Enumeration Problems on Lattices

by

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Thesis presented in partial fulfilment of the requirements for the degree of Master of Science in Mathematics in the Faculty of Science at Stellenbosch University

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Declaration

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Abstract

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The main objective of our study is enumerating spanning trees $\tau(G)$ and perfect matchings $\mathrm{PM}(G)$ on graphs G and lattices $\mathcal{L}$. We demonstrate two methods of enumerating spanning trees of any connected graph, namely the matrix-tree theorem and as a special value of the Tutte polynomial $T(G; x, y)$.

We present a general method for counting spanning trees on lattices in $d \geq 2$ dimensions. In particular we apply this method on the following regular lattices with $d = 2$: rectangular, triangular, honeycomb, kagomé, diced, $9 \cdot 3$ lattice and its dual lattice to derive a explicit formulas for the number of spanning trees of these lattices of finite sizes.

Regarding the problem of enumerating of perfect matchings, we prove Cayley's theorem which relates the Pfaffian of a skew symmetric matrix to its determinant. Using this and defining the Pfaffian orientation on a planar graph, we derive explicit formula for the number of perfect matchings on the following planar lattices; rectangular, honeycomb and triangular.

For each of these lattices, we also determine the bulk limit or thermodynamic limit, which is a natural measure of the rate of growth of the number of spanning trees $\tau(\mathcal{L})$ and the number of perfect matchings $\mathrm{PM}(\mathcal{L})$.

An algorithm is implemented in the computer algebra system SAGE to count the number of spanning trees as well as the number of perfect matchings of the lattices studied.

Uittreksel

Telprobleme op roosters

("Enumeration Problems on Lattices")

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Die hoofdoel van ons studie is die aftelling van spanbome $\tau(G)$ en volkome afparings $\mathrm{PM}(G)$ in grafieke G en roosters $\mathcal{L}$. Ons beskou twee metodes om spanbome in 'n samehangende grafiek af te tel, naamlik deur middel van die matriks-boom-stelling, en as 'n spesiale waarde van die Tutte polinoom $T(G; x, y)$.

Ons behandel 'n algemene metode om spanbome in roosters in $d \geq 2$ dimensies af te tel. In die besonder pas ons hierdie metode toe op die volgende reguliere roosters met $d = 2$: reghoekig, driehoekig, heuningkoek, kagomé, blokkies, $9 \cdot 3$ rooster en sy duale rooster. Ons bepaal eksplisiete formules vir die aantal spanbome in hierdie roosters van eindige grootte.

Wat die aftelling van volkome afparings aanbetref, gee ons 'n bewys van Cayley se stelling wat die Pfaffiaan van 'n skeefsimmetriese matriks met sy determinant verbind. Met behulp van hierdie stelling en Pfaffiaanse oriënterings van planare grafieke bepaal ons eksplisiete formules vir die aantal volkome afparings in die volgende planare roosters: reghoekig, driehoekig, heuningkoek.

Vir elk van hierdie roosters word ook die "grootmaat limiet" (of termodinamiese limiet) bepaal, wat 'n natuurlike maat vir die groeitempo van die aantaal spanbome $\tau(\mathcal{L})$ en die aantal volkome afparings $\mathrm{PM}(\mathcal{L})$ voorstel.

'n Algoritme is in die rekenaaralgebra-stelsel SAGE geimplementeer om die aantal spanboome asook die aantal volkome afparings in die toepaslike roosters af te tel.

Acknowledgements

I am most grateful to God for his divine providence throughout the entire period of this thesis. It is purely His grace that has brought me this far.

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Dedication

To my dearest mother, Veronica Adinorkie Adi.

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Nomenclature

Sets

$\mathbb{N}$	Natural numbers.
$\mathbb{Z}$	Integers.
$\mathbb{R}$	Real numbers.
$M_n(\mathbb{Z})$	$[a_{ij}]_{1\leq i,j\leq n}$, $a_{ij} \in \mathbb{Z}$.
S_n	Symmetric group of order n
$\mathcal{P}(X)$	Power set of X.
$\mathcal{U}_x$	The set of all neighbourhoods of x
$[n]$	n element set, i.e., $\{1, 2, \ldots, n\}$.
$V(G)$	Vertex set of the graph G.
$E(G)$	Edge set of the graph G.
$N_G(v)$	Open Neighbourhood of vertex v.
$N_G[v]$	Closed Neighbourhood of vertex v.

Graphs

G^*	Dual planar graph.
$G - e$	Edge deletion.
G/e	Edge contraction.
C_n	Cycle graph.
P_n	Path graph.
E_n	Empty graph.
W_n	Wheel graph.
K_n	Complete graph.
$K_{n,m}$	Complete bipartite graph.

Functions

$v(G)$	Order of the graph G.
$e(G)$	Size of the graph G.
$\deg(v)$	Degree of a vertex.
$\kappa(G)$	Connectivity of the graph G.
$\chi(G)$	Chromatic number.

$\det(A)$	Determinant of the matrix A.
$\tau(G)$	Number of spanning trees of the graph G.
$\mathrm{PM}(G)$	Number of perfect matchings of the graph G.

Matrices

$\mathrm{A}(G)$	Adjacency matrix of the graph G.
$\mathrm{L}(G)$	Laplacian matrix of the graph G.

Polynomials

$P(G;k)$	Chromatic polynomial of the graph G.
$T(G;x,y)$	Tutte polynomial of the graph G.

Chapter 1

Introduction

The study of discrete structures and their relations — the branch of mathematics called graph theory — has evolved since the days of the Swiss mathematician Leonard Euler and his study of the seven bridges of Königsberg in 1735. Together with other branches of mathematics like combinatorics — the art of counting, a wide range of combinatorial enumeration problems on graphs, e.g. counting spanning trees, matchings, perfect matching, or independent sets have been studied. These problems are not only of purely mathematical interest, but also occur naturally in statistical physics and other fields of science. Our study will focus mainly on counting spanning trees and perfect matchings.

The question of counting spanning trees of a graph was first answered by the German physicist Gustav Kirchhoff in his study of electrical networks in 1847 [17]. It is also answered by the famous matrix-tree theorem — one of the classical theorems in algebraic graph theory, which provides a means of counting the spanning trees of a connected graph in terms of eigenvalues or determinants of associated matrices.

Another method by which spanning trees of a graph can also be counted is by using the Tutte polynomial of the graph which is defined by a deletion-contraction recurrence equation of the edges of a graph. The Tutte polynomial contains several pieces of information of a graph and is a generalisation of counting problems related to graph colouring. We shall study planar lattices and relate the Tutte polynomial of a planar graph to the Tutte polynomial of its dual planar representation.

The second chapter is devoted to the matrix-tree theorem and the Tutte polynomial of a graph. We shall also proof a corollary of the matrix-tree theorem that expresses the number of spanning trees of a connected graph as a product of the non-zero eigenvalues of the Laplacian of the graph divided by the total number of vertices.

In the third chapter, we will use the above mentioned theorems to derive explicit formulas for the number of spanning trees of some regular planar lattices, namely the rectangular lattice, the triangular lattice and its dual the honeycomb lattice, the kagomé lattice and its dual the diced lattice and lastly the lattice of nanogons and triangles and its dual lattice. We shall also give the asymptotic growth of the number of spanning trees of these lattices. These results are also obtained in [22].

In the final chapter we define the Pfaffian of a real skew symmetric matrix and the Pfaffian orientation of a planar graph and use Cayley's theorem which expresses the Pfaffian of any real skew symmetric matrix as the square root of its determinant to give derive explicit formulas to count the number of perfect matchings on the rectangular lattice, triangular lattice and its dual the honeycomb lattice.

Chapter 2

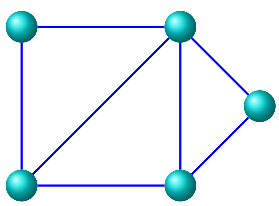
Fundamental Concepts

2.1 Introduction

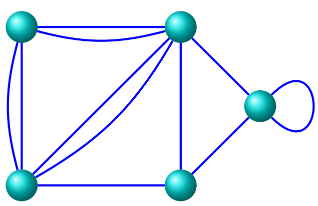
In this chapter, we will discuss some basic definitions as well as some theorems which will aid in our study. In particular, we will focus our attention on two methods of counting combinatorial objects such as spanning trees given a graph G. These two methods of counting are the matrix tree theorem and the Tutte polynomial. Since our study is mainly on lattices, we also study planar graphs and count some combinatorial substructures such as spanning trees, spanning forests and connected subgraphs of connected planar graphs. Furthermore, we give the special values of the Tutte polynomial that are used in counting these combinatorial substructures.

Definition 2.1.1. *A* ***graph*** G *is an ordered pair* (V, E) *of sets, where* V *is the set of vertices of* G *and* E *is a subset of the set* $V^{(2)}$ *of unordered pairs of* V *called the edges of* G*.*

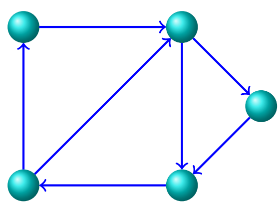
For simplicity, we abbreviate an edge $\{a, b\} \in E$ by ab (or ba) and we denote the order $|V|$ and size $|E|$ of the graph G by $v(G)$ and $e(G)$ respectively. The graphs that will be studied are *simple graphs,* i.e., graphs whose edge set is a set of unordered pairs of distinct vertices, or simply, graphs with no loops or multiple edges. *Multigraphs* on the other hand, are graphs whose edge set is a multi-set or has repeated elements. A collection of several edges with the same end vertices is called *multiple edges* or *parallel lines* and if an edge connects a vertex to itself, it is called a *loop.* Moreover, if the edges of a graph are oriented, i.e., the elements of the edge set are ordered pairs of vertices, then the graph is called a *digraph.* For example:



(a) Simple graph.



(b) Multigraph.



(c) Digraph.

Figure 2.1: A simple graph, a multigraph and a digraph.

For each of the above graphs, we can also define a function $\omega : V \times V \to \mathbb{R}$ (i.e., we assign a numerical value to each edge), in which case $G(V, E, \omega)$ is said to be a *weighted graph*.

Definition 2.1.2. *A graph G is said to be* ***connected*** *if every pair of vertices can be joined by a path. Informally, a graph is said to be* ***connected*** *if one can pick up the entire graph by just picking only one vertex.*

The graphs in Figure 2.1 are *connected* graphs.

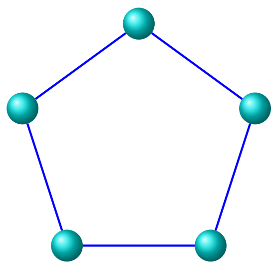
2.2 Some Terminologies in Graph Theory

If $uv \in E$ is an edge, then the vertices u and v are said to be *incident* with the edge uv and they are also said to be *adjacent* or neighbouring vertices. In this case, u and v are called the end vertices of the edge uv. The *open neighbourhood of a vertex* v, denoted by $N_G(v)$, is the set of vertices adjacent to v.

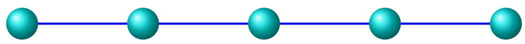
$$N_G(v) := \{x \in V \mid xv \in E\}.$$

The *closed neighbourhood of a vertex* v, denoted by $N[v]$, is the set $\{v\} \cup N_G(v)$. The *degree of a vertex* v, denoted by $\deg(v)$, is the number of edges incident with v. If the underlying graph is *simple*, then $\deg(v) = |N_G(v)|$.

A *walk* in a graph is defined as a sequence of (not necessarily distinct) vertices $v_1, \ldots, v_k$, such that $v_i v_{i+1} \in E$ for $i = 1, \ldots, k-1$. If the vertices in a walk are distinct, then it is called a *path*. If only the edges are distinct, then it is called a *trail*. So then, every *path* is a *trail* but the converse is not true. A *closed path* or *cycle* is a path $v_1, \ldots, v_k$ with an additional edge $v_k v_1$ such that $k \geq 3$. Every trail that begins and ends at the same vertex is called a *closed trail* or a *circuit*. The graph C_n is a cycle on n vertices and the graph P_n is a path on n vertices.



(a) C_5.



(b) P_5.

Figure 2.2: The graphs C_5 and P_5.

2.3 Trees and their Properties

Definition 2.3.1. *A* ***tree*** *T is a connected acyclic graph, i.e., a connected graph with no cycles. If a vertex $v \in T$ has degree $\deg(v) = 1$, then v is called a* ***leaf*** .

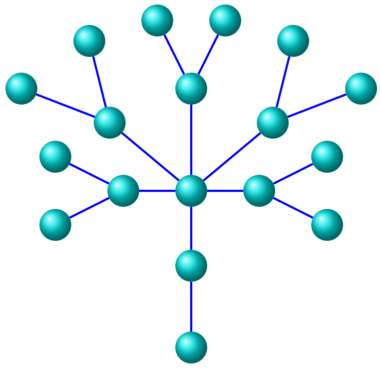


Figure 2.3: A tree.

The following theorem lists some properties of trees.

Theorem 2.3.2. *Let T be an arbitrary graph of order n. Then the following statements are equivalent:*

(i) *T is a tree;*

(ii) *T is a minimal connected graph, that is, T is connected and for any edge $uv \in E(T)$, $T - uv$ is a disconnected graph. (That is to say, T is connected and every edge in T is a so-called bridge.);*

(iii) *T is a maximal acyclic graph; that is, T is acyclic and for any non-adjacent vertices u and v in $V(T)$, $T + uv$ contains a cycle.*

Proof.
$(i) \Rightarrow (ii)$ & $(i) \Rightarrow (iii)$: Suppose T is a tree and $uv \in E(T)$, then the graph $T - uv$ does not contain a path $up_1p_2 \dots p_kv$, since T is acyclic. Therefore, removing the edge uv renders T disconnected. Hence, T is a minimal connected graph. Similarly, if we choose non-adjacent vertices u and v of T, then there is a path $up_1p_2 \dots p_kv$ in T and so, $T + uv$ will contain the cycle $up_1p_2 \dots p_kv$. Hence T is a maximal acyclic graph.

$(ii) \Rightarrow (i)$ & $(ii) \Rightarrow (iii)$: Let us suppose that T is a minimal connected graph. If T contains a cycle $up_1p_2 \dots p_kv$, then $T - uv$ is still connected which will contradict the minimal connectivity of T. Hence T is an acyclic graph which is a tree. Also, since $(i) \Rightarrow (iii)$ from the first part of the proof, then by transitivity $(ii) \Rightarrow (iii)$.

$(iii) \Rightarrow (i)$ & $(iii) \Rightarrow (ii)$: Suppose T is a maximal acyclic graph, then if the vertices u and v belong to different components, then $T + uv$ will not create a cycle $up_1p_2 \dots p_kv$, since otherwise the path $up_1p_2 \dots p_kv$ is in T. Hence T is a tree. Also, since $(i) \Rightarrow (ii)$ from the first part of the proof, then by transitivity $(iii) \Rightarrow (ii)$.

□

Definition 2.3.3. *A **forest** is an acyclic graph. In particular, a **forest** with one connected component is a tree.*

Corollary 2.3.4. *A tree T of order n has size $n - 1$; a forest F of order n with k components has size $n - k$.*

Proof.
We prove this corollary by induction on n. Suppose T is a tree of order $n = 1$, then $|E(T)| = 0$ and the corollary holds. Now assume that it holds for all trees of order less than n and let T be a tree of order n. Let $uv \in E(T)$, then $T - uv$ is a disconnected graph (since by Definition 2.3.1, T is acyclic) with two connected components T_1 and T_2 which are trees themselves with orders n_1 and n_2, respectively, where $n = n_1 + n_2$. Since n_1 and n_2 are both less than n, by the induction hypothesis, $|E(T_1)| = n_1 - 1$ and $|E(T_2)| = n_2 - 1$. Notice now that the edge set of T, $E(T)$, is a disjoint union of $E(T_1)$, $E(T_2)$ and $\{uv\}$. That is,

$$\begin{aligned} E(T) &= E(T_1) \cup E(T_2) \cup \{uv\} \\ |E(T)| &= |E(T_1)| + |E(T_2)| + |\{uv\}| \\ &= n_1 - 1 + n_2 - 1 + 1 \\ &= n_1 + n_2 - 1 \\ &= n - 1, \quad \text{since}\, n = n_1 + n_2. \end{aligned}$$

This completes the first part of the corollary.

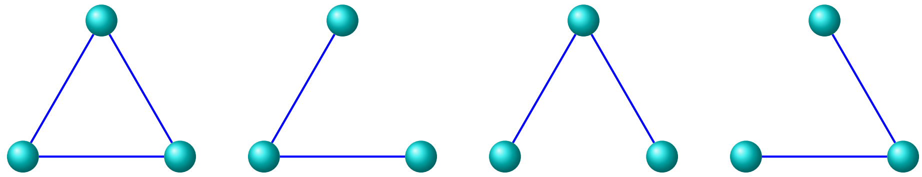
Now let F be a forest of order n with k components, say $F = \{T_1, T_2, \ldots, T_k\}$. Then by Definition 2.3.1, each T_i, for $i = 1, 2, \ldots, k$ is a tree. Let n_i be the order of each tree T_i for $i = 1, 2, \ldots, k$. Then from the first part of the proof, $|E(T_i)| = n_i - 1$ for each T_i. The edge set of the forest F is a disjoint union of the edge sets of each tree, $E(T_i)$. That is,

$$\begin{aligned} E(F) &= \bigcup_{i=1}^{k} E(T_i) \\ |E(F)| &= |E(T_1)| + |E(T_2)| + \cdots + |E(T_k)| \\ &= n_1 - 1 + n_2 - 1 + \cdots + n_k - 1 \\ &= \sum_{i=1}^{k} n_i - \underbrace{(1 + 1 + \cdots + 1)}_{k\,\text{times}} \\ &= n - k. \end{aligned}$$

□

Definition 2.3.5. *A graph T is said to be a* ***subgraph*** *of a graph G if $V(T) \subseteq V(G)$ and $E(T) \subseteq E(G)$. In this case, we write $T \subseteq G$ and say: T is contained in G. If, however, T contains* ***all edges*** *of G that join two vertices in $V(T)$, then T is said to be the* ***subgraph induced*** *or by $V(T)$. In this case, we write $T = \langle V(T) \rangle$. T is called a spanning subgraph if $V(T) = V(G)$. In particular, T is a* ***spanning tree*** *of G if T is a tree and $V(T) = V(G)$.*

For example, Figure 2.4 below shows the *complete graph* (i.e., a graph with every vertex adjacent to all other vertices) K_3 and its spanning trees.



(a) K_3.

Figure 2.4: The complete graph K_3 and its spanning trees.

Subgraphs of a graph $G = (V, E)$ can be obtained by deleting edges and vertices of G. If $W \subset V$, then we denote by $G \setminus W$ the subgraph of G obtained by deleting the vertices in W and *all the edges incident with them*. In addition, W is called a *vertex cut set* or simply *cut set* if $G \setminus W$ is disconnected and we call a vertex that forms a cut set on its own a *cut vertex*. Similarly, if $F \subset E$, then $G \setminus F = (V, E \setminus F)$ is called a *spanning subgraph* of G obtained by *deleting all the edges* in F. Observe that every simple graph is a spanning subgraph of a complete graph. In addition, F is called an *edge cut set* if $G - F$ is disconnected and we call such an edge that forms an edge cut set on its own a *cut edge* or a *bridge*. If $W = \{w\}$ and $F = \{uv\}$ (i.e., they are singletons), then the notation is simplified to $G - w$ and $G - uv$, respectively.

The *connectivity* of G, denoted $\kappa(G)$, is the minimum size of a cut set of G. A graph is k-connected if $k \leq \kappa(G)$. If G is connected but not complete of order n then $1 \leq \kappa(G) \leq n-2$. If G is disconnected then $\kappa(G) = 0$, and if G is complete, $\kappa(G) = n-1$.

Definition 2.3.6. *Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We say G is **isomorphic** to G' and write $G \cong G'$ if there exists a bijection $\varphi : V(G) \to V(G')$ such that $uv \in E \Leftrightarrow \varphi(u)\varphi(v) \in E'$ for all $u, v \in V$. We call such a map φ an **isomorphism**; if $G = G'$ then φ is called an **automorphism**.*

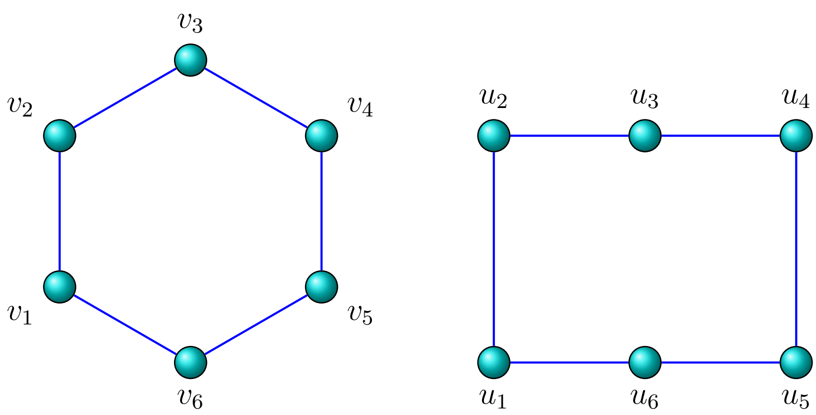


Figure 2.5: Isomorphic graphs.

2.4 Graphs and Matrices

In this section, we define some important matrices which will aid in our study. For a graph $G = (V, E)$, we denote v_i for the i-th vertex and e_j for the j-th edge.

Definition 2.4.1. *The **adjacency matrix** of a graph G of order n, denoted by A, is the $n \times n$ matrix whose (i, j)-th entry is the number of edges joining vertex i to vertex j. If G is a simple graph i.e., $E(G) \subseteq \{\{u, v\}|\ u, v \in V(G), u \neq v\}$ then the (i, j)-th entry of A is defined as*

$$[\mathrm{A}]_{i,j} := \begin{cases} 1 & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

If G is an oriented graph, then the elements of the edge set are ordered pairs of vertices and the (i, j)-th entry of A is defined as

$$[\mathrm{A}]_{i,j} := \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G), \\ -1 & \text{if } (v_j, v_i) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.4.2. *The **incidence matrix** of a graph G of order $n = |V(G)|$ and size $m = |E(G)|$, denoted by M, is the $n \times m$ matrix whose (i, j)-th entry is defined as*

$$[\mathrm{M}]_{i,j} := \begin{cases} 1 & \text{if } v_i \text{ is incident to } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

If G is an oriented graph, then the (i, j)-th entry of M is defined as

$$[\mathrm{M}]_{i,j} := \begin{cases} 1 & \text{if } v_i \text{ is the tail of } e_j, \\ -1 & \text{if } v_i \text{ is the head of } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.4.3. *The **degree matrix** of a graph G of order n is the $n \times n$ diagonal matrix D whose (i, j)-th entry is defined as*

$$[\mathrm{D}]_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.4.4. *Let G be a simple graph with adjacency matrix A and degree matrix D. The **Laplacian matrix** L is defined as*

$$\mathrm{L} = \mathrm{D} - \mathrm{A},$$

with entries defined by

$$[\mathrm{L}]_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j, \\ -1 & \text{if } i \neq j \text{ and } v_i v_j \in E, \\ 0 & \text{if } i \neq j \text{ and } v_i v_j \notin E. \end{cases}$$

Definition 2.4.5. *Given a matrix* N*, the* (i, j)*-th* ***cofactor*** *of* N*, denoted by* C_{ij}*, is defined to be*

$$C_{ij} := (-1)^{i+j} \det(\mathrm{N}(i|j)),$$

where $\det(\mathrm{N}(i|j))$ *is the determinant of the sub-matrix formed by deleting row* i *and column* j *of* N*.*

2.5 Counting Spanning Trees

The most important method of counting spanning trees dates back to the German physicist, Gustav Kirchhoff in his study of electrical networks in 1847. He was the first to count the number of spanning trees given any labelled graph [17]. This is the famous matrix-tree theorem which demonstrates a wonderful connection between spanning trees and some associated matrices defined in Section 2.4.

Theorem 2.5.1 (Matrix Tree Theorem). *Given any connected labelled graph* G *with adjacency matrix* A *and degree matrix* D*, the number of spanning trees of* G*, denoted by* $\tau(G)$*, is given by*

$$\tau(G) = C_{ij} := (-1)^{i+j} \det(\mathrm{L}(i|j)) \; for \; i, j = 1, \ldots, n. \tag{2.5.2}$$

Proof. (Sketch)
Let G be an arbitrary connected labelled graph of order n and size m with adjacency matrix A, degree matrix D, and Laplacian matrix L. The following steps gives an outline of the proof of the matrix tree theorem:

(a) Suppose G to be an oriented graph with incidence matrix N. Then the matrix product, $\mathrm{NN^T}$ equals the Laplacian matrix L of G.

(b) Now let $T \subseteq G$ with n vertices and $n-1$ edges and r be an arbitrary integer such that $1 \leq r \leq n$, and let N' be the $(n-1) \times (n-1)$ sub-matrix formed by deleting row r and all the columns that do not correspond to edges in T. If T is a tree, then $|\det(\mathrm{N}')| = 1$, otherwise $\det(\mathrm{N}') = 0$.

(c) Now we investigate the cofactors of the Laplacian matrix $\mathrm{NN^T} = \mathrm{L}$. Since $\mathrm{rank(L)} \leq n-1$, we consider two cases:

   (i) Case I: If $\mathrm{rank(L)} < n-1$, then all $(n-1)\times(n-1)$ submatrices have determinant zero.

   (ii) Case II: If $\mathrm{rank(L)} = n-1$, then $\det(\ker \mathrm{L}) = 1$ and $\ker \mathrm{L} = \langle [1, 1, \ldots, 1]^{\mathrm{T}} \rangle$ since all row sums are 0. Therefore the adjoint of the Laplacian matrix adj(L) must be of the form

$$\mathrm{adj(L)} = c \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \text{ (where } c \text{ is a constant)},$$

which implies that the cofactors of L are all the same. That is

$$\begin{aligned} C_{ij} &= (-1)^{(i+j)} \det(\mathrm{NN}^{\mathrm{T}}(i|j)) \\ &= (-1)^{(i+j)} \det(\mathrm{N}_i \mathrm{N}_j^{\mathrm{T}}). \end{aligned}$$

(d) Since all cofactors are the same, we choose $i = j = 1$. Using the Cauchy-Binet formula, we partition the determinant of $\mathrm{N}_1\mathrm{N}_1^{\mathrm{T}}$ into singular and non-singular matrices. And so we have

$$C_{11} = \sum_{\mathrm{N}_1 \text{ non-singular}} \det(\mathrm{N}_1[[m]|\mathrm{K}])^2 + \sum_{\mathrm{N}_1 \text{ singular}} \det(\mathrm{N}_1[[m]|\mathrm{K}])^2.$$

Step (b) tells us that all $(n-1) \times (n-1)$ submatrices that correspond to a spanning tree of G contribute ± 1 to the sum (i.e., all non-singular submatrices), while all the others (i.e., singular submatrices) contribute 0. Hence,

$$C_{11} = \sum_{\mathrm{N}_1 \text{ non-singular}} 1 = \tau(G).$$

And since all cofactors are the same,

$$C_{ij} = (-1)^{i+j} \det[\mathrm{NN}^{\mathrm{T}}(i|j)] = \tau(G).$$

A detailed proof of the matrix tree theorem can be found in [15] and [21]. □

Next, we show a relation between the number of spanning trees and eigenvalues of the Laplacian matrix L.

Corollary 2.5.3. *Let G be a connected labelled graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, and let* L *be the Laplacian matrix of G. Then the number of spanning trees of G, denoted by $\tau(G)$, is given by*

$$\tau(G) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i,$$

where the λ_i $(i = 1, \ldots, n-1)$ are the non-zero eigenvalues of L.

Proof.
Let L be the Laplacian matrix of the graph G. The characteristic polynomial of L is

$$\begin{aligned} P(\lambda) &= \det(\mathrm{I}\lambda - \mathrm{L}) \\ &= \lambda^n - c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} - \cdots \pm c_1\lambda \mp c_0, \end{aligned} \tag{2.5.4}$$

where the coefficient c_k is the sum of all determinants obtained from L by removing k rows and the associated columns [19]. In particular, c_1 is the sum of the cofactors corresponding to the main diagonal of L. And since all cofactors of L are the same, namely the number $\tau(G)$ of spanning trees of G, this implies that

$$c_1 = n \times \tau(G). \tag{2.5.5}$$

From equation (2.5.4),

$$\begin{aligned} P(\lambda) &= (\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n), \qquad \left[\lambda_n = 0,\ \text{since}\det(\mathrm{L}) = 0 = \prod_{i=1}^{n}\lambda_i\right] \\ &= \lambda(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_{n-1}) \\ &= \lambda^n - \lambda^{n-1}\sum_{i=1}^{n-1}\lambda_i + \cdots \pm \lambda\prod_{i=1}^{n-1}\lambda_i. \end{aligned} \tag{2.5.6}$$

Comparing equations (2.5.6), (2.5.5) and (2.5.4),

$$c_1 = n \times \tau(G) = \prod_{i=1}^{n-1}\lambda_i,$$

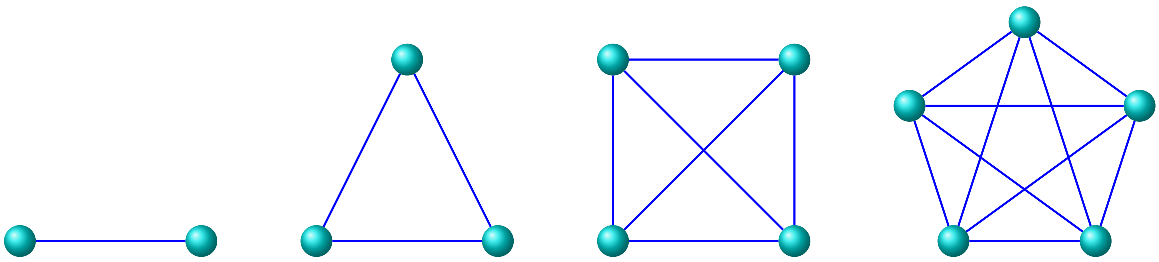
$$\tau(G) = \frac{1}{n}\prod_{i=1}^{n-1}\lambda_i.$$

□

Next, we apply the Matrix Tree Theorem 2.5.1 and Corollary 2.5.3 to obtain Arthur Cayley's explicit formula for counting the number of spanning trees of a complete graph.

2.5.1 The Matrix Tree Theorem on Complete Graphs

Definition 2.5.7. *A graph G is said to be **complete** if every vertex is adjacent to all other vertices. A complete graph of order n is denoted by K_n, and it is an $(n-1)$-regular graph. That is, for every vertex $v \in V(K_n)$,* $\deg(v) = n-1$.



(a) K_2. (b) K_3. (c) K_4. (d) K_5.

Figure 2.6: Complete graphs.

Given a complete graph K_n, each vertex is adjacent to all other vertices and each vertex has degree $n-1$. So by Definition 2.4.4 the Laplacian matrix L for K_n is given by

$$\mathrm{L} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{array}\begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & \cdots & v_n \end{array} \\ \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix} \end{array}.$$

Let L_1 be the $(n-1) \times (n-1)$ matrix obtained from L by removing the first row and the first column of L. Then,

$$L_1 = \begin{array}{c} \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{array} \begin{array}{c} \begin{array}{cccc} v_2 & v_3 & \cdots & v_n \end{array} \\ \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix} \end{array}.$$

We now find the eigenvalues of L_1:

$$L_1 - nI = \begin{array}{c} \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{array} \begin{array}{c} \begin{array}{cccc} v_2 & v_3 & \cdots & v_n \end{array} \\ \begin{bmatrix} -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 \end{bmatrix} \end{array}.$$

$L_1 - nI$ is a rank one matrix, so n is an eigenvalue with multiplicity $\geq \det(\ker(L_1 - nI)) = (n-1) - 1 = n-2$. Also, we know that

$$\operatorname{tr}(L_1) = (n-1)(n-1) = (n-1)^2 = \sum_{i=1}^{n-1} \lambda_i,$$

so the $(n-1)^{\text{th}}$ eigenvalue, λ_{n-1}, is

$$\begin{aligned} \lambda_{n-1} &= \operatorname{tr}(L_1) - (\text{all the other eigenvalues}) \\ &= \sum_{i=1}^{n-1} \lambda_i - \sum_{i=1}^{n-2} \lambda_i \\ &= (n-1)^2 - n(n-2) \\ &= 1. \end{aligned}$$

Therefore, the number of spanning trees of the complete graph K_n of order n is

$$\tau(K_n) = \det(L_1) = \prod_{i=1}^{n-1} \lambda_i = n^{n-2} \cdot 1 = n^{n-2},$$

which was first proved by Arthur Cayley in 1889 [8].

2.6 Planar Graphs

In studying enumeration problems on lattices, it is quite natural to study planar graphs since most lattices can be regarded as planar graphs. Before this is done, we will need some ideas and tools from topology, which are stated without proofs.

2.6.1 Topological Prerequisites

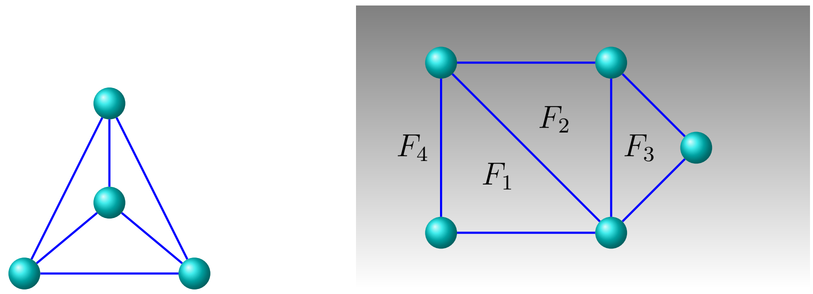
We now give some topological ideas relevant in the study of planar graphs. A *straight line segment* in the Euclidean plane is a subset of $\mathbb{R}^2$ that has the form $\{p + \lambda(p - q)\}$ for distinct points $p, q \in \mathbb{R}$. A *polygon* P is a subset of $\mathbb{R}^2$, which is the union of finitely many straight line segments and is homeomorphic to the unit circle, S^1, the set of points $\{(x, y) \in \mathbb{R}^2 | \ x^2 + y^2 = 1\}$. A *polygonal arc* or an *arc* is a subset of $\mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the closed unit interval $[0, 1]$. The images of 0 and 1 under such a homeomorphism are the *endpoints* of this polygonal arc. If P is an arc with endpoints x and y then we denote the point set $P \setminus \{x, y\}$, the *interior* of P by P°.[12]

Let $O \subseteq \mathbb{R}^2$ be an open set. Being linked by an arc defines an equivalence relation on O and the corresponding equivalence classes are themselves open; they are called the *regions* of O. A closed set $X \subseteq \mathbb{R}^2$ is said to separate O if $O \setminus X$ has more than one region [12]. The set of points $\{y \in \mathbb{R}^2 | \forall \ U \in \mathcal{U}_y, \ U \cap X \neq \emptyset, \ U \cap (\mathbb{R}^2 \setminus X) \neq \emptyset\}$ is called the *boundary* of the set $X \subset \mathbb{R}^2$ where $\mathcal{U}_y = \{U \subseteq \mathbb{R}^2 | \ U$ is a neighbourhood of $y\}$.

Theorem 2.6.1 (Jordan Curve Theorem for Polygons). *For every polygon $P \subseteq \mathbb{R}^2$, the set $\mathbb{R}^2 \setminus P$ has exactly two regions, of which only one is bounded. And each of the two regions has the entire P as its boundary.*

Proof.
We refer the reader to [20] and [25] for a detailed proof of the theorem. □

Definition 2.6.2. *A graph G is said to be* ***planar*** *if it can be drawn in the plane with vertices represented by points and edges represented by continuous curves, such that pairs of edges intersect only at vertices. A drawing of a planar graph G in the plane such that edges intersect only at the vertices is called a* ***planar embedding*** *or a* ***planar representation*** *of the graph G.*



(a) Planar embedding of K_4.

(b) Planar embedding of a graph of order 5.

Figure 2.7: Some examples of planar graphs.

A planar embedding of a graph G divides the plane into regions that are called faces. For example, in Figure 2.7b, the faces are F_1, F_2, F_3, F_4. We denote a planar graph G by $G = (V, E, F)$, where V is the vertex set, E is the edge set and F is the set of regions or faces.

Definition 2.6.3. *Let G be a planar graph and F a face in G. Then the* ***bound degree*** *of F denoted by $b(F)$ is the number of edges that bounds the face F.*

In Figure 2.7b, $b(F_1) = b(F_2) = b(F_3) = 3, b(F_4) = 5$.

Theorem 2.6.4 (Euler's Formula). *Let G be a connected planar graph with n vertices, m edges and f faces. Then*

$$n - m + f = 2. \tag{2.6.5}$$

Proof.
We prove the theorem by induction on the number of edges. For $m = 0, G$ must be a single vertex, such that $n = 1, m = 0$ and $f = 1$, so the formula (2.6.5) holds.

Now we assume the theorem holds for all planar graphs with $|E| \leq m-1$ and assume $|E| = m$. Then there are two cases to consider:

- Case 1: Suppose G is a tree, then by Corollary 2.3.4, $m = n - 1$ and $f = 1$, since the planar representation of any tree has only one region. Thus $n - m + f = n - (n-1) + 1 = 2$, and the result holds.

- Case 2: Suppose G is not a tree, then G contains at least one cycle, say C. Choose this cycle and let e be an edge in this cycle. Comparing the two graphs $G - e$ and G, we see that $|V(G)| = |V(G-e)| = n$, and also the graph $G-e$ has one edge less than the graph G. Since the two regions that are separated by e in G are merged when e is removed, $G-e$ has one face less than G. We can now apply the induction hypothesis on $G - e$,

$$n - (m-1) + (f-1) = 2 \ \Rightarrow \ n - m + f = 2.$$

This completes the proof. □

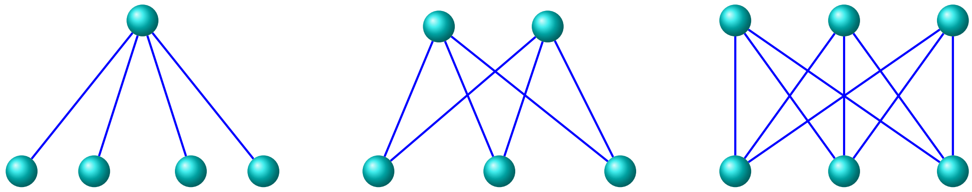
We now use Euler's formula (2.6.5) to establish that certain graphs are non-planar.

Definition 2.6.6. *A graph G is* ***bipartite*** *if its vertex set can be partitioned into two disjoint sets X and Y such that every edge in G has one end vertex in X and the other in Y. The sets X and Y are said to be* ***partite sets.***

Definition 2.6.7. *A bipartite graph with partite sets X and Y is called a* ***complete bipartite graph*** *if its edge set is of the form*

$$E = \{xy \mid x \in X, y \in Y\},$$

that is, if every possible connection of a vertex of X with a vertex of Y is present in the graph. Such a graph is denoted by $K_{|X|,|Y|}$.[15]



(a) $K_{1,4}$ (b) $K_{2,3}$ (c) $K_{3,3}$

Figure 2.8: Complete bipartite graphs

Theorem 2.6.8. *The complete bipartite graph $K_{3,3}$ is nonplanar.*

Proof.
We prove this by contradiction. Suppose $K_{3,3}$ is planar, then by definition it has a planar embedding. The planar embedding has $n = 3$ vertices, $m = 9$ edges, and by Euler's formula, it must have 5 faces, since $K_{3,3}$ is connected. Let the set of faces be $F = \{F_i | i = 1, \dots, 5\}$, and consider the sum

$$C = \sum_{F_i \in F} b(F_i),$$

running over all regions of G. Now as G is planar, every edge $e \in G$ bounds at most two regions in G, and so $C \leq 2m = 18$. On the other hand, each region of G has at least 4 edges (since G is bipartite, so it has no triangles), so we have $C \geq 4f = 20$. We have reached a contradiction thus $K_{3,3}$ is nonplanar. □

Theorem 2.6.9. *If G is a planar graph with $n \geq 3$ vertices and m edges, then $m \leq 3n-6$. Furthermore, if $m = 3n-6$, then $b(F) = 3$ for all faces F, and G is said to be a maximal planar graph (i.e., no edge can be added without losing the planarity) and the embedding is called a triangulation.*

Proof.
Let the set of faces be $F = \{F_i | i = 1, \dots, f\}$. Now consider the sum

$$C = \sum_{F_i \in F} b(F_i),$$

running over all faces of G. We saw in the proof of Theorem 2.6.8 that $C \leq 2m$. Also since $n \geq 3$, each region must be bounded by at least 3 edges, so we have $C \geq 3f$. Combining the two equations with Euler's formula (2.6.5), we have

$$3f \leq 2m \Rightarrow 3(2 + m - n) \leq 2m \Rightarrow m \leq 3n - 6.$$

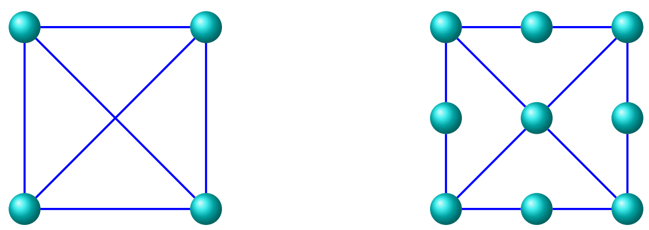
Clearly for equality to hold, we must have $3f = 2m$, which indicates that all faces are triangles, and this completes the proof. □

Theorem 2.6.10. *The complete graph K_5 is nonplanar.*

Proof.
The complete graph K_5 has 5 vertices by definition and 10 edges since this is equivalent to counting the number of pairs in a 5-element set, so $m = \binom{5}{2} = 10$. By Theorem 2.6.9, a planar graph of order 5 can have at most 9 edges. Hence K_5 is nonplanar. □

So far, we have proved that the graphs $K_{3,3}$ and K_5 are nonplanar. An obvious question one can ask is: how can we generally decide whether a graph is planar or not? The answer to this is due to Kuratowski who used a clever means to characterize planar graphs in 1930. But before we state his theorem, we use some of the concepts we have seen so far to add to the list of nonplanar graphs. Suppose G is a graph that contains either $K_{3,3}$ or K_5 as a subgraph; then G must be nonplanar since any planar embedding of G must contain a planar embedding of either $K_{3,3}$ or K_5. And we have seen from Theorems 2.6.8 and 2.6.10 that both $K_{3,3}$ and K_5 are nonplanar.

Definition 2.6.11. *Let G be a graph. A* ***subdivision*** *of an edge $e \in G$ results from replacing a path for e. We say that a graph H is a* ***subdivision*** *of G if H can be obtained from G by a finite sequence of subdivisions.*



(a) K_4.

(b) A subdivision of K_4.

Figure 2.9: The graph K_4 and a subdivision of K_4

Theorem 2.6.12. *A graph G is planar if and only if every subdivision of G is planar.*

Proof.
Essentially trivial (planar embeddings can be obtained by adding or removing vertices on the subdivided edges). □

From Theorem 2.6.12, next on our list of nonplanar graphs are all graphs containing a subdivision of either $K_{3,3}$ or K_5. Now, Kuratowski tells us that there are no other nonplanar graphs apart from these. So one can say that the list of nonplanar graphs mainly centres around two main graphs $K_{3,3}$ and K_5.

Theorem 2.6.13 (Kuratowski's Theorem). *A graph G is planar if and only if it contains no subdivision of $K_{3,3}$ or K_5.*

Proof. We refer the reader to [3] and [6], for the proof of this theorem. □

2.7 Counting on Planar Graphs

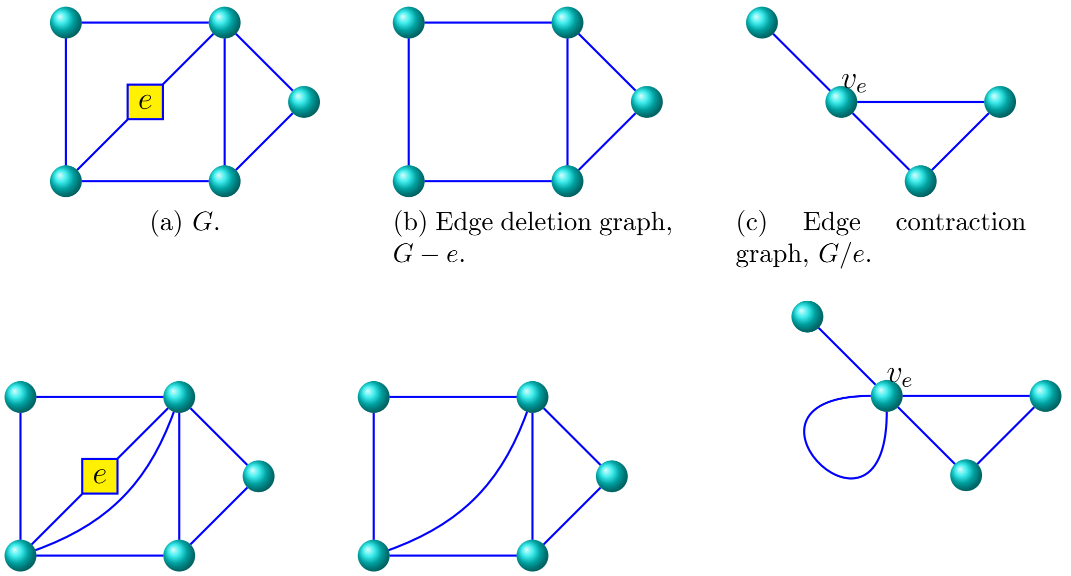
In this section, we count spanning trees on planar graphs. We first need some tools to help us in counting on planar graphs.

2.7.1 Deleting and Contraction

Let $G = (V, E)$ be a graph and $e \in E$. Recall from Section 2.3 that $G - e$ is the subgraph obtained by *deleting* or *removing* the edge e from E, so $G - e = (V, E \setminus e)$. We define the graph G/e obtained from G by *contracting* the edge e with end vertices, say x and y, into a new vertex v_e, such that v_e is adjacent to all the former neighbours of x and of y. If $G/e = (V', E')$ then the vertex set is $V' := (V \setminus \{x, y\}) \cup \{v_e\}$ (where v_e is the 'new' vertex, i.e., $v_e \notin V \cup E$) and the edge set

$$E' = \Big\{vw \in E \mid \{v, w\} \cap \{x, y\} = \emptyset\Big\} \cup \Big\{v_e w \mid xw \in E \setminus \{e\} \text{ or } yw \in E \setminus \{e\}\Big\}.$$

Figure 2.10 below shows the deletion and contraction of the edge e in a simple and multi-graph G of order 5.



(a) G.

(b) Edge deletion graph, $G - e$.

(c) Edge contraction graph, G/e.

(d) G.

(e) Edge deletion graph, $G - e$.

(f) Edge contraction graph, G/e.

Figure 2.10: Deleting and contracting of edge e in a simple graph and a multi graph

2.7.2 Duality

Let $G = (V, E, F)$ be a planar graph. We construct the *dual* planar graph $G^* = (V^*, E^*, F^*)$ as follows. For each face, $F_i \in G$, place a point v_i^* in the interior of F_i. The set of all points v_i^* for $i = 1, \ldots, f$ is the set of vertices of G^*. For any two vertices v_i^* and v_j^* with $i \neq j$ in V^*, if the faces F_i and F_j in G share an edge $e \in G$, then we draw an edge e^* called the *dual edge*, across e with v_i^* and v_j^* as end vertices. The dual edges form the edge set of G^*. Note that if e is a cut edge or a bridge of G, then $F_i = F_j$ so the dual edge e^* becomes a loop in G^* with multiple edges; conversely, if e is a loop in G, then the dual edge e^* is a cut edge or a bridge of G^*. This can always be done in such a way that the resulting dual graph G^* is again planar and always connected, since

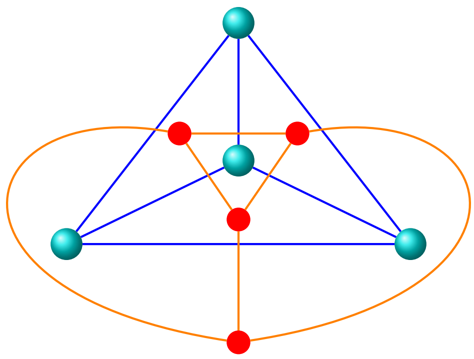


Figure 2.11: Planar embedding of K_4 and its dual planar graph K_4^*.

even if G is disconnected, the intersection of all the faces of the connected components of G is the exterior face, which shares a boundary with some faces in every connected component, so we can always find a path from the vertex associated with the exterior face to every other face. Figure 2.11 is a planar embedding of the complete graph K_4 and a dual planar graph K_4^*.

Note that $G \leftrightarrow G^*$ is an involution among connected planar graphs if we identify $V^* = F$, $E^* = E$ and $F^* = V$. The following facts should be clear from the terms defined in Section 2.2 and Definition 2.6.3:

(i) For the dual planar graph G^*, $\deg(v_i^*) = b(F_i)$ for $i = 1, \ldots, f$.

(ii) For the planar graph G, $\deg(v_j) = b(F_j^*)$ for $j = 1, \ldots, n$.

Theorem 2.7.1. *Let $G = (V, E, F)$ be a connected planar graph with an edge e such that e is not a loop or a cut edge. Then,*

$$(G - e)^* = G^*/e^*, \tag{2.7.2}$$

and

$$(G/e)^* = G^* - e^*. \tag{2.7.3}$$

Proof.
Let $G = (V, E, F)$ be a connected planar graph with $V = \{v_i \mid i \in [n]\}$, $E = \{e_j \mid j \in [m]\}$ and $F = \{f_k \mid k \in [q]\}$, and let $G^* = (V^*, E^*, F^*)$ be the dual planar graph of G^* with $V^* = \{f_i^* \mid i \in [q]\}$, $E^* = \{e_j^* \mid j \in [m]\}$ and $F^* = \{v_k^* \mid k \in [n]\}$. Pick $e_j \in E$ for some j such that e_j is neither a loop nor a cut edge of G. Then e_j is an edge boundary of two distinct faces, say f_1 and f_2 in G. Now deleting the edge e_j in G results in the amalgamation of the faces f_1 and f_2 into a single face, say f. Any face f_k, adjacent to f_1 or f_2 is now adjacent to the face f in $G - e$ and all other faces with their respective adjacencies are not affected by the deletion of the edge e_j from G. Correspondingly, in the dual $(G - e)^*$, we see that the two vertices, $f_1^*, f_2^* \in G^*$ corresponding to the faces f_1 and f_2 in G are now replaced by a single vertex f^* in $(G - e_j)^*$ and all other vertices f_i^* of G^* are vertices of $(G - e)^*$. Also, any vertex f_i^* with $i \in \{3, 4, \ldots, q\}$ in G^* adjacent to f_1^* and f_2^* is also adjacent to f^* in $(G - e_j)^*$ and adjacencies between vertices of $(G - e)^*$ other than the new vertex f^* are also adjacencies in G^*. But this whole process corresponds

to the contraction of the edge $e_j^* \in G^*$ with end vertices f_1^* and f_2^* to the new vertex f^*. Thus $(G-e)^* = G^*/e^*$ if e is neither a loop nor a cut edge. And this completes the first part of the proof.

For the second part of the proof we proceed as follows. Since G is connected, then $G \leftrightarrow G^*$ is an involution, thus the dual of the dual planar graph G is the original planar graph G. That is, $G^{**} = G$. Now note that since the edge $e \in G$ is not a loop, its corresponding dual edge $e^* \in G^*$ will not be a cut edge hence the graph $G^* - e^*$ is connected. So from the first part of the proof, equation (2.7.2) tells us that

$$(G^* - e^*)^* = G^{**}/e^{**} = G/e,$$

which completes the proof of the second statement. □

2.7.3 Graph Colouring

Definition 2.7.4. *Let G be a graph. A k-**vertex colouring** or simply a k-**colouring** of the graph G is a function $c : V(G) \to S$, where S is the set of k colours. If k is the smallest positive integer for which G has a k-**colouring**, $c : V(G) \to [k]$, with no two adjacent vertices assigned with the same colour, then the colouring c is called a **proper** k-**colouring** of G and k is said to be the **chromatic number** of G denoted by $\chi(G)$. If $\chi(G) = k$ then G is k-**chromatic** and if $\chi(G) \leq k$ then we say G is k-**colourable** .*

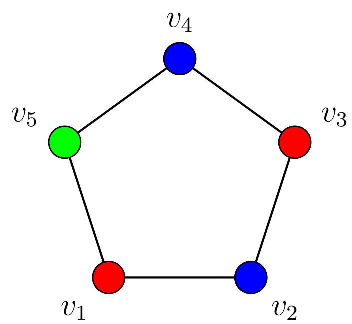


Figure 2.12: Vertex colouring of C_5.

One can consider a proper k-colouring of any graph G as a partition $\{V_1, V_2, \ldots, V_k\}$ of V into k *independent sets*, i.e., sets of vertices of G that are pairwise non-adjacent, called the *colour classes*. In Figure 2.12 above, we see that $\chi(C_5) = 3$.

Definition 2.7.5. *Let G be a graph. An **edge colouring** of the graph G is a function $c : E(G) \to S$, such that if $e, f \in E(G)$ are adjacent, then $c(e) \neq c(f)$. If k is the smallest positive integer for which G has an edge colouring, $c : V(G) \to [k]$, then k is the **edge chromatic number** or **chromatic index** of G denoted by $\chi^{'}(G)$.*

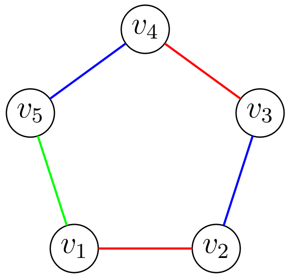


Figure 2.13: Edge colouring of C_5.

Definition 2.7.6. *Let G be a graph with edge set $E = \{e_1, e_2, \ldots, e_m\}$. We define the* ***line graph*** *$L(G)$ of G as the graph such that:*

- *every vertex $v_{e_i} \in L(G)$ represents an edge $e_i \in E$ for $i = 1, \ldots, m$; and*
- *two vertices $v_{e_i}, v_{e_j} \in L(G)$ are adjacent if and only if the edges e_i and e_j are adjacent in G for $i \neq j$ and $i, j = 1, \ldots, m$.*

By Definition 2.7.6, an edge colouring of a graph G can also be thought of as a vertex colouring of the line graph $L(G)$. And so, $\chi^{'}(G) = \chi(L(G))$.

Example 2.7.7.
Below is a list of the chromatic numbers for some classes of graphs.

(i) $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

(ii) $\chi(W_n) = \begin{cases} 3 & \text{if } n \text{ is even,} \\ 4 & \text{if } n \text{ is odd.} \end{cases}$

(iii) $\chi(P_n) = \begin{cases} 2 & \text{if } n \geq 2, \\ 1 & \text{if } n = 1. \end{cases}$

(iv) $\chi(K_n) = n$.

(v) $\chi(E_n) = 1$.

(vi) $\chi(K_{n,m}) = 2$.

2.7.4 Chromatic Polynomials

The chromatic polynomial was first defined for planar graphs only by George David Birkhoff in 1912 when he studied the Four Colour Problem (i.e., whether the countries on any given map can be coloured with four or fewer colours in such a way that adjacent countries are coloured differently). The chromatic polynomial provides a way of counting the number of distinct colourings of a graph. By distinct colouring, we mean that, if G is a graph with vertices labelled $v_1, v_2, \ldots, v_n$ then two colourings C_1 and C_2 are regarded as distinct if at least one vertex v_i is assigned a colour in C_1 different from the colour it is assigned in C_2. For example,

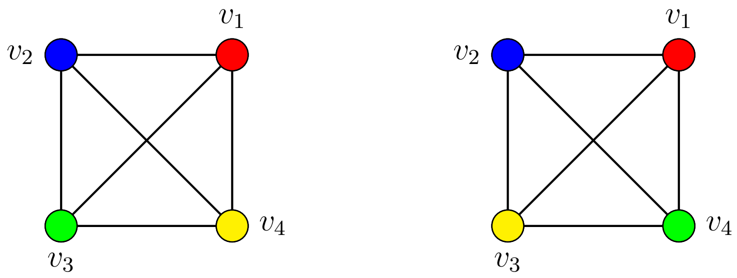


Figure 2.14: Two different colourings of K_4.

Definition 2.7.8. *The* ***chromatic polynomial*** *denoted by $P(G; k)$ is defined to be the number of different vertex-colourings of a graph G using at most k colours.*

For example, if the graph G is the empty graph E_n, then each vertex can be coloured in k ways since E_n is a disconnected graph. By definition, E_n has n vertices so, $P(E_n; k) = k^n$. Also for a complete graph K_n, since each vertex is adjacent to all the other vertices, each vertex must have a different colour; see for example Figure 2.14. The vertex v_1 can be coloured in k ways, vertex v_2 can be coloured in $k-1$ etc, and the last vertex v_n can be coloured in $k-n+1$. So we see that a colouring of K_n corresponds to an n-permutation of the colour set $\{1, 2, \ldots, k\}$. Thus we have

$$P(K_n; k) = \begin{cases} k(k-1)\cdots(k-n+1) = \prod_{i=0}^{n-1}(k-i) = k^{\underline{n}} & \text{if } k \geq n, \\ 0 & \text{if } k < n. \end{cases}$$

We call $k^{\underline{n}}$ the *falling factorials* of length n. We observe that a graph G is k-colourable if and only if $P(G; k) > 0$, or equivalently $P(G; k) > 0$ if and only if $\chi(G) \leq k$.

Theorem 2.7.9. *Let G be a graph and e be an edge such that e is not a loop. Then the chromatic polynomial of G is computed recursively as*

$$P(G; x) = P(G-e; x) - P(G/e; x). \tag{2.7.10}$$

Proof.
Let G be a graph and e be any edge in G with end vertices u and w. Now we consider how many x-colourings of $G-e$ are there supposing we colour u and w with the same colour. Let $c : V(G-e) \to [x]$ be such a colouring where $c(u) = c(w)$. Then, c can also be thought of as a colouring of the graph G/e with $c(u) = c(w) = c(v_e)$, where v_e is the new vertex formed by contracting the edge e. Similarly, any x-colouring $c^{'} : V(G/e) \to [x]$ of G/e can also be thought of as a colouring of $G-e$ with $c^{'}(v_e) = c^{'}(u) = c^{'}(w)$. Thus, there are $P(G/e; x)$ such x-colourings of the graph $G-e$.

Now we consider how many x-colourings of $G-e$ are there supposing the vertices u and w are assigned with different colours. Let $\sigma : V(G-e) \to [x]$ be such a colouring where $\sigma(u) \neq \sigma(w)$. Then σ can also be thought as a colouring of G with $\sigma(u) \neq \sigma(w)$. Similarly, any x-colouring, $\sigma^{'} : V(G) \to [x]$ of G can also be thought as a colouring of $G-e$ with $\sigma^{'}(u) \neq \sigma^{'}(w)$. Thus, there are $P(G; x)$ x-colourings of the graph $G-e$.

Hence the total number of x-colourings of the graph $G-e$ is

$$P(G-e; x) = P(G/e; x) + P(G; x) \Rightarrow P(G; x) = P(G-e; x) - P(G/e; x),$$

which is indeed a polynomial and this completes the proof. □

Remark 2.7.11.

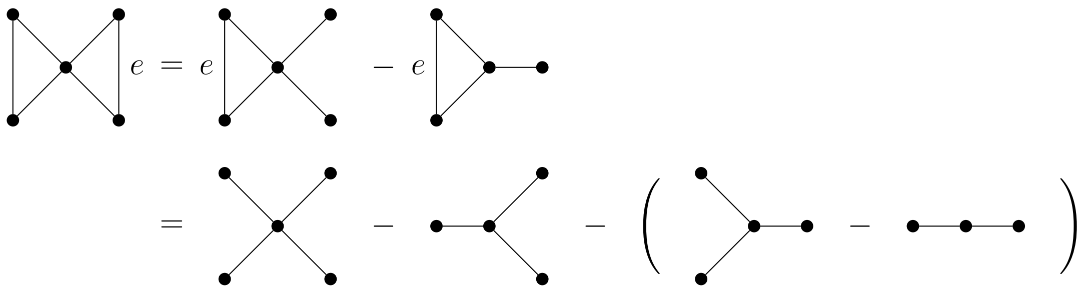
(a) The chromatic polynomials can be used in two ways:

  (i) A repeated application of the recursion $P(G;x) = P(G-e;x) - P(G/e;x)$ expresses $P(G;x)$ as an integer linear combination of chromatic polynomials of empty graphs.

  (ii) A repeated application of the recursion $P(G-e;x) = P(G/e;x) + P(G;x)$ expresses $P(G;x)$ as an integer linear combination of chromatic polynomials of complete graphs.

(b) Using induction, one can show that if T is a tree of order n, then $P(T;x) = x(x-1)^{n-1}$.

Example 2.7.12.
We determine the chromatic polynomial $P(G;x)$ for the graph G below by a "graphical" approach, singling out the edge e at each stage.



Since all the graphs are trees, we use $P(T;x) = x(x-1)^{n-1}$

$$
\begin{aligned}
&= x(x-1)^4 - x(x-1)^3 - x(x-1)^3 + x(x-1)^2 \\
&= x(x-1)^4 - 2x(x-1)^3 + x(x-1)^2 \\
&= x(x-1)^2[(x-1)^2 - 2(x-1) + 1] \\
&= x(x-1)^2(x-2)^2.
\end{aligned}
$$

Thus the graph G has $P(G;3) = 12$ 3-colourings and $P(G;4) = 144$ 4-colourings.

The next theorem by Whitney [27] gives some properties of the chromatic polynomial of any given graph.

Theorem 2.7.13 (Whitney). *Let $G = (V,E)$ be a graph of order n with no loops. Then the chromatic polynomial $P(G;x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ in x with $a_i \in \mathbb{Z}$ for $i = 0,1,\ldots,n$, has the following properties:*

(i) $\deg(P(G;x)) = n$,

(ii) the coefficients $a_0, a_1, \ldots, a_{n-\kappa(G)}$ are non-zero alternating in sign with $a_0 = 1, |a_1| = e(G)$ and $a_i = 0$ for $i > n - c(G)$, where $c(G)$ is the number of connected components of G.

Proof.
We refer the reader to [27] or [26] for the proof of the theorem. □

2.7.5 Tutte Polynomials

The Tutte polynomial $T(G;x,y) = \sum_{i,j} t_{ij}x^i y^j$, is a generalization of the chromatic polynomial defined as a function in two variables with non-negative integral coefficients.

Definition 2.7.14. *The **Tutte polynomial** of a graph $G = (V,E)$ denoted by $T(G;x,y)$ is defined recursively as follows:*

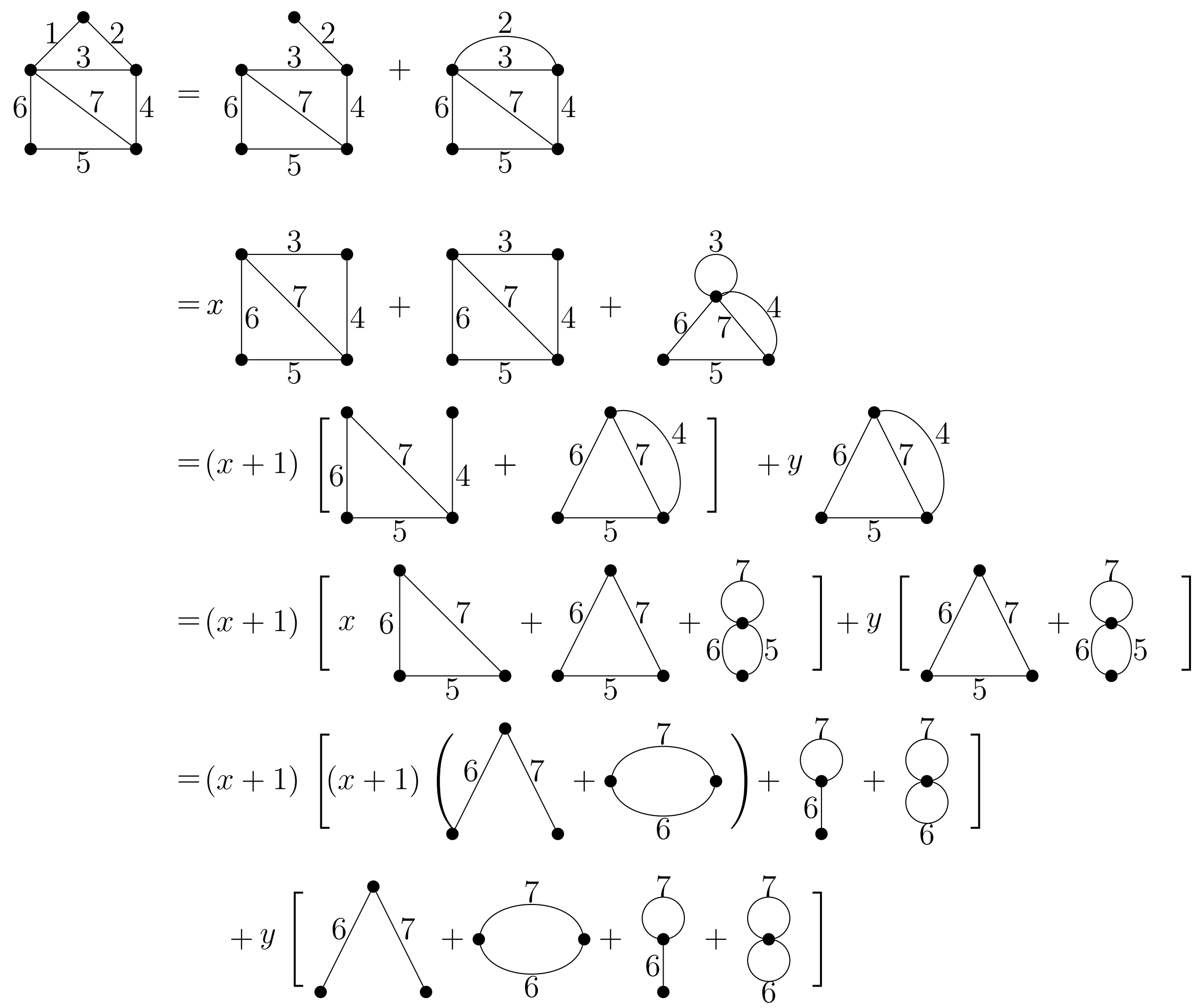
$$T(G;x,y) = \begin{cases} xT(G/e;x,y) & \textit{if } e \textit{ is a bridge,} \\ yT(G-e;x,y) & \textit{if } e \textit{ is a loop,} \\ T(G-e;x,y) + T(G/e;x,y) & \textit{if } e \textit{ is neither a bridge nor a loop,} \end{cases} \tag{2.7.15}$$

together with the condition

$$T(E_n) = 1, \tag{2.7.16}$$

where E_n is the edgeless graph with n vertices.

Example 2.7.17.
We compute the Tutte polynomial of a graph G by deleting the edges in the order $1,2,\ldots,7$.



$$
\begin{aligned}
&= (x+1)\ \left[(x+1)\ (x^2+x+y)+xy+y^2\right]+y(x^2+x+y+xy+y^2)\\
&=\ (x+1)^2(x^2+x+y)+(x+1)(xy+y^2)+y(x^2+x+y+xy+y^2)\\
&=\ x^4+3x^3+y^3+3x^2+2y^2+3x^2y+2xy^2+4xy+x+y.
\end{aligned}
$$

Remark 2.7.18.
If G is a union of disjoint graphs, $G_1, G_2, \ldots, G_k$, i.e., $G = \bigcup_{i=1}^{k} G_i$, then the Tutte polynomial satisfies

$$T(G;x,y)=\prod_{i=1}^{k} T(G_i;x,y).$$

Theorem 2.7.19. *Let G be a planar graph with G^* as its dual planar representation. Then for the Tutte polynomial, we have*

$$T(G^*;x,y)=T(G;y,x). \tag{2.7.20}$$

Proof.
Let G be a planar graph with G^* as it dual planar representation. We need to verify if equation (2.7.20) satisfies Definition 2.7.14. We consider the following cases:

(i) If G is the edgeless graph E_n then clearly, $T(G^*;x,y)=T(G;y,x)=1$.

(ii) If e^* is a bridge in G^* then e is a loop in G hence we have

$$\begin{aligned} T(G^*;x,y) &= xT(G^*/e^*;x,y) \\ &= xT((G-e)^*;x,y) \quad \text{(by equation (2.7.2)} \\ &= xT((G-e);y,x) \quad \text{(taking the dual)} \\ &= xT(G;y,x). \end{aligned}$$

Similarly if e^* is a loop in G^* then e is a bridge in G and we have

$$\begin{aligned} T(G^*;x,y) &= yT(G^*-e^*;x,y) \\ &= yT((G/e)^*;x,y) \quad \text{(by equation (2.7.3)} \\ &= yT(G/e;y,x) \quad \text{(taking the dual)} \\ &= yT(G;y,x). \end{aligned}$$

(iii) If e^* is neither a bridge nor a loop in G^*, then e is also neither a bridge nor a loop in G. Thus we have

$$\begin{aligned} T(G^*;x,y) &= T(G^*-e^*;x,y) + T(G^*/e^*;x,y) \\ &= T((G/e)^*;x,y) + T((G-e)^*;y,x) \quad \text{(by equations (2.7.2) and (2.7.3))} \\ &= T((G/e);y,x) + T(G-e;x,y) \quad \text{(taking the dual)} \\ &= T(G;y,x). \end{aligned}$$

This completes the proof. □

Definition 2.7.21. *Let $G = (V,E)$ be a graph and let $\mathcal{A} = \mathcal{P}(E)$ be the power set of E, i.e., $\mathcal{A} = \{A_i \subseteq E \mid i \in [2^{e(G)}]\}$. We denote by $G_{\mathcal{A}} = \{G_{A_i} = (V,A_i) \mid A_i \in \mathcal{A},\ i \in [2^{e(G)}]\}$ the set of all spanning subgraphs of G. Then for each $A_i \in \mathcal{A}$, we define the **rank** of the graph (V,A_i) by*

$$r(A_i) = |V| - c(G_{A_i}) \tag{2.7.22}$$

where $c(G_{A_i})$ is the number of connected components of the graph $G_{A_i} = (V,A_i)$.

From Definition 2.7.21 we see that $0 \le r(A_i) \le |A_i|$ with

$$\begin{aligned} &r(A_i) = 0 \Leftrightarrow A_i = \emptyset, \\ &r(A_i) = |A_i| \Leftrightarrow G_{A_i} \text{ is a forest.} \end{aligned} \tag{2.7.23}$$

Also, if $A \subseteq B$, then $r(A) \le r(B)$ and

$$r(B) = r(E) \Leftrightarrow c(G_B) = c(G). \tag{2.7.24}$$

Lastly, we set $r(G) = r(E) = |V| - c(G)$.

Definition 2.7.25. *Let $G = (V,E)$ be a graph. Then the **Whitney rank generating function** of G is defined as the two variable polynomial*

$$R(G;u,v) = \sum_{A \subseteq E} u^{(r(G)-r(A))} v^{(|A|-r(A))}. \tag{2.7.26}$$

Theorem 2.7.27. *Let $G = (V, E)$ be a graph. Then the Tutte polynomial $T(G; x, y)$ is uniquely defined, with*

$$T(G; x, y) = R(G; x-1, y-1) = \sum_{A \subseteq E} (x-1)^{(r(G)-r(A))}(y-1)^{(|A|-r(A))}. \qquad (2.7.28)$$

Proof.
In equation (2.7.26), we set $u = x - 1$ and $v = y - 1$. Then it suffices to show that the following identities hold for $R(G; u; v)$ with $x = u + 1$ and $y = v + 1$:

$$R(G; u, v) = \begin{cases} 1 & \text{if } E = \emptyset, \\ (u+1)R(G/e; u, v) & \text{if } e \text{ is a bridge,} \\ (v+1)R(G-e; u, v) & \text{if } e \text{ is a loop,} \\ R(G-e; u, v) + R(G/e; u, v) & \text{if } e \text{ is neither a bridge nor a loop.} \end{cases}$$

Case 1:
If $E = \emptyset$, then $\mathcal{P}(E) = \{\emptyset\}$ and so $A = E$ hence $c(A) = c(G) = v(G) \Rightarrow r(G) = r(A) = 0$. Furthermore, $|A| = 0$ and thus

$$\begin{aligned} R(G; u, v) &= \sum_{A \subseteq E} u^{(r(G)-r(A))} v^{(|A|-r(A))} \\ &= \sum_{A \subseteq E} 1 \cdot 1 \\ &= 1. \end{aligned}$$

For cases 2 and 3 we check how the rank changes upon edge deletion and edge contraction. Let $e \in A$ and define r' and r'' to be the rank of $A \setminus e \subset E$ in $G - e$ and G/e respectively.

Case 2:
If e is a bridge, then $c(G-e) = c(G)+1$ and so from equation (2.7.22), $r''(G/e) = r(G)-1$. Also, $c(G_A - e) = c(G_A) + 1$ and so

$$r''(A \setminus e) = \begin{cases} r(A) - 1 & \text{if } e \in A, \\ r(A) & \text{if } e \notin A. \end{cases} \qquad (2.7.29)$$

Now consider the sum in equation (2.7.26) over all subsets of E, $\mathcal{P}(E)$. Then we can partition the members of $\mathcal{P}(E)$ whether they do or do not contain the cut edge e. Thus we have

$$\begin{aligned} R(G; u, v) &= \sum_{A \subseteq E} u^{(r(G)-r(A))} v^{(|A|-r(A))} \\ &= \sum_{A \subseteq E \setminus e} u^{(r(G)-r(A))} v^{(|A|-r(A))} + \sum_{A : e \in A} u^{(r(G)-r(A))} v^{(|A|-r(A))} \\ &= \sum_{A \subseteq E \setminus e} u^{(r''(G/e)-r''(A)+1)} v^{(|A|-r''(A))} + \sum_{A : e \in A} u^{(r''(G/e)+1-(r''(A \setminus e)+1))} v^{(|A|-(r''(A \setminus e)+1))} \\ &= u \sum_{A \subseteq E \setminus e} u^{(r''(G/e)-r''(A))} v^{(|A|-r''(A))} + \sum_{B = A \setminus e} u^{(r''(G/e)-r''(B))} v^{(|B|-r''(B))} \\ &= (u+1)R(G/e; u, v). \end{aligned}$$

Case 3:
If e is a loop then $c(G-e)=c(G)$ and so $r^{'}(G\setminus e)=r(G)$. Also, $c(G_A-e)=c(G_A)$ and so,

$$r^{'}(A\setminus e)=\begin{cases} r(A)-1 & \text{if } e\in A,\\ r(A) & \text{if } e\notin A.\end{cases}$$

Again we can partition the members of $\mathcal{P}(E)$ by whether they do or do not contain the loop e. Thus we have

$$\begin{aligned} R(G;u,v) &= \sum_{A\subseteq E} u^{(r(G)-r(A))}v^{(|A|-r(A))}\\ &= \sum_{A\subseteq E\setminus e} u^{(r(G)-r(A))}v^{(|A|-r(A))} + \sum_{A:e\in A} u^{(r(G)-r(A))}v^{(|A|-r(A))}\\ &= \sum_{A\subseteq E\setminus e} u^{(r^{'}(G\setminus e)-r^{'}(A))}v^{(|A|-r^{'}(A))} + \sum_{A:e\in A} u^{(r^{'}(G\setminus e)-r^{'}(A\setminus e))}v^{(|A|-r^{'}(A\setminus e))}\\ &= \sum_{A\subseteq E\setminus e} u^{(r^{'}(G\setminus e)-r^{'}(A))}v^{(|A|-r^{'}(A))} + v\sum_{B=A\setminus e} u^{(r^{'}(G\setminus e)-r^{'}(B))}v^{(|B|-r'(B))}\\ &= (v+1)R(G\setminus e;u,v). \end{aligned}$$

Case 4:
If e is neither a loop nor a bridge; for the contraction G/e, $c(G_A/e)=c(G_A)$ and

$$\begin{aligned} r^{''}(A\setminus e) &= |V|-1-c(G_A/e)\\ &= |V|-c(G_A)-1\\ &= r(A)-1 \end{aligned} \tag{2.7.30}$$

for any $A\subseteq E$ that contains e. For the deletion $G\setminus e$,

$$\begin{aligned} r^{'}(A\setminus e) &= |V|-c(G_A)\\ &= r(A) \end{aligned} \tag{2.7.31}$$

for any $A\subseteq E\setminus e$. Now using equations (2.7.30) and (2.7.31) and applying the same partition, we have

$$\begin{aligned} R(G;u,v) &= \sum_{A\subseteq E} u^{(r(G)-r(A))}v^{(|A|-r(A))}\\ &= \sum_{A\subseteq E\setminus e} u^{(r(G)-r(A))}v^{(|A|-r(A))} + \sum_{A:e\in A} u^{(r(G)-r(A))}v^{(|A|-r(A))}\\ &= \sum_{A\subseteq E\setminus e} u^{(r^{'}(G\setminus e)-r^{'}(A))}v^{(|A|-r^{'}(A))} + \sum_{B=A\setminus e} u^{(r^{''}(G/e)+1-(r^{''}(B)+1))}v^{(|B|+1-(r^{''}(B)+1))}\\ &= R(G\setminus e;u,v)+R(G/e;u,v). \end{aligned}$$

This completes the proof. $\square$

Remark 2.7.32.
Let us conclude this chapter with some remarks. The Tutte polynomial can be used in counting various combinatorial structures. From equation (2.7.28) the following computations can be made given a connected graph G. We apply these to Example 2.7.17:

(i) $T(G;1,1) = R(G;0,0) = \tau(G)$, where $\tau(G)$ is the number of spanning trees of G. Since as $x, y \to 1$, the only A's contributing to the sum in equation (2.7.28) are those for which $c(A) = c(G) = 1$ and with no cycles. For example, the Tutte polynomial of the graph G in example 2.7.17 is

$$T(G;x,y) = x^4 + 3x^3 + y^3 + 3x^2 + 2y^2 + 3x^2y + 2xy^2 + 4xy + x + y, \text{ thus}$$
$$T(G;1,1) = 21 \text{ (The Matrix Tree Theorem 2.5.1 also gives the same result).}$$

(ii) $T(G;2,1) = R(G;1,0)$ counts the number of spanning forests and for our example, $T(G;2,1) = 82$, hence, G has 82 spanning forests.

(iii) $T(G;1,2) = R(G;0,1)$ counts the number of connected spanning subgraphs and for our example, $T(G(5,7);2,1) = 48$, hence, G has 48 connected spanning subgraphs.

Chapter 3

Lattices and Spanning Trees

3.1 Introduction

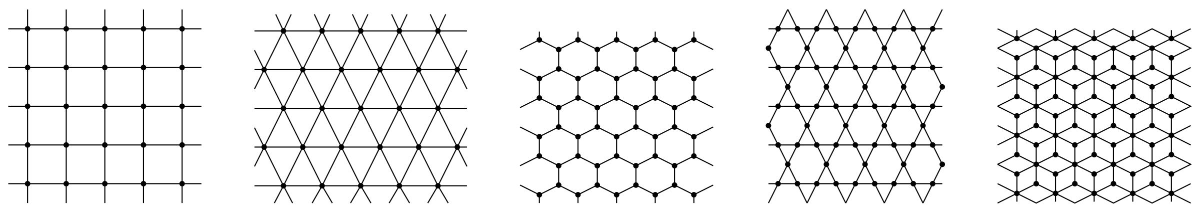
In this chapter, we discuss lattices such as the rectangular grid, the triangular lattice with its dual, the honeycomb lattice, the kagomé with its dual the diced lattice and the $9 \cdot 3$ lattice and its dual. For each of these lattices, we will derive an explicit formula for counting their spanning trees. We also introduce the term bulk limit or thermodynamic limit which measures the asymptotic growth for the number of spanning trees.

Definition 3.1.1. *Let $\mathbb{V}$ be a finite-dimensional vector space over $\mathbb{R}$. Then $(\mathbb{V}, +)$ is an abelian group. A subgroup $\mathcal{L}$ of $\mathbb{V}$ is said to be* ***discrete*** *if there is a positive constant ϵ such that if $x \in \mathcal{L}$ and $x \neq 0$, then $\langle x, x\rangle \geq \epsilon$. A* ***lattice*** *$\mathcal{L}$ is defined as a discrete additive subgroup of $\mathbb{V}$. The rank of a lattice is the dimension of its span. If $\mathcal{M}$ is a* ***sublattice*** *of $\mathcal{L}$ then $\mathcal{M}$ is a subgroup of $\mathcal{L}$ and we say $\mathcal{M}$ is a* ***full*** *sublattice of $\mathcal{L}$ if* $\mathrm{rank}(\mathcal{M}) = \mathrm{rank}(\mathcal{L})$ *[13].*

From now onwards, we will think of our "lattices" as follows:

- Graphs on a lattice (or a finite portion thereof) in the sense of Definition 3.1.1 with vertices in $\mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_d,\ v_1, v_2, \ldots, v_d \in \mathbb{R}^d$.
- The graph should be invariant under translation in all d directions of the lattice.

Example 3.1.2. Below are some examples of lattices.



(a) Square Grid $\mathcal{L}_{sq}$.

(b) Triangular Lattice $\mathcal{L}_{tri}$.

(c) Honeycomb Lattice $\mathcal{L}_{hc}$ (dual of $\mathcal{L}_{tri}$).

(d) Kagomé Lattice $\mathcal{L}_{kag}$.

(e) Diced Lattice $\mathcal{L}_{diced}$ (dual of $\mathcal{L}_{kag}$).

Figure 3.1: Regular planar lattices.

3.2 Counting Spanning Trees on Lattices

In this section, we would like to count the number of spanning trees of the square lattice $\mathcal{L}_{rec}$, the triangular lattice $\mathcal{L}_{tri}$ and the kagomé lattice $\mathcal{L}_{kag}$. Recall that Theorem 2.7.19 connects the Tutte polynomial $T(\mathcal{L}^*; x, y)$ and $T(\mathcal{L}; y, x)$. We note that counting spanning trees of the lattice $\mathcal{L}$ will not result in counting the spanning trees of its dual lattice $\mathcal{L}^*$ since in the dual lattice $\mathcal{L}^*$ the vertex in the unbounded face causes $\mathcal{L}^*$ to have more edges than $\mathcal{L}$. But asymptotically, one can consider the number of spanning trees as equal. That is $\tau(\mathcal{L}) \approx \tau(\mathcal{L}^*)$. We will start with the rectangular lattice. Corollary 2.5.3 gives a relation between the number of spanning trees and the non-zero eigenvalues of the Laplacian matrix L, of a graph. With this in mind, we first want to compute the eigenvalues and eigenvectors of the cycle graph C_n and the path graph P_n which will be vital in counting the number of spanning trees of a square lattice $\mathcal{L}_{sq}$.

3.3 Eigenvalues and Eigenvectors of C_n and P_n

In this section, we will derive an explicit formula for the eigenvalues and the eigenvectors of the adjacency matrix $\mathrm{A}(C_n)$ and the Laplacian matrix $\mathrm{L}(C_n)$ of the cycle graph C_n as well as the eigenvalues and the eigenvectors of the adjacency matrix $\mathrm{A}(P_n)$ and the Laplacian matrix $\mathrm{L}(P_n)$ of the path graph P_n. We will start with the cycle graph C_n and then proceed to the path graph P_n.

3.3.1 Eigenvalues and Eigenvectors of C_n

We first consider a cycle graph C_n on n vertices. Let $V = \{v_0, v_1, \dots, v_{n-1}\}$ be the vertex set of C_n and $E = \{v_i v_{i+1} \mid v_i \in V,\ i = 0, 1, \dots, n-1,\ v_n = v_0\}$ be the edge set of C_n. We can view the vertices of vertices of C_n corresponding to integers modulo n (i.e. with labels $0, 1, \dots, n-1$) with edges $\{i, i+1\}$ also computed modulo n. The cycle graph C_n is a 2-regular graph whose adjacency matrix $\mathrm{A}(C_n)$ and Laplacian matrix $\mathrm{L}(C_n)$ are *circulant* (i.e., a square matrix whose successive rows are obtained by cyclic permutations of its first row) and defined as follows:

$$\mathrm{A}(C_n) = \begin{array}{cc} & \begin{array}{ccccc} v_0 & v_1 & v_2 & \cdots & v_{n-1} \end{array} \\ \begin{array}{c} v_0 \\ v_1 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{array} & \begin{bmatrix} 0 & 1 & 0 & \dots & 1 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & 1 \\ 1 & \dots & 0 & 1 & 0 \end{bmatrix} \end{array} \text{ and } \mathrm{L}(C_n) = \begin{array}{cc} & \begin{array}{ccccc} v_0 & v_1 & v_2 & \cdots & v_{n-1} \end{array} \\ \begin{array}{c} v_0 \\ v_1 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{array} & \begin{bmatrix} 2 & -1 & 0 & \dots & -1 \\ -1 & 2 & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & -1 & 2 & -1 \\ -1 & \dots & 0 & -1 & 2 \end{bmatrix} \end{array}.$$

Lemma 3.3.1. *The Laplacian matrix of the cycle graph* $\mathrm{L}(C_n)$ *has eigenvectors* $\boldsymbol{x}_k = [x_k(0), x_k(1), \dots, x_k(n-1)]^{\mathrm{T}}$ *and* $\boldsymbol{y}_k = [y_k(0), y_k(1), \dots, y_k(n-1)]^{\mathrm{T}}$ *where*

$$x_k(j) = \cos\left(\frac{2\pi jk}{n}\right) \quad \textit{and} \tag{3.3.2a}$$

$$y_k(j) = \sin\left(\frac{2\pi jk}{n}\right), \tag{3.3.2b}$$

for $0 \le k \le \frac{n}{2}$, $j = 0, 1, \ldots, n-1$, $\boldsymbol{y}_0$ *and* $\boldsymbol{y}_{n/2}$ *(which are is zero vectors) are ignored. The corresponding eigenvalues for* $\boldsymbol{x}_k$ *and* $\boldsymbol{y}_k$ *are*

$$\lambda_k = 2 - 2\cos\left(\frac{2k\pi}{n}\right), \quad (k = 0, \ldots, n-1). \tag{3.3.3}$$

Proof.
The Laplacian of C_n is

$$\mathrm{L}(C_n) = \begin{array}{c} \\ {\scriptstyle 0} \\ {\scriptstyle 1} \\ \vdots \\ {\scriptstyle n-2} \\ {\scriptstyle n-1} \end{array} \begin{array}{c} \begin{array}{ccccc} {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle \ldots} & {\scriptstyle n-1} \end{array} \\ \begin{bmatrix} 2 & -1 & 0 & \ldots & -1 \\ -1 & 2 & -1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & -1 & 2 & -1 \\ -1 & \ldots & 0 & -1 & 2 \end{bmatrix} \end{array}.$$

Let λ_k $(k = 0, 1, \ldots, n-1)$ be the eigenvalues of $\mathrm{L}(C_n)$. Then λ_k must satisfy the eigenvalue-eigenvector equation

$$\mathrm{L}(C_n)\boldsymbol{x}_k = \lambda_k \boldsymbol{x}_k.$$

Expanding the above equation we get

$$\begin{array}{ccccccc} 2x_k(0) & - & x_k(1) & - & x_k(n-1) & = & \lambda_k x_k(0) \\ 2x_k(1) & - & x_k(0) & - & x_k(2) & = & \lambda_k x_k(1) \\ 2x_k(2) & - & x_k(1) & - & x_k(3) & = & \lambda_k x_k(2) \\ \vdots & - & \vdots & - & \vdots & = & \vdots \\ 2x_k(n-2) & - & x_k(n-3) & - & x_k(n-1) & = & \lambda_k x_k(n-2) \\ 2x_k(n-1) & - & x_k(n-2) & - & x_k(0) & = & \lambda_k x_k(n-1). \end{array} \tag{3.3.4}$$

We can write the left-hand side of equation (3.3.4) as

$$(\mathrm{L}(C_n)\boldsymbol{x}_k)(j) = 2x_k(j) - x_k(j-1) - x_k(j+1).$$

Substituting equation (3.3.2a) into the above equation and expanding the cosine function we have

$$\begin{aligned} (\mathrm{L}(C_n)\boldsymbol{x}_k)(j) &= 2\cos\left(\frac{2\pi kj}{n}\right) - \cos\left(\frac{2\pi k(j-1)}{n}\right) - \cos\left(\frac{2\pi k(j+1)}{n}\right) \\ &= 2\cos\left(\frac{2\pi kj}{n}\right) - \cos\left(\frac{2\pi kj}{n}\right)\cos\left(\frac{2\pi k}{n}\right) - \sin\left(\frac{2\pi kj}{n}\right)\sin\left(\frac{2\pi k}{n}\right) \\ &\quad - \cos\left(\frac{2\pi kj}{n}\right)\cos\left(\frac{2\pi k}{n}\right) + \sin\left(\frac{2\pi kj}{n}\right)\sin\left(\frac{2\pi k}{n}\right) \\ &= 2\cos\left(\frac{2\pi kj}{n}\right) - \cos\left(\frac{2\pi kj}{n}\right)\cos\left(\frac{2\pi k}{n}\right) - \cos\left(\frac{2\pi kj}{n}\right)\cos\left(\frac{2\pi k}{n}\right) \\ &= 2x_k(j) - 2x_k(j)\cos\left(\frac{2\pi k}{n}\right) \\ &= \left[2 - 2\cos\left(\frac{2\pi k}{n}\right)\right] x_k(j) \qquad (3.3.5) \\ &= \lambda_k \boldsymbol{x}_k. \end{aligned}$$

Similarly, we show that

$$\mathrm{L}(C_n)\boldsymbol{y}_k = \lambda_k \boldsymbol{y}_k \tag{3.3.6}$$

and following a similar procedure as done above, we write the left-hand side of equation (3.3.5) as

$$(\mathrm{L}(C_n)\boldsymbol{y}_k)(j) = 2y_k(j) - y_k(j-1) - y_k(j+1).$$

Substituting equation (3.3.2b) into the above equation and expanding the sine function we have

$$\begin{aligned}
(\mathrm{L}(C_n)\boldsymbol{y}_k)(j) &= 2\sin\left(\frac{2\pi kj}{n}\right) - \sin\left(\frac{2\pi k(j-1)}{n}\right) - \sin\left(\frac{2\pi k(j+1)}{n}\right) \\
&= 2\sin\left(\frac{2\pi kj}{n}\right) - \sin\left(\frac{2\pi kj}{n}\right)\cos\left(\frac{2\pi k}{n}\right) + \sin\left(\frac{2\pi k}{n}\right)\cos\left(\frac{2\pi kj}{n}\right) \\
&\quad - \sin\left(\frac{2\pi kj}{n}\right)\cos\left(\frac{2\pi k}{n}\right) - \sin\left(\frac{2\pi k}{n}\right)\cos\left(\frac{2\pi kj}{n}\right) \\
&= 2\sin\left(\frac{2\pi kj}{n}\right) - \sin\left(\frac{2\pi kj}{n}\right)\cos\left(\frac{2\pi k}{n}\right) - \sin\left(\frac{2\pi kj}{n}\right)\cos\left(\frac{2\pi k}{n}\right) \\
&= 2y_k(j) - 2y_k(j)\cos\left(\frac{2\pi k}{n}\right) \\
&= \left[2 - 2\cos\left(\frac{2\pi k}{n}\right)\right] y_k(j) \qquad (3.3.7) \\
&= \lambda_k \boldsymbol{y}_k.
\end{aligned}$$

From equations (3.3.5) and (3.3.7) we see clearly that

$$\begin{aligned}
\boldsymbol{x}_k &= [x_k(0), x_k(1), \ldots, x_k(n-1)]^{\mathrm{T}}, \text{ with } x_k(j) = \cos\left(\frac{2\pi jk}{n}\right) \text{ and} \\
\boldsymbol{y}_k &= [y_k(0), y_k(1), \ldots, y_k(n-1)]^{\mathrm{T}}, \text{ with } y_k(j) = \sin\left(\frac{2\pi jk}{n}\right)
\end{aligned}$$

for $k \le \frac{n}{2}$ and $j = 0, 1, \ldots, n-1$ are indeed eigenvectors. We note that for $0 < k < \frac{n}{2}$, the eigenvalue λ_k has multiplicity 2 by making use of the symmetry of the cosine function. Therefore, we let k run from 0 to $n-1$ instead and

$$\lambda_k = 2 - 2\cos\left(\frac{2\pi k}{n}\right).$$

For n odd, we ignore the all zeros vector $\boldsymbol{y}_0 = \mathbf{0}_n$, $\boldsymbol{x}_0 = \mathbf{1}_n$ is the all ones eigenvector associated with the eigenvalue $\lambda_0 = 0$ and for $1 \le k < \frac{n}{2}$, $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$ give $(n-1)$ other eigenvectors.
For n even, again we ignore the all zeros vector $\boldsymbol{y}_0 = \boldsymbol{y}_{\frac{n}{2}} = \mathbf{0}_n$, $\boldsymbol{x}_0 = \mathbf{1}_n$ is again the all ones eigenvector associated with the eigenvalue $\lambda_0 = 0$, for $1 \le k < \frac{n}{2}$, $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$ give $(n-2)$ eigenvectors and lastly, $\boldsymbol{x}_{\frac{n}{2}}$ corresponds to the eigenvalue $\lambda_{\frac{n}{2}} = 4$. This completes the proof. □

The eigenvalues $\lambda_k^{'}$ $(k = 0, \ldots, n-1)$ of $\mathrm{A}(C_n)$ are obtained from the eigenvalues of $\mathrm{L}(C_n)$ by a shift of 2 since C_n is a 2-regular graph. Thus we have,

$$\lambda_k^{'} = 2\cos\left(\frac{2k\pi}{n}\right) \ (k = 0, \ldots, n-1),$$

with the same eigenvectors in equation (3.3.2).

Remark 3.3.8.

(i) In general, for any d-regular graph, the eigenvalues λ_k of the Laplacian matrix are obtained from the eigenvalues $\lambda_k^{'}$ of its adjacency matrix by a shift of the degree d. That is,

$$\lambda_k = d - \lambda_k^{'}, \quad (k = 0, \ldots, n-1).$$

In particular for C_n,

$$\lambda_k = 2 - \lambda_k^{'}, \quad (k = 0, \ldots, n-1).$$

(ii) From equation (3.3.2) if we let $\zeta = e^{\frac{2\pi \mathrm{i}}{n}} = \cos\left(\frac{2\pi}{n}\right) + \mathrm{i}\sin\left(\frac{2\pi}{n}\right)$ such that $\zeta^n = 1$, then $\boldsymbol{v}_k = [1, \zeta^k, \zeta^{2k}, \ldots, \zeta^{(n-1)k}]^{\mathrm{T}}$ $(1 \le k \le n)$ is also an eigenvector of $\mathrm{A}(C_n)$ and $\mathrm{L}(C_n)$ since it is a linear combination of the eigenvectors $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$.

(iii) Using Corollary 2.5.3, we obtain the number of spanning trees of a cycle graph C_n as

$$\tau(C_n) = \frac{1}{n}\prod_{k=1}^{n-1}\left[2 - 2\cos\left(\frac{2k\pi}{n}\right)\right]$$

since $\boldsymbol{x}_0 = \mathbf{1}_n$ is the all ones eigenvector corresponding to the 0 eigenvalue.

3.3.2 Eigenvalues and Eigenvectors of P_n

We now compute the eigenvalues and eigenvectors of a path graph P_n on n vertices. The adjacency matrix $\mathrm{A}(P_n)$ and the Laplacian matrix $\mathrm{L}(P_n)$ are

$$\mathrm{A}(P_n) = \begin{array}{c c} & \begin{array}{ccccc} v_1 & v_2 & v_3 & \cdots & v_n \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{array} & \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 & 1 \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix} \end{array} \text{ and } \mathrm{L}(P_n) = \begin{array}{c c} & \begin{array}{ccccc} v_1 & v_2 & v_3 & \cdots & v_n \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{array} & \begin{bmatrix} 1 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & -1 & 2 & -1 \\ 0 & \ldots & 0 & -1 & 1 \end{bmatrix} \end{array}.$$

The eigenvalues of the adjacency matrix $\mathrm{A}(P_n)$ are computed as follows. Let $c_n(x)$ be the characteristics polynomial. That is,

$$c_n(x) = \det[x\mathrm{I}_n - \mathrm{A}(P_n)] = \det\begin{bmatrix} x & -1 & 0 & \ldots & 0 \\ -1 & x & -1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & -1 & x & -1 \\ 0 & \ldots & 0 & -1 & x \end{bmatrix}.$$

Using the first row to compute the determinant, we arrive at the the recurrence

$$c_n(x) = xc_{n-1}(x) - c_{n-2}(x), \quad c_0(x) = 1,\ c_1(x) = x. \tag{3.3.9}$$

We follow the four steps outlined in the third chapter of [1] to solve the recurrence equation (3.3.9).

Step 1:

We first write the recurrence equation (3.3.9) as a single equation with the initial conditions. For $n < 0$, $c_n(x) = 0$ and this satisfies equation (3.3.9). But for $n = 0$, $c_0(x) = 1$ while the right-hand side is 0. So the complete recurrence equation is therefore

$$c_n(x) = xc_{n-1}(x) - c_{n-2}(x) + [n = 0]. \tag{3.3.10}$$

The last term in equation (3.3.10) is defined as follows:

$$[Q] = \begin{cases} 1 & \text{if } Q \text{ is true} \\ 0 & \text{if } Q \text{ is false.} \end{cases}$$

Step 2:

Express equation (3.3.10) as an equation between generation functions where lowering an index corresponds to multiplication by a power of z.

$$\begin{aligned} C(z) = \sum_{n\geq 0} c_n(x)z^n &= x\sum_{n\geq 0} c_{n-1}(x)z^n - \sum_{n\geq 0} c_{n-2}(x)z^n + \sum_{n\geq 0}[n=0]z^n \\ &= xzC(z) - z^2C(z) + 1 \end{aligned} \tag{3.3.11}$$

Step 3:

We solve equation (3.3.11). That is,

$$C(z) = \frac{1}{z^2 - xz + 1}. \tag{3.3.12}$$

Step 4:

We express the right-hand side of (3.3.12) as a generating function and compare the coefficients. Writing equation (3.3.12) in terms of partial fractions we have

$$C(z) = \frac{1}{z^2 - xz + 1} = \frac{1}{(1-\alpha z)(1-\beta z)} = \frac{a}{(1-\alpha z)} + \frac{b}{(1-\beta z)},$$

where

$$\alpha = \frac{x + \sqrt{x^2-4}}{2}, \qquad \beta = \frac{x - \sqrt{x^2-4}}{2},$$

$$a = \frac{\alpha}{\alpha - \beta} = \frac{x + \sqrt{x^2-4}}{2\sqrt{x^2-4}}, \quad \text{and} \quad b = \frac{\beta}{\beta - \alpha} = -\frac{x - \sqrt{x^2-4}}{2\sqrt{x^2-4}}.$$

Therefore,

$$C(z) = \sum_{n\geq 0} c_n(x)z^n = \frac{x + \sqrt{x^2-4}}{2\sqrt{x^2-4}} \sum_{n\geq 0} \left(\frac{x + \sqrt{x^2-4}}{2}\right)^n z^n - \frac{x - \sqrt{x^2-4}}{2\sqrt{x^2-4}} \sum_{n\geq 0} \left(\frac{x - \sqrt{x^2-4}}{2}\right)^n z^n.$$

Comparing coefficients we see that the characteristics polynomial of the adjacency matrix $\mathrm{A}(P_n)$ is

$$c_n(x) = \frac{1}{\sqrt{x^2-4}}\left[\left(\frac{x+\sqrt{x^2-4}}{2}\right)^{n+1} - \left(\frac{x-\sqrt{x^2-4}}{2}\right)^{n+1}\right]. \tag{3.3.13}$$

We now find the roots of equation (3.3.13) which are precisely all the eigenvalues of A. If μ is an eigenvalue of the matrix $\mathrm{A}(P_n)$ then μ is a root of the characteristic polynomial $c_n(x)$ and $c_n(\mu) = 0$. Therefore equation (3.3.13) becomes

$$\left(\frac{\mu+\sqrt{\mu^2-4}}{\mu-\sqrt{\mu^2-4}}\right)^{n+1} = 1 = e^{2\pi \mathrm{i}}.$$

Hence,

$$\left(\frac{\mu+\sqrt{\mu^2-4}}{\mu-\sqrt{\mu^2-4}}\right) = e^{\frac{2k\pi \mathrm{i}}{n+1}} \text{ with } k = 1, 2, \ldots, n. \tag{3.3.14}$$

We exclude the case $k = 0$ since that will cause the zero in the numerator to cancel with the zero in the denominator. From equation (3.3.14) we have,

$$\begin{aligned}
(\mu+\sqrt{\mu^2-4})^2 &= 4e^{\frac{2k\pi \mathrm{i}}{n+1}} \\
\mu+\sqrt{\mu^2-4} &= \pm 2e^{\frac{k\pi \mathrm{i}}{n+1}} \\
\sqrt{\mu^2-4} &= \pm 2e^{\frac{k\pi \mathrm{i}}{n+1}} - \mu \\
\mu^2 - 4 &= \mu^2 \pm 4\mu e^{\frac{k\pi \mathrm{i}}{n+1}} + 4e^{\frac{2k\pi \mathrm{i}}{n+1}} \\
\mu &= \pm\frac{e^{\frac{2k\pi \mathrm{i}}{n+1}}+1}{e^{\frac{k\pi \mathrm{i}}{n+1}}} \\
&= \pm\left(e^{\frac{k\pi \mathrm{i}}{n+1}} + e^{-\frac{k\pi \mathrm{i}}{n+1}}\right) \\
&= \pm 2\cos\frac{k\pi}{n+1}.
\end{aligned}$$

Therefore the eigenvalues of $\mathrm{A}(P_n)$ are

$$\mu_k = 2\cos\frac{k\pi}{n+1} \;\; (k = 1, 2, \ldots, n) \tag{3.3.15}$$

since

$$\cos\frac{k\pi}{n+1} = -\cos\frac{(n+1-k)\pi}{n+1} \;\; (k = 1, 2, \ldots, n).$$

Next we compute the eigenvalues of the Laplacian, $\mathrm{L}(P_n)$.

Lemma 3.3.16. *The Laplacian matrix* $\mathrm{L}(P_n)$ *has the same eigenvalues as* $\mathrm{L}(C_{2n})$*, namely*

$$\mu_k = 2 - 2\cos\left(\frac{k\pi}{n}\right) \quad (k = 0, \ldots, n-1),$$

with associated eigenvectors $\boldsymbol{u}_k = [v_k(1), v_k(2), \ldots, v_k(n)]^{\mathrm{T}}$

$$v_k(j) = \cos\left(\frac{kj\pi}{n} - \frac{\pi k}{2n}\right), \quad (j = 1, \ldots, n).$$

Proof.
To see this, we consider P_n as a result of folding C_{2n} in such a way that no vertex of C_{2n} is fixed. That is we treat the path graph P_n as a quotient of the cycle graph C_{2n} by identifying vertex j of P_n with vertices j and $2n + 1 - j$ of C_{2n}. Then it suffices to find an eigenvector $\boldsymbol{v}_k = [v_k(1), v_k(2), \ldots, v_k(2n)]^{\mathrm{T}}$ of C_{2n} such that $v_k(j) = v_k(2n+1-j)$ for all vertices j of C_{2n}. Then $\boldsymbol{u}_k = [v_k(1), v_k(2), \ldots, v_k(n)]^{\mathrm{T}}$ will be an eigenvector of $\mathrm{L}(P_n)$. We first show that $\boldsymbol{u}$ is indeed an eigenvector of $\mathrm{L}(P_n)$. Let μ be an eigenvalue of $\mathrm{L}(P_n)$. Then μ must satisfy the eigenvalue-eigenvector equation of $\mathrm{L}(P_n)$ which is

$$\mathrm{L}(P_n)\boldsymbol{u}_k = \mu_k \boldsymbol{u}_k$$

$$\begin{bmatrix} 1 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & -1 & 2 & -1 \\ 0 & \ldots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_k(1) \\ v_k(2) \\ \vdots \\ v_k(n-1) \\ v_k(n) \end{bmatrix} = \mu_k \begin{bmatrix} v_k(1) \\ v_k(2) \\ \vdots \\ v_k(n-1) \\ v_k(n) \end{bmatrix}. \tag{3.3.17}$$

Expanding equation (3.3.17), we have

$$v_k(1) - v_k(2) = \mu v_k(1) \quad \text{for the first row} \tag{3.3.18}$$
$$-v_k(j-1) + 2v_k(j) - v_k(j+1) = \mu v_k(j) \quad (j = 2, \ldots, n-1) \tag{3.3.19}$$
$$-v_k(n-1) + v_k(n) = \mu v_k(n) \quad \text{for the last row.} \tag{3.3.20}$$

From equation (3.3.19) we have a circulant sub-matrix for $i = 2, \ldots, n-1$. Thus it follows from the case of a cycle graph as done above that our choice of $\boldsymbol{u}$ satisfies the eigenvalue-eigenvector equation for $\mathrm{L}(P_n)$. It remains to check if equations (3.3.18) and (3.3.20) also satisfy the eigenvalue-eigenvector equation. To see this, we note that $v_k(1) = v_k(2n)$ and $v_k(n) = v_k(n+1)$. Thus from the two equations we have:

$$\begin{aligned} \mu v_k(1) &= v_k(1) - v_k(2) \\ &= -v_k(1) + v_k(1) + v_k(1) - v_k(2) \\ &= -v_k(2n) + 2v_k(1) - v_k(2). \end{aligned}$$

Similarly,

$$\begin{aligned} \mu v_k(n) &= -v_k(n-1) + v_k(n) \\ &= -v_k(n-1) + v_k(n) + v_k(n) - v_k(n) \\ &= -v_k(n-1) + 2v_k(n) - v_k(n+1). \end{aligned}$$

Thus, our $\boldsymbol{u}_k$ satisfies the eigenvalue-eigenvector equation for $\mathrm{L}(P_n)$.

We check if there exists an eigenvector $\boldsymbol{v}_k$ of $\mathrm{L}(C_{2n})$ satisfying $v_k(j) = v_k(2n+1-j)$ for every vertex j of C_{2n} so that we can obtain our eigenvector μ from it. Let

$$v_k(j) = \cos\left(\frac{kj\pi}{n} - \frac{\pi k}{2n}\right).$$

Then,

$$\begin{aligned}
v_k(2n+1-j) &= \cos\left(\frac{k(2n+1-j)\pi}{n} - \frac{\pi k}{2n}\right) \\
&= \cos\left[\frac{k\pi}{n}\left(\frac{4n+2-2j-1}{2}\right)\right] \\
&= \cos\left[\frac{4n\pi k}{2n} - \left(\frac{2j\pi k - \pi k}{2n}\right)\right] \\
&= \cos\left[2\pi k - \left(\frac{2j\pi k - \pi k}{2n}\right)\right] \\
&= \cos\left(\frac{kj\pi}{n} - \frac{\pi k}{2n}\right) \qquad (3.3.21) \\
&= v_k(j).
\end{aligned}$$

So our condition $v_k(j) = v_k(2n+1-j)$ is satisfied. Expanding the cosine function in equation (3.3.21), we have

$$\begin{aligned}
v_k(j) &= \cos\left(\frac{2\pi kj}{2n} - \frac{\pi k}{2n}\right) \\
&= \cos\left(\frac{\pi k}{2n}\right)\cos\left(\frac{2kj\pi}{2n}\right) + \sin\left(\frac{\pi k}{2n}\right)\sin\left(\frac{2kj\pi}{2n}\right).
\end{aligned}$$

If we let

$$x_k(j) = \cos\left(\frac{2kj\pi}{2n}\right) \quad \text{and} \quad y_k(j) = \sin\left(\frac{2kj\pi}{2n}\right),$$

then $\boldsymbol{x}_k = [x_k(0), x_k(1), \ldots, x_k(n-1)]^{\mathrm{T}}$ and $\boldsymbol{y}_k = [y_k(0), y_k(1), \ldots, y_k(n-1)]^{\mathrm{T}}$ and the eigenvector $\boldsymbol{v}_k$ is generated by the eigenvectors $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$ which are the eigenvectors of $\mathrm{L}(C_{2n})$. It follows from Lemma 3.3.1 that the eigenvalues are

$$\mu_k = 2 - 2\cos\left(\frac{2k\pi}{2n}\right) = 2 - 2\cos\left(\frac{k\pi}{n}\right) \quad (k = 0, \ldots, n-1). \qquad (3.3.22)$$

This completes the proof. □

Remark 3.3.23.
Using Corollary 2.5.3, the number of spanning trees of a path graph P_n is

$$\tau(P_n) = \frac{1}{n}\prod_{k=1}^{n-1}\left[2 - 2\cos\left(\frac{k\pi}{n}\right)\right] = 1,$$

since P_n is by itself a tree and $\boldsymbol{x}_0 = \mathbf{1}_n$ is the all ones eigenvector corresponding to the 0 eigenvalue.

3.4 Spanning trees on Rectangular Lattice $\mathcal{L}_{N_1\times N_2}$

We now want to use the results from the previous section to count the number of spanning trees of a rectangular lattice $\mathcal{L}_{N_1\times N_2}$. We first write the Laplacian matrix $\mathrm{L}(\mathcal{L}_{N_1\times N_2})$ in a form suitable for computing its eigenvalues. Let $\mathcal{L}$ be any lattice on n vertices in d dimensions. Then $\mathcal{L}$ can be decomposed into a hypercubic array of $N_1 \times N_2 \times \cdots \times N_d$ unit cells, with each cell containing ν vertices such that $n = \nu \prod_{i=1}^{d} N_i$. We label the cells by coordinates $\boldsymbol{n} = \{n_1, n_2, \ldots, n_d\}$, where $n_i = 0, 1, \ldots, N_i - 1$ $(i = 1, 2, \ldots, d)$. Similarly, we label the vertices in each unit cell as $1, 2, \ldots, \nu$ and define $a(\boldsymbol{n}, \boldsymbol{n}^{'})$ to be the $\nu \times \nu$ cell vertex adjacency matrix which describes the connectivity between the vertices of the unit cells $\boldsymbol{n}$ and $\boldsymbol{n}^{'}$ [22]. That is,

$$a_{ij}(\boldsymbol{n}, \boldsymbol{n}^{'}) = \begin{cases} 1 & \text{if vertex } i \text{ in cell } \boldsymbol{n} \text{ and vertex } j \text{ in cell } \boldsymbol{n}^{'} \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.1}$$

Assuming our lattice $\mathcal{L}$ is wrapped around a torus, we have $a(\boldsymbol{n}, \boldsymbol{n}^{'}) = a(\boldsymbol{n} - \boldsymbol{n}^{'}, 0)$ thus we write $a(\boldsymbol{n}) = a(n_1, n_2, \ldots, n_d)$ for $a(\boldsymbol{n}, 0)$. For example in the two dimensional case,

$$\begin{aligned} a\big((x_1, y_1), (x_2, y_2)\big) &= a\big((0,0), (x_1 - x_2, y_1 - y_2)\big) \\ &= a\big((x_1 - x_2, y_1 - y_2)\big). \end{aligned}$$

We illustrate the general formulation with an example. Here we consider a simpler form of the rectangular lattice $\mathcal{L}_{3\times 4}$ with 12 unit cells, and each cell contains a single vertex, i.e., $\nu = 1$. Also, $N_1 = 3$, $N_2 = 4$ and $n = \nu \cdot N_1 \cdot N_2 = 12$. For every vertex in a cell assuming periodic boundary conditions for $k_1 = 0, 1, \ldots, N_1 - 1$ and $k_2 = 0, 1, \ldots, N_2 - 1$, we have the 1×1 cell vertex adjacency matrix with connectivity described as follows:

$$\begin{aligned} a(0,0) &= \big[a_{11}\big((0,0),(0,0)\big)\big] &= 0, \\ a(0,-1) &= \big[a_{11}\big((0,0),(0,-1)\big)\big] &= 1, \\ a(0,1) &= \big[a_{11}\big((0,0),(0,1)\big)\big] &= 1, \\ a(-1,0) &= \big[a_{11}\big((0,0),(-1,0)\big)\big] &= 1, \\ a(1,0) &= \big[a_{11}\big((0,0),(1,0)\big)\big] &= 1. \end{aligned}$$

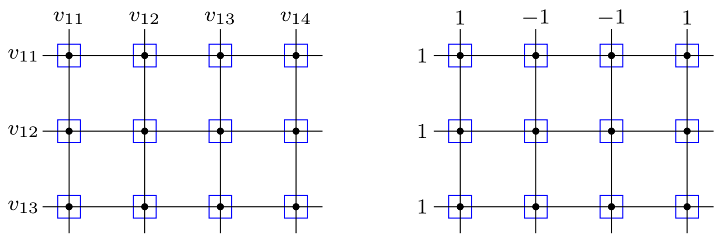


Figure 3.2: A rectangular lattice $\mathcal{L}_{3\times 4}$ on 12 vertices.

The rectangular lattice $\mathcal{L}_{3\times 4}$ in Figure 3.2 is the cartesian product of the path graphs P_3 and P_4. That is, $\mathcal{L}_{3\times 4} = P_3 \times P_4$. The adjacency matrix and the laplacian matrix of the graphs P_3 and P_4 are as follows:

$$\mathrm{A}(P_3) = \begin{array}{c c} & \begin{array}{ccc} v_{11} & v_{12} & v_{13} \end{array} \\ \begin{array}{c} v_{11} \\ v_{12} \\ v_{13} \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}, \qquad \mathrm{L}(P_3) = \begin{array}{c c} & \begin{array}{ccc} v_{11} & v_{12} & v_{13} \end{array} \\ \begin{array}{c} v_{11} \\ v_{12} \\ v_{13} \end{array} & \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array},$$
$$\mathrm{A}(P_4) = \begin{array}{c c} & \begin{array}{cccc} v_{11} & v_{12} & v_{13} & v_{14} \end{array} \\ \begin{array}{c} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}, \text{ and } \mathrm{L}(P_4) = \begin{array}{c c} & \begin{array}{cccc} v_{11} & v_{12} & v_{13} & v_{14} \end{array} \\ \begin{array}{c} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \end{array} & \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}. \tag{3.4.2}$$

Using equations (3.3.21) and (3.3.22) we have the eigenvalues of $\mathrm{L}(P_3)$ and $\mathrm{L}(P_4)$ with their associated eigenvectors as follows:

$\mathrm{L}(P_3)$		$\mathrm{L}(P_4)$	
Eigenvalues	Eigenvectors	Eigenvalues	Eigenvectors
$\mu_0 = 0$	$\boldsymbol{u}_0 = [1, 1, 1]^{\mathrm{T}}$	$\mu_0' = 0$	$\boldsymbol{v}_0 = [1, 1, 1, 1]^{\mathrm{T}}$
$\mu_1 = 1$	$\boldsymbol{u}_1 = [1, 0, -1]^{\mathrm{T}}$	$\mu_1' = 2 - \sqrt{2}$	$\boldsymbol{v}_1 = [-1, 1-\sqrt{2}, -1+\sqrt{2}, 1]^{\mathrm{T}}$
$\mu_2 = 3$	$\boldsymbol{u}_2 = [1, -2, 1]^{\mathrm{T}}$	$\mu_2' = 2$	$\boldsymbol{v}_2 = [1, -1, -1, 1]^{\mathrm{T}}$
		$\mu_3' = 2 + \sqrt{2}$	$\boldsymbol{v}_3 = [-1, 1+\sqrt{2}, -1-\sqrt{2}, 1]^{\mathrm{T}}$

Table 3.1: Eigenvalues and Eigenvectors of $\mathrm{L}(P_3)$ and $\mathrm{L}(P_4)$.

Before we proceed we need the definition of the tensor product of matrices.

Definition 3.4.3. *If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is a $p \times q$ matrix then the* ***Kronecker or tensor product*** *of A and B denoted $A \otimes B$ is the $mp \times nq$ matrix defined as:*

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

Using equation (3.4.2) and Definition 3.4.3, the adjacency matrix and the Laplacian matrix of rectangular lattice $\mathcal{L}_{3\times 4}$ in Figure 3.2 can be written as follows:

$$\mathrm{A}(\mathcal{L}_{3\times 4}) = (\mathrm{I}_3 \otimes \mathrm{A}(P_4)) + (\mathrm{A}(P_3) \otimes \mathrm{I}_4) \quad = \begin{array}{c c} & \begin{array}{c|c|c} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left[\begin{array}{c|c|c} \mathrm{A}(P_4) & \mathrm{I}_4 & \mathbf{0}_4 \\ \hline \mathrm{I}_4 & \mathrm{A}(P_4) & \mathrm{I}_4 \\ \hline \mathbf{0}_4 & \mathrm{I}_4 & \mathrm{A}(P_4) \end{array}\right] \end{array}$$

and

$$\mathrm{L}(\mathcal{L}_{3\times 4}) = (\mathrm{I}_3 \otimes \mathrm{L}(P_4)) + (\mathrm{L}(P_3)\otimes \mathrm{I}_4) = \begin{array}{c} \\ {\scriptstyle 0} \\ {\scriptstyle 1} \\ {\scriptstyle 2} \end{array} \begin{array}{c|c|c} {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 2} \\ \end{array} \left[\begin{array}{c|c|c} \ell(0,0) & -\mathrm{I}_4 & \mathbf{0}_4 \\ \hline -\mathrm{I}_4 & \ell(1,1) & -\mathrm{I}_4 \\ \hline \mathbf{0}_4 & -\mathrm{I}_4 & \ell(0,0) \end{array}\right]$$

where I_4 is the 4×4 identity matrix, $\mathbf{0}_4$ is the 4×4 zero matrix,

$$\ell(0,0) = \begin{array}{c} \\ {\scriptstyle v_{11}} \\ {\scriptstyle v_{12}} \\ {\scriptstyle v_{13}} \\ {\scriptstyle v_{14}} \end{array} \begin{array}{c} \begin{array}{cccc} {\scriptstyle v_{11}} & {\scriptstyle v_{12}} & {\scriptstyle v_{13}} & {\scriptstyle v_{14}} \end{array} \\ \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \end{array} \quad \text{and} \quad \ell(1,1) = \begin{array}{c} \\ {\scriptstyle v_{11}} \\ {\scriptstyle v_{12}} \\ {\scriptstyle v_{13}} \\ {\scriptstyle v_{14}} \end{array} \begin{array}{c} \begin{array}{cccc} {\scriptstyle v_{11}} & {\scriptstyle v_{12}} & {\scriptstyle v_{13}} & {\scriptstyle v_{14}} \end{array} \\ \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix} \end{array}.$$

For any rectangular lattice in general, $\mathcal{L}_{N_1\times N_2}$ we have

$$\mathrm{L}(\mathcal{L}_{N_1\times N_2}) = (\mathrm{I}_{N_1} \otimes \mathrm{L}(P_{N_2})) + (\mathrm{L}(P_{N_1})\otimes \mathrm{I}_{N_2}). \tag{3.4.4}$$

We compute the eigenvalues and eigenvectors of the Laplacian $\mathrm{L}(\mathcal{L}_{3\times 4})$.

Lemma 3.4.5. *Let $A \in M_{N_1}(\mathbb{Z})$ with eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{N_1-1}$ associated with the eigenvectors $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N_1-1}$ and let $B \in M_{N_2}(\mathbb{Z})$ with eigenvalues $\mu_0, \mu_1, \ldots, \mu_{N_2-1}$ associated with the eigenvectors $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{N_2-1}$. Then the eigenvalues and eigenvectors of $(\mathrm{I}_{N_2}\otimes A) + (B \otimes \mathrm{I}_{N_1})$ are $\lambda_{k_1} + \mu_{k_2}$ and $\boldsymbol{v}_{k_2}\otimes \boldsymbol{u}_{k_1}$ respectively for $k_1 = 0, 1, \ldots, N_1 - 1$ and $k_2 = 0, 1, \ldots, N_2 - 1$.*

Proof.
From the eigenvalue-eigenvector equation we have

$$\begin{aligned} A\boldsymbol{u}_{k_1} &= \lambda_{k_1}\boldsymbol{u}_{k_1} \quad (k_1 = 0, \ldots, N_1 - 1) \quad \text{and} \\ B\boldsymbol{v}_{k_2} &= \mu_{k_2}\boldsymbol{v}_{k_2} \quad (k_2 = 0, \ldots, N_2 - 1). \end{aligned}$$

Therefore,

$$\begin{aligned} ((\mathrm{I}_{N_2}\otimes A) + (B\otimes \mathrm{I}_{N_1}))(\boldsymbol{v}_{k_2}\otimes \boldsymbol{u}_{k_1}) &= (\mathrm{I}_{N_2}\otimes A)(\boldsymbol{v}_{k_2}\otimes \boldsymbol{u}_{k_1}) + (B\otimes \mathrm{I}_{N_1})(\boldsymbol{v}_{k_2}\otimes \boldsymbol{u}_{k_1}) \\ &= (\mathrm{I}_{N_2}\boldsymbol{v}_{k_2}\otimes A\boldsymbol{u}_{k_1}) + (B\boldsymbol{v}_{k_2}\otimes \mathrm{I}_{N_1}\boldsymbol{u}_{k_1}) \\ &= (\boldsymbol{v}_{k_2}\otimes \lambda_{k_1}\boldsymbol{u}_{k_1}) + (\mu_{k_2}\boldsymbol{v}_{k_2}\otimes \boldsymbol{u}_{k_1}) \\ &= \lambda_{k_1}(\boldsymbol{v}_{k_2}\otimes \boldsymbol{u}_{k_1}) + \mu_{k_2}(\boldsymbol{v}_{k_2}\otimes \boldsymbol{u}_{k_1}) \\ &= (\lambda_{k_1} + \mu_{k_2})(\boldsymbol{v}_{k_2}\otimes \boldsymbol{u}_{k_1}). \end{aligned}$$

□

Using Lemma 3.4.5, we compute the eigenvalues $\mathrm{L}(\mathcal{L}_{N_1\times N_2})$ by taking all possible sums $\mu_{k_1} + \mu'_{k_2}$ ($k_1 = 0, 1, \ldots, N_1 - 1$ and $k_2 = 0, 1, \ldots, N_2 - 1$) of the eigenvalues μ_{k_1} of $\mathrm{L}(P_{N_1})$ and the eigenvalues μ'_{k_2} of $\mathrm{L}(P_{N_2})$. The eigenvectors of $\mathrm{L}(\mathcal{L}_{N_1\times N_2})$ are also computed by taking all possible tensor products of the eigenvectors of $\mathrm{L}(P_{N_1})$ and the eigenvectors of

$\mathrm{L}(P_{N_2})$ as indicated in Figure 3.2 for the case where $\mu_0 = 0$ with eigenvector $\boldsymbol{u}_0 = [1, 1, 1]^{\mathrm{T}}$ and $\mu_2^{'} = 2$ with eigenvector $\boldsymbol{v}_2 = [1, -1, -1, 1]^{\mathrm{T}}$. That is,

$$\begin{array}{cc} & \begin{array}{cccc|cccc|cccc} v_{11} & v_{12} & v_{13} & v_{14} & v_{21} & v_{22} & v_{23} & v_{24} & v_{31} & v_{32} & v_{33} & v_{34} \end{array} \\ \begin{array}{c} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ \hline v_{21} \\ v_{22} \\ v_{23} \\ v_{24} \\ \hline v_{31} \\ v_{32} \\ v_{33} \\ v_{34} \end{array} & \left[\begin{array}{cccc|cccc|cccc} 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 2 \end{array}\right] \end{array} \left[\begin{array}{c} 1 \\ -1 \\ -1 \\ 1 \\ \hline 1 \\ -1 \\ -1 \\ 1 \\ \hline 1 \\ -1 \\ -1 \\ 1 \end{array}\right] = (0+2) \left[\begin{array}{c} 1 \\ -1 \\ -1 \\ 1 \\ \hline 1 \\ -1 \\ -1 \\ 1 \\ \hline 1 \\ -1 \\ -1 \\ 1 \end{array}\right]$$

and thus, $\mu_0 + \mu_2^{'} = 0 + 2 = 2$ is an eigenvalue of $\mathrm{L}(\mathcal{L}_{3\times 4})$ with eigenvector $\boldsymbol{u}_0 \otimes \boldsymbol{v}_2$. Equation (3.3.22) gives us all the eigenvalues μ_{k_1} of $\mathrm{L}(P_{N_1})$ and eigenvalues $\mu_{k_2}^{'}$ of $\mathrm{L}(P_{N_2})$. That is,

$$\begin{aligned} \mu_{k_1} &= 2 - 2\cos\left(\frac{k_1\pi}{N_1}\right) \qquad (k_1 = 0, 1, \ldots, N_1 - 1), \\ \mu_{k_2}^{'} &= 2 - 2\cos\left(\frac{k_2\pi}{N_2}\right) \qquad (k_2 = 0, 1, \ldots, N_2 - 1). \end{aligned}$$

Let $\lambda_k = \mu_{k_1} + \mu_{k_2}^{'}$ be all the eigenvalues of $\mathrm{L}(\mathcal{L}_{N_1\times N_2})$ for $k = 0, 1, \ldots, n-1$. Then, by Corollary 2.5.3, the number of spanning trees of the rectangular lattice $\mathcal{L}_{N_1\times N_2}$ is

$$\begin{aligned} \tau(\mathcal{L}_{N_1\times N_2}) &= \frac{1}{n}\prod_{k=1}^{n-1}\lambda_k \\ &= \frac{1}{N_1N_2}\prod_{\substack{k_1=0 \\ (k_1,k_2)\neq(0,0)}}^{N_1-1}\prod_{k_2=0}^{N_2-1}\left(\mu_{k_1} + \mu_{k_2}^{'}\right) \\ &= \frac{1}{N_1N_2}\prod_{\substack{k_1=0 \\ (k_1,k_2)\neq(0,0)}}^{N_1-1}\prod_{k_2=0}^{N_2-1}\left[2 - 2\cos\left(\frac{k_1\pi}{N_1}\right) + 2 - 2\cos\left(\frac{k_2\pi}{N_2}\right)\right] \\ &= \frac{1}{N_1N_2}\prod_{\substack{k_1=0 \\ (k_1,k_2)\neq(0,0)}}^{N_1-1}\prod_{k_2=0}^{N_2-1}\left[4 - 2\left(\cos\left(\frac{k_1\pi}{N_1}\right) + \cos\left(\frac{k_2\pi}{N_2}\right)\right)\right]. \end{aligned} \tag{3.4.6}$$

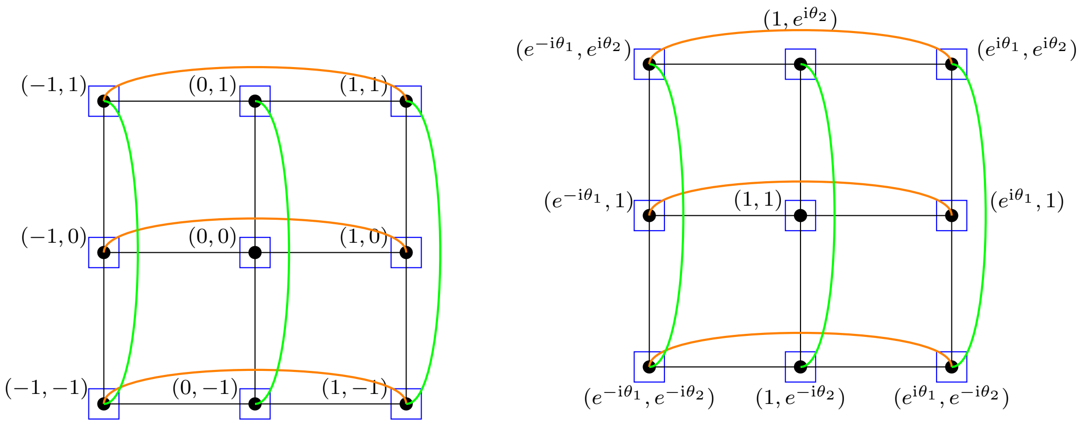
Example 3.4.7.
Using equation (3.4.6), we compute the number of spanning trees of the rectangular lattice

$\mathcal{L}_{3\times 4}$ in Figure 3.2 as:

$$\begin{aligned}
\tau(\mathcal{L}_{3\times 4}) &= \frac{1}{3\cdot 4} \prod_{\substack{k_1=0 \\ (k_1,k_2)\neq(0,0)}}^{2} \prod_{k_2=0}^{3} \left[4 - 2\left(\cos\left(\frac{k_1\pi}{3}\right) + \cos\left(\frac{k_2\pi}{4}\right)\right)\right] \\
&= \frac{1}{12}\Big[(2-\sqrt{2})\cdot 2\cdot(2+\sqrt{2})\cdot 1\cdot(3-\sqrt{2})\cdot 3\cdot(3+\sqrt{2})\cdot 3\cdot(5-\sqrt{2})\cdot 5\cdot(5+\sqrt{2})\Big] \\
&= \frac{28980}{12} \\
&= 2415.
\end{aligned}$$

We used the computer algebra system SAGE to write a program to compute $\tau(\mathcal{L}_{3\times 4})$. See Appendix C.1 for this.

We can also count the number of spanning trees of a rectangular lattice $\mathcal{L}_{N_1\times N_2}$, in which we introduce new edges such that one can wrap it around a torus to get a 4-regular lattice $\mathcal{L}^*_{N_1\times N_2}$. That is, we join the end vertices of the path graphs P_{N_1} on the vertical axis and the end vertices of the P_{N_2} on the horizontal axis. For example, consider the figure below:



(a) The lattice $\mathcal{L}^*_{N_1\times N_2}$ on a torus. (b) The lattice $\mathcal{L}^*_{N_1\times N_2}$ with coordinates on a torus.

Figure 3.3: Rectangular lattice $\mathcal{L}^*_{N_1\times N_2}$ on a torus with eigenvalues.

From Figure 3.3 we see that the rectangular lattice $\mathcal{L}^*_{N_1\times N_2}$ is just the cartesian product of two cycle graphs, say C_{N_1} and C_{N_2}. That is, $\mathcal{L}^*_{N_1\times N_2} = C_{N_1} \times C_{N_2}$. We know from equation (3.3.3) that the eigenvalues of C_{N_1} and C_{N_2} are

$$\mu_{k_1} = 2 - 2\cos\left(\frac{2k_1\pi}{N_1}\right), \quad (k_1 = 0, \ldots, N_1 - 1)$$

and

$$\lambda_{k_2} = 2 - 2\cos\left(\frac{2k_2\pi}{N_2}\right), \quad (k_2 = 0, \ldots, N_2 - 1)$$

with eigenvectors

$$\boldsymbol{u}_{k_1} = [1, \zeta^{k_1}, \zeta^{2k_1}, \ldots, \zeta^{(N_1-1)k_1}]^{\mathrm{T}}, \quad (k_1 = 0, \ldots, N_1 - 1)$$

and

$$\boldsymbol{v}_{k_2} = [1, \zeta^{k_2}, \zeta^{2k_2}, \ldots, \zeta^{(N_2-1)k_2}]^{\mathrm{T}}, \quad (k_2 = 0, \ldots, N_2 - 1)$$

respectively. The entries of the eigenvectors correspond to the vertices of our lattice $\mathcal{L}^*_{N_1\times N_2}$ as shown in Figure 3.3b. That is, without loss of generality, if we consider the cell $(0,0)$ to have a value 1 in the first entry of the eigenvector $\boldsymbol{u}_0$ and a value 1 in the first entry of the eigenvector $\boldsymbol{v}_0$ then the values for the (ℓ_1, ℓ_2)-th cell are obtained from the $(0,0)$-cell by multiplying by $e^{\frac{\mathrm{i}2\pi\ell_1}{N_1}} e^{\frac{\mathrm{i}2\pi\ell_2}{N_2}}$. The lattice is invariant under translation and so for each (ℓ_1, ℓ_2) cell, it follows from equation (3.4.1) that the cell vertex adjacency matrix are as follows:

$$a(0,0) = 0,\ a(0,-1) = a(0,1) = a(-1,0) = a(1,0) = 1. \tag{3.4.8}$$

The Laplacian of our finite rectangular $\mathrm{L}(\mathcal{L}^*_{N_1\times N_2})$ is an $N_1 \times N_1$ block circulant matrix with each block of size $N_2 \times N_2$. That is

$$\mathrm{L}(\mathcal{L}^*_{N_1\times N_2}) = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & \cdots & N_1 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ \vdots \\ N_1-1 \\ N_1 \end{array} & \left[\begin{array}{c|c|c|c|c} \Lambda & -\mathrm{I}_{N_2} & \mathbf{0}_{N_2} & \ldots & -\mathrm{I}_{N_2} \\ \hline -\mathrm{I}_{N_2} & \Lambda & -\mathrm{I}_{N_2} & \ldots & \mathbf{0}_{N_2} \\ \hline \vdots & \ddots & \ddots & \ddots & \vdots \\ \hline \mathbf{0}_{N_2} & \ldots & -\mathrm{I}_{N_2} & \Lambda & -\mathrm{I}_{N_2} \\ \hline -\mathrm{I}_{N_2} & \ldots & \mathbf{0}_{N_2} & -\mathrm{I}_{N_2} & \Lambda \end{array}\right] \end{array}$$

where I_{N_2} is the $N_2 \times N_2$ identity matrix, $\mathbf{0}_{N_2}$ is the $N_2 \times N_2$ zero matrix and

$$\Lambda = \begin{array}{cc} & \begin{array}{ccccc} v_1 & v_2 & v_3 & \cdots & v_{N_2} \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_{N_2-1} \\ v_{N_2} \end{array} & \left[\begin{array}{ccccc} 4 & -1 & 0 & \ldots & -1 \\ -1 & 4 & -1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & -1 & 4 & -1 \\ -1 & \ldots & 0 & -1 & 4 \end{array}\right] \end{array}.$$

We now find the eigenvalues of $\mathrm{L}(\mathcal{L}^*_{N_1\times N_2})$. We see from Figure 3.3 that, the coordinate $(0,0)$ in Figure 3.3a corresponds to $(1,1)$ in Figure 3.3b. Thus, the (k_1, k_2)-cell in Figure 3.3a corresponds to the $(e^{\mathrm{i}k_1\theta_1}, e^{\mathrm{i}k_2\theta_2})$ in Figure 3.3b where $\theta_1 = \frac{2\pi\ell_1}{N_1}$ and $\theta_2 = \frac{2\pi\ell_2}{N_2}$ with $0 \le \ell_1 \le N_1 - 1$ and $0 \le \ell_2 \le N_2 - 1$. Therefore we see that the (k_1, k_2)-th cell in Figure 3.3a corresponds to the cell obtained by multiplying the cell $(1,1)$ by $e^{\mathrm{i}(k_1\theta_1 + k_2\theta_2)}$ in Figure 3.3b. Let λ be an eigenvalue of $\mathrm{L}(\mathcal{L}^*_{N_1\times N_2})$. Since our lattice $\mathcal{L}^*_{N_1\times N_2}$ is regular, then for each cell (k_1, k_2) in $\mathcal{L}^*_{N_1\times N_2}$, the eigenvalue-eigenvector equation of the Laplacian $\mathrm{L}(\mathcal{L}^*_{N_1\times N_2})$ is

$$\begin{aligned}\lambda e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} &= (4-a(0,0))e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - a(-1,0)e^{\mathrm{i}((k_1-1)\theta_1+k_2\theta_2)} - a(1,0)e^{\mathrm{i}((k_1+1)\theta_1+k_2\theta_2)}\\ &\quad - a(0,-1)e^{\mathrm{i}(k_1\theta_1+(k_2-1)\theta_2)} - a(0,1)e^{\mathrm{i}(k_1\theta_1+(k_2+1)\theta_2)}\\ &= \left[4-a(0,0)-a(-1,0)e^{-\mathrm{i}\theta_1}-a(1,0)e^{\mathrm{i}\theta_1}-a(0,-1)e^{-\mathrm{i}\theta_2}-a(0,1)e^{\mathrm{i}\theta_2}\right]e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}.\end{aligned}$$

Let

$$\mathrm{M}(\theta_1,\theta_2) = 4-a(0,0)-a(-1,0)e^{-\mathrm{i}\theta_1}-a(1,0)e^{\mathrm{i}\theta_1}-a(0,-1)e^{-\mathrm{i}\theta_2}-a(0,1)e^{\mathrm{i}\theta_2}.$$

Substituting equation (3.4.8) we obtain

$$\begin{aligned}\mathrm{M}(\theta_1,\theta_2) &= 4-(e^{-\mathrm{i}\theta_1}+e^{\mathrm{i}\theta_1}+e^{-\mathrm{i}\theta_2}+e^{\mathrm{i}\theta_2})\\ &= 4-(2\cos\theta_1+2\cos\theta_2)\end{aligned}$$

and the eigenvalue-eigenvector equation becomes

$$\begin{aligned}\lambda e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} &= \mathrm{M}(\theta_1,\theta_2)e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}\\ &= \left[4-(2\cos\theta_1+2\cos\theta_2)\right]e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}\\ &= \left[4-\left(2\cos\left(\frac{2\pi\ell_1}{N_1}\right)+2\cos\left(\frac{2\pi\ell_2}{N_2}\right)\right)\right]e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}\end{aligned}$$

Comparing the left-hand side and the right-hand side of the above equation, we see that the eigenvalues are

$$\begin{aligned}\lambda &= \mathrm{M}\left(\frac{2\pi\ell_1}{N_1},\frac{2\pi\ell_2}{N_2}\right)\\ &= 4-\left(2\cos\left(\frac{2\pi\ell_1}{N_1}\right)+2\cos\left(\frac{2\pi\ell_2}{N_2}\right)\right)\end{aligned}$$

whose associated eigenvector has entries $e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}$ $(k_1 = 0,1,\ldots,N_1-1,\ k_2 = 0,1,\ldots,N_2-1)$ which is the tensor product $\boldsymbol{u}_{k_1}\otimes\boldsymbol{v}_{k_2}$. Hence, we have all the eigenvalues of $\mathrm{L}(\mathcal{L}^*_{N_1\times N_2})$. In particular, for $k_1 = k_2 = 0$ we have the all ones eigenvector corresponding to the zero eigenvalue. Thus from Corollary 2.5.3 the number of spanning trees of the rectangular lattice $\mathcal{L}_{N_1\times N_2}$ wrapped around the torus is

$$\begin{aligned}\tau(\mathcal{L}^*_{N_1\times N_2}) &= \frac{1}{N_1N_2}\prod_{\substack{\ell_1=0\\(\ell_1,\ell_2)\neq(0,0)}}^{N_1-1}\prod_{\ell_2=0}^{N_2-1}\lambda_{(\ell_1,\ell_2)}\\ &= \frac{1}{N_1N_2}\prod_{\substack{\ell_1=0\\(\ell_1,\ell_2)\neq(0,0)}}^{N_1-1}\prod_{\ell_2=0}^{N_2-1}\left[4-\left(2\cos\left(\frac{2\ell_1\pi}{N_1}\right)+2\cos\left(\frac{2\ell_2\pi}{N_2}\right)\right)\right]\\ &= \frac{1}{N_1N_2}\prod_{\substack{\ell_1=0\\(\ell_1,\ell_2)\neq(0,0)}}^{N_1-1}\prod_{\ell_2=0}^{N_2-1}\left[4-2\left(\cos\left(\frac{2\ell_1\pi}{N_1}\right)+\cos\left(\frac{2\ell_2\pi}{N_2}\right)\right)\right]. \qquad (3.4.9)\end{aligned}$$

3.4.1 Asymptotic Growth of $\tau(\mathcal{L}_{N_1\times N_2})$

Lattices are studied frequently in physics, and a natural question that one can ask is given a lattice $\mathcal{L}$ on n vertices, how does the number of spanning trees $\tau(\mathcal{L})$ grow as $n\to\infty$? Let G be a graph of order n, the number of spanning trees $\tau(G)$ has asymptotic exponential growth. That is,

$$\tau(G)\approx \exp(nz_{\mathcal{L}}) \quad \text{as} \quad n\to\infty. \tag{3.4.10}$$

where $z_{\mathcal{L}}$ is called the bulk limit. We define the bulk limit as follows:

Definition 3.4.11. *Let $\mathcal{L}$ be a two-dimensional planar lattice with $\mathcal{L}^*$ as its dual. We define the* ***bulk limit*** *or* ***thermodynamic limit*** *denoted by $z_{\mathcal{L}}$ as*

$$z_{\mathcal{L}} = \lim_{|v(G)|\to\infty} \frac{\log\tau(G)}{|v(G)|}. \tag{3.4.12}$$

We also define the ratio of the bulk limits of $\mathcal{L}^*$ and $\mathcal{L}$ as

$$\nu_{\mathcal{L}} = \lim_{|v(G)|\to\infty} \frac{|v(G^*)|}{|v(G)|} \tag{3.4.13}$$

such that $\nu_{\mathcal{L}}\nu_{\mathcal{L}^*}=1$. Here, G is a finite section of $\mathcal{L}$ with G^* as its planar dual and $|v(G^*)| = |e(G)|-|v(G)|+1$, i.e., the number faces of G (disregarding the unbounded face).

Using equations (3.4.12) and (3.4.13) we have

$$z_{\mathcal{L}^*} = \frac{z_{\mathcal{L}}}{\nu_{\mathcal{L}}}. \tag{3.4.14}$$

As examples which we shall verify in the subsequent sections to come, we have

$$\nu_{\mathcal{L}_{hc}} = \frac{1}{2}, \quad \nu_{\mathcal{L}_{kag}} = 1, \quad \nu_{\mathcal{L}_{9\cdot 3}} = \frac{1}{2} \tag{3.4.15}$$

where $\mathcal{L}_{9\cdot 3}$ is the lattice with nanogons and triangles shown in Figure 3.8. We now compute the bulk limit of a rectangular lattice $\mathcal{L}_{N_1\times N_2}$. From equation (3.4.6) we have

$$\tau(\mathcal{L}_{N_1\times N_2}) = \frac{1}{N_1N_2}\prod_{\substack{k_1=0\\(k_1,k_2)\neq(0,0)}}^{N_1-1}\prod_{k_2=0}^{N_2-1}\left[4-2\left(\cos\left(\frac{k_1\pi}{N_1}\right)+\cos\left(\frac{k_2\pi}{N_2}\right)\right)\right]$$

$$\log\tau(\mathcal{L}_{N_1\times N_2}) = \log\frac{1}{N_1}+\log\frac{1}{N_2}+\sum_{\substack{k_1=0\\(k_1,k_2)\neq(0,0)}}^{N_1-1}\sum_{k_2=0}^{N_2-1}\log\left[4-2\left(\cos\left(\frac{k_1\pi}{N_1}\right)+\cos\left(\frac{k_2\pi}{N_2}\right)\right)\right]$$

$$\frac{\log\tau(\mathcal{L}_{N_1\times N_2})}{N_1N_2} = \frac{1}{N_1N_2}\left(\log\frac{1}{N_1}+\log\frac{1}{N_2}+\sum_{\substack{k_1=0\\(k_1,k_2)\neq(0,0)}}^{N_1-1}\sum_{k_2=0}^{N_2-1}\log\left[4-2\left(\cos\left(\frac{k_1\pi}{N_1}\right)+\cos\left(\frac{k_2\pi}{N_2}\right)\right)\right]\right). \tag{3.4.16}$$

As $N_1,N_2\to\infty$, $\frac{1}{N_1N_2}\log\frac{1}{N_1}\to 0$ and $\frac{1}{N_1N_2}\log\frac{1}{N_2}\to 0$. Since the cosine function is continuous, the sum converges to an integral. Let $x=\frac{k_2}{N_2}$ and $y=\frac{k_1}{N_1}$. Then x and y

ranges from 0 (since k_1 and k_2 starts from 0) to 1 (since $\frac{N_1-1}{N_1} \to 1$ as $N_1 \to \infty$ and $\frac{N_2-1}{N_2} \to 1$ as $N_2 \to \infty$). Finally, $\mathrm{d}x = \frac{1}{N_2}$, $\mathrm{d}y = \frac{1}{N_1}$ and equation (3.4.16) now becomes,

$$\begin{aligned} z_{\mathcal{L}_{N_1 \times N_2}} = \lim_{N_1, N_2 \to \infty} \frac{\log \tau(\mathcal{L}_{N_1 \times N_2})}{N_1 N_2} &= \int_0^1 \int_0^1 \log(4 - 2\cos \pi x - 2\cos \pi y)\ \mathrm{d}x\ \mathrm{d}y \\ &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \log(4 - 2\cos x - 2\cos y)\ \mathrm{d}x\ \mathrm{d}y \\ &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \log(2 + 2 - 2\cos x - 2\cos y)\ \mathrm{d}x\ \mathrm{d}y. \end{aligned} \tag{3.4.17}$$

The integral improper and has a logarithmic singularity at the point $(0,0)$ hence it converges. The generalized version of the integral above

$$f(u,v) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \log(u + v - u\cos x - v\cos y)\ \mathrm{d}x\ \mathrm{d}y$$

is evaluated in Appendix A with

$$\begin{aligned} f(u,v) &= \frac{4}{\pi}\left[\left(\frac{u}{v}\right)^{\frac{1}{2}} - \frac{1}{3^2}\left(\frac{u}{v}\right)^{\frac{3}{2}} + \frac{1}{5^2}\left(\frac{u}{v}\right)^{\frac{5}{2}} - \frac{1}{7^2}\left(\frac{u}{v}\right)^{\frac{7}{2}} + \dots\right] + \log\left(\frac{v}{2}\right) \qquad (3.4.18) \\ &= \frac{4}{\pi}\left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}\left(\frac{u}{v}\right)^{k+\frac{1}{2}}\right] + \log\left(\frac{v}{2}\right). \end{aligned}$$

Comparing equations (3.4.17) and (3.4.18), we set $u = 2$ and $v = 2$ and we obtain

$$\begin{aligned} z_{\mathcal{L}_{N_1 \times N_2}} &= f(2,2) \\ &= \frac{4}{\pi}\left[1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots\right] + \log\left(\frac{2}{2}\right) \\ &= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \\ &= \frac{4}{\pi} G, \qquad (3.4.19) \\ &= 1.166\ 243\ 616 \dots \end{aligned}$$

where

$$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

is called the *Catalan constant.*

Therefore from equation (3.4.10), the number of spanning trees of a rectangular lattice $\mathcal{L}_{N_1 \times N_2}$ is asymptotically

$$\tau(\mathcal{L}_{N_1 \times N_2}) \approx \exp\left(\frac{4N_1N_2}{\pi} G\right) \approx 3.2099^{N_1 N_2}.$$

Remark 3.4.20.
If one wraps the rectangular lattice $\mathcal{L}_{N_1 \times N_2}$ on a torus, from equation (3.4.9), the bulk limit becomes

$$z_{\mathcal{L}_{N_1 \times N_2}} = \lim_{N_1, N_2 \to \infty} \frac{\log \tau(\mathcal{L}_{N_1 \times N_2})}{N_1 N_2} = \int_0^1 \int_0^1 \log(4 - 2\cos 2\pi x - 2\cos 2\pi y)\ \mathrm{d}x\ \mathrm{d}y.$$

Using the generalization of this integral in Appendix A we get the same result in equation (3.4.18).

3.5 Spanning trees on Triangular Lattice $\mathcal{L}_{tri}$

In this section, we will derive an explicit formula for the number of spanning trees of a triangular lattice $\mathcal{L}_{tri}$ as well as its bulk limit $z_{\mathcal{L}_{tri}}$. Note that the dual of the triangular lattice, $\mathcal{L}^*_{tri}$, is the honeycomb lattice $\mathcal{L}_{hc}$. We recall from Theorem 2.7.19 that the Tutte polynomial satisfies

$$T(\mathcal{L}^*_{tri}; x, y) = T(\mathcal{L}_{hc}; x, y) = T(\mathcal{L}_{tri}; y, x),$$

and substituting $x = y = 1$ gives us the number of spanning trees (see Remark 2.7.32). Thus we see that counting the number of spanning trees of a finte triangular lattice $\mathcal{L}_{tri}$ also results in counting the number of spanning trees of its dual, the honeycomb lattice $\mathcal{L}_{hc}$ which is also finite. That is,

$$\tau(\mathcal{L}_{tri}) = \tau(\mathcal{L}_{hc}). \tag{3.5.1}$$

The triangular lattice $\mathcal{L}_{tri}$ can be regarded as an $N_1 \times N_2$ rectangular lattice $\mathcal{L}_{N_1 \times N_2}$ with an additional edge added in the same way to every unit square of the rectangular lattice $\mathcal{L}_{N_1 \times N_2}$. For example

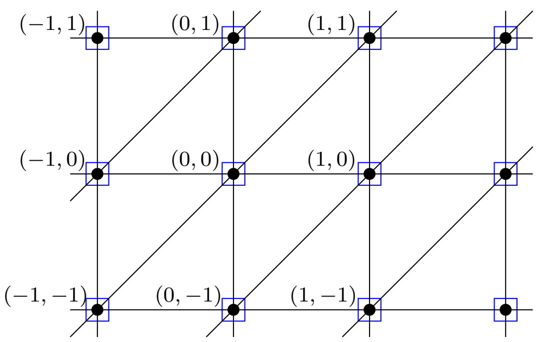
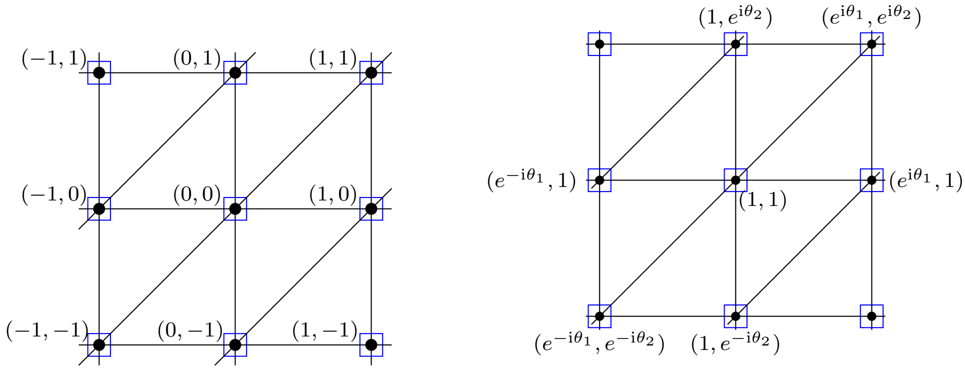


Figure 3.4: A triangular lattice $\mathcal{L}_{tri}$ on 12 vertices.

We now derive an explicit formula for counting the number of spanning trees in a finite triangular lattice $\mathcal{L}_{tri}$ of order n. We follow the method that we applied to the rectangular lattice in the previous section to compute the number of spanning trees $\tau(\mathcal{L}_{tri})$ of a finite triangular lattice. Here each cell contain a single vertex so $\nu = 1$ and we have N_1 vertices in the vertical direction, N_2 vertices in the horizontal direction and $n = N_1 N_2$. Wrapping the triangular lattice around a torus, we obtain a 6-regular lattice. From Figure 3.4 we see that, each unit cell contains a single vertex and since the lattice is invariant under translation, the 1×1 vertex cell adjacency matrix of the $(0, 0)$ cell is the same as all the other cells (k_1, k_2) $(k_1 = 0, 1, \ldots, N_1 - 1,\ k_2 = 0, 1, \ldots, N_2 - 1)$.

$$a(0,0) = 0,\ a(-1,0) = a(1,0) = a(0,-1) = a(0,1) = a(-1,-1) = a(1,1) = 1. \tag{3.5.2}$$



(a) Labelling cells (k_1, k_2).   (b) Labelling cells $(e^{\mathrm{i}k_1\theta_1}, e^{\mathrm{i}k_2\theta_2})$.

Figure 3.5: Triangular lattice $\mathcal{L}_{tri}$ with eigenvectors.

We construct eigenvectors as follows: The values in cell (k_1, k_2) are obtained from the values in cell $(0, 0)$ by multiplying by a factor $e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}$ where $\theta_1 = \frac{2\pi\ell_1}{N_1}$ and $\theta_2 = \frac{2\pi\ell_2}{N_2}$. Therefore, we also label cell (k_1, k_2) by $(e^{\mathrm{i}k_1\theta_1}, e^{\mathrm{i}k_2\theta_2})$ as indicated in Figure 3.5b.
Let λ be an eigenvalue of $\mathrm{L}(\mathcal{L}_{tri})$. Since the triangular lattice $\mathcal{L}_{tri}$ is regular, then for each cell (k_1, k_2) the eigenvalue-eigenvector equation of the Laplacian $\mathrm{L}(\mathcal{L}_{tri})$ is

$$\begin{aligned}
\lambda e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} &= (6 - a(0,0))e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - a(-1,-1)e^{\mathrm{i}((k_1-1)\theta_1+(k_2-1)\theta_2)} - a(0,-1)e^{\mathrm{i}(k_1\theta_1+(k_2-1)\theta_2)} \\
&\quad - a(-1,0)e^{\mathrm{i}((k_1-1)\theta_1+k_2\theta_2)} - a(1,0)e^{\mathrm{i}((k_1+1)\theta_1+k_2\theta_2)} - a(0,1)e^{\mathrm{i}(k_1\theta_1+(k_2+1)\theta_2)} \\
&\quad - a(1,1)e^{\mathrm{i}((k_1+1)\theta_1+(k_2+1)\theta_2)} \\
&= \big[6 - a(0,0) - a(-1,-1)e^{-\mathrm{i}(\theta_1+\theta_2)} - a(0,-1)e^{-\mathrm{i}\theta_2} - a(-1,0)e^{-\mathrm{i}\theta_1} - a(1,0)e^{\mathrm{i}\theta_1} \\
&\quad - a(0,1)e^{\mathrm{i}\theta_2} - a(1,1)e^{\mathrm{i}(\theta_1+\theta_2)}\big]e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}.
\end{aligned}$$

Substituting equation (3.5.2) into the above equation we obtain

$$\begin{aligned}
\lambda e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} &= \big[6 - \big(e^{-\mathrm{i}(\theta_1+\theta_2)} + e^{-\mathrm{i}\theta_2} + e^{-\mathrm{i}\theta_1} + e^{\mathrm{i}\theta_1} + e^{\mathrm{i}\theta_2} + e^{\mathrm{i}(\theta_1+\theta_2)}\big)\big]e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} \\
&= \big[6 - \big(e^{-\mathrm{i}\theta_1} + e^{\mathrm{i}\theta_1} + e^{-\mathrm{i}\theta_2} + e^{\mathrm{i}\theta_2} + e^{-\mathrm{i}(\theta_1+\theta_2)} + e^{\mathrm{i}(\theta_1+\theta_2)}\big)\big]e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} \\
&= \big[6 - \big(2\cos\theta_1 + 2\cos\theta_2 + 2\cos(\theta_1+\theta_2)\big)\big]e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}
\end{aligned}$$

Let $\mathrm{M}(\theta_1, \theta_2) = 6 - (2\cos\theta_1 + 2\cos\theta_2 + 2\cos(\theta_1+\theta_2))$ then the above equation now becomes

$$\lambda e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} = \mathrm{M}(\theta_1, \theta_2)e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}. \tag{3.5.3}$$

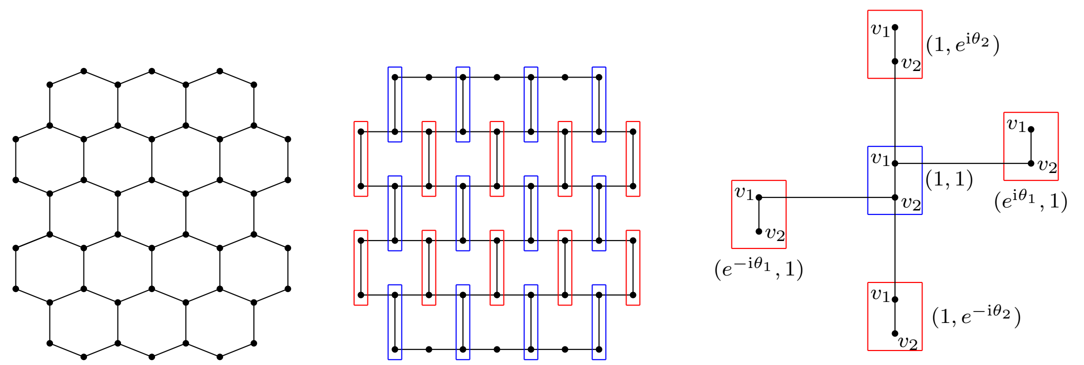
Comparing both sides of equation (3.5.3) we have

$$\begin{aligned}
\lambda_{(\ell_1,\ell_2)} &= \mathrm{M}\left(\frac{2\pi\ell_1}{N_1}, \frac{2\pi\ell_2}{N_2}\right) \\
&= 6 - \left(2\cos\left(\frac{2\pi\ell_1}{N_1}\right) + 2\cos\left(\frac{2\pi\ell_2}{N_2}\right) + 2\cos\left(\frac{2\pi\ell_1}{N_1} + \frac{2\pi\ell_2}{N_2}\right)\right).
\end{aligned} \tag{3.5.4}$$

Again from equation (3.5.4) if $\ell_1 = \ell_2 = 0$, we have the zero eigenvalue, $\lambda_{(0,0)}$ associated with the all ones eigenvector. Therefore from Corollary 2.5.3 the number of spanning trees of a finite triangular lattice is

$$\begin{aligned}
\tau(\mathcal{L}_{tri}) &= \frac{1}{N_1N_2}\prod_{\substack{\ell_1=0\\(\ell_1,\ell_2)\neq(0,0)}}^{N_1-1}\prod_{\ell_2=0}^{N_2-1}\lambda_{(\ell_1,\ell_2)}\\
&= \frac{1}{N_1N_2}\prod_{\substack{\ell_1=0\\(\ell_1,\ell_2)\neq(0,0)}}^{N_1-1}\prod_{\ell_2=0}^{N_2-1}\left[6-\left(2\cos\left(\frac{2\ell_1\pi}{N_1}\right)+2\cos\left(\frac{2\ell_2\pi}{N_2}\right)+2\cos\left(\frac{2\ell_1\pi}{N_1}+\frac{2\ell_2\pi}{N_2}\right)\right)\right]\\
&= \frac{1}{N_1N_2}\prod_{\substack{\ell_1=0\\(\ell_1,\ell_2)\neq(0,0)}}^{N_1-1}\prod_{\ell_2=0}^{N_2-1}\left[6-2\left(\cos\left(\frac{2\ell_1\pi}{N_1}\right)+\cos\left(\frac{2\ell_2\pi}{N_2}\right)+\cos\left(\frac{2\ell_1\pi}{N_1}+\frac{2\ell_2\pi}{N_2}\right)\right)\right]. \qquad (3.5.5)
\end{aligned}$$

Remark 3.5.6.
We have seen that for a finite honeycomb lattice, $\tau(\mathcal{L}_{hc}) = \tau(\mathcal{L}_{tri})$ provided they are of the same size. In counting the number of spanning trees of a finite honeycomb lattice, we consider the honeycomb lattice as a square lattice with the unit cells containing two vertices (i.e., $\nu = 2$) with labels 1 and 2 as shown in Figure 3.6b and 3.6c below.



(a) A finite honeycomb lattice. (b) Honeycomb lattice in a "brick wall" form. (c) Labelling cells $(e^{ik_1\theta_1}, e^{ik_2\theta_2})$.

Figure 3.6: Honeycomb lattices $\mathcal{L}_{hc}$.

Wrapping the lattice around a torus we have a 3-regular lattice, and the 2×2 vertex cell adjacency matrices (i.e., the matrix that describes the connectivity between the vertices in the cells) of the $(0, 0)$-cell are the same as those of the other cells. Using Figure 3.6c we have:

$$\begin{aligned}
&a(0,0) = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}, \quad a(-1,0) = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{array}, \quad a(1,0) = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array},\\
&a(0,-1) = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{array}, \quad a(0,1) = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}.
\end{aligned} \qquad (3.5.7)$$

Let λ be an eigenvalue of the Laplacian $\mathrm{L}(\mathcal{L}_{hc})$ then for each vertex v_1, v_2 in the cell (k_1, k_2) (see Figure 3.6c) eigenvalue-eigenvector equation is as follows

$$\lambda v_1 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} = 3v_1 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - v_2 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - v_2 e^{\mathrm{i}((k_1+1)\theta_1+k_2\theta_2)} - v_2 e^{\mathrm{i}(k_1\theta_1+(k_2+1)\theta_2)}$$
$$\lambda v_2 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} = -v_1 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - v_1 e^{\mathrm{i}((k_1+1)\theta_1+k_2\theta_2)} - v_1 e^{\mathrm{i}(k_1\theta_1+(k_2+1)\theta_2)} + 3v_2 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}.$$

where $\theta_i = \frac{2\pi\ell_i}{N_i}$ $(i = 1, 2)$. The above equation can also be written in a matrix form as

$$\begin{aligned}\begin{bmatrix}\lambda v_1\\ \lambda v_2\end{bmatrix} e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} &= \Big[(3\mathrm{I}_2 - a(0,0))e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - a(-1,0)e^{\mathrm{i}((k_1-1)\theta_1+k_2\theta_2)} - a(1,0)e^{\mathrm{i}((k_1+1)\theta_1+k_2\theta_2)} \\ &\quad - a(0,-1)e^{\mathrm{i}(k_1\theta_1+(k_2-1)\theta_2)} - a(0,1)e^{\mathrm{i}(k_1\theta_1+(k_2+1)\theta_2)}\Big] \cdot \begin{bmatrix}v_1\\ v_2\end{bmatrix} \\ &= \Big[\big[3\mathrm{I}_2 - a(0,0) - a(-1,0)e^{-\mathrm{i}\theta_1} - a(0,1)e^{\mathrm{i}\theta_1} - a(0,-1)e^{-\mathrm{i}\theta_2} \\ &\quad - a(0,1)e^{\mathrm{i}\theta_2}\big]e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}\Big] \cdot \begin{bmatrix}v_1\\ v_2\end{bmatrix}.\end{aligned}$$

Now let

$$\mathrm{M}(\theta_1, \theta_2) = 3\mathrm{I}_2 - a(0,0) - a(-1,0)e^{-\mathrm{i}\theta_1} - a(1,0)e^{\mathrm{i}\theta_1} - a(0,-1)e^{-\mathrm{i}\theta_2} - a(0,1)e^{\mathrm{i}\theta_2}.$$

Then substituting equation (3.5.7) into the above we obtain the 2×2 matrix

$$\mathrm{M}(\theta_1, \theta_2) = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 3 & -(1+e^{\mathrm{i}\theta_1}+e^{\mathrm{i}\theta_2}) \\ -(1+e^{-\mathrm{i}\theta_1}+e^{-\mathrm{i}\theta_2}) & 3 \end{bmatrix} \end{array}. \tag{3.5.8}$$

So the eigenvalue-eigenvector equation now becomes

$$\begin{bmatrix}\lambda v_1\\ \lambda v_2\end{bmatrix} e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} = \mathrm{M}(\theta_1,\theta_2)e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} \cdot \begin{bmatrix}v_1\\ v_2\end{bmatrix}.$$

Comparing both sides of the equation and using the fact that the determinant is equal to the product of the eigenvalues we have two eigenvectors and eigenvalues for each $(\theta_1, \theta_2) \neq (0, 0)$ and the product of the two eigenvalues is

$$\begin{aligned}\det(\mathrm{M}(\theta_1,\theta_2)) &= 9 - (3 + e^{-\mathrm{i}\theta_1} + e^{\mathrm{i}\theta_1} + e^{-\mathrm{i}\theta_2} + e^{\mathrm{i}\theta_2} + e^{-\mathrm{i}(\theta_1-\theta_2)} + e^{\mathrm{i}(\theta_1-\theta_2)}) \\ &= 6 - 2(\cos\theta_1 + \cos\theta_2 + \cos(\theta_1 - \theta_2)) \\ &= 6 - 2(\cos\theta_1 + \cos\theta_2 + \cos(\theta_1 + \theta_2)).\end{aligned}$$

From equation (3.5.8), setting $\theta_1 = \theta_2 = 0$ we have

$$\mathrm{M}(0,0) = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \end{array}$$

which has eigenvalues $\lambda_1 = 6$ and $\lambda_2 = 0$. Putting everything together and using Corollary 2.5.3 we have

$$\begin{aligned}
\tau(\mathcal{L}_{hc}) &= \frac{\lambda_1}{\nu N_1 N_2} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} \lambda_{(\ell_1,\ell_2)} \\
&= \frac{6}{2N_1N_2} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} \left[6 - 2\left(\cos\left(\frac{2\ell_1\pi}{N_1}\right) + \cos\left(\frac{2\ell_2\pi}{N_2}\right) + \cos\left(\frac{2\ell_1\pi}{N_1} + \frac{2\ell_2\pi}{N_2}\right)\right)\right] \\
&= \frac{3}{N_1N_2} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} \left[6 - 2\left(\cos\left(\frac{2\ell_1\pi}{N_1}\right) + \cos\left(\frac{2\ell_2\pi}{N_2}\right) + \cos\left(\frac{2\ell_1\pi}{N_1} + \frac{2\ell_2\pi}{N_2}\right)\right)\right]. \qquad (3.5.9)
\end{aligned}$$

Example 3.5.10.
As an example, we compute the number of spanning trees of the triangular lattice in Figure 3.4 of order 12. From equation (3.5.4) we have

$$\begin{aligned}
\tau(\mathcal{L}_{tri}) &= \frac{1}{12} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{2} \prod_{\ell_2=0}^{3} \left[6 - 2\left(\cos\left(\frac{2\ell_1\pi}{3}\right) + \cos\left(\frac{\ell_2\pi}{2}\right) + \cos\left(\frac{2\ell_1\pi}{3} + \frac{\ell_2\pi}{2}\right)\right)\right] \\
&= \frac{1}{12}\left[4 \cdot 8 \cdot 4 \cdot 6 \cdot (7+\sqrt{3}) \cdot 8 \cdot (7-\sqrt{3}) \cdot 6 \cdot (7-\sqrt{3}) \cdot 8 \cdot (7+\sqrt{3})\right] \\
&= \frac{624033792}{12} \\
&= 52002816.
\end{aligned}$$

We used the computer algebra system SAGE to write a program to compute $\tau(\mathcal{L}_{tri})$. See Appendix C.2 for this.

3.5.1 Asymptotic Growth of $\tau(\mathcal{L}_{tri})$

We compute the bulk limit of the triangular lattice z_{tri}. From equation (3.5.5) we have

$$\tau(\mathcal{L}_{tri}) = \frac{1}{N_1N_2} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} \left[6 - 2\left(\cos\left(\frac{2\ell_1\pi}{N_1}\right) + \cos\left(\frac{2\ell_2\pi}{N_2}\right) + \cos\left(\frac{2\ell_1\pi}{N_1} + \frac{2\ell_2\pi}{N_2}\right)\right)\right]$$

$$\log\tau(\mathcal{L}_{tri}) = \log\frac{1}{N_1} + \log\frac{1}{N_2} + \sum_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1} \sum_{\ell_2=0}^{N_2-1} \log\left[6 - 2\left(\cos\left(\frac{2\ell_1\pi}{N_1}\right) + \cos\left(\frac{2\ell_2\pi}{N_2}\right) + \cos\left(\frac{2\ell_1\pi}{N_1} + \frac{2\ell_2\pi}{N_2}\right)\right)\right]$$

$$\frac{\log\tau(\mathcal{L}_{tri})}{N_1N_2} = \frac{1}{N_1N_2}\left(\log\frac{1}{N_1} + \log\frac{1}{N_2} + \sum_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1} \sum_{\ell_2=0}^{N_2-1} \log\left[6 - 2\left(\cos\left(\frac{2\ell_1\pi}{N_1}\right) + \cos\left(\frac{2\ell_2\pi}{N_2}\right) + \cos\left(\frac{2\ell_1\pi}{N_1} + \frac{2\ell_2\pi}{N_2}\right)\right)\right]\right) \qquad (3.5.11)$$

Again the cosine function is continuous so the sum converges to an integral as $N_1, N_2 \to \infty$, $\frac{1}{N_1N_2}\log\frac{1}{N_1} \to 0$ and $\frac{1}{N_1N_2}\log\frac{1}{N_2} \to 0$. Let $y = \frac{\ell_1}{N_1}$ and $x = \frac{\ell_2}{N_2}$. Then x and y ranges from 0 (since ℓ_1 and ℓ_2 starts from 0) to 1 (since $\frac{N_1-1}{N_1} \to 1$ as $N_1 \to \infty$ and $\frac{N_2-1}{N_2} \to 1$ as $N_2 \to \infty$). Lastly, $\mathrm{d}x = \frac{1}{N_2}$ and $\mathrm{d}y = \frac{1}{N_1}$ and equation (3.5.11) now becomes

$$\begin{aligned} z_{\mathcal{L}_{tri}} = \lim_{N_1,N_2\to\infty} \frac{\log \tau(\mathcal{L}_{tri})}{N_1 N_2} &= \int_0^1 \int_0^1 \log\left[6 - 2\cos 2\pi x - 2\cos 2\pi y - 2\cos(2\pi x + 2\pi y)\right] \mathrm{d}x\, \mathrm{d}y \\ &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log\left[6 - 2\cos x - 2\cos y - 2\cos(x+y)\right] \mathrm{d}x\, \mathrm{d}y \\ &= \frac{3\sqrt{3}}{\pi}\left[1 - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{13^2} - \cdots\right] \qquad (3.5.12) \\ &= 1.615\ 329\ 736\ 097\ldots. \end{aligned}$$

Equation (3.5.12) is obtained using two different approaches (algebraic and graphical) in [28]. Lastly, using equations (3.4.9), (3.4.12) we obtain the bulk limit of the honeycomb lattice $\mathcal{L}_{hc}$ as

$$z_{\mathcal{L}_{hc}} = \frac{1}{2} z_{\mathcal{L}_{tri}} = 0.8076648680485.$$

From equation (3.4.10), the number of spanning trees of the triangular lattice $\mathcal{L}_{tri}$ is asymptotically

$$\tau(\mathcal{L}_{tri}) \approx \exp\left(N_1 N_2 z_{\mathcal{L}_{tri}}\right) \approx 5.02955^{N_1 N_2}$$

and the number of spanning trees of the honeycomb lattice $\mathcal{L}_{tri}$ is asymptotically

$$\tau(\mathcal{L}_{hc}) \approx \exp\left(\nu N_1 N_2 z_{\mathcal{L}_{hc}}\right) = \exp\left(2 N_1 N_2 z_{\mathcal{L}_{hc}}\right) \approx 5.02955^{N_1 N_2}.$$

3.6 Spanning Trees on Kagomé Lattice $\mathcal{L}_{kag}$

We now want to count the number of spanning trees in a finite kagomé lattice. The kagomé lattice can be considered as a square grid of unit cells with each cell containing 3 vertices forming a triangle as shown in Figure 3.7. Thus $\nu = 3$. Wrapping the kagomé lattice $\mathcal{L}_{kag}$ around a torus, we obtain a 4-regular lattice and the 3×3 cell vertex adjacency matrices are defined as follows:

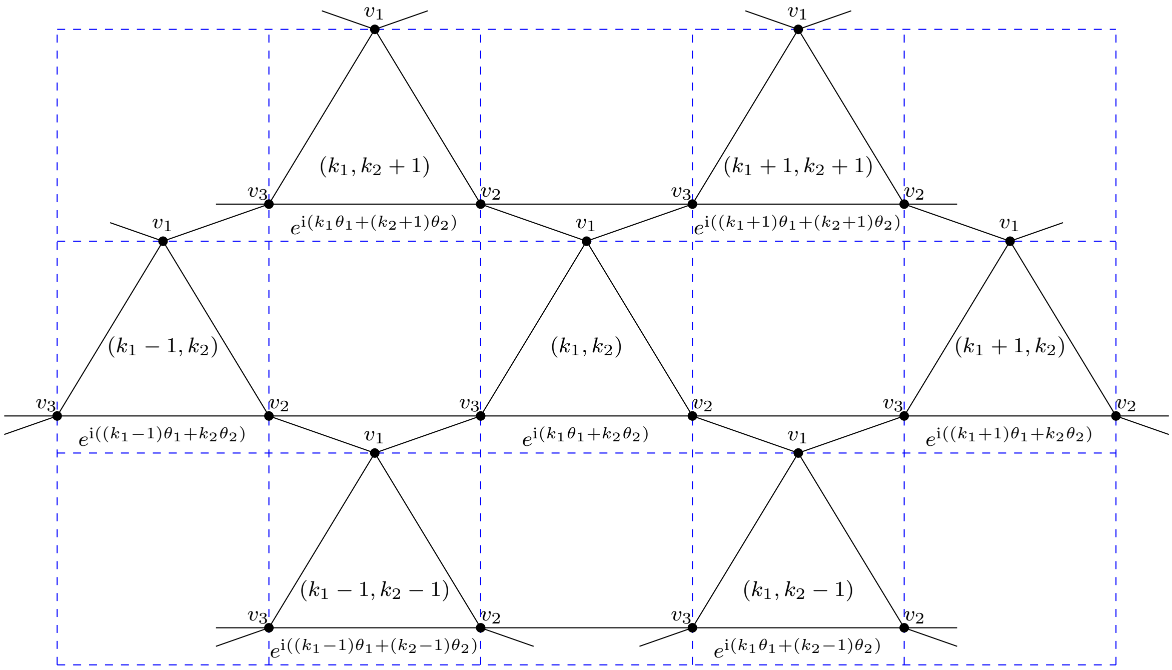


Figure 3.7: A finite kagomé lattice $\mathcal{L}_{kag}$ with cells labelled.

$$a(0,0) = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}, \quad a(-1,0) = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}, \quad a(1,0) = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$a(0,-1) = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \quad a(0,1) = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \quad a(-1,-1) = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array},$$

$$a(1,1) = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}. \tag{3.6.1}$$

Again we wrap the kagomé lattice $\mathcal{L}_{kag}$ around a torus we have periods N_1 in the horizontal direction and N_2 in the other direction ("right slanted" $\nwarrow$) as indicated in Figure 3.7. Thus the cells in the horizontal direction corresponds to a N_1-th root of unity and the cells in the other direction corresponds to a N_2-th root of unity. Let $\theta_1 = \frac{2\pi\ell_1}{N_1}$ and $\theta_2 = \frac{2\pi\ell_2}{N_2}$ and let λ be an associated eigenvalue of the Laplacian $\mathrm{L}(\mathcal{L}_{kag})$. For the vertices v_1, v_2, v_3 in the cell (k_1, k_2) (see Figure 3.7), the eigenvalue-eigenvector equations are as follows:

$$
\begin{aligned}
\lambda v_1 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} &= 4v_1 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - v_2 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - v_2 e^{\mathrm{i}(k_1\theta_1+(k_2+1)\theta_2)} - v_3 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}\\
&\quad - v_3 e^{\mathrm{i}((k_1+1)\theta_1+(k_2+1)\theta_2)}\\
\lambda v_2 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} &= -v_1 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - v_1 e^{\mathrm{i}(k_1\theta_1+(k_2-1)\theta_2)} + 4v_2 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - v_3 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}\\
&\quad - v_3 e^{\mathrm{i}((k_1+1)\theta_1+k_2\theta_2)}\\
\lambda v_3 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} &= -v_1 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - v_1 e^{\mathrm{i}((k_1-1)\theta_1+(k_2-1)\theta_2)} - v_2 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - v_2 e^{\mathrm{i}((k_1-1)\theta_1+k_2\theta_2)}\\
&\quad + 4v_3 e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}.
\end{aligned}
$$

The above equation can be written in a matrix form as

$$
\begin{aligned}
\begin{bmatrix}\lambda v_1\\ \lambda v_2\\ \lambda v_3\end{bmatrix} e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} &= \Big[(4\mathrm{I}_3 - a(0,0))e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - a(-1,0)e^{\mathrm{i}((k_1-1)\theta_1+k_2\theta_2)} - a(1,0)e^{\mathrm{i}((k_1+1)\theta_1+k_2\theta_2)}\\
&\quad - a(0,-1)e^{\mathrm{i}(k_1\theta_1+(k_2-1)\theta_2)} - a(0,1)e^{\mathrm{i}(k_1\theta_1+(k_2+1)\theta_2)} - a(-1,-1)e^{\mathrm{i}((k_1-1)\theta_1+(k_2-1)\theta_2)}\\
&\quad - a(1,1)e^{\mathrm{i}((k_1+1)\theta_1+(k_2+1)\theta_2)}\Big] \cdot \begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}\\
&= \Big[\big[4\mathrm{I}_3 - a(0,0) - a(-1,0)e^{-\mathrm{i}\theta_1} - a(1,0)e^{\mathrm{i}\theta_1} - a(0,-1)e^{-\mathrm{i}\theta_2} - a(0,1)e^{\mathrm{i}\theta_2}\\
&\quad - a(-1,-1)e^{-\mathrm{i}(\theta_1+\theta_2)} - a(1,1)e^{\mathrm{i}(\theta_1+\theta_2)}\big]e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}\Big] \cdot \begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}. \qquad (3.6.2)
\end{aligned}
$$

Now let

$$
\begin{aligned}
\mathrm{M}(\theta_1,\theta_2) = 4\mathrm{I}_3 &- a(0,0) - a(-1,0)e^{-\mathrm{i}\theta_1} - a(1,0)e^{\mathrm{i}\theta_1} - a(0,-1)e^{-\mathrm{i}\theta_2} - a(0,1)e^{\mathrm{i}\theta_2}\\
&- a(-1,-1)e^{-\mathrm{i}(\theta_1+\theta_2)} - a(1,1)e^{\mathrm{i}(\theta_1+\theta_2)}.
\end{aligned}
$$

Substituting equation (3.6.1) into the above equation we obtain the 3×3 matrix

$$
\mathrm{M}(\theta_1,\theta_2) = \begin{array}{c} \\ v_1\\ v_2\\ v_3\end{array}\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3\end{array}\\ \begin{bmatrix} 4 & -(1+e^{\mathrm{i}\theta_2}) & -(1+e^{\mathrm{i}(\theta_1+\theta_2)})\\ -(1+e^{-\mathrm{i}\theta_2}) & 4 & -(1+e^{\mathrm{i}\theta_1})\\ -(1+e^{-\mathrm{i}(\theta_1+\theta_2)}) & -(1+e^{-\mathrm{i}\theta_1}) & 4 \end{bmatrix}\end{array}. \qquad (3.6.3)
$$

Equation (3.6.2) now becomes

$$
\begin{bmatrix}\lambda v_1\\ \lambda v_2\\ \lambda v_3\end{bmatrix} e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} = \mathrm{M}(\theta_1,\theta_2)e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} \begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}.
$$

Again we use the fact that the determinant of a matrix is the product of its eigenvalues. Finding the determinant of $\mathrm{M}(\theta_1,\theta_2)$ using the first row we have (for $(\theta_1,\theta_2) \neq (0,0)$)

$$
\begin{aligned}
\det(\mathrm{M}(\theta_1,\theta_2)) &= 56 - 8\cos\theta_1 - (10 + 2\cos\theta_1 + 10\cos\theta_2 + 2\cos(\theta_1+\theta_2))\\
&\quad - (10 + 2\cos\theta_1 + 2\cos\theta_2 + 10\cos(\theta_1+\theta_2))\\
&= 12[3 - (\cos\theta_1 + \cos\theta_2 + \cos(\theta_1+\theta_2))]. \qquad (3.6.4)
\end{aligned}
$$

From equation (3.6.3), setting $\theta_1 = \theta_2 = 0$ we have

$$\mathrm{M}(0,0) = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \end{array}$$

which has eigenvalues $\lambda_1 = \lambda_2 = 6$ and $\lambda_3 = 0$. Putting everything together and using Corollary 2.5.3, the number of spanning trees of the kagomé lattice $\tau(\mathcal{L}_{kag})$ is

$$\begin{aligned}
\tau(\mathcal{L}_{kag}) &= \frac{\lambda_1 \cdot \lambda_2}{\nu N_1 N_2} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} \det\left(\mathrm{M}\left(\frac{2\pi\ell_1}{N_1}, \frac{2\pi\ell_2}{N_2}\right)\right) \\
&= \frac{6^2}{3N_1N_2} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} 12\left[3 - \left(\cos\left(\frac{2\pi\ell_1}{N_1}\right) + \cos\left(\frac{2\pi\ell_2}{N_2}\right) + \cos\left(\frac{2\pi\ell_1}{N_1} + \frac{2\pi\ell_2}{N_2}\right)\right)\right] \\
&= \frac{12}{N_1N_2} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} \left[36 - 12\left(\cos\left(\frac{2\pi\ell_1}{N_1}\right) + \cos\left(\frac{2\pi\ell_2}{N_2}\right) + \cos\left(\frac{2\pi\ell_1}{N_1} + \frac{2\pi\ell_2}{N_2}\right)\right)\right]. \qquad (3.6.5)
\end{aligned}$$

Remark 3.6.6.
The dual of the kagomé lattice is the diced lattice $\mathcal{L}_{diced}$ and from Theorem 2.7.19 the Tutte polynomial satisfies

$$T(\mathcal{L}^*_{kag}; x, y) = T(\mathcal{L}_{diced}; x, y) = T(\mathcal{L}_{kag}; y, x).$$

Setting $x = y = 1$ we see that

$$\tau(\mathcal{L}_{kag}) = \tau(\mathcal{L}^*_{kag}) = \tau(\mathcal{L}_{diced}).$$

Example 3.6.7.
We compute the number of spanning trees of a finite kagomé lattice with $N_1 = 3$ and $N_2 = 4$.

$$\begin{aligned}
\tau(\mathcal{L}_{kag}) &= \frac{36}{3\cdot 3\cdot 4} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{2} \prod_{\ell_2=0}^{3} \left[36 - 12\left(\cos\left(\frac{2\pi\ell_1}{3}\right) + \cos\left(\frac{\pi\ell_2}{2}\right) + \cos\left(\frac{2\pi\ell_1}{3} + \frac{\pi\ell_2}{2}\right)\right)\right] \\
&= 24\cdot 48\cdot 24\cdot 36\cdot(42+6\sqrt{3})\cdot 48\cdot(42-6\sqrt{3})\cdot 36\cdot(42-6\sqrt{3})\cdot 48\cdot(42+6\sqrt{3}) \\
&= 226397622582116352.
\end{aligned}$$

We used the computer algebra system SAGE to write a program to compute $\tau(\mathcal{L}_{kag})$. See Appendix C.3 for this.

3.6.1 Asymptotic Growth of $\tau(\mathcal{L}_{kag})$

We compute the thermodynamic limit of the kagomé lattice. From equation (3.6.5) we have

$$\tau(\mathcal{L}_{kag}) = \frac{12}{N_1N_2}\prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1}\prod_{\ell_2=0}^{N_2-1}\left[36-12\left(\cos\left(\frac{2\pi\ell_1}{N_1}\right)+\cos\left(\frac{2\pi\ell_2}{N_2}\right)+\cos\left(\frac{2\pi\ell_1}{N_1}+\frac{2\pi\ell_2}{N_2}\right)\right)\right]$$

$$\frac{\log\tau(\mathcal{L}_{kag})}{3N_1N_2} = \frac{1}{3N_1N_2}\left(\log 12+\log\frac{1}{N_1}+\log\frac{1}{N_2}+\sum_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1}\sum_{\ell_2=0}^{N_2-1}\log\left[36-12\left(\cos\left(\frac{2\ell_1\pi}{N_1}\right)+\cos\left(\frac{2\ell_2\pi}{N_2}\right)+\cos\left(\frac{2\ell_1\pi}{N_1}+\frac{2\ell_2\pi}{N_2}\right)\right)\right]\right).$$

Using the same argument as before as $N_1, N_2 \to \infty$ we have

$$\begin{aligned}
z_{\mathcal{L}_{kag}} &= \frac{1}{3}\int_0^1\int_0^1 \log\big(36-12\cos 2\pi x-12\cos 2\pi y-12\cos(2\pi x+2\pi y)\big)\ \mathrm{d}x\ \mathrm{d}y \\
&= \frac{1}{3\cdot 4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\log\big(36-12\cos x-12\cos y-12\cos(x+y)\big)\ \mathrm{d}x\ \mathrm{d}y \\
&= \frac{1}{3\cdot 4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\log\big[6\big(6-2\cos x-2\cos y-2\cos(x+y)\big)\big]\mathrm{d}x\ \mathrm{d}y \\
&= \frac{1}{3\cdot 4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\big[\log 6+\log\big(6-2\cos x-2\cos y-2\cos(x+y)\big)\big]\mathrm{d}x\ \mathrm{d}y \\
&= \frac{1}{3}\big(\log 6+z_{\mathcal{L}_{tri}}\big) \\
&= 1.135\ 696\ 401\ 775\ldots. \\
&= z_{diced}, \quad \text{since from equation (3.4.13) } \nu_{\mathcal{L}_{kag}}=1.
\end{aligned}$$

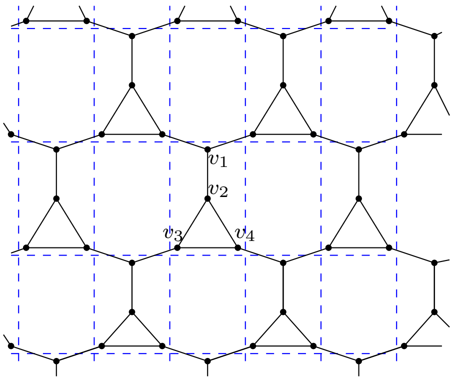
From equation (3.4.10), the number of spanning trees of the kagomé lattice $\mathcal{L}_{kag}$ is asymptotically,

$$\tau(\mathcal{L}_{kag}) \approx \exp\left(\nu N_1N_2 z_{\mathcal{L}_{kag}}\right) = \exp\left(3N_1N_2 z_{\mathcal{L}_{kag}}\right) \approx 30.17728^{N_1N_2} = \tau(\mathcal{L}_{diced})$$

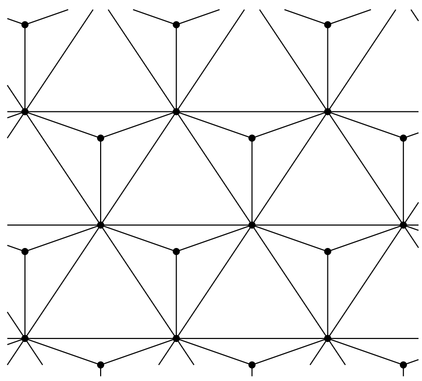
since $\nu_{\mathcal{L}_{kag}} = 1$.

3.7 Spanning Trees on a Lattice with Nanogons and Triangles $\mathcal{L}_{9.3}$

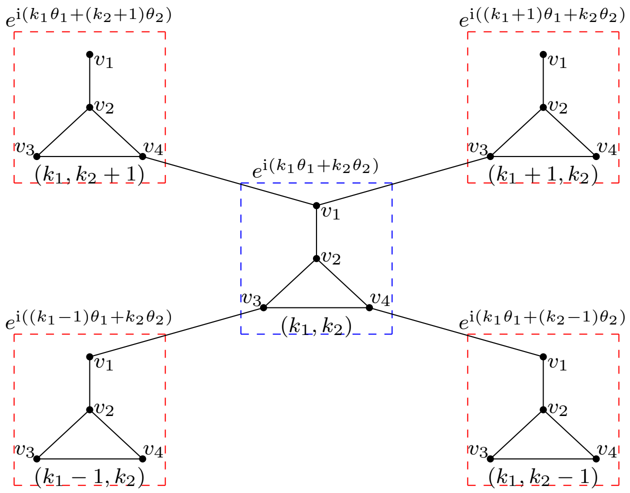
We follow a similar method discussed above to derive an explicit formula for the number of spanning trees on the $9\cdot 3$ lattice $\mathcal{L}_{9.3}$ and its dual $\mathcal{L}^*_{9.3}$. Here we treat the $9\cdot 3$ lattice as a square of unit cells with each cell containing 4 vertices as shown in the Figure 3.8c



(a) A finite $9 \cdot 3$ lattice $\mathcal{L}_{9\cdot 3}$.



(b) Dual of a finite $9 \cdot 3$ lattice $\mathcal{L}^*_{9\cdot 3}$.



(c) $9 \cdot 3$ lattice with coordinates of the cells.

Figure 3.8: A $9 \cdot 3$ lattice $\mathcal{L}_{9\cdot 3}$.

Thus $\nu = 4$ and from Figure 3.8c and using equation (3.4.1) the vertex cell adjacency matrices are as follows:

$$a(0,0) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}, \qquad a(-1,0) = a(1,0)^{\mathrm{T}} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$a(0,-1) = a(0,1)^{\mathrm{T}} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}. \tag{3.7.1}$$

Let λ be an eigenvalue of $\mathrm{L}(\mathcal{L}_{9\cdot 3})$. Then the eigenvalue-eigenvector equations for the

(k_1, k_2) cell become,

$$\begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \\ \lambda v_4 \end{bmatrix} e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} = \Big[(3\mathrm{I}_4 - a(0,0))e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} - a(-1,0)e^{\mathrm{i}((k_1-1)\theta_1+k_2\theta_2)} - a(1,0)e^{\mathrm{i}((k_1+1)\theta_1+k_2\theta_2)}$$

$$- a(0,-1)e^{\mathrm{i}(k_1\theta_1+(k_2-1)\theta_2)} - a(0,1)e^{\mathrm{i}(k_1\theta_1+(k_2+1)\theta_2)}\Big] \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$$= \Big[\big[3\mathrm{I}_4 - a(0,0) - a(-1,0)e^{-\mathrm{i}\theta_1} - a(0,1)e^{\mathrm{i}\theta_1} - a(0,-1)e^{-\mathrm{i}\theta_2}$$

$$- a(0,1)e^{\mathrm{i}\theta_2}\big]e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)}\Big] \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

Let

$$\mathrm{M}(\theta_1, \theta_2) = 3\mathrm{I}_4 - a(0,0) - a(-1,0)e^{-\mathrm{i}\theta_1} - a(0,1)e^{\mathrm{i}\theta_1} - a(0,-1)e^{-\mathrm{i}\theta_2} - a(0,1)e^{\mathrm{i}\theta_2}.$$

Substituting equation (3.7.1) into the above equation we have the 4×4 matrix

$$\mathrm{M}(\theta_1, \theta_2) = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 3 & -1 & -e^{\mathrm{i}\theta_1} & -e^{\mathrm{i}\theta_2} \\ -1 & 3 & -1 & -1 \\ -e^{-\mathrm{i}\theta_1} & -1 & 3 & -1 \\ -e^{-\mathrm{i}\theta_2} & -1 & -1 & 3 \end{bmatrix} \end{array}. \tag{3.7.2}$$

Thus the eigenvalue-eigenvector equation now becomes

$$\begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \\ \lambda v_4 \end{bmatrix} e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} = \mathrm{M}(\theta_1, \theta_2)e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

Again using the fact that the determinant of a matrix is the product of its eigenvalues we have (for $(\theta_1, \theta_2) \neq (0,0)$)

$$\begin{aligned} \det(\mathrm{M}(\theta_1, \theta_2)) &= 24 - 8(\cos\theta_1 + \cos\theta_2 + \cos(\theta_1 - \theta_2)) \\ &= 24 - 8(\cos\theta_1 + \cos\theta_2 + \cos(\theta_1 + \theta_2)). \end{aligned}$$

Setting $\theta_1 = \theta_2 = 0$ we have,

$$\mathrm{M}(0,0) = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \end{array}$$

which has eigenvalues $\lambda_1 = 4, \lambda_2 = 4, \lambda_3 = 4$ and $\lambda_4 = 0$. Thus the number of spanning trees of the $9 \cdot 3$ lattice $\tau(\mathcal{L}_{9\cdot 3})$ is

$$\begin{aligned}
\tau(\mathcal{L}_{9\cdot 3}) &= \frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3}{\nu N_1 N_2} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} \det\left(\mathrm{M}\left(\frac{2\pi\ell_1}{N_1}, \frac{2\pi\ell_2}{N_2}\right)\right) \\
&= \frac{16}{N_1 N_2} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} \left[24 - 8\left(\cos\left(\frac{2\pi\ell_1}{N_1}\right) + \cos\left(\frac{2\pi\ell_2}{N_2}\right) + \cos\left(\frac{2\pi\ell_1}{N_1} + \frac{2\pi\ell_2}{N_2}\right)\right)\right]. \qquad (3.7.3)
\end{aligned}$$

Example 3.7.4.
We compute the number of spanning trees of a finite $9 \cdot 3$ lattice with $N_1 = 3$ and $N_2 = 4$.

$$\begin{aligned}
\tau(\mathcal{L}_{9\cdot 3}) &= \frac{16}{3 \cdot 4} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{2} \prod_{\ell_2=0}^{3} \left[24 - 8\left(\cos\left(\frac{2\pi\ell_1}{N_1}\right) + \cos\left(\frac{2\pi\ell_2}{N_2}\right) + \cos\left(\frac{2\pi\ell_1}{N_1} + \frac{2\pi\ell_2}{N_2}\right)\right)\right] \\
&= \frac{4}{3}\Big[16 \cdot 32 \cdot 16 \cdot 24 \cdot 4(7+\sqrt{3}) \cdot 32 \cdot -4(-7+\sqrt{3}) \cdot 24 \cdot -4(-7+\sqrt{3}) \cdot 32 \cdot 4(7+\sqrt{3})\Big] \\
&= 3489849906561024.
\end{aligned}$$

We used the computer algebra system SAGE to write a program to compute $\tau(\mathcal{L}_{9\cdot 3})$. See Appendix C.4 for this.

3.7.1 Asymptotic Growth of $\tau(\mathcal{L}_{9\cdot 3})$

We compute the thermodynamic limit of the $9 \cdot 3$ lattice $\mathcal{L}_{9\cdot 3}$. From equation (3.7.3) we have

$$\tau(\mathcal{L}_{9\cdot 3}) = \frac{16}{N_1 N_2} \prod_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} \left[24 - 8\left(\cos\left(\frac{2\pi\ell_1}{N_1}\right) + \cos\left(\frac{2\pi\ell_2}{N_2}\right) + \cos\left(\frac{2\pi\ell_1}{N_1} - \frac{2\pi\ell_2}{N_2}\right)\right)\right]$$

$$\frac{\log\tau(\mathcal{L}_{9\cdot 3})}{4N_1N_2} = \frac{1}{4N_1N_2}\left(\log 16 + \log\frac{1}{N_1} + \log\frac{1}{N_2} + \sum_{\substack{\ell_1=0 \\ (\ell_1,\ell_2)\neq(0,0)}}^{N_1-1} \sum_{\ell_2=0}^{N_2-1} \log\left[24 - 8\left(\cos\left(\frac{2\ell_1\pi}{N_1}\right) + \cos\left(\frac{2\ell_2\pi}{N_2}\right) + \cos\left(\frac{2\ell_1\pi}{N_1} + \frac{2\ell_2\pi}{N_2}\right)\right)\right]\right).$$

Again using the same argument as before, as $N_1, N_2 \to \infty$ we have

$$\begin{aligned}
z_{\mathcal{L}_{9\cdot 3}} &= \frac{1}{4}\int_0^1\int_0^1 \log\big(24 - 8\cos 2\pi x - 8\cos 2\pi y - 8\cos(2\pi x + 2\pi y)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{4\cdot 4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log\big(24 - 8\cos x - 8\cos y - 8\cos(x+y)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{4\cdot 4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log\big[4\big(6 - 2\cos x - 2\cos y - 2\cos(x+y)\big)\big]\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{4\cdot 4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \big[\log 4 + \log\big(6 - 2\cos x - 2\cos y - 2\cos(x+y)\big)\big]\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{4}\big(\log 4 + z_{\mathcal{L}_{tri}}\big) \\
&= 0.750406024304223\ldots \\
&= \frac{1}{2} z_{\mathcal{L}^*_{9\cdot 3}},
\end{aligned}$$

since $z_{\mathcal{L}^*_{9\cdot 3}} = 2z_{\mathcal{L}_{9\cdot 3}}$ from equations (3.4.12) and (3.4.13). Thus $z_{\mathcal{L}^*_{9\cdot 3}} = 1.500\,812\,048\,608\ldots$.

Lastly from equation (3.4.10), the number of spanning trees of the $9\cdot 3$ lattice $\mathcal{L}_{9\cdot 3}$ is asymptotically,

$$\tau(\mathcal{L}_{9\cdot 3}) \approx \exp\left(\nu N_1 N_2 z_{\mathcal{L}_{9\cdot 3}}\right) = \exp\left(4N_1N_2 z_{\mathcal{L}_{9\cdot 3}}\right) \approx 20.11818^{N_1N_2}$$

and the number of spanning trees of the dual of the $9\cdot 3$ lattice $\mathcal{L}^*_{9\cdot 3}$ is

$$\tau(\mathcal{L}^*_{9\cdot 3}) \approx \exp\left(\nu N_1 N_2 z_{\mathcal{L}^*_{9\cdot 3}}\right) = \exp\left(2N_1N_2 z_{\mathcal{L}^*_{9\cdot 3}}\right) \approx 20.11818^{N_1N_2}.$$

Remark 3.7.5.
In conclusion, for a general planar lattice $\mathcal{L}$ in d dimensions one proceeds in a similar fashion as shown to obtain the number of spanning trees of the lattice $\mathcal{L}$ as

$$\tau(\mathcal{L}) = \frac{\Lambda}{\nu N_1\cdots N_d}\prod_{\substack{\ell_1=0 \\ (\boldsymbol{\ell}\neq\mathbf{0})}}^{N_1-1}\cdots\prod_{\ell_d=0}^{N_d-1}\det\left(\mathrm{M}\left(\frac{2\pi\ell_1}{N_1},\ldots,\frac{2\pi\ell_d}{N_d}\right)\right) \tag{3.7.6}$$

where $\boldsymbol{\ell} = (\ell_1, \ell_2, \ldots, \ell_d)$, $\mathbf{0} = (\underbrace{0,0,\ldots,0}_{d\text{ times}})$, and $\Lambda = \begin{cases} 1 & \text{if } \nu = 1, \\ \lambda_1\cdots\lambda_{\nu-1} & \text{if } \nu > 1. \end{cases}$

Chapter 4

Lattices and Perfect Matchings

4.1 Introduction

A combinatorial structure that corresponds to the configuration of a physical system is a perfect matching. In statistical mechanics, one is interested counting the number of all such possible configurations. The solution to this problem provides a good insight to the related physical system under study. One way to count the perfect matchings of a graph is to manually count all possible configurations. However, the task becomes complicated for graphs with large order and size and in fact it is known that for a general graph, the problem of counting all perfect matchings is a #P-complete problem [24]. Thus no efficient algorithm is known.

However, for planar graphs, this is not so since one can always define a Pfaffian orientation on a planar graph and using Cayley's theorem we are able to count all the perfect matchings of our planar graph. The goal of this chapter is to derive a closed expression which counts the exact number of perfect matchings on the rectangular lattice and its asymptotic formula. Furthermore, we will also derive an asymptotic formula for the number of perfect matchings on the triangular lattice and its dual, the honeycomb lattice.

Definition 4.1.1. *A **matching** M in a graph $G = (V, E)$, is a set of independent edges i.e. pairwise non-adjacent edges. The end vertices of each edge $e \in M$, are said to be **matched** under M and also said to be **covered** or **saturated** by M. If M covers every vertex in G then M is said to be a **perfect matching** or a **dimer covering**.*

Definition 4.1.2. *Let $G = (V, E)$ be a graph and $\mathcal{M} = \{M_i \mid i \in \mathbb{N}\}$ be the set of all matchings in G. Then we say M is a **maximum matching** if $M = \max\{M_i \mid i \in \mathbb{N}\}$ i.e., M has the largest cardinality. We say a matching is a **maximal matching** if it cannot be enlarged by the addition of any edge.*

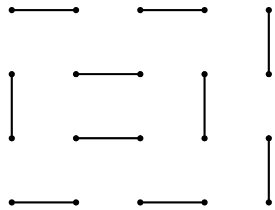


Figure 4.1: Perfect matchings on a rectangular lattice $\mathcal{L}_{4\times 5}$.

A perfect matching is a largest possible matching of a graph since it covers all the vertices of the graph. It is easy to see also that a graph has no perfect matching if it has odd order. Not all graphs have perfect matchings, however all graphs do have maximal matchings.

4.2 Pfaffian

Consider a weighted oriented graph $G = (V, E, \omega)$ with $V = \{v_1, v_2, \ldots, v_n\}$ and weight function $\omega : V \times V \to \mathbb{R}$. Using Definition 2.4.1 we define the entries of the adjacency matrix of G as follows

$$\mathrm{A}(G) = [a_{ij}]_{1\leq i,j\leq n},$$
$$\text{where } a_{ij} = \begin{cases} \omega(e), & \text{if } e = (v_i, v_j) \\ -\omega(e), & \text{if } e = (v_j, v_i) \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\mathrm{A}(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & \cdots & v_n \end{array} \\ \begin{bmatrix} 0 & a_{1,2} & a_{1,3} & \ldots & a_{1,n} \\ -a_{1,2} & 0 & a_{2,3} & \ldots & a_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -a_{1,n-1} & \ldots & -a_{n-2,n-1} & 0 & a_{n-1,n} \\ -a_{1,n} & \ldots & -a_{n-2,n} & -a_{n-1,n} & 0 \end{bmatrix} \end{array}. \tag{4.2.1}$$

We see from equation (4.2.1) that $\mathrm{A}(G)^{\mathrm{T}} = -\mathrm{A}(G)$. We call such a matrix a *real skew-symmetric matrix*. If n is odd, then

$$\det \mathrm{A}(G) = \det[\mathrm{A}(G)^{\mathrm{T}}] = \det[-\mathrm{A}(G)] = (-1)^n \det \mathrm{A}(G) = -\det \mathrm{A}(G), \tag{4.2.2}$$

and hence $\det \mathrm{A}(G) = 0$. But what happens when n is even? For example, when $n = 4$, we have

$$\det \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix} = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2. \tag{4.2.3}$$

We see from equation (4.2.3) that individual terms on the right-hand side are of the form $a_{i_1j_1}a_{i_2j_2}$ with $\{i_1j_1, i_2j_2\} = \{1, 2, 3, 4\}$ and also with different signs. This leads to the following idea. Let n be a positive integer; then any partition of $\{1, 2, \ldots, 2n\}$ into n unordered pairs is a perfect matching π on $\{1, 2, \ldots, 2n\}$, where

$$\pi = i_1j_1, i_2j_2, \ldots, i_nj_n \text{ with } i_k < j_k \ \forall k.$$

If we let Π to be the set of all such partitions of $\{1, 2, \ldots, 2n\}$ then there are $(2n-1)!!$ different perfect matchings altogether since the first element has $2n-1$ possible choices, after which the next element will have $2n-3$ possible choices and so on. The $(n-1)$-th element will have 3 choices and the n-th element has only one choice. The double factorial is defined as follows:

$$(2n-1)!! = \prod_{k=1}^{n}(2k-1) = \frac{(2n)!}{2^n n!}. \tag{4.2.4}$$

Given a skew-symmetric $2n \times 2n$ matrix A, we use the notation

$$a_\pi = \prod_{k=1}^{n} a_{i_kj_k}.$$

Definition 4.2.5. *Let S_n be a symmetric group. Then for any permutation $\sigma \in S_n$, we define the* ***number of inversions*** *of σ denoted $\operatorname{inv}(\sigma)$ as the number of pairs (i, j) such that $i < j$ but j occurs before i in the permutation.*

Example 4.2.6.
Let and $\sigma = \{2\ 5\ 1\ 3\ 4\}$. The inversions of σ as follows: $(1, 2), (1, 5), (3, 5), (4, 5)$. Therefore, σ has four inversions, i.e., $\operatorname{inv}(\sigma) = 4$.

Definition 4.2.7. *Let $\sigma \in S_n$. Then we define the* ***sign*** *of σ denoted $\operatorname{sgn}(\sigma)$ as*

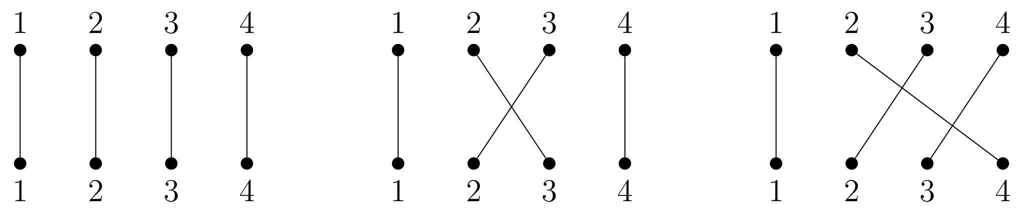
$$\operatorname{sgn}(\sigma) = (-1)^{\operatorname{inv}(\sigma)}. \tag{4.2.8}$$

For example the sign of the permutation in Example (4.2.6) is

$$\operatorname{sgn}(\sigma) = (-1)^{\operatorname{inv}(\sigma)} = (-1)^4 = 1.$$

Note that a partition of $\{1, 2, \ldots, 2n\}$ into n unordered pairs can also be interpreted as a permutation of $\{1, 2, \ldots, 2n\}$. Also, the inversion of a permutation can also defined as the number of crossings in the partition π since each inversion (i, j) corresponds to a crossing in π. We illustrate this with an example with the case $n = 2$. From equation (4.2.4), there are 3 perfect matchings of $\{1, 2, 3, 4\}$ as indicated in equation (4.2.3) with inversions $(2, 4)$ and $(3, 4)$ for the partition $\{14, 23\}$, $(2, 3)$ for the partition $\{13, 24\}$, and the partition $\{12, 34\}$ has no inversion.

Example 4.2.9.



$\text{inv}(12,34) = 0$ $\quad$ $\text{inv}(13,24) = 1$ $\quad$ $\text{inv}(14,23) = 2$

$\text{sgn}(12,34) = 1$ $\quad$ $\text{sgn}(13,24) = -1$ $\quad$ $\text{sgn}(14,23) = 1$

Note that, the signs of the partitions in Example 4.2.9 correspond exactly to the signs in equation (4.2.3). This is a theorem due the British mathematician Arthur Cayley who proved it in 1847. He also introduced the term Pfaffian of a real skew symmetric matrix and related it to the determinant [7]. Other proofs are also known, for example [2] and [14].

Definition 4.2.10. *Let $A = [a_{ij}]_{1\le i,j\le n}$ be a real skew-symmetric $n\times n$ matrix. We define the **Pfaffian** $\text{Pf}(A)$ as*

$$\text{Pf}(A) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \sum_{\pi\in\Pi} \big(\text{sgn}(\pi)\big) a_\pi & \text{if } n \text{ is even} \end{cases} \tag{4.2.11}$$

where

$$a_\pi = \prod_{k=1}^{n/2} a_{i_k \pi(i_k)}.$$

We note that since the matrix A is skew-symmetric i.e., $a_{ij} = -a_{ji}$ then the Pfaffian $\text{Pf}(A)$ is well-defined.

Theorem 4.2.12 (Cayley). *Let $A = [a_{ij}]_{1\le i,j\le n}$ be a real skew-symmetric $n\times n$-matrix. Then*

$$\det A = [\text{Pf}(A)]^2. \tag{4.2.13}$$

Proof.
If n is odd, then the determinant vanishes since $\det(A) = -\det(A)$ by equation (4.2.2). Now we assume n to be even. The usual definition of the determinant is

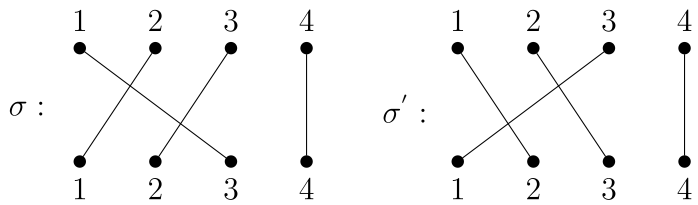
$$\det A = \sum_{\sigma\in S_n} \big(\text{sgn}(\sigma)\big) a_\sigma = \sum_{\sigma\in S_n} \big(\text{sgn}(\sigma)\big) \prod_{i=1}^{n} a_{i\sigma(i)}. \tag{4.2.14}$$

We want to sieve out the terms in equation (4.2.14) which cancel each other. We know that every permutation $\sigma \in S_n$ can be decomposed into a product of even and odd disjoint cycles. Let P be the subset of all permutations $\sigma \in S_n$ with at least one cycle of odd length. For every $\sigma \in P$, $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ $(m \le n)$ we arrange the cycles in such a way that σ_1 is the cycle of odd length with the least element. We define $\psi : P \to P$ as

$$\psi(\sigma) = \sigma' = \sigma_1^{-1}\sigma_2\cdots\sigma_m$$

where σ_1^{-1} is the inverse of the cycle σ_1. Clearly, $\psi(\sigma^{'}) = \sigma$, so ψ is an involution. Now for such pairs $\sigma, \sigma^{'} \in P$, $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\sigma^{'})$ since they have the same cycle type and so the terms a_σ and $a_{\sigma'}$ cancel out in the determinant equation (4.2.14). We illustrate this with an example.

Example 4.2.15.
Consider the following permutations of S_4; Let $\sigma = \sigma_1\sigma_2 = (132)(4)$ then $\psi(\sigma) = \sigma^{'} = \sigma_1^{-1}\sigma_2 = (123)(4)$.



From the diagram we see that both permutations ($\sigma = 3124$ and $\sigma^{'} = 2314$) have 2 inversions, namely, $(1, 3)$ and $(2, 3)$ for σ and $(1, 2)$ and $(1, 3)$ for 2314 hence $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\sigma^{'}) = 1$. In the determinant equation (4.2.14) we have

$$\begin{aligned}
\det A &= \cdots + \big(\mathrm{sgn}(\sigma)\big)a_\sigma + \big(\mathrm{sgn}(\sigma^{'})\big)a_{\sigma'} + \ldots \\
&= \cdots + \big(\mathrm{sgn}(\sigma)\big)\prod_{i=1}^{4} a_{i\sigma(i)} + \big(\mathrm{sgn}(\sigma^{'})\big)\prod_{i=1}^{4} a_{i\sigma'(i)} + \ldots \\
&= \cdots + a_{1\sigma(1)} + a_{2\sigma(2)} + a_{3\sigma(3)} + a_{4\sigma(4)} + a_{1\sigma'(1)} + a_{2\sigma'(2)} + a_{3\sigma'(3)} + a_{4\sigma'(4)} + \ldots \\
&= \cdots + a_{13}a_{21}a_{32}a_{44} + a_{12}a_{23}a_{31}a_{44} + \ldots \\
&= \cdots + a_{13}a_{12}a_{23}a_{44} - a_{12}a_{23}a_{13}a_{44} + \ldots
\end{aligned}$$

Thus we see that all ψ-pairs of P vanish in equation (4.2.14) and we are now left with

$$\det A = \sum_{\sigma\in\Omega} \big(\mathrm{sgn}(\sigma)\big)a_\sigma = \sum_{\sigma\in\Omega} \big(\mathrm{sgn}(\sigma)\big)\prod_{i=1}^{n} a_{i\sigma(i)} \tag{4.2.16}$$

where $\Omega \subset S_n$ is the set of all permutations whose cycles are all of *even* length. By Definition (4.2.10),

$$\begin{aligned}
[\mathrm{Pf}(A)]^2 &= \left(\sum_{\pi_1\in\Pi} \big(\mathrm{sgn}(\pi_1)\big)a_{\pi_1}\right)\left(\sum_{\pi_2\in\Pi} \big(\mathrm{sgn}(\pi_2)\big)a_{\pi_2}\right) \\
&= \sum_{\pi_1,\pi_2\in\Pi} \big(\mathrm{sgn}(\pi_1)\cdot\mathrm{sgn}(\pi_2)\big)a_{\pi_1}a_{\pi_2} \\
&= \sum_{\pi_1,\pi_2\in\Pi} \big(\mathrm{sgn}(\pi_1)\cdot\mathrm{sgn}(\pi_2)\big)\prod_{k_1=1}^{n/2} a_{i_{k_1}\pi_1(i_{k_1})}\prod_{k_2=1}^{n/2} a_{i_{k_2}\pi_2(i_{k_2})}
\end{aligned} \tag{4.2.17}$$

Comparing equations (4.2.16) and (4.2.17) we must now show that:

(i) $a_{\pi_1}a_{\pi_2} = a_\sigma$. i.e., $\displaystyle\prod_{k_1=1}^{n/2} a_{i_{k_1}\pi_1(i_{k_1})}\prod_{k_2=1}^{n/2} a_{i_{k_2}\pi_2(i_{k_2})} = \prod_{i=1}^{n} a_{i\sigma(i)}$.

(ii) $\text{sgn}(\pi_1) \cdot \text{sgn}(\pi_2) = \text{sgn}(\sigma)$.

Thus we must find a one-to-one correspondence from the remaining terms Ω of $\det(A)$ in equation (4.2.16)(which are the permutations whose decomposition consist of only even cycles) to $[\text{Pf}(A)]^2$ in equation (4.2.17). We define the bijection $\varphi : (\pi_1, \pi_2) \mapsto \sigma$ as

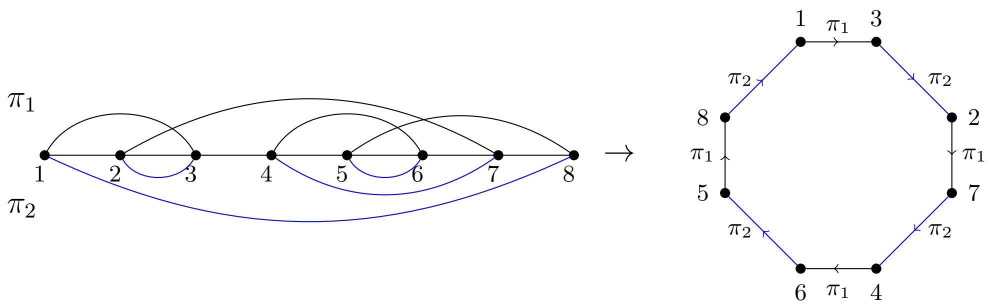
$$\big(\text{sgn}(\pi_1)\big)a_{\pi_1} \cdot \big(\text{sgn}(\pi_2)\big)a_{\pi_2} = \big(\text{sgn}(\sigma)\big)a_\sigma$$

where $\pi_1, \pi_2 \in \Pi$ which are given by

$$\pi_1 = \{i_1j_1, i_2j_2, \ldots, i_{n/2}j_{n/2}\} \text{ and } \pi_2 = \{i_1^{'}j_1^{'}, i_2^{'}j_2^{'}, \ldots, i_{n/2}^{'}j_{n/2}^{'}\}$$

as follows. We first choose any number i_1 from the set $\{1, 2, \ldots, n\}$ then the number j_1 must be fixed since it pairs up with i_1 in π_1. However, j_1 also appears in π_2 so we set $j_1 = i_1^{'}$ in π_2. Then $j_1^{'}$ is fixed and it can either be the number i_1 in π_1 or another number from the set $\{1, 2, \ldots, n\} \setminus \{i_1, i_1^{'}\}$. We assume the second case so we set $j_1^{'} = i_2$ and repeat the process until $j_{n/2}^{'} = i_1$. This procedure results in edges alternating between the perfect matchings π_1 and π_2 resulting in an even oriented cycle which corresponds to permutations with only even cycles. Applying the process repeatedly, we obtain a permutation with only even cycles. We illustrate this with an example.

Example 4.2.18.
$\pi_1 = \{13, 27, 46, 58\}$ and $\pi_2 = \{32, 74, 65, 81\}$



Next we claim that
the permutation σ in the determinant term is given by $\pi_2\pi_1^{-1}$.
Suppose the sign of the partition π_1 is given by:

$$\text{sgn}(\pi_1) = \text{sgn}\begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ i_1 & j_1 & \ldots & i_{n/2} & j_{n/2} \end{pmatrix} = (-1)^{m_1}$$

and the sign of the partition π_2 is given by:

$$\text{sgn}(\pi_2) = \text{sgn}\begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ j_1 & j_1^{'} & \ldots & j_{n/2} & j_{n/2}^{'} \end{pmatrix} = (-1)^{m_2}.$$

Then,

$$\begin{aligned}\pi_2\pi_1^{-1} &= \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ j_1 & j_1^{'} & \dots & j_{n/2} & j_{n/2}^{'} \end{pmatrix} \begin{pmatrix} i_1 & j_1 & \dots & i_{n/2} & j_{n/2} \\ 1 & 2 & \dots & n-1 & n \end{pmatrix} \\ &= \begin{pmatrix} i_1 & j_1 & \dots & i_{n/2} & j_{n/2} \\ j_1 & j_1^{'} & \dots & j_{n/2} & j_{n/2}^{'} \end{pmatrix} \\ &= \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} \\ &= \sigma. \end{aligned}$$

Now since $\operatorname{sgn}(\pi_1^{-1}) = \operatorname{sgn}(\pi_1)$, it follows from equation (4.2.8) that

$$\operatorname{sgn}(\sigma) = \operatorname{sgn}\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = (-1)^{m_1} \cdot (-1)^{m_2} = (-1)^{m_1+m_2}.$$

This proves our claim and thus completes the proof. □

4.2.1 Pfaffian Orientation

One possible conclusion that one can make from Cayley's theorem is that the Pfaffian of a matrix can be used to count the number of perfect matchings of a graph. This can be seen as follows. Let $G = (V, E)$ be an oriented simple graph with $V = \{v_1, v_2, \dots, v_n\}$ and $v(G) = |V| = n$ is even. Let $\mathrm{PM}(G)$ be the number of perfect matchings of G. Note that as G is oriented, then the elements of the edge set E are ordered pairs. Let $\mathrm{A}(G)$ be the adjacency matrix of G:

$$\mathrm{A}(G) = [a_{ij}]_{1\le i,j\le n}$$
$$\text{where, } a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E \\ -1, & \text{if } (v_j, v_i) \in E \\ 0, & \text{otherwise.} \end{cases}$$

The adjacency matrix $\mathrm{A}(G)$ is clearly a skew-symmetric matrix and the Pfaffian of $\mathrm{A}(G)$ is

$$\mathrm{Pf}[\mathrm{A}(G)] = \sum_{\pi} \big(\operatorname{sgn}(\pi)\big) a_\pi$$

where π runs over all matchings of $V = \{v_1, v_2, \dots, v_n\}$ and the perfect matchings are those for which

$$a_\pi = \prod_{k=1} a_{k\pi(k)} = \pm 1.$$

The question now is to place an orientation on the graph G such that $\big(\operatorname{sgn}(\pi)\big)a_\pi$ is always $+1$ or always -1. If this question can be answered, then $|\mathrm{Pf}(A)| = \mathrm{PM}(G) \Rightarrow \mathrm{PM}(G)^2 = \mathrm{Pf}(A)^2 = \det(A)$ by Theorem (4.2.12). Hence we have

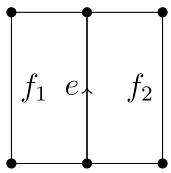
$$\mathrm{PM}(G) = \sqrt{\det(A)}$$

which counts the number of perfect matchings in G. Recall from the proof of Cayley's theorem that $\Omega \subset S_n$ contains all permutations whose cycles are all of even length. So for any permutation $\sigma \in \Omega$, let $\sigma = \sigma_1\sigma_2\cdots\sigma_r$ be the cycle decomposition with $a_\sigma \neq 0$. Then since each cycle σ_i has even length, $\text{sgn}(\sigma) = (-1)^r$. Therefore, we require that $a_\sigma = a_{\sigma_1}a_{\sigma_2}\cdots a_{\sigma_r} = (-1)^r$ which holds if $a_{\sigma_i} = -1$ for all i.

Definition 4.2.19. *Let G be an oriented graph. We say G has a **Pfaffian orientation** if for any $\sigma \in \Omega$ with cycle decomposition $\sigma = \sigma_1\sigma_2\cdots\sigma_r$, $a_\sigma = a_{\sigma_1}a_{\sigma_2}\cdots a_{\sigma_r} = (-1)^r$ or $a_\sigma = 0$.*

Our task now is to find a Pfaffian orientation given any graph. In general, this task is very difficult since the class of graphs that have a Pfaffian orientation is not known, but for planar graphs, one can always find such an orientation. So we restrict ourselves again to the study of planar graphs.

Let $G = (V, E, F)$ be a simple connected oriented planar graph and let e be an edge in G that is not a bridge. Then e is a boundary of two different faces of G. Now consider a face $f \in F$ and let B_f be the set of all boundary edges of f that are not bridges. We say that $e \in B_f$ is *clockwise* oriented if the face f lies on the right side of e as one moves along e in the direction indicated by the orientation. For example in the figure below,

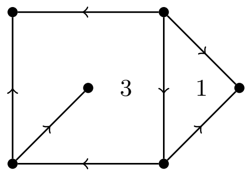


e is clockwise oriented in B_{f_2} and counter-clockwise in B_{f_1}. We do not take bridges into account.

Lemma 4.2.20. *Let $G = (V, E, F)$ be a simple connected planar graph. Then there always exists an orientation such that for any face f in G (except possibly for the outer face) the number of clockwise oriented edges is odd.*

Proof.
We prove the lemma by induction on the number of faces $|F|$ of G.
Base step:
If $|F| = 1$ then G must be a tree since G is connected and so all the edges are bridges which can always be oriented such that the number of clockwise oriented edges is odd.
Inductive hypothesis and inductive step:
Suppose $|F| = k$ with $k > 1$ and assume the lemma holds for $|F| \leq k - 1$. There must exist at least a face f in G incident to an edge e that is not a bridge such that e is a boundary of f and the outer unbounded face. Removing the edge e from G we obtain the graph $G \setminus e$ which merges the face f with the outer unbounded face. By the induction hypothesis, the graph $G \setminus e$ satisfies the lemma and we can now orient the edge e in such a way that the condition is also satisfied for the face f. The bridges are oriented arbitrarily. This completes the proof. □

Example 4.2.21.
Below is a simple planar graph with such an orientation described in Lemma (4.2.20).



From the diagram above, the square has 3 clockwise oriented edges in the square and 1 in the triangle.

Lemma 4.2.22. *Let $G = (V, E, F)$ be a simple connected planar graph with no bridges, and oriented according to Lemma (4.2.20). Then for any cycle C in G, the number of edges oriented clockwise is of opposite parity to the number of vertices in the interior of C.*

Proof.
Let C be any cycle in G and $G^{'} = (V^{'}, E^{'}, F^{'})$ be the subgraph of G containing the cycle C (i.e., both the interior of C and the boundary of C). Applying Euler's formula in equation (2.6.5) to $G^{'}$ (disregarding the outer unbounded face) we get

$$|V^{'}| - |E^{'}| + |F^{'}| = 1. \tag{4.2.23}$$

Let the number of edges of C be ℓ. (Note that this is also equal to the number of vertices of C.) Suppose that there are v vertices, e edges, and f faces within the interior of C. Then, $|V^{'}| = v + \ell$, $|E^{'}| = e + \ell$ and $|F^{'}| = f$. Substituting these into equation (4.2.23) we get

$$\begin{aligned} v + \ell - e - \ell + f &= 1 \\ v - e + f &= 1. \end{aligned} \tag{4.2.24}$$

Let c_i be the number of edges oriented clockwise in the i-th face and c be the number of edges oriented clockwise in the cycle C. Now we to show that c is of opposite parity to v (i.e., $c + v \equiv 1 \pmod 2$).

For each c_i $(i = 1, \ldots, f)$, $c_1 \equiv 1 \pmod 2$ since by assumption each c_i is odd and so

$$f = \sum_{i=1}^{f} c_i \pmod 2. \tag{4.2.25}$$

Looking at equation (4.2.25), one can do more by splitting the sum as follows. Any edge in the interior of C is counted as clockwise exactly only once and this gives us the total number of edges in the interior of C which is e. Moreover, the orientation of any edge on C is the same on C and the same on the interior faces they are incident to. Hence the number of edges oriented clockwise does not change and this is the number c. Therefore we write the sum as

$$\sum_{i=1}^{f} c_i = c + e$$

and from equation (4.2.25) we obtain

$$f \equiv c + e \pmod 2.$$

From equation (4.2.24), $f = 1 + e - v$. Substituting this into the above equation we get

$$\begin{aligned} 1 + e - v &\equiv c + e \pmod 2 \\ 1 - v &\equiv c \pmod 2 \\ v + c &\equiv 1 \pmod 2 \end{aligned}$$

which shows that the number of vertices v in the interior of C is of opposite parity to the number of edges c oriented clockwise and this completes the proof. □

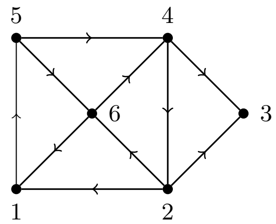
Theorem 4.2.26. *Let $G = (V, E, F)$ be a simple connected planar graph without bridges then the orientation given in Lemma* (4.2.20) *is Pfaffian.*

Proof.
We refer the reader to [1] for the proof of this theorem. □

Corollary 4.2.27. *Let G be an simple connected planar graph without bridges with a Pfaffian orientation and* A *be its oriented adjacency matrix. Then by Cayley's theorem* (4.2.12)*, the number of perfect matchings in G is*

$$\mathrm{PM}(G) = \sqrt{\det \mathrm{A}}. \tag{4.2.28}$$

Example 4.2.29.
As an example, we compute the number of perfect matchings in the graph G below.



The oriented adjacency matrix is

$$\mathrm{A} = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 0 & -1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 \end{bmatrix} \end{array},$$

which has determinant $\det(\mathrm{A}) = 16$, and so $\mathrm{PM}(G) = 4$. Therefore G has a total of 4 perfect matchings.

4.3 Perfect Matchings on Rectangular Lattice $\mathcal{L}_{N_1 \times N_2}$

An application of counting perfect matchings in statistical physics is the well known *dimer* problem. A *dimer* is a molecule which consist of two atoms bonded together. In graph theoretic terms, the atoms are the vertices and the bonds between the atoms are the edges connecting exactly two vertices. The dimer problem is to determine the number of ways to cover all the vertices with dimers such that every vertex is occupied by only one dimer. Thus a dimer configuration of a given graph is a perfect matching.

Our goal in this section is to derive a formula for counting the number perfect matchings $\mathrm{PM}(\mathcal{L}_{N_1\times N_2})$ of the rectangular lattice $\mathcal{L}_{N_1\times N_2}$ using the methods discussed in the previous section. We will also measure the asymptotic growth of $\mathrm{PM}(\mathcal{L}_{N_1\times N_2})$ as $N_1, N_2 \to \infty$.

Given a rectangular lattice $\mathcal{L}_{N_1\times N_2}$ with N_2 even, one can define a Pfaffian orientation on $\mathcal{L}_{N_1\times N_2}$ as illustrated in Figure 4.2. Note that any interior face in Figure 4.2 has 1 or 3 clockwise edges; thus the orientation is Pfaffian.

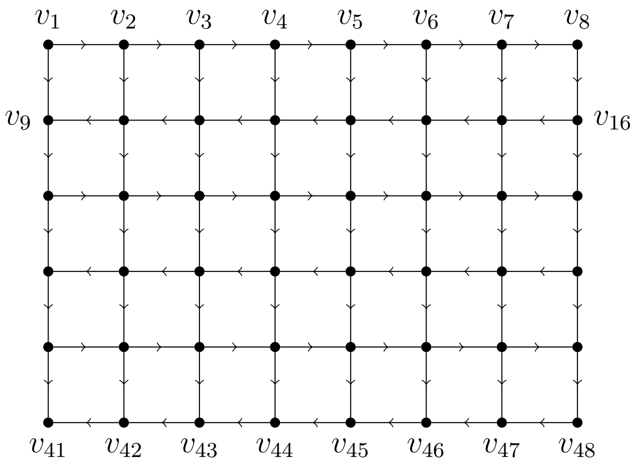


Figure 4.2: A rectangular lattice $\mathcal{L}_{6\times 8}$ with Pfaffian orientation.

We now go ahead and derive the formula for counting the number of perfect matchings of $\mathcal{L}_{N_1\times N_2}$. Label the first row of the oriented lattice $\mathcal{L}_{N_1\times N_2}$ as $v_1, v_2, \ldots, v_{N_2}$, the second row as $v_{N_2+1}, \ldots, v_{2N_2}$, and proceed this way until all the vertices in $\mathcal{L}_{N_1\times N_2}$ are labelled as illustrated in Figure 4.2. Let A be the $N_2 \times N_2$ oriented adjacency matrix of the first row. Then

$$\mathrm{A} = \begin{array}{c c} & \begin{array}{ccccc} v_1 & v_2 & v_3 & \cdots & v_{N_2} \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_{N_2-1} \\ v_{N_2} \end{array} & \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ -1 & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & -1 & 0 & 1 \\ 0 & \ldots & 0 & -1 & 0 \end{bmatrix} \end{array}.$$

Let I_{N_2} be the $N_2 \times N_2$ identity matrix which represents the connection between each row and let $\mathbf{0}_{N_2}$ be the $N_2 \times N_2$ zero matrix. Then the full oriented adjacency matrix

$\mathrm{A}(\mathcal{L}_{N_1\times N_2})$ is an $N_1 \times N_1$ block matrix with each block of size $N_2 \times N_2$. That is

$$\mathrm{A}(\mathcal{L}_{N_1\times N_2}) = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ N_1-1 \\ N_1 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \dots & N_1 \end{array} \\ \left[\begin{array}{c|c|c|c|c} \mathrm{A} & \mathrm{I}_{N_2} & \mathbf{0}_{N_2} & \dots & \mathbf{0}_{N_2} \\ \hline -\mathrm{I}_{N_2} & -\mathrm{A} & \mathrm{I}_{N_2} & \dots & \mathbf{0}_{N_2} \\ \hline \vdots & \ddots & \ddots & \ddots & \vdots \\ \hline \mathbf{0}_{N_2} & \dots & -\mathrm{I}_{N_2} & \mp\mathrm{A} & \mathrm{I}_{N_2} \\ \hline \mathbf{0}_{N_2} & \dots & \mathbf{0}_{N_2} & -\mathrm{I}_{N_2} & \pm\mathrm{A} \end{array}\right] \end{array}.$$

Considering the blocks in $\mathrm{A}(\mathcal{L}_{N_1\times N_2})$ as the rows, we multiply the odd rows by -1, multiply all columns except for the first column by -1 and lastly multiply all the rows except for the first row by -1. We now obtain

$$\mathrm{A}'(\mathcal{L}_{N_1\times N_2}) = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ N_1-1 \\ N_1 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \dots & N_1 \end{array} \\ \left[\begin{array}{c|c|c|c|c} -\mathrm{A} & \mathrm{I}_{N_2} & \mathbf{0}_{N_2} & \dots & \mathbf{0}_{N_2} \\ \hline \mathrm{I}_{N_2} & -\mathrm{A} & \mathrm{I}_{N_2} & \dots & \mathbf{0}_{N_2} \\ \hline \vdots & \ddots & \ddots & \ddots & \vdots \\ \hline \mathbf{0}_{N_2} & \dots & \mathrm{I}_{N_2} & -\mathrm{A} & \mathrm{I}_{N_2} \\ \hline \mathbf{0}_{N_2} & \dots & \mathbf{0}_{N_2} & \mathrm{I}_{N_2} & -\mathrm{A} \end{array}\right] \end{array}. \tag{4.3.1}$$

If we let B be the $N_1 \times N_1$ matrix given by

$$\mathrm{B} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

and I_{N_1} be the $N_1 \times N_1$ identity matrix then $\mathrm{A}(\mathcal{L}_{N_1\times N_2})$ can be written as a sum of two tensor products:

$$\mathrm{A}'(\mathcal{L}_{N_1\times N_2}) = -\mathrm{I}_{N_1} \otimes \mathrm{A} + \mathrm{B} \otimes \mathrm{I}_{N_2}. \tag{4.3.2}$$

Corollary 4.2.27 gives us the total number of perfect matchings is

$$\left[\mathrm{PM}(\mathcal{L}_{N_1\times N_2})\right]^2 = \det\left(\mathrm{A}'(\mathcal{L}_{N_1\times N_2})\right) = \det\left(\mathrm{A}(\mathcal{L}_{N_1\times N_2})\right). \tag{4.3.3}$$

From equation (4.3.2) and (4.3.3),

$$\det\left(\mathrm{A}(\mathcal{L}_{N_1\times N_2})\right) = \det\left(\mathrm{B} \otimes \mathrm{I}_{N_2} - \mathrm{I}_{N_1} \otimes \mathrm{A}\right).$$

We know that the determinant $\det\big(\mathrm{A}(\mathcal{L}_{N_1\times N_2})\big)$ is the product of its eigenvalues. Now if we suppose that B has eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{N_1-1}$ associated with the eigenvectors $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N_1-1}$ and A has eigenvalues $\mu_0, \mu_1, \ldots, \mu_{N_2-1}$ associated with the eigenvectors $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{N_2-1}$ then by Lemma 3.4.5 the eigenvalues and eigenvectors of $(\mathrm{B}\otimes \mathrm{I}_{N_2} - \mathrm{I}_{N_1}\otimes \mathrm{A})$ are $\lambda_{\ell_1} - \mu_{\ell_2}$ and $\boldsymbol{u}_{\ell_1}\otimes \boldsymbol{v}_{\ell_2}$ respectively for $\ell_1 = 0, 1, \ldots, N_1-1$ and $\ell_2 = 0, 1, \ldots, N_2-1$. The question is now to find the eigenvalues of the matrices A and B. Equation (3.3.15) gives us the eigenvalues of B namely

$$\lambda_{\ell_1} = 2\cos\left(\frac{\ell_1\pi}{N_1+1}\right) \quad (\ell_1 = 1, 2, \ldots, N_1)$$

and the eigenvalues of A are also

$$\mu_{\ell_2} = 2\mathrm{i}\cos\left(\frac{\ell_2\pi}{N_2+1}\right) \quad (\ell_2 = 1, 2, \ldots, N_2) \ \text{ see Appendix B.}$$

Using the fact that the determinant of a matrix equals the product of its eigenvalues we have

$$\begin{aligned}
\det\big(\mathrm{A}(\mathcal{L}_{N_1\times N_2})\big) &= \det\big(\mathrm{B}\otimes \mathrm{I}_{N_2} - \mathrm{I}_{N_1}\otimes \mathrm{A}\big)\\
&= \prod_{\ell_1=1}^{N_1}\prod_{\ell_2=1}^{N_2}(\lambda_{\ell_1} - \mu_{\ell_2})\\
&= \prod_{\ell_1=1}^{N_1}\prod_{\ell_2=1}^{N_2}\left[2\cos\left(\frac{\ell_1\pi}{N_1+1}\right) - 2\mathrm{i}\cos\left(\frac{\ell_2\pi}{N_2+1}\right)\right].
\end{aligned}$$

The square of the determinant is

$$\left[\det\big(\mathrm{A}(\mathcal{L}_{N_1\times N_2})\big)\right]^2 = \prod_{\ell_1=1}^{N_1}\prod_{\ell_2=1}^{N_2}\left[4\cos^2\left(\frac{\ell_1\pi}{N_1+1}\right) + 4\cos^2\left(\frac{\ell_2\pi}{N_2+1}\right)\right]. \tag{4.3.4}$$

From equation (4.3.3), the total number of perfect matchings $\mathrm{PM}(\mathcal{L}_{N_1\times N_2})$, will be the fourth root of the above expression. That is

$$\mathrm{PM}(\mathcal{L}_{N_1\times N_2}) = \prod_{\ell_1=1}^{N_1}\prod_{\ell_2=1}^{N_2}\left[4\cos^2\left(\frac{\ell_1\pi}{N_1+1}\right) + 4\cos^2\left(\frac{\ell_2\pi}{N_2+1}\right)\right]^{\frac{1}{4}}. \tag{4.3.5}$$

We simplify equation (4.3.5) further by considering two cases:
Case 1 (N_1 even):
If N_1 is even then we have

$$\cos\left(\frac{(N_1+1-\ell_1)\pi}{N_1+1}\right) = -\cos\left(\frac{\ell_1\pi}{N_1+1}\right), \ \cos\left(\frac{(N_2+1-\ell_2)\pi}{N_2+1}\right) = -\cos\left(\frac{\ell_2\pi}{N_2+1}\right)$$

for $\ell_1 = 1, \ldots, \frac{N_1}{2}$, $\ell_2 = 1, \ldots, \frac{N_2}{2}$ and equation (4.3.4) now becomes

$$\left[\det\big(\mathrm{A}(\mathcal{L}_{N_1\times N_2})\big)\right]^2 = 4^{N_1N_2}\prod_{\ell_1=1}^{N_1/2}\prod_{\ell_2=1}^{N_2/2}\left[\cos^2\left(\frac{\ell_1\pi}{N_1+1}\right) + \cos^2\left(\frac{\ell_2\pi}{N_2+1}\right)\right]^4.$$

Hence

$$\mathrm{PM}(\mathcal{L}_{N_1\times N_2}) = 4^{\frac{N_1N_2}{4}} \prod_{\ell_1=1}^{N_1/2}\prod_{\ell_2=1}^{N_2/2}\left[\cos^2\left(\frac{\ell_1\pi}{N_1+1}\right)+\cos^2\left(\frac{\ell_2\pi}{N_2+1}\right)\right]. \tag{4.3.6}$$

Example 4.3.7.
As an example, we compute the total number of perfect matchings on a chessboard. Its dimension is 8×8. From equation (4.3.6) we have

$$\begin{aligned}\mathrm{PM}(\mathcal{L}_{8\times 8}) &= 4^{16}\prod_{\ell_1=1}^{4}\prod_{\ell_2=1}^{4}\left[\cos^2\left(\frac{\ell_1\pi}{9}\right)+\cos^2\left(\frac{\ell_2\pi}{9}\right)\right]\\ &= 12,988,816.\end{aligned}$$

See Appendix C.5 for a detailed computation of $\mathrm{PM}(\mathcal{L}_{8\times 8})$ using the computer algebra system SAGE.

Case 2 (N_1 odd):
We note that N_1-1 and N_1+1 are even. Also,

$$\cos\frac{\frac{N_1+1}{2}\pi}{N_1+1} = \cos\frac{(N_1+1)\pi}{2(N_1+1)} = \cos\frac{\pi}{2} = 0.$$

Thus it follows from equation (4.3.6) that

$$\mathrm{PM}(\mathcal{L}_{N_1\times N_2}) = 4^{\frac{N_1N_2}{4}} \prod_{\ell_1=1}^{(N_1-1)/2}\prod_{\ell_2=1}^{N_2/2}\left[\cos^2\left(\frac{\ell_1\pi}{N_1+1}\right)+\cos^2\left(\frac{\ell_2\pi}{N_2+1}\right)\right]\prod_{\ell_2=1}^{N_2/2}\cos\left(\frac{\ell_2\pi}{N_2+1}\right). \tag{4.3.8}$$

We claim that the last term in equation (4.3.8) is

$$\prod_{\ell_2=1}^{N_2/2}\cos\left(\frac{\ell_2\pi}{N_2+1}\right) = 4^{-\frac{N_2}{4}}. \tag{4.3.9}$$

Proof.
Let $\omega = e^{\frac{2\pi \mathrm{i}}{N_2+1}}$ be the (N_2+1)-st root of unity. Then

$$\begin{aligned}\omega^{\ell_2} &= e^{\frac{2\pi \mathrm{i}\ell_2}{N_2+1}}\ (\ell_2 = 1,2,\ldots,N_2+1)\\ &= \cos\left(\frac{2\pi\ell_2}{N_2+1}\right)+\mathrm{i}\sin\left(\frac{2\pi\ell_2}{N_2+1}\right)\end{aligned}$$

and

$$\cos\left(\frac{2\pi\ell_2}{N_2+1}\right) = \frac{\omega^{\ell_2}+\omega^{-\ell_2}}{2}.$$

Using the cosine identity for double angles (i.e., $\cos 2\theta = 2\cos^2\theta - 1$) we write the above equation as

$$\cos^2\left(\frac{\pi\ell_2}{N_2+1}\right) = \frac{1}{2}\left[\cos\left(\frac{2\pi\ell_2}{N_2+1}\right)+1\right] = \frac{\omega^{\ell_2}+\omega^{-\ell_2}+2}{4}$$

and taking the square root and then the product we get

$$\cos\left(\frac{\ell_2\pi}{N_2+1}\right) = 4^{-\frac{1}{2}}\left(\omega^{\ell_2}+\omega^{-\ell_2}+2\right)^{\frac{1}{2}}$$
$$\prod_{\ell_2=1}^{N_2/2}\cos\left(\frac{\ell_2\pi}{N_2+1}\right) = 4^{-\frac{N_2}{4}}\prod_{\ell_2=1}^{N_2/2}\left(\omega^{\ell_2}+\omega^{-\ell_2}+2\right)^{\frac{1}{2}}.$$

Comparing the above equation to equation (4.3.9) we have to show that

$$\prod_{\ell_2=1}^{N_2/2}\left(\omega^{\ell_2}+\omega^{-\ell_2}+2\right)^{\frac{1}{2}} = 1.$$

Since ω is the (N_2+1)-st root of unity this implies that $\omega^{-\ell_2} = \omega^{N_2+1-\ell_2}$ and so we write the above equation without the square root as

$$\prod_{\ell_2=1}^{N_2/2}\left(\omega^{\ell_2}+\omega^{-\ell_2}+2\right) = \prod_{\ell_2=1}^{N_2/2}\left(\omega^{\ell_2}+1\right)\left(\omega^{-\ell_2}+1\right) = \prod_{\ell_2=1}^{N_2}\left(\omega^{\ell_2}+1\right).$$

Finally, $1+z+z^2+\cdots+z^{N_2} = (z-\omega)(z-\omega^2)\cdots(z-\omega^{N_2})$. Setting $z=-1$ yields

$$1 = (-1-\omega)(-1-\omega^2)\cdots(-1-\omega^{N_2}) = (-1)^{N_2}\prod_{\ell_2=1}^{N_2}\left(\omega^{\ell_2}+1\right) = \prod_{\ell_2=1}^{N_2}\left(\omega^{\ell_2}+1\right)$$

since N_2 is even. This proves our claim that

$$\prod_{\ell_2=1}^{N_2/2}\cos\left(\frac{\ell_2\pi}{N_2+1}\right) = 4^{-\frac{N_2}{4}}.$$

Substituting this into equation (4.3.8) we get

$$\mathrm{PM}(\mathcal{L}_{N_1\times N_2}) = 4^{\frac{(N_1-1)N_2}{4}}\prod_{\ell_1=1}^{(N_1-1)/2}\prod_{\ell_2=1}^{N_2/2}\left[\cos^2\left(\frac{\ell_1\pi}{N_1+1}\right)+\cos^2\left(\frac{\ell_2\pi}{N_2+1}\right)\right]. \tag{4.3.10}$$

□

In summary we have proved the theorem below which is due to Fisher, Kastelyn and Temperley.

Theorem 4.3.11 (Fisher-Kastelyn-Temperley)**.** *Let $\mathcal{L}_{N_1\times N_2}$ be a rectangular lattice with N_2 even, then the number of perfect matchings is*

$$\mathrm{PM}(\mathcal{L}_{N_1\times N_2}) = 4^{\left\lfloor\frac{N_1}{2}\right\rfloor\frac{N_2}{2}}\prod_{\ell_1=1}^{\lfloor N_1/2\rfloor}\prod_{\ell_2=1}^{N_2/2}\left[\cos^2\left(\frac{\ell_1\pi}{N_1+1}\right)+\cos^2\left(\frac{\ell_2\pi}{N_2+1}\right)\right]. \tag{4.3.12}$$

4.3.1 Asymptotic Growth of $\mathrm{PM}(\mathcal{L}_{N_1\times N_2})$

We now want to determine the behaviour of $\mathrm{PM}(\mathcal{L}_{N_1\times N_2}))$ as $N_1, N_2 \to \infty$. Our discussion suggests that $\mathrm{PM}(\mathcal{L}_{N_1\times N_2})$ grows exponentially in N_1N_2 so we look at

$$c = \lim_{N_1,N_2\to\infty} \frac{\log \mathrm{PM}(\mathcal{L}_{N_1\times N_2})}{N_1N_2}. \tag{4.3.13}$$

From equation (4.3.12)

$$\frac{\log \mathrm{PM}(\mathcal{L}_{N_1\times N_2})}{N_1N_2} = \frac{1}{N_1N_2}\left(\left\lfloor \frac{N_1}{2}\right\rfloor \frac{N_2}{2}\log 4 + \sum_{\ell_1=1}^{\lfloor N_1/2\rfloor}\sum_{\ell_2=1}^{N_2/2}\log\left[\cos^2\left(\frac{\ell_1\pi}{N_1+1}\right)+\cos^2\left(\frac{\ell_2\pi}{N_2+1}\right)\right]\right). \tag{4.3.14}$$

As $N_1, N_2 \to \infty$, the first term in equation (4.3.14) approaches $\frac{\log 4}{4}$ and since the cosine function is continuous, the sum converges to an integral. Let $x = \frac{\ell_1}{N_1+1}$ and $y = \frac{\ell_2}{N_2+1}$ then x and y ranges from 0 (since $\frac{1}{N_1+1}, \frac{1}{N_2+1} \to 0$) to $\frac{1}{2}$ (since $\frac{\lfloor N_1/2\rfloor}{N_1+1}, \frac{N_2/2}{N_2+1} \to \frac{1}{2}$.) Lastly, $\mathrm{d}x = \frac{1}{N_1+1}, \mathrm{d}y = \frac{1}{N_2+1}$ and equation (4.3.13) now becomes

$$\begin{aligned}
c = \lim_{N_1,N_2\to\infty} \frac{\log \mathrm{PM}(\mathcal{L}_{N_1\times N_2})}{N_1N_2} &= \frac{\log 4}{4} + \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \log\left(\cos^2\pi x + \cos^2\pi y\right)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{\log 2}{2} + \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \log\left(\frac{1}{2}+\frac{1}{2}\cos 2\pi x + \frac{1}{2}+\frac{1}{2}\cos 2\pi y\right)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{\log 2}{2} + \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi} \log\left(\frac{1}{2}+\frac{1}{2}\cos x + \frac{1}{2}+\frac{1}{2}\cos y\right)\,\mathrm{d}x\,\mathrm{d}y. \qquad (4.3.15)
\end{aligned}$$

The generalized version of the above integral

$$f(u,v) = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\log(u+v+u\cos x+v\cos y)\,\mathrm{d}x\,\mathrm{d}y$$

is evaluated in Appendix A with

$$\begin{aligned}
f(u,v) &= \frac{4}{\pi}\left[\left(\frac{u}{v}\right)^{\frac{1}{2}} - \frac{1}{3^2}\left(\frac{u}{v}\right)^{\frac{3}{2}} + \frac{1}{5^2}\left(\frac{u}{v}\right)^{\frac{5}{2}} - \frac{1}{7^2}\left(\frac{u}{v}\right)^{\frac{7}{2}} + \dots\right] + \log\left(\frac{v}{2}\right) \qquad (4.3.16)\\
&= \frac{4}{\pi}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\left(\frac{u}{v}\right)^{k+\frac{1}{2}}\right] + \log\left(\frac{v}{2}\right)
\end{aligned}$$

Comparing equations (4.3.15) and (4.3.16), we set $u = v = \frac{1}{2}$ and we obtain

$$\begin{aligned}
c &= \frac{\log 2}{2} + \frac{1}{4}f\left(\frac{1}{2},\frac{1}{2}\right) \\
&= \frac{\log 2}{2} + \frac{1}{\pi}\left[1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots\right] + \frac{1}{4}\log\left(\frac{1}{4}\right) \\
&= \frac{\log 2}{2} + \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} - \frac{\log 2}{2} \\
&= \frac{G}{\pi}, \\
&= 0.291560904\dots
\end{aligned}$$

where

$$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

is the *Catalan constant*. Therefore from equation (4.3.13), the number of perfect matchings of a rectangular lattice $\mathcal{L}_{N_1 \times N_2}$ is asymptotically

$$\mathrm{PM}(\mathcal{L}_{N_1 \times N_2}) \approx e^{\frac{N_1 N_2 G}{\pi}} \approx 1.3385^{N_1 N_2}.$$

4.4 Perfect Matchings on Honeycomb Lattice $\mathcal{L}_{hc}$

In this section, we derive an asymptotic formula for the number of perfect matchings on a honeycomb lattice $\mathcal{L}_{hc}$. We assume our lattice is wrapped around a torus of dimension $N_1 \times N_2$. We consider the honeycomb lattice as square lattice of unit cells with each unit cell containing 2 vertices. As shown in Figure 4.3, the left slanted ($\nearrow$) direction corresponds to N_1-th roots of unity and the righted slanted ($\nwarrow$) direction corresponds to N_2-th roots of unity. Set $\theta_1 = \frac{2\pi\ell_1}{N_1}$ and $\theta_2 = \frac{2\pi\ell_2}{N_2}$ for $\ell_1 \in \{0, 1, \ldots, N_1 - 1\}$ and $\ell_2 \in \{0, 1, \ldots, N_2 - 1\}$. The oriented vertex cell adjacency matrix is defined as

$$a_{ij}(\boldsymbol{n}, \boldsymbol{n}') = \begin{cases} 1 & \text{if vertex } i \text{ in cell } \boldsymbol{n} \text{ and vertex } j \text{ in cell } \boldsymbol{n}' \text{ are adjacent with orientation } i \to j, \\ -1 & \text{if vertex } i \text{ in cell } \boldsymbol{n} \text{ and vertex } j \text{ in cell } \boldsymbol{n}' \text{ are adjacent with orientation } j \to i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4.1}$$

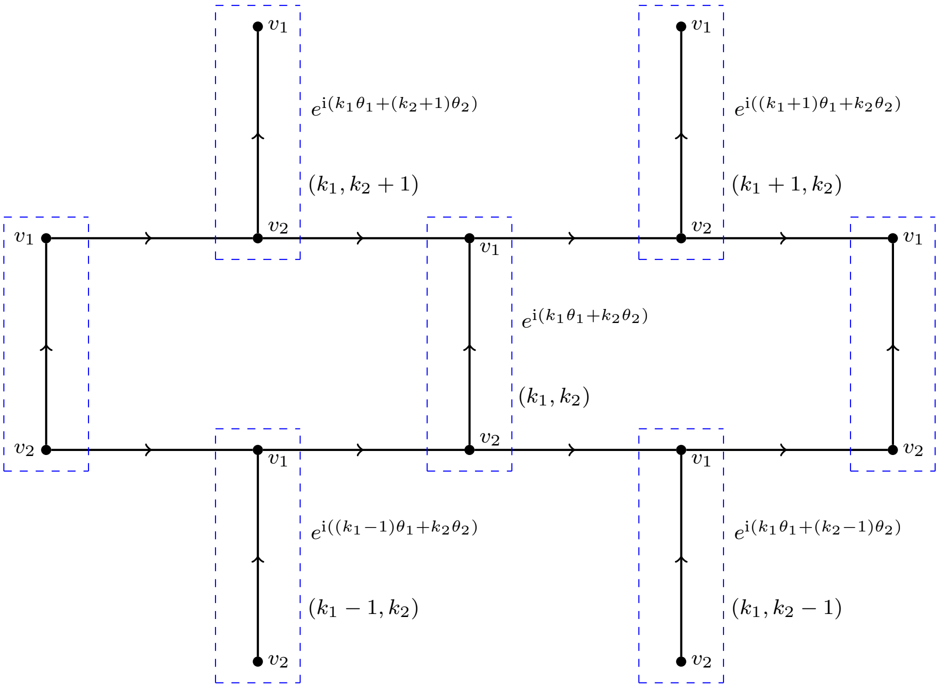


Figure 4.3: Pfaffian orientation and unit cells of the honeycomb lattice.

From Figure 4.3 the oriented vertex cell adjacency matrices are defined as follows

$$
\begin{aligned}
&a(0,0) = \begin{array}{cc} & \begin{matrix} v_1 & v_2 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \end{matrix} & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{array}, \quad
a(-1,0) = \begin{array}{cc} & \begin{matrix} v_1 & v_2 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \end{matrix} & \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \end{array}, \quad
a(1,0) = \begin{array}{cc} & \begin{matrix} v_1 & v_2 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \end{matrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}, \\
&a(0,-1) = \begin{array}{cc} & \begin{matrix} v_1 & v_2 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{array}, \quad
a(0,1) = \begin{array}{cc} & \begin{matrix} v_1 & v_2 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \end{matrix} & \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \end{array}.
\end{aligned} \tag{4.4.2}
$$

Let λ be an eigenvalue of the oriented adjacency matrix $\mathrm{A}(\mathcal{L}_{hc})$ then for the cell (k_1, k_2) (see Figure 4.3) the eigenvalue-eigenvector equation is as follows

$$
\begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix} e^{\mathrm{i}(k_1\theta_1 + k_2\theta_2)} = \begin{array}{cc} & \begin{matrix} \qquad v_1 \qquad & \qquad v_2 \qquad \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \end{matrix} & \begin{bmatrix} 0 & -1 + e^{\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_2} \\ 1 - e^{-\mathrm{i}\theta_1} + e^{-\mathrm{i}\theta_2} & 0 \end{bmatrix} \end{array} e^{\mathrm{i}(k_1\theta_1 + k_2\theta_2)} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{4.4.3}
$$

Let

$$
\mathrm{M}(\theta_1, \theta_2) = \begin{array}{cc} & \begin{matrix} \qquad v_1 \qquad & \qquad v_2 \qquad \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \end{matrix} & \begin{bmatrix} 0 & -1 + e^{\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_2} \\ 1 - e^{-\mathrm{i}\theta_1} + e^{-\mathrm{i}\theta_2} & 0 \end{bmatrix} \end{array}.
$$

Using the oriented vertex cell adjacency matrices in equation (4.4.1), we can also write $\mathrm{M}(\theta_1, \theta_2)$ as

$$
\mathrm{M}(\theta_1, \theta_2) = a(0,0) + a(-1,0)e^{-\mathrm{i}\theta_1} + a(1,0)e^{\mathrm{i}\theta_1} + a(0,-1)e^{-\mathrm{i}\theta_2} + a(0,1)e^{\mathrm{i}\theta_2}.
$$

Comparing both sides of equation (4.4.3) and using the fact that the determinant is equal to the product of the eigenvalues we find that the product of the two eigenvalues associated with a pair (ℓ_1, ℓ_2) is

$$
\begin{aligned}
\lambda &= \det\big(\mathrm{M}(\theta_1, \theta_2)\big) \\
&= 3 - (e^{-\mathrm{i}\theta_1} + e^{\mathrm{i}\theta_1}) + (e^{-\mathrm{i}\theta_2} + e^{\mathrm{i}\theta_2}) - (e^{-\mathrm{i}(\theta_1 - \theta_2)} + e^{\mathrm{i}(\theta_1 - \theta_2)}) \\
&= 3 - 2\cos\theta_1 + 2\cos\theta_2 - 2\cos(\theta_1 - \theta_2) \\
&= 3 - 2\cos\left(\frac{2\pi\ell_1}{N_1}\right) + 2\cos\left(\frac{2\pi\ell_2}{N_2}\right) - 2\cos\left(\frac{2\pi\ell_1}{N_1} + \frac{2\pi\ell_2}{N_2}\right).
\end{aligned}
$$

Again using the fact that the determinant of a matrix equals the product of all of its eigenvalues we have

$$
\begin{aligned}
\det\big(\mathrm{A}(\mathcal{L}_{hc})\big) &= \prod_{\ell_1=0}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} \lambda_{(\ell_1,\ell_2)} \\
&= \prod_{\ell_1=0}^{N_1-1} \prod_{\ell_2=0}^{N_2-1} \left[3 - 2\left(\cos\left(\frac{2\pi\ell_1}{N_1}\right) - \cos\left(\frac{2\pi\ell_2}{N_2}\right) + \cos\left(\frac{2\pi\ell_1}{N_1} + \frac{2\pi\ell_2}{N_2}\right)\right)\right].
\end{aligned}
$$

The asymptotic formula for $\mathrm{PM}(\mathcal{L}_{hc})$ is obtained as follows: As $N_1, N_2 \to \infty$ we define

$$
c_{hc} = \lim_{N_1, N_2 \to \infty} \frac{\log\left[\det\big(\mathrm{A}(\mathcal{L}_{hc})\big)\right]}{2N_1N_2}. \tag{4.4.4}
$$

Using a similar argument as done in the case of the rectangular lattice we have

$$\begin{aligned} c_{hc} &= \frac{1}{2}\int_0^1\int_0^1 \log\left[3 - 2\big(\cos 2\pi x - \cos 2\pi y + \cos(2\pi x + 2\pi y)\big)\right]\, \mathrm{d}x\ \mathrm{d}y \\ &= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log\left[3 - 2\big(\cos x - \cos y + \cos(x + y)\big)\right]\, \mathrm{d}x\ \mathrm{d}y \\ &= 0.323065947\ldots \end{aligned}$$

We refer the reader to [29] as to how the value of c_{hc} is obtained. Therefore from equation (4.5.2), the number of perfect matchings of the honeycomb lattice $\mathcal{L}_{hc}$ of dimension $N_1 \times N_2$ is asymptotically

$$\mathrm{PM}(\mathcal{L}_{hc}) \approx e^{2\cdot c_{hc}N_1N_2} \approx 1.90815^{N_1N_2}.$$

4.5 Perfect Matchings on Triangular Lattice $\mathcal{L}_{tri}$

We follow a similar procedure as done above to derive an asymptotic formula for counting the number of perfect matchings on a triangular lattice $\mathcal{L}_{tri}$. Figure 4.4 defines the Pfaffian orientation on the triangular lattice which is considered as a square lattice of unit cells with each unit cell containing two vertices.

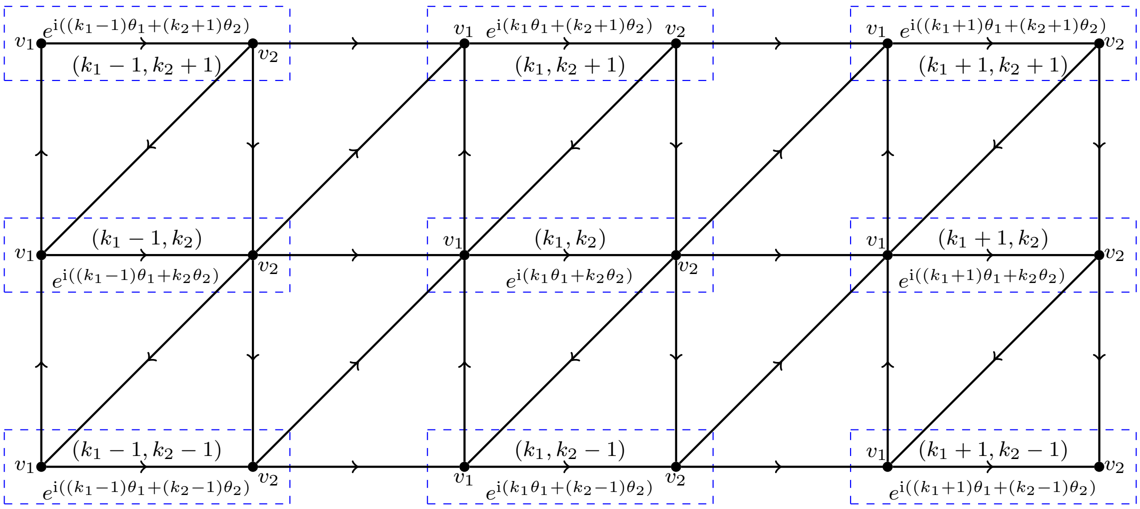


Figure 4.4: Pfaffian orientation and unit cells of the triangular lattice.

Let λ be an eigenvalue of the oriented adjacency matrix $\mathrm{A}(\mathcal{L}_{tri})$ then for the cell (k_1, k_2) (see Figure 4.4) eigenvalue-eigenvector equation in a matrix form is as follows

$$\begin{bmatrix}\lambda v_1\\ \lambda v_2\end{bmatrix} e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} = \begin{array}{c} \\ v_1 \\ v_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \qquad\qquad v_1 \qquad\qquad & \qquad\qquad v_2 \qquad\qquad \end{array} \\ \begin{bmatrix} -e^{-\mathrm{i}\theta_2} + e^{\mathrm{i}\theta_2} & 1 - e^{-\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_2} - e^{-\mathrm{i}(\theta_1+\theta_2)} \\ -1 + e^{\mathrm{i}\theta_1} + e^{-\mathrm{i}\theta_2} + e^{\mathrm{i}(\theta_1+\theta_2)} & e^{-\mathrm{i}\theta_2} - e^{\mathrm{i}\theta_2} \end{bmatrix} \end{array} e^{\mathrm{i}(k_1\theta_1+k_2\theta_2)} \cdot \begin{bmatrix} v_1\\ v_2\end{bmatrix}. \tag{4.5.1}$$

Let

$$\mathrm{M}(\theta_1, \theta_2) = \begin{array}{c} \\ v_1 \\ v_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \qquad\qquad v_1 \qquad\qquad & \qquad\qquad v_2 \qquad\qquad \end{array} \\ \begin{bmatrix} -e^{-\mathrm{i}\theta_2} + e^{\mathrm{i}\theta_2} & 1 - e^{-\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_2} - e^{-\mathrm{i}(\theta_1+\theta_2)} \\ -1 + e^{\mathrm{i}\theta_1} + e^{-\mathrm{i}\theta_2} + e^{\mathrm{i}(\theta_1+\theta_2)} & e^{-\mathrm{i}\theta_2} - e^{\mathrm{i}\theta_2} \end{bmatrix} \end{array}.$$

Then the eigenvalues are

$$\begin{aligned}
\lambda &= \det\big(\mathrm{M}(\theta_1,\theta_2)\big) \\
&= 6 - (e^{-\mathrm{i}\theta_1} + e^{\mathrm{i}\theta_1}) - (e^{-\mathrm{i}2\theta_2} + e^{\mathrm{i}2\theta_2}) + (e^{-\mathrm{i}(\theta_1+2\theta_2)} + e^{\mathrm{i}(\theta_1+2\theta_2)}) \\
&= 6 - 2\cos\theta_1 - 2\cos 2\theta_2 + 2\cos(\theta_1 + 2\theta_2) \\
&= 6 - 2\left[\cos\left(\frac{2\pi\ell_1}{N_1}\right) + \cos\left(\frac{4\pi\ell_2}{N_2}\right) - \cos\left(\frac{2\pi\ell_1}{N_1} + \frac{4\pi\ell_2}{N_2}\right)\right].
\end{aligned}$$

Again using the fact that the determinant of a matrix equals the product of all of its eigenvalues we have

$$\begin{aligned}
\det\big(\mathrm{A}(\mathcal{L}_{tri})\big) &= \prod_{\ell_1=0}^{N_1-1}\prod_{\ell_2=0}^{N_2-1} \lambda_{(\ell_1,\ell_2)} \\
&= \prod_{\ell_1=0}^{N_1-1}\prod_{\ell_2=0}^{N_2-1} \left[6 - 2\left(\cos\left(\frac{2\pi\ell_1}{N_1}\right) + \cos\left(\frac{4\pi\ell_2}{N_2}\right) - \cos\left(\frac{2\pi\ell_1}{N_1} + \frac{4\pi\ell_2}{N_2}\right)\right)\right].
\end{aligned}$$

The asymptotic formula for $\mathrm{PM}(\mathcal{L}_{tri})$ is obtained as follows: As $N_1, N_2 \to \infty$ we define

$$c_{tri} = \lim_{N_1,N_2\to\infty} \frac{\log\big[\det\big(\mathrm{A}(\mathcal{L}_{tri})\big)\big]}{2N_1N_2}. \tag{4.5.2}$$

Using a similar argument as done above we have

$$\begin{aligned}
c_{tri} &= \frac{1}{2}\int_0^1\int_0^1 \log\big[6 - 2\big(\cos 2\pi x + \cos 4\pi y - \cos(2\pi x + 4\pi y)\big)\big]\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log\big[6 - 2\big(\cos x + \cos 2y - \cos(x + 2y)\big)\big]\,\mathrm{d}x\,\mathrm{d}y \\
&= 0.857189074\ldots
\end{aligned}$$

The value of c_{tri} is be obtained from [29]. Therefore from equation (4.5.2), the number of perfect matchings of the triangular lattice $\mathcal{L}_{tri}$ of dimension $N_1 \times N_2$ is asymptotically

$$\mathrm{PM}(\mathcal{L}_{tri}) \approx e^{2\cdot c_{tri}N_1N_2} \approx 5.55322^{N_1N_2}.$$

Chapter 5

Conclusion

In this work, we demonstrate two methods of counting the number of spanning trees of any connected graph G; namely, as a special value of the Tutte polynomial $T(G; x, y)$ with $x = y = 1$ and the matrix-tree theorem which states that the number of spanning trees of any connected graph G is given by any cofactor of its Laplacian matrix.

We also prove that given any connected graph, the number of spanning trees can be expressed as a ratio of the product of the non-zero eigenvalues of the Laplacian matrix to the total number of vertices of the graph. Moreover, since our study concentrates on lattices, we study planar graphs because most lattices can be regarded as planar graphs. In addition we define the dual representation G^* of a planar graph G and prove that given any connected planar graph G, the Tutte polynomial

$$T(G^*; x, y) = T(G; y; x).$$

Using these results, we derive explicit formulas for the number of spanning trees of a finite section of following regular planar lattices; the rectangular lattice, the triangular lattice and its dual the honeycomb lattice, the kagomé lattice and dual the diced lattice and lastly the lattice with nanogons and triangles and its dual lattice. For each of these lattices, we also measure the thermodynamic limit or the bulk limit which is a measure of the rate of growth of the number of spanning trees.

Lastly, we prove Cayley's theorem which gives a relation between the Pfaffian of a skew-symmetric matrix and its determinant. Cayley's theorem gives us a solution to the dimer problem in statistical physics. The basic task of the dimer problem is to determine the number of ways to cover all the vertices of a lattice with dimers such that every vertex in our lattice is occupied by exactly one dimer. In graph theoretic terms, a dimer configuration of a given is a perfect matching. In other to count the total number of perfect matchings we first find a Pfaffian orientation of a graph which in general, is a difficult task since the class of graphs that admit a Pfaffian orientation is not known. But for planar graphs, one can always find a Pfaffian orientation. Using this results, we derive explicit formula for the number of perfect matchings on the rectangular lattice as well as its asymptotic growth. We also derive an asymptotic formula for the number of perfect matchings on the triangular lattice and its dual the honeycomb lattice.

Appendices

Appendix A

Integral Evaluation

To evaluate the integral below

$$z_{\mathcal{L}_{m\times n}} = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \log(2+2-2\cos x-2\cos y)\ \mathrm{d}x\ \mathrm{d}y,$$

we consider the following. Let

$$f(u,v) = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \log(u+v-u\cos x-v\cos y)\ \mathrm{d}x\ \mathrm{d}y. \tag{A.0.1}$$

Taking the u-derivative we have

$$\begin{aligned}\frac{\partial f}{\partial u} &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{1-\cos x}{u+v-u\cos x-v\cos y}\ \mathrm{d}x\ \mathrm{d}y\\ &= \frac{1}{\pi}\int_0^\pi (1-\cos x)\ \mathrm{d}x\cdot\frac{1}{\pi}\int_0^\pi\frac{\mathrm{d}y}{u+v-u\cos x-v\cos y}.\end{aligned} \tag{A.0.2}$$

We evaluate the integral below using Weierstrass substitution. Let

$$I = \frac{1}{\pi}\int_0^\pi\frac{\mathrm{d}y}{u+v-u\cos x-v\cos y} \tag{A.0.3}$$

and let $\alpha = u+v-u\cos x$. Then equation (A.0.3) becomes

$$I = \frac{1}{\pi}\int_0^\pi\frac{\mathrm{d}y}{\alpha-v\cos y}. \tag{A.0.4}$$

Using Weierstrass substitution, we have

$$\begin{aligned}&t=\tan\left(\frac{y}{2}\right) &&\cos\left(\frac{y}{2}\right)=\frac{1}{\sqrt{1+t^2}} &&\sin\left(\frac{y}{2}\right)=\frac{t}{\sqrt{1+t^2}}\\ &\cos y=\frac{1-t^2}{1+t^2} &&\mathrm{d}y=\frac{2\ \mathrm{d}t}{1+t^2} &&\frac{1}{\alpha-v\cos y}=\frac{1+t^2}{(\alpha-v)+(\alpha+v)t^2}.\end{aligned}$$

Substituting the equations above into equation (A.0.3) we have

$$\begin{aligned}
I &= \frac{1}{\pi}\int_0^\infty \frac{2\ \mathrm{d}t}{1+t^2}\cdot\frac{1+t^2}{(\alpha-v)+(\alpha+v)t^2} \\
&= \frac{2}{\pi}\int_0^\infty \frac{\mathrm{d}t}{(\alpha-v)+(\alpha+v)t^2} \\
&= \frac{2}{\pi}\left[\frac{1}{\sqrt{(\alpha^2-v^2)}}\arctan\left(\frac{\sqrt{\alpha+v}}{\sqrt{\alpha-v}}t\right)\Bigg|_0^\infty\right] \\
&= \frac{2}{\pi}\left[\frac{1}{\sqrt{(\alpha^2-v^2)}}\left(\frac{\pi}{2}-0\right)\right] \\
&= \frac{1}{\sqrt{(\alpha^2-v^2)}} \\
&= \frac{1}{\left[(u+v-u\cos x)^2-v^2\right]^{\frac{1}{2}}}.
\end{aligned}$$

Thus, equation (A.0.2) becomes

$$\frac{\partial f}{\partial u} = \frac{1}{\pi}\int_0^\pi \frac{(1-\cos x)\ \mathrm{d}x}{\left[(v+u(1-\cos x))^2-v^2\right]^{\frac{1}{2}}}. \tag{A.0.5}$$

We make another substitution. Let $\omega = 1-\cos x$ then $\mathrm{d}\omega = \sin x\ \mathrm{d}x$. Since $\sin^2 x = 1-\cos^2 x = (1-\cos x)(1+\cos x)$, $1+\cos x = 2-\omega$, $\sin x = (2\omega-\omega^2)^{\frac{1}{2}}$ then $(1-\cos x)\ \mathrm{d}x = \frac{\omega\ \mathrm{d}\omega}{[\omega(2-\omega)]^{\frac{1}{2}}}$. The denominator in equation (A.0.5) also becomes

$$\begin{aligned}
\left[(v+u(1-\cos x))^2-v^2\right]^{\frac{1}{2}} &= \left[(v+u\omega)^2-v^2\right]^{\frac{1}{2}} \\
&= \left[v^2+2uv\omega+u^2\omega^2-v^2\right]^{\frac{1}{2}} \\
&= [u\omega(u\omega+2v)]^{\frac{1}{2}}
\end{aligned}$$

Substituting them into equation (A.0.5) we have

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \frac{1}{\pi}\int_0^2 \frac{\omega\ \mathrm{d}\omega}{\left[\omega^2 u(2-\omega)(u\omega+2v)\right]^{\frac{1}{2}}} \\
&= \frac{1}{\pi}\int_0^2 \frac{\mathrm{d}\omega}{\left[u(2-\omega)(u\omega+2v)\right]^{\frac{1}{2}}}.
\end{aligned} \tag{A.0.6}$$

Using the identity

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\arctan g(\omega) = \frac{g'(\omega)}{1+[g(\omega)]^2},$$

we write the denominator of equation (A.0.5) as

$$\frac{1}{[u(2-\omega)(u\omega+2v)]^{\frac{1}{2}}} = -\frac{2}{u}\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\arctan\left[\frac{u(2-\omega)}{u\omega+2v}\right]^{\frac{1}{2}}\right).$$

Substituting into equation (A.0.5) we have

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \frac{1}{\pi}\int_0^2 -\frac{2}{u}\ \mathrm{d}\omega\ \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\arctan\left[\frac{u(2-\omega)}{u\omega+2v}\right]^{\frac{1}{2}}\right) \\
&= -\frac{2}{\pi u}\int_0^2 \mathrm{d}\left(\arctan\left[\frac{u(2-\omega)}{u\omega+2v}\right]^{\frac{1}{2}}\right) \\
&= -\frac{2}{\pi u}\left(\arctan\left[\frac{u(2-\omega)}{u\omega+2v}\right]^{\frac{1}{2}}\right)\Bigg|_0^2 \\
&= -\frac{2}{\pi u}\left[0-\arctan\left(\frac{u}{v}\right)^{\frac{1}{2}}\right] \\
&= \frac{2}{\pi u}\arctan\left(\frac{u}{v}\right)^{\frac{1}{2}}. \qquad \text{(A.0.7)}
\end{aligned}$$

We write arctan as a power series using the identity

$$\arctan x = \int \frac{\mathrm{d}x}{1+x^2} = \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\ \mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\ \mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

Therefore using the identity above, we can write equation (A.0.7) as

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \frac{2}{\pi u}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{u}{v}\right)^{n+\frac{1}{2}}\right] \\
&= \frac{2}{\pi u}\left[\left(\frac{u}{v}\right)^{\frac{1}{2}} - \frac{1}{3}\left(\frac{u}{v}\right)^{\frac{3}{2}} + \frac{1}{5}\left(\frac{u}{v}\right)^{\frac{5}{2}} - \frac{1}{7}\left(\frac{u}{v}\right)^{\frac{7}{2}} + \dots\right]. \qquad \text{(A.0.8)}
\end{aligned}$$

Integrating each term in equation (A.0.8) with respect to u we have

$$\begin{aligned}
f(u,v) &= \frac{4}{\pi}\left[\left(\frac{u}{v}\right)^{\frac{1}{2}} - \frac{1}{3^2}\left(\frac{u}{v}\right)^{\frac{3}{2}} + \frac{1}{5^2}\left(\frac{u}{v}\right)^{\frac{5}{2}} - \frac{1}{7^2}\left(\frac{u}{v}\right)^{\frac{7}{2}} + \dots\right] + h(v) \\
&= \frac{4}{\pi}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left(\frac{u}{v}\right)^{n+\frac{1}{2}}\right] + h(v). \qquad \text{(A.0.9)}
\end{aligned}$$

From equations (A.0.1) and (A.0.9) setting $u=0$ we get

$$\begin{aligned}
f(0,v) = h(v) &= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\log(v - v\cos y)\ \mathrm{d}x\ \mathrm{d}y \\
&= \frac{1}{\pi^2}\int_0^{\pi}\mathrm{d}x\int_0^{\pi}\log[v(1-\cos y)]\ \mathrm{d}y \\
&= \frac{1}{\pi}\int_0^{\pi}\log v\ \mathrm{d}y + \frac{1}{\pi}\int_0^{\pi}\log(1-\cos y)\ \mathrm{d}y \\
&= \log v + \frac{1}{\pi}(-\pi\log 2) \\
&= \log\left(\frac{v}{2}\right).
\end{aligned}$$

Substituting into equation (A.0.9) we get

$$f(u,v) = \frac{4}{\pi}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left(\frac{u}{v}\right)^{n+\frac{1}{2}}\right] + \log\left(\frac{v}{2}\right). \tag{A.0.10}$$

Appendix B

Computing Eigenvalues

Let

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & -1 & 0 & 1 \\ 0 & \dots & 0 & -1 & 0 \end{bmatrix}. \tag{B.0.1}$$

The eigenvalues of the matrix A are computed as follows. Let $f_n(x)$ be the characteristic polynomial. That is,

$$f_n(x) = \det[x\mathrm{I}_n - \mathrm{A}] = \det \begin{bmatrix} x & -1 & 0 & \dots & 0 \\ 1 & x & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & x & -1 \\ 0 & \dots & 0 & 1 & x \end{bmatrix}.$$

Using the first row to compute the determinant, we arrive at the the recurrence

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x), \quad f_0(x) = 1,\ f_1(x) = x. \tag{B.0.2}$$

We follow the four steps outlined in the third chapter of [1] to solve the recurrence equation (B.0.2).
Step 1:
We first write the recurrence equation (B.0.2) as a single equation with the initial conditions. For $n < 0$, $f_n(x) = 0$ and this satisfies equation (B.0.2). But for $n = 0$, $f_0(x) = 1$ while the right-hand side is 0. So the complete recurrence equation is therefore

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x) + [n = 0]. \tag{B.0.3}$$

The last term in equation (B.0.3) is defined as follows:

$$[Q] = \begin{cases} 1 & \text{if } Q \text{ is true} \\ 0 & \text{if } Q \text{ is false.} \end{cases}$$

Step 2:
We express equation (B.0.3) as an equation between generation functions. If we let $F(z) = \sum_{n\geq 0} f_n(x) z^n$, then

$$\begin{aligned} F(z) &= x\sum_{n\geq 0} f_{n-1}(x)z^n + \sum_{n\geq 0} f_{n-2}(x)z^n + \sum_{n\geq 0}[n=0]z^n \\ &= xz\sum_{n\geq 1} f_{n-1}(x)z^{n-1} + z^2\sum_{n\geq 2} f_{n-2}(x)z^{n-2} + \sum_{n\geq 0}[n=0]z^n \\ &= xz\sum_{n\geq 0} f_n(x)z^n + z^2\sum_{n\geq 0} f_n(x)z^n + \sum_{n\geq 0}[n=0]z^n \\ &= xzF(z) + z^2F(z) + 1. \end{aligned} \tag{B.0.4}$$

Step 3:
We solve equation (B.0.4). That is,

$$F(z) = \frac{1}{1 - xz - z^2}. \tag{B.0.5}$$

Step 4:
We express the right-hand side of (B.0.5) as a generating function and compare the coefficients. That is,

$$\begin{aligned} F(z) &= \frac{1}{1-xz-z^2} = \frac{1}{(1-\alpha z)(1-\beta z)} = \frac{a}{(1-\alpha z)} + \frac{b}{(1-\beta z)} \qquad \text{(B.0.6)} \\ \sum_{n\geq 0} f_n z^n &= a\sum_{n\geq 0}\alpha^n z^n + b\sum_{n\geq 0}\beta^n z^n \\ &= \sum_{n\geq 0}(a\alpha^n + b\beta^n)z^n. \end{aligned}$$

Comparing coefficients, we have

$$f_n(x) = a\alpha^n + b\beta^n \tag{B.0.7}$$

Solving equation (B.0.6) using partial fractions we get

$$\begin{aligned} \alpha &= \frac{x+\sqrt{x^2+4}}{2}, & & \beta = \frac{x-\sqrt{x^2+4}}{2}, \\ a &= \frac{\alpha}{\alpha-\beta} = \frac{x+\sqrt{x^2+4}}{2\sqrt{x^2+4}}, & \text{and} \quad & b = \frac{\beta}{\beta-\alpha} = -\frac{x-\sqrt{x^2+4}}{2\sqrt{x^2+4}}. \end{aligned}$$

Substituting the above into equation (B.0.7) we have

$$f_n(x) = \frac{1}{\sqrt{x^2+4}}\left[\left(\frac{x+\sqrt{x^2+4}}{2}\right)^{n+1} - \left(\frac{x-\sqrt{x^2+4}}{2}\right)^{n+1}\right]. \tag{B.0.8}$$

The roots of equation (B.0.8) are precisely all the eigenvalues of matrix A in equation (B.0.2). If λ is an eigenvalue of A then λ is a root of the characteristic polynomial $f_n(x)$

and $f_n(\lambda) = 0$. Therefore equation (B.0.8) becomes

$$\left(\frac{\lambda + \sqrt{\lambda^2+4}}{\lambda - \sqrt{\lambda^2+4}}\right)^{n+1} = 1 = e^{2\pi \mathrm{i}}.$$

Hence,

$$\left(\frac{\lambda + \sqrt{\lambda^2+4}}{\lambda - \sqrt{\lambda^2+4}}\right) = e^{\frac{2k\pi \mathrm{i}}{n+1}} \text{ with } k = 1, 2, \ldots, n. \tag{B.0.9}$$

We exclude the case $k = 0$ since that will cause the zero in the numerator to cancel with the zero in the denominator From equation (B.0.9) we have

$$\begin{aligned}
(\lambda + \sqrt{\lambda^2+4})^2 &= -4e^{\frac{2k\pi \mathrm{i}}{n+1}} \\
\lambda + \sqrt{\lambda^2+4} &= \pm 2\mathrm{i}e^{\frac{k\pi \mathrm{i}}{n+1}} \\
\sqrt{\lambda^2+4} &= \pm 2\mathrm{i}e^{\frac{k\pi \mathrm{i}}{n+1}} - \lambda \\
\lambda^2 + 4 &= \lambda^2 \pm 4\lambda\mathrm{i}e^{\frac{k\pi \mathrm{i}}{n+1}} - 4e^{\frac{2k\pi \mathrm{i}}{n+1}} \\
\lambda &= \pm\mathrm{i}\frac{e^{\frac{2k\pi \mathrm{i}}{n+1}} + 1}{e^{\frac{k\pi \mathrm{i}}{n+1}}} \\
&= \pm\mathrm{i}\left(e^{\frac{k\pi \mathrm{i}}{n+1}} + e^{-\frac{k\pi \mathrm{i}}{n+1}}\right) \\
&= \pm 2\mathrm{i}\cos\frac{k\pi}{n+1}.
\end{aligned}$$

The eigenvalues of A are

$$\lambda_k = 2\mathrm{i}\cos\frac{k\pi}{n+1} \ (k = 1, 2, \ldots, n) \tag{B.0.10}$$

since

$$\cos\frac{k\pi}{n+1} = -\cos\frac{(n+1-k)\pi}{n+1} \quad (k = 1, 2, \ldots, n).$$

Appendix C

Computer Algebra System

We use the computer algebra system SAGE to write a program to count the number of spanning trees for the lattices that are studied in this thesis.

C.1 Spanning Trees on a Rectangular Lattice

This program computes the total number of spanning trees of a rectangular lattice $\mathcal{L}_{N_1\times N_2}$. It is based on the formula

$$\tau(\mathcal{L}_{N_1\times N_2}) = \frac{1}{N_1N_2}\prod_{\substack{k_1=0 \\ (k_1,k_2)\neq(0,0)}}^{N_1-1}\prod_{k_2=0}^{N_2-1}\left[4-2\left(\cos\left(\frac{k_1\pi}{N_1}\right)+\cos\left(\frac{k_2\pi}{N_2}\right)\right)\right].$$

As an example, we use the program to compute the spanning trees of $\mathcal{L}_{3\times 4}$. Note that $\nu = \Lambda = 1$.

```
def spanning_trees_rec_lat(N1,N2,nu,Lambda,torus_wrap):
    """
        Objective of program:
            This program computes the exact number of spanning trees on a
            rectangular lattice of dimension N1 x N2.

        Inputs:
            N1 --- Dimension of the rectangular lattice in one direction
                   (positive integer).
            N2 --- Dimension of the rectangular lattice in the other
                   direction (positive integer).
            nu --- Number of vertices in the each cell of the rectangular
                   lattice (nu = 1).
            Lambda --- Non-zero eigenvalues of M(0,0) (Lambda = 1).
            torus_wrap --- whether lattice is wrapped round a torus.
                           (The value is either one or zero).
        Output:
            Exact number of spanning trees on the rectangular lattice
            of dimension N1 x N2 (not wrapped around a torus if
            torus_wrap = 0) and (wrapped around a torus if torus_wrap = 1).

        Example:
            N1 = 3
```

```
            N2 = 4
            nu = 1
            Lambda = 1
            torus_wrap = 0
            spanning_trees_rec_lat(N1,N2,mu,Lambda,0)
            sp_rec_lat = 2415.
    """
    if (torus_wrap !=0)&(torus_wrap !=1):
        print "The last variable must either be 0 or 1."

    if torus_wrap == 0:
        pdt = 1
        for k1 in range(N1):
            for k2 in range(N2):
                if k1 == 0:
                    pdt1 = 1
                    for k2 in range(1,N2):
                        pdt1 = pdt1 * (4 - 2 *(cos(2*pi*k1/N1) +  cos(2*pi*k2/N2)))
                else:
                    pdt = pdt * (4 - 2 *(cos(2*pi*k1/N1) + cos(2*pi*k2/N2)))
        sp_rec_lat = (Lambda/(nu * N1 * N2)) * (pdt1 * pdt)
        print "The number of spanning trees of the rectangular lattice of dimension"\
        ,N1,"x",N2,"wrapped around a torus is","%0.0f"%sp_rec_lat

    if torus_wrap == 1:
        pdt = 1
        for k1 in range(N1):
            for k2 in range(N2):
                if k1 == 0:
                    pdt1 = 1
                    for k2 in range(1,N2):
                        pdt1 = pdt1 * (4 - 2 *(cos(pi*k1/N1) +  cos(pi*k2/N2)))
                else:
                    pdt = pdt * (4 - 2 *(cos(pi*k1/N1) + cos(pi*k2/N2)))
        sp_rec_lat = (Lambda/(nu * N1 * N2)) * (pdt1 * pdt)
        print "The number of spanning trees on the rectangular lattice of dimension"\
        ,N1,"x",N2,"not wrapped around a torus is","%0.0f"%sp_rec_lat.full_simplify()

spanning_trees_rec_lat(3,4,1,1,1)
The number of spanning trees of the rectangular lattice of dimension 3 x
4 not wrapped around a torus is 2415
```

C.2 Spanning Trees on a Triangular Lattice

This program computes the total number of spanning trees of a triangular lattice $\mathcal{L}_{tri}$ wrapped around a torus. It is based on the formula

$$\tau(\mathcal{L}_{tri}) = \frac{1}{N_1 N_2} \prod_{k_1=0}^{N_1-1} \prod_{\substack{k_2=0 \\ (k_1,k_2)\neq(0,0)}}^{N_2-1} \left[6 - 2\left(\cos\left(\frac{2k_1\pi}{N_1}\right) + \cos\left(\frac{2k_2\pi}{N_2}\right) + \cos\left(\frac{2k_1\pi}{N_1} + \frac{2k_2\pi}{N_2}\right)\right)\right].$$

As an example, we use the program to compute the spanning trees of $\mathcal{L}_{tri}$ with $N_1 = 3,\ N_2 = 4,\ \nu = \Lambda = 1$.

```
def spanning_trees_tri_lat(N1,N2,nu,Lambda):
    """
    Objective of program:
        This program computes the exact number of spanning trees
        on a triangular lattice of dimension N1 x N2
        wrapped around a torus.

    Inputs:
        N1 --- Dimension of the triangular lattice in one direction
               (positive integer).
        N2 --- Dimension of the triangular lattice in the other
               direction (positive integer).
        nu --- Number of vertices in the each cell of the triangular
               lattice (nu = 1).
        Lambda --- Non-zero eigenvalues of M(0,0).

    Output:
        Exact number of spanning trees on the triangular lattice
        of dimension N1 x N2 wrapped around a torus.

    Example:
        N1 = 3
        N2 = 4
        nu = 1
        Lambda = 1
        spanning_trees_tri_lat(N1,N2,mu,Lambda)
        sp_tri_lat = 52002816
    """
    pdt = 1
    for k1 in range(N1):
        for k2 in range(N2):
            if k1 == 0:
                pdt1 = 1
                for k2 in range(1,N2):
                    pdt1 = pdt1 * (6 - 2 *(cos(2*pi*k1/N1) + cos(2*pi*k2/N2)
                           + cos(2*k1*pi/N1 + 2*k2*pi/N2)))
            else:
                pdt = pdt * (6 - 2 *(cos(2*pi*k1/N1) + cos(2*pi*k2/N2)
                      + cos(2*k1*pi/N1 + 2*k2*pi/N2)))
    sp_tri_lat = (Lambda/(nu * N1 * N2)) * (pdt1 * pdt)
    print "The number of spanning trees of the triangular lattice of dimension"\
        ,N1,"x",N2,"wrapped around a torus is","%0.0f"%sp_tri_lat.full_simplify()

spanning_trees_tri_lat(3,4,1,1)
The number of spanning trees of the triangular lattice of dimension 3 x
4 wrapped around a torus is 52002816
```

C.3 Spanning Trees on a Kagomé Lattice

This program computes the total number of spanning trees of a kagomé lattice $\mathcal{L}_{kag}$ wrapped around a torus. The program is based on the formula

$$\tau(\mathcal{L}_{kag}) = \frac{36}{3N_1N_2} \prod_{\substack{k_1=0 \\ (k_1,k_2)\neq(0,0)}}^{N_1-1} \prod_{k_2=0}^{N_2-1} 12 \left[3 - \left(\cos\left(\frac{2\pi k_1}{N_1}\right) + \cos\left(\frac{2\pi k_2}{N_2}\right) + \cos\left(\frac{2\pi k_1}{N_1} + \frac{2\pi k_2}{N_2}\right)\right)\right].$$

As an example, we use the program to compute the spanning trees of $\mathcal{L}_{kag}$ with $N_1 = 3,\ N_2 = 4,\ \nu = 3,\ \Lambda = 36$.

```
def spanning_trees_kag_lat(N1,N2,nu,Lambda):
    """
    Objective of program:
        This program computes the exact number of spanning
        trees on a kagome lattice of dimension N1 x N2
        wrapped round a torus.

    Inputs:
        N1 --- Dimension of kagome lattice in one direction
               (positive integer).
        N2 --- Dimension of kagome lattice in the other
               direction (positive integer).
        nu --- Number of vertices in each cell of the
               kagome lattice; (nu = 3).
        Lambda --- Non-zero eigenvalues of M(0,0); (Lamda = 36).

    Output:
        Exact number of spanning trees on the kagome lattice
        of dimension N1 x N2 wrapped around a torus.

    Example:
        N1 = 3
        N2 = 4
        nu = 3
        Lambda = 36
        sp_kag_lat = spanning_trees_kag_lat(N1,N2,nu,Lambda)
        sp_kag_lat = 226397622582116352
    """

    pdt = 1
    for k1 in range(N1):
        for k2 in range(N2):
            if k1 == 0:
                pdt1 = 1
                for k2 in range(1,N2):
                    pdt1 = pdt1 * (36 - 12 *(cos(2*pi*k1/N1) + cos(2*pi*k2/N2)
                           + cos(2*k1*pi/N1 + 2*k2*pi/N2)))
            else:
                pdt = pdt * (36 - 12 *(cos(2*pi*k1/N1) + cos(2*pi*k2/N2)
                      + cos(2*k1*pi/N1 + 2*k2*pi/N2)))
    sp_kag_lat = (Lambda/(nu * N1 * N2)) * (pdt1 * pdt)
    print "The number of spanning trees of the kagome lattice of dimension"\
     ,N1,"x",N2,"wrapped around a torus is","%0.0f"%sp_kag_lat.full_simplify()
```

```
spanning_trees_kag_lat(3,4,3,36)
The number of spanning trees of the kagome lattice of dimension 3 x 4
wrapped around a torus is 226397622582116352
```

C.4 Spanning Trees on a $9 \cdot 3$ Lattice

This program computes the total number of spanning trees of a $9 \cdot 3$ lattice $\mathcal{L}_{9 \cdot 3}$ wrapped around a torus. The program is based on the formula

$$\tau(\mathcal{L}_{9\cdot 3}) = \frac{64}{4N_1N_2} \prod_{\substack{k_1=0 \\ (k_1,k_2)\neq(0,0)}}^{N_1-1} \prod_{k_2=0}^{N_2-1} \left[24 - 8\left(\cos\left(\frac{2\pi k_1}{N_1}\right) + \cos\left(\frac{2\pi k_2}{N_2}\right) + \cos\left(\frac{2\pi k_1}{N_1} + \frac{2\pi k_2}{N_2}\right)\right)\right].$$

As an example, we use the program to compute the spanning trees of $\mathcal{L}_{9\cdot 3}$ with $N_1 = 3,\ N_2 = 4,\ \nu = 4,\ \Lambda = 64$.

```
def spanning_trees_9_3_lat(N1,N2,nu,Lambda):
    """
    Objective of program:
        This program computes the exact number of spanning trees on a
        9-3 lattice wrapped around a torus of dimension N1 x N2.

    Inputs:
        N1 --- Dimension of 9-3 lattice in one direction
               (positive integer).
        N2 --- Dimension of 9-3 lattice in the other
               direction (positive integer).
        nu --- Number of vertices in each cell of the 9-3;
               lattice (nu = 4).
        Lambda --- Non-zero eigenvalues of M(0,0); (Lambda = 64).

    Output:
        Exact number of spanning trees on the 9-3 lattice of
        dimension N1 x N2 wrapped around a torus.

    Example:
        N1 = 3
        N2 = 4
        nu = 4
        Lambda = 64
        sp_9_3_lat = spanning_trees_9_3_lat(N1,N2,nu,Lambda)
        sp_9_3_lat = 3489849906561024
    """

    pdt = 1
    for k1 in range(N1):
        for k2 in range(N2):
            if k1 == 0:
                pdt1 = 1
                for k2 in range(1,N2):
                    pdt1 = pdt1 * (24 - 8 *(cos(2*pi*k1/N1) +  cos(2*pi*k2/N2)
                           + cos(2*k1*pi/N1 + 2*k2*pi/N2)))
```

```
            else:
                pdt = pdt * (24 - 8 *(cos(2*pi*k1/N1) + cos(2*pi*k2/N2)
                      + cos(2*k1*pi/N1 + 2*k2*pi/N2)))
    sp_9_3_lat = (Lambda/(nu * N1 * N2)) * (pdt1 * pdt)
    print "The number of spanning trees of the 9-3 lattice of dimension"\
        ,N1,"x",N2,"wrapped around a torus is","%0.0f"%sp_9_3_lat.full_simplify()

spanning_trees_9_3_lat(3,4,4,64)
The number of spanning trees of the 9-3 lattice of dimension 3 x 4
wrapped around a torus is 3489849906561024
```

C.5 Perfect Matchings on a Rectangular Lattice

This program computes the exact number of perfect matchings on a rectangular lattice $\mathcal{L}_{N_1\times N_2}$. It is based on the formula

$$\mathrm{PM}(\mathcal{L}_{N_1\times N_2}) = 4^{\lfloor\frac{N_1}{2}\rfloor\frac{N_2}{2}}\prod_{k_1=1}^{\lfloor N_1/2\rfloor}\prod_{k_2=1}^{N_2/2}\left[\cos^2\left(\frac{k_1\pi}{N_1+1}\right)+\cos^2\left(\frac{k_2\pi}{N_2+1}\right)\right]$$

which is due to Fisher, Kasteleyn and Temperley. It is normally called the FKT algorithm. As an example, we use the program to compute the total number of perfect matchings on the chessboard $\mathrm{PM}(\mathcal{L}_{8\times 8})$.

```
def perfect_matchings_rec_lat(N1,N2):
    """
     Objective of program:
        This program computes the exact number of perfect matchings
        on a rectangular lattice of dimension N1 x N2.
        It is based on a theorem by Fisher, Kasteleyn and Temperley.

     Inputs:
         N1 --- dimension of the rectangular lattice in one
                direction (positive integer).
         N2 --- dimension of the rectangular lattice in the
                other direction (positive even integer).

     Output:
         Exact number of perfect matchings on the rectangular
         lattice of dimension N1 x N2.

    Example:
        N1 = 8
        N2 = 8
        perfect_matchings_rec_lat(N1,N2)
        12988816
    """
    if ((N1 % 2) != 0) & ((N2 % 2) != 0):
        print "One of your entries must be an even integer"
    else:
        pdt = 1
        for i in range(1,floor(N1/2)+1):
            for j in range(1,(N2/2)+1):
                pdt = pdt * ((cos(i*pi/(N1+1)))^2 + (cos(j*pi/(N2+1)))^2)
```

```
        pm_rec_lat = 4^(floor(N1/2)*N2/2)*pdt
        print "The number of perfect matchings on the rectangular lattice of dimension"\
        ,N1,"x",N2,"is","%0.0f"%pm_rec_lat.full_simplify()

perfect_matchings_rec_lat(8,8)
The number of perfect matchings on the rectangular lattice of dimension 8 x 8 is 12988816
```

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