On the $\langle r, s \rangle$-domination number of a graph

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Declaration

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Abstract

The (classical) domination number of a graph is the cardinality of a smallest subset of its vertex set with the property that each vertex of the graph is in the subset or adjacent to a vertex in the subset. Since its introduction to the literature during the early 1960s, this graph parameter has been researched extensively and finds application in the generic facility location problem where a smallest number of facilities must be located on the vertices of the graph, at most one facility per vertex, so that there is at least one facility in the closed neighbourhood of each vertex of the graph.

The placement constraint in the above application may be relaxed in the sense that multiple facilities may possibly be located at a vertex of the graph and the adjacency criterion may be strengthened in the sense that a graph vertex may possibly be required to be adjacent to multiple facilities. More specifically, the number of facilities that can possibly be located at the i-th vertex of the graph may be restricted to at most $r_i \geq 0$ and it may be required that there should be at least $s_i \geq 0$ facilities in the closed neighbourhood of this vertex. If the graph has n vertices, then these restriction and sufficiency specifications give rise to a pair of vectors $\boldsymbol{r} = [r_1, \ldots, r_n]$ and $\boldsymbol{s} = [s_1, \ldots, s_n]$. The smallest number of facilities that can be located on the vertices of a graph satisfying these generalised placement conditions is called the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the graph. The classical domination number of a graph is therefore its $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number in the special case where $\boldsymbol{r} = [1, \ldots, 1]$ and $\boldsymbol{s} = [1, \ldots, 1]$.

The exact values of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number, or at least upper bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number, are established analytically in this dissertation for arbitrary graphs and various special graph classes in the general case, in the case where the vector $\boldsymbol{s}$ is a step function and in the balanced case where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$.

A linear algorithm is put forward for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a tree, and two exponential-time (but polynomial-space) algorithms are designed for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph. The efficiencies of these algorithms are compared to one another and to that of an integer programming approach toward computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph.

Uittreksel

Die (klassieke) dominasiegetal van 'n grafiek is die grootte van 'n kleinste deelversameling van die grafiek se puntversameling met die eienskap dat elke punt van die grafiek in die deelversameling is of naasliggend is aan 'n punt in die deelversameling. Sedert die verskyning van hierdie grafiekparameter in the literatuur gedurende die vroeë 1960s, is dit deeglik nagevors en vind dit neerslag in die generiese plasingstoepassing waar 'n kleinste getal fasiliteite op die punte van die grafiek geplaas moet word, hoogstens een fasiliteit per punt, sodat daar minstens een fasiliteit in die geslote buurpuntversameling van elke punt van die grafiek is.

Die plasingsbeperking in die bogenoemde toepassing mag egter verslap word in die sin dat meer as een fasiliteit potensieel op 'n punt van die grafiek geplaas kan word en verder mag die naasliggendheidsvereiste verhoog word in die sin dat 'n punt van die grafiek moontlik aan veelvuldige fasiliteite naasliggend moet wees. Gestel dat die getal fasiliteite wat op die i-de punt van die grafiek geplaas mag word, beperk word tot hoogstens $r_i \geq 0$ en dat hierdie punt minstens $s_i \geq 0$ fasiliteite in die geslote buurpuntversameling daarvan moet hê. Indien die grafiek n punte bevat, gee hierdie plasingsbeperkings en -vereistes aanleiding tot die paar vektore $\boldsymbol{r} = [r_1, \ldots, r_n]$ en $\boldsymbol{s} = [s_1, \ldots, s_n]$. Die kleinste getal fasiliteite wat op die punte van 'n grafiek geplaas kan word om aan hierdie veralgemeende voorwaardes te voldoen, word die $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-dominasiegetal van die grafiek genoem. Die klassieke dominasiegetal van 'n grafiek is dus die $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-dominasiegetal daarvan in die spesiale geval waar $\boldsymbol{r} = [1, \ldots, 1]$ en $\boldsymbol{s} = [1, \ldots, 1]$.

In hierdie verhandeling word die eksakte waardes van, of minstens grense op, die $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-dominasiegetal van arbitrêre grafieke of spesiale klasse grafieke analities bepaal vir die algemene geval, vir die geval waar $\boldsymbol{s}$ 'n trapfunksie is, en vir die gebalanseerde geval waar $\boldsymbol{r} = [r, \ldots, r]$ en $\boldsymbol{s} = [s, \ldots, s]$.

'n Lineêre algoritme word ook daargestel vir die berekening van die $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-dominasiegetal van 'n boom, en twee eksponensiële-tyd (maar polinoom-ruimte) algoritmes word ontwerp vir die berekening van die $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-dominasiegetal van 'n arbitrêre grafiek. Die doeltreffendhede van hierdie algoritmes word met mekaar vergelyk en ook met dié van 'n heeltallige programmeringsbenadering tot die bepaling van die $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-dominasiegetal van 'n grafiek.

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Table of Contents

List of Figures

List of Tables

List of Algorithms

CHAPTER 1

Introduction

Contents

1.1 Problem description

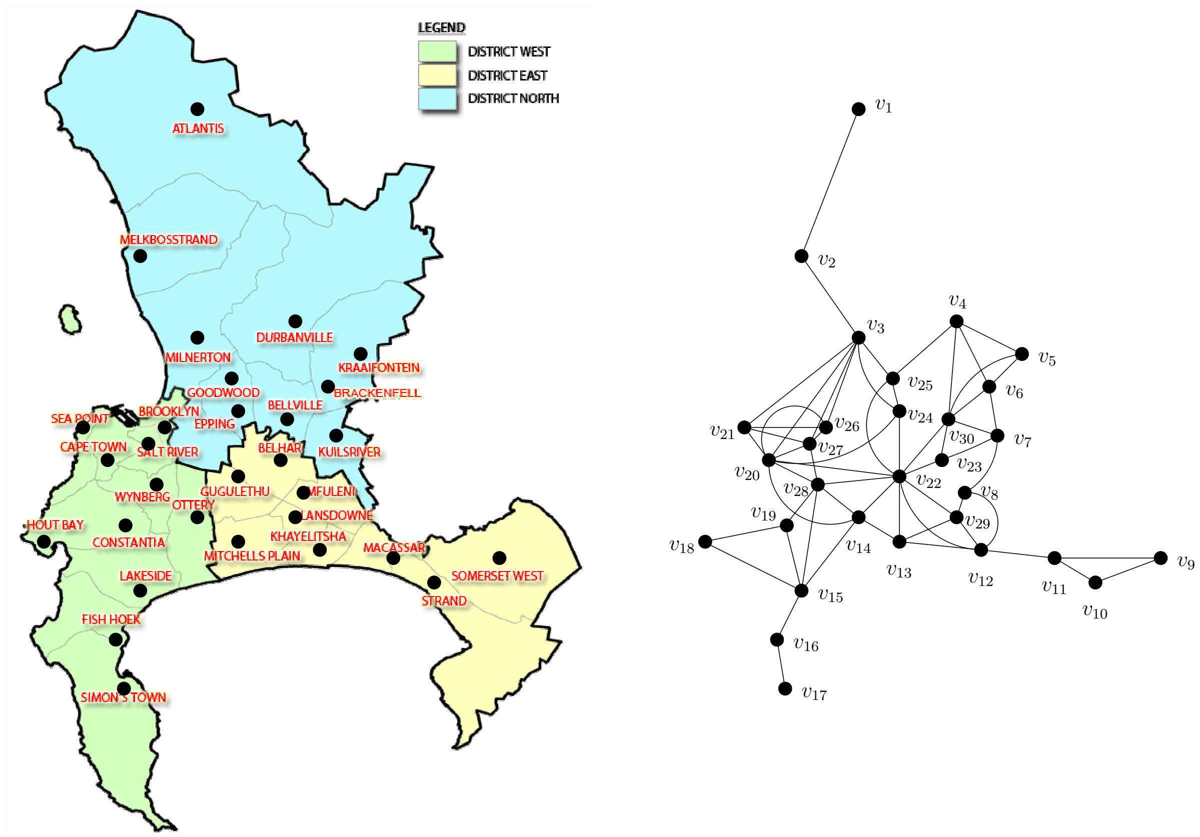
Suppose the City of Cape Town has to purchase new fire trucks and seeks an optimal placement of these trucks in such a manner that the number of trucks (and hence the total purchase cost) is minimised, subject to a specified quality of service delivery.

The map in Figure 1.1a) [17] shows the locations of the thirty fire and rescue stations in the municipality of the City of Cape Town. To determine where the fire trucks must be placed, the constraints of each fire station have to be considered. Specifically, each station has a certain capacity in terms of how many of these fire trucks can be operated by the station. For this specific example, the stations are classified into four categories:

1. stations with too few personnel to man a fire truck (these are typically substations),
2. small stations capable of operating a single truck each,
3. medium stations capable of operating two trucks each, and
4. large stations capable of operating three trucks each.

If a station requires service by more trucks than it can physically host, the station should have access to a sufficient number of fire trucks in its closed neighbourhood, that is, all the stations that are in reach within some threshold response time. For this specific example, the number of trucks required in the closed neighbourhood of a fire station depends on the size of the area it serves, as well as the predominant type of housing in the area surrounding the fire station.

The facility location problem is modelled by a graph on thirty vertices, in which vertices represent the fire stations and in which two vertices are joined by an edge when it is possible to travel between the associated locations in less than twenty minutes, see Figure 1.1b). Two constraints are therefore associated with vertex v_i of the graph: a capacity restriction on the number of trucks that can be placed at the corresponding station, denoted by r_i, and a sufficient number of fire trucks required in its closed neighbourhood, denoted by s_i, as listed in Table 1.1.



a) b)

FIGURE 1.1: *The placement of fire and rescue stations in the City of Cape Town and the underlying graph.*

The smallest number of fire trucks that may be placed on the graph, such that both constraints are met, is 18. The number of fire trucks assigned in this optimal facility location to vertex v_i, denoted by f_i, is given in Table 1.1 for all $i = 1, \ldots, 30$.

The capacity constraint values of the vertices can be combined into a vector $\boldsymbol{r} = [r_1, \ldots, r_{30}]$. Similarly, the vector $\boldsymbol{s} = [s_1, \ldots, s_{30}]$ describes the sufficient numbers of trucks in the closed neighbourhoods of the vertices.

In the graph theoretic literature, the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph of order n is defined as the minimum number of units of some commodity that may be placed on the vertices of the graph such that there are at most r_i units at vertex v_i and such that at least s_i units are placed in the closed neighbourhood of vertex v_i, where $\boldsymbol{r} = [r_1, \ldots, r_n]$ and $\boldsymbol{s} = [s_1, \ldots, s_n]$. Hence, in the above example the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the graph in Figure 1.1b) was computed. Even for a relatively small instance, like the example above, it is not easy to compute the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph. The aim in this dissertation is to simplify the task of computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph by establishing good upper bounds on this graph parameter, by establishing exact values of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number for certain graph classes in closed form and by designing and implementing algorithms capable of computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph as rapidly as possible.

The concept of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination was first introduced by Cockayne [20] in 2007. The balanced case where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ was independently studied by Rubalcaba and Slater

Vertex	Location	r_i	s_i	f_i
v_1	Atlantis	2	3	2
v_2	Melkbosstrand	1	1	1
v_3	Milnerton	2	2	1
v_4	Durbanville	1	3	1
v_5	Kraaifontein	1	2	1
v_6	Brackenfell	1	1	0
v_7	Kuilsriver	1	1	0
v_8	Mfuleni	1	3	1
v_9	Somerset West	0	2	0
v_{10}	Strand	2	2	2
v_{11}	Macassar	0	3	0
v_{12}	Khayelitsha	1	3	1
v_{13}	Mitchell's Plain	1	3	0
v_{14}	Ottery	1	2	0
v_{15}	Lakeside	2	2	2

Vertex	Location	r_i	s_i	f_i
v_{16}	Fish Hoek	1	2	1
v_{17}	Simon's Town	1	2	1
v_{18}	Houtbay	1	2	0
v_{19}	Constantia	1	3	0
v_{20}	Cape Town	3	1	0
v_{21}	Sea Point	1	1	0
v_{22}	Gugulethu	1	3	1
v_{23}	Belhar	2	2	0
v_{24}	Epping	1	2	0
v_{25}	Goodwood	3	2	0
v_{26}	Brooklyn	0	1	0
v_{27}	Salt River	2	2	0
v_{28}	Wynberg	2	1	1
v_{29}	Lansdowne	1	3	1
v_{30}	Bellville	2	1	1
	Total			18

TABLE 1.1: *The constraints on the fire and rescue stations, in Figure 1.1, and a placement of the minimum number of trucks such that all the location constraints are met.*

[72] in the same year. Burger and Van Vuuren [11] established the first lower and upper bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph in 2008 and also presented exact values of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number for certain graph classes in the balanced case where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$. A quadratic-time algorithm for calculating the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a tree (also for the balanced case) was proposed by Cockayne [19].

1.2 Dissertations objectives

The following objectives are pursued in this dissertation:

I To *document* the relevant results that have appeared in the literature on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of simple, undirected graphs.

II To *establish* upper bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph that are as tight as possible.

III To *propose* bounds on or values for the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of various infinite graph classes in closed form, including complete graphs, cycles, paths, cartesian products, circulants and biparite graphs.

IV To *design* and *implement* algorithms for computing the exact value of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph and to *analyse* the time and space complexities of these algorithms.

1.3 Dissertation layout

This dissertation contains a total of seven chapters (including this introduction chapter) and two appendices.

The second chapter of the dissertation contains descriptions of the basic concepts and definitions related to multisets, graph theory, the probabilistic method and complexity theory. These notions are used in the remainder of the dissertation.

In the third chapter, a literature review is provided on $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination and its special cases, that is, (classical) domination, k-tuple domination and $\{k\}$-domination. In §3.1–3.3 these special cases of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination are reviewed separately, each time providing general results on the relevant parameter, known exact values of the parameter for certain graph classes as well as a discussion on the algorithmic complexity of the computation problem associated with the parameter. The chapter closes with a discussion on the literature related to the general case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination in §3.4. This discussion centres around general bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number and exact values of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination numbers for certain graph classes. An overview of known algorithmic approaches is also provided.

Various novel upper bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph are established in Chapter 4. The chapter opens with a discussion on general bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph. Upper bounds for the balanced case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination (*i.e.* where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$) are presented in §4.2. This includes an upper bound in terms of the minimum degree of the graph as well as four bounds established by employing probabilistic arguments. The chapter closes with a comparison of the newly established upper bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph in §4.3.

The focus of Chapter 5 falls on the following special classes of graphs: complete graphs, cycles, paths, cartesian products, circulants and bipartite graphs. For each of these classes novel upper bounds on or exact values of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number are proposed.

Chapter 6 opens with a new linear-time algorithm for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a tree. Furthermore, three exact algorithms for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph and an integer programming formulation of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem are presented in this chapter. More specifically, a new branch-and-bound algorithm and a new branch-and-reduce algorithm are introduced, while a known dynamic programming algorithm for solving the set multicover problem is adapted from [48] for $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination. These algorithms, together with the integer programming formulation, are then compared by considering their execution times for specific graph classes.

The dissertation closes in §7.1 with a summary of the work presented within, an appraisal of the contributions of the dissertation in §7.2, as well as some ideas with respect to future work on the theory of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination in §7.3.

CHAPTER 2

Preliminary concepts and notation

Contents

This chapter opens with the definition of a multiset and an overview of four operations on multisets in §2.1. In §2.2 certain fundamentals from graph theory are reviewed and this is followed by an discussion on the probabilistic method in §2.3. The chapter concludes with a review of the basic notions from complexity theory in §2.4.

2.1 Basic notions in the theory of multisets

The notation related to multisets specified in [79] is adopted in this dissertation. A *multiset* $\mathcal{A}$ over A is a pair $\langle A, f\rangle$, where A is a set and $f : A \mapsto \mathbb{N}_0$ is a function, with $f(a)$ indicating the number of times the element a appears in the multiset $\mathcal{A}$, called the *multiplicity* of a in $\mathcal{A}$. Let $\mathcal{A} = \langle A, f\rangle$ and $\mathcal{B} = \langle A, g\rangle$ be two multisets. The *sum* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $A \uplus B$, is the multiset $\mathcal{C} = \langle A, h\rangle$, where $h(a) = f(a) + g(a)$ for all $a \in A$. The *removal* of $\mathcal{B}$ from $\mathcal{A}$, denoted by $\mathcal{A} \ominus \mathcal{B}$, is the multiset $\mathcal{C} = \langle A, h\rangle$, where $h(a) = \max\{f(a) - g(a), 0\}$ for all $a \in A$. The *deletion* of $\mathcal{B}$ from $\mathcal{A}$, denoted by $\mathcal{A} - \mathcal{B}$, produces the multiset $\mathcal{C} = \langle A, h\rangle$, where $h(a) = 0$ for all $a \in A$ where $g(a) \neq 0$, and $h(a) = f(a)$ otherwise. The *intersection* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \cap \mathcal{B}$, is the multiset $\mathcal{C} = \langle A, h\rangle$, where $h(a) = \min\{f(a), g(a)\}$ for all $a \in A$.

Let $A = \{1, 2, 3, 4\}$ and consider the multisets $\mathcal{A} = \{1, 2, 2, 3, 3, 3, 4, 4\}$ and $\mathcal{B} = \{1, 1, 3, 4, 4, 4\}$ over A as examples. The multisets may also be expressed as $\mathcal{A} = \langle A, f\rangle$ and $\mathcal{B} = \langle A, g\rangle$, where $f(i) = a_i$ and $g(i) = b_i$ for $\boldsymbol{a} = [a_1, a_2, a_3, a_4] = [1, 2, 3, 2]$ and $\boldsymbol{b} = [b_1, b_2, b_3, b_4] = [2, 0, 1, 3]$. The sum of the multisets $\mathcal{A}$ and $\mathcal{B}$ is the multiset $A \uplus B = \{1, 1, 1, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4\}$, while the removal of $\mathcal{B}$ from $\mathcal{A}$ is $\mathcal{A} \ominus \mathcal{B} = \{2, 2, 3, 3\}$. Deletion of $\mathcal{B}$ from $\mathcal{A}$, produces the multiset $\mathcal{A} - \mathcal{B} = \{2, 2\}$, while the intersection of $\mathcal{A}$ and $\mathcal{B}$ is $\mathcal{A} \cap \mathcal{B} = \{1, 3, 4, 4\}$.

2.2 Basic notions from graph theory

A *simple graph* $G = (V, E)$ is a nonempty, finite set $V(G)$ of elements, called *vertices*, together with a (possibly empty) set $E(G)$ of two-element subsets of $V(G)$, called *edges*. An edge between $u \in V(G)$ and $v \in V(G)$ is denoted by uv. The *order* of G is the cardinality of the *vertex set* $V(G)$, while its *size* is the cardinality of the *edge set* $E(G)$. A graph is called a *trivial graph* when its order is 1, while any graph with more than one vertex is called a *non-trivial graph*.

Consider, as example, the graph G_1 in Figure 2.1 a). The graph has vertex set $V(G_1) = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E(G_1) = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_2v_4\}$. Therefore, G_1 is a graph of order 5 and size 6.

2.2.1 Adjacency and neighbourhoods

If $uv \in E(G)$, then the vertices u and v are *adjacent* in G, and the edge uv and the vertex u (or v) are *incident* in G. The *complement* $\overline{G}$ of a graph G has vertex set $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. That is, two vertices in G are adjacent in G if and only if they are not adjacent in $\overline{G}$. The graph G_2 in Figure 2.1 b) is the complement of G_1.

The vertices in $V(G)$ that are adjacent to $v \in V(G)$ are called the *neighbours* of v in G. The *open neighbourhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set $\{u \in V(G) \mid uv \in E(G)\}$, while the *closed neighbourhood* of a vertex $v \in V(G)$, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. Let $S \subseteq V(G)$. Then the set $\bigcup_{s\in S} N_G[s]$ is called the *closed neighbourhood of* S and is denoted by $N_G[S]$, while the set $N_G[S] - S$ is called the *open neighbourhood of* S and is denoted by $N_G(S)$.

The open neighbourhood of the vertex v_4 in the graph G_1 in Figure 2.1 a) is the set of all vertices of G_1 adjacent to v_4, that is $N_{G_1}(v_4) = \{v_1, v_2\}$, while the closed neighbourhood $N[v_4]$ is $N(v_4) \cup \{v_4\} = \{v_1, v_2, v_4\}$. The open and closed neighbourhoods of the set $S = \{v_2, v_3\}$ in the graph G_1 is $N_{G_1}(S) = \{v_1, v_4\}$ and $N_{G_1}[S] = \{v_1, v_2, v_3, v_4\}$, respectively.

2.2.2 Vertex degrees

The *degree* of a vertex $v \in V(G)$, denoted by $\deg_G(v)$, is the number of neighbours of v in G, that is $\deg_G(v) = |N_G(v)|$. The *minimum degree* of G, denoted by $\delta(G)$, is the minimum degree of all the vertices of G, while the *maximum degree* of G, denoted by $\Delta(G)$, is the maximum degree of all the vertices of G. An *isolated vertex* $v \in V(G)$ has $\deg_G(v) = 0$, while a *universal vertex* has $\deg_G(v) = |V(G)| - 1$. A vertex of degree 1 is called an *end-vertex* of G and a vertex adjacent to an end-vertex is called a *support vertex*. A graph of order n is called a *complete*

a) G_1 b) $G_2 = \overline{G_1}$

FIGURE 2.1: *a) A graphical representation of a graph G_1 with vertex set $V(G_1) = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E(G_1) = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_2v_4\}$. b) A graphical representation of the complement $\overline{G_1}$ of the graph G_1.*

graph, denoted by K_n, if every vertex in $V(G)$ is a universal vertex of the graph. A graph of order n is an *edgeless graph*, denoted by $\overline{K_n}$, if every vertex of $V(G)$ is isolated. A *regular graph* is a graph whose vertices all have the same degree.

The graph G_1 in Figure 2.1 has minimum degree $\delta(G_1) = 1$ and maximum degree $\Delta(G_1) = 4$. The vertex v_1 is a universal vertex, since $deg_{G_1}(v_1) = 4 = |V(G_1)| - 1$, while v_5 is an end-vertex. Finally, the vertex v_1 is an isolated vertex in G_2.

Whenever the graph is clear from the context, the sets $V(G)$, $E(G)$, $N_G(v)$ and $N_G[v]$ are abbreviated to V, E, $N(v)$ and $N[v]$, respectively. This applies throughout the dissertation; therefore the numbers $\deg_G(v)$, $\delta(G)$, $\Delta(G)$ and a parameter $\pi(G)$ may become $\deg(v)$, δ, Δ and π if there is no ambiguity.

2.2.3 Graph isomorphisms

Two graphs G and H are *isomorphic* if there exists a bijection $\phi : V(G) \mapsto V(H)$, called an *isomorphism*, which preserves adjacency, *i.e.* $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. The fact that two graphs G and H are isomorphic is expressed by writing $G \cong H$. An *automorphism* of a graph G is an isomorphism from G onto itself. A graph G is *vertex-transitive* if, for any vertices u and v of G, there exists an automorphism ϕ of G such that $\phi(u) = v$. G is *edge-transitive* if, for any two edges u_1v_1 and u_2v_2 of G, there exists an automorphism ϕ of G such that $\phi(u_1v_1) = u_2v_2$.

a) G_3 b) G_4

FIGURE 2.2: *Two isomorphic graphs.*

Consider the two graphs in Figure 2.2. Define the bijection $\phi : V(G_3) \mapsto V(G_4)$ such that $\phi(v_1) = u_2$, $\phi(v_2) = u_4$, $\phi(v_3) = u_1$ and $\phi(v_4) = u_3$. Then ϕ is an isomorphism form G_3 to G_4 and hence $G_3 \cong G_4$.

2.2.4 Subgraphs

A *subgraph* H of a graph G is a graph for which $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. When $V(H) = V(G)$ and $E(H) \subseteq E(G)$, the subgraph H is called a *spanning subgraph* of G. If H is a subgraph of G in which every two vertices in $V(H)$ are adjacent, then H is a *complete subgraph* of G, also known as a *clique*. For any nonempty subset $S \subseteq V(G)$, the *induced subgraph* of S in G, denoted by $G\langle S\rangle$, is the subgraph of G with vertex set S and edge set $\{uv \in E(G) \mid u, v \in S\}$. A graph G is called *H-free* if G contains no induced subgraph isomorphic to H.

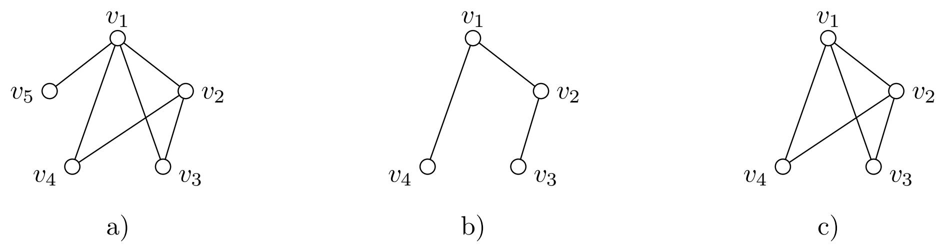


FIGURE 2.3: *a) The graph G_1. b) A subgraph of G_1. c) The induced subgraph $G_1\langle\{v_1, v_2, v_3, v_4\}\rangle$.*

The graph in Figure 2.3 c) is the induced subgraph of $\{v_1, v_2, v_3, v_4\}$ in G_1.

2.2.5 Connectedness

A *walk* of length k is an alternating sequence $W = v_0, e_1, v_1, e_2, v_2, e_3, \ldots, v_{n-1}, e_n, v_n$ of vertices and edges with $e_i = v_{i-1}v_i$ for all $i = 1, \ldots, n$. If $v_0 = x$ and $v_n = y$, then W is called an *x-y walk* of length n. A *path* is a walk whose vertices are all distinct. A graph of order n that consists of only a path is called the *path of order n* and is denoted by P_n. If $v_0 = v_n$ in the walk W and all the other vertices are distinct, the walk W is called a *cycle* of order or length n. A graph of order n that consists of only a cycle is called the *cycle of order n* and is denoted by C_n. A graph G is *connected* if, for any pair of vertices x and y of G, there exists an x-y path in G; otherwise G is *disconnected*. A *maximal connected subgraph* of a graph G is a subgraph that is connected and is not a subgraph of any larger connected subgraph of G. A subgraph H of a graph G is called a *component* of G if H is a maximal connected subgraph of G.

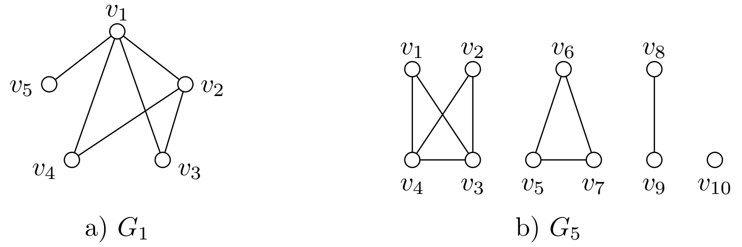


FIGURE 2.4: *a) A connected graph. b) A disconnected graph with 4 components.*

The graph G_1 in Figure 2.4 a) is a connected graph since there exists a v_i-v_j path in G_1 for any pair of vertices v_i and v_j. The graph G_5 in Figure 2.4 b) is disconnected and has four components.

2.2.6 Special graphs

The vertex set of a *multipartite graph* can be partitioned into k sets $V_1, \ldots, V_k$, called *partite sets*, in such a way that there is no adjacency between vertices of the same partite set. A *complete multipartite graph* is a multipartite graph with partite sets $V_1, \ldots, V_k$ such that every vertex in V_i is adjacent to every vertex in V_j for all $1 \le i, j \le k$ and $i \ne j$. If $|V_i| = r_i$ for all $i = 1, \ldots, k$, then the complete multipartite graph is denoted by $K_{r_1,\ldots,r_k}$. A (complete) multipartite graph with two partite sets is called a *(complete) bipartite graph*. A *star* is a complete bipartite graph of the form $K_{1,s}$.

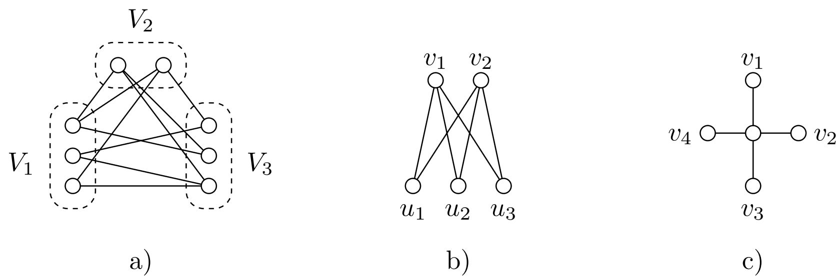


FIGURE 2.5: *a) A multipartite graph with partite sets V_1, V_2 and V_3. b) The complete bipartite graph $K_{2,3}$. c) The star $K_{1,4}$.*

A multipartite graph with three partite sets V_1, V_2 and V_3 is depicted in Figure 2.5 a). Figure 2.5 also contains the complete bipartite graph $K_{2,3}$ and the star $K_{1,4}$.

The *circulant* $G = C_n \langle a_1, a_2, \ldots, a_k \rangle$ with $0 < a_1 < a_2 < \cdots < a_k < n$ is defined as a graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G) = \{\{i, i+j\} \mid i = 1, 2, \ldots, n \text{ and } j = a_1, a_2, \ldots, a_k\}$, where arithmetic is performed modulo n. Note that $C_n \langle 1 \rangle$ is the cycle C_n and that $K_n = C_n \langle 1, 2, \ldots, \lfloor n/2 \rfloor \rangle$. Two further examples of circulants are given in Figure 2.6.

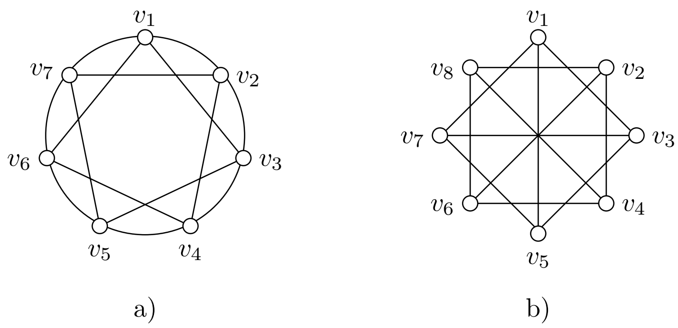


FIGURE 2.6: *a) The circulant $C_7 \langle 1, 2 \rangle$. b) The circulant $C_8 \langle 2, 4 \rangle$.*

A *tree* of order n is a connected graph with no cycles. A *spanning tree* of a graph G is a spanning subgraph of G that is a tree. All the end-vertices of a tree are called *leaves*. A *rooted tree* is a tree in which exactly one vertex r is specified and called the *root* of the tree. Let T be a rooted tree with root r and let v be the neighbour of u on the unique path from r to u. Then u is a *child* of v and v is the *parent* of u.

A spanning tree T_1 of the circulant $C_7 \langle 1, 2 \rangle$ is shown in Figure 2.7. If T_1 is rooted at v_1, then v_6 is the parent of v_4 and v_5, while v_4 and v_5 are children of v_6.

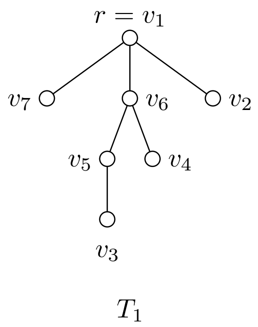


FIGURE 2.7: *A spanning tree of the circulant $C_7 \langle 1, 2 \rangle$ rooted at v_1.*

2.2.7 Operations on graphs

The *cartesian product* $G \Box H$ of two graphs G and H has vertex set $V(G) \times V(H)$. Two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G \Box H$ if and only if $u_1 = v_1$ and $u_2 v_2 \in E(H)$, or $u_2 = v_2$ and $u_1 v_1 \in E(G)$. The cartesian product $G \Box H$ may be constructed by placing a copy of G at each vertex of H and adding the appropriate edges (see Figure 2.8 for an example). For $g \in V(G)$, the subgraph ${}^{g}H = \{g\} \times H$ of $G \Box H$ is called an H-fibre of $G \Box H$. Similarly, a G-fibre is the subgraph $G^h = G \times \{h\}$ for $h \in V(H)$. It is clear that all G-fibres are isomorphic to G and all H-fibres are isomorphic to H. The ${}^{u_1}C_3$-fibre of $C_4 \Box C_3$ is highlighted in grey in Figure 2.8.

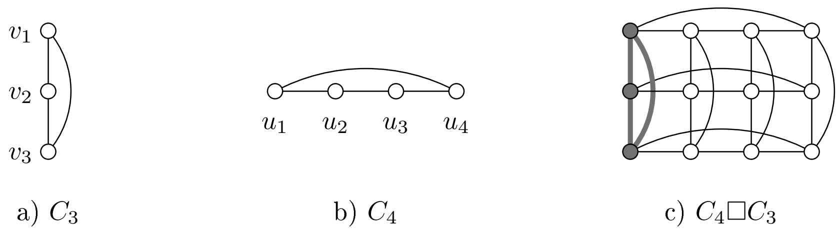


FIGURE 2.8: *a) A cycle of order 3. b) A cycle of order 4. c) The cartesian product of C_3 and C_4.*

The *corona* of two graphs G and H is the graph formed from one copy of G and $|V(G)|$ copies of H, where the i-th vertex of G is adjacent to every vertex in the i-th copy of H. The corona of two graphs G and H is denoted by $G \circ H$. The corona of C_4 and C_3 is illustrated in Figure 2.9.

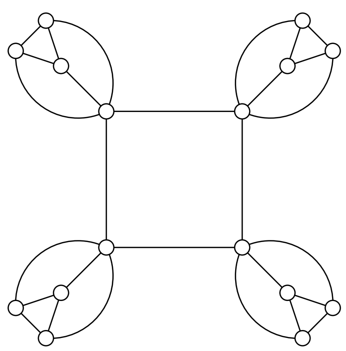


FIGURE 2.9: *The graph $C_4 \circ C_3$.*

2.2.8 Packings and dominating sets

A subset S of the vertex set of a graph G is a *2-packing* of G if, for each pair of vertices $u, v \in S$, $N[u] \cap N[v] = \emptyset$. The *packing number* of a graph G is the cardinality of a largest 2-packing of G and is denoted by $\rho(G)$.

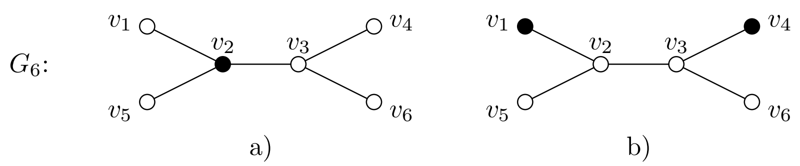


FIGURE 2.10: *The solid vertices denote two maximal 2-packings of G_6. The maximal 2-packing in a) is of minimum cardinality, while the maximal 2-packing in b) is of maximum cardinality.*

Two maximal 2-packings are illustrated in Figure 2.10. The largest cardinality of a 2-packing for G_6 is 2 (see Figure 2.10 b)) and therefore $\rho(G_6) = 2$.

A *dominating set* S of a graph G is a subset of $V(G)$ such that $N[S] = V(G)$, *i.e.* every vertex of $V(G)$ is either in S or adjacent to at least one vertex of S. Observe that domination is a super-hereditary property, since every superset of a dominating set is again dominating. It follows that a dominating set S of G is minimal dominating if and only if $S - \{s\}$ is not dominating for every $s \in S$. The *domination number* of G, denoted by $\gamma(G)$, and the *upper domination number* of G, denoted by $\Gamma(G)$, are respectively the smallest and largest cardinalities of a minimal dominating set of G.

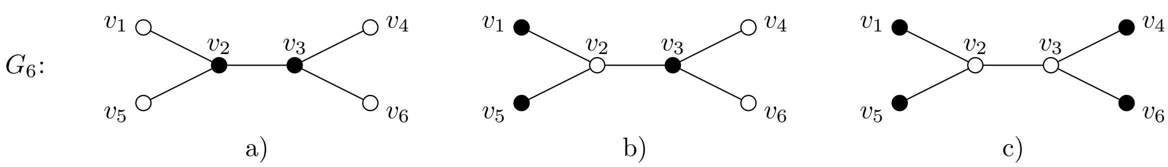


FIGURE 2.11: *The solid vertices denote three minimal dominating sets of G_6. The minimal dominating set in a) is of minimum cardinality while the minimal dominating set in c) is of maximum cardinality.*

Consider the graph G_6 in Figure 2.11. The solid vertices denote three dominating sets of G_6. Note that all three of these sets are minimal dominating sets. The dominating set in Figure 2.11 a) is of minimum cardinality and so $\gamma(G_6) = 2$. The upper domination number of G_6 is $\Gamma(G_6) = 4$ as illustrated by the minimal dominating set of maximum cardinality in Figure 2.11 c).

2.2.9 $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-Domination

Cockayne [20] introduced a general framework for domination in graphs. Let $\boldsymbol{r}$ and $\boldsymbol{s}$ be n-vectors of non-negative integers. An *$\boldsymbol{r}$-function* of a graph G of order n, with $V(G) = \{v_1, \ldots, v_n\}$, is a function $f : V(G) \mapsto \mathbb{N}_0$ satisfying $f(v_i) \leq r_i$ for all $i = 1, \ldots, n$. For every $v \in V(G)$, let $f[v] = \sum_{u \in N[v]} f(u)$. Furthermore, let $f(S) = \sum_{v \in S} f(v)$ for any subset S of $V(G)$. An $\boldsymbol{r}$-function f is called *$\boldsymbol{s}$-dominating* if $f[v_i] \geq s_i$ for each $i = 1, \ldots, n$. The *weight* of an $\boldsymbol{r}$-function f is $|f| = \sum_{v \in V} f(v) = f(V(G))$. The smallest weight of an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G is called the *$\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number* of G and is denoted by $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$. An $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G exists if and only if $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for all $i = 1, \ldots, n$.

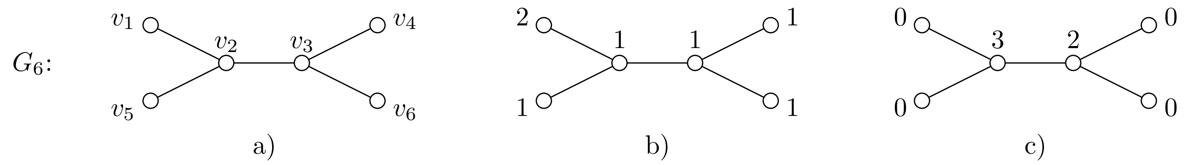


FIGURE 2.12: *The graph G_6 of order 6 and two 6-vectors $\boldsymbol{r} = [2, 1, 3, 2, 1, 1]$ and $\boldsymbol{s} = [3, 2, 4, 3, 2, 1]$. An $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G_6 of weight 7 is depicted in a), while an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G_6 of minimum weight, namely 5, is shown in b).*

Consider the graph G_6 in Figure 2.12 together with the 6-vectors $\boldsymbol{r} = [2, 1, 3, 2, 1, 1]$ and $\boldsymbol{s} = [3, 2, 4, 3, 2, 1]$. Two $\boldsymbol{s}$-dominating $\boldsymbol{r}$-functions of G_6 are shown in Figure 2.12. The function in a) has a weight of 7 while the function in b) has a weight of 5. The function in Figure 2.12 b) has the smallest possible weight that an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G_6 can achieve and therefore $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G_6) = 5$.

Note that if $\boldsymbol{r} = \boldsymbol{s} = [1, \ldots, 1]$, then the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number is the (classical) domination number, $\gamma(G)$. Other special cases of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination include *$\{k\}$-domination* and *$k$-tuple domination* which are the cases where $\boldsymbol{r} = \boldsymbol{s} = [k, \ldots, k]$, and where $\boldsymbol{r} = [1, \ldots, 1]$ and $\boldsymbol{s} = [k, \ldots, k]$, respectively, for some $k \in \mathbb{N}_0$. The $\{k\}$-domination number of a graph G is denoted by $\gamma_{\{k\}}(G)$, while the k-tuple domination number of a graph G is denoted by $\gamma_{\times k}(G)$.

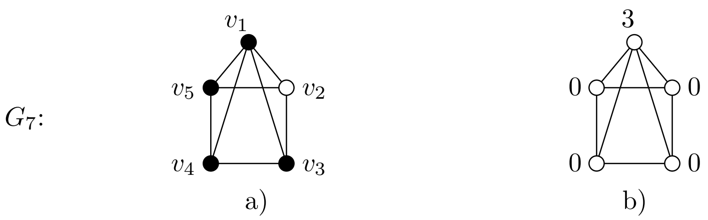


FIGURE 2.13: *The solid vertices in a) denote a minimum 3-tuple dominating set of G_7, while a $\{3\}$-dominating function of G_7 of minimum weight 3 is shown in b).*

The solid vertices in Figure 2.13 a) denote a 3-tuple dominating set of cardinality 4 of the graph G_7; this set is of minimum cardinality and therefore $\gamma_{\times 3}(G_7) = 4$. The $\{3\}$-dominating function shown in Figure 2.13 b) is of minimum weight and hence $\gamma_{\{3\}}(G_7) = 3$.

2.3 The probabilistic method

The *probabilistic method* is a non-constructive method for proving the existence of structures with certain desired properties. Define, as in [44], a *sample space* as the finite set Ω and denote an element of Ω by v. Furthermore, let $\Pr : \mathcal{P}(\Omega) \mapsto [0, 1]$ be a function that maps the powerset of the sample space onto the interval $[0, 1]$ such that $\Pr(\Omega) = 1$, $\Pr(\overline{A}) = 1 - \Pr(A)$ for any subset $A \subseteq \Omega$ and $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ for any two disjoint subsets A and B of Ω.

The sample space Ω together with the function Pr is called a *probability space*. An *event* A in the probability space is a subset of Ω and the probability of an event A is defined as $\Pr(A) = \sum_{v \in A} \Pr(v)$.

A *random variable* X on the sample space Ω is a real-valued function on Ω, that is $X : \Omega \mapsto \mathbb{R}$. If X is a random variable defined on a probability space $\Omega = \{v_1, \ldots, v_n\}$ where $\Pr(v_i) = p_i$ for all $i = 1, \ldots, n$, then the *expected value* of X is defined as $E[X] = \sum_{i=1}^{n} p_i X(v_i)$. Note that if $E[X] = x$, then there exists at least one element $v \in \Omega$ such that $X(v) \geq x$ and at least one element $u \in \Omega$ such that $X(u) \leq x$.

If $X_1, \ldots, X_r$ are random variables on a sample space Ω and $X = c_1X_1 + \cdots + c_rX_r$, then the *linearity of expectation* states that $E[X] = c_1E[X_1] + \cdots + c_rE[X_r]$. Note that the linearity of expectation does not depend on the independence or otherwise of the variables $X_1, \ldots, X_r$.

The probabilistic method is mainly used in two ways in graph theory. First, one may make use of expected values since any random variable assumes at least one value that is not smaller and at least one value that is not larger than its expectation. Secondly, if it can be shown that a random element from a universe satisfies a certain property P with positive probability, then there must exist an element in the universe that satisfies property P.

The proof of the following proposition illustrates the use of the probabilistic method to show that every graph G has a bipartite subgraph containing more than half of the edges of G.

Proposition 2.1 (De Vos [23]). *Every graph G has a bipartite subgraph H for which $|E(H)| \geq 1/2|E(G)|$.*

Proof. Let $G = (V, E)$ be a simple graph and form a set X by picking every element $v \in V(G)$ independently at random with $\Pr(v \in X) = 1/2$. Let $Y \subseteq V(G)\backslash X$ be the set of vertices with at least one neighbour in X. Furthermore, let H be the subgraph containing all the vertices in X and Y together with the set of edges $\{uv \mid u \in X, v \in Y \text{ or } v \in X, u \in Y\}$. Hence, H is a bipartite subgraph of G and for any edge $e = uv \in E(G)$,

$$\Pr(e \in E(H)) = \Pr(u \in X \text{ and } v \notin X) + \Pr(u \notin X \text{ and } v \in X) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}.$$

For every edge $e \in E(G)$, let χ_e be the indicator random variable such that $\chi_e = 1$ if $e \in E(H)$ and $\chi_e = 0$ if $e \notin E(H)$. Then

$$E[|E(H)|] = \sum_{e \in E(G)} E[\chi_e] = \sum_{e \in E(G)} \Pr(e \in E(H)) = \frac{1}{2}|E(G)|$$

and so there exists at least one bipartite subgraph H of G for which $|E(H)| \geq 1/2|E(G)|$. ■

2.4 Basic notions from complexity theory

Algorithmic complexity is the field of study focussing on measures of the efficiency of algorithms, either by estimating the number of operations executed by an algorithm, called the *time complexity* T, or by estimating the amount of memory required by an algorithm during execution, called the *space complexity* S. If n is the input size of the algorithm, then T and S are typically expressed as functions of n [44].

2.4.1 $\mathcal{O}$-notation

The time complexity $T(n)$ of an algorithm may be measured in terms of the number of basic operations required to execute the algorithm. In many algorithmic implementations it is rather difficult to enumerate exactly the number of basic operations performed or the amount of memory expended during the execution of the algorithm. Instead it is often sufficient to determine a worst-case estimate for these numbers, resulting in the so-called *worst-case complexity* of the algorithm. Moreover, *asymptotic upper bounds* are often used to describe the worst-case behaviour of the functions $T(n)$ and $S(n)$.

A function $g(n)$ is an asymptotic upper bound for the function $f(n)$ as $n \to \infty$, denoted by $f(n) = \mathcal{O}(g(n))$, if there exist constants $c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N}$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$. Consider, for example, an algorithm with space complexity $S(n) = n^2 + 2$ and let $g(n) = n^2$. Then $S(n) = \mathcal{O}(n^2)$ since $0 \leq S(n) \leq 3/2g(n)$ for all $n \geq 2$.

The $\mathcal{O}^*$-notation is similar to the usual big $\mathcal{O}$-notation, but it suppresses all polynomial factors [84]. For example, if an algorithm has time complexity $T(n) = \mathcal{O}(2^n \cdot n^2)$, then $T(n) = \mathcal{O}^*(2^n)$.

A *polynomial-time algorithm* is an algorithm with a worst-case time complexity of $\mathcal{O}(n^a)$ for some constant $a \in \mathbb{R}^+$, where n denotes the input size of the algorithm. An algorithm that is not a polynomial-time algorithm is called an *exponential-time algorithm*. A *polynomial-space algorithm* and an *exponential-space algorithm* is defined similarly.

2.4.2 The decision problem associated with an optimisation problem

A computational problem formulated as a question with only two possible solutions, `yes` or `no`, is called a *decision problem*. An *optimisation problem* is a computational problem that asks for the minimisation or maximisation of the value of some parameter. An optimisation problem may be formulated as a decision problem by introducing a parameter k and (possibly repeatedly) asking whether the optimal value of the optimisation problem is at most or at least equal to k, by varying the value of k appropriately [34].

Consider, for example, the optimisation problem of calculating the minimum cardinality of a dominating set S of a graph G. The related decision problem may be formulated as follows.

Decision Problem 2.1 (Dominating Set).

Instance: *A graph $G = (V, E)$ and a positive integer $k \leq |V(G)|$.*

Question: *Does there exist a dominating set $S \subseteq V(G)$ of G such that $|S| \leq k$?*

2.4.3 Complexity classes

An algorithm *accepts* a decision problem if the output to any instance of the problem is `yes`. The complexity class **P** is the class of all decision problems that may be accepted in polynomial time, that is, there exists an algorithm that accepts the problem in at most bn^c steps, where n is the input size to the algorithm and b and c are positive constants.

An algorithm *verifies* a decision problem if the output to an instance of the problem together with some additional information (known as a certificate) is `yes`. The complexity class **NP** is the class of all decision problems that may be verified in polynomial time. Note that the definition of the class **NP** does not include problems for which the answer is `no`. In fact, the complexity class **co-NP** comprises all problems whose complement is in the class **NP**. It is not known whether **co-NP** = **NP** or **co-NP** $\neq$ **NP**.

Let D_1 and D_2 be two decision problems. The problem D_1 is *polynomial time reducible* to D_2, denoted by $D_1 \leq_p D_2$, if there exists a function $f : D_1 \mapsto D_2$, computable in polynomial time, such that x is an instance of D_1 if and only if $f(x)$ is an instance of D_2.

Decision Problem 2.2 (Set Cover).

Instance: *A universe* $\mathcal{U}$*, a family* $\mathcal{S}$ *of subsets of* $\mathcal{U}$ *and a positive integer* $k \leq |V(G)|$.

Question: *Does there exist a subset* $S' \subseteq \mathcal{S}$ *such that* $\bigcup_{S \in S'} S = \mathcal{U}$ *with* $|S'| \leq k$*?*

To show that Decision Problem 2.1 is polynomial time reducible to Decision Problem 2.2 an instance of Decision Problem 2.1 must be transformed in polynomial time to an instance of Decision Problem 2.2. Let $G = (V, E)$ be a graph of order n and let G and $k \in \mathbb{N}$ be an instance of Decision Problem 2.1. Define an instance of Decision Problem 2.2 as the universe $\mathcal{U} = \{v_1, \ldots, v_n\}$ and the family of sets $\mathcal{S} = \{N_G[v_i] \mid v_i \in V(G)\}$. Then $D = \{v_{i_1}, \ldots, v_{i_k}\}$ is a dominating set of G if and only if $\bigcup_{S \in S'} S = \mathcal{U}$ for $S' = \{N_G[v_i] \mid v_i \in D\}$. Since this transformation is computable in polynomial time, Decision Problem 2.1 is polynomial time reducible to Decision Problem 2.2.

A problem D is said to be **NP**-complete if $D \in \mathbf{NP}$ and D' is polynomial time reducible to D for every $D' \in \mathbf{NP}$. Since $D_1 \leq_p D_3$ when $D_1 \leq_p D_2$ and $D_2 \leq_p D_3$, it is sufficient to show that there exists an **NP**-complete problem that is polynomial time reducible to $D \in \mathbf{NP}$ in order to prove that D is **NP**-complete. For example, to show that Decision Problem 2.2 is **NP**-complete, it is necessary to first show that it is in **NP**. Given a family of sets S' it can be checked in polynomial time whether $|S'| \leq k$ and whether S' covers $\mathcal{U}$, hence Decision Problem 2.2 is in **NP**. Decision Problem 2.1 is **NP**-complete [31] and is polynomial time reducible to Decision Problem 2.2, therefore Decision Problem 2.2 is **NP**-complete.

2.5 Chapter summary

The chapter opened with a brief introduction to the theory of multisets in §2.1. In §2.2 the necessary definitions of some basic concepts in graph theory were given. Various concepts related to adjacency, vertex degrees and graph isomorphisms were introduced, along with definitions of subgraphs, various operations on graphs and some special classes of graphs. The section closed with a definition of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination and its special cases.

The probabilistic method was introduced in §2.3 and illustrated by means of an example. The chapter concluded with a number of basic notions from complexity theory in §2.4 that are necessary to analyse and compare graph algorithms.

CHAPTER 3

Literature review

Contents

A literature review on topics related to graph domination is provided in this chapter. Each of the special cases of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination is considered separately and for each of these special cases, this review is presented in three parts, namely general results and bounds, special graph classes and algorithms.

3.1 Domination

Recall that domination is a super-hereditary property implying that every superset of a dominating set of a graph G is again a dominating set of G. The following proposition provides a characterisation of minimal dominating sets.

Proposition 3.1 (Ore [64]). *A dominating set S of a graph G is a minimal dominating set of G if and only if $N[s] - N[S - \{s\}] \neq \emptyset$ for every $s \in S$.* ■

3.1.1 General bounds on the domination number of a graph

There are numerous bounds on the classical domination number, the most obvious upper bound being the number of vertices in the graph. Since at least one vertex is needed to dominate a graph it is therefore clear that $1 \leq \gamma(G) \leq n$ for a graph G of order n. Both of these bounds are sharp; the upper bound is achieved only when $G = \overline{K_n}$ and the lower bound when G contains a universal vertex (a vertex of degree $n-1$). This upper bound can be improved considerably for graphs without isolated vertices. If G is a graph without isolated vertices, then the complement of any minimal dominating set of G is also a dominating set of G. The following bound therefore follows immediately.

Theorem 3.1 (Ore [64]). *If G is a graph of order n without isolated vertices, then $\gamma(G) \leq n/2$.* ■

Graphs that achieve this bound were independently characterized by Payan and Kuong [65] in 1982, and by Fink *et al.* [28] in 1985 as graphs of which the components are the cycle C_4 or the corona $H \circ K_1$, where H is any connected graph.

Ore's result of Theorem 3.1 holds for graphs with minimum degree at least 1. McCuaig and Shepherd [60] focused on graphs with minimum degree at least 2 and managed to improve the upper bound to two fifths of the order of the graph if the graph is not one of a number of forbidden graphs.

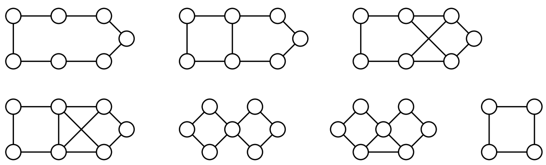


FIGURE 3.1: *The family of forbidden graphs in Theorem 3.2.*

Theorem 3.2 (McCuaig & Shepherd [60]). *If G is a connected graph of order n with minimum degree $\delta \geq 2$ and $G \notin \mathcal{A}$, where $\mathcal{A}$ is the family of graphs shown in Figure 3.1, then $\gamma(G) \leq 2n/5$.* ■

The bound in Theorem 3.2 can be improved even further for graphs with minimum degree at least 3 — this time without any forbidden graphs.

Theorem 3.3 (Reed [68]). *If G is a connected graph of order n with minimum degree $\delta \geq 3$, then $\gamma(G) \leq 3n/8$.* ■

A conjecture stated in [43] proposes that for any graph G with minimum degree at least $k \in \mathbb{N}$, $\gamma(G) \leq kn/(3k-1)$. This conjecture was settled for $\delta < 7$ in [50], [76], [87] and [89]. For the case where $\delta \geq 7$ Harant *et al.* provided the following better bound by using the probabilistic method.

Theorem 3.4 (Harant *et al.* [39]). *For any graph G of order n and minimum degree $\delta \geq 1$,*

$$\gamma(G) \leq n\left(1 - \delta\left(\frac{1}{\delta+1}\right)^{1+1/\delta}\right). \quad ■$$

Recently, Henning *et al.* [46] showed that Reed's bound of Theorem 3.3 holds for most graphs with minimum degree 2, with the exclusion of, among others, a family of forbidden graphs. To describe these forbidden graphs the following definitions are required.

Let G be a graph that contains a path $v_1u_1u_2v_2$ on four vertices such that $\deg_G(u_1) = \deg_G(u_2) = 2$. A *type-1 G-reducible graph* is a graph obtained from G by deleting $\{u_1, u_2\}$ and by identifying v_1 and v_2. If there is a path $x_1w_1w_2w_3x_2$ on 5 vertices in G such that $\deg(w_2) = 2$ and $N_G(w_1) = N_G(w_3) = \{x_1, x_2, w_2\}$, then the graph obtained by deleting $\{w_1, w_2, w_3\}$ and adding the edge x_1x_2, if this edge is not already present, is called a *type-2 G-reducible graph*.

A vertex $x \in V(G)$ is called a *bad-cut vertex* if $G - x$ contains a component C_x which is an induced cycle of order 4, where x is adjacent to at least one and at most three vertices on C_x. The number of bad-cut vertices in G is denoted by $bc(G)$.

A cycle of order 5 in G is called a *special cycle* if u and v are consecutive vertices and at least one of u or v has degree 2 in G. The maximum number of vertex-disjoint special cycles in G that contain no bad-cut vertices is denoted by $sc(G)$.

Let $\mathcal{F}_4$ be the set containing the cycle of order 4, that is, $\mathcal{F}_4 = \{C_4\}$. Now define the family $\mathcal{F}_i$ for $i > 4$ and $i \equiv 1 \pmod 3$ as follows. A graph G belongs to $\mathcal{F}_i$ if an only if $\delta \geq 2$ and there exists a type-1 or type-2 G-reducible graph that belongs to $\mathcal{F}_{i-3}$. Note that $\mathcal{F}_7 = \mathcal{A}$, the family of graphs forbidden in Theorem 3.2. Let $\mathcal{F}_{\leq 13} = \mathcal{F}_4 \cup \mathcal{F}_7 \cup \mathcal{F}_{10} \cup \mathcal{F}_{13}$. The family $\mathcal{F}$ of forbidden graphs is all the graphs in $\mathcal{F}_{\leq 13}$ that contain no bad-cut vertices, that is, $\mathcal{F} = \{G \in \mathcal{F}_{\leq 13} \mid bc(G) = 0\}$. Then the following bound holds.

Theorem 3.5 (Henning *et al.* [46]). *If G is a connected graph of order n with minimum degree $\delta \geq 2$, then $G \in \mathcal{F}$ or*

$$\gamma(G) \leq \frac{1}{8}(3|V(G)| + sc(G) + bc(G)). \qquad \blacksquare$$

It therefore follows that if a connected graph G of order n with minimum degree at least 2 has no special cycles and no bad-cut vertices, then $\gamma(G) \leq 3n/8$.

The earliest bound on $\gamma(G)$ determined by the probabilistic method is due to Alon and Spencer and dates back to 1992.

Theorem 3.6 (Alon & Spencer [1]). *For any graph G of order n and minimum degree $\delta \geq 1$,*

$$\gamma(G) \leq \frac{n(1 + \ln(\delta + 1))}{\delta + 1}.$$

Proof. Let $G = (V, E)$ be a simple graph of order n with minimum degree δ and form a set X by picking every element $v \in V(G)$ independently at random with $\Pr(v \in X) = p$ where $0 < p \leq 1$. Let Y be the set of all the vertices of G not adjacent to any vertices in X. The set $S = X \cup Y$ is a dominating set of G and hence $\gamma(G) \leq E[|S|]$. By the linearity of expectation it follows that

$$\gamma(G) \leq E[|S|] = E[|X|] + E[|Y|].$$

Since the random variable $|X|$ can be written as the sum of n indicator random variables χ_v for $v \in V(G)$ where $\chi_v = 1$ if $v \in X$ and $\chi_v = 0$ if $v \notin X$, it follows that

$$E[|X|] = \sum_{v \in V(G)} E[\chi_v] = \sum_{v \in V(G)} \Pr(v \in X) = np.$$

Similarly,

$$E[|Y|] = \sum_{v \in V(G)} \Pr(v \in Y) \leq \sum_{v \in V(G)} (1-p)^{\delta+1} = n(1-p)^{\delta+1}$$

and hence

$$\gamma(G) \leq E[|X|] + E[|Y|] \leq np + n(1-p)^{\delta+1}. \tag{3.1}$$

Furthermore, since $1 - p \leq e^{-p}$, it follows that $\gamma(G) \leq np + ne^{-p(\delta+1)}$. The right-hand side of this expression is minimised when

$$p = \frac{\ln(\delta+1)}{\delta+1}$$

and therefore

$$\gamma(G) \leq \frac{n(1+\ln(\delta+1))}{\delta+1}$$

as claimed. ∎

The actual minimum for the righthand side of (3.1) occurs when $p = 1-(1+\delta)^{-1/\delta}$; substituting this value of p into (3.1) therefore produces a better bound, namely, the bound in Theorem 3.4. Although the difference between the two bounds is substantial for small values of δ, the bounds are asymptotically equivalent to $n \ln \delta/\delta$ for large values of δ.

Walikar *et al.* [83] obtained the following lower bound on the domination number in terms of the maximum degree of a graph.

Theorem 3.7 (Walikar *et al.* [83]). *For any graph G of order n and maximum degree Δ,*

$$\gamma(G) \geq \left\lceil \frac{n}{1+\Delta} \right\rceil. \quad ∎$$

The next lower bound on the domination number follows as a result of the relationship between the packing number $\rho(G)$ and the domination number $\gamma(G)$ of a graph G.

Theorem 3.8 (Haynes *et al.* [43]). *For any graph G, $\rho(G) \leq \gamma(G)$.* ∎

The following bound on the domination number is in terms of the order and the size of the graph.

Theorem 3.9 (Berge [3]). *For any graph G of order n and size m, $\gamma(G) \geq n - m$.* ∎

3.1.2 Values of the domination number for special graph classes

Determining expressions for the domination numbers of paths, cycles, complete graphs and multipartite graphs are quite straight forward. The domination number of a path of length n is $\lceil n/3 \rceil$, while the remaining three graph classes achieve the lower bound in Theorem 3.7. Hence, for cycles $\gamma(P_n) = \gamma(C_n) = \lceil n/(1+\Delta) \rceil = \lceil n/3 \rceil$ and for complete multipartite graphs $\gamma(K_{r_1,\ldots,r_k}) = \lceil \sum_{i=1}^{k} r_i/(1+\Delta) \rceil = 2$ if $r_i > 1$ for $i = 1, \ldots, k$, while the domination number of a complete graph is 1.

The domination number of a circulant of the form $C_n \langle 1, 2, \ldots, r \rangle$ for $n \geq 3$ and $1 \leq r \leq \lfloor n/2 \rfloor$ achieves the lower bound in Theorem 3.7 [35], that is $\gamma(C_n \langle 1, 2, \ldots, r \rangle) = \lceil n/(1+2r) \rceil$.

Grobler [35] also established the following exact values for circulants of the form $G = C_n \langle 1, 3, \ldots, 2r-1 \rangle$, where $1 \leq r \leq \lfloor n/2 \rfloor$.

Proposition 3.2 (Grobler [35]). *Let $G = C_n\langle 1, 3, \ldots, 2r-1\rangle$ be a circulant, where $1 \leq r \leq (n-1)/2$, and let $n = (2r+1)m + q$ for some integer m, where $0 \leq q \leq 2r$.*

1. *If $q = 0$, then $\gamma(G) = m$.*
2. *If $q = 2$, then $\gamma(G) = m + 1$.*
3. *If q is odd, then $\gamma(G) = m + 1$.* ■

By using dynamic programming and periodicity, Spalding [77] determined the domination number of $C_n\langle a_1, \ldots, a_k\rangle$ for all possible subsets $\{a_1, \ldots, a_k\}$ of $\{1, \ldots, 9\}$. There are 512 such non-isomorphic circulants and their values are listed in [77, p. 112–174].

The main open problem related to the domination number of the cartesian product of graphs is one posed by Vizing [82]. In 1963 Vizing conjectured that the domination number of the cartesian product of two graphs is at least the product of the domination numbers of the two graphs.

Conjecture 3.1 (Vizing [82]). *For any graphs G and H,*

$$\gamma(G \square H) \geq \gamma(G)\gamma(H). \tag{3.2}$$

A graph G is said to satisfy Vizing's Conjecture if the inequality (3.2) holds for every graph H. The first significant breakthrough with respect to settling Vizing's conjecture is due to Barcalkin and German [2]. They provided a large class of graphs, called BG-graphs, that satisfy Vizing's conjecture. A graph is called a *decomposable graph* if it may be covered by $\gamma(G)$ cliques. If G is a spanning graph of a decomposable graph G' such that $\gamma(G) = \gamma(G')$, then G is a BG-graph and it satisfies Vizing's conjecture. The class of BG-graphs includes all graphs G for which $\gamma(G) = 2$ or $\gamma(G) = \rho(G)$. The class of BG-graphs were extended to Type $\mathcal{X}$ graphs by Hartnell and Rall [42] in 1995. This new class of graphs also included all graphs G for which $\gamma(G) - 1 = \rho(G)$. Recently Brešar and Rall [9] introduced the concept of fair reception which also generalises the class of BG-graphs. It was shown by Brešar [6] and Sun [78] that all graphs with domination number 3 satisfy Vizing's conjecture. Clark and Suen [18] used what is called the *double-projection argument* to show that

$$\gamma(G \square H) \geq \frac{1}{2}\gamma(G)\gamma(H) \tag{3.3}$$

for all graphs G and H. Vizing also established an upper bound in terms of the order of the graphs and their domination numbers.

Proposition 3.3 (Vizing [82]). *For any graph G of order n_G and any graph H of order n_H,*

$$\gamma(G \square H) \leq \min\{\gamma(G)n_H, \gamma(H)n_G\}.$$ ■

In 1991 El-Zahar and Pareek [25] proposed the following lower bound on the domination number of a cartesian product in terms of the orders of the graphs.

Proposition 3.4 (El-Zahar & Pareek [25]). *For any graph G of order n_G and any graph H of order n_H, $\gamma(G \square H) \geq \min\{n_G, n_H\}$.* ■

A survey of results established and progress made towards settling Vizing's conjecture may be found in [8].

This section closes with a summary of known results for the domination number of cartesian products of complete graphs, paths and cycles. The domination number of the cartesian product of two complete graphs was settled by Grobler [35] in 1998.

Proposition 3.5 (Grobler [35]). *$\gamma(K_n \Box K_m) = n$ for any $m \geq n \geq 2$.* ■

The first results on the domination numbers of the cartesian products of paths were established in 1983 by Jacobson and Kinch [49]. They obtained exact values of $\gamma(P_n \Box P_m)$ for the cases $n = 1, 2, 3, 4$. The values of $\gamma(P_n \Box P_m)$ for $n = 1, 2, 3$ were also established independently by Cockayne *et al.* [22] in 1985. In 1993 Chang and Clark [15] found exact values for $\gamma(P_5 \Box P_m)$ and $\gamma(P_6 \Box P_m)$. The cases where $n = 7, \ldots, 15$ were settled by Spalding and Fischer [77] in 1998. The next result contains a summary of the values of the domination number of $P_n \Box P_m$ for $n = 2, \ldots, 15$ and for all $m \geq 2$.

Theorem 3.10. *Let $m \geq n \geq 2$. Then*

$$\gamma(P_n \Box P_m) = \begin{cases} \left\lceil \frac{m+1}{2} \right\rceil & \text{if } n = 2 \\ \left\lceil \frac{3m+1}{4} \right\rceil & \text{if } n = 3 \\ m & \text{if } n = 4 \text{ and } m \neq 5, 6, 9 \\ m+1 & \text{if } n = 4 \text{ and } m = 5, 6, 9 \\ \left\lceil \frac{6m+4}{5} \right\rceil & \text{if } n = 5 \text{ and } m \neq 7 \\ \left\lceil \frac{6m+4}{5} \right\rceil - 1 & \text{if } n = 5 \text{ and } m = 7 \\ \left\lceil \frac{10m+4}{7} \right\rceil & \text{if } n = 6 \text{ and } m \not\equiv 3 \pmod 7 \text{ or } m = 3 \\ \left\lceil \frac{10m+4}{7} \right\rceil + 1 & \text{if } n = 6 \text{ and } m \equiv 3 \pmod 7 \text{ and } m \neq 3 \\ \left\lceil \frac{5m+1}{3} \right\rceil & \text{if } n = 7 \\ \left\lceil \frac{15m+7}{8} \right\rceil & \text{if } n = 8 \\ \left\lceil \frac{23m+10}{11} \right\rceil & \text{if } n = 9 \\ \left\lceil \frac{30m+15}{13} \right\rceil & \text{if } n = 10 \text{ and } m \not\equiv 10 \pmod{13} \text{ and } m \neq 13, 16 \\ \left\lceil \frac{30m+15}{13} \right\rceil - 1 & \text{if } n = 10 \text{ and } m \equiv 10 \pmod{13} \text{ or } m = 13, 16 \\ \left\lceil \frac{38m+22}{15} \right\rceil & \text{if } n = 11 \text{ and } m \neq 11, 18, 20, 22, 23 \\ \left\lceil \frac{38m+22}{15} \right\rceil - 1 & \text{if } n = 11 \text{ and } m = 11, 18, 20, 22, 23 \\ \left\lceil \frac{80m+38}{29} \right\rceil & \text{if } n = 12 \\ \left\lceil \frac{98m+54}{33} \right\rceil & \text{if } n = 13 \text{ and } m \not\equiv 13, 16, 18, 19 \pmod{33} \\ \left\lceil \frac{98m+54}{33} \right\rceil - 1 & \text{if } n = 13 \text{ and } m \equiv 13, 16, 18, 19 \pmod{33} \\ \left\lceil \frac{35m+20}{11} \right\rceil & \text{if } n = 14 \text{ and } m \not\equiv 7 \pmod{22} \\ \left\lceil \frac{35m+20}{11} \right\rceil - 1 & \text{if } n = 14 \text{ and } m \equiv 7 \pmod{22} \\ \left\lceil \frac{44m+28}{13} \right\rceil & \text{if } n = 15 \text{ and } m \not\equiv 5 \pmod{26} \\ \left\lceil \frac{44m+28}{13} \right\rceil - 1 & \text{if } n = 15 \text{ and } m \equiv 5 \pmod{26}. \end{cases}$$ ■

Chang [14] conjectured the following result for the cartesian product of paths of order at least 16.

Conjecture 3.2 (Chang [14]). *For all $m, n \geq 16$,*

$$\gamma(P_n \Box P_m) = \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4.$$ ■

To motivate this conjecture, Chang proposed constructions of dominating sets for $P_n \Box P_m$ achieving this value.

Proposition 3.6 (Chang [14]). *For all* $8 \leq n \leq m$,

$$\gamma(P_n \Box P_m) \leq \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4.$$

■

In 2004 Guichard [36] showed that $\gamma(P_n \Box P_m) \geq \lfloor (n+2)(m+2)/5 \rfloor - 9$ for $m \geq n \geq 16$ and in 2011 Gonçlaves *et al.* [33] proved Conjecture 3.2.

Since the graph $P_n \Box P_m$ is a spanning graph of both $P_n \Box C_m$ and $C_n \Box C_m$, the domination numbers of $C_n \Box C_m$ and $P_n \Box C_m$ are bounded from below by $\lceil mn/5 \rceil$ and from above by $\lfloor (n+2)(m+2)/5 \rfloor - 4$ when $m, n \geq 16$.

In 2011 Nandi *et al.* [62] established the exact values of the domination number of $P_n \Box C_m$ for the cases $n = 2, 3$ or 4, as well as an upper and lower bound on the domination number of $P_5 \Box C_m$. Their results are summarised below.

Theorem 3.11 (Nandi *et al.* [62]). *Let* $n \leq 5$. *Then*

$$\gamma(P_n \Box C_m) = \begin{cases} \lceil \frac{m+1}{2} \rceil & \text{if } n = 2 \text{ and } m \not\equiv 0 \pmod 4 \\ \frac{m}{2} & \text{if } n = 2 \text{ and } m \equiv 0 \pmod 4 \\ \lceil \frac{3m}{4} \rceil & \text{if } n = 3 \\ m & \text{if } n = 4 \text{ and } m \neq 3, 5, 9 \\ m+1 & \text{if } n = 4 \text{ and } m = 3, 5, 9. \end{cases}$$

Also, $\gamma(P_5 \Box C_3) = 4$, $\gamma(P_5 \Box C_4) = 5$ *and* $\gamma(P_5 \Box C_5) = 7$. *Furthermore, if* $m \geq 6$, *then*

$$m + \lceil m/5 \rceil \leq \gamma(P_5 \Box C_m) \leq m + \lceil m/4 \rceil.$$

■

Klavžar and Seifter [52] proposed the first results on the domination number of the cartesian product of cycles by finding exact values for $\gamma(C_n \Box C_m)$ in the cases where $n = 3$ or 4 as well as a partial result for $n = 5$. The case where $n = 5$ and $m \equiv 3 \pmod 5$ was settled by Xiang *et al.* [86]. El-Zahar and Shaheen [26, 27] established results for $\gamma(C_n \Box C_m)$ in the cases where $n = 6, 7$ or 9, as well as partial results for $n = 8$, and Shaheen [75] extended this work to $n = 10$ in 2000. All of these results are summarised in the following proposition.

Proposition 3.7. *Let $m \geq n$ with $n \leq 10$. Then*

$$\gamma(C_n \Box C_m) = \begin{cases} \lceil \frac{3m}{4} \rceil & \text{if } n = 3 \\ m & \text{if } n = 4 \\ m & \text{if } n = 5 \text{ and } m \equiv 0 \pmod 5 \\ m+1 & \text{if } n = 5 \text{ and } m \equiv 1,2,4 \pmod 5 \\ m+2 & \text{if } n = 5 \text{ and } m \equiv 3 \pmod 5 \\ \lceil \frac{4m}{3} \rceil & \text{if } n = 6 \text{ and } m \equiv 0,1,4 \pmod 6 \text{ or } m \equiv 5 \pmod{18} \\ \lceil \frac{4m}{3} \rceil + 1 & \text{if } n = 6 \text{ and } m \equiv 2,3,5 \pmod 6 \text{ and } m \not\equiv 5 \pmod{18} \\ \lceil \frac{3m}{2} \rceil & \text{if } n = 7 \text{ and } m \equiv 0,5,9 \pmod{14} \\ \lceil \frac{3m}{2} \rceil + 1 & \text{if } n = 7 \text{ and } m \equiv 1,3,4,6,7,10,11,13 \pmod 6 \\ \lceil \frac{3m}{2} \rceil + 2 & \text{if } n = 7 \text{ and } m \equiv 2,8,12 \pmod 6 \\ \lceil \frac{9m}{5} \rceil & \text{if } n = 8 \text{ and } m \equiv 0,4,9 \pmod{10} \\ & \text{or } m \equiv 13,18,22,23,31,32 \pmod{40} \\ 2m & \text{if } n = 9 \text{ and } m \neq 11,13 \\ 2m+1 & \text{if } n = 9 \text{ and } m = 11,13 \\ 2m & \text{if } n = 10 \text{ and } 0 \pmod 5 \\ 2m+2 & \text{if } n = 10 \text{ and } 1,2,4 \pmod 5 \\ 2m+4 & \text{if } n = 10 \text{ and } 3 \pmod 5. \end{cases}$$

■

For the cases where $n = 8$ and $m \not\equiv 0,4,9 \pmod{10}$ and $m \not\equiv 13,18,22,23,31,32 \pmod{40}$ El-Zahar and Shaheen [27] showed that $\lceil \frac{9m}{5} \rceil \leq \gamma(C_8 \Box C_m) \leq \lceil \frac{9m}{5} \rceil + 1$.

The values of the domination number of $C_n \Box P_m$ were determined for $n \leq 12$ in a computer algorithmic study by Hare and Hare [41]. These results for $n \leq 5$, were later also established analytically by Hare and Hare.

Theorem 3.12 (Hare & Hare [41]). *Let $n \leq 5$. Then*

$$\gamma(C_n \Box P_m) = \begin{cases} \lceil \frac{3m}{4} \rceil + 1 & \text{if } n = 3 \text{ and } m \equiv 0 \pmod 4 \\ \lceil \frac{3m}{4} \rceil & \text{if } n = 3 \text{ and } m \not\equiv 0 \pmod 4 \\ m & \text{if } n = 4 \\ m+2 & \text{if } n = 5 \text{ and } m \geq 5. \end{cases}$$

■

The values obtained in the computer algorithmic study by Hare and Hare are summarised in the final proposition of this section.

Proposition 3.8 (Hare & Hare [41]).

$$\gamma(C_n \Box P_m) = \begin{cases} m + \left\lceil \frac{m+2}{3} \right\rceil & \textit{if } n = 6 \textit{ and } 2 \leq m \leq 100 \\ m + 1 + \left\lceil \frac{m+1}{2} \right\rceil & \textit{if } n = 7 \textit{ and } 5 \leq m \leq 100 \\ 2m - \left\lfloor \frac{m-1}{5} \right\rfloor & \textit{if } n = 8 \textit{ and } m \equiv 5 \pmod{10} \textit{ with } 5 \leq m \leq 500 \\ m + 1 - \left\lfloor \frac{m-1}{5} \right\rfloor & \textit{if } n = 8 \textit{ and } m \not\equiv 5 \pmod{10} \textit{ with } 5 \leq m \leq 500 \\ 2m + 2 & \textit{if } n = 9 \textit{ and } 4 \leq m \leq 100 \\ 2m + 4 & \textit{if } n = 10 \textit{ and } 10 \leq m \leq 100 \\ 2m + 2 + \left\lceil \frac{m+2}{3} \right\rceil & \textit{if } n = 11 \textit{ and } 12 \leq m \leq 84 \\ 2m + 3 + \left\lceil \frac{m+1}{2} \right\rceil & \textit{if } n = 12 \textit{ and } 11 \leq m \leq 16. \end{cases}$$

■

3.1.3 Algorithms for computing the domination number of a graph

The problem of computing the domination number of a graph has the following decision problem associated with it.

Decision Problem 3.1 (Dominating Set).

Instance: *A graph $G = (V, E)$ and a positive integer $k \leq |V(G)|$.*

Question: *Does there exist a dominating set $S \subseteq V(G)$ of G such that $|S| \leq k$?*

Decision Problem 3.1 was first shown to be **NP**-complete by Garey and Johnson [31] in 1979. Cockayne *et al.* [21] showed, however, that when Decision Problem 3.1 is restricted to trees, it may be solved in polynomial time. The problem may be reduced to the well-known set cover decision problem, formulated below.

Decision Problem 3.2 (Set Cover).

Instance: *A universe $\mathcal{U}$, a family $\mathcal{S}$ of subsets of $\mathcal{U}$ and a positive integer $k \leq |V(G)|$.*

Question: *Does there exist a subset $S' \subseteq \mathcal{S}$ such that $\bigcup_{S \in S'} S = \mathcal{U}$ with $|S'| \leq k$?*

Given a set $\mathcal{U} = \{1, \ldots, n\}$, called a *universe*, and a family of sets $\mathcal{S}$ such that $\bigcup_{S \in \mathcal{S}} S = \mathcal{U}$, a collection of elements of $\mathcal{S}$ that cover $\mathcal{U}$ is called a *set cover* of $\mathcal{U}$.

The objective of the related opimisation problem, *minimum set cover problem* (MSC), is to find a set cover of $\mathcal{U}$ of minimum cardinality. Consider, as an example, the universe $\mathcal{U} = \{1, 2, 3, 4\}$ and the family of sets $\mathcal{S} = \{\{1, 2\}, \{2, 3, 4\}, \{1, 3\}, \{2, 4\}\}$. The subset $\{\{1, 2\}, \{1, 3\}, \{2, 4\}\}$ of $\mathcal{S}$ forms a cover of $\mathcal{U}$ since $\{1, 2\} \cup \{1, 3\} \cup \{2, 4\} = \mathcal{U}$. This is, however, not a minimum set cover since $\mathcal{U}$ is also covered by the family of sets $\{\{1, 2\}, \{2, 3, 4\}\}$.

To translate Decision Problem 3.1 to Decision Problem 3.2, let the universe $\mathcal{U}$ be the vertex set of G and let the family of sets $\mathcal{S}$ consist of the closed neighbourhoods $N[v]$ of v, for all $v \in V(G)$. The set cover problem may be solved by the trivial branch-and-reduce algorithm given in pseudo-code form as Algorithm 3.1.

The currently fastest exact algorithm for computing the domination number of an arbitrary graph follows a more sophisticated branch-and-reduce approach and solves the domination problem in $\mathcal{O}(1.4969^n)$ time [80]. Rooij and Bodleander [80] made use of the set cover problem to model

Algorithm 3.1: MSC($\mathcal{S}$,$\mathcal{U}$) A trivial set cover algorithm

Input : A set cover instance ($\mathcal{S}$,$\mathcal{U}$).
Output: A minimum set cover of ($\mathcal{S}$,$\mathcal{U}$).

if $\mathcal{S} = \emptyset$ *and* $\mathcal{U} \neq \emptyset$ **then**
  return False
else
  if $\mathcal{S} = \emptyset$ **then**
    return $\emptyset$
  else
    Let $S \in \mathcal{S}$ be a set of maximum cardinality in $\mathcal{S}$.
    $A_1 = \{S\} \cup \text{MSC}(\{S' \backslash S \mid S' \in \mathcal{S} \backslash \{S\}\}, \mathcal{U} \backslash S)$
    $A_2 = \text{MSC}(\mathcal{S} \backslash \{S\}, \mathcal{U})$
return *The smallest family of sets from* A_1 *and* A_2.

Decision Problem 3.1 and used the *measure-and-conquer* approach, introduced by Fomin *et al.* [29], to improve the basic branch-and-reduce approach of Algorithm 3.1.

Note that the minimum dominating set problem can also be formulated as the following integer programming problem. Let $\boldsymbol{x} = [x_1, \ldots, x_n]$ be the characteristic vector for a dominating set S of a graph G of order n. Then the objective is to

$$\text{minimise} \qquad \sum_{i=1}^{n} x_i \tag{3.4}$$

subject to the constraints

$$\sum_{v_j \in N[v_i]} x_i \geq 1, \qquad i = 1, \ldots, n, \tag{3.5}$$

$$x_i \in \{0, 1\}, \quad i = 1, \ldots, n. \tag{3.6}$$

3.2 k-Tuple domination

Another special case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination is k-tuple domination, namely the case where $\boldsymbol{r} = [1, \ldots, 1]$ and $\boldsymbol{s} = [k, \ldots, k]$. Note that the k-tuple domination number exists only when $k \leq \delta+1$. Harary and Haynes [40] introduced k-tuple domination in 2000 by defining a *k-tuple dominating set D* of G as a subset of $V(G)$ such that $|N[v] \cap D| \geq k$ for all $v \in V(G)$.

3.2.1 General bounds on the k-tuple domination number of a graph

Harary and Haynes [40] presented the following generalisation of the lower bound in Theorem 3.7.

Theorem 3.13 (Harary & Haynes [40]). *If $k \in \mathbb{N}$ and G is a graph of order n with maximum degree Δ and minimum degree $\delta \geq k-1$, then*

$$\gamma_{\times k}(G) \geq \frac{kn}{\Delta+1}. \qquad \blacksquare$$

By counting the number of edges between the vertices of G and a k-tuple dominating set of G, Harary and Haynes proposed the following lower bound on $\gamma_{\times k}(G)$ in terms of the order and size of G.

Theorem 3.14 (Harary & Haynes [40]). *If $k \in \mathbb{N}$ and G is a graph of order n and size m with minimum degree $\delta \geq k-1$, then*

$$\gamma_{\times k}(G) \geq \frac{2kn - 2m}{k+1}. \qquad \blacksquare$$

If $k = 2$, then the k-tuple domination number is referred to as the *double domination number* of a graph. Harary and Haynes [40] established the following upper bound on the double domination number of a graph with minimum degree at least 2.

Theorem 3.15 (Harary & Haynes [40]). *If G is a graph of order n with minimum degree $\delta \geq 2$, then*

$$\gamma_{\times 2}(G) \leq \begin{cases} \lfloor n/2 \rfloor + \gamma(G) & \textit{if } n = 3, 5 \\ \lfloor n/2 \rfloor + \gamma(G) - 1 & \textit{otherwise.} \end{cases} \qquad \blacksquare$$

Henning [45] showed that the double domination number of a graph of order n with minimum degree at least 2 is, in fact, bounded from above by $3n/4$.

Theorem 3.16 (Henning [45]). *If $G \neq C_5$ is a connected graph of order n with minimum degree $\delta \geq 2$, then $\gamma_{\times 2}(G) \leq 3n/4$.* $\blacksquare$

In 2005 Henning and Harant [38] used the probabilistic method to show that

$$\gamma_{\times 2}(G) \leq n \left(\frac{\ln(1+d) + \ln \delta + 1}{\delta} \right)$$

for any graph G of order n, where $d = \frac{1}{n} \sum_{v \in V} \mathrm{d}_v$. This idea was further generalised in 2007 to k-tuple domination with the following conjecture posed by Rautenbach and Volkmann [67].

Conjecture 3.3 (Rautenbach & Volkmann [67]). *If $k \in \mathbb{N}$ and G is a graph of order n and minimum degree $\delta \geq k$, then*

$$\gamma_{\times k}(G) \leq \frac{n}{\delta + 2 - k} \left(\ln(\delta + 2 - k) + \ln \left(\sum_{v \in V(G)} \binom{\mathrm{d}_v + 1}{k-1} \right) - \ln n + 1 \right).$$

The special case of Conjecture 3.3 where $k = 3$ was established by Rautenbach and Volkmann [67] themselves, while the general conjecture was proven to be correct by Chang [12], Xu *et al.* [88] and Zverovich [90] independently in 2008.

Rautenbach and Volkmann [67] also established another upper bound on k-tuple domination, namely

$$\gamma_{\times k}(G) \leq \frac{n}{\delta + 1} \left(k \ln((\delta + 1)) + \sum_{i=0}^{k-1} \frac{(k-i)}{i!(\delta+1)^{k-i-1}} \right) \tag{3.7}$$

for any graph G of order n with minimum degree δ satisfying $2k \leq (\delta + 1)/\ln(\delta + 1)$.

Recently Gagarin *et al.* [30] improved both of the aforementioned bounds by proposing the following bound, generalised from Theorems 3.4 and 3.6, which is independent of the degree sequence of G.

Theorem 3.17 (Gagarin *et al.* [30]). *Let* $\delta' = \delta - k + 1$ *and* $\tilde{b}_t = \binom{\delta+1}{t}$. *For any graph* G *of order* n *and minimum degree* $\delta \geq k$,

$$\gamma_{\times k}(G) \leq n\left(1 - \frac{\delta'}{\tilde{b}_{k-1}^{1/\delta'}(1+\delta')^{1+1/\delta'}}\right).$$

■

The next probabilistic bound, again independent of the degree sequence of G, was proposed by Przybyło [66] in 2013.

Theorem 3.18 (Przybyło [66]). *If* $k \in \mathbb{N}$ *and* G *is any graph of order* n *with minimum degree* $\delta \geq k - 1$ *satisfying*

$$k \leq \frac{\delta + 2 - k}{\ln(\delta + 2 - k) + 1},$$

then

$$\gamma_{\times k}(G) \leq n\left(\sum_{i=1}^{k} \frac{\ln(\delta + 2 - i) + 1}{\delta + 2 - i}\right).$$

■

Note that there exist graphs for which this bound is better than the bound in Conjecture 3.3.

3.2.2 Values of the k-tuple domination number for special graph classes

The value of $\gamma_{\times 2}(C_n)$ was established by Harary and Haynes [40], showing that $\gamma_{\times 2}(C_n) = \lceil 2n/3 \rceil$ for $n \geq 3$. Using a similar approach, Goddard and Henning [32] showed that $\gamma_{\times 2}(P_n) = \lceil 2(n+1)/3 \rceil$ for $n \geq 2$. Clearly, $\gamma_{\times k}(K_n) = k$ for $k \leq n$. Haynes *et al.* [43] observed that $\gamma_{\times 2}(K_{a,b}) = 4$ for $a, b \geq 3$. The following lower bound on the k-tuple domination number of a bipartite graph was established in 2013.

Proposition 3.9 (Kazemi [51]). *If* G *is a bipartite graph with minimum degree* $\delta \geq k - 1 \geq 1$, *then* $\gamma_{\times k} \geq 2k - 2$, *with equality being achieved if and only if* $G = K_{k-1,k-1}$.

Mehri and Mirnia [61] established the values of $\gamma_{\times k}(P_n \Box P_m)$ for the cases $n = 2, 3$ or 4, and for $k = 3$ (note that $k \leq 3$).

Proposition 3.10 (Mehri & Mirnia [61]). $\gamma_{\times 3}(P_2 \Box P_m) = \left\lceil \frac{3(m+1)}{2} \right\rceil$ *for all* $m \geq 3$. ■

Mehri and Mirnia also proposed the following result on the 3-tuple domination number of $P_3 \Box P_m$.

Proposition 3.11 (Mehri & Mirnia [61]). $\gamma_{\times 3}(P_3 \Box P_m) = \left\lceil \frac{9m - \lfloor \frac{m-2}{5} \rfloor + 4}{4} \right\rceil$ *for all* $m \geq 3$. ■

The value of $\gamma_{\times 3}(P_4 \Box P_m)$ for $m \geq 4$ is also due to Mehri and Mirnia and is presented next.

Proposition 3.12 (Mehri & Mirnia [61]). *For all* $m \geq 4$,

$$\gamma_{\times 3}(P_4 \Box P_m) = \left\lceil \frac{11m - \lfloor \frac{m-4}{5} \rfloor + 9}{4} \right\rceil.$$

■

3.2.3 Algorithms for computing the k-tuple domination number of a graph

The problem of computing the k-tuple domination number of a graph is associated with the following decision problem.

Decision Problem 3.3 (k-Tuple Dominating Set).

Instance: *A graph $G = (V, E)$ with minimum degree δ, an integer $k \leq \delta + 1$ and a positive integer $\ell \leq |V(G)|$.*

Question: *Does there exist a k-tuple dominating set $S \subseteq V(G)$ of G such that $|S| \leq \ell$?*

Decision Problem 3.3 is **NP**-complete since Decision Problem 3.1, which is **NP**-complete, is a special case of Decision Problem 3.3. Liao and Chang [57], however, provided a linear-time algorithm for solving the 2-tuple dominating set problem for trees.

3.3 $\{k\}$-Domination

Another special case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination is $\{k\}$-domination, that is the case where $\boldsymbol{r} = \boldsymbol{s} = [k, \ldots, k]$. Domke *et al.* [24] established the relationship $k\rho(G) \leq \gamma_{\{k\}}(G) \leq k\gamma(G)$, between the $\{k\}$-domination number $\gamma_{\{k\}}(G)$, the domination number $\gamma(G)$ and the packing number $\rho(G)$ of any graph G, while Rubalcaba and Slater [72] noted that $\gamma_{\times k}(G) \geq \gamma_{\{k\}}(G)$ for any graph G.

3.3.1 Values of the $\{k\}$-domination number for special graph classes

The $\{k\}$-domination numbers of cycles and paths were establish by Lee and Chang [56] in 2008. They showed that $\gamma_{\{k\}}(C_n) = \lceil kn/3 \rceil$ and $\gamma_{\{k\}}(P_n) = k\lceil n/3 \rceil$ for any $k, n \in \mathbb{N}$. It is easy to see that the $\{k\}$-domination number of any complete graph is k.

The earliest Vizing-like result for $\{k\}$-domination is due to Brešar *et al.* [7] and follows from inequality (3.3). This result states that for any graphs G and H,

$$\gamma(G \Box H) \geq \frac{1}{2k^2} \gamma_{\{k\}}(G) \gamma_{\{k\}}(H).$$

Generalising the result of Clark and Suen [18], Brešar *et al.* also established the following relationship between the $\{k\}$-domination number of the cartesian product of two graphs and the $\{k\}$-domination numbers of the two graphs.

Theorem 3.19 (Brešar *et al.* [7]). *For any graphs G and H and any $k \in \mathbb{N}$,*

$$\gamma_{\{k\}}(G \Box H) \geq \frac{1}{k(k+1)} \gamma_{\{k\}}(G) \gamma_{\{k\}}(H). \qquad \blacksquare$$

Recently Choudhary *et al.* [16] used similar techniques to those of Clark and Suen [18] together with specific properties of binary matrices to improve the above inequality as follows.

Theorem 3.20 (Choudhary *et al.* [16]). *For any graphs G and H and any $k \in \mathbb{N}$,*

$$\gamma_{\{k\}}(G \Box H) \geq \frac{1}{2k} \gamma_{\{k\}}(G) \gamma_{\{k\}}(H). \qquad \blacksquare$$

In 2009 Hou and Lu [47] established the following lower bound on $\gamma_{\{k\}}(G \Box H)$ in terms of their packing numbers and $\{k\}$-domination numbers.

Theorem 3.21 (Hou & Lu [47]). *For any graphs G and H and any $k \in \mathbb{N}$,*

$$\gamma_{\{k\}}(G \Box H) \geq \max\{\rho(G)\gamma_{\{k\}}(H), \rho(H)\gamma_{\{k\}}(G)\}.$$ ■

Hou and Lu [47] also formulated the following conjecture which is a generalisation of Vizing's conjecture.

Conjecture 3.4 (Hou & Lu [47]). *For any graphs G and H and any $k \in \mathbb{N}$,*

$$\gamma_{\{k\}}(G \Box H) \geq \frac{1}{k}\gamma_{\{k\}}(G)\gamma_{\{k\}}(H).$$

Hou and Lu [47] showed that Conjecture 3.4 is true for $k = 2$ and for all graphs G for which $\rho(G) = \gamma(G)$.

3.3.2 Algorithms for computing the $\{k\}$-domination number of a graph

The problem of computing the $\{k\}$-domination number of a graph is associated with the following decision problem.

Decision Problem 3.4 ($\{k\}$-Dominating Set).

Instance: *A graph $G = (V, E)$ and positive integers k and $\ell \leq |V(G)|$.*

Question: *Does there exist an $\boldsymbol{k}$-dominating $\boldsymbol{k}$-function f of G such that $|f| \leq \ell$?*

Similar to Decision Problem 3.3, it is clear that Decision Problem 3.4 is also **NP**-complete. There are no exact algorithms in the literature specifically for computing the $\{k\}$-domination number of an arbitrary graph. There are, however, a number of approximation algorithms in [4, 5, 53, 81] for computing bounds on the $\{k\}$-domination number of an arbitrary graph.

3.4 $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-Domination

Consider the *graph triple* $(G, \boldsymbol{r}, \boldsymbol{s})$ together with an $\boldsymbol{r}$-function f. As in [20], let

$$\begin{aligned}
P_f &= \{v \in V(G) \mid f(v) > 0\}, \\
A_f &= \{v_i \in V(G) \mid f[v_i] \leq s_i\}, \\
B_f &= \{v_i \in V(G) \mid f[v_i] = s_i\} \text{ and} \\
C_f &= \{v_i \in V(G) \mid f[v_i] \geq s_i\}.
\end{aligned}$$

In terms of these subsets of $V(G)$, the $\boldsymbol{r}$-function f is $\boldsymbol{s}$-dominating if $V(G) = C_f$. Cockayne [20] showed that an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function f is minimal if and only if $N[v] \cap B_f \neq \emptyset$ for all $v \in P_f$.

The $\boldsymbol{r}$-function f of a graph G is called an $\boldsymbol{s}$*-packing* if $V(G) = A_f$. The $\boldsymbol{s}$-packing property is hereditary and the largest weight of a maximal $\boldsymbol{s}$-packing $\boldsymbol{r}$-function of G is denoted by $\rho_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$.

3.4.1 General bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph

The earliest lower bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph G is in terms of $\rho_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$ and was established by Shepherd, as cited in [20], in 2006.

Theorem 3.22 (Shepherd, as cited in [20]). *Let $(G, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple where $\boldsymbol{r} = [r_1, \dots, r_n]$ and $\boldsymbol{s} = [s_1, \dots, s_n]$ satisfy $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for $i = 1, \dots, n$. If $s_i \geq 1$ for $i = 1, \dots, n$, then $\rho_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$.* ■

In [11] Burger and Van Vuuren defined, for a graph triple $(G, \boldsymbol{r}, \boldsymbol{s})$ satisfying $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for all $i = 1, \dots, n$, the *constraint difference* T_j and *constraint slackness* T_j^* of a vertex $v_j \in V(G)$ as

$$T_j = \left(\sum_{v_k \in N[v_j]} r_k \right) - s_j, \qquad j = 1, \dots, n$$

and

$$T_j^* = \min \left\{ \min_{v_k \in N[v_j]} \{T_k\}, r_j \right\}, \qquad j = 1, \dots, n,$$

respectively. The constraint slackness T_j^* may be interpreted as the maximum difference between r_j and the value of any $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G at $v_j \in V(G)$. Let $\boldsymbol{T^*} = [T_1^*, \dots, T_n^*]$.

A graph triple $(G, \boldsymbol{r}, \boldsymbol{s})$ is called *non-reducible* if $\boldsymbol{T^*} = \boldsymbol{r}$ and *reducible* otherwise. If $(G, \boldsymbol{r}, \boldsymbol{s})$ is non-reducible, then $s_j \leq \sum_A r_k$, where $A \subseteq V(G)$ with $|A| = \deg(v_j)$, for each $j = 1, \dots, n$. The following result follows from the definition of a reducible graph triple.

Proposition 3.13. *Let $\boldsymbol{T^*}$ be the constraint slackness of the graph triple $(G, \boldsymbol{r}, \boldsymbol{s})$ where $\boldsymbol{r} = [r_1, \dots, r_n]$ and $\boldsymbol{s} = [s_1, \dots, s_n]$ satisfy $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for $i = 1, \dots, n$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) = \gamma_{\boldsymbol{r}'}^{\boldsymbol{s}'}(G) + \sum_{j=1}^{n} (r_j - T_j^*),$$

where

$$s_j' = s_j - \sum_{v_k \in N[v_j]} (r_k - T_k^*), \qquad j = 1, \dots, n$$

and $\boldsymbol{r}' = \mathbf{T}^*$. ■

The following result provides an upper bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph in terms of its constraint slackness and a generalisation of the lower bound in Theorem 3.7.

Theorem 3.23 (Burger & Van Vuuren [11]). *Let $T_1^*, \dots, T_m^*$ be the constraint slackness associated with the vertices of a packing of cardinality m of a graph triple $(G, \boldsymbol{r}, \boldsymbol{s})$ where G is a graph of order n with maximum degree Δ, and where $\boldsymbol{r} = [r_1, \dots, r_n]$ and $\boldsymbol{s} = [s_1, \dots, s_n]$ satisfy $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for $i = 1, \dots, n$. Then*

$$\frac{\sum_{i=1}^{n} s_i}{\Delta + 1} \leq \gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \sum_{i=1}^{n} r_i - \sum_{j=1}^{m} T_j^*.$$ ■

The special case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination where $\boldsymbol{r} = [r, \dots, r]$ and $\boldsymbol{s} = [s, \dots, s]$, called the *balanced case*, was studied in [72, 73]. In [72] it was noted that

$$\gamma_{\times k}(G) = \gamma_{\boldsymbol{1}}^{\boldsymbol{k}}(G) \geq \gamma_{\boldsymbol{2}}^{\boldsymbol{k}}(G) \geq \cdots \geq \gamma_{\boldsymbol{k}}^{\boldsymbol{k}}(G) = \gamma_{\{k\}}(G) \tag{3.8}$$

for any graph G with minimum degree δ and $k \leq \delta + 1$. Also note that $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) = \gamma_{\boldsymbol{s}}^{\boldsymbol{s}}(G)$ in the balanced case when $r \geq s$.

3.4.2 Values of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number for special graph classes

Exact values of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number for special graph classes are only known for the balanced case. Burger and Van Vuuren [11] determined the value of $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}$, in this case, for cycles and paths. The $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a cycle C_n of order n is presented below.

Proposition 3.14 (Burger & Van Vuuren [11]). *Let $(C_n, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $s \leq 3r$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_n) = \left\lceil \frac{sn}{3} \right\rceil. \qquad \blacksquare$$

The value of $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}$ of a path P_n of order n is given next.

Proposition 3.15 (Burger & Van Vuuren [11]). *Let $(P_n, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $r \leq s \leq 2r$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_n) = \begin{cases} \frac{sn}{3} + s - r & \textit{if} \quad n \equiv 0 \pmod 3 \\ s\left\lceil \frac{n}{3} \right\rceil & \textit{if} \quad n \equiv 1, 2 \pmod 3. \end{cases} \qquad \blacksquare$$

Rubalcaba and Slater [72] established the following lower bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a tree.

Proposition 3.16 (Rubalcaba and Slater [72]). *Let $(T, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $r \leq s \leq 2r$. Then $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(T) \geq s\gamma(T)$, for any tree T of order n.* $\blacksquare$

If $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ are two n-vectors, the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a complete graph is s [11]. Burger and Van Vuuren also established the following bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of complete bipartite graphs.

Proposition 3.17 (Burger & Van Vuuren [11]). *Let $(K_{m,n}, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $s \leq (n+1)r$ for $m \geq n$. If $s > nr$, then $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_{m,n}) = m(s - nr) + nr$. Otherwise*

$$\left\lceil \frac{ns(m-1) + ms(n-1)}{nm - 1} \right\rceil \leq \gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_{m,n}) \leq \left\lceil \frac{ns(m-1) + ms(n-1)}{nm - 1} \right\rceil + 1. \qquad \blacksquare$$

The following result provides an upper bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the cartesian product of cycles.

Proposition 3.18 (Burger & Van Vuuren [11]). *Let $(C_n \square C_m, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $s \leq 5r$ for $n \geq m \geq 4$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_n \square C_m) \leq \left\lceil \frac{mns}{5} \right\rceil + g(n, m, s) + g(m, n, s) + h(n, m, s),$$

where

$$g(x, y, s) = \begin{cases} 0 & \textit{if } s \equiv 0 \pmod 5 \textit{ or } y \equiv 0 \pmod 5 \\ \left\lceil \frac{2(x-1)}{5} \right\rceil & \textit{if } ys \equiv 3 \pmod 5 \\ \left\lceil \frac{x-1}{5} \right\rceil & \textit{otherwise} \end{cases}$$

and

$$h(n, m, s) = \begin{cases} \left\lceil \frac{2(m-1)}{5} \right\rceil - \left\lceil \frac{m-1}{5} \right\rceil & \textit{if } m, n, s \equiv 2, 3 \pmod 5 \\ & \textit{or } s \equiv 2 \pmod 5 \textit{ and } m \equiv n \equiv 1 \pmod 5 \\ & \textit{or } s \equiv 3 \pmod 5 \textit{ and } m \equiv n \equiv 4 \pmod 5 \\ 0 & \textit{otherwise.} \end{cases} \qquad \blacksquare$$

3.4.3 Algorithms for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph

The problem of computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph is associated with the following decision problem.

Decision Problem 3.5 ($\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-Dominating Set).

Instance: *A graph triple $(G, \boldsymbol{r}, \boldsymbol{s})$ and a positive integer $k \leq |V(G)|$.*

Question: *Does there exist an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function f of G such that $|f| \leq k$?*

Similar to Decision Problems 3.3 and 3.4, it is clear that Decision Problem 3.5 is also **NP**-complete.

The minimum dominating set problem may be reduced to the set cover problem, as was shown in §3.1.3. However, since a vertex may be covered more than once in the context of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination, the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-problem cannot be reduced to the set cover problem. It is nevertheless possible to reduce the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem to the so-called *set multicover with multiplicity constraints* (SMCM) problem.

Given a universe $\mathcal{U}$ of n elements and a family of sets $\mathcal{S}$, where each set $S \in \mathcal{S}$ is a subset of $\mathcal{U}$, a *set multicover with multiplicity constraints* is a sub-family $\mathcal{S}'$ of $\mathcal{S}$ such that each element $i \in \mathcal{U}$ is covered at least b_i times and in which each set $S_j \in \mathcal{S}$ is used at most d_j times. The vector $\boldsymbol{b} = [b_1, \ldots, b_n]$ is called the *coverage requirement vector* and the vector $\boldsymbol{d} = [d_1, \ldots, d_n]$ is called the *multiplicity constraint vector*.

Decision Problem 3.6 (Set multicover with multiplicity constraints).

Instance: *A universe $\mathcal{U}$ of elements, a family $\mathcal{S}$ of subsets of $\mathcal{U}$, coverage requirement vector $\boldsymbol{b}$, multiplicity constraint vector $\boldsymbol{d}$ and a positive integer $k \leq |\mathcal{S}|$.*

Question: *Does $\mathcal{U}$ have a set multicover with multiplicity constraints of size at most k?*

The objective of the related optimisation problem, the *minimum SMCM*, is to find a set multicover with multiplicity constraints of $\mathcal{U}$ of minimum cardinality. Consider, as an example, the universe $\mathcal{U} = \{1, 2, 3, 4\}$ and the family of sets $\mathcal{S} = \{\{1, 2\}, \{2, 3, 4\}, \{1, 3\}, \{2, 4\}\}$, this time with added coverage requirement vector $\boldsymbol{b} = [3, 2, 3, 1]$ and multiplicity constraints vector $\boldsymbol{d} = [3, 1, 2, 2]$. The family of sets $\{\{1, 2\}, \{1, 3\}, \{1, 3\}, \{2, 3, 4\}\}$ forms a minimum set multicover with multiplicity constraints of $\mathcal{U}$.

To reduce Decision Problem 3.5 to Decision Problem 3.6, take the universe $\mathcal{U}$ as the vertex set of G and let the family of sets $\mathcal{S}$ be the set of closed neighbourhoods $N[v]$ of $v \in V(G)$. Furthermore, let $\boldsymbol{r}$ be the multiplicity constraint vector and $\boldsymbol{s}$ the coverage requirement vector.

Research in designing heuristic and approximation algorithms for the SMCM problem and for covering integer programs, which may also be applied to SMCM problems, is widespread [37, 53, 54, 55]. However, the only known exact algorithm for solving the SMCM problem, and thus also the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem, appears in [48] (see Algorithm 3.2) and solves the SMCM problem in $\mathcal{O}^*((b + 1)^n)$ time using $\mathcal{O}^*((b + 1)^n)$ space, where $b = \max_i\{b_i\}$. After trial implementation it was observed that this algorithm is impractical, because of its excessive memory usage.

Another algorithm for $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination was presented by Cockayne [19] in 2007. This algorithm is able to compute the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a tree in polynomial-time.

Algorithm 3.2: DSMCM($\mathcal{S}, \boldsymbol{r}, \boldsymbol{s}$) A dynamic programming-based algorithm for the set multicover with multiplicity constraint problem (Hua *et al.* [48])

Input : A set multicover instance $\mathcal{S}$ with coverage requirement vector $\boldsymbol{s}$ and multiplicity constraint vector $\boldsymbol{r}$.

Output: The minimum $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination number $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}$.

```
Define an initial vertex v_0 with label (∅, 0, ..., 0). This vertex is called the level 0 vertex.
Set H(v_0) = 0.
for i ← 1 to n do
    Define (s + 1)^n vertices with labels (S_i, 0, ..., 0) to (S_i, s, ..., s) for set S_i ∈ 𝒮 where
    s = max s_i and set H(v) = ∞ for all vertices v = (S_i, y_1, ..., y_n). All these vertices are
    called level i vertices.
    for j ← 1 to r_i do
        For each vertex v = (S_{i−1}, y_1, ..., y_n) with H(v) ≠ ∞, add a directed edge with
        edge weight j to u = (S_i, y_1 + jq_1, ..., y_n + jq_n). Here q_k = 1 if k ∈ S_i otherwise
        q_k = 0. Note that if y_k + jq_k ≥ s_k, then set y_k + jq_k = s_k.
        if H(v) + j < H(u) then
            H(u) ← H(v) + j
    Remove all level i vertices and their incident edges.
return H(v) for the vertex v = (S_n, s_1, ..., s_n). If H(v) = ∞, then there does not exist a
multicover and hence no s-dominating r-function.
```

3.5 Chapter summary

A literature review of $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination and its special cases, that is, (classical) domination, k-tuple domination and $\{k\}$-domination, was provided in this chapter. General results on the (classical) domination number were presented in §3.1; this included a characterisation of minimality as well as upper and lower bounds on the domination number. Exact values of the domination number for certain graph classes was presented, including those for paths, cycles, complete (bipartite) graphs and the cartesian product of graphs. The section concluded with an overview of an algorithmic approach for calculating the domination number of a graph.

In §3.2, k-tuple domination was considered. Known bounds on and exact values of the k-tuple domination number for certain graph classes were presented together with a short discussion on the algorithmic complexity of the k-tuple domination number problem.

Bounds on the $\{k\}$-domination number were presented in §3.3 as well as exact values of the $\{k\}$-domination number for certain graph classes.

The chapter closed with a discussion on the literature related to the general case of $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination in §3.4. This included general bounds on the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination number, exact values of the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination numbers for paths, cycles and complete graphs, as well as bounds on the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination number of complete bipartite graphs and the cartesian product of cycles. It was finally shown that the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination problem can be reduced to the set multicover with multiplicity constraints problem and a dynamic programming algorithm for solving the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination problem was presented.

CHAPTER 4

Bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number

Contents

Various upper bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph are established in this chapter. The chapter opens with a discussion on general bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph, after which the balanced case (*i.e.* where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$) is considered. The chapter closes with a comparison of the newly established upper bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph.

4.1 General bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number

Consider the graph triple $(G, \boldsymbol{r}, \boldsymbol{s})$ where $\boldsymbol{r} = [r_1, \ldots, r_n]$ and $\boldsymbol{s} = [s_1, \ldots, s_n]$ satisfy $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for $i = 1, \ldots, n$ and let f be an $\boldsymbol{r}$-function with $f(v_i) = r_i$ for $i = 1, \ldots, n$. The function f is most certainly an $\boldsymbol{s}$-dominating function and hence $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \sum_{i=1}^{n} r_i$. This bound is sharp and is achieved only when $\sum_{v_j \in N[v_i]} r_j = s_i$ for all $i = 1, \ldots, n$. Burger and Van Vuuren [11] improved on this bound by incorporating packings, as described in §3.4.1. In the classical domination setting this bound reduces to the bound $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq n$ for a graph G of order n. Ore [64], however, improved this bound on the domination number of a graph G to half its order if G has no isolated vertices, that is when $\delta + 1 \geq 2$. To prove this bound, Ore made use of the fact that the complement of a minimal dominating set of a graph G with no isolated vertices is itself also a dominating set of G. If a similar approach is used in an attempt to generalise this bound to $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination (*i.e.* the value $\sum_{i=1}^{n} r_i/2$), the question arises whether there exists an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function f such that the function g defined as $g(v_i) = r_i - f(v_i)$ for $i = 1, \ldots, n$ is also $\boldsymbol{s}$-dominating. For this to be true, only graph triples for which $\sum_{v_j \in N[v_i]} r_j \geq 2s_i$ for $i = 1, \ldots, n$ can be considered.

It turns out, perhaps surprisingly, that $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \not\leq \sum_{i=1}^{n} r_i/2$ in general. For a counterexample, consider the Petersen graph P and take $\boldsymbol{r} = \boldsymbol{3}$ and $\boldsymbol{s} = \boldsymbol{6}$. Then $\sum_{i=1}^{10} r_i/2 = 15$, while $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P) = 16$. An example of an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P of minimum weight is shown in Figure 4.1.

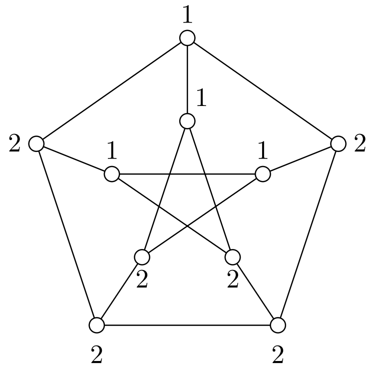


FIGURE 4.1: *A* $\mathbf{6}$*-dominating* $\mathbf{3}$*-function of the Petersen graph.*

Even if the constraint on $\boldsymbol{r}$ and $\boldsymbol{s}$ is changed to $\sum_{v_j \in N[v_i]} r_j > 2s_i$, a result similar to Ore's is not possible. As an example, consider the complete multipartite graph $K_{4,4,4}$ with vectors $\boldsymbol{r} = [5,3,1,2,1,3,5,4,3,3,2,5]$ and $\boldsymbol{s} = [15,14,13,13,12,13,14,13,13,13,12,14]$. Then $\sum_{i=1}^{12} r_i/2 = 18.5$, while $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_{4,4,4}) = 19$. An example of an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of $K_{4,4,4}$ of minimum weight is shown in Figure 4.2.

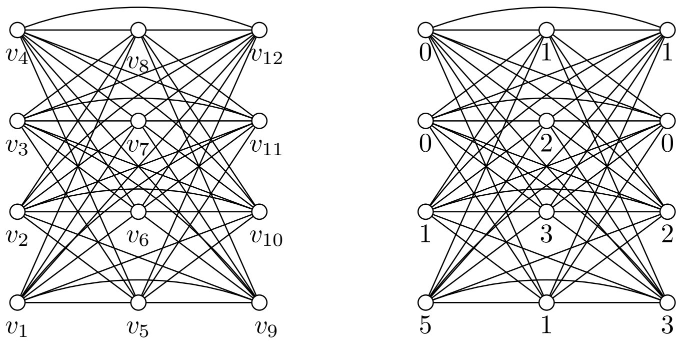


FIGURE 4.2: *An* $\boldsymbol{s}$*-dominating* $\boldsymbol{r}$*-function of* $K_{4,4,4}$.

If $\min_i\{r_i\} \geq \max_i\{s_i\}$, then the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph may be bounded as follows.

Proposition 4.1. *Let* $(G, \boldsymbol{r}, \boldsymbol{s})$ *be a graph triple where* $\boldsymbol{r} = [r_1, \ldots, r_n]$ *and* $\boldsymbol{s} = [s_1, \ldots, s_n]$ *satisfy* $\sum_{v_j \in N[v_i]} r_j \geq s_i$ *for* $i = 1, \ldots, n$*. Furthermore, let* $r_{\min} = \min_i\{r_i\}$ *and* $s_{\max} = \max_i\{s_i\}$*. If* $r_{\min} \geq s_{\max}$*, then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) = \gamma_{\boldsymbol{s}'}^{\boldsymbol{s}}(G),$$

where $\boldsymbol{s}' = [s_{\max}, \ldots, s_{\max}]$.

Proof. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{s}'$-function of G of minimum weight. Since $r_{\min} \geq s_{\max}$, f is also an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G and it follows that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq |f| = \gamma_{\boldsymbol{s}'}^{\boldsymbol{s}}(G). \tag{4.1}$$

Now let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G of minimum weight such that there exists a $v \in V$ for which $f(v) > s_{\max}$. The function

$$f'(u) = \begin{cases} s_{\max} & \text{if } u = v \\ f(u) & \text{if } u \neq v \end{cases}$$

is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G with $|f'| < |f|$. This contradicts the minimality of f and therefore $f(v) \leq s_{\max}$ for all $v \in V$. It follows that f is an $\boldsymbol{s}$-dominating $\boldsymbol{s}'$-function and so

$$\gamma_{\boldsymbol{s}'}^{\boldsymbol{s}}(G) \leq |f| = \gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G). \tag{4.2}$$

The desired result follows by a combination of (4.1) and (4.2). ■

By using a probabilistic approach, it is possible to establish the following upper bound on the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination number of an arbitrary graph.

Theorem 4.1. *Let $(G, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where G is a graph of order n with minimum degree δ, and where the vectors $\boldsymbol{r} = [r_1, \ldots, r_n]$ and $\boldsymbol{s} = [s_1, \ldots, s_n]$ satisfy $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for every $v_i \in V$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \frac{\sum_{i=1}^{n} r_i}{2} + \sum_{i=1}^{n}\sum_{j=1}^{s_i} j \sum_{k=0}^{\mathrm{d}_i+1} (-1)^k \binom{\mathrm{d}_i+1}{k}\binom{j+\mathrm{d}_i - k(r_{\max}^i+1)}{\mathrm{d}_i}(r_i+1)^{-\mathrm{d}_i-1},$$

where $r_{\max}^i = \max_{v_j \in N[v_i]}\{r_j\}$ and $\mathrm{d}_i = \deg(v_i)$.

Proof. Define the function f on V by assigning values in $\{0, \ldots, r_i\}$ to each vertex $v_i \in V$ independently with probability $p = \frac{1}{r_i+1}$. Suppose $f[v_i] = j_i$ for each vertex $v_i \in V$. Since $\sum_{v_j \in N[v_i]} r_j \geq s_i$, there exists a function g_i such that $g_i[v_i] = s_i - j_i$. Let h be a function on V defined as $h(v_j) = \max\{f(v_j) + \sum_{i=1}^{n} g_i(v_j), r_j\}$. Then h is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function and it follows that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq E\left(|h|\right) \leq E(|f|) + \sum_{i=1}^{n} E(|g_i|) \leq \sum_{i=1}^{n} E(f(v_i)) + \sum_{i=1}^{n}\sum_{j=1}^{s_i} j\Pr(|g_i| = j). \tag{4.3}$$

First consider the expected value

$$E(f(v)) = \sum_{j=0}^{r_v} j\Pr(f(v) = j) = \sum_{j=0}^{r_v} jp = \frac{r_v(r_v+1)}{2}p = \frac{r_v}{2} \tag{4.4}$$

of $f(v)$. Let a denote the number of non-negative integer solutions of the equation $x_1 + \cdots + x_k = i$ with $x_j \leq r_j$. Then a is clearly less than the number of non-negative integer solutions of the equation $x_1 + \cdots + x_k = i$ with $x_j \leq r_{\max}$, where $r_{\max} = \max_j\{r_j\}$. Then a is the coefficient of z^i in the generating function $\left((1 - z^{r_{\max}+1})/(1-z)\right)^{\mathrm{d}_v}$ [59] and hence

$$\Pr(|g_v| = i) = ap^{\mathrm{d}_v+1} \leq \sum_{k=0}^{\mathrm{d}_v+1} (-1)^k \binom{\mathrm{d}_v+1}{k}\binom{i+\mathrm{d}_v - k(r_{\max}^v+1)}{\mathrm{d}_v}(r_v+1)^{-\mathrm{d}_v-1}. \tag{4.5}$$

The desired bound on $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$ follows by substituting (4.4)–(4.5) into (4.3). ■

4.2 Bounds on the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination number for the balanced case

For the balanced case of $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination, the bound $\sum_{i=1}^{n} r_i = rn$ can always be improved since $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \gamma_{\boldsymbol{r}'}^{\boldsymbol{s}}(G)$ where $\boldsymbol{r}' = [r-1, \ldots, r-1]$.

Proposition 4.2. *Let $(G, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where G is a graph of order n with minimum degree δ, and where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $r(\delta+1) \geq s$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \left\lceil \frac{s}{\delta+1} \right\rceil n.$$

Proof. For any graph triple $(G, \boldsymbol{r}, \boldsymbol{s})$, $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \gamma_{\boldsymbol{r}'}^{\boldsymbol{s}}(G)$ where $\boldsymbol{r}' = [r-1, \ldots, r-1]$ [72]. Hence

$$\gamma_{\boldsymbol{s}}^{\boldsymbol{s}}(G) \leq \gamma_{\boldsymbol{s}-\boldsymbol{1}}^{\boldsymbol{s}}(G) \leq \cdots \leq \gamma_{\boldsymbol{r}_{\min}}^{\boldsymbol{s}}(G),$$

where $r_{\min} = \lceil s/(\delta+1) \rceil$. Furthermore, since the function f, defined as $f(v) = r_{\min}$ for every $v \in V(G)$, is an $\boldsymbol{s}$-dominating function of G, it follows that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \gamma_{\boldsymbol{r}_{\min}}^{\boldsymbol{s}}(G) \leq \left\lceil \frac{s}{\delta+1} \right\rceil n. \qquad \blacksquare$$

The bound in Proposition 4.2 is sharp and is achieved only when G is a regular graph and $r = s/(\delta+1)$. If $(G, \boldsymbol{r}, \boldsymbol{s})$ is a graph triple for which $r(\delta+1) > 2s$, then a result similar to Ore's bound in Theorem 3.1, follows from Proposition 4.2.

Corollary 4.1. *Let $(G, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where G is a graph of order n with minimum degree δ, and where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $r(\delta+1) > 2s$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \frac{rn}{2}.$$

Proof. Since $r(\delta+1) > 2s$, it follows that $\lceil s/(\delta+1) \rceil \leq r/2$ and the bound follows immediately from Proposition 4.2. $\blacksquare$

The above bound, however, offers no improvement over the bound in Proposition 4.2. The bound in Proposition 4.2 can be improved by employing the same idea as that of Burger and Van Vuuren [11] in Theorem 3.23.

Let $(G, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple where G is a graph of order n with minimum degree δ, and where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $r(\delta+1) \geq s$. Furthermore, let $S = \{v_i \mid \deg(v_i) = \delta\}$. Define the vector $\boldsymbol{T}^\delta$, called the *balanced constraint slackness*, as

$$T_i^\delta = \begin{cases} \left\lceil \frac{s}{\delta+1} \right\rceil (\delta+1) - s & \text{if } v_i \in N[S] \text{ and} \\ \left\lceil \frac{s}{\delta+1} \right\rceil & \text{otherwise.} \end{cases}$$

Theorem 4.2. *Let $T_1^\delta, \ldots, T_m^\delta$ be the balanced constraint slackness associated with the vertices of a packing P of cardinality p of a graph triple $(G, \boldsymbol{r}, \boldsymbol{s})$, where G is a graph of order n with minimum degree δ and where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $r(\delta+1) \geq s$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \left\lceil \frac{s}{\delta+1} \right\rceil n - \sum_{i=1}^{p} T_i^\delta.$$

Proof. Let f be an $\boldsymbol{r}$-function such that

$$f(v) = \begin{cases} \lceil s/(\delta+1) \rceil - T_i^\delta & \text{if } v_i \in P \\ \lceil s/(\delta+1) \rceil & \text{otherwise.} \end{cases}$$

To show that f is an $\boldsymbol{s}$-dominating function, first consider a vertex $v \in P$. If $v \in N[S]$, then $\deg_G(v) \geq \delta$ and v is not adjacent to any other vertex in P. Therefore,

$$\begin{aligned} f[v] &= f(v) + \sum_{u \in N(v)} f(u) \\ &= s - \lceil s/(\delta+1) \rceil \delta + \sum_{u \in N(v)} \lceil s/(\delta+1) \rceil \\ &\geq s - \lceil s/(\delta+1) \rceil \delta + \lceil s/(\delta+1) \rceil \delta \\ &= s. \end{aligned}$$

If $v \notin N[S]$, then $\deg_G(v) \geq \delta + 1$. Therefore,

$$\begin{aligned} f[v] &= f(v) + \sum_{u \in N(v)} f(u) \\ &= 0 + \sum_{u \in N(v)} \lceil s/(\delta+1) \rceil \\ &\geq \lceil s/(\delta+1) \rceil (\delta+1) \\ &= s. \end{aligned}$$

Now consider a vertex $v \notin P$. Then v is adjacent to at most one vertex in P. If $v \notin N[P]$, then clearly

$$f[v] \geq \lceil s/(\delta+1) \rceil (\delta+1) = s.$$

If $v \in N[P]$, then assume that v is adjacent to $w \in P \cap N[S]$. It follows that

$$\begin{aligned} f[v] &= f(w) + \sum_{\substack{u \,\in\, N[v] \\ u \,\neq\, w}} f(u) \\ &\geq s - \lceil s/(\delta+1) \rceil \delta + \lceil s/(\delta+1) \rceil \delta \\ &= s. \end{aligned}$$

If $w \in P \backslash N[S]$, then $\deg_G(v) \geq \delta + 1$. Therefore,

$$\begin{aligned} f[v] &= f(w) + \sum_{\substack{u \,\in\, N[v] \\ u \,\neq\, w}} f(u) \\ &\geq 0 + \lceil s/(\delta+1) \rceil (\delta+1) \\ &= s. \end{aligned}$$

Therefore f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G and so

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) = |f| \leq \left\lceil \frac{s}{\delta+1} \right\rceil n - \sum_{i=1}^{p} T_i^{\delta}.$$

■

The next proposition provides an upper bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph G with maximum degree Δ when $s \geq \Delta + 1$.

Proposition 4.3. *Let $(G, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where G is a graph of order n with minimum degree δ and maximum degree Δ, and where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $r(\delta+1) \geq s$. For vectors $\boldsymbol{r}' = [r', \ldots, r']$ and $\boldsymbol{s}' = [s', \ldots, s']$, where $r' = r - \lfloor s/\Delta+1 \rfloor$ and $s' = s - (\delta + 1) \lfloor s/\Delta+1 \rfloor$, it follows that*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq n \left\lfloor \frac{s}{\Delta+1} \right\rfloor + \gamma_{\boldsymbol{r}'}^{\boldsymbol{s}'}(G).$$

Proof. Let g be an $\boldsymbol{s}'$-dominating $\boldsymbol{r}'$-function of G of minimum weight and define the function f such that $f(v) = \lfloor s/\Delta+1 \rfloor + g(v)$. Since $g(v) \leq r' = r - \lfloor s/\Delta+1 \rfloor$, it follows that $f(v) \leq r$. Furthermore, $g[v] \geq s' = s - (\delta+1) \lfloor s/\Delta+1 \rfloor$ and therefore

$$f[v] = \sum_{u \in N[v]} f(v) = (\mathrm{d}_v + 1) \lfloor s/\Delta+1 \rfloor + g[v] \geq s.$$

Hence f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G and the desired bound follows. ■

Next it is examined whether a probabilistic approach can produce better upper bounds on the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination number of a graph than the bound in Proposition 4.2. For the first three bounds a set is constructed by randomly choosing vertices from a multiset containing r copies of each vertex $v \in V$. The following result shows how such a set can be used to find an upper bound on $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$. Let $\mathrm{d}_v = \deg_G(v)$.

Lemma 4.1. *Let $(G, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple where G is a graph of order n with minimum degree δ and where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ for some $r, s \in \mathbb{N}$ such that $r(\delta+1) \geq s$. Let U be the multiset over V containing r copies of each vertex $v \in V$ and form a multiset D by picking every element $u \in U$ independently at random with $\Pr(u \in D) = p$ for some $0 < p \leq 1$. Let $f(u)$ denote the resulting multiplicity of u in D and let $D_i = \{v \in V \mid \sum_{u \in N[v]} f(u) = i\}$ for $i = 0, \ldots, s-1$. Then*

$$\Pr(v \in D_i) = \binom{r(\mathrm{d}_v+1)}{i} p^i (1-p)^{r(\mathrm{d}_v+1)-i}$$

and

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq E(|D|) + \sum_{i=0}^{s-1} (s-i) E(|D_i|)$$

for each $v \in V$ and $i = 1, \ldots, s-1$.

Proof. Since $v \in D_i$ only if exactly i of the neighbours of v are in D, the probability that v is in D_i follows readily. Now, since $r(\delta+1) \geq s$ there exists a subset $D_i' \subseteq U \ominus D$ such that $\sum_{u \in N[v]} g(u) \geq s - i$ for every $v \in D_i$, where $g(u)$ denotes the resulting multiplicity of u in D_i'. Thus $|D_i'| \leq (s-i)|D_i|$.

Let $A = D \uplus (\uplus_{i=0}^{s-1} D_i')$ and define the function h such that $h(v)$ is the minimum of the multiplicity of v in A and r. Then h is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G and it follows that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq E(|h|) \leq E(|A|) \leq E(|D|) + \sum_{i=0}^{s-1} E(|D_i'|) \leq E(|D|) + \sum_{i=0}^{s-1} (s-i) E(|D_i|)$$

by the linearity of expectation. ■

The next result is a generalisation of the bound on the k-tuple domination number (3.7) in §3.2.1.

Theorem 4.3. *Let $(G, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where G is a graph of order n with minimum degree δ, and where $\boldsymbol{r} = [r, \dots, r]$ and $\boldsymbol{s} = [s, \dots, s]$ satisfy*

$$\frac{r(\delta+1)}{\ln(r(\delta+1))} \geq 2s. \tag{4.6}$$

Then

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \frac{n}{\delta+1}\left(s\ln(r(\delta+1)) + \sum_{i=0}^{s-1} \frac{(s-i)(r)^{i-s}}{i!(\delta+1)^{s-i-1}}\right).$$

Proof. Let

$$p = \frac{s\ln(r(\delta+1))}{r(\delta+1)} + \epsilon$$

where $\epsilon \geq 0$ such that $0 < p \leq 1$. Furthermore, let D, D_i and D_i' be defined as in Lemma 4.1. For each $v \in V$ and $i = 1, \dots, s-1$ it follows that

$$\Pr(v \in D_i) = \binom{r(\mathrm{d}_v+1)}{i} p^i (1-p)^{r(\mathrm{d}_v+1)-i}.$$

Furthermore, since $p \leq 1/2$ and $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$,

$$\begin{aligned}
\Pr(v \in D_i) &\leq \binom{r(\mathrm{d}_v+1)}{i}(1-p)^{r(\mathrm{d}_v+1)} \\
&\leq \frac{r^i(\mathrm{d}_v+1)^i}{i!}(1-p)^{r(\mathrm{d}_v+1)} \\
&\leq \frac{r^i(\mathrm{d}_v+1)^i}{i!}e^{-pr(\mathrm{d}_v+1)} \\
&= \frac{1}{i!}e^{-pr(\mathrm{d}_v+1)+i\ln(r(\mathrm{d}_v+1))}.
\end{aligned}$$

Note that

$$p \geq \frac{s\ln(r(\delta+1))}{r(\delta+1)} \geq \frac{s-1}{r(\delta+1)} \geq \frac{s-1}{r(\mathrm{d}_v+1)}$$

and that the function $h(\mathrm{d}_v) = -pr(\mathrm{d}_v+1) + i\ln(r(\mathrm{d}_v+1))$ is monotonically decreasing in d_v since

$$h'(\mathrm{d}_v) = -pr + \frac{ir}{r(\mathrm{d}_v+1)} \leq -pr + \frac{(s-1)}{\mathrm{d}_v+1} \leq 0.$$

It follows that

$$\Pr(v \in D_i) \leq \frac{1}{i!}e^{-pr(\delta+1)+i\ln(r(\delta+1))} \leq \frac{1}{i!}e^{-s\ln(r(\delta+1))+i\ln(r(\delta+1))} = \frac{r^{i-s}}{i!}(\delta+1)^{i-s}$$

and hence, by Lemma 4.1,

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) &\leq E(|D|) + \sum_{i=0}^{s-1}(s-i)E(|D_i|) \\
&\leq \sum_{u\in U}\Pr(u \in D) + \sum_{i=0}^{s-1}(s-i)\sum_{v\in V}\Pr(v \in D_i) \\
&\leq rnp + n\sum_{i=0}^{s-1}\frac{(s-i)r^{i-s}}{i!}(\delta+1)^{i-s} \\
&= \frac{n}{\delta+1}\left(s\ln(r(\delta+1)) + \sum_{i=0}^{s-1}\frac{(s-i)r^{i-s}}{i!(\delta+1)^{s-i-1}}.\right)
\end{aligned}$$

■

In the classical domination setting (*i.e.* where $r = s = 1$), the bound in Theorem 4.3 reduces to the bound in Theorem 3.6. Note also that in the case of the k-tuple domination number (*i.e.* where $r = 1$ and $s = k$), the bound in Theorem 4.3 reduces to (3.7) in §3.2.1. This bound outperforms the bound in Proposition 4.2 if $\delta \gg s$.

The next result is a generalisation of Conjecture 3.3 and uses the same methods as employed in [88].

Theorem 4.4. *Let* $(G, \boldsymbol{r}, \boldsymbol{s})$ *be a graph triple where* G *is a graph of order* n *with minimum degree* δ*, where* $\boldsymbol{r} = [r, \dots, r]$ *and* $\boldsymbol{s} = [s, \dots, s]$ *satisfy* $r(\delta+1) \geq s$*. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \frac{rn}{r(\delta+1) - s + 1}\left(\ln(r(\delta+1) - s + 1) + \ln\left(\sum_{v \in V} \binom{r(\mathrm{d}_v + 1)}{s-1}\right) - \ln r - \ln n + 1\right).$$

Proof. Let D, D_i and D_i' be defined as in Lemma 4.1 and let $m_i = \sum_{v \in V} \binom{r(\mathrm{d}_v+1)}{i}$. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) &\leq E(|D|) + \sum_{i=0}^{s-1} (s-i) E(|D_i|) \\
&\leq \sum_{u \in U} \Pr(u \in D) + \sum_{i=0}^{s-1} (s-i) \sum_{v \in V} \Pr(v \in D_i) \\
&\leq rnp + \sum_{i=0}^{s-1} (s-i) \sum_{v \in V} \binom{r(\mathrm{d}_v+1)}{i} p^i (1-p)^{r(\mathrm{d}_v+1)-i} \\
&\leq rnp + \sum_{i=0}^{s-1} (s-i) \sum_{v \in V} \binom{r(\mathrm{d}_v+1)}{i} p^i (1-p)^{r(\delta+1)-i} \\
&= rnp + \sum_{i=0}^{s-1} (s-i) m_i p^i (1-p)^{r(\delta+1)-i} \\
&= rnp + (1-p)^{r(\delta+1)-s+1} \sum_{i=0}^{s-1} (s-i) m_i p^i (1-p)^{s-1-i}.
\end{aligned}$$

Now let $h(p) = \sum_{i=0}^{s-1} (s-i) m_i p^i (1-p)^{s-1-i}$. To show that $h(p)$ is monotonically increasing in $0 \leq p \leq 1$, it is shown that

$$\begin{aligned}
h'(p) = &\left\{-s(s-1) m_0 (1-p)^{s-2} + (s-1) m_1 (1-p)^{s-2}\right\} \\
&+ \left\{-(s-1)(s-2) m_1 p^1 (1-p)^{s-3} + 2(s-2) m_2 p^1 (1-p)^{s-3}\right\} \\
&\vdots \\
&+ \left\{-(s-i)(s-i-1) m_i p^i (1-p)^{s-i-2} + (i+1)(s-i-1) m_{i+1} p^i (1-p)^{s-i-2}\right\} \\
&\vdots \\
&+ \left\{-2 m_{s-2} p^{s-2} + (s-1) m_{s-1} p^{s-2}\right\} \\
= &\sum_{i=0}^{s-2} \left\{-(s-i)(s-i-1) m_i p^i (1-p)^{s-i-2} + (i+1)(s-i-1) m_{i+1} p^i (1-p)^{s-i-2}\right\} \\
= &\sum_{i=0}^{s-2} \left(-(s-i) m_i + (i+1) m_{i+1}\right)(s-i-1) p^i (1-p)^{s-i-2}
\end{aligned}$$

is nonnegative. It suffices to show that $-(s-i)m_i + (i+1)m_{i+1} \geq 0$ for $0 \leq i \leq s-2$. For $0 \leq i \leq s-2$,

$$\begin{aligned}(i+1)m_{i+1} - (s-i)m_i &= (i+1)\sum_{v\in V}\binom{r(\mathrm{d}_v+1)}{i+1} - (s-i)\sum_{v\in V}\binom{r(\mathrm{d}_v+1)}{i}\\ &= \sum_{v\in V}\left\{(i+1)\binom{r(\mathrm{d}_v+1)}{i+1} - (s-i)\binom{r(\mathrm{d}_v+1)}{i}\right\}\\ &= \sum_{v\in V}\binom{r(\mathrm{d}_v+1)}{i}(r(\mathrm{d}_v+1)-i-s+i)\\ &= \sum_{v\in V}\binom{r(\mathrm{d}_v+1)}{i}(r(\mathrm{d}_v+1)-s)\,.\end{aligned}$$

Since $r(\mathrm{d}_v+1) \geq r(\delta+1) \geq s$ it is clear that $-(s-i)m_i + (i+1)m_{i+1} \geq 0$. Furthermore, since $1-x \leq e^{-x}$ for all $x \in \mathbb{R}$, it follows that

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) &\leq rnp + (1-p)^{r(\delta+1)-s+1}h(p)\\ &\leq rnp + (1-p)^{r(\delta+1)-s+1}h(1)\\ &= rnp + (1-p)^{r(\delta+1)-s+1}m_{s-1}\\ &\leq rnp + m_{s-1}e^{-p(r(\delta+1)-s+1)}.\end{aligned}$$

The expression $rnp + m_{s-1}e^{-p(r(\delta+1)-s+1)}$ is minimised for

$$p = \frac{1}{r(\delta+1)-s+1}(\ln(r(\delta+1)-s+1) + \ln m_{s-1} - \ln r - \ln n),$$

from which the desired bound follows. ■

As with Theorem 4.3, the bound in Theorem 4.4 reduces to the bound in Theorem 3.6 in the classical domination setting (*i.e.* where $r = s = 1$). Note also that in the case of the k-tuple domination number (*i.e.* where $r = 1$ and $s = k$) this bound reduces to the bound in Conjecture 3.3. Similar to the bound in Theorem 4.3, this bound outperforms the bound in Proposition 4.2 when $\delta \gg s$. When the difference between the average degree and minimum degree of a graph is small, then this bound performs better than the bound in Theorem 4.3.

The result of the next theorem is also a generalisation of those of Theorems 3.4 and 3.6, but does not depend on the degree sequence of G.

Theorem 4.5. *Let $(G, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple where G is a graph of order n with minimum degree δ where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $r(\delta+1) \geq s$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq rn\left(1 - d\left(\frac{r}{m_{s-1}}\right)^{1/d}\left(\frac{1}{d+1}\right)^{1+1/d}\right),$$

where $m_i = \binom{r(\delta+1)}{i}$ and $d = r(\delta+1)-s$.

Proof. For each $v \in V$, let $N'(v)$ denote the set containing δ vertices of $N(v)$ and let $N'[v] = N'(v) \cup \{v\}$. Let U be the multiset over V containing r copies of each vertex $v \in V$ and form a multiset D by picking every element $u \in U$ independently at random with $\Pr(u \in D) = p$ for some $0 < p \leq 1$. Let $f(u)$ denote the resulting multiplicity of u in D and let $D_i = \{v \in V \mid \sum_{u \in N'[v]} f(u) = i\}$ for $i = 0, \ldots, s-1$.

Since $v \in D_i$ only if exactly i of the vertices in $N'[v]$ are in D,

$$\Pr(v \in D_i) = \binom{r(\delta+1)}{i} p^i (1-p)^{r(\delta+1)-i}.$$

Now, since $r(\delta+1) \geq s$ there exists a subset $D'_i \subseteq U \ominus D$ such that $\sum_{u \in N'[v]} g(u) \geq s - i$ for every $v \in D_i$, where $g(u)$ denotes the resulting multiplicity of u in D'_i. Thus, $|D'_i| \leq (s-i)|D_i|$. Let $A = D \uplus (\uplus_{i=0}^{s-1} D'_i)$ and define the function h such that $h(v)$ is the minimum of the multiplicity of v in A and r. Then h is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G and it follows that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq E(|h|) \leq E(|A|) \leq E(|D|) + \sum_{i=0}^{s-1} E(|D'_i|) \leq E(|D|) + \sum_{i=0}^{s-1} (s-i) E(|D_i|)$$

by the linearity of expectation. Therefore,

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) &\leq E(|D|) + \sum_{i=0}^{s-1} (s-i) E(|D_i|) \\
&\leq \sum_{u \in U} \Pr(u \in D) + \sum_{i=0}^{s-1} (s-i) \sum_{v \in V} \Pr(v \in D_i) \\
&\leq rnp + n \sum_{i=0}^{s-1} (s-i) m_i p^i (1-p)^{r(\delta+1)-i} \\
&= rnp + n(1-p)^{r(\delta+1)-s+1} \sum_{i=0}^{s-1} (s-i) m_i p^i (1-p)^{s-1-i}.
\end{aligned}$$

Now let $h(p) = \sum_{i=0}^{s-1} (s-i) m_i p^i (1-p)^{s-1-i}$. As in the proof of Theorem 4.4, it is possible to show that $h(p)$ is monotonically increasing over the interval $0 \leq p \leq 1$. Hence

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) &\leq rnp + (1-p)^{r(\delta+1)-s+1} h(p) \\
&\leq rnp + (1-p)^{r(\delta+1)-s+1} h(1) \\
&= rnp + (1-p)^{d+1} m_{s-1}.
\end{aligned}$$

The expression $rnp + (1-p)^{d+1} m_{s-1}$ is minimised for

$$p = 1 - \left(\frac{r}{m_{s-1}(d+1)} \right)^{1/d},$$

which shows that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq rn \left(1 - d \left(\frac{r}{m_{s-1}} \right)^{1/d} \left(\frac{1}{d+1} \right)^{1+1/d} \right). \qquad \blacksquare$$

In the case of the k-tuple domination number (*i.e.* where $r = 1$ and $s = k$), the bound in Theorem 4.5 reduces to the bound in Theorem 3.17. This bound performs better than the bound in Proposition 4.2 when $\delta \gg s$ or when the difference between s and $r(\delta+1)$ is very small. The bound in Theorem 4.5 always outperforms the bounds in Theorems 4.4 and 4.3.

For the next bound a function is constructed where each function value is chosen randomly from the set $\{0, \ldots, r\}$.

Theorem 4.6. *Let $(G, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple where G is a graph of order n with minimum degree δ and where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ for some $r, s \in \mathbb{N}$ such that $r(\delta+1) \geq s$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq \frac{rn}{2} + \sum_{i=0}^{s-1}(s-i) \sum_{v \in V} \sum_{k=0}^{\mathrm{d}_v+1} (-1)^k \binom{\mathrm{d}_v+1}{k} \binom{i+\mathrm{d}_v - k(r+1)}{\mathrm{d}_v} (r+1)^{-\mathrm{d}_v-1}.$$

Proof. Consider the function f on V which assigns values in $\{0, \ldots, r\}$ to each vertex $v \in V$ independently with probability $p = \frac{1}{r+1}$. Let $D_i = \{v \in V \mid f[v] = i\}$ for $i = 0, \ldots, s-1$. Since $r(\delta+1) \geq s$, there exists a function g_i such that $g_i[v] \geq s-i$ for each $v \in D_i$ and with $|g_i| \leq (s-i)|D_i|$.

Let a denote the number of non-negative integer solutions of the equation $x_1 + \cdots + x_{\mathrm{d}_v+1} = i$ with $x_j \leq r$ for all $j = 1, \ldots, d_v+1$. Then a is the coefficient of z^i in the generating function $\left((1-z^{r+1})/(1-z)\right)^{\mathrm{d}_v}$ [59] and hence

$$a = \sum_{k=0}^{\mathrm{d}_v+1} (-1)^k \binom{\mathrm{d}_v+1}{k} \binom{i+\mathrm{d}_v-k(r+1)}{\mathrm{d}_v}.$$

It follows that $\Pr(v \in D_i) = ap^{\mathrm{d}_v+1}$.

Let h be a function on V such that $h(v) = \max\{f(v) + \sum_{i=0}^{s-1} g_i(v), r_v\}$. Then h is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function and it follows that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq E\left(|h|\right) \leq E(|f|) + \sum_{i=0}^{s-1} E(|g_i|) \leq \sum_{v \in V} E(f(v)) + \sum_{i=0}^{s-1} (s-i) E(|D_i|). \tag{4.7}$$

Similar to the calculation of $E(|f|)$ in the proof of Theorem 4.1, it follows that

$$E(f(v)) = \sum_{j=0}^{r} j\Pr(f(v) = j) = \frac{r(r+1)}{2} p = \frac{r}{2}. \tag{4.8}$$

To complete the bound, the expected value of the size of D_i is

$$\begin{aligned} E(|D_i|) &= \sum_{v \in V} \Pr(v \in D_i) \\ &= \sum_{v \in V} ap^{\mathrm{d}_v+1} \\ &= \sum_{v \in V} \sum_{k=0}^{\mathrm{d}_v+1} (-1)^k \binom{\mathrm{d}_v+1}{k} \binom{i+\mathrm{d}_v-k(r+1)}{\mathrm{d}_v} (r+1)^{-\mathrm{d}_v-1}. \end{aligned} \tag{4.9}$$

The bound follows by substituting (4.8)–(4.9) into (4.7). ■

In the classical domination setting (*i.e.* where $r = s = 1$), the bound in Theorem 4.6 reduces to a bound strictly larger than $n/2$ for a graph of order n. However, in the case of the k-tuple domination number (*i.e.* where $r = 1$ and $s = k$), the bound outperforms the bounds in [67, 88, 90] for large values of k. Note that there exist graphs for which the bound in Theorem 4.6 performs better than the bound in Theorem 3.17. The bound in Theorem 4.6 performs better than the bound in Proposition 4.2 for large values of s and for graphs in which the minimum degree δ and the average degree differ significantly.

4.3 A comparison of the upper bounds

In this section three graphs are used to illustrate the relative performances of the upper bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number established in §4.2. For the case where $k = 1$, the bound in Theorem 3.18 is included for the sake of comparison. To illustrate the effectiveness of these bounds, the lower bound in Theorem 3.23 is also included in all the comparisons.

First consider the randomly generated graph G_1, whose adjacency matrix is provided on the accompanying compact disk, of order 50 with minimum degree $\delta = 5$, maximum degree $\Delta = 40$ and an average degree of 22.6. Figure 4.3 depicts the bounds of §4.2 for three values of r over all possible s values. In the case of k-tuple domination, where $r = 1$, the bound in Theorem 4.6 is the superior of the bounds in §4.2 for all values of s, except for the special case where $s = 1$. When $r = 10$ or $r = 20$, the bound in Theorem 4.5 performs best for $s = 1$ and $s = 2$, while the bound in Theorem 4.6 performs the best when $s > r(\delta + 1)/2$. For all the other values, the bound in Proposition 4.2 is the superior bound.

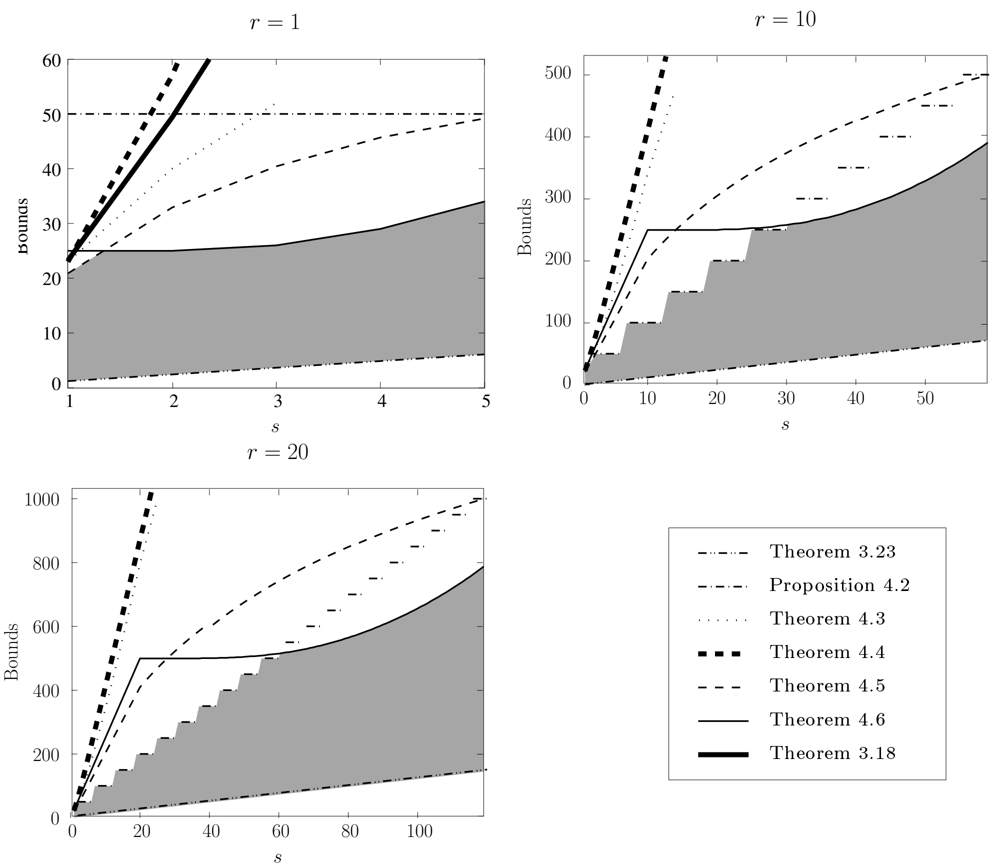


FIGURE 4.3: *A comparison of the bounds in §4.2 for the random graph G_1 of order 50 with $\delta = 5$, $\Delta = 40$ and average degree $= 22.6$. Illustrated above are the cases where $r = 1$, $r = 10$ and $r = 20$.*

Secondly, consider the randomly generated graph G_2, whose adjacency matrix is provided on the accompanying compact disk, of order 100 with minimum degree $\delta = 19$, maximum degree $\Delta = 43$ and an average degree of 30. Figure 4.4 depicts the bounds of §4.2 for three values of r over all possible s values. In the case of k-tuple domination, where $r = 1$, the bound in Theorem 4.5 is the superior of the bounds for values of $s \leq 4$ or $s \geq 13$, while the bound in Theorem 4.6 performs best for $4 < s < 13$. When $r = 5$ or $r = 10$, the bound in Proposition 4.2 outperforms the bounds in §4.2 in most cases. The bound in Theorem 4.6 performs better than the bound in Proposition 4.2 for a few values of s, while the bound in Theorem 4.5 performs best when $\lceil s/(\delta + 1)\rceil = r$.

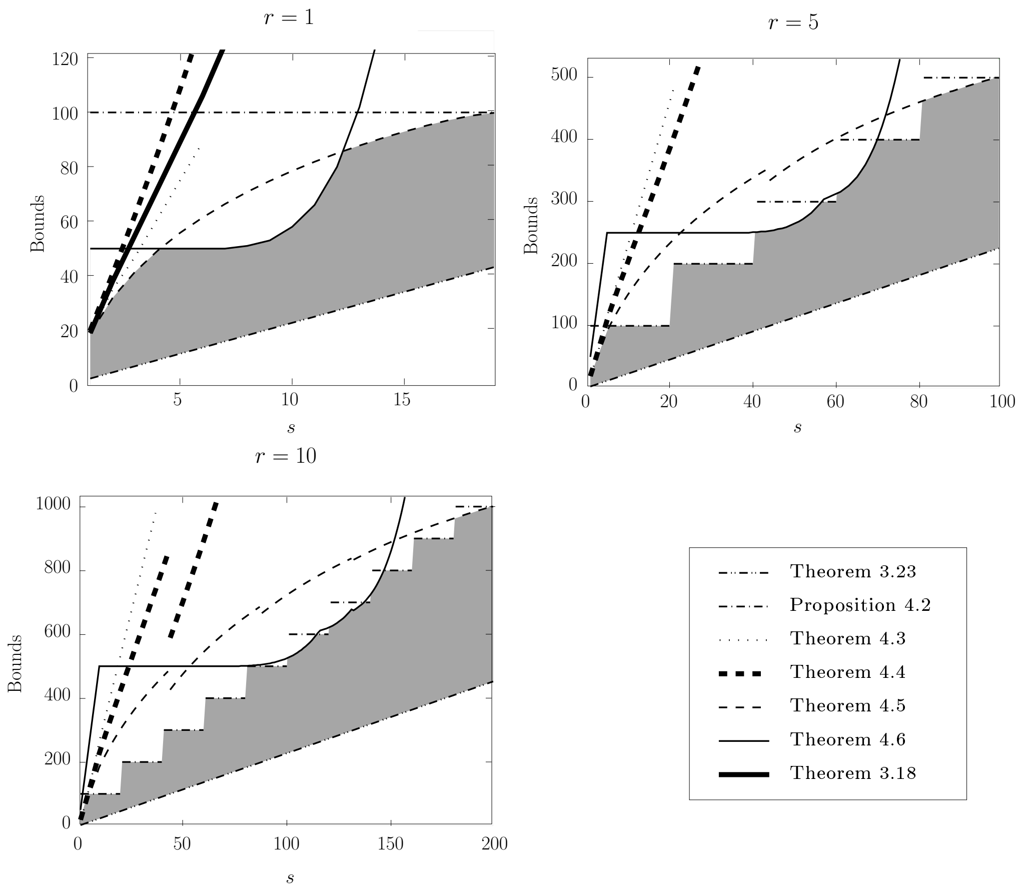


FIGURE 4.4: *A comparison of the bounds in §4.2 for the random graph G_2 of order 100 with $\delta = 19$, $\Delta = 43$ and average degree $= 30$. Illustrated above are the cases where $r = 1$, $r = 5$ and $r = 10$.*

Finally, consider the 10-regular graph G_3, whose adjacency matrix is provided on the accompanying compact disk, with minimum degree $\delta = 10$. Figure 4.5 depicts the bounds of §4.2 for three values of r over all possible s values. In the case of k-tuple domination, where $r = 1$, the bound in Theorem 4.5 performs best, except for the cases where $s = 3, 4$; in these cases the bound in Theorem 4.6 performs better. In the cases where $r = 10$ and $r = 20$, the bounds in Theorems 4.3 and 4.4 are not shown since the bound in Theorem 4.5 always outperforms these bounds. When $r = 10$ or $r = 20$, the use of Proposition 4.3 renders a considerable improvement on these bounds. The bound in Proposition 4.2 performs the best for most values of s, with the bound in Theorem 4.5 outperforming the other bounds when the difference between s and $c(\delta + 1)$ is small for $c \in \mathbb{N}$.

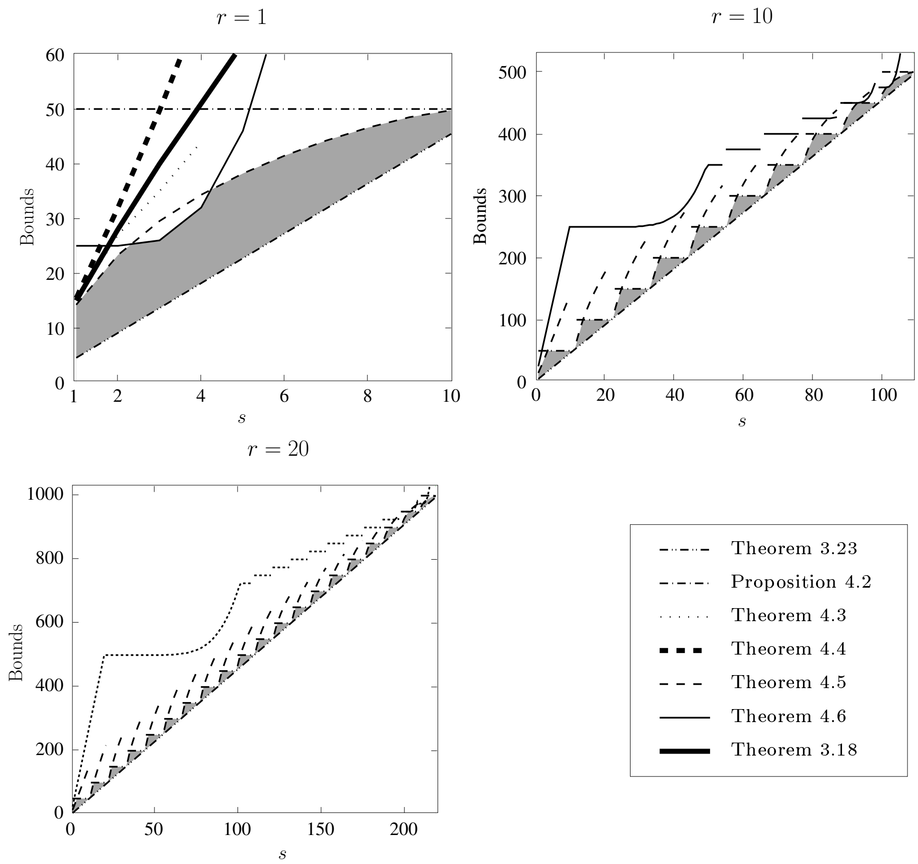


FIGURE 4.5: *A comparison of the bounds in §4.2 for the 10-regular graph G_3 of order 50. Illustrated above are the cases where $r = 1$, $r = 10$ and $r = 20$.*

4.4 Chapter summary

An upper bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph triple $(G, \boldsymbol{r}, \boldsymbol{s})$ was presented in §4.1. In that section it was also demonstrated why it is sufficient only to consider graph triples for which $r_{\min} \leq s_{\max}$, where $r_{\min} = \min_i\{r_i\}$ and $s_{\max} = \max_i\{s_i\}$.

In §4.2 only the balanced case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination was considered. An upper bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number was established in terms of the minimum degree of a graph, after which the probabilistic method was used to establish improvements on this bound. In total, four bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number were established in §4.2 for the balanced case, using the probabilistic method.

The bounds of §4.2 were finally compared for three randomly generated graphs in §4.3 so as to illustrate the relative performances of these bounds.

CHAPTER 5

Special graph classes

Contents

In this chapter the focus falls on the following special classes of graphs: complete graphs, cycles, paths, cartesian products, circulants and bipartite graphs. For each of these classes upper bounds on or exact values of the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination number are proposed. Note that, in view of Proposition 4.1, the only graph triples $(G, \boldsymbol{r}, \boldsymbol{s})$ considered in this chapter are those for which $\min_i\{r_i\} \leq \max_i\{s_i\}$.

5.1 Complete graphs

First consider the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination number of a complete graph.

Proposition 5.1. *Let $(K_n, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where $\boldsymbol{r} = [r_1, \ldots, r_n]$ and $\boldsymbol{s} = [s_1, \ldots, s_n]$ satisfy $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for $i = 1, \ldots, n$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_n) = \max\{s_1, \ldots, s_n\}.$$

Proof. Let $s_j = \max\{s_1, \ldots, s_n\}$ and suppose $e \in \mathbb{N}_0$ such that $s_j = \sum_{i \in A} r_i + e$ for some subset $A \subseteq \{1, 2, \ldots, n\}$. Then the function

$$f(v_i) \quad = \quad \begin{cases} r_i & \text{if } i \in A \\ e & \text{for one element of } \{1, \ldots, n\} \backslash A \\ 0 & \text{otherwise} \end{cases}$$

is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of K_n and hence $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_n) \leq |f| = s_j$.

Now assume that $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_n) < s_j$. There then exists an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function f of minimum weight such that $|f| < s_j$. But

$$\sum_{v_k \in N[v_j]} f(v_k) = \sum_{v_k \in V(K_n)} f(v_k) = |f| < s_j,$$

which is impossible. Hence $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_n) = s_j$. ■

This result correlates with the result by Burger and Van Vuuren [11] in §3.4.2 for the balanced case where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$.

5.2 Cycles

Suppose that the vertices of the n-cycle C_n are labelled $v_1, \ldots, v_n$ and let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of C_n of minimum weight. Then it follows, for any three consecutive vertices of C_n, that $f(v_{i-1}) + f(v_i) + f(v_{i+1}) \geq s_i$.

Since $N[v_1] = N[v_2] = N[v_3]$ for C_3, the value of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the triangle C_3 follows easily.

Proposition 5.2. *Let $(C_3, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple where $\boldsymbol{r} = [r_1, r_2, r_3]$ and $\boldsymbol{s} = [s_1, s_2, s_3]$ satisfy $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for $i = 1, 2, 3$. Then $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_3) = \max\{s_1, s_2, s_3\}$.* ■

A lower bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the 4-cycle C_4 is given next.

Proposition 5.3. *Let $(C_4, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where $\boldsymbol{r} = [r_1, \ldots, r_4]$ and $\boldsymbol{s} = [s_1, \ldots, s_4]$ satisfy $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for all $i = 1, \ldots, 4$. Then*

$$\begin{aligned} \gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4) \geq \max\{&s_1, s_2, s_3, s_4, s_1 + s_3 - r_2 - r_4, s_1 + s_4 - r_1 - r_4, s_2 + s_3 - r_2 - r_3, \\ &s_2 + s_4 - r_1 - r_3, \lceil (s_1 + s_2 + s_3 + s_4)/3 \rceil\}. \end{aligned}$$

Proof. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of C_4 of minimum weight. Then

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4) = \underbrace{f(v_{i-1}) + f(v_i) + f(v_{i+1})}_{\geq s_i} + \underbrace{f(v_{i+2})}_{\geq 0} \geq s_i$$

for $i = 1, 2, 3, 4$, where the indices are taken modulo 4.

Furthermore,

$$\begin{aligned} \gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4) = &\underbrace{f(v_1) + f(v_2)}_{\geq \max\{s_1 - r_4, s_2 - r_3\}} + \underbrace{f(v_3) + f(v_4)}_{\geq \max\{s_3 - r_2, s_4 - r_1\}} \\ &\geq \max\{s_1 + s_3 - r_2 - r_4, s_1 + s_4 - r_1 - r_4, s_2 + s_3 - r_2 - r_3, s_2 + s_4 - r_1 - r_3\}. \end{aligned}$$

The fact that $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4) \geq \lceil \frac{s_1 + s_2 + s_3 + s_4}{3} \rceil$ follows from Theorem 3.23. ■

Considering the results for cycles of order 3 and 4, it would seem that the number of cases to consider in the proofs grow considerably for cycles of larger orders. In an attempt at establishing a more general result, a step function is considered for the $\boldsymbol{s}$ vector.

Proposition 5.4. *Let* $\left(C_n, \boldsymbol{r}, \boldsymbol{s}_m^k\right)$ *be a graph triple, where* $\boldsymbol{r} = [r, \ldots, r]$ *and*

$$\boldsymbol{s}_m^k = [\overbrace{s+k, \ldots, s+k}^{m \text{ times}}, s, \ldots, s]$$

satisfy $r \geq \max\left\{s, \left\lceil \frac{s}{3} \right\rceil + k, \left\lceil \frac{2s+k}{3} \right\rceil, \left\lceil \frac{s+k}{2} \right\rceil\right\}$ *for some natural numbers* k *and* $m < n$. *Then*

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = \begin{cases} \max\left\{\left\lceil \frac{sn+mk}{3} \right\rceil, (s+k)\left\lfloor \frac{n}{3} \right\rfloor + \left\lceil \frac{s+k}{2} \right\rceil\right\} & \text{if } n \equiv 2 \pmod 3 \text{ and } m \neq n-1, \\ \max\left\{\left\lceil \frac{sn+mk}{3} \right\rceil, s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{s}{2} \right\rceil\right\} & \text{if } n, m \equiv 2 \pmod 3, \\ \max\left\{\left\lceil \frac{sn+mk}{3} \right\rceil, s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil\right\} & \text{otherwise.} \end{cases}$$

Proof. It follows from Theorem 3.23 that

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) \geq \left\lceil \frac{\sum_{i=1}^n s_i}{3} \right\rceil = \left\lceil \frac{sn+mk}{3} \right\rceil.$$

Let $n = 3\ell$ for some $\ell \in \mathbb{N}_0$ and consider the following three subcases.

If $m = 3a$ for some $a \in \mathbb{N}_0$, then

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = \underbrace{f(v_1) + \cdots + f(v_m)}_{\geq a(s+k)} + \underbrace{f(v_{m+1}) + \cdots + f(v_n)}_{\geq s(\ell-a)} \geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil.$$

If $m = 3a+1$ for some $a \in \mathbb{N}_0$, then

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = \underbrace{f(v_n) + f(v_1) + \cdots + f(v_{m+1})}_{\geq (a+1)(s+k)} + \underbrace{f(v_{m+2}) + \cdots + f(v_{n-1})}_{\geq s(\ell-(a+1))} \geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil.$$

Finally, if $m = 3a+2$ for some $a \in \mathbb{N}_0$, then

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = \underbrace{f(v_1) + \cdots + f(v_{m+1})}_{\geq (a+1)(s+k)} + \underbrace{f(v_{m+2}) + \cdots + f(v_n)}_{\geq s(\ell-(a+1))} \geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil.$$

Note that $\left\lceil \frac{sn+mk}{3} \right\rceil \leq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil$ for $n \equiv 0 \pmod 3$ and any $s, m, k \in \mathbb{N}_0$.

For an upper bound, the construction in [11] is used to find an $\boldsymbol{s}_m^k$-dominating $\mathbf{r}$-function f. Let f be the $\mathbf{r}$-function satisfying $\sum_{i=1}^3 f(v_{3j+i}) = s$ for all $j = 0, \ldots, \ell-1$ and $\left\lceil \frac{s}{3} \right\rceil \geq f(v_{3j+1}) \geq f(v_{3j+2}) \geq f(v_{3j+3}) \geq \left\lfloor \frac{s}{3} \right\rfloor$ for all $j = 0, \ldots, \ell-1$. Now, let f' be an $\mathbf{r}$-function such that $f'(v_{3j+2}) = f(v_{3j+2}) + k$ for $j = 0, \ldots, \left\lceil \frac{m}{3} \right\rceil$, while $f'(v) = f(v)$ for all the other vertices of C_n. Then f' is $\boldsymbol{s}_m^k$-dominating and hence

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = |f'| \leq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil.$$

This concludes the case where $n = 3\ell$ for some $\ell \in \mathbb{N}_0$.

Next suppose $n = 3\ell+1$ for some $\ell \in \mathbb{N}_0$ and consider the following three subcases.

If $m = 3a$ for some $a \in \mathbb{N}_0$, then

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = \underbrace{f(v_1) + \cdots + f(v_m)}_{\geq a(s+k)} + \underbrace{f(v_{m+1}) + \cdots + f(v_{n-1})}_{\geq s(\ell-a)} + \underbrace{f(v_n)}_{\geq 0} \geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil.$$

If $m = 3a + 1$ for some $a \in \mathbb{N}_0$, then

$$\begin{aligned}\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) &= \underbrace{f(v_n) + f(v_1) + \cdots + f(v_{m+1})}_{\geq (a+1)(s+k)} + \underbrace{f(v_{m+2}) + \cdots + f(v_{n-2})}_{\geq s(\ell-(a+1))} + \underbrace{f(v_{n-1})}_{\geq 0} \\ &\geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil.\end{aligned}$$

Finally, if $m = 3a + 2$ for some $a \in \mathbb{N}_0$, then

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = \underbrace{f(v_1) + \cdots + f(v_{m+1})}_{\geq (a+1)(s+k)} + \underbrace{f(v_{m+2}) + \cdots + f(v_{n-1})}_{\geq s(\ell-(a+1))} + \underbrace{f(v_n)}_{\geq 0} \geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil.$$

To establish an upper bound, it is first determined which of the two bounds $\left\lceil \frac{sn+mk}{3} \right\rceil$ or $s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil$ is larger. Let $m = 3a + b$ for some $a \in \mathbb{N}_0$ and $b \in \{0, 1, 2\}$.

If $b = 0$, then $\left\lceil \frac{sn+mk}{3} \right\rceil \geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil$. For an upper bound, the construction in [11] is used to find an $\boldsymbol{s}_m^k$-dominating $\mathbf{r}$-function f. Let f be the $\mathbf{r}$-function satisfying $\sum_{i=1}^{3} f(v_{3j+i}) = s$ for all $j = 0, \ldots, \ell - 1$ and $\left\lceil \frac{s}{3} \right\rceil \geq f(v_{3j+1}) \geq f(v_{3j+2}) \geq f(v_{3j+3}) \geq \left\lfloor \frac{s}{3} \right\rfloor$ for all $j = 0, \ldots, \ell - 1$ with $f(v_n) = f(v_1)$. Now, let f' be an $\mathbf{r}$-function such that $f'(v_{3j+2}) = f(v_{3j+2}) + k$ for $j = 0, \ldots, a-1$, while $f'(v) = f(v)$ for all the other vertices of C_n. Then f' is $\boldsymbol{s}_m^k$-dominating and hence

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = |f'| \leq \left\lceil \frac{sn}{3} \right\rceil + \frac{mk}{3} = \left\lceil \frac{sn+mk}{3} \right\rceil.$$

In the cases where $b \neq 0$, first assume that $\left\lceil \frac{sn+mk}{3} \right\rceil \leq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil$. This inequality implies that $\left\lceil \frac{s+bk}{3} \right\rceil \leq k$ which, in turn, implies that $k \geq \frac{s}{3-b}$.

If $b = 1$, then $k \geq \frac{s}{2}$. Let f be an $\mathbf{r}$-function such that $f(v_{3j+1}) = \left\lfloor \frac{s}{2} \right\rfloor$ for $j = 1, \ldots, \ell$ and $f(v_{3j+2}) = \left\lceil \frac{s}{2} \right\rceil$ for $j = 0, \ldots, \ell - 1$. Furthermore, let $f(v_{3j+3}) = 0$ for $j = a, \ldots, \ell$ and $f(v_{3j+3}) = k$ for $j = 0, \ldots, a - 1$ and set $f(v_1) = k$. Since $k > \frac{s}{2}$, it follows that f is an $\boldsymbol{s}_m^k$-dominating function of C_n. Hence

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = |f| \leq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil.$$

If $b = 2$, then $k \geq s$ and an $\mathbf{r}$-function f can be defined as follows. Let $f(v_{3j+2}) = s$ for $j = 0, \ldots, \ell - 1$, $f(v_{3j+1}) = k$ for $j = 0, \ldots, a$ and for any other vertex v, set $f(v) = 0$. Since $k > s$, it follows that f is an $\boldsymbol{s}_m^k$-dominating function of C_n. Hence

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = |f| \leq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil.$$

Now consider the possibility that $\left\lceil \frac{sn+mk}{3} \right\rceil > s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil$, so that $\left\lceil \frac{s+bk}{3} \right\rceil > k$. Let f be the $\mathbf{r}$-function satisfying $\sum_{i=1}^{3} f(v_{3j+i}) = s$ for all $j = 1, \ldots, \ell-1$ and $f(v_{3j+2}) = \left\lceil \left(s + k - \left\lceil \frac{s+bk}{3} \right\rceil\right)/2 \right\rceil$, $f(v_{3j+3}) = \left\lfloor \left(s + k - \left\lceil \frac{s+bk}{3} \right\rceil\right)/2 \right\rfloor$ and $f(v_{3j+4}) = \left\lceil \frac{s+bk}{3} \right\rceil - k$ for all $j = 0, \ldots, \ell - 1$ with $f(v_1) = \left\lceil \frac{s+bk}{3} \right\rceil$. Now, let f' be an $\mathbf{r}$-function such that $f'(v_{3j+1}) = f(v_{3j+1}) + k$ for $j = 1, \ldots, a$ and $f'(v) = f(v)$ for all the other vertices C_n. Then f' is a $\boldsymbol{s}_m^k$-dominating function and

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = |f'| \leq s\ell + ka + \left\lceil \frac{s+bk}{3} \right\rceil = \left\lceil \frac{sn+mk}{3} \right\rceil.$$

Finally, suppose $n = 3\ell + 2$ for some $\ell \in \mathbb{N}_0$ and consider the following four subcases.

If $m = 3a$ for some $a \in \mathbb{N}_0$, then

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = \underbrace{f(v_1) + \cdots + f(v_m)}_{\geq a(s+k)} + \underbrace{f(v_{m+1}) + \cdots + f(v_{n-2})}_{\geq s(\ell-a)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq 0} \geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil.$$

If $m = 3a + 1$ for some $a \in \mathbb{N}_0$, $a \neq \ell$, then

$$\begin{aligned}\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) &= \underbrace{f(v_n) + f(v_1) + \cdots + f(v_{m+1})}_{\geq (a+1)(s+k)} + \underbrace{f(v_{m+2}) + \cdots + f(v_{n-3})}_{\geq s(\ell-(a+1))} + \underbrace{f(v_{n-2}) + f(v_{n-1})}_{\geq 0} \\ &\geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil.\end{aligned}$$

If $m = 3\ell + 1$, then

$$\begin{aligned}2\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) &= \underbrace{f(v_1)}_{\geq 0} + \underbrace{f(v_2) + \cdots + f(v_{n-1})}_{\geq \ell(s+k)} + \underbrace{f(v_1) + f(v_2) + f(v_n)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_n)}_{\geq \ell(s+k)} \\ &\geq 2(s+k)\left\lfloor \frac{n}{3} \right\rfloor + s + k.\end{aligned}$$

Therefore,

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) \geq (s+k)\left\lfloor \frac{n}{3} \right\rfloor + \left\lceil \frac{s+k}{2} \right\rceil.$$

If $m = 3a + 2$, then

$$\begin{aligned}2\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) =& \underbrace{f(v_1) + \cdots + f(v_{m+1})}_{\geq (a+1)(s+k)} + \underbrace{f(v_{m+2}) + \cdots + f(v_{n-2})}_{\geq s(\ell-a-1)} \\ &+ \underbrace{f(v_{n-1})}_{\geq 0} + \underbrace{f(v_n) + f(v_1) + \cdots + f(v_m)}_{\geq (a+1)(s+k)} + \underbrace{f(v_{m+1}) + \cdots + f(v_n)}_{\geq s(\ell-a)} \\ \geq& 2s\left\lfloor \frac{n}{3} \right\rfloor + 2k\left\lceil \frac{m}{3} \right\rceil + s.\end{aligned}$$

Therefore,

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) \geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{s}{2} \right\rceil.$$

Let $m = 3a + b$ for some $a \in \mathbb{N}_0$ and $b \in \{0, 1, 2\}$.

If $b = 0$, then $\lceil \frac{sn+mk}{3} \rceil \geq s\lfloor \frac{n}{3} \rfloor + k\lceil \frac{m}{3} \rceil$. For an upper bound, the construction given in [11] is used to find an $\boldsymbol{s}_m^k$-dominating $\mathbf{r}$-function f. Let f be the $\mathbf{r}$-function satisfying $\sum_{i=1}^{3} f(v_{3j+i}) = s$ for all $j = 0, \ldots, \ell - 1$ and $\lceil \frac{s}{3} \rceil \geq f(v_{3j+1}) \geq f(v_{3j+2}) \geq f(v_{3j+3}) \geq \lfloor \frac{s}{3} \rfloor$ for all $j = 0, \ldots, \ell - 1$ with $f(v_{n-1}) = f(v_1)$ and $f(v_n) = f(v_2)$. Now, let f' be an $\mathbf{r}$-function such that $f'(v_{3j+2}) = f(v_{3j+2}) + k$ for $j = 0, \ldots, a - 1$ and $f'(v) = f(v)$ for all the other vertices of C_n. Then f' is $\boldsymbol{s}_m^k$-dominating and

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = |f'| \leq \left\lceil \frac{sn}{3} \right\rceil + \frac{mk}{3} = \left\lceil \frac{sn+mk}{3} \right\rceil.$$

If $b = 1$ and $\lceil \frac{sn+mk}{3} \rceil \geq s\lfloor \frac{n}{3} \rfloor + k\lceil \frac{m}{3} \rceil$, then $\lceil (2s+k)/3 \rceil \geq k$. Let f be an $\mathbf{r}$-function such that $f(v_{3j+1}) = s - \lceil (2s+k)/3 \rceil$, $f(v_{3j+2}) = 2\lceil (2s+k)/3 \rceil - s$ and $f(v_{3j+3}) = s - \lceil (2s+k)/3 \rceil$ for

$j = 0, \ldots, \ell - 1$ with $f(v_{n-1}) = s - \lceil (2s + k)/3 \rceil$ and $f(v_n) = 2\lceil (2s + k)/3 \rceil - s$. Now, let f' be an $\mathbf{r}$-function such that $f'(v_{3j+2}) = f(v_{3j+3}) + k$ for $j = 0, \ldots, a - 1$ and $f'(v) = f(v)$ for all the other vertices of C_n. Then f' is an $\boldsymbol{s}_m^k$-dominating function of C_n and

$$\begin{aligned}\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = |f| &\leq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + \left\lceil \frac{2s+k}{3} \right\rceil \\ &= \left\lceil \frac{sn+mk}{3} \right\rceil.\end{aligned}$$

If $b = 1$ and $\left\lceil \frac{sn+mk}{3} \right\rceil \leq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil$, then $\lceil (2s + k)/3 \rceil \leq k$. Let f be an $\mathbf{r}$-function such that $f(v_{3j+2}) = s$ for $j = 0, \ldots, \ell - 1$, $f(v_{3j+3}) = k$ for $j = 0, \ldots, a - 1$ and $f(v_n) = k$, otherwise let $f(v) = 0$. Since $k \geq s$, it follows that f is an $\boldsymbol{s}_m^k$-dominating function of C_n. Hence

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = |f| \leq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil.$$

If $m = n - 1$ and $\left\lceil \frac{s+k}{2} \right\rceil \geq \left\lceil \frac{2s+k}{3} \right\rceil$, then let f be the $\mathbf{r}$-function such that $f(v_{3j+1}) = 0$, $f(v_{3j+2}) = \lceil (s + k)/2 \rceil$ and $f(v_{3j+3}) = \lfloor (s + k)/2 \rfloor$ for $j = 0, \ldots, \ell - 1$ with $f(v_{n-1}) = 0$ and $f(v_n) = \lceil (s + k)/2 \rceil$. Then f is $\boldsymbol{s}_m^k$-dominating and

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = |f| \leq (s+k)\left\lfloor \frac{n}{3} \right\rfloor + \lceil (s+k)/2 \rceil.$$

If $\left\lceil \frac{s+k}{2} \right\rceil < \left\lceil \frac{2s+k}{3} \right\rceil$, then let f be the $\mathbf{r}$-function such that $f(v_{3j+1}) = s - \lceil (2s+k)/3 \rceil$, $f(v_{3j+2}) = 2\lceil (2s+k)/3 \rceil$ and $f(v_{3j+3}) = s+k-\lceil (2s+k)/3 \rceil$ for $j = 0, \ldots, \ell-1$ with $f(v_{n-1}) = s-\lceil (2s+k)/3 \rceil$ and $f(v_n) = 2\lceil (2s + k)/3 \rceil$. Then f is $\boldsymbol{s}_m^k$-dominating and

$$\begin{aligned}\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = |f| &\leq (s+k)\left\lfloor \frac{n}{3} \right\rfloor + \lceil (2s+k)/3 \rceil \\ &= \left\lceil \frac{sn+mk}{3} \right\rceil.\end{aligned}$$

If $b = 2$ and $\lceil (2s + 2k)/3 \rceil \leq \lceil s/2 \rceil + k$, then $k \geq \lceil s/2 \rceil$. Let f be the $\mathbf{r}$-function such that $f(v_{3j+1}) = k$, $f(v_{3j+2}) = \lceil s/2 \rceil$ and $f(v_{3j+3}) = \lfloor s/2 \rfloor$ for $j = 0, \ldots, a$. Also, let $f(v_{3j+1}) = 0$, $f(v_{3j+2}) = \lceil s/2 \rceil$ and $f(v_{3j+3}) = \lfloor s/2 \rfloor$ for $j = a + 1, \ldots, \ell - 1$ with $f(v_{n-1}) = 0$ and $f(v_n) = \lceil s/2 \rceil$. Then f is $\boldsymbol{s}_m^k$-dominating and

$$\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = |f| \leq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{s}{2} \right\rceil.$$

If $\lceil (2s + 2k)/3 \rceil > \lceil s/2 \rceil + k$, then let f be the $\mathbf{r}$-function such that $f(v_1) = \lceil (2s + 2k)/3 \rceil - \lceil (s + k - \lceil (s + k)/3 \rceil)/2 \rceil$ and $f(v_n) = \lceil (s + k - \lceil (s + k)/3 \rceil)/2 \rceil$. Furthermore, let $f(v_{3j+2}) = \lceil (s + k - \lceil (s + k)/3 \rceil)/2 \rceil$, $f(v_{3j+3}) = \lceil (s + k)/3 \rceil$ and $f(v_{3j+4}) = \lfloor (s + k - \lceil (s + k)/3 \rceil)/2 \rfloor$ for $j = 0, \ldots, a - 1$. Also, let $f(v_{3j+2}) = \lceil (s + k - \lceil (s + k)/3 \rceil)/2 \rceil$, $f(v_{3j+3}) = \lceil (s + k)/3 \rceil$ and $f(v_{3j+4}) = s - \lceil (s + k)/3 \rceil - \lceil (s + k - \lceil (s + k)/3 \rceil)/2 \rceil$ for $j = a, \ldots, \ell - 1$. Then f is $\boldsymbol{s}_m^k$-dominating and

$$\begin{aligned}\gamma_{\mathbf{r}}^{\boldsymbol{s}_m^k}(C_n) = |f| &\leq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + \left\lceil \frac{2s+2k}{3} \right\rceil \\ &= \left\lceil \frac{sn+mk}{3} \right\rceil.\end{aligned}$$

■

5.3 Paths

Suppose that the vertices of the path P_n of order n are labelled $v_1, \ldots, v_n$ and that v_1 and v_n are the end vertices of the path. Furthermore, let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then $f(v_1) \geq s - f(v_2) \geq s - r$ and similarly $f(v_n) \geq s - r$. Also, for any three consecutive vertices of P_n it follows that $f(v_{i-1}) + f(v_i) + f(v_{i+1}) \geq s_i$.

The $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a path of order 3 is easily established, as shown below.

Proposition 5.5. *Let $(P_3, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where $\boldsymbol{r} = [r_1, r_2, r_3]$ and $\boldsymbol{s} = [s_1, s_2, s_3]$ satisfy $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for $i = 1, 2, 3$. Then $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) = \max\{s_1, s_2, s_3, s_1 + s_3 - r_2\}$.*

Proof. It is first shown that $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3)$ is at least as large as s_1, s_2, s_3 and $s_1 + s_3 - r_2$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_3 of minimum weight. Then

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) = \underbrace{f(v_1) + f(v_2)}_{\geq s_1} + \underbrace{f(v_3)}_{\geq 0} \geq s_1.$$

It easily follows that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) = \underbrace{f(v_1) + f(v_2) + f(v_3)}_{\geq s_2} \geq s_2.$$

Similarly,

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) = \underbrace{f(v_1)}_{\geq 0} + \underbrace{f(v_2) + f(v_3)}_{\geq s_3} \geq s_3.$$

Moreover,

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) = \underbrace{f(v_1) + f(v_2)}_{\geq s_1} + \underbrace{f(v_3)}_{\geq s_3 - r_2} \geq s_1 + s_3 - r_2.$$

For the upper bound, first consider the case where $s_i = \max\{s_1, s_2, s_3, s_1 + s_3 - r_2\}$ for $i \in \{1, 3\}$. Let f be an $\boldsymbol{r}$-function of P_3 such that $f(v_i) = s_i - r_2$, $f(v_2) = r_2$ and $f(v_j) = 0$ for $j \neq i, 2$. Then f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_3 and so $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) \leq s_i$.

If $s_2 = \max\{s_1, s_2, s_3, s_1 + s_3 - r_2\}$ and $s_i \geq r_2 + r_i$ for $i = 1$ or 3, then let f be an $\boldsymbol{r}$-function of P_3 such that $f(v_i) = r_i$, $f(v_2) = r_2$ and $f(v_j) = s_2 - r_i - r_2$ for $j \neq i, 2$. If $s_i < r_2 + r_i$ and $s_i \leq r_2$ for $i = 1, 3$, then let f be an $\boldsymbol{r}$-function of P_3 such that $f(v_j) = s_2 - r_2$, $f(v_2) = r_2$ and $f(v_i) = 0$ for $j \neq i, 2$. If $s_i < r_2 + r_i$ and $s_i > r_2$ for $i = 1$ or 3, then let f be an $\boldsymbol{r}$-function of P_3 such that $f(v_i) = s_i - r_2$, $f(v_2) = r_2$ and $f(v_j) = s_2 - s_i$ for $j \neq i, 2$. In all three cases f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_3 and so $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) \leq s_2$.

Finally, if $s_1 + s_3 - r_2 = \max\{s_1, s_2, s_3, s_1 + s_3 - r_2\}$, then let f be an $\boldsymbol{r}$-function of P_3 such that $f(v_1) = s_1 - r_2$, $f(v_2) = r_2$ and $f(v_3) = s_3 - r_2$. Then f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_3 and so $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) \leq s_1 + s_3 - r_2$. ■

The $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a path of order 4 is considered next.

Proposition 5.6. *Let $(P_4, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where $\boldsymbol{r} = [r_1, \ldots, r_4]$ and $\boldsymbol{s} = [s_1, \ldots, s_4]$ satisfy $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for all $i = 1, 2, 3, 4$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_4) = \max\{s_2, s_3, s_1 + s_4, s_1 + s_3 - r_2, s_2 + s_4 - r_3, s_2 + s_3 - r_2 - r_3\}. \quad ■$$

The proof of Proposition 5.6 is similar to that of Proposition 5.5 and may be found in Appendix A. Considering the results for paths of orders 3 and 4, it would seem that the number of cases to consider in the proofs grow considerably for paths of larger orders. In an attempt at establishing a more general result, a step function is considered for the $\boldsymbol{s}$ vector. Firstly, consider a vector with one step.

Proposition 5.7. *Let* $\left(P_n, \boldsymbol{r}, \boldsymbol{s}_m^k\right)$ *be a graph triple, where* $\boldsymbol{r} = [r, \ldots, r]$ *and*

$$\boldsymbol{s}_m^k = [\overbrace{s+k, \ldots, s+k}^{m \text{ times}}, s, \ldots, s]$$

satisfy $2r \geq s+k$ *for some natural number* k *and* $m < n$. *Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_m^k}(P_n) = s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + g(s,k,r)$$

where

$$g(s,k,r) = \begin{cases} s+k-r & \text{if } n, m \equiv 0 \pmod 3 \\ \max\{s+k-r, k\} & \text{if } n \equiv 0 \pmod 3 \text{ and } m \equiv 1,2 \pmod 3 \\ \max\{s+k-r, s\} & \text{if } n \equiv 1 \pmod 3 \text{ and } m \equiv 0 \pmod 3 \\ s & \text{if } n \equiv 2 \pmod 3 \text{ and } m \equiv 0 \pmod 3 \\ s+k & \text{otherwise.} \end{cases}$$

Proof. Let f be an $\boldsymbol{s}_m^k$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. First consider the case where $n = 3\ell$ for some $\ell \in \mathbb{N}_0$.

If $m = 3a + b$ for some $a \in \mathbb{N}_0$ and $b \in \{0,1,2\}$, then

$$\begin{aligned} \gamma_{\boldsymbol{r}}^{\boldsymbol{s}_m^k}(P_n) &= \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{3a+1})}_{\geq a(s+k)} + \underbrace{f(v_{3a+2}) + \cdots + f(v_{n-2})}_{\geq s(\ell-a-1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\ &\geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + s + k - r. \end{aligned} \tag{5.1}$$

Also, if $b \in \{1,2\}$, then

$$\begin{aligned} \gamma_{\boldsymbol{r}}^{\boldsymbol{s}_m^k}(P_n) &= \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{3a+2})}_{\geq a(s+k)} + \underbrace{f(v_{3a+3}) + \cdots + f(v_{n-1})}_{\geq s(\ell-a-1)} + \underbrace{f(v_{n-1})}_{\geq 0} \\ &\geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + k. \end{aligned} \tag{5.2}$$

Note that the expression on the right-hand side of (5.2) is larger than that in (5.1) when $s < r$.

To find an upper bound on $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_m^k}(P_n)$, an $\boldsymbol{s}_m^k$-dominating $\boldsymbol{r}$-function is constructed. If $m = 3a$ and $s < r$, then let f be an $\boldsymbol{r}$-function such that $f(v_1) = s+k-r$, $f(v_{n-1}) = s$ and $f(v_n) = 0$. Furthermore let $f(v_{3c+2}) = r$, $f(v_{3c+3}) = s+k-r$ and $f(v_{3c+4}) = 0$ for $c = 0, \ldots, a-1$. Also, let $f(v_{3c+2}) = s$ and $f(v_{3c+3}) = f(v_{3c+4}) = 0$ for $c = a, \ldots, \ell-2$. Then f is $\boldsymbol{s}_m^k$-dominating and $|f| = s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + s + k - r$.

If $m \neq 3a$ and $s < r$, then let f be an $\boldsymbol{r}$-function such that $f(v_{3c+1}) = s+k-r$, $f(v_{3c+2}) = r$ and $f(v_{3c+3}) = 0$ for $c = 0, \ldots, a$. Also, let $f(v_{3c+1}) = f(v_{3c+3}) = 0$ and $f(v_{3c+2}) = s$ for $c = a+1, \ldots, \ell-1$. It follows that f is an $\boldsymbol{s}_m^k$-dominating function and $|f| = s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + k$.

If $s \geq r$, then let f be an $\boldsymbol{r}$-function such that $f(v_1) = s + k - r$, $f(v_2) = r$ and $f(v_n) = s - r$. Furthermore let $f(v_{3c+3}) = 0$, $f(v_{3c+4}) = s + k - r$ and $f(v_{3c+5}) = r$ for $c = 0, \ldots, a - 1$. Also, let $f(v_{3c+3}) = 0$, $f(v_{3c+4}) = s - r$ and $f(v_{3c+5}) = r$ for $c = a, \ldots, \ell - 2$. Then f is $\boldsymbol{s}_m^k$-dominating and $|f| = s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + s + k - r$.

It therefore follows that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_m^k}(P_n) \leq \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + s + k - r & \text{if } m = 3a \text{ and} \\ \left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + \max\{s + k - r, k\} & \text{if } m \neq 3a. \end{cases} \tag{5.3}$$

The result for the case where $n \equiv 0 \pmod 3$ follows from a combination of (5.1)–(5.3).

Next consider the case where $n = 3\ell + 1$ for some $\ell \in \mathbb{N}_0$.

If $m = 3a$ for some $a \in \mathbb{N}_0$, then

$$\begin{aligned} \gamma_{\boldsymbol{r}}^{\boldsymbol{s}_m^k}(P_n) &= \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{3a+1})}_{\geq a(s+k)} + \underbrace{f(v_{3a+2}) + \cdots + f(v_n)}_{\geq s(\ell - a)} \\ &\geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + s + k - r \end{aligned} \tag{5.4}$$

and also

$$\begin{aligned} \gamma_{\boldsymbol{r}}^{\boldsymbol{s}_m^k}(P_n) &= \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{3a-1})}_{\geq (a-1)(s+k)} + \underbrace{f(v_{3a}) + \cdots + f(v_{n-2})}_{\geq s(\ell - a)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\ &\geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + s. \end{aligned} \tag{5.5}$$

Note that the expression on the right-hand side of (5.5) is larger than in (5.4) when $k < r$.

If $m = 3a + b$ for some $a \in \mathbb{N}_0$ and $b \in \{1, 2\}$, then

$$\begin{aligned} \gamma_{\boldsymbol{r}}^{\boldsymbol{s}_m^k}(P_n) &= \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{3a+2})}_{\geq a(s+k)} + \underbrace{f(v_{3a+3}) + \cdots + f(v_n)}_{\geq s(\ell - a)} \\ &\geq s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + s + k. \end{aligned} \tag{5.6}$$

If $m = 3a$ and $k < r$, then let f be an $\boldsymbol{r}$-function such that $f(v_1) = \min\{r, s\}$, $f(v_2) = \max\{s + k - r, k\}$ and let $f(v_{3c+3}) = \min\{r, s\}$ and $f(v_{3c+4}) = \max\{s - r, 0\}$ for $c = 0, \ldots, \ell - 1$. Also, let $f(v_{3c+2}) = k$ for $c = 1, \ldots, a - 1$ with $f(v) = 0$ otherwise. Then f is $\boldsymbol{s}_m^k$-dominating and $|f| = s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + s$.

Now, if $m = 3a$ and $k \geq r$, then $s \leq r$. Let f be the $\boldsymbol{r}$-function where $f(v_1) = s + k - r$ and let $f(v_{3c+2}) = r$, $f(v_{3c+3}) = 0$ and $f(v_{3c+4}) = s + k - r$ for $c = 0, \ldots, a - 1$. Also, let $f(v_{3c+2}) = 0$, $f(v_{3c+3}) = s$ and $f(v_{3c+4}) = 0$ for $c = a, \ldots, \ell - 1$. Then f is $\boldsymbol{s}_m^k$-dominating and $|f| = s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + s + k - r$.

If $m \neq 3a$, then let f be an $\boldsymbol{r}$-function such that $f(v_{3c+1}) = \max\{s + k - r, 0\}$ and $f(v_{3c+2}) = \min\{r, s + k\}$ for $c = 0, \ldots, a$ and $f(v_{3c+3}) = 0$ for $c = 0, \ldots, a - 1$. Also, let $f(v_{3c+2}) = 0$ and $f(v_{3c+3}) = \max\{s - r, 0\}$ for $c = a + 1, \ldots, \ell - 1$ and let $f(v_{3c+4}) = \min\{r, s\}$ for $c = a, \ldots, \ell - 1$. Then f is $\boldsymbol{s}_m^k$-dominating and $|f| = s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + s + k - r$.

It therefore follows that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_m^k}(P_n) \leq \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + \max\{s + k - r, s\} & \text{if } m = 3a \text{ and} \\ \left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + s + k & \text{if } m \neq 3a. \end{cases} \tag{5.7}$$

The result for the case where $n \equiv 1 \pmod 3$ follows from a combination of (5.4)–(5.7).

Finally, consider the case where $n = 3\ell + 2$ for some $\ell \in \mathbb{N}_0$.

If $m = 3a$ for some $a \in \mathbb{N}_0$, then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_m^k}(P_n) &= \underbrace{f(v_1) + \cdots + f(v_{3a})}_{\geq a(s+k)} + \underbrace{f(v_{3a+1}) + \cdots + f(v_{n-2})}_{\geq s(\ell-a)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\ &\geq s\left\lfloor\frac{n}{3}\right\rfloor + k\left\lfloor\frac{m}{3}\right\rfloor + s. \end{aligned} \tag{5.8}$$

If $m = 3a + b$ for some $a \in \mathbb{N}_0$ and $b \in \{1, 2\}$, then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_m^k}(P_n) &= \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{3a+2})}_{\geq a(s+k)} + \underbrace{f(v_{3a+3}) + \cdots + f(v_n)}_{\geq s(\ell-a)} \\ &\geq s\left\lfloor\frac{n}{3}\right\rfloor + k\left\lfloor\frac{m}{3}\right\rfloor + s + k. \end{aligned} \tag{5.9}$$

If $m = 3a$, then let f be an $\boldsymbol{r}$-function such that $f(v_{3c+1}) = \min\{r, s\}$ and $f(v_{3c+3}) = 0$ for $c = 0, \ldots, \ell - 1$. Also, let $f(v_{3c+2}) = \max\{s + k - r, k\}$ for $c = 0, \ldots, a - 1$ and $f(v_{3c+2}) = \max\{s - r, 0\}$ for $c = a, \ldots, \ell - 1$. Finally, let $f(v_{n-1}) = \min\{r, s\}$ and $f(v_n) = \max\{s - r, 0\}$. Then f is $\boldsymbol{s}_m^k$-dominating and $|f| = s\left\lfloor\frac{n}{3}\right\rfloor + k\left\lfloor\frac{m}{3}\right\rfloor + s$.

If $m \neq 3a$, then let f be an $\boldsymbol{r}$-function such that $f(v_{3c+1}) = \min\{r, s\}$ and $f(v_{3c+3}) = 0$ for $c = 0, \ldots, \ell - 1$. Also, let $f(v_{3c+2}) = \max\{s+k-r, k\}$ for $c = 0, \ldots, a$ and $f(v_{3c+2}) = \max\{s-r, 0\}$ for $c = a + 1, \ldots, \ell - 1$. Finally, let $f(v_{n-1}) = \min\{r, s\}$ and $f(v_n) = \max\{s - r, 0\}$. Then f is $\boldsymbol{s}_m^k$-dominating and $|f| = s\left\lfloor\frac{n}{3}\right\rfloor + k\left\lfloor\frac{m}{3}\right\rfloor + s + k$.

It therefore follows that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_m^k}(P_n) \leq \begin{cases} \left\lfloor\frac{n}{3}\right\rfloor + k\left\lfloor\frac{m}{3}\right\rfloor + s & \text{if } m = 3a \text{ and} \\ \left\lfloor\frac{n}{3}\right\rfloor + k\left\lfloor\frac{m}{3}\right\rfloor + s + k & \text{if } m \neq 3a. \end{cases} \tag{5.10}$$

The result for the case where $n \equiv 2 \pmod 3$ follows from a combination of (5.8)–(5.10). ■

When $\boldsymbol{s}$ is a step function with two steps, the cases where $n \equiv 0, 1, 2 \pmod 3$ are considered separately. The proofs of the next three results are similar to that of Proposition 5.7 and may be found in Appendix A. First consider the case where $n \equiv 0 \pmod 3$.

Proposition 5.8. *Let $\left(P_n, \boldsymbol{r}, \boldsymbol{s}_{m,p}^{k,\ell}\right)$ be a graph triple, where $\boldsymbol{r} = [r, \ldots, r]$ and*

$$\boldsymbol{s}_{m,p}^{k,\ell} = [\overbrace{s+k, \ldots, s+k}^{m \text{ times}}, \overbrace{s+\ell, \ldots, s+\ell}^{p \text{ times}}, s, \ldots, s]$$

satisfy $2r \geq s + k$ and $3r \geq s + \ell$ for $m, p < n$, $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$. If $n \equiv 0 \pmod 3$, then

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = s\left\lfloor\frac{n}{3}\right\rfloor + k\left\lfloor\frac{m}{3}\right\rfloor + \ell\left\lfloor\frac{p}{3}\right\rfloor + g(s, k, l, r)$$

where

$$g(s,k,l,r) = \begin{cases} \max\{s+k-r, s+\ell-r\} & \text{if } m,p \equiv 0 \pmod 3 \\ \max\{s+k-r, s+\ell-r, \ell\} & \text{if } m \equiv 0 \text{ and } p \equiv 1 \pmod 3 \\ \max\{s+k-r, s+\ell-r, \ell, s+2\ell-2r\} & \text{if } m \equiv 0 \text{ and } p \equiv 2 \pmod 3 \\ \max\{s+k-r, k, k-\ell, s+\ell-2r\} & \text{if } m \equiv 1 \text{ and } p \equiv 0 \pmod 3 \\ \max\{s+k-r, k, \ell, k-\ell\} & \text{if } m,p \equiv 1 \pmod 3 \\ \max\{s+k+\ell-r, s+k-r, k, \ell, s+2\ell-2r\} & \text{if } m \equiv 1 \text{ and } p \equiv 2 \pmod 3 \\ \max\{s+k-r, k, k-\ell, s+k+\ell-2r, \ell-r\} & \text{if } m \equiv 2 \text{ and } p \equiv 0 \pmod 3 \\ \max\{s+k+\ell-r, s+k-r, k, \ell\} & \text{if } m \equiv 2 \text{ and } p \equiv 1 \pmod 3 \\ \max\{s+k+\ell-r, k, k+\ell\} & \text{if } m,p \equiv 2 \pmod 3. \end{cases}$$

■

Now consider the case where $n \equiv 1 \pmod 3$.

Proposition 5.9. *Let $\left(P_n, \boldsymbol{r}, \boldsymbol{s}_{m,p}^{k,\ell}\right)$ be a graph triple, where $\boldsymbol{r} = [r,\dots,r]$ and*

$$\boldsymbol{s}_{m,p}^{k,\ell} = [\overbrace{s+k,\dots,s+k}^{m \text{ times}}, \overbrace{s+\ell,\dots,s+\ell}^{p \text{ times}}, s,\dots,s]$$

satisfy $2r \geq s+k$ and $3r \geq s+\ell$ for $m,p < n$, $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$. If $n \equiv 1 \pmod 3$, then

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = s\left\lfloor\frac{n}{3}\right\rfloor + k\left\lfloor\frac{m}{3}\right\rfloor + \ell\left\lfloor\frac{p}{3}\right\rfloor + g(s,k,l,r)$$

where

$$g(s,k,l,r) = \begin{cases} \max\{s, s+k-r, s+\ell-r\} & \text{if } m,p \equiv 0 \pmod 3 \\ \max\{s+k-r, s+\ell, s+k-\ell-2r\} & \text{if } m \equiv 0 \pmod 3 \text{ and } p \equiv 1 \pmod 3 \\ \max\{s+k-r, s+\ell, s+2\ell-2r\} & \text{if } m \equiv 0 \pmod 3 \text{ and } p \equiv 2 \pmod 3 \\ \max\{s+\ell-r, s+k, s+k-\ell-r\} & \text{if } m \equiv 1 \pmod 3 \text{ and } p \equiv 0 \pmod 3 \\ \max\{s+\ell-r, s+k, \ell\} & \text{if } m,p \equiv 1 \pmod 3 \\ \max\{s+k+\ell-r, s+k, \ell, s+2\ell-2r\} & \text{if } m \equiv 1 \pmod 3 \text{ and } p \equiv 2 \pmod 3 \\ \max\{s+k, s+k+\ell-2r, \ell-r\} & \text{if } m \equiv 2 \pmod 3 \text{ and } p \equiv 0 \pmod 3 \\ \max\{s+k+\ell-r, s+k, \ell\} & \text{if } m \equiv 2 \pmod 3 \text{ and } p \equiv 1 \pmod 3 \\ \max\{s+k+\ell, s+2\ell-2r\} & \text{if } m,p \equiv 2 \pmod 3. \end{cases}$$

■

Finally, consider the case where $n \equiv 2 \pmod 3$.

Proposition 5.10. *Let $\left(P_n, \boldsymbol{r}, \boldsymbol{s}_{m,p}^{k,\ell}\right)$ be a graph triple where $\boldsymbol{r} = [r,\dots,r]$ and*

$$\boldsymbol{s}_{m,p}^{k,\ell} = [\overbrace{s+k,\dots,s+k}^{m \text{ times}}, \overbrace{s+\ell,\dots,s+\ell}^{p \text{ times}}, s,\dots,s]$$

satisfy $2r \geq s+k$ and $3r \geq s+\ell$ for $m,p < n$, $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$. If $n \equiv 2 \pmod 3$, then

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = s\left\lfloor\frac{n}{3}\right\rfloor + k\left\lfloor\frac{m}{3}\right\rfloor + \ell\left\lfloor\frac{p}{3}\right\rfloor + g(s,k,l,r)$$

where

$$g(s,k,l,r) = \begin{cases} \max\{s, s+k-r, s+\ell-r, s+k-\ell-r, 2s+\ell-2r\} & \text{if } m,p \equiv 0 \pmod 3 \\ \max\{s, s+k-r, s+\ell, s+k-\ell-r\} & \text{if } m \equiv 0 \pmod 3 \text{ and } p \equiv 1 \pmod 3 \\ \max\{s+k-r, s+\ell, 2s+k-2r, 2(s+\ell-r)\} & \text{if } m \equiv 0 \pmod 3 \text{ and } p \equiv 2 \pmod 3 \\ \max\{s+k, s+\ell-r, s+k-\ell\} & \text{if } m \equiv 1 \pmod 3 \text{ and } p \equiv 0 \pmod 3 \\ \max\{s+k, s+\ell, s+k-\ell-r\} & \text{if } m,p \equiv 1 \pmod 3 \\ \max\{s+k, s+\ell, s+k+\ell-r, s+2\ell-2r\} & \text{if } m \equiv 1 \pmod 3 \text{ and } p \equiv 2 \pmod 3 \\ \max\{s+k, s+\ell-r, s+k-\ell-r, 2s+k+\ell-2r\} & \text{if } m \equiv 2 \pmod 3 \text{ and } p \equiv 0 \pmod 3 \\ \max\{s+k, \ell, s+k+\ell-r, 2s+k+\ell-2r\} & \text{if } m \equiv 2 \pmod 3 \text{ and } p \equiv 1 \pmod 3 \\ \max\{s+k, s+k+\ell\} & \text{if } m,p \equiv 2 \pmod 3. \end{cases}$$

■

5.4 Cartesian Products

This section opens with Vizing-like results for the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination number of the product of two graphs. Only the balanced case where $\boldsymbol{r} = [r, \dots, r]$ and $\boldsymbol{s} = [s, \dots, s]$ is considered in this section.

Since $\gamma_{\{s\}}(G) \leq \gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$, it follows immediately from Theorem 3.19 that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G \Box H) \geq \frac{1}{s(s+1)} \gamma_{\{s\}}(G)\gamma_{\{s\}}(H).$$

The approach of Brešar *et al.* [7] in the proof of Theorem 3.19 may be generalised.

Proposition 5.11. *Let $(G, \boldsymbol{r}, \boldsymbol{s})$ and $(H, \boldsymbol{r}, \boldsymbol{s})$ be graph triples, where G and H are graphs with minimum degrees δ_G and δ_H, respectively, and where $\boldsymbol{r} = [r, \dots, r]$ and $\boldsymbol{s} = [s, \dots, s]$ satisfy $r(\min\{\delta_G, \delta_H\} + 1) \geq s$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G \Box H) \geq \frac{1}{s+1} \gamma(G)\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(H).$$

Proof. Let $S = \{u_1, \dots, u_{\gamma(G)}\}$ be a minimum dominating set of G and partition the vertex set of G into $\gamma(G)$ sets $\{\pi_1 \dots, \pi_{\gamma(G)}\}$ such that $\{u_i\} \subseteq \pi_i \subseteq N[u_i]$. Furthermore, let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of $G \Box H$ of minimum weight. Clearly, the set $D = \{v \in V(G \Box H) \mid f(v) \geq 1\}$ forms a dominating set of $G \Box H$.

For each $i = 1, \dots, \gamma(G)$, let $H_i = \pi_i \times V(H)$ and define the *cell* C_w^i as $\pi_i \times \{w\}$. Furthermore, define the *vertical neighbourhood* of the cell C_w^i as $V_w^i = \pi_i \times N_H[w]$. The cell C_w^i is called *vertically s-undominated* if $f(V_w^i) \leq s - 1$ and *vertically s-dominated* otherwise.

For $i = 1, \dots, \gamma(G)$, let k_i denote the number of s-undominated cells in H_i and define the $\boldsymbol{r}$-function f_i as follows. For each $w \in V(H)$, if C_w^i is vertically s-dominated, then let $f_i(w) = \min\{f(C_w^i), r\}$. Otherwise, if C_w^i is vertically s-undominated, then let $f_i(w) = r$. If $f_i[w] < s$ then add to the function values of the neighbours of w such that $f_i[w] = s$. This is possible since $r(\delta_H + 1) \geq s$. Then f_i is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of H and therefore

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(H) \leq |f_i| \leq f(H_i) + sk_i.$$

Summing over i,

$$\gamma(G)\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(H) \leq f(V(G \Box H)) + s \sum_{i=1}^{\gamma(G)} k_i. \tag{5.11}$$

If a cell C_w^i is vertically s-undominated, then since f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of $G \Box H$ it follows that $C_w^i \subseteq N[D \cap V_w]$. Therefore, every vertex in C_w^i is dominated by a vertex in $D \cap V_w$. On the other hand, by definition, every vertex in C_w^i is dominated by (u_i, w). Hence, if m_w denotes the number of vertically s-undominated cells in V_w, then $|S| = \gamma(G) \leq |D \cap V_w| + |S| - m_w$, so that $m_w \leq |D \cap V_w|$. It follows that

$$\sum_{i=1}^{\gamma(G)} k_i = \sum_{w \in V(H)} m_w \leq \sum_{w \in V(H)} |D \cap V_w| = |D| \leq f(V(G \Box H)). \tag{5.12}$$

It therefore follows by inequalities (5.11) and (5.12) that

$$\gamma(G)\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(H) \leq (s+1)f(V(G \Box H)) = (s+1)\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G \Box H).$$

■

To find a bound on $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G\Box H)$ in terms of $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$ and $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(H)$, a relationship between $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$ and $\gamma(G)$ needs to be established. By combining Theorem 3.7 and Proposition 4.2 it follows that $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq (\Delta+1)\lceil s/(\delta+1)\rceil\gamma(G)$.

Corollary 5.1. *Let $(G, \boldsymbol{r}, \boldsymbol{s})$ and $(H, \boldsymbol{r}, \boldsymbol{s})$ be graph triples, where G is a graph with minimum degree δ_G and maximum degree Δ_G, and where H is a graph with minimum degree δ_H. If $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $r(\min\{\delta_G, \delta_H\}+1) \geq s$, then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(H) \leq (s+1)(\Delta_G+1)\left\lceil \frac{s}{\delta_G+1}\right\rceil \gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G\Box H). \qquad \blacksquare$$

To improve on the bound in Corollary 5.1, the notion of a k-partition, as introduced by Choudhary *et al.* [16], is employed. Let $\{P_1, \ldots, P_t\}$ be a multiset of subsets of a set A. Then $P^A = \{P_1, \ldots, P_t\}$ is a *k-partition* of A if each element of A is present in exactly k of the sets $P_1, \ldots, P_t$.

Theorem 5.1. *Let $(G, \boldsymbol{r}, \boldsymbol{s})$ and $(H, \boldsymbol{r}, \boldsymbol{s})$ be graph triples, where G and H are graphs with minimum degree δ_G and δ_H, respectively, and where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $r(\min\{\delta_G, \delta_H\}+1) \geq s$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G\Box H) \geq \frac{1}{2s}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(H).$$

Proof. Let f_G be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G of minimum weight. Furthermore, let $S = \{u_1, \ldots, u_{\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)}\}$ be the multiset containing $f_G(v_i)$ copies of v_i and let $\pi^G = \{\pi_1 \ldots, \pi_{\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)}\}$ be an s-partition of G such that $\{u_i\} \subseteq \pi_i \subseteq N[u_i]$. Then $\pi^G \times V(H)$ forms an s-partition of $G\Box H$. Moreover, let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of $G\Box H$ of minimum weight and let D be the multiset containing $f(gh)$ copies of $gh \in V(G\Box H)$.

As in Proposition 5.11, let $H_i = \pi_i \times V(H)$ and $C_w^i = \pi_i \times \{w\}$ for each $i = 1, \ldots, \gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$. Also, let $V_w^i = \pi_i \times N_H[w]$ be the vertical neighbourhood of the cell C_w^i. The cell C_w^i is called vertically s-undominated if $f(V_w^i) \leq s-1$ and vertically s-dominated otherwise.

For $i = 1, \ldots, \gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$, let k_i denote the number of s-undominated cells in H_i and define the $\boldsymbol{r}$-function f_i as follows. For each $w \in V(H)$, if C_w^i is vertically s-dominated, then let $f_i(w) = \min\{f(C_w^i), r\}$. Otherwise, if C_w^i is vertically s-undominated, then let $f_i(w) = r$. If $f_i[w] < s$, then add to the function values of the neighbours of w such that $f_i[w] = s$. This is possible since $r(\delta_H+1) \geq s$. Then f_i is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of H and therefore

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(H) \leq |f_i| \leq f(H_i) + sk_i.$$

Summing over i,

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(H) \leq s|f| + s\sum_{i=1}^{\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)} k_i. \tag{5.13}$$

If the cell C_w^i is vertically s-undominated, then since f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of $G\Box H$, it follows that $C_w^i \subseteq N[D \cap V_w]$, where $V_w = \pi^G \times \{w\}$. Therefore, every vertex in C_w^i is dominated by a vertex in $D \cap V_w$. On the other hand, by definition, every vertex in C_w^i is dominated by (u_i, w). Hence, if m_w denotes the number of vertically s-undominated cells in V_w, then $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \leq |D \cap V_w| + \gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) - m_w$, so that $m_w \leq |D \cap V_w|$. It follows that

$$\sum_{i=1}^{\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)} k_i = \sum_{w\in V(H)} m_w \leq \sum_{w\in V(H)} |D \cap V_w| = |D| = |f|. \tag{5.14}$$

It therefore follows by inequalities (5.13) and (5.14) that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(H) \leq 2s|f| = 2s\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G\Box H).$$ ■

The following lemma is similar to the results by Klavžar and Seifter [52] for dominating sets of $P_n\Box P_m$ and $C_n\Box C_m$ and will aid in the calculation of the domination numbers of $P_n\Box K_m$ and $C_n\Box K_m$.

If S is a dominating set of $G = H\Box F$, then define $c_i(S) = |S \cap H^{v_i}|$ and $d_i(S) = |S \cap {}^{v_i}F|$.

Lemma 5.1. *Let $H_1 = P_n\Box K_m$ and $H_2 = C_n\Box K_m$ for $m, n \geq 3$. Then there exists a dominating set S of H_i of minimum cardinality such that $d_j(S) \leq m - 1$ for every $j \leq n$ and $i = 1, 2$.*

Proof. First consider the graph $H_1 = P_n\Box K_m$ and let S be a dominating set of H_1 of minimum cardinality. Suppose that $d_i(S) = m$ for k copies of K_m, with $1 \leq k \leq n$. A dominating set S' such that $d_i(S') = m$ for only $k-1$ copies of K_m can now be constructed with $|S| = |S'|$.

If $d_1(S) = m$, then it follows from the minimality of S that $d_2(S) = 0$. The dominating set S' is now defined as $S' = S\backslash\{v_{1,1}\} \cup \{v_{1,2}\}$.

If $d_n(f) = m$, then the dominating set S' may be constructed analogously.

Assume that $d_\ell(S) = m$ for some $\ell < n$. If $d_{\ell-1}(S) = d_{\ell+1}(S) = 0$, then let $S' = S\backslash\{v_{1,\ell}, v_{2,\ell}\} \cup \{v_{1,\ell-1}, v_{2,\ell+1}\}$.

Suppose now that $d_{\ell-1}(S) > 0$ or $d_{\ell+1}(S) > 0$. Without loss of generality assume that $d_{\ell-1}(S) > 0$ and that $v_{j,\ell-1} \in S$. It follows from the minimality of S that $v_{j,\ell+1} \notin S$ and that $d_{\ell+1}(S) < m - 1$. The dominating set S' may now be defined as $S' = S\backslash\{v_{j,\ell}\} \cup \{v_{j,\ell+1}\}$.

In all the cases above, S' is a dominating set of minimum cardinality with $d_i(S') = m$ for only $k - 1$ copies of K_m. The continuation of this process will produce a dominating set S of H_1 of minimum cardinality with $d_i(S) \leq m - 1$ for all $i \leq n$.

The proof for the graph $H_2 = C_n\Box K_m$ is similar to the argument above, except that the cases $d_1(S) = m$ and $d_n(S) = m$ do not have to be considered separately. ■

The domination numbers of the graphs $P_n\Box K_m$ and $C_n\Box K_m$ follow immediately from Propositions 3.3 and 3.4 when $m \geq n$. The following proposition caters for all values of $m, n \geq 5$.

Proposition 5.12. *Let $H_1 = P_n\Box K_m$ and $H_2 = C_n\Box K_m$ for some natural numbers $m, n \geq 5$. Then $\gamma(H_1) = \gamma(H_2) = n$.*

Proof. If $m \geq n$, it follows immediately from Propositions 3.3 and 3.4 that $\gamma(H_1) = \gamma(H_2) = n$ for $i = 1, 2$.

Let $m < n$ and let S be a dominating set of H_i as defined in Lemma 5.1. If $d_i(S) \geq 1$ for all i, then $\gamma(H_i) = |S| \geq n$. Hence, let $\widetilde{Y} = \{Y_{i_1}, Y_{i_2}, \ldots, Y_{i_s}\}$ be the set of copies of K_m with $i_1 < i_2 < \cdots < i_s$ such that $d_{i_k}(S) = 0$. Since $d_i(S) \leq m - 1$ for all $i \leq n$, it follows that $i_k - i_\ell \geq 2$ for any $1 \leq k, \ell \leq s$ and therefore $s \leq \frac{n}{2}$.

First consider the graph $H_2 = C_n\Box K_m$. If $s = \frac{n}{2}$, then it is clear that n is an even number. Without loss of generality, assume that $Y_2, Y_4, \ldots, Y_n$ are the copies of K_m with $d_i(S) = 0$. Since S is a dominating set, it follows that $d_{2i-1}(S) + d_{2i+1}(S) \geq m$ for each $i = 1, 2, \ldots, \frac{n}{2}$, and so

$$\sum_{i=1}^{\frac{n}{2}} d_{2i-1}(S) + d_{2i+1}(S) \geq \frac{mn}{2}.$$

Therefore,

$$|S| \geq \left(\frac{mn}{2}\right)/2 = \frac{mn}{4}.$$

Since $m \geq 5$, it follows that $\gamma(H_2) = |S| > n$.

If $s < \frac{n}{2}$, then there are indices i_j and i_{j+1} in $\widetilde{Y}$ such that $i_{j+1} - i_j \geq 3$. Let $\widetilde{Y}_j$ consist of a maximal sequence of elements of $\widetilde{Y}$ such that their indices differ by exactly two. Then $\bigcup_{j=1}^{\ell} \widetilde{Y}_j$ is a partition of $\widetilde{Y}$. Furthermore, define G_j as $\widetilde{Y}_j$ together with all the copies of K_m that dominate $\widetilde{Y}_j$. If there exists a copy of K_m, say Y_i, such that $d_i(S) > 0$ and Y_i is not adjacent to any of the elements in $\widetilde{Y}$, then $G_j = Y_i$ for $\ell < j \leq k$. Hence $\bigcup_{j=1}^{k} G_j$ is a partition of $V(H_2)$, as illustrated in Figure 5.1.

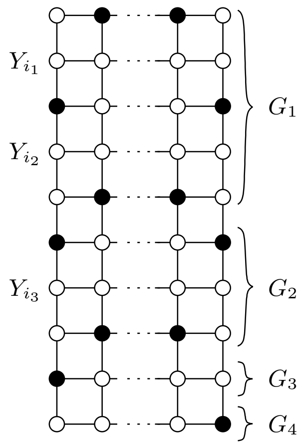


FIGURE 5.1: *A partition of $C_n \Box K_m$. Note that some of the edges have been omitted to simplify the drawing. Furthermore, the black vertices are the vertices in S and the white vertices are those not in S.*

It is now shown that $|G_i \cap S|$ is more than the number of copies of K_m in G_i. It is clear that G_i contains an odd number of copies of K_m, say $2t+1$, for some integer $t \geq 0$. Hence G_i contains t copies of K_m such that $d_{2j}(S) = 0$. Assume these copies are Y_{2j} for $j = 1, 2, \ldots, t$. Since S is a dominating set, it follows that $d_{2j-1}(S) + d_{2j+1}(S) \geq m$ for each $j = 1, 2, \ldots, t$, and so

$$\sum_{j=1}^{t} d_{2j-1}(S) + d_{2j+1}(S) \geq mt.$$

Since $m \geq 5$ it follows that

$$|G_i \cap S| \geq \left(\frac{mt + d_1(S) + d_{2t+1}(S)}{2}\right) \geq \frac{mt+2}{2} > 2t+1.$$

But because $\bigcup_{i=1}^{k} G_i$ is a partition of H_2, it follows that $|S| > n$. This contradicts Proposition 3.3 and therefore $d_i(S) \geq 1$ for all $i \leq n$. Hence $\gamma(H_2) = n$.

Now consider the graph $H_1 = P_n \Box K_m$. Since $d_i(S) \leq m - 1$, it follows that $d_1(S) \neq 0$ and $d_n(S) \neq 0$. Therefore $s < n/2$ and it may be shown in a similar way as above that $\gamma(H_1) = |S| > n$, which contradicts Proposition 3.3. It is therefore concluded that $d_i(S) \geq 1$ for all $i \leq n$ and hence $\gamma(H_1) = n$. ∎

5.5 Circulants

Determining the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination number of an arbitrary circulant is a difficult problem. However, if the simplest circulant that is not a cycle is considered, then it is possible to calculate the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination number for the balanced case.

Proposition 5.13. *Let $(C_n\langle 1,2\rangle, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$ satisfy $r(\delta + 1) \geq s$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_n\langle 1,2\rangle) = \left\lceil \frac{sn}{5} \right\rceil.$$

Proof. It follows from Theorem 3.23 that $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_n\langle 1,2\rangle) \geq sn/5$. For the upper bound, suppose the vertices of $C_n\langle 1,2\rangle$ are labelled $v_1, \ldots, v_n$ and let $n = 5\ell + j$ for some $\ell \in \mathbb{N}_0$ and $j \in \{0,1,2,3,4\}$. If $n, s \equiv 2 \pmod 5$ or $n, s \equiv 3 \pmod 5$, then let f be an $\boldsymbol{r}$-function of $C_n\langle 1,2\rangle$ satisfying $\sum_{i=1}^{5} f(v_{5k+i}) = s$ and $\lceil s/5\rceil \geq f(v_{3k+1}) \geq f(v_{3k+3}) \geq f(v_{3k+4}) \geq f(v_{3k+5}) \geq f(v_{3k+2}) \geq \lfloor s/5 \rfloor$ for all $k = 0, \ldots, \ell - 1$ and $f(v_{3\ell+i}) = f(v_i)$ for all $i = 1, \ldots, j$. Otherwise, let f be an $\boldsymbol{r}$-function of $C_n\langle 1,2\rangle$ satisfying $\sum_{i=1}^{5} f(v_{5k+i}) = s$ and $\lceil s/5\rceil \geq f(v_{3k+1}) \geq f(v_{3k+2}) \geq f(v_{3k+3}) \geq f(v_{3k+4}) \geq f(v_{3k+5}) \geq \lfloor s/5 \rfloor$ for all $k = 0, \ldots, \ell - 1$ and $f(v_{3\ell+i}) = f(v_i)$ for all $i = 1, \ldots, j$. Then f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function and it is easy to verify that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_n\langle 1,2\rangle) \leq |f| = s\ell + \left\lceil \frac{sj}{5} \right\rceil = \left\lceil \frac{sn}{5} \right\rceil. \qquad \blacksquare$$

5.6 Bipartite graphs

First consider the star $K_{1,n}$ with partite sets $N = \{v_{n+1}\}$ and $M = \{v_1, \ldots, v_n\}$.

Proposition 5.14. *Let $(K_{1,n}, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where $\boldsymbol{r} = [r_1, \ldots, r_{n+1}]$ and $\boldsymbol{s} = [s_1, \ldots, s_{n+1}]$ satisfy $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for $i = 1, \ldots, n+1$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_{1,n}) = \max\left\{ s_{n+1}, r_{n+1} + \sum_{i=1}^{n} \max\{s_i - r_{n+1}, 0\} \right\}.$$

Proof. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of $K_{1,n}$ of minimum weight. Note that the closed neighbourhood of v_{n+1} is the vertex set of $K_{1,n}$ and therefore

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_{1,n}) = f(v_1) + \cdots + f(v_{n+1}) \geq s_{n+1}.$$

Since $f(v_i) \geq \max\{s_i - r_{n+1}, 0\}$ for all $i = 1, \ldots, n$, it follows for any $j \in \{1, \ldots, n\}$ that

$$\begin{aligned} \gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_{1,n}) &= f(v_j) + f(v_{n+1}) + \sum_{\substack{i=1 \\ i \neq j}}^{n} f(v_i) \geq s_j + \sum_{\substack{i=1 \\ i \neq j}}^{n} \max\{s_i - r_{n+1}, 0\} \\ &\geq r_{n+1} + \sum_{i=1}^{n} \max\{s_i - r_{n+1}, 0\}. \end{aligned}$$

For the upper bound, first consider the case where $s_{n+1} = \max\{s_{n+1}, r_{n+1} + \sum_{i=1}^{n} \max\{s_i - r_{n+1}, 0\}\}$ and suppose $e \in \mathbb{N}_0$ such that $s_{n+1} = \sum_{i \in A} r_i + e$ for some subset $A \subseteq \{1, 2, \ldots, n+1\}$. Then the function

$$f(v_i) = \begin{cases} r_i & \text{if } i \in A \\ e & \text{for one element of } \{1,\ldots,n+1\}\backslash A \\ 0 & \text{otherwise} \end{cases}$$

is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of $K_{1,n}$ and with $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_{1,n}) \leq |f| = s_{n+1}$.

If $r_{n+1} + \sum_{i=1}^{n} \max\{s_i - r_{n+1}, 0\} \geq s_{n+1}$, then let f be an $\boldsymbol{r}$-function of $K_{1,n}$ such that $f(v_{n+1}) = r_{n+1}$ and $f(v_i) = \max\{s_i - r_{n+1}, 0\}$ for $i = 1,\ldots,n$. Then f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of $K_{1,n}$ and with $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_{1,n}) \leq |f| = r_{n+1} + \sum_{i=1}^{n} \max\{s_i - r_{n+1}, 0\}$. ■

Now consider complete bipartite graphs $K_{m,n}$ with $m \geq n$ and partite sets N and M. Let $N_r = \{v_i \in N \mid r_i \text{ is odd}\}$ and $M_r = \{v_i \in M \mid r_i \text{ is odd}\}$, with $n_r = |N_r|$ and $m_r = |M_r|$.

Proposition 5.15. *Let $(K_{m,n}, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where $\boldsymbol{r} = [r_1,\ldots,r_n]$ and $\boldsymbol{s} = [s_1,\ldots,s_n]$ satisfy $\sum_{v_j \in N[v_i]} r_j > 2s_i$ for $i = 1,\ldots,n$. Then*

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_{m,n}) \leq \frac{1}{2}\sum_{i=1}^{n} r_i.$$

Proof. Define an $\boldsymbol{r}$-function

$$f(v_i) = \begin{cases} r_i/2 & \text{if } v_i \notin N_r \cup M_r \\ \lfloor r_i/2 \rfloor & \text{for } \lceil n_r/2 \rceil \text{ vertices of } N_r \text{ and } \lceil m_r/2 \rceil \text{ vertices of } M_r \\ \lceil r_i/2 \rceil & \text{otherwise.} \end{cases}$$

Let $X = \sum_{v_i \in M} f(v_i)$ and $Y = \sum_{v_i \in N} f(v_i)$. Then

$$X = \sum_{v_i \in M\backslash M_r} f(v_i) + \sum_{v_i \in M_r} f(v_i) = \sum_{v_i \in M\backslash M_r} r_i/2 + \sum_{v_i \in M_r} \lfloor r_i/2 \rfloor + \lfloor m_r/2 \rfloor$$

and similarly

$$Y = \sum_{v_i \in N\backslash N_r} r_i/2 + \sum_{v_i \in N_r} \lfloor r_i/2 \rfloor + \lfloor n_r/2 \rfloor.$$

It is first shown that f is $\boldsymbol{s}$-dominating. Let $v_j \in V(K_{m,n})$. Without loss of generality, assume that $v_j \in M$ and consider the case where $v_j \notin M_r$. It follows that $f(v_j) = r_j/2$ and that

$$\begin{aligned} f[v_j] &= f(v_j) + Y \\ &= r_j/2 + \sum_{v_i \in N\backslash N_r} r_i/2 + \sum_{v_i \in N_r} \lfloor r_i/2 \rfloor + \lfloor n_r/2 \rfloor \\ &\geq r_j/2 + \sum_{v_i \in N} r_i/2 - 1/2 \\ &> s_j - 1/2. \end{aligned}$$

Now consider the case where $v_j \in M_r$ and let $f(v_j) = \lfloor r_i/2 \rfloor$. Then

$$\begin{aligned} f[v_j] &= f(v_j) + X \\ &= \lfloor r_i/2 \rfloor + \sum_{v_i \in M\backslash M_r} r_i/2 + \sum_{v_i \in M_r} \lfloor r_i/2 \rfloor + \lfloor m_r/2 \rfloor \\ &\geq r_j/2 + \sum_{v_i \in N} r_i/2 - 1 \\ &> s_j - 1. \end{aligned}$$

Finally, let $f(v_j) = \lceil r_i/2 \rceil$. Then

$$\begin{aligned} f[v_j] &= f(v_j) + X \\ &= \lceil r_i/2 \rceil + \sum_{v_i \in M \setminus M_r} r_i/2 + \sum_{v_i \in M_r} \lfloor r_i/2 \rfloor + \lfloor m_r/2 \rfloor \\ &\geq r_j/2 + \sum_{v_i \in N} r_i/2 \\ &> s_j. \end{aligned}$$

Therefore, $f[v_i] \geq s_i$ for all $i = 1, \ldots, n$ and so f is $\boldsymbol{s}$-dominating. It follows that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(K_{m,n}) \leq |f| = X + Y \leq \frac{1}{2} \sum_{i=1}^{n} r_i.$$ ■

5.7 Chapter summary

Exact values of and bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of certain graph classes were established in this chapter. The exact value of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a complete graph K_n for any n-vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ was determined in §5.1. In §5.2, the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of C_3 was determined, as well as a lower bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of C_4. This section also contains a result on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of C_n when $\boldsymbol{s}$ is a step function with one step. The $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a path was considered in §5.3. Exact values of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of P_3 and P_4 were established for any n-vectors $\boldsymbol{r}$ and $\boldsymbol{s}$, as well as values of $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_n)$ when $\boldsymbol{s}$ is a step function with either one or two steps.

A bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the product of two graphs was established in terms of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination numbers of the two graphs. The section also contains results on the domination number of $P_n \square K_n$ and $C_n \square K_n$. The $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the circulant $C_n\langle 1, 2 \rangle$ was established in §5.5 and the chapter closed in §5.6 with an upper bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a bipartite graph.

CHAPTER 6

Algorithms for the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number

Contents

This chapter opens with a generalisation of a quadratic-time algorithm for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of tree [19] from the balanced case to the case where the entries of $\boldsymbol{r}$ and $\boldsymbol{s}$ may be distinct. Furthermore, three exact algorithms for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph G and an integer programming formulation of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem are presented and compared in this chapter. More specifically, a new branch-and-bound algorithm and a new branch-and-reduce algorithm are introduced, while a dynamic programming algorithm for solving the set multicover problem [48] is adapted for $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination. These algorithms, together with the integer programming formulation, are then compared by considering their execution times for specific graph classes.

6.1 A linear algorithm for the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a tree

A linear algorithm for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a tree is presented in this section. The algorithm was inspired by the quadratic-time algorithm in [19] for the special case where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$.

The following property of integer pairs is required for a theoretical justification of the algorithm. Let r_k, r_ℓ, s_k and c be positive integers such that $s_k \leq r_k + r_\ell + c$. Furthermore, let $P_{k,\ell}$ be the set of all ordered pairs (a, b) of non-negative integers satisfying $0 \leq a \leq r_k$ and $b - a \leq r_\ell$.

Lemma 6.1. *Let $(x_i, y_i) \in P_{i,\ell}$ for $i = 1, \ldots, j$ and let r_ℓ, r_p, s_ℓ, s_p and z be positive integers such that $\sum_{i=1}^{j} x_i \geq s_l - r_l - r_p$, $s_\ell \leq \sum_{i=1}^{j} r_i + r_\ell + r_p$ and $s_p \leq r_p + r_l + z$. Furthermore, let $M = \max_i\{y_i - x_i\}$ and $\Sigma = \sum_{i=1}^{j} x_i$. If $y = \max\{M, s_\ell - \Sigma\}$ and $x = \max\{M, y - r_p, s_p - r_p - z\}$, then $(x, y) \in P_{\ell,p}$.*

Proof. By hypothesis and the definition of x and M it is clear that $0 \leq M \leq r_\ell$ and $0 \leq M \leq x$. Furthermore, $x \geq y - r_p$, that is $y - x \leq r_p$. If $x = M$ or $x = s_p - r_p - z$, then it follows that $x \leq r_\ell$. If $x = y - r_p$ and $y = M$, then $x = M - r_p \leq r_\ell$. If $x = y - r_p$ and $y = s_\ell - \Sigma$, then $x = s_\ell - \Sigma - r_p \leq s_\ell - s_l + r_l + r_p - r_p \leq r_\ell$. It now follows that $(x, y) \in P_{\ell,p}$. ■

Let $\mathcal{L}(T) = \{u_1, \ldots, u_j\}$ be the set of leaves of a rooted tree T and let h be any function mapping $\mathcal{L}(T)$ to $\mathbb{N} \times \mathbb{N}$ such that $h(u_i) \in P_{u_i,p}$, where p is the parent of u_i for $i = 1, \ldots, j$. Suppose that $h(u_i) = (x_i, y_i)$. Algorithm 6.1 computes $\mu(T, h)$, the minimum weight of an $\boldsymbol{r}$-function f of T satisfying

$$f(u_i) \geq x_i \text{ for } u_i \in \mathcal{L}(T), \tag{6.1}$$

$$f[u_i] \geq y_i \text{ for } u_i \in \mathcal{L}(T), \tag{6.2}$$

$$f[v] \geq s_v \text{ for } v \in V(T) - \mathcal{L}(T)\text{[1]}. \tag{6.3}$$

The algorithm also returns the $\boldsymbol{r}$-function f; this function is called a $\mu(T, h)$-function in this section for the sake of brevity. Executing this algorithm with the function $h = h^*$, where $h^*(u) = (\max\{0, s_u - r_p\}, s_u)$ for every $u \in \mathcal{L}$ and where p is the parent of u, produces an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of minimum weight $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(T) = \mu(T, h^*)$ for T.

Algorithm 6.1: TreeDomination$(\boldsymbol{r}, \boldsymbol{s}, T, h)$

Input : A tree T with root $w \notin \mathcal{L}(T)$, the coverage requirement vector $\boldsymbol{s}$, the multiplicity constraint vector $\boldsymbol{r}$ and a function h labelling the leaves of T.

Output: An $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function x of minimum weight $\mu(T, h)$.

if $\boldsymbol{r}$ *is* $\boldsymbol{s}$*-dominating* **then**
  $\mu \leftarrow 0$
  $\boldsymbol{x} \leftarrow \boldsymbol{r}$
  foreach $u \in \mathcal{L}(T)$ **do**
    $(x_u, y_u) \leftarrow h(u)$
    Label u with $(\max\{x_u, s_p - \sum_{v_j \in N[p], v_j \neq u} x_j\}, y_u)$, where p is the parent of vertex u
  while T *is not a star* **do**
    Pick support vertex v with $C(v) = \{u_1, \ldots, u_k\} \subseteq \mathcal{L}(T)$
    $M \leftarrow \max_i\{y_{u_i} - x_{u_i}\}$
    $\Sigma \leftarrow \sum_i x_{u_i}$
    $y_v \leftarrow \max\{M, s_v - \Sigma\}$
    $x_v \leftarrow \max\{M, y_v - r_p, s_p - \sum_{v_j \in N[p], v_j \neq v} x_{v_j}\}$
    $\mu \leftarrow \mu + \Sigma$
    $T \leftarrow T - C(v)$
  `//` `At this stage` w `is the only unlabelled vertex,` T `is a star with centre` w `and` $C(w) = \{u_1, \ldots, u_k\}$.
  $M \leftarrow \max_i\{y_{u_i} - x_{u_i}\}$
  $\Sigma \leftarrow \sum_i x_{u_i}$
  $x_w \leftarrow \max\{M, s_w - \Sigma\}$
  $\mu \leftarrow \mu + \Sigma + x_w$
return x *and* μ

---

[1] The value s_v is the entry of the vector $\boldsymbol{s}$ associated with the vertex v. Similarly, r_v is the entry of $\boldsymbol{r}$ associated with the vertex v.

Suppose that the tree T, of order at least 3, is rooted at w. Let $\mathcal{S}(T)$ be the set of support vertices v such that the set of children $C(v)$ of v satisfies $C(v) \subseteq \mathcal{L}(T)$. For $v \in \mathcal{S}(T)$, let $T' = T - C(v)$. Note that $\mathcal{L}(T') = (\mathcal{L}(T) - C(v)) \cup \{v\}$ and that if $\mathcal{S}(T) = \{w\}$, then T is a star with centre w.

Algorithm 6.1 starts by assigning a label $(x_v, y_v) = (r_v, 0)$ to every vertex $v \in V(T)$. The algorithm then possibly decreases each x-value until a $\mu(T,h)$-function is obtained. The leaves of the tree are considered first by computing a label for every leaf $u \in \mathcal{L}(T)$ according to the function value $h(u) = (x_u, y_u)$. The x-value of the label is increased when $x < s_p - \sum_{v_j \in N[p], v_j \neq v} x_{v_j}$, where p is the parent of the leaf u. A label (x, y) is then calculated for some vertex $v \in \mathcal{S}(T)$ from the labels of $C(v)$ such that $(x, y) \in P_{v,p}$, where p is the parent of v. For the first iteration of the algorithm it is assumed that $\mu(T,h) = \sum_{u \in \mathcal{L}(T)} x_u$. The set $C(v)$ is removed to form a new tree T' in which the set of labels of $\mathcal{L}(T')$ is defined by the function h'. The process is repeated with the tree T' and the function h', where $\mu(T', h') = \sum_{u \in \mathcal{L}(T')} x_u + \mu(T,h)$. These iterations stop when $\mathcal{S}(T) = \{w\}$. At this stage w is the only unlabelled vertex. For every vertex $u \in V(T) - \{w\}$, let $f(u) = x_u$, where (x_u, y_u) is the label of u. A $\mu(T,h)$-function is found by computing a value for $f(w)$.

The following two results show that the value of $\mu(T,h)$ and the construction of a $\mu(T,h)$-function is calculated correctly in Algorithm 6.1.

Let $C(v) = \{u_1, \ldots, u_j\}$ for $v \in \mathcal{S}(T)$ and let $h(u_i) = (x_i, y_i) \in P_{u_i,v}$ for all $i = 1, \ldots, j$. Furthermore, let y, M and Σ be defined as in Lemma 6.1 and let $x_v = \max\{M, y_v - r_p, s_p - \sum_{v_j \in N[p], v_j \neq v} x_{v_j}\}$, where x_{v_j} is either equal to r_{v_j} or equal to the function value of a $\mu(T,h)$-function. Note that v is a leaf of $T' = T - C(v)$. Define $h' : \mathcal{L}(T') \mapsto \mathbb{N} \times \mathbb{N}$ by $h'(v) = (x, y)$ and $h'(u) = h(u), u \in \mathcal{L}(T') - \{v\}$.

Proposition 6.1. *Let T be a tree and let h and h' be the functions defined above. If $T' = T - C(v)$, then $\mu(T,h) = \Sigma + \mu(T', h')$.*

Proof. Let f' be a $\mu(T', h')$-function. Define f on $V(T)$ by $f(u_i) = x_i$ for all $i = 1, \ldots, j$ and let $f(u) = f'(u),\ u \in \mathcal{V}(T) - C(v)$.

It is shown that f is a $\mu(T,h)$-function by showing that f satisfies (6.1), (6.2) and (6.3). Since $(x_i, y_i) \in P_{u_i,v}$, the function f satisfies (6.1). Furthermore, $f[u_i] = f(u_i) + f(v) = f(u_i) + f'(v) \geq x_i + x \geq x_i + M \geq x_i + (y_i - x_i) = y_i$, for all $i = 1, \ldots, j$, so that f satisfies (6.2). Finally, $f[v] = f'[v] + \Sigma \geq y + \Sigma \geq (s_v - \Sigma) + \Sigma = s_v$, so that f satisfies (6.3). Therefore f is a $\mu(T,h)$-function and

$$\mu(T,h) \leq |f| = |f'| + \Sigma = \mu(T', h') + \Sigma. \tag{6.4}$$

Now let f be a $\mu(T,h)$-function and define the function f' such that $f'(v) = f(v) = x_v$ for $v \in V(T')$. To show that f' is a $\mu(T', h')$-function it is only necessary to show that $f'(v) \geq x$ and $f'[v] \geq y$, since v is a leaf of T'. To show that $f'(v) \geq x$, consider the following three cases.

Case 1: $x = M$. Assume that $f'(v) = f(v) < x$. Furthermore, let $M = y_k - x_k$ for $0 \leq k \leq j$. Then $f[u_k] = x_k + f(v) < x_k + M = x_k + (y_k - x_k) = y_k$. This contradiction shows that $f'(v) \geq x$.

Case 2: $x = y - r_p$, where p is the parent of v. Since $y - r_p \geq M$, it follows that $y > M$ and so $y = s_v - \Sigma$. Assume that $f'(v) = f(v) < x = y - r_p$. Then $f[v] = \Sigma + f(v) + f(p) < \Sigma + (y - r_p) + r_p = \Sigma + s_v - \Sigma = s_v$. This contradiction shows that $f'(v) \geq x$.

Case 3: $x = s_p - \sum_{v_j \in N[p], v_j \neq v} x_{v_j}$. Assume that $f'(v) = f(v) < x$. Then

$$f[p] = f(v) + \sum_{\substack{v_j \in N[p], \\ v_j \neq v}} f(v_j) < x + \sum_{\substack{v_j \in N[p], \\ v_j \neq v}} f(v_j) = s_p - \sum_{\substack{v_j \in N[p], \\ v_j \neq v}} x_{v_j} + \sum_{\substack{v_j \in N[p], \\ v_j \neq v}} x_{v_j} = s_p.$$

This contradiction shows that $f'(v) \geq x$ and so (6.1) is satisfied in all three cases.

To show that $f'[v] \geq y$, first assume that $y = M$. It follows that $f'[v] \geq f'(v) \geq x \geq M = y$. Secondly, let $y = s_v - \Sigma$ and suppose that $f'[v] < y$. Then $f[v] = \Sigma + f'[v] < \Sigma + y = \Sigma + s_v - \Sigma = s_v$. This contradiction shows that $f'[v] \geq y$.

Since f' satisfies (6.1), (6.2) and (6.3), f' is a $\mu(T', h')$-function. Hence

$$\mu(T', h') \leq |f'| = |f| - \Sigma = \mu(T, h) - \Sigma. \tag{6.5}$$

The desired result follows from (6.4) and (6.5). ■

The part of the algorithm where the remaining tree is a star and where w is the only unlabelled vertex in T is considered in the following result.

Proposition 6.2. *Let S be a star with centre w with $\mathcal{L}(S) = \{u_1, \ldots, u_j\}$ for $j \geq 2$. Furthermore, let $h(u_i) = (x_i, y_i)$ for all $i = 1, \ldots, j$ and let M and Σ be defined as in Lemma 6.1. Furthermore, let $x = \max\{M, s_w - \Sigma\}$. Then $\mu(S, h) = max\{M + \Sigma, s_w\}$.*

Proof. Let f be any $\mu(S, h)$-function and assume that $f(u_i) = x_i + \alpha_i$ with $0 \geq \alpha_i \geq r_{u_i} - x_i$ for all $i = 1, \ldots, j$. Let $M = y_k - x_k$ for $0 \leq k \leq j$. Since f is a $\mu(S, h)$-function, $f[u_k] = f(u_k) + f(w) \geq y_k$ and it follows that $f(w) \geq y_k - x_k - \alpha_k = M - \alpha_k$. Furthermore, $f[w] = f(w) + \sum_{i=1}^{j} f(u_i) \geq M - \alpha_k + \Sigma + \sum_{i=1}^{j} \alpha_i \geq M + \Sigma$.

Since f is a $\mu(S, h)$-function, $f[w] \geq s_w$ and it follows that

$$\mu(S, h) = |f| \geq \max\{M + \Sigma, s_w\}. \tag{6.6}$$

Define the function f such that $f(u_i) = x_i$ for all $i = 1, \ldots, j$ and $f(w) = y$. To show that f is a $\mu(S, h)$-function it is shown that f satisfies (6.1), (6.2) and (6.3). Note that $x_i \geq s_w - r_w - \sum_{j \leq n, j \neq i} x_j$ for each $i = 1, \ldots, n$, showing that $s_w - \Sigma \leq r_w$. The function f clearly satisfies (6.1). Furthermore, for all $i = 1, \ldots, k$, $f[u_i] = f(u_i) + f(w) \geq x_i + y \geq x_i + M \geq x_i + (y_i - x_i) = y_i$, so that f satisfies (6.2). Finally, $f[w] = f(w) + \Sigma = y + \Sigma \geq (s_w - \Sigma) + \Sigma = s_w$, so that f satisfies (6.3). Therefore f is a $\mu(S, h)$-function and it follows that

$$\mu(S, h) \leq |f| = \max\{\Sigma + M, s_w\}. \tag{6.7}$$

The result follows by a combination of (6.6) and (6.7) ■

The working of Algorithm 6.1 is illustrated by computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the tree T, rooted at w, in Figure 6.1 a) for $\boldsymbol{r} = [4, 2, 3, 5, 2, 2, 3, 3, 4, 2, 3, 2]$ and $\boldsymbol{s} = [2, 2, 2, 2, 2, 6, 2, 2, 2, 8, 2, 4]$. Recall that the algorithm uses the function $h^*(u) = (x^*, y^*) = (\max\{0, s_u - r_p\}, s_u)$ for $u \in \mathcal{L}(T)$ as input to find the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of T.

At first each vertex v is labelled $(r_v, 0)$, as depicted in Figure 6.1 b), these labels are then updated during the execution of the algorithm. The algorithm first considers the leaves of the tree, starting with v_1. The vertex v_1 receives the label $(\max\{0, -3\}, 2) = (0, 2)$. Similarly v_2 is labelled $(0, 2)$. At vertex v_3, the case where $x^* < s_p - \sum_{v_j \in N[p], v_j \neq v} x_{v_j}$ is encountered for the first time. Therefore, v_3 receives an x-value of $x_{v_3} = \max\{0, 6 - 4\} = 2$ producing the label $(2, 2)$. The remaining leaves are labelled in a similar way, where vertex v_8 is the only other vertex for which $x^* < s_p - \sum_{v_j \in N[p], v_j \neq v} x_{v_j}$. The labels of the leaves are shown in Figure 6.1 c).

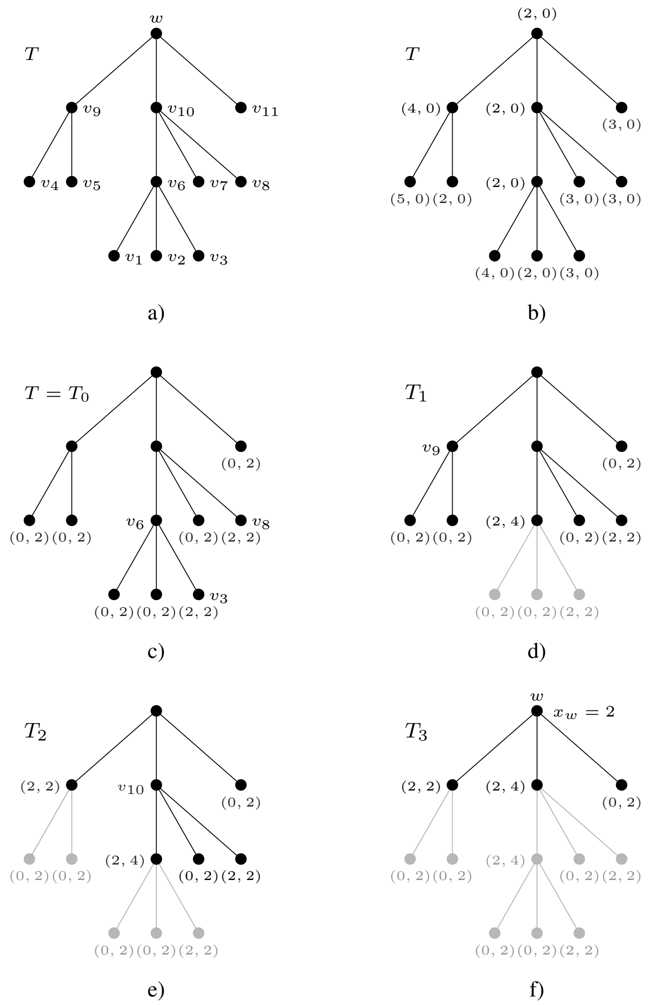


FIGURE 6.1: *An illustration of the working of Algorithm 6.1 when computing the value of* $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(T)$ *for the tree in a), where* $\boldsymbol{r} = [4, 2, 3, 5, 2, 2, 3, 3, 4, 2, 3, 2]$ *and* $\boldsymbol{s} = [2, 2, 2, 2, 2, 6, 2, 2, 2, 8, 2, 4]$. *The algorithm returns the* $\boldsymbol{s}$*-dominating* $\boldsymbol{r}$*-function* $\boldsymbol{f} = [0, 0, 2, 0, 0, 2, 0, 2, 2, 2, 0, 2]$ *of weight* $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(T) = 12$.

Next the label of vertex v_6 is determined from the labels of $C(v_6)$. For v_6, the algorithm computes $M = 2$, $\Sigma = 2$, $y = 4$ and $x = \max\{2, 2, -4\} = 2$. The set $C(v_6)$ is now deleted to form the subtree T_1, depicted by the black edges and vertices in Figure 6.1 d). The subtrees T_2 and T_3 produced by further iterations of the algorithm are all depicted by black vertices and edges, while the deleted vertices and edges are depicted in grey in Figure 6.1 e–f). The vertex v_6 is a leaf of T_1 and has the label $(2, 4)$. Next, the algorithm considers v_9 and assigns the label $(2, 4)$ to the vertex. The label $(2, 4)$ is assigned to v_{10} in T_2. Since T_3 is a star, the algorithm computes $x_w = \max\{M, s_w - \Sigma\} = \max\{2, 4 - 4\} = 2$. Let $f(u) = x_u$ for every vertex $u \in V(T)$. Then f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of T with weight $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(T) = 12$.

Table 6.1 contains results obtained by applying Algorithm 6.1 to random trees of orders 25, 50, 100 and 200. The algorithm was implemented in `Mathematica 7` [85] and considered five random graphs of each order. For each graph the algorithm was executed for five different pairs of $\boldsymbol{r}$ and $\boldsymbol{s}$ vectors. The vector $\boldsymbol{s}$ was taken as a randomly generated vector with values between one and five, while the vector $\boldsymbol{r}$ was taken as the first random $\boldsymbol{s}$-dominating vector with values between one and six. The average time is measured in seconds.

n	Average $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}$	Average time
25	40.40	0.063
50	77.24	0.260
100	160.40	1.268
200	303.80	7.215

TABLE 6.1: *Results obtained by Algorithm 6.1 for random trees of orders 25, 50, 100 and 200.*

The worst-case time and space complexities of Algorithm 6.1 are considered next.

Proposition 6.3 (Worst-case time and space complexities of Algorithm 6.1). *If the input to Algorithm 6.1 is a tree T of order n, then the algorithm computes the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of T in $\mathcal{O}(n)$ time using $\mathcal{O}(n)$ space.*

Proof. Algorithm 6.1 considers each vertex of T only once. Since each of the steps in which the values of M, Σ, x, y and μ are computed may be performed in $\mathcal{O}(1)$ time, the worst-case time complexity of Algorithm 6.1 is $\mathcal{O}(n)$.

The algorithm stores five n-vectors, that is, the vectors $\boldsymbol{r}$ and $\boldsymbol{s}$, the labels $\boldsymbol{x}$ and $\boldsymbol{y}$, and the tree T. The values of M, Σ, x, y and μ may be stored in $\mathcal{O}(1)$ space, resulting in a space complexity of $\mathcal{O}(n)$. ■

6.2 An exponential dynamic programming approach

The minimum dominating set problem may be reduced to the set cover problem, as demonstrated in §3.1.3. However, since a vertex may be covered more than once in the context of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination, the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-problem cannot be reduced to the set cover problem, but it is possible to reduce the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem to the *set multicover with multiplicity constraints* (SMCM) problem.

Hua *et al.* [48] designed the first exact algorithm for solving the SMCM problem. In this section their algorithm is adapted for the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem and the working of this algorithm is illustrated by means of an example.

Recall that in the SMCM problem, a universe $\mathcal{U}$ of n elements and a family of sets $\mathcal{S}$ are given, where each set $S \in \mathcal{S}$ is a subset of $\mathcal{U}$. The objective is to find a sub-family $\mathcal{S}'$ of $\mathcal{S}$ such that each element $i \in \mathcal{U}$ is covered at least b_i times, provided that each set $S_j \in \mathcal{S}$ is used at most d_j times. The vector $\boldsymbol{b}$ is called the coverage requirement vector and the vector $\boldsymbol{d}$ is called the multiplicity constraint vector.

Let $(\mathcal{S},\mathcal{U})$ be a SMCM problem instance with coverage requirement vector $\boldsymbol{s}$ and multiplicity constraint vector $\boldsymbol{r}$, where $\mathcal{U} = V(G)$ and $\mathcal{S} = \{N_G[i] \mid i \in \mathcal{U}\}$ for some graph G. Algorithm 6.2 takes the family of sets $\mathcal{S}$ and the vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ as input and defines an initial vertex v_0 with label $(\emptyset, 0, \ldots, 0)$, this vertex is called the *level 1 vertex*.

The algorithm then creates vertices with labels $(S_i, 0, \ldots, 0)$ to $(S_i, s, \ldots, s)$ for each $i \in \mathcal{U}$. These vertices are called *level i vertices*. Let $\mathcal{V}$ be the set of all level i vertices for $i = 0, \ldots, n$ and define a function $H : \mathcal{V} \mapsto \mathbb{N}$. Initially the function value of v_0 is set to $H(v_0) = 0$, while $H(u) = \infty$ for every $u \in \mathcal{V} \backslash \{v_0\}$. The function values are updated during execution of the algorithm such that the function value $H(u)$ for vertex $u = (S_i, y_1, \ldots, y_n)$ indicates the number of sets that are required for the element i to be covered y_i times. If $H(u) = \infty$, then there does not exist a sub-family of $\mathcal{S}$ in which element i is covered y_i times.

The algorithm now constructs a directed bipartite graph between the vertices on level i and the vertices on level $i-1$ by adding an edge of weight j between every vertex v on level $i-1$ for which $H(v) \neq \infty$ and the vertex $u = (S_i, y_1 + jq_1, \ldots, y_n + jq_n)$ on level i for all $0 \leq j \leq r_i$. Here $q_k = 1$ if $k \in S_i$ or else $q_k = 0$. If $y_k + jq_k \geq s_k$, then the value of $y_k + jq_k$ is set to s_k; this corrects the typographical error made in [48] where $y_k + jq_k$ is compared to $s = \max s_i$. The function value $H(u)$ is changed to $H(v) + j$ whenever $H(v) + j < H(u)$.

After all the iterations have been completed, the algorithm returns $H(v)$ for the vertex $v = (S_n, s_1, \ldots, s_n)$, that is the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the graph G or the value ∞ if no $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function exists.

This process is equivalent to constructing all the levels at once and finding a path of minimum weight between the vertices $v_0 = (\emptyset, 0, \ldots, 0)$ and $v = (S_n, s_1, \ldots, s_n)$.

Algorithm 6.2: DSMCM($\mathcal{S}, \boldsymbol{r}, \boldsymbol{s}$) A dynamic programming-based algorithm for the set multicover with multiplicity constraint problem (Hua *et al.* [48])

Input : A set multicover instance $\mathcal{S}$ with coverage requirement vector $\boldsymbol{r}$ and the multiplicity constraint vector $\boldsymbol{s}$.

Output: The minimum $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}$.

```
Define an initial vertex v_0 with label (∅, 0, ..., 0). This vertex is called the level 0 vertex.
Set H(v_0) = 0.
for i ← 1 to n do
    Define (s+1)^n vertices with labels (S_i, 0, ..., 0) to (S_i, s, ..., s) for set S_i ∈ S where
    s = max s_i and set H(v) = ∞ for all vertices v = (S_i, y_1, ..., y_n). All these vertices are
    called level i vertices.
    for j ← 1 to r_i do
        For each vertex v = (S_{i-1}, y_1, ..., y_n) with H(v) ≠ ∞, add a directed edge with
        edge weight j to u = (S_i, y_1 + jq_1, ..., y_n + jq_n). Here q_k = 1 if k ∈ S_i otherwise
        q_k = 0. Note that if y_k + jq_k ≥ s_k, then set y_k + jq_k = s_k.
        if H(v) + j < H(u) then
            H(u) ← H(v) + j
    Remove all level i vertices and their incident edges.
return H(v) for the vertex v = (S_n, s_1, ..., s_n). If H(v) = ∞, then there does not exist a
multicover and hence no s-dominating r-function.
```

The working of Algorithm 6.2 is illustrated by means of an example. Consider a cycle of length four with constraint vectors $\boldsymbol{r} = [1,1,2,1]$ and $\boldsymbol{s} = [2,1,3,2]$, and let $s = \max_i\{s_i\} = 3$. Together with $\boldsymbol{r}$ and $\boldsymbol{s}$, the family of sets $\mathcal{S} = \{S_1 = \{1,2,4\}, S_2 = \{1,2,3\}, S_3 = \{2,3,4\}, S_4 = \{1,3,4\}\}$ is taken as input for Algorithm 6.2. For each level $i \in \{1,2,3,4\}$, the algorithm creates $(s+1)^n = 4^4 = 256$ vertices. Figure 6.2 only shows the vertices that have incoming or outgoing edges. During the first iteration levels 0 and 1 are considered. Since $r_1 = 1$, the initial vertex $v_0 = (\emptyset, 0, 0, 0, 0)$ is joined to two vertices on level 1, the vertices $(S_1, 0+0\times 1, 0+0\times 1, 0+0\times 0, 0+0\times 1) = (S_1, 0, 0, 0, 0)$ and $(S_1, 0+1\times 1, 0+1\times 1, 0+1\times 0, 0+1\times 1) = (S_1, 1, 1, 0, 1)$. The vertex $(S_1, 0, 0, 0, 0)$ represents the case where S_1 is not included in the multicover and the vertex $(S_1, 1, 1, 0, 1)$ represents the case where S_1 is included once in the multicover. These vertices obtain a function value $H(u)$ equal to the weight of the incoming edge, zero and one, respectively. During the next iteration the vertices on levels 1 and 2 are considered. Since $r_2 = 1$, each of the vertices on level 1 is joined to two vertices on level 2. Furthermore, since $S_2 = \{1,2,3\}$ it follows that $\boldsymbol{q} = [1,1,1,0]$ and therefore the vertex $(S_1, 0, 0, 0, 0)$ is joined to $(S_2, 0, 0, 0, 0)$ and

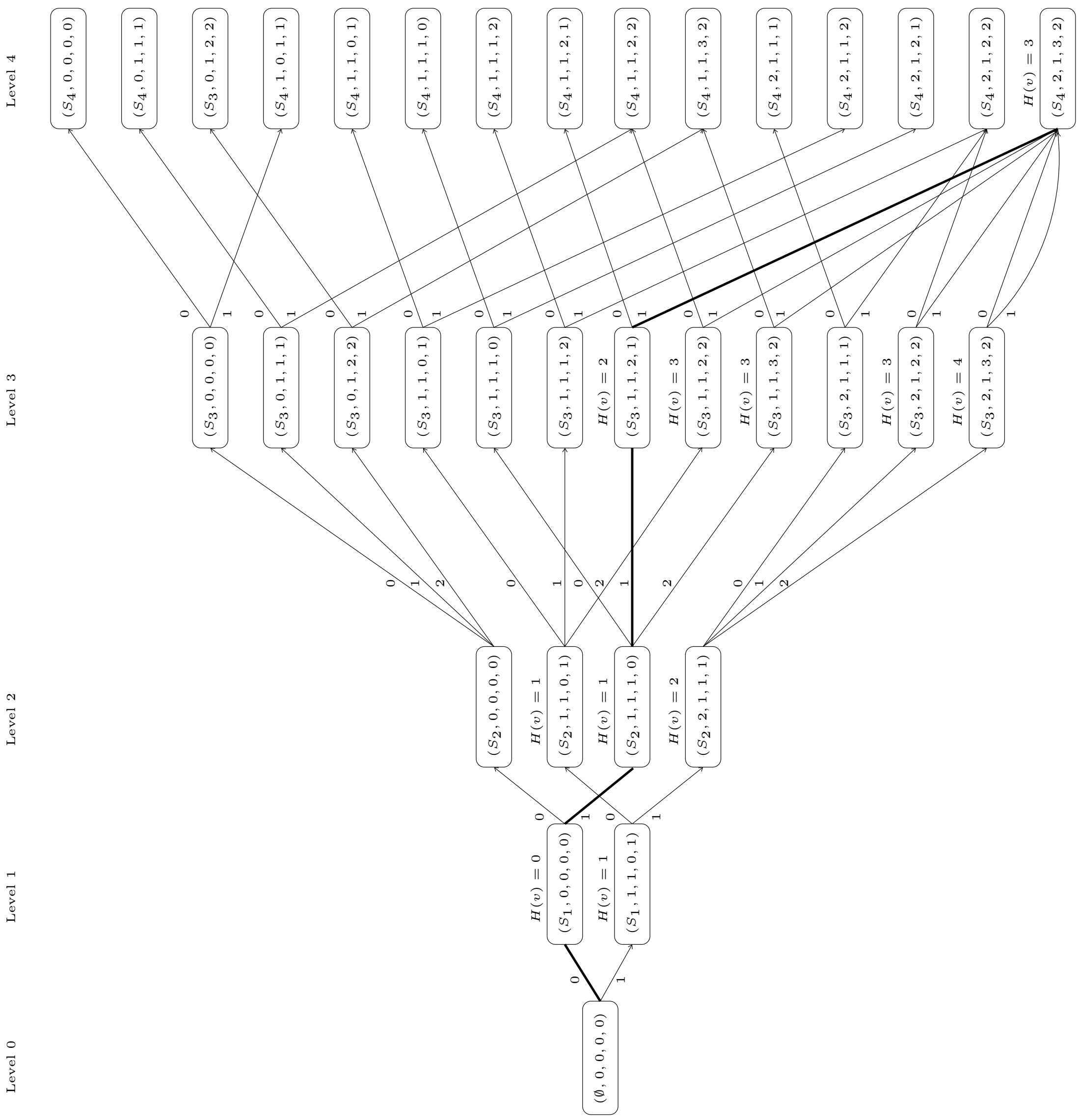


FIGURE 6.2: *A part of the graph produced by Algorithm 6.2 when computing* $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4)$, *where* $\boldsymbol{r} = [1, 1, 2, 1]$ *and* $\boldsymbol{s} = [2, 1, 3, 2]$. *The algorithm returns the minimum* $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$*-domination number* $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4) = 3$.

$(S_2, 1, 1, 1, 0)$. The vertex $(S_1, 1, 1, 0, 1)$ is joined to $(S_2, 1, 1, 0, 1)$ and $(S_2, 2, 1, 1, 1)$. Note that the third entry of $(S_2, 2, 1, 1, 1)$ is not 2, but $1 = s_2$ since $1 + 1 \times 1 > s_2$. The process is repeated for two more iterations. The algorithm then returns the value $H(v) = 3$ of the vertex $v = (S_4, 2, 1, 3, 2)$ and therefore $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4) = 3$.

Proposition 6.4 (Worst-case time and space complexities of Algorithm 6.2). (Hua *et al.* [48]) *If the input to Algorithm 6.2 is a graph G of order n and the coverage requirement vector $\boldsymbol{s} = [s_1, \ldots, s_n]$, then the algorithm computes the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of G in $\mathcal{O}^*((s+1)^n)$ time using $\mathcal{O}^*((s+1)^n)$ space, where $s = \max_i\{s_i\}$.* ■

6.3 An improved branch-and-bound approach

This section contains a description of a novel branch-and-bound approach toward computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph G.

A pseudo-code listing of the recursive approach toward determining an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function h of smallest weight for a graph G is given in Algorithm 6.3, such that $|h| \geq |f|$ where f is a given $\boldsymbol{r}$-function of G. The algorithm takes the function f as input, where f is represented by a vector $\boldsymbol{f} = [f_1, \ldots, f_n]$ in which the number of units at vertex v_i is indicated in position i of the vector. It then produces an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function $\boldsymbol{h} = \boldsymbol{f} + \boldsymbol{g}$ of weight Ψ as output. Initially Ψ is set to either a theoretical upper bound on the parameter $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}$ or the weight of a known $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G. The value of Ψ is updated by Algorithm 6.3 as the algorithm finds $\boldsymbol{s}$-dominating $\boldsymbol{r}$-functions of smaller weight. If no $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function exists, the algorithm returns the zero vector.

Algorithm 6.3: DominatingFunction($\boldsymbol{f}$)

Input : An $\boldsymbol{r}$-function $\boldsymbol{f}$ of G indicating the number of units per vertex, in the form of a vector.

Output: A minimal dominating function $\boldsymbol{h} = \boldsymbol{f} + \boldsymbol{g}$ of G of weight Ψ such that the index of every non-zero entry in $\boldsymbol{g}$ is larger than or equal to the largest index of any non-zero entry of $\boldsymbol{f}$.

if $\boldsymbol{f}$ *is* $\boldsymbol{s}$*-dominating* **then**
    $\Psi \leftarrow |\boldsymbol{f}|$
    $\boldsymbol{h} \leftarrow \boldsymbol{f}$
else
    $\boldsymbol{R} \leftarrow$ a vector indicating how many units can still be added to a vertex and where the index of every non-zero entry is larger or equal to the largest index of any non-zero entry of f
    forall the $v_i \in \boldsymbol{R}$, $v_i \neq 0$ **do**
        if $|\boldsymbol{f}| + 1 < \Psi$ **then**
            $f_i \leftarrow f_i + 1$
            `DominatingFunction(`$\boldsymbol{f}$`)`
return $\boldsymbol{h}$

Furthermore, to ensure that the same function $\boldsymbol{h} = \boldsymbol{f} + \boldsymbol{g}$ is not considered more than once during the course of execution of the algorithm the indices of the non-zero entries of $\boldsymbol{g}$ must be at least as large as the largest index of any non-zero entry of $\boldsymbol{f}$. The algorithm generates a search tree by branching on the addition of all possible units to v_i, calling itself with the function

$$f_j = \begin{cases} f_j + 1 & \text{if } j = i \\ f_j & \text{otherwise} \end{cases}$$

as input at each branch.

The root of the search tree is generated during a call to Algorithm 6.3 in Algorithm 6.4 with $\boldsymbol{f} = [0, \ldots, 0]$. This call occurs in Step 5 of Algorithm 6.4. The input parameters to Algorithm 6.4 are the graph G and the vectors $\boldsymbol{r}$ and $\boldsymbol{s}$. These parameters are defined globally, outside the recursive algorithm `DominatingFunction(`$\boldsymbol{f}$`)` and are never altered during the construction of the search tree. An upper bound Ψ on $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$ is determined in Algorithm 6.4. The value of Ψ is equal to one more than the minimum of a theoretical upper bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of G (see §3.4) and the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a spanning tree of G (calculated by Algorithm 6.1). Since Algorithm 6.3 only considers functions of weight strictly smaller than Ψ, the addition of 1 to the upper bound ensures that an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function is found. Algorithm 6.4 yields as output a minimum $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G.

Note that there may not exist a spanning tree such that the vector $\boldsymbol{r}$ is $\boldsymbol{s}$-dominating; in such a case the upper bound is taken as one more than the theoretical upper bound.

Algorithm 6.4: $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G, \boldsymbol{r}, \boldsymbol{s})$

Input : A graph G with vertex set $\{v_0, \ldots, v_{n-1}\}$ and constraint vectors $\boldsymbol{r}$ and $\boldsymbol{s}$.
Output: A smallest $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G.

$T \leftarrow$ a spanning tree of G
$\Psi_i \leftarrow$ a theoretical upper bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of G for $i = 1, \ldots, k$ for some $k \in \mathbb{N}$
$\Psi \leftarrow \min\{\Psi_1, \ldots, \Psi_k, \texttt{TreeDom}(\boldsymbol{r}, \boldsymbol{s}, T, h^*)\} + 1$
if $\boldsymbol{r}$ *is* $\boldsymbol{s}$*-dominating* **then**
  $h \leftarrow$ `DominatingFunction(`$[0, \ldots, 0]$`)`
return h

As an example of the working of Algorithm 6.4, the search tree constructed by the algorithm when computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4)$ of the cycle C_4 for $\boldsymbol{r} = [1, 1, 2, 3]$ and $\boldsymbol{s} = [2, 1, 3, 2]$ is shown in Figure 6.3. For a theoretical upper bound, consider Proposition 3.13 and note that $\boldsymbol{T^*}{=}\boldsymbol{r}$. Now consider the packing $M = \{v_4\}$ of G and the spanning tree $T = G - e$, where $e = \{v_1 v_4\}$, of G. The global variable Ψ may therefore be initialised as $\Psi = \min\{\sum_{i=1}^{n} r_i - \sum_{v_j \in M} r_j, \texttt{TreeDom}(\boldsymbol{r}, \boldsymbol{s}, T, h^*)\} + 1 = \min\{7 - 3, 4\} + 1 = 5$.

Since $\boldsymbol{r}$ is $\boldsymbol{s}$-dominating, Algorithm 6.4 calls Algorithm 6.3 with $\boldsymbol{f} = [0, 0, 0, 0]$. A unit can be added to every vertex of $\boldsymbol{f}$ and hence $\boldsymbol{R} = [1, 1, 2, 3]$, as shown at the root of the tree in Figure 6.3. Algorithm 6.3 then calls itself with the function $\boldsymbol{f} = [1, 0, 0, 0]$ as input. Since $\boldsymbol{f}$ is not an $\boldsymbol{s}$-dominating function, the vector $\boldsymbol{R} = [0, 1, 2, 3]$ is considered. Since $2 = |\boldsymbol{f}| + 1 < \Psi = 5$, Algorithm 6.3 calls itself again with the function $\boldsymbol{f} = [1, 1, 0, 0]$ as input. It is still possible to add units to vertices v_3 and v_4, and the vector $\boldsymbol{R} = [0, 0, 2, 3]$ is therefore considered. Furthermore, $3 = |\boldsymbol{f}| + 1 < \Psi = 5$ and hence Algorithm 6.3 calls itself again with $\boldsymbol{f} = [1, 1, 1, 0]$ as input. Since $4 = |\boldsymbol{f}| + 1 < \Psi = 5$ and $\boldsymbol{R} = [0, 0, 1, 3]$, Algorithm 6.3 calls itself yet again with $\boldsymbol{f} = [1, 1, 2, 0]$ as input, as shown in the node labelled [a]. Since this instance of $\boldsymbol{f}$ is an $\boldsymbol{s}$-dominating function, $\Psi = |\boldsymbol{f}| = 4$ and the algorithm now returns to the previous level of recursion. Since $|\boldsymbol{f}| + 1 \geq \Psi$, the algorithm falls back another level of recursion, calling itself with the function $\boldsymbol{f} = [1, 1, 0, 1]$ as input. The search tree is bounded at this point, indicated by the symbol '$\times$' in Figure 6.3, since $4 = |\boldsymbol{f}| + 1 \not< \Psi = 4$. The algorithm again drops back a level of recursion, calling itself with the function $\boldsymbol{f} = [1, 0, 1, 0]$ as input.

This process continues, as shown in Figure 6.3, uncovering a smaller $\boldsymbol{s}$-dominating function at the node labelled [b] and being bounded at the nodes labelled '$\times$'. Since no smaller $\boldsymbol{s}$-dominating function than $[0,1,1,1]$ is found during the remainder of the search, the algorithm returns the $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function $[0,1,1,1]$ of weight $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4) = 3$.

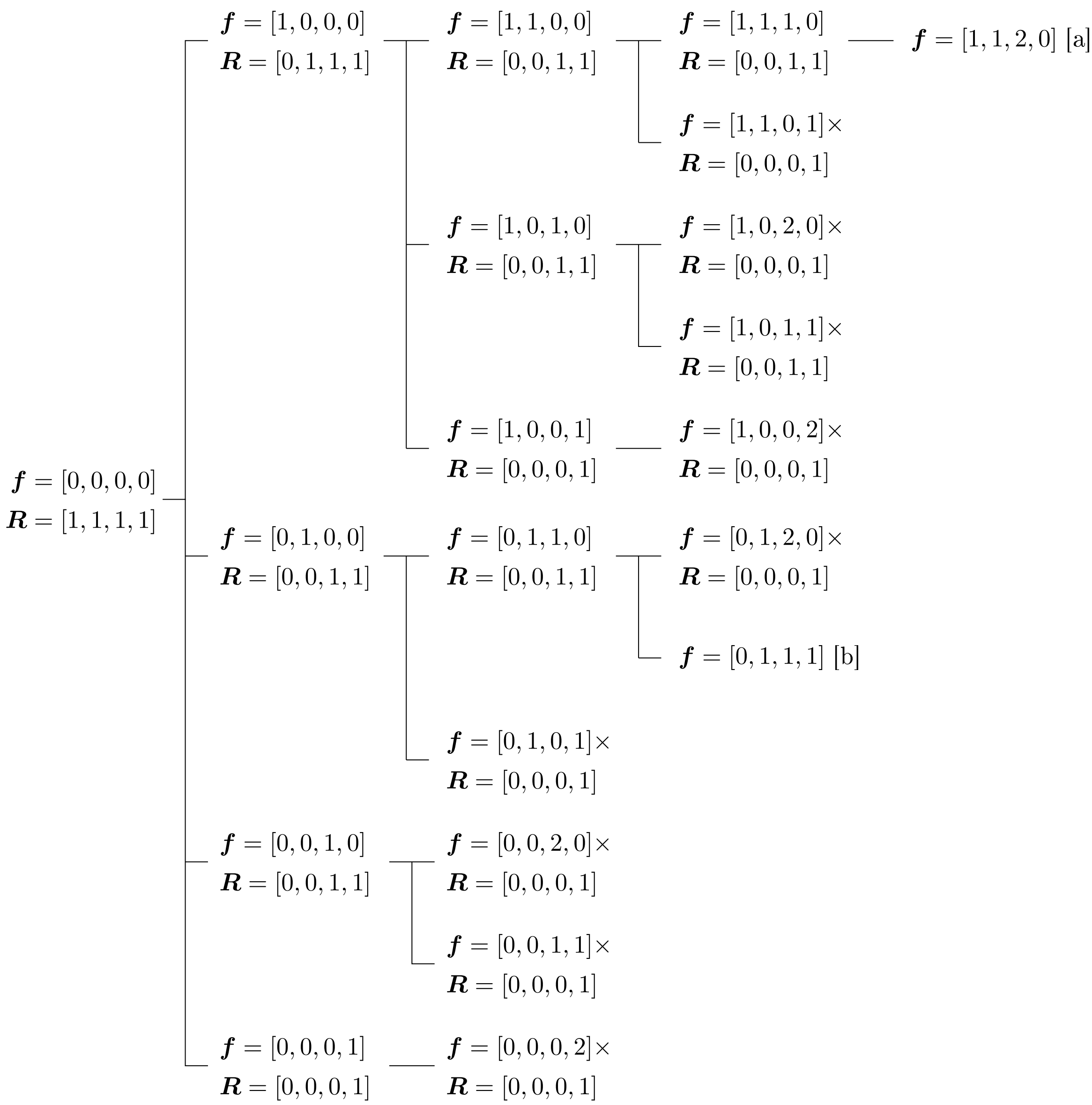


FIGURE 6.3: *The search tree produced by Algorithm 6.3 when computing $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4)$, where $\boldsymbol{r} = [1,1,2,3]$ and $\boldsymbol{s} = [2,1,3,2]$. The algorithm returns the $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function $[0,1,1,1]$ of weight $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4) = 3$.*

Proposition 6.5 (Worst-case time and space complexities of Algorithm 6.4). *If the input to Algorithm 6.4 is a graph of order n, the coverage requirement vector $\boldsymbol{s} = [s_1, \ldots, s_n]$ and the multiplicity constraint vector $\boldsymbol{r} = [r_1, \ldots, r_n]$, then Algorithm 6.4 computes the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of G in $\mathcal{O}^*((r+1)^n)$ time using $\mathcal{O}(n^2 + n))$ space, where $r = \max_i\{r_i\}$.*

Proof. Step 1 of Algorithm 6.4 may be performed in $\mathcal{O}(n)$ time. Algorithm 6.4 then calls Algorithm 6.3 once, after which Algorithm 6.3 calls itself recursively $\prod_{i=1}^{n}(r_i + 1) - 1$ times, since the algorithm considers all possible combinations of the number of units placed at each vertex. The desired result follows since steps 1–3 and 5 of Algorithm 6.3 may be performed in $\mathcal{O}(n^2)$, $\mathcal{O}(1)$, $\mathcal{O}(1)$ and $\mathcal{O}(n)$ times, respectively.

To determine the space complexity, first consider the input variables. The algorithm stores the adjacency matrix of the graph G in $\mathcal{O}(n^2)$ space and three n-vectors, that is, the vectors $\boldsymbol{r}$ and $\boldsymbol{s}$, and the spanning tree T. The values of Ψ and Ψ_i for $i = 1, \ldots, k$ may be stored in $\mathcal{O}(1)$ space and the algorithm $\texttt{TreeDom}(\boldsymbol{r}, \boldsymbol{s}, T, h^*)$ uses $\mathcal{O}(n)$ space, see Proposition 6.3. Algorithm 6.4 calls Algorithm 6.3 in step 5. Algorithm 6.3 stores three n-vectors, that is, the vectors $\boldsymbol{f}$, $\boldsymbol{h}$ and $\boldsymbol{R}$. The occupied space is reused in the subsequent calls to the algorithm in step 9. This results in a space complexity of $\mathcal{O}(n^2 + n)$. ■

6.4 A further improved branch-and-reduce approach

The novel algorithm presented in this section is based on work by Fomin *et al.* [29] and by Van Rooij and Bodlaender [80] who solved the minimum dominating set problem by reducing it to the set cover problem. Since the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem may be reduced to the SMCM problem, a similar approach to that of [29, 80] may be used to solve the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem.

As mentioned in [63], the SMCM problem may also be described as a set cover problem with a universe $\mathcal{U}$ containing b_i copies of the element i for all $i \in \{1, \ldots, n\}$ and a family of sets $\mathcal{S}$, where each set $S_j \in \mathcal{S}$ is a subset of the set $\{1, \ldots, n\}$ and appears d_j times in $\mathcal{S}$. Then the objective is to find a sub-family $\mathcal{S}'$ of $\mathcal{S}$ that covers $\mathcal{U}$. This alternative description of the SMCM problem is used throughout this section.

Recall that the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem may be reduced to the SMCM problem by taking the universe $\mathcal{U}$ as the vertex set of the graph G and the family of sets $\mathcal{S}$ as the closed neighbourhoods of the vertices of G. Furthermore, take the coverage requirement vector $\boldsymbol{b}$ as $\boldsymbol{s}$ and take the multiplicity constraint vector $\boldsymbol{d}$ as $\boldsymbol{r}$.

The proposed algorithm is similar to the standard recursive algorithm for the set cover problem presented in [29, 80] (see §3.4), except for an alteration to account for the use of multisets. The algorithm is called with a universe consisting of s_i copies of v_i for each $i \in \{1, \ldots, n\}$ and the multiset of sets $\mathcal{S}$ consisting of r_i copies of the neighbourhood of v_i for each $i \in \{1, \ldots, n\}$ as input.

Algorithm 6.5 takes a universe $\mathcal{U}$ and a multiset of sets $\mathcal{S}$ as input. If both $\mathcal{U}$ and $\mathcal{S}$ are empty, the algorithm returns the empty set, while if $\mathcal{S}$ is empty and $\mathcal{U}$ is not empty, the algorithm terminates. When $\mathcal{S}$ is not empty, the algorithm chooses an element $S \in \mathcal{S}$ of maximum cardinality and largest multiplicity in $\mathcal{S}$ and two subproblems are considered within a branching paradigm: one where S is part of the multicover and one where S is not part of the multicover. In the case where S is part of the multicover, the elements of S are removed from $\mathcal{U}$ (since $\mathcal{U}$ is a multiset this means that the multiplicities of the elements of S in $\mathcal{U}$ decrease by one) and S is removed from $\mathcal{S}$. Furthermore, for every $S' \in \mathcal{S}$, all the elements in S' not in the universe are removed from $\mathcal{S}$ as well. In the case where S is not part of the multicover, S is deleted from $\mathcal{S}$. The algorithm recursively solves both subproblems and returns the smallest multicover found by the recursive calls. The algorithm terminates if no set $S \in \mathcal{S}$ remains to branch on.

The working of Algorithm 6.5 is illustrated in Figure 6.4 which contains a part of the branching search tree constructed by the algorithm when computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4)$ of the cycle C_4 for $\boldsymbol{r} = [1, 1, 2, 3]$ and $\boldsymbol{s} = [2, 1, 3, 2]$.

Algorithm 6.5 is called with the family of sets $\mathcal{S} = \{\{1, 2, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{2, 3, 4\}, \{1, 3, 4\},$

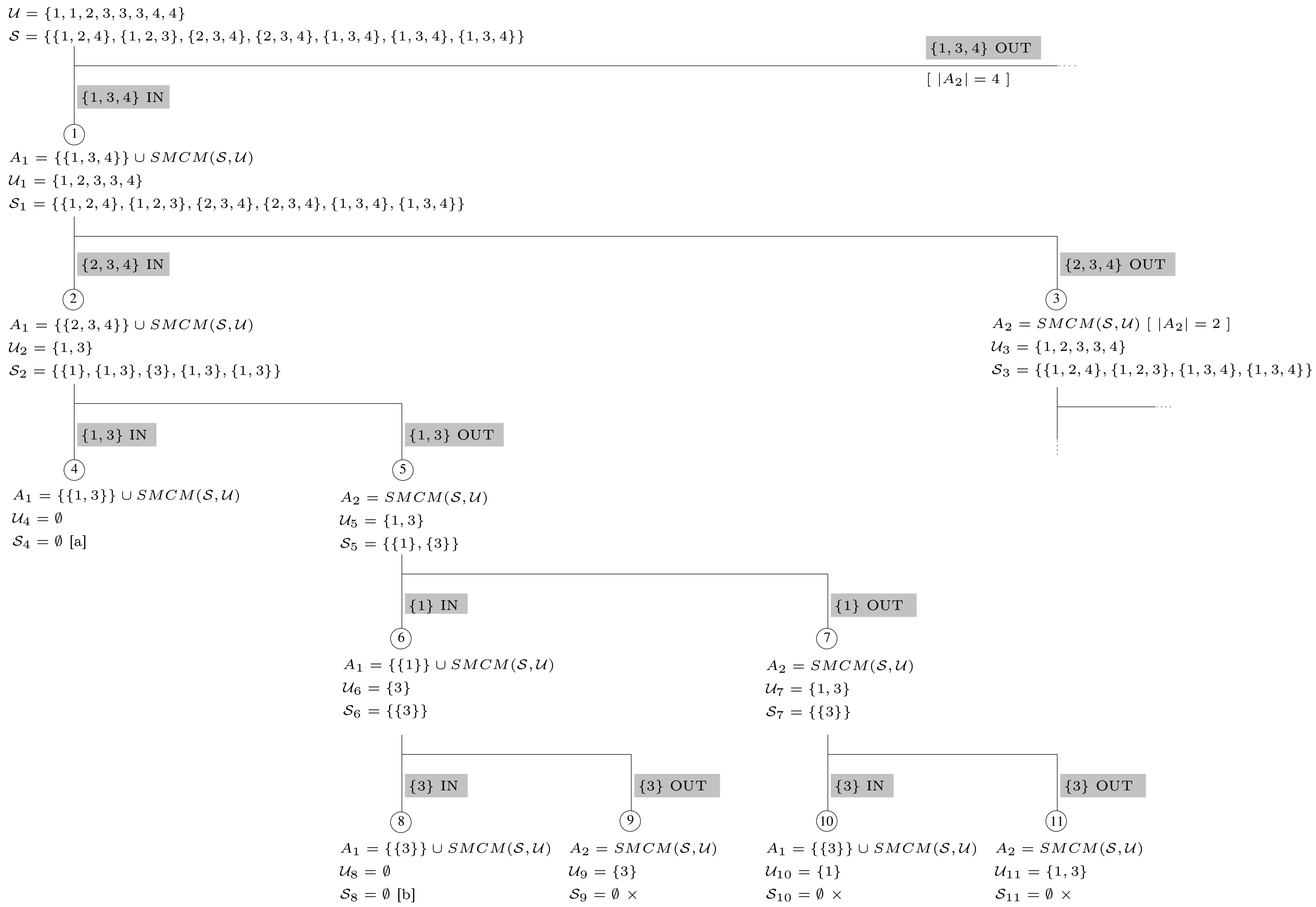


FIGURE 6.4: *Part of the branching search tree produced by Algorithm 6.5 when computing* $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4)$ *for the 4-cycle* C_4*, where* $\boldsymbol{r} = [1, 1, 2, 3]$ *and* $\boldsymbol{s} = [2, 1, 3, 2]$. *The algorithm returns the minimum set multicover* $\{\{1, 3, 4\}, \{2, 3, 4\}, \{1, 3\}\}$ *which can be translated to either the* $\boldsymbol{s}$*-dominating* $\boldsymbol{r}$*-function* $[0, 1, 1, 1]$ *or* $[0, 0, 1, 2]$ *of weight* $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4) = 3$. *In the figure, [a] and [b] denote the end-vertices of the branches that produce multicovers for the universe, while* $\times$ *denotes branches that do not yield multicovers for the universe.*

Algorithm 6.5: SMCM($\mathcal{S}$,$\mathcal{U}$) A set multicover with multiplicity constraint algorithm

Input : A set multicover instance ($\mathcal{S}$,$\mathcal{U}$).
Output: A minimum set multicover of ($\mathcal{S}$,$\mathcal{U}$).

if $\mathcal{S} = \emptyset$ *and* $\mathcal{U} \neq \emptyset$ **then**
  return False
else
  if $\mathcal{S} = \emptyset$ **then**
    return $\emptyset$
  else
    Let $S \in \mathcal{S}$ be a set of maximum cardinality and largest multiplicity in $\mathcal{S}$.
    $A_1 = \{S\} \uplus \text{SMCM}(\{S' \cap (\mathcal{U} \ominus S) | S' \in \mathcal{S} \ominus \{S\}\}, \mathcal{U} \ominus S)$
    $A_2 = \text{SMCM}(\mathcal{S} - \{S\}, \mathcal{U})$
return *The smallest multiset from* A_1 *and* A_2

$\{1,3,4\}, \{1,3,4\}\}$ and the universe $\mathcal{U} = \{1,1,2,3,3,3,4,4\}$ as input. Since all the sets in $\mathcal{S}$ have cardinality 3, the algorithm branches on the set with the largest multiplicity, $S = \{1,3,4\}$. Two multisets, A_1 and A_2, are constructed. The multiset A_1 consists of the set $S = \{1,3,4\}$ and the output of Algorithm 6.5, when called with the family of sets $\mathcal{S}_1 = \{\{1,2,4\}, \{1,2,3\}, \{2,3,4\}, \{2,3,4\}, \{1,3,4\}, \{1,3,4\}\}$ and the universe $\mathcal{U}_1 = \{1,2,3,3,4\}$ as input. The universe $\mathcal{U}_1$ is formed by deleting each element of S once from the original universe. Since $\mathcal{U}_1$ still contains all the vertices of the cycle C_4, only S is removed from $\mathcal{S}$ to form the new family of sets. The multiset A_2 is the output of Algorithm 6.5, when called with the family of sets from which all the copies of S have been removed, *i.e.* $\mathcal{S} = \{\{1,2,4\}, \{1,2,3\}, \{2,3,4\}, \{2,3,4\}\}$ and the universe $\mathcal{U} = \{1,1,2,3,3,3,4,4\}$ as input. Figure 6.4 only depicts the branch of the search tree in which A_1 is constructed.

To construct A_1 at the node labelled 1, the algorithm is called with $\mathcal{S}_1 = \{\{1,2,4\}, \{1,2,3\}, \{2,3,4\}, \{2,3,4\}, \{1,3,4\}, \{1,3,4\}\}$ and $\mathcal{U}_1 = \{1,2,3,3,4\}$ as input. Since $\{2,3,4\}$ and $\{1,3,4\}$ have the same cardinality and multiplicity in $\mathcal{S}$, either one can be chosen as S; in this example $S = \{2,3,4\}$. The algorithm now constructs the two multisets A_1 and A_2 as shown at the nodes labelled 2 and 3 in Figure 6.4. The multiset A_1 consists of the set $S = \{2,3,4\}$ and the output of Algorithm 6.5 when called with the universe $\mathcal{U}_2$ and the family of sets $\mathcal{S}_2$ as input, where $\mathcal{U}_2 = \{1,3\}$ is formed by deleting each element of S once from the universe $\mathcal{U}_1$ at node 1. The multiset $\mathcal{S}_2 = \{\{1\}, \{1,3\}, \{3\}, \{1,3\}, \{1,3\}\}$ is formed by removing one copy of S and deleting every element of $S' \in \mathcal{S}_1$ that is not in $\mathcal{U}_2$. With this call of the algorithm the multisets A_1 and A_2 at nodes 4 and 5 in Figure 6.4 are constructed with the choice of $S = \{1,3\}$.

At node 4 in Figure 6.4 the multiset A_1 consists of the set $S = \{1,3\}$ and the output of Algorithm 6.5 when called with both the universe and the family of sets equal to the empty set as input. The algorithm returns the empty set and produces the first multicover at the node labelled [a]. The multiset A_2 at node 5 is the output of Algorithm 6.5 when called with the universe $\mathcal{U}_5 = \{1,3\}$ and $\mathcal{S}_5 = \{\{1\}, \{3\}\}$ as input, where the latter is formed by deleting all the copies of S from $\mathcal{S}_2$.

This branch leads to four leaves of the search tree (nodes 8 to 11) of which only one yields a multicover of the universe, the node labelled [b]. The leaves of the search tree labelled with the symbol '$\times$' indicate branches that do not yield multicovers of the universe.

To find a minimum multicover, consider the cardinalities of the multisets A_1 and A_2. The multiset with smallest cardinality is chosen at each branch. In Figure 6.4 it is shown that A_2 at node 3 has cardinality 2 and that A_2, where $\{1,3,4\}$ is not in the multicover, has cardinality 4. It is easy to see that the branch ending at the node labelled [a] provides a minimum set multicover $\{\{1,3,4\},\{2,3,4\},\{1,3\}\}$. This multicover can be translated to one of the $\boldsymbol{s}$-dominating $\boldsymbol{r}$-functions $[0,1,1,1]$ or $[0,0,1,2]$ (in vector form).

To improve upon the efficiency of Algorithm 6.5 the following definitions are required. Let $(\mathcal{S},\mathcal{U})$ be an SMCM instance with coverage requirement vector $\boldsymbol{b}$ and multiplicity constraint vector $\boldsymbol{d}$. The entry b_i indicates the multiplicity of $i \in \{1,\ldots,n\}$ in $\mathcal{U}$. Furthermore, let $\mathcal{S}(i) = \{S \in \mathcal{S} \mid i \in S\}$ be the family of sets in $\mathcal{S}$ in which the element i occurs and define the multiplicity f_i to be the number of elements of $\mathcal{S}$ that contain i, so that $f_i = |\mathcal{S}(i)|$.

First consider the multiset A_2 at node 7 in Figure 6.4. It is immediately clear that this branch cannot form a multicover of $\mathcal{U}_7$ since there is no element 1 in $\mathcal{S}_7$, but there is an element 1 in the universe $\mathcal{U}_7$, in other words $b_1 > f_1$. In general it is clear that an SMCM instance cannot yield a multicover if there exists an element $i \in \mathcal{U}$ for which $b_i > f_i$. This observation inspires the first of two reduction rules:

Reduction Rule 1. (No multicover)

if *there exists an element* $i \in \mathcal{U}$ *for which* $b_i > f_i$ ***then***
  return false
end if

Now consider the multiset A_2 at node 5 in Figure 6.4. The only way to cover $\mathcal{U}_5$ is by choosing both elements in $\mathcal{S}_5$. In general, if $b_i = f_i$ for some element $i \in \mathcal{U}$, then $\mathcal{S}(i)$ must be part of the multicover of $\mathcal{U}$. This observation inspires the second reduction rule.

Reduction Rule 2. (Unique elements)

if *there exists an element* $i \in \mathcal{U}$ *for which* $b_i = f_i$ ***then***
  return $\mathcal{S}(i) \uplus SMCM(\{S' \cap (\mathcal{U} \ominus \mathcal{S}(i)) | S' \in \mathcal{S} \ominus \mathcal{S}(i)\}, \mathcal{U} \ominus \mathcal{S}(i))$
end if

The formulation of Algorithm 6.5 can be improved since it is no longer necessary to test whether every computed candidate multicover in fact covers all of $\mathcal{U}$. Upon incorporation of the two reduction rules above, Algorithm 6.6 is obtained.

The worst-case time and space complexities of Algorithm 6.6 are the same as the branch-and-bound algorithm Algorithm 6.4.

Proposition 6.6 (Worst-case time and space complexities of Algorithm 6.6). *If the input to Algorithm 6.6 is a graph G of order n, the coverage requirement vector $\boldsymbol{s} = [s_1,\ldots,s_n]$ and the multiplicity constraint vector $\boldsymbol{r} = [r_1,\ldots,r_n]$, then Algorithm 6.6 computes the $\langle \boldsymbol{r},\boldsymbol{s}\rangle$-domination number of G in $\mathcal{O}^*((r+1)^n)$ time using $\mathcal{O}(n^2+n)$ space, where $r = \max_i\{r_i\}$.*

Proof. Algorithm 6.6 calls itself recursively at most $\prod_{i=1}^{n}(r_i+1)-1$ times, since the algorithm considers all possible combinations of sets in $\mathcal{S}$. The desired result follows since the remaining steps of the algorithm may be performed in polynomial time.

To determine the space complexity, first consider the input variables. The universe $\mathcal{U}$ may be stored as an n-vector in which each entry indicates the multiplicity of an element in the universe. Furthermore, the multiset of sets $\mathcal{S}$ may be stored as a list of length n together with a vector, indicating the multiplicity of each element, of size n. Each element of $\mathcal{S}$ is of size at most n. The occupied space is reused in the subsequent calls to the algorithm in steps 6, 11 and 12. The

Algorithm 6.6: An improved version of Algorithm 6.5 incorporating Reduction Rules 1 and 2

Input : A set multicover instance $(\mathcal{S},\mathcal{U})$.
Output: A minimum set multicover of $(\mathcal{S},\mathcal{U})$.

```
b_i ← the multiplicity of i ∈ {1, ..., n} in U
f_i ← the multiplicity of i ∈ {1, ..., n} in S
if there exists an element i ∈ U for which b_i > f_i then
    return False
if there exists an element i ∈ U for which b_i = f_i then
    return S(i) ⊎ SMCM({S' ∩ (U ⊖ S(i))|S' ∈ S ⊖ S(i)},U ⊖ S(i))
if S = ∅ then
    return ∅
else
    Let S ∈ S be a set of maximum cardinality and largest multiplicity in S.
    A_1 = {S} ⊎ SMCM({S' ∩ (U ⊖ S)|S' ∈ S ⊖ {S}},U ⊖ S)
    A_2 = SMCM(S − {S},U)
return The smallest multiset of A_1 and A_2
```

algorithm also stores two n-vectors $\boldsymbol{b}$ and $\boldsymbol{f} = [f_1, \ldots, f_n]$, while the sets A_1 and A_2 may each occupy at most the same amount of space as the multiset of sets $\mathcal{S}$. This results in a space complexity of $\mathcal{O}(n^2 + n)$. ■

Since the inclusion of the two reduction rules is not reflected in the time complexity in Proposition 6.6, the execution times of the algorithm before and after the inclusion of the reduction rules is compared in Table 6.2. The algorithm was implemented in `Mathematica 7` [85] and cycles of order at most ten were considered for the comparison. For each graph the algorithm was executed for five different pairs of $\boldsymbol{r}$ and $\boldsymbol{s}$ vectors. The vector $\boldsymbol{s}$ was taken as a randomly generated vector with values between zero and six, while the vector $\boldsymbol{r}$ was taken as the first random $\boldsymbol{s}$-dominating vector containing fewer than two zero entries.

		Algorithm 6.5		Algorithm 6.6		Difference		Percentage decrease	
Graph	Ave $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}$	Ave no of Calls	Ave Time	Ave no of Calls	Ave Time	Ave no of Calls	Ave Time	Ave no of Calls	Ave Time
C_3	4.60	10	0.00	10	0.00	0	0	0	0
C_4	5.20	207	0.05	40	0.02	167	0.03	74.12%	51.33%
C_5	6.40	708	0.14	104	0.03	604	0.11	82.87%	70.58%
C_6	7.40	2 105	0.45	181	0.06	1 925	0.39	88.45%	80.71%
C_7	8.40	6 132	1.29	376	0.13	5 756	1.17	92.45%	88.90%
C_8	9.40	9 617	2.03	423	0.14	9 195	1.89	95.77%	93.47%
C_9	11.80	57 937	12.44	978	0.33	56 959	12.11	97.60%	96.30%
C_{10}	12.40	166 693	37.85	1 757	0.63	164 935	37.22	98.05%	96.96%

TABLE 6.2: *A comparison between the SMCM algorithm with reduction rules (Algortihm 6.5) and the algorithm without reduction rules (Algorithm 6.6).*

Table 6.2 provides the average execution times and average number of calls of Algorithms 6.5 and 6.6 as well as the difference in and percentage decrease in the execution time and number of calls for the two algorithms. The average time is measured in seconds. Except for the 3-cycle, both the execution times and number of calls decrease considerably upon inclusion of the reduction rules. A percentage decrease of nearly ninety seven percent in the execution time was observed for a cycle of length ten.

6.5 A very fast integer programming approach

The $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem may also be formulated as an integer programming problem, as demonstrated in this section.

Define the nonnegative integer decision variable $f(v_i)$ as the number of units placed at the vertex v_i of a graph of order n. Then the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem may be formulated as an integer programming problem in which the objective is to

$$\text{minimise} \qquad \sum_{i=1}^{n} f(v_i) \tag{6.8}$$

subject to the constraints

$$f(v_i) \leq r_i, \quad i = 1, \ldots, n, \tag{6.9}$$

$$\sum_{v_j \in N[v_i]} f(v_j) \geq s_i, \quad i = 1, \ldots, n. \tag{6.10}$$

The off-the-shelf software suite `LINGO` [58] of Lindo Systems was used to solve instances of the resulting integer programming problem. To solve this integer programming problem `LINGO` first solves the linear programming relaxation of (6.8)–(6.10) and then uses a branch-and-bound method to find an integer solution. `LINGO` also uses a pre-integer solver which employs heuristics and constraint cuts to reformulate the model so as to speed up the execution of the branch-and-bound process [74]. For the numerical comparisons at the end of this chapter the pre-integer solver was switched off.

Consider again determining the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of $G = C_4$ by means of the integer programming formulation (6.8)–(6.10) for $\boldsymbol{r} = [1, 2, 2, 3]$ and $\boldsymbol{s} = [2, 3, 3, 2]$. First the linear relaxation of (6.8)–(6.10) is solved. `LINGO` returns a minimum of three and a half, as indicated at the root of the tree in Figure 6.5, which is clearly not an integer solution. At this point, $f(1)$ is arbitrarily chosen to branch on and it is clear that, at the minimum, $f(1)$ is either at most 0 or at least 1. The original problem is now replaced by two subproblems, one with the additional constraint that $f(1) \leq 0$ and the other with the additional constraint that $f(1) \geq 1$.

In the branch where $f(1) \leq 0$, the linear relaxation of the subproblem is once again solved and it still produces a non-integer solution. The problem is now replaced by two problems, one where $f(2) \leq 1$ and one where $f(2) \geq 2$. The subproblem at the branch where $f(2) \leq 1$ has no feasible solution, while the other subproblem returns an integer solution of weight 4. Since appending a constraint can only increase the value of the objective function of a solution, it is clear from the solution at the root of the search tree that an optimal integer solution must have weight at least 4. It is therefore not necessary to consider any other options after the solution of weight 4 was reached at the node labelled [a].

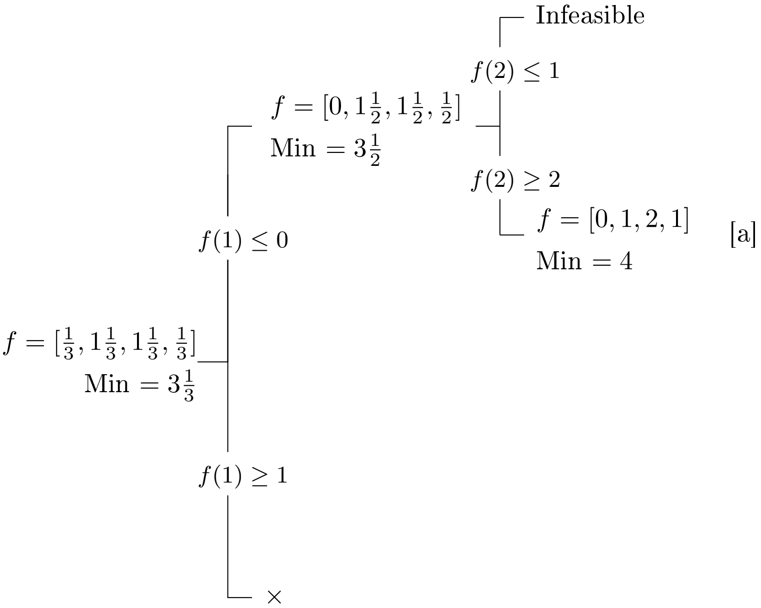


FIGURE 6.5: *The search tree produced by the branch-and-bound method when* `LINGO` *computes* $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4)$*, where* $\boldsymbol{r} = [1, 2, 2, 3]$ *and* $\boldsymbol{s} = [2, 3, 3, 2]$*. The algorithm returns the* $\boldsymbol{s}$*-dominating* $\boldsymbol{r}$*-function* $[1, 2, 0, 1]$ *of weight* $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(C_4) = 4$*.*

6.6 Algorithmic comparison

The algorithms in §6.3 and §6.4 and the integer programming formulation discussed in §6.5 are compared in this section by considering their execution times for grids in the plane and grids on a torus of order less than 25, circulants of order at most ten and wrapped ladders of orders at most 25. Note that Algorithm 6.2 is not included in this comparison due to its excessive memory usage.

Algorithms 6.4 and 6.6 were implemented in Wolfram's `Mathematica 7` [85], while the integer programming model was implemented in `LINGO` [58]. All the tests in this section were executed on an Intel(R) Core(TM)2 Duo 3.16 GHz processor with 3.23 GB RAM running on `Microsoft Windows XP`.

For each graph the algorithms were executed for five different pairs of $\boldsymbol{r}$ and $\boldsymbol{s}$ vectors. The vector $\boldsymbol{s}$ was randomly generated to contain values between zero and six, while the vector $\boldsymbol{r}$ was taken as the first random $\boldsymbol{s}$-dominating vector containing values between zero and five with fewer than two zero entries. Only non-reducible graph triples were used in the tests. In the tables in this section the column labelled *Ave no of calls* contains the average number of times the algorithm calls itself recursively. For the integer programming formulation the average number of iterations refers to the average number of fundamental operations performed by `LINGO`'s solver, while the average number of steps refers to the average number of branches in the branch-and-bound tree [74]. All times reported in this section were measured in seconds. A time limit of thirty minutes was placed on the execution times of all the algorithms. If this time limit was reached for all five iterations, the phrase 'Time out' is reported, while if the time limit was reached for at least three pairs of $\boldsymbol{r}$ and $\boldsymbol{s}$ vectors, an asterisk is added to the phrase. Finally, if the time limit was reached for at most two pairs of $\boldsymbol{r}$ and $\boldsymbol{s}$ vectors, the average of the remaining results is marked with an asterisk.

			Branch-and-bound (§6.3)		Branch-and-reduce (§6.4)		Integer programming (§6.5)		
Graph	n	Ave γ_r^s	Ave no of calls	Ave time	Ave no of calls	Ave time	Ave no of iterations	Ave no of steps	Ave time
$C_3\Box C_3$	9	7.80	6 935	1.04	1 757	0.46	26.20	0.40	0.01
$C_3\Box C_4$	12	10.60	319 637	54.78	15 895	4.39	26.00	0.20	0.01
$C_3\Box C_5$	15	13.00	*Time out	*Time out	173 686	51.91	42.00	0.40	0.01
$C_3\Box C_6$	18	15.40	Time out	Time out	646 620	206.93	44.80	1.00	0.01
$C_3\Box C_7$	21	16.20	Time out	Time out	*Time out	*Time out	76.80	3.00	0.01
$C_3\Box C_8$	24	21.80	Time out	Time out	Time out	Time out	86.60	2.40	0.01
$C_4\Box C_4$	16	13.40	Time out	Time out	361,163	110.48	55.60	0.40	0.01
$C_4\Box C_5$	20	17.60	Time out	Time out	3 135 977	1 031.88	58.40	1.00	0.01
$C_4\Box C_6$	24	21.20	Time out	Time out	Time out	Time out	57.00	1.60	0.01
$C_5\Box C_5$	25	23.20	Time out	Time out	Time out	Time out	82.00	1.60	0.01

TABLE 6.3: *Results obtained by the algorithms in §6.3, §6.4 and §6.5 for grid graphs on a torus ($C_n\Box C_m$) of orders not exceeding 25.*

Considering the results for small grid graphs on a torus in Table 6.3 it is clear that the integer programming model is by far the superior of the three methods. The integer programming model solved all instances in at most 0.02 seconds and never required more than 8 branches in the branch-and-bound tree to solve the problem. In stark contrast, the branch-and-bound algorithm only managed to solve instances of order at most 15 in less than half an hour, while the branch-and-reduce did slightly better by solving instances of order at most 21 in less than half an hour.

			Branch-and-bound (§6.3)		Branch-and-reduce (§6.4)		Integer programming (§6.5)		
Graph	n	Ave γ_r^s	Ave no of calls	Ave time	Ave no of calls	Ave time	Ave no of iterations	Ave no of steps	Ave time
$P_2\Box P_2$	4	4.40	53	0.01	43	0.02	0.00	0.00	0.01
$P_2\Box P_3$	6	6.80	1 261	0.17	169	0.06	0.00	0.00	0.00
$P_2\Box P_4$	8	8.80	6 516	0.93	378	0.13	0.00	0.00	0.01
$P_2\Box P_5$	10	11.80	291 979	44.66	4 048	1.43	0.00	0.00	0.01
$P_2\Box P_6$	12	13.20	1 450 769	246.35	9 774	3.38	2.20	0.00	0.01
$P_2\Box P_7$	14	13.80	*4 168 809	*796.12	9 918	3.48	0.00	0.00	0.01
$P_2\Box P_8$	16	16.80	*Time out	*Time out	74 068	27.53	0.00	0.00	0.01
$P_2\Box P_9$	18	20.40	Time out	Time out	344 119	131.02	0.00	0.00	0.01
$P_2\Box P_{10}$	20	20.80	Time out	Time out	667 247	262.02	0.00	0.00	0.01
$P_2\Box P_{11}$	22	23.80	Time out	Time out	1 958 566	800.93	3.40	0.00	0.01
$P_2\Box P_{12}$	24	27.00	Time out	Time out	3 416 211	1428.73	3.60	0.00	0.01
$P_3\Box P_3$	9	9.40	22 984	3.39	1 102	0.37	0.00	0.00	0.00
$P_3\Box P_4$	12	13.60	1 186 399	200.34	4 647	1.68	6.60	0.00	0.01
$P_3\Box P_5$	15	15.80	Time out	Time out	33 013	12.88	11.00	0.00	0.01
$P_3\Box P_6$	18	24.00	Time out	Time out	220 755	85.72	18.00	0.00	0.01
$P_3\Box P_7$	21	20.60	Time out	Time out	*738 587	*304.00	13.80	0.00	0.01
$P_3\Box P_8$	24	25.60	Time out	Time out	*Time out	*Time out	23.80	0.00	0.01
$P_4\Box P_4$	16	16.40	Time out	Time out	227,560	86.60	15.80	0.00	0.02
$P_4\Box P_5$	20	21.00	Time out	Time out	1 135 803	480.65	23.60	0.00	0.01
$P_4\Box P_6$	24	23.40	Time out	Time out	*Time out	*Time out	24.40	0.00	0.01
$P_5\Box P_5$	25	22.80	Time out	Time out	*Time out	*Time out	23.80	0.00	0.01

TABLE 6.4: *Results obtained by the algorithms in §6.3, §6.4 and §6.5 for grid graphs in the plane ($P_n\Box P_m$) of orders not exceeding 25.*

For small grid graphs in the plane it was once again the integer programming formulation that performed the best, as may be seen in Table 6.4. The integer programming formulation solved all the instances in at most 0.02 seconds, while the other two algorithms failed to solve all the instances in less than half an hour. The branch-and-reduce algorithm managed to solve all the instances of order 20 and less in less than thirty minutes and solved at least two of the five instances for orders between 20 and 25 in less than half an hour. The branch-and-bound algorithm did significantly worse by only solving instances of order at most 16 in less than half an hour.

			Branch-and-bound (§6.3)		Branch-and-reduce (§6.4)		Integer programming (§6.5)		
Graph	n	Ave $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}$	Ave no of calls	Ave time	Ave no of calls	Ave time	Ave no of iterations	Ave no of steps	Ave time
$P_2\square C_3$	6	5.80	240	0.04	141	0.05	0.00	0.00	0.01
$P_2\square C_4$	8	9.20	11 897	1.66	1 646	0.55	1.20	0.00	0.01
$P_2\square C_5$	10	10.00	82 113	12.52	2 066	0.68	0.00	0.00	0.01
$P_2\square C_6$	12	13.20	1 496 861	252.16	9 946	3.40	6.00	0.00	0.01
$P_2\square C_7$	14	15.60	*Time out	*Time out	40 280	14.63	6.00	0.00	0.01
$P_2\square C_8$	16	17.40	Time out	Time out	189 943	70.87	10.40	0.00	0.01
$P_2\square C_9$	18	19.80	Time out	Time out	3 057 477	115.11	12.60	0.00	0.02
$P_2\square C_{10}$	20	22.40	Time out	Time out	*706 409	*284.74	9.00	0.00	0.01
$P_2\square C_{11}$	22	23.20	Time out	Time out	*2 054 543	*812.96	16.60	0.00	0.02
$P_2\square C_{12}$	24	25.60	Time out	Time out	Time out	Time out	12.00	0.00	0.02

TABLE 6.5: *Results obtained by the algorithms in §6.3, §6.4 and §6.5 for wrapped ladders ($P_2\square C_m$) of orders not exceeding 25.*

Considering the results for small wrapped ladders in Table 6.5 it is clear that the branch-and-bound algorithm solved less than half of the instances in less than thirty minutes, while the branch-and-reduce algorithm solved all but eight instances in less than half an hour. The integer programming formulation outperformed the other two algorithms by solving all the instances in less than 0.02 seconds.

For all non-isomorphic, connected circulants of order at most 10 the branch-and-reduce algorithm outperformed the branch-and-bound algorithm in 66% of the instances. It is evident form Table 6.6 that the integer programming formulation performs the best. It solved all the instances in at most 0.01 seconds.

6.7 Chapter summary

A linear algorithm for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of tree was presented in §6.1. This is a generalisation of the algorithm in [19] for the special case where $\boldsymbol{r} = [r, \dots, r]$ and $\boldsymbol{s} = [s, \dots, s]$. The correct working of the algorithm was established by two theorems and illustrated with an example.

An exact algorithm designed by Hua *et al.* [48] was adapted for $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination and was illustrated by means of an example in §6.2. Two exact, polynomial-space algorithms for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph were introduced in this chapter. The first algorithm follows a branch-and-bound approach and was illustrated by means of an example in §6.3. In §6.4 an algorithm using a branch-and-reduce approach was introduced and was also illustrated by means of an example. The $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem was formulated as an integer programming problem in §6.5. The branch-and-bound method adopted by the software suite `LINGO` to solve an integer programming problem was illustrated by means of an example in §6.5.

The chapter closed with a number of numerical tests comparing the execution times of the algorithms in §6.3, §6.4 and §6.5 for four different classes of small graphs; that is, grids in the plane ($P_n\square P_m$) and grids on a torus ($C_n\square C_m$) of orders at most 25, circulants of orders at

		Branch-and-bound (§6.3)		Branch-and-reduce (§6.4)		Integer programming (§6.5)		
Graph	Ave γ_r^s	Ave no of calls	Ave time	Ave no of calls	Ave time	Ave no of iterations	Ave no of steps	Ave time
$C_3\langle 1\rangle$	4.20	23	0.01	9	0.00	0.00	0.00	0.00
$C_4\langle 1\rangle$	4.80	96	0.02	34	0.02	0.00	0.00	0.00
$C_4\langle 1,2\rangle$	3.80	21	0.01	9	0.00	0.00	0.00	0.01
$C_5\langle 1\rangle$	6.00	196	0.03	115	0.03	0.00	0.00	0.01
$C_5\langle 1,2\rangle$	5.20	116	0.02	11	0.01	0.00	0.00	0.00
$C_6\langle 1\rangle$	8.20	1 108	0.15	91	0.03	0.00	0.00	0.01
$C_6\langle 1,2\rangle$	5.80	604	0.09	303	0.10	3.60	0.00	0.01
$C_6\langle 1,3\rangle$	6.60	360	0.06	276	0.09	0.00	0.00	0.01
$C_6\langle 1,3\rangle$	5.60	291	0.05	12	0.01	0.00	0.00	0.01
$C_7\langle 1\rangle$	9.20	7 063	0.95	365	0.12	0.00	0.00	0.00
$C_7\langle 1,2\rangle$	6.40	1 495	0.21	592	0.20	5.00	0.00	0.00
$C_7\langle 1,2,3\rangle$	5.80	516	0.08	13	0.01	0.00	0.00	0.01
$C_8\langle 1\rangle$	9.60	14 057	1.96	679	0.22	0.00	0.00	0.00
$C_8\langle 1,2\rangle$	6.40	2 933	0.43	884	0.30	4.60	0.00	0.01
$C_8\langle 1,3\rangle$	7.40	3 438	0.49	1 102	0.37	7.40	0.00	0.00
$C_8\langle 1,4\rangle$	9.00	11 580	1.62	2 395	0.78	8.00	0.20	0.01
$C_8\langle 1,2,3\rangle$	6.00	1 784	0.27	1 326	0.43	4.60	0.00	0.01
$C_8\langle 1,2,4\rangle$	6.20	1 548	0.23	1 518	0.49	4.60	0.00	0.00
$C_8\langle 1,3,4\rangle$	6.20	1 624	0.24	171	0.07	0.00	0.00	0.01
$C_8\langle 1,2,3,4\rangle$	5.80	724	0.12	13	0.02	0.00	0.00	0.01
$C_9\langle 1\rangle$	11.80	127 227	18.43	1 517	0.51	0.00	0.00	0.00
$C_9\langle 1,2\rangle$	7.20	10 684	1.58	2 028	0.69	7.80	0.00	0.01
$C_9\langle 1,3\rangle$	8.40	10 845	1.61	3 568	1.13	10.60	0.00	0.01
$C_9\langle 1,2,3\rangle$	6.40	3 116	0.48	2 233	0.73	6.60	0.00	0.01
$C_9\langle 1,2,4\rangle$	6.00	4 646	0.70	3 458	1.15	5.80	0.00	0.01
$C_9\langle 1,2,3,4\rangle$	6.00	1 242	0.20	13	0.02	0.00	0.00	0.01
$C_{10}\langle 1\rangle$	9.60	72 008	11.02	822	0.28	0.00	0.00	0.01
$C_{10}\langle 1,2\rangle$	9.60	50 524	7.76	2 116	0.74	7.80	0.00	0.00
$C_{10}\langle 1,3\rangle$	11.20	99 376	15.25	6 212	2.09	8.00	0.00	0.01
$C_{10}\langle 1,4\rangle$	8.60	19 698	3.05	4 266	1.41	11.00	0.00	0.00
$C_{10}\langle 1,5\rangle$	10.80	82 131	12.59	2 047	0.66	5.00	0.00	0.01
$C_{10}\langle 2,5\rangle$	9.20	34 391	5.28	930	0.31	2.40	0.00	0.01
$C_{10}\langle 1,2,3\rangle$	6.60	5 368	0.86	4 801	1.62	6.00	0.00	0.01
$C_{10}\langle 1,2,4\rangle$	7.00	7 827	1.23	4 336	1.46	8.00	0.00	0.01
$C_{10}\langle 1,2,5\rangle$	7.00	13 367	2.08	7 554	2.50	13.40	0.00	0.00
$C_{10}\langle 1,3,5\rangle$	8.00	13 321	2.06	2 447	0.91	12.80	0.00	0.00
$C_{10}\langle 1,4,5\rangle$	8.40	23 670	3.68	539	0.22	0.00	0.00	0.01
$C_{10}\langle 2,4,5\rangle$	7.40	10 159	1.59	4 052	1.46	13.20	0.00	0.00
$C_{10}\langle 1,2,3,4\rangle$	6.20	3 259	0.53	2 469	0.83	3.80	0.00	0.01
$C_{10}\langle 1,2,3,5\rangle$	6.00	3 984	0.64	3 807	1.25	4.40	0.00	0.01
$C_{10}\langle 1,2,4,5\rangle$	6.00	3 104	0.50	4 579	1.51	4.20	0.00	0.01
$C_{10}\langle 1,2,3,4,5\rangle$	5.60	1 910	0.33	12	0.02	0.00	0.00	0.00

TABLE 6.6: *Results obtained by the algorithms in §6.3, §6.4 and §6.5 for all non-isomorphic, connected circulants of orders $n \leq 10$.*

most 10 and wrapped ladders ($P_n \Box C_m$) of orders at most 25. The results show that the integer programming model runs incomparably faster than the two exact algorithms in §6.3 and §6.4, solving the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination number of every tested graph in a time not exceeding 0.02 seconds. The branch-and-reduce algorithm in §6.4 managed to solve the $\langle \boldsymbol{r}, \boldsymbol{s}\rangle$-domination number for graphs of orders up to 20 in less than thirty minutes, while the branch-and-bound algorithm in §6.3 reaches the thirty minute time out mark for graphs of order more than fifteen.

The execution-times of the algorithms in §6.3–6.4 may be shortened dramatically, up to as many as 100 times [10], if implemented in a low-level programming language such as C instead of a high-level language such as Mathematica. However, this might not be enough to perform better than Lingo when solving the integer programming problem of §6.5.

CHAPTER 7

Conclusion

Contents

This dissertation closes with a summary of the work presented within, an appraisal of the contributions of the dissertation as well as some ideas with respect to future work on the theory of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination.

7.1 Dissertation summary

The dissertation opened in Chapter 1 with a brief introduction to the problem of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination and closed with a statement on the objectives pursued in the dissertation and a description of the organisation of material in the dissertation.

The basic mathematical concepts underlying the novel work in this dissertation were introduced in Chapter 2. This included a brief introduction to the theory of multisets in §2.1, some basic concepts from graph theory as well as an introduction to $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination and its special cases in §2.2. The probabilistic method was reviewed briefly in §2.3 and the chapter closed with discussion on a number of basic notions from complexity theory in §2.4.

A literature review on $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination and its special cases, that is, (classical) domination, k-tuple domination and $\{k\}$-domination, was provided in Chapter 3, in fulfilment of Dissertation Objective I of §1.2. General results on the (classical) domination number were reviewed in §3.1, and a summary of known exact values of the domination number followed for certain graph classes such as paths, cycles, complete (bipartite) graphs and the cartesian product of graphs. An overview of a standard algorithmic approach towards computing the domination number of a graph was also presented. In §3.2, known bounds on and exact values of the k-tuple domination number were summarised for certain graph classes and this was accompanied by a short discussion on the algorithmic complexity of the k-tuple domination number computation problem. Known bounds on the $\{k\}$-domination number were reviewed in §3.3 and an overview on known values of the $\{k\}$-domination number followed for certain graph classes.

The chapter closed with a discussion on the literature related to the general case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination in §3.4. This discussion centred around general bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number and exact values of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number for certain graph classes. It was finally shown that the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem may be reduced to the well-known set multicover with multiplicity constraints problem and a known dynamic programming algorithm for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number was described.

A new upper bound on the general case of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph triple $(G, \boldsymbol{r}, \boldsymbol{s})$ was presented in Chapter 4. It was demonstrated in §4.1 why it is sufficient only to consider graph triples for which $\min_i\{r_i\} \leq \max_i\{s_i\}$. In §4.2, the focus shifted to the balanced case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination, that is where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$. A new upper bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number was established for an arbitrary graph in this special case in terms of the minimum degree of the graph, after which the probabilistic method was used to obtain improvements on this bound. In total, four novel bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph were therefore established in §4.2 (for the balanced case) in fulfilment of Dissertation Objective II of §1.2. The bounds of §4.2 were finally compared for three randomly generated graphs so as to illustrate the relative performances of these bounds.

In Chapter 5, exact values of and bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number were established for certain infinite graph classes, in fulfilment of Dissertation Objective III of §1.2. The exact value of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number (for any n-vectors $\boldsymbol{r}$ and $\boldsymbol{s}$) was determined for complete graphs, stars, 3-cycles and paths of orders 3 and 4. The chapter also contained a result on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an n-cycle when $\boldsymbol{s}$ is a step function with one step, as well as values of $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_n)$ for a path P_n of order n when $\boldsymbol{s}$ is a step function with either one or two steps. A bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the product of two graphs was also obtained in terms of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination numbers of the two graphs. Furthermore, the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the circulant $C_n\langle 1, 2 \rangle$ was established and the chapter closed with an upper bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a bipartite graph.

Four algorithmic approaches towards computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number were put forward in Chapter 6, in fulfilment of Dissertation Objective IV of §1.2. First, a new linear algorithm for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a tree was presented. Thereafter, two novel exact, polynomial-space algorithms were introduced for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph. The first algorithm follows a branch-and-bound approach, while the second algorithm adopts a branch-and-reduce approach. The $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem was also formulated as an integer programming problem. This integer programming problem was solved using the software suite `LINGO` [58] of Lindo Systems while the first three algorithms mentioned above were implemented in Wolfram's `Mathematica` [85]. The execution times of the four above-mentioned algorithms were compared for four different classes of small graphs. It was found that the integer programming solution approach is incomparably faster than the remaining two exact algorithms. Furthermore, of the remaining two algorithms the branch-and-reduce algorithm outperformed the branch-and-bound algorithm convincingly.

7.2 Appraisal of dissertation contributions

The main contributions of this dissertation are threefold. The first contribution centres around upper bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph. Three previous upper bounds in [88, 90, 30] on the k-tuple domination number of an arbitrary graph were generalised in this dissertation to the balanced case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination. A probabilistic approach was also adopted to established a new, fourth bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination of an arbitrary graph. For the special case of k-tuple domination, this bound offers improvement on the existing bounds for graphs where the minimum degree of the graph differs significantly from its average degree.

This work has been submitted for publication [71]. All four of the above novel bounds only hold for the balanced case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$. One of these bounds, however, was successfully generalised to hold for all possible $\boldsymbol{r}$ and $\boldsymbol{s}$ values, but it was very difficult to measure the performance of this bound (in other words, it is not clear whether this is a good bound or whether it grows too quickly as the graph order increases).

The second contribution involves establishing exact values of or upper bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination numbers of certain graph classes. It was only possible to obtain exact values of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number for any n-vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ for a small number of graph classes. Even for very simple graph structures, such as cycles and paths, the general case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination remains elusive. A Vizing-like result was established for the balanced case of $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination, showing that the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the product of two graphs is at least $2s$ times larger than the product of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination numbers of the two graphs. In the case of classical domination (*i.e.* where $\boldsymbol{r} = \boldsymbol{s} = [1, \ldots, 1]$), this bound reduces to what was until recently the best lower bound on the domination number of the cartesian product of two graphs in terms of the domination numbers of the two graphs [18]. A generalisation of Vizing's conjecture was given in Conjecture 3.4 for $\{k\}$-domination and it seems that this expectation may be generalised further to $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination by conjecturing that the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the product of two graphs is at least s times larger than the product of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination numbers of the two graphs.

The final contribution involves various algorithmic approaches towards computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph. This contribution firstly includes a generalisation of work by Cockayne [19] in 2007, who introduced a quadratic-time algorithm for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a tree for the balanced case where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$. In this dissertation a linear algorithm was presented for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a tree for any n-vectors $\boldsymbol{r} = [r_1, \ldots, r_n]$ and $\boldsymbol{s} = [s_1, \ldots, s_n]$ — thus improving on both the time complexity and scope of the algorithm in [19]. This work has also been submitted for publication [69].

The algorithmic contribution of this dissertation also includes an exact algorithm for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph. The algorithm was inspired by the work of Fomin *et al.* [29] and Van Rooij and Bodleander [80] who solved the problem of computing the domination number of a graph by reducing it to the set cover problem. A similar approach was adopted in this dissertation, since the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem may be reduced to the *set multicover with multiplicity constraints* problem. The algorithm follows a branch-and-bound approach and employs two reduction rules to speed up its execution time. This work has been accepted for publication [70]. Van Rooij and Bodleander [80] used a measure-and-conquer analysis as a design tool to refine their algorithm for the set cover problem and produced seven reduction rules for their basic algorithm. The two reduction rules proposed in this dissertation were two obvious choices and it may therefore be worthwhile to consider a measure-and-conquer analysis to improve on the current algorithm. Although such an improvement will probably reduce the execution time of the algorithm, this improvement might not be enough to outperform the integer programming solution approach of §6.5 in terms of execution time.

7.3 Future work

In this section, seven open questions related to the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph are posed and three suggestions are made with respect to possible future research emanating from the work presented in this dissertation.

In §4.1 it was observed that, in general, $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \not\leq \sum_{i=1}^{n} r_i/2$, even when $\sum_{v_j \in N[v_i]} r_j \geq 2s_i$ for all $i = 1, \ldots, n$. But what will be the result of relaxing the restriction on the relationship between $\boldsymbol{r}$ and $\boldsymbol{s}$, or when an upper bound of the form $a \sum_{i=1}^{n} r_i$ is sought for some value of $a > 1/2$?

Question 7.1. *Is it possible to find a linear upper bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph in terms of the sum $\sum_{i=1}^{n} r_i$ when there is a restriction on the relationship between $\boldsymbol{r}$ and $\boldsymbol{s}$, such as that $\sum_{v_j \in N[v_i]} r_j \geq cs_i$ for all $i = 1, \ldots, n$ and some $c \geq 1$?*

Unfortunately no new lower bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph were established in this dissertation. This is, however, certainly an area worth pursuing. As in the case of lower bounds on the domination number of graphs, it may be possible to establish lower bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph in terms of the order and size of the graph.

Question 7.2. *Is it possible to improve on the lower bound $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G) \geq \frac{\sum_{i=1}^{n} s_i}{\Delta+1}$?*

In §5.5, the exact value of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the circulant $C_n\langle 1, 2 \rangle$ was established for the balanced case where $\boldsymbol{r} = [r, \ldots, r]$ and $\boldsymbol{s} = [s, \ldots, s]$. From extensive numerical experimentation it would seem that the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the circulant $C_n\langle 1, k \rangle$, where $1 \leq k \leq \lfloor n/2 \rfloor$, either achieves the lower bound in Theorem 3.23 or is one more than this lower bound.

Question 7.3. *What is the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of circulants of the form $C_n\langle 1, k \rangle$ where $1 \leq k \leq \lfloor n/2 \rfloor$?*

In view of the recent success in finding a closed-form formula for the domination number of the cartesian product of paths, it may be worthwhile investigating the possibility of establishing exact values for or good bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the cartesian products of a combination of paths, cycles and complete graphs.

Question 7.4. *What is the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the following cartesian products: $P_n \square G$, $C_n \square G$ and $K_n \square G$, where $G \in \{P_m, C_m, K_m\}$?*

In the proof of Theorem 5.1, the notion of an s-partition of G played an important role. Choudhary *et al.* [16] made use of k-partitions for both G and H in their proof of a bound on the $\{k\}$-domination number of the cartesian product of G and H. Might the partitioning of both graphs (instead of just one of the graphs) bring new insights into establishing better bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of the cartesian product $G \square H$? Is it possible that such an approach might be used to attempt proving Vizing's conjecture?

Question 7.5. *Consider the cartesian product $G \square H$. Will the partitioning of both G and H into k-partitions provide an improved bound on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of $G \square H$?*

Van Rooij and Bodleander [80] employed a measure-and-conquer analysis as a design tool to improve their algorithm for computing the domination number of a graph. The measure-and-conquer analysis differs from classical analyses, since it makes use of a non-standard measure. Classical analyses typically rely on integer measures of the size of an instance, such as the number of vertices in a graph. Recall, however, that the algorithm of Van Rooij and Bodleander [80] is for the set cover problem with an instance consisting of a universe and a set of subsets of the universe. They used a measure that depends on the cardinality of the sets as well as the number of sets that contain a certain element of the universe.

Question 7.6. *Is it possible to improve on Algorithm 6.6 by adopting a measure-and-conquer analysis as a design tool? How will the results differ between using the same measure as Van Rooij and Bodleander [80] or using a measure that also depends on the multiplicity of elements in the universe and/or the multiplicity of a set in the set of sets?*

There exist linear-time algorithms for computing the domination numbers of interval graphs and permutation graphs. The question therefore arises whether it is possible to design linear-time algorithms for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination numbers of these graph classes and whether there exist other classes of graphs for which this is possible.

Question 7.7. *For which classes of graphs, other than trees, is it possible to design a linear-time algorithm for computing the* $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$*-domination number?*

The subtle complexities of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination problem, the scarcity of results in the literature on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of an arbitrary graph for any n-vectors $\boldsymbol{r}$ and $\boldsymbol{s}$, and the fact that the algorithms proposed in Chapter 6 are rather slow (even for relatively small graphs), lead the author to believe that it may be desirable to pursue heuristic algorithms for establishing bounds on the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph.

Suggestion 7.1. *Design heuristic algorithms for computing good bounds on the* $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$*-domination number of an arbitrary graph instance.*

The notion of criticality with respect to $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination has not yet been considered. It would indeed be interesting to consider how the removal of vertices or edges will influence the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number and the existence of $\boldsymbol{s}$-dominating $\boldsymbol{r}$-functions of a graph.

Suggestion 7.2. *Consider how the* $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$*-domination number of a graph* G *will change when vertices or edges are removed from* G*.*

The upper $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of graph G is the maximum weight of a *minimal* $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of G and is denoted by $\Gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(G)$. Chang *et al.* [13] provided an upper bound on the upper k-tuple domination number of an r-regular graph and showed that the decision problem associated with the problem of computing the upper k-tuple domination problem of a graph is **NP**-complete, even when the computation is restricted to the classes of bipartite graphs and chordal graphs.

Suggestion 7.3. *Determine upper and lower bounds on the upper* $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$*-domination number of a graph. Determine values of the upper* $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$*-domination number for special graph classes.*

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APPENDIX A

On the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a path

This appendix contains the proofs of Propositions 5.6 and 5.8–5.10.

Proposition 5.6. Let $(P_4, \boldsymbol{r}, \boldsymbol{s})$ be a graph triple, where $\boldsymbol{r} = [r_1, \ldots, r_4]$ and $\boldsymbol{s} = [s_1, \ldots, s_4]$ satisfy $\sum_{v_j \in N[v_i]} r_j \geq s_i$ for all $i = 1, 2, 3, 4$. Then

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_4) = \max\{s_2, s_3, s_1 + s_4, s_1 + s_3 - r_2, s_2 + s_4 - r_3, s_2 + s_3 - r_2 - r_3\}.$$

Proof. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_4 of minimum weight. To establish the lower bound, note that

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) = \underbrace{f(v_1) + f(v_2) + f(v_3)}_{\geq s_2} + \underbrace{f(v_4)}_{\geq 0} \geq s_2$$

and

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) = \underbrace{f(v_1)}_{\geq 0} + \underbrace{f(v_2) + f(v_3) + f(v_4)}_{\geq s_3} \geq s_3.$$

Furthermore,

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) &= \underbrace{f(v_1) + f(v_2)}_{\geq \max\{s_1, s_2 - r_3\}} + \underbrace{f(v_3) + f(v_4)}_{\geq \max\{s_4, s_3 - r_2\}} \\ &\geq \max\{s_1 + s_4, s_1 + s_3 - r_2, s_2 + s_4 - r_3, s_2 + s_3 - r_2 - r_3\}.\end{aligned}$$

For the upper bound, first consider the case where $s_2 = \max\{s_2, s_3, s_1 + s_4, s_1 + s_3 - r_2, s_2 + s_4 - r_3, s_2 + s_3 - r_2 - r_3\}$ and $s_2 \geq r_2 + r_3$. Let f be an $\boldsymbol{r}$-function of P_4 such that $f(v_1) = \max\{s_2 - r_2 - r_3, s_1 - r_2\}$, $f(v_2) = r_2$, $f(v_3) = \min\{r_3, s_2 - s_1\}$ and $f(v_4) = 0$. If $s_2 < r_2 + r_3$, then let f be an $\boldsymbol{r}$-function of P_4 such that $f(v_1) = \min\{r_1, s_2 - r_2 - s_4, s_2 - s_3\}$, $f(v_2) = r_2$, $f(v_3) = \max\{s_2 - r_1 - r_2, s_4, s_3 - r_2\}$ and $f(v_4) = 0$. In both cases f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_3 and so $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) \leq s_2$.

If $s_3 = \max\{s_2, s_3, s_1 + s_4, s_1 + s_3 - r_2, s_2 + s_4 - r_3, s_2 + s_3 - r_2 - r_3\}$ and $s_3 \geq r_2 + r_3$, then let f be an $\boldsymbol{r}$-function of P_4 such that $f(v_1) = 0$, $f(v_2) = \min\{r_2, s_3 - s_4\}$, $f(v_3) = r_3$ and $f(v_4) = \max\{s_3 - r_3 - r_2, s_4 - r_2\}$. If $s_3 < r_2 + r_3$, then let f be an $\boldsymbol{r}$-function of P_4 such that $f(v_1) = 0$, $f(v_2) = \max\{s_3 - r_4 - r_3, s_1, s_2 - r_3\}$, $f(v_3) = r_3$ and $f(v_4) = \min\{r_4, s_3 - r_3 - s_1, s_3 - s_2\}$. In both cases f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_3 and so $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) \leq s_3$.

If $s_1 + s_4 = \max\{s_2, s_3, s_1 + s_4, s_1 + s_3 - r_2, s_2 + s_4 - r_3, s_2 + s_3 - r_2 - r_3\}$, then $s_3 \leq s_4 + r_2$ and $s_2 \leq s_1 + r_3$. Let f be an $\boldsymbol{r}$-function of P_4 such that $f(v_1) = s_1 - r_2$, $f(v_2) = r_2$, $f(v_3) = r_3$ and $f(v_4) = s_4 - r_3$. Then f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_4 and so $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) \leq s_1 + s_4$.

If $s_1 + s_3 - r_2 = \max\{s_2, s_3, s_1 + s_4, s_1 + s_3 - r_2, s_2 + s_4 - r_3, s_2 + s_3 - r_2 - r_3\}$, then $s_1 \geq r_2$, $s_4 \leq s_3 - r_2$ and $s_2 \leq s_1 + r_3$. Let f be an $\boldsymbol{r}$-function of P_4 such that $f(v_1) = s_1 - r_2$, $f(v_2) = r_2$, $f(v_3) = \min\{r_3, s_3 - r_2\}$ and $f(v_4) = \max\{0, s_3 - r_3 - r_2\}$. Then f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_4 and so $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) \leq s_1 + s_3 - r_2$.

Similarly, if $s_4 + s_2 - r_3 = \max\{s_2, s_3, s_1 + s_4, s_1 + s_3 - r_2, s_2 + s_4 - r_3, s_2 + s_3 - r_2 - r_3\}$, then $s_4 \geq r_3$, $s_1 \leq s_2 - r_3$ and $s_3 \leq s_4 + r_2$. Let f be an $\boldsymbol{r}$-function of P_4 such that $f(v_1) = \max\{0, s_2 - r_2 - r_3\}$, $f(v_2) = \min\{r_2, s_2 - r_3\}$, $f(v_3) = r_3$ and $f(v_4) = s_4 - r_3$. Then f is an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_4 and so $\gamma_{\boldsymbol{r}}^{\boldsymbol{s}}(P_3) \leq s_1 + s_3 - r_2$. ■

Let $\boldsymbol{s}$ be a step function with two steps. To obtain the value of the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a path of order n, the cases where $n \equiv 0, 1, 2 \pmod 3$ are considered separately.

Proposition 5.8. Let $\left(P_n, \boldsymbol{r}, \boldsymbol{s}_{m,p}^{k,\ell}\right)$ be a graph triple, where $\boldsymbol{r} = [r, \ldots, r]$ and

$$\boldsymbol{s}_{m,p}^{k,\ell} = [\overbrace{s+k, \ldots, s+k}^{m \text{ times}}, \overbrace{s+\ell, \ldots, s+\ell}^{p \text{ times}}, s, \ldots, s]$$

satisfy $2r \geq s + k$ and $3r \geq s + \ell$ for $m, p < n$, $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$. If $n \equiv 0 \pmod 3$, then

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + \ell\left\lfloor \frac{p}{3} \right\rfloor + g(s,k,l,r)$$

where

$$g(s,k,l,r) = \begin{cases} \max\{s+k-r, s+\ell-r\} & \text{if } m \equiv 0 \text{ and } p \equiv 0 \pmod 3 \\ \max\{s+k-r, s+\ell-r, \ell\} & \text{if } m \equiv 0 \text{ and } p \equiv 1 \pmod 3 \\ \max\{s+k-r, s+\ell-r, \ell, s+2\ell-2r\} & \text{if } m \equiv 0 \text{ and } p \equiv 2 \pmod 3 \\ \max\{s+k-r, k, k-\ell, s+\ell-2r\} & \text{if } m \equiv 1 \text{ and } p \equiv 0 \pmod 3 \\ \max\{s+k-r, k, \ell, k-\ell\} & \text{if } m, p \equiv 1 \pmod 3 \\ \max\{s+k+\ell-r, s+k-r, k, \ell, s+2\ell-2r\} & \text{if } m \equiv 1 \text{ and } p \equiv 2 \pmod 3 \\ \max\{s+k-r, k, k-\ell, s+k+\ell-2r, \ell-r\} & \text{if } m \equiv 2 \text{ and } p \equiv 0 \pmod 3 \\ \max\{s+k+\ell-r, s+k-r, k, \ell\} & \text{if } m \equiv 2 \text{ and } p \equiv 1 \pmod 3 \\ \max\{s+k+\ell-r, k, k+\ell\} & \text{if } m, p \equiv 2 \pmod 3. \end{cases}$$

Proof. First consider the case where $m \equiv 0 \pmod 3$ and the three subcases where $p \equiv 0, 1, 2 \pmod 3$.

Case 1: $m \equiv 0 \pmod 3$.

Case 1a: $p \equiv 0 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned} \gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = &\underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor - 1)} + \underbrace{f(v_m) + f(v_{m+1})}_{\geq s-r+\max\{k,\ell\}} \\ &+ \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p+1})}_{\geq (s+l)\lfloor p/3 \rfloor} + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\ \geq &s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s - r + \max\{k, \ell\}. \end{aligned} \tag{A.1}$$

If $s+k-r = \max\{s+k-r, s+\ell-r\}$, then $s+k \geq r$ and $s+\ell \leq 2r$. Furthermore, if $s \geq r$, then $k \leq r$. Let f_1 be an $\boldsymbol{r}$-function such that $f_2(v_1) = s+k-r$, $f_2(v_{n-1}) = r$ and $f_2(v_n) = s-r$. Furthermore, let $f_1(v_{3c+2}) = r$, $f_1(v_{3c+3}) = 0$ and $f_1(v_{3c+4}) = s+k-r$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $f_1(v_{3c+2}) = \max\{0, \ell-k+r\}$, $f_1(v_{3c+3}) = \min\{s+\ell, s+k-r\}$ and $f_1(v_{3c+4}) = 0$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally, let $f_1(v_{3c+2}) = r$, $f_1(v_{3c+3}) = s-r$ and $f_1(v_{3c+4}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 2$. Then f_1 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_1| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s - r + k. \tag{A.2}$$

If $s < r$, then let f_2 be an $\boldsymbol{r}$-function such that $f_2(v_1) = s+k-r$, $f_2(v_{n-1}) = s$ and $f_2(v_n) = 0$. Furthermore, let $f_2(v_{3c+2}) = r$, $f_2(v_{3c+3}) = 0$ and $f_2(v_{3c+4}) = s+k-r$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $f_2(v_{3c+2}) = \min\{r, s+\ell\}$, $f_2(v_{3c+3}) = 0$ and $f_2(v_{3c+4}) = \max\{0, s+\ell-r\}$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally, let $f_2(v_{3c+2}) = s$, $f_2(v_{3c+3}) = 0$ and $f_2(v_{3c+4}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 2$. Then f_2 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_2| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s - r + k. \tag{A.3}$$

If $s+\ell-r = \max\{s+k-r, s+\ell-r\}$, then $s+\ell \geq r$. Let f_3 be an $\boldsymbol{r}$-function such that $f_3(v_1) = \max\{0, s+k-r\}$, $f_3(v_2) = \min\{r, s+k\}$, $f_3(v_m) = \max\{s+\ell-2r, 0\}$ and $f_3(v_{m+1}) = \min\{r, s+\ell-r\}$. Furthermore, let $f_3(v_{3c}) = 0$, $f_3(v_{3c+1}) = \max\{0, s+k-r\}$ and $f_3(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $f_3(v_{3c+2}) = r$, $f_3(v_{3c+3}) = \max\{s+\ell-2r, 0\}$ and $f_3(v_{3c+4}) = \min\{r, s+\ell-r\}$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally, let $f_3(v_{3c+1}) = \min\{r, s\}$, $f_3(v_{3c+2}) = \max\{0, s-r\}$ and $f_3(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 2$ and let $f_3(v_{n-1}) = \min\{r, s\}$ and $f_3(v_n) = \max\{0, s-r\}$. Then f_3 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_3| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s - r + \ell. \tag{A.4}$$

The result for the Case 1a follows from a combination of (A.1)–(A.4).

Case 1b: $p \equiv 1 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{m+1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p})}_{\geq (s+l)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p+1}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r.
\end{aligned} \tag{A.5}$$

Also,

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor - 1)} + \underbrace{f(v_m) + \cdots + f(v_{m+p+1})}_{\geq (s+l)(\lfloor p/3 \rfloor + 1)} \\
& + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_n)}_{\geq \max\{s-r, 0\}} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell + \max\{s-r, 0\}.
\end{aligned} \tag{A.6}$$

If $s+k-r = \max\{s+k-r, s+\ell-r, \ell\}$, then $s+k \geq r$ and $s+\ell \leq 2r$. Let f_4 be an $\boldsymbol{r}$-function such that $f_3(v_1) = s+k-r$, $f_3(v_{n-1}) = \min\{r, s\}$ and $f_3(v_n) = \max\{0, s-r\}$. Furthermore, let $f_4(v_{3c+2}) = r$, $f_4(v_{3c+3}) = 0$ and $f_4(v_{3c+4}) = s+k-r$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $f_4(v_{3c+2}) = \min\{r, s, s+\ell\}$, $f_4(v_{3c+3}) = \max\{0, s-r, \ell-r\}$ and $f_4(v_{3c+4}) = \max\{0, \min\{r, \ell\}\}$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally, let $f_4(v_{3c+2}) = \min\{r, s\}$, $f_4(v_{3c+3}) = \max\{0, s-r\}$ and $f_4(v_{3c+4}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 2$ and let $f_4(v_n) = 0$. Then f_4 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_4| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r. \tag{A.7}$$

If $s+\ell-r = \max\{s+k-r, s+\ell-r, \ell\}$, then $s+\ell \geq r$ and $s \geq r$. Let f_5 be an $\boldsymbol{r}$-function such that $f_5(v_1) = s+k-r$, $f_5(v_2) = r$, $f_5(v_m) = \max\{s+\ell-2r, 0\}$ and $f_5(v_{m+1}) = \min\{r, s+\ell-r\}$. Furthermore, let $f_5(v_{3c}) = 0$, $f_5(v_{3c+1}) = s+k-r$ and $f_5(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $f_5(v_{3c+2}) = r$, $f_5(v_{3c+3}) = \max\{s+\ell-2r, 0\}$ and $f_5(v_{3c+4}) = \min\{r, s+\ell-r\}$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally, let $f_5(v_{3c+1}) = r$, $f_5(v_{3c+2}) = s-r$ and $f_5(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 2$ and let $f_5(v_{n-1}) = r$ and $f_5(v_n) = s-r$. Then f_5 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_5| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell - r. \tag{A.8}$$

If $\ell = \max\{s+k-r, s+\ell-r, \ell\}$, then $s \leq r$. Let f_6 be an $\boldsymbol{r}$-function such that $f_6(v_1) = \max\{0, s+k-r\}$, $f_6(v_2) = \min\{r, s+k\}$ and $f_6(v_n) = 0$. Furthermore, let $f_6(v_{3c}) = 0$, $f_6(v_{3c+1}) = \max\{0, s+k-r\}$ and $f_6(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $f_6(v_{3c}) = \min\{r, \max\{0, \ell-r\}\}$, $f_6(v_{3c+1}) = \min\{r, \ell\}$ and $f_6(v_{3c+2}) = \max\{s+\ell-2r, s\}$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally, let $f_6(v_{3c}) = 0$, $f_6(v_{3c+1}) = 0$ and $f_6(v_{3c+2}) = s$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then f_6 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_6| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell. \tag{A.9}$$

The result for the Case 1b follows from a combination of (A.5)–(A.9).

Case 1c: $p \equiv 2 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{m+1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p-1})}_{\geq (s+l)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r.
\end{aligned} \tag{A.10}$$

Also,

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor - 1)} + \underbrace{f(v_m) + \cdots + f(v_{m+p})}_{\geq (s+l)(\lfloor p/3 \rfloor + 1)} \\
& + \underbrace{f(v_{m+p+1}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_n)}_{\geq \max\{s-r, 0\}} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell + \max\{s-r, 0\}
\end{aligned} \tag{A.11}$$

and finally,

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_{m-1})}_{\geq (s+k)(\lfloor m/3\rfloor -1)} + \underbrace{f(v_m)+f(v_{m+1})+f(v_{m+2})}_{\geq s+\ell} + \underbrace{f(v_{m+3})}_{\geq s+\ell-2r} \\ &+ \underbrace{f(v_{m+4})+\cdots+f(v_{m+p+1})}_{\geq (s+l)\lfloor p/3\rfloor} + \underbrace{f(v_{m+p+2})+\cdots+f(v_n)}_{\geq s(\lfloor n/3\rfloor - \lfloor m/3\rfloor - \lfloor p/3\rfloor -1)} \\ \geq& s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + 2\ell - 2r. \end{aligned} \tag{A.12}$$

If $s+k-r = \max\{s+k-r, s+\ell-r, \ell, s+2\ell-2r\}$, then $s+k \geq r$ and $s+\ell \leq 2r$ and the function f_4 is also an $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating function for the case where $p \equiv 2 \pmod 3$.

If $s+\ell-r = \max\{s+k-r, s+\ell-r, \ell, s+2\ell-2r\}$, then $s+\ell \geq r$, $s \geq r$ and $\ell \leq r$. Let f_7 be an $\boldsymbol{r}$-function such that $f_7(v_1) = s+k-r$, $f_7(v_2) = r$ and $f_7(v_n) = s-r$. Furthermore, let $f_7(v_{3c}) = 0$, $f_7(v_{3c+1}) = s+k-r$ and $f_7(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3\rfloor - 1$. Also, let $f_7(v_{3c}) = s-r$, $f_7(v_{3c+1}) = \ell$ and $f_7(v_{3c+2}) = r$ for $c = \lfloor m/3\rfloor, \ldots, \lfloor m/3\rfloor + \lfloor p/3\rfloor$. Finally, let $f_7(v_{3c}) = s-r$, $f_7(v_{3c+1}) = 0$ and $f_7(v_{3c+2}) = r$ for $c = \lfloor m/3\rfloor + \lfloor p/3\rfloor + 1, \ldots, \lfloor n/3\rfloor - 1$. Then f_7 is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_7| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + \ell - r. \tag{A.13}$$

If $\ell = \max\{s+k-r, s+\ell-r, \ell, s+2\ell-2r\}$, then $s \leq r$ and $s+\ell \leq 2r$. Let f_8 be an $\boldsymbol{r}$-function such that $f_8(v_1) = \max\{0, s+k-r\}$, $f_8(v_2) = \min\{r, s+k\}$ and $f_8(v_n) = 0$. Furthermore, let $f_8(v_{3c}) = 0$, $f_8(v_{3c+1}) = \max\{0, s+k-r\}$ and $f_8(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3\rfloor - 1$. Also, let $f_8(v_{3c}) = 0$, $f_8(v_{3c+1}) = \min\{r, \ell\}$ and $f_8(v_{3c+2}) = \max\{s, s+\ell-r\}$ for $c = \lfloor m/3\rfloor, \ldots, \lfloor m/3\rfloor + \lfloor p/3\rfloor$. Finally, let $f_8(v_{3c}) = 0$, $f_8(v_{3c+1}) = 0$ and $f_8(v_{3c+2}) = s$ for $c = \lfloor m/3\rfloor + \lfloor p/3\rfloor + 1, \ldots, \lfloor n/3\rfloor - 1$. Then f_8 is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_8| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + \ell. \tag{A.14}$$

If $s+2\ell-2r = \max\{s+k-r, s+\ell-r, \ell, s+2\ell-2r\}$, then $\ell \geq r$ and $s+\ell \geq 2r$. Let f_9 be an $\boldsymbol{r}$-function such that $f_9(v_1) = \max\{0, s+k-r\}$, $f_9(v_2) = \min\{r, s+k\}$ and $f_9(v_m) = s+\ell-2r$. Furthermore, let $f_9(v_{3c}) = 0$, $f_9(v_{3c+1}) = \max\{0, s+k-r\}$ and $f_9(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3\rfloor - 1$. Also, let $f_9(v_{3c+1}) = r$, $f_9(v_{3c+2}) = r$ and $f_9(v_{3c+3}) = s+\ell-2r$ for $c = \lfloor m/3\rfloor, \ldots, \lfloor m/3\rfloor + \lfloor p/3\rfloor$. Finally, let $f_9(v_{3c+1}) = 0$, $f_9(v_{3c+2}) = \min\{r, s\}$ and $f_9(v_{3c+3}) = \max\{0, s-r\}$ for $c = \lfloor m/3\rfloor + \lfloor p/3\rfloor + 1, \ldots, \lfloor n/3\rfloor - 1$. Then f_9 is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_9| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + 2\ell - 2r. \tag{A.15}$$

The result for the Case 1c follows from a combination of (A.4)–(A.15).

Now consider the case where $m \equiv 1 \pmod 3$ and the three subcases where $p \equiv 0, 1, 2 \pmod 3$.

Case 2: $m \equiv 1 \pmod 3$.

Case 2a: $p \equiv 0 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2)+\cdots+f(v_m)}_{\geq (s+k)\lfloor m/3\rfloor} + \underbrace{f(v_{m+1})+\cdots+f(v_{m+p})}_{\geq (s+l)\lfloor p/3\rfloor} \\ &+ \underbrace{f(v_{m+p+1})+\cdots+f(v_{n-2})}_{\geq s(\lfloor n/3\rfloor - \lfloor m/3\rfloor - \lfloor p/3\rfloor -1)} + \underbrace{f(v_{n-1})+f(v_n)}_{\geq s} \\ \geq& s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k - r. \end{aligned} \tag{A.16}$$

It also follows that

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_{m+1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+2})+\cdots+f(v_{m+p+1})}_{\geq (s+l)\lfloor p/3 \rfloor} \\
&+ \underbrace{f(v_{m+p+2})+\cdots+f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_n)}_{\geq 0} \\
\geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell \lfloor p/3 \rfloor + k
\end{aligned} \tag{A.17}$$

and

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_{m+1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+2})}_{\geq 0} + \underbrace{f(v_{m+3})+\cdots+f(v_{m+p-1})}_{\geq (s+l)(\lfloor p/3 \rfloor - 1)} \\
&+ \underbrace{f(v_{m+p})+\cdots+f(v_n)}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor)} \\
\geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell \lfloor p/3 \rfloor + k - \ell.
\end{aligned} \tag{A.18}$$

Finally,

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)+\cdots+f(v_{m-1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_m)+f(v_{m+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+2})+\cdots+f(v_{m+p+1})}_{\geq (s+l)\lfloor p/3 \rfloor} \\
&+ \underbrace{f(v_{m+p+2})+\cdots+f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_n)}_{\geq s-r} \\
\geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell \lfloor p/3 \rfloor + s + \ell - 2r.
\end{aligned} \tag{A.19}$$

If $s+k-r = \max\{s+k-r, k, k-\ell, s+\ell-2r\}$, then $s \geq r$ and $s+\ell \geq r$. Let f_{10} be an $\boldsymbol{r}$-function such that $f_{10}(v_1) = s+k-r$, $f_{10}(v_2) = r$ and $f_{10}(v_n) = s-r$. Furthermore, let $f_{10}(v_{3c}) = 0$, $f_{10}(v_{3c+1}) = s+k-r$ and $f_{10}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $f_{10}(v_{3c}) = \min\{r, s+\ell-r\}$, $f_{10}(v_{3c+1}) = \max\{s+\ell-2r, 0\}$ and $f_{10}(v_{3c+2}) = r$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally, let $f_{10}(v_{3c}) = s-r$, $f_{10}(v_{3c+1}) = 0$ and $f_{10}(v_{3c+2}) = r$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then f_{10} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_{10}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r. \tag{A.20}$$

Now, if $k = \max\{s+k-r, k, k-\ell, s+\ell-2r\}$, then $s \leq r$ and $\ell \geq 0$. Let f_{11} be an $\boldsymbol{r}$-function such that $f_{11}(v_1) = \min\{r, k\}$, $f_{11}(v_2) = \max\{s+k-r, s\}$ and $f_{11}(v_n) = 0$. Furthermore, let $f_{11}(v_{3c}) = \max\{0, k-r\}$, $f_{11}(v_{3c+1}) = \min\{r, k\}$ and $f_{11}(v_{3c+2}) = s$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $f_{11}(v_{3c}) = \max\{\ell-r, \ell-k, 0\}$, $f_{11}(v_{3c+1}) = \min\{r, \ell, k\}$ and $f_{11}(v_{3c+2}) = s$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally, let $f_{11}(v_{3c}) = 0$, $f_{11}(v_{3c+1}) = 0$ and $f_{11}(v_{3c+2}) = s$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then f_{11} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_{11}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + k. \tag{A.21}$$

Now, if $k-\ell = \max\{s+k-r, k, k-\ell, s+\ell-2r\}$, then $\ell \leq 0$ and $s+\ell \leq r$. Let f_{12} be an $\boldsymbol{r}$-function such that $f_{12}(v_{3c+1}) = \max\{s+k-r, 0\}$, $f_{12}(v_{3c+2}) = \min\{r, s+k\}$ and $f_{12}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor$. Furthermore, let $f_{12}(v_{3c+1}) = 0$, $f_{12}(v_{3c+2}) = s+\ell$ and $f_{12}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally, let $f_{12}(v_{3c+1}) = 0$, $f_{12}(v_{3c+2}) = \min\{r, s\}$ and

$f_{12}(v_{3c+3}) = \max\{s-r, 0\}$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 1$. Then f_{12} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_{12}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + k - \ell. \tag{A.22}$$

Finally, if $s+\ell-2r = \max\{s+k-r, k, k-\ell, s+\ell-2r\}$, then $s+\ell \geq 2r$. Let f_{13} be an $\boldsymbol{r}$-function such that $f_{13}(v_{3c+1}) = \max\{s+k-r, 0\}$, $f_{13}(v_{3c+2}) = \min\{r, s+k\}$ and $f_{13}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Furthermore, let $f_{13}(v_m) = s+\ell-2r$ and $f_{13}(v_{m+1}) = r$ and let $f_{13}(v_{3c}) = r$, $f_{13}(v_{3c+1}) = s+\ell-2r$ and $f_{13}(v_{3c+2}) = r$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally, let $f_{13}(v_{3c}) = \max\{s-r, 0\}$, $f_{13}(v_{3c+1}) = 0$ and $f_{13}(v_{3c+2}) = \min\{r, s\}$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$ and let $f(v_n) = \max\{s-r, 0\}$. Then f_{13} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_{13}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell - 2r. \tag{A.23}$$

The result for the Case 2a follows from a combination of (A.16)–(A.23).

Case 2b: $p \equiv 1 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_m)}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+1}) + \cdots + f(v_{m+p-1})}_{\geq (s+l)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r
\end{aligned} \tag{A.24}$$

and

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m+1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p})}_{\geq (s+l)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p+1}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_n)}_{\geq 0} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + k.
\end{aligned} \tag{A.25}$$

Also,

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1) + \cdots + f(v_{m-1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_m) + \cdots + f(v_{m+p+1})}_{\geq (s+l)(\lfloor p/3 \rfloor + 1)} \\
& + \underbrace{f(v_{m+p+2}) + \cdots + f(v_n)}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell
\end{aligned} \tag{A.26}$$

and finally,

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m+1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p-3})}_{\geq (s+l)(\lfloor p/3 \rfloor - 1)} \\
& + \underbrace{f(v_{m+p-2}) + f(v_{m+p-1})}_{\geq 0} + \underbrace{f(v_{m+p}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + k - \ell.
\end{aligned} \tag{A.27}$$

If $s+k-r = \max\{s+k-r, k, \ell, k-\ell\}$, then $s \geq r$, $s+\ell \geq r$ and $s+k \geq r$. Let f_{14} be an $\boldsymbol{r}$-function such that $f_{14}(v_1) = s+k-r$, $f_{14}(v_2) = r$ and $f_{14}(v_n) = s-r$. Furthermore, let $f_{14}(v_{3c}) = 0$, $f_{14}(v_{3c+1}) = s+k-r$ and $f_{14}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $f_{14}(v_{3c}) = s-r$ and $f_{14}(v_{3c+2}) = r$ for $c = \lfloor m/3 \rfloor+1, \ldots, \lfloor n/3 \rfloor-1$. Finally, let $f_{14}(v_{3c+1}) = \ell$ for $c = \lfloor m/3 \rfloor+1, \ldots, \lfloor m/3 \rfloor+\lfloor p/3 \rfloor$ and let $f_{14}(v_{3c+1}) = 0$ for $c = \lfloor m/3 \rfloor+\lfloor p/3 \rfloor+1, \ldots, \lfloor n/3 \rfloor-1$. Then f_{14} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_{14}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r. \tag{A.28}$$

If $k = \max\{s+k-r, k, \ell, k-\ell\}$, then $s \leq r$, $\ell \geq 0$ and $s+k \geq s+\ell$. Let f_{15} be an $\boldsymbol{r}$-function such that $f_{15}(v_1) = \max\{0, s+k-r\}$, $f_{15}(v_2) = \min\{r, s+k\}$ and $f_{15}(v_n) = 0$. Furthermore, let $f_{15}(v_{3c}) = 0$, $f_{15}(v_{3c+1}) = \max\{0, s+k-r\}$ and $f_{15}(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $f_{15}(v_{3c}) = 0$, $f_{15}(v_{3c+1}) = \max\{0, s+\ell-r\}$ and $f_{15}(v_{3c+2}) = \min\{r, s+\ell\}$ for $c = \lfloor m/3 \rfloor+1, \ldots, \lfloor p/3 \rfloor$. Finally, let $f_{15}(v_{3c}) = 0$, $f_{15}(v_{3c+1}) = 0$ and $f_{15}(v_{3c+2}) = s$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then f_{15} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_{15}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k. \tag{A.29}$$

If $\ell = \max\{s+k-r, k, \ell, k-\ell\}$, then $s+\ell \geq s+k$. If $s \geq r$, then let f_{16} be an $\boldsymbol{r}$-function such that $f_{16}(v_{3c+1}) = \max\{s+k-r, 0\}$, $f_{16}(v_{3c+2}) = \min\{r, s+k\}$ and $f_{16}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Furthermore, let $f_{16}(v_{3c+1}) = \min\{\ell, r\}$, $f_{16}(v_{3c+2}) = r$ and $f_{16}(v_{3c+3}) = \max\{s-r, s+\ell-2r\}$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor+\lfloor p/3 \rfloor$. Finally, let $f_{16}(v_{3c+1}) = 0$, $f_{16}(v_{3c+2}) = r$ and $f_{16}(v_{3c+3}) = s-r$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then f_{16} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_{16}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell. \tag{A.30}$$

Now, if $s < r$, then let f_{17} be an $\boldsymbol{r}$-function such that $f_{16}(v_{3c+1}) = \max\{s+k-r, 0\}$, $f_{16}(v_{3c+2}) = \min\{r, s+k\}$ and $f_{16}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Furthermore, let $f_{16}(v_{3c+1}) = \min\{r, \max\{0, s+\ell-r\}\}$, $f_{16}(v_{3c+2}) = \min\{r, s+\ell\}$ and $f_{16}(v_{3c+3}) = \max\{0, s+\ell-2r\}$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally, let $f_{17}(v_{3c+1}) = 0$, $f_{17}(v_{3c+2}) = s$ and $f_{17}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then f_{17} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_{17}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell. \tag{A.31}$$

If $k-\ell = \max\{s+k-r, k, k-\ell\}$, then $\ell \leq 0$ and $s+\ell \leq r$. The function f_{12} is also an $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating function for the case where $p \equiv 1 \pmod 3$.

The result for the Case 2b follows from a combination of (A.22) and (A.24)–(A.31).

Case 2c: $p \equiv 2 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = & \underbrace{f(v_1)}_{\geq \max\{s+k-r,0\}} + \underbrace{f(v_2) + \cdots + f(v_m)}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+1}) + \cdots + f(v_{m+p+1})}_{\geq (s+l)(\lfloor p/3 \rfloor+1)} \\ & + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 2)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\ \geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \max\{s+k+\ell-r, \ell\} \end{aligned} \tag{A.32}$$

and

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = \underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_{m+1})}_{\geq (s+k)\lfloor m/3\rfloor} + \underbrace{f(v_{m+2})+\cdots+f(v_{m+p-1})}_{\geq (s+l)\lfloor p/3\rfloor}$$
$$+ \underbrace{f(v_{m+p})+\cdots+f(v_{n-1})}_{\geq s(\lfloor n/3\rfloor-\lfloor m/3\rfloor-\lfloor p/3\rfloor-1)} + \underbrace{f(v_n)}_{\geq \max\{s-r,0\}}$$
$$\geq s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + \max\{s+k-r,k\}. \tag{A.33}$$

Finally,

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = \underbrace{f(v_1)+\cdots+f(v_{m-1})}_{\geq (s+k)\lfloor m/3\rfloor} + \underbrace{f(v_m)+f(v_{m+1})+f(v_{m+2})}_{\geq s+\ell} + \underbrace{f(v_{m+3})}_{\geq s+\ell-2r}$$
$$+ \underbrace{f(v_{m+4})+\cdots+f(v_{m+p+1})}_{\geq (s+l)\lfloor p/3\rfloor} + \underbrace{f(v_{m+p+2})+\cdots+f(v_{n-2})}_{\geq s(\lfloor n/3\rfloor-\lfloor m/3\rfloor-\lfloor p/3\rfloor-2)} + \underbrace{f(v_{n-1})+f(v_n)}_{\geq s}$$
$$\geq s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + 2\ell - 2r. \tag{A.34}$$

If $s+k+\ell-r = \max\{s+k+\ell-r, s+k-r, k, \ell, s+2\ell-2r\}$, then $s+k \geq s+\ell-r$, $s+\ell \geq r$ and $s+k \geq r$. Let f_{18} be an $\boldsymbol{r}$-function such that $f_{18}(v_1) = s+k-r$, $f_{18}(v_{n-1}) = \min\{r,s\}$ and $f_{18}(v_n) = \max\{s-r,0\}$. Furthermore, let $f_{18}(v_{3c+2}) = r$, $f_{18}(v_{3c+3}) = 0$ and $f_{18}(v_{3c+4}) = s+k-r$ for $c = 1,\ldots,\lfloor m/3\rfloor$. Also, let $f_{18}(v_{3c+2}) = r$, $f_{18}(v_{3c+3}) = \min\{r, s+\ell-r\}$ and $f_{18}(v_{3c+4}) = \max\{s+\ell-2r, 0\}$ for $c = \lfloor m/3\rfloor+1,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor+1$. Finally, let $f_{18}(v_{3c+2}) = \min\{r,s\}$, $f_{18}(v_{3c+3}) = \max\{s-r,0\}$ and $f_{18}(v_{3c+4}) = 0$ for $c = \lfloor m/3\rfloor+\lfloor p/3\rfloor+2,\ldots,\lfloor n/3\rfloor-1$. Then f_{18} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_{18}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s+k+\ell-r. \tag{A.35}$$

If $s+k-r = \max\{s+k+\ell-r, s+k-r, k, \ell, s+2\ell-2r\}$, then $\ell \leq 0$ and $s \geq r$. Let f_{19} be an $\boldsymbol{r}$-function such that $f_{19}(v_1) = s+k-r$, $f_{19}(v_2) = r$ and $f_{19}(v_n) = s-r$. Furthermore, let $f_{19}(v_{3c}) = 0$, $f_{19}(v_{3c+1}) = s+k-r$ and $f_{19}(v_{3c+2}) = r$ for $c = 1,\ldots,\lfloor m/3\rfloor$. Also, let $f_{19}(v_{3c}) = 0$, $f_{19}(v_{3c+1}) = \max\{s+\ell-r,0\}$ and $f_{19}(v_{3c+2}) = \min\{r, s+\ell\}$ for $c = \lfloor m/3\rfloor + 1,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally, let $f_{19}(v_{3c}) = 0$, $f_{19}(v_{3c+1}) = s-r$ and $f_{19}(v_{3c+2}) = r$ for $c = \lfloor m/3\rfloor+\lfloor p/3\rfloor+2,\ldots,\lfloor n/3\rfloor-1$. Then f_{19} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_{19}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s+k-r. \tag{A.36}$$

If $k = \max\{s+k+\ell-r, s+k-r, k, \ell, s+2\ell-2r\}$, then $s+\ell \leq r$, $s+k \geq s+\ell$ and $s \leq r$. Let f_{20} be an $\boldsymbol{r}$-function such that $f_{20}(v_{3c+1}) = \max\{s+k-r, 0\}$, $f_{20}(v_{3c+2}) = \min\{r, s+k\}$ and $f_{20}(v_{3c+3}) = 0$ for $c = 0,\ldots,\lfloor m/3\rfloor$. Also, let $f_{20}(v_{3c+1}) = 0$, $f_{20}(v_{3c+2}) = s+\ell$ and $f_{20}(v_{3c+3}) = 0$ for $c = \lfloor m/3\rfloor+1,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally, let $f_{20}(v_{3c+1}) = 0$, $f_{20}(v_{3c+2}) = s$ and $f_{20}(v_{3c+3}) = 0$ for $c = \lfloor m/3\rfloor+\lfloor p/3\rfloor+1,\ldots,\lfloor n/3\rfloor-1$. Then f_{20} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_{20}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + k. \tag{A.37}$$

If $\ell = \max\{s+k+\ell-r, s+k-r, k, \ell, s+2\ell-2r\}$, then $s+k \leq r$, $s+k \leq s+\ell$, $s \leq r$ and $s+\ell \leq 2r$. Let f_{21} be an $\boldsymbol{r}$-function such that $f_{21}(v_{3c+1}) = 0$, $f_{21}(v_{3c+2}) = s+k$ and $f_{21}(v_{3c+3}) = 0$ for $c = 0,\ldots,\lfloor m/3\rfloor-1$. Also, let $f_{21}(v_{3c+1}) = 0$, $f_{21}(v_{3c+2}) = \min\{r, s+\ell\}$ and $f_{21}(v_{3c+3}) = \max\{s+\ell-r, 0\}$ for $c = \lfloor m/3\rfloor,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally, let $f_{21}(v_{3c+1}) = 0$,

$f_{21}(v_{3c+2}) = s$ and $f_{21}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then f_{21} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_{21}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell. \tag{A.38}$$

If $s+2\ell-2r = \max\{s+k+\ell-r, s+k-r, k, \ell, s+2\ell-2r\}$, then $s+\ell \geq 2r$ and $s+k \leq s+\ell-r$. Let f_{22} be an $\boldsymbol{r}$-function such that $f_{22}(v_{3c+1}) = \max\{s+k-r, 0\}$, $f_{22}(v_{3c+2}) = \min\{r, s+k\}$ and $f_{22}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Furthermore, let $f_{22}(v_{3c+1}) = s+\ell-2r$, $f_{22}(v_{3c+2}) = r$ and $f_{22}(v_{3c+3}) = r$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Also, let $f_{22}(v_{m+p+1}) = s+\ell-2r$ and $f_{22}(v_{3c+4}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 2$. Finally, let $f_{22}(v_{3c+2}) = \min\{r, s\}$ and $f_{22}(v_{3c+3}) = \max\{s-r, 0\}$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then f_{22} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_{22}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell - 2r. \tag{A.39}$$

The result for the Case 2b follows from a combination of (A.32)–(A.39).

Finally consider the case where $m \equiv 2 \pmod 3$ and the three subcases where $p \equiv 0, 1, 2 \pmod 3$.

Case 3: $m \equiv 2 \pmod 3$.

Case 3a: $p \equiv 0 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = {} & \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_m)}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+1}) + \cdots + f(v_{m+p-1})}_{\geq (s+l)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r
\end{aligned} \tag{A.40}$$

and

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = {} & \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_m)}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+1}) + \cdots + f(v_{m+p})}_{\geq (s+l)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p+1}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_n)}_{\geq 0} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + k.
\end{aligned} \tag{A.41}$$

It also follows that

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = {} & \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_m)}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+1}) + f(v_{m+2})}_{\geq 0} + \underbrace{f(v_{m+3}) + \cdots + f(v_{m+p-1})}_{\geq (s+l)(\lfloor p/3 \rfloor - 1)} \\
& + \underbrace{f(v_{m+p}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + k - \ell
\end{aligned} \tag{A.42}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = &\underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2)+\cdots+f(v_{m-1})}_{\geq(s+k)\lfloor m/3\rfloor} + \underbrace{f(v_{m+1})+f(v_{m+2})}_{\geq s+\ell-r}\\ &+ \underbrace{f(v_{m+3})+\cdots+f(v_{m+p-1})}_{\geq(s+\ell)\lfloor p/3\rfloor} + \underbrace{f(v_{m+p})+\cdots+f(v_n)}_{\geq s(\lfloor n/3\rfloor-\lfloor m/3\rfloor-\lfloor p/3\rfloor-1)}\\ \geq &s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s+k+\ell-2r. \qquad \text{(A.43)}\end{aligned}$$

Finally,

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = &\underbrace{f(v_1)+\cdots+f(v_{m-2})}_{\geq(s+k)\lfloor m/3\rfloor} + \underbrace{f(v_{m-1})}_{\geq 0} + \underbrace{f(v_m)+f(v_{m+1})}_{\geq s+\ell-r}\\ &+ \underbrace{f(v_{m+2})+\cdots+f(v_{m+p-1})}_{\geq(s+\ell)\lfloor p/3\rfloor} + \underbrace{f(v_{m+p})+\cdots+f(v_n)}_{\geq s(\lfloor n/3\rfloor-\lfloor m/3\rfloor-\lfloor p/3\rfloor-1)}\\ \geq &s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + \ell - r. \qquad \text{(A.44)}\end{aligned}$$

If $s+k-r = \max\{s+k-r, k, k-\ell, s+k+\ell-2r, \ell-r\}$, then $s \geq r$, $s+k \geq r$, $\ell \leq r$ and $s+\ell \geq r$. The function f_{14} is also an $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating function for the case where $m \equiv 2 \pmod 3$ and $p \equiv 0 \pmod 3$.

If $k = \max\{s+k-r, k, k-\ell, s+k+\ell-2r, \ell-r\}$, then $s \leq r$, $\ell \geq 0$ and $s+\ell \leq 2r$. The function f_{15} is also an $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating function for the case where $m \equiv 2 \pmod 3$ and $p \equiv 0 \pmod 3$.

If $k-\ell = \max\{s+k-r, k, k-\ell, s+k+\ell-2r, \ell-r\}$, then $\ell \leq 0$ and $s+\ell \leq r$. The function f_{12} is also an $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating function for the case where $m \equiv 2 \pmod 3$ and $p \equiv 0 \pmod 3$.

If $s+k+\ell-2r = \max\{s+k-r, k, k-\ell, s+k+\ell-2r, \ell-r\}$, then $\ell \geq r$, $s+k \geq r$ and $s+\ell \geq 2r$. Let f_{23} be an $\boldsymbol{r}$-function such that $f_{23}(v_1) = s+k-r$, $f_{23}(v_2) = r$ and $f_{23}(v_{m+p+1}) = s+\ell-2r$. Furthermore, let $f_{23}(v_{3c}) = 0$, $f_{23}(v_{3c+1}) = s+k-r$ and $f_{23}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3\rfloor$. Also, let $f_{23}(v_{3c}) = s+\ell-2r$, $f_{23}(v_{3c+1}) = r$ and $f_{23}(v_{3c+2}) = r$ for $c = \lfloor m/3\rfloor+1, \ldots, \lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally, let $f_{23}(v_{3c+1}) = 0$, $f_{23}(v_{3c+2}) = \min\{r,s\}$ and $f_{23}(v_{3c+3}) = \max\{s-r, 0\}$ for $c = \lfloor m/3\rfloor+\lfloor p/3\rfloor+1, \ldots, \lfloor n/3\rfloor-1$. Then f_{23} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_{23}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s+k+\ell-2r. \qquad \text{(A.45)}$$

If $\ell-r = \max\{s+k-r, k, k-\ell, s+k+\ell-2r, \ell-r\}$, then $s+k \leq r$, $\ell \geq r$ and $s+k \leq s+\ell-r$. Let f_{24} be an $\boldsymbol{r}$-function such that $f_{24}(v_{3c+2}) = s+k$ and $f_{24}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3\rfloor-1$ and $f_{24}(v_{3c+1}) = 0$ for $c = 0, \ldots, \lfloor m/3\rfloor$. Also, let $f_{24}(v_{3c+2}) = \min\{r, s+\ell-r\}$ and $f_{24}(v_{3c+3}) = \min\{s+\ell-2r, 0\}$ for $c = \lfloor m/3\rfloor, \ldots, \lfloor m/3\rfloor+\lfloor p/3\rfloor$ and let $f_{24}(v_{3c+1}) = r$ for $c = \lfloor m/3\rfloor+1, \ldots, \lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally, let $f_{24}(v_{3c+1}) = 0$, $f_{24}(v_{3c+2}) = s$ and $f_{24}(v_{3c+3}) = 0$ for $c = \lfloor m/3\rfloor+\lfloor p/3\rfloor+1, \ldots, \lfloor n/3\rfloor-1$. Then f_{24} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_{24}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + \ell - r. \qquad \text{(A.46)}$$

The result for the Case 3a follows from a combination of (A.22), (A.28), (A.29) and (A.40)–(A.46).

Case 3b: $p \equiv 1 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_m)}_{\geq (s+k)\lfloor m/3\rfloor} + \underbrace{f(v_{m+1})+\cdots+f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3\rfloor} \\ &+ \underbrace{f(v_{m+p})+\cdots+f(v_{n-1})}_{\geq s(\lfloor n/3\rfloor-\lfloor m/3\rfloor-\lfloor p/3\rfloor-1)} + \underbrace{f(v_n)}_{\geq \max\{s-r,0\}} \\ \geq& s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + \max\{s+k-r,k\} \end{aligned} \tag{A.47}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)}_{\geq \max\{s+k-r,0\}} + \underbrace{f(v_2)+\cdots+f(v_{m-1})}_{\geq (s+k)\lfloor m/3\rfloor} \\ &+ \underbrace{f(v_m)+\cdots+f(v_{m+p+1})}_{\geq (s+\ell)(\lfloor p/3\rfloor+1)} + \underbrace{f(v_{m+p+2})+\cdots+f(v_{n-2})}_{\geq s(\lfloor n/3\rfloor-\lfloor m/3\rfloor-\lfloor p/3\rfloor-2)} + \underbrace{f(v_{n-1})+f(v_n)}_{\geq s} \\ \geq& s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + \max\{s+k+\ell-r,\ell\}. \end{aligned} \tag{A.48}$$

If $s+k+\ell-r = \max\{s+k+\ell-r, s+k-r, k, \ell\}$, then $\ell \geq 0$, $s+k \geq r$ and $s+\ell \geq r$. Let f_{25} be an $\boldsymbol{r}$-function such that $f_{25}(v_{3c+1}) = s+k-r$ and $f_{25}(v_{3c+2}) = r$ for $c = 0, \ldots, \lfloor m/3\rfloor$ and $f_{25}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3\rfloor - 1$. Also, let $f_{25}(v_{3c}) = \max\{s+\ell-2r, 0\}$ and $f_{25}(v_{3c+1}) = \min\{r, s+\ell-r\}$ for $c = \lfloor m/3\rfloor+1, \ldots, \lfloor m/3\rfloor+\lfloor p/3\rfloor+1$ and $f_{25}(v_{3c+2}) = r$ for $c = \lfloor m/3\rfloor+1, \ldots, \lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally, let $f_{25}(v_{3c+2}) = \min\{r, s\}$ and $f_{25}(v_{3c+3}) = \max\{s-r, 0\}$ for $c = \lfloor m/3\rfloor+\lfloor p/3\rfloor+1, \ldots, \lfloor n/3\rfloor-1$ and $f_{25}(v_{3c+1}) = 0$ for $c = \lfloor m/3\rfloor+\lfloor p/3\rfloor+1, \ldots, \lfloor n/3\rfloor-2$. Then f_{25} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_{25}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s+k+\ell-r. \tag{A.49}$$

If $s+k-r = \max\{s+k+\ell-r, s+k-r, k, \ell\}$, then $s \geq r$ and $\ell \leq 0$. Let f_{26} be an $\boldsymbol{r}$-function such that $f_{26}(v_1) = s+k-r$, $f_{26}(v_2) = r$ and $f_{26}(v_n) = s-r$. Furthermore, let $f_{26}(v_{3c}) = 0$, $f_{26}(v_{3c+1}) = s+k-r$ and $f_{26}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3\rfloor$. Also, let $f_{26}(v_{3c}) = 0$, $f_{26}(v_{3c+1}) = \max\{s+\ell-r, 0\}$ and $f_{26}(v_{3c+2}) = \max\{r, s+\ell\}$ for $c = \lfloor m/3\rfloor+1, \ldots, \lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally, let $f_{26}(v_{3c}) = 0$, $f_{26}(v_{3c+1}) = s-r$ and $f_{26}(v_{3c+2}) = r$ for $c = \lfloor m/3\rfloor+\lfloor p/3\rfloor+1, \ldots, \lfloor n/3\rfloor-1$. Then f_{26} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_{26}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s+k-r. \tag{A.50}$$

If $k = \max\{s+k+\ell-r, s+k-r, k, \ell\}$, then $s \leq r$, $s+k \geq s+\ell$ and $s+\ell \leq r$. Let f_{27} be an $\boldsymbol{r}$-function such that $f_{27}(v_{3c+1}) = \max\{s+k-r, 0\}$, $f_{27}(v_{3c+2}) = \min\{r, s+k\}$ and $f_{27}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3\rfloor$. Also, let $f_{27}(v_{3c+1}) = 0$, $f_{27}(v_{3c+2}) = s+\ell$ and $f_{27}(v_{3c+3}) = 0$ for $c = \lfloor m/3\rfloor+1, \ldots, \lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally, let $f_{27}(v_{3c+1}) = 0$, $f_{27}(v_{3c+2}) = s$ and $f_{27}(v_{3c+3}) = 0$ for $c = \lfloor m/3\rfloor+\lfloor p/3\rfloor+1, \ldots, \lfloor n/3\rfloor-1$. Then f_{27} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|f_{27}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + k. \tag{A.51}$$

If $\ell = \max\{s+k+\ell-r, s+k-r, k, \ell\}$, then $s+k \leq r$, $s+k \leq s+\ell$ and $s \leq r$. Let f_{28} be an $\boldsymbol{r}$-function such that $f_{23}(v_1) = 0$, $f_{23}(v_{n-1}) = s$ and $f_{23}(v_n) = 0$. Furthermore, let $f_{28}(v_{3c+2}) = s+k$, $f_{28}(v_{3c+3}) = 0$ and $f_{28}(v_{3c+4}) = 0$ for $c = 0, \ldots, \lfloor m/3\rfloor - 1$. Also, let $f_{28}(v_{3c+2}) = \min\{r, s+\ell\}$, $f_{28}(v_{3c+3}) = \min\{\max\{s+\ell-r, 0\}, r\}$ and $f_{28}(v_{3c+4}) = \max\{0, s+$

$\ell - 2r\}$ for $c = \lfloor m/3 \rfloor, \dots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally, let $f_{28}(v_{3c+2}) = s$, $f_{28}(v_{3c+3}) = 0$ and $f_{28}(v_{3c+4}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \dots, \lfloor n/3 \rfloor - 2$. Then f_{28} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_{28}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell. \tag{A.52}$$

The result for the Case 3b follows from a combination of (A.47)–(A.52).

Case 3c: $p \equiv 2 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) =& \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_m)}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+1}) + \cdots + f(v_{m+p+1})}_{\geq (s+\ell)(\lfloor p/3 \rfloor + 1)} \\
&+ \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 2)} + \underbrace{f(v_n)}_{\geq \max\{s-r,0\}} \\
\geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \max\{s + k + \ell - r, k + \ell\}
\end{aligned} \tag{A.53}$$

and

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) =& \underbrace{f(v_1) + \cdots + f(v_{m+1})}_{\geq (s+k)(\lfloor m/3 \rfloor + 1)} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\
&+ \underbrace{f(v_{m+p}) + \cdots + f(v_n)}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} \\
\geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + k.
\end{aligned} \tag{A.54}$$

If $s + k + \ell - r = \max\{s + k + \ell - r, k, k + \ell\}$, then $s \geq r$ and $s + \ell \geq r$ and the function f_{25} is also an $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating function for the case where $p \equiv 2 \pmod 3$.

If $k = \max\{s + k + \ell - r, k, k + \ell\}$, then $\ell \leq 0$ and $s + \ell \leq r$. Let f_{29} be an $\boldsymbol{r}$-function such that $f_{29}(v_{3c+1}) = \max\{s+k-r, 0\}$, $f_{29}(v_{3c+2}) = \min\{r, s+k\}$ and $f_{29}(v_{3c+3}) = 0$ for $c = 0, \dots, \lfloor m/3 \rfloor$. Also, let $f_{29}(v_{3c+1}) = 0$, $f_{29}(v_{3c+2}) = s + \ell$ and $f_{29}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + 1, \dots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally, let $f_{29}(v_{3c+1}) = 0$, $f_{29}(v_{3c+2}) = \min\{r, s\}$ and $f_{29}(v_{3c+3}) = \max\{s - r, 0\}$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \dots, \lfloor n/3 \rfloor - 1$. Then f_{29} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_{29}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + k. \tag{A.55}$$

If $k + \ell = \max\{s + k + \ell - r, k, k + \ell\}$, then $s \leq r$ and $\ell \geq 0$. Let f_{30} be an $\boldsymbol{r}$-function such that $f_{30}(v_1) = \max\{s + k - r, 0\}$, $f_{30}(v_2) = \min\{r, s + k\}$ and $f_{30}(v_n) = 0$. Furthermore, let $f_{30}(v_{3c}) = 0$, $f_{30}(v_{3c+1}) = \max\{s+k-r, 0\}$ and $f_{30}(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \dots, \lfloor m/3 \rfloor$. Also, let $f_{30}(v_{3c}) = \max\{\ell - r, 0\}$, $f_{30}(v_{3c+1}) = \min\{r, \ell\}$ and $f_{30}(v_{3c+2}) = s$ for $c = \lfloor m/3 \rfloor + 1, \dots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1$. Finally, let $f_{30}(v_{3c}) = 0$, $f_{30}(v_{3c+1}) = 0$ and $f_{30}(v_{3c+2}) = s$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 2, \dots, \lfloor n/3 \rfloor - 1$. Then f_{30} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|f_{30}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + k + \ell. \tag{A.56}$$

The result for the Case 3c follows from a combination of (A.53)–(A.56). ■

The $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a path of order n, where $n \equiv 1 \pmod 3$ and $\boldsymbol{s}$ is a step function with two steps is considered next.

Proposition 5.9. Let $\left(P_n, \boldsymbol{r}, \boldsymbol{s}_{m,p}^{k,\ell}\right)$ be a graph triple, where $\boldsymbol{r} = [r, \ldots, r]$ and

$$\boldsymbol{s}_{m,p}^{k,\ell} = [\overbrace{s+k, \ldots, s+k}^{m \text{ times}}, \overbrace{s+\ell, \ldots, s+\ell}^{p \text{ times}}, s, \ldots, s]$$

satisfy $2r \geq s+k$ and $3r \geq s+\ell$ for $m, p < n$, $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$. If $n \equiv 1 \pmod 3$, then

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + \ell\left\lfloor \frac{p}{3} \right\rfloor + g(s,k,l,r)$$

where

$$g(s,k,l,r) = \begin{cases} \max\{s, s+k-r, s+\ell-r\} & \text{if } m,p \equiv 0 \pmod 3 \\ \max\{s+k-r, s+\ell, s+k-\ell-2r\} & \text{if } m \equiv 0 \pmod 3 \text{ and } p \equiv 1 \pmod 3 \\ \max\{s+k-r, s+\ell, s+2\ell-2r\} & \text{if } m \equiv 0 \pmod 3 \text{ and } p \equiv 2 \pmod 3 \\ \max\{s+\ell-r, s+k, s+k-\ell-r\} & \text{if } m \equiv 1 \pmod 3 \text{ and } p \equiv 0 \pmod 3 \\ \max\{s+\ell-r, s+k, \ell\} & \text{if } m,p \equiv 1 \pmod 3 \\ \max\{s+k+\ell-r, s+k, \ell, s+2\ell-2r\} & \text{if } m \equiv 1 \pmod 3 \text{ and } p \equiv 2 \pmod 3 \\ \max\{s+k, s+k+\ell-2r, \ell-r\} & \text{if } m \equiv 2 \pmod 3 \text{ and } p \equiv 0 \pmod 3 \\ \max\{s+k+\ell-r, s+k, \ell\} & \text{if } m \equiv 2 \pmod 3 \text{ and } p \equiv 1 \pmod 3 \\ \max\{s+k+\ell, s+2\ell-2r\} & \text{if } m,p \equiv 2 \pmod 3. \end{cases}$$

Proof. First consider the case where $m \equiv 0 \pmod 3$ and the three subcases where $p \equiv 0, 1, 2 \pmod 3$.

Case 1: $m \equiv 0 \pmod 3$.

Case 1a: $p \equiv 0 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned} \gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = &\underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_{m-1})}_{\geq (s+k)(\lfloor m/3\rfloor -1)} + \underbrace{f(v_m)+f(v_{m+1})}_{\geq s-r+\max\{k,\ell\}} \\ &+ \underbrace{f(v_{m+2})+\cdots+f(v_{m+p+1})}_{\geq (s+\ell)\lfloor p/3\rfloor} + \underbrace{f(v_{m+p+2})+\cdots+f(v_n)}_{\geq s(\lfloor n/3\rfloor - m/3 - \lfloor p/3\rfloor)} \\ \geq & s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell p/3 + \max\{s+k-r, s+\ell-r\} \end{aligned} \tag{A.57}$$

and

$$\begin{aligned} \gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = &\underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_{m-1})}_{\geq (s+k)(\lfloor m/3\rfloor -1)} + \underbrace{f(v_m)+\cdots+f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3\rfloor} \\ &+ \underbrace{f(v_{m+p})+\cdots+f(v_{n-2})}_{\geq s(\lfloor n/3\rfloor - m/3 - \lfloor p/3\rfloor)} + \underbrace{f(v_{n-1})+f(v_n)}_{\geq s} \\ \geq & s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell p/3 + s. \end{aligned} \tag{A.58}$$

If $s = \max\{s, s+k-r, s+\ell-r\}$, then $k, \ell \leq r$ and $s+\ell \leq 2r$. Let g_1 be an $\boldsymbol{r}$-function such that $g_1(v_1) = \min\{r, s\}$ and $g_1(v_2) = \max\{k, s+k-r\}$. Furthermore, let $g_1(v_{3c}) = \min\{r, s\}$ and $g_1(v_{3c+1}) = \max\{0, s-r\}$ for $c = 1, \ldots, \lfloor n/3\rfloor - 1$. Also, let $g_1(v_{3c+2}) = k$ for $c = 1, \ldots, \lfloor m/3\rfloor - 1$, $g_1(v_{3c+2}) = \ell$ for $c = \lfloor m/3\rfloor, \ldots, \lfloor m/3\rfloor + \lfloor p/3\rfloor - 1$ and $g_1(v_{3c+2}) = 0$ for $c = \lfloor m/3\rfloor + \lfloor p/3\rfloor, \ldots, \lfloor n/3\rfloor - 1$. Finally let $g_1(v_{n-1}) = \min\{s, r\}$ and $g_1(v_n) = \max\{0, s-r\}$. Then g_1 is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_1| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell p/3 + s. \tag{A.59}$$

If $s+k-r = \max\{s, s+k-r, s+\ell-r\}$, then $s+k \geq r$, $k \geq \ell, r$ and $s \leq r$. Let g_2 be an $\boldsymbol{r}$-function such that $g_2(v_{3c+1}) = s+k-r$, $g_2(v_{3c+2}) = r$ and $g_2(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Furthermore let $g_2(v_{3c+1}) = s+k-r$, $g_2(v_{3c+2}) = \ell + r - k$ and $g_2(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally let $g_2(v_{m+p+1}) = s+k-r$ and let $g_2(v_{3c+2}) = 0$, $g_2(v_{3c+3}) = s$ and $g_2(v_{3c+4}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 2$. Then g_2 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_2| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell p/3 + s + k - r. \tag{A.60}$$

If $s+\ell-r = \max\{s, s+k-r, s+\ell-r\}$, then $s+\ell \geq r$, $\ell \geq r$ and $s+\ell \geq s+k$. Let g_3 be an $\boldsymbol{r}$-function such that $g_3(v_1) = \max\{0, s+k-r\}$, $g_3(v_2) = \min\{r, s+k\}$, $g_3(v_{m+p}) = \max\{0, s+\ell-2r\}$ and $g_3(v_{m+p+1}) = \min\{r, s+\ell-r\}$. Furthermore, let $g_3(v_{3c}) = 0$, $g_3(v_{3c+1}) = \max\{0, s+k-r\}$ and $g_3(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $g_3(v_{3c}) = \max\{0, s+\ell-2r\}$, $g_3(v_{3c+1}) = \min\{r, s+\ell-r\}$ and $g_3(v_{3c+2}) = r$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally let $g_3(v_{3c+2}) = 0$, $g_3(v_{3c+3}) = \min\{r, s\}$ and $g_3(v_{3c+4}) = \max\{0, s-r\}$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 2$. Then g_3 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_3| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell p/3 + s + \ell - r. \tag{A.61}$$

The result for the Case 1a follows from a combination of (A.57)–(A.61).

Case 1b: $p \equiv 1 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{m+1})}_{\geq (s+k)(m/3)} \\
& + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p})}_{\geq (s+\ell)\lfloor p/3 \rfloor} + \underbrace{f(v_{m+p+1}) + \cdots + f(v_n)}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor)} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r
\end{aligned} \tag{A.62}$$

and

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor - 1)} + \underbrace{f(v_m) + \cdots + f(v_{m+p+1})}_{\geq (s+\ell)(\lfloor p/3 \rfloor + 1)} \\
& + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell.
\end{aligned} \tag{A.63}$$

Finally,

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor - 1)} + \underbrace{f(v_m) + f(v_{m+1})}_{\geq s+k-r} + \underbrace{f(v_{m+2})}_{\geq 0} \\
& + \underbrace{f(v_{m+3}) + \cdots + f(v_{m+p-2})}_{\geq (s+\ell)(p/3-1)} + \underbrace{f(v_{m+p-1})}_{\geq 0} + \underbrace{f(v_{m+p}) + f(v_{m+p+1})}_{\geq s-r} \\
& + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - \ell - 2r.
\end{aligned} \tag{A.64}$$

If $s+k-r = \max\{s+k-r, s+\ell, s+k-\ell-2r\}$, then $s+k \geq r$, $r \geq -\ell$ and $s+\ell \leq r$ since $s+k \leq 2r$. Let g_4 be an $\boldsymbol{r}$-function such that $g_4(v_{3c+1}) = s+k-r$, $g_4(v_{3c+2}) = r$ and $g_4(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$ and let $g_4(v_{m+1}) = s+k-r$. Furthermore let $g_4(v_{3c+2}) = 0$, $g_4(v_{3c+3}) = 0$ and $g_4(v_{3c+4}) = s+\ell$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally let $g_4(v_{3c+2}) = 0$, $g_4(v_{3c+3}) = \min\{r, s\}$ and $g_4(v_{3c+4}) = \max\{0, s-r\}$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 1$. Then g_4 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_4| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell p/3 + s + k - r. \tag{A.65}$$

If $s+\ell = \max\{s+k-r, s+\ell, s+k-\ell-2r\}$, then $s+\ell \geq s+k-r$. Let g_5 be an $\boldsymbol{r}$-function such that $g_5(v_1) = \max\{0, s+k-r\}$, $g_5(v_2) = \min\{r, s+k\}$, $g_5(v_{n-1}) = \min\{r, s\}$ and $g_5(v_n) = \max\{0, s-r\}$. Furthermore let $g_5(v_{3c}) = 0$, $g_5(v_{3c+1}) = \max\{0, s+k-r\}$ and $g_5(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $g_5(v_{3c}) = \min\{\max\{0, s+\ell-r\}, r\}$, $g_5(v_{3c+1}) = \min\{r, s+\ell\}$ and $g_5(v_{3c+2}) = \max\{0, s+\ell-2r\}$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_5(v_{3c}) = \min\{r, s\}$, $g_5(v_{3c+1}) = \max\{0, s-r\}$ and $g_5(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_5 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_5| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell p/3 + s + \ell. \tag{A.66}$$

If $s+k-\ell-2r = \max\{s+k-r, s+\ell, s+k-\ell-2r\}$, then $s+\ell \leq s-r$ showing that $s \geq r$. Let g_6 be an $\boldsymbol{r}$-function such that $g_6(v_1) = s+k-r$, $g_6(v_{m+2}) = 0$, $g_6(v_{m+p-1}) = 0$ and $g_6(v_n) = s-r$. Furthermore let $g_6(v_{3c+2}) = r$, $g_6(v_{3c+3}) = 0$ and $g_6(v_{3c+4}) = s+k-r$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_6(v_{3c}) = 0$, $g_6(v_{3c+1}) = s+\ell$ and $g_6(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally let $g_6(v_{3c+1}) = s-r$, $g_6(v_{3c+2}) = 0$ and $g_6(v_{3c+3}) = r$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 1$. Then g_6 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_6| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell p/3 + s + k - \ell - 2r. \tag{A.67}$$

The result for the Case 1a follows from a combination of (A.62)–(A.67). *Case 1c:* $p \equiv 2 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = & \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{m+1})}_{\geq (s+k)(m/3)} \\
& + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} + \underbrace{f(v_{m+p}) + \cdots + f(v_n)}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor)} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r
\end{aligned} \tag{A.68}$$

and

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = & \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor - 1)} + \underbrace{f(v_m) + \cdots + f(v_{m+p})}_{\geq (s+\ell)(\lfloor p/3 \rfloor + 1)} \\
& + \underbrace{f(v_{m+p+1}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell.
\end{aligned} \tag{A.69}$$

Finally,

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =&\underbrace{f(v_1)+f(v_2)}_{\geq s+k}+\underbrace{f(v_3)+\cdots+f(v_{m-1})}_{\geq(s+k)(\lfloor m/3\rfloor-1)}+\underbrace{f(v_m)+f(v_{m+1})}_{\geq s+\ell-r}+\underbrace{f(v_{m+2})+f(v_{m+3})}_{\geq s+\ell-r}\\&+\underbrace{f(v_{m+4})+\cdots+f(v_{m+p+1})}_{\geq(s+\ell)\lfloor p/3\rfloor}+\underbrace{f(v_{m+p+2})+\cdots+f(v_{n-1})}_{\geq s(\lfloor n/3\rfloor-m/3-\lfloor p/3\rfloor-1)}+\underbrace{f(v_n)}_{\geq 0}\\&\geq s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell\lfloor p/3\rfloor+s+2\ell-2r.\end{aligned} \tag{A.70}$$

If $s+k-r=\max\{s+k-r,s+\ell,s+2\ell-2r\}$, then $s+k\geq r$ and $s+\ell\leq r$ since $s+k\leq 2r$. The function g_4 is also an $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating function for the case where $p\equiv 2 \pmod 3$.

If $s+\ell=\max\{s+k-r,s+\ell,s+2\ell-2r\}$, then $\ell\leq 2r$. The function g_5 is also an $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating function for the case where $p\equiv 2 \pmod 3$ except when $s+\ell\geq 2r$ and $s\leq r$. When $s+\ell\geq 2r$ and $s\leq r$ let g_7 be an $\boldsymbol{r}$-function such that $g_7(v_1)=\max\{0,s+k-r\}$, $g_7(v_2)=\min\{r,s+k\}$, $g_7(v_{n-1})=s$ and $g_7(v_n)=0$. Furthermore let $g_7(v_{3c})=0$, $g_7(v_{3c+1})=\max\{0,s+k-r\}$ and $g_7(v_{3c+2})=\min\{r,s+k\}$ for $c=1,\ldots,\lfloor m/3\rfloor-1$. Also, let $g_7(v_{3c})=s$, $g_7(v_{3c+1})=r$ and $g_7(v_{3c+2})=\ell-r$ for $c=\lfloor m/3\rfloor,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally let $g_7(v_{3c})=s$, $g_5(v_{3c+1})=0$ and $g_7(v_{3c+2})=0$ for $c=\lfloor m/3\rfloor+\lfloor p/3\rfloor+1,\ldots,\lfloor n/3\rfloor-1$. Then g_7 is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_7|=s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell p/3+s+\ell. \tag{A.71}$$

If $s+2\ell-2r=\max\{s+k-r,s+\ell,s+2\ell-2r\}$, then $\ell\geq 2r$ and $s\leq r$. Let g_8 be an $\boldsymbol{r}$-function such that $g_8(v_1)=\max\{0,s+k-r\}$, $g_8(v_2)=\min\{r,s+k\}$, $g_8(v_{m+p+1})=s+\ell-2r$ and $g_8(v_n)=0$. Furthermore let $g_8(v_{3c})=0$, $g_8(v_{3c+1})=\max\{0,s+k-r\}$ and $g_8(v_{3c+2})=\min\{r,s+k\}$ for $c=1,\ldots,\lfloor m/3\rfloor-1$. Also, let $g_8(v_{3c})=s+\ell-2r$, $g_8(v_{3c+1})=r$ and $g_8(v_{3c+2})=r$ for $c=\lfloor m/3\rfloor,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally let $g_8(v_{3c+1})=0$, $g_8(v_{3c+2})=0$ and $g_8(v_{3c+3})=s$ for $c=\lfloor m/3\rfloor+\lfloor p/3\rfloor+1,\ldots,\lfloor n/3\rfloor-1$. Then g_8 is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_8|=s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell p/3+s+2\ell-2r. \tag{A.72}$$

The result for the Case 1c follows from a combination of (A.65), (A.66), (A.57)–(A.72).

Now consider the case where $m\equiv 1 \pmod 3$ and the three subcases where $p\equiv 0,1,2 \pmod 3$.

Case 2: $m\equiv 1 \pmod 3$.

Case 2a: $p\equiv 0 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =&\underbrace{f(v_1)+\cdots+f(v_{m-1})}_{\geq(s+k)\lfloor m/3\rfloor}+\underbrace{f(v_m)+f(v_{m+1})}_{\geq s-r+\ell}\\&+\underbrace{f(v_{m+2})+\cdots+f(v_{m+p+1})}_{\geq(s+\ell)\lfloor p/3\rfloor}+\underbrace{f(v_{m+p+2})+\cdots+f(v_{n-2})}_{\geq s(\lfloor n/3\rfloor-m/3-\lfloor p/3\rfloor-1)}+\underbrace{f(v_{n-1})+f(v_n)}_{\geq s}\\&\geq s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell p/3+s+\ell-r\end{aligned} \tag{A.73}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_{m+1})}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_{m+2})+\cdots+f(v_{m+p+1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ &+ \underbrace{f(v_{m+p+2})+\cdots+f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1})+f(v_n)}_{\geq s} \\ \geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell p/3 + s + k. \end{aligned} \tag{A.74}$$

Finally,

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_{m+1})}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_{m+2})+\cdots+f(v_{m+p-2})}_{\geq (s+\ell)(\lfloor p/3 \rfloor - 1)} \\ &+ \underbrace{f(v_{m+p-1})}_{\geq 0} + \underbrace{f(v_{m+p})+\cdots+f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor)} + \underbrace{f(v_n)}_{\geq s-r} \\ \geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell p/3 + s + k - \ell - r. \end{aligned} \tag{A.75}$$

If $s+k = \max\{s+k, s+\ell-r, s+k-\ell-r\}$, then $\ell + r \geq 0$. First consider the case where $s+k \geq s+\ell$. Let g_9 be an $\boldsymbol{r}$-function such that $g_9(v_1) = \max\{0, s+k-r\}$, $g_9(v_2) = \min\{r, s+k\}$, $g_9(v_{n-1}) = \min\{r, s\}$ and $g_9(v_n) = \max\{0, s-r\}$. Furthermore let $g_9(v_{3c}) = 0$, $g_9(v_{3c+1}) = \max\{0, s+k-r\}$ and $g_9(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_9(v_{3c}) = 0$, $g_9(v_{3c+1}) = \max\{0, s+\ell-r\}$ and $g_9(v_{3c+2}) = \min\{r, s+\ell\}$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_9(v_{3c}) = \min\{r, s\}$, $g_9(v_{3c+1}) = \max\{0, s-r\}$ and $g_9(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_9 is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_9| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k. \tag{A.76}$$

If $s+\ell > s+k$, then let g_{10} be an $\boldsymbol{r}$-function such that $g_{10}(v_1) = \max\{0, s+k-r\}$, $g_{10}(v_2) = \min\{r, s+k\}$, $g_{10}(v_{n-1}) = \min\{r, s\}$ and $g_{10}(v_n) = \max\{0, s-r\}$. Furthermore let $g_{10}(v_{3c}) = 0$, $g_{10}(v_{3c+1}) = \max\{0, s+k-r\}$ and $g_{10}(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{10}(v_{3c}) = \min\{r, s+\ell\}$, $g_{10}(v_{3c+1}) = \max\{0, s+\ell-2r\}$ and $g_{10}(v_{3c+2}) = \min\{r, \max\{0, s+\ell-r\}\}$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{10}(v_{3c}) = \min\{r, s\}$, $g_{10}(v_{3c+1}) = \max\{0, s-r\}$ and $g_{10}(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{10} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_{10}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k. \tag{A.77}$$

If $s+\ell-r = \max\{s+k, s+\ell-r, s+k-\ell-r\}$, then $s+\ell \geq r$. Note that, if $s+\ell \leq 2r$, then $s+k \leq r$. Let g_{11} be an $\boldsymbol{r}$-function such that $g_{11}(v_{3c+1}) = \max\{0, s+k-r\}$, $g_{11}(v_{3c+2}) = \min\{r, s+k\}$ and $g_{11}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$ and let $g_{11}(v_m) = \max\{0, s+\ell-2r\}$ and $g_{11}(v_{m+1}) = \min\{r, s+\ell-r\}$. Also, let $g_{11}(v_{3c}) = r$, $g_{11}(v_{3c+1}) = \max\{0, s+\ell-2r\}$ and $g_{11}(v_{3c+2}) = \min\{r, s+\ell-r\}$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{11}(v_{3c}) = \min\{r, s\}$, $g_{11}(v_{3c+1}) = \max\{0, s-r\}$ and $g_{11}(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$ and let $g_{11}(v_{n-1}) = \min\{r, s\}$ and $g_{11}(v_n) = \max\{0, s-r\}$. Then g_{11} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_{11}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell - r. \tag{A.78}$$

If $s+k-\ell-r = \max\{s+k, s+\ell-r, s+k-\ell-r\}$, then $\ell \leq 0$ and $s-r \geq s+\ell$. Let g_{12} be an $\boldsymbol{r}$-function such that $g_{12}(v_{3c+1}) = \max\{0, s+k-r\}$, $g_{12}(v_{3c+2}) = \min\{r, s+k\}$ and $g_{12}(v_{3c+3}) = 0$

for $c = 0, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{12}(v_{3c+1}) = \max\{0, s+\ell-r\}$, $g_{12}(v_{3c+2}) = \min\{r, s+\ell\}$ and $g_{12}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally let $g_{12}(v_{3c+1}) = s - r$, $g_{12}(v_{3c+2}) = 0$ and $g_{12}(v_{3c+3}) = r$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 1$ and let $g_{12}(v_n) = s - r$. Then g_{12} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_{12}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - \ell - r. \tag{A.79}$$

The result for the Case 2a follows from a combination of (A.73)–(A.79).

Case 2b: $p \equiv 1 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m+1})}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ &+ \underbrace{f(v_{m+p+1}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\ \geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k \end{aligned} \tag{A.80}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_m) + \cdots + f(v_{m+p+1})}_{\geq (s+\ell)(\lfloor p/3 \rfloor + 1)} \\ &+ \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_n)}_{\geq s-r} \\ \geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell - r. \end{aligned} \tag{A.81}$$

Finally,

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_m) + \cdots + f(v_{m+p+1})}_{\geq (s+\ell)(\lfloor p/3 \rfloor + 1)} \\ &+ \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} \\ \geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell. \end{aligned} \tag{A.82}$$

Let $s + k = \max\{s+k, s+\ell-r, \ell\}$ and first consider the case where $s \geq r$. Let g_{13} be an $\boldsymbol{r}$-function such that $g_{13}(v_1) = s + k - r$, $g_{13}(v_2) = r$, $g_{13}(v_{n-1}) = r$ and $g_{13}(v_n) = s - r$. Furthermore let $g_{13}(v_{3c}) = 0$, $g_{13}(v_{3c+1}) = s + k - r$ and $g_{13}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{13}(v_{3c}) = \min\{\max\{0, s+\ell-r\}, r\}$, $g_{13}(v_{3c+1}) = \max\{0, s+\ell-2r\}$ and $g_{13}(v_{3c+2}) = \min\{r, s+\ell\}$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{13}(v_{3c}) = r$, $g_{13}(v_{3c+1}) = s - r$ and $g_{13}(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{13} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_{13}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k. \tag{A.83}$$

If $s < r$ and $\ell \geq 0$, then let g_{14} be an $\boldsymbol{r}$-function such that $g_{14}(v_1) = \max\{0, s+k-r\}$, $g_{14}(v_2) = \min\{r, s+k\}$, $g_{14}(v_{n-1}) = s$ and $g_{14}(v_n) = 0$. Furthermore let $g_{14}(v_{3c}) = 0$, $g_{14}(v_{3c+1}) = \max\{0, s+k-r\}$ and $g_{14}(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{14}(v_{3c}) = s$, $g_{14}(v_{3c+1}) = \max\{0, \ell - r\}$ and $g_{14}(v_{3c+2}) = \min\{r, \ell\}$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$.

Finally let $g_{14}(v_{3c}) = s$, $g_{14}(v_{3c+1}) = 0$ and $g_{14}(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{14} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{14}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k. \tag{A.84}$$

If $s < r$ and $\ell < 0$, then let g_{15} be an $\boldsymbol{r}$-function such that $g_{15}(v_1) = \max\{0, s+k-r\}$, $g_{15}(v_2) = \min\{r, s+k\}$, $g_{15}(v_{n-1}) = s$ and $g_{15}(v_n) = 0$. Furthermore let $g_{15}(v_{3c}) = 0$, $g_{15}(v_{3c+1}) = \max\{0, s+k-r\}$ and $g_{15}(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{15}(v_{3c}) = 0$, $g_{15}(v_{3c+1}) = s + \ell$ and $g_{15}(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{15}(v_{3c}) = s$, $g_{15}(v_{3c+1}) = 0$ and $g_{15}(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{15} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{15}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k. \tag{A.85}$$

If $s + \ell - r = \max\{s + k, s + \ell - r, \ell\}$, then $s \geq r$ and $s + \ell \geq 2r$. Let g_{16} be an $\boldsymbol{r}$-function such that $g_{16}(v_n) = s - r$. Furthermore let $g_{16}(v_{3c+1}) = s + k - r$, $g_{16}(v_{3c+2}) = r$ and $g_{16}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $g_{16}(v_{3c+1}) = r$, $g_{16}(v_{3c+2}) = s + \ell - 2r$ and $g_{16}(v_{3c+3}) = r$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{16}(v_{3c+1}) = s - r$, $g_{16}(v_{3c+2}) = 0$ and $g_{16}(v_{3c+3}) = r$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{16} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{16}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell - r. \tag{A.86}$$

If $\ell = \max\{s + k, s + \ell - r, \ell\}$, then $s + k \geq s + \ell - r$. Let g_{17} be an $\boldsymbol{r}$-function such that $g_{17}(v_n) = 0$. Furthermore let $g_{17}(v_{3c+1}) = \max\{0, s + k - r\}$, $g_{17}(v_{3c+2}) = \min\{r, s + k\}$ and $g_{17}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $g_{17}(v_{3c}) = \min\{r, \ell\}$, $g_{17}(v_{3c+1}) = \max\{0, \ell - r\}$ and $g_{17}(v_{3c+2}) = s$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{17}(v_{3c}) = 0$, $g_{17}(v_{3c+1}) = 0$ and $g_{17}(v_{3c+2}) = s$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{17} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{17}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell. \tag{A.87}$$

The result for the Case 2b follows from a combination of (A.80)–(A.87).

Case 2c: $p \equiv 2 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = &\underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m+1})}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ &+ \underbrace{f(v_{m+p}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\ \geq &s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k\end{aligned} \tag{A.88}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = &\underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m+1})}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ &+ \underbrace{f(v_{m+p}) + f(v_{m+p+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+p+2}) + \cdots + f(v_n)}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} \\ \geq &s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k + \ell - r.\end{aligned} \tag{A.89}$$

It also follows that

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_m) + \cdots + f(v_{m+p})}_{\geq (s+\ell)(\lfloor p/3 \rfloor + 1)} \\ &+ \underbrace{f(v_{m+p+1}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_n)}_{\geq 0} \\ \geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell \lfloor p/3 \rfloor + \ell \end{aligned} \tag{A.90}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_m) + f(v_{m+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ &+ \underbrace{f(v_{m+p}) + f(v_{m+p+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+p+2}) + \cdots + f(v_n)}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} \\ \geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell \lfloor p/3 \rfloor + s + 2\ell - 2r. \end{aligned} \tag{A.91}$$

If $s+k+\ell-r = \max\{s+k+\ell-r, s+k, s+2\ell-2r, \ell\}$, then $\ell \geq r$, $s+k \geq r$ and $\ell - k \leq r$. Let g_{18} be an $\boldsymbol{r}$-function such that $g_{18}(v_1) = s+k-r$, $g_{18}(v_2) = r$, $g_{18}(v_{m+p}) = \ell - k$ and $g_{18}(v_{m+p+1}) = s+k-r$. Furthermore let $g_{18}(v_{3c}) = 0$, $g_{18}(v_{3c+1}) = s+k-r$ and $g_{18}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{18}(v_{3c}) = \ell - k$, $g_{18}(v_{3c+1}) = s+k-r$ and $g_{18}(v_{3c+2}) = r$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{18}(v_{3c+2}) = 0$, $g_{18}(v_{3c+3}) = \min\{r, s\}$ and $g_{18}(v_{3c+4}) = \max\{0, s-r\}$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{18} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_{18}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k + \ell - r. \tag{A.92}$$

If $s+k = \max\{s+k+\ell-r, s+k, s+2\ell-2r, \ell\}$, then $\ell \leq r$. Let g_{19} be an $\boldsymbol{r}$-function such that $g_{19}(v_1) = \max\{0, s+k-r\}$, $g_{19}(v_2) = \min\{r, s+k\}$, $g_{19}(v_{n-1}) = \min\{r, s\}$ and $g_{19}(v_n) = \max\{0, s-r\}$. Furthermore let $g_{19}(v_{3c}) = 0$, $g_{19}(v_{3c+1}) = \max\{0, s+k-r\}$ and $g_{19}(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{19}(v_{3c}) = \min\{r, s\}$, $g_{19}(v_{3c+1}) = \max\{0, s-r\}$ and $g_{19}(v_{3c+2}) = \ell$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{19}(v_{3c+2}) = \min\{r, s\}$, $g_{19}(v_{3c+3}) = \max\{0, s-r\}$ and $g_{19}(v_{3c+4}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{19} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_{19}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k. \tag{A.93}$$

If $s+2\ell-2r = \max\{s+k+\ell-r, s+k, s+2\ell-2r, \ell\}$, then $s+\ell \geq 2r$ and $s+\ell-r \geq s+k$. Let g_{20} be an $\boldsymbol{r}$-function such that $g_{20}(v_{m+p+1}) = s+\ell-2r$. Furthermore let $g_{20}(v_{3c+1}) = \max\{0, s+k-r\}$, $g_{20}(v_{3c+2}) = \min\{r, s+k\}$ and $g_{20}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $g_{20}(v_{3c+1}) = s+\ell-2r$, $g_{20}(v_{3c+2}) = r$ and $g_{20}(v_{3c+3}) = r$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{20}(v_{3c+2}) = 0$, $g_{20}(v_{3c+3}) = \min\{r, s\}$ and $g_{20}(v_{3c+4}) = \max\{0, s-r\}$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{20} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_{20}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + 2\ell - 2r. \tag{A.94}$$

If $\ell = \max\{s+k+\ell-r, s+k, s+2\ell-2r, \ell\}$, then $s+\ell \leq 2r$ and $s+k \leq r$. First consider the case where $\ell > r$. Let g_{21} be an $\boldsymbol{r}$-function such that $g_{21}(v_n) = 0$. Furthermore let $g_{21}(v_{3c+1}) = 0$, $g_{21}(v_{3c+2}) = s+k$ and $g_{21}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $g_{21}(v_{3c+1}) = 0$,

$g_{21}(v_{3c+2}) = r$ and $g_{21}(v_{3c+3}) = s + \ell - r$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{21}(v_{3c+1}) = 0$, $g_{21}(v_{3c+2}) = 0$ and $g_{21}(v_{3c+3}) = s$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{21} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{21}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell. \tag{A.95}$$

Now let $\ell \leq r$ and let g_{22} be an $\boldsymbol{r}$-function such that $g_{22}(v_n) = 0$. Furthermore let $g_{22}(v_{3c+1}) = 0$, $g_{22}(v_{3c+2}) = s + k$ and $g_{22}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $g_{22}(v_{3c+1}) = 0$, $g_{22}(v_{3c+2}) = \ell$ and $g_{22}(v_{3c+3}) = s$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{22}(v_{3c+1}) = 0$, $g_{22}(v_{3c+2}) = 0$ and $g_{22}(v_{3c+3}) = s$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{22} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{22}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell. \tag{A.96}$$

The result for the Case 2c follows from a combination of (A.88)–(A.96).

Finally consider the case where $m \equiv 2 \pmod 3$ and the three subcases where $p \equiv 0, 1, 2 \pmod 3$.

Case 3: $m \equiv 2 \pmod 3$.

Case 3a: $p \equiv 0 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = & \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_m)}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_{m+1}) + \cdots + f(v_{m+p})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p+1}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - p/3 - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell p/3 + s + k
\end{aligned} \tag{A.97}$$

and

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = & \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_m) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p}) + f(v_{m+p+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_n)}_{\geq 0} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell p/3 + s + k + \ell - 2r.
\end{aligned} \tag{A.98}$$

Finally,

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = & \underbrace{f(v_1) + \cdots + f(v_{m-2})}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_{m-1})}_{\geq 0} + \underbrace{f(v_m) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p}) + f(v_{m+p+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_n)}_{\geq 0} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell p/3 + \ell - r.
\end{aligned} \tag{A.99}$$

If $s+k = \max\{s+k, s+k+\ell-2r, \ell-r\}$, then $\ell \leq 2r$ and $s+k \geq \ell-r$. First consider the case where $s + k \leq r$ and let g_{23} be an $\boldsymbol{r}$-function such that $g_{23}(v_1) = 0$, $g_{23}(v_2) = s + k$, $g_{23}(v_{n-1}) = s$ and $g_{23}(v_n) = 0$. Furthermore let $g_{23}(v_{3c}) = 0$, $g_{23}(v_{3c+1}) = 0$ and $g_{23}(v_{3c+2}) = s + k$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{23}(v_{3c}) = s$, $g_{23}(v_{3c+1}) = \min\{r, \ell\}$ and $g_{23}(v_{3c+2}) = \max\{0, \ell - r\}$

for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{23}(v_{3c}) = s$, $g_{23}(v_{3c+1}) = 0$ and $g_{23}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{23} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{23}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k. \tag{A.100}$$

If $s + k > r$ and $s \geq r$, then let g_{24} be an $\boldsymbol{r}$-function such that $g_{24}(v_1) = s + k - r$, $g_{24}(v_2) = r$, $g_{24}(v_{n-1}) = r$ and $g_{24}(v_n) = s - r$. Furthermore let $g_{24}(v_{3c}) = 0$, $g_{24}(v_{3c+1}) = s + k - r$ and $g_{24}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{24}(v_{3c}) = \min\{r, s + \ell\}$, $g_{24}(v_{3c+1}) = \max\{0, s + \ell - 2r\}$ and $g_{24}(v_{3c+2}) = \max\{0, \min\{r, s + \ell - r\}\}$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{24}(v_{3c}) = r$, $g_{24}(v_{3c+1}) = s - r$ and $g_{24}(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{24} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{24}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k. \tag{A.101}$$

If $s + k > r$ and $s < r$, then let g_{25} be an $\boldsymbol{r}$-function such that $g_{25}(v_1) = s + k - r$, $g_{25}(v_2) = r$, $g_{25}(v_{n-1}) = s$ and $g_{25}(v_n) = 0$. Furthermore let $g_{25}(v_{3c}) = 0$, $g_{25}(v_{3c+1}) = s + k - r$ and $g_{25}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{25}(v_{3c}) = s$, $g_{25}(v_{3c+1}) = \max\{0, \ell - r\}$ and $g_{25}(v_{3c+2}) = \min\{r, \ell\}$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{25}(v_{3c}) = s$, $g_{25}(v_{3c+1}) = 0$ and $g_{25}(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{25} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{25}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k. \tag{A.102}$$

If $s + k + \ell - 2r = \max\{s + k, s + k + \ell - 2r, \ell - r\}$, then $\ell \geq 2r$, $s + k \geq r$ and $s \leq r$. Let g_{26} be an $\boldsymbol{r}$-function such that $g_{26}(v_1) = s + k - r$, $g_{26}(v_2) = r$, $g_{26}(v_{m+p+1}) = s + \ell - 2r$ and $g_{26}(v_{m+p+2}) = 0$. Furthermore let $g_{26}(v_{3c}) = 0$, $g_{26}(v_{3c+1}) = s + k - r$ and $g_{26}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{26}(v_{3c}) = s + \ell - 2r$, $g_{26}(v_{3c+1}) = r$ and $g_{26}(v_{3c+2}) = r$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{26}(v_{3c+2}) = 0$, $g_{26}(v_{3c+3}) = s$ and $g_{26}(v_{3c+4}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{26} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{26}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k + \ell - 2r. \tag{A.103}$$

If $\ell - r = \max\{s + k, s + k + \ell - 2r, \ell - r\}$, then $s + k \leq r$ and $l \geq r$. Let g_{27} be an $\boldsymbol{r}$-function such that $g_{27}(v_1) = 0$. Furthermore let $g_{27}(v_{3c+1}) = s + k$, $g_{27}(v_{3c+2}) = 0$ and $g_{27}(v_{3c+3}) = 0$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{27}(v_{3c+1}) = \ell - r$, $g_{27}(v_{3c+2}) = s$ and $g_{27}(v_{3c+3}) = r$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{27}(v_{m+p}) = \ell - r$, $g_{27}(v_{m+p+1}) = s$ and $g_{27}(v_{m+p+2}) = 0$, and let $g_{27}(v_{3c+2}) = 0$, $g_{27}(v_{3c+3}) = s$ and $g_{27}(v_{3c+4}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{27} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{27}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell - r. \tag{A.104}$$

The result for the Case 3a follows from a combination of (A.97)–(A.104).

Case 3b: $p \equiv 1 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = &\underbrace{f(v_1) + f(v_2)}_{\geq s+k-r} + \underbrace{f(v_3) + \cdots + f(v_m)}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_{m+1}) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\
&+ \underbrace{f(v_{m+p}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k
\end{aligned} \tag{A.105}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)}_{\geq s+k} + \underbrace{f(v_2) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3\rfloor)} + \underbrace{f(v_m) + \cdots + f(v_{m+p+1})}_{\geq (s+\ell)(\lfloor p/3\rfloor+1)} \\ &+ \underbrace{f(v_{m+p+2}) + \cdots + f(v_n)}_{\geq s(\lfloor n/3\rfloor - \lfloor m/3\rfloor - \lfloor p/3\rfloor - 1)} \\ \geq& s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k + \ell - r. \end{aligned} \tag{A.106}$$

Finally,

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)}_{\geq 0} + \underbrace{f(v_2) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3\rfloor)} + \underbrace{f(v_m) + \cdots + f(v_{m+p+1})}_{\geq (s+\ell)(\lfloor p/3\rfloor+1)} \\ &+ \underbrace{f(v_{m+p+2}) + \cdots + f(v_n)}_{\geq s(\lfloor n/3\rfloor - \lfloor m/3\rfloor - \lfloor p/3\rfloor - 1)} \\ \geq& s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + \ell. \end{aligned} \tag{A.107}$$

If $s+k = \max\{s+k, s+k+\ell-r, \ell\}$, then $s+k \geq \ell$ and $\ell \leq r$. First consider the case where $\ell \geq 0$ and let g_{28} be an $\boldsymbol{r}$-function such that $g_{28}(v_1) = \max\{0, s+k-r\}$, $g_{28}(v_2) = \min\{r, s+k\}$, $g_{28}(v_{n-1}) = \min\{r, s\}$ and $g_{28}(v_n) = \max\{0, s-r\}$. Furthermore let $g_{28}(v_{3c}) = 0$, $g_{28}(v_{3c+1}) = \max\{0, s+k-r\}$ and $g_{28}(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3\rfloor$. Also, let $g_{28}(v_{3c}) = \min\{r, s\}$, $g_{28}(v_{3c+1}) = \max\{0, s-r\}$ and $g_{28}(v_{3c+2}) = \ell$ for $c = \lfloor m/3\rfloor + 1, \ldots, \lfloor m/3\rfloor + \lfloor p/3\rfloor$. Finally let $g_{28}(v_{3c}) = \min\{r, s\}$, $g_{28}(v_{3c+1}) = \max\{0, s-r\}$ and $g_{28}(v_{3c+2}) = 0$ for $c = \lfloor m/3\rfloor + \lfloor p/3\rfloor + 1, \ldots, \lfloor n/3\rfloor - 1$. Then g_{28} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_{28}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k. \tag{A.108}$$

Now, if $\ell \leq 0$, then let g_{29} be an $\boldsymbol{r}$-function such that $g_{29}(v_1) = \max\{0, s+k-r\}$, $g_{29}(v_2) = \min\{r, s+k\}$, $g_{29}(v_{n-1}) = \min\{r, s\}$ and $g_{29}(v_n) = \max\{0, s-r\}$. Furthermore let $g_{29}(v_{3c}) = 0$, $g_{29}(v_{3c+1}) = \max\{0, s+k-r\}$ and $g_{29}(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3\rfloor$. Also, let $g_{29}(v_{3c}) = \min\{r, s+\ell\}$, $g_{29}(v_{3c+1}) = \max\{0, s+\ell-r\}$ and $g_{29}(v_{3c+2}) = 0$ for $c = \lfloor m/3\rfloor + 1, \ldots, \lfloor m/3\rfloor + \lfloor p/3\rfloor$. Finally let $g_{29}(v_{3c}) = \min\{r, s\}$, $g_{29}(v_{3c+1}) = \max\{0, s-r\}$ and $g_{29}(v_{3c+2}) = 0$ for $c = \lfloor m/3\rfloor + \lfloor p/3\rfloor + 1, \ldots, \lfloor n/3\rfloor - 1$. Then g_{29} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_{29}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k. \tag{A.109}$$

If $s+k+\ell-r = \max\{s+k, s+k+\ell-r, \ell\}$, then $s+k \geq r$, $\ell \geq r$ and $s+\ell \geq r$. Let g_{30} be an $\boldsymbol{r}$-function such that $g_{30}(v_1) = s+k-r$, $g_{30}(v_2) = r$, $g_{30}(v_{m+p}) = \max\{0, s+\ell-2r\}$ and $g_{30}(v_{m+p+1}) = \min\{r, s+\ell-r\}$. Furthermore let $g_{30}(v_{3c}) = 0$, $g_{30}(v_{3c+1}) = s+k-r$ and $g_{30}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3\rfloor$. Also, let $g_{30}(v_{3c}) = \max\{0, s+\ell-2r\}$, $g_{30}(v_{3c+1}) = \min\{r, s+\ell-r\}$ and $g_{30}(v_{3c+2}) = r$ for $c = \lfloor m/3\rfloor+1, \ldots, \lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally let $g_{30}(v_{3c+2}) = 0$, $g_{30}(v_{3c+3}) = \min\{r, s\}$ and $g_{30}(v_{3c+4}) = \max\{0, s-r\}$ for $c = \lfloor m/3\rfloor+\lfloor p/3\rfloor+1, \ldots, \lfloor n/3\rfloor-1$. Then g_{30} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_{30}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k + \ell - r. \tag{A.110}$$

If $\ell = \max\{s+k, s+k+\ell-r, \ell\}$, then $s+k \leq r$, $s+k \leq \ell$ and $s \leq r$. First consider the case where $\ell \leq r$ and let g_{31} be an $\boldsymbol{r}$-function such that $g_{31}(v_1) = 0$. Furthermore let

$g_{31}(v_{3c+2}) = s+k$, $g_{31}(v_{3c+3}) = 0$ and $g_{31}(v_{3c+4}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $g_{31}(v_{3c+2}) = \ell$, $g_{31}(v_{3c+3}) = 0$ and $g_{31}(v_{3c+4}) = s$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{31}(v_{3c+2}) = 0$, $g_{31}(v_{3c+3}) = s$ and $g_{31}(v_{3c+4}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{31} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_{31}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell. \tag{A.111}$$

Now if $\ell > r$, then let g_{32} be an $\boldsymbol{r}$-function such that $g_{32}(v_1) = 0$. Furthermore let $g_{32}(v_{3c+2}) = s+k$, $g_{32}(v_{3c+3}) = 0$ and $g_{32}(v_{3c+4}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $g_{32}(v_{3c+2}) = r$, $g_{32}(v_{3c+3}) = \min\{r, s+\ell-r\}$ and $g_{32}(v_{3c+4}) = \max\{0, s+\ell-2r\}$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{32}(v_{3c+2}) = 0$, $g_{32}(v_{3c+3}) = s$ and $g_{32}(v_{3c+4}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{32} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_{32}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell. \tag{A.112}$$

The result for the Case 3b follows from a combination of (A.105)–(A.112).

Case 3c: $p \equiv 2 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = &\underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_m)}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_{m+1}) + \cdots + f(v_{m+p+1})}_{\geq (s+\ell)(\lfloor p/3 \rfloor + 1)} \\
&+ \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 2)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq &s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k + \ell
\end{aligned} \tag{A.113}$$

and

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = &\underbrace{f(v_1)}_{\geq 0} + \underbrace{f(v_2) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor)} + \underbrace{f(v_m) + f(v_{m+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\
&+ \underbrace{f(v_{m+p}) + f(v_{m+p+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 2)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq &s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + 2\ell - 2r.
\end{aligned} \tag{A.114}$$

If $s+k+\ell = \max\{s+k+\ell, s+2\ell-2r\}$, then $\ell - k \leq 2r$. First consider the case where $\ell \leq 0$ and let g_{33} be an $\boldsymbol{r}$-function such that $g_{33}(v_1) = \max\{0, s+k-r\}$, $g_{33}(v_2) = \min\{r, s+k\}$, $g_{33}(v_{n-1}) = \min\{r, s\}$ and $g_{33}(v_n) = \max\{0, s-r\}$. Furthermore let $g_{33}(v_{3c}) = 0$, $g_{33}(v_{3c+1}) = \max\{0, s+k-r\}$ and $g_{33}(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{33}(v_{3c}) = \min\{r+\ell, s+\ell\}$, $g_{33}(v_{3c+1}) = \max\{0, s-r\}$ and $g_{33}(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1$. Finally let $g_{33}(v_{3c}) = \min\{r, s\}$, $g_{33}(v_{3c+1}) = \max\{0, s-r\}$ and $g_{33}(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 2, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{33} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|g_{33}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k + \ell. \tag{A.115}$$

Now let $\ell > 0$ and assume $s+k \leq r$. Let g_{34} be an $\boldsymbol{r}$-function such that $g_{34}(v_1) = 0$, $g_{34}(v_2) = s+k$, $g_{34}(v_{n-1}) = s$ and $g_{34}(v_n) = 0$. Furthermore let $g_{34}(v_{3c}) = 0$, $g_{34}(v_{3c+1}) = 0$ and $g_{34}(v_{3c+2}) = s+k$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{34}(v_{3c}) = \min\{r, \ell-k\}$, $g_{34}(v_{3c+1}) = \max\{0, \ell-k-r\}$ and $g_{34}(v_{3c+2}) = s+k$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1$. Finally let

$g_{34}(v_{3c}) = s$, $g_{34}(v_{3c+1}) = 0$ and $g_{34}(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 2, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{34} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{34}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k + \ell. \tag{A.116}$$

If $s+k > r$, then let g_{35} be an $\boldsymbol{r}$-function such that $g_{35}(v_1) = s+k-r$, $g_{35}(v_2) = r$, $g_{35}(v_{n-1}) = \min\{r, s\}$ and $g_{35}(v_n) = \max\{0, s-r\}$. Furthermore let $g_{35}(v_{3c}) = 0$, $g_{35}(v_{3c+1}) = s+k-r$ and $g_{35}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $g_{35}(v_{3c}) = \min\{\max\{0, s+\ell-r\}, r\}$, $g_{35}(v_{3c+1}) = \max\{0, s+\ell-2r\}$ and $g_{35}(v_{3c+2}) = \min\{r, s+\ell\}$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1$. Finally let $g_{35}(v_{3c}) = \min\{r, s\}$, $g_{35}(v_{3c+1}) = \max\{0, s-r\}$ and $g_{35}(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 2, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{35} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{35}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k + \ell. \tag{A.117}$$

If $s+2\ell-2r = \max\{s+k+\ell, s+2\ell-2r\}$, then $r \geq s+\ell-2r \geq s+k$. Let g_{35} be an $\boldsymbol{r}$-function such that $g_{35}(v_1) = 0$, $g_{35}(v_{m+p+1}) = s+\ell-2r$, $g_{35}(v_{n-1}) = s$ and $g_{35}(v_n) = 0$. Furthermore let $g_{35}(v_{3c+1}) = s+k$, $g_{35}(v_{3c+2}) = 0$ and $g_{35}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $g_{35}(v_{3c+1}) = s+\ell-2r$, $g_{35}(v_{3c+2}) = r$ and $g_{35}(v_{3c+3}) = r$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $g_{35}(v_{3c}) = s$, $g_{35}(v_{3c+1}) = 0$ and $g_{35}(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 2, \ldots, \lfloor n/3 \rfloor - 1$. Then g_{35} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|g_{35}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + 2\ell - 2r. \tag{A.118}$$

The result for the Case 3c follows from a combination of (A.113)–(A.118). ■

The next result calculate the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a path of order n, where $n \equiv 2 \pmod 3$ and $\boldsymbol{s}$ is a step function with two steps.

Proposition 5.10. Let $\left(P_n, \boldsymbol{r}, \boldsymbol{s}^{k,\ell}_{m,p}\right)$ be a graph triple where $\boldsymbol{r} = [r, \ldots, r]$ and

$$\boldsymbol{s}^{k,\ell}_{m,p} = [\overbrace{s+k, \ldots, s+k}^{m \text{ times}}, \overbrace{s+\ell, \ldots, s+\ell}^{p \text{ times}}, s, \ldots, s]$$

satisfy $2r \geq s+k$ and $3r \geq s+\ell$ for $m, p < n$, $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$. If $n \equiv 2 \pmod 3$, then

$$\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = s\left\lfloor \frac{n}{3} \right\rfloor + k\left\lfloor \frac{m}{3} \right\rfloor + \ell\left\lfloor \frac{p}{3} \right\rfloor + g(s,k,l,r)$$

where

$$g(s,k,l,r) = \begin{cases} \max\{s, s+k-r, s+\ell-r, s+k-\ell-r, 2s+\ell-2r\} & \text{if } m, p \equiv 0 \pmod 3 \\ \max\{s, s+k-r, s+\ell, s+k-\ell-r\} & \text{if } m \equiv 0 \pmod 3 \text{ and } p \equiv 1 \pmod 3 \\ \max\{s+k-r, s+\ell, 2s+k-2r, 2(s+\ell-r)\} & \text{if } m \equiv 0 \pmod 3 \text{ and } p \equiv 2 \pmod 3 \\ \max\{s+k, s+\ell-r, s+k-\ell\} & \text{if } m \equiv 1 \pmod 3 \text{ and } p \equiv 0 \pmod 3 \\ \max\{s+k, s+\ell, s+k-\ell-r\} & \text{if } m, p \equiv 1 \pmod 3 \\ \max\{s+k, s+\ell, s+k+\ell-r, s+2\ell-2r\} & \text{if } m \equiv 1 \pmod 3 \text{ and } p \equiv 2 \pmod 3 \\ \max\{s+k, s+\ell-r, s+k-\ell-r, 2s+k+\ell-2r\} & \text{if } m \equiv 2 \pmod 3 \text{ and } p \equiv 0 \pmod 3 \\ \max\{s+k, \ell, s+k+\ell-r, 2s+k+\ell-2r\} & \text{if } m \equiv 2 \pmod 3 \text{ and } p \equiv 1 \pmod 3 \\ \max\{s+k, s+k+\ell\} & \text{if } m, p \equiv 2 \pmod 3. \end{cases}$$

Proof. First consider the case where $m \equiv 0 \pmod 3$ and the three subcases where $p \equiv 0, 1, 2 \pmod 3$.

Case 1: $m \equiv 0 \pmod 3$.

Case 1a: $p \equiv 0 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = &\underbrace{f(v_1) + \cdots + f(v_m)}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+1}) + \cdots + f(v_{m+p})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ &+ \underbrace{f(v_{m+p+1}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\ \geq &s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s \end{aligned} \tag{A.119}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = &\underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{m+1})}_{\geq (s+k)m/3} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p+1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ &+ \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor)} + \underbrace{f(v_n)}_{\geq 0} \\ \geq &s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r. \end{aligned} \tag{A.120}$$

It also follows that

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = &\underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor - 1)} + \underbrace{f(v_m) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ &+ \underbrace{f(v_{m+p}) + f(v_{m+p+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor)} + \underbrace{f(v_n)}_{\geq 0} \\ \geq &s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell - r \end{aligned} \tag{A.121}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = &\underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{m+1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p-2})}_{\geq (s+\ell)(\lfloor p/3 \rfloor - 1)} \\ &+ \underbrace{f(v_{m+p-1})}_{\geq 0} + \underbrace{f(v_{m+p}) + \cdots + f(v_n)}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor + 1)} \\ \geq &s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - \ell - r. \end{aligned} \tag{A.122}$$

Finally,

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = &\underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor - 1)} + \underbrace{f(v_m) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ &+ \underbrace{f(v_{m+p}) + f(v_{m+p+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor)} + \underbrace{f(v_n)}_{\geq s-r} \\ \geq &s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + 2s + \ell - 2r. \end{aligned} \tag{A.123}$$

If $s = \max\{s, s+k-r, s+\ell-r, s+k-\ell-r, 2s+\ell-2r\}$, then $k, \ell \leq r$, $k-\ell \leq r$ and $s+\ell \leq 2r$. Fisrt consdier the case where $\ell \geq 0$ and let h_1 be an $\boldsymbol{r}$-function such that $h_1(v_{3c+1}) = \min\{r, s\}$, $h_1(v_{3c+2}) = \max\{k, s-r+k\}$ and $h_1(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Furthermore let $h_1(v_{3c+1}) = \min\{r, s\}$, $h_1(v_{3c+2}) = \max\{\ell, s-r+\ell\}$ and $h_1(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Also, let $h_1(v_{3c+1}) = \min\{r, s\}$, $h_1(v_{3c+2}) = \max\{0, s-r\}$ and $h_1(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 1$. Finally let $h_1(v_{n-1}) = \min\{r, s\}$ and $h_1(v_n) = \max\{0, s-r\}$ Then h_1 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_1| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s. \tag{A.124}$$

Now assume that $\ell < 0$ and let h_2 be an $\boldsymbol{r}$-function such that $h_2(v_{3c+1}) = \min\{r, s+\ell\}$, $h_2(v_{3c+2}) = \max\{k-\ell, s+k-r\}$ and $h_2(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Furthermore let $h_2(v_{3c+1}) = \min\{r, s+\ell\}$, $h_2(v_{3c+2}) = \max\{0, s+\ell-r\}$ and $h_2(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Also, let $h_2(v_{3c+1}) = \min\{r, s\}$, $h_2(v_{3c+2}) = \max\{0, s-r\}$ and $h_2(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 1$. Finally let $h_2(v_{n-1}) = \min\{r, s\}$ and $h_2(v_n) = \max\{0, s-r\}$. Then h_2 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_2| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s. \tag{A.125}$$

If $s+k-r = \max\{s, s+k-r, s+\ell-r, s+k-\ell-r, 2s+\ell-2r\}$, then $s+k \geq s+\ell$, $s+k \geq r$, $s \leq r$, $k \geq r$ and $\ell \geq 0$. Let h_3 be an $\boldsymbol{r}$-function such that $h_3(v_1) = s+k-r$ and $h_3(v_n) = 0$. Furthermore let $h_3(v_{3c+2}) = r$, $h_3(v_{3c+3}) = k-r$ and $h_3(v_{3c+4}) = s$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $h_3(v_{3c+2}) = \min\{r, \ell\}$, $h_3(v_{3c+3}) = \max\{0, \ell-r\}$ and $h_3(v_{3c+4}) = s$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally let $h_3(v_{3c+2}) = 0$, $h_3(v_{3c+3}) = 0$ and $h_3(v_{3c+4}) = s$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 1$. Then h_3 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_3| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r. \tag{A.126}$$

If $s+\ell-r = \max\{s, s+k-r, s+\ell-r, s+k-\ell-r, 2s+\ell-2r\}$, then $s+k \leq s+\ell$, $s+\ell \geq r$, $s \leq r$ and $\ell \geq r$. Let h_4 be an $\boldsymbol{r}$-function such that $h_4(v_1) = \max\{0, s+k-r\}$, $h_4(v_2) = \min\{r, s+k\}$, $h_4(v_m) = \max\{0, s+\ell-2r\}$ and $h_4(v_{m+1}) = \min\{r, s+\ell-r\}$. Furthermore let $h_4(v_{3c}) = 0$, $h_4(v_{3c}) = \max\{0, s+k-r\}$ and $h_4(v_{3c}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $h_4(v_{3c+2}) = r$, $h_4(v_{3c+3}) = \max\{0, s+\ell-2r\}$ and $h_4(v_{3c+4}) = \min\{r, s+\ell-r\}$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally let $h_4(v_{3c+2}) = 0$, $h_4(v_{3c+3}) = 0$ and $h_4(v_{3c+4}) = s$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 1$ and let $h_4(v_n) = 0$. Then h_4 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_4| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell - r. \tag{A.127}$$

If $s+k-\ell-r = \max\{s, s+k-r, s+\ell-r, s+k-\ell-r, 2s+\ell-2r\}$, then $\ell \leq 0$ and since $s+k-r \geq s+\ell$ it follows that $s+k \geq r$ and $s+\ell \leq r$. Let h_5 be an $\boldsymbol{r}$-function such that $h_5(v_{m+1}) = s+k-r$ and $h_5(v_{m+2}) = 0$,. Furthermore let $h_5(v_{3c+1}) = s+k-r$, $h_5(v_{3c+2}) = r$ and $h_5(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $h_5(v_{3c}) = 0$, $h_5(v_{3c+1}) = s+\ell$ and $h_5(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally let $h_5(v_{3c}) = \min\{r, s\}$, $h_5(v_{3c+1}) = \max\{0, s-r\}$ and $h_5(v_{3c+2}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor$. Then h_5 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_5| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - \ell - r. \tag{A.128}$$

If $2s+\ell-2r = \max\{s, s+k-r, s+\ell-r, s+k-\ell-r, 2s+\ell-2r\}$, then $s+\ell \geq 2r$ and $s \geq r$. Let h_6 be an $\boldsymbol{r}$-function such that $h_6(v_1) = s+k-r$, $h_6(v_2) = r$, $h_6(v_m) = s+\ell-2r$

and $h_6(v_{m+1}) = r$. Furthermore let $h_6(v_{3c}) = 0$, $h_6(v_{3c}) = s + k - r$ and $h_6(v_{3c}) = r$ for $c = 1, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $h_6(v_{3c+2}) = r$, $h_6(v_{3c+3}) = s + \ell - 2r$ and $h_6(v_{3c+4}) = r$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$-1. Finally let $h_6(v_{3c+2}) = s - r$, $h_6(v_{3c+3}) = 0$ and $h_6(v_{3c+4}) = r$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 1$ and let $h_6(v_n) = s - r$. Then h_6 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_6| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + 2s + \ell - 2r. \tag{A.129}$$

The result for the Case 1a follows from a combination of (A.119)–(A.129).

Case 1b: $p \equiv 1 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = & \underbrace{f(v_1) + \cdots + f(v_m)}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+1}) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s
\end{aligned} \tag{A.130}$$

and

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = & \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{m+1})}_{\geq (s+k)m/3} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p+1}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor)} + \underbrace{f(v_n)}_{\geq 0} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r.
\end{aligned} \tag{A.131}$$

Also,

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = & \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m-1})}_{\geq (s+k)(\lfloor m/3 \rfloor - 1)} + \underbrace{f(v_m) + \cdots + f(v_{m+p+1})}_{\geq (s+\ell)(\lfloor p/3 \rfloor + 1)} \\
& + \underbrace{f(v_{m+p+2}) + \cdots + f(v_n)}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor)} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell
\end{aligned} \tag{A.132}$$

and finally,

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = & \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{m+1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+2})}_{\geq 0} + \underbrace{f(v_{m+3}) + \cdots + f(v_{m+p-2})}_{\geq (s+\ell)(\lfloor p/3 \rfloor - 1)} \\
& + \underbrace{f(v_{m+p-1})}_{\geq 0} + \underbrace{f(v_{m+p}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - \ell - r.
\end{aligned} \tag{A.133}$$

If $s = \max\{s, s+k-r, s+\ell, s+k-\ell-r\}$, then $k \leq r$, $k - \ell \leq r$ and $\ell \leq 0$. The function h_2 is also an $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating function for the case where $p \equiv 1 \pmod 3$.

If $s + k - r = \max\{s, s+k-r, s+\ell, s+k-\ell-r\}$, then $s + k \geq r$, $s + \ell \leq r$, $\ell \geq 0$ and $s \leq r$. Let h_7 be an $\boldsymbol{r}$-function such that $h_7(v_1) = s + k - r$ and $h_7(v_n) = 0$. Furthermore let $h_7(v_{3c+2}) = r$, $h_7(v_{3c+3}) = 0$ and $h_7(v_{3c+4}) = s + k - r$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $h_7(v_{3c+2}) = 0$, $h_7(v_{3c+3}) = 0$ and $h_7(v_{3c+4}) = s+\ell$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Finally let $h_7(v_{3c+2}) = 0$, $h_7(v_{3c+3}) = 0$ and $h_7(v_{3c+4}) = s$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 1$. Then h_7 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_7| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r. \tag{A.134}$$

If $s+\ell = \max\{s, s+k-r, s+\ell, s+k-\ell-r\}$, then $s+\ell \geq s+k-r$ and $\ell \geq 0$. Let h_8 be an $\boldsymbol{r}$-function such that $h_8(v_1) = \max\{0, s+k-r\}$ and $h_8(v_2) = \min\{r, s+k\}$. Furthermore $h_8(v_{3c}) = 0$, $h_8(v_{3c+1}) = \max\{0, s+k-r\}$ and $h_8(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $h_8(v_{3c}) = \max\{0, s+\ell-2r\}$, $h_8(v_{3c+1}) = \min\{r, s+\ell\}$ and $h_8(v_{3c+2}) = \min\{\max\{0, s+\ell-r\}, r\}$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $h_8(v_{3c}) = 0$, $h_8(v_{3c+1}) = \min\{r, s\}$ and $h_8(v_{3c+2}) = \max\{0, s-r\}$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then h_8 is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_8| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell. \tag{A.135}$$

If $s+k-\ell-r = \max\{s, s+k-r, s+\ell, s+k-\ell-r\}$, then $s+k \geq r$, $s+\ell \leq r$ and $\ell \leq 0$. The function h_5 is also an $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating function for the case where $p \equiv 1 \pmod 3$.

The result for the Case 1b follows from a combination of (A.125), (A.128) and (A.130)–(A.135).

Case 1c: $p \equiv 2 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{m+1})}_{\geq (s+k)m/3} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ & + \underbrace{f(v_{m+p}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor)} + \underbrace{f(v_n)}_{\geq 0} \\ \geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - r \end{aligned} \tag{A.136}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1) + \cdots + f(v_m)}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+1}) + \cdots + f(v_{m+p+1})}_{\geq (s+\ell)(\lfloor p/3 \rfloor + 1)} \\ & + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\ \geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell. \end{aligned} \tag{A.137}$$

Also,

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = {} & \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{m+1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ & + \underbrace{f(v_{m+p}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - m/3 - \lfloor p/3 \rfloor)} + \underbrace{f(v_n)}_{\geq s-r} \\ \geq {} & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + 2s + k - 2r \end{aligned} \tag{A.138}$$

and finally,

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = &\underbrace{f(v_1)+f(v_2)}_{\geq s+k}+\underbrace{f(v_3)+\cdots+f(v_{m-1})}_{\geq(s+k)(\lfloor m/3\rfloor-1)}+\underbrace{f(v_m)+f(v_{m+1})}_{\geq s+\ell-r}+\underbrace{f(v_{m+2})+\cdots+f(v_{m+p-1})}_{\geq(s+\ell)\lfloor p/3\rfloor}\\ &+\underbrace{f(v_{m+p})+f(v_{m+p+1})}_{\geq s+\ell-r}+\underbrace{f(v_{m+p+2})+\cdots+f(v_{n-2})}_{\geq s(\lfloor n/3\rfloor-m/3-\lfloor p/3\rfloor-1)}+\underbrace{f(v_{n-1})+f(v_n)}_{\geq s}\\ &\geq s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell\lfloor p/3\rfloor+2(s+\ell-r). \end{aligned} \tag{A.139}$$

If $s+k-r=\max\{s+k-r,s+\ell,2s+k-2r,2(s+\ell-r)\}$, then $s+k\geq r$, $s+\ell\leq r$ and $s\leq r$. The function h_7 is also an $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating function for the case where $p\equiv 2 \pmod 3$.

If $s+\ell=\max\{s+k-r,s+\ell,2s+k-2r,2(s+\ell-r)\}$, then $s+\ell\geq s+k-r$ and $s+\ell\leq 2r$. First consider the case where $s\leq r$ and let h_9 be an $\boldsymbol{r}$-function such that $h_9(v_{3c+1})=\max\{0,s+k-r\}$, $h_9(v_{3c+2})=\min\{r,s+k\}$ and $h_9(v_{3c+3})=0$ for $c=0,\ldots,\lfloor m/3\rfloor-1$. Also, let $h_9(v_{3c+1})=\min\{r,s+\ell\}$, $h_9(v_{3c+2})=\max\{0,s+\ell-r\}$ and $h_9(v_{3c+3})=0$ for $c=\lfloor m/3\rfloor,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally let $h_9(v_{3c+1})=s$, $h_9(v_{3c+2})=0$ and $h_9(v_{3c+3})=0$ for $c=\lfloor m/3\rfloor+\lfloor p/3\rfloor+1,\ldots,\lfloor n/3\rfloor-1$, and let $h_9(v_{n-1})=s$ and $h_9(v_n)=0$. Then h_9 is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|h_9|=s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell\lfloor p/3\rfloor+s+\ell. \tag{A.140}$$

Now consider the case where $s>r$. Let h_{10} be an $\boldsymbol{r}$-function such that $h_{10}(v_{m+1})=\max\{s+k-r,s+\ell-r\}$ and $h_{10}(v_{m+2})=\min\{r,\ell-k+r\}$. Furthermore $h_{10}(v_{3c+1})=s+k-r$, $h_{10}(v_{3c+2})=r$ and $h_{10}(v_{3c+2})=0$ for $c=0,\ldots,\lfloor m/3\rfloor-1$. Also, let $h_{10}(v_{3c})=0$, $h_{10}(v_{3c+1})=\min\{r,\ell+r\}$ and $h_{10}(v_{3c+2})=\max\{s-r,s+\ell-r\}$ for $c=\lfloor m/3\rfloor,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor-1$. Finally let $h_{10}(v_{3c})=0$, $h_{10}(v_{3c+1})=r$ and $h_{10}(v_{3c+2})=s-r$ for $c=\lfloor m/3\rfloor+\lfloor p/3\rfloor,\ldots,\lfloor n/3\rfloor-1$. Then h_{10} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|h_{10}|=s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell\lfloor p/3\rfloor+s+\ell. \tag{A.141}$$

If $2s+k-2r=\max\{s+k-r,s+\ell,2s+k-2r,2(s+\ell-r)\}$, then $s\geq r$ and $\ell\leq 0$. Let h_{11} be an $\boldsymbol{r}$-function such that $h_{11}(v_1)=s+k-r$ and $h_{11}(v_2)=s-r$. Furthermore let $h_{11}(v_{3c+2})=r$, $h_{11}(v_{3c+3})=0$ and $h_{11}(v_{3c+4})=s+k-r$ for $c=0,\ldots,\lfloor m/3\rfloor-1$. Also, let $h_{11}(v_{3c+2})=\ell-k+r$, $h_{11}(v_{3c+3})=0$ and $h_{11}(v_{3c+4})=s+k-r$ for $c=\lfloor m/3\rfloor,\ldots,\lfloor m/3\rfloor+\lceil\lfloor p/3\rfloor/2\rceil-1$. Furthermore let $h_{11}(v_{3c+2})=s-r$, $h_{11}(v_{3c+3})=0$ and $h_{11}(v_{3c+4})=\ell+r$ for $c=\lfloor m/3\rfloor+\lceil\lfloor p/3\rfloor/2\rceil,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor-1$. Finally let $h_{11}(v_{3c+2})=s-r$, $h_{11}(v_{3c+3})=0$ and $h_{11}(v_{3c+4})=r$ for $c=\lfloor m/3\rfloor+\lfloor p/3\rfloor,\ldots,\lfloor n/3\rfloor-1$. Then h_{11} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|h_{11}|=s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell\lfloor p/3\rfloor+2s+k-2r. \tag{A.142}$$

If $2(s+\ell-r)=\max\{s+k-r,s+\ell,2s+k-2r,2(s+\ell-r)\}$, then $s+\ell\geq 2r$. Let h_{12} be an $\boldsymbol{r}$-function such that $h_{12}(v_{3c+1})=\max\{0,s+k-r\}$, $h_{12}(v_{3c+2})=\min\{r,s+k\}$ and $h_{12}(v_{3c+3})=0$ for $c=0,\ldots,\lfloor m/3\rfloor-2$. Furthermore, let $h_{12}(v_{m-2})=\max\{0,s+k-r\}$, $h_{12}(v_{m-1})=\min\{r,s+k\}$ and $h_{12}(v_m)=s+\ell-2r$. Also, let $h_{12}(v_{3c+1})=r$, $h_{12}(v_{3c+2})=r$ and $h_{12}(v_{3c+3})=s+\ell-2r$ for $c=\lfloor m/3\rfloor,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally let $h_{12}(v_{3c+1})=\min\{r,s\}$, $h_{12}(v_{3c+2})=\max\{0,s-r\}$ and $h_{12}(v_{3c+3})=0$ for $c=\lfloor m/3\rfloor+\lfloor p/3\rfloor+1,\ldots,\lfloor n/3\rfloor-1$, and let $h_{12}(v_{n-1})=\min\{r,s\}$ and $h_{12}(v_n)=\max\{0,s-r\}$. Then h_{12} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|h_{12}|=s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell\lfloor p/3\rfloor+2(s+\ell-r). \tag{A.143}$$

The result for the Case 1c follows from a combination of (A.134), (A.136)–(A.148).

Now consider the case where $m \equiv 1 \pmod 3$ and the three subcases where $p \equiv 0, 1, 2 \pmod 3$.

Case 2: $m \equiv 1 \pmod 3$.

Case 2a: $p \equiv 0 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_{m+1})}_{\geq (s+k)\lfloor m/3\rfloor} + \underbrace{f(v_{m+2})+\cdots+f(v_{m+p+1})}_{\geq (s+\ell)\lfloor p/3\rfloor} \\ &+ \underbrace{f(v_{m+p+2})+\cdots+f(v_n)}_{\geq s(\lfloor n/3\rfloor-\lfloor m/3\rfloor-p/3)} \\ \geq& s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k \end{aligned} \tag{A.144}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)+\cdots+f(v_{m-1})}_{\geq (s+k)\lfloor m/3\rfloor} + \underbrace{f(v_m)+f(v_{m+1})}_{\geq s+\ell-r} \\ &+ \underbrace{f(v_{m+2})+\cdots+f(v_{m+p+1})}_{\geq (s+\ell)\lfloor p/3\rfloor} + \underbrace{f(v_{m+p+2})+\cdots+f(v_n)}_{\geq s(\lfloor n/3\rfloor-\lfloor m/3\rfloor-p/3)} \\ \geq& s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + \ell - r. \end{aligned} \tag{A.145}$$

Finally,

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_{m+1})}_{\geq (s+k)\lfloor m/3\rfloor} + \underbrace{f(v_{m+2})}_{\geq 0} + \underbrace{f(v_{m+3})+\cdots+f(v_{m+p-11})}_{\geq (s+\ell)(\lfloor p/3\rfloor-1)} \\ &+ \underbrace{f(v_{m+p+2})+\cdots+f(v_{n-2})}_{\geq s(\lfloor n/3\rfloor-\lfloor m/3\rfloor-p/3)} + \underbrace{f(v_{n-1})+f(v_n)}_{\geq s} \\ \geq& s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k - \ell. \end{aligned} \tag{A.146}$$

If $s+k = \max\{s+k, s+\ell-r, s+k-\ell\}$, then $\ell \geq 0$ and $\ell - k \leq r$. Let h_{13} be an $\boldsymbol{r}$-function such that $h_{13}(v_1) = \max\{0, s+k-r\}$ and $h_{13}(v_2) = \min\{r, s+k\}$. Furthermore let $h_{13}(v_{3c}) = 0$, $h_{13}(v_{3c+1}) = \max\{0, s+k-r\}$ and $h_{13}(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3\rfloor$. Also, let $h_{13}(v_{3c}) = \max\{0, \ell-k\}$, $h_{13}(v_{3c+1}) = \max\{0, \min\{s+k-r, s+\ell-r\}\}$ and $h_{13}(v_{3c+2}) = \min\{r, s+k, s+\ell\}$ for $c = \lfloor m/3\rfloor+1, \ldots, \lfloor m/3\rfloor+\lfloor p/3\rfloor$. Finally let $h_{13}(v_{3c}) = 0$, $h_{13}(v_{3c+1}) = \min\{r, s\}$ and $h_{13}(v_{3c+2}) = \max\{0, s-r\}$ for $c = \lfloor m/3\rfloor + \lfloor p/3\rfloor + 1, \ldots, \lfloor n/3\rfloor$. Then h_{13} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|h_{13}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k. \tag{A.147}$$

If $s+\ell-r = \max\{s+k, s+\ell-r, s+k-\ell\}$, then $\ell - k \geq r$ implying that $\ell \geq r$. Also note that if $s \geq r$, then $s+\ell \geq 2r$. Let h_{14} be an $\boldsymbol{r}$-function such that $h_{14}(v_{3c+1}) = \max\{0, s+k-r\}$, $h_{14}(v_{3c+2}) = \min\{r, s+k\}$ and $h_{14}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3\rfloor - 1$. Furthermore let $h_{14}(v_{3c+1}) = \min\{r, s+\ell-r\}$, $h_{14}(v_{3c+2}) = \max\{0, s+\ell-2r\}$ and $h_{14}(v_{3c+3}) = r$ for $c = \lfloor m/3\rfloor, \ldots, \lfloor m/3\rfloor + \lfloor p/3\rfloor - 1$. Also, let $h_{14}(v_{m+p}) = \min\{r, s+\ell-r\}$ and $h_{14}(v_{m+p+1}) = \max\{0, s+\ell-2r\}$. Finally let $h_{14}(v_{3c+3}) = 0$, $h_{14}(v_{3c+4}) = \min\{r, s\}$ and $h_{14}(v_{3c+5}) = \max\{0, s-r\}$ for $c = \lfloor m/3\rfloor + \lfloor p/3\rfloor, \ldots, \lfloor n/3\rfloor - 1$. Then h_{14} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|h_{14}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + \ell - r. \tag{A.148}$$

If $s+k-\ell = \max\{s+k, s+\ell-r, s+k-\ell\}$, then $\ell \leq 0$. Let h_{15} be an $\boldsymbol{r}$-function such that $h_{15}(v_{3c+1}) = \max\{0, s+k-r\}$, $h_{15}(v_{3c+2}) = \min\{r, s+k\}$ and $h_{15}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor$. Furthermore let $h_{15}(v_{3c+1}) = \max\{0, s+\ell-r\}$, $h_{15}(v_{3c+2}) = \min\{r, s+\ell\}$ and $h_{15}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor - 1$. Also, let $h_{15}(v_{3c+1}) = \min\{r, s\}$, $h_{15}(v_{3c+2}) = \max\{0, s-r\}$ and $h_{15}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor, \ldots, \lfloor n/3 \rfloor - 1$. Finally let $h_{15}(v_{n-1}) = \min\{r, s\}$ and $h_{15}(v_n) = \max\{0, s-r\}$. Then h_{15} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|h_{15}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - \ell. \tag{A.149}$$

The result for the Case 2a follows from a combination of (A.144)–(A.149).

Case 2b: $p \equiv 1 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m+1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\
&+ \underbrace{f(v_{m+p+1}) + \cdots + f(v_n)}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor)} \\
\geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k
\end{aligned} \tag{A.150}$$

and

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1) + \cdots + f(v_{m-1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_m) + f(v_{m+1})}_{\geq s+\ell-r} \\
&+ \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p+1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} + \underbrace{f(v_{m+p+2}) + \cdots + f(v_n)}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor)} \\
\geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell - r.
\end{aligned} \tag{A.151}$$

Finally,

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_{m+1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+2})}_{\geq 0} + \underbrace{f(v_{m+3}) + \cdots + f(v_{m+p-2})}_{\geq (s+\ell)(\lfloor p/3 \rfloor - 1)} \\
&+ \underbrace{f(v_{m+p-1})}_{\geq 0} + \underbrace{f(v_{m+p}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor -} + \underbrace{f(v_n)}_{\geq s-r} \\
\geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - \ell - r.
\end{aligned} \tag{A.152}$$

If $s+k = \max\{s+k, s+\ell, s+k-\ell-r\}$, then $s+\ell \leq 2r$. Let h_{16} be an $\boldsymbol{r}$-function such that $h_{16}(v_{3c+1}) = \max\{0, s+k-r\}$, $h_{16}(v_{3c+2}) = \min\{r, s+k\}$ and $h_{16}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor$. Furthermore let $h_{16}(v_{3c+1}) = \max\{0, s+\ell-r\}$, $h_{16}(v_{3c+2}) = \min\{r, s+\ell\}$ and $h_{16}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Also, let $h_{16}(v_{3c+1}) = \min\{r, s\}$, $h_{16}(v_{3c+2}) = \max\{0, s-r\}$ and $h_{16}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Finally let $h_{16}(v_{n-1}) = \min\{r, s\}$ and $h_{16}(v_n) = \max\{0, s-r\}$. Then h_{16} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|h_{16}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k. \tag{A.153}$$

If $s+\ell = \max\{s+k, s+\ell, s+k-\ell-r\}$, then let h_{17} be an $\boldsymbol{r}$-function such that $h_{17}(v_{3c+1}) = \max\{0, s+k-r\}$, $h_{17}(v_{3c+2}) = \min\{r, s+k\}$ and $h_{17}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor -$

1. Furthermore let $h_{17}(v_{3c+1}) = \min\{r, \max\{0, s+\ell-r\}\}$, $h_{17}(v_{3c+2}) = \min\{r, s+\ell\}$ and $h_{17}(v_{3c+3}) = \max\{0, s+\ell-2r\}$ for $c = \lfloor m/3\rfloor, \ldots, \lfloor m/3\rfloor + \lfloor p/3\rfloor$. Also, let $h_{17}(v_{3c+1}) = \min\{r, s\}$, $h_{17}(v_{3c+2}) = \max\{0, s-r\}$ and $h_{17}(v_{3c+3}) = 0$ for $c = \lfloor m/3\rfloor + \lfloor p/3\rfloor + 1, \ldots, \lfloor n/3\rfloor - 1$. Finally let $h_{17}(v_{n-1}) = \min\{r, s\}$ and $h_{17}(v_n) = \max\{0, s-r\}$. Then h_{17} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_{17}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + \ell. \tag{A.154}$$

If $s+k-\ell-r = \max\{s+k, s+\ell, s+k-\ell-r\}$, then $s+\ell \leq s-r$ implying that $\ell \leq 0$, $s+\ell \leq r$ and $s \geq r$. Let h_{18} be an $\boldsymbol{r}$-function such that $h_{18}(v_{3c+1}) = s+k-r$, $h_{18}(v_{3c+2}) = r$ and $h_{18}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3\rfloor$. Furthermore let $h_{18}(v_{3c+1}) = 0$, $h_{18}(v_{3c+2}) = s+\ell$ and $h_{18}(v_{3c+3}) = 0$ for $c = \lfloor m/3\rfloor + 1, \ldots, \lfloor m/3\rfloor + \lfloor p/3\rfloor - 1$. Also, let $h_{18}(v_{3c+2}) = s-r$, $h_{18}(v_{3c+3}) = 0$ and $h_{18}(v_{3c+4}) = r$ for $c = \lfloor m/3\rfloor + \lfloor p/3\rfloor + 1, \ldots, \lfloor n/3\rfloor - 1$. Finally let $h_{18}(v_{m+p-1}) = 0$ and $h_{18}(v_n) = s-r$. Then h_{18} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_{18}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k - \ell - r. \tag{A.155}$$

The result for the Case 2b follows from a combination of (A.150)–(A.155).

Case 2c: $p \equiv 2 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = &\underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_{m+1})}_{\geq (s+k)\lfloor m/3\rfloor} + \underbrace{f(v_{m+2})+\cdots+f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3\rfloor}\\ &+ \underbrace{f(v_{m+p})+\cdots+f(v_n)}_{\geq s(\lfloor n/3\rfloor - \lfloor m/3\rfloor - \lfloor p/3\rfloor)}\\ \geq &s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k \end{aligned} \tag{A.156}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = &\underbrace{f(v_1)+\cdots+f(v_{m-1})}_{\geq (s+k)\lfloor m/3\rfloor} + \underbrace{f(v_m)+\cdots+f(v_{m+p})}_{\geq (s+\ell)(\lfloor p/3\rfloor+1)}\\ &+ \underbrace{f(v_{m+p+1})+\cdots+f(v_{n-2})}_{\geq s(\lfloor n/3\rfloor - \lfloor m/3\rfloor - \lfloor p/3\rfloor - 1)} + \underbrace{f(v_{n-1})+f(v_n)}_{\geq s}\\ \geq &s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + \ell. \end{aligned} \tag{A.157}$$

Also,

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = &\underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_{m+1})}_{\geq (s+k)\lfloor m/3\rfloor} + \underbrace{f(v_{m+2})+\cdots+f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3\rfloor}\\ &+ \underbrace{f(v_{m+p})+f(v_{m+p+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+p+2})+\cdots+f(v_{n-1})}_{\geq s(\lfloor n/3\rfloor - \lfloor m/3\rfloor - \lfloor p/3\rfloor - 1)} + \underbrace{f(v_n)}_{\geq 0}\\ \geq &s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k + \ell - r \end{aligned} \tag{A.158}$$

and finally,

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = &\underbrace{f(v_1)+\cdots+f(v_{m-1})}_{\geq (s+k)\lfloor m/3\rfloor} + \underbrace{f(v_m)+f(v_{m+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+2})+\cdots+f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3\rfloor}\\ &+ \underbrace{f(v_{m+p})+f(v_{m+p+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+p+2})+\cdots+f(v_{n-1})}_{\geq s(\lfloor n/3\rfloor - \lfloor m/3\rfloor - \lfloor p/3\rfloor - 1)} + \underbrace{f(v_n)}_{\geq 0}\\ \geq &s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + 2\ell - 2r. \end{aligned} \tag{A.159}$$

If $s+k=\max\{s+k, s+\ell, s+k+\ell-r, s+2\ell-2r\}$, then $s+\ell \leq 2r$ and $\ell \leq r$. The function h_{16} is also an $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating function for the case where $p \equiv 2 \pmod 3$.

If $s+\ell=\max\{s+k, s+\ell, s+k+\ell-r, s+2\ell-2r\}$, then $k \leq r$. Fist consider the case where $s \geq r$ and let h_{19} be an $\boldsymbol{r}$-function such that $h_{19}(v_{3c+1})=s+k-r$, $h_{19}(v_{3c+2})=r$ and $h_{19}(v_{3c+3})=0$ for $c=0,\ldots,\lfloor m/3\rfloor-1$. Furthermore let $h_{19}(v_{3c+1})=\min\{r, s+\ell-r\}$, $h_{19}(v_{3c+2})=r$ and $h_{19}(v_{3c+3})=\max\{0, s+\ell-2r\}$ for $c=\lfloor m/3\rfloor,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor$. Also, let $h_{19}(v_{3c+1})=r$, $h_{19}(v_{3c+2})=s-r$ and $h_{19}(v_{3c+3})=0$ for $c=\lfloor m/3\rfloor+\lfloor p/3\rfloor+1,\ldots,\lfloor n/3\rfloor-1$. Finally let $h_{19}(v_{n-1})=r$ and $h_{19}(v_n)=s-r$. Then h_{19} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_{19}|=s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell\lfloor p/3\rfloor+s+\ell. \tag{A.160}$$

Now assume that $s<r$ and let h_{20} be an $\boldsymbol{r}$-function such that $h_{20}(v_{3c+1})=s$, $h_{20}(v_{3c+2})=k$ and $h_{20}(v_{3c+3})=0$ for $c=0,\ldots,\lfloor m/3\rfloor-1$. Furthermore let $h_{20}(v_{3c+1})=s$, $h_{20}(v_{3c+2})=\min\{r,\ell\}$ and $h_{20}(v_{3c+3})=\max\{0,\ell-r\}$ for $c=\lfloor m/3\rfloor,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor$. Also, let $h_{20}(v_{3c+1})=s$, $h_{20}(v_{3c+2})=0$ and $h_{20}(v_{3c+3})=0$ for $c=\lfloor m/3\rfloor+\lfloor p/3\rfloor+1,\ldots,\lfloor n/3\rfloor-1$. Finally let $h_{20}(v_{n-1})=s$ and $h_{20}(v_n)=0$. Then h_{20} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_{20}|=s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell\lfloor p/3\rfloor+s+\ell. \tag{A.161}$$

If $s+k+\ell-r=\max\{s+k, s+\ell, s+k+\ell-r, s+2\ell-2r\}$, then $s \leq r$, $\ell \geq r$, $k \geq r$ and $\ell-k \leq r$. Let h_{21} be an $\boldsymbol{r}$-function such that $h_{21}(v_{3c+2})=r$, $h_{21}(v_{3c+3})=0$ and $h_{21}(v_{3c+4})=s+k-r$ for $c=0,\ldots,\lfloor m/3\rfloor-1$. Furthermore let $h_{21}(v_{3c+2})=r$, $h_{21}(v_{3c+3})=\ell-k$ and $h_{21}(v_{3c+4})=s+k-r$ for $c=\lfloor m/3\rfloor,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor$. Also, let $h_{21}(v_{3c+2})=0$, $h_{21}(v_{3c+3})=0$ and $h_{21}(v_{3c+4})=s$ for $c=\lfloor m/3\rfloor+\lfloor p/3\rfloor+1,\ldots,\lfloor n/3\rfloor-1$. Finally let $h_{21}(v_1)=s+k-r$ and $h_{21}(v_n)=0$. Then h_{21} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_{21}|=s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell\lfloor p/3\rfloor+s+k+\ell-r. \tag{A.162}$$

If $s+2\ell-2r=\max\{s+k, s+\ell, s+k+\ell-r, s+2\ell-2r\}$, then $\ell \geq 2r$, $s+\ell-r \geq s+k$ and $s \leq r$. Let h_{22} be an $\boldsymbol{r}$-function such that $h_{22}(v_{3c+1})=\max\{0, s+k-r\}$, $h_{22}(v_{3c+2})=\min\{r, s+k\}$ and $h_{22}(v_{3c+3})=0$ for $c=0,\ldots,\lfloor m/3\rfloor-1$. Furthermore let $h_{22}(v_{3c+1})=s+\ell-2r$, $h_{22}(v_{3c+2})=r$ and $h_{22}(v_{3c+3})=r$ for $c=\lfloor m/3\rfloor,\ldots,\lfloor m/3\rfloor+\lfloor p/3\rfloor$. Also, let $h_{22}(v_{3c+2})=0$, $h_{22}(v_{3c+3})=0$ and $h_{22}(v_{3c+4})=s$ for $c=\lfloor m/3\rfloor+\lfloor p/3\rfloor+1,\ldots,\lfloor n/3\rfloor-1$. Finally let $h_{22}(v_{m+p+1})=s+\ell-2r$ and $h_{22}(v_n)=0$. Then h_{22} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_{22}|=s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell\lfloor p/3\rfloor+s+k-\ell-r. \tag{A.163}$$

The result for the Case 2c follows from a combination of (A.156)–(A.163).

Finally consider the case where $m \equiv 2 \pmod 3$ and the three subcases where $p \equiv 0,1,2 \pmod 3$.

Case 3: $m \equiv 2 \pmod 3$.

Case 3a: $p \equiv 0 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) =& \underbrace{f(v_1)+\cdots+f(v_{m+1})}_{\geq (s+k)(\lfloor m/3\rfloor+1)}+\underbrace{f(v_{m+2})+\cdots+f(v_{m+p+1})}_{\geq (s+\ell)\lfloor p/3\rfloor} \\ &+\underbrace{f(v_{m+p+2})+\cdots+f(v_{n-2})}_{\geq s(\lfloor n/3\rfloor-\lfloor m/3\rfloor-\lfloor p/3\rfloor-1)}+\underbrace{f(v_{n-1})+f(v_n)}_{\geq s} \\ \geq& s\lfloor n/3\rfloor+k\lfloor m/3\rfloor+\ell\lfloor p/3\rfloor+s+k \end{aligned} \tag{A.164}$$

and

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = & \underbrace{f(v_1)}_{\geq 0} + \underbrace{f(v_2) + \cdots + f(v_{m-1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_m) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p}) + f(v_{m+p+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell - r. \qquad (A.165)
\end{aligned}$$

Also,

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = & \underbrace{f(v_1) + \cdots + f(v_{m+1})}_{\geq (s+k)(\lfloor m/3 \rfloor + 1)} + \underbrace{f(v_{m+2})}_{\geq 0} + \underbrace{f(v_{m+3}) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)(\lfloor p/3 \rfloor - 1)} \\
& + \underbrace{f(v_{m+p}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor)} + \underbrace{f(v_n)}_{\geq s-r} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k - \ell - r \qquad (A.166)
\end{aligned}$$

and finally

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) = & \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{m-1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_m) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p}) + f(v_{m+p+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + 2s + k + \ell - 2r. \qquad (A.167)
\end{aligned}$$

If $s+k = \max\{s+k, s+\ell-r, s+k-\ell-r, 2s+k+\ell-2r\}$, then $\ell + r \geq 0$ and $s + \ell \leq 2r$. The function h_{13} is also an $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating function for the case where $m \equiv 2 \pmod 3$ and $p \equiv 0 \pmod 3$.

If $s+\ell-r = \max\{s+k, s+\ell-r, s+k-\ell-r, 2s+k+\ell-2r\}$, then $s+\ell \geq r$, $\ell - k \geq r$ and $s+k \leq r$. Let h_{23} be an $\boldsymbol{r}$-function such that $h_{23}(v_1) = 0$, $h_{23}(v_{n-1}) = s$ and $h_{23}(v_n) = 0$. Furthermore let $h_{23}(v_{3c+2}) = s+k$, $h_{23}(v_{3c+3}) = 0$ and $h_{23}(v_{3c+4}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$ and let $h_{23}(v_m) = s+k$ and $h_{23}(v_{m+1}) = \ell - k - r$. Also, let $h_{23}(v_{3c+1}) = r$, $h_{23}(v_{3c+2}) = s+k$ and $h_{23}(v_{3c+3}) = \ell - k - r$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $h_{23}(v_{3c+1}) = s$, $h_{23}(v_{3c+2}) = 0$ and $h_{23}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then h_{23} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|h_{23}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + \ell - r. \qquad (A.168)$$

If $s+k-\ell-r = \max\{s+k, s+\ell-r, s+k-\ell-r, 2s+k+\ell-2r\}$, then $s+k \geq r$, $s+\ell \leq r$ and $\ell \leq 0$. The function h_{18} is also an $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating function for the case where $m, p \equiv 1 \pmod 3$.

If $2s+k+\ell-2r = \max\{s+k, s+\ell-r, s+k-\ell-r, 2s+k+\ell-2r\}$, then $s+\ell \geq 2r$ and $s \geq r$. Let h_{24} be an $\boldsymbol{r}$-function such that $h_{24}(v_1) = s+k-r$, $h_{24}(v_2) = r$ and $h_{24}(v_n) = s-r$. Furthermore let $h_{24}(v_{3c}) = 0$, $h_{24}(v_{3c+1}) = s+k-r$ and $h_{24}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3 \rfloor$. Also, let $h_{24}(v_{3c}) = s+\ell-2r$, $h_{24}(v_{3c+1}) = r$ and $h_{24}(v_{3c+2}) = r$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$ and let $h_{24}(v_{m+p+1}) = s+\ell-2r$ and $h_{24}(v_{m+p+2}) = r$. Finally let $h_{24}(v_{3c+2}) = s-r$, $h_{24}(v_{3c+3}) = 0$ and $h_{24}(v_{3c+4}) = r$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then h_{24} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_{24}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + 2s + k + \ell - 2r. \tag{A.169}$$

The result for the Case 3a follows from a combination of (A.155), (A.164)–(A.169).

Case 3b: $p \equiv 1 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) =& \underbrace{f(v_1) + \cdots + f(v_{m+1})}_{\geq (s+k)(\lfloor m/3 \rfloor + 1)} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ &+ \underbrace{f(v_{m+p+1}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\ \geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k \end{aligned} \tag{A.170}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) =& \underbrace{f(v_1)}_{\geq 0} + \underbrace{f(v_2) + \cdots + f(v_{m-1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_m) + \cdots + f(v_{m+p+1})}_{\geq (s+\ell)(\lfloor p/3 \rfloor + 1)} \\ &+ \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_n)}_{\geq 0} \\ \geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell. \end{aligned} \tag{A.171}$$

It also follows that

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) =& \underbrace{f(v_1) + f(v_2)}_{\geq s+k} + \underbrace{f(v_3) + \cdots + f(v_m)}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ &+ \underbrace{f(v_{m+p}) + f(v_{m+p+1})}_{\geq s+\ell-r} + \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor)} + \underbrace{f(v_n)}_{\geq 0} \\ \geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k + \ell - r \end{aligned} \tag{A.172}$$

and finally, that

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) =& \underbrace{f(v_1)}_{\geq s+k-r} + \underbrace{f(v_2) + \cdots + f(v_{m-1})}_{\geq (s+k)\lfloor m/3 \rfloor} + \underbrace{f(v_m) + \cdots + f(v_{m+p+1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\ &+ \underbrace{f(v_{m+p+2}) + \cdots + f(v_{n-1})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_n)}_{\geq s-r} \\ \geq& s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + 2s + k + \ell - 2r. \end{aligned} \tag{A.173}$$

If $s+k = \max\{s+k, \ell, s+k+\ell-r, 2s+k+\ell-2r\}$, then $\ell \leq r$ and $s+\ell \leq 2r$. First consider the case where $s \geq r$ and let h_{25} be an $\boldsymbol{r}$-function such that $h_{25}(v_{3c+1}) = s+k-r$, $h_{25}(v_{3c+2}) = r$ and $h_{25}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor$. Furthermore let $h_{25}(v_{3c+1}) = \max\{0, s+\ell-r\}$, $h_{25}(v_{3c+2}) = \min\{r, s+\ell\}$ and $h_{25}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Also, let $h_{25}(v_{3c+1}) = r$, $h_{25}(v_{3c+2}) = s-r$ and $h_{25}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Finally let $h_{25}(v_{n-1}) = r$ and $h_{25}(v_n) = s-r$. Then h_{25} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_{25}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k. \tag{A.174}$$

If $s < r$, then let h_{26} be an $\boldsymbol{r}$-function such that $h_{26}(v_{3c+1}) = \max\{0, s+k-r\}$, $h_{26}(v_{3c+2}) = \min\{r, s+k\}$ and $h_{26}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor$. Furthermore let $h_{26}(v_{3c+1}) = \min\{s, s+\ell\}$, $h_{26}(v_{3c+2}) = \max\{0, \ell\}$ and $h_{26}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Also, let $h_{26}(v_{3c+1}) = s$, $h_{26}(v_{3c+2}) = 0$ and $h_{26}(v_{3c+3}) = 0$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Finally let $h_{26}(v_{n-1}) = s$ and $h_{26}(v_n) = 0$. Then h_{26} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_{26}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k. \tag{A.175}$$

If $\ell = \max\{s+k, \ell, s+k+\ell-r, 2s+k+\ell-2r\}$, then $s+k \leq r$. Let h_{27} be an $\boldsymbol{r}$-function such that $h_{27}(v_1) = 0$ and $h_{27}(v_n) = 0$. Furthermore let $h_{27}(v_{3c+2}) = s+k$, $h_{27}(v_{3c+3}) = 0$ and $h_{27}(v_{3c+4}) = 0$ for $c = 0, \ldots, \lfloor m/3 \rfloor - 1$. Also, let $h_{27}(v_{3c+2}) = \min\{r, \ell\}$, $h_{27}(v_{3c+3}) = \max\{0, \ell - r\}$ and $h_{27}(v_{3c+4}) = s$ for $c = \lfloor m/3 \rfloor, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $h_{27}(v_{3c+2}) = 0$, $h_{27}(v_{3c+3}) = 0$ and $h_{27}(v_{3c+4}) = s$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then h_{27} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_{27}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + \ell. \tag{A.176}$$

If $s+k+\ell-r = \max\{s+k, \ell, s+k+\ell-r, 2s+k+\ell-2r\}$, then $s+k \geq r$, $\ell \geq r$ and $s \leq r$. Let h_{28} be an $\boldsymbol{r}$-function such that $h_{28}(v_1) = s+k-r$, $h_{28}(v_2) = r$ and $h_{28}(v_n) = 0$. Furthermore let $h_{28}(v_{3c}) = 0$, $h_{28}(v_{3c+1}) = s+k-r$ and $h_{28}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3 \rfloor$ and let $h_{28}(v_{m+1}) = 0$ and $h_{28}(v_{m+2}) = s+\ell-r$. Also, let $h_{28}(v_{3c+2}) = r$, $h_{28}(v_{3c+3}) = 0$ and $h_{28}(v_{3c+4}) = s+\ell-r$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $h_{28}(v_{3c+2}) = 0$, $h_{28}(v_{3c+3}) = 0$ and $h_{28}(v_{3c+4}) = s$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then h_{28} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_{28}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k + \ell - r. \tag{A.177}$$

If $2s+k+\ell-2r = \max\{s+k, \ell, s+k+\ell-r, 2s+k+\ell-2r\}$, then $s+\ell \geq 2r$ and $s \geq r$. Let h_{29} be an $\boldsymbol{r}$-function such that $h_{29}(v_1) = s+k-r$, $h_{29}(v_2) = r$ and $h_{29}(v_n) = s-r$. Furthermore let $h_{29}(v_{3c}) = 0$, $h_{29}(v_{3c+1}) = s+k-r$ and $h_{29}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3 \rfloor$ and let $h_{29}(v_{m+1}) = s+\ell-2r$ and $h_{29}(v_{m+2}) = r$. Also, let $h_{29}(v_{3c+2}) = r$, $h_{29}(v_{3c+3}) = s+\ell-2r$ and $h_{29}(v_{3c+4}) = r$ for $c = \lfloor m/3 \rfloor + 1, \ldots, \lfloor m/3 \rfloor + \lfloor p/3 \rfloor$. Finally let $h_{29}(v_{3c+2}) = s-r$, $h_{29}(v_{3c+3}) = 0$ and $h_{29}(v_{3c+4}) = r$ for $c = \lfloor m/3 \rfloor + \lfloor p/3 \rfloor + 1, \ldots, \lfloor n/3 \rfloor - 1$. Then h_{29} is $\boldsymbol{s}^{k,\ell}_{m,p}$-dominating and

$$|h_{29}| = s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + 2s + k + \ell - 2r. \tag{A.178}$$

The result for the Case 3b follows from a combination of (A.170)–(A.178).

Case 3c: $p \equiv 2 \pmod 3$. Let f be an $\boldsymbol{s}$-dominating $\boldsymbol{r}$-function of P_n of minimum weight. Then

$$\begin{aligned}
\gamma_{\boldsymbol{r}}^{\boldsymbol{s}^{k,\ell}_{m,p}}(P_n) = & \underbrace{f(v_1) + \cdots + f(v_{m+1})}_{\geq (s+k)(\lfloor m/3 \rfloor + 1)} + \underbrace{f(v_{m+2}) + \cdots + f(v_{m+p-1})}_{\geq (s+\ell)\lfloor p/3 \rfloor} \\
& + \underbrace{f(v_{m+p}) + \cdots + f(v_{n-2})}_{\geq s(\lfloor n/3 \rfloor - \lfloor m/3 \rfloor - \lfloor p/3 \rfloor - 1)} + \underbrace{f(v_{n-1}) + f(v_n)}_{\geq s} \\
\geq & s\lfloor n/3 \rfloor + k\lfloor m/3 \rfloor + \ell\lfloor p/3 \rfloor + s + k
\end{aligned} \tag{A.179}$$

and

$$\begin{aligned}\gamma_{\boldsymbol{r}}^{\boldsymbol{s}_{m,p}^{k,\ell}}(P_n) =& \underbrace{f(v_1)+f(v_2)}_{\geq s+k} + \underbrace{f(v_3)+\cdots+f(v_m)}_{\geq (s+k)\lfloor m/3\rfloor} + \underbrace{f(v_m)+\cdots+f(v_{m+p+1})}_{\geq (s+\ell)(\lfloor p/3\rfloor+1)}\\ &+ \underbrace{f(v_{m+p+2})+\cdots+f(v_n)}_{\geq s(\lfloor n/3\rfloor-\lfloor m/3\rfloor-\lfloor p/3\rfloor-1)}\\ \geq& s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k + \ell. \end{aligned} \tag{A.180}$$

If $s+k = \max\{s+k, s+k+\ell\}$, then $\ell \leq 0$. Let h_{30} be an $\boldsymbol{r}$-function such that $h_{30}(v_{3c+1}) = \max\{0, s+k-r\}$, $h_{30}(v_{3c+2}) = \min\{r, s+k\}$ and $h_{30}(v_{3c+3}) = 0$ for $c = 0, \ldots, \lfloor m/3\rfloor$. Furthermore let $h_{30}(v_{3c+1}) = \max\{0, s+\ell-r\}$, $h_{30}(v_{3c+2}) = \min\{r, s+\ell\}$ and $h_{30}(v_{3c+3}) = 0$ for $c = \lfloor m/3\rfloor+1, \ldots, \lfloor m/3\rfloor + \lfloor p/3\rfloor$. Also, let $h_{30}(v_{3c+1}) = \min\{r, s\}$, $h_{30}(v_{3c+2}) = \max\{0, s-r\}$ and $h_{30}(v_{3c+3}) = 0$ for $c = \lfloor m/3\rfloor + \lfloor p/3\rfloor + 1, \ldots, \lfloor n/3\rfloor - 1$. Finally let $h_{30}(v_{n-1}) = \min\{r, s\}$ and $h_{30}(v_n) = \max\{0, s-r\}$. Then h_{30} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|h_{30}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k. \tag{A.181}$$

If $s+k+\ell = \max\{s+k, s+k+\ell\}$, then $\ell \geq 0$. First assume that $s \leq r$ and let h_{31} be an $\boldsymbol{r}$-function such that $h_{31}(v_1) = \max\{0, s+k-r\}$ and $h_{31}(v_2) = \min\{r, s+k\}$. Furthermore let $h_{31}(v_{3c}) = 0$, $h_{31}(v_{3c+1}) = \max\{0, s+k-r\}$ and $h_{31}(v_{3c+2}) = \min\{r, s+k\}$ for $c = 1, \ldots, \lfloor m/3\rfloor$. Also, let $h_{31}(v_{3c}) = \min\{r, \ell\}$, $h_{31}(v_{3c+1}) = \max\{0, \ell-r\}$ and $h_{31}(v_{3c+2}) = s$ for $c = \lfloor m/3\rfloor+1, \ldots, \lfloor m/3\rfloor+\lfloor p/3\rfloor+1$. Finally let $h_{31}(v_{3c}) = 0$, $h_{31}(v_{3c+1}) = s$ and $h_{31}(v_{3c+2}) = 0$ for $c = \lfloor m/3\rfloor + \lfloor p/3\rfloor + 2, \ldots, \lfloor n/3\rfloor$. Then h_{31} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|h_{31}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k + \ell. \tag{A.182}$$

Now assume that $s > r$. Let h_{32} be an $\boldsymbol{r}$-function such that $h_{32}(v_1) = s+k-r$ and $h_{32}(v_2) = r$. Furthermore let $h_{32}(v_{3c}) = 0$, $h_{32}(v_{3c+1}) = s+k-r$ and $h_{32}(v_{3c+2}) = r$ for $c = 1, \ldots, \lfloor m/3\rfloor$. Also, let $h_{32}(v_{3c}) = \max\{0, s+\ell-2r\}$, $h_{32}(v_{3c+1}) = \min\{r, s+\ell-r\}$ and $h_{32}(v_{3c+2}) = r$ for $c = \lfloor m/3\rfloor+1, \ldots, \lfloor m/3\rfloor+\lfloor p/3\rfloor+1$. Finally let $h_{32}(v_{3c}) = 0$, $h_{32}(v_{3c+1}) = r$ and $h_{32}(v_{3c+2}) = s-r$ for $c = \lfloor m/3\rfloor + \lfloor p/3\rfloor + 2, \ldots, \lfloor n/3\rfloor$. Then h_{32} is $\boldsymbol{s}_{m,p}^{k,\ell}$-dominating and

$$|h_{32}| = s\lfloor n/3\rfloor + k\lfloor m/3\rfloor + \ell\lfloor p/3\rfloor + s + k + \ell. \tag{A.183}$$

The result for the Case 3c follows from a combination of (A.179)–(A.183). ■

APPENDIX B

Contents of the accompanying compact disc

A brief description of the contents of the compact disc included with the dissertation is given in this appendix. The compact disc contains computer implementations of all the algorithms described and analysed in the dissertation as well as the graph triples $(G, \boldsymbol{r}, \boldsymbol{s})$ used in the comparison of the algorithms in §6.6. The code and graph triples enable the reader to reproduce the algorithmic results reported in this dissertation.

The implementation of the integer programming model was achieved in the software suite `LINGO` [58] of Lindo Systems, while the remaining algorithms were implemented in Wolfram's `Mathematica` [85]. Detail on the compilation and usage of the computer code are provided on the compact disc. The compact disc also contains the adjacency matrices of the three randomly generated graphs that were used to compare the upper bounds in §4.3. The compact disc contains the following four directories:

Algorithms for the (r,s)-domination number. This directory contains four subdirectories, namely "Linear tree algorithm", "Branch-and-bound algorithm", "Branch-and-reduce algorithm" and "Integer programming model" which, in turn, contain the relevant algorithmic implementations for computing the $\langle \boldsymbol{r}, \boldsymbol{s} \rangle$-domination number of a graph.

Graph triples used for comparison of algorithms. This directory contains four PDF files listing the $\boldsymbol{r}$ and $\boldsymbol{s}$ vectors used to compare the execution times of the algorithms in Chapter 6.

Repository. Text files are provided in this directory containing the adjacency matrices of the three randomly generated graphs that were used to compare the upper bounds in §4.3.

Dissertation. This directory contains an electronic copy of the dissertation.