-1)$-tuple $(v_2, \ldots, v_r)$. It is a simple exercise to check that this association defines an isomorphism $\iota : G_U \xrightarrow{\sim} F^{r-1}$.

Define $\Gamma_U := \Gamma \cap U$. We compute the action of some element $\iota^{-1}(v_2, \ldots, v_r) \in \Gamma_U$ on $\omega = (\omega_1, \tilde{\omega})$ as $(\omega_1 - (v_2\omega_2 + \cdots + v_r\omega_r), \tilde{\omega})$. Let $\Lambda_U = \iota(\Gamma_U) \subset F^{r-1}$, viewed as a group of column vectors.

It is clearly $\mathbb{F}_q$-linear and since Γ is commensurable with $\mathrm{GL}_r(A)$, also $\iota(\Gamma_U)$ is commensurable with A^{r-1} and hence $\tilde{\omega}\Lambda_U \subset \mathbb{C}_\infty$ is strongly discrete in $\mathbb{C}_\infty$. By Proposition 2.2.3 the exponential function $e_{\tilde{\omega}\Lambda_U}$ defines an isomorphism $\tilde{\omega}\Lambda_U \backslash \mathbb{C}_\infty \xrightarrow{\sim} \mathbb{C}_\infty$.

Note that if $\gamma \in \Gamma_U$, then $j(\gamma, \omega) = \omega_r = 1 = \det \gamma$, and hence that any weak modular form for Γ is invariant under Γ_U. Any weak modular form for Γ thus descends to a function on the quotient space $\Gamma_U \backslash \Omega^r$. By Proposition 2.6.9, the existence of this quotient space is guaranteed if Γ_U acts discontinuously on Ω^r, which follows from Lemma 3.1.6.

The quotient morphism is essentially the restriction of the map $\mathcal{E} : \mathbb{C}_\infty \times \Omega^{r-1} \to \mathbb{C}_\infty \times \Omega^{r-1}$ defined by $(\omega_1, \tilde{\omega}) \mapsto (e_{\tilde{\omega}\Lambda_U}(\omega_1), \tilde{\omega})$ to the subset $\Omega^r \subset \mathbb{C}_\infty \times \Omega^{r-1}$, since it defines an isomorphism between $\Gamma_U \backslash \Omega^r$ and its image. This is an isomorphism as groups and as rigid analytic spaces. Thus any Γ_U invariant function f factorizes through $e_{\tilde{\omega}\Lambda_U}$ and hence any Γ_U invariant function holomorphic on Ω^r is a function of the variables $e_{\tilde{\omega}\Lambda_U}(\omega_1)$, ω_2, $\ldots$, ω_{r-1}:

$$f(\omega) = \bar{f}(e_{\tilde{\omega}\Lambda_U}(\omega_1), \tilde{\omega}).$$

However, we would like a function to remain bounded as $\omega_1 \to \infty$, so the multiplicative inverse of $e_{\tilde{\omega}\Lambda_U}(\omega_1)$ would be a better choice. Since $\omega_1 \notin$

$\tilde{\omega}\Lambda_U \otimes_A F_\infty$, the function $e_{\tilde{\omega}\Lambda_U}(\omega_1)$ is never zero, and it would make sense to do this. However, it will be of benefit later on if we use a constant multiple of the inverse, instead of the inverse itself.

Definition 3.2.1. *Let the* parameter at the cusp ∞ *for* Γ *be the function*

$$u_{\tilde{\omega}}(\omega_1) := e_{\bar{\pi}\tilde{\omega}\Lambda_U}(\bar{\pi}\omega_1)^{-1} = \bar{\pi}^{-1} e_{\tilde{\omega}\Lambda_U}(\omega_1)^{-1},$$

where $\bar{\pi}$ is the Carlitz period. Note that it is a function of both ω_1 and $\tilde{\omega}$ and that it depends on Γ.

Remark. With this normalization we can define rationality of modular forms when $A = \mathbb{F}_q[t]$ in a simple way. Böckle pointed out that for other A, we shall need a different normalization.

Lemma 3.2.2. *Let $\Lambda \subset F^r$ be a projective A-module. For any n, there exists a constant c depending only on n such that for any $\omega \in \Omega_n$, there are at most c elements $\lambda \in \omega\Lambda$ such that $|\lambda| < |\omega|$.*

(In fact, we may choose $c = q^{nr}$.)

Proof. Any λ is of the form $a_1\omega_1 + \cdots + a_r\omega_r$, and we must have $|\lambda| \geq |\pi|^{-n}|\omega|$ by definition of Ω_n. In particular, for any λ, we have $|\pi^n\lambda| \geq |\omega|$, hence for every class in $(A/\pi^n)^r$, there can be at most one representative λ satisfying $|\lambda| < |\omega|$. Hence, the number of λ such that $|\lambda| < |\omega|$ is bounded by some constant depending only on n. $\square$

Corollary 3.2.3. *For every n, there exists a constant R_n depending only on n such that for any $\omega \in \Omega_n$ and any z such that $|z| \leq |\omega|$ we have $|e_{\omega\Lambda}(z)| < R_n$.*

Proof. Note that $e_{\omega\Lambda}(z) = z\prod_{\lambda\in\omega\Lambda}\left(1 - \frac{z}{\lambda}\right)$ and this product can be split up into three factors: where $|\lambda| < |z|$, where $|\lambda| = |z|$ and where $|\lambda| > |z|$. Those factors where $|z| < |\lambda|$ have absolute value 1 and the factors where $|z| = |\lambda|$ have absolute value less than or equal to 1. Hence

$$|e_{\omega\Lambda}(z)| \leq |z| \prod_{|\lambda|<|z|} \left|\frac{z}{\lambda}\right|.$$

Since $\omega \in \Omega_n$, we have $|\lambda| \geq |\pi|^n \cdot |\omega|$, so each factor satisfies $\left|\frac{z}{\lambda}\right| \leq |\pi|^{-n}$. By Lemma 3.2.2 the number of such λ's is bounded by some constant depending on n, yielding the result. $\square$

Proposition 3.2.4 ([BP]). *The map*

$$\Gamma_U\backslash\Omega^r \to \mathbb{C}_\infty \times \Omega^{r-1}$$

$$[(\omega_1, \tilde{\omega})] \mapsto (u_{\tilde{\omega}}(\omega_1), \tilde{\omega})$$

defines a rigid analytic isomorphism of $\Gamma_U\backslash\Omega^r$ onto a subset of $\mathbb{C}_\infty \times \Omega^{r-1}$ of the form

$$\mathcal{U}' = \mathcal{U} \smallsetminus (\{0\} \times \Omega^{r-1})$$

where $\mathcal{U}$ is an open neighbourhood of $(\{0\} \times \Omega^{r-1})$.

Moreover, for every $n \in \mathbb{N}$ there exists $r_n > 0$ such that $B(0, r_n) \times \Omega_n^{r-1} \subset \mathcal{U}$.

Proof. This map is exactly the isomorphism $\mathcal{E}$ described above followed by the map $z \mapsto (\bar{\pi}z)^{-1}$ on the first coordinate $e_{\tilde{\omega}\Lambda_U}(\omega_1)$, which is itself an isomorphism, since it is never 0. It remains to show that the image is of the required form.

Since $e_{\tilde{\omega}\Lambda_U}$ defines an isomorphism $\Gamma_U\backslash\mathbb{C}_\infty \xrightarrow{\sim} \mathbb{C}_\infty$, the required image consists of elements of the form

$$(e_{\bar{\pi}\tilde{\omega}\Lambda_U}(\bar{\pi}z)^{-1}, \tilde{\omega}) \in \mathbb{C}_\infty \times \Omega^{r-1}$$

such that $z \notin \tilde{\omega}\Lambda_U \otimes_A F_\infty$. Hence, it is enough to show that for fixed n and all $\tilde{\omega} \in \Omega_n^{r-1}$, the quantity $e_{\tilde{\omega}\Lambda_U}(z)$ is bounded from above for $z \in \tilde{\omega}\Lambda_U \otimes_A F_\infty$.

Since $F_\infty = \mathbb{F}_q((\pi))$ is a discrete valuation ring, any $z \in F_\infty$ can be written in the form $a + z_0$, where $a \in A$ and $|z_0| < 1$. Thus, if we denote $|\tilde{\omega}| := \max\{|\omega_2|, \ldots, |\omega_r|\}$, then $F_\infty\tilde{\omega}\Lambda_U \subset \tilde{\omega}\Lambda_U + B(0, |\tilde{\omega}|)$, and we may write $\omega_1 = \omega_0 + \lambda$, where $\lambda \in \tilde{\omega}\Lambda_U$ and $|\omega_0| < |\tilde{\omega}|$. Then by Corollary 3.2.3, the result follows. □

Proposition 3.2.5 ([BP]). *Any Γ_U invariant function f holomorphic on Ω^r can be written in the form*

$$f(\omega) = \sum_{n\in\mathbb{Z}} f_n(\tilde{\omega}) u_{\tilde{\omega}}(\omega_1)^n$$

where each f_n is a function holomorphic on Ω^{r-1}. Moreover, each f_n is uniquely determined.

Proof. This now follows by simply putting together Proposition 3.2.4 and Proposition 2.6.16. □

Remark. Like in the classical case, the u-expansion does not define a series that converges everywhere. However, Proposition 3.2.5 implies that it does converge for ω in a set of the form $\bigcup_{n\geq 1} B(0, r_n) \times \Omega_n^{r-1}$ as required by Proposition 2.6.16.

Let $\tilde{\Gamma}$ be the subgroup of Γ consisting of matrices γ with first row $(1, 0, \ldots, 0)$ and first column $(1, 0, \ldots, 0)^T$. By abuse of notation we shall also denote by $\tilde{\Gamma}$ the group of $(r-1) \times (r-1)$ matrices obtained by deleting the first row and first column. In the following propositions we shall need to compute the parameter $u_{\tilde{\omega}}(\omega_1)$ after some $\gamma \in \mathrm{GL}_r(A)$ has been applied to ω. Therefore we write $\widetilde{\gamma\omega}$ for the last $r-1$ entries of $\gamma\omega$ and $\pi_1(\gamma\omega)$ for its first entry.

Lemma 3.2.6. *Let $\gamma \in \mathrm{GL}_r(A)$ have first row $(1, 0, \ldots, 0)$ and first column $(1, 0, \ldots, 0)^T$ as described above. There exists a constant k depending only on γ such that if $\omega \in B(0, r_n) \times \Omega_n^{r-1}$, then $\gamma\omega \in B(0, r_n \left|j(\gamma, \omega)\right|^{-1}) \times \Omega_{2n+k}^{r-1}$.*

Proof. We have $\omega \in \Omega_n$ if and only if $|\omega|_i \geq |\pi|^n |\omega|$. If this is the case, then by Lemma 3.1.1 we have

$$\frac{|\gamma\omega|_i}{|\gamma\omega|} \geq \frac{c_3 |\omega|_i^2}{c_2 |\omega|^2} \geq |\pi|^{2n+k}$$

for some constant k depending only on γ. Hence $\gamma\omega \in \Omega_{2n+k}$. Performing this in rank $r-1$, implies that if $\tilde{\omega} \in \Omega_n^{r-1}$, then $\widetilde{\gamma\omega} \in \Omega_{2n+k}^{r-1}$.

Assume that $\omega \in B(0, r_n) \times \Omega_n^{r-1}$. Then $|u_{\tilde{\omega}}(\omega_1)| < r_n$, implying that $|u_{\widetilde{\gamma\omega}}(\pi_1(\gamma\omega))| = |j(\gamma, \omega)^{-1} u_{\tilde{\omega}}(\omega_1)| < r_n \left|j(\gamma, \omega)\right|^{-1}$. □

Proposition 3.2.7. *Let f be a weak modular form of weight k and type m for Γ and let it have the expansion given in Proposition 3.2.5. Then for every $n \in \mathbb{Z}$, the function $f_n : \Omega^{r-1} \to \mathbb{C}_\infty$ is a weak modular form of weight $k-n$ and type m for $\tilde{\Gamma}$.*

Proof. Note that since f is a weak modular form for Γ, it automatically satisfies the modularity condition for all $\tilde{\gamma} \in \tilde{\Gamma}$. Let $h \in \Gamma_U$. If we write

$$h = \begin{pmatrix} 1 & 0 \\ v & \mathrm{id} \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\gamma} \end{pmatrix},$$

where $\tilde{\gamma}$ and id are $(r-1) \times (r-1)$ matrices, and v is a $(r-1) \times 1$ matrix, then

$$h\gamma = \begin{pmatrix} 1 & 0 \\ v & \tilde{\gamma} \end{pmatrix}.$$

Therefore $u_{\widetilde{\gamma\omega}}(\pi_1(\gamma\omega))$ and $u_{\tilde{\omega}}(\omega_1)$ are related by

$$u_{\widetilde{\gamma\omega}}(\pi_1(\gamma\omega)) = u_{j(\tilde{\gamma},\tilde{\omega})^{-1}\tilde{\omega}}(j(\tilde{\gamma},\tilde{\omega})^{-1}\omega_1) = j(\tilde{\gamma},\tilde{\omega})u_{\tilde{\omega}}(\omega_1).$$

Now we may calculate

$$\begin{aligned}\sum_{n\in\mathbb{Z}} f_n(\tilde{\omega})u_{\tilde{\omega}}(\omega_1)^n &= f(\omega) = f[\gamma]_{k,m}(\omega)\\ &= j(\gamma,\omega)^{-k}(\det\gamma)^{-m} f(\gamma\omega)\\ &= j(\gamma,\omega)^{-k}(\det\gamma)^{-m}\sum_{n\in\mathbb{Z}} f_n(\widetilde{\gamma\omega})u_{\widetilde{\gamma\omega}}(\pi_1(\gamma\omega))^n\\ &= \sum_{n\in\mathbb{Z}} j(\gamma,\omega)^{-k}(\det\gamma)^{-m} f_n(\widetilde{\gamma\omega})j(\gamma,\omega)^n u_{\tilde{\omega}}(\omega_1)^n.\end{aligned}$$

Suppose that the left-hand side converges on the neighbourhood $\bigcup_{n\geq 1} B(0,r_n)\times \Omega_n^{r-1}$. (By Proposition 3.2.5 it does converge on some neighbourhood of this form.) Then, by Lemma 3.2.6, the right-hand side converges on $\bigcup_{n\geq 1} B(0,r_n j(\gamma,\omega)^{-1})\times \Omega_{2n+k}^{r-1}$. Therefore this equality takes place on a neighbourhood of this form.

By the uniqueness of the f_n we have the equality $f_n(\tilde{\omega}) = f_n(\widetilde{\gamma\omega})j(\tilde{\gamma},\tilde{\omega})^{n-k}(\det\tilde{\gamma})^{-m}$ for all $\tilde{\gamma}\in\tilde{\Gamma}$, i.e. that f_n is a weak modular form of weight $k-n$ and type m for $\tilde{\Gamma}$. $\square$

Definition 3.2.8. *Let f be a Γ_U invariant function with the Laurent expansion from Proposition 3.2.5. The* order at infinity *of f is* $\operatorname{ord}_{\Gamma_U}(f) = \inf\{n\in\mathbb{Z} \mid f_n(\tilde{\omega})\neq 0$ *for some* $\tilde{\omega}\}$. *We say that f is* meromorphic at infinity *w.r.t.* Γ *if* $\operatorname{ord}_{\Gamma_U}(f)\neq -\infty$ *and that f is* holomorphic at infinity *w.r.t.* Γ *if* $\operatorname{ord}_{\Gamma_U}(f)\geq 0$.

This definition may be stated in a number of slightly different ways. The ones we shall consider involve growth conditions of f as $|\omega|_i$ tends to infinity. Let us say that f remains bounded along "vertical lines" if for any fixed $\tilde{\omega}$ there exist $N,R>0$ such that $|\omega|_i > R \Rightarrow |f(\omega)| < N$. If for any $N>0$ there exists an $R>0$ with this property, we say that f tends to 0 along vertical lines. Analogously, we say that f remains bounded (resp. tends to 0) along "vertical strips" if for any $\tilde{z}\in\Omega^{r-1}$ there exists a neighbourhood $U\ni\tilde{z}$ in Ω^{r-1} and $N,R>0$ such that $|\omega|_i > R, \tilde{\omega}\in U \Rightarrow |f(\omega)| < N$ (resp. if for all $N>0$ there exists $R>0$ with this property).

Proposition 3.2.9 ([BP]). *Let f be a weak modular form for Γ. The following conditions are equivalent:*

(a) *f is holomorphic at infinity;*

(b) *f remains bounded in every vertical line;*

(c) *f remains bounded in every vertical strip.*

Moreover $\mathrm{ord}_{\Gamma_U}(f) \geq 1$ *if and only if f tends to 0 in every vertical line (equivalently every vertical strip).*

Proof. From the proof of Proposition 3.2.4, for every $\tilde{\omega} \in \Omega^{r-1}$, there is a neighbourhood $\tilde{V} \subset \Omega^{r-1}$ of $\tilde{\omega}$ and an affinoid open $V \subset \mathbb{C}_\infty \times \tilde{V}$ for which there is an isomorphism $\Gamma_U \backslash V \xrightarrow{\sim} B'(0, R(\tilde{\omega})) \times \tilde{V}$, where $B'(z, r)$ denotes the punctured disc with centre z and radius r, and $R(\tilde{\omega}) = \bar{\pi}^{-1}\rho(\tilde{\omega})^{-1}$ is a locally constant function.

A Γ_U invariant holomorphic function $f : \Omega^r \to \mathbb{C}_\infty$ induces a function $F : B'(0, R(\tilde{\omega})) \times \tilde{V} \to \mathbb{C}_\infty$. Fixing $\tilde{\omega}$ for the moment, we see that the expansion of f in terms of $u_{\tilde{\omega}}(\omega_1)$ is exactly the Laurent expansion of F at 0. Hence f is meromorphic at infinity if and only if, for every $\tilde{\omega}$, F has a pole at 0, and f is holomorphic at infinity if and only if, for every $\tilde{\omega}$, F extends to a function on $B(0, R(\tilde{\omega}))$. By Proposition 2.6.8, F extends to a function on $B(0, R(\tilde{\omega}))$ if and only if is bounded on $B(0, R(\tilde{\omega}))$.

(a)$\Rightarrow$(c): Now, if f is holomorphic at infinity, then for every $\tilde{\omega}$, $F(u, \tilde{\omega})$ is bounded for u in some neighbourhood of 0. However, this is the same as f being bounded, for every fixed $\tilde{\omega}$, on the "vertical line" where $\tilde{\omega}$ is fixed and $|\omega|_i \to \infty$.

Actually, by the proof of Proposition 3.2.4, if f is holomorphic at infinity, then every $\tilde{\omega}$ has a neighbourhood $\tilde{W}$ on which ρ is constant, and hence F is bounded on $B'(0, R(\tilde{\omega})) \times \tilde{W}$. So f is not only bounded in "vertical lines", but also in "vertical strips".

(c)$\Rightarrow$(b) is obvious.

(b)$\Rightarrow$(a): If f is bounded on every such line, then for every $\tilde{\omega}$, F is bounded on $W \smallsetminus \{0\} \times \{\tilde{\omega}\}$ for some neighbourhood W of 0. Then by Proposition 2.6.8 F extends to a holomorphic function on $W \times \{\tilde{\omega}\}$. This means that F has a power series expansion with no negative terms of $u_{\tilde{\omega}}(\omega_1)$, implying that f is holomorphic at infinity.

□

There are other cusps than the one at infinity and modular forms need to exhibit boundedness at each cusp. Rather than define expansions at every cusp, we study the expansions at infinity of the functions $f[\delta]_{k,m}$ for various $\delta \in \mathrm{GL}_r(F)$.

Define P to be the parabolic subgroup of $\mathrm{GL}_r(F)$ consisting of matrices of the form

$$\begin{pmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{pmatrix}.$$

Proposition 3.2.10 ([BP])**.** *Let $f : \Omega^r \to \mathbb{C}_\infty$ be a Γ_U invariant function and let $\delta \in P(F)$. Then $f[\delta]_{k,m}$ is a function invariant under $\Gamma_{\delta,U} := (\delta^{-1}\Gamma\delta) \cap U(F)$. Moreover*

$$\mathrm{ord}_{\Gamma_{\delta,U}}(f[\delta]_{k,m}) = \mathrm{ord}_{\Gamma_U}(f).$$

In particular $f[\delta]_{k,m}$ is meromorphic (resp. holomorphic) at infinity w.r.t. Γ if and only if f is meromorphic (resp. holomorphic) at infinity w.r.t. Γ.

Proof. Let us assume first that δ is of the form

$$\begin{pmatrix} 1 & 0 \\ \beta & \mathrm{id} \end{pmatrix}.$$

Then $\Gamma_{\delta,U} = \Gamma_U$ and $\delta\omega = (\omega_1 - \tilde{\omega}\beta, \tilde{\omega})$ and $\det\delta = 1 = j(\delta,\omega)$. The u parameter for $f[\delta]_{k,m}$ may now be written in terms of the parameter for f as follows:

$$\begin{aligned} u_{\tilde{\omega}}(\omega_1 - \tilde{\omega}\beta) &= \left(e_{\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(\bar{\pi}\omega_1) - e_{\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(\bar{\pi}\tilde{\omega}\beta)\right)^{-1} \\ &= u_{\tilde{\omega}}(\omega_1)\left(1 - u_{\tilde{\omega}}(\omega_1) e_{\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(\bar{\pi}\tilde{\omega}\beta)\right)^{-1}. \end{aligned}$$

Then the expansion for $f[\delta]_{k,m}$ is

$$\begin{aligned} f[\delta]_{k,m}(\omega) &= \sum_{n\in\mathbb{Z}} f_n(\tilde{\omega}) u_{\tilde{\omega}}(\omega_1 - \tilde{\omega}\beta)^n \\ &= \sum_{n\in\mathbb{Z}} f_n(\tilde{\omega}) u_{\tilde{\omega}}(\omega_1)^n \left(1 - u_{\tilde{\omega}}(\omega_1) e_{\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(\bar{\pi}\tilde{\omega}\beta)\right)^{-n}, \end{aligned}$$

proving the statement for δ of this form.

Now suppose that δ is of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \tilde{\delta} \end{pmatrix},$$

where $\alpha \in F^\times$ and $\tilde{\delta} \in \mathrm{GL}_{r-1}(F)$. If we let $\tilde{\Lambda}_\delta := \iota(\Gamma_{\delta,U}) = \tilde{\delta}^{-1}\Lambda_U\alpha$, then $\det\delta = \alpha\det\tilde{\delta}$ and $j(\delta,\omega) = j(\tilde{\delta},\tilde{\omega})$, implying that $\delta\omega = j(\tilde{\delta},\tilde{\omega})^{-1}\left(\alpha^{-1}\omega_1, \tilde{\omega}\tilde{\delta}^{-1}\right)$.

Hence the expansion of $f[\delta]_{k,m}$ must be in terms of $u_{\delta,\tilde{\omega}}(\omega_1) := e_{\bar{\pi}\tilde{\omega}\tilde{\Lambda}_\delta}(\bar{\pi}\alpha^{-1}\omega_1)^{-1} = e_{\bar{\pi}\tilde{\omega}\tilde{\delta}^{-1}\Lambda_U\alpha}(\bar{\pi}\omega_1)^{-1}$. We have

$$\begin{aligned} u_{\frac{\tilde{\omega}\tilde{\delta}^{-1}}{j(\tilde{\delta},\tilde{\omega})}}\left(\frac{\alpha^{-1}\omega_1}{j(\tilde{\delta},\tilde{\omega})}\right) &= e_{\bar{\pi}\frac{\tilde{\omega}\tilde{\delta}^{-1}}{j(\tilde{\delta},\tilde{\omega})}\Lambda_U}\left(\bar{\pi}\frac{\alpha^{-1}\omega_1}{j(\tilde{\delta},\tilde{\omega})}\right)^{-1} \\ &= \alpha\cdot j(\tilde{\delta},\tilde{\omega})\cdot e_{\bar{\pi}\tilde{\omega}\tilde{\delta}^{-1}\Lambda_U\alpha}(\bar{\pi}\omega_1)^{-1} \\ &= \alpha\cdot j(\tilde{\delta},\tilde{\omega})\cdot u_{\delta,\tilde{\omega}}(\omega_1). \end{aligned}$$

and it follows that the expansion of $f[\delta]_{k,m}$ is

$$\begin{aligned} & j(\tilde{\delta},\tilde{\omega})^{-k}(\det\tilde{\delta})^{-m}f(\delta\omega) \\ = \ & j(\tilde{\delta},\tilde{\omega})^{-k}\det(\tilde{\delta})^{-m}f\left(\frac{\omega_1\alpha^{-1}}{j(\tilde{\delta},\tilde{\omega})},\frac{\tilde{\omega}\tilde{\delta}^{-1}}{j(\tilde{\delta},\tilde{\omega})}\right) \\ = \ & j(\tilde{\delta},\tilde{\omega})^{-k}\det(\tilde{\delta})^{-m}\sum_{n\in\mathbb{Z}}f_n\left(\frac{\tilde{\omega}\tilde{\delta}^{-1}}{j(\tilde{\delta},\tilde{\omega})}\right)u_{\frac{\tilde{\omega}\tilde{\delta}^{-1}}{j(\tilde{\delta},\tilde{\omega})}}\left(\frac{\alpha^{-1}\omega_1}{j(\tilde{\delta},\tilde{\omega})}\right)^n \\ = \ & j(\tilde{\delta},\tilde{\omega})^{-k}\det\tilde{(}\delta)^{-m}\sum_{n\in\mathbb{Z}}f_n\left(\frac{\tilde{\omega}\tilde{\delta}^{-1}}{j(\tilde{\delta},\tilde{\omega})}\right)\alpha^n\cdot j(\tilde{\delta},\tilde{\omega})^n\cdot u_{\delta,\tilde{\omega}}(\omega_1)^n. \end{aligned}$$

The statement for δ of this form follows from this equality.

Finally, by writing an arbitrary $\delta\in P(F)$ as a product of matrices of the two forms above, the statement follows in full generality.

□

Proposition 3.2.11 ([BP]). *Let $\Gamma_1<\Gamma$ be arithmetic subgroups of $\mathrm{GL}_r(F)$ and set $\Gamma_{1,U}:=\Gamma_U\cap\Gamma_1$. Let $f:\Omega^r\to\mathbb{C}_\infty$ be a Γ_U invariant function. Then*

$$\mathrm{ord}_{\Gamma_{1,U}}(f)=[\Gamma_U:\Gamma_{1,U}]\mathrm{ord}_{\Gamma_U}(f).$$

In particular f is meromorphic (resp. holomorphic) at infinity w.r.t. Γ if and only if f is meromorphic (resp. holomorphic) at infinity w.r.t. Γ_1.

Proof. Note that $\Lambda_{1,U}=\iota(\Gamma_{1,U})$ is a subgroup of $\Lambda_U=\iota(\Gamma_U)$ of index $[\Lambda_U:\Lambda_{1,U}]=[\Gamma_U:\Gamma_{1,U}]$. Also note that both $\Lambda_{1,U}$ and Λ_U are $\mathbb{F}_p$-linear subsets of F^{r-1}, and hence that for any $\tilde{\omega}$ the sets $\tilde{\omega}\Lambda_U$ and $\tilde{\omega}\Lambda_{1,U}$ are $\mathbb{F}_p$-linear subsets of $\mathbb{C}_\infty$. Let the index be p^d, and let $\Phi_{\tilde{\omega}}$ be the polynomial from Proposition 2.2.5 for which $e_{\bar{\pi}\tilde{\omega}\Lambda_U}=\Phi_{\tilde{\omega}}\circ e_{\bar{\pi}\tilde{\omega}\Lambda_{1,U}}$. We know that $\Phi_{\tilde{\omega}}$ is $\mathbb{F}_p$-linear, and

$$\Phi_{\tilde{\omega}}(z)=e_{e_{\bar{\pi}\tilde{\omega}\Lambda_{1,U}}(\bar{\pi}\tilde{\omega}\Lambda_U)}(z)=:\sum_{i=0}^{d}\Phi_{\tilde{\omega},i}z^{p^i},$$

and that the polynomial depends on $\tilde{\omega}$. By the choice of d we have $\Phi_{\tilde{\omega},d} \neq 0$. We may now compute

$$\begin{aligned}
u_{\tilde{\omega}}(\omega_1) &= e_{\bar{\pi}\tilde{\omega}\Lambda_U}(\bar{\pi}\omega_1)^{-1} = \left(\Phi_{\tilde{\omega}}(e_{\bar{\pi}\tilde{\omega}\Lambda_{1,U}}(\bar{\pi}\omega_1))\right)^{-1} \\
&= \left(\Phi_{\tilde{\omega}}\left(\frac{1}{u_{1,\tilde{\omega}}(\omega_1)}\right)\right)^{-1} = \left(\sum_{i=0}^{d} \Phi_{\tilde{\omega},i} u_{1,\tilde{\omega}}(\omega_1)^{-p^i}\right)^{-1} \\
&= \frac{u_{1,\tilde{\omega}}(\omega_1)^{p^d}}{\Phi_{\tilde{\omega},d}} \left(1 + \sum_{i=0}^{d-1} \frac{\Phi_{\tilde{\omega},i}}{\Phi_{\tilde{\omega},d}} u_{1,\tilde{\omega}}(\omega_1)^{p^d - p^i}\right)^{-1}.
\end{aligned}$$

We may expand the last factor as a geometric series, which contains only positive powers of $u_{1,\tilde{\omega}}$. So $\mathrm{ord}_{\Gamma_U}(f) = n$ if and only if the expansion of f in terms of $u_{\tilde{\omega}}$ starts with $f_n(\tilde{\omega})u_{\tilde{\omega}}(\omega_1)^n$ if and only if the expansion of f in terms of $u_{1,\tilde{\omega}}$ starts with $f_n(\tilde{\omega})\frac{u_{1,\tilde{\omega}}(\omega_1)^{np^d}}{\Phi^n_{\tilde{\omega},d}}$ if and only if $\mathrm{ord}_{\Gamma_1,U}(f) = np^d$, proving the proposition. □

3.3 Modular forms

3.3.1 Analytic modular forms

Definition 3.3.1. *A function $f : \Omega^r \to \mathbb{C}_\infty$ is a* modular form *of weight k and type m for Γ if f is a weak modular form of weight k and type m for Γ and $f[\delta]_{k,m}$ is holomorphic at infinity for all $\delta \in \mathrm{GL}_r(F)$. It is said to be a* cusp form *(resp.* double cusp form*) if furthermore $\mathrm{ord}_{\Gamma_U}(f[\delta]_{k,m}) \geq 1$ (resp. ≥ 2) for all $\delta \in \mathrm{GL}_r(F)$.*

Proposition 3.3.2. *Let $D = \{\delta_i\}_i$ be a set of representatives for the double coset $\Gamma\backslash\mathrm{GL}_r(F)/P(F)$ and let f be a weak modular form for Γ.*

(a) *If $f[\delta]_k$ is holomorphic at infinity for each $\delta \in D$, then f is a modular form.*

(b) *Let $h(A)$ be the class number of A. If $\Gamma = \mathrm{GL}_r(A)$, then D has $h(A)$ elements.*

(c) *In general, D is a finite set.*

Thus, the condition of holomorphy at infinity only needs to be checked for finitely many $\delta \in \mathrm{GL}_r(F)$. In particular, if $A = \mathbb{F}_q[t]$ and $\Gamma = \mathrm{GL}_r(A)$, then f is a modular form if and only if it is holomorphic at infinity.

Proof. (a) follows from the Γ-invariance of f and Proposition 3.2.10.

For (b) recall that the quotient of algebraic groups GL_r/P is isomorphic to projective space $\mathbb{P}^{r-1}$. Therefore $\mathrm{GL}_r(F)/P(F) \cong \mathbb{P}^{r-1}(F)$. We now show that the map $\mathrm{GL}_r(A)\backslash\mathbb{P}^{r-1}(F)$ to the class group $\mathrm{Cl}(A)$ of A defined by $[x_1 : \cdots : x_r] \mapsto [\sum_{i=1}^r x_i A]$ is a bijection.

First we check that the map is well-defined. Firstly two different representatives in $\mathbb{P}^{r-1}(F)$ differ by a factor of some $a \in F$. Therefore the ideals $\sum_{i=1}^r x_i A$ differ by a factor $a \in F$, and thus lie in the same ideal class. Now suppose that $\gamma \in \mathrm{GL}_r(A)$ and that $\gamma[x_1 : \cdots : x_r] = [y_1 : \cdots : y_r]$. Since $\gamma \in \mathrm{GL}_r(A)$, each $y_j \in \sum_{i=1}^r x_i A$, and that each $x_j \in \sum_{i=1}^r y_i A$. This implies that the ideals $\sum_{i=1}^r x_i A$ and $\sum_{i=1}^r y_i A$ are the same.

The map is clearly surjective, so it remains to check injectivity. Suppose that $[x_1 : \cdots : x_r]$ and $[y_1 : \cdots : y_r]$ satisfy $\sum_{i=1}^r x_i A = \sum_{i=1}^r y_i A$ (if these ideals differ by a constant factor, we may renormalize one of the elements to make the ideals equal). Denote these ideals by M. Consider the maps

$$\begin{aligned} p_x &: A^r \to M, \quad (a_1, \dots, a_r) \mapsto \textstyle\sum_{i=1}^r a_i x_i; \quad \text{and} \\ p_y &: A^r \to M, \quad (a_1, \dots, a_r) \mapsto \textstyle\sum_{i=1}^r a_i y_i. \end{aligned}$$

Both are surjective and fit into short exact sequences

$$0 \to \ker p_x \to A^r \to M \to 0, \tag{3.1}$$

(and similar for y). Since $\mathrm{GL}_r(A)$ consists of exactly the automorphisms of A^r, the problem becomes that of lifting the identity on M to an automorphism of A^r in the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & \ker p_x & \to & A^r & \to & M & \to & 0 \\ & & & & & & || & & \\ 0 & \to & \ker p_y & \to & A^r & \to & M & \to & 0. \end{array}$$

Since M is projective (as an A-module), and the identity map $\mathrm{id}_M : M \to M$ is surjective, there is a map $s : M \to A^r$ such that $p_x \circ s = \mathrm{id}_M$. This means that the short exact sequence (3.1) splits. Therefore $\ker p_x \cong A^r/M$, and similarly $\ker p_y \cong A^r/M$. Thus we may choose an isomorphism $\ker p_x \xrightarrow{\sim} \ker p_y$. Writing $A^r = \ker p_x \oplus M$ (resp. $A^r = \ker p_y \oplus M$) in the following diagram:

$$\begin{array}{ccccc} \ker p_x & \to & \ker p_x \oplus M & \leftarrow & M \\ \downarrow & & & & \downarrow \\ \ker p_y & \to & \ker p_y \oplus M & \leftarrow & M, \end{array}$$

means that there is a unique morphism $\ker p_x \oplus M \to \ker p_y \oplus M$ making the whole diagram commute. But by the Five Lemma, if the two morphisms on the side are isomorphisms, then the arrow in the middle is an isomorphism as well.

This gives us the automorphism of A^r we were after and proves injectivity.

Now for a congruence subgroup Γ of a general stabilizer (not necessarily $\mathrm{GL}_r(A)$), the group $\Gamma \cap \mathrm{GL}_r(A)$ is of finite index in $\mathrm{GL}_r(A)$. But D can have at most as many elements as this index, proving (c). □

As in the classical case, the modular forms of a given weight and a given type form a $\mathbb{C}_\infty$-vector space. Furthermore, by Lemma 3.1.3 (a) the product of a modular form of weight k_1 and type m_1 with a modular form of weight k_2 and type m_2 gives a modular form of weight $k_1 + k_2$ and type $m_1 + m_2$. The weight and type define a double grading on this ring.

Definition 3.3.3. *Denote by $\mathcal{M}_{k,m}(\Gamma)$ the space of modular forms of weight k and type m for Γ. We shall omit m if the type is 0. Also denote by $\mathcal{M}(\Gamma)$ the doubly graded ring of modular forms for Γ.*

Remark. Since m occurs only in the power of the determinant of some $\gamma \in \Gamma$, it only depends on the order of $\det \Gamma \subseteq \mathbb{F}_q^*$. In particular, it makes sense to consider spaces $\mathcal{M}_{k,m}(\Gamma)$ for all $k \in \mathbb{Z}$, and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$.

3.3.2 Algebraic modular forms

Let $K \subset \mathrm{GL}_r(\hat{A})$ be a fine open compact subset and denote by $\mathcal{L}_K$ the dual of the relative Lie algebra of $\bar{E} \to \bar{M}_K^r$. (Recall that $E \to M_K^r$ is the universal Drinfeld module and that $\bar{E}$ is its extension to $\bar{M}_K^r$, which is unique by Theorem 2.8.4.) It is an invertible sheaf, so we may speak of $\mathcal{L}_K^k$.

Definition 3.3.4. *([Pi] Definition 5.4) For any integer k we define the space of* algebraic modular forms *of weight k as*

$$\mathcal{M}_k^{alg}(K) = H^0\left(\bar{M}_K^r, \mathcal{L}_{K(N)}^k\right)^K \quad \text{where } K(N) \subset K \text{ is fine.}$$

(This is the subring of K-invariant elements under the $\mathrm{GL}_r(\hat{A})$ action described in [Pi].) In [Pi] it is shown that this definition does not depend on the choice of N.

The ring of algebraic modular forms is the graded ring

$$\mathcal{M}^{alg}(K) = \bigoplus_{k \geq 0} \mathcal{M}_k(K).$$

Also denote by $\mathcal{W}_k^{alg}(K) = H^0(M_K^r, \mathcal{L}_{K(N)}^k)^K$ the space of algebraic weak modular forms of weight k and by $\mathcal{W}_k(\Gamma)$ the space of analytic weak modular forms of weight k for Γ.

Theorem 3.3.5 ([BP])**.** *Let S be a set of representatives for the double coset $\mathrm{GL}_r(F)\backslash\mathrm{GL}_r(\mathbb{A}_F^f)/K$, and for each $g \in S$ set $\Gamma_g = gKg^{-1} \cap \mathrm{GL}_r(F)$. Then there are isomorphisms of $\mathbb{C}_\infty$-vector spaces:*

$$\mathbb{C}_\infty \otimes_F \mathcal{W}_k^{alg}(K) \xrightarrow{\sim} \bigoplus_{g\in S} \mathcal{W}_k(\Gamma_g).$$

$$\mathbb{C}_\infty \otimes_F \mathcal{M}_k^{alg}(K) \xrightarrow{\sim} \bigoplus_{g\in S} \mathcal{M}_k(\Gamma_g).$$

3.4 Examples of modular forms

From now on we fix a projective A-submodule $\Lambda \subset F^r$ and denote by $\mathrm{GL}_A(\Lambda)$ its stabilizer in $\mathrm{GL}_r(F)$. We also impose the condition on Λ that $\Lambda = A \times \tilde{\Lambda}$, i.e. that the first coordinate can be separated from the others. We shall view elements of $\Lambda \subset F^r$ as column vectors, and hence $\omega\Lambda$ can be interpreted as a lattice in $\mathbb{C}_\infty$.

3.4.1 Eisenstein series

Perhaps the simplest example of a modular form is the Eisenstein series. We shall provide two examples — Eisenstein series for the full modular group $\mathrm{GL}_A(\Lambda)$ and Eisenstein series for congruence subgroups $\Gamma(N)$.

Eisenstein series for $\mathrm{GL}_A(\Lambda)$

Let k be a positive integer and define

$$E^k(\omega) := {\sum_{\lambda\in\omega\Lambda}}' \lambda^{-k}.$$

For any $N > 0$, there are only finitely many elements of Λ in any ball of radius N. Therefore this infinite sum converges for any $\omega \in \Omega^r$. Moreover, on any Ω_n, this convergence is uniform, implying that E^k is holomorphic on Ω^r.

The elements $\gamma \in \mathrm{GL}_A(\Lambda)$ have the property that lattices $\omega\gamma^{-1}\Lambda = \omega\Lambda$. This means that $\gamma\omega\Lambda = j(\gamma,\omega)^{-1}\omega\Lambda$ and ultimately that

$$E^k[\gamma]_{k,0}(\omega) = j(\gamma,\omega)^{-k} \sum_{\lambda \in j(\gamma,\omega)^{-1}\omega\Lambda} \lambda^{-k} = E^k(\omega),$$

proving that E_k is a weak modular form of weight k and type 0 for $\mathrm{GL}_A(\Lambda)$. We shall see in section 3.5.1 that the u-expansion has no terms with a negative exponent of u, which means that it is a modular form.

Eisenstein series for $\Gamma(N)$

Definition 3.4.1. *Let $N \subset A$ be an ideal. The group $\Gamma(N) \subset \mathrm{GL}_A(\Lambda)$ defined by*

$$\Gamma(N) = \ker\left(\mathrm{GL}_A(\Lambda) \to \mathrm{GL}_A(\Lambda/N\Lambda)\right)$$

is called a principal congruence subgroup *of $\mathrm{GL}_A(\Lambda)$.*

We may adapt the definition of the previous Eisenstein series in order to obtain modular forms for more general arithmetic groups. Let $N \subset A$ be a non-zero ideal, and let $[v] \in N^{-1}\Lambda/\Lambda$ be a non-zero residue class with $v = (v_1, \ldots, v_r) \in N^{-1}\Lambda/\Lambda$ a representative for it (i.e. $[v] = v + \Lambda$). Define

$$E^k_{[v]}(\omega) := \sum_{\lambda \in \omega\cdot[v]} \lambda^{-k} = \sum_{(a_1,\ldots,a_r)\in\Lambda} ((v_1 + a_1)\omega_1 + \cdots + (v_r + a_r))^{-k}.$$

By our assumption that $\Lambda = A \times \tilde{\Lambda}$, there is an isomorphism $\iota_1 : N^{-1}\Lambda/\Lambda \xrightarrow{\sim} (N^{-1}/A) \times (N^{-1}\tilde{\Lambda}/\tilde{\Lambda})$. For $[z] \in (N^{-1}/A)$, set $|[z]| := |N| \cdot \min\{|z_0| : z_0 \in [z]\}$.

Proposition 3.4.2 ([BP])**.** *Let $\omega \in \Omega^r$, let $[v] \in N^{-1}\Lambda/\Lambda$ be a non-zero residue class and let $v \in [v]$ be a representative, as before.*

(a) $E^1_{[v]}(\omega) = e_{\omega\Lambda}(\omega v)^{-1}$.

(b) $E^k_{[v]}[\gamma]_k(\omega) = E^k_{[\gamma^{-1}v]}(\omega)$ *for all $\gamma \in \mathrm{GL}_A(\Lambda)$. In particular, $E^k_{[v]}$ is a weak modular form for $\Gamma(N)$.*

(c) $\mathrm{ord}_{\Gamma(N)_U}(E^1_{[v]}) = (|[v_1]| \cdot q^{\deg N})^{r-1}$.

(d) *Each $E^k_{[v]}$ is holomorphic at infinity.*

(e) *$E^k_{[v]}$ is a modular form of weight k for $\Gamma(N)$.*

Proof. The statement in (a) is Lemma 2.5.1 with $z = \omega v$ and lattice $\omega\Lambda$. For (b) we calculate that for any $\gamma \in \mathrm{GL}_A(\Lambda)$,

$$\begin{aligned} E^k_{[v]}[\gamma]_k(\omega) &= j(\gamma,\omega)^{-k} \sum_{\substack{(a_1,\ldots,a_r)\in N^{-1}\Lambda \\ a\equiv v \pmod{\Lambda}}} \left(\omega\gamma^{-1}\cdot a\right)^{-k} \\ &= \sum_{\substack{(a_1,\ldots,a_r)\in N^{-1}\Lambda \\ a\equiv v \pmod{\Lambda}}} (\omega\gamma^{-1}\cdot a)^{-k} \\ &= \sum_{\substack{(a_1,\ldots,a_r)\in \gamma^{-1}N^{-1}\Lambda \\ a\equiv \gamma^{-1}v \pmod{\Lambda}}} (\omega\cdot a)^{-k} \\ &= E^k_{\gamma^{-1}[v]}(\omega). \end{aligned}$$

In order to prove (c) we need, to some extent, to calculate the u expansion of $E^k_{[v]}(\omega)$. We have

$$\begin{aligned} E^k_{[v]}(\omega) &= \sum_{(a_1,\ldots,a_r)\in\Lambda} (\omega\cdot(a+v))^{-k} \\ &= \bar{\pi}^k \sum_{a_1\in A}\sum_{(a_2,\ldots,a_r)\in\tilde{\Lambda}} (\bar{\pi}(a_1+v_1)\omega_1 + \bar{\pi}\tilde{\omega}\tilde{v} + \bar{\pi}\tilde{\omega}\tilde{a})^{-k} \\ &= \bar{\pi}^k \sum_{a_1\in A} \left(P_{k,\bar{\pi}\tilde{\omega}\tilde{\Lambda}}\left(\bar{\pi}(a_1+v_1)\omega_1 + \bar{\pi}\tilde{\omega}\tilde{v} + \bar{\pi}\tilde{\omega}\tilde{a}\right)^{-1}\right) \\ &= \bar{\pi}^k \sum_{a_1\in A} P_{k,\bar{\pi}\tilde{\omega}\tilde{\Lambda}}\left((e_{\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(\bar{\pi}(a_1+v_1)\omega_1) + e_{\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(\bar{\pi}\tilde{\omega}\tilde{v}))^{-1}\right) \end{aligned} \tag{3.2}$$

Let us write $\tilde{E}^1_{[\tilde{v}]}(\tilde{\omega}) := e_{\tilde{\omega}\tilde{\Lambda}}(\tilde{\omega}\tilde{v})^{-1}$ for the rank $r-1$ Eisenstein series. Then $e_{\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(\bar{\pi}\tilde{\omega}\tilde{v})^{-1} = \bar{\pi}^{-1}\tilde{E}^1_{[\tilde{v}]}(\tilde{\omega})$.

Let $n \in N$. Then for some ideal $M \subset A$ we have $nA = MN$ and $N^{-1} = \frac{1}{n}M$. Since $a_1 + v_1 \in N^{-1} \subset F$, it is of the form m/n where $m \in M$ and $n \in N$. Suppose that $a_1 + v_1 \neq 0$. Then, since $M \subset mA$, we have $(a_1+v_1)^{-1}A = \frac{n}{m}A = \frac{1}{m}nA = \frac{1}{m}MN \supset N$. Hence $(a_1+v_1)^{-1}\tilde{\Lambda} \supset N\tilde{\Lambda}$ and hence by Proposition 2.2.5 (b), $e_{\bar{\pi}\tilde{\omega}(a_1+v_1)^{-1}\tilde{\Lambda}}(x) = \Phi_{a_1+v_1,\tilde{\omega}}(e_{\bar{\pi}\tilde{\omega}N\tilde{\Lambda}}(x))$ for some polynomial $\Phi_{a_1+v_1,\tilde{\omega}}$.[2] Note that this stays true if we make the convention that $\Phi_0 = 0$. Let us mention at this point that if $a_1 + v_1 \neq 0$ then $\Phi_{a_1+v_1,\tilde{\omega}}$ has degree $|(a_1+v_1)^{-1}/N|^{r-1}$ and linear coefficient 1 and that

[2] Actually, by Proposition 2.2.5, this polynomial is the exponential function associated to the finite $\mathbb{F}_q$-linear set $e_{\bar{\pi}\tilde{\omega}N\tilde{\Lambda}}(\bar{\pi}\tilde{\omega}(a_1+v_1)^{-1}\tilde{\Lambda})$.

its leading coefficient is a function which is nowhere zero on Ω^{r-1}. Now

$$\begin{aligned} e_{\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(\bar{\pi}(a_1+v_1)\omega_1) &= (a_1+v_1)e_{\bar{\pi}\tilde{\omega}(a_1+v_1)^{-1}\tilde{\Lambda}}(\bar{\pi}\omega_1) \\ &= (a_1+v_1)\Phi_{a_1+v_1,\tilde{\omega}}(e_{\bar{\pi}\tilde{\omega}N\tilde{\Lambda}}(\bar{\pi}\omega_1)) \\ &= (a_1+v_1)\Phi_{a_1+v_1,\tilde{\omega}}(u^{-1}). \end{aligned} \tag{3.3}$$

For simplicity, denote $P_{k,\bar{\pi}\tilde{\omega}\tilde{\Lambda}}$ by P_k and $\Phi_{a_1+v_1,\tilde{\omega}}$ by $\Phi_{a_1+v_1}$ and even Φ when a_1+v_1 is understood. Together equations (3.2) and (3.3) yield

$$E^k_{[v]}(\omega) = \bar{\pi}^k \sum_{a_1\in A} P_k\left(\left((a_1+v_1)\Phi_{a_1+v_1}(u^{-1}) + \tilde{E}^1_{[\tilde{v}]}(\tilde{\omega})^{-1}\right)^{-1}\right). \tag{3.4}$$

If $v_1 \in A$, then there is a term where $a_1+v_1=0$, which must be the term $\bar{\pi}^k P_k\left(\tilde{E}^1_{[\tilde{v}]}\right) = \tilde{E}^k_{[\tilde{v}]}(\tilde{\omega})$. It is not divisible by u and we shall shortly see that all the other terms are divisible by u. Hence in this case $\mathrm{ord}_{\Gamma(N)_U}\left(E^1_{[v]}(\omega)\right) = 0$.

If $v_1 \notin A$, then a generic term is of the form $\bar{\pi}^k P_k((c_d u^{-d} + \cdots + c_0)^{-1})$ where the argument is a polynomial of degree $d = |(a_1+v_1)^{-1}/N|^{r-1}$ in u^{-1} with coefficients being functions on Ω^{r-1}. The argument in this function can be rewritten as $u^d(c_d+\cdots+c_0u^d)^{-1}$ where $(c_d+\cdots+c_0u^d)$ is invertible since c_d is a function which is nowhere zero on Ω^{r-1}. Such a term thus has u-order $|(a_1+v_1)^{-1}/N|^{r-1}$. Hence $\mathrm{ord}_{\Gamma(N)}\left(E^1_{[v]}(\omega)\right) = \min_{a_1\in A}\left\{|(a_1+v_1)^{-1}/N|^{r-1}\right\} \geq 0$ so that $E^1_{[v]}(\omega)$ is holomorphic at infinity.

Noting that the Goss polynomial $P_k(X)$ is divisible by X (Proposition 2.5.2 (c)), equation (3.4) immediately implies that for any k we have $\mathrm{ord}_{\Gamma(N)_U}\left(E^k_{[v]}(\omega)\right) \geq \mathrm{ord}_{\Gamma(N)_U}\left(E^1_{[v]}(\omega)\right)$, implying (d).

To show (e), it remains to show that $E^k_{[v]}[\delta]_k$ is holomorphic at infinity for every $\delta \in \mathrm{GL}_r(F)$. This will follow from the fact that $E^k_{[v]}[\delta]_k$ is a linear combination of Eisenstein series of higher level, for each of which (c) is true.

Note that if $c \in F^\times$, then $E^k_{[v]}[c\cdot\mathrm{id}]_k = c^k E^k_{[v]}$, so we assume from now on that δ^{-1} has entries in A. Then $\delta^{-1}\Lambda \subseteq \Lambda$, but is of finite index, since $\delta \in \mathrm{GL}_r(F)$ is an automorphism of F^r. Therefore there exists a principal ideal $mA \subset A$ which annihilates the finite A-module $\Lambda/\delta^{-1}\Lambda$ and hence $(mA)^r \subseteq \delta^{-1}\Lambda \subseteq \Lambda$.

Since Λ contains finitely many cosets modulo $m\Lambda$, the set $\delta^{-1}\Lambda$ also contains only finitely many cosets, and $\delta^{-1}(\Lambda+v)$ also contains finitely many cosets modulo $m\Lambda$. Let $u_1, \dots, u_n$ be representatives for these cosets. Then

$$\begin{aligned} E^k_{[v]}[\delta]_k(\omega) &= \sum_{a\in\Lambda} \big(\omega\cdot\delta^{-1}(a+v)\big)^{-k} \\ &= \sum_{i=1}^{n}\sum_{a\in m\Lambda} (\omega\cdot(a+u_i))^{-k} \\ &= \sum_{i=1}^{n}\sum_{a\in\Lambda} (\omega\cdot(ma+u_i))^{-k} \\ &= \sum_{i=1}^{n}\sum_{a\in\Lambda} m^{-k}\left(\omega\cdot\left(a+\frac{u_i}{m}\right)\right)^{-k} \\ &= \sum_{i=1}^{n} m^{-k}E^k_{[\frac{u_i}{m}]}(\omega). \end{aligned}$$

In this case each E^k is an Eisenstein series for $\Gamma(mN^{-1})$, hence is holomorphic at infinity, implying that $E^k_{[v]}[\delta]_k(\omega)$ is holomorphic at infinity. This completes the proof of (e). □

3.4.2 Coefficient forms

Let $\varphi := \varphi^{\omega\Lambda}$ be the Drinfeld module associated to $\omega\Lambda$. Then, for $a \in A$, we have the equality

$$\varphi_a(X) = \sum_{i=0}^{r\deg a} g_i(a,\omega)X^{q^i}, \tag{3.5}$$

where $g_0(a,\omega) = a$ and for all $\omega \in \Omega^r$ we have $g_{r\deg a}(a,\omega) \neq 0$. When $A = \mathbb{F}_q[t]$ and $a = t$ we shall suppress mention of a and write simply $g_i(\omega)$. Later we shall need the convention that $g_0(a,\omega) = a$ even when $\deg a = 0$.

Proposition 3.4.3 ([BP]). *Let $N \subset A$ be an ideal, let $a \in N$ and let $[v] \in N^{-1}\Lambda/\Lambda$ be a non-zero residue class. Then*

(a) $\varphi_a(E^1_{[v]}(\omega)^{-1}) = 0$; *and*

(b) $\varphi_a(X) = aX \prod'_{[v]\in a^{-1}\Lambda/\Lambda} (1 - XE^1_{[v]})$.

Proof. We have

$$\begin{aligned}\varphi_a(E^1_{[v]}(\omega)^{-1}) &= \varphi_a(e_{\omega\Lambda}(\omega\cdot v)) \quad \text{by Lemma 2.5.1}\\ &= e_{\omega\Lambda}(a\omega\cdot v) \quad \text{by Proposition 2.2.5}\end{aligned}$$

proving (a), since $av \in aN^{-1}\Lambda \supset \Lambda$ and (b) follows from this case since both polynomials have the same roots, degree and coefficient of X. □

The coefficients vary with ω and thus define functions on Ω^r. The functions $g_i(a,\omega)$ are called *coefficient forms*. From Proposition 3.4.3, they are polynomials in the weight 1 Eisenstein series $E^1_{[v]}(\omega)$. Since these are modular forms for $\Gamma(a)$, each $g_i(a,\omega)$ is also a modular form for $\Gamma(N)$.

Proposition 3.4.3 allows us to generalize the definition of coefficient forms to all ideals $N \subset A$. Set

$$\varphi_N(X) = X \sideset{}{'}\prod_{[v]\in N^{-1}\Lambda/\Lambda} (1 - E^1_{[v]}(\omega)X).$$

The roots of this polynomial are

$$\{0\} \cup \left\{ E^1_{[v]}(\omega)^{-1} \;\middle|\; 0 \neq [v] \in N^{-1}\Lambda/\Lambda \right\} = \{e_{\omega A^r}(\omega v) \,|\, v \in [v] \in N^{-1}\Lambda/\Lambda\}$$

which forms an $\mathbb{F}_q$-linear set, which means that the polynomial itself is $\mathbb{F}_q$-linear, since it is the exponential function of its zero set. We define the *normalized coefficient forms* with respect to N as the coefficients of the polynomial

$$\varphi_N(X) = \sum_{i=0}^{r\deg N} \mathfrak{g}_i(N,\omega)X^{q^i}.$$

Note that the usual coefficient forms can be related to normalized coefficient forms by the formula $g_i(a,\omega) = a\mathfrak{g}_i(aA,\omega)$. In particular we deduce that $\mathfrak{g}_0(N,\omega) = 1$ for every N.

Proposition 3.4.4 ([BP])**.** *For every ideal $N \subset A$ and every $i = 1,\ldots,r\deg N$, the function $\mathfrak{g}_i(N,\omega)$ defines a modular form of weight $q^i - 1$ and type 0 for* $\mathrm{GL}_A(\Lambda)$.

Proof. Since $\mathfrak{g}_i$ is a homogeneous polynomial of weight $q^i - 1$ in the modular forms $E^1_{[v]}(\omega)$, it too is a modular form of weight $q^i - 1$ for $\Gamma(N)$. However, by Proposition 3.4.2 (b), any element of $\mathrm{GL}_A(\Lambda)$ permutes the forms $E^1_{[v]}$, showing that $\mathfrak{g}_i$ is a modular form for $\mathrm{GL}_A(\Lambda)$. □

Of special interest to us will be the discriminant function.

Definition 3.4.5. *We call the coefficient form $g_{r\deg a}(a,\omega)$ of highest weight the* Drinfeld discriminant function for *a and denote it by $\Delta(a,\omega)$. Again, if $A=\mathbb{F}_q[t]$, we denote $\Delta(\omega):=\Delta(t,\omega)=g_r(\omega)$. In that case we call it simply the rank r* Drinfeld discriminant function. *To avoid confusion, in this work we never write $\Delta(N,\omega)$ for the normalized coefficient form of highest weight. Instead we shall use the usual notation $\mathfrak{g}_{r\deg N}(N,\omega)$ in that case.*

We shall see that $\Delta(\omega)$ is the type 0 modular form of lowest weight which is a cusp form for $\mathrm{GL}_r(\mathbb{F}_q[t])$.

3.4.3 Relations between modular forms

The main goal of this section is to obtain some relations between the Eisenstein series for $\mathrm{GL}_A(\Lambda)$ and the coefficient forms. In doing so we reach a secondary goal by mentioning other examples of modular forms for $\mathrm{GL}_A(\Lambda)$, namely the coefficients of the exponential function. This is also a convenient place for the definition of some quantities that are of arithmetic interest.

(The contents of this section is essentially in [Ge3], though in the form of relations between coefficients where Λ is a fixed lattice.)

Definition 3.4.6. *For every integer $n>0$ we define:*

(a) $[n]:=t^{q^n}-t\in\mathbb{F}_q[t]$;

(b) $D_n:=[n][n-1]^q\cdots[1]^{q^{n-1}}$;

(c) $L_n:=[n][n-1]\cdots[1]$.

For each $\omega\in\Omega^r$, there is the exponential function $e_{\omega\Lambda}(X)=\sum_{i\geq 0}e_i(\omega\Lambda)X^{q^i}$ from Section 2.2. As ω varies, the coefficients $e_i(\omega):=e_i(\omega\Lambda)$ vary, and it turns out that for each i, the function $e_i(\omega)$ is a modular form of weight q^i-1 and type 0 for $\mathrm{GL}_r(A)$. It should not be hard to show this directly, but we shall proceed to prove this by obtaining some relations between $(e_i(\omega))_{0\leq i\leq k}$ and $(g_i(a,\omega))_{0\leq i\leq k}$.

Proposition 3.4.7. *Let A be a Drinfeld ring and let $\Lambda\subset\mathbb{C}_\infty$ be an A-lattice. Let $a\in A\smallsetminus\mathbb{F}_q$ and*

$$e_\Lambda(X)=\sum_{n\geq 0}e_nX^{q^n}\quad \textit{and}\quad \varphi_a^\Lambda(X)=\sum_{n=0}^{r\deg a}g_nX^{q^n}.$$

Then

$$\left(a^{q^k}-a\right)e_k=\sum_{i=1}^{r\deg a}g_ie_{k-i}^{q^i}.$$

Proof. The relation follows by comparing coefficients of X^{q^k} in equation (2.3): $e_\Lambda(aX) = \varphi_a^\Lambda(e_\Lambda(X))$. □

Corollary 3.4.8. (a) *For every $i \geq 0$, we have $e_i(\omega) \in F[g_1(a,\omega), \ldots, g_{r\deg a}(a,\omega)]$.*

(b) *If $A = \mathbb{F}_q[t]$, then for every $i \geq 0$, we have $D_i e_i(\omega) \in A[g_1(\omega), \ldots, g_r(\omega)]$.*

Proof. Both (a) and (b) follow from simple inductions. We outline the induction for (b). We have

$$e_1(\omega) = [1]^{-1} g_1(\omega)$$

so that $D_1 e_1(\omega) = g_1(\omega)$. Taking $a = t$ in equation 3.4.7 we get that

$$\left(t^{q^k} - t\right) e_k(\omega) = \sum_{i=1}^{r\deg a} g_i(\omega) e_{k-i}(\omega)^{q^i}.$$

Noting that $D_i = [i] D_{i-1}^q$, we see that if $D_i e_i(\omega) \in A\left[g_1(\omega), \ldots, g_r(\omega)\right]$, then $D_{i+1} e_i(\omega)^q \in A\left[g_1(\omega), \ldots, g_r(\omega)\right]$ and by induction $D_{i+j} e_i(\omega)^{q^j} \in A\left[g_1(\omega), \ldots, g_r(\omega)\right]$.

Now note that equation 3.4.7 can be rewritten as

$$D_k e_k(\omega) = D_{k-1}^q \sum_{i=1}^{r} g_i(\omega) e_{k-i}(\omega)^{q^i}$$

and (b) easily follows. □

Corollary 3.4.9. *The functions $e_n(\omega)$ are modular forms of weight $q^n - 1$ and type 0 for $\mathrm{GL}_A(\Lambda)$.*

The functions $e_i(\omega)$ and $g_i(a,\omega)$ occur as the coefficients of important power series. It turns out that the Eisenstein series also occur as coefficients of such a series.

Lemma 3.4.10. *The power series expansion for the function $\frac{X}{e_{\omega\Lambda}(X)}$ is*

$$\frac{X}{e_{\omega\Lambda}(X)} = 1 - \sum_{k\geq 1} E^k(\omega) X^k.$$

Proof.

$$\begin{aligned}
\frac{X}{e_{\omega\Lambda}(X)} &= \sum_{\lambda\in\omega\Lambda} \frac{X}{X-\lambda} = 1 - \sum_{\lambda\in\omega\Lambda}{}' \frac{\frac{X}{\lambda}}{1-\frac{X}{\lambda}} \\
&= 1 - \sum_{\lambda\in\omega\Lambda}{}' \sum_{k\geq 1} \left(\frac{X}{\lambda}\right)^k = 1 - \sum_{k\geq 1} \left(\sum_{\lambda\in\omega\Lambda}{}' \lambda^{-k}\right) X^k \\
&= 1 - \sum_{k\geq 1} E^k(\omega) X^k.
\end{aligned}$$

□

We can use this to prove that there are also relations between the coefficient forms and the Eisenstein series.

Proposition 3.4.11. *Let A be a Drinfeld ring and let $a \in A$. Then*

$$\left(a - a^{q^k}\right) E^{q^k-1}(\omega) = \sum_{i=0}^{k-1} E^{q^i-1}(\omega) g_{k-i}(a,\omega)^{q^i}.$$

It will be of benefit for us to break the *proof* into two lemmas, together implying Proposition 3.4.11. Let $\beta_n(\omega)$ be the coefficient of X^{q^n} in the expansion of the logarithm function for the lattice $\omega\Lambda$.

Lemma 3.4.12. *The following relations exist between $\beta_k(\omega)$, the coefficient forms and the functions $e_n(\omega)$:*

(a) $\beta_k + \sum_{i=1}^{k} \beta_{k-i}(\omega)^{q^i} e_i(\omega) = 0$;

(b) $a\beta_k(\omega) = \sum_{i=0}^{k} \beta_i(\omega) g_{i-k}(a,\omega)^{q^i}$.

Proof. By comparing coefficients of X^{q^k} in the equality $X = e_\Lambda(\log_\Lambda(X))$, we get (a). Similarly, we use the equalities (true for arbitrary Λ)

$$a\log_\Lambda(X) = \log_\Lambda \circ e_\Lambda(a\log_\Lambda(X)) = \log_\Lambda(\Phi_a(e_\Lambda(\log_\Lambda(X)))) = \log_\Lambda(\Phi_a(X)),$$

when (b) follows by comparing coefficients of X^{q^k}. □

Lemma 3.4.13. *For $k \geq 1$, we have $\beta_k(\omega) = -E^{q^k-1}(\omega)$.*

Proof. By definition of $E^k(\omega)$ we have $E^{pk}(\omega) = E^k(\omega)^p$ for every k, and hence that $E^{q^k-q^i}(\omega) = E^{q^{k-i}-1}(\omega)^{q^i}$. Again we compare the coefficient X^{q^k} of power series

$$X = e_{\omega\Lambda}(X) \cdot \frac{X}{e_{\omega\Lambda}(X)} = \left(\sum_{i\geq 0} e_i(\omega) X^{q^i}\right)\left(1 - \sum_{j\geq 0} E^j(\omega)\right),$$

yielding

$$e_k(\omega) - \sum_{i=0}^{k-1} E^{q^k-q^i}(\omega) e_i(\omega) = 0.$$

When $k = 1$ this yields $E^{q-1}(\omega) = e_1(\omega) = -\beta_1(\omega)$ (by Lemma 3.4.12 (a)). Assuming that $\beta_i(\omega) = -E^{q^i-1}(\omega)$ for $i = 1, \ldots, k-1$, we get

$$E^{q^k-1}(\omega) = e_k(\omega) - \sum_{i=1}^{k-1} E^{q^k-q^i} e_i(\omega) = e_k(\omega) + \sum_{i=1}^{k} \beta_{k-i}^{q^i} e_i(\omega) = -\beta_k,$$

again making use of Lemma 3.4.12 (a). □

Corollary 3.4.14. (a) *For every $k \geq 0$, we have $E^{q^k-1}(\omega) \in F[g_1(a,\omega), \dots, g_{r \deg a}(a,\omega)]$.*

(b) *If $A = \mathbb{F}_q[t]$, then for every $k \geq 0$, we have $L_k E^{q^k-1}(\omega) \in A[g_1(\omega), \dots, g_r(\omega)]$.*

The proof is almost identical to that of Corollary 3.4.8 and is omitted.

Example. Suppose that $r \geq 2$ and $A = \mathbb{F}_q[t]$. Then we have

- $[1]e_1(\omega) = g_1(\omega)$;
- $[2]e_2(\omega) = g_1(\omega)e_1(\omega)^q + g_2 = [1]^{-q}g_1^{q+1} + g_2$;
- $-[1]E^{q-1}(\omega) = -g_1(\omega) \Rightarrow [1]E^{q-1}(\omega) = g_1(\omega)$ and hence $E^{q-1}(\omega) = e_1(\omega)$;
- $-[2]E^{q^2-1}(\omega) = -g_2(\omega) + E^{q-1}(\omega)g_1(\omega) \Rightarrow g_2(\omega) = [1]^q E^{q-1}(\omega)^{q+1} + [2]E^{q^2-1}(\omega)$, generalizing this formula, which was known for $r = 2$, to all $r \geq 2$.

We have seen that many of the modular forms can be written as polynomials in the coefficient forms. By the following theorem appearing in [BP], we see that this is true more generally when $A = \mathbb{F}_q[t]$.

Theorem 3.4.15 ([BP])**.** *Let $A = \mathbb{F}_q[t]$.*

(a) *The graded ring of modular forms of type 0 for $\mathrm{GL}_r(A)$ is*

$$\mathcal{M}(\mathrm{GL}_r(A)) = \mathbb{C}_\infty[g_1(\omega), \dots, g_r(\omega)].$$

Moreover, the coefficient forms $g_1(\omega), \dots, g_r(\omega)$ are algebraically independent.

(b) *The graded ring of modular forms for $\Gamma(t)$ is generated by the weight one Eisenstein series for $\Gamma(t)$.*

Proof. This follows from Theorem 3.3.5 and [Pi] Theorem 8.2. □

3.5 Computation of certain u-expansions

The computations will be significantly simplified by using $u := u_{\tilde{\omega}}(\omega_1)$ throughout. In these computations and in what follows we shall often come across expressions of the form $\varphi_a(u^{-1})$, where φ is a Drinfeld module and u is the parameter at infinity. In order to handle such expressions we make the following definition which resembles the one in Gekeler [Ge2].

Definition 3.5.1. *Let Λ be a lattice of rank d and φ be the associated Drinfeld module. We define*

$$f_a^{\Lambda}(X) := X^{q^{d\deg a}}\varphi_a(X^{-1}),$$

the reciprocal polynomial of $\varphi(X)$.

In most cases the lattice will be the lattice $\bar{\pi}\tilde{\omega}\Lambda_U$ that defines the parameter $u_{\bar{\pi}\tilde{\omega}}(\bar{\pi}\omega_1)$ at infinity. In this case we shall write simply $f_a(u)$. Our computations will only involve modular forms for $\mathrm{GL}_r(A)$ or for principal congruence subgroups $\Gamma(N)$. In those cases, $\Lambda_U = \tilde{\Lambda}$ and $\Lambda_U = N\tilde{\Lambda}$, respectively. Hence, from now on, we use the expression in terms of $\tilde{\Lambda}$ instead. For the following computations we write

$$\varphi_a^{\bar{\pi}\tilde{\omega}\Lambda_U}(X) = aX + \bar{\pi}^{1-q}\tilde{g}_1(a,\tilde{\omega})X^q + \cdots + \bar{\pi}^{1-q^{(r-1)\deg a}}\tilde{g}_{(r-1)\deg a}(a,\tilde{\omega})X^{q^{(r-1)\deg a}},$$

where the $\tilde{g}_i$ are rank $r-1$ coefficient forms.

Lemma 3.5.2. *The polynomial $f_a(u)$ is invertible in $\mathbb{C}_\infty[\omega_2,\ldots,\omega_r][\![u]\!]$, i.e. $\frac{1}{f_a(u)} = \sum_{n\geq 0} c_n(\tilde{\omega})u^n$ for some functions $c_i : \Omega^{r-1} \to \mathbb{C}_\infty$ holomorphic on Ω^{r-1}.*

Proof. Recall that the leading coefficient $\bar{\pi}^{1-q^{(r-1)\deg a}}g_{(r-1)\deg a}(a,\tilde{\omega})$ of $\varphi_a^{\bar{\pi}\tilde{\omega}\Lambda_U}(X)$ is nowhere zero on Ω^{r-1}. We may then calculate $f_a(u) = \bar{\pi}^{1-q^{(r-1)\deg a}}g_{(r-1)\deg a}(a,\tilde{\omega}) + \cdots + au^{q^{(r-1)\deg a}-1}$, where the constant term $\bar{\pi}^{1-q^{(r-1)\deg a}}g_{(r-1)\deg a}(a,\tilde{\omega})$ has no zeros on Ω^{r-1}. Therefore $f_a(u)^{-1}$ has a geometric series expansion of the form

$$\bar{\pi}^{1-q^{(r-1)\deg a}}g_{(r-1)\deg a}(a,\tilde{\omega})^{-1} + \sum_{n\geq 1}\left(-\frac{f_a(u) - \bar{\pi}^{1-q^{(r-1)\deg a}}g_{(r-1)\deg a}(a,\tilde{\omega})}{g_{(r-1)\deg a}(a,\tilde{\omega})}\right)^n.$$

The numerator consists only of functions that are holomorphic on Ω^{r-1}, while the inverse of the denominator is also holomorphic, since $g_{(r-1)\deg a}(a,\tilde{\omega})$ has no zeroes on Ω^{r-1}. □

We shall often encounter expressions of the form $\frac{u^{q^{(r-1)\deg a}}}{f_a(u)}$. Lemma 3.5.2 shows that this is a power series in u starting with $\bar{\pi}^{1-q^{(r-1)\deg a}}g_{(r-1)\deg a}(a,\tilde{\omega})^{-1}u^{q^{(r-1)\deg a}}$. We shall denote this expression by u_a. In particular,

$$\Phi_a^{\bar{\pi}\tilde{\omega}\Lambda_U}(e_{\bar{\pi}\tilde{\omega}\Lambda_U}(\bar{\pi}\omega_1))^{-1} = u_a.$$

3.5.1 The u-expansion of Eisenstein series for $\mathbf{GL}_r(A)$

We now calculate the u-expansion of the Eisenstein series. This serves two goals. Firstly, it will prove that the Eisenstein series are indeed modular forms, and secondly it will give us a concrete example of a u-expansion of a modular form.

Recall our convention that we denote rank $r-1$ objects with tildes. So, by $\tilde{E}^k(\tilde{\omega})$ we mean the rank $r-1$ Eisenstein series evaluated at $\tilde{\omega}$. For this section we also assume that $\Lambda = A \times \tilde{\Lambda}$. The transformation of the sum to a sum involving the lattice $\bar{\pi}\tilde{\Lambda}$ may seem strange, but it is necessary to arrive at an expression involving the parameter $u_{\tilde{\omega}}(\omega_1)$.

$$
\begin{aligned}
E^k(\omega) &= \sum_{\lambda\in\tilde{\omega}\tilde{\Lambda}}{}' \lambda^{-k} + \sum_{a\in A}{}' \left(\sum_{\lambda\in\tilde{\omega}\tilde{\Lambda}} (a\omega_1+\lambda)^{-k} \right) \\
&= \tilde{E}^k(\tilde{\omega}) + \bar{\pi}^k \sum_{a\in A}{}' \left(\sum_{\lambda\in\tilde{\omega}\tilde{\Lambda}} (\bar{\pi}a\omega_1+\bar{\pi}\lambda)^{-k} \right) \\
&= \tilde{E}^k(\tilde{\omega}) + \bar{\pi}^k \sum_{a\in A}{}' \left(\sum_{\lambda\in\bar{\pi}\tilde{\omega}\tilde{\Lambda}} (\bar{\pi}a\omega_1+\lambda)^{-k} \right) \\
&= \tilde{E}^k(\tilde{\omega}) + \bar{\pi}^k \sum_{a\in A}{}' P_{k,\bar{\pi}\tilde{\omega}\tilde{\Lambda}} \left(\sum_{\lambda\in\bar{\pi}\tilde{\omega}\tilde{\Lambda}} (\bar{\pi}a\omega_1+\lambda)^{-1} \right) \\
&= \tilde{E}^k(\tilde{\omega}) + \bar{\pi}^k \sum_{a\in A_+} \sum_{\zeta\in\mathbb{F}_q^\times} P_{k,\bar{\pi}\tilde{\omega}\tilde{\Lambda}} \left(e_{\bar{\pi}\tilde{\Lambda}}(\zeta\bar{\pi}a\omega_1)^{-1} \right) \\
&= \tilde{E}^k(\tilde{\omega}) + \bar{\pi}^k \sum_{a\in A_+} \sum_{\zeta\in\mathbb{F}_q^\times} P_{k,\bar{\pi}\tilde{\omega}\tilde{\Lambda}} \left(\zeta^{-1}\varphi_a^{\bar{\pi}\tilde{\Lambda}}(e_{\bar{\pi}\tilde{\Lambda}}(\bar{\pi}\omega_1))^{-1} \right)
\end{aligned}
$$

Note that $\sum_{\zeta\in\mathbb{F}_q^\times}\zeta^k = 0$ if k is not divisible by $q-1$, and $\sum_{\zeta\in\mathbb{F}_q^\times}\zeta^k = -1$ if k is divisible by $q-1$. It is thus clear that only the terms in $P_{k,\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(X)$ for which the exponent of X is divisible by $q-1$ will contribute to the sum. On the other hand the exponents of the non-zero terms in $P_{k,\bar{\pi}\tilde{\omega}\tilde{\Lambda}}$ are all congruent to k modulo $q-1$ by Proposition 2.5.2 (g), so either nothing contributes, or everything does. Assuming from now on that $q-1$ divides k, the last expression simplifies to

$$
\tilde{E}^k(\tilde{\omega}) + \bar{\pi}^k \sum_{a\in A_+} -P_{k,\bar{\pi}\tilde{\omega}\tilde{\Lambda}} \left(\varphi_a^{\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(e_{\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(\bar{\pi}\omega_1))^{-1} \right) = \tilde{E}^k(\tilde{\omega}) - \bar{\pi}^k \sum_{a\in A_+} P_{k,\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(u_a).
$$

Proposition 3.5.3. *When $q-1 \mid k$, the u-expansion of the Eisenstein series $E^k(\omega)$ is*
$$E^k(\omega) = \tilde{E}^k(\tilde{\omega}) - \bar{\pi}^k \sum_{a \in A_+} P_{k,\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(u_a).$$

Proof. This is simply a summary of the preceding calculation. $\square$

Note incidentally that this is a sum indexed over the set A_+ instead of over $\mathbb{Z}$. Expansions of this form are called A-expansions or non-standard expansions. One may argue that since we have replaced $\mathbb{Z}$ by A in many cases, this should be a better expansion than the u-expansion. This is true at least to the point that Hecke eigenvalues are easier to read off from A-expansions than from u-expansions. However, not every modular form (nor even every eigenform) has an A-expansion, so they are still limited in their application.

In rank 2, there are other forms with known A-expansions, e.g. the discriminant function. In [Pe], Petrov has recently constructed an infinite family of forms with such expansions. It would be interesting to see if this extends to the higher rank case.

In the rest of this section we determine which coefficients are possibly non-zero. These results are applicable, not only to Eisenstein series, but to all modular forms for $\mathrm{GL}_A(\Lambda)$. They follow the corresponding results of Gekeler [Ge2] in the rank 2 case quite closely.

Lemma 3.5.4. *Suppose that f is a modular form for $\mathrm{GL}_A(\Lambda)$ and that it has the expansion $f(\omega) = \sum_{n\geq 0} f_n(\tilde{\omega})u^n$ and that for some m, the function $f_m(\tilde{\omega})$ is not identically 0. Then $q-1 \mid m$.*

Proof. Note that, since coefficients forms are modular forms for $\mathrm{GL}_A(\Lambda)$, they must be invariant under the action of scalar matrices cI, hence the u-expansion must be invariant as well. This means that if u^m occurs in the expansion of a coefficient form, then $c^m u^m = u^m$ for all $c \in \mathbb{F}_q^\times$, or equivalently, $q-1 \mid m$. $\square$

Proposition 3.5.5. *Let f be one of the following modular forms for $\mathrm{GL}_A(\Lambda)$: an Eisenstein series $E^{q^k-1}(\omega)$ or a coefficient form $g_i(a,\omega)$ for any $a \in A \smallsetminus \mathbb{F}_q$. Let the expansion of f be $f(\omega) = \sum_{n\geq 0} f_n(\tilde{\omega})u^n$. If $f_m(\tilde{\omega})$ is not identically 0, then $q-1 \mid m$ and $m \equiv -1, 0 \pmod q$.*

Proof. The fact that $q-1 \mid m$ is contained in Lemma 3.5.4. Abbreviate by $(*)$ the property that if $f_m(\tilde{\omega})$ is non-zero, then $m \equiv -1, 0 \pmod q$. Our strategy is to systematically obtain $(*)$ for various power series related to Eisenstein series.

The polynomial $f_a(u)$ viewed as a power series satisfies $(*)$, since its non-zero terms have exponents $q^{\deg a} - q^m$ for $m = 0, \ldots, \deg a - 1$. The polynomial u^{q^k-1} trivially satisfies $(*)$. Now suppose that $\deg a \geq 1$ and $k \geq 1$. Then $u_a^{q^k-1}$ satisfies $(*)$, since it equals $u_a^{q^k} u_a^{-1} = u_a^{q^k} u^{-q^{\deg a}} f_a(u)$, where the first two factors are q-th powers and the last has already been shown. By Proposition 2.5.2 (f), the Goss polynomial $P_{q^k-1}(X)$ has exponents divisible by q, except for the leading term which is X^{q^k-1}, and hence $P_{q^k-1}(u_a)$ satisfies $(*)$, since all exponents except $u_a^{q^k-1}$ are divisible by q, and for $u_a^{q^k-1}$ it was just shown.

Looking at the equation from Proposition 3.5.3, one clearly sees that any Eisenstein series satisfies $(*)$. The statement when $f = g_i$ is a coefficient form follows from the relation in Proposition 3.4.11 and by induction for $i = 1, \ldots, r$. $\square$

3.5.2 The u-expansion of Eisenstein series for principal congruence subgroups

As before, we let $N \subset A$ be an ideal and $[v] \in N^{-1}\Lambda/\Lambda$ be a congruence class. During the proof of Proposition 3.4.2 we obtained the formula (3.4). Since $E^k_{[v]}$ is simply a polynomial in $E^1_{[v]}$, we restrict ourselves to the case of weight 1 Eisenstein series in this section. In general it seems hard to do better than (3.4), but we shall calculate the first non-zero coefficient of this expansion in some special cases. In fact, even though everything up to equation (3.7) is true for general N, throughout this section we assume that $N = nA$ is principal.

During the proof of Proposition 3.4.2 we saw that if $v_1 \in A$, then there is a constant term $\bar{\pi} P_1\left(\bar{\pi}^{-1}\tilde{E}^1_{[\tilde{v}]}\right) = \tilde{E}^1_{[\tilde{v}]}$, so in that case the first non-zero coefficient is easy to obtain. Otherwise there is a unique a_1 such that $|a_1 + v_1| < 1$. Hence the unique term with lowest u-order corresponds to this a_1 and is

$$\left((a_1 + v_1)\Phi_{a_1+v_1}(u^{-1}) + \tilde{E}^1_{[\tilde{v}]}(\tilde{\omega})^{-1}\right)^{-1}, \tag{3.6}$$

where the polynomial $\Phi_{a_1+v_1}(X)$ is the exponential function associated to the finite $\mathbb{F}_q$-linear set $L = e_{\bar{\pi}N\tilde{\omega}\tilde{\Lambda}}(\bar{\pi}\tilde{\omega}(a_1+v_1)^{-1}\tilde{\Lambda})$. (This was the definition of $\Phi_{a_1+v_1}(X)$ during the proof of Proposition 3.4.2.) We may choose v_1 so that $a_1 = 0$. Hence, the u-expansion of this term is the multiplicative inverse of a polynomial $c_d u^{-d} + \cdots + c_0$ in u^{-1}, where $d = q^{(r-1)\deg nv_1}$, $c_0 = \tilde{E}^1_{[\tilde{v}]}(\tilde{\omega})$ and

$$
\begin{aligned}
(3.7)\qquad c_d &= v_1 \prod'_{z \in \bar{\pi} v_1^{-1} \tilde{\omega}\tilde{\Lambda}/\bar{\pi} N \tilde{\omega}\tilde{\Lambda}} e_{\bar{\pi} N \tilde{\omega}\tilde{\Lambda}}(z)^{-1}. \\
&= v_1 \prod'_{z \in v_1^{-1} \tilde{\omega}\tilde{\Lambda}/N \tilde{\omega}\tilde{\Lambda}} \left(n\bar{\pi} e_{\tilde{\omega}\tilde{\Lambda}}\left(\frac{z}{n}\right)\right)^{-1} \\
&= n^{1-d}\bar{\pi}^{1-d} v_1 \prod'_{z \in (nv_1)^{-1} \tilde{\omega}\tilde{\Lambda}/\tilde{\omega}\tilde{\Lambda}} e_{\tilde{\omega}\tilde{\Lambda}}(z)^{-1} \\
&= n^{1-d}\bar{\pi}^{1-d} v_1 \prod'_{z \in (nv_1)^{-1} \tilde{\omega}\tilde{\Lambda}/\tilde{\omega}\tilde{\Lambda}} \tilde{E}^1_{[z]}(\tilde{\omega}) \\
&= n^{-d}\bar{\pi}^{1-d} \tilde{g}_{(r-1)\deg(nv_1)}(nv_1, \tilde{\omega}) \\
(3.8)\qquad &= n^{-d}\bar{\pi}^{1-d} \tilde{\Delta}(nv_1, \tilde{\omega}).
\end{aligned}
$$

Proposition 3.5.6. *Let $n \in A$ be non-constant, let $[v] \in n^{-1}\Lambda/\Lambda$ be a residue class, let $\bar{v} = (v_1, \tilde{v}) \in [v]$ such that $|\bar{v}| < 1$ and let $d = q^{(r-1)\deg nv_1}$. Then*

$$
E^1_{[v]}(\omega) = \begin{cases} \tilde{E}^1_{[\tilde{v}]}(\tilde{\omega}) + O(u) & \text{if } v_1 = 0 \\ n^d \bar{\pi}^d \tilde{\Delta}(nv_1, \tilde{\omega})^{-1} u^d + \text{higher terms} & \text{if } v_1 \neq 0. \end{cases}
$$

Proof. The case where $v_1 = 0$ was shown above. During the proof of Proposition 3.4.2 (equation (3.2)) it was also shown that if $v_1 \neq 0$, then the unique term containing the lowest power of u is $\bar{\pi}$ times the multiplicative inverse of $(c_d u^{-d} + \cdots + c_0)$ which is $\bar{\pi} u^d (c_d + \cdots + c_0 u^d)^{-1} = \bar{\pi} u^d c_d^{-1}(1 + O(u))$, since c_d is a function which is nowhere 0 on Ω^{r-1}. By the computation leading up to equation (3.8), the Proposition follows. $\square$

3.5.3 The u-expansion of Coefficient Forms

The goal of this section is to investigate the u-expansions of the coefficient forms. We shall use the previous section where we computed the expansions of Eisenstein series for principal congruence subgroups. Since we restricted ourselves to principal ideals N, we also make that assumption now. (Again, everything up to equation (3.10) is true for general N, but for simplicity we make this assumption now.) One thing to note before starting our calculations is that the parameters at infinity are not the same. Let us start by relating them. In this section, let us denote the parameter for $\mathrm{GL}_r(A)$ by u and the parameter for $\Gamma(N)$ by u_N.

We have $\Gamma(N) \subset \mathrm{GL}_A(\Lambda)$ and thus $\Gamma(N)_U \subset \mathrm{GL}_A(\Lambda)_U$. The index $[\mathrm{GL}_A(\Lambda)_U : \Gamma(N)_U] = |N^{-1}/A|^{r-1} = q^{(r-1)\deg N}$. Furthermore, the function u is both $\mathrm{GL}_r(A)_U$ and $\Gamma(N)_U$ invariant. By Proposition 3.2.11, $\mathrm{ord}_{\Gamma(N)_U}(u) = |N^{-1}/A|^{r-1}\,\mathrm{ord}_{\mathrm{GL}_A(\Lambda)_U}(u)$.

From the proof of Proposition 3.2.11, there is an $\mathbb{F}_p$-linear polynomial $\Phi_{\tilde{\omega}}$ such that $e_{\bar{\pi}\tilde{\omega}\tilde{\Lambda}} = \Phi_{\tilde{\omega}} \circ e_{N\bar{\pi}\tilde{\omega}\tilde{\Lambda}}$, which in this case is $\mathbb{F}_q$-linear since $\Gamma(N)_U$ can be viewed as a $\mathbb{F}_q$-subvector space of $\mathrm{GL}_A(\Lambda)_U$. Supposing that $\Phi_{\tilde{\omega}}(X) = \sum_{i=0}^{d} \Phi_{\tilde{\omega},i} X^{q^i}$, the proof of Proposition 3.2.11 gives the relation

$$u = \frac{u_N^{q^d}}{\Phi_{\tilde{\omega},d}} \left(1 + \sum_{i=0}^{d-1} \frac{\Phi_{\tilde{\omega},i}}{\Phi_{\tilde{\omega},d}} u_N^{q^d - q^i}\right)^{-1}.$$

In particular, we can relate the first terms in the expansions of a form in terms of u and u_N respectively. It is perhaps worth mentioning more explicitly what the polynomial $\Phi_{\tilde{\omega}}$ is. From Proposition 2.2.5, we know that

$$\Phi_{\tilde{\omega}}(X) = X \prod_{\alpha \in N^{-1}\Lambda/\Lambda}{}' \left(1 - \frac{X}{e_{\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(\bar{\pi}\tilde{\omega}\alpha)}\right),$$

which is exactly the polynomial whose coefficients are the normalized coefficient forms $\bar{\pi}^{1-q^i}\tilde{\mathfrak{g}}_i(N, \tilde{\omega})$ of rank $r-1$. Let us finish this comparison by putting all of this together in one formula (where we set $d := (r-1)\deg N$):

$$u = \frac{u_N^{d}}{\bar{\pi}^{1-q^d}\tilde{\mathfrak{g}}_d(N, \tilde{\omega})} \left(1 + \sum_{i=0}^{(r-1)\deg N - 1} \frac{\bar{\pi}^{1-q^i}\tilde{\mathfrak{g}}_i(N, \tilde{\omega})}{\bar{\pi}^{1-q^d}\tilde{\mathfrak{g}}_d(N, \tilde{\omega})} u_N^{q^d - q^i}\right)^{-1}. \tag{3.9}$$

By definition, the normalized coefficient forms are symmetric polynomials in the Eisenstein series of weight 1. More explicitly,

$$\mathfrak{g}_i(N, \omega) = \sum_{\substack{S \subset (N^{-1}/A)^r \\ |S| = q^i}} \prod_{[v] \in S} E^1_{[v]}(\omega).$$

This allows us to compute the first term of most coefficient forms. By Proposition 3.5.6, the minimum power of u will occur when we choose S to contain only classes $[v]$ where $v_1 \in A$ or, if this is not possible, as many such $[v]$ as possible. In particular, when $i \le (r-1)\deg N$, the sum contains terms where we can choose $S \subset 0 \times N^{-1}\tilde{\Lambda}/\tilde{\Lambda}$ and hence $\mathfrak{g}_i(N, \omega)$ is not a cusp form.

In this case we have

$$
\begin{aligned}
\mathfrak{g}_i(N,\omega) &= \sum_{\substack{S\subset 0\times N^{-1}\tilde\Lambda/\tilde\Lambda\\ |S|=q^i}} \prod_{[v]\in S} E^1_{[v]}(\omega)+O(u_N)\\
&= \sum_{\substack{S\subset 0\times N^{-1}\tilde\Lambda/\tilde\Lambda\\ |S|=q^i}} \prod_{[v]\in S} \tilde E^1_{[\tilde v]}(\tilde\omega)+O(u_N)\\
&= \tilde{\mathfrak{g}}_i(N,\tilde\omega)+O(u), \qquad (3.10)
\end{aligned}
$$

since $\mathfrak{g}_i(N,\omega)$ is a modular form for $\mathrm{GL}_A(\Lambda)$ and hence has an expansion in terms of u.

For $i=(r-1)\deg N+j$, $(1\le j\le \deg N)$ the unique coefficient of the lowest power of u_N will be determined by multiplying $E^1_{[v]}(\omega)$ for those $[v]$ that have the lowest absolute value. More explicitly, let $S'\subset S$ be the set $\{v_1 : |v_1|\le q^j\}\times N^{-1}\tilde\Lambda/\tilde\Lambda$. Then the term with lowest power of u_n will be that of $\prod_{[v]\in S'} E^1_{[v]}(\omega)$. This allows us to calculate $\mathrm{ord}_{\Gamma(N)_U}(g_i(\omega))$ and the leading coefficient for these i. We shall content ourselves with doing this for $\Delta(\omega)$.[3]

$$
\begin{aligned}
\Delta(\omega) &= t\sideset{}{'}\prod_{[v]\in(t^{-1}A/A)^r} E^1_{[v]}(\omega)\\
&= t\sideset{}{'}\prod_{\substack{[v]\in(t^{-1}A/A)^r\\ v_1=0}} E^1_{[v]}(\omega)\prod_{\substack{[v]\in(t^{-1}A/A)^r\\ v_1\neq 0}} E^1_{[v]}(\omega)
\end{aligned}
$$

The first product is

$$
\sideset{}{'}\prod_{[v]\in 0\times(t^{-1}A/A)^{r-1}}\left(\tilde E^1_{[\tilde v]}(\tilde\omega)+O(u_t)\right)=\frac{1}{t}\tilde\Delta(\tilde\omega)+O(u_t)
$$

and the second is

$$
\begin{aligned}
&\prod_{\substack{[v]\in(t^{-1}A/A)^r\\ v_1\neq 0}}\left(t^{q^{(r-1)\deg tv_1}}\bar\pi^{q^{(r-1)\deg tv_1}}\tilde g_{(r-1)\deg tv_1}(tv_1,\tilde\omega)^{-1}u_t+\text{higher terms}\right)\\
&= \prod_{v_1\in\mathbb{F}_q^\times}\prod_{[\tilde v]\in(t^{-1}A/A)^{r-1}}(\bar\pi v_1^{-1}u_t+\text{higher terms})\\
&= \bar\pi^{(q-1)q^{r-1}}\left(\prod_{v_1\in\mathbb{F}_q^\times}(v_1^{-1}u_t)^{q^{r-1}}\right)+\text{higher terms}\\
&= -\bar\pi^{(q-1)q^{r-1}}u_t^{(q-1)q^{r-1}}+\text{higher terms},
\end{aligned}
$$

[3]Recall our notation from Definition 3.4.5 that $\Delta(\omega)$ is the rank r Drinfeld discriminant function $g_r(t,\omega)$ when $A=\mathbb{F}_q[t]$.

since $\deg tv_1 = 0$ and since $\prod_{v_1\in\mathbb{F}_q^\times}(v_1)^{-q^{r-1}} = \prod_{v_1\in\mathbb{F}_q^\times} v_1 = -1$. By equation (3.9) we have that $u = \frac{u_t^{q^{r-1}}}{\bar\pi^{1-q^r}\tilde\Delta(\tilde\omega)}$ + higher terms, meaning that

$$\Delta(\omega) = -\bar\pi^{q-1}\tilde\Delta(\tilde\omega)^q u^{q-1} + O(u^q). \tag{3.11}$$

We shall confirm this expansion by giving a product formula for $\Delta(\omega)$.

3.5.4 A product formula for the discriminant function

Assume in this section that $A = \mathbb{F}_q[t]$. In [Ge1], Gekeler gave a product formula for the rank 2 discriminant function. Hamahata generalized this to a product formula for the general rank r discriminant function. However, Hamahata's expansion is in terms of r different parameters $e_{\bar\pi A}(\bar\pi\omega_i)^{-1}$, for $i = 1,\ldots,r$. In this section we give a different expansion in terms of the parameter $u_{\tilde\omega}(\omega_1)$ at infinity. The exposition follows that of Gekeler's original very closely.

Theorem 3.5.7 (Gekeler [Ge1])**.** *The rank 2 Drinfeld discriminant function has the product expansion*

$$\Delta(\omega) = -\bar\pi^{q^2-1}u^{q-1}\prod_{a\in A}{}' f_a(u)^{q^2-1}.$$

Remark. This formula does not contradict equation (3.11), since the rank 1 discriminant function $\tilde\Delta$ should be the leading coefficient of $\varphi_t^A(X)$ associated to the lattice A. Using Proposition 2.4.5 and the fact that $\varphi_t^{\bar\pi A}(X) = tX + X^q$, we obtain $\tilde\Delta = \bar\pi^{q-1}$ in this case.

Note that this product makes sense, since the expansion of $f_a(u)$ is of the form $c + O(u^{q^{\deg a} - q^{\deg a - 1}})$ where $c\in\mathbb{F}_q^\times$. The first term can be calculated and the product converges. If we were to take this formula in general rank we find that $f_a(u)$ has constant coefficient $\bar\pi^{q^{(r-1)\deg a}-1}\tilde\Delta(a,\tilde\omega)^{-1}$ and the product would not make sense.

Definition 3.5.8. *When* $\deg a \geq 1$, *set* $h_a(X) := \bar\pi^{1-q^{(r-1)\deg a}}\tilde\Delta(a,\tilde\omega)^{-1}f_a(X)$. *When* $\deg a = 0$, *set* $h_a(X) = 1$.

Note that division by 0 does not occur, since $\tilde\Delta(a,\tilde\omega)$ is never 0 on Ω^{r-1}.

Lemma 3.5.9. *If* $\deg a \geq 1$, *then the polynomial* $h_a(X)$ *is of the form* $1 + O\left(X^{q^{(r-1)\deg a} - q^{(r-1)\deg a - 1}}\right)$.

Proof. If $\varphi_a^{\tilde{\Lambda}}(X) = aX + \cdots + \bar{\pi}^{q^{(r-1)\deg a}-1}\tilde{\Delta}(a,\tilde{\omega})X^{q^{(r-1)\deg a}}$ then $f_a(X) = \bar{\pi}^{q^{(r-1)\deg a}-1}\tilde{\Delta}(a,\tilde{\omega}) + g_{(r-1)\deg a-1}(a,\tilde{\omega})X^{q^{(r-1)\deg a}-q^{(r-1)\deg a-1}} + \cdots + aX^{q^{(r-1)\deg a}-1}$ when clearly the statement holds. □

Remark. The powers of $\bar{\pi}$ that appear in Lemma 3.5.9 are there because the coefficient forms g_i and the polynomials f_a and h_a are defined with respect to different lattices ($\tilde{\omega}\tilde{\Lambda}$ and $\bar{\pi}\tilde{\omega}\tilde{\Lambda}$ respectively). In the rest of this section when this phenomenon occurs it is for the same reason.

The product formula we shall derive is similar to Theorem 3.5.7, but with $h_a(u)$ instead of $f_a(u)$. The key lies in the following lemmas:

Lemma 3.5.10.

$$\Delta(\omega) = t\bar{\pi}^{q^r-1} \sideset{}{'}\prod_{\alpha\in(t^{-1}A/A)^r} e_{\bar{\pi}\omega A^r}(\bar{\pi}\omega\alpha)^{-1}.$$

Proof. We have

$$\begin{aligned}
\Delta(\omega) &= t \sideset{}{'}\prod_{[v]\in(t^{-1}A/A)^r} E^1_{[v]}(\omega) \quad \text{by Proposition 3.4.3 (b)}\\
&= t \sideset{}{'}\prod_{[v]\in(t^{-1}A/A)^r} e_{\omega A^r}(\omega v)^{-1} \quad \text{by Proposition 3.4.2 (a)}\\
&= t \sideset{}{'}\prod_{[v]\in(t^{-1}A/A)^r} \bar{\pi} e_{\bar{\pi}\omega A^r}(\bar{\pi}\omega v)^{-1} \quad \text{by Proposition 2.2.5 (a)}\\
&= t\bar{\pi}^{q^r-1} \sideset{}{'}\prod_{\alpha\in(t^{-1}A/A)^r} e_{\bar{\pi}\omega A^r}(\bar{\pi}\omega\alpha)^{-1}.
\end{aligned}$$

□

Lemma 3.5.11. *Let $\tilde{\Lambda} = \omega_2 A + \cdots + \omega_r A$ be a lattice of rank $r-1$ and set $\Lambda = \omega_1 A + \tilde{\Lambda}$. Then*

$$e_\Lambda(X) = e_{\tilde{\Lambda}}(X) \sideset{}{'}\prod_{a\in A} \frac{e_{\tilde{\Lambda}}(X) + e_{\tilde{\Lambda}}(a\omega_1)}{e_{\tilde{\Lambda}}(a\omega_1)}.$$

Proof. We calculate

$$\begin{aligned}
e_\Lambda(X) &= X\prod_{\lambda\in\Lambda}{}'\left(1-\frac{X}{\lambda}\right)\\
&= X\prod_{\lambda\in\tilde\Lambda}{}'\left(1-\frac{X}{\lambda}\right)\prod_{a\in A}{}'\prod_{\lambda\in\tilde\Lambda}\left(1-\frac{X}{\lambda-a\omega_1}\right)\\
&= e_{\tilde\Lambda}(X)\prod_{a\in A}{}'\frac{X+a\omega_1}{a\omega_1}\prod_{\lambda\in\tilde\Lambda}{}'\left(\frac{1-\frac{X+a\omega_1}{\lambda}}{1-\frac{a\omega_1}{\lambda}}\right)\\
&= e_{\tilde\Lambda}(X)\prod_{a\in A}{}'\frac{e_{\tilde\Lambda}(a\omega_1+X)}{e_{\tilde\Lambda}(a\omega_1)}.
\end{aligned}$$

□

Lemma 3.5.12. *Let φ be a rank d Drinfeld A-module such that $\varphi_t(X) = tX+\cdots+DX^{q^d}$ and fix $z_0\in\mathbb{C}_\infty$. Then, as polynomials, we have the equality*

$$D\prod_{\varphi_t(z)=\varphi_t(z_0)}(X-z)=\varphi_t(X-z_0).$$

Proof. Note that $\varphi_t(z)=\varphi_t(z_0)$ if and only if $z-z_0$ is a root of $\varphi_t(X)$ if and only if z is a root of $\varphi_t(X-z_0)$. Hence the polynomials have the same set of roots. Moreover, the degree on both sides is q^d (the left since that is the number of pre-images of $\varphi_t(z)$ under φ_t), the leading coefficients are both equal to D and the right hand side has only simple roots. This implies that the polynomials are equal. □

For simplicity write φ for the Drinfeld module and $e(X)$ for the exponential function associated to the lattice $\bar\pi\tilde\omega\tilde\Lambda$. Following the same argument as in Gekeler [Ge1], we calculate the product

$$\prod_{\alpha\in(t^{-1}A/A)^r}{}'\frac{e(\bar\pi a\omega_1)}{e(\bar\pi\omega\alpha)+e(\bar\pi a\omega_1)}$$

for any fixed $a\in A$, $a\neq 0$. Note that $e(\bar\pi a\omega_1)=\varphi_a(e(\bar\pi\omega_1))$ and that the set $\{e(\bar\pi\omega_1\alpha)\,|\,\alpha\in(t^{-1}A/A)^r\}$ is the inverse image of the set $\{c\cdot e(\bar\pi\omega_1)\,|\,c\in\mathbb{F}_q\}$ under φ_t. Therefore

$$\begin{aligned}
&\prod_{\alpha\in(t^{-1}A/A)^r}(X+e(\bar\pi\omega\alpha))\\
&=\prod_{c\in\mathbb{F}_q}\bar\pi^{q^{r-1}-1}\tilde\Delta(\tilde\omega)^{-1}\varphi_t\left(X-c\cdot e(\bar\pi t^{-1}\omega_1)\right)\\
&=\prod_{c\in\mathbb{F}_q}\bar\pi^{q^{r-1}-1}\tilde\Delta(\tilde\omega)^{-1}\left(\varphi_t(X)-c\cdot e(\bar\pi\omega_1)\right)
\end{aligned}$$

and hence (replacing X with $e(\pi\omega_1)$)

$$\begin{aligned}
&\prod_{\alpha\in(t^{-1}A/A)^r}(e(\bar\pi\omega\alpha)+e(\bar\pi a\omega_1))\\
&=\prod_{c\in\mathbb{F}_q}\bar\pi^{q^{r-1}-1}\tilde\Delta(\tilde\omega)^{-1}\left(\varphi_t(e(\bar\pi a\omega_1))-c\cdot e(\bar\pi\omega_1)\right)\\
&=\prod_{c\in\mathbb{F}_q}\bar\pi^{q^{r-1}-1}\tilde\Delta(\tilde\omega)^{-1}\varphi_{at-c}\left(e(\bar\pi\omega_1)\right)
\end{aligned}\tag{3.12}$$

Hence

$$\begin{aligned}
\prod_{\alpha\in(t^{-1}A/A)^r}{}' \frac{e(\bar\pi a\omega_1)}{e(\bar\pi\omega\alpha)+e(\bar\pi a\omega_1)} &= \prod_{\alpha\in(t^{-1}A/A)^r}\frac{e(\bar\pi a\omega_1)}{e(\bar\pi\omega\alpha)+e(\bar\pi a\omega_1)}\\
&=\frac{\varphi_a(e(\bar\pi\omega_1))^{q^r}}{\prod_{c\in\mathbb{F}_q}\bar\pi^{q^{r-1}-1}\tilde\Delta(\tilde\omega)^{-1}\varphi_{at-c}(e(\bar\pi\omega_1))}\\
&=\frac{f_a(u)^{q^r}}{\prod_{c\in\mathbb{F}_q}\bar\pi^{q^{r-1}-1}\tilde\Delta(\tilde\omega)^{-1}f_{at-c}(u)}
\end{aligned}$$

When $\deg a\geq 1$, this becomes

$$\frac{\left(\bar\pi^{1-q^{(r-1)\deg a}}\tilde\Delta(a,\tilde\omega)\right)^{q^r}h_a(u)^{q^r}}{\prod_{c\in\mathbb{F}_q}\bar\pi^{1-q^{(r-1)(\deg a+1)}}\tilde\Delta(at-c,\tilde\omega)\bar\pi^{q^{r-1}-1}\tilde\Delta(\tilde\omega)^{-1}h_{at-c}(u)}$$

, which is equal to

$$=\frac{h_a(u)^{q^r}}{\prod_{c\in\mathbb{F}_q}h_{at-c}(u)},\tag{3.13}$$

(since $\tilde\Delta(at-c,\tilde\omega)=\tilde\Delta(\tilde\omega)\tilde\Delta(a,\tilde\omega)^{q^{r-1}}$) and when $a\in\mathbb{F}_q^\times$ it becomes

$$\frac{a^{q^r}}{\prod_{c\in\mathbb{F}_q}h_{at-c}(u)}=\frac{a}{\prod_{c\in\mathbb{F}_q}h_{at-c}(u)}.$$

Lemma 3.5.13. *The following infinite products are equal:*

$$\prod_{\substack{a\in A\\ \deg a\geq 1}} h_a(u) = \prod_{a\in A}{}' \prod_{c\in\mathbb{F}_q} h_{at-c}(u).$$

Proof. This is essentially an exercise in showing that these products converge as functions on some neighbourhood of the cusp at infinity. Once this is known, each product is taken over the same index set with the same factors and we are done. More precisely, it is enough to show that for any n, there exists $r_n > 0$ such that each product has radius of convergence r_n for every $\tilde{\omega} \in \Omega_n^{r-1}$. This will guarantee uniform convergence on any Ω_n^{r-1}.

By definition we have $\varphi_a(X) = aX \prod_{\varphi_a(\alpha)=0} \left(1 - \frac{X}{\alpha}\right)$, and hence by definition $h_a(X) = \prod_{\varphi_a(\alpha)=0}(1-\alpha X)$. In these expressions, α runs over the elements of $e_{\tilde{\omega}\tilde{\Lambda}}(a^{-1}\tilde{\omega}\tilde{\Lambda})$. Now, by Corollary 3.2.3, we may bound the α's by a universal d valid for all $\omega \in \Omega_n$. Also let $\varepsilon < 1$, let $|X| \leq \varepsilon/d$ and denote by s_i $(i = 1, \ldots, q^{r-1})$ the i-th symmetric polynomial in the α. Note that $h_a(X) = 1 + \sum_{i\geq 1}^{q^{(r-1)\deg a}} s_i X^i$. By our assumption that each $\alpha \leq d$, we now have $s_i \leq d^i$ and hence $s_i X^i < \varepsilon^i$. Since $h_a(X)$ has zero coefficients for X^i for $i < q^{(r-1)\deg a} - q^{(r-1)\deg a - 1}$, we have $|h_a(X) - 1| < \varepsilon^{q^{(r-1)\deg a} - q^{(r-1)\deg a-1}}$. When $\deg a \to \infty$, this tends to 0, implying that the product is convergent on the ball $X < \varepsilon/d$. $\square$

It remains to calculate the factor

$$\prod_{\alpha\in(t^{-1}A/A)^r}{}' e(\bar{\pi}\omega\alpha).$$

Once again we may break it up into two parts, where $\alpha_1 = 0$ and where $\alpha_1 \neq 0$. We have

$$\begin{aligned}
&\prod_{\alpha_1\in\mathbb{F}_q}{}' \prod_{\tilde{\alpha}\in(t^{-1}A/A)^{r-1}} e\left(\bar{\pi}\frac{\alpha_1\omega_1}{t} + \bar{\pi}\tilde{\omega}\tilde{\alpha}\right)\\
&= \prod_{\alpha_1\in\mathbb{F}_q}{}' \prod_{\tilde{\alpha}\in(t^{-1}A/A)^{r-1}} \left(e\left(\bar{\pi}\frac{\alpha_1\omega_1}{t}\right) + e(\bar{\pi}\tilde{\omega}\tilde{\alpha})\right)\\
&= \prod_{\alpha_1\in\mathbb{F}_q}{}' \bar{\pi}^{q^{r-1}-1}\tilde{\Delta}(\tilde{\omega})^{-1}\varphi_t\left(e\left(\bar{\pi}\frac{\alpha_1\omega_1}{t}\right)\right) \qquad \text{by Lemma 3.5.12}\\
&= \bar{\pi}^{(q^{r-1}-1)(q-1)}\tilde{\Delta}(\tilde{\omega})^{1-q} \prod_{\alpha_1\in\mathbb{F}_q}{}' \alpha_1 \cdot e(\bar{\pi}\omega_1)\\
&= -\bar{\pi}^{(q^{r-1}-1)(q-1)}\tilde{\Delta}(\tilde{\omega})^{1-q}u^{1-q}, \qquad (3.14)
\end{aligned}$$

while

$$t \prod_{\tilde{\alpha}\in(t^{-1}A/A)^{r-1}}' e(\bar{\pi}\tilde{\omega}\tilde{\alpha})^{-1} = \bar{\pi}^{1-q^{r-1}}\tilde{\Delta}(\tilde{\omega}) \tag{3.15}$$

by Lemma 3.5.10, but applied in rank $r-1$.

Theorem 3.5.14.

$$\Delta(\omega) = -\tilde{\Delta}(\tilde{\omega})^q \bar{\pi}^{q-1} u^{q-1} \prod_{a\in A}' h_a(u)^{q^r-1}.$$

Proof. The proof is essentially contained in the section preceding this Theorem, but we outline the argument. By Lemmas 3.5.10 and 3.5.11 we have

$$\Delta(\omega) = t\bar{\pi}^{q^r-1} \prod_{\alpha\in(t^{-1}A/A)^r}' \left(e(\bar{\pi}\omega\alpha) \prod_{a\in A}' \frac{e(\bar{\pi}\omega\alpha)+e(a\bar{\pi}\omega_1)}{e(a\bar{\pi}\omega_1)} \right).$$

Each factor can be simplified to equation (3.13), and after taking the product over all $a \in A$, Lemma 3.5.13 tells us that the denominator cancels with exactly one $\prod_{a\in A} h_a(u)$ in the numerator. By equations (3.14) and (3.15), the product $\prod'_{\alpha\in(t^{-1}A/A)^r} e(\bar{\pi}\omega\alpha)^{-1}$ is $-t^{-1}\bar{\pi}^{q-q^r}\tilde{\Delta}(\tilde{\omega})^q u^{q-1}$. Putting this together yields the theorem. □

3.6 Rational Modular Forms

In [Ge2], Gekeler showed that some modular forms have the property that all its coefficients lie in F. For us, the definition will be slightly more complicated, since the coefficients are functions, not elements of $\mathbb{C}_\infty$. Recalling that these functions are themselves weak modular forms, we shall proceed to inductively define a rational modular form as one where the coefficients are all rational weak modular forms. We restrict ourselves to the case $A = \mathbb{F}_q[t]$ and $\Gamma = \mathrm{GL}_r(A)$.

3.6.1 Rational modular forms in rank 2

Let us recall in this section a few results from [Ge2] in order to get a feeling for which forms are rational. In this case modular forms are functions of one variable and their u-expansions have constant coefficients. Furthermore the parameter u is $e_L(\omega_1)^{-1}$, where L is the Carlitz lattice. So we start by investigating certain quantities related to the Carlitz module.

Proposition 3.6.1. (a) *The coefficients of the Carlitz exponential function are*

$$e_n(\bar{\pi}A) = D_n^{-1}.$$

(b) *The coefficients of the Carlitz logarithm function are*

$$\beta_n(\bar{\pi}A) = L_n^{-1}.$$

(c) *The rank 1 Eisenstein series* $E^k(\bar{\pi}A) := \sum'_{\lambda\in\bar{\pi}A} \lambda^{-k} \in F$ *is rational.*

Proof. For (a) we use the relation from Proposition 3.4.7, which in the case of the Carlitz module $\varphi(X) = tX + X^q$ becomes

$$(t^{q^k} - t)e_k = e_{k-1}^q.$$

Since $e_0 = 1$, the statement follows by an easy induction.

Similarly, we prove (b) by induction using the relation from 3.4.12 (a), while (c) follows from (a) and Lemma 3.4.10. □

By Propositions 2.5.2 (h) and 3.6.1 (a), the Goss polynomial $P_{k,\bar{\pi}A}(X)$ has rational coefficients. The u-expansion of the rank 2 Eisenstein series from Proposition 3.5.3 thus becomes:

$$E^k(\omega) = \bar{\pi}^k E^k(\bar{\pi}A) - \bar{\pi}^k \sum_{a\in A_+} P_{k,\bar{\pi}A}(u_a).$$

Furthermore, we can calculate the polynomial f_a (and thus u_a) more precisely. If φ is the Carlitz module, then $\varphi_t(X) = tX + X^q$ and hence $\varphi_a(X) = aX + \cdots + a_d X^{q^{\deg a}} \in A[X]$, where $a_d \in \mathbb{F}_q$ is the leading coefficient of a. Then $f_a(X) = a_d + \cdots + aX^{q^{\deg a}-1} \in A[X]$ with constant coefficient invertible in A. That means that the power series $\frac{u^{q^{\deg a}}}{f_a(u)}$ also has coefficients in A.

Proposition 3.6.2. *For every $k \geq 1$, the modular form $\bar{\pi}^{1-q^k} L_k E^{q^k-1}(\omega)$ has a u-expansion with coefficients in A.*

Proof. By Proposition 2.5.2 (f), we have

$$P_{q^k-1}(X) = \sum_{i=0}^{k-1} \beta_i X^{q^k-q^i} = \sum_{i=0}^{k-1} L_i^{-1} X^{q^k-q^i}.$$

Then since u_a has coefficients in A it remains to note that $L_k/L_j \in A$ when $j < k$. □

Corollary 3.6.3. *The following modular forms have integral u-expansions:*

(a) $\bar{\pi}^{1-q} g_1(\omega) \in A[\![u]\!]$;

(b) $\bar{\pi}^{1-q^2} g_2(\omega) \in A[\![u]\!]$.

Proof. (a) follows from Proposition 3.6.2, while for (b) the relations $g_1(\omega) = [1]E^{q-1}$ and $g_2(\omega) = [1]^q E^{q-1}(\omega)^{q+1} + [2]E^{q^2-1}(\omega)$ from Proposition 3.4.11 yield

$$\begin{aligned}\bar{\pi}^{1-q^2} g_2(\omega) &= \bar{\pi}^{1-q^2}\left([1]^q E^{q-1}(\omega)^{q+1} + [2]E^{q^2-1}(\omega)\right)\\ &= [1]^q\left([1]^{-1} - \sum_{a\in A_+} u_a^{q-1}\right)^{q+1} - [1]^{-1} + [2]\sum_{a\in A_+}\left([1]^{-1}u_a^{q^2-q} - u_a^{q^2-1}\right),\end{aligned}$$

where the only possible non-integral term is the constant term, but it cancels to 0. □

3.6.2 Rationality in higher rank

We would like to define a modular form to be rational if its coefficients are rational. However, the coefficients are not in general modular forms, since they may fail to be holomorphic at infinity. Hence we need to extend the definition to weak modular forms.

Definition 3.6.4. *We say that a weak modular form f of rank 2 is a* rational weak modular form *if in the Fourier expansion at infinity (Proposition 3.2.5)*

$$f(\omega) = \sum_{n\in\mathbb{Z}} f_n u^n,$$

the coefficients f_n are all elements of F.

We say that a weak modular form f of rank r is a rational weak modular form *if in the Fourier expansion at infinity*

$$f(\omega) = \sum_{n\in\mathbb{Z}} f_n(\tilde{\omega}) u^n,$$

the functions f_n are all rational weak modular forms of rank $r-1$.

A modular form which is also a rational weak modular form is called a rational modular form.

Remark. Note that, by the definition, the sum, difference and product of rational modular forms are rational modular forms (of course of possibly different weights). The quotient is also a rational weak modular form if no division by 0 occurs on Ω^r. By Proposition 3.3.2, it is also enough to consider expansions at one cusp, since we are considering only modular forms for $\mathrm{GL}_r(\mathbb{F}_q[t])$.

Theorem 3.6.5. *The modular forms $\bar{\pi}^{1-q^i} g_i(a,\omega)$, $\bar{\pi}^{-k}E^k(\omega)$ and $\bar{\pi}^{1-q^k} e_k(\tilde{\Lambda})$ are rational modular forms.*

Proof. By Corollary 3.4.8 and Corollary 3.4.14 it suffices to prove that $\bar{\pi}^{-k}E^k(\omega)$ is a rational modular form for every k. We now proceed by induction on the rank.

For $r = 2$ this was proved in Proposition 3.6.2. Now suppose that it is true for forms of rank $r-1$. By Proposition 3.5.3 we have the expansion

$$\bar{\pi}^{-k}E^k(\omega) = \bar{\pi}^{-k}\tilde{E}^k(\tilde{\omega}) - \sum_{a\in A_+} P_{k,\bar{\pi}\tilde{\Lambda}}(u_a).$$

By the induction hypothesis, the constant coefficient is a rational modular form of weight k. By Proposition 2.5.2 (h) the coefficients of the polynomial $P_{k,\bar{\pi}\tilde{\Lambda}}$ lie in the ring $\mathbb{F}_q[e_i(\bar{\pi}\tilde{\Lambda})_i]$. Since $e_i(\bar{\pi}\tilde{\Lambda}) = \bar{\pi}^{1-q^i}e_i(\tilde{\Lambda})$ these functions are all rational modular forms, hence the coefficients of $P_{k,\bar{\pi}\tilde{\Lambda}}$ are all rational modular forms.

Recall that $u_a = \frac{u^{q^{(r-1)\deg a}}}{f_a(u)}$ where $f_a(u) = u^{q^{(r-1)\deg a}}\varphi_a^{\bar{\pi}\tilde{\Lambda}}(u^{-1})$. Thus the coefficients of $f_a(u)$ are the coefficient forms $\tilde{g}_i(a,\bar{\pi}\tilde{\omega}) = \bar{\pi}^{1-q^i}\tilde{g}_i(a,\tilde{\omega})$ (by Proposition 2.4.5), which are rational modular forms by our induction hypothesis. We have $f_a(u) = \Delta(a,\bar{\pi}\tilde{\Lambda}) + ur(u)$, where $r(u)$ is a polynomial with rational modular forms as coefficients. Thus

$$\frac{u^{q^{(r-1)\deg a}}}{f_a(u)} = \frac{u^{q^{(r-1)\deg a}}}{\tilde{\Delta}(a,\bar{\pi}\tilde{\Lambda})}\left(1+\sum_{n\geq 0}\left(-\frac{ur(u)}{\tilde{\Delta}(a,\bar{\pi}\tilde{\Lambda})}\right)^n\right),$$

meaning that u_a has a u-expansion with coefficients rational weak modular forms. (They are holomorphic on Ω^{r-1} since Δ has no zeros on Ω^{r-1}, but not necessarily modular forms since division by Δ takes place.

Finally note that if $P_{k,\bar{\pi}\tilde{\Lambda}}$ has rational modular forms as coefficients and u_a has a u-expansion consisting of rational weak modular forms, then $P_{k,\bar{\pi}\tilde{\Lambda}}(u_a)$ has a u-expansion with rational weak modular forms as coefficients. $\square$

In order to study properties of modular forms under reduction modulo an ideal of A, we also need to say when they are *integral*, not just rational. We may define integrality inductively in the same way as rationality was defined.

In the rest of this section, we adopt the following normalization to ease notation: Replace $g_i(\omega)$ by $\bar{\pi}^{1-q^i} g_i(\omega)$ and hence $\Delta(\omega)$ by $\bar{\pi}^{1-q^r}\Delta(\omega)$. Then the functions $g_i(\omega)$ and $\Delta(\omega)$ are all rational modular forms. We also adopt the notation $E_k(\omega) := \bar{\pi}^{1-q^k} E^{q^k-1}(\omega)$ for Eisenstein series of weight $q^k - 1$. Lastly we shall also often suppress the arguments ω and $\tilde{\omega}$ and rely on the tildes to indicate the rank of the functions (in formulas they will either be rank r or rank $r-1$).

Example. If $r = 2$, Proposition 3.6.2 tells us that g_1 and g_2 are integral modular forms.

Before continuing with the rank 3 case, let us make a few observations.

Proposition 3.6.6. (a) *If the rank i discriminant functions are integral modular forms for $2 \leq i \leq r$, then the multiplicative inverse of the rank r discriminant function Δ^{-1} is an integral weak modular form of weight $1 - q^r$.*

Now suppose that the rank $r-1$ coefficient forms $\tilde{g}_i$ are integral modular forms. Then the following holds:

(b) *The expression u_a is a power series in u with integral weak modular forms as coefficients.*

(c) *Δ is an integral modular form of weight $q^r - 1$.*

(d) *g_1 and g_2 are integral modular forms*

Proof. By equation (3.11) (and Theorem 3.5.14) the expansion of $\Delta(\omega)$ is of the form $-\tilde{\Delta}^q u^{q-1} + \cdots$ (after the normalization discussed directly before the Example). Hence the expansion of $\Delta(\omega)^{-1}$ will be a geometric series of the form $-\tilde{\Delta}(\tilde{\omega})^{-q} u^{1-q}(1 + \cdots)$. Since $\Delta(\omega)$ is an integral modular form, the claim would follow if $\tilde{\Delta}(\tilde{\omega})$ was an integral weak modular form. Claim (a) now follows by induction, since we know from Theorem 3.5.7 that the rank 2 Drinfeld discriminant function has -1 as its first non-zero coefficient.

By definition the coefficients of the polynomials u_a are quotients of polynomials in the coefficient forms $\tilde{g}_i$ by powers of the discriminant $\tilde{\Delta}$. By assumption the coefficient forms are all integral, and we have just shown that the inverse of the discriminant function is integral, proving (b).

By the product formula in Theorem 3.5.14, Δ is a product of expressions of the form u_a and $\tilde{\Delta}$, proving (c).

By Propositions 3.4.11 and 3.5.3 we have $g_1 = [1]E_1 = \tilde{g}_1 - [1]\sum_{a\in A_+} u_a^{q-1}$ which is an integral modular form by (b) and the assumption on $\tilde{g}_1$.

Also by Proposition 3.4.11 we have $g_2 = [2]E_2 - g_1^q E_1 = [2]E_2 - [1]^q E_1^{q+1}$, while by Propositions 3.5.3 and 2.5.2 (f) we have

$$E_1 = \tilde{E}_1 - \sum_{a \in A_+} u_a^{q-1} \quad \text{and} \quad E_2 = \tilde{E}_2 - \sum_{a \in A_+} \left(u_a^{q^2-1} - \tilde{E}_1 u_a^{q^2-q} \right).$$

Now note that $\sum u_a^{q^2-1}$ has integral coefficients and that $[2]\tilde{E}_1 \sum u_a^{q^2-q}$ has integral coefficients, since $[2]\tilde{E}_1 = \frac{[2]}{[1]}\tilde{g}_1$. Also note that $[1]^q \left(\tilde{E}_1 - \sum u_a^{q-1}\right)^{q+1} = \frac{1}{[1]}\left(\tilde{g}_1 - [1]\sum u_a^{q-1}\right)^{q+1}$, every term in the binomial expansion of $(\tilde{g}_1 - [1]\sum u_a^{q-1})^{q+1}$ except $\tilde{g}_1^{q+1}$ has a factor $[1]$, and thus remains integral after division by $[1]$.

Hence, modulo A we have $g_2 \equiv [2]\tilde{E}_2 - \frac{1}{[1]}\tilde{g}_1{}^{q+1} = \tilde{g}_2$, which is integral. This completes the proof of (d). □

Corollary 3.6.7. *The rank 3 coefficient forms $g_1(\omega)$, $g_2(\omega)$, $g_3(\omega) = \Delta(\omega)$ are integral modular forms. The rank 4 coefficient forms $g_1(\omega)$, $g_2(\omega)$ and $g_4(\omega) = \Delta(\omega)$ are integral modular forms.*

Proof. The statements in rank 3 are corollaries of Proposition 3.6.6 (c) and (d) and Proposition 3.6.2, while the statements in rank 4 are corollaries of Proposition 3.6.6 (c) and (d) and what was just shown in rank 3. □

It is plausible that all coefficient forms turn out to be integral modular forms. This would allow us to study their reductions modulo certain ideals. Proposition 3.6.6 lays a good foundation for a possible inductive argument to prove this, but a complete proof has been elusive so far.

Chapter 4

Hecke operators

Hecke operators play a large role in the theory of classical modular forms. In the function field case one might say that their study has not been so fruitful, since one does not get the same powerful results. However, it is still interesting to study them and to see where they differ from classical Hecke operators and why.

Hecke operators are averaging linear transformations $T : \mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$, where we take $\Gamma_1 = \Gamma_2$ in most cases. In the classical case one then distinguishes those forms which are eigenforms for these operators. There are various interesting results in the classical case, which one can hope to generalize to the function field case. For example, in the classical case, the eigenvalues of an eigenform can be directly read from the coefficients of its Fourier expansion and thus no two eigenforms can have the same set of eigenvalues. This is false for Drinfeld modular forms. Another difference with the classical case is that Hecke operators for Drinfeld modular forms are completely multiplicative.

We shall not discuss all these questions, but rather content ourselves with developing the basic theory of Hecke operators and by computing some examples.

4.1 Hecke Rings

The theory of general Hecke operators was developed in the first half of the twentieth century and a good account is given by Shimura in [Sh]. Here he proceeds to define Hecke operators for subgroups of an arbitrary group G. For simplicity, we shall immediately assume that $A = \mathbb{F}_q[t]$ and set $G = \mathrm{GL}_r(F)$.

Let $G = \mathrm{GL}_r(F)$. Recall that two subgroups $\Gamma_1, \Gamma_2 \subset G$ are said to be commensurable if $\Gamma_1 \cap \Gamma_2$ is of finite index in both Γ_1 and Γ_2. This defines an equivalence relation $\sim$ on the set of subgroups of G. Indeed, the only non-trivial part is to check transitivity. Suppose that $\Gamma_1 \sim \Gamma_2$ and $\Gamma_2 \sim \Gamma_3$. Then $[\Gamma_1 \cap \Gamma_2 : \Gamma_1 \cap \Gamma_2 \cap \Gamma_3] \leq [\Gamma_2 : \Gamma_2 \cap \Gamma_3]$. So $[\Gamma_1 \cap \Gamma_2 \cap \Gamma_3]$ is of finite index in $\Gamma_2 \cap \Gamma_3$ and by symmetry of finite index in $\Gamma_1 \cap \Gamma_2$. Since these are of finite index in Γ_1 and Γ_3, respectively, we have that $\Gamma_1 \cap \Gamma_2 \cap \Gamma_3$ is of finite index in both Γ_1 and Γ_3. Then also $\Gamma_1 \cap \Gamma_3$ must be of finite index.

We note that by definition a congruence subgroup of $\mathrm{GL}_r(A)$ contains some $\Gamma(N)$ which is defined as the kernel of the map $\mathrm{GL}_r(A) \to \mathrm{GL}_r(A/N)$. Hence $\Gamma(N)$, and indeed every congruence subgroup, is of finite index in $\mathrm{GL}_r(A)$ and hence is commensurable with it. Thus all the congruence subgroups fall in the same commensurability class.

Lemma 4.1.1. *For every $g \in \mathrm{GL}_r(F)$ and any congruence subgroup Γ', the group $g\Gamma' g^{-1}$ is also a congruence subgroup.*

Proof. We can find $m \in A$ such that $\Gamma(m) \subset \Gamma'$ and both mg and mg^{-1} have coefficients in A. Then $g^{-1}\Gamma(m^3)g \subset g^{-1}(I + m^3 M_r(A))g = I + m \cdot mg^{-1} \cdot M_r(A) \cdot mg \subset I + mM_r(A)$. Noting that the determinant of any matrix in $g\Gamma' g^{-1}$ is an element of $\mathbb{F}_q^\times$ allows us to deduce that $g^{-1}\Gamma(m^3)g \subset \Gamma(m)$. Then $\Gamma(m^3) \subset g\Gamma(m)g^{-1} \subset g\Gamma' g^{-1}$. □

Hecke operators will be based on double cosets of the form $\Gamma_1 \alpha \Gamma_2$, where $\alpha \in \mathrm{GL}_r(F)$ and Γ_1 and Γ_2 are congruence subgroups of $\mathrm{GL}_r(A)$. This implies that $\alpha \Gamma_1 \alpha^{-1} \sim \Gamma_2$. A double coset like this can always be written as a union of left cosets or as a union of right cosets.

Proposition 4.1.2. *If $\alpha \in \mathrm{GL}_r(F)$, then the double coset $\Gamma_1 \alpha \Gamma_2$ can be written as a disjoint union of $[\Gamma_1 : \Gamma_1 \cap \alpha^{-1}\Gamma_2 \alpha]$ left cosets or as a disjoint union of $[\Gamma_2 : \Gamma_2 \cap \alpha \Gamma_1 \alpha^{-1}]$ right cosets.*

Proof. [Sh] Proposition 3.1. □

Now let R_{12} be the free $\mathbb{Z}$-module on expressions $\Gamma_1 \alpha \Gamma_2$, where $\alpha \in \mathrm{GL}_r(F)$. We can define a weighting on R_{12} by defining $\deg(\Gamma_1 \alpha \Gamma_2)$ to be the number of right cosets from Proposition 4.1.2, and then extend it linearly to R_{12}. (Note that it can also be done with left cosets, but we shall only consider right cosets here.)

We can now define a multiplication map $R_{12} \times R_{23} \to R_{13}$, which is well-defined and associative. If we write the following double cosets as unions of right cosets

$$\Gamma_1 \alpha \Gamma_2 = \bigcup_i \Gamma_1 \alpha_i, \quad \text{and} \quad \Gamma_2 \beta \Gamma_3 = \bigcup_j \Gamma_2 \beta_j,$$

then their product should be related to $\Gamma_1\alpha\Gamma_2\beta\Gamma_3 = \cup_j \Gamma_1\alpha\Gamma_2\beta_j = \cup_{i,j}\Gamma_1\alpha_i\beta_j$. The latter can be reinterpreted as a union of double cosets $\Gamma_1\gamma\Gamma_3$. But for the product to be associative we need to count them with multiplicity. So we define $\Gamma_1\alpha\Gamma_2 \times \Gamma_2\beta\Gamma_3$ as the sum of double cosets $\Gamma_1\gamma\Gamma_3$, where such a term is taken with multiplicity $\#\{(i,j) \,|\, \Gamma_1\alpha_i\beta_i = \Gamma_1\gamma\}$. For more details, e.g. that this is well defined, we refer to [Sh] Chapter 3.1.

Proposition 4.1.3. *Let $x \in \Gamma_1\alpha\Gamma_2$, $y \in \Gamma_2\beta\Gamma_3$ and $z \in \Gamma_3\gamma\Gamma_4$. Then*

(a) $\deg(x \times y) = \deg(x)\deg(y)$ *and*

(b) $(x \times y) \times z = x \times (y \times z)$.

Proof. [Sh] Propositions 3.3 and 3.4. □

Now let us assume that $\Gamma := \Gamma_1 = \Gamma_2 = \Gamma_3$ and that α lies in some semigroup S such that $\Gamma \subset S \subset \mathrm{GL}_r(F)$. Define $R(\Gamma, S)$ as the free $\mathbb{Z}$-module on expressions $\Gamma\alpha\Gamma$ where $\alpha \in S$. The multiplication operator defined above defines a ring structure on $R(\Gamma, S)$. When $\Gamma = \Gamma(N)$, then matrix transposition defines an anti-isomorphism of Γ. Then [Sh] Proposition 3.8 implies that $R(\Gamma, S)$ is commutative.

Before ending this section, let us quickly mention how all of this can be adapted to the case where A is not a principal ideal domain. There are essentially two ways this can be done. In the first way, instead of $\mathrm{GL}_r(A)$, we should take $\mathrm{GL}_A(\Lambda)$ for various projective modules Λ. However, in this case only the operators that preserve the relevant component are present. To obtain all the operators we should replace $\mathrm{GL}_r(A)$ with $G = \mathrm{GL}_r(\mathbb{A}_F^f)$, the adelic points of the general linear group and replace $\Gamma(N)$ by $K(N) = \ker(G \to \mathrm{GL}_r(N\hat{A}))$. If this is done, we should get exactly the same definitions as the algebraic Hecke operators defined in [Pi].

4.2 The Hecke Ring for $\Gamma = \mathbf{GL}_r(A)$

Our goal in this section is to study the Hecke ring $R(\Gamma, S)$ when $\Gamma = \mathrm{GL}_r(A)$ and $S = \mathrm{GL}_r(F) \cap M_{r\times r}(A)$, the semigroup of matrices with entries in A and non-zero determinant. The results in this section might not have been known, but the proofs are direct translations of those found in [Sh] into the function field situation. We assume throughout that A is a principal ideal domain.

We shall need the notion of lattices in a vector space $V = F^r$. They are very similar to lattices in $\mathbb{C}_\infty$, hence the same terminology. If there is confusion we shall distinguish between them by saying either "a lattice in V" or "a lattice in $\mathbb{C}_\infty$." However, there will not be much confusion, since the former is used only in this chapter, and this chapter uses that notion almost exclusively.

Definition 4.2.1. *A* lattice in V *is a rank r projective A-submodule of V.*

Theorem 4.2.2 (Elementary Divisors)**.** *Let L be a free module over A and $M \subseteq L$ a non-zero finitely generated submodule. Then there exists a basis $\mathcal{B}$ of L, a subset $\{e_1, \ldots, e_m\} \subseteq \mathcal{B}$ and elements $a_1, \ldots, a_m \in A$ such that:*

(a) *$\{a_1e_1, \ldots, a_me_m\}$ is a basis of M; and*

(b) *$a_i \mid a_{i+1}$ for $i = 1, \ldots, m-1$.*

Moreover, the a_i are determined uniquely (up to multiplication by a unit) satisfying these conditions.

Proof. [La] III. Theorem 7.8. □

For lattices $L_1 \subset L_2$ of the same rank, the Theorem of Elementary Divisors tells us that the quotient L_2/L_1 is of the form $\prod_{i=1}^{r}(A/a_iA)$ where $a_i \in A$ and $a_i \mid a_{i+1}$ for each i. We shall denote this ordered set of elements $(a_1, \ldots, a_r)$ by $\{L_2 : L_1\}$. Also denote by $[L_1 : L_2] := a_1a_2 \cdots a_r$ the product of these elements. Note that this notation replaces the usual notation for the index, which is now $q^{\deg[L_1:L_2]}$ instead.

Lemma 4.2.3. *The diagonal matrices* $\mathrm{diag}[a_1, \ldots, a_r]$ *where $a_1, \ldots, a_r \in A$ and $a_i \mid a_{i+1}$ for $i = 1, \ldots, r-1$, form a set of representatives for the double coset* $\Gamma\backslash S/\Gamma$.

Proof. Let $g \in S$ and consider its action on the standard lattice $L = A^r$. Clearly $g(L) \subseteq L$ and $g(L)$ is also of rank r, since $\det g \neq 0$. Hence $g(L)$ is a lattice of finite index in L. By the Theorem of Elementary Divisors, there exist a basis $e_1, \ldots, e_r$ of L and $a_1, \ldots, a_r$ such that $g(L) = a_1e_1A + \cdots + a_re_rA$. If P is the matrix that changes the basis from the standard basis to $(e_1, \ldots, e_r)$, then this means that $P^{-1}gP = \mathrm{diag}[a_1, \ldots, a_r]$ and since $P \in \mathrm{GL}_r(A)$, we conclude that $\Gamma g\Gamma = \Gamma\mathrm{diag}[a_1, \ldots, a_r]\Gamma$. □

By Lemma 4.2.3 we know the double cosets have representatives of the form $\mathrm{diag}[a_1, \ldots, a_r]$, and hence that the Hecke algebra is generated by double cosets of the form $\Gamma\alpha\Gamma$ where $\alpha = \mathrm{diag}[a_1, \ldots, a_r]$.

Definition 4.2.4. *Define the following double cosets:*

(a) *For any r-tuple $(a_i) \in A^r$ such that $a_i \mid a_{i+1}$ for every i, let* $T(a_1,\ldots,a_r) := \Gamma\alpha\Gamma$, *where* $\alpha = diag[a_1,\ldots,a_r]$.

(b) *For any prime $p \in A$ and $i = 1,\ldots,r$, let* $T_{p,i}^{(r)} := T(\underbrace{1,\ldots,1}_{r-i}\underbrace{p,\ldots,p}_{i})$.

Later when p is fixed, we shall write simply $T_i^{(r)}$.

(c) *For any $n \in A$, let $T_n := \sum T(a_1,\ldots,a_r)$ where the sum is taken over all r-tuples $(a_i) \in A^r$ such that $a_i \mid a_{i+1}$ for every i and $n = a_1 \cdots a_r$.*

In order to study these operators by using lattices, we would like to make the association $\Gamma\delta \mapsto A^r\delta$ between right cosets of Γ in $\Gamma\alpha\Gamma$ and lattices in F^r, but as it stands this is not well-defined. From now on we denote the standard lattice A^r by L.

Lemma 4.2.5. *Let $M, N \subset L$ be lattices. Then $\{L : M\} = \{L : N\}$ if and only if $M\alpha = N$ for some $\alpha \in \mathrm{GL}_r(A)$.*

Proof. Since an element of $\mathrm{GL}_r(A)$ preserves any lattice, the "if" part is clear. Now, suppose that $\{L : M\} = \{L : N\} = (a_1, a_2, \ldots, a_r)$. By the definition of this index there exist $2r$ elements $u_1,\ldots,u_r,v_1,\ldots,v_r \in A$ such that $L = u_1A + \cdots + u_rA = v_1A + \cdots + v_rA$, $M = a_1u_1A + \cdots + a_ru_rA$ and $N = a_1v_1A + \cdots + a_rv_rA$. We define α by letting $u_i\alpha = v_i$ for $i = 1,\ldots,r$. This means that $L\alpha = L$ (and hence $\alpha \in \mathrm{GL}_r(A)$) and $M\alpha = N$. □

Lemma 4.2.6. *Suppose that $\Gamma\alpha\Gamma = T(a_1,\ldots,a_r)$. The association $\Gamma\delta \mapsto L\delta$ defines a bijection between the right cosets $\Gamma\delta$ in $\Gamma\alpha\Gamma$ and the lattices $M \subset L$ such that $\{L : M\} = (a_1,\ldots,a_r)$.*

Proof. Assume, without loss of generality, that $\alpha = \mathrm{diag}[a_1,\ldots,a_r]$. If $\Gamma\delta = \Gamma\alpha\beta$ $(\beta \in \Gamma)$, then $\{L : L\delta\} = \{L : L\alpha\beta\} = \{L : L\alpha\} = (a_1,\ldots,a_r)$ by Lemma 4.2.5.

Conversely, if $\{L : M\} = (a_1,\ldots,a_r)$, then by Lemma 4.2.5 $M = L\alpha\beta$ for some $\beta \in \Gamma$. Since $\beta \in \Gamma$, we have $\Gamma\alpha\beta \subset \Gamma\alpha\Gamma$. The fact that this correspondence defines a bijection follows by noting that $\Gamma\delta = \Gamma\delta'$ if and only if $L\delta = L\delta'$. □

Corollary 4.2.7. *The degree of $T(a_1,\ldots,a_r)$ (the number of right cosets in $\Gamma diag[a_1,\ldots,a_r]\Gamma$) equals the number of lattices $M \subset L$ such that $\{L : M\} = (a_1,\ldots,a_r)$.*

Proposition 4.2.8. *The multiplication of two double cosets is given as follows: $\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma = \sum_\gamma c_\gamma \Gamma\gamma\Gamma$, where c_γ equals the number of lattices M such that $\{L : M\} = \{L : L\beta\}$ and $\{M : L\gamma\} = \{L : L\alpha\}$.*

Proof. Let $\Gamma\alpha\Gamma = \cup_i \Gamma\alpha_i$ and $\Gamma\beta\Gamma = \cup_j \Gamma_j$ be disjoint unions of these double cosets. By the way multiplication is defined, we have

$$c_\gamma = \#\{(i,j) \,|\, \Gamma\alpha_i\beta_j = \Gamma\gamma\} = \#\{(i,j) \,|\, L\alpha_i\beta_j = L\gamma\}.$$

Given γ and j, note that i is uniquely determined by $\Gamma\alpha_i = \Gamma\gamma\beta_j^{-1}$. If $L\alpha_i\beta_j = L\gamma$, set $M = L\beta_j$. Then $\{L : M\} = \{L : L\beta_j\} = \{L : L\beta\}$ and $\{M : L\gamma\} = \{L\beta_j : L\alpha_i\beta_j\} = \{L : L\alpha_i\} = \{L : L\alpha\}$.

Conversely, suppose that M is a lattice such that $\{L : M\} = \{L : L\beta\}$ and $\{M : L\gamma\} = \{L : L\alpha\}$. By Lemma 4.2.6 this means that there is a unique β_j such that $M = L\beta_j$. Then $\{L : L\gamma\beta_j^{-1}\} = \{L\beta_j : L\gamma\} = \{L : L\alpha\}$. Now, by Lemma 4.2.6, we must have $L\gamma\beta_j^{-1} = L\alpha_i$ for some i. Then $L\gamma = L\alpha_i\beta_j$. $\square$

Proposition 4.2.9. *These double cosets satisfy the following multiplication laws:*

(a) $T(a_1b_1, \ldots, a_rb_r) = T(a_1, \ldots, a_r)T(b_1, \ldots, b_r)$ *if* $\gcd(a_r, b_r) = 1$.

(b) $T(ca_1, \ldots, ca_r) = T(c, \ldots, c)T(a_1, \ldots, a_r)$ *for every* $c \in A$ *and every* r*-tuple* (a_i).

Proof. Note that if $\gcd(a_r, b_r) = 1$, then $\gcd(a_i, b_i) = 1$ for every $i = 1, \ldots, r$. Let $\alpha = \mathrm{diag}[a_1, \ldots, a_r]$ and $\beta = \mathrm{diag}[b_1, \ldots, b_r]$ and let $\gamma \in \Gamma\alpha\Gamma\beta\Gamma$. Suppose for a contradiction that there exist two lattices M and M' such that both $\{L : M\} = \{L : M'\} = \{L : L\beta\}$ and $\{M : L\gamma\} = \{M'L\gamma\} = \{L : L\alpha\}$. By the Second Isomorphism Theorem, the indices $[M+M' : M] = [M' : M\cap M']$. (Here, and in the remainder of this proof, the index refers to the usual concept and not the one defined in terms of $\{\cdot : \cdot\}$.) Since $M + M' \subset L$, the left hand side must be a divisor of $[L : M] = \det(\beta)$ and since $L\gamma \subset M \cap M'$, the right hand side is a divisor of $[L : L\alpha] = \det(\alpha)$. Since $\det(\alpha)$ and $\det(\beta)$ are relatively prime, both quantities must be 1, implying that $M = M'$. This means that if $\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma = \sum_\gamma c_\gamma \Gamma\gamma\Gamma$, then each $c_\gamma = 1$.

On the other hand, if $\gamma \in \Gamma\alpha\Gamma\beta\Gamma$, then we can find a lattice M satisfying these properties. Then $L\gamma \subset M \subset L$, and the quotient $L/L\gamma \cong L/M \oplus M/L\gamma \cong L/L\alpha \oplus L/L\beta \cong L/L\alpha\beta$, since $\gcd(\det(\alpha), \det(\beta)) = 1$.

Part (b) follows from the fact that the matrix $c \cdot \mathrm{id}$ commutes with every element of Γ. $\square$

Proposition 4.2.9 allows us to focus our study to elements of the form $T(p^{e_1}, \ldots, p^{e_r})$ where $p \in A$ is prime and $e_1 \le \cdots \le e_r$. Let $R_p^{(r)}(\Gamma, S)$ be the subalgebra of $R(\Gamma, S)$ generated by elements of the form $T(p^{e_1}, \ldots, p^{e_r})$. It turns out that $R_p^{(r)}(\Gamma, S)$ is generated by the elements $T_{p,i}^{(r)}$. We prove this by induction on the rank, explaining why the superscript is included in the notation. From now on fix a prime p. Then A/pA is a finite field with $q^{\deg p}$ elements.

Proposition 4.2.10. *Let $c_k^{(r)}$ be the number of k-dimensional (A/pA)-subvector spaces of $(A/pA)^r$. Then*

$$\deg\left(T_k^{(r)}\right) = c_k^{(r)} = \frac{(q^{r\deg p}-1)(q^{r\deg p}-q^{\deg p})\cdots(q^{r\deg p}-q^{(k-1)\deg p})}{(q^{k\deg p}-1)(q^{k\deg p}-q^{\deg p})\cdots(q^{k\deg p}-q^{(k-1)\deg p})}.$$

Proof. By Corollary 4.2.7 the degree of $T_k^{(r)}$ is the number of lattices $M \subset L$ such that $\{L : M\} = (\underbrace{1,\ldots,1}_{r-k}\underbrace{p,\ldots,p}_{k})$. In that case $pL \subset M \subset L$ and M/pL is a $n-k$-dimensional subspace of $L/pL \cong (A/pA)^r$. On the other hand, for every $n-k$-dimensional subspace W of $(A/pA)^r$, there is a unique M such that $M/pL = K$.

The formula for $c_k^{(i)}$ is standard and well-known. See for example [Ro] I.1 Exercise 21. $\square$

Let $\pi : R_p^{(r+1)} \to R_p^{(r)}$ be the $\mathbb{Z}$-linear map

$$\{\pi(T(1,p^{e_1},\ldots,p^{e_r})) = T(p^{e_1},\ldots,p^{e_r}), \pi(T(P^{e_0},\ldots,p^{e_r})) = 0 \quad \text{for } e_0 > 0.$$

Lemma 4.2.11. *The map π is a surjective ring homomorphism with kernel $T_{r+1}^{(r+1)}R_p^{(r+1)}$.*

Proof. Surjectivity is clear and the fact that the kernel is as stated, follows from Proposition 4.2.9 (b). Set $a' = (1,p^{a_1},\ldots,p^{a_r})$, $b' = (1,p^{b_1},\ldots,p^{b_r})$, $d' = (1,p^{d_1},\ldots,p^{d_r})$ and $a = (p^{a_1},\ldots,p^{a_r})$, $b = (p^{b_1},\ldots,p^{b_r})$, $d = (p^{d_1},\ldots,p^{d_r})$ and suppose that $a \cdot b$ contains the term d with coefficient c_d and that $a' \cdot b'$ contains the term d' with coefficient c'_d. It suffices to prove that for any d we have $c_d = c'_d$.

Let $L = u_1A + \cdots + u_rA$, $L' = u_0A + L$ and $N = p^{d_1}u_1A + \cdots p^{d_r}u_rA$, $N' = u_0A + N$. Then $\{L : N\} = d$ and $\{L' : N'\} = d'$. By Corollary 4.2.7 we know that $c_d = \#\{M \,|\, \{L : M\} = b, \{M : N\} = a\}$ and $c'_d = \#\{M' \,|\, \{L' : M'\} = b', \{M' : N'\} = a'\}$. Let M' be a lattice for which $\{L' : M'\} = b'$ and $\{M' : N'\} = a'$ and set $M = M' \cap L$. Then $\{L : M\} = b$ and $\{M : N\} = a$. On the other hand let M be a lattice in $u_1F + \cdots + u_rA$ for which $\{L : M\} = b$ and $\{M : N\} = a$ and set $M' = u_0A + M$. Then $\{L' : M'\} = b'$ and $\{M' : N'\} = a'$, providing the one-to-one correspondence that proves $c'_d = c_d$.

Now we know that $T(a)\cdot T(b) = \sum_d c_d T(d)$ and $T(a')\cdot T(b') = \sum_{d'} c'_d T(d') + T(p,\ldots,p)x$ for some element $x \in R_p^{(n+1)}$. (The last term appears, since there are lattices M' with the desired indices that are not of the form $u_0 A + M$, but these are necessarily contained in pL.) Then

$$\pi(T(a')\cdot T(b')) = \pi\left(\sum_{d'} c'_d T(d')\right) + \pi(T(p,\ldots,p)x) = \sum_d c_d T(d) = T(a)\cdot T(b).$$

$\square$

Definition 4.2.12. *For any element of $R_p(\Gamma, S)$ define $w(\sum c_\alpha \Gamma\alpha\Gamma) := \max\{z \,|\, p^z = \det\alpha,\ c_\alpha \neq 0\}$. Call an element $\sum c_\alpha \Gamma\alpha\Gamma$* homogeneous *if $\det\alpha$ is the same for every α where $c_\alpha \neq 0$.*

The elements $\Gamma\alpha\Gamma$ and $T(p^{e_1},\ldots,p^{e_r})$ are trivially homogeneous, while it is also clear that the product of any two homogeneous elements is homogeneous.

Theorem 4.2.13. *The ring $R_p^{(r)}$ is the polynomial ring in r algebraically independent elements $T_i^{(r)}$ for $i = 1,\ldots,r$.*

Proof. The proof is by induction on r. If $r = 1$, Proposition 4.2.9 (b) implies that $T(p^e) = T(p)^e$. Now assume that $r \geq 2$ and that the result is true for r. To prove that every element of the form $T(p^{e_0},\ldots,p^{e_r})$ is a polynomial in the required elements, we proceed by induction on $w(T(p^{e_0},\ldots,p^{e_r})) = e_0 + \cdots + e_r$. Note that if this is zero, then $e_0 = \cdots = e_r = 0$ and $T(1,\ldots,1) = 1 \in \mathbb{Z}$.

If $e_0 > 0$, then by Proposition 4.2.9 (b), we have the equality $T(p^{e_0},\ldots,p^{e_r}) = T(p,\ldots,p)^{e_0} T(1, p^{e_1-e_0},\ldots,p^{e_r-e_0})$, so it suffices to handle the case where $e_0 = 0$. Let $X = T(1, p^{e_1},\ldots,p^{e_r})$. By the induction hypothesis, the element $\pi(X) = T(p^{e_1},\ldots,p^{e_r}) \in R_p^{(r)}$ can be represented as a polynomial

$$\Phi(T_1^{(r)},\ldots,T_r^{(r)}) = \sum_k u_k M_k(T_1^{(r)},\ldots,T_r^{(r)}),$$

where the M_k are monomials and $u_k \in \mathbb{Z}$. Note that since X is homogeneous, this polynomial must be homogeneous in the sense that $w(M_k)$ is the same for every k. Now consider the element $Y = \Phi\left(T_1^{(r+1)},\ldots,T_r^{(r+1)}\right) \in R_p^{(r+1)}$. Note that since X is homogeneous, it too is homogeneous. We have $\pi(X) = \pi(Y)$, implying by Lemma 4.2.11 that $X - Y = T_{r+1}^{(r+1)} \cdot Z$ for some $Z \in R_p^{(r+1)}$. Consequently, $X - Y$ is homogeneous, and hence Z is homogeneous, but $w(Z) < w(X)$. By our induction hypothesis, Z is a polynomial in the elements $T_i^{(r)}$ $(i = 1,\ldots,r)$.

For algebraic independence, suppose that there is some polynomial relation between the $T_i^{(r)}$:

$$\Phi\left(T_1^{(r+1)},\ldots,T_r^{(r+1)}\right)=\sum_{d=m}^{n}\left(T_{r+1}^{(r+1)}\right)^d\Phi_d\left(T_1^{(r+1)},\ldots,T_r^{(r+1)}\right)=0,$$

where $\Phi_m,\Phi_n\neq 0$. Since $T_{r+1}^{(r+1)}$ is not a zero divisor (by Proposition 4.2.9 (b)), this means that

$$\sum_{d=m}^{n}\left(T_{r+1}^{(r+1)}\right)^{d-m}\Phi_d\left(T_1^{(r+1)},\ldots,T_r^{(r+1)}\right)=0$$

Thus $0=\pi(0)=\pi\left(\Phi_d\left(T_1^{(r+1)},\ldots,T_r^{(r+1)}\right)\right)=\Phi_d\left(T_1^{(r)},\ldots,T_r^{(r)}\right)$, giving a contradiction with the assumption that $\Phi_m\neq 0$. □

As in the classical case, we shall obtain a recurrence relation for the elements T_{p^k} by interpreting the formal power series $\sum_{k=0}^{\infty}T_{p^k}X^k$ as the multiplicative inverse of $\sum_{i=0}^{r}(-1)^i q^{\frac{1}{2}i(i-1)\deg p}T_i^{(n)}X^i$. To do this we need two lemmas. Recall that $c_i^{(k)}$ was defined in Proposition 4.2.10.

Lemma 4.2.14. *If we make the conventions that $c_i^{(k)}=0$ when $i>k$ and $c_0^{(0)}=1$, then*

$$T_i^{(r)}X^i\left(\sum_{j=0}^{\infty}T_{p^j}X^j\right)=\sum_{k=0}^{r}c_i^{(k)}\left(\sum_{1\leq d_1\leq\cdots\leq d_k}T(1,\ldots,1,p^{d_1},\ldots,p^{d_k})X^{d_1+\cdots+d_k}\right).$$

Proof. For any k and any k-tuple $(d_1,\ldots,d_k)$ we shall compute the coefficient $m(d_1,\ldots,d_k)$ of $T(1,\ldots,1,p^{d_1},\ldots,p^{d_k})$ in the product $T_i^{(r)}X^i\left(\sum_{j=0}^{\infty}T_{p^j}X^j\right)$. Note that for any such k-tuple only the term $T_{p^j}X^j$ where $i+j=d_1+\cdots+d_k$ can contribute to this coefficient. Let N be a fixed lattice such that $\{L:N\}=(1,\ldots,1,p^{d_1},\ldots,p^{d_k})$. By Proposition 4.2.8 we can compute

$$m(d_1,\ldots,d_k)=\sum_{\alpha}\#\{M\,|\,\{L:M\}=(\underbrace{1,\ldots,1}_{r-i}\underbrace{p,\ldots,p}_{i}),\ \{M:N\}=\{L:L\alpha\}\},$$

the sum ranging over $\Gamma\alpha\Gamma$ where $\det(\alpha)=p^m$ and $\alpha\in\Delta$. If $N\subset M$ then there exists $\alpha\in S$ such that $\{M:N\}=\{L:L\alpha\}$ and $\det(\alpha)=[M:N]=p^m$. Thus

$$\begin{aligned} m(d_1,\ldots,d_k)=\#\Big\{&\text{lattices } M\,\Big|\,[M:N]=p^m,\\ &\{L:M\}=(\underbrace{1,\ldots,1}_{r-i}\underbrace{p,\ldots,p}_{i})\Big\}. \end{aligned}\tag{4.1}$$

If $L = u_1A + \cdots + u_rA$ and $N = u_1A + \cdots + u_{r-k}A + p^{d_1}u_{r-k+1}A + \cdots + p^{d_k}u_rA$, then $pL + N = u_1A + \cdots + u_{r-k}A + pu_{r-k+1} + \cdots + pu_rA$, and hence $L/(pL+N) \cong (A/pA)^k$. Furthermore, if M satisfies the conditions in equation (4.1), then $L/M \cong (A/pA)^i$. If $i > k$, then there is no M satisfying the conditions, and hence $m(d_1, \ldots, d_k) = 0$. If $i \leq k$, then $M/(pL+N) \cong (A/pA)^{k-i}$ is a subspace of $L/(pL+N) \cong (A/pA)^k$. Conversely, for every such subspace, there exists a unique lattice M satisfying the conditions in equation 4.1. Therefore $m(d_1, \ldots, d_k) = c_i^{(k)}$. □

Lemma 4.2.15. *For any $k > 0$, we have*

$$\sum_{i=0}^{k} (-1)^i q^{\frac{1}{2}i(i-1)\deg p} c_i^{(k)} = 0.$$

Proof. Let $\Phi(X) = (X-1)(X-q^{\deg p})\cdots(X-q^{(k-1)\deg p})$ be a polynomial of degree k, and consider the polynomial

$$\Psi(X) = \sum_{i=0}^{k-1} \frac{\Phi(X)}{\Phi'(q^{i\deg p})(X-q^{i\deg p})}.$$

The polynomial Ψ has degree strictly less than k and for $j = 0, 1, \ldots, k-1$ we have $\Phi'(q^{j\deg p}) = (q^{j\deg p}-1)(q^{j\deg p}-q^{\deg p})\cdots(q^{j\deg p}-q^{(j-1)\deg p})(q^{j\deg p}-q^{(j+1)\deg p})\cdots(q^{j\deg p}-q^{(k-1)\deg p})$ which is also the value of $\frac{\Phi(X)}{(X-q^{j\deg p})}$ at $X = q^{j\deg p}$. Therefore $\Psi(X) = 1$ for k values $X = q^{j\deg p}$, $j = 0, 1, \ldots, k-1$ and hence $\Psi(X)$ is identically 1.

Then also

$$1 = \Psi(q^{k\deg p}) = \sum_{i=0}^{k-1} \frac{(q^{k\deg p}-1)(q^{k\deg p}-q^{\deg p})\cdots(q^{k\deg p}-q^{(k-1)\deg p})}{\Phi'(q^{i\deg p})(q^{k\deg p}-q^{i\deg p})},$$

where $\Phi'(q^{i\deg p}) = (q^{i\deg p}-1)(q^{i\deg p}-q^{\deg p})\cdots(q^{i\deg p}-q^{(i-1)\deg p})(q^{i\deg p}-p^{i+1})\cdots(q^{i\deg p}-q^{(k-1)\deg p})$. Each factor $(q^{i\deg p}-q^{(i+t)\deg p}) = -\frac{q^{k\deg p}-q^{(k-t)\deg p}}{q^{(k-i-t)\deg p}}$ (for $t = 1, \ldots, k-1-i$), so that

$$\begin{aligned} 1 &= \sum_{i=0}^{k-1} \frac{(q^{k\deg p}-1)(q^{k\deg p}-q^{\deg p})\cdots(q^{k\deg p}-q^{(i-1)\deg p})p^{\frac{1}{2}(k-i)(k-i-1)}}{(q^{i\deg p}-1)(q^{i\deg p}-q^{\deg p})\cdots(q^{i\deg p}-q^{(i-1)\deg p})} \\ &= \sum_{i=0}^{k-1} q^{\frac{1}{2}(k-i)(k-i-1)\deg p} c_i^{(k)}. \end{aligned}$$

Replacing $k-i$ by j, this becomes $1 = \sum_{j=1}^{k} q^{\frac{1}{2}j(j-1)\deg p} c_j^{(k)}$. □

Theorem 4.2.16. *The operators T_n satisfy the following:*

(a) *If $n = p$ is prime, then $T_p = T_1^{(r)} = T(1, \ldots, 1, p)$,*

(b) *If $\gcd(m, n) = 1$, then $T_{mn} = T_m T_n$.*

(c)
$$T_{p^k} = \sum_{i=1}^{r} (-1)^{i+1} q^{\frac{1}{2}i(i-1)\deg p} T_i^{(r)} T_{p^{k-i}}.$$

Proof. The definition of T_n immediately gives (a), while (b) follows from Proposition 4.2.9 (a). The statement of (c) will follow by comparing coefficients of X^k in the equality of formal power series

$$\left(\sum_{m=0}^{\infty} T_m X^m\right)\left(\sum_{i=0}^{r} (-1)^i q^{\frac{1}{2}i(i-1)\deg p} T_i^{(r)} X^i\right) = 1,$$

which is true, since by Lemma 4.2.14, this power series is

$$\sum_{i=0}^{r} (-1)^i q^{\frac{1}{2}i(i-1)\deg p} \sum_{k=0}^{r} c_i^{(k)} \sum_{1 \le d_1 \le \cdots \le d_k} T(1, \ldots, 1, p^{d_1}, \ldots, p^{d_k}) X^{d_1 + \cdots + d_k}.$$

By 4.2.15 the coefficients of X^k where $k \geq 1$ sum to 0, proving the identity. $\square$

4.3 Hecke Operators on Drinfeld modular forms

In this section we show how the abstract Hecke ring $R(\Gamma, S)$ can be represented in the space of Drinfeld modular forms of weight k. To do this, we associate to an element $\Gamma\alpha\Gamma$ (or more generally $\Gamma_1\alpha\Gamma_2$ for congruence subgroups Γ_1 and Γ_2) a linear operator $\mathcal{M}_k(\Gamma) \to \mathcal{M}_k(\Gamma)$ (resp. $\mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$).

Definition 4.3.1. *Let $\Gamma_1, \Gamma_2 \subset \mathrm{GL}_r(A)$ be congruence subgroups and let $\alpha \in \mathrm{GL}_r(F)$. Define the* weight k $[\Gamma_1\alpha\Gamma_2]$ double coset operator *as the linear operator $[\Gamma_1\alpha\Gamma_2] : \mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$ defined by*

$$f[\Gamma_1\alpha\Gamma_2]_k = \sum_j f[\beta_j]_k,$$

where $\Gamma_1\alpha\Gamma_2 = \cup_j \Gamma_1\beta_j$ is a disjoint union of right cosets.

First note that the definition does not depend on the choice of β_j, since $\Gamma_1\beta_j = \Gamma_1\beta_j'$ if and only if $\beta_j' = \alpha\beta$ for some $\alpha \in \Gamma_1$, implying that $f[\beta_j']_k = f[\alpha]_k[\beta_j]_k = f[\beta_j]_k$, since f is a modular form with respect to Γ_1.

To show that we end up with a weak modular form for Γ_2, note that for any $\gamma \in \Gamma_2$ we have

$$(f[\Gamma_1\alpha\Gamma_2]_k)[\gamma]_k = \sum_j f[\beta_j]_k[\gamma]_k = \sum f[\beta_j\gamma]_k = f[\Gamma_1\alpha\Gamma_2]_k,$$

since if $\{\beta_j\}$ is a set of right coset representatives of $\Gamma_1\alpha\Gamma_2$, then so is $\{\beta_j\gamma\}$.

Lastly, note that $f[\Gamma_1\alpha\Gamma_2]_k[\delta]_k$ is holomorphic at infinity for each $\delta \in \mathrm{GL}_r(F)$, since for any modular form f and any $\delta \in \mathrm{GL}_r(F)$, the function $f[\delta]_k$ is holomorphic at infinity. This means that the image is contained in $\mathcal{M}_k(\Gamma_2)$. By the same reasoning, if f is a cusp form, then $f[\Gamma_1\alpha\Gamma_2]$ is a cusp form.

Since Drinfeld modular forms take values in a field of characteristic p, many of the results from the previous section become much easier than in the classical case, or in the case of characteristic 0 valued automorphic forms. In particular, Theorem 4.2.16 becomes

$$T_{p^k} = T_1^{(r)} T_{p^{k-1}} = T_p T_{p^{k-1}},$$

which by induction implies that $T_{p^k} = (T_p)^k$ for all primes $p \in A$ and all $k \in \mathbb{Z}_{\geq 0}$. Together with Theorem 4.2.16 (b), we have complete multiplicativity of Hecke operators for $\Gamma = \mathrm{GL}_r(A)$.

Now, let us compute the action of the Hecke operators on modular forms, starting with T_p. A set of right coset representatives for $\Gamma\alpha\Gamma$ where $\alpha = \mathrm{diag}[1, \ldots, 1, p]$ is the set of β ranging over matrices of the form

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & p & & \\ & & * & \ddots & \\ & & * & & 1 \end{pmatrix}$$

which are almost diagonal with 1's on the diagonal, except for one entry which is p; and the only other non-zero entries are in the column below that p. These entries can also be chosen to have degree less than $\deg p$. Indeed, on any matrix with determinant p one can perform row reduction (since Γ acts on the left) until you get a matrix with only 1's and one p on the diagonal.

Completing the row reduction then shows that any matrix is in a coset $\Gamma\beta$ for some β of the form stated.

By definition, for such a β, we have $f[\beta]_k(\omega) = j(\beta,\omega)^{-k} f(\beta\omega) = j(\beta,\omega)^{-k}$ $f(j(\beta,\omega)^{-1}\omega \cdot \beta^{-1})$. Next, note that if

$$\beta = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & p & & \\ & & b_i & \ddots & \\ & & b_r & & 1 \end{pmatrix}$$

then

$$\beta^{-1} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1/p & & \\ & & -b_i/p & \ddots & \\ & & -b_r/p & & 1 \end{pmatrix}, \tag{4.2}$$

(i ranging from $m+1$ to r, where p appears in column m). Denote $\omega_{m,\beta} = \frac{\omega_m - b_1\omega_{m+1} - \cdots - b_r\omega_r}{p}$. Then, if $m < r$, we have

$$f[\beta]_k(\omega) = f((\omega_1, \ldots, \omega_{m-1}, \omega_{m,\beta}, \omega_{m+1}, \ldots, \omega_r)),$$

since $j(\beta,\omega) = 1$, and if $m = r$, we have

$$f[\beta]_k(\omega) = p^k f((p\omega_1, \ldots, p\omega_{r-1}, \omega_r)),$$

since $j(\beta,\omega) = 1/p$.

Proposition 4.3.2. *For each Hecke operator T_p, every Eisenstein series $E^k(\omega)$ is an eigenform for T_p with eigenvalue p^k.*

Proof. By definition of T_p we have

$$\begin{aligned} T_pE^k(\omega) &= \sum_\beta E^k(\omega) \\ &= \sum_{m=1}^{r-1} \sum_{b_{m+1},\ldots,b_r \in (A/pA)} E^k((\omega_1, \ldots, \omega_{m-1}, \omega_{m,\beta}, \omega_{m+1}, \ldots, \omega_r)) \\ &\quad + p^k E^k((p\omega_1, \ldots, p\omega_{r-1}, \omega_r)) \end{aligned}$$

By definition of $E^k(\omega)$, it is the sum of λ^{-k} as λ ranges over the elements of the lattice $\omega\Lambda$. Hence $E^k((\omega_1, \dots, \omega_{m-1}, \omega_{m,\beta}, \omega_{m+1}, \dots, \omega_r))$ is the sum over λ^{-k} where λ ranges over elements of the lattice $\Lambda_\beta := \omega_1 A + \cdots + \omega_{m-1}A + \omega_{m,\beta}A + \omega_{m+1}A + \cdots + \omega_r A$, which contains $\omega\Lambda$ with $[\Lambda_\beta : \omega\Lambda] = p$. Moreover, as m ranges through 1 to $r-1$ and $b_{m+1}, \dots, b_r$ range over all elements of (A/pA), Λ_β ranges over all such lattices, except $\Lambda_p := \omega_1 A + \cdots + \omega_{r-1}A + \frac{1}{p}\omega_r A$. However,

$$\begin{aligned} p^k E^k((p\omega_1, \dots, p\omega_{r-1}, \omega_r)) &= p^k \sum_{\lambda \in p\Lambda_p} \lambda^{-k} \\ &= \sum_{\lambda \in \Lambda_p} \lambda^{-k}. \end{aligned}$$

Hence $T_p E^k(\omega)$ is the sum of λ as λ ranges through all lattices that contain $\omega\Lambda$ with index p. Note that some terms get repeated, namely those where $\lambda \in \omega\Lambda$. Moreover, each such term gets repeated $1 + q^{\deg p} + q^{2\deg p} + \cdots + q^{(r-1)\deg p}$ times (for column m, there are $r-m$ b_i's, each chosen in $q^{\deg p}$ ways). In characteristic p, this is the same as just counting it once.

Hence the sum includes each $\lambda \in \frac{1}{p}\omega\Lambda$ exactly once and

$$T_p E^k(\omega) = \sum_{\lambda \in \omega\Lambda} \left(\frac{\lambda}{p}\right)^{-k} = p^k E^k(\omega).$$

□

Proposition 4.3.3. *The space of cusp forms of weight $q^r - 1$ and type 0 is one-dimensional and generated by the discriminant function Δ.*

Proof. Suppose that there is some non-zero cusp form f of weight $q^r - 1$ that is not a multiple of Δ. Then, by Theorem 3.4.15 (a), and since $\Delta(\omega)$ is already of weight $q^r - 1$, it must be a polynomial in the coefficient forms $g_1(\omega), \dots, g_{r-1}(\omega)$. Suppose that it is of the form $f(\omega) = \sum_{(e_i)} g_1(\omega)^{e_1} \cdots g_{r-1}(\omega)^{e_{r-1}}$. Then, by equation (3.10), the constant term in its u-expansion is $f_0(\tilde{\omega}) = \sum_{(e_i)} \tilde{g}_1(\tilde{\omega})^{e_1} \cdots \tilde{g}_{r-1}(\tilde{\omega})^{e_{r_1}}$, which is identically 0 by assumption that it is a cusp form. However, by Theorem 3.4.15, the rank $r-1$ coefficient forms $\tilde{g}_1(\tilde{\omega}), \dots, \tilde{g}_{r-1}(\tilde{\omega})$ are algebraically independent, implying that the polynomial that defines f must be the zero polynomial. Then f must actually be identically 0 itself, which is a contradiction. □

Proposition 4.3.4. *The discriminant function $\Delta(\omega)$ is an eigenform of T_p with associated eigenvalue p^{q-1}.*

Proof. By Proposition 4.3.3, it follows that Δ is an eigenform. It remains to calculate the eigenvalue. We do this by calculating the first non-zero coefficient of $T_p\Delta$, since we already know the first non-zero coefficient of Δ to be $-\bar{\pi}^{q-1}\tilde{\Delta}(\omega)^q u^{q-1}$. We show that the β (among double coset representatives from equation 4.2) with non-trivial column m contribute only to higher terms when $m > 1$ and that there is a nice formula for those β where the non-trivial column is column 1.

First suppose that $2 \leq m \leq r-1$. Then $\Delta[\beta]_{q^r-1}(\omega)$ has an expansion in terms of $e_{\pi\Lambda_\beta}(\pi\omega_1)^{-1}$, where Λ_β is the lattice $\omega_2 A + \cdots + \omega_{m-1}A + \omega_{m,\beta}A + \omega_{m+1}A + \cdots + \omega_r A$. This lattice contains $\tilde{\Lambda}$ with $[\Lambda_\beta : \tilde{\Lambda}] = p$, and hence by Proposition 2.2.5 (b) there is an $\mathbb{F}_q$-linear polynomial Φ of degree $q^{\deg p}$ such that $e_{\pi\Lambda_\beta}(\pi\omega_1) = \Phi(e_{\pi\tilde{\Lambda}}(\pi\omega_1))$. Suppose that $\Phi(X) = X + \cdots + DX^{q^{\deg p}}$. Remembering that $e_{\bar{\pi}\tilde{\omega}\tilde{\Lambda}}(\omega_1) = u^{-1}$, we then have

$$e_{\pi\Lambda_\beta}(\pi\omega_1)^{-1} = \left(\Phi\left(u^{-1}\right)\right)^{-1} = \frac{u^{q^{\deg p}}}{D}(1+\cdots)^{-1},$$

the last factor being an inverse which is a power series expansion in u. Each term in the expansion of $\Delta[\beta]_k(\omega)$ thus has only terms with powers of u greater than $q-1$ appearing.

Essentially the same thing can be done for $m = r$, since then $\omega_2 A + \cdots + \omega_{r-1}A + \frac{\omega_r}{p}A$ is again a lattice that contains $\tilde{\Lambda}$ with the same index as before. The factor $j(\beta,\omega)^{-k} = p^k$ plays no role in the end.

We are left to compute what happens when we take the sum over all β where the non-trivial column is the first. We have

$$\begin{aligned}
\sum_\beta \Delta[\beta]_{q^r-1}(\omega) &= \sum_{b_2,\ldots,b_r\in(A/pA)} f\left(\omega_{1,\beta},\omega_2,\ldots,\omega_r\right) \\
&= \sum_{b_2,\ldots,b_r}\sum_{n\geq 0} f_n(\tilde{\omega})u_{\tilde{\omega}}\left(\frac{\omega_1 - b_2\omega_2 - \cdots - b_r\omega_r}{p}\right)^n \\
&= \sum_{n\geq 0} f_n(\tilde{\omega})\sum_{b_2,\ldots,b_r} e_{\pi\tilde{\omega}\tilde{\Lambda}}\left(\pi\frac{\omega_1 - b_2\omega_2 - \cdots - b_r\omega_r}{p}\right)^{-n}
\end{aligned}$$

The internal sum will turn out to be exactly over elements of the $\mathbb{F}_q$-linear set $\Lambda_p := e_{\pi\tilde{\omega}\tilde{\Lambda}}\left(\frac{\pi}{p}\tilde{\omega}\tilde{\Lambda}\right)$ and this allows us to write it as a Goss polynomial.

$$\begin{aligned}
&\sum_{b_2,\ldots,b_r} e_{\pi\tilde{\omega}\tilde{\Lambda}}\left(\pi\frac{\omega_1 - b_2\omega_2 - \cdots - b_r\omega_r}{p}\right)^{-n} \\
&= \sum_{b_2,\ldots,b_r}\left(e_{\pi\tilde{\omega}\tilde{\Lambda}}(\pi\omega_1/p) - e_{\pi\tilde{\omega}\tilde{\Lambda}}\left(\frac{b_2\omega_2 + \cdots + b_r\omega_r}{p}\right)\right)^{-n} \\
&= \sum_{\lambda\in\Lambda_p}\left(e_{\pi\tilde{\omega}\tilde{\Lambda}}(\pi\omega_1/p) - \lambda\right)^{-n} \\
&= P_{n,\Lambda_p}\left(\sum_{\lambda\in\Lambda_p}\left(e_{\pi\tilde{\omega}\tilde{\Lambda}}(\pi\omega_1/p) - \lambda\right)^{-1}\right) \\
&= P_{n,\Lambda_p}\left(e_{\Lambda_p}\left(e_{\pi\tilde{\omega}\tilde{\Lambda}}(\pi\omega_1/p)\right)^{-1}\right)
\end{aligned}$$

By Proposition 2.2.5 (b), $e_{\Lambda_p}\left(e_{\pi\tilde{\omega}\tilde{\Lambda}}(\pi\omega_1/p)\right) = e_{\frac{\pi}{p}\tilde{\omega}}(\pi\omega_1/p) = p^{-1}u^{-1}$. Hence

$$\sum_{\beta}\Delta[\beta]_{q^r-1}(\omega) = \sum_{n\geq 0} f_n(\tilde{\omega})P_{n,\Lambda_p}(pu).$$

By Lemma 3.5.4, every n for which f_n is not identically 0 is divisible by $q-1$, and by Proposition 2.5.2 (g), every term of P_{n,Λ_p} with non-zero coefficient must then also be divisible by $q-1$. Thus the coefficients of u^i for $i = 1,\ldots,q-2$ are 0.

If $n = q-1$, then $P_{q-1,\Lambda_p}(X) = X^{q-1}$ (Proposition 2.5.2 (e)), hence there is a term $-\bar{\pi}^{q-1}p^{q-1}\tilde{\Delta}(\tilde{\omega})^q u^{q-1}$. To complete the proof we show that for $n \geq q$, the polynomial $P_{n,\Lambda_p}(X)$ has no X^{q-1} term. By Proposition 3.5.5, the only n that appear are congruent to 0 or -1 modulo q. If $q \mid n$, then by Proposition 2.5.2 (d) there can clearly be no term with exponent $q-1$. So suppose that $n = mq-1$, where $m \geq 2$. Then, by Proposition 2.5.2 (i), $X^2P'_{n,\Lambda_p}(X) = (q-1)P_{mq,\Lambda_p}(X) = -\left(P_{m,\Lambda_p}(X)\right)^q$. By Proposition 2.5.2 (c), if $m \geq 2$, then $P_{m,\Lambda_p}(X)$ has no X term, implying that the least non-zero term of $P'_{n,\Lambda_p}(X)$ must be at least X^{2q-2}, and hence that $P_{n,\Lambda_p}(X)$ can have no X^{q-1} term. □

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