BIVARIATE BOX SPLINES AND SURFACE SUBDIVISION

by

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Thesis presented in partial fulfilment of the requirements for the degree of ***Master of Science*** *at the University of Stellenbosch*

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Abstract

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The main purpose of this thesis is to construct subdivision surfaces by using bivariate subdivision rules, in particular the Butterfly subdivision scheme, which is an interpolatory subdivision rule, by which we preserve initial control points at all steps of the iterative process until we get the limit surface as a good approximant for a refined control net.

Taking the existence of bivariate refinable functions as given, our study focuses on the classical $2\mathrm{I}_2$-refinable bivariate box splines, which are 2-dimensional extensions of the univariate cardinal B-splines, which can be used as basis functions for bivariate subdivision.

The first two chapters are concerned with the analytical definition of box splines, as well as their properties, as studied in Sections 1.7 - 1.8. In the second Chapter, in particular, in Section 2.1, the refinability of bivariate box splines is studied by means of Fourier transforms. Also, tensor products are used to construct bivariate box splines. In Sections 2.3 - 2.6, the notion of bivariate subdivision and the issue of subdivision convergence, in particular a necessary condition for such convergence, namely the sum rule condition, are presented. The link between subdivision and the cascade algorithm is introduced in Section 2.4, in order to generate a bivariate refinable function, which in turn plays a key role in subdivision surface construction.

Having dealt with the bivariate box splines, we subsequently, in Chapter 3, focus our attention on an algebraic approach for the construction of bivariate subdivision schemes. In particular, we algebraically construct the interpolatory Butterfly subdivision symbol, from the bivariate box spline basis of an ideal $\mathcal{I}$ or its power $\mathcal{I}^k$, in the ring of polynomials Π, characterised by a vanishing condition at three of the corners of the box $[-1, 1]^2$. It is shown, in Section 3.4, that the Butterfly subdivision symbol can be expressed as $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$- combinations of normalised box spline symbols.

In Chapter 4, we give the convergence and smoothness analysis of the interpolatory Butterfly subdivision scheme by using the contractivity of subdivision operators. We also give the convergence analysis of a new interpolatory scheme, as obtained by using the Courant hat function factorization of the Laurent polynomial identity satisfying the interpolatory condition, and for which a limiting function is obtained from cascade convergence. Furthermore, a suitable interval range for the tension parameter w, that preserves the agreement between the refined control net and the C^1 limit surface for this subdivision scheme, is also investigated.

The thesis is concluded, in Chapter 5, with meaningful results obtained from the study and also by a discussion on unresolved issues that are left for future research.

Uittreksel

TWEEVERANDERLIKE BOKSLATFUNKSIES EN OPPERVLAKSUBDIVISIE

(" Bivariate box splines and Surface Subdivision ")

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OPSOMMING

Die hoofdoel van hierdie tesis is die konstruksie van subdivisie-oppervlakke met behulp van tweeveranderlike subdivisiereëls, in die besonder die Butterfly subdivisieskema , wat 'n interpolerende subdivisiereël is , waarin die aanvanklike kontrolepunte by elke stap van die iteratiewe proses gepreserveer word totdat die limietoppervlak as 'n goeie benadering van die verfynde kontrolenet verkry word.

Deur die bestaan van tweeveranderlike verfynbare funksies as gegewe te aanvaar, fokus ons studie op die klassieke $2I_2$-verfynbare tweeveranderlike bokslatfunksies , wat 2-dimensionele ekstensies van die eenveranderlike kardinale B-latfunksies is, en wat gebruik kan word as basisfunksies vir tweeveranderlike subdivisie.

Die eerste twee hoofstukke het te doen met die analitiese definisie van bokslatfunksies, waarvan die eienskappe bestudeer word in Afdelings 1.7 - 1.8. In die tweede hoofstuk, in Afdeling 2.1, word die verfynbaarheid van tweeveranderlike bokslatfunksies bestudeer met behulp van Fourier transforms. Verder word tensorprodukte gebruik vir die konstruksie van tweeveranderlike bokslatfunksies. In Afdelings 2.3 -2.6 word die begrip van tweeverandelike subdivisie en die kwessie van subdivisiekonvergensie , en in die besonder 'n nodige voorwaarde vir subdivisiekonvergensie , naamlik die somreël voorwaarde , gegee.

Na afhandeling van die tweeveranderlike bokslatfunksies , fokus ons vervolgens , in Hoofstuk 3, ons aandag op 'n algebraïese benadering tot die konstruksie van tweeveranderlike subdivisieskemas. In die besonder konstrueer ons algebraïes die interpolerende Butterfly subdivisiesimbool , vanuit die tweeveranderlike bokslatfunksiebasis van 'n ideaal $\mathcal{I}$, of die produk $\mathcal{I}^k$ daarvan , in die ring van polinome Π , gekarakteriseer deur 'n nulvoorwaarde

by drie van die hoekpunte van die boks $[-1,1]^2$. Daar word aangetoon , in Afdeling 3.4, dat die Butterfly subdivisiesimbool uitgedruk kan word as $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$- kombinasies van genormaliseerde bokslatfunksiesimbole.

In Hoofstuk 4 gee ons die konvergensie - en gladheidsanalise van die interpolerende Butterfly subdivisieskema deur gebruik te maak van die kontraktiwiteit van subdivisie-operatore. Ons gee ook die konvergensie - analise van 'n nuwe interpolerende skema , soos verkry deur gebruik te maak van die Courant hoedfunksiefaktorisering van die Laurent polinoomidentiteit wat die interpolerende voorwaarde bevredig , en waarvoor 'n limietfunksie verkry word vanuit kaskade-algoritmekonvergensie. Daarby word'n geskikte intervalgebried vir die spanningparameter w , wat die ooreenkoms tussen die verfynde kontrolenet en die C^1-limietoppervlak preserveer, ondersoek.

Die tesis word afgesluit , in Hooftuk 5, met betekenisvolle resultate verkry in die studie, asook deur 'n bespreking van onopgeloste kwessies wat vir verdere navorsing gelaat word.

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Dedications

To my family [especially to my sister, Hadas]

Contents

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Nomenclature

Symbols

$\mathbb{Z}$ Set of integers.

$\mathbb{C}$ Set of complex numbers.

$\mathbb{R}$ Set of real numbers.

$\mathbb{N}$ Set of natural numbers.

$\mathbb{Z}_+$ Set of non-negative integers.

$\mathbb{N}^3$ Set of natural numbers in 3-tuples.

$\mathbb{N}_0^3$ $\mathbb{N}^3 \cup \{(0,0,0)\}$

$\mathbb{R}^d$ The set $\{\mathbf{x} = (x_1, \ldots, x_d) : x_j \in \mathbb{R},\ j = 1, \ldots, d\}$.

$\mathbb{Z}^d$, $\mathbb{C}^d$ Set of integer, complex d-tuples with $d \in \mathbb{N}$ respectively.

$\mathbb{Z}^{m\times n}$ The set of all matrices of order $m \times n$ with entries from $\mathbb{Z}$.

$\ell(\mathbb{Z})$ The linear space of bi-infinite real-valued sequences in $\mathbb{Z}$, that is,

$\mathbf{c} \in \ell(\mathbb{Z}) \Leftrightarrow \mathbf{c} = \{c_j\}_{j\in\mathbb{Z}} \subseteq \mathbb{R}$.

$\ell_0(\mathbb{Z})$ The linear space of finitely supported bi-infinite real-valued sequences in $\ell(\mathbb{Z})$.

$\ell(\mathbb{Z}^d)$ The linear space of bi-infinite real-valued sequences in $\mathbb{Z}^d$, that is,

$\mathbf{c} \in \ell(\mathbb{Z}^d) \Leftrightarrow \mathbf{c} = \{\mathbf{c_j}\}_{\mathbf{j}\in\mathbb{Z}^d} \subseteq \mathbb{R}^d$ with $d \in \mathbb{N}$.

$\ell_0(\mathbb{Z}^d)$ The subspace of finitely supported sequences in $\ell(\mathbb{Z}^d)$ with $d \in \mathbb{N}$.

$M(\mathbb{R})$ The linear space of real-valued functions in $\mathbb{R}$ with $d \in \mathbb{N}$, that is, $f : \mathbb{R} \to \mathbb{R}$.

$M_0(\mathbb{R})$ The subspace of finitely supported functions in $M(\mathbb{R})$.

$M(\mathbb{R}^d)$ The linear space of real-valued functions in $\mathbb{R}^d$, that is, $f : \mathbb{R}^d \to \mathbb{R}$.

$M_0(\mathbb{R}^d)$ The subspace of finitely supported functions in $M(\mathbb{R}^d)$ with $d \in \mathbb{N}$.

supp $(\mathbf{c})$ The support of a sequence $\mathbf{c} \in \ell_0(\mathbb{Z}^2)$, that is, $\mathrm{supp}(\mathbf{c}) = \{\mathbf{j} \in \mathbb{Z}^2 : c_{\mathbf{j}} \neq 0\}$

supp^c $(\mathbf{c})$ The support of a compactly supported sequence $\mathbf{c} \in \ell_0(\mathbb{Z}^2)$.

$\overline{\mathcal{A}}$ The closure of a set $\mathcal{A}$, that is, $\overline{\mathcal{A}}$ consists of $\mathcal{A}$ together with all limit points of $\mathcal{A}$. In other words, $\overline{\mathcal{A}}$ is an intersection of all closed sets containing $\mathcal{A}$.

$\text{supp}\,(f)$ The support of a function $f \in M_0(\mathbb{R}^2)$, that is, the smallest closed set containing $\{\mathbf{x} \in \mathbb{R}^2 : f(\mathbf{x}) \neq \mathbf{0}\}$, in other words, $\left(\text{supp}\,(f) = \overline{\{\mathbf{x} : \; f(\mathbf{x}) \neq 0\}}\right)$.

$C(\mathbb{R})$ The subspace of continuous real-valued functions in $M(\mathbb{R})$, that is,

$f : \mathbb{R} \to \mathbb{R}$ is continuous.

$C_0(\mathbb{R})$ The subspace of finitely supported functions in $C(\mathbb{R})$.

$C(\mathbb{R}^d)$ The subspace of continuous functions in $M(\mathbb{R}^d)$ for $d = 2, 3, \ldots$.

$C^{-1}(\mathbb{R}^d)$ The subspace of piecewise continuous functions in $M(\mathbb{R}^d)$ with $d \in \mathbb{N}$.

$C_u(\mathbb{R}^2)$ The linear space of uniformly bounded functions in $C(\mathbb{R}^2)$.

$C_0(\mathbb{R}^d)$ The subspace of finitely supported continuous functions in $M_0(\mathbb{R}^d)$ for $d = 2, 3, \ldots$.

$C_0^\infty(\mathbb{R}^d)$ The linear space of infinitely differentiable functions in $C(\mathbb{R}^d)$ with $d = 2, 3, \ldots$.

$C^\alpha(\mathbb{R}^d)$ The subset of α times differentiable functions in $C(\mathbb{R}^d)$ with $\alpha \in \mathbb{N}_0$ and $d = 1, 2, 3, \ldots$.

$C_0^\alpha(\mathbb{R})$ The subset of finitely supported functions in $C_0(\mathbb{R})$ with $\alpha \in \mathbb{N}_0$.

$C_0^\alpha(\mathbb{R}^d)$ The subset of finitely supported functions in $C_0(\mathbb{R}^\mathrm{d})$ with $d = 2, 3, \ldots$.

$\ell_\infty(\mathbb{Z}^d)$ The linear subspace of bounded sequences in $\ell(\mathbb{Z}^d)$ with $d = 2, 3, \ldots$, that is, the set $\left\{c_\mathbf{j} \in \ell(\mathbb{Z}^d) \subseteq \mathbb{R} : \sup_\mathbf{j} |c_\mathbf{j}| < \infty\right\}$ of bounded sequences.

$||\cdot||_\infty$ The sup-norm for the linear space $\ell_\infty(\mathbb{Z}^2)$ [or $C_u(\mathbb{R}^2)$], that is,

$||\mathbf{c}||_\infty := \sup_{\mathbf{j}\in\mathbb{Z}^2} |\mathbf{c}_\mathbf{j}|$, $\mathbf{c} \in \ell_\infty(\mathbb{Z}^2)$, or $\left(||f||_\infty := \sup_{\mathbf{x}\in\mathbb{R}^2} ||f(\mathbf{x})||, \; f \in C_u(\mathbb{R}^2).\right)$

$\sum_j$ and $\sum_\mathbf{j}$ The summations $\sum_{j\in\mathbb{Z}}$ and $\sum_{\mathbf{j}\in\mathbb{Z}^d}$ with $d \in \mathbb{N}$ respectively.

$\sum_{i,j}$ The summation $\sum_{i,j\in\mathbb{Z}}$.

$\sup_\mathbf{j}$ and $\sup_\mathbf{x}$ The suprema over all $\mathbf{j} \in \mathbb{Z}^d$ and $\mathbf{x} \in \mathbb{R}^d$ with $d = 2, 3, \ldots$ respectively.

ϕ The refinable function, that is, function satisfying the dilation equation given by $\phi(\cdot) = \sum_\mathbf{j} p_\mathbf{j}\, \phi(A \cdot - \mathbf{j})$, for $\mathbf{j} \in \mathbb{Z}^d$, $d \in \mathbb{N}$.

ϕ^I The interpolatory refinable function, that is, a refinable function satisfying $\phi^I(\mathbf{0}) = 1$ for $\mathbf{0} \in \mathbb{Z}^d$ and $\phi^I(\mathbf{j}) = 0$, for $\mathbf{j} \in \mathbb{Z}^d \backslash \{\mathbf{0}\}$.

B_m	The cardinal B-spline of order m.
δ	The delta sequence defined by $\delta_{\mathbf{0}} = 1$ and $\delta_{\mathbf{j}} = 0$ for $\mathbf{j} \neq \mathbf{0} \in \mathbb{Z}^d$ with $d = 2, 3, \ldots$.
c	The scalar control points $c_j \in \mathbb{R}$ with $j \in \mathbb{Z}$.
$\mathbf{c}$	The control points in $\mathbf{c}_{i,j} \in \mathbb{R}^3$ with grid indexing points $(i, j) \in \mathbb{Z}^2$.
φ	The limiting function after subdivision, that is, $\phi_{\mathbf{p}} = \mathbb{S}_{\mathbf{p}}^{\infty}\mathbf{c}$.
$\mathbf{j}^T$	The transpose of $\mathbf{j} := (j_1, j_2, \ldots, j_d) \in \mathbb{Z}^d$ for $d \in \mathbb{N}$.
A	The dilation matrix A, that is, a $d \times d$ invertible matrix with integer entries.
$\mathbf{p}$	The refinement mask in $\ell_0(\mathbb{Z}^2)$.
$\mathbf{p}^I$	The interpolatory refinement mask in $\ell_0(\mathbb{Z}^2)$, that is, $\mathbf{p}^I = \{p^I_{i,j} \in \mathbb{R} : i, j \in \mathbb{R}\}$.
$\mathbf{z}$	$\mathbf{z} := (z_1, \ldots, z_d)$, with $d = 2, 3, \ldots$.
P	The mask symbol associated with refinement mask $\mathbf{p} \in \ell_0(\mathbb{Z}^d)$, that is, the Laurent polynomial $\sum_{\mathbf{j} \in \mathbb{Z}^d} p_{\mathbf{j}}\, \mathbf{z}^{\mathbf{j}}$.
P^I	The interpolatory mask symbol associated with the interpolatory refinement mask $\mathbf{p} \in \ell_0(\mathbb{Z}^d)$ with $d = 2, 3, \ldots$.
$\mathcal{D}$	Direction matrix, which is a matrix of non-zero arbitrary direction vectors.
$\mathcal{D}_n$	The direction matrix that is defined by $\mathcal{D}_n := \{\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^n\}$ where $\mathbf{e}^i$, $i = 1, 2, \ldots, n$ are possibly either one of $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, $\mathbf{e}_3 = (1, 1)$, or $\mathbf{e}_4 = (1, -1)$.
$\mathbb{B}_{\mathcal{D}}$	The corresponding box spline associated with $\mathcal{D}$.
$\mathbb{B}_{\mathcal{D}_n}$	The corresponding box spline associated with direction set $\mathcal{D}_n$.
$\tilde{\mathbb{B}}_{\mathcal{D}}(z_1, z_2)$	The normalised box spline symbols associated with $\mathcal{D}$.
$\mathbf{j} \in \mathbb{Z}^d$	The d-tuple $\mathbf{j} := (j_1, \ldots, j_d) \in \mathbb{Z}^d$.
$\mathcal{G}_k$	The k-directional mesh.
$\lfloor x \rfloor$	The largest integer $\leq x$.
$\lceil x \rceil$	The smallest integer $\geq x$.
$\mathbb{S}_{\mathbf{p}}$	The subdivision operator associated with the mask $\mathbf{p} \in \ell_0(\mathbb{Z}^2)$.
$\mathbb{S}_{\mathbf{p}}^r$	The subdivision operator associated with the mask $\mathbf{p}$, applied r times .
$\mathbf{c}^{(r)}$	The resulting sequence after applying $\mathbb{S}_{\mathbf{p}}^r$ to a sequence $\mathbf{c} \in \ell(\mathbb{Z}^2)$.
I_2	The dilation matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
I_d	The $d \times d$ square identity matrix.

Z' The vanishing set, that is, $\{(1,-1),(-1,-1),(-1,1)\}$

Π_k^2 The space of bivariate polynomials of total degree k, that is, the linear space of functions of the form $f(x,y) = \sum_{0\le i+j\le k} a_{i,j}\, x^i\, y^j$.

$\mathcal{I}$ The polynomial ideal $\mathcal{I} := \left\{ f \in \Pi : f(\varepsilon_1, \varepsilon_2) = 0, \quad \text{for} \quad (\varepsilon_1, \varepsilon_2) \in \mathrm{Z}' \right\}$.

$\mathcal{I}^k$ The k^{th} power ideal of an ideal $\mathcal{I} = \langle f_1, \ldots, f_s \rangle$.

$\mathcal{I}_k$ The set of generators for the power ideal $\mathcal{I}^k$.

$\mathbb{A}$ The ring of polynomials over field $\mathbb{K}$, that is, $\mathbb{A} := \mathbb{K}[x_1, \ldots, x_n]$.

$\mathcal{R}$ The ring of Laurent polynomials over field $\mathbb{K}$, that is, $\mathcal{R} := \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

$(\mathcal{B}, ||\cdot||)$ A normed linear space $\mathcal{B}$ with norm $\|\cdot\|$.

Chapter 1

Interpolatory bivariate refinable functions

1.1 Introduction

Refinement equations play an essential role in numerous application areas of science and engineering, in particular in wavelet analysis and geometric modelling. In this thesis, we study bivariate refinable functions which are functions that can be expressed as a linear combination of their dilation by a dilation matrix.

Refinable functions can be characterized by their refinement masks, and these masks play a key role in the analysis of refinable functions, which are useful as basis functions in subdivision schemes for the construction of a curve or surface from the initial set of data (control) points.

The mask symbol corresponding to a given refinement mask is the Laurent polynomial whose coefficients are given by the mask. This Laurent polynomials serve as a useful tool in subdivision analysis.

We shall give particular attention to specifically interpolatory bivariate refinable functions. These are refinable functions that assume the value 1 at the origin and vanish at all other integer pairs.

We study characterizations of refinement masks of these refinable functions with their associated mask symbols by taking into consideration integer dilation matrix A. We give a particular attention to the dilation matrix $A = 2\mathrm{I}_2$.

Following the work in [FdV11], well-known polynomial identities are studied based on results from [CCJZ11] to characterize the masks in terms of box spline symbols. An algebraic approach for studying mask symbols for constructing subdivision schemes will be given.

We proceed to introduce interpolatory subdivision schemes, which are schemes that preserve initial data points at all steps of the iterative process.

By taking the existence of interpolatory bivariate refinable functions as given, we further investigate the refinable box splines as basis functions for subdivision analysis, since subdivision of box splines is the cornerstone of many popular multivariate subdivision schemes.

The concepts of a subdivision algorithm and a cascade algorithm will be introduced. Furthermore, by using tensor products of univariate refinable functions, the interlink between the subdivision algorithm and numerical experimentation of the cascade algorithm will be implemented to generate a bivariate refinable function, which in turn plays a key role in the construction of the desirable smooth surface. Also, we note that the convergence of the cascade algorithm implies convergence of a subdivision scheme, which in turn is strongly bonded to the existence of interpolatory bivariate refinable function.

After that, as the major part of our work, interpolatory surface subdivision schemes come into play as they are basic tools for generating 3D objects by starting from control meshes and implementing the subdivision process until a fine smooth surface is obtained from the coarser ones. The control points in the control mesh define the shape of the surface and hence the control mesh plays a key role in the description of the surface.

The objective of this study is to implement the refinable box splines for constructing bivariate subdivision schemes, especially the Butterfly interpolatory subdivision scheme. A special focus will be given to an algebraic verification of the Butterfly subdivision scheme from the normalized box spline symbols which act as generators of some polynomial ideal $\mathcal{I}$ and its power $\mathcal{I}^k$. That is, the Laurent mask symbol of the Butterfly subdivision scheme can be expressed as $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$-combinations of normalised box spline polynomial symbols, $\{\mathbb{B}_{\mathcal{D}}(z_1, z_2) : (z_1, z_2) \in \mathbb{C}^2 \backslash (0,0)\}$. Furthermore, the convergence analysis of this subdivision scheme as well as a newly obtained interpolatory subdivision scheme, will be further shown, by using the contractivity property of subdivision operators.

1.2 Notations and general concepts

We shall denote the set of integers by $\mathbb{Z}$, the set of natural numbers by $\mathbb{N}$, the set of non-negative integers by $\mathbb{Z}_+$, the set of real numbers by $\mathbb{R}$, and the set of complex numbers by $\mathbb{C}$. Also, for $d \in \mathbb{N}$, the symbols $\mathbb{Z}^d$, $\mathbb{R}^d$ and $\mathbb{C}^d$ will denote the set of ordered d-tuples with respect to the integers, real numbers and complex numbers, respectively. In particular, our study concentrates on the case $d = 2$, so that the symbols $\mathbb{Z}^2$, $\mathbb{R}^2$ and $\mathbb{C}^2$ denote the set of ordered pairs with respect to integers, real numbers and complex numbers, respectively. Note that $\mathbb{Z}^1 = \mathbb{Z}$, $\mathbb{R}^1 = \mathbb{R}$ and $\mathbb{C}^1 = \mathbb{C}$.

We use $\ell(\mathbb{Z}^d)$ to denote the linear space of all real-valued sequences, $\mathbf{c} = \left\{c_{\mathbf{j}} \in \mathbb{R} : \mathbf{j} \in \mathbb{Z}^d\right\}$, whereas $\ell_0(\mathbb{Z}^d)$ is used to represent the subspace of finitely supported sequences, with support given by $\operatorname{supp} \mathbf{c} := \left\{\mathbf{j} \in \mathbb{Z}^d : c_{\mathbf{j}} \neq 0\right\}$.

Analogously, for the linear space $M(\mathbb{R}^d)$ of all real valued d-variate functions f on $\mathbb{R}^d$, the set of finitely supported functions constitute a linear subspace denoted by $M_0(\mathbb{R}^d)$, with support $\operatorname{supp}(f)$, which is the smallest closed set containing $\{\mathbf{x} : f(\mathbf{x}) \neq 0\}$.

The symbol $C(\mathbb{R}^d)$ denotes the subspace of continuous functions in $M(\mathbb{R}^d)$, whereas $C_0(\mathbb{R}^d)$ denotes the subspace of continuous functions in $M_0(\mathbb{R}^d)$. Also, we shall denote by $C^{-1}(\mathbb{R}^d)$ the subspace of piecewise continuous functions in $M(\mathbb{R}^d)$. We shall write $\sum\limits_{\mathbf{j}}$ for $\sum\limits_{\mathbf{j} \in \mathbb{Z}^d}$.

To begin our discussion, it is important to start by introducing the concept of a refinable function, which is a cornerstone for the study of subdivision and wavelet analysis.

Definition 1.2.1. *For any invertible $d \times d$-matrix A with entries in $\mathbb{N}$, if $\phi \in M_0(\mathbb{R}^d)$ and $\{p_{\boldsymbol{j}}\} \in \ell_0(\mathbb{Z}^d)$ are such that*

$$\phi(\boldsymbol{x}) = \sum_{\boldsymbol{j}} p_{\boldsymbol{j}}\, \phi(A\boldsymbol{x} - \boldsymbol{j}), \quad \boldsymbol{x} \in \mathbb{R}^d, \tag{1.2.1}$$

we say that ϕ is A-refinable with refinement mask $\{p_{\boldsymbol{j}}\}$, and we call equation (1.2.1) the refinement (or dilation) equation. The matrix A is known as the dilation matrix.

Note that an A-refinable function ϕ is a self-similar function in the sense that ϕ can be expressed as a linear combination of the shifts of its own dilation (scaling) with dilation factor A. There is a unique bijective relation between a refinable function ϕ and its corresponding refinement mask $\{p_{\mathbf{j}}\}$, as was established in [CMD91]. This means that if ϕ and $\{p_{\mathbf{j}}\}$ are as in (1.2.1), and the function ϕ satisfies

$$\phi(\mathbf{x}) = \sum_{\mathbf{j}} q_{\mathbf{j}}\, \phi(A\mathbf{x} - \mathbf{j}), \quad \mathbf{x} \in \mathbb{R}^d \tag{1.2.2}$$

for some sequence $\{q_{\mathbf{j}}\} \in \ell_0(\mathbb{Z}^d)$, then $\{q_{\mathbf{j}}\} = \{p_{\mathbf{j}}\}$.

Definition 1.2.2. *For a refinement mask $\{p_{\boldsymbol{j}}\} \in \ell_0(\mathbb{Z}^d)$ as in Definition 1.2.1, the Laurent polynomial*

$$P(\boldsymbol{z}) := \sum_{\boldsymbol{j}} p_{\boldsymbol{j}}\, \boldsymbol{z}^{\boldsymbol{j}}\,, \quad \boldsymbol{z} \in \mathbb{C}^d \backslash \{\boldsymbol{0}\}, \tag{1.2.3}$$

where $\boldsymbol{z}^{\boldsymbol{j}} := z_1^{j_1} z_2^{j_2} \dots z_d^{j_d}$, is called the corresponding refinement mask symbol.

Definition 1.2.3. *An A-refinable function $\phi \in M_0(\mathbb{R}^d)$, as in Definition 1.2.1, and satisfying, moreover, the condition*

$$\phi(\boldsymbol{j}) = \delta_{\boldsymbol{j}}\,, \quad \boldsymbol{j} \in \mathbb{Z}^d, \tag{1.2.4}$$

with $\{\delta_{\boldsymbol{j}}\}$ denoting the Kronecker delta sequence given by

$$\delta_{\boldsymbol{j}} := \begin{cases} 1, & \text{if} \quad \boldsymbol{j} = \boldsymbol{0}; \\ 0, & \text{if} \quad \boldsymbol{j} \neq \boldsymbol{0}, \end{cases} \tag{1.2.5}$$

is called an interpolatory refinable function.

In other words, an interpolatory refinable function assumes the value 1 at the origin and vanishes at all other integer d-tuples.

We proceed to study refinement masks associated with interpolatory refinable functions.

1.3 Interpolatory refinement masks

The refinement mask of an interpolatory refinable function satisfies the following necessary condition.

Proposition 1.3.1. *Let $\phi \in M_0(\mathbb{R}^d)$ denote an interpolatory refinable function with refinement mask $\{p_{\boldsymbol{j}}\} \in \ell_0(\mathbb{Z}^d)$. Then*

$$p_{A\boldsymbol{j}^T} = \delta_{\boldsymbol{j}}, \quad \boldsymbol{j} \in \mathbb{Z}^d. \tag{1.3.1}$$

Proof. From (1.2.1) and (1.2.4), we have that, for $\mathbf{j} \in \mathbb{Z}^d$,

$$\begin{aligned} \delta_{\mathbf{j}} = \phi(\mathbf{j}) &= \sum_{\mathbf{k} \in \mathbb{Z}^d} p_{\mathbf{k}}\, \phi(A\mathbf{j}^T - \mathbf{k}^T) \\ &= \sum_{\mathbf{k} \in \mathbb{Z}^d} p_{\mathbf{k}}\, \delta_{A\mathbf{j}^T - \mathbf{k}^T} = p_{A\mathbf{j}^T}. \end{aligned}$$

□

We proceed to establish the following necessary condition on refinement masks with non-zero integral over $\mathbb{R}^d$, by extending the proof in [Rab10], where the dilation matrix $A = 2\mathrm{I}_2$ was considered.

Proposition 1.3.2. *Let $\phi \in M_0(\mathbb{R}^d)$ denote an integrable A-refinable function with non-zero integral over $\mathbb{R}^d$, and with refinement mask $\{p_{\boldsymbol{j}}\} \in \ell_0(\mathbb{Z}^d)$. Then*

$$\sum_{\boldsymbol{j}} p_{\boldsymbol{j}} = |\det\ A|. \tag{1.3.2}$$

Proof. Suppose that the dilation matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1d} \\ a_{21} & a_{22} & \dots & a_{2d} \\ \vdots & \vdots & \dots & \vdots \\ a_{d1} & a_{d2} & \dots & a_{dd} \end{bmatrix}. \tag{1.3.3}$$

Then, by writing $p_{\mathbf{j}} = p_{j_1, j_2, \dots, j_d}$, where $\mathbf{j} := (j_1, j_2, \dots, j_d) \in \mathbb{Z}^d$, we can integrate the refinement equation (1.2.1) to obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} \phi(x_1, x_2, \dots, x_d)\, \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_d \\ &= \sum_{j_1, \dots, j_d} p_{j_1, j_2, \dots j_d} \int_{\mathbb{R}^d} \phi\left[A(x_1, \dots, x_d)^T - (j_1, j_2, \dots, j_d)^T\right] \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_d. \end{aligned} \tag{1.3.4}$$

Since, for $\mathbf{j} \in \mathbb{Z}^d$, the variable transformation

$$(\xi_1, \xi_2, \dots, \xi_d) := A(x_1, x_2, \dots, x_d)^T - (j_1, j_2, \dots, j_d)^T$$

has, according to (1.3.3), the Jacobian

$$\mathrm{J}(x_1, \dots, x_d) := \begin{vmatrix} \frac{\partial \xi_1}{\partial x_1} & \frac{\partial \xi_1}{\partial x_2} & \dots & \frac{\partial \xi_1}{\partial x_d} \\ \frac{\partial \xi_2}{\partial x_1} & \frac{\partial \xi_2}{\partial x_2} & \dots & \frac{\partial \xi_2}{\partial x_d} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial \xi_d}{\partial x_1} & \frac{\partial \xi_d}{\partial x_2} & \dots & \frac{\partial \xi_d}{\partial x_d} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1d} \\ a_{21} & a_{22} & \dots & a_{2d} \\ \vdots & \vdots & \dots & \vdots \\ a_{d1} & a_{d2} & \dots & a_{dd} \end{vmatrix} = \det A,$$

we have

$$\int_{\mathbb{R}^d} \phi\left[A(x_1, x_2, \ldots, x_d)^T - (j_1, j_2, \ldots, j_d)^T\right] |\det A| \, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_d$$
$$= \int_{\mathbb{R}^d} \phi(\xi_1, \ldots, \xi_d) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \ldots \mathrm{d}\xi_d. \tag{1.3.5}$$

We deduce from (1.3.4) and (1.3.5) that

$$\int_{\mathbb{R}^d} \phi(x_1, x_2, \ldots, x_d) \, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_d$$
$$= \left[\sum_{j_1 \ldots j_d} p_{j_1, j_2, \ldots j_d} \frac{1}{|\det A|}\right] \int_{\mathbb{R}^d} \phi(x_1, x_2, \ldots, x_d) \, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_d, \tag{1.3.6}$$

after having noted also from the invertibility of A that $\det A \neq 0$, and thereby completing our proof of (1.3.2), after having noted the fact that the integral of ϕ is non-zero over $\mathbb{R}^d$. □

In view of Propositions 1.3.1 and 1.3.2, the existence of an interpolatory refinable function ϕ with non-zero integral over $\mathbb{R}^d$ necessitates for the corresponding refinement mask $\{p_{\mathbf{j}}\}$ to satisfy the conditions

$$\left.\begin{aligned} p_{A\mathbf{j}^T} &= \delta_{\mathbf{j}}, \quad \mathbf{j} \in \mathbb{Z}^d; \\ \sum_{\mathbf{j}} p_{\mathbf{j}} &= |\det A|. \end{aligned}\right\} \tag{1.3.7}$$

For the case $d = 2$, according to (1.2.3) in Definition 1.2.2, the refinement mask symbol P corresponding to a refinement mask $\{p_{\mathbf{j}}\} \in \ell_0(\mathbb{Z}^2)$ is given by

$$P(z_1, z_2) = \sum_{j,k} p_{j,k} \, z_1^j \, z_2^k, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}. \tag{1.3.8}$$

Observe from (1.3.8) that, for $d = 2$, the necessary conditions (1.3.7) on a refinement mask $\{p_{\mathbf{j}}\}$ has the equivalent mask symbol formulation

$$\left.\begin{array}{l} \text{The constant term in } P(z_1, z_2) \text{ is } 1; \\ P \text{ has no term in } z_1^{\gamma_1} z_2^{\gamma_2} \text{ such that } (\gamma_1, \gamma_2) := A(i,j)^T \neq (0,0) \text{ for some } (i,j) \in \mathbb{Z}^2 \,; \\ P(1,1) = \displaystyle\sum_{i,j \in \mathbb{Z}} p_{i,j} = |\det A|. \end{array}\right\} \tag{1.3.9}$$

Observe that, for the dilation matrix $A = 2\mathrm{I}_2$, the conditions (1.3.7) on the refinement mask $\{p_{\mathbf{j}}\}$ is given by

$$\left.\begin{aligned} p_{2i,2j} &= \delta_{i,j}, \quad (i,j) \in \mathbb{Z}^2; \\ \sum_{i,j \in \mathbb{Z}} p_{i,j} &= 4, \end{aligned}\right\} \tag{1.3.10}$$

whereas the condition (1.3.9) on an interpolatory mask symbol P is given by

$$\left.\begin{array}{l} \text{The constant term in } P(z_1, z_2) \text{ is } 1; \\ P \text{ has no term in } z_1^{2\gamma_1} z_2^{2\gamma_2} \text{ for any } (\gamma_1, \gamma_2) \in \mathbb{Z}^2 \backslash \{(0,0)\}; \\ \displaystyle\sum_{i,j \in \mathbb{Z}} p_{i,j} = 4. \end{array}\right\} \tag{1.3.11}$$

1.4 Cardinal B-splines

Cardinal B-splines constitute a class of univariate refinable functions that can be defined recursively as follows.

Definition 1.4.1. *Let $A \subseteq \mathbb{R}$. The characteristic function χ_A on a set A is defined by*

$$\chi_A(x) := \begin{cases} 1, & \text{if} \quad x \in A; \\ 0, & \text{if} \quad x \in \mathbb{R}\backslash A. \end{cases} \tag{1.4.1}$$

Definition 1.4.2. *The cardinal B-splines $\{B_m : m \in \mathbb{N}\}$ are defined recursively by*

$$\left. \begin{aligned} &B_1 := \chi_{[0,1]}; \\ &B_m(x) = \int_0^1 B_{m-1}(x-t)\; dt, \quad x \in \mathbb{R}, \quad m = 2, 3, \ldots . \end{aligned} \right\} \tag{1.4.2}$$

For any $m \in \mathbb{N}$, we call B_m the cardinal B-spline of order m.

In general, cardinal splines are defined as follows.

Definition 1.4.3. *For any $m \in \mathbb{N}$, a cardinal spline of order m is a piecewise polynomial of degree at most m, which is in $C^{m-2}(\mathbb{R})$, and with break points, or knots on $\mathbb{Z}$. The space of all cardinal splines of order m is denoted by $\mathcal{S}_m$, that is,*

$$\mathcal{S}_m := \left\{ f \in M(\mathbb{R}) : f|_{[j,j+1]} \in \Pi_{m-1},\ j \in \mathbb{Z}\,;\ f \in C^{m-2}(\mathbb{R}) \right\}. \tag{1.4.3}$$

The following properties of cardinal B-splines B_m are proved in [dVC10].

Theorem 1.4.4. *For $m \in \mathbb{N}$, the m^{th}-order cardinal B-spline B_m satisfies the following properties:*

(i)

$$\operatorname{supp} B_m = [0, m]; \tag{1.4.4}$$

(ii)

$$B_m(x) > 0, \quad x \in (0, m); \tag{1.4.5}$$

(iii)

$$B_m(\cdot - j) \in \mathcal{S}_m, \quad j \in \mathbb{Z}; \tag{1.4.6}$$

(iv)

$$B_m(x) = \frac{1}{(m-1)!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (x-j)_+^{m-1}, \quad x \in \mathbb{R}, \tag{1.4.7}$$

where

$$x_+^k := \begin{cases} x^k, & \text{if} \quad x \geq 0; \\ 0, & \text{if} \quad x < 0; \end{cases}$$

(v)

$$B_{m+1}(x) = \frac{x}{m}\, B_m(x) + \frac{m+1-x}{m}\, B_m(x-1), \quad x \in \mathbb{R}; \tag{1.4.8}$$

(vi)

$$B_m'(x) = B_{m-1}(x) - B_{m-1}(x-1),\ x \in \mathbb{R} \quad \textit{if} \quad m \geq 3; \tag{1.4.9}$$

(vii)

$$B_m(m-x) = B_m(x), \quad x \in \mathbb{R}; \tag{1.4.10}$$

(viii)

$$\int_{\mathbb{R}} B_m(x)\, \mathrm{d}x = 1; \tag{1.4.11}$$

(ix)

$$\sum_j B_m(x-j) = 1, \quad x \in \mathbb{R}; \tag{1.4.12}$$

(x)

$$B_m(x) = \frac{1}{2^{m-1}} \sum_{j=0}^{m} \binom{m}{j}\, B_m(2x-j), \quad x \in \mathbb{R}, \tag{1.4.13}$$

that is, B_m is 2-refinable with refinement mask $\boldsymbol{p}^m = \{p_j^m : j \in \mathbb{Z}\}$ given by

$$p_j^m := \frac{1}{2^{m-1}} \binom{m}{j}, \quad j \in \mathbb{Z}. \tag{1.4.14}$$

(xi) The sequence $\{B_m(\cdot - j) : j \in \mathbb{Z}\}$ is a basis for $\mathcal{S}_m$, in the sense that, for $f \in \mathcal{S}_m$, there exists a unique sequence $\{c_j\} \in \ell(\mathbb{Z})$ such that

$$f(x) = \sum_j c_j\, B_m(\cdot - j), \quad \cdot \in \mathbb{R}. \tag{1.4.15}$$

According to (1.4.3),(1.4.4) and (1.4.6), the m^{th}-order cardinal B-spline B_m is a piecewise polynomial of degree $m-1$ on its support interval $[0, m]$, with breakpoints at the integers $\{1, \ldots, m-1\}$, if $m \geq 2$. By applying the formula (1.4.7), we obtain, for $m = 2, 3, 4$, the explicit formulations

-

$$B_2(x) = h(x) := \begin{cases} x, & x \in [0,1); \\ 2-x, & x \in [1,2); \\ 0, & x \in \mathbb{R} \backslash [0,2), \end{cases} \tag{1.4.16}$$

- $$\tilde{B}_2(x) = \tilde{h}(x) := B_2(x+1) = \begin{cases} 1+x, & x \in [-1,0); \\ 1-x, & x \in [0,1); \\ 0, & x \in \mathbb{R}\backslash[-1,1), \end{cases} \tag{1.4.17}$$

- $$B_3(x) = \begin{cases} \frac{1}{2}x^2, & x \in [0,1); \\ -x^2+3x-\frac{3}{2}, & x \in [1,2); \\ \frac{1}{2}x^2-3x+\frac{9}{2}, & x \in [2,3); \\ 0, & x \in \mathbb{R}\backslash[0,3), \end{cases} \tag{1.4.18}$$

- $$B_4(x) = \begin{cases} \frac{1}{6}x^3, & x \in [0,1); \\ -\frac{1}{2}x^3+2x^2-2x+\frac{2}{3}, & x \in [1,2); \\ \frac{1}{2}x^3-4x^2+10x-\frac{22}{3}, & x \in [2,3); \\ \frac{1}{6}(4-x)^3, & x \in [3,4); \\ 0, & x \in \mathbb{R}\backslash[0,4). \end{cases} \tag{1.4.19}$$

Graphs of B_2, B_3 and B_4 are given in Figures 1.1-1.3.

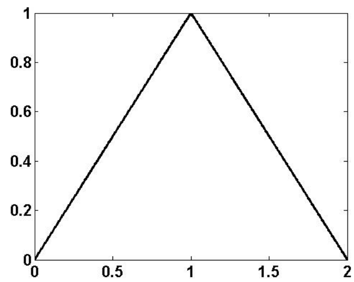

Figure 1.1: *Hat function $B_2 = h$*

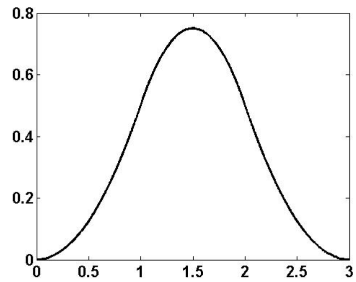

Figure 1.2: *Quadratic B-spline B_3*

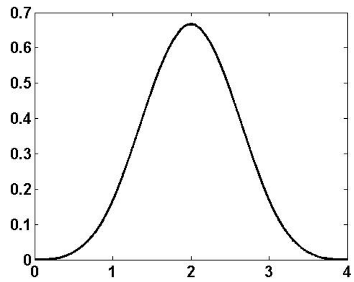

Figure 1.3: *Cubic B-spline B_4*

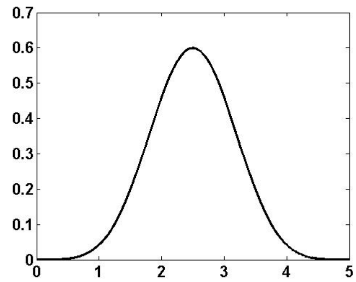

Figure 1.4: *Quintic B- spline B_5*

We proceed in Section 1.5 to introduce a bivariate version of cardinal B-splines, namely box splines.

1.5 Box splines

An important class of bivariate refinable functions are provided by box splines which we proceed to introduce in this section. We shall rely on the Haar function (or roof function)

$$\mathbb{B}_1(\mathbf{x}) := \boldsymbol{\chi}_{[0,1)^2}(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in [0,1)^2; \\ 0, & \mathbf{x} \in \mathbb{R}^2 \backslash [0,1)^2, \end{cases} \tag{1.5.1}$$

from which we will generate higher order box splines.

To this end, let $\{\mathbf{e}_j : j = 1, \ldots, 4\} \subset \mathbb{R}^2$ be the set of direction vectors defined by

$$\mathbf{e}_1 := (1,0); \quad \mathbf{e}_2 := (0,1); \quad \mathbf{e}_3 := \mathbf{e}_1 + \mathbf{e}_2 = (1,1); \quad \mathbf{e}_4 := \mathbf{e}_1 - \mathbf{e}_2 = (1,-1). \tag{1.5.2}$$

We introduce the notion of a direction set

$$\mathcal{D} = \left\{ \underbrace{\mathbf{e}_1, \ldots, \mathbf{e}_1}_{\mathrm{k}}, \underbrace{\mathbf{e}_2, \ldots, \mathbf{e}_2}_{\ell}, \underbrace{\mathbf{e}_3, \ldots, \mathbf{e}_3}_{\mathrm{m}}, \underbrace{\mathbf{e}_4, \ldots, \mathbf{e}_4}_{\mathrm{p}} \right\}, \tag{1.5.3}$$

where k and ℓ are positive integers, m and p are non-negative integers, in terms of which we then define the integer

$$n := \mathrm{k} + \ell + \mathrm{m} + \mathrm{p}. \tag{1.5.4}$$

To facilitate our discussion, we relabel the direction vectors in (1.5.3), and define the direction set sequence $\{\mathcal{D}_r \subset \mathcal{D} : r = 2, \ldots, n\}$ by

$$\mathcal{D}_r := \left\{ \mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^r \right\}, \; r = 2, \ldots, n. \tag{1.5.5}$$

Observe that then $\mathcal{D}_2 = \{\mathbf{e}_1, \mathbf{e}_2\}$.

With the definition

$$[\mathcal{D}_r] := \left\{ \sum_{j=1}^{r} t_j \, \mathbf{e}^j \,:\, 0 \leq t_j < 1, \; j = 1, 2, \ldots, r, \quad r = 2, \ldots, n \right\}, \tag{1.5.6}$$

it then follows that $[\mathcal{D}_2] = [0,1)^2$.

Definition 1.5.1. *For a given sequence* $\{\mathcal{D}_r : r = 2, 3, \ldots\}$ *of direction sets, the corresponding box splines* $\{\mathbb{B}_r(\mathbf{x}) := \mathbb{B}(\mathbf{x}|\mathcal{D}_r), \; r = 2, 3, \ldots\}$ *are defined recursively by means of*

$$\left. \begin{aligned} &\mathbb{B}(\mathbf{x}|\mathcal{D}_2) := \mathbb{B}_1(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2; \\ &\mathbb{B}_r(\mathbf{x}) = \mathbb{B}(\mathbf{x}|\mathcal{D}_r) := \int_0^1 \mathbb{B}(\mathbf{x} - t\,\mathbf{e}^r|\mathcal{D}_{r-1})\,\mathrm{d}t, \quad \mathbf{x} \in \mathbb{R}^2, \; r = 3, 4, \ldots, \end{aligned} \right\} \tag{1.5.7}$$

where $\mathbb{B}_1$*, the Haar function is given by* (1.5.1)*.*

For a given direction set $\mathcal{D}$ *as in* (1.5.3)*, we shall write*

$$\left. \begin{aligned} &\mathbb{B}_{k,\ell,m,p} := \mathbb{B}(\cdot|\mathcal{D}); \\ &\mathbb{B}_{k,\ell} := \mathbb{B}_{k,\ell,0,0}; \\ &\mathbb{B}_{k,\ell,m} := \mathbb{B}_{k,\ell,m,0}, \end{aligned} \right\} \tag{1.5.8}$$

with, in particular, $\mathbb{B}_{1,1} := \mathbb{B}_1$. *Also,* $\mathbb{B}_2 := \mathbb{B}_{1,1,1}$, *the Courant hat function, for the choices* $\mathbf{e}^1 := \mathbf{e}_1$; $\mathbf{e}^2 := \mathbf{e}_2$; $\mathbf{e}^3 := \mathbf{e}_3$; *whereas* $\mathbb{B}_3 := \mathbb{B}_{1,1,2}$, *for the choices* $\mathbf{e}^1 := \mathbf{e}_1$; $\mathbf{e}^2 := \mathbf{e}_2$; $\mathbf{e}^3 := \mathbf{e}_3$; *and* $\mathbb{B}_4 := \mathbb{B}_{1,1,1,1}$, *the Zwart-Powell function, for the choices* $\mathbf{e}^1 := \mathbf{e}_1$; $\mathbf{e}^2 := \mathbf{e}_2$; $\mathbf{e}^3 := \mathbf{e}_3$; $\mathbf{e}^4 := -\mathbf{e}_4$.

Observe from the function (1.5.7) that permuting the direction vectors $\{\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^r\}$ does not change the box splines $\mathbb{B}_r$.

1.6 Examples

We proceed to explicitly compute specific box splines.

1.6.1 The Haar box spline $\mathbb{B}_1$

The Haar box spline is defined, according to (1.5.1), (1.5.7) and (1.5.8), by

$$\mathbb{B}_1(x,y) := \mathbb{B}_{1,1}(x,y) = \begin{cases} 1, \text{ if } & (x,y) \in [0,1)^2; \\ 0, \text{ if } & (x,y) \in \mathbb{R}^2 \backslash [0,1)^2, \end{cases} \tag{1.6.1}$$

from which we see that $\mathbb{B}_1$ is a piecewise constant polynomial function which is discontinuous around the boundary of its support whereas $\mathbb{B}_1$ assumes the value one in the interior $(0,1)^2$ of its support. We proceed to show that $\mathbb{B}_1$ is a refinable function with its

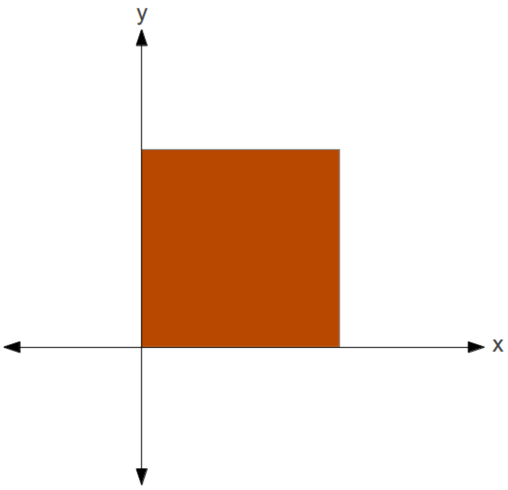

Figure 1.5: *The support of* $\mathbb{B}_1$

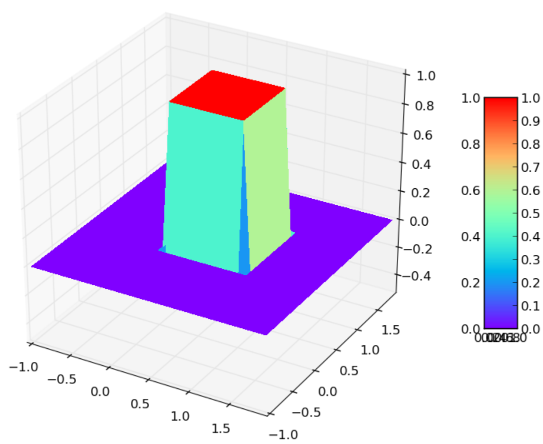

Figure 1.6: *The Haar box spline* $\mathbb{B}_1$

refinement mask $\{p^1_{\mathbf{j}}\}$ given by

$$\left.\begin{array}{l} p^1_{0,0} = p^1_{1,1} = p^1_{0,1} = p^1_{1,0} = 1; \\ p^1_{i,j} = 0, \quad (i,j) \notin \mathbb{Z}^2 \backslash \{(0,0),(0,1),(1,0),(1,1)\}, \end{array}\right\} \tag{1.6.2}$$

where the values of the refinement mask $\{p^1_{\mathbf{j}}\}$ are graphically illustrated in Figure 1.7 below. To this end, we first observe from (1.6.1) that

$$\mathbb{B}_1(2x,2y) = \begin{cases} 1, & (x,y) \in [0,\frac{1}{2})^2; \\ 0, & (x,y) \notin [0,\frac{1}{2})^2, \end{cases};$$
$$\mathbb{B}_1(2x-1,2y) = \begin{cases} 1, & (x,y) \in [\frac{1}{2},1) \times [0,\frac{1}{2}); \\ 0, & (x,y) \notin [\frac{1}{2},1) \times [0,\frac{1}{2}), \end{cases};$$
$$\mathbb{B}_1(2x,2y-1) = \begin{cases} 1, & (x,y) \in [0,\frac{1}{2}) \times [\frac{1}{2},1); \\ 0, & (x,y) \notin [0,\frac{1}{2}) \times [\frac{1}{2},1), \end{cases};$$
$$\mathbb{B}_1(2x-1,2y-1) = \begin{cases} 1, & (x,y) \in [\frac{1}{2},1)^2; \\ 0, & (x,y) \notin [\frac{1}{2},1)^2. \end{cases} \tag{1.6.3}$$

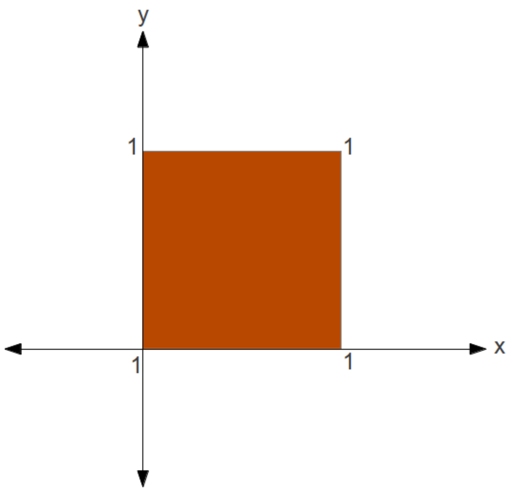

Figure 1.7: *The refinement mask* $\{p^1_{\mathbf{j}}\}$

Since the support $[0,1]^2$ of the Haar box spline $\mathbb{B}_1$ can be partitioned into four disjoint regions $[0,\frac{1}{2})^2, [\frac{1}{2},1] \times [0,\frac{1}{2}), [0,\frac{1}{2}] \times [\frac{1}{2},1)$ and $[\frac{1}{2},1]^2$, it follows from (1.6.1) and (1.6.3) that, for $x,y \in \mathbb{R}$,

$$\mathbb{B}_1(x,y) = \mathbb{B}_1(2x,2y) + \mathbb{B}_1(2x-1,2y) + \mathbb{B}_1(2x,2y-1) + \mathbb{B}_1(2x-1,2y-1). \tag{1.6.4}$$

Hence, $\mathbb{B}_1$ is $2\mathrm{I}_2$-refinable with refinement mask $\{p_{i,j} : i,j \in \mathbb{Z}\}$ given by (1.6.2). It then follows from (1.3.8) and (1.6.2) that the corresponding refinement mask symbol is given by

$$P_1(z_1,z_2) = (1 + z_1 + z_2 + z_1 z_2) = (1+z_1)(1+z_2), \quad z_1,z_2 \in \mathbb{C}. \tag{1.6.5}$$

Therefore, $P_1(z_1,z_2) = (1+z_1)(1+z_2)$ is the refinement symbol for the box spline $\mathbb{B}_1$.

1.6.2 The Courant hat function $\mathbb{B}_2$

As defined in (1.5.8) of Definition 1.5.1, the Courant hat function $\mathbb{B}_2$ is given by

$$\mathbb{B}_2(x,y) := \mathbb{B}_{1,1,1}(x,y) = \int_0^1 \mathbb{B}_1(x-t,y-t)\,dt, \quad (x,y) \in \mathbb{R}^2. \tag{1.6.6}$$

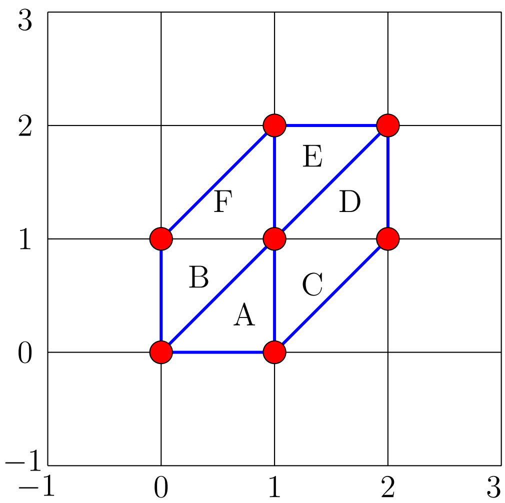

Figure 1.8: *The support of* $\mathbb{B}_2$

As illustrated graphically in Figure 1.8, let the regions A, . . . , F be defined by

$$\left.\begin{array}{l} \mathrm{A} := \left\{(x,y) \in [0,1)^2 : x \geq y\right\}; \\ \mathrm{B} := \left\{(x,y) \in [0,1)^2 : x \leq y\right\}; \\ \mathrm{C} := \{(x,y) \in [1,2) \times [0,1) : x-1 \leq y\}; \\ \mathrm{D} := \left\{(x,y) \in [1,2)^2 : x \geq y\right\}; \\ \mathrm{E} := \left\{(x,y) \in [1,2)^2 : x < y\right\}; \\ \mathrm{F} := \{(x,y) \in [0,1) \times [1,2) : x+1 \geq y\}, \end{array}\right\} \tag{1.6.7}$$

according to which the regions A, . . . , F are disjoint and form a partition of the hexagonal region $[\mathcal{D}_3]$, as defined by

$$[\mathcal{D}_3] := \left\{\sum_{j=1}^{3} t_j\, \mathbf{e}^j : 0 \leq t_j < 1,\ j = 1,2,3\right\}, \tag{1.6.8}$$

that is, $[\mathcal{D}_3] = \mathrm{A} \cup \mathrm{B} \cup \mathrm{C} \cup \mathrm{D} \cup \mathrm{E} \cup \mathrm{F}$.

We claim that

$$(x-t, y-t) \in [0,1)^2,\ t \in [0,1) \Leftrightarrow (x,y) \in [\mathcal{D}_3]. \tag{1.6.9}$$

To prove (1.6.9), suppose first that $(x,y) \in [\mathcal{D}_3]$, that is, there exist $t_1, t_2, t_3 \in [0,1)$ such that

$$(x,y) = t_1(1,0) + t_2(0,1) + t_3(1,1),$$

or equivalently,

$$\left\{\begin{array}{l} x = t_1 + t_3; \\ y = t_2 + t_3, \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} x - t_3 = t_1; \\ y - t_3 = t_2, \end{array}\right\} \tag{1.6.10}$$

and it follows that, with $t := t_3 \in [0,1)$, we have $(x-t, y-t) \in [0,1)^2$. Conversely, suppose $(x-t, y-t) \in [0,1)^2$ for some $t \in [0,1)$, that is,

$$\left.\begin{array}{l} 0 \leq x - t < 1; \\ 0 \leq y - t < 1, \end{array}\right\} \tag{1.6.11}$$

so that

$$\left.\begin{aligned} x-t&=u;\\ y-t&=v, \end{aligned}\right\} \tag{1.6.12}$$

for $u, v \in [0,1)$. The definitions $t_1 := u$; $t_2 := v$; $t_3 := t$, then yields (1.6.10), and it follows that $(x,y) \in [\mathcal{D}_3]$, thereby completing our proof of (1.6.9).

It follows from (1.6.9) and (1.6.1) that

$$\mathbb{B}_1(x-t, y-t) = 0, \quad (x,y) \notin [\mathcal{D}_3], \quad t \in [0,1), \tag{1.6.13}$$

and thus, by using also (1.6.6), we obtain

$$\mathbb{B}_2(x,y) = 0, \quad (x,y) \notin [\mathcal{D}_3]. \tag{1.6.14}$$

We proceed, by using (1.6.6) and (1.6.1), to calculate $\mathbb{B}_2(x,y)$ on each of the disjoint regions $\mathrm{A}, \ldots, \mathrm{F}$ in Figure 1.8, as follows:

$$\begin{aligned}
(x,y) \in \mathrm{A} \Rightarrow \mathbb{B}_2(x,y) &= \int_0^y 1\,dt = y;\\
(x,y) \in \mathrm{B} \Rightarrow \mathbb{B}_2(x,y) &= \int_0^x 1\ dt = x;\\
(x,y) \in \mathrm{C} \Rightarrow \mathbb{B}_2(x,y) &= \int_{x-1}^y 1\ dt = 1+y-x;\\
(x,y) \in \mathrm{D} \Rightarrow \mathbb{B}_2(x,y) &= \int_x^2 1\,\mathrm{d}t = 2-x;\\
(x,y) \in \mathrm{E} \Rightarrow \mathbb{B}_2(x,y) &= \int_{y-1}^1 1\,\mathrm{d}t = 2-y;\\
(x,y) \in \mathrm{F} \Rightarrow \mathbb{B}_2(x,y) &= \int_{y-1}^x 1\,\mathrm{d}t = 1+x-y,
\end{aligned}$$

and thus

$$\mathbb{B}_2(x,y) = \begin{cases} y, & (x,y) \in \mathrm{A};\\ x, & (x,y) \in \mathrm{B};\\ 1+y-x, & (x,y) \in \mathrm{C};\\ 2-x, & (x,y) \in \mathrm{D};\\ 2-y, & (x,y) \in \mathrm{E};\\ 1+x-y, & (x,y) \in \mathrm{F};\\ 0, & (x,y) \in \mathbb{R}^2 \backslash [\mathcal{D}_3]. \end{cases} \tag{1.6.15}$$

Observe from (1.6.15) that the Courant hat function $\mathbb{B}_2$ is a piecewise linear bivariate polynomial, with also

$$\mathbb{B}_2(i,j) = \begin{cases} 1, & (i,j) = (1,1);\\ 0, & (i,j) \in \mathbb{Z}^2 \backslash \{(1,1)\}. \end{cases} \tag{1.6.16}$$

A graph of $\mathbb{B}_2$ is shown in Figure 1.9. Moreover, we see from (1.6.15) that

$$\mathbb{B}_2 \in C(\mathbb{R}^2) \setminus C^1(\mathbb{R}^2). \tag{1.6.17}$$

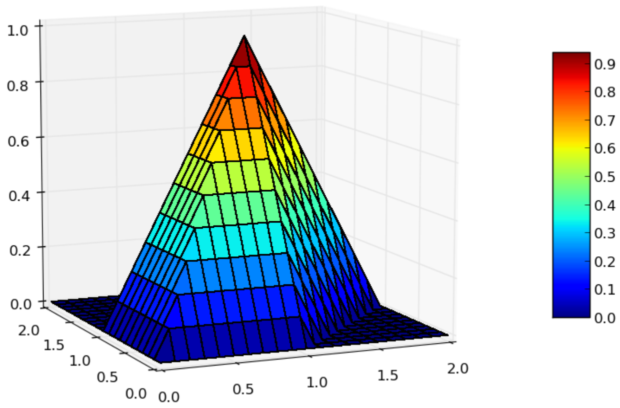

Figure 1.9: *The Courant hat function,* $\mathbb{B}_2$

We proceed to show that $\mathbb{B}_2$ is a $2I_2$-refinable function.

To this end, we use (1.6.6) and (1.6.4) to obtain, for any $(x, y) \in \mathbb{R}^2$,

$$
\begin{aligned}
\mathbb{B}_2(x,y) \quad &= \int_0^1 \mathbb{B}_1(x-t,y-t)\,\mathrm{d}t = \int_0^1 \sum_{i,j\in\mathbb{Z}} p^1_{i,j}\,\mathbb{B}_1(2(x-t,y-t)-(i,j))\,\mathrm{d}t \\
&= \int_0^1 \mathbb{B}_1(2x-2t,2y-2t)\,\mathrm{d}t + \int_0^1 \mathbb{B}_1(2x-2t-1,2y-2t)\,\mathrm{d}t \\
&+ \int_0^1 \mathbb{B}_1(2x-2t,2y-2t-1)\,\mathrm{d}t + \int_0^1 \mathbb{B}_1(2x-2t-1,2y-2t-1)\,\mathrm{d}t \\
&= \left\{\int_0^{\frac{1}{2}} \mathbb{B}_1(2x-2t,2y-2t)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \mathbb{B}_1(2x-2t,2y-2t)\,\mathrm{d}t\right\} \\
&\quad + \left\{\int_0^{\frac{1}{2}} \mathbb{B}_1(2x-2t-1,2y-2t)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \mathbb{B}_1(2x-2t-1,2y-2t)\,\mathrm{d}t\right\} \\
&\quad + \left\{\int_0^{\frac{1}{2}} \mathbb{B}_1(2x-2t,2y-2t-1)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \mathbb{B}_1(2x-2t,2y-2t-1)\,\mathrm{d}t\right\} \\
&\quad + \left\{\int_0^{\frac{1}{2}} \mathbb{B}_1(2x-2t-1,2y-2t-1)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \mathbb{B}_1(2x-2t-1,2y-2t-1)\,\mathrm{d}t\right\} \\
&= \left\{\frac{1}{2}\int_0^1 \mathbb{B}_1(2x-t,2y-t)\,\mathrm{d}t + \frac{1}{2}\int_0^1 \mathbb{B}_1(2x-t-1,2y-t-1)\,\mathrm{d}t\right\} \\
&\quad + \left\{\frac{1}{2}\int_0^1 \mathbb{B}_1(2x-t-1,2y-t)\,\mathrm{d}t + \frac{1}{2}\int_0^1 \mathbb{B}_1(2x-t-2,2y-t-1)\,\mathrm{d}t\right\} \\
&\quad + \left\{\frac{1}{2}\int_0^1 \mathbb{B}_1(2x-t,2y-t-1)\,\mathrm{d}t + \frac{1}{2}\int_0^1 \mathbb{B}_1(2x-t-1,2y-t-2)\,\mathrm{d}t\right\} \\
&\quad + \left\{\frac{1}{2}\int_0^1 \mathbb{B}_1(2x-t-1,2y-t-1)\,\mathrm{d}t + \frac{1}{2}\int_{\frac{1}{2}}^1 \mathbb{B}_1(2x-t-2,2y-t-2)\,\mathrm{d}t\right\} \\
&= \frac{1}{2}\left\{\mathbb{B}_2(2x,2y) + 2\mathbb{B}_2(2x-1,2y-1)\right\} + \frac{1}{2}\left\{\mathbb{B}_2(2x-1,2y) + \mathbb{B}_2(2x-2,2y-1)\right\} \\
&\quad + \frac{1}{2}\left\{\mathbb{B}_2(2x,2y-1) + \mathbb{B}_2(2x,2y-2)\right\} + \frac{1}{2}\left\{\mathbb{B}_2(2x-1,2y-1) + \mathbb{B}_2(2x-2,2y-2)\right\},
\end{aligned}
$$

and thus,

$$\mathbb{B}_2(x,y) = \frac{1}{2}\left[\mathbb{B}_2(2x,2y) + \mathbb{B}_2(2x,2y-1) + \mathbb{B}_2(2x-1,2y) + 2\mathbb{B}_2(2x-1,2y-1)\right.$$
$$\left. + \mathbb{B}_2(2x-1,2y-2) + \mathbb{B}_2(2x-2,2y-1) + \mathbb{B}_2(2x-2,2y-2)\right], \ (x,y) \in \mathbb{R}^2, \tag{1.6.18}$$

which in turn implies that the box spline $\mathbb{B}_2$ is $2\mathrm{I}_2$-refinable with refinement mask $\{p^2_{\mathbf{j}}\}$ given by

$$\left.\begin{array}{l} p^2_{1,1} = 1, \ p^2_{0,0} = p^2_{0,1} = p^2_{1,0} = p^2_{2,1} = p^2_{1,2} = p^2_{2,2} = \dfrac{1}{2}; \\ p^2_{i,j} = 0, \quad (i,j) \notin \{(0,0),(0,1),(1,0),(1,1),(1,2),(2,1),(2,2)\}, \end{array}\right\} \tag{1.6.19}$$

where the values of the refinement mask $\{p^2_{\mathbf{j}}\}$ are graphically illustrated in Figure 1.10.

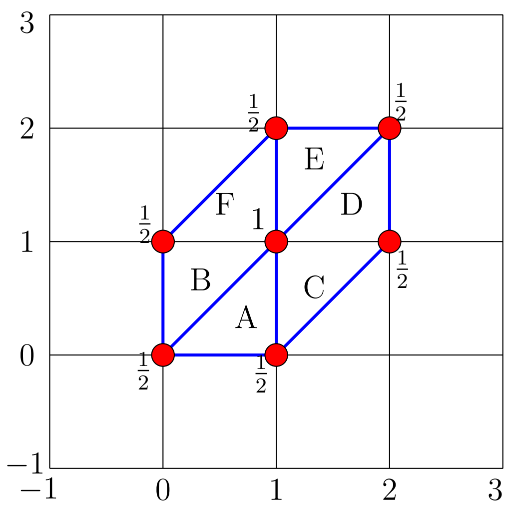

Figure 1.10: *The refinement mask* $\{p^2_{\mathbf{j}}\}$

Observe from (1.6.15) and (1.6.19) that the support of $\mathbb{B}_2$ and its refinement mask $\{p^2_{\mathbf{j}}\}$ agree, in the sense that

$$\left.\begin{array}{l} \operatorname{supp} \mathbb{B}_2 = \overline{[\mathcal{D}_3]}; \\ \operatorname{supp} p_{\mathbf{j}} = [\mathcal{D}_3]|_{\mathbb{Z}^2}. \end{array}\right\} \tag{1.6.20}$$

It follows from (1.3.8) and (1.6.19) that the corresponding mask symbol is given by

$$\begin{aligned} P_2(z_1,z_2) &= \left(\frac{1}{2} + \frac{1}{2}z_1 + \frac{1}{2}z_2 + z_1z_2 + \frac{1}{2}z_1^2z_2 + \frac{1}{2}z_1z_2^2 + \frac{1}{2}z_1^2z_2^2\right) \\ &= \frac{1}{2}(1+z_1)(1+z_2)(1+z_1z_2) = \left(\frac{1+z_1z_2}{2}\right) P_1(z_1,z_2), \end{aligned} \tag{1.6.21}$$

with P_1 denoting the symbol of the Haar box spline $\mathbb{B}_1$, as given in (1.6.5).

Therefore, $P_2(z_1,z_2) = \left(\dfrac{1+z_1z_2}{2}\right) P_1(z_1,z_2)$ is the symbol for the Courant hat function $\mathbb{B}_2$.

1.6.3 The shifted Courant hat function $\widehat{\mathbb{B}}_2$

The shifted Courant hat function is defined by

$$\widehat{\mathbb{B}}_2(x,y) := \mathbb{B}_2(x+1,y+1), \quad (x,y)\in\mathbb{R}^2, \tag{1.6.22}$$

with $\mathbb{B}_2$ denoting the Courant hat function, according to which, together with (1.6.15) and (1.6.7) we have the explicit formulation

$$\widehat{\mathbb{B}}_2(x,y) = \begin{cases} 1+y, & (x,y)\in\widehat{\mathrm{A}}; \\ 1+x, & (x,y)\in\widehat{\mathrm{B}}; \\ 1+y-x, & (x,y)\in\widehat{\mathrm{C}}; \\ 1-x, & (x,y)\in\widehat{\mathrm{D}}; \\ 1-y, & (x,y)\in\widehat{\mathrm{E}}; \\ 1+x-y, & (x,y)\in\widehat{\mathrm{F}}; \\ 0, & (x,y)\in\mathbb{R}^2\backslash\overline{[\widehat{\mathcal{D}_3}]}, \end{cases} \tag{1.6.23}$$

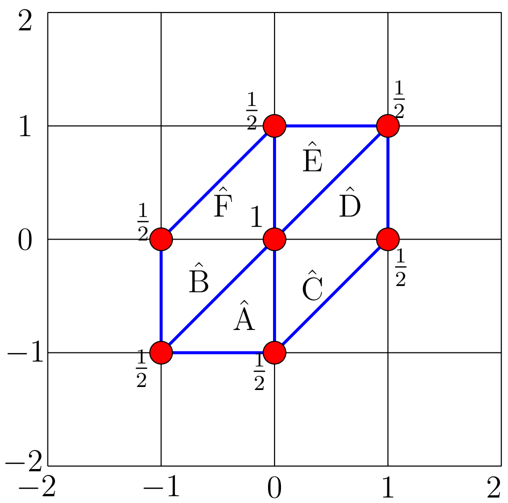

Figure 1.11: *The support of the shifted Courant hat function* $\widehat{\mathbb{B}}_2$

where

$$[\widehat{\mathcal{D}_3}] := \widehat{\mathrm{A}}\cup\widehat{\mathrm{B}}\cup\widehat{\mathrm{C}}\cup\widehat{\mathrm{D}}\cup\widehat{\mathrm{E}}\cup\widehat{\mathrm{F}}, \tag{1.6.24}$$

with the disjoint regions $\widehat{\mathrm{A}},\ldots,\widehat{\mathrm{F}}$ defined by

$$\left.\begin{aligned} \widehat{\mathrm{A}} &:= \left\{(x,y)\in[-1,0)^2 : x\geq y\right\}; \\ \widehat{\mathrm{B}} &:= \left\{(x,y)\in[-1,0)^2 : x\leq y\right\}; \\ \widehat{\mathrm{C}} &:= \{(x,y)\in[0,1)\times[-1,0) : x-1\leq y\}; \\ \widehat{\mathrm{D}} &:= \left\{(x,y)\in[0,1)^2 : y\leq x\right\}; \\ \widehat{\mathrm{E}} &:= \left\{(x,y)\in[0,1)^2 : y\geq x\right\}; \\ \widehat{\mathrm{F}} &:= \{(x,y)\in[-1,0)\times[0,1) : x+1\geq y\}, \end{aligned}\right\} \tag{1.6.25}$$

as graphically illustrated in Figure 1.12.

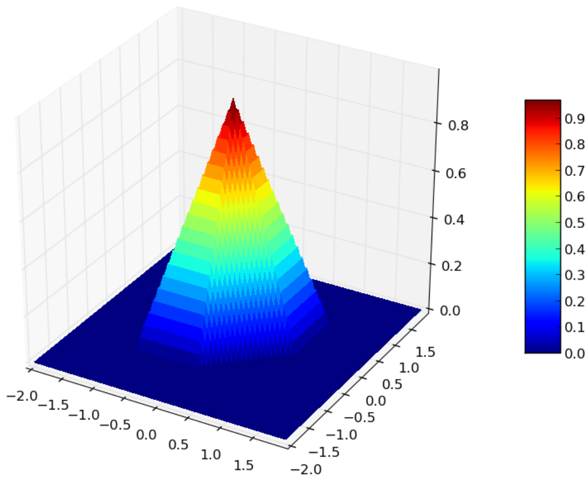

Figure 1.12: *The shifted Courant hat function* $\widehat{\mathbb{B}}_2$

Analogous to $\mathbb{B}_2$ in Section 1.6.2, we find that $\widehat{\mathbb{B}}_2$ satisfies the following properties:

- $\widehat{\mathbb{B}}_2(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbb{R}^2 \backslash \{\hat{\mathcal{D}}_3\};$
- $\widehat{\mathbb{B}}_2(\mathbf{j}) = \delta_{\mathbf{j}}, \quad \mathbf{j} \in \mathbb{Z}^2;$
- $\widehat{\mathbb{B}}_2$ is $2\mathrm{I}_2$-refinable function with refinement mask $\{p_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^2}$ given by

$$\left.\begin{array}{ll} \hat{p}^2_{0,0} = 1, & \hat{p}^2_{1,1} = \hat{p}^2_{0,-1} = \hat{p}^2_{0,1} = \hat{p}^2_{1,0} = \hat{p}^2_{-1,0} = \hat{p}^2_{-1,-1} = \dfrac{1}{2}; \\ \hat{p}^2_{i,j} = 0, & (i,j) \notin \{(0,0),(0,1),(1,0),(-1,0),(0,-1),(1,1),(-1,-1)\}. \end{array}\right\} \tag{1.6.26}$$

  Hence $\widehat{\mathbb{B}}_2$ is an interpolatory refinable function. The function $\widehat{\mathbb{B}}_2$ is graphically illustrated in Figure 1.12.

  The values of the refinement mask $\{\hat{p}^2_{\mathbf{j}}\}$, as given in (1.6.26), are graphically illustrated in Figure 1.11.

- The continuity order of the shifted Courant hat function:

$$\widehat{\mathbb{B}}_2 \in C(\mathbb{R}^2) \setminus C^1(\mathbb{R}^2). \tag{1.6.27}$$

We note from (1.3.8) and (1.6.26) that the corresponding refinement mask symbol is given by

$$\widehat{P}_2(z_1, z_2) = \frac{1}{2}(1 + z_1)\,(1 + z_2)\,(1 + z_1 z_2)\, z_1^{-1} z_2^{-1}, \tag{1.6.28}$$

and thus,

$$\widehat{P}_2(z_1, z_2) = \left(\frac{1 + z_1^{-1} z_2^{-1}}{2}\right) P_1(z_1, z_2), \tag{1.6.29}$$

where $P_1(z_1, z_2)$ is the symbol for the box spline $\mathbb{B}_1$, as given in (1.6.5). The refinement symbol $\hat{P}_2$ will be used as a basic factor for generating the Butterfly subdivision mask symbol, which will be discussed in Chapters 4 and 5.

1.7 Bivariate box spline properties with respect to meshes

In this section, we derive some properties of bivariate box splines.

First, we introduce the concept of a k-directional mesh.

Definition 1.7.1. *For any integer $k \geq 2$, a k-directional mesh is a set of vectors*

$$\mathcal{G}_k := \left\{ \underbrace{\mathbf{d}^1, \ldots, \mathbf{d}^1}_{m_1}, \ldots, \underbrace{\mathbf{d}^k, \ldots, \mathbf{d}^k}_{m_k} \right\}, \tag{1.7.1}$$

where $\mathbf{d}^i := (\alpha_i, \beta_i) \in \mathbb{Z}^2$, $\alpha_i \in \mathbb{R}$, $\beta_i \geq 0$, with $\alpha_i\beta_j \neq \alpha_j\beta_i$ for $i \neq j$, and with multiplicities $m_i \geq 1$ for $i = 1, \ldots, k$. Also, $n := \sum_{i=1}^{k} m_i$.

We proceed to provide examples for $k = 2, 3, 4$.

The 2-directional mesh $\mathcal{G}_2$

For $k = 2$, the choices $\mathbf{d}^1 := \mathbf{e}_1$; $\mathbf{d}^2 := \mathbf{e}_2$, yield the 2-directional grid $\mathcal{G}_2$, which in fact corresponds to the rectangular $\mathbb{Z}^2$ grid in $\mathbb{R}^2$ as illustrated in Figure 1.13. Note that here $n = 2$.

The 3-directional mesh $\mathcal{G}_3$

For $k = 3$, the choices $\mathbf{d}^1 := \mathbf{e}_1$; $\mathbf{d}^2 := \mathbf{e}_2$; $\mathbf{d}^3 := \mathbf{e}_3$, yield the 3-directional grid $\mathcal{G}_3$ which in fact corresponds to the type-I triangular $\mathbb{Z}^2$ grid in $\mathbb{R}^2$ as illustrated in Figure 1.14. Note that here $n = 3$.

The 4-directional mesh $\mathcal{G}_4$

For $k = 4$, the choices $\mathbf{d}^1 := \mathbf{e}_1$; $\mathbf{d}^2 := \mathbf{e}_2$; $\mathbf{d}^3 := \mathbf{e}_3$ and $\mathbf{d}^4 := -\mathbf{e}_4$ as in (1.5.2), yield the 3-directional grid $\mathcal{G}_3$ which in fact corresponds to the type-II triangular $\mathbb{Z}^2$ grid in $\mathbb{R}^2$ as illustrated in Figure 1.15. Note that here $n = 4$.

The following bivariate box spline result holds with respect to the 3-directional mesh $\mathcal{G}_3$.

Theorem 1.7.2. *[Han00] For $k, \ell \in \mathbb{N}$ and $m \in \mathbb{Z}_+$, the box spline $\mathbb{B}_{k,\ell,m}$ is a piecewise bivariate polynomial of total degree at most $\gamma := k + \ell + m - 2$ relative to the 3-directional mesh $\mathcal{G}_3$. Furthermore, $\mathbb{B}_{k,\ell,m}$ is $(\gamma - k)$ times continuously differentiable across each horizontal line in the mesh; $(\gamma - \ell)$ times continuously differentiable across each vertical line and $(\gamma - m)$ times continuously differentiable across each diagonal line of the corresponding type-I triangular $\mathbb{Z}^2$ grid.*

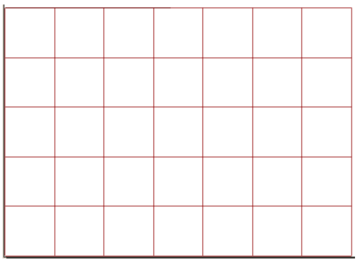

Figure 1.13: *The 2-directional mesh* $\mathcal{G}_2$

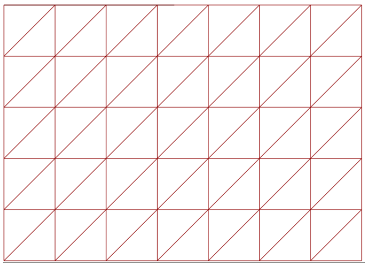

Figure 1.14: *The 3-directional mesh* $\mathcal{G}_3$ *(type I-triangular grid)*

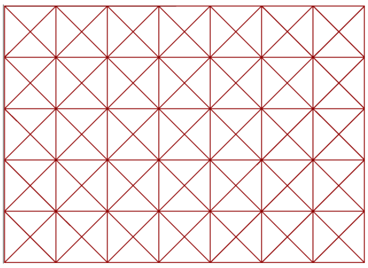

Figure 1.15: *The 4-directional mesh* $\mathcal{G}_4$*(type II-triangular grid)*

Examples

(a) According to the results of Section 1.6.2, the Courant hat function $\mathbb{B}_2 := \mathbb{B}_{1,1,1}$, with direction set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as in (1.5.2), is a linear bivariate polynomial in each triangle of its support. Furthermore, it is continuous across each vertical line, horizontal line grid, and diagonal grid line.

Observe that these results are consistent with Theorem 1.7.2, in the notation of which we have here $\gamma = 1 + 1 + 1 - 2 = 1$.

(b) According to Theorem 1.7.2, the bivariate box spline $\mathbb{B}_3 := \mathbb{B}_{1,1,2}$, with direction set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_3\}$, is a quadratic bivariate polynomial in each triangle as in Figure 1.14. Also, since here $\gamma = 1 + 1 + 2 - 2 = 2$, $\mathbb{B}_3$ is continuously differentiable once across each vertical line and horizontal line grid, but only continuous across the diagonal grid lines. As illustrated graphically in Figure 1.16, let the regions $\Omega_1, \ldots, \Omega_{10}$ be

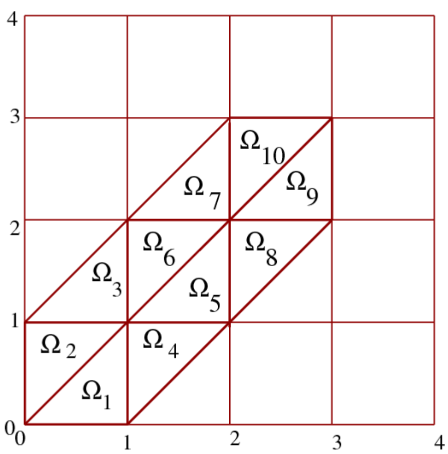

Figure 1.16: *The support of* $\mathbb{B}_{1,1,2}$

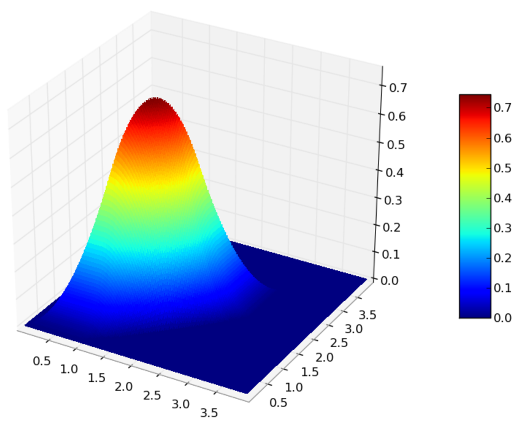

Figure 1.17: *The box spline* $\mathbb{B}_{1,1,2}$

defined by

$$\left.\begin{aligned}
\Omega_1 &:= \left\{(x,y) \in [0,1)^2 : x \geq y\right\},\\
\Omega_2 &:= \left\{(x,y) \in [0,1)^2 : x \leq y\right\},\\
\Omega_3 &:= \{(x,y) \in [0,1) \times [1,2) : x+1 \geq y\},\\
\Omega_4 &:= \{(x,y) \in [1,2) \times [0,1) : x-1 \leq y\},\\
\Omega_5 &:= \left\{(x,y) \in [1,2)^2 : x \geq y\right\},\\
\Omega_6 &:= \left\{(x,y) \in [1,2)^2 : x < y\right\},\\
\Omega_7 &:= \{(x,y) \in [1,2) \times [2,3) : y \leq x+1\},\\
\Omega_8 &:= \{(x,y) \in [2,3) \times [1,2) : y \geq x-1\},\\
\Omega_9 &:= \left\{(x,y) \in [2,3)^2 : y \leq x\right\},\\
\Omega_{10} &:= \left\{(x,y) \in [2,3)^2 : y \geq x\right\},
\end{aligned}\right\} \tag{1.7.2}$$

according to which the regions Ω_i, $i \in \{1, 2, \ldots, 10\}$ are disjoint, and form a partition of the region

$$[\mathcal{D}_4] := \left\{\sum_{j=0}^{4} t_j\, \mathbf{e}^j, \quad 0 \leq t_j < 1,\ j = 0, \ldots, 4\right\}, \tag{1.7.3}$$

that is, $[\mathcal{D}_4] := \bigcup_{i=1}^{10} \Omega_i$. By using equation (1.5.7), we calculate the formula

$$\mathbb{B}_3(x,y) := \mathbb{B}_{1,1,2}(x,y) = \begin{cases}
\frac{1}{2}y^2, & (x,y) \in \Omega_1;\\
\frac{1}{2}x^2, & (x,y) \in \Omega_2;\\
\frac{1}{2}x^2 - \frac{1}{2}y^2 + y - \frac{1}{2}, & (x,y) \in \Omega_3;\\
-\frac{1}{2}x^2 + \frac{1}{2}y^2 + x - \frac{1}{2}, & (x,y) \in \Omega_4;\\
-\frac{1}{2}x^2 - \frac{1}{2}y^2 + x + 2y - \frac{3}{2}, & (x,y) \in \Omega_5;\\
-\frac{1}{2}x^2 - \frac{1}{2}y^2 + 2x + y - \frac{3}{2}, & (x,y) \in \Omega_6;\\
-\frac{1}{2}x^2 + \frac{1}{2}y^2 + 2x - 3y + \frac{5}{2}, & (x,y) \in \Omega_7;\\
\frac{1}{2}x^2 - \frac{1}{2}y^2 - 3x + 2y + \frac{5}{2}, & (x,y) \in \Omega_8;\\
\frac{1}{2}x^2 - 3x + \frac{9}{2}, & (x,y) \in \Omega_9;\\
\frac{1}{2}y^2 - 3y + \frac{9}{2}, & (x,y) \in \Omega_{10};\\
0, & (x,y) \in \mathbb{R}^2 \backslash [\mathcal{D}_4].
\end{cases}$$

We proceed to show that $\mathbb{B}_3 := \mathbb{B}_{1,1,2}$ is a $2\mathrm{I}_2$-refinable function. To this end, we use (1.6.6) and (1.6.18), to obtain, for any $(x,y) \in \mathbb{R}^2$,

$$\begin{aligned}
\mathbb{B}_3(x,y) &:= \int_0^1 \mathbb{B}_2(x-t, y-t)\,\mathrm{d}t = \int_0^1 \sum_{i,j\in\mathbb{Z}} p^2_{i,j}\, \mathbb{B}_2(2(x-t,y-t)-(i,j))\,\mathrm{d}t\\
&= \int_0^1 \mathbb{B}_2(2x-2t-1, 2y-2t-1)\,\mathrm{d}t + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x-2t, 2y-2t)\,\mathrm{d}t\\
&\quad + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x-2t, 2y-2t-1)\,\mathrm{d}t + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x-2t-1, 2y-2t)\,\mathrm{d}t\\
&\quad + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x-2t-2, 2y-2t-1)\,\mathrm{d}t + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x-2t-1, 2y-2t-2)\,\mathrm{d}t\\
&\quad + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x-2t-2, 2y-2t-2)\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
\mathbb{B}_3(x,y) &= \int_0^{\frac{1}{2}} \mathbb{B}_2(2x-2t-1,2y-2t-1)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \mathbb{B}_2(2x-2t-1,2y-2t-1)\,\mathrm{d}t \\
&\quad + \frac{1}{2}\left\{\int_0^{\frac{1}{2}} \mathbb{B}_2(2x-2t,2y-2t)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \mathbb{B}_2(2x-2t,2y-2t)\,\mathrm{d}t\right\} \\
&\quad + \frac{1}{2}\left\{\int_0^{\frac{1}{2}} \mathbb{B}_2(2x-2t,2y-2t-1)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \mathbb{B}_2(2x-2t,2y-2t-1)\,\mathrm{d}t\right\} \\
&\quad + \frac{1}{2}\left\{\int_0^{\frac{1}{2}} \mathbb{B}_2(2x-2t-1,2y-2t)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \mathbb{B}_2(2x-2t-1,2y-2t)\,\mathrm{d}t\right\} \\
&\quad + \frac{1}{2}\left\{\int_0^{\frac{1}{2}} \mathbb{B}_2(2x-2t-2,2y-2t-1)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \mathbb{B}_2(2x-2t-2,2y-2t-1)\,\mathrm{d}t\right\} \\
&\quad + \frac{1}{2}\left\{\int_0^{\frac{1}{2}} \mathbb{B}_2(2x-2t-1,2y-2t-2)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \mathbb{B}_2(2x-2t-1,2y-2t-2)\,\mathrm{d}t\right\} \\
&\quad + \frac{1}{2}\left\{\int_0^{\frac{1}{2}} \mathbb{B}_2(2x-2t-2,2y-2t-2)\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \mathbb{B}_2(2x-2t-2,2y-2t-2)\,\mathrm{d}t\right\} \\
&= \frac{1}{2}\int_0^1 \mathbb{B}_2(2x-t-1,2y-t-1)\,\mathrm{d}t + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x-t-2,2y-t-2)\,\mathrm{d}t \\
&\quad + \frac{1}{4}\left\{\int_0^1 \mathbb{B}_2(2x-t,2y-t)\,\mathrm{d}t + \int_0^1 \mathbb{B}_2(2x-t-1,2y-t-1)\,\mathrm{d}t\right\} \\
&\quad + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x-t,2y-t-1)\,\mathrm{d}t + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x-t-1,2y-t-2)\,\mathrm{d}t \\
&\quad + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x-t-1,2y-t)\,\mathrm{d}t + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x-t-2,2y-t-1)\,\mathrm{d}t \\
&\quad + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x-t-2,2y-t-1)\,\mathrm{d}t + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x-t-3,2y-t-2)\,\mathrm{d}t \\
&\quad + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x-t-1,2y-t-2)\,\mathrm{d}t + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x-t-2,2y-t-3)\,\mathrm{d}t \\
&\quad + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x-t-2,2y-t-2)\,\mathrm{d}t, \\
&= \frac{3}{4}\left[\mathbb{B}_3(2x-1,2y-1) + \mathbb{B}_3(2x-2,2y-2)\right] + \frac{1}{2}\left[\mathbb{B}_3(2x-1,2y-2)\right. \\
&\quad \left. + \mathbb{B}_3(2x-1,2y-2)\right] + \frac{1}{4}\left[\mathbb{B}_3(2x,2y) + \mathbb{B}_3(2x-1,2y) + \mathbb{B}_3(2x-1,2y-2)\right. \\
&\quad \left. + \mathbb{B}_3(2x-2,2y-1) + \mathbb{B}_3(2x-3,2y-2) + \mathbb{B}_3(2x-2,2y-3) + \mathbb{B}_3(2x-3,2y-3)\right],
\end{aligned}
\tag{1.7.4}
$$

It follows from (1.7.4) that the box spline $\mathbb{B}_3$ is $2\mathrm{I}_2$-refinable with refinement mask $\{p^3_{\mathbf{j}}\}$ given by

$$
\left.
\begin{aligned}
& p^3_{1,1} = p^3_{2,2} = \frac{3}{4};\; p^3_{2,1} = p^3_{2,1} = \frac{1}{2}; \\
& p^3_{0,1} = p^3_{1,0} = p^3_{0,0} = p^3_{3,2} = p^3_{2,3} = p^3_{3,3} = \frac{1}{4}; \\
& p^3_{i,j} = 0, \quad (i,j) \notin \{(0,0),(1,0),(0,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3)\}.
\end{aligned}
\right\}
\tag{1.7.5}
$$

According to (1.3.8) and (1.7.5), the corresponding refinement mask symbol is given by

$$\begin{aligned} P_3(z_1,z_2) &= \Big(\frac{1}{4}+\frac{1}{4}z_1+\frac{1}{4}z_2+\frac{1}{2}z_1^2z_2+\frac{1}{2}z_1z_2^2+\frac{1}{4}z_1^3z_2^2+\frac{1}{4}z_1^2z_2^3+\frac{1}{4}z_1^2z_2^3 \\ &\quad +\frac{3}{4}z_1^2z_2^2+\frac{1}{4}z_1^3z_2^3+\frac{3}{4}z_1z_2\Big) \\ &= \frac{1}{4}(1+z_1)(1+z_2)(1+z_1z_2)^2 = \left(\frac{1+z_1z_2}{2}\right)P_2(z_1,z_2), \end{aligned}$$

from (1.6.21), with P_2 denoting the symbol of $\mathbb{B}_2$.

Hence, $P_3(z_1,z_2) = \left(\frac{1+z_1z_2}{2}\right)P_2(z_1,z_2)$ is the symbol for the box spline $\mathbb{B}_2$.

Observe from (1.7.5) that $\{p_{i,j} : i,j \in \mathbb{Z}\}$ is not interpolatory but symmetric, that is, $p_{i,j} = p_{j,i}$ for $i,j \in \mathbb{Z}$.

Next, we quote a result from [Han00] for box splines with respect to the 4-directional mesh

$\mathcal{G}_4$.

Theorem 1.7.3. *[Han00] For $k,\ell \in \mathbb{N}$ and $m,p \in \mathbb{Z}_+$, the box spline $\mathbb{B}_{k,\ell,m,p}$ is a piecewise bivariate polynomial of total degree at most $\gamma := k+\ell+m+p-2$ relative to the 4-directional mesh ($\mathcal{G}_4$-mesh). Furthermore, $\mathbb{B}_{k,\ell,m,p}$ is $(\gamma-k)$ times continuously differentiable across each horizontal line in the mesh; $(\gamma-\ell)$ times continuously differentiable across each vertical line ; $(\gamma-m)$ times continuously differentiable across each of the positively sloped diagonal lines; and $(\gamma-p)$ times continuously differentiable across each of the negatively sloped diagonal lines of the corresponding type-II triangular $\mathbb{Z}^2$ grid.*

Examples

(a) The well-known prototype example for the 4-directional mesh $\mathcal{G}_4$ is the box spline $\mathbb{B}_4 := \mathbb{B}_{1,1,1,1}$ associated with $\mathcal{D}_4 = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,-\mathbf{e}_4\}$.

As illustrated graphically in Figure 1.18, the regions $\Omega_1,\ldots,\Omega_{24}$ are disjoint, and form a partition of the octagonal region $[\mathcal{D}_4]$, as defined by

$$[\mathcal{D}_4] := \left\{\sum_{j=1}^{4}\mathbf{e}^j\,t_j : 0 \le t_j < 1,\ j=1,2,3,4\right\}, \tag{1.7.6}$$

that is, $[\mathcal{D}_4] = \bigcup_{i=1}^{24}\Omega_i$, as illustrated in Figure 1.18. Similar to the above examples, the graph of $\mathbb{B}_4$ is shown in Figure 1.19.

Next, we demonstrate the $2\mathrm{I}_2$-refinability of $\mathbb{B}_4$ by first noting that

$$\mathbb{B}_4(x,y) := \int_0^1 \mathbb{B}_2(x+t,y-t)\,\mathrm{d}t, \tag{1.7.7}$$

and then using (1.7.7) and (1.6.18), to obtain, for any $(x,y) \in \mathbb{R}^2$,

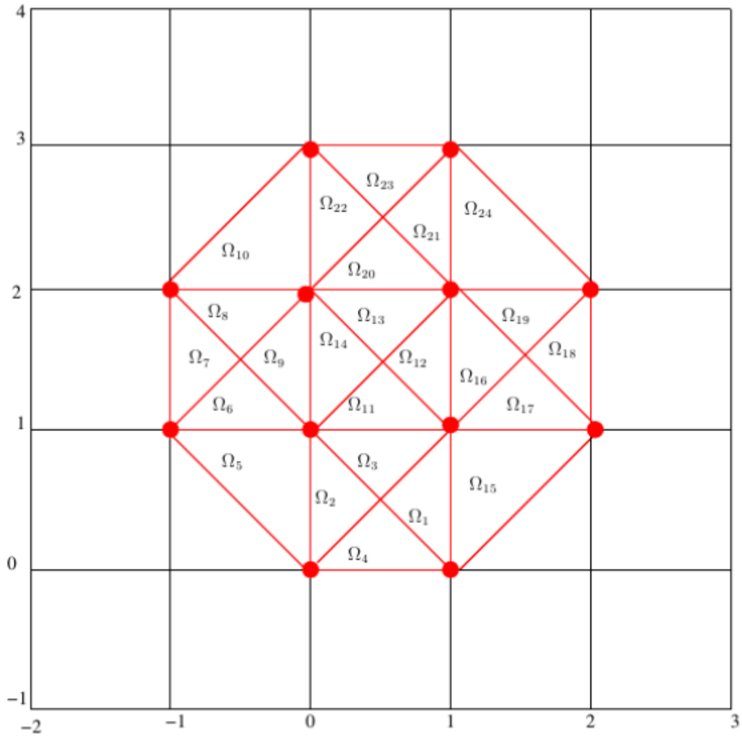

Figure 1.18: *The support of* $\mathbb{B}_4$

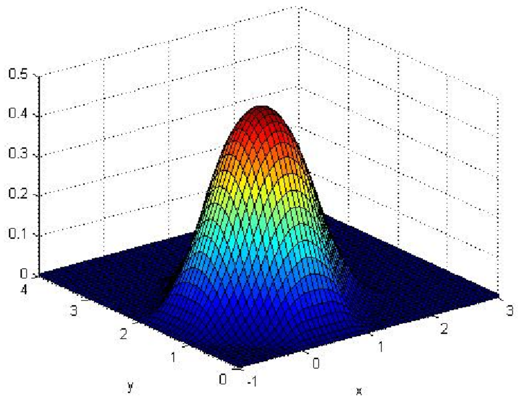

Figure 1.19: *Graph of the box spline* $\mathbb{B}_4$ *from [vdB08]*

$$
\begin{aligned}
\mathbb{B}_4(x,y) := \mathbb{B}_{1111}(x,y) &= \int_0^1 \mathbb{B}_2(x+t, y-t)\,\mathrm{d}t \\
&= \int_0^1 \sum_{i,j} p_{i,j}^2\, \mathbb{B}_2(2(x+t, y-t) - (i,j))\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^1 \mathbb{B}_2(2x+2t, 2y-2t)\,\mathrm{d}t \quad + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x+2t-1, 2y-2t)\,\mathrm{d}t \\
&\quad + \int_0^1 \mathbb{B}_2(2x+2t, 2y-2t-1)\,\mathrm{d}t \quad + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x+2t-1, 2y-2t-1)\,\mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x+2t-2, 2y-2t-1)\,\mathrm{d}t + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x+2t-1, 2y-2t-2)\,\mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x+2t-2, 2y-2t-2)\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{B}_4(x,y) &= \frac{1}{4}\int_0^1 \mathbb{B}_2(2x+t,2y-t)\,\mathrm{d}t + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x+1+t,2y-1-t)\,\mathrm{d}t \\
&\quad + \frac{3}{4}\int_0^1 \mathbb{B}_2(2x+t,2y-1-t)\,\mathrm{d}t + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x-1+t,2y-2-t)\,\mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x+1+t,2y-2-t)\,\mathrm{d}t + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x+t-1,2y-1-t)\,\mathrm{d}t \\
&\quad + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x+t,2y-2-t)\,\mathrm{d}t + \frac{1}{2}\int_0^1 \mathbb{B}_2(2x-2+t,2y-1-t)\,\mathrm{d}t \\
&\quad + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x+t,2y-3-t)\,\mathrm{d}t + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x-2+t,2y-2-t)\,\mathrm{d}t \\
&\quad + \frac{1}{4}\int_0^1 \mathbb{B}_2(2x-1+t,2y-3-t)\,\mathrm{d}t \\
&= \frac{1}{4}\left[\mathbb{B}_4(2x,2y) + \mathbb{B}_4(2x+1,2y-1) + \mathbb{B}_4(2x-1,2y) + \mathbb{B}_4(2x-1,2y-1)\right. \\
&\quad + \mathbb{B}_4(2x,2y-2) + \mathbb{B}_4(2x-2,2y-1) + \mathbb{B}_4(2x,2y-3) + \mathbb{B}_4(2x-2,2y-2) \\
&\quad \left.+ \mathbb{B}_4(2x-1,2y-3)\right] + \frac{3}{4}\left[\mathbb{B}_4(2x,2y-1) + \mathbb{B}_4(2x-1,2y-2)\right] \\
&\quad + \frac{1}{2}\mathbb{B}_4(2x+1,2y-2).
\end{aligned}
\tag{1.7.8}
$$

Thus, the box spline $\mathbb{B}_4$ as given in (1.7.8) is $2\mathrm{I}_2$-refinable with refinement mask $\{p^4_{\mathbf{j}}\}$ given by

$$
\left.
\begin{aligned}
& p^4_{-1,2} = \frac{1}{2};\; p^4_{0,1} = p^4_{1,2} = \frac{3}{4}; \\
& p^4_{0,0} = p^4_{2,1} = p^4_{1,0} = p^4_{2,2} = p^4_{0,2} = p^4_{0,3} = p^4_{1,1} = p^4_{-1,1} = p^4_{2,2} = \frac{1}{4}; \\
& p^4_{i,j} = 0, \quad (i,j) \notin \{(1,1),(0,0),(-1,1),(1,0),(2,1),(0,1),(0,2),(-1,2), \\
& \qquad (1,3),(0,3),(1,2),(2,2)\}.
\end{aligned}
\right\}
\tag{1.7.9}
$$

It then follows from (1.3.8) and (1.7.9) that the corresponding refinement mask symbol in its factorised form is given by

$$
P_4(z_1,z_2) = \frac{1}{4}\,(1+z_1)(1+z_2)(1+z_1z_2)(1+z_1z_2^{-1}) = \left(\frac{1+z_1z_2^{-1}}{2}\right) P_2(z_1,z_2),
$$

from (1.6.21), with P_2 denoting the refinement symbol of $\mathbb{B}_2$.

Therefore, $P_4(z_1,z_2) = \left(\frac{1+z_1z_2^{-1}}{2}\right) P_2(z_1,z_2)$ is the symbol for the box spline $\mathbb{B}_4$.

Observe from (1.7.9) that $\{p^4_{i,j} : i,j \in \mathbb{Z}\}$ is not interpolatory, but symmetric, that is, $p_{i,j} = p_{j,i}$ for $i,j \in \mathbb{Z}$.

As stated in, e.g., [dBHR93, EU10, CK02], the box spline $\mathbb{B}_4$ is known as the Zwart-Powell element. As graphically illustrated in Figure 1.18, the Zwart-Powell element is supported on an octagonal support $[-1,2]\times[0,3]$. We also note that here, in the notation of Theorem 1.7.3, that $\gamma = 1+1+1+1-2 = 2$, and thus, $\mathbb{B}_4$ is a piecewise quadratic bivariate polynomial and once continuously differentiable, that is, it is in $C^1(\mathbb{R}^2)$ across the grid lines according to Theorem 1.7.3. It is therefore smoother than the piecewise quadratic bivariate box spline $\mathbb{B}_3 := \mathbb{B}_{1,1,2}$ but its support is considerably larger. It has also a nice symmetric property.

(b) The box spline $\mathbb{B}_{2,1,1,1}$ associated with $\mathcal{D}_5 = \{\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ with $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ as in (1.5.2), is a piecewise cubic bivariate polynomial according to Theorem 1.7.3, and is again continuously differentiable on $\mathbb{R}$.

1.8 Properties of bivariate box splines

In this section, we investigate further properties of the bivariate box splines of Section 1.5. All of the properties given here are from [dVC10], where the proofs are left as exercises. We shall provide the full derivations below. We introduce the following preliminaries.

For integers $k > 0$, $\ell > 0$, $m \geq 0$, $p \geq 0$ in the box spline notation introduced in the first line of (1.5.8), we define the integer

$$n^* := \min\{k+\ell+m, k+\ell+p, k+m+p, \ell+m+p\} - 2, \tag{1.8.1}$$

from which together with (1.5.4), it then follows that

$$n^* = (k+\ell+m+p) - \max\{k,\ell,m,p\} - 2 = n - \max\{k,\ell,m,p\} - 2. \tag{1.8.2}$$

For any $k \in \mathbb{N}$, we shall write $C^k(\mathbb{R}^2)$ for the space of real-valued functions f such that f and all its partial derivatives up to order k are continuous on $\mathbb{R}^2$. We need the following lemma for proving the continuity order of box splines as in Theorem 1.8.2 below.

Lemma 1.8.1. *Let f be a function of two variables. If the partial derivatives $f_x \in C^m(\mathbb{R}^2)$ and $f_y \in C^m(\mathbb{R}^2)$, then $f \in C^{m+1}(\mathbb{R}^2)$.*

The following properties are satisfied by bivariate box splines.

Theorem 1.8.2. *For $d \in \{2,3,4\}$, let $\mathcal{G}_d$ denote the d-directional mesh with vertices in $\mathbb{Z}^2$ as described in Section 1.7. Then the box spline $\mathbb{B}_{k,\ell,m,p} := \mathbb{B}(.|\mathcal{D}_n)$, as obtained from (1.5.7), where the positive integer n is defined in (1.5.3), satisfies the following properties:*

(**a**) *The positivity property*

$$\mathbb{B}(\mathbf{x}|\mathcal{D}_n) > 0, \quad \mathbf{x} \in \operatorname{Int}[\mathcal{D}_n] \tag{1.8.3}$$

holds.

(**b**) *(i) The restriction of $\mathbb{B}_{k,\ell}$ to each square of the $\mathcal{G}_2$-mesh is a polynomial in $\Pi^2_{k+\ell-2}$;*

(ii) The restriction of $\mathbb{B}_{k,\ell,m}$ to each triangle of the $\mathcal{G}_3$-mesh is a polynomial in $\Pi^2_{k+\ell+m-2}$;

(iii) The restriction of $\mathbb{B}_{k,\ell,m,p}$ to each triangle of the $\mathcal{G}_4$-mesh is a polynomial in Π^2_n where $n := k+\ell+m+p$ as in (1.5.4).

(**c**) *The continuity condition*

$$\mathbb{B}(.|\mathcal{D}_n) \in C^{n^*}(\mathbb{R}^2), \tag{1.8.4}$$

where n^* is as given in (1.8.1), holds.

(**d**) *The support property*

$$supp\, \mathbb{B}(\mathbf{x}|\mathcal{D}_n) = \overline{[\mathcal{D}_n]}, \tag{1.8.5}$$

where $[\mathcal{D}_n]$ *is defined as in* (1.5.6)*, holds.*

(**e**) *The partition of unity property*

$$\sum_{\boldsymbol{j}\in\mathbb{Z}^2} \mathbb{B}(\mathbf{x}-\boldsymbol{j}|\mathcal{D}_n) = 1, \quad \mathbf{x}\in\mathbb{R}^2, \tag{1.8.6}$$

holds.

(**f**) *The unit integral condition*

$$\int_{\mathbb{R}^2} \mathbb{B}(\mathbf{x}|\mathcal{D}_n)\,\mathrm{d}\mathbf{x} = 1 \tag{1.8.7}$$

holds.

Proof. Our proofs of (**a**) – (**f**) are all by induction.

(**a**) **The positivity of box splines**

To show the positivity of box splines, we apply induction with respect to n. First, observe from (1.5.1) that the positivity condition (1.8.3) holds for $n = 1$. Suppose now that (1.8.3) holds for a fixed integer $n \in \mathbb{N}$. Our inductive proof of (**a**) will be complete if we can show that $\mathbb{B}(\mathbf{x}|\mathcal{D}_{n+1}) > 0$, $\mathbf{x} \in \mathrm{Int}[\mathcal{D}_{n+1}]$. To this end, let $\mathbf{x} \in \mathrm{Int}(\mathcal{D}_{n+1})$, from which it follows from (1.5.6) that

$$\mathbf{x} - t\,\mathbf{e}^{n+1} \in \mathrm{Int}(\mathcal{D}_n),\ t \in (0,1),$$

and thus, from the inductive hypothesis (1.8.3),

$$\mathbb{B}(\mathbf{x}|\mathcal{D}_{n+1}) = \int_0^1 \mathbb{B}(\mathbf{x} - t\,\mathbf{e}^{n+1}|\mathcal{D}_n)\,\mathrm{d}t > 0.$$

(**b**)(i) The restriction of box splines to $\mathcal{G}_d$- directional meshes with d=2,3,4.

Case 1 In order to show that $\mathbb{B}_{k,\ell}|_{\mathcal{G}_2} \in \Pi^2_{k+\ell-2}$, we employ induction on both k and ℓ respectively.

First, let $k = 1$ and apply the inductive argument with respect to ℓ.

Observe that the box spline $\mathbb{B}_{1,1}$ as in (1.5.1) is a constant function with degree zero (and is parallel to the xy-plane). The restriction of $\mathbb{B}_{1,1}$ to $\mathcal{G}_2$-mesh gives polynomials in $\Pi^2_{1+1-2} = \Pi^2_0$, that is, the linear space of bivariate constant polynomials.

Suppose it holds true, by induction hypothesis, that $\mathbb{B}_{k,\ell}|_{\mathcal{G}_2} \in \Pi^2_{\ell-1}$ for some fixed non-negative integers k and ℓ. Then by using (1.5.7) together with (1.5.2), for $\mathbf{x} \in \mathbb{R}^2$, we have

$$\mathbb{B}_{k,\ell+1}(\mathbf{x}) = \int_0^1 \mathbb{B}_{k,\ell}(\mathbf{x} - t\,\mathbf{e}_2)\,\mathrm{d}t = \int_0^1 \mathbb{B}_{k,\ell}(x, y-t)\,\mathrm{d}t = \int_{y-1}^{y} \mathbb{B}_{k,\ell}(x,t)\,\mathrm{d}t, \quad t\in[0,1]. \tag{1.8.8}$$

Then, by applying the Fundamental Theorem of Calculus on equation (1.8.8), we get

$$\frac{\partial \mathbb{B}_{k,\ell+1}}{\partial y} = \underbrace{\mathbb{B}_{k,\ell}(x,y)}_{\in \Pi^2_{k+\ell-2}} - \underbrace{\mathbb{B}_{k,\ell}(x,y-1)}_{\in \Pi^2_{k+\ell-2}}.$$

Hence, $\dfrac{\partial \mathbb{B}_{k,\ell+1}}{\partial y} \in \Pi^2_{k+\ell-2}$ and up on integration with respect to y, we get an increment in the degree of piecewise polynomial $\mathbb{B}_{k,\ell+1}|_{\mathcal{G}_2}$ by one, that is,

$$\frac{\partial \mathbb{B}_{k,\ell+1}}{\partial y} \in \Pi^2_{k+(\ell+1)-2} = \Pi^2_{k+\ell-1}.$$

Case 2 Let ℓ be arbitrary but fixed. Then we proceed by induction on k. First, observe from (1.5.1) that the restriction of $\mathbb{B}_{1,1}$ to $\mathcal{G}_2$-mesh gives polynomials in $\Pi^2_{1+1-2} = \Pi^2_0$. Suppose now that, by induction hypothesis, that $\mathbb{B}_{k,\ell}$ restricted to $\mathcal{G}_2$-mesh is in $\Pi^2_{k+\ell-2}$.

Now, by using (1.5.7) and (1.5.2) same way as in Case 1, we get

$$\mathbb{B}_{k+1,\ell}(\mathbf{x}) = \int_0^1 \mathbb{B}_{k,\ell}(x-t,y)\,\mathrm{d}t = \int_{x-1}^{x} \mathbb{B}_{k,\ell}(t,y)\,\mathrm{d}t, \quad t \in [0,1]. \tag{1.8.9}$$

which, after implementing the Fundamental Theorem of Calculus on equation (1.8.9), becomes

$$\frac{\partial \mathbb{B}_{k+1,\ell}}{\partial y} = \underbrace{\mathbb{B}_{k,\ell}(x,y)}_{\in \Pi^2_{k+\ell-2}} - \underbrace{\mathbb{B}_{k,\ell}(x-1,y)}_{\in \Pi^2_{k+\ell-2}}.$$

Thus, $\dfrac{\partial \mathbb{B}_{k+1,\ell}}{\partial y} \in \Pi^2_{k+\ell-2}$ and upon integration with respect to x, we get an increment for the degree of piecewise polynomial $\mathbb{B}_{k+1,\ell}|_{\mathcal{G}_2}$ by one, that is,

$$\frac{\partial \mathbb{B}_{k+1,\ell}}{\partial x} \in \Pi^2_{k+(\ell+1)-2} = \Pi^2_{k+\ell-1}.$$

A similar proof using induction argument is implemented for proving the remaining cases, including the case with triangulations, $\mathcal{G}_3$ and $\mathcal{G}_4$.

(c) The continuity condition

Now, in order to prove the continuity condition, that is, $\mathbb{B}(.|\mathcal{D}_n) \in C^{n^*}(\mathbb{R}^2)$ with n^* given in (1.8.2), we use induction on each of the multiplicities: k, ℓ, m, p .

Observe, from (1.7.7) and Theorem 1.7.3, that the Zwart-Powell element $\mathbb{B}_4$ is C^2-quadratic function on 4-directional mesh, that is, non-zero on an octagonal support contained in $[-1,2] \times [0,3]$. So using (1.8.2) with $k = \ell = m = p = 1$, we see that it holds for $n^* = 1$.

Suppose now, by inductive argument, that (1.8.4) holds for a fixed integer $n^* \in \mathbb{N}$ where n^* is as in (1.8.1).

Then it remains to show 4-possible cases of induction: k, ℓ, m and p by using (1.5.7) on each of the cases together with the Fundamental Theorem of Calculus, that is, $\mathbb{B}_{k,\ell,m,p} \in C^{n^*}(\mathbb{R}^2)$.

(i) **Induction on** k

$$\begin{aligned}\mathbb{B}_{k+1,\ell,m,p}(\mathbf{x}) &= \int_0^1 \mathbb{B}_{k,\ell,m,p}(\mathbf{x} - t\,\mathbf{e}_1)\,\mathrm{d}t, \quad \mathbf{x} := (x, y) \in \mathbb{R}^2 \\ &= \int_0^1 \mathbb{B}_{k,\ell,m,p}(x - t, y)\,\mathrm{d}t = \int_{x-1}^{x} \mathbb{B}_{k,\ell,m,p}(t, y)\,\mathrm{d}t. \end{aligned} \tag{1.8.10}$$

Using the Fundamental Theorem of Calculus on equation (1.8.10), we get

$$\frac{\partial \mathbb{B}_{k+1,\ell,m,p}}{\partial x}(\mathbf{x}) = \underbrace{\mathbb{B}_{k,\ell,m,p}(x, y)}_{\in C^{n^*}(\mathbb{R}^2)} - \underbrace{\mathbb{B}_{k,\ell,m,p}(x - 1, y)}_{\in C^{n^*}(\mathbb{R}^2)}. \tag{1.8.11}$$

Equation (1.8.11) in turn implies that $\dfrac{\partial \mathbb{B}_{k+1,\ell,m,p}}{\partial x} \in C^{n^*}(\mathbb{R}^2)$. Next, by using similar argument, we make induction on the remaining multiplicities, namely, ℓ, m, p and it is shown as follows.

(ii) **Induction on** ℓ

$$\begin{aligned}\mathbb{B}_{k,\ell+1,m,p}(\mathbf{x}) &= \int_0^1 \mathbb{B}_{k,\ell,m,p}(\mathbf{x} - t\,\mathbf{e}_2)\,\mathrm{d}t = \int_0^1 \mathbb{B}_{k,\ell,m,p}(x, y - t)\,\mathrm{d}t \\ &= \int_{y-1}^{y} \mathbb{B}_{k,\ell,m,p}(x, t)\,\mathrm{d}t. \end{aligned} \tag{1.8.12}$$

Applying the Fundamental Theorem of Calculus on equation (1.8.12) yields

$$\frac{\partial \mathbb{B}_{k,\ell+1,m,p}}{\partial y}(x, y) = \underbrace{\mathbb{B}_{k,\ell,m,p}(x, y)}_{\in C^{n^*}(\mathbb{R}^2)} - \underbrace{\mathbb{B}_{k,\ell,m,p}(x, y - 1)}_{\in C^{n^*}(\mathbb{R}^2)}. \tag{1.8.13}$$

Equation (1.8.13) in turn implies that $\dfrac{\partial \mathbb{B}_{k,\ell+1,m,p}}{\partial y} \in C^{n^*}(\mathbb{R}^2)$. A similar inductive approach can be implemented to show that it also holds for m and p. In other words, it can be, by induction on m and p, shown that

$$\frac{\partial \mathbb{B}_{k,\ell,m+1,p}}{\partial x} \in C^{n^*}(\mathbb{R}^2) \quad \text{and} \frac{\partial \mathbb{B}_{k,\ell,m+1,p}}{\partial y} \in C^{n^*}(\mathbb{R}^2).$$

Now, since the box spline $\mathbb{B} := \mathbb{B}_{k,\ell,m,p}$ with $\dfrac{\partial \mathbb{B}}{\partial x} \in C^{n^*}(\mathbb{R}^2)$ and $\dfrac{\partial \mathbb{B}}{\partial y} \in C^{n^*}(\mathbb{R}^2)$, we apply Lemma 1.8.1 together with

$$n^* + 1 = |k + \ell + m + p| - \max\{k, \ell, m, p\} - 1$$

in order to check that $\mathbb{B} \in C^{n^*+1}(\mathbb{R}^2)$. That is, since $\dfrac{\partial \mathbb{B}}{\partial x} \in C^{n^*}(\mathbb{R}^2)$ with $k = 0, 1, 2, \ldots, n^*$, we have that

$$\mathbb{B}, \frac{\partial \mathbb{B}}{\partial x}, \frac{\partial}{\partial y}\left(\frac{\partial \mathbb{B}}{\partial x}\right), \frac{\partial^2 \mathbb{B}}{\partial x^2}, \ldots, \frac{\partial^{n^*}}{\partial x^k \partial y^{n^*-k}}\left(\frac{\partial \mathbb{B}}{\partial x}\right) \in C^0(\mathbb{R}^2), \tag{1.8.14}$$

and also if $\dfrac{\partial \mathbb{B}_{k\ell mp}}{\partial y} \in C^{n^*}(\mathbb{R}^2)$, then

$$\mathbb{B}, \frac{\partial \mathbb{B}}{\partial y}, \frac{\partial}{\partial x}\left(\frac{\partial \mathbb{B}}{\partial y}\right), \frac{\partial^2 \mathbb{B}}{\partial y^2}, \ldots, \frac{\partial^{n^*}}{\partial x^k \partial y^{n^*-k}}\left(\frac{\partial \mathbb{B}}{\partial y}\right) \in C^0(\mathbb{R}^2) \tag{1.8.15}$$

with $k = 0, 1, 2, \ldots, n^*$ holds. Thus, from (1.8.14) and (1.8.15), we get

$$\mathbb{B}, \frac{\partial \mathbb{B}}{\partial x}, \frac{\partial}{\partial x}\left(\frac{\partial \mathbb{B}}{\partial y}\right), \ldots, \frac{\partial^{n^*+1} \mathbb{B}}{\partial x^k \partial y^{n^*-k}} \in C^0 \quad \text{where} \quad k = 0, 1, 2, \ldots, n^* + 1.$$

Hence, $\mathbb{B}_{k,\ell,m,p} \in C^{n^*+1}(\mathbb{R}^2)$. This completes the induction argument.

$\square$

(d) The support property

We proceed by induction on n. From the definition of the box spline $\mathbb{B}_{1,1}$ in (1.5.1), we have that

$$\mathbb{B}_{1,1}(\mathbf{x}) := \chi_{[0,1)^2}(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in [0,1)^2; \\ 0, & \text{otherwise}\ . \end{cases}$$

From the definition of support, that is, $\operatorname{supp}^c \mathbb{B}(x|\mathcal{D}_2) = \overline{\{\mathbf{x} \in \mathbb{R}^2 : \mathbb{B}(x|\mathcal{D}_2) \neq 0\}}$, it is obvious that $\mathbb{B}(\mathbf{x}|\mathcal{D}_2) \neq 0$ only when $x \in \overline{[0,1)}^2 = [0,1]^2 = \{t_1\, \mathbf{e}^1 + t_2\, \mathbf{e}^2,\ t_j \in [0,1]\}$. Hence, the base case

$$\operatorname{supp}^c \mathbb{B}(\mathbf{x}|\mathcal{D}_2) = \left\{\mathbf{x} : \mathbf{x} = \sum_{i=1}^{2} t_i\, \mathbf{e}^i;\ 0 \leq t_i \leq 1\right\} = [\mathcal{D}_2]$$

trivially holds.

Assume, by induction argument, that it holds true for n, that is, equivalently:

$$\operatorname{supp}^c \mathbb{B}(\mathbf{x}|\mathcal{D}_n) = [\mathcal{D}_n].$$

We need to show that

$$\operatorname{supp}^c \mathbb{B}(\mathbf{x}|\mathcal{D}_{n+1}) = [\mathcal{D}_{n+1}] = \left\{\sum_{i=1}^{n+1} t_i\, \mathbf{e}^i,\ 0 \leq t_i \leq 1\right\}.$$

That is,

$$\operatorname{supp}^c \mathbb{B}(\mathbf{x}|\mathcal{D}_{n+1}) = [\mathcal{D}_{n+1}] \Leftrightarrow \operatorname{supp}^c \mathbb{B}(\mathbf{x}|\mathcal{D}_{n+1}) \subseteq [\mathcal{D}_{n+1}] \bigcap [\mathcal{D}_{n+1}] \subseteq \operatorname{supp}^c \mathbb{B}(\mathbf{x}|\mathcal{D}_{n+1}). \tag{1.8.16}$$

The first part of (1.8.16) proceeds as follows.

We let

$$\mathbf{x} \in \operatorname{supp}^c \mathbb{B}(\cdot|\mathcal{D}_{n+1}), \Rightarrow \int_0^1 \mathbb{B}(\mathbf{x} - t_{n+1}\, \mathbf{e}^{n+1}|\mathcal{D}_n)\, \mathrm{d}t_{n+1} \neq 0$$

and this implies that there exists $t_{n+1} \in (0,1)$ where $\mathbb{B}(\mathbf{x} - t_{n+1}\mathbf{e}^{n+1}|\mathcal{D}_n) \neq 0$, that is,

$$\begin{aligned}
\mathbf{x} \in \operatorname{supp}^c \mathbb{B}(\cdot|\mathcal{D}_{n+1}) &\Rightarrow \mathbf{x} \in \operatorname{supp}^c \mathbb{B}(\cdot|\mathcal{D}_{n+1}) \\
&\Rightarrow \mathbf{x} - t_{n+1}\,\mathbf{e}^{n+1} \in \operatorname{supp}^c \mathbb{B}(\cdot|\mathcal{D}_n) \\
&\Rightarrow \mathbf{x} - t_{n+1}\mathbf{e}^{n+1} = \sum_{i=1}^{n} t_i\,\mathbf{e}^i, \text{ for some } t_i \in [0,1] \\
&\Rightarrow \mathbf{x} = \sum_{i=1}^{n} t_i\,\mathbf{e}^i + t_{n+1}\,\mathbf{e}^{n+1}, \quad 0 \leq t_i \leq 1 \\
&\Rightarrow \mathbf{x} = \sum_{i=1}^{n+1} t_i\,\mathbf{e}^i, \quad 0 \leq t_i \leq 1 \\
&\Rightarrow \mathbf{x} \in [\mathcal{D}_{n+1}].
\end{aligned}$$

Thus,

$$\operatorname{supp}^c \mathbb{B}(\mathbf{x}|\mathcal{D}_{n+1}) \subseteq [\mathcal{D}_{n+1}]. \tag{1.8.17}$$

The other part of (1.8.16) proceeds as follows. We also let

$$\begin{aligned}
\mathbf{x} \in [\mathcal{D}_{n+1}] \Rightarrow \mathbf{x} &= \sum_{i=1}^{n+1} t_i\,\mathbf{e}^i, \quad 0 \leq t_i \leq 1 \\
&\Rightarrow \mathbf{x} - t_{n+1}\,\mathbf{e}^{n+1} = \sum_{i=1}^{n} t_i\,\mathbf{e}^i,\ 0 \leq t_i \leq 1 \\
&\Rightarrow \mathbf{x} - t_{n+1}\,\mathbf{e}^{n+1} \in \operatorname{supp}^c \mathbb{B}(\cdot|\mathcal{D}_n) \\
&\Rightarrow \mathbb{B}(\mathbf{x} - t_{n+1}\,\mathbf{e}^{n+1}|\mathcal{D}_n) \neq 0
\end{aligned} \tag{1.8.18}$$

Since $\mathbb{B}(\cdot|\mathcal{D}_n)$ is continuous and $\mathbb{B}(\mathbf{x}|\mathcal{D}_n) > 0$ for all $\mathbf{x}$, then there exists a neighbourhood $\mathrm{N} := (t_{n+1} - \varepsilon, t_{n+1} + \varepsilon)$ such that $\mathbb{B}(\mathbf{x} - k\,\mathbf{e}^{n+1}|\mathcal{D}_n) > 0$ for $k \in \mathrm{N}$.

Hence, equation (1.8.18) is equivalent to

$$\int_0^1 \mathbb{B}(\mathbf{x} - k\,\mathbf{e}^{n+1}|\mathcal{D}_n)\,\mathrm{d}k \neq 0 \Leftrightarrow \mathbb{B}(\mathbf{x}|\mathcal{D}_{n+1}) \neq 0 \Rightarrow \mathbf{x} \in \operatorname{supp}^c \mathbb{B}(\cdot|\mathcal{D}_{n+1}).$$

Thus,

$$[\mathcal{D}_{n+1}] \subseteq \operatorname{supp}^c \mathbb{B}(\mathbf{x}|\mathcal{D}_{n+1}). \tag{1.8.19}$$

From (1.8.17) and (1.8.19), we deduce that $\operatorname{supp}^c \mathbb{B}(\mathbf{x}|\mathcal{D}_{n+1}) = [\mathcal{D}_{n+1}]$.

Thus, $\operatorname{supp}^c \mathbb{B}(\mathbf{x}|\mathcal{D}_{n+1}) = [\mathcal{D}_{n+1}]$ and this completes the proof.

(e) **The partition of unity property**

The inductive proof for the partition of unity property proceeds as follows.

Observe, from the definition of $\mathbb{B}_1$ as in (1.5.1), that

$$\sum_{\mathbf{j} \in \mathbb{Z}^2} \mathbb{B}(\mathbf{x} - \mathbf{j}|\mathcal{D}_2) = 1, \quad \mathbf{x} \in \mathbb{R}^2 \quad \text{holds.}$$

Suppose it holds true for n, that is, $\sum_{\mathbf{j}\in\mathbb{Z}^2}\mathbb{B}(\mathbf{x}-\mathbf{j}|\mathcal{D}_n)=1$ for all $\mathbf{x}\in\mathbb{R}^2$. We need to show that it holds for $n+1$.

Since $\mathcal{D}_{n+1}=\mathcal{D}_n\bigcup\{\mathbf{e}^{n+1}\}$, with $\mathbf{e}^{n+1}\in\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$, as in (1.5.2), we apply induction to each of the possible cases.

If $\mathbf{e}^{n+1}=\mathbf{e}_1$, and by setting $\acute{x}:=x-t\in\mathbb{R}$ with $t\in[0,1]$, we have

$$\begin{aligned}
\sum_{(i,j)\in\mathbb{Z}^2}\mathbb{B}\left((x,y)-(i,j)|\mathcal{D}_{n+1}\right) &= \sum_{(i,j)\in\mathbb{Z}^2}\int_0^1\mathbb{B}\left((x,y)-(i,j)-(t,0)|\mathcal{D}_n\right)\,\mathrm{d}t\\
&= \sum_{(i,j)\in\mathbb{Z}^2}\int_0^1\mathbb{B}(x-t-i,y-j|\mathcal{D}_n)\,\mathrm{d}t\\
&= \sum_{(i,j)\in\mathbb{Z}^2}\int_0^1\mathbb{B}(\acute{x}-i,y-j|\mathcal{D}_n)\,\mathrm{d}t\\
&= \int_0^1\sum_{(i,j)\in\mathbb{Z}^2}\mathbb{B}(\acute{x}-i,y-j|\mathcal{D}_n)\,\mathrm{d}t\\
&= \int_0^1 1\,\mathrm{d}t=1.
\end{aligned}$$

If $\mathbf{e}^{n+1}=\mathbf{e}_2$, and by setting $\acute{y}:=y-t\in\mathbb{R}$ with $t\in[0,1]$, we have

$$\begin{aligned}
\sum_{(i,j)\in\mathbb{Z}^2}\mathbb{B}\left((x,y)-(i,j)|\mathcal{D}_{n+1}\right) &= \sum_{(i,j)\in\mathbb{Z}^2}\int_0^1\mathbb{B}\left((x,y)-(i,j)-(0,t)|\mathcal{D}_n\right)\,\mathrm{d}t\\
&= \sum_{(i,j)\in\mathbb{Z}^2}\int_0^1\mathbb{B}\left(x-i,y-t-j|\mathcal{D}_n\right)\,\mathrm{d}t\\
&= \sum_{(i,j)\in\mathbb{Z}^2}\int_0^1\mathbb{B}(x-i,\acute{y}-j|\mathcal{D}_n)\,\mathrm{d}t\\
&= \int_0^1\sum_{(i,j)\in\mathbb{Z}^2}\mathbb{B}(x-i,\acute{y}-j|\mathcal{D}_n)\,\mathrm{d}t\\
&= \int_0^1 1\,\mathrm{d}t=1.
\end{aligned}$$

The case with $\mathbf{e}^{n+1}=\mathbf{e}_3$ and $\mathbf{e}^{n+1}=\mathbf{e}_4$ is shown in a similar fashion.

(f) The unit integral property

$$\int_{\mathbb{R}^2}\mathbb{B}(\mathbf{x}|\mathcal{D}_n)\,\mathrm{d}\mathbf{x}=\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbb{B}(x,y|\mathcal{D}_n)\,\mathrm{d}x\,\mathrm{d}y=1$$

.

Proof. We begin with the base case. For $n=2$, it holds from (1.5.1) that

$$\int_{\mathbb{R}^2}\mathbb{B}(\mathbf{x}|\mathcal{D}_2)\,\mathrm{d}\mathbf{x}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\boldsymbol{\chi}_{[0,1)^2}\,\mathrm{d}x\,\mathrm{d}y=\int_0^1\int_0^1 1\,\mathrm{d}x\,\mathrm{d}y=1.$$

By induction argument, assume that it holds true for n, that is,

$$\int_{\mathbb{R}^2} \mathbb{B}(\mathbf{x}|\mathcal{D}_n)\mathrm{d}\mathbf{x} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{B}(x, y|\mathcal{D}_n)\, \mathrm{d}x\, \mathrm{d}y = 1.$$

Then, we need to check whether it is true for $n+1$.

If $\mathbf{e}^{n+1} = \mathbf{e}_1$, and applying the induction argument by setting $\acute{x} := x - t \in \mathbb{R}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathbb{B}(\mathbf{x}|\mathcal{D}_{n+1})\, \mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^2} \left[\int_0^1 \mathbb{B}(\mathbf{x} - t\,\mathbf{e}_1|\mathcal{D}_n)\, \mathrm{d}t\right] \mathrm{d}\mathbf{x} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\int_0^1 \mathbb{B}((x, y) - t\,\mathbf{e}_1|\mathcal{D}_n)\, \mathrm{d}t\right) \mathrm{d}x\, \mathrm{d}y \\
&= \int_0^1 \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{B}((x, y) - t\,\mathbf{e}_1|\mathcal{D}_n)\, \mathrm{d}x\, \mathrm{d}y\right] \mathrm{d}t \\
&= \int_0^1 \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{B}((x - t, y)|\mathcal{D}_n)\, \mathrm{d}x\, \mathrm{d}y\right) \mathrm{d}t = \int_0^1 1\, \mathrm{d}t = 1.
\end{aligned}$$

If $\mathbf{e}^{n+1} = \mathbf{e}_2$, and applying the induction argument by setting $\acute{y} := y - t \in \mathbb{R}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathbb{B}(\mathbf{x}|\mathcal{D}_{n+1})\, \mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^2} \left[\int_0^1 \mathbb{B}(\mathbf{x} - t\,\mathbf{e}_2|\mathcal{D}_n)\, \mathrm{d}t\right] \mathrm{d}\mathbf{x} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\int_0^1 \mathbb{B}((x, y) - t\,\mathbf{e}_2|\mathcal{D}_n)\, \mathrm{d}t\right) \mathrm{d}x\, \mathrm{d}y \\
&= \int_0^1 \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{B}((x, y) - t\,\mathbf{e}_2|\mathcal{D}_n)\, \mathrm{d}x\, \mathrm{d}y\right] \mathrm{d}t \\
&= \int_0^1 \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{B}(x, y - t|\mathcal{D}_n)\, \mathrm{d}x\, \mathrm{d}y\right) \mathrm{d}t = \int_0^1 1\, \mathrm{d}t = 1.
\end{aligned}$$

The case with $\mathbf{e}^{n+1} = \mathbf{e}_3$ and $\mathbf{e}^{n+1} = \mathbf{e}_4$ can be shown in a similar fashion. □

The following result in [dVC10] is a special case of Theorem 1.8.2 (**d**), which is about the support of a box spline in the bivariate case.

Corollary 1.8.3. *For $n \in \mathbb{N}$ with $n := k + \ell + m + p$ as in (1.5.4), the support of a box spline $\mathbb{B}(\mathbf{x}|\mathcal{D}_n)$ is in general a closed polygonal region $k, \ell > 0$, $m, p \geq 0$ with the following restrictions.*

(a) The box spline $\mathbb{B}_{k,\ell}$ is supported on the rectangle $[0, k] \times [0, \ell]$ with $m = p = 0$.

(b) The box spline $\mathbb{B}_{k,\ell,m}$ is supported on a closed hexagonal region with $\ell > 0$ and $m > 0$.

(c) The box spline $\mathbb{B}_{k,\ell,m,p}$ is supported on a closed octagonal region if $m, p > 0$.

Proof. The proof is immediate from the proof of Theorem 1.8.2 (**d**). □

Remark

The support of bivariate box splines in general are built from the union of smaller triangular pieces in $\mathbb{Z}^2$-grid. If we denote the set $\Omega \subseteq \mathbb{Z}^2$ on the grid in which the support encompasses, then Ω is bounded, polygonal and convex set with its boundary along a 3/4-directional grid itself. The following result in [dVC10] relates univariate B-splines with box splines.

Proposition 1.8.4. *Given $k, m \in \mathbb{N}$, then the box spline*

$$\mathbb{B}_{k,m}(\mathbf{x}) = \mathbb{B}(\mathbf{x}|\{\underbrace{\mathbf{e}_1, \ldots, \mathbf{e}_1}_{k}, \underbrace{\mathbf{e}_2, \ldots, \mathbf{e}_2}_{m}\}) = B_k(x)\, B_m(y)$$

where $\mathbf{x} := (x, y) \in \mathbb{R}^2$ and B_k and B_m are cardinal B-splines.

Proof. We use inductive argument on both k and m respectively. For fixed k, we first apply the inductive argument on m. For $m = 1$ and $k = 1$ and from the definition of $\mathbb{B}_1$ in (1.5.1), we have that

$$\mathbb{B}_1(x, y) = \mathbb{B}(\mathbf{x}|\mathcal{D}_2) = \boldsymbol{\chi}_{[0,1)^2}(x, y) = \boldsymbol{\chi}_{[0,1)}(x)\, \boldsymbol{\chi}_{[0,1)}(y) = B_1(x)\, B_1(y). \tag{1.8.20}$$

Suppose it holds true for an arbitrary m. Then, for fixed k and from the definition of $\mathbb{B}_1$ in (1.5.1), we have that

$$\begin{aligned}\mathbb{B}_{k,m+1}(\mathbf{x}) &= \int_0^1 \mathbb{B}_{k,m}(\mathbf{x} - t\,\mathbf{e}_2|\mathcal{D}_n)\, \mathrm{d}t = \int_0^1 B_k(x) B_m(y - t)\, \mathrm{d}t \\ &= B_k(x) \int_0^1 B_m(y - t)\, \mathrm{d}t = B_k(x)\, B_{m+1}(y).\end{aligned} \tag{1.8.21}$$

Similarly, for fixed m and from the definition of $\mathbb{B}_1$ in (1.5.1), we have that

$$\begin{aligned}\mathbb{B}_{k+1,m}(\mathbf{x}) &= \int_0^1 \mathbb{B}_{k,m}(\mathbf{x} - t\,\mathbf{e}_1|\mathcal{D}_n)\, \mathrm{d}t = \int_0^1 B_k(x - t) B_m(y)\, \mathrm{d}t \\ &= B_m(y) \int_0^1 B_k(x - t)\, \mathrm{d}t = B_{k+1}(x)\, B_m(y).\end{aligned} \tag{1.8.22}$$

From (1.8.21) and (1.8.22), we deduce that $\mathbb{B}_{k,m}(x, y) = B_k(x)\, B_m(y)$. □

1.8.1 The general setting of box splines

The following results are some generalized facts about multivariate box splines. More information with greater detail is given in [dBHR93, CK02, Jia00].

Let $\mathcal{D} = \{\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^n\} \subset \mathbb{Z}^d \backslash \{\mathbf{0}\}$ be a multi-set of non-zero vectors in $\mathbb{R}^d$.

(a) A box spline $\mathbb{B}_{\mathcal{D}} := \mathbb{B}(\cdot|\mathcal{D})$ associated with direction matrix $\mathcal{D} \in \mathbb{Z}^{d\times n}$ is a distribution in the space of all continuous functions on $\mathbb{R}^d$, that is,

$\mathbb{B}_{\mathcal{D}} : C(\mathbb{R}^d) \to \mathbb{R}$ satisfies the following condition:

$$\begin{aligned}\langle \mathbb{B}_{\mathcal{D}}, f\rangle :&= \int_{\mathbb{R}^d} \mathbb{B}_{\mathcal{D}}(\mathbf{x})\, f(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_{[0,1)^n} f(\mathcal{D}.\mathbf{t})\, \mathrm{d}\mathbf{t} \\ &= \int_{[0,1)^n} f(\mathbf{e}^1\, t_1 + \ldots + \mathbf{e}^n\, t_n)\, \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n \quad \text{for } f \in C_0^\infty(\mathbb{R}^d), \\ &\quad \mathbf{x} \in \mathbb{R}^d, \quad \mathbf{t} = (t_1, t_2, \ldots, t_n)^T.\end{aligned} \tag{1.8.23}$$

The value of the box spline is $\mathbb{B}_{\mathcal{D}}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$.

(b) The restriction of box splines $\mathbb{B}(\cdot|\mathcal{D})$ to their regional support pieces results in a piecewise polynomial of total order $m = n - d + 1$ (exact total degree $n - d$) where their compact support is given by

$$\operatorname{supp}^c \mathbb{B}_{\mathcal{D}} = \left\{ \sum_{j=1}^{n} \mathbf{e}^j \, t_j : 0 \le t_j \le 1, \; j = 1, \dots, n \right\}.$$

(c) The centre of the support of a box spline $\mathbb{B}(\mathbf{x}|\mathcal{D})$ is the sum of the column vectors in $\mathcal{D}$, that is, $\mathbf{x}_{\mathcal{D}} := \frac{1}{2} \sum_{k=1}^{n} \mathbf{e}_k$. The box splines in general are symmetric about the center of their support.

(d) The linear space of all box splines are built from the shifts of the box spline $\mathbb{B}_{\mathcal{D}}$ on $\mathbb{Z}^d$, that is,

$$\mathcal{S}_{\mathbb{B}_{\mathcal{D}}} := \operatorname{span}\left[\mathbb{B}_{\mathcal{D}}(\cdot - \mathbf{j})\right] \quad \text{for } \mathbf{j} \in \mathbb{Z}^d. \tag{1.8.24}$$

(e) The sequence $\{\mathbb{B}_{\mathcal{D}}(. - \mathbf{j})\}_{\mathbf{j} \in \mathbb{Z}^d}$ is linearly independent if and only if the direction matrix $\mathcal{D}$ is uni-modular which means that all square sub-matrices of $\mathcal{D}$ have determinant ± 1, that is, $\det C = \{+1, -1\}, \forall\, C \in \mathcal{D}$.

(f) $\mathbb{B}_{\mathcal{D}}$ satisfies refinement equation (1.2.1) with dilation matrix $2\mathrm{I}_d$, $d \in \mathbb{Z}_+$, that is,

$$\mathbb{B}_{\mathcal{D}}(\mathbf{x}) = \sum_{\mathbf{j} \in \mathbb{Z}^d} p_{\mathbf{j}}^{\mathcal{D}} \, \mathbb{B}_{\mathcal{D}}(2\mathbf{x} - \mathbf{j}), \quad \mathbf{x} \in \mathbb{R}^d, \tag{1.8.25}$$

where $p_{\mathbf{j}}^{\mathcal{D}} \in \ell_0(\mathbb{Z}^d)$ is obtained from the coefficients of the symbol $\mathcal{P}^{\mathcal{D}}(\mathbf{z})$ given by

$$\mathcal{P}^{\mathcal{D}}(\mathbf{z}) = 2^d \prod_{v \in \mathcal{D}} \left[\frac{1 + \mathbf{z}^v}{2} \right] \tag{1.8.26}$$

for $\mathbf{z} := (z_1, \dots, z_d)$, $z_i \in \mathbb{C}$, $i = 1, \dots, d$, and v runs through all the columns of the $d \times n$ matrix with rank d, $\mathcal{D} \in \mathbb{Z}^{d \times n}$, with $n \ge d$.

In the following section, the role of direction sets (or matrices) associated with box splines will be introduced, for instance, to study the continuity exponent, to know the degree of polynomial pieces and to construct the subdivision algorithm as given in [Pra85, She96].

1.9 Direction matrices and box splines

The following linear algebraic notions are used to study box splines with associated direction matrices $\mathcal{D}$. We denote by $\mathbb{Z}^{d \times n}$ the class of matrices with integer entries of d-rows and n-columns, and $\#\mathcal{D}$ denotes the number of columns of the matrix $\mathcal{D} \in \mathbb{Z}^{d \times n}$.

We write $\mathcal{Z} \subset \mathcal{D} = [\mathbf{e}^1, \dots, \mathbf{e}^n]$ if $\mathcal{Z} = [\mathbf{e}^{i_1}, \dots, \mathbf{e}^{i_k}]$ for some $1 \le i_1 \le i_2 < \dots < i_k \le \#\mathcal{D}$, and $\mathcal{D} \backslash \mathcal{Z} \in \mathbb{R}^{d \times (n-k)}$ is a matrix where the columns of $\mathcal{Z}$ have been removed from $\mathcal{D}$ up to the multiplicity in which they occur in $\mathcal{Z}$.

For the matrix $\mathcal{D} = [\mathbf{e}^1, \dots, \mathbf{e}^n] \in \mathbb{Z}^{d \times n}$, the following are important concepts associated with direction matrices.

Remark: Let $X = (x_1, \dots, x_n)^T$ be a real column vector.

(a) The column span (the range) of the matrix $\mathcal{D}$ is defined to be

$$\operatorname{ran}(\mathcal{D}) = \operatorname{span}(\mathcal{D}) := \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} = \mathcal{D}\mathbf{v}, \text{ for some } \mathbf{v} \in \mathbb{R}^n\right\}. \tag{1.9.1}$$

(b) The kernel of the matrix $\mathcal{D}$ is defined to be

$$\ker \mathcal{D} := \left\{\mathbf{x} \in \mathbb{R}^d : \mathcal{D}\mathbf{x} = \mathbf{0}, \quad \mathbf{0} := (0, 0, \ldots, 0)^T \in \mathbb{R}^d\right\}. \tag{1.9.2}$$

(c) A matrix $\mathcal{Z} \subset \mathcal{D}$ is said to be a *spanning matrix* if $\operatorname{ran}(\mathcal{Z}) = \operatorname{ran}(\mathcal{D})$.

(d) $\chi(\mathcal{D})$ is defined as the set consisting of all bases that span $\mathcal{D}$.

(e) The rank of matrix $\mathcal{D}$ is defined to be

$$\mathbf{k} := k(\mathcal{D}) = \dim(\operatorname{ran}(\mathcal{D})). \tag{1.9.3}$$

(f) The set $A(\mathcal{D})$ is defined to be

$$A(\mathcal{D}) := \left\{\mathcal{Z} \subset \mathcal{D} : \operatorname{span}(\mathcal{D}\backslash\mathcal{Z}) \neq \mathbb{R}^d\right\}. \tag{1.9.4}$$

(g) The continuity integer r of $\mathcal{D}$ is defined by

$$r = r(\mathcal{D}) := \min\{\#\mathcal{Z} : \mathcal{Z} \in A(\mathcal{D})\} - 2 := k - 2 \tag{1.9.5}$$

where k is the rank of $\mathcal{D}$ as in (1.9.3).

We proceed to study the smoothness order of box splines based on direction matrices.

Proposition 1.9.1. *According to [dBHR93], the box spline $\mathbb{B}_{\mathcal{D}} := \mathbb{B}(\cdot|\mathcal{D})$, with associated direction matrix $\mathcal{D}$, satisfies $\mathbb{B}_{\mathcal{D}} \in C^r(ran(\mathcal{D}))$, where $r := r(\mathcal{D})$ is as in (1.9.5) and the range of $\mathcal{D}$, $ran(\mathcal{D})$, is usually either $\mathbb{R}^2$ or a subspace of $\mathbb{R}^2$.*

Example

Consider the box spline $\mathbb{B}_{2,2,1}$ with direction matrix

$$\mathcal{D} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} := [\mathrm{v}_1\mathrm{v}_2\mathrm{v}_3\mathrm{v}_4\mathrm{v}_5]$$

in which case $\operatorname{ran}(\mathcal{D}) = \mathbb{R}^2$. Then, using Proposition 1.9.1, together with equation (1.9.1) and (1.9.3), we get

$$\begin{aligned}\chi(\mathcal{D}) &= \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\} \\ &= \{[\mathrm{v}_1\mathrm{v}_4], [\mathrm{v}_1\mathrm{v}_5], [\mathrm{v}_2\mathrm{v}_3], [\mathrm{v}_2\mathrm{v}_4], [\mathrm{v}_2\mathrm{v}_5], [\mathrm{v}_3\mathrm{v}_5]\}\}.\end{aligned}$$

Furthermore, using (1.9.4), we have that

$$A(\mathcal{D}) = \{[\mathrm{v}_1\mathrm{v}_2\mathrm{v}_5], [\mathrm{v}_3\mathrm{v}_4\mathrm{v}_5], [\mathrm{v}_1\mathrm{v}_3\mathrm{v}_4\mathrm{v}_5], [\mathrm{v}_1\mathrm{v}_2\mathrm{v}_4\mathrm{v}_5], [\mathrm{v}_2\mathrm{v}_3\mathrm{v}_4\mathrm{v}_5], [\mathrm{v}_1\mathrm{v}_2\mathrm{v}_3\mathrm{v}_4], \mathcal{D}\}.$$

Thus, as in (1.9.5), the continuity exponent r is given by

$$\begin{aligned} r = r(\mathcal{D}) &:= \min\{\#\mathcal{Z} : \mathcal{Z} \in A(\mathcal{D})\} - 2 \\ &= 3 - 2 = 1. \end{aligned} \tag{1.9.6}$$

Hence, the box spline $\mathbb{B}_{2,2,1}$ is of continuity order $C^1(\mathbb{R}^2)$.

Remark: The way of computing the smoothness class using Proposition 1.9.1 only holds for box splines, and more details are given in [EU10, dBHR93, Pra85]. Getting the smoothness exponent by using this approach is more beneficial than using the direct approach via calculus, which involves integrating box splines in their appropriate directions to enhance the smoothness exponent. Hence, the approach based on Proposition 1.9.1 is a more efficient method since it thoroughly relies on direction matrices associated with box splines.

We finalize this chapter by introducing our objective as follows. As our main work, we give attention to the structure of the refinement mask of an interpolatory Butterfly subdivision polynomial, which is expressed in terms of box spline symbols. We recall that the symbol of a d-variate box spline $\mathbb{B}(\cdot|\mathcal{D})$ with its associated direction matrix $\mathcal{D}$ as given in (1.8.26). Then we basically focus on the interpolatory Butterfly subdivision scheme with its algebraic verification, according to which, the mask symbol of this subdivision scheme is expressed in terms of its box spline constituents, by using (1.8.26) as follows:

$$\begin{aligned}\mathcal{P}_w(z_1,z_2) = 4z_1^{-3}\, z_2^{-3}\, [&7z_1\, z_2\, \mathbb{B}_{2,2,2}(z_1,z_2) - 2z_1\, \mathbb{B}_{1,3,3}(z_1,z_2) - 2z_2\, \mathbb{B}_{3,1,3}(z_1,z_2) \\ &-2z_1\, z_2\, \mathbb{B}_{3,3,1}(z_1,z_2)]\,, \end{aligned} \tag{1.9.7}$$

where $\mathbb{B}_{\mathrm{a,b,c}}(z_1,z_2)$, with $\mathrm{a,b,c} \in \mathbb{N}$, are normalised box spline symbols which are given by

$$\mathbb{B}_{\mathrm{a,b,c}}(z_1,z_2) := \left(\frac{1+z_1}{2}\right)^{\mathrm{a}} \left(\frac{1+z_2}{2}\right)^{\mathrm{b}} \left(\frac{1+z_1 z_2}{2}\right)^{\mathrm{c}}, \tag{1.9.8}$$

with $(z_1,z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$.

Chapter 2

Box splines and subdivision

In this chapter, we study subdivision (refinement) of bivariate box splines by using Fourier analogy as discussed in [dVC10]. We provide study tensor product B-splines as a simple way of constructing bivariate box splines and we give their illustrative graphs. In Section 2.3, bivariate subdivision schemes, especially the interpolatory ones will be discussed. We also study the link between the subdivision algorithm and the cascade algorithms, with the view to establish the existence of interpolatory bivariate refinable functions. Refinement rules for constructing surfaces will be introduced at the end of this section.

2.1 The refinability of box splines

In this section, we apply results from Fourier analysis to investigate the refinability of bivariate box splines.

Definition 2.1.1. *For any piecewise continuous compactly supported function $F : \mathbb{R}^2 \to \mathbb{C}$, its Fourier transform $\hat{F} : \mathbb{R}^2 \to \mathbb{C}$ is defined by*

$$\hat{F}(\mathbf{w}) := \int_{\mathbb{R}^2} e^{-i\mathbf{x}\cdot\mathbf{w}} F(\mathbf{x})\, d\mathbf{x}, \quad \mathbf{w} \in \mathbb{R}^2, \tag{2.1.1}$$

where $\mathbf{x} := (x_1, x_2)$, $\mathbf{w} := (w_1, w_2)$ and $\mathbf{x} \cdot \mathbf{w} = x_1\, w_1 + x_2\, w_2$.

For our result in Theorem 2.1.4 below, we shall require the following lemma from [dVC10] that provides an alternative formulation of bivariate box splines.

Lemma 2.1.2. *For $\mathbb{B}(\mathbf{x}|\mathcal{D}_n)$ as in (1.5.7),(1.5.5) and $n \geq 2$, the following holds:*

$$\int_{\mathbb{R}^2} \mathbb{B}(\mathbf{x}|\mathcal{D}_n)\, f(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_{[0,1]^n} f\left(\sum_{i=1}^{n} t_i\, \boldsymbol{e}^i\right) \mathrm{d}t_1 \ldots \mathrm{d}t_n \quad \textit{for all} \quad f \in C(\mathbb{R}^2). \tag{2.1.2}$$

Proof. The proof is by induction.

For $n = 2$, by applying (1.5.1) and the definition in (1.5.7), we have that, for any $f \in C(\mathbb{R}^2)$,

$$\begin{aligned}\int_{\mathbb{R}^2} \mathbb{B}(\mathbf{x}|\mathcal{D}_2)\, f(\mathbf{x})\, \mathrm{d}\mathbf{x} &= \int_{[0,1]^2} f(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_{[0,1]^2} f(t_1, t_2)\, \mathrm{d}t_1\, \mathrm{d}t_2 \\ &= \int_{[0,1]^2} f(t_1\, \mathbf{e}^1 + t_2\, \mathbf{e}^2)\, \mathrm{d}t_1\, \mathrm{d}t_2,\end{aligned} \tag{2.1.3}$$

where the **e**-vectors in this case are as in (1.5.2).

For $m \geq 3$, since $\mathcal{D}_m = \mathcal{D}_{m-1} \cup \{\mathbf{e}^m\}$, it follows from definition as in (1.5.7) that

$$\mathbb{B}(\mathbf{x}|\mathcal{D}_m) = \int_0^1 \mathbb{B}(\mathbf{x} - t_m\, \mathbf{e}^m|\mathcal{D}_{m-1})\, \mathrm{d}t_m.$$

Hence, applying a simple change of variables of integration and the induction hypothesis consecutively, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} \mathbb{B}(\mathbf{x}|\mathcal{D}_m)\, f(\mathbf{x})\, \mathrm{d}\mathbf{x} &= \int_0^1 \left\{ \int_{\mathbb{R}^2} \mathbb{B}(\mathbf{x} - t_m \mathbf{e}^m|\mathcal{D}_{m-1})\, f(\mathbf{x})\, \mathrm{d}\mathbf{x} \right\} \mathrm{d}t_m \\ &= \int_0^1 \left\{ \int_{\mathbb{R}^2} \mathbb{B}(\mathbf{x}|\mathcal{D}_{m-1})\, f(\mathbf{x} + t_m\, \mathbf{e}^m)\, \mathrm{d}\mathbf{x} \right\} \mathrm{d}t_m \\ &= \int_0^1 \left\{ \int_{[0,1]^{m-1}} f\left(\sum_{i=1}^{m-1} t_i\, \mathbf{e}^i + t_m\, \mathbf{e}^m \right) \mathrm{d}t_1 \ldots \mathrm{d}t_{m-1} \right\} \mathrm{d}t_m \\ &= \int_{[0,1]^m} f\left(\sum_{i=1}^{m} t_i\, \mathbf{e}^i \right) \mathrm{d}t_1 \ldots \mathrm{d}t_m.\end{aligned} \tag{2.1.4}$$

□

Theorem 2.1.3. *The Fourier transform of the box spline $\mathbb{B}(\mathbf{x}|\mathcal{D}_n)$ is given explicitly by*

$$\hat{\mathbb{B}}(\mathbf{w}|\mathcal{D}_n) = \left(\frac{1 - e^{-iw_1}}{iw_1} \right)^k \left(\frac{1 - e^{-iw_2}}{iw_2} \right)^\ell \left(\frac{1 - e^{-i(w_1+w_2)}}{i(w_1 + w_2)} \right)^m \left(\frac{1 - e^{-i(w_1-w_2)}}{i(w_1 - w_2)} \right)^p, \tag{2.1.5}$$

with k, ℓ, m, p denoting the multiplicities of $\{(1,0), (0,1), (1,1), (1,-1)\}$, respectively, that constitute $\mathcal{D}_n$, with n given in (1.5.4).

Proof. The formula (2.1.5) follows directly by choosing $f(\mathbf{x}) = e^{-i\mathbf{x}.\mathbf{w}}$ in (2.1.2) of Lemma 2.1.2, so that

$$\begin{aligned}&\int_{[0,1]^n} e^{-i(t_1\, \mathbf{e}_1 + \ldots + t_n\, \mathbf{e}_n)\cdot \mathbf{w}}\, dt_1 \ldots \mathrm{d}t_n = \int_{[0,1]^n} e^{-i(X+Y+Z+W)\cdot \mathbf{w}}\, dt_1 \ldots \mathrm{d}t_n \\ &= \left(\int_0^1 e^{-itw_1} \mathrm{d}t \right)^k \left(\int_0^1 e^{-itw_2} \mathrm{d}t \right)^\ell \left(\int_0^1 e^{-it(w_1+w_2)} \mathrm{d}t \right)^m \left(\int_0^1 e^{-it(w_1-w_2)} \mathrm{d}t \right)^p \\ &= \left(\frac{1 - e^{-iw_1}}{iw_1} \right)^k \left(\frac{1 - e^{-iw_2}}{iw_2} \right)^\ell \left(\frac{1 - e^{-i(w_1+w_2)}}{i(w_1 + w_2)} \right)^m \left(\frac{1 - e^{-i(w_1-w_2)}}{i(w_1 - w_2)} \right)^p,\end{aligned}$$

where $X := \sum_{j=1}^{k} t_j\, \mathbf{e}_1$, $\quad Y := \sum_{j=k+1}^{k+\ell} t_j\, \mathbf{e}_2$, $\quad Z := \sum_{j=k+\ell+1}^{k+\ell+m} t_j\, \mathbf{e}_3$, $\quad W := \sum_{j=k+\ell+m+1}^{n} t_j\, \mathbf{e}_4$. □

The following result represents a useful property of Fourier transform, which is a key tool for proving refinability of refinable functions, in particularity, box splines. We provide the general proof for d-variate refinable functions. The case $d = 2$ is given in [dVC10].

Theorem 2.1.4. *For any $d \in \mathbb{N}$, let A be an invertible $d \times d$ matrix and denote by A^{-T} the transpose of the inverse of A. Then, for any $\boldsymbol{b} \in \mathbb{R}^d$, the Fourier transform of $G(\mathbf{x}) := \mathrm{g}(A\mathbf{x} - \boldsymbol{b})$, where g is any piecewise continuous function with compact support in $\mathbb{R}^d$, is given by*

$$\hat{G}(\mathbf{w}) = \frac{e^{-i\boldsymbol{b}\cdot A^{-T}\mathbf{w}}}{|\det\ A|}\hat{\mathrm{g}}(A^{-T}\mathbf{w}), \quad \mathbf{w} \in \mathbb{R}^d. \tag{2.1.6}$$

Proof. Since the Jacobian determinant corresponding to the transformation

$\mathbf{y} = A\mathbf{x} - \mathbf{b}$ is given by det A, we have that

$$\begin{aligned}
\hat{G}(\mathbf{w}) &= \int_{\mathbb{R}^d} e^{-i\mathbf{x}\cdot\mathbf{w}}\mathrm{g}(A\mathbf{x} - \mathbf{b})\,\mathrm{d}\mathbf{x} \\
&= \frac{1}{|\det A|}\int_{\mathbb{R}^d} e^{-i(A^{-1}\mathbf{y}+A^{-1}\mathbf{b})\cdot\mathbf{w}}\mathrm{g}(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&= \frac{e^{-i(A^{-1}\mathbf{b})\cdot\mathbf{w}}}{|\det A|}\int_{\mathbb{R}^d} e^{-i(A^{-1}\mathbf{y})\cdot\mathbf{w}}\mathrm{g}(\mathbf{y})\,\mathrm{d}\mathbf{y} \\
&= \frac{e^{-i\mathbf{b}\cdot(A^{-T}\mathbf{w})}}{|\det A|}\hat{\mathrm{g}}(A^{-T}\mathbf{w}),
\end{aligned} \tag{2.1.7}$$

where we have used the fact that

$$(A^{-1}\mathbf{y})\cdot\mathbf{w} = \mathbf{y}\cdot A^{-T}\mathbf{w}, \quad \text{for} \quad \mathbf{w},\mathbf{y} \in \mathbb{R}^d.$$

$\square$

Now, in our case, for $d = 2$, the result (2.1.6) of Theorem 2.1.4 can be applied to the refinement equation

$$\phi(\mathbf{x}) = \sum_{\mathbf{j}\in\mathbb{Z}^2} p_{\mathbf{j}}\,\phi(A\mathbf{x} - \mathbf{j}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{2.1.8}$$

to yield

$$\hat{\phi}(\mathbf{w}) = \left(\frac{1}{|\det A|}\sum_{\mathbf{j}} p_{\mathbf{j}}\,e^{-i\mathbf{j}\cdot A^{-T}\mathbf{w}}\right)\hat{\phi}(A^{-T}\mathbf{w}), \quad \mathbf{w} \in \mathbb{R}^2, \tag{2.1.9}$$

which is therefore the Fourier transform formulation of the refinement equation (2.1.8).

Since any box spline $\mathbb{B}(\cdot|\mathcal{D}_n)$ with $n \geq 2$, is a compactly supported piecewise continuous function according to Theorem 1.8.2 [(**a**), (**c**), (**d**)], we can apply the formula in (2.1.1) and Theorem 2.1.3 to determine the dilation matrix A that satisfies the lattice refinement property $\mathbb{Z}^2 \subseteq A^{-1}\mathbb{Z}^2$ for which the box spline $\mathbb{B}(\cdot|\mathcal{D}_n)$ is refinable with respect to the dilation matrix A, and with refinement mask $\{p_{\mathbf{j}}\}$, provided that

$$\frac{1}{|\det A|}\sum_{\mathbf{j}} p_{\mathbf{j}}\,e^{-i\mathbf{j}\cdot A^{-T}\mathbf{w}} = \frac{\hat{\mathbb{B}}(\mathbf{w}|\mathcal{D}_n)}{\hat{\mathbb{B}}(A^{-T}\mathbf{w}|\mathcal{D}_n)} \tag{2.1.10}$$

is a Laurent polynomial in z_1 and z_2, with $z_1 := e^{-iw_1/2}$ and $z_2 := e^{-iw_2/2}$, where $\mathbf{w} := (w_1, w_2)$, so that equation (2.1.10) is a necessary and sufficient condition for the refinability of box splines $\mathbb{B}(\cdot|\mathcal{D}_n)$ with dilation matrix A and a finitely supported refinement mask $\{p_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^2}$. Next, we mention a result that holds for box splines, $\mathbb{B}(\cdot|\mathcal{D}_n)$ in general.

Proposition 2.1.5. *The box spline $\mathbb{B}(\cdot|\mathcal{D}_n) := \mathbb{B}_{k,\ell,m,p}$ is $2\mathrm{I}_2$-refinable with associated two-scale Laurent mask symbol given by*

$$\sum_{\mathbf{j}\in\mathbb{Z}^2} p_{\mathbf{j}}\, \boldsymbol{z}^{\mathbf{j}} = \left(\frac{1+z_1}{2}\right)^k \left(\frac{1+z_2}{2}\right)^\ell \left(\frac{1+z_1z_2}{2}\right)^m \left(\frac{1+z_1z_2^{-1}}{2}\right)^p, \tag{2.1.11}$$

where $\boldsymbol{z} := (z_1, z_2) = (e^{-iw_1/2}, e^{-iw_2/2})$, with $\mathbf{w} := (w_1, w_2)$.

Proof. The proof proceeds by letting $\mathbf{z}^{\mathbf{j}} := z_1^{j_1} z_2^{j_2}$, where $\mathbf{z} := (z_1, z_2) \in \mathbb{C}^2\backslash\{(0,0)\}$ and $\mathbf{j} := (j_1, j_2) \in \mathbb{Z}^2$. Applying (2.1.3), together with Theorem 2.1.4, we get

$$\begin{aligned} P(\mathbf{z}|2\mathrm{I}_2, \mathcal{D}_n) &:= \frac{1}{|\det\ 2\mathrm{I}_2|} \sum_{\mathbf{j}\in\mathbb{Z}^2} p_{\mathbf{j}}\, \mathbf{z}^{\mathbf{j}} \\ &= \frac{1}{4} \sum_{j_1,j_2\in\mathbb{Z}} p_{j_1,j_2}\, z_1^{j_1} z_2^{j_2} \\ &= \frac{1}{4} \sum_{\mathbf{j}\in\mathbb{Z}^2} p_{\mathbf{j}}\, e^{-i\mathbf{j}\cdot A^{-1}\mathbf{w}}. \end{aligned}$$

From equation (2.1.3) and (1.5.4), we have that

$$\begin{aligned} P(\mathbf{z}|2\mathrm{I}_2, \mathcal{D}_n) &= 2^{-n} \left(\frac{1-e^{-iw_1}}{1-e^{-iw_1/2}}\right)^k \left(\frac{1-e^{-iw_2}}{1-e^{-iw_2/2}}\right)^\ell \left(\frac{1-e^{-i(w_1+w_2)}}{1-e^{-i(w_1+w_2)/2}}\right)^m \left(\frac{1-e^{-i(w_1-w_2)}}{1-e^{-i(w_1-w_2)/2}}\right)^p \\ &= 2^{-n}\left(1+e^{-iw_1/2}\right)^k \left(1+e^{-iw_2/2}\right)^\ell \left(1+e^{-iw_1/2}e^{-iw_2/2}\right)^m \left(1-e^{-iw_1/2}e^{-iw_2/2}\right)^p \\ &= \left(\frac{1+z_1}{2}\right)^k \left(\frac{1+z_2}{2}\right)^\ell \left(\frac{1+z_1z_2}{2}\right)^m \left(\frac{1+z_1z_2^{-1}}{2}\right)^p \\ &= 2^{-n} \sum_{i_1=0}^{k}\sum_{i_2=0}^{\ell}\sum_{i_3=0}^{m}\sum_{i_4=0}^{p} \binom{k}{i_1}\binom{\ell}{i_2}\binom{m}{i_3}\binom{p}{i_4} z_1^{i_1+i_3+i_4} z_2^{i_2+i_3-i_4}, \end{aligned} \tag{2.1.12}$$

which is a Laurent polynomial in $z_1 := e^{-iw_1/2}$ and $z_2 := e^{-iw_2/2}$. Hence, according to (2.1.10), the box spline $\mathbb{B}(\cdot|\mathcal{D}_n) := \mathbb{B}_{k,\ell,m,p}$ is $2\mathrm{I}_2$-refinable. □

The refinement mask $\{p_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^2} = \{p_{j_1,j_2}\}$ of the box spline $\mathbb{B}_{k,\ell,m,p}(\cdot)$ with dilating factor $2\mathrm{I}_2$, namely

$$\mathbb{B}_{k,\ell,m,p}(\mathbf{x}) = \sum_{\mathbf{j}\in\mathbb{Z}^2} p_{\mathbf{j}}\, \mathbb{B}_{k,\ell,m,p}(2\mathbf{x}-\mathbf{j}), \tag{2.1.13}$$

is determined by multiplying (2.1.12) by $\det(2\mathrm{I}_2) = 4$ and then changing $i_1+i_3+i_4, i_2+i_3-i_4$ to j_1, j_2, respectively, to arrive at

$$\sum_{j_1,j_2\in\mathbb{Z}} p_{j_1,j_2} z_1^{j_1} z_2^{j_2}. \tag{2.1.14}$$

The explicit formulation of refinement rules (subdivision) now becomes easy once the refinement masks $\{p_{j_1,j_2}\}$ are known, that is, we multiply (2.1.14) by $z_1^{-\lfloor (k+m+p)/2 \rfloor} \times z_2^{-\lfloor (\ell+m-p)/2 \rfloor}$ to get the centered positive masks:

$$\sum_{j_1,j_2} \tilde{p}_{j_1,j_2}\, z_1^{j_1} z_2^{j_2} = z_1^{-\lfloor (k+m+p)/2 \rfloor} \times z_2^{-\lfloor (\ell+m-p)/2 \rfloor} \sum_{j_1,j_2} p_{j_1,j_2}\, z_1^{j_1}\, z_2^{j_2}. \tag{2.1.15}$$

By applying (2.1.15), we extract the refinement masks for box splines. For instance, using (2.1.11) with dilation matrix $A = 2\mathrm{I}_2$, the box spline $\mathbb{B}_{2,2,2}$ acts as basis function for generating the Loop subdivision scheme and its Laurent polynomial representation for subdivision, as given by

$$P(z_1, z_2) = \left(\frac{1+z_1}{2}\right)^2 \left(\frac{1+z_2}{2}\right)^2 \left(\frac{1+z_1 z_2}{2}\right)^2. \tag{2.1.16}$$

Equation (2.1.16) can be centered by $z_1^{-2} z_2^{-2}$ to give a refinement (subdivision) mask, which is given as the matrix

$$\{\tilde{p}_{j_1,j_2} : -1 \le j_1, j_2 \le 3\} = \frac{1}{64}\begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 2 & 6 & 6 & 2 \\ 1 & 6 & 10 & 6 & 1 \\ 2 & 6 & 6 & 2 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{bmatrix}. \tag{2.1.17}$$

This mask is used for triangulation of control meshes via the Loop subdivision scheme. As it is fully based on the box spline $\mathbb{B}_{2,2,2}$, it is sometimes known as a box spline scheme.

In the previous chapter, we have seen that the ZP element $\mathbb{B}_4$ is $2\mathrm{I}_2$-refinable. In the next result, we further check that the box spline $\mathbb{B}_4$ is refinable with respect to Quincunx dilation $Q := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ by using the Fourier result as in (2.1.10).

Proposition 2.1.6. *The ZP element $\mathbb{B}_4$ is Q-refinable, which is also known as Quincunx-refinable.*

Proof. Using (2.1.1),(2.1.5) and (2.1.10), the proof proceeds as follows.

$$\begin{aligned} \frac{\hat{\mathbb{B}}(\mathbf{w}|\mathcal{D}_4)}{\hat{\mathbb{B}}(Q^{-T}\mathbf{w}|\mathcal{D}_4)} &= \frac{1}{2}\sum_{j,k} p_{j,k}\, e^{-i(j,k)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-T}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}} \\ &= \frac{1}{2}\sum_{j,k} p_{j,k}\, e^{-\frac{i}{2}[(-j+k)w_1 + (j+k)w_2]} \\ &= \frac{1}{2}\sum_{j,k} p_{j,k}\, [e^{-\frac{i}{2}w_1}]^{-j}[e^{-\frac{i}{2}w_1}]^{k}[e^{-\frac{i}{2}w_2}]^{j}[e^{-\frac{i}{2}w_2}]^{k} \\ &= \frac{1}{2}\sum_{j,k} p_{j,k}\, z_1^{-j+k} z_2^{j+k}, \end{aligned} \tag{2.1.18}$$

where $z_1 := e^{-\frac{i}{2}w_1}$ and $z_2 := e^{-\frac{i}{2}w_2}$, respectively. But, according to (2.1.5), with the case $n = 4$, we have that

$$\hat{\mathbb{B}}(w_1, w_2|\mathcal{D}_4) = \left(\frac{1-e^{-iw_1}}{iw_1}\right)\left(\frac{1-e^{-iw_2}}{iw_2}\right)\left(\frac{1-e^{-i(w_1+w_2)}}{i(w_1+w_2)}\right)\left(\frac{1-e^{-i(w_1-w_2)}}{i(w_1-w_2)}\right). \tag{2.1.19}$$

Applying (2.1.19), we obtain

$$\begin{aligned}\hat{\mathbb{B}}\left[\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix} w_1 \\ w_2\end{pmatrix}\right] &= \hat{\mathbb{B}}\left(\frac{-w_1+w_2}{2}, \frac{w_1+w_2}{2}\right) \\ &= \left(\frac{1-e^{-iw_1}}{iw_1}\right)\left(\frac{1-e^{-iw_2}}{iw_2}\right)\left(\frac{1-e^{-i(w_1+w_2)/2}}{i(w_1+w_2)}\right)\left(\frac{1-e^{-i(w_1-w_2)/2}}{i(w_1-w_2)}\right).\end{aligned} \tag{2.1.20}$$

Now, taking the ratio between (2.1.19) and (2.1.20), we obtain

$$\begin{aligned}\frac{\hat{\mathbb{B}}(\mathbf{w}|\mathcal{D}_4)}{\hat{\mathbb{B}}(Q^{-T}\mathbf{w}|\mathcal{D}_4)} &= \left(\frac{1+e^{-\frac{i}{2}(w_1+w_2)}}{2}\right)\left(\frac{1+e^{-\frac{i}{2}(w_1-w_2)}}{2}\right), \\ &= \left(\frac{1+z_1z_2}{2}\right)\left(\frac{1+z_1z_2^{-1}}{2}\right).\end{aligned} \tag{2.1.21}$$

which is indeed a Laurent polynomial in $z_1 := e^{-\frac{i}{2}w_1}$ and $z_2 := e^{-\frac{i}{2}w_2}$, respectively. Hence, according to (2.1.10), the box spline $\mathbb{B}_4$ is Quincunx refinable with its refinement mask obtained by multiplying (2.1.18) by $|\det A| = 2$ and from the coefficients of the symbol in (2.1.21), we get

$$p_{00} = p_{1,1} = p_{2,0} = \frac{1}{2};\ p_{1,-1} = -\frac{1}{2}. \tag{2.1.22}$$

□

The refinability of 4-directional box splines in a general sense is described by the following result.

Theorem 2.1.7. *The 4-directional box spline $\mathbb{B}_{k,\ell,m,p}$, with the conditions $k = m$ and $\ell = p$, is Q-refinable where* $Q := \begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix}$.

Proof. Applying (2.1.10) and Theorem 2.1.4, we obtain the Laurent polynomial

$$\begin{aligned}P(\mathbf{z}|Q,\mathcal{D}_n) &= \frac{1}{|\det Q|}\sum_{\mathbf{j}} p_{\mathbf{j}}\,\mathbf{z}^{\mathbf{j}} = \frac{1}{2}\sum_{j,k} p_{j,k}\, z_1^j\, z_2^k \\ &= \left(\frac{1-e^{-iw_1}}{1-e^{-i(w_1+w_2)/2}} \times \frac{w_1+w_2}{2w_1}\right)^k \left(\frac{1-e^{-iw_2}}{1-e^{-i(w_1-w_2)/2}} \times \frac{w_1-w_2}{2w_2}\right)^\ell \\ &\quad \left(\frac{1-e^{-i(w_1+w_2)}}{1-e^{-iw_1}} \times \frac{w_1}{w_1+w_2}\right)^m \left(\frac{1-e^{-i(w_1-w_2)}}{1-e^{-iw_2}} \times \frac{w_2}{w_1-w_2}\right)^p.\end{aligned} \tag{2.1.23}$$

Now, observe from (2.1.23) that P is a Laurent polynomial in $z_1 = e^{-\frac{i}{2}w_1}$ and

$z_2 = e^{-\frac{i}{2}w_2}$ if and only if $k = m$ and $\ell = p$. This is satisfied when (2.1.23) becomes

$$P(\mathbf{z}|Q, \mathcal{D}_n) = \left(\frac{1 + z_1 z_2}{2}\right)^k \left(\frac{1 + z_1 z_2^{-1}}{2}\right)^\ell .$$

Hence, the refinement mask $\mathbf{p} = \{p_{i,j} : i, j \in \mathbb{Z}\}$ can be computed by multiplying (2.1.23) by $|\det Q| = 2$, or

$$\sum_{j,k} p_{j,k}\, z_1^j z_2^k = 2P(\mathbf{z}|Q, \mathcal{D}_n) = 2^{-(k+\ell)+1} \sum_{i=0}^{k} \sum_{j=0}^{\ell} \binom{k}{i} \binom{\ell}{j} z_1^{i+j} z_2^{i-j},$$

so that

$$p_{\mathbf{j}} := p_{j_1,j_2} = 2^{-(k+\ell)+1} \binom{k}{(j_1 + j_2)/2} \binom{\ell}{(j_1 - j_2)/2}, ;$$

where $\binom{r}{k} := 0$ for $k < 0$, $k > r$ or $k \notin \mathbb{Z}$, according to which

$$p_{\mathbf{j}} = \begin{cases} 2^{-(k+\ell)+1} \binom{k}{i} \binom{\ell}{j}; & \mathbf{j} := Q \begin{bmatrix} i \\ j \end{bmatrix} \in Q\mathbb{Z}^2, \\ 0, & \mathbf{j} \notin Q\mathbb{Z}^2. \end{cases} \tag{2.1.24}$$

Thus, from (2.1.8), (2.1.9) and (2.1.10), the box spline $\mathbb{B}_{k,\ell,m,p}$ is Q-refinable with its refinement mask as in (2.1.24) for $k = m$ and $\ell = p$. □

2.1.1 Box spline subdivision in terms of refinement mask symbols

In this subsection, we only give the intuitive notion of subdivision of box splines in terms of refinement mask symbols. (For details, see the next chapter.)

Given a set of control coefficients $\mathbf{c}^{\mathrm{k}-1}$ on the coarse grid $\frac{1}{2^{\mathrm{k}-1}}\mathbb{Z}^2$, the following algorithm computes a set of control coefficients $\mathbf{c}^{\mathrm{k}}$ on the refined grid $\frac{1}{2^{\mathrm{k}}}\mathbb{Z}^2$ as follows:

- Construct the generating function $C^{\mathrm{k}-1}[\mathrm{x}, \mathrm{y}]$ from $\mathbf{c}^{\mathrm{k}-1}$. Up-sample $C^{\mathrm{k}-1}[\mathrm{x}, \mathrm{y}]$ to yield $C^{\mathrm{k}-1}[\mathrm{x}^2, \mathrm{y}^2]$. Set $C^{\mathrm{k}}[\mathrm{x}, \mathrm{y}] = 4\, C^{\mathrm{k}-1}[\mathrm{x}^2, \mathrm{y}^2]$.
- For each direction vector $\{(\mathrm{a}, \mathrm{b})\} \in \mathcal{D}$, update $C^{\mathrm{k}}[\mathrm{x}, \mathrm{y}]$ via the recurrence

  $$C^{\mathrm{k}}(\mathrm{x}, \mathrm{y}) = \left(\frac{1 + \mathrm{x}^{\mathrm{a}}\, \mathrm{y}^{\mathrm{b}}}{2}\right) C^{\mathrm{k}-1}(\mathrm{x}, \mathrm{y}).$$

  Each multiplication by $\left(\frac{1 + \mathrm{x}^{\mathrm{a}}\, \mathrm{y}^{\mathrm{b}}}{2}\right)$ corresponds to midpoint averaging on $\mathbf{c}^{\mathrm{k}}$ in the direction (a, b).
- Extract the coefficients $\mathbf{c}^{\mathrm{k}}$ of the generating function $C^{\mathrm{k}}[\mathrm{x}, \mathrm{y}]$.

2.2 Tensor products

Tensor products could be regarded as means of constructing bivariate refinable functions, especially the bivariate tensor product box splines from the univariate B splines discussed in Section 1.4.

Definition 2.2.1. *Let ϕ_1 and ϕ_2 be univariate refinable functions with corresponding refinement mask symbols $P_1(z) = \sum_j p_j^{[1]} z^j$ and $P_2(z) = \sum_k p_k^{[2]} z^k$, $z \in \mathbb{C}\backslash\{0\}$ respectively. Then the the tensor product Φ is defined as*

$$\Phi(\mathrm{x}, \mathrm{y}) := \phi_1(\mathrm{x})\, \phi_2(\mathrm{y}), \quad (\mathrm{x}, \mathrm{y}) \in \mathbb{R}^2. \tag{2.2.1}$$

Theorem 2.2.2. *Let ϕ_1 and ϕ_2 be two univariate refinable functions with respective mask sequences $\{p_j^1\}$ and $\{p_j^2\}$. Then the tensor product*

$$\Phi(x, y) = \phi_1(x)\, \phi_2(y)$$

satisfies the following :

(i) *Refinability property :*

Φ is a bivariate $2\mathrm{I}_2$-refinable function with refinement equation

$$\Phi(\mathbf{x}) = \sum_{\boldsymbol{j}} p_{\boldsymbol{j}}\, \Phi(2\mathbf{x} - \boldsymbol{j}), \mathbf{x} \in \mathbb{R}^2,$$

where

$$p_{\boldsymbol{j}} := p_{j_1}^{[1]}\, p_{j_2}^{[2]}, \quad \boldsymbol{j} := (j_1, j_2) \in \mathbb{Z}^2. \tag{2.2.2}$$

(ii) *Normalization (scaling) property :*

$$\int_{\mathbb{R}^2} \Phi(\mathbf{x})\, \mathrm{d}\mathbf{x} = 1.$$

(iv) *Interpolatory property:*

If the refinable functions ϕ_1 and ϕ_2 are both interpolatory as in (1.2.3), then so is the tensor product Φ.

(iv) *Partition of unity property:*

If ϕ_1 and ϕ_2 satisfy the partition of unity condition, then so does Φ, that is,

$$\sum_{\boldsymbol{j}} \Phi(\mathbf{x} - \boldsymbol{j}) = 1, \quad \textit{for} \quad \boldsymbol{x} \in \mathbb{R}^2.$$

Proof. (**i**) From (2.2.1) and the refinability of ϕ_1 and ϕ_2 together with (2.2.2), we obtain for any $\mathbf{x} =: (x, y) \in \mathbb{R}^2$

$$\begin{aligned}
\sum_{\mathbf{j}} p_{\mathbf{j}}\, \Phi(2\mathbf{x} - \mathbf{j}) &= \sum_{j_1} \sum_{j_2} p_{j_1}^{[1]}\, p_{j_2}^{[2]}\, \phi_1(2x - j_1)\phi_2(2y - j_2) \\
&= \left[\sum_{j_1} p_{j_1}^{[1]}\, \phi_1(2x - j_1)\right] \left[\sum_{j_2} p_{j_2}^{[2]}\, \phi_2(2x - j_2)\right] \\
&= \phi_1(x)\, \phi_2(y) = \Phi(\mathbf{x}),
\end{aligned}$$

which completes the proof of (**i**).

(**ii**) Since $\phi_1(\mathrm{x})$ and $\phi_2(\mathrm{y})$ are scaling functions[1], it is immediate from (2.2.1) that

$$\int_{\mathbb{R}^2} \Phi(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi(\mathrm{x},\mathrm{y})\, d\mathrm{x}\, d\mathrm{y} = \left[\int_{\mathbb{R}} \phi_1(\mathrm{x})\, d\mathrm{x}\right] \left[\int_{\mathbb{R}} \phi_2(\mathrm{y})\, d\mathrm{y}\right] = 1.$$

(**iii**)

Preservation of the interpolatory property of the tensor product refinable function Φ is given as follows. We let $\mathbf{j} := (\mathrm{k}, \ell) \in \mathbb{Z}^2$, then

$$\Phi(\mathbf{j}) = \Phi(\mathrm{k}, \ell) := \phi_1(\mathrm{k})\, \phi_2(\ell) = \delta_{\mathrm{k}}\, \delta_\ell = \delta_{\mathrm{k},\ell} = \delta_{\mathbf{j}},$$

so that Φ is interpolatory.

(**iv**)

To check the partition of unity property of Φ, set $\mathbf{x} := (\mathrm{x}, \mathrm{y}) \in \mathbb{R}^2$ and $\mathbf{j} := (\mathrm{k}, \ell) \in \mathbb{Z}^2$. Then we have

$$\begin{aligned}
\sum_{\mathbf{j}} \Phi(\mathbf{x} - \mathbf{j}) &= \sum_{\mathrm{k},\ell} \sum_{\mathrm{i},\mathrm{j}} \mathbf{p}_{\mathrm{i},\mathrm{j}}\, \Phi\left(2(\mathrm{x}-\mathrm{k}) - \mathrm{i}, 2(\mathrm{y}-\ell) - \mathrm{j}\right) \\
&= \sum_{\mathrm{k},\ell} \sum_{\mathrm{i},\mathrm{j}} p^{[1]}_{\mathrm{i}}\, p^{[2]}_{\mathrm{j}}\, \phi_1(2(\mathrm{x}-\mathrm{k}) - \mathrm{i})\, \phi_2(2(\mathrm{y}-\ell) - \mathrm{j}) \\
&= \sum_{\mathrm{k},\ell} \left(\sum_{\mathrm{i}} p^{[1]}_{\mathrm{i}}\, \phi_2(2(\mathrm{x}-\mathrm{k}) - \mathrm{i})\right) \left(\sum_{\mathrm{j}} p^{[2]}_{\mathrm{j}}\, \phi_2(2(\mathrm{y}-\ell) - \mathrm{j})\right) \\
&= \sum_{\mathrm{k},\ell} \phi_1(\mathrm{x}-\mathrm{k})\, \phi_2(\mathrm{y}-\ell) = \sum_{\mathrm{k}} \phi_1(\mathrm{x}-\mathrm{k}) \sum_{\ell} \phi_2(\mathrm{y}-\ell) = 1,
\end{aligned}$$

which shows that Φ satisfies the partition of unity property. □

Next, we give the Laurent polynomial representation for a refinement mask of tensor product refinable function in terms of its constituents.

Remark: Let $\tilde{P}_1$, $\tilde{P}_2$ and P be the mask symbols corresponding to the masks $p^{[1]}_{\mathrm{k}}, p^{[2]}_{\ell}$ and $p_{\mathrm{k},\ell}$ in Theorem 2.2.2. Then it follows from (2.2.2), for any $z_1, z_2 \in \mathbb{C}\backslash\{0\}$, that

$$P(z_1, z_2) = \sum_{\mathrm{k},\ell} p_{\mathrm{k},\ell}\, z_1^{\mathrm{k}}\, z_2^{\ell} = \left(\sum_{\mathrm{k}} p^{[1]}_{\mathrm{k}}\, z_1^{\mathrm{k}}\right) \left(\sum_{\ell} p^{[2]}_{\ell}\, z_2^{\ell}\right) := \tilde{P}_1(z_1) \cdot \tilde{P}_2(z_2). \tag{2.2.3}$$

Examples

(**a**) Consider the shifted hat function $\tilde{h} \in C_0(\mathbb{R})$ as given (1.4.17), which is interpolatory, $2\mathrm{I}_2$-refinable function supported on $[-1, 1]$ with refinement mask $p_{\mathbf{j}} = \{\frac{1}{2}, 1, \frac{1}{2}\}$, and corresponding mask symbol $\tilde{P}_{\tilde{h}}$ given by $\tilde{P}_{\tilde{h}}(z) = \frac{1}{2} z^{-1}(1+z)^2$. It follows from Theorem 2.2.2 that $\Phi = \tilde{h}.\tilde{h} \in C_0(\mathbb{R}^2)$ is interpolatory with associated refinement mask symbol given by

$$\tilde{P}_{\tilde{h}}(z_1, z_2) = \frac{1}{4} z_1^{-1} z_2^{-1} (1+z_1)^2 (1+z_2)^2, \quad (z_1, z_2) \in \mathbb{Z}^2\backslash\{(0,0)\}. \tag{2.2.4}$$

[1] If the refinable function $\phi(x)$ satisfies $\int_{\mathbb{R}} \phi(\mathrm{x})\, dx = 1$, then it is known as a scaling function.

(**b**) Another prominent interplatory refinable function is the Deslauriers-Dubuc (DD) function with refinement mask given by $\{p_j : j \in [-3,3]\} = \left\{-\frac{1}{16}, 0, \frac{9}{16}, 1, \frac{9}{16}, 0, -\frac{1}{16}\right\}$, and its corresponding mask symbol given by

$$P(z) = -\frac{1}{16}z^{-3} + \frac{9}{16}z^{-1} + 1 + \frac{9}{16}z - \frac{1}{16}z^{3}, \quad z \in \mathbb{C}\backslash\{0\}. \tag{2.2.5}$$

By using tensor products and (2.2.5), we generate the bivariate version of DD-function, the tensored Deslauriers-Dubuc function, with graph shown in Figure 2.4. Also, the graphs of the linear, quadratic and cubic tensored spline functions are given in Figures 3.2 and 3.10.

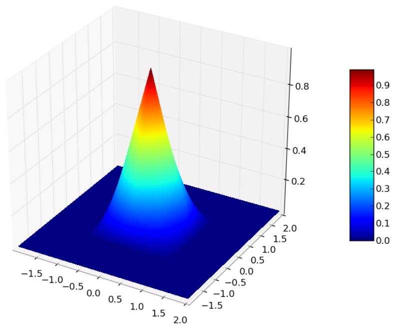

Figure 2.1: *The tensored hat function*

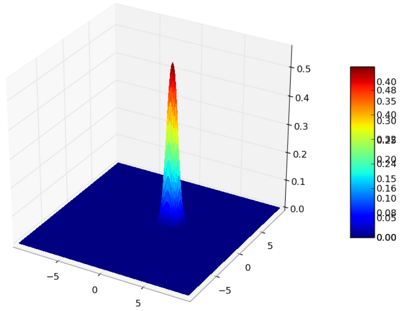

Figure 2.2: *The bilinear quadratic function*

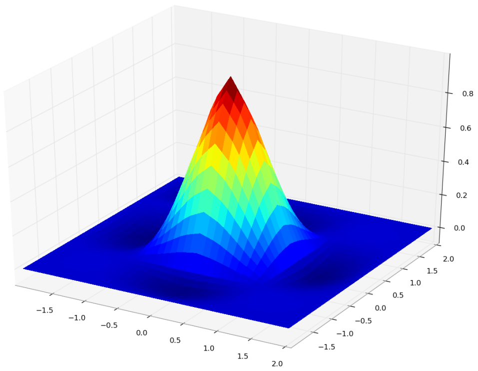

Figure 2.3: *The tensored Deslauriers-Dubuc function*

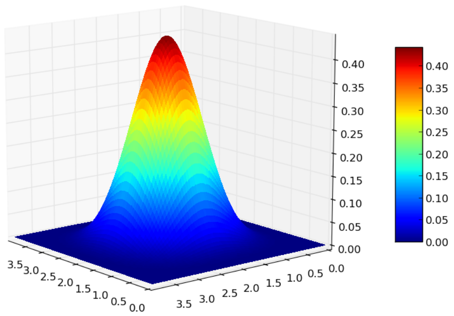

Figure 2.4: *The tensored cubic spline function*

2.3 Bivariate subdivision schemes

In this section, we study bivariate subdivision schemes and their properties.

Definition 2.3.1. *Given $p = \{p_{\boldsymbol{j}}\}_{\boldsymbol{j}\in\mathbb{Z}^2} \in \ell_0(\mathbb{Z}^2)$ and an invertible 2×2 dilation matrix A, the subdivision operator $\mathbb{S}_{\boldsymbol{p}} : \ell(\mathbb{Z}^2) \to \ell(\mathbb{Z}^2)$ is defined for any sequence $\boldsymbol{c} \in \ell(\mathbb{Z}^2)$ by*

$$(\mathbb{S}_{\boldsymbol{p}}\boldsymbol{c})_{\boldsymbol{j}} = \sum_{\boldsymbol{k}} p_{\boldsymbol{j}-A\boldsymbol{k}^T}\, \boldsymbol{c}_{\boldsymbol{k}^T}, \quad \boldsymbol{j} \in \mathbb{Z}^2. \tag{2.3.1}$$

Let $\mathbf{c} = \{\mathbf{c_j} : \mathbf{j} \in \mathbb{Z}^2\} \subset \mathbb{R}^3$ denote a finitely supported control point sequence. Then repeated application of the subdivision operator $\mathbb{S}_\mathbf{p}$ yields the algorithm

$$\left.\begin{aligned} \mathbf{c}^0_\mathbf{j} &:= \mathbf{c_j}; \\ \mathbf{c}^r_\mathbf{j} &:= (\mathbb{S}_\mathbf{p}\mathbf{c}^{r-1})_\mathbf{j} = (\mathbb{S}^r_\mathbf{p}\mathbf{c})_\mathbf{j}, \quad \mathbf{j} \in \mathbb{Z}^2,\ r = 1, 2, \ldots, \end{aligned}\right\} \tag{2.3.2}$$

which is equivalent to

$$\mathbf{c}^r = \mathbb{S}^r_\mathbf{p}\mathbf{c}\,; \quad \mathbb{S}^{r+1}_\mathbf{p}\mathbf{c} = \mathbb{S}_\mathbf{p}(\mathbb{S}^r_\mathbf{p}\mathbf{c}), \qquad r = 1, 2, \ldots, \tag{2.3.3}$$

with the convention that $\mathbb{S}^0_\mathbf{p}$ is the identity operator on $\ell(\mathbb{Z}^2)$, and $\mathbb{S}^1_\mathbf{p} = \mathbb{S}_\mathbf{p}$.

For a given control point sequence $\mathbf{c} = \{\mathbf{c_j} \in \mathbb{R}^3 : \mathbf{j} \in \mathbb{Z}^2\}$, we define the generating function

$$\mathbf{C}(z_1, z_2) := \sum_{i,j\in\mathbb{Z}} \mathbf{c}_{i,j}\, z_1^i\, z_2^j, \tag{2.3.4}$$

with $\mathbf{z} := (z_1, z_2) \in \mathbb{C}^2\backslash\{(0,0)\}$, and let $A = 2\mathrm{I}_2$. Applying equation (2.3.1) and using (2.3.4), as well as (1.3.8), we deduce that, for any $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$,

$$(\mathbb{S}_\mathbf{p}\mathbf{c})(z_1, z_2) = \mathbf{C}(z_1^2, z_2^2)\, P(z_1, z_2), \quad \text{with} \quad \mathbf{z}^2 := (z_1^2, z_2^2), \tag{2.3.5}$$

where $\mathbf{C}(z_1^2, z_2^2)$ is the up-sampled version of the control data $\mathbf{c}$ and $P(z_1, z_2)$ is the subdivision polynomial as in (1.3.8). We provide a proof for equation (2.3.5) in the following result.

Theorem 2.3.2. *The bivariate subdivision formula in terms of mask symbols is as given in* (2.3.5).

Proof.

$$\begin{aligned} (\mathbb{S}_\mathbf{p}\mathbf{c})\,(\mathbf{z}) &= \sum_\mathbf{j} (\mathbb{S}_\mathbf{p}\mathbf{c})_\mathbf{j}\, \mathbf{z}^\mathbf{j} \\ &= \sum_\mathbf{j} \left(\sum_\mathbf{k} p_{\mathbf{j}-2\mathrm{I}_2\mathbf{k}^T}\, \mathbf{c}_{\mathbf{k}^T} \right) \mathbf{z}^\mathbf{j} \\ &= \sum_\mathbf{k} \left(\sum_\mathbf{j} p_{\mathbf{j}-2\mathrm{I}_2\mathbf{k}^T}\, \mathbf{z}^\mathbf{j} \right) \mathbf{c}_{\mathbf{k}^T} \\ &= \sum_\mathbf{k} \left(\sum_\mathbf{j} p_\mathbf{j}\, \mathbf{z}^{\mathbf{j}+2\mathrm{I}_2\mathbf{k}^T} \right) \mathbf{c}_{\mathbf{k}^T} \\ &= \left[\sum_\mathbf{k} \mathbf{c}_{\mathbf{k}^T}\, \mathbf{z}^{2\mathrm{I}_2\mathbf{k}^T} \right] \left[\sum_\mathbf{j} p_\mathbf{j}\, \mathbf{z}^\mathbf{j} \right] \\ &= \left[\sum_{k1,k2} \mathbf{c}_{\mathbf{k}^T} z_1^{2k1} z_2^{2k2} \right] \left[\sum_\mathbf{j} p_\mathbf{j}\, z_1^{j_1}\, z_2^{j_2} \right] \\ &= \mathbf{C}\left(z_1^2, z_2^2\right) P\left(z_1, z_2\right), \end{aligned}$$

$\square$

that is, we proceed to rewrite equation (2.3.5) in terms of sub-masks of the mask p.

To this end, we let

$$\mathcal{E} := \{0,1\}^2 = \{0,1\} \times \{0,1\} = \{(0,0),(0,1),(1,0),(1,1)\} = \mathbb{Z}_2^2, \tag{2.3.6}$$

which is a complete set of representatives of the distinct cosets of $\mathbb{Z}^2/2\mathbb{Z}^2$, with

$$\mathbb{Z}^2 = \bigcup_{\gamma\in\mathcal{E}} \left[\gamma + (2\mathrm{I}_2)\mathbb{Z}^2\right]$$

for $\gamma, \beta \in \mathcal{E}$, so that also $[\gamma + (2\mathrm{I}_2)\mathbb{Z}^2] \cap [\beta + (2\mathrm{I}_2)\mathbb{Z}^2] = \emptyset$. Note that $\mathcal{E}$ is the set of extreme points given by vertices of the unit square $[0,1]^2$ containing $\mathbf{0} := (0,0)$ and $\mathbf{1} := (1,1)$. Then, the four sub-masks $\{p_{\mathbf{e}} : \mathbf{e} \in \mathcal{E}\}$, with associated symbols $\{P_{\mathbf{e}}(\mathbf{z}) : \mathbf{e} \in \mathcal{E}\}$, are defined by

$$P_{\mathbf{e}}(\mathbf{z}) = \sum_{\mathbf{j}\in\mathbb{Z}^2} p_{\mathbf{e}+2\mathbf{j}}\, \mathbf{z}^{\mathbf{j}}, \quad \mathbf{e} \in \mathcal{E}, \tag{2.3.7}$$

where $\mathbf{z}^{\mathbf{j}} := z_1^{j_1}\, z_2^{j_2}$, with $j_1, j_2 \in \mathbb{Z}$.

Using the standard decomposition result

$$P(\mathbf{z}) = \sum_{\mathbf{e}\in\mathcal{E}} \mathbf{z}^{\mathbf{e}}\, P_{\mathbf{e}}(\mathbf{z}^2), \tag{2.3.8}$$

we can rewrite equation (2.3.5) as

$$\begin{aligned}(\mathbb{S}_{\mathbf{p}}\mathbf{c})(z_1,z_2) = {} & \mathbf{C}(z_1^2,z_2^2)\, P_{(0,0)}(z_1^2,z_2^2) + z_2\, \mathbf{C}(z_1^2,z_2^2)\, P_{(0,1)}(z_1^2,z_2^2) \\ & + z_1\, \mathbf{C}(z_1^2,z_2^2)\, P_{(1,0)}(z_1^2,z_2^2) + z_1\, z_2\, \mathbf{C}(z_1^2,z_2^2)\, P_{(1,1)}(z_1^2,z_2^2).\end{aligned} \tag{2.3.9}$$

Equivalently, (2.3.9) is given compactly as

$$(\mathbb{S}_{\mathbf{p}}\mathbf{c})(\mathbf{z}) = \sum_{\mathbf{e}\in\mathcal{E}} \mathbf{z}^{\mathbf{e}}\, \mathbf{C}(\mathbf{z}^2)\, P_{\mathbf{e}}(\mathbf{z}^2), \tag{2.3.10}$$

where $\mathbf{z} := (z_1, z_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$.

Equation (2.3.9) shows that a subdivision step is the result of convolution of the input initial data $\mathbf{c}$ by its sub-mask $p_{\mathbf{e}}$ followed by up-sampling and multiplication by $\mathbf{z}^{\mathbf{e}}$, to produce $\mathbb{S}_{\mathbf{p}}\mathbf{c}$.

A natural question one might ask with respect to the algorithm (2.3.3) is whether, after making repeated iterations of subdivision operator, these sets of control points will converge to a curve or a surface, and will there be a condition that assures such convergence. In order to facilitate this convergence analysis in the norm $\|\mathbf{c}\ \|_\infty := \sup\limits_{\mathbf{j}\in\mathbb{Z}^2} |\mathbf{c}_{\mathbf{j}}|$, we introduce the following criterion for subdivision convergence.

Definition 2.3.3. *The subdivision operator $\mathbb{S}_{\boldsymbol{p}}$ defined in (2.3.1) by a non-trivial sequence $\boldsymbol{p} := \{p_{i,j}\} \in \ell_0(\mathbb{Z}^2)$ is said to provide a convergent subdivision scheme for $\boldsymbol{c} := \{\ \boldsymbol{c}_{\boldsymbol{j}}^0 \in \mathbb{R} |\ \boldsymbol{j} \in \mathbb{Z}^2\} \in \ell_\infty(\mathbb{Z}^2)$, if there exists a non-trivial function $\phi_{\boldsymbol{p}} \in C(\mathbb{R}^2)$ such that*

$$\lim_{r\to\infty} \left|\phi_{\boldsymbol{p}}(2^{-r}\boldsymbol{j}) - p_{\boldsymbol{j}}^{[r]}\right| = 0, \quad r \to \infty, \tag{2.3.11}$$

where $r = 1, 2, \dots,$ *and*

$$\boldsymbol{p}_{\boldsymbol{j}}^{[r]} := (\mathbb{S}_{\boldsymbol{p}}^{r}\delta)_{\boldsymbol{j}}, \tag{2.3.12}$$

with $\boldsymbol{\delta} := \{\delta_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^2\}$ *denoting the Kronecker delta sequence as in* (1.2.5). *We call* $\phi_{\mathbf{p}}$ *the limit function corresponding to* $\mathbb{S}_{\boldsymbol{p}}$*, or basis function associated with the subdivision operator* $\mathbb{S}_{\mathbf{p}}$*, and we shall sometimes write* $\mathbb{S}_{\boldsymbol{p}}^{\infty}\boldsymbol{c}^0 := \phi_{\boldsymbol{p}}$*. Also,* $\mathbb{S}_{\boldsymbol{p}}$ *is said to be a convergent subdivision scheme of order* m*, or in short,* $\mathbb{S}_{\boldsymbol{p}}$ *is* $C^m SS$*, if* $\mathbb{S}_{\boldsymbol{p}}^{\infty}\boldsymbol{c} \in C^m(\mathbb{R}^2)$.

The following result introduces a simple necessary condition for $\mathbb{S}_{\mathbf{p}}$ to be convergent. Our proof uses a similar argument as in [KLY07].

Theorem 2.3.4. *Suppose* $\mathbb{S}_{\boldsymbol{p}}$ *is a convergent subdivision scheme with mask* $\{p_{\boldsymbol{j}}\}$ *and dilation matrix* $A = 2\mathrm{I}_2$*. Then for every* $\gamma \in \mathcal{E}$ *as in* (2.3.6)*, we have that*

$$\sum_{\boldsymbol{j}\in\mathbb{Z}^2} p_{\gamma-2\boldsymbol{j}} = 1. \tag{2.3.13}$$

Proof. Let $\mathbf{c}^0$ be the initial control data, and let $\gamma \in \mathcal{E}$ as in (2.3.6) be arbitrarily fixed. Then, from the definition of subdivision convergence as in (2.3.11), then there exists $\phi \in C(\mathbb{R}^2)$ such that

$$\begin{aligned} &\lim_{k\to\infty}\sup_{\mathbf{j}\in\mathbb{Z}^2}\left|(\mathbb{S}_{\mathbf{p}}^k\mathbf{c}^0)_{\mathbf{j}} - \phi_{\mathbf{p}}[(2\mathrm{I}_2)^{-k}\mathbf{j}]\right| = 0 \\ \Leftrightarrow &\lim_{k\to\infty}\sup_{\alpha_1,\alpha_2\in\mathbb{Z}}\left|(\mathbb{S}_{\mathbf{p}}^k\mathbf{c}^0)_{\alpha_1,\alpha_2} - \phi_{\mathbf{p}}(2^{-k}\alpha_1, 2^{-k}\alpha_2)\right| = 0. \end{aligned} \tag{2.3.14}$$

Let $\mathbf{x}_0 := (x_0, y_0) \in \mathbb{R}^2$ such that $\phi_{\mathbf{p}}(\mathbf{x}_0) \neq 0$. Since $\phi_{\mathbf{p}}$ is continuous, there exists an open neighbourhood K of $\mathbf{x_0}$ such that $\phi_{\mathbf{p}}(\mathrm{K}) \neq 0$. Since also

$$\mathrm{Q} := \left\{(2^{-m}\alpha_1, 2^{-m}\alpha_2) : (\alpha_1, \alpha_2) \in \mathbb{Z}^2,\ m \in \mathbb{Z}\right\}$$

is dense in $\mathbb{R}^2$, we have that $\mathrm{Q} \cap \mathrm{K} \neq \emptyset$, that is, there exists $m \in \mathbb{Z}$ and $(\beta_1, \beta_2) \in \mathbb{Z}^2$ such that $\phi_{\mathbf{p}}(2^{-m}\beta_1, 2^{-m}\beta_2) \neq \emptyset$. For sufficiently large k, we therefore have that

$$\begin{aligned} (\mathbb{S}_{\mathbf{p}}^k\mathbf{c}^0)_{2^{k-m}\alpha_1+\gamma_1, 2^{k-m}\alpha_2+\gamma_2} &= \sum_{\beta_1,\beta_2} p_{2^{k-m}\alpha_1+\gamma_1-2\beta_1, 2^{k-m}\alpha_2+\gamma_2-2\beta_2}\,(\mathbb{S}_{\mathbf{p}}^{k-1}\mathbf{c}^0)_{\beta_1,\beta_2} \\ &= \sum_{\beta_1,\beta_2} p_{2(2^{k-m-1}\alpha_1-\beta_1)+\gamma_1, 2(2^{k-m-1}\alpha_2-\beta_2)+\gamma_2}\,(\mathbb{S}_{\mathbf{p}}^{k-1}\mathbf{c}^0)_{\beta_1,\beta_2} \\ &= \sum_{\beta_1,\beta_2} p_{2\beta_1+\gamma_1, 2\beta_2+\gamma_2}\,(\mathbb{S}_{\mathbf{p}}^{k-1}\mathbf{c}^0)_{2^{k-m-1}\alpha_1-\beta_1, 2^{k-m-1}\alpha_2-\beta_2}. \end{aligned}$$

Referring to equation (2.3.14), we deduce that

$$\begin{aligned} &\phi_{\mathbf{p}}(2^{-m}\alpha_1 + 2^{-k}\gamma_1, 2^{-m}\alpha_2 + 2^{-k}\gamma_2) - \sum_{\beta_1,\beta_2} p_{2\beta_1+\gamma_1,2\beta_2+\gamma_2}\,\phi_{\mathbf{p}}(2^{-m}\alpha_1 - 2^{-k+1}\beta_1, 2^{-m}\alpha_2 - 2^{-k+1}\beta_2) \\ &= \phi_{\mathbf{p}}(2^{-m}\alpha_1 + 2^{-k}\gamma_1, 2^{-m}\alpha_2 + 2^{-k}\gamma_2) - (\mathbb{S}_{\mathbf{p}}^k\mathbf{c}^0)_{2^{k-m}\alpha_1+\gamma_1, 2^{k-m}\alpha_2+\gamma_2} + \\ &\sum_{\beta_1,\beta_2} p_{2\beta_1+\gamma_1,2\beta_2+\gamma_2}\left[(\mathbb{S}_{\mathbf{p}}^{k-1}\mathbf{c}^0)_{2^{k-m-1}\alpha_1-\beta_1, 2^{k-m-1}\alpha_2-\beta_2} - \phi_{\mathbf{p}}(2^{-m}\alpha_1 - 2^{-k+1}\beta_1, 2^{-m}\alpha_2 + 2^{-k+1}\beta_2)\right]. \end{aligned}$$

By taking $k \to \infty$, and using the uniform convergence of $\mathbb{S}$, as well as the continuity of $\phi_{\mathbf{p}}$, we obtain

$$\phi_{\mathbf{p}}(2^{-m}\alpha_1, 2^{-m}\alpha_2) = \sum_{\beta_1,\beta_2} p_{2\beta_1+\gamma_1, 2\beta_2+\gamma_2}\, \phi_{\mathbf{p}}(2^{-m}\alpha_1, 2^{-m}\alpha_2).$$

Since also $\phi_{\mathbf{p}}(2^{-m}\alpha_1, 2^{-m}\alpha_2) \neq 0$, we deduce that

$$\sum_{\beta_1,\beta_2} p_{2\beta_1+\gamma_1, 2\beta_2+\gamma_2} = 1, \quad \text{for} \quad (\gamma_1,\gamma_2) \in \mathcal{E} \text{ as in (2.3.11)}, \tag{2.3.15}$$

and thereby completing our proof. □

Theorem 2.3.5. *For a given sequence $\boldsymbol{p} = \{p_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^2\} \in \ell_0$, with supp $\{p_{\boldsymbol{j}}\} = [\mu,\nu] \times [\alpha,\beta]|_{\mathbb{Z}^2}$, let the sequence $\{p^r_{i,j}\}$, $r = 1, 2, \ldots$ be defined by (2.3.12).*

Then:

(a) The recursive formulation

$$p^{[1]}_{i,j} = p_{i,j}, \quad p^{[r]}_{i,j} = \sum_{k,\ell} p_{k,\ell}\, p^{[r-1]}_{i-2^{r-1}k, j-2^{r-1}\ell} \tag{2.3.16}$$

is satisfied for $r = 2, 3, \ldots$.

(b) For any sequence $\boldsymbol{c} = \{\boldsymbol{c}_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^2\} \in \ell(\mathbb{Z}^2)$ of control points and $r = 1, 2, \ldots$, the subdivision process in (2.3.3) can be formulated as the up-sampling convolution operation

$$(\mathbb{S}^r_{\boldsymbol{p}}\boldsymbol{c})_{i,j} = \sum_{k,\ell} p^{[r]}_{i-2^r k, j-2^r \ell}\, \boldsymbol{c}_{k,\ell}, \quad i,j \in \mathbb{Z}. \tag{2.3.17}$$

(c) If, moreover, $\{p_{\boldsymbol{j}} : \boldsymbol{j} \in \mathbb{Z}^2\}$ satisfies the sum rule condition as in (2.3.13), the condition

$$\sum_{k,\ell} p_{i-2^r k, j-2^r \ell} = 1, \quad (i,j) \in \mathbb{Z}^2, \tag{2.3.18}$$

is satisfied for $r = 1, 2, \ldots$.

Proof. The inductive proof is a direct extension of the one given for the univariate case in [dVC10], Theorem 4.2.1. □

2.4 Cascade operators

Definition 2.4.1. *For a given dilation matrix A and a sequence $\boldsymbol{p} \in \ell_0(\mathbb{Z}^d)$, where $d \in \mathbb{N}$, the cascade operator $C_{\boldsymbol{p}} : M(\mathbb{R}^d) \to M(\mathbb{R}^d)$ is given by*

$$(C_{\boldsymbol{p}} g)(\mathbf{x}) := \sum_{\boldsymbol{j}\in\mathbb{Z}^d} p_{\boldsymbol{j}}\, g(A\mathbf{x} - \boldsymbol{j}), \quad \mathbf{x} \in \mathbb{R}^d, \quad g \in M(\mathbb{R}^d). \tag{2.4.1}$$

We note that $C_{\mathbf{p}}g = g$ if and only if g satisfies the refinement equation

$$g(\mathbf{x}) = \sum_{\mathbf{j}} p_{\mathbf{j}}\, g(\mathbf{x} - \mathbf{j}), \quad \mathbf{x} \in \mathbb{R}^d. \tag{2.4.2}$$

As in the univariate case in [dVC10], the resulting cascade algorithm $C_{\mathbf{p}}$ then generates a sequence of functional iterates $\{h_r : r = 1, 2, \ldots\}$ obtained by means of the following recursion relation:

$$h_0 := h\,; \quad h_r := C_{\mathbf{p}} h_{r-1} = C_{\mathbf{p}}^r h\,, \quad r = 1, 2, \ldots, \tag{2.4.3}$$

where the function $h \in M(\mathbb{R}^d)$ is an initializer of the algorithm. Equation (2.4.3) is equivalently expressed as $h_r := C_{\mathbf{p}} h_{r-1} = C_{\mathbf{p}}^r h, \quad h_{r+1} := C_{\mathbf{p}} h_r$, where

$$C_{\mathbf{p}}^0 h = h\,; \quad C_{\mathbf{p}}^{r+1} h = C_{\mathbf{p}}\left(C_{\mathbf{p}}^r h\right), \quad r = 1, 2, \ldots. \tag{2.4.4}$$

Here, for the case $d = 2$, one can take the initial function $h(\mathbf{x})$ as the tensor product of the symmetric hat function in $\mathbb{R}^2$ with itself, that is,

$$h(\mathbf{x}) = h_0(\mathbf{x}) := \prod_{i=1}^{2} \tilde{B}_2(x_i), \quad \mathbf{x} := (x_1, x_2) \in \mathbb{R}^2, \tag{2.4.5}$$

where $\tilde{B}_2$ is the B-spline of degree 1 with $t \in \mathbb{R}$ as in (1.4.16).

Definition 2.4.2. *The cascade operator $C_{\boldsymbol{p}}$ in (2.4.1) converges on a set $X \subset C_0(\mathbb{R}^d)$ if, for any initial function $h \in X$, for instance as in (2.4.5), there exists a function $g \in C(\mathbb{R}^d)$ such that*

$$\lim_{r\to\infty} \left\|C_{\boldsymbol{p}}^r h - g\right\|_\infty = 0, \quad r = 1, 2, \ldots. \tag{2.4.6}$$

The limit function g will be denoted by $C_{\boldsymbol{p}}^\infty h$.

The following result provides a key relationship between subdivision schemes and the cascade algorithm. It is an extension of the result for the case $d = 2$ given in [FdV11].

Proposition 2.4.3. *Suppose that $\boldsymbol{p}^I = \{p_{\boldsymbol{j}}^I\}$ is an interpolatory mask and let A be any arbitrary integer dilation matrix. Then, for any sequence $\boldsymbol{c} \in \ell(\mathbb{Z}^d)$, and for any $g \in C(\mathbb{R}^d)$, we have that*

$$\sum_{\boldsymbol{j}} (\mathbb{S}_{\boldsymbol{p}^I}^r \boldsymbol{c})_{\boldsymbol{j}}\, g(A^r\mathbf{x} - \boldsymbol{j}) = \sum_{\boldsymbol{j}} \boldsymbol{c}_{\boldsymbol{j}}\, (C_{\boldsymbol{p}^I}^r g)(\mathbf{x} - \boldsymbol{j}) \quad \textit{for} \quad \mathbf{x} \in \mathbb{R}^d \textit{ and } \quad r = 1, 2, \ldots. \tag{2.4.7}$$

In particular, if the delta sequence $\boldsymbol{\delta}$, as given in (1.2.5), is chosen as the sequence $\boldsymbol{c}$, then we get, for any $g \in M(\mathbb{R}^d)$,

$$(\mathcal{C}_{\boldsymbol{p}^I}^r g)(\mathbf{x}) = \sum_{\boldsymbol{j}} (\mathbb{S}_{\boldsymbol{p}^I}^r \delta)_{\boldsymbol{j}}\, g(A^r\mathbf{x} - \boldsymbol{j}) \quad \textit{for} \quad \mathbf{x} \in \mathbb{R}^d, \quad r = 1, 2, \ldots. \tag{2.4.8}$$

Proof. Let $g \in M(\mathbb{R}^d)$ and $\mathbf{c} \in \ell(\mathbb{Z}^d)$. We observe, using (2.3.3) and (2.4.4), that (2.4.7) trivially holds for $r = 0$.

Now, apply (2.3.3) together with (2.3.1) and (2.4.1), to obtain, for any $\mathbf{x} \in \mathbb{R}^d$, and $r \in \mathbb{N}$,

$$\begin{aligned}
\sum_{\mathbf{j}} (\mathbb{S}^r_{\mathbf{p}^I}\mathbf{c})_{\mathbf{j}}\, g(A^r\mathbf{x}-\mathbf{j}) &= \sum_{\mathbf{j}} \left[\sum_{\mathbf{k}} p^I_{\mathbf{j}-A\mathbf{k}^T} (\mathbb{S}^{r-1}_{\mathbf{p}^I}\mathbf{c})_{\mathbf{k}}\right] g(A^r\mathbf{x}-\mathbf{j}) \\
&= \sum_{\mathbf{k}} (\mathbb{S}^{r-1}_{\mathbf{p}^I}\mathbf{c})_{\mathbf{k}} \sum_{\mathbf{j}} p^I_{\mathbf{j}-A\mathbf{k}^T}\, g(A^r\mathbf{x}-\mathbf{j}) \\
&= \sum_{\mathbf{k}} (\mathbb{S}^{r-1}_{\mathbf{p}^I}\mathbf{c})_{\mathbf{k}} \sum_{\mathbf{j}} p^I_{\mathbf{j}}\, g(A^r\mathbf{x}-A\mathbf{k}^T-\mathbf{j}) \\
&= \sum_{\mathbf{k}} (\mathbb{S}^{r-1}_{\mathbf{p}^I}\mathbf{c})_{\mathbf{k}} \sum_{\mathbf{j}} p^I_{\mathbf{j}}\, g(A(A^{r-1}\mathbf{x}-\mathbf{k}^T)-\mathbf{j}) \\
&= \sum_{\mathbf{k}} (\mathbb{S}^{r-1}_{\mathbf{p}^I}\mathbf{c})_{\mathbf{k}} (C_{\mathbf{p}^I} g)(A^{r-1}\mathbf{x}-\mathbf{k}^T) \\
&\;\;\vdots \\
&= \sum_{\mathbf{k}} (\mathbb{S}^0_{\mathbf{p}^I}\mathbf{c})_{\mathbf{k}^T} (C^r_{\mathbf{p}^I} g)(\mathbf{x}-\mathbf{k}^T) \\
&= \sum_{\mathbf{k}^T} \mathbf{c}_{\mathbf{k}}\, (C^r_{\mathbf{p}^I} g)(\mathbf{x}-\mathbf{k}^T) \\
&= \sum_{\mathbf{j}} \mathbf{c}_{\mathbf{j}}\, (C^r_{\mathbf{p}^I} g)(\mathbf{x}-\mathbf{j}), \qquad (2.4.9)
\end{aligned}$$

by virtue of (2.3.3), thereby showing that (2.4.7) holds. Replacing $\mathbf{c}$ by $\boldsymbol{\delta}$ in (2.4.7) yields

$$\sum_{\mathbf{j}} (\mathbb{S}^r_{\mathbf{p}^I}\delta)_{\mathbf{j}}\, g(A^r . -\mathbf{j}) = \sum_{\ell} \delta_\ell\, (C^r_{\mathbf{p}^I} g)(. - \ell) = (C^r_{\mathbf{p}^I} g)(\mathbf{x}), \qquad (2.4.10)$$

and thereby completing our proof of (2.4.8). □

Next, we state the following result which will be used to prove subdivision convergence.

Lemma 2.4.4. *For a mask sequence $\boldsymbol{p} = \{p_{i,j}\} \in \ell_0(\mathbb{Z}^2)$ supported on some rectangular region $\mathrm{R} := [\mu,\nu] \times [\alpha,\beta]|_{\mathbb{Z}^2}$, the sequence $\{h_r : r = 0,1,2,\ldots\}$ generated by the cascade algorithm satisfies the following:*

(i) For $r = 1,2,\ldots,$

$$h_r(x,y) = \sum_{i,j} p^{[r]}_{i,j}\, h(2^r x - i, 2^r y - j). \qquad (2.4.11)$$

(ii) For $r = 1,2,\ldots,$

$$h_r\left(\frac{i}{2^r}, \frac{j}{2^r}\right) = p^{[r]}_{i,j}. \qquad (2.4.12)$$

Proof. (i) First, observe from (2.4.1),(2.4.3) and (2.3.16) that, for $\mathbf{x} \in \mathbb{R}^2$,

$$\begin{aligned}
h_1(x,y) &= \sum_{i,j} p_{i,j}\, h_0(2x-i, 2y-j) \\
&= \sum_{i,j} p^{[1]}_{i,j}\, h(2x-i, 2y-j), \qquad (2.4.13)
\end{aligned}$$

and hence (2.4.11) holds for $r = 1$.
Proceeding inductively, we use (2.3.16), the induction hypothesis, (2.4.1) and (2.4.3), to obtain, for $\mathbf{x} \in \mathbb{R}^2$,

$$\begin{aligned}
\sum_{i,j} p^{[r+1]}_{i,j}\, h(2^{r+1}x - i, 2^{r+1}y - j) &= \sum_{i,j} \left[\sum_{k,\ell} p_{k,\ell}\, p^{[r]}_{i-2^r k, j-2^r \ell}\, h(2^{r+1}x - i, 2^{r+1}y - j) \right] \\
&= \sum_{k,\ell} p_{k,\ell} \left[\sum_{i,j} p^{[r]}_{i-2^r k, j-2^r \ell}\, h(2^{r+1}x - i, 2^{r+1}y - j) \right] \\
&= \sum_{k,\ell} p_{k,\ell} \sum_{i,j} p^{[r]}_{i,j}\, h\left(2^{r+1}x - (i + 2^r k), 2^{r+1}y - (j + 2^r \ell)\right) \\
&= \sum_{k,\ell} p_{k,\ell} \left[\sum_{i,j} p^{[r]}_{i,j}\, h(2^r(2x - k) - i, 2^r(2y - \ell) - j) \right] \\
&= \sum_{k,\ell} p_{k,\ell}\, h_r(2x - k, 2y - \ell) = (C_{\mathbf{p}} h_r)(x) = h_{r+1}(x), \qquad (2.4.14)
\end{aligned}$$

which completes the induction proof of (2.4.11).

(ii) For $k, \ell \in \mathbb{Z}$, and $r \in \mathbb{N}$, by setting $(x, y) = \left(\frac{k}{2^r}, \frac{\ell}{2^r} \right)$ in (2.4.11), and using $h(i, j) = \delta_{i,j}$, $i, j \in \mathbb{Z}$; we obtain $h_r \left(\frac{k}{2^r}, \frac{\ell}{2^r} \right) = p^{[r+1]}_{k,\ell}$, and thereby completing the proof of (2.4.12).

□

Next, we show that convergence of the cascade algorithm implies subdivision convergence for the bivariate case.

Theorem 2.4.5. *Let $p = \{p_{i,j}\} \in \ell_0(\mathbb{Z}^2)$ be a mask sequence supported in a finite rectangular region $\mathrm{R} = [\mu, \nu] \times [\alpha, \beta]|_{\mathbb{Z}^2}$ satisfying the sum rule condition as in (2.3.13). If the corresponding cascade algorithm $C_{\mathbf{p}}$ in (2.4.3) is convergent with limit function $h_{\mathbf{p}}$, then the subdivision scheme $\mathbb{S}_{\mathbf{p}}$ is convergent with limit function $\phi_{\mathbf{p}} := h_p$, with, moreover,*

$$\sup_{i,j} \left| \phi_{\mathbf{p}} \left(\frac{i}{2^r}, \frac{j}{2^r} \right) - p^{[r]}_{i,j} \right| \leq \| h_{\mathbf{p}} - h_r \|_\infty , \qquad (2.4.15)$$

for $r = 1, 2, \ldots$.

Proof. Since the point $\left(\frac{i}{2^r}, \frac{j}{2^r} \right) \in \mathbb{R}^2$ for a fixed r and $\mathbb{Z}^2 \subseteq \mathbb{R}^2$, we have that, for $r = 1, 2, \ldots,$

$$\sup_{(i,j)\in\mathbb{Z}^2} \left| h_{\mathbf{p}} \left(\frac{i}{2^r}, \frac{j}{2^r} \right) - h_r \left(\frac{i}{2^r}, \frac{j}{2^r} \right) \right| \leq \sup_{(x,y)\in\mathbb{R}^2} \left| h_{\mathbf{p}}(x, y) - h_r(x, y) \right| , \qquad (2.4.16)$$

and thus, by applying also (2.4.12), we get

$$\sup_{i,j} \left| h_{\mathbf{p}} \left(\frac{i}{2^r}, \frac{j}{2^r} \right) - p^{[r]}_{i,j} \right| \leq \| h_{\mathbf{p}} - h_r \|_\infty . \qquad (2.4.17)$$

From cascade convergence, the right hand side of equation (2.4.17) tends to 0 as $r \to \infty$. Hence the subdivision algorithm converges to the non-trivial function $\phi_p := h_{\mathbf{p}}$. This completes the proof. □

Remark

For $d \in \mathbb{N}$, convergence of the cascade algorithm on $C_0(\mathbb{R}^d)$ therefore implies the associated subdivision scheme to be convergent.

2.5 Sum rules-in terms of symbols

In this section, we obtain a necessary condition for subdivision convergence, namely the sum rule of order 1, as given by

$$P_{\mathbf{e}}(\mathbf{1}) = \sum_{\mathbf{j}\in\mathbb{Z}^2} p_{\mathbf{e}+2\mathbf{j}}, \quad \forall \mathbf{e} \in \mathcal{E}, \tag{2.5.1}$$

with $P_{\mathbf{e}}$ and $\mathcal{E}$ given as in, respectively, (2.3.7) and (2.3.6). According to [CCJZ11], the convergence of a bivariate subdivision scheme is determined by the properties of the infinite product $\prod_{j=0}^{\infty} \frac{1}{4} P(\mathbf{z}^{2^{-j}})$, with $\mathbf{z}$ restricted to the torus

$$T := \left\{ (e^{ix}, e^{iy}) : x, y \in \mathbb{R} \right\}. \tag{2.5.2}$$

In order to switch to real variables $\boldsymbol{\xi} := (\xi_1, \xi_2)$, we set $z_j := e^{-i\pi\xi_j}$, with $j = 1, 2$, so that the set $\mathcal{E} = \{0,1\}^2$ as in (2.3.6) transforms to the set

$$\mathrm{Z} = \mathrm{Z}_{\mathcal{E}} = \left\{ \varepsilon = \mathrm{e}^{-\mathrm{i}\pi\mathbf{e}} : \mathbf{e} \in \mathcal{E} \right\} = \{-1, 1\}^2 \tag{2.5.3}$$

of vertices of the square $[-1,1]^2$, and the necessary condition (2.5.1) takes the equivalent form

$$P(1,1) = 4\,; \quad P(\varepsilon_1, \varepsilon_2) = 0 \;\text{ for }\; (\varepsilon_1, \varepsilon_2) \in \mathrm{Z}' := \mathrm{Z}\backslash\{(1,1)\}. \tag{2.5.4}$$

For this reason, we call Z' the zero set, and the condition in (2.5.4) is known as the sum rule condition of order one, to be denoted by Z_1.

2.6 Interpolatory subdivision

Recall from Definition 1.2.4 that ϕ^I is a $2\mathrm{I}_2$-interpolatory bivariate refinable function if it satisfies the refinement equation (1.2.1), together with the condition

$$\phi^I(0,0) = 1\,; \quad \phi^I(i,j) = 0, \quad (i,j) \in \mathbb{Z}^2\backslash\{(0,0)\}, \tag{2.6.1}$$

whereas, according to (1.3.10), its corresponding interpolatory mask $\{p^I_{\mathbf{j}}\}$ must satisfy

$$p^I_{0,0} = 1 \quad ; \quad p^I_{2i,2j} = 0, \quad i, j \in \mathbb{Z}\backslash\{0\}. \tag{2.6.2}$$

If, moreover, ϕ^I is continuous, we shall call ϕ^I a fundamental refinable function, as in [Jia95, HJ98].

By considering the dilation matrix $A = 2\mathrm{I}_2$, we observe that the interpolatory condition as in (1.3.7) together with (2.3.1) takes the form

$$\mathbf{c}_{\mathbf{j}} := (\mathbb{S}_{\mathbf{p}^I}\mathbf{c})_{2\mathrm{I}_2\,\mathbf{j}^T} = (\mathbb{S}_{\mathbf{p}^I}\mathbf{c})_{2i,2j}, \quad \mathbf{j} = (i,j) \in \mathbb{Z}^2. \tag{2.6.3}$$

In this case, using (2.3.1), together with (2.3.3), and by induction on $r = 1, 2, \ldots,$ we also have that

$$\mathbf{c}^r_{\mathbf{j}} := \mathbf{c}^{r+1}_{2\mathrm{I}_2\,\mathbf{j}^T} = \mathbf{c}^{r+1}_{2i,2j}, \quad \mathbf{j} \in \mathbb{Z}^2, \tag{2.6.4}$$

which means that, at each level of iteration, the subdivision scheme keeps all points at the previous iteration step and hence it is interpolatory.

Theorem 2.6.1. *Suppose that a refinement mask $\boldsymbol{p} = \{p_{i,j}\}$ satisfies*

$$\sum_{i,j} p_{i,j} = 4. \tag{2.6.5}$$

Then $\mathbf{p}^I$ is interpolatory if and only if the corresponding mask symbol P^I, as in (1.2.3), satisfies

$$P^I(z_1, z_2) + P^I(-z_1, z_2) + P^I(z_1, -z_2) + P^I(-z_1, -z_2) = 4, \tag{2.6.6}$$

for any $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0, 0)\}$.

Proof. For any $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0, 0)\}$, we have from (1.2.3) that

$$\begin{aligned} P^I(z_1, z_2) + P^I(-z_1, z_2) &= \Bigg[\sum_{i,j} p^I_{2i,j}\, z_1^{2i} z_2^j + \sum_{i,j} p^I_{2i+1,j}\, z_1^{2i+1} z_2^j \\ &\quad + \sum_{i,j} p^I_{2i,j}\, z_1^{2i} z_2^j - \sum_{i,j} p^I_{2i+1,j}\, z_1^{2i+1} z_2^j\Bigg] \\ &= 2\sum_{i,j} p^I_{2i,j}\, z_1^{2i} z_2^j \\ &= 2\sum_{i,j} p^I_{2i,2j}\, z_1^{2i} z_2^{2j} + 2\sum_{i,j} p^I_{2i,2j+1}\, z_1^{2i} z_2^{2j+1}, \end{aligned} \tag{2.6.7}$$

and similarly,

$$P^I(z_1, -z_2) + P^I(-z_1, -z_2) = \left[2\sum_{i,j} p^I_{2i,2j}\, z_1^{2i} z_2^{2j} - 2\sum_{i,j} p^I_{2i,2j+1}\, z_1^{2i} z_2^{2j+1}\right]. \tag{2.6.8}$$

By adding (2.6.7) and (2.6.8), we get

$$P^I(z_1, z_2) + P^I(-z_1, z_2) + P^I(z_1, -z_2) + P^I(-z_1, -z_2) = 4\sum_{i,j\in\mathbb{Z}} p^I_{2i,2j}\, z_1^{2i}\, z_2^{2j}, \tag{2.6.9}$$

for any $(z_1, z_2) \in \mathbb{C}^2 \backslash \{(0, 0)\}$. □

According to Theorem 2.6.1, for a bivariate refinable function ϕ^I to satisfy the interpolatory condition $\phi^I(\mathbf{j}) = \delta_{\mathbf{j}}$, $\mathbf{j} \in \mathbb{Z}^2$, we have

$$P^I(z_1, z_2) + P^I(-z_1, z_2) + P^I(z_1, -z_2) + P^I(-z_1, -z_2) = 4. \tag{2.6.10}$$

This is the necessary condition for existence of an interpolatory bivariate refinable function.

Also, observe that the necessary condition for convergence of interpolatory subdivision schemes is closely related to equation 2.5.1, according to the following result.

Proposition 2.6.2. *Assuming that an interpolatory refinable function ϕ^I exists and satisfies the refinement equation (1.2.1), with a finitely supported refinement mask $\boldsymbol{p}^I$ and dilation matrix $A = 2\mathrm{I}_2$, then the refinement mask $\boldsymbol{p}^I$ must satisfy*

$$\sum_{\mathbf{j}\in\mathbb{Z}^2} p_{\mathbf{j}} = 4 \Leftrightarrow \sum_{i,j} p^I_{2i,2j} = \sum_{i,j} p^I_{2i+1,2j+1} = \sum_{i,j} p^I_{2i,2j+1} = \sum_{i,j} p^I_{2i+1,2j} = 1, \tag{2.6.11}$$

and equation (2.6.2) holds as well.

The condition in (2.6.11) is equivalent to the one in (2.5.1), which is the sum rule condition. It is a known fact that (2.6.11) is a necessary condition for the convergence of a subdivision scheme associated with refinement mask $\mathbf{p}$, that is, subdivision scheme convergence implies that (2.6.11) holds. Moreover, if the subdivision scheme converges, then the interpolatory mask given in (2.6.2) is equivalent to the interpolatory property that $\phi^I(0,0) = 1$ and $\phi^I(i,j) = 0$ for $i,j \in \mathbb{Z}\backslash\{0\}$ as pointed in Proposition 1.3.1.

Remark: In this study, we did not focus on how to construct interpolatory bivariate refinable functions from refinement masks. In this regard, one can consult [HJ97], [HJ98], [Jia00] and [RS96] for more details. In the next section, we give some results on the convergence of bivariate subdivision schemes.

2.7 Subdivision scheme convergence

Assuming that an interpolatory refinable function exists, we concentrate our focus next on the convergence of the corresponding interpolatory subdivision scheme. Here, we recall the fact that a dilation matrix A is a linear map that defines a bijection from the set of rational pairs $\mathbb{Q}^2$ into itself, so that the dyadic set Ξ, which is defined by

$$\Xi := \left\{A^{-r}\mathbf{j}^T \;:\; \mathbf{j}\in\mathbb{Z}^2,\; r=1,2,\ldots\right\} = \left\{\left(\frac{i}{2^r},\frac{j}{2^r}\right),\; r=1,2,\ldots,\; i,j\in\mathbb{Z}\right\}, \tag{2.7.1}$$

is dense in $\mathbb{R}^2$. Using the Laurent polynomial formulation for the control sequence $\mathbf{c}^k$, which is

$$\mathbf{C}^r(z_1,z_2) = \sum_{i,j} \mathbf{c}_{i,j} z_1^i\, z_2^j, \tag{2.7.2}$$

we again apply (2.3.5) inductively, as was done in [DHL11], to obtain, in the notation of (2.7.2),

$$\begin{aligned} C^{k+n}(\mathbf{z}) &= P(\mathbf{z})\,P(\mathbf{z}^2)\;\ldots P(\mathbf{z}^{2^{n-1}})\,C^k(\mathbf{z}^{2^n}) := P^{[n]}(\mathbf{z})\,C^k(\mathbf{z}^{2^n}) \\ \Leftrightarrow C^{k+n}(z_1,z_2) &= P(z_1,z_2)\,P(z_1^2,z_2^2)\ldots P(z_1^{2^{n-1}},z_2^{2^{n-1}})\,C^k(z_1^{2^n},z_2^{2^n}) \\ &:= P^{[n]}(z_1,z_2)\,C^k(z_1^{2^n},z_2^{2^n}), \quad \mathbf{z} := (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\}, \end{aligned} \tag{2.7.3}$$

where $P(z_1,z_2)$ is as in (1.2.3). Comparing the coefficients of equal powers of $\mathbf{z}$ on both sides of (2.7.3), yields 2^{2n} different rules mapping C^k to C^{k+n}, which are determined by the coefficients of

$$\begin{aligned} P^{[n]}(\mathbf{z}) &= \prod_{j=0}^{n-1} P(\mathbf{z}^{2^j}) = \sum_{j\in\mathbb{Z}} p_j^{[n]}\,\mathbf{z}^j, \\ \Leftrightarrow P^{[n]}(\mathbf{z}) &= P(\mathbf{z})\,P(\mathbf{z}^2)\ldots P(\mathbf{z}^{2^{n-1}}) = \sum_{\mathbf{j}\in\mathbb{Z}^2} p_{\mathbf{j}}^{[n]}\mathbf{z}^{\mathbf{j}}, \end{aligned} \tag{2.7.4}$$

and thus,

$$\mathbf{c}^{k+n}_{\gamma+2^n\alpha} = \sum_{\beta\in\mathbb{Z}^2} p^{[n]}_{\gamma+2^n\beta}\,\mathbf{c}^k_{\alpha-\beta}, \quad \gamma \in \{0,1,\ldots,2^n-1\}^2\,. \tag{2.7.5}$$

The norm of the operator $\mathbb{S}^n_{\mathbf{p}}$ is therefore given by

$$\left\|\mathbb{S}^n_{\mathbf{p}}\right\|_\infty := \max\left\{\sum_{\beta\in\mathbb{Z}^2}\left|p^{[n]}_{\gamma+2^n\beta}\right|,\ \gamma\in\{0,1,\ldots,2^n-1\}^2\right\}. \tag{2.7.6}$$

This equation will help us to study the contractivity condition for subdivision convergence.

Our next result extends a convergence result in univariate interpolatory subdivision, as proved in [Olo10], to the bivariate case.

We give the equivalent bivariate analogue based on the univariate approach, as follows.

Theorem 2.7.1. *Let ϕ^I denote an interpolatory $2\mathrm{I}_2$-refinable function with refinement mask $\{p_{i,j}\in\mathbb{R}:\ i,j\in\mathbb{Z}\}$. The interpolatory subdivision scheme given in (2.6.3) and (2.6.4) then satisfies, for any control point sequence $\boldsymbol{c} := \{\boldsymbol{c}_{i,j}\}\in\ell(\mathbb{Z}^2)$, with $\boldsymbol{c}_{i,j}\in\mathbb{R}^3$, $i,j\in\mathbb{Z}$,*

$$\boldsymbol{c}^r_{i,j} = \varphi_{\boldsymbol{c}}\left(\frac{i}{2^r},\frac{j}{2^r}\right), \quad i,j\in\mathbb{Z}, \tag{2.7.7}$$

with

$$\varphi_c(x,y) := \sum_{i,j} c_{i,j}\,\phi^I(x-i,y-j), \quad r=0,1,2,\ldots. \tag{2.7.8}$$

Also, $\varphi_{\boldsymbol{c}}:\mathbb{R}^2\to\mathbb{R}^3$ is a continuous function on $\mathbb{R}^2$, and $\varphi_{\boldsymbol{c}}$ represents the limit surface of interpolatory subdivision scheme (2.6.3), in the sense that, for any $(x,y)\in\mathbb{R}^2$, and a sequence $\{(i_r,j_r),\ r=0,1,2,\ldots\}\subseteq\mathbb{Z}^2$ such that

$$\left\|(x,y)-\left(\frac{i_r}{2^r},\frac{j_r}{2^r}\right)\right\|_\infty \to 0\,, \quad r\to\infty, \tag{2.7.9}$$

we have

$$\left|\varphi_{\boldsymbol{c}}(x,y)-\boldsymbol{c}^r_{i,j}\right| \to 0, \quad r\to\infty. \tag{2.7.10}$$

Proof. First, we use (2.7.8), together with the refinement equation for $\phi_{\mathbf{c}}$, and eventually

(2.6.1), to obtain

$$\begin{aligned}
\varphi_{\mathbf{c}}\left(\frac{i}{2^r},\frac{j}{2^r}\right) &= \sum_{k,\ell}\mathbf{c}^0_{k,\ell}\,\phi^I\left(\frac{i}{2^r}-k,\frac{j}{2^r}-\ell\right)\\
&= \sum_{k,\ell}\mathbf{c}^0_{k,\ell}\sum_{q,s}p_{q,s}\,\phi^I\left(\frac{i}{2^{r-1}}-2k-q,\frac{j}{2^{r-1}}-2\ell-s\right)\\
&= \sum_{k,\ell}\mathbf{c}^0_{k,\ell}\sum_{q,s}p_{q-2k,s-2\ell}\,\phi^I\left(\frac{i}{2^{r-1}}-q,\frac{j}{2^{r-1}}-s\right)\\
&= \sum_{q,s}\left[p_{q-2k,s-2\ell}\,\mathbf{c}^0_{k,\ell}\right]\,\phi^I\left(\frac{i}{2^{r-1}}-q,\frac{j}{2^{r-1}}-s\right)\\
&= \sum_{q,s}\mathbf{c}^1_{q,s}\,\phi^I\left(\frac{i}{2^{r-1}}-q,\frac{j}{2^{r-1}}-s\right)\\
&\;\;\vdots\\
&= \sum_{q,s}\mathbf{c}^r_{q,s}\,\phi^I\left(i-q,j-s\right)\\
&= \sum_{q,s}\mathbf{c}^r_{q,s}\,\delta_{i-q,j-s}\\
&= \mathbf{c}^r_{i,j},
\end{aligned}$$

which proves (2.7.7). Since ϕ^I is compactly supported, the summation in (2.7.8) is over a finite number of indices i,j for any fixed $\mathbf{x}:=(x,y)\in\mathbb{R}^2$. Since ϕ^I is continuous on each $\mathbf{x}\in\mathbb{R}^2$, it follows that $\varphi_{\mathbf{c}}$ is continuous on $\mathbb{R}^2$.

Let $\mathbf{x}:=(x,y)\in\mathbb{R}^2$ be fixed. Since the dyadic set Ξ as in (2.7.1), is dense in $\mathbb{R}^2$, there exists a sequence of points $\{(i_r,j_r)\,,\ r=0,1,2,\ldots\}$ such that (2.7.10) is satisfied.

By using (2.7.7), we get, for any $r=0,1,2,\ldots,$

$$\left|\varphi_{\mathbf{c}}(x,y)-\mathbf{c}^r_{(i,j)}\right| = \left|\varphi_{\mathbf{c}}(x,y)-\varphi_{\mathbf{c}}\left(\frac{i}{2^r},\frac{i}{2^r}\right)\right| \to 0, \quad r\to\infty, \tag{2.7.11}$$

and thereby completing our proof. □

As an example, we choose $\phi(\mathbf{x}):=\prod_{i=1}^{2}\tilde{B}_2(x_i)$, where $\tilde{B}_2$ is defined as in (1.4.17), and it follows that, for any $\mathbf{k}:=(k,\ell)\in\mathbb{Z}^2$, we have

$$\varphi_{\mathbf{c}}(\mathbf{k})=\varphi_{\mathbf{c}}(k,\ell)=\sum_{i,j}\mathbf{c}_{i,j}\,h(k-i,\ell-j)=\sum_{i,j}\mathbf{c}_{i,j}\,\delta_{k-i,\ell-j}=\mathbf{c}_{\mathbf{k}}.$$

That is, if the control point sequence $\mathbf{c}:=\{\mathbf{c}_{i,j}\}\in\ell(\mathbb{Z}^2)$, then $\varphi_{\mathbf{c}}$ is the bilinear interpolant of the sequence $\{\mathbf{c}_{i,j}\in\mathbb{R}^3 : (i,j)\in\mathbb{Z}^2\}$, and according to (2.7.8),(2.7.9),(2.7.10), the sequences $\{\mathbf{c}^{(r)}_{i,j}\}_{r=0,1,2,\ldots}$ fills up the limit surface.

Corollary 2.7.2. *Suppose that $\mathbb{B}_{\mathcal{D}}$ is an interpolatory box spline with its interpolatory mask $\boldsymbol{p}^I\in\ell_0(\mathbb{Z}^2)$, and with dilation matrix $A=2\mathrm{I}_2$. Then, for any initial sequence*

$\boldsymbol{c} \in \ell(\mathbb{Z}^2)$, the corresponding interpolatory subdivision scheme is convergent, with the limit function φ defined by

$$\varphi_{\boldsymbol{c}}(\mathbf{x}) = \sum_{\boldsymbol{j}} \boldsymbol{c}_{\boldsymbol{j}} \mathbb{B}_{\mathcal{D}}(x - \boldsymbol{j}), \quad \mathbf{x} \in \mathbb{R}^2, \quad \boldsymbol{j} \in \mathbb{Z}^2. \tag{2.7.12}$$

Example

Consider the bivariate shifted box spline $\hat{\mathbb{B}}_2$ given in (1.6.22), with corresponding interpolatory mask $\tilde{\mathbf{p}}^I$, given by, as in (1.6.26)

$$\left.\begin{array}{ll} \tilde{p}^2_{0,0} = 1, & \tilde{p}^2_{1,1} = \tilde{p}^2_{0,-1} = \tilde{p}^2_{0,1} = \tilde{p}^2_{1,0} = \tilde{p}^2_{-1,0} = \tilde{p}^2_{-1,-1} = \dfrac{1}{2}; \\ \tilde{p}^2_{i,j} = 0, & (i,j) \notin \{(0,0),(0,1),(1,0),(-1,0),(0,-1),(1,1),(-1,-1)\}. \end{array}\right\} \tag{2.7.13}$$

Implementing Corollary 2.7.2, the subdivision scheme $\mathbb{S}^{(2)}_{\tilde{\mathbf{p}}^I}$ is convergent. Therefore, we can see that, for any initial sequence $\mathbf{c} \in \ell(\mathbb{Z}^2)$, the limit function $\varphi = \mathbb{S}^{\infty}_{\tilde{\mathbf{p}}}\mathbf{c}$ exists.

2.8 Surface refinement rules

In order to construct a subdivision surface, we take grid indexing points from $\mathbb{Z}^2$ and control points from $\mathbb{R}^3$, and then apply the subdivision operator (2.3.1). Since we consider schemes where each component of the surface is a scalar function generated by the same subdivision scheme, it is sufficient to analyse control points in $\mathbb{R}$, as pointed out in [DHL11].

The set of control vectors $\{\mathbf{c}^k_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^2} \in \mathbb{R}$ generated by the binary subdivision scheme as in (2.3.1) is given by

$$\mathbf{c}^k_{\mathbf{j}} = \sum_{\mathbf{i}\in\mathbb{Z}^2} p_{\mathbf{j}-2\mathbf{i}}\, \mathbf{c}^{k-1}_{\mathbf{i}}, \quad \mathbf{j} \in \mathbb{Z}^2,\ k = 1, 2, \ldots. \tag{2.8.1}$$

For surface construction based on (2.8.1), we have four different refinement rules depending on the parity of each component of the vector $\mathbf{j} \in \mathbb{Z}^2$. The surface refinement rules, according to equation (2.3.1) and (2.3.3), are given by

$$\mathbf{c}^{k+1}_{\gamma+2\alpha} = \sum_{\beta\in\mathbb{Z}^2} p_{\gamma+2\alpha-2\beta}\, \mathbf{c}^k_{\beta} = \sum_{\beta\in\mathbb{Z}^2} p_{\gamma+2\beta}\, \mathbf{c}^k_{\alpha-\beta}, \quad \gamma \in \mathcal{E}, \tag{2.8.2}$$

where $\mathcal{E}$ is as in (2.3.6). Equation (2.8.2) is given explicitly by

$$\left.\begin{array}{l} \mathbf{c}^{k+1}_{2\alpha_1,2\alpha_2} = \displaystyle\sum_{\beta_1,\beta_2} p_{2\alpha_1-\beta_1,2\alpha_2-\beta_2}\ \mathbf{c}^k_{\beta_1,\beta_2}; \\ \mathbf{c}^{k+1}_{2\alpha_1,2\alpha_2+1} = \displaystyle\sum_{\beta_1,\beta_2} p_{2\alpha_1-\beta_1,2\alpha_2+1-\beta_2}\ \mathbf{c}^k_{\beta_1,\beta_2}; \\ \mathbf{c}^{k+1}_{2\alpha_1+1,2\alpha_2} = \displaystyle\sum_{\beta_1,\beta_2} p_{2\alpha_1+1-\beta_1,2\alpha_2-\beta_2}\ \mathbf{c}^k_{\beta_1,\beta_2}; \\ \mathbf{c}^{k+1}_{2\alpha_1+1,2\alpha_2+1} = \displaystyle\sum_{\beta_1,\beta_2} p_{2\alpha_1+1-\beta_1,2\alpha_2+1-\beta_2}\ \mathbf{c}^k_{\beta_1,\beta_2}. \end{array}\right\} \tag{2.8.3}$$

We shall implement equation (2.8.3) to study the Butterfly subdivision scheme in the next chapter.

Chapter 3

Algebraic study of subdivision schemes

In this chapter, we study bivariate subdivision schemes by mainly focusing on the well known interpolatory Butterfly surface subdivision scheme. In particular, we employ an alternative algebraic approach, following the work of [CCJZ11], to construct subdivision polynomials for a class of bivariate subdivision schemes, including the Butterfly subdivision scheme, from bivariate box splines.

The motive of introducing algebraic concepts is to systematically study and construct bivariate subdivision schemes from the box splines of Chapter 1. These box splines act as generators for surface subdivision schemes.

3.1 Algebraic preliminaries

3.1.1 Preliminary definitions

We begin with basic definitions and we quote some important results.

Definition 3.1.1. *Let $\mathfrak{R}$ be a commutative ring. A Laurent polynomial over a ring $\mathfrak{R}$ is a finite linear combination of Laurent monomials with coefficients in $\mathfrak{R}$. The collection of all Laurent polynomials in the variables $z_1, \ldots, z_n$ with coefficients in $\mathfrak{R}$ is denoted by $\mathfrak{R}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, and is a commutative ring.*

Definition 3.1.2. *Let $\mathbb{A} := \mathbb{K}[z_1, \ldots, z_n]$ be a polynomial ring in n variables. A subset $\mathcal{I} \subseteq \mathbb{A}$ is said to be an ideal if it satisfies*

(a) $0 \in \mathcal{I}$*;*

(b) If $f, g \in \mathcal{I}$*, then* $f + g \in \mathcal{I}$*;*

(c) If $f \in \mathcal{I}$ *and* $h \in \mathbb{A}$*, then* $fh = hf \in \mathcal{I}$*.*

We also introduce the following for our purpose.

- Let $\mathbb{K}$ be a field, and let $f_1, \ldots, f_r \in \mathbb{A}$. We define the affine variety $\mathbf{V}(f_1, \ldots, f_r) \subset \mathbb{K}^n$ by

$$\bar{\mathbf{a}} := (a_1, \ldots, a_n) \in \mathbf{V}(f_1, \ldots, f_r) \Leftrightarrow f_i(\bar{\mathbf{a}}) = f_i(a_1, \ldots, a_n) = 0,$$

for $i = 1, 2, \ldots, r$.

- We define the set of polynomials that vanishes in a given set $S \subset \mathbb{C}^2$ by,

$$\mathbb{I}(S) := \{f \in \mathbb{C}[z_1, z_2] : f(z_1, z_2) = 0, \, \forall (a_1, a_2) \in S\}\,.$$

We note that $\mathbb{I}(S)$ is an ideal.

- The ideal generated by a finite set of polynomials $\{f_1, \ldots, f_s\}$, defined as

$$\Big\langle f_1, \ldots, f_s \Big\rangle := \left\{ f \big| f = \sum_{i=1}^{s} g_i \, f_i \,, \; g_i \in \mathbb{C}[z_1, z_2] \right\}. \tag{3.1.1}$$

We note that an ideal $\mathcal{I}$ is said to be finitely generated if it can be expressed as in (3.1.1) for some finite set of polynomials $\{f_1, \ldots, f_s\}$.

The following important results for our discussion are proved in [CLO07], and will be used later in this Chapter.

Proposition 3.1.3. *[CLO07]*

Given $f_1, \ldots, f_r \in \mathbb{A}$ and $g_1, \ldots, g_s \in \mathbb{A}$. If $\langle f_1, \ldots, f_r\rangle = \langle g_1, \ldots, g_s\rangle$, then

$$\boldsymbol{V}(f_1, \ldots, f_r) = \boldsymbol{V}(g_1, \ldots, g_s). \tag{3.1.2}$$

Lemma 3.1.4. *[CLO07]*

Let $\mathcal{V} := \boldsymbol{V}(f_1, \ldots, f_r)$ and $\mathcal{W} := \boldsymbol{W}(g_1, \ldots, g_k)$ be affine varieties. Then

$$\mathcal{V} \cap \mathcal{W} = \boldsymbol{V}(f_1, \ldots, f_r, g_1, \ldots, g_k), \tag{3.1.3}$$

and thus the vanishing ideal is given $\mathbb{I}(\mathcal{V} \cap \mathcal{W}) = \langle f_1, \ldots, f_r, g_1, \ldots, g_k\rangle$.

Also,

$$\mathcal{V} \cup \mathcal{W} = \boldsymbol{V}(f_i g_j, \; i = 1, \ldots, r, \; j = 1, \ldots, k), \tag{3.1.4}$$

and thus the vanishing ideal is given $\mathbb{I}(\mathcal{V} \cup \mathcal{W}) = \langle \big\{ f_i g_j \bigm| 1 \le i \le r; \; 1 \le j \le k \big\} \rangle$.

Lemma 3.1.5. *[CLO07]*

Given two ideals $\mathcal{I}$ and $\mathcal{J}$ with respective affine varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ given by $\mathbb{I}(\mathcal{V}_1) = \langle f_1, \ldots, f_r\rangle$ and $\mathbb{I}(\mathcal{V}_2) = \langle g_1, \ldots, g_m\rangle$, it holds that

$$\mathbb{I}(\mathcal{V}_1 \cup \mathcal{V}_2) = \langle \{f_i g_j | 1 \le i \le r; \; 1 \le j \le m\}\rangle. \tag{3.1.5}$$

Definition 3.1.6. *A ring $\mathcal{R}$ is Noetherian if it satisfies the ascending chain condition (ACC) on ideals, that is, given any chain $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \ldots \subseteq \mathcal{I}_k \subseteq \ldots$, there exists a positive integer p such that $\mathcal{I}_p = \mathcal{I}_n$ for all $p \ge n$.*

Theorem 3.1.7. *(Hilbert Basis Theorem) [CLO07]*

Every ideal $\mathcal{I}$ in $\mathbb{A}$ has a finite generating set, that is, $\mathcal{I} = \langle h_1, \ldots, h_s\rangle$ for some $h_1, \ldots, h_s \in \mathcal{I}$.

Definition 3.1.8. *Let $f_1, \ldots, f_s$ be polynomials in $\mathbb{C}[z_1, z_2]$. Let the set $\boldsymbol{V}$ be given by $\boldsymbol{V}(f_1, \ldots, f_s) := \{(a_1, a_2) \in \mathbb{C}^2 : f_i(a_1, a_2) = 0,\ 1 \leq i \leq s\}$. We call $\boldsymbol{V}(f_1, \ldots, f_s)$ the affine (algebraic) variety defined by $f_1, \ldots, f_s$.*

Next, we introduce the notion of monomial orderings.

Definition 3.1.9. *Let $\mathbb{K}$ be a field, and let X be the set of monomials in $\mathbb{A}$. A monomial ordering is a total ordering $\succ$ on X which satisfies*

(i) $\mathbf{a} \succ \mathbf{b} \Rightarrow \mathbf{ac} \succ \mathbf{bc}$, *for all* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in X$.

(ii) $\mathbf{a} \succ 1$ *for all* $\mathbf{a} \in X$.

That is, for any $\mathbf{a} := z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n} \in \mathbb{A}$, there is an associated n-tuple $(\alpha_1, \ldots, \alpha_n)$, that helps us to compare and order monomials in the sense that $\boldsymbol{z}^{\boldsymbol{\alpha}} \in X \Leftrightarrow \boldsymbol{\alpha} \in \mathbb{N}^n$. In other words, an admissible ordering establishes a one-to-one correspondence between $\mathbb{N}^n$ and the monomials $\boldsymbol{z}^{\boldsymbol{\alpha}} := z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ in $\mathbb{A}$, that is, $[\alpha_1 \leftrightarrow z_1^{\alpha_1}, \ldots, \alpha_n \leftrightarrow z_n^{\alpha_n}]$.

We also note that a total ordering $\succ$ on $\mathbb{N}^n$ is admissible if $\forall \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}^n$, $\boldsymbol{\alpha} \succ \boldsymbol{\beta} \Rightarrow \boldsymbol{\alpha} + \boldsymbol{\gamma} \succ \boldsymbol{\beta} + \boldsymbol{\gamma}$ with $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ n-tuples in $\mathbb{N}^n$.

The following definition introduces different types of orderings for monomials.

Definition 3.1.10. *Let α and β be in $\mathbb{N}^2$. The following orderings are valid for monomials:*

(a) ***Lexicographic order****: $\alpha \succ_{lex} \beta$ if the left-most non-zero entry in $\alpha - \beta$ is positive.*

(b) ***Graded Lex order (degree lexicographic ordering):*** *$\alpha \succ_{grlex} \beta$ if $|\alpha| \succ |\beta|$ or ($|\alpha| = |\beta|$ and $\alpha \succ_{lex} \beta$). We use the singular command* ***dp*** *for the degree lexicographical ordering as it is computationally preferable to other orderings.*

(c) ***Graded reverse Lex order****: $\alpha \succ_{grevlex} \beta$ if $|\alpha| \succ |\beta|$ and the right-most non-zero entry in $\alpha - \beta$ is negative.*

Definition 3.1.11. *Assume that an arbitrary admissible ordering $\succ$ is fixed on $\mathbb{K}^2$. Given a non-zero polynomial $f(\boldsymbol{z}) := \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}} \in \mathbb{K}[z_1, z_2]$ with $\boldsymbol{z}^{\boldsymbol{\alpha}} := z_1^{\alpha_1} z_2^{\alpha_2}$, $\boldsymbol{\alpha} := (\alpha_1, \alpha_2) \in \mathbb{K}^2$, we define*

(a) The multi-degree of f as multideg(f) $= \max\{\boldsymbol{t} \in \mathbb{N}^2 : a_{\boldsymbol{t}} \neq 0\}$ (the maximum is taken with respect to $\succ$);

(b) The leading monomial of f as :

$$LM(f) = \boldsymbol{x}^{multideg(f)};$$

(c) The leading coefficient of f as :

$$LC(f) = a_{multideg(f)};$$

(d) *The leading term of f as :*

$$LT(f) = LC(f) \cdot LM(f);$$

(e) *The leading ideal is defined by*

$$L(\mathcal{I}) := \langle \{LT(f) | f \in \mathcal{I}\} \rangle.$$

We also note that $L(\mathcal{I})$ carries plenty of information on $\mathcal{I}$. For instance, it is needed to get an ideal bases.

3.1.1.1 Gröbner basis and Buchberger's algorithm

In this subsection, we introduce the concept of a Gröbner Basis of an ideal $\mathcal{I}$ in $\mathbb{A}$.

Definition 3.1.12. *A set $\{g_1, \ldots, g_s\}$ in an ideal $\mathcal{I}$ of polynomials is said to be a Gröbner basis for $\mathcal{I}$ if the leading terms $LT(g_i)$ generate $L(\mathcal{I})$.*

Definition 3.1.13. *Let $f, g \in \mathbb{A}$ be non-zero polynomials.*

(i) *If multideg $f = \alpha$ and multideg $g = \beta$, then let $\gamma = (\gamma_1, \ldots, \gamma_n)$, where $\gamma_i = \max(\alpha_i, \beta_i)$ for each i. We find x^γ as follows:*

$$x^\gamma = LCM(x^\alpha, x^\beta) = LCM\left(LM(f), LM(g)\right).$$

(ii) *The S-polynomial of f and g is the combination*

$$Spoly(f, g) := \frac{x^\gamma}{LT(f)} \cdot f - \frac{x^\gamma}{LT(\boldsymbol{g})} \cdot g. \tag{3.1.6}$$

Next, we give a characterization of a Gröbner basis. The following proposition helps to check whether a given family G is a Gröbner basis or not.

Proposition 3.1.14. *Let $\boldsymbol{I}$ be a polynomial ideal of $\mathbb{A}$. A given basis $G := \{g_1, \ldots, g_s\}$ of $\boldsymbol{I}$ is a Gröbner basis if and only if, for all $i \neq j$, the remainder $\overline{S[g_i, g_j]}^G$ of the division of $S[g_i, g_j]$ by G is zero.*

Next, we present Buchberger's algorithm to compute a Gröbner basis of an ideal. We used a singular program, the code with its output of which is given in Appendix A. We will show that, for a given ideal $\mathbf{I} = \mathcal{I}(f_1, \ldots, f_r)$, one can always find a Gröbner basis $G := (g_1, \ldots, g_s)$ of $\mathbf{I}$.

Algorithm 1 Buchberger's algorithm

Input: A family $\mathcal{F} := (f_1, \ldots, f_m) \subset \mathbb{K}[z_1, z_2] \backslash \{\mathbf{0}\}$ of polynomials
Output: A Gröbner basis $G := (f_1, \ldots, f_m, f_{m+1}, \ldots, f_r)$ of the ideal $\mathcal{I} := \langle f_1, \ldots, f_m \rangle$.

(a) Compute all S-polynomials $Spoly(f_i, f_j)$ for $f_i, f_j \in \mathcal{F}$.

(b) Compute the remainders of these S-polynomials when divided by $\mathcal{F}$.

(c) If there are non-zero remainders, set

$$\mathcal{F} := \mathcal{F} \cup \{ \text{ non-zero remainders} \},$$

and continue with step 1.

(d) If all the S-polynomials reduce to zero, then we set $G := \mathcal{F}$. This process will be stationary because of the fact that $\mathbb{K}[z_1, z_2]$ is Noetherian.

In Section 3.2, following the work in [CCJZ11, Sau07], we study bivariate subdivision schemes with their associated mask symbols. A special focus is given on the verification of the Butterfly subdivision scheme in Section 3.4.

3.2 Algebraic approach to bivariate subdivision

First, we consider the ring of bivariate Laurent polynomials, say, $\mathcal{R} := \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$ and an ideal $\mathcal{I}$ in $\mathcal{R}$ in such a way that $\mathcal{I}$ contains a set of certain Laurent polynomial symbols with some special properties. By multiplying each element in $\mathcal{R}$ by an appropriate monomial factor $\mathbf{z}^{\boldsymbol{\alpha}} := z_1^{\alpha_1} z_2^{\alpha_2}$; $\boldsymbol{\alpha} := (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, which is a unit in $\mathcal{R}$, we obtain an element in the ring of polynomials in 2-variables, that is, $\Pi := \mathbb{C}[z_1, z_2]$. This results in a shift of the support of a mask with the advantage that such a shift does not influence properties of a bivariate subdivision scheme such as smoothness, regularity, approximation order and the sum rule condition. Using this advantage, we study the structure of the mask symbols of bivariate subdivision schemes by showing that they are $\mathbb{C}[z_1, z_2]$-linear combinations of shifted box spline generators of some polynomial ideal $\mathcal{I}$ with the associated direction set of vectors as columns of certain uni-modular matrices.[1]

Before we proceed, we need to recall the necessary condition for subdivision convergence given by the sum rule condition of order k (where the case with k = 1 is given in (2.6.11), that is, the sum rule of order 1).

As a preliminary, we recall that the necessary condition for subdivision convergence, that is, the sum rule condition of order k.

In the univariate case, the sum rule condition of order $\mathrm{k} \in \mathbb{N}$ for a mask sequence $\{p_j\} \in \ell_0(\mathbb{Z})$ is given by

$$\left.\begin{array}{l} \beta_\ell := \sum_j (2j)^\ell \, p_{2j} = \sum_j (2j+1)^\ell \, p_{2j+1}, \quad \ell = 0, \ldots, \mathrm{k}-1; \\ \text{with } \beta_0 = 1. \end{array}\right\} \tag{3.2.1}$$

[1]Uni-modular matrices are matrices having square sub-matrices with determinant ± 1.

The following result was proved in [dVC10]:

A sequence $\{p_j\} \in \ell_0(\mathbb{Z})$ satisfies the sum rule condition of order $\mathrm{k} \in \mathbb{N}$ if and only if the Laurent polynomial symbol $\mathrm{P}(z) := \frac{1}{2}\sum_j p_j\, z^j$ is given by the formulation

$$\mathrm{P}(z) = \left(\frac{1+z}{2}\right)^{\mathrm{k}} \mathrm{R}(z), \tag{3.2.2}$$

for some Laurent polynomial R such that $\mathrm{R}(1) = 1$.

Now observe that (3.2.2) implies

$$(\mathrm{D}^j \mathrm{P})(-1) = 0,\ j = 0, \ldots, \mathrm{k}-1, \tag{3.2.3}$$

where $\mathrm{D}^j P$ is the j^{th}-derivative of P. Based on (3.2.3), the extension from the univariate to the bivariate case of the sum rule condition of order $\mathrm{k} \in \mathbb{N}$ is as follows.

Definition 3.2.1. *Let $\mathcal{E} := \{0,1\}^2$, the complete set of representatives of $\mathbb{Z}^2/2\mathbb{Z}^2$, given by vertices of the unit square containing $\boldsymbol{0} := (0,0)$ and $\boldsymbol{1} := (1,1)$ as in (2.3.6). We construct the set $\mathrm{Z} = \mathrm{Z}_{\mathcal{E}} := \left\{\varepsilon_{\mathrm{j}} := \mathrm{e}^{-\mathrm{i}\pi\gamma_{\mathrm{j}}} : (\gamma_1, \gamma_2) \in \mathcal{E},\ \mathrm{j} = 1, 2\right\} = \{-1, 1\}^2$ as in (2.5.3). The mask symbol $\mathrm{P}(z_1, z_2)$ is said to satisfy the sum rule condition of order k, or condition Z_{k}, $\mathrm{k} \in \mathbb{N}$, if*

$$\mathrm{P}(\boldsymbol{1}) = 4,\ \textit{and} \quad (\mathrm{D}^{\boldsymbol{j}}\mathrm{P})(\varepsilon_1, \varepsilon_2) = 0, \quad (\varepsilon_1, \varepsilon_2) \in \mathrm{Z}' := \mathrm{Z}\backslash\{\boldsymbol{1}\},\ |\boldsymbol{j}| < \mathrm{k},$$

where $(\mathrm{D}^{\boldsymbol{j}}\mathrm{P})(\varepsilon_1, \varepsilon_2)$ is the $\boldsymbol{j}^{th}$-directional derivative of P at $(\varepsilon_1, \varepsilon_2)$, that is,

$$\mathrm{D}^{\boldsymbol{j}}\mathrm{P} = \frac{\partial^{(\mathrm{j}_1+\mathrm{j}_2)}\mathrm{P}}{\partial {\mathrm{z}_1}^{\mathrm{j}_1}\, \partial {\mathrm{z}_2}^{\mathrm{j}_2}}, \quad \mathrm{j}_1, \mathrm{j}_2 \in \mathbb{N}. \tag{3.2.4}$$

As in (2.5.1), the sum rule of order 1 is obtained by setting $\boldsymbol{j} = \boldsymbol{0}$ in (3.2.4), that is,

$$\mathrm{P}(\boldsymbol{1}) = 4\,, \quad \textit{and} \quad \mathrm{P}(\varepsilon_1, \varepsilon_2) = 0, \quad \textit{for} \quad (\varepsilon_1, \varepsilon_2) \in \mathrm{Z}'. \tag{3.2.5}$$

It is a known fact that a sum rule condition as in (3.2.4) and (3.2.5) must be satisfied for some $k \in \mathbb{N}$ by the subdivision polynomial of a convergent subdivision scheme, and we further study the structure of the mask symbols of such a subdivision scheme, since the mask symbol of a convergent subdivision scheme gives important information about the limit function of a subdivision scheme.

Since all convergent bivariate subdivision schemes must satisfy Z_1, we begin with a characterization of the polynomial ideal

$$\mathcal{I} := \left\{f \in \Pi : f(\varepsilon_1, \varepsilon_2) = 0, \quad \text{for} \quad (\varepsilon_1, \varepsilon_2) \in \mathrm{Z}'\right\}. \tag{3.2.6}$$

Hence, the ideal $\mathcal{I}$ is the set of all polynomials satisfying

$$P(\varepsilon_1, \varepsilon_2) = 0 \quad \text{for all} \quad (\varepsilon_1, \varepsilon_2) \in \mathrm{Z}',$$

that is,

$$P(\mathbf{z}) \in \mathcal{I} \Leftrightarrow P(\varepsilon_1, \varepsilon_2) = 0 \quad \text{for all} \quad (\varepsilon_1, \varepsilon_2) \in \mathrm{Z}'. \tag{3.2.7}$$

We recall the fact that the product $\mathcal{I}_1 \cdot \mathcal{I}_2$ of two ideals $\mathcal{I}_1 = \big\langle \{a_j : j = 1, \ldots, n\} \big\rangle$ and $\mathcal{I}_2 = \big\langle \{b_r : r = 1, \ldots, m\} \big\rangle$ in a commutative ring is generated by the point-wise products of the corresponding generating sets, that is, ,

$$\mathcal{I}_1 \cdot \mathcal{I}_2 := \langle \{a_j b_r : j = 1, \ldots, n;\ r = 1 \ldots, m\} \rangle. \tag{3.2.8}$$

Using (3.2.8), it follows that the power ideal $\mathcal{I}^{\mathrm{k}}$ of $\mathcal{I}$ equivalently given by

$$\mathcal{I}^{\mathrm{k}} = \Big\{ f \in \Pi : (\mathrm{D}^{\mathbf{j}} f(\varepsilon_1, \varepsilon_2) = 0 \quad \text{for} \quad (\varepsilon_1, \varepsilon_2) \in \mathbb{Z}',\ |\mathbf{j}| \leq k \Big\}. \tag{3.2.9}$$

In order to compute the ideal $\mathcal{I}$ in (3.2.6), we proceed as follows:

By using Lemma 3.1.5, and since the ideal $\mathcal{I}$ comprises Laurent polynomials which vanish on $\mathrm{Z}' := \mathrm{Z} \backslash \{\mathbf{1}\}$, we write the vanishing set Z' as a union of the irreducible affine varieties $\mathcal{V}_1 := \{(1, -1)\}$, $\mathcal{V}_2 := \{(-1, 1)\}$, $\mathcal{V}_3 := \{(-1, -1)\}$, where

$$\mathrm{Z}' = \{(1, -1)\} \cup \{(-1, 1)\} \cup \{(-1, -1)\} := \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \tag{3.2.10}$$

where $\mathcal{V}_i$'s are irreducible. The vanishing ideals for each of the irreducible affine varieties are given by

$$\begin{aligned} \mathbb{I}(\mathcal{V}_1) &= \langle z_1 + 1, z_2 - 1 \rangle = \langle f_1, f_2 \rangle; \\ \mathbb{I}(\mathcal{V}_2) &= \langle z_1 - 1, z_2 + 1 \rangle = \langle g_1, g_2 \rangle; \\ \mathbb{I}(\mathcal{V}_3) &= \langle z_1 + 1, z_2 + 1 \rangle = \langle h_1, h_2 \rangle. \end{aligned} \tag{3.2.11}$$

Then, applying Lemma 3.1.4, we get

$$\begin{aligned} \mathbb{I}\,(\mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3) \quad = \quad & \big\langle f_i g_j h_t \big| 1 \leq i \leq 2,\ 1 \leq j \leq 2,\ 1 \leq t \leq 2 \big\rangle \\ = & \Bigg\langle (z_1^2 - 1)(z_2 + 1), (z_2^2 - 1)(z_1 + 1), \\ & (z_1 + 1)^2 (z_1 - 1), (z_1 + 1)^2 (z_2 + 1), \\ & (z_1^2 - 1)(z_2 - 1), (z_2 + 1)^2 (z_2 - 1), \\ & (z_1 + 1)(z_2 + 1)^2, (z_2^2 - 1)(z_1 - 1) \Bigg\rangle. \end{aligned} \tag{3.2.12}$$

Equation (3.2.12) can be simplified to minimal generators, that is, a minimal set of polynomials which generate the same ideal, using a Gröbner basis with grlex ordering (**dp**) to yield

$$\mathcal{I} := \mathbb{I}\,(\mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3) = \big\langle z_1^2 - 1, z_2^2 - 1, (z_1 + 1)(z_2 + 1) \big\rangle. \tag{3.2.13}$$

According to [CCJZ11], the polynomial ideal

$$\mathcal{J} = \{ f \in \Pi : f(\varepsilon_1, \varepsilon_2) = 0, \quad \text{for} \quad (\varepsilon_1, \varepsilon_2) \in Z \} \tag{3.2.14}$$

has set of generators $\langle z_1^2 - 1, z_2^2 - 1 \rangle$; i.e.,

$$\mathcal{J} = \langle z_1^2 - 1, z_2^2 - 1 \rangle. \tag{3.2.15}$$

It can be checked by using the fact that for any two varieties, $\mathcal{V}_1$ and $\mathcal{V}_2$,

$$\mathcal{V}_1 \subset \mathcal{V}_2 \Rightarrow \mathcal{I}(\mathcal{V}_2) \subset \mathcal{I}(\mathcal{V}_1)$$

or alternatively, applying Groebner basis computation also leads to equation (3.2.15).

Our next result is a particular case of the one mentioned in [MS04] for the multivariate case.

Proposition 3.2.2. *The ideal $\mathcal{I}$ consisting of the mask symbol of a bivariate convergent subdivision scheme $S_{\mathbf{p}}$ can be generated by the polynomials*

$$\left(\frac{1+z_1}{2}\frac{1+z_2}{2}\right), \left(\frac{1+z_1}{2}\frac{z_1+z_2}{2}\right), \left(\frac{1+z_2}{2}\frac{z_1+z_2}{2}\right),$$

that is,

$$\mathcal{I} = \left\langle \frac{1+z_1}{2}\frac{1+z_2}{2}, \frac{1+z_1}{2}\frac{z_1+z_2}{2}, \frac{1+z_2}{2}\frac{z_1+z_2}{2} \right\rangle. \tag{3.2.16}$$

Proof. Let us denote (3.2.16) by

$$\tilde{\mathcal{I}} = \left\langle \frac{1+z_1}{2}\frac{1+z_2}{2}, \frac{1+z_1}{2}\frac{z_1+z_2}{2}, \frac{1+z_2}{2}\frac{z_1+z_2}{2} \right\rangle.$$

Then one can show that the generators of $\mathcal{I}$ are contained in $\tilde{\mathcal{I}}$, and the generators of $\tilde{\mathcal{I}}$ are also contained in $\mathcal{I}$. We rewrite, for $z_1, z_2 \in \mathbb{C}\backslash\{0\}$, z_1^2-1, z_2^2-1 and $(z_1+1)(z_2+1)$ in terms of elements of generators in $\tilde{\mathcal{I}}$, that is,

$$\left.\begin{aligned}
z_1^2-1 &= -4\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)+4\left(\frac{1+z_1}{2}\right)\left(\frac{z_1+z_2}{2}\right);\\
z_2^2-1 &= -4\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)+4\left(\frac{1+z_2}{2}\right)\left(\frac{z_1+z_2}{2}\right);\\
(z_1+1)(z_2+1) &= 4\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right).
\end{aligned}\right\} \tag{3.2.17}$$

It follows from equation (3.2.17) that $\mathcal{I} \subseteq \tilde{\mathcal{I}}$.

Conversely, we write

$$\left.\begin{aligned}
\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right) &= \frac{1}{4}(1+z_1)(1+z_2);\\
\left(\frac{1+z_1}{2}\right)\left(\frac{z_1+z_2}{2}\right) &= \frac{1}{4}(z_2^2-1)+\frac{1}{4}(1+z_1)(1+z_2);\\
\left(\frac{1+z_1}{2}\right)\left(\frac{z_1+z_2}{2}\right) &= \frac{1}{4}(z_1^2-1)+\frac{1}{4}(1+z_1)(1+z_2),
\end{aligned}\right\} \tag{3.2.18}$$

which yields $\tilde{\mathcal{I}} \subseteq \mathcal{I}$. This shows that $\mathcal{I} = \tilde{\mathcal{I}}$. □

Next, we establish an ideal $\mathcal{I}$ that is related to set of mask symbols of a convergent bivariate subdivision scheme.

Proposition 3.2.3. *The ideal $\mathcal{I}$ in (3.2.6) is generated by $(1-z_1^2), (1-z_2^2)$ and $\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)$, that is,*

$$\mathcal{I} = \left\langle (1-z_1^2), (1-z_2^2), \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right) \right\rangle. \tag{3.2.19}$$

Proof. The proof is straightforward as is direct consequence of (3.2.13). □

Remark

The bivariate subdivision polynomials (or mask symbols) can be represented as a special class of Laurent polynomials in $\mathcal{R}$ with the property of the sum rule condition. Each element $\mathcal{L}$ of the class of Laurent polynomials associated to a well defined subdivision scheme must be spanned by the generators of the ideal $\mathcal{I}$ in (3.2.19). A well defined bivariate subdivision scheme $\mathrm{P}(z_1, z_2)$ can be written as $\mathbb{C}[z_1, z_2]$-linear combination of the polynomials $1 - z_1^2, 1 - z_2^2, \frac{1}{4}(1 + z_1)(1 + z_2)$, with

$$\mathrm{P}(z_1, z_2) \in \left\langle 1-z_1^2, 1-z_2^2, \frac{1}{4}(1+z_1)(1+z_2)\right\rangle = \left\langle \frac{1+z_1}{2}\frac{1+z_2}{2}, \frac{1+z_1}{2}\frac{z_1+z_2}{2}, \frac{1+z_2}{2}\frac{z_1+z_2}{2}\right\rangle.$$

3.3 Box splines and bivariate subdivision schemes

Consider the box spline $\mathbb{B}_{\mathcal{D}}$ with direction matrix $\mathcal{D} := [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are as in (1.5.5), which correspond to the non-zero elements of $\mathcal{E}$ as in (2.3.6). Recall that a matrix $\mathcal{D}$ is called uni-modular if its square sub-matrix ϑ have determinant ± 1. For technical use, we define the normalised Laurent polynomials,

$$\mathrm{r}_{\mathbf{a}}(\mathbf{z}) := \frac{1}{2}(1 + \mathbf{z}^{\mathbf{a}}), \quad \text{and} \quad \mathrm{s}_{\mathbf{a}}(\mathbf{z}) := \frac{1}{2}(1 - \mathbf{z}^{\mathbf{a}}), \tag{3.3.1}$$

where $\mathbf{z}^{\mathbf{a}} := z_1^{a_1} z_2^{a_2}$, $\mathbf{a} := (a_1, a_2) \in \mathbb{Z}^2$. Observe that $\mathrm{r}_{\mathbf{a}}(\mathbf{z}) + \mathrm{s}_{\mathbf{a}}(\mathbf{z}) = 1$, $\mathbf{a} \in \mathbb{Z}^2$, and the following useful identity also holds;

$$\mathrm{s}_{\mathbf{b}}(\mathbf{z})\, \mathrm{r}_{\mathbf{a}}(\mathbf{z}) + \mathrm{s}_{\mathbf{a}}(\mathbf{z})\, \mathrm{r}_{\mathbf{b}}(\mathbf{z}) = \mathrm{s}_{\mathbf{a}+\mathbf{b}}(\mathbf{z}) = 1 - \mathrm{r}_{\mathbf{a}+\mathbf{b}}(\mathbf{z}), \quad \text{for} \quad \mathbf{a}, \mathbf{b} \in \mathbb{Z}^2. \tag{3.3.2}$$

The identity (3.3.2) shows that the polynomials $\mathrm{r}_{\mathbf{a}}, \mathrm{r}_{\mathbf{b}}, \mathrm{r}_{\mathbf{a}+\mathbf{b}}$ generate the ring of Laurent polynomials $\mathcal{R}$, since they generate a unit, that is, $\mathcal{R} = \langle \mathrm{r}_{\mathbf{a}}(\mathbf{z}), \mathrm{r}_{\mathbf{b}}(\mathbf{z}), \mathrm{r}_{\mathbf{a}+\mathbf{b}}(\mathbf{z})\rangle$. For any sub-matrix ϑ of $\mathcal{D}$, we define the normalized polynomial

$$\mathrm{q}_{\vartheta}(\mathbf{z}) := \prod_{\theta \in \vartheta} \mathrm{r}_{\theta}(\mathbf{z}), \tag{3.3.3}$$

where θ runs through all columns of ϑ.

Observe that if ϑ is uni-modular, then the polynomial $4\mathrm{q}_{\vartheta}(\mathbf{z})$ is the symbol of the corresponding degree zero box spline, with mask symbols

$$\left(4 \cdot \frac{1+z_1}{2} \cdot \frac{1+z_2}{2}\right), \left(4 \cdot \frac{1+z_1}{2} \cdot \frac{1+z_1 z_2^{-1}}{2}\right), \left(4 \cdot \frac{1+z_1}{2} \cdot \frac{1+z_1 z_2^{-1}}{2}\right).$$

Using the columns of ϑ, and from (3.3.1), we have

$$\mathrm{r}_{(1,0)}(\mathbf{z}) = \frac{1}{2}(1 + z_1), \quad \mathrm{r}_{(0,1)}(\mathbf{z}) = \frac{1}{2}(1 + z_2), \quad \text{and} \quad \mathrm{r}_{(1,1)}(\mathbf{z}) = \frac{1}{2}(1 + z_1 z_2). \tag{3.3.4}$$

The following lemma introduces a useful property of normalized box spline symbols [2].

Lemma 3.3.1. *For any given triple* $(a, b, c) \in \mathbb{N}_0^3$*, we have that*

$$\langle \tilde{\mathbb{B}}_{\mathrm{a+1,b,c}}, \tilde{\mathbb{B}}_{\mathrm{a,b+1,c}}, \tilde{\mathbb{B}}_{\mathrm{a,b,c+1}}\rangle = \langle \tilde{\mathbb{B}}_{\mathrm{a,b,c}}\rangle, \tag{3.3.5}$$

where $\tilde{\mathbb{B}}_{\mathrm{a,b,c}}(z_1, z_2)$ *is given in* (1.9.8).

[2] A box spline symbol $\mathrm{P}(z_1, z_2)$ is said to be normalised if it satisfies $\mathrm{P}(1, 1) = 1$.

Proof. Let $\tilde{\mathbb{B}}_{\mathrm{a,b,c}}$ be the 3-directional box spline given by $\mathbb{B}_{\mathrm{a,b,c}} := 4\tilde{\mathbb{B}}_{\mathrm{a,b,c}}$. The identity (3.3.2) can be written equivalently as

$$\frac{1}{2}(1 - z_2)\,\tilde{\mathbb{B}}_{1,0,0}(z_1, z_2) + \frac{1}{2}(1 - z_1)\,\tilde{\mathbb{B}}_{0,1,0}(z_1, z_2) + \tilde{\mathbb{B}}_{0,0,1}(z_1, z_2) = 1.$$

Multiplying both sides by $\tilde{\mathbb{B}}_{\mathrm{a,b,c}}$ then leads to the required result. □

Remark 3.3.2. *The family of 3-directional normalized box spline symbols is a partially ordered set with respect to inclusion relation, and it is closed under multiplication, since*

$$\tilde{\mathbb{B}}_{\mathrm{a,b,c}} \cdot \tilde{\mathbb{B}}_{\mathrm{d,e,f}} = \tilde{\mathbb{B}}_{\mathrm{a+d,b+e,c+f}}.$$

The following result is the bivariate version of the one given in [CCJZ11] for the multivariate case.

Theorem 3.3.3. *The mask symbol of a convergent bivariate subdivision scheme $\mathbb{S}_{\boldsymbol{p}}$ can be written in the form*

$$\mathrm{P}(\boldsymbol{z}) = 4\sum_{\vartheta} \lambda_\vartheta \cdot \mathrm{r}_\vartheta(\boldsymbol{z}) \cdot \mathrm{q}_\vartheta(\boldsymbol{z}), \tag{3.3.6}$$

where $\mathrm{r}_\vartheta(\boldsymbol{z})$ is a Laurent polynomial satisfying $\mathrm{r}_\vartheta(\boldsymbol{1}) = 1$, where $\boldsymbol{1} := (1,1)$ and $\lambda_\vartheta \in \mathbb{R}$, with $\sum_{\vartheta} \lambda_\vartheta = 1$. The sum runs over all uni-modular 2×2 sub-matrices ϑ of $\mathcal{D}$.

Our next result from [CCJZ11], introduces generators for the power ideal $\mathcal{I}^k$, which are the mask symbols of certain normalised three-directional box splines.

Theorem 3.3.4. *The k^{th} -power $\{\mathcal{I}^{\mathrm{k}} : \mathrm{k} \in \mathbb{N}\}$ of an ideal $\mathcal{I}$ is generated by the set of three directional box spline symbols given by*

$$\mathcal{I}_{\mathrm{k}} := \left\{\tilde{\mathbb{B}}_{\mathrm{b,b,a}}, \tilde{\mathbb{B}}_{\mathrm{b,a,b}}, \tilde{\mathbb{B}}_{\mathrm{a,b,b}} : a = 0, 1, 2, \ldots, \lfloor \frac{1}{2}\mathrm{k} \rfloor, \ \textit{and} \quad \mathrm{b} = \mathrm{k} - \mathrm{a}\right\}, \tag{3.3.7}$$

where $\tilde{\mathbb{B}}_{\mathrm{a,b,c}}(z_1, z_2)$ is given in (1.9.8).

Proof. The proof is by induction on k and is given in [CCJZ11]. □

We note that, for the set $\mathcal{I}_{\mathrm{k}}$ as in (3.3.7), the set of generators for $\mathcal{I}^{\mathrm{k}}$ is the minimal set that generates every element in $\mathcal{I}^{\mathrm{k}}$.

For instance, fixing $\mathrm{k} = 1, 2, 3, 4$ and applying (3.3.7), we get the following set of generators of $\mathcal{I}_{\mathrm{k}}$:

$$\left.\begin{aligned}
\mathcal{I}_1 &:= \left\{\tilde{\mathbb{B}}_{1,1,0}, \tilde{\mathbb{B}}_{1,0,1}, \tilde{\mathbb{B}}_{0,1,1}\right\};\\
\mathcal{I}_2 &:= \left\{\tilde{\mathbb{B}}_{0,2,2}, \tilde{\mathbb{B}}_{1,1,1}, \tilde{\mathbb{B}}_{2,2,0}, \tilde{\mathbb{B}}_{2,0,2}\right\};\\
\mathcal{I}_3 &:= \left\{\tilde{\mathbb{B}}_{0,3,3}, \tilde{\mathbb{B}}_{2,2,1}, \tilde{\mathbb{B}}_{2,1,2}, \tilde{\mathbb{B}}_{1,2,2}, \tilde{\mathbb{B}}_{3,3,0}, \tilde{\mathbb{B}}_{3,0,3}\right\};\\
\mathcal{I}_4 &:= \left\{\tilde{\mathbb{B}}_{1,3,3}, \tilde{\mathbb{B}}_{2,2,2}, \tilde{\mathbb{B}}_{4,4,0}, \tilde{\mathbb{B}}_{4,0,4}, \tilde{\mathbb{B}}_{0,4,4}, \tilde{\mathbb{B}}_{3,3,1}, \tilde{\mathbb{B}}_{3,1,3}\right\},
\end{aligned}\right\} \tag{3.3.8}$$

where $\tilde{\mathbb{B}}_{\mathrm{a,b,c}}(z_1, z_2)$, with $\mathrm{a, b, c} \in \mathbb{Z}_+$, are as in (1.9.8).

We mention the following remarks based on the above discussion.

Remarks

(**a**) The set $\mathcal{I}_{\mathrm{k}}$ of generators for the power ideal $\mathcal{I}^{\mathrm{k}}$ is symmetric in the sense that it is invariant under permutation of two variables, and that the indices of the generators can be interchanged cyclically.

(**b**) The order of polynomial reproduction of a bivariate subdivision scheme perfectly matches with smoothness of the constituent box splines corresponding to the generators in $\mathcal{I}_{\mathrm{k}}$ that make up the subdivision scheme.

We proceed to quote a key result from [CCJZ11].

Theorem 3.3.5. *(Decomposition) A bivariate subdivision scheme $\mathbb{S}_{\mathbf{p}}$ satisfies the sum rule condition if and only if its mask symbol can be written in the form*

$$\mathrm{P}(z_1, z_2) = \sum_{\tilde{\mathbb{B}}_{\mathrm{a,b,c}} \in \mathcal{I}_{\mathrm{k}}} \lambda_{\mathrm{a,b,c}}\, \mathrm{q}_{\mathrm{a,b,c}}(z_1, z_2)\, \tilde{\mathbb{B}}_{\mathrm{a,b,c}}(z_1, z_2), \tag{3.3.9}$$

with $\sum \lambda_{\mathrm{a,b,c}} = 1$, where $\tilde{\mathbb{B}}_{\mathrm{a,b,c}}$ is the 3-directional normalised box spline as in (1.9.8), and where $\mathrm{q}_{a,b,c}(z_1, z_2)$ are normalised Laurent polynomials, that is, $\mathrm{q}_{\mathrm{a,b,c}}(\boldsymbol{1}) = 1$.

In (3.3.9), ‘k’ refers to the order of polynomial reproduction of the subdivision operator $\mathbb{S}_{\mathbf{p}}$ and the set $\mathcal{I}_{\mathrm{k}}$ has the advantage of classifying as well as constructing bivariate subdivision schemes.

From the discussion up to now, we observe that any bivariate subdivision symbol can be decomposed into 3-directional box splines, $\tilde{\mathbb{B}}_{\mathrm{a,b,c}}$. For instance, we consider 4-directional box splines, which can be decomposed into pieces of 3-directional ones, that is, the mask symbol associated to a 4-directional box spline can be rewritten as a convex combination of shifts of 3-directional normalised box spline symbols, as given below:

$$\begin{aligned}
\mathbb{B}_{\mathrm{a,b,c,d}}(z_1, z_2) &= 4\left(\frac{1+z_1}{2}\right)^{\mathrm{a}}\left(\frac{1+z_2}{2}\right)^{\mathrm{b}}\left(\frac{1+z_1 z_2}{2}\right)^{\mathrm{c}}\left(\frac{1+z_1 z_2^{-1}}{2}\right)^{\mathrm{d}} \\
&= \frac{1}{2^{\mathrm{d}}}\left(1+\frac{z_1}{z_2}\right)^{\mathrm{d}} \tilde{\mathbb{B}}_{\mathrm{a,b,c}}(z_1, z_2) = 2^{-\mathrm{d}} \sum_{\ell=0}^{\mathrm{d}} \binom{\mathrm{d}}{\ell}\left(\frac{z_1}{z_2}\right)^{\ell} \tilde{\mathbb{B}}_{\mathrm{a,b,c}}(z_1, z_2) \\
&= \sum_{\ell=0}^{\mathrm{d}} \lambda_{\mathrm{d},\ell}\, z_1^{\ell}\, z_2^{-\ell}\, \tilde{\mathbb{B}}_{\mathrm{a,b,c}}(z_1, z_2),
\end{aligned} \tag{3.3.10}$$

where $\lambda_{\mathrm{d},\ell} := 2^{-\mathrm{d}}\binom{\mathrm{d}}{\ell}$. By using the identity

$$(z_1 + z_2) = (1 + z_1)(1 + z_2) - (1 + z_1 z_2),$$

we write

$$\tilde{\mathbb{B}}_{0,0,0,1}(z_1, z_2) = \frac{1}{z_2}\left[\frac{z_1 + z_2}{2}\right] = \frac{1}{z_2}\left[2\tilde{\mathbb{B}}_{1,1,0}(z_1, z_2) - \tilde{\mathbb{B}}_{0,0,1}(z_1, z_2)\right].$$

Thus, for 4-directional box splines, we have that

$$\begin{aligned}\mathbb{B}_{\mathrm{a,b,c,d}}(z_1,z_2) &= \mathbb{B}_{\mathrm{a,b,c}}(z_1,z_2)\frac{1}{z_2^{\mathrm{d}}}\left(2\tilde{\mathbb{B}}_{1,1,0}(z_1,z_2)-\tilde{\mathbb{B}}_{0,0,1}(z_1,z_2)\right)^{\mathrm{d}}\\ &= \sum_{\ell=0}^{\mathrm{d}}\frac{2^{\ell}(-1)^{\mathrm{d}-\ell}}{z_2^{\mathrm{d}}}\binom{\mathrm{d}}{\ell}\tilde{\mathbb{B}}_{\mathrm{a}+\ell,\mathrm{b}+\ell,\mathrm{c}+\mathrm{d}-\ell}(z_1,z_2).\end{aligned}$$

3.4 The Butterfly subdivision scheme

The above mentioned algebraic results are key tools to verify the interpolatory Butterfly subdivision scheme on regular triangulations. This subdivision scheme is derived systematically by using Theorem 3.3.5 with set of generators of box spline symbols, and by taking their convex combination, and finally doing appropriate normalizations. According to [DLG90, KLY07, DHL11], the Laurent polynomial of the Butterfly subdivision scheme is given by

$$\mathcal{P}_w(z_1,z_2)=\sum_{i=-3}^{3}\sum_{j=-3}^{3}p_{i,j}\,z_1^i\,z_2^j=\frac{1}{2}(1+z_1)(1+z_2)(1+z_1z_2)(1-w\,\mathrm{r}(z_1,z_2))(z_1z_2)^{-1}, \tag{3.4.1}$$

where

$$\begin{aligned}r(z_1,z_2) = {}& 2z_1^{-2}z_2^{-1}+2z_1^{-1}z_2^{-2}-4z_1^{-1}z_2^{-1}-4z_1^{-1}-4z_2^{-1}+2z_1^{-1}z_2\\ &+2z_1z_2^{-1}+12-4z_1-4z_2-4z_1z_2+2z_1^2z_2+2z_1z_2^2,\end{aligned} \tag{3.4.2}$$

with $r(z_1,z_2)=r(z_2,z_1)=r(z_1^{-1},z_2^{-1})$, and $r(\mathbf{1})=0$, $r(\varepsilon)=16$, $\forall\varepsilon\in \mathrm{Z}\backslash\{\mathbf{1}\}$, as in (2.5.4).

Equation (3.4.1), together with (3.4.2), can be expressed in terms of its box spline constituents by

$$\begin{aligned}\frac{1}{4}z_1^3z_2^3\,\mathcal{P}_w(z_1,z_2) \quad = \quad & \Big\{7z_1z_2\,\tilde{\mathbb{B}}_{2,2,2}(z_1,z_2)-2z_1\,\tilde{\mathbb{B}}_{1,3,3}(z_1,z_2)\\ & -2z_2\,\tilde{\mathbb{B}}_{3,1,3}(z_1,z_2)-2z_1z_2\,\tilde{\mathbb{B}}_{3,3,1}(z_1,z_2)\Big\},\end{aligned} \tag{3.4.3}$$

where $\tilde{\mathbb{B}}_{\mathrm{a,b,c}}$ is as in (1.9.8). To verify this result, we choose the set of generators from $\mathcal{I}_4$ for the power ideal $\mathcal{I}^4$ as given, according to (3.3.8), by

$$\left\{\tilde{\mathbb{B}}_{2,2,2}(z_1,z_2),\tilde{\mathbb{B}}_{3,1,3}(z_1,z_2),\tilde{\mathbb{B}}_{3,3,1}(z_1,z_2),\tilde{\mathbb{B}}_{1,3,3}(z_1,z_2)\right\}.$$

Now, by choosing the appropriate normalised q-symbols:

$$\left(\frac{7+6z_1z_2}{13}\right),\ z_2,z_1,\left(\frac{1+z_1+z_2}{3}\right),$$

and by keeping the convex combination of the coefficients, we get

$$\begin{aligned}z_1^3z_2^3\,\mathcal{P}_w(z_1,z_2) = 4\Bigg[&26\cdot\left(\frac{7+6z_1z_2}{13}\right)\tilde{\mathbb{B}}_{3,3,1}(z_1,z_2)-2z_2\,\tilde{\mathbb{B}}_{3,1,3}(z_1,z_2)\\ &-2z_1\,\tilde{\mathbb{B}}_{1,3,3}(z_1,z_2)-21\left(\frac{1+z_1+z_2}{3}\right)\tilde{\mathbb{B}}_{2,2,2}(z_1,z_2)\Bigg].\end{aligned} \tag{3.4.4}$$

In order to rewrite equation (3.4.4), a simplified expression that preserves both convexity of coefficients and normalization of the symbols is established.

To attain this, we expand equation (3.4.4) to a more simplified form, that is,

$$\begin{aligned}
&26\left(\frac{7+6z_1z_2}{13}\right)\tilde{\mathbb{B}}_{3,3,1}(z_1,z_2)-21\left(\frac{1+z_1+z_2}{3}\right)\tilde{\mathbb{B}}_{2,2,2}(z_1,z_2)\\
&=2(7+6z_1z_2)\,\tilde{\mathbb{B}}_{3,3,1}(z_1,z_2)-7(1+z_1+z_2)\,\tilde{\mathbb{B}}_{2,2,2}(z_1,z_2)\\
&=2\,[7(1+z_1z_2)-z_1z_2]\,\tilde{\mathbb{B}}_{3,3,1}(z_1,z_2)-7(1+z_1+z_2)\tilde{\mathbb{B}}_{2,2,2}(z_1,z_2)\\
&=2\left(14\left(\frac{1+z_1z_2}{2}\right)\tilde{\mathbb{B}}_{3,3,1}(z_1,z_2)\right)-2z_1z_2\,\tilde{\mathbb{B}}_{3,3,1}(z_1,z_2)\\
&\quad-7(1+z_1+z_2)\,\tilde{\mathbb{B}}_{2,2,2}(z_1,z_2)\\
&=7(1+z_1+z_2+z_1z_2)\,\tilde{\mathbb{B}}_{2,2,2}(z_1,z_2)-7(1+z_1+z_2)\,\tilde{\mathbb{B}}_{2,2,2}(z_1,z_2)\\
&\quad-2z_1z_2\,\tilde{\mathbb{B}}_{3,3,1}(z_1,z_2)\\
&=7z_1z_2\,\tilde{\mathbb{B}}_{2,2,2}(z_1,z_2)-2z_1z_2\,\tilde{\mathbb{B}}_{3,3,1}(z_1,z_2).
\end{aligned}\tag{3.4.5}$$

Substituting (3.4.5) into (3.4.4), we get

$$\begin{aligned}
z_1^3\,z_2^3\,\mathcal{P}_w(z_1,z_2) \;=\; 4\Big[&7z_1z_2\,\tilde{\mathbb{B}}_{2,2,2}(z_1,z_2)-2z_1z_2\,\tilde{\mathbb{B}}_{3,3,1}(z_1,z_2)-2z_2\,\tilde{\mathbb{B}}_{3,1,3}(z_1,z_2)\\
&-2z_1\,\tilde{\mathbb{B}}_{1,3,3}(z_1,z_2)\Big].
\end{aligned}\tag{3.4.6}$$

Now, simplifying the expression in (3.4.6), by using (1.9.8), to its equivalence form:

$$\begin{aligned}
\mathcal{P}_w(z_1,z_2) \;=\; &4\Big[7z_1z_2\,\tilde{\mathbb{B}}_{2,2,2}(z_1,z_2)-2z_1\,\tilde{\mathbb{B}}_{1,3,3}(z_1,z_2)-2z_2\,\tilde{\mathbb{B}}_{3,1,3}(z_1,z_2)\\
&-2z_1z_2\,\tilde{\mathbb{B}}_{3,3,1}(z_1,z_2)\Big]\\
=\;&-\frac{1}{16}\left[z_1^3z_2^2+z_1^2z_2^3+z_1^3z_2+z_1z_2^3+z_1^2z_2^{-1}+z_1^{-1}z_2^2+z_1^{-1}z_2^2\right.\\
&\left.+z_1z_2^{-2}+z_1^{-2}z_2+z_1^{-3}z_2^{-1}+z_1^{-1}z_2^{-3}+z_1^{-2}z_2^{-3}+z_1^{-3}z_2^{-2}\right]\\
&+\frac{1}{8}[z_1^2z_2+z_1z_2^2+z_1z_2^{-1}+z_1^{-1}z_2+z_1^{-1}z_2^{-2}+z_1^{-2}z_2^{-1}]\\
&+\frac{1}{2}[z_1^{-1}+z_2^{-1}+z_1z_2+z_1+z_2+z_1^{-1}z_2^{-1}]+1,
\end{aligned}\tag{3.4.7}$$

which is the Butterfly subdivision scheme with tension parameter $w=\frac{1}{16}$, and with its refinement mask given below in (3.4.9). The grid points in the set M for which the refinement mask $\mathbf{p}^I$ will be non-zero are given by

$$\mathrm{M}=(0,0)\cup\mathrm{M}^1\cup\mathrm{M}^2\cup\mathrm{M}^3,$$

where

$$\begin{aligned}
\mathrm{M}^1 &:= \{(1,0),(0,1),(1,1),(-1,0),(0,-1),(-1,-1)\}\,;\\
\mathrm{M}^2 &:= \{(1,-1),(2,1),(1,2),(-1,-2),(-1,1)(-2,-1)\}\,;\\
\mathrm{M}^3 &:= \{(3,2),(2,3),(3,1),(1,3),(2,-1),(-2,-3),(-1,2),(1,-2),\\
&\quad\;\;(-2,1),(-3,-1),(-1,-3),(-3,-2)\}\,.
\end{aligned}\tag{3.4.8}$$

Thus, in terms of the subdivision polynomial, the mask symbol representation of this

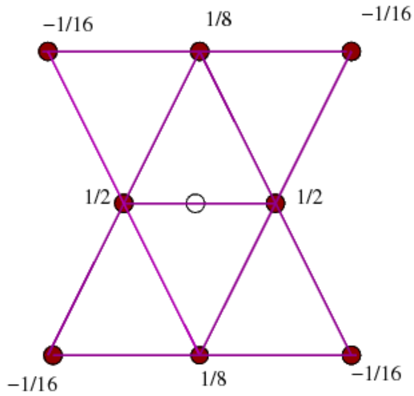

Figure 3.1: *Stencil of the Butterfly subdivision scheme*

subdivision scheme is given by

$$\mathrm{P}(z_1, z_2) = \sum_{(j,k)\in \mathrm{M}} p_{(j,k)}\, z_1^j\, z_2^k,$$

where

$$\begin{cases} p_{(0,0)} & = 1, \quad \text{as it is interpolatory;} \\ p_{(j,k)} & = \frac{1}{2}, \quad \text{if} \quad (j,k) \in \mathrm{M}^1; \\ p_{(j,k)} & = \frac{1}{8}, \quad \text{if} \quad (j,k) \in \mathrm{M}^2; \\ p_{(j,k)} & = -\frac{1}{16}, \quad \text{if} \quad (j,k) \in \mathrm{M}^3. \end{cases} \tag{3.4.9}$$

as shown in Figure 3.1.

Note that (3.4.7), together with (3.4.9), gives the matrix representation of mask coefficients $\{p_{i,j} : -3 \le i, j \le 3\}$ as

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{16} & -\frac{1}{16} & 0 \\ 0 & 0 & -\frac{1}{16} & 0 & \frac{1}{8} & 0 & -\frac{1}{16} \\ 0 & -\frac{1}{16} & \frac{1}{8} & \frac{1}{2} & \frac{1}{2} & \frac{1}{8} & -\frac{1}{16} \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{16} & \frac{1}{8} & \frac{1}{2} & \frac{1}{2} & \frac{1}{8} & -\frac{1}{16} & 0 \\ -\frac{1}{16} & 0 & \frac{1}{8} & 0 & -\frac{1}{16} & 0 & 0 \\ 0 & -\frac{1}{16} & -\frac{1}{16} & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.4.10}$$

Thus, one can view the Butterfly subdivision symbol algebraically as

$$\mathcal{P}_w(z_1, z_2) \in \left\langle \left\{ \tilde{\mathbb{B}}_{2,2,2}(z_1, z_2),\ \tilde{\mathbb{B}}_{1,3,3}(z_1, z_2),\ \tilde{\mathbb{B}}_{3,1,3}(z_1, z_2), \tilde{\mathbb{B}}_{3,3,1}(z_1, z_2) \right\} \right\rangle. \tag{3.4.11}$$

By applying Remark 3.3.2 and (3.4.7), we observe that each of the generating box splines can be factored in terms of the normalised Courant hat function, that is,

$\mathcal{P}_w(z_1, z_2) \in \langle \tilde{\mathbb{B}}_{1,1,1} \rangle$, and hence, we can write

$$\mathcal{P}_w(z_1, z_2) = \tilde{\mathbb{B}}_{1,1,1}(z_1, z_2)\, \mathbf{b}(z_1, z_2). \tag{3.4.12}$$

Consequently,

$$\begin{aligned} \mathbf{b}(z_1, z_2) \quad = \quad & z_1^{-3} z_2^{-2} \Big[28\, z_1^{-1} z_2^{-1} \tilde{\mathbb{B}}_{1,1,1}(z_1, z_2) - 8 z_1^{-2} z_2^{-2}\, \tilde{\mathbb{B}}_{0,2,2}(z_1, z_2) \\ & -8 z_1^{-3} z_2^{-2}\, \tilde{\mathbb{B}}_{2,0,2}(z_1, z_2) - 8 z_1^{-1} z_2^{-1}\, \tilde{\mathbb{B}}_{2,2,0}(z_1, z_2) \Big]. \end{aligned} \tag{3.4.13}$$

Writing (3.4.13) in a more usable way, we have that

$$\tilde{Q}(z_1,z_2) := z_1^3 z_2^2\,\mathbf{b}(z_1,z_2) = 28z_1^2z_2^2\,\tilde{\mathbb{B}}_{1,1,1}(z_1,z_2) - 8z_1\,\tilde{\mathbb{B}}_{0,2,2}(z_1,z_2) - 8z_2\,\tilde{\mathbb{B}}_{2,0,2}(z_1,z_2) - 8z_1^2z_2\,\tilde{\mathbb{B}}_{2,2,0}(z_1,z_2). \quad (3.4.14)$$

One can observe that the symbol $\tilde{Q}(z_1,z_2)$ in equation (3.4.14) does not define a convergent subdivision scheme according to Theorem 3.3.5, although each of the summands in $\tilde{Q}(z_1,z_2)$ does correspond to a convergent subdivision scheme. Besides, the Butterfly subdivision scheme $\mathcal{P}_w$ obtained using the algebraic approach is an interpolatory subdivision scheme defined on a triangular grid with a single tension parameter w, but none of the summands in the affine combination satisfy this property. Details of computational analysis of smoothness and convergence of this subdivision scheme is given in the next chapter.

3.4.1 Some features

We mention the following basic features of the Butterfly subdivision scheme:

- The mask plots of the Butterfly subdivision scheme as in [KLY07] on the integer grid $\mathbb{Z}^2$ are displayed in Figures 3.2 -3.4.

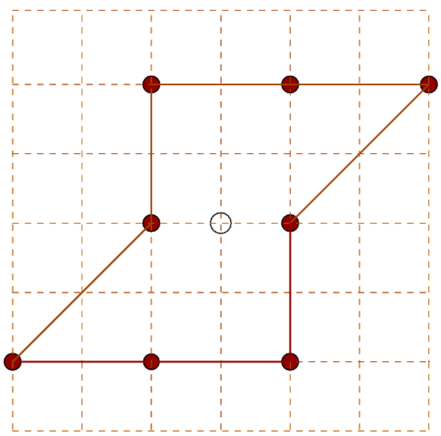

Figure 3.2: *Odd-Even mask*

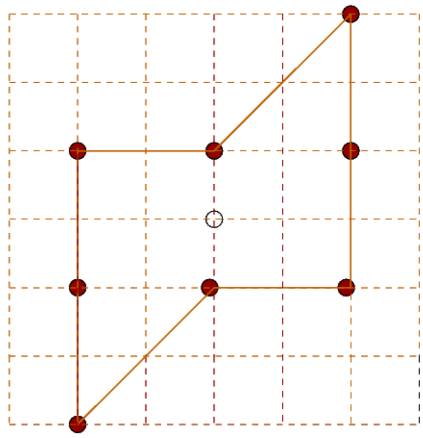

Figure 3.3: *Even-Odd mask*

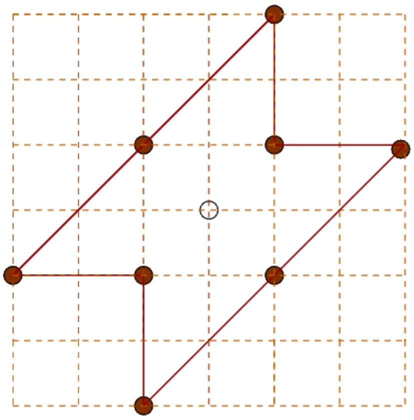

Figure 3.4: *Odd-Odd mask*

- The matrix version of subdivision polynomial of the Butterfly subdivision scheme as in (3.4.7) with $w = \frac{1}{16}$ is given by

$$P(z_1, z_2) = \frac{1}{16}\begin{pmatrix} z_1^{-3} & z_1^{-2} & z_1^{-1} & z_1^{0} & z_1^{1} & z_1^{2} & z_1^{3}\end{pmatrix} \times \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 2 & 8 & 8 & 2 & -1 \\ -1 & 0 & 8 & 16 & 8 & 0 & 0 \\ -1 & 2 & 8 & 8 & 2 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} z_2^{-3} \\ z_2^{-2} \\ z_2^{-1} \\ z_2^{0} \\ z_2^{1} \\ z_2^{2} \\ z_2^{3}\end{pmatrix}. \tag{3.4.15}$$

- From (2.3.3), (2.8.2) together with (3.4.10), we note that the subdivision formulation of the Butterfly subdivision scheme is given explicitly by

$$\left.\begin{aligned}
\mathbf{c}^{k+1}_{2i+1,2j} &= \frac{1}{2}\left(\mathbf{c}^k_{i,j} + \mathbf{c}^k_{i+1,j}\right) + 2w\left(\mathbf{c}^k_{i,j-1} + \mathbf{c}^k_{i+1,j+1}\right) \\
&\quad - w\left(\mathbf{c}^k_{i-1,j-1} + \mathbf{c}^k_{i+1,j-1} + \mathbf{c}^k_{i,j+1} + \mathbf{c}^k_{i+2,j+1}\right); \\
\mathbf{c}^{k+1}_{2i,2j+1} &= \frac{1}{2}\left(\mathbf{c}^k_{i,j} + \mathbf{c}^k_{i,j+1}\right) + 2w\left(\mathbf{c}^k_{i-1,j} + \mathbf{c}^k_{i+1,j+1}\right) \\
&\quad - w\left(\mathbf{c}^k_{i-1,j-1} + \mathbf{c}^k_{i-1,j+1} + \mathbf{c}^k_{i+1,j} + \mathbf{c}^k_{i+1,j+2}\right); \\
\mathbf{c}^{k+1}_{2i+1,2j+1} &= \frac{1}{2}\left(\mathbf{c}^k_{i,j} + \mathbf{c}^k_{i+1,j+1}\right) + 2w\left(\mathbf{c}^k_{i+1,j} + \mathbf{c}^k_{i,j+1}\right) \\
&\quad - w\left(\mathbf{c}^k_{i,j-1} + \mathbf{c}^k_{i-1,j} + \mathbf{c}^k_{i+2,j+1} + \mathbf{c}^k_{i+1,j+2}\right); \\
\mathbf{c}^{k+1}_{2i,2j} &= \mathbf{c}^k_{i,j}.
\end{aligned}\right\} \tag{3.4.16}$$

- The corresponding basic limit function for the Butterfly subdivision mask as in (3.4.9) is displayed below in Figure 3.5.

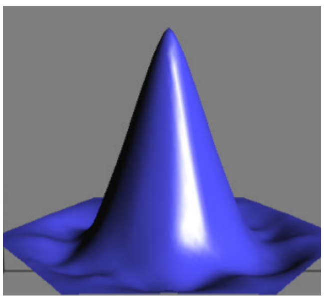

Figure 3.5: *The Butterfly basis function* $[w = \frac{1}{16}]$.

From equation (3.4.10), we observe that the stencil satisfies the conditions that for any $k \in \mathbb{Z}$,

$$\sum_j (-1)^j\, p_{j,k} = \sum_j (-1)^j\, p_{k,j} = \sum_j (-1)^j\, p_{j,j+k} = 0. \tag{3.4.17}$$

Equation (3.4.17) means that the stencil of the Butterfly subdivision scheme satisfies the the sum of even and odd masks along every vertical or horizontal line are the same. This fact induces a factorization of the corresponding Laurent polynomial of the Butterfly subdivision scheme.

- The following is an example of Butterfly surface obtained from control nets in $\mathbb{R}^3$. The Matlab source code for the following figures have been done by Thomas Yu.

Example 3.4.1. *Tetrahedron*

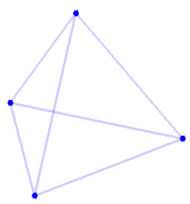

Figure 3.6: 1*st iteration*

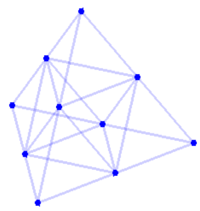

Figure 3.7: 2*nd iteration*

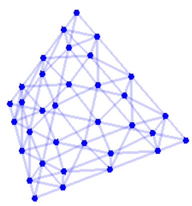

Figure 3.8: 3*rd iteration*

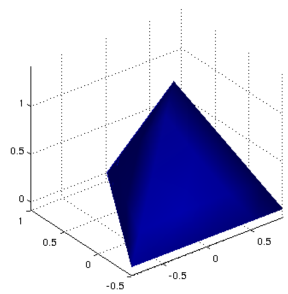

Figure 3.9: *Last iteration*

Example 3.4.2. *Torus*

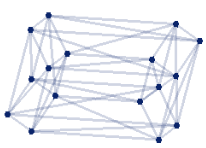

Figure 3.10: 1*st iteration*

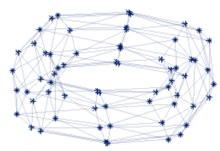

Figure 3.11: 2*nd iteration*

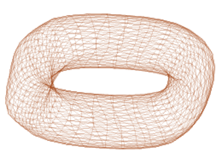

Figure 3.12: 4*th iteration*

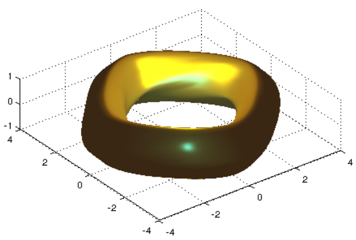

Figure 3.13: *Last iteration*

3.4.2 The polynomial reproduction property

In this subsection, we will show that the Butterfly subdivision scheme reproduces polynomials of up to degree 3 for the tension parameter $\mathrm{w} = \frac{1}{16}$. The following result proves the polynomial reproducibility of the Butterfly subdivision scheme.

Theorem 3.4.3. *The Butterfly subdivision scheme* (3.4.1) *with* $\mathrm{w} = \frac{1}{16}$ *reproduces cubic bivariate polynomials, in the sense that*

$$\sum_{(k,\ell)\in\mathbb{Z}^2} p_{i-2k,j-2\ell}\, f(k,\ell) = f\left(\frac{i}{2},\frac{j}{2}\right), \quad i,j\in\mathbb{Z},\ f\in\Pi_3^2, \tag{3.4.18}$$

where the sequence $\{p_{i,j}\}$ *is defined by* (3.4.1), *with* $\mathrm{w} = \frac{1}{16}$.

Proof. It will suffice to prove that (3.4.18) is satisfied by the monomial $f(x,y) = x^\alpha y^\beta$, where α and β are non-negative integers, with $\alpha+\beta \le 3$.

Since $i,j \in \{-3,-2,\ldots,2,3\}$ and $i-2k, j-2\ell \in \{-3,-2,\ldots,2,3\}$, we have that

$$k \in \left\{\lceil\frac{1}{2}(i-3)\rceil, \lceil\frac{1}{2}(i-3)\rceil+1,\ldots,\lfloor\frac{1}{2}(i+3)\rfloor\right\}$$

and

$$\ell \in \left\{\lceil\frac{1}{2}(j-3)\rceil, \lceil\frac{1}{2}(j-3)\rceil+1,\ldots,\lfloor\frac{1}{2}(j+3)\rfloor\right\}.$$

It follows that the possible integer pairs are given by

$$\begin{aligned}[k\times\ell] \;=\; &\Big\{(\lceil\tfrac{1}{2}(i-3)\rceil, \lceil\tfrac{1}{2}(j-3)\rceil),\ldots,(\lceil\tfrac{1}{2}(i-3)\rceil, \lfloor\tfrac{1}{2}(j+3)\rfloor),\\ &(\lceil\tfrac{1}{2}(i-3)\rceil+1, \lceil\tfrac{1}{2}(j-3)\rceil),\ldots,(\lceil\tfrac{1}{2}(i-3)\rceil+1, \lfloor\tfrac{1}{2}(j+3)\rfloor)\\ &\ldots,(\lfloor\tfrac{1}{2}(i+3)\rfloor, \lceil\tfrac{1}{2}(j-3)\rceil),\ldots,(\lfloor\tfrac{1}{2}(i+3)\rfloor, \lfloor\tfrac{1}{2}(j+3)\rfloor)\Big\}.\end{aligned} \tag{3.4.19}$$

Consider the following cases:

- **Case I**: Consider the monomial $f(x,y) = x^\alpha$ with $\alpha = 0$, or $\alpha = 1$, or $\alpha = 2$ or $\alpha = 3$.

  (a) For $i = 2m+1$ and $j = 2n+1$, we have to show that

$$\sum_{(k,\ell)\in\mathbb{Z}^2} p_{(2m+1)-2k,(2n+1)-2\ell}\, f(k,\ell) = f\left(\frac{2m+1}{2},\frac{2n+1}{2}\right). \tag{3.4.20}$$

$$\begin{aligned}
\sum_{(k,\ell)\in\mathbb{Z}^2} p_{(2m+1)-2k,(2n+1)-2\ell}\, f(k,\ell) &= p_{3,3}\, f(m-1,n-1) + p_{3,1}\, f(m-1,n) \\
&\quad + p_{3,-1}\, f(m-1,n+1) + p_{3,-3}\, f(m-1,n+2) \\
&\quad + p_{1,3}\, f(m,n-1) + p_{1,1}\, f(m,n) + p_{1,-1}\, f(m,n+1) \\
&\quad + p_{1,-3}\, f(m,n+2) + p_{-1,3}\, f(m+1,n-1) \\
&\quad + p_{-1,1}\, f(m+1,n) + p_{-1,-3}\, f(m+1,n+2) \\
&\quad + p_{-3,3}\, f(m+2,n-1) + p_{-3,1}\, f(m+2,n) \\
&\quad + p_{-3,-1}\, f(m+2,n+1) + p_{-3,3}\, f(m+2,n+2) \\
&= -\frac{1}{16}\left[(m-1)^\alpha + m^\alpha + m^\alpha + (m+2)^\alpha\right] \\
&\quad + \frac{1}{2} m^\alpha + \frac{1}{8}\left[m^\alpha + m^\alpha\right]. \qquad (3.4.21)
\end{aligned}$$

For fixed values $\alpha \in \{0,1,2,3\}$ and from (3.4.21), we now obtain :

For $\alpha = 0$,

$$\sum_{(k,\ell)} p_{(2m+1)-2k,(2n+1)-2\ell}\, f(k,\ell) = 1 = f\left(\frac{2m+1}{2}, \frac{2n+1}{2}\right).$$

For $\alpha = 1$,

$$\sum_{(k,\ell)} p_{(2m+1)-2k,(2n+1)-2\ell}\, f(k,\ell) = \left(m+\frac{1}{2}\right) = f\left(\frac{m+1}{2}, \frac{n+1}{2}\right).$$

For $\alpha = 2$,

$$\sum_{(k,\ell)} p_{(2m+1)-2k,(2n+1)-2\ell}\, f(k,\ell) = \left(m+\frac{1}{2}\right)^2 = f\left(\frac{m+1}{2}, \frac{n+1}{2}\right).$$

For $\alpha = 3$,

$$\sum_{(k,\ell)} p_{(2m+1)-2k,2n-2\ell}\, f(k,\ell) = = \left(m+\frac{1}{2}\right)^3 = f\left(\frac{m+1}{2}, \frac{n+1}{2}\right).$$

(b) For $i = 2m+1$ and $j = 2n$, with $f(x,y) = x^\alpha$ with $\alpha = 0,1,2,3$,

$$\begin{aligned}
\sum_{(k,\ell)} p_{(2m+1)-2k,2n-2\ell}\, f(k,\ell) &= p_{3,2}\,(m-1)^\alpha + p_{3,0}\,(m-1)^\alpha + p_{3,-2}\,(m-1)^\alpha \\
&\quad + p_{1,2}\, m^\alpha + p_{1,0}\, m^\alpha + p_{1,-2}\, m^\alpha + p_{-3,2}\,(m+2)^\alpha \\
&\quad + p_{-3,0}\,(m+2)^\alpha + p_{-3,-2}\,(m+2)^\alpha \\
&= -\frac{1}{16}\left[(m-1)^\alpha + m^\alpha + (m+1)^\alpha + (m+2)^\alpha\right] \\
&\quad + \frac{1}{2}\left[m^\alpha + (m+1)^\alpha\right] + \frac{1}{8}\left[m^\alpha + (m+1)^\alpha\right]
\end{aligned}$$

For $\alpha = 0$,

$$\sum_{(k,\ell)} p_{(2m+1)-2k,2n-2\ell}\, f(k,\ell) = 1.$$

For $\alpha = 1$,

$$\sum_{(\mathrm{k},\ell)} p_{(2\mathrm{m}+1)-2\mathrm{k},2\mathrm{n}-2\ell}\, \mathrm{f}(\mathrm{k},\ell) = \left(\mathrm{m} + \frac{1}{2}\right).$$

For $\alpha = 2$,

$$\sum_{(\mathrm{k},\ell)} p_{(2\mathrm{m}+1)-2\mathrm{k},2\mathrm{n}-2\ell}\, \mathrm{f}(\mathrm{k},\ell) = \left(\mathrm{m}^2 + \mathrm{m} + \frac{1}{4}\right) = \left(\mathrm{m} + \frac{1}{2}\right)^2 = \mathrm{f}\left(\frac{\mathrm{m}+1}{2}, \mathrm{n}\right).$$

For $\alpha = 3$,

$$\begin{aligned}\sum_{(\mathrm{k},\ell)} p_{(2\mathrm{m}+1)-2\mathrm{k},2\mathrm{n}-2\ell}\, \mathrm{f}(\mathrm{k},\ell) &= (m^3 + \frac{3}{2}m^2 + \frac{3}{4}m + \frac{1}{8}) \\ &= \left(\mathrm{m} + \frac{1}{2}\right)^3 = \mathrm{f}\left(\frac{\mathrm{m}+1}{2}, \mathrm{n}\right). \qquad (3.4.22)\end{aligned}$$

(c) For $\mathrm{i} = 2\mathrm{m}$, and $\mathrm{j} = 2\mathrm{n}$ with $\mathrm{f}(\mathrm{x},\mathrm{y}) = \mathrm{x}^\alpha$, $\alpha = 0$, or $\alpha = 1$ or $\alpha = 2$ or $\alpha = 3$.

$$\begin{aligned}\sum_{(\mathrm{k},\ell)} p_{2\mathrm{m}-2\mathrm{k},2\mathrm{n}-2\ell}\, \mathrm{f}(\mathrm{k},\ell) &= p_{2,2}\,(\mathrm{m}-1)^\alpha + p_{2,0}\,(\mathrm{m}-1)^\alpha + p_{2,-2}\,(\mathrm{m}-1)^\alpha + p_{0,2}\,\mathrm{m}^\alpha \\ &\quad + p_{0,0}\,\mathrm{m}^\alpha + p_{0,-2}\,\mathrm{m}^\alpha + p_{-2,2}\,(\mathrm{m}+1)^\alpha \\ &\quad + p_{-2,0}\,(\mathrm{m}+1)^\alpha + p_{-2,-2}\,(\mathrm{m}+1)^\alpha \\ &= -\frac{1}{16}\,[(m-1)^\alpha + m^\alpha + (m+1)^\alpha + (m+2)^\alpha] \\ &\quad + \frac{1}{2}\,[\mathrm{m}^\alpha + (\mathrm{m}+1)^\alpha] + \frac{1}{8}\,[\mathrm{m}^\alpha + (\mathrm{m}+1)^\alpha]. \qquad (3.4.23)\end{aligned}$$

For $\alpha = 0$,

$$\sum_{(\mathrm{k},\ell)\in\mathbb{Z}^2} p_{2\mathrm{m}+1-2\mathrm{k},2\mathrm{n}-2\ell}\, \mathrm{f}(\mathrm{k},\ell) = 1.$$

For $\alpha = 1$,

$$\sum_{(\mathrm{k},\ell)\in\mathbb{Z}^2} p_{2\mathrm{m}-2\mathrm{k},2\mathrm{n}-2\ell}\, \mathrm{f}(\mathrm{k},\ell) = \mathrm{m}.$$

For $\alpha = 2$,

$$\sum_{(\mathrm{k},\ell)\in\mathbb{Z}^2} p_{2\mathrm{m}-2\mathrm{k},2\mathrm{n}-2\ell}\, \mathrm{f}(\mathrm{k},\ell) = \mathrm{m}^2.$$

For $\alpha = 3$,

$$\sum_{(\mathrm{k},\ell)\in\mathbb{Z}^2} p_{2\mathrm{m}-2\mathrm{k},2\mathrm{n}-2\ell}\, \mathrm{f}(\mathrm{k},\ell) = \mathrm{m}^3 = \mathrm{f}\left(\frac{2\mathrm{m}}{2}, \frac{2\mathrm{n}}{2}\right).$$

Thus, from the above, we conclude that

$$\sum_{(\mathrm{k},\ell)\in\mathbb{Z}^2} p_{2\mathrm{m}-2\mathrm{k},2\mathrm{n}-2\ell}\, \mathrm{f}(\mathrm{k},\ell) = \mathrm{f}\left(\frac{\mathrm{i}}{2}, \frac{\mathrm{j}}{2}\right) \quad \text{for every} \quad \mathrm{i},\mathrm{j} \in \mathbb{Z}.$$

<u>Case II:</u> For the monomial $\mathrm{f(x,y) = x^\alpha y^\beta}$ with $(\alpha,\beta) \in \{(1,2),(1,1)\}$.

(d) For $\mathrm{i = 2m+1}$ and $\mathrm{j = 2n}$, we have that

$$\begin{aligned}
\sum_{(\mathrm{k},\ell)} p_{(\mathrm{2m+1})-\mathrm{2k},\mathrm{2n}-2\ell}\, \mathrm{f(k},\ell) &= p_{3,2}\,\mathrm{f(m-1,n-1)} + p_{3,0}\,\mathrm{f(m-1,n)} \\
&\quad + p_{3,-2}\,\mathrm{f(m-1,n+1)} + p_{1,2}\,\mathrm{f(m,n-1)} + p_{1,0}\,\mathrm{f(m,n)} \\
&\quad + p_{1,-2}\,\mathrm{f(m,n+1)} + p_{-3,2}\,\mathrm{f(m+2,n-1)} \\
&\quad + p_{-3,0}\,\mathrm{f(m+2,n)} + p_{-3,-2}\,\mathrm{f(m+2,n+1)} \\
&= -\frac{1}{16}\,\mathrm{f(m-1,n-1)} + \frac{1}{8}\,\mathrm{f(m,n-1)} + \frac{1}{2}\,\mathrm{f(m,n)} \\
&\quad -\frac{1}{16}\,\mathrm{f(m,n+1)} - \frac{1}{16}\,\mathrm{f(m+1,n-1)} + \frac{1}{2}\,\mathrm{f(m+1,n)} \\
&\quad +\frac{1}{8}\,\mathrm{f(m+1,n+1)} - \frac{1}{16}\,\mathrm{f(m+2,n+1)}. \qquad (3.4.24)
\end{aligned}$$

With $(\alpha,\beta) \in \{(1,2)\}$ and by using (3.4.24), we have $\mathrm{f(x,y) = xy^2}$, so that

$$\sum_{(k,\ell)} p_{(\mathrm{2m+1})-\mathrm{2k},\mathrm{2n}-2\ell}\, \mathrm{f(k},\ell) = \mathrm{mn^2} + \frac{1}{2}\mathrm{n^2} = \mathrm{f}\left(\mathrm{m} + \frac{1}{2}, \mathrm{n}\right).$$

$$\begin{aligned}
\sum_{(k,\ell)} p_{(2m+1)-2k,2n-2\ell}\, f(k,\ell) &= -\frac{1}{16}(m-1)(n-1)^2 + \frac{1}{8}m(n-1)^2 \\
&\quad +\frac{1}{2}mn^2 - \frac{1}{16}m(n+1)^2 - \frac{1}{16}(m+1)(n-1)^2 \\
&\quad +\frac{1}{2}(m+1)n^2 + \frac{1}{8}(m+1)(n+1)^2 \\
&\quad -\frac{1}{16}(m+2)(n+1)^2 \\
&= mn^2 + \frac{1}{2}n^2 = f\left(m+\frac{1}{2}, n\right) \qquad (3.4.25)
\end{aligned}$$

By symmetry of the monomials, it also holds true with $\mathrm{f(x,y) = x^2\,y}$.

With $(\alpha,\beta) \in \{(1,1)\}$, and by using (3.4.24), we have $\mathrm{f(x,y) = xy}$ that

$$\begin{aligned}
\sum_{(k,\ell)} p_{(2m+1)-2k,2n-2\ell}\, f(k,\ell) &= -\frac{1}{16}(m-1)(n-1) + \frac{1}{8}m(n-1) + \frac{1}{2}mn \\
&\quad -\frac{1}{16}m(n+1) - \frac{1}{16}(m+1)(n-1) + \frac{1}{2}(m+1)n \\
&\quad +\frac{1}{8}(m+1)(n+1) - \frac{1}{16}(m+2)(n+1) \\
&= \left(m+\frac{1}{2}\right)n = f\left(m+\frac{1}{2}, n\right) \qquad (3.4.26)
\end{aligned}$$

(e) For $\mathrm{i} = 2\mathrm{m}$, and $\mathrm{j} = 2\mathrm{n}$, with $\mathrm{f(x, y)} = \mathrm{x}^{\alpha}\mathrm{y}^{\beta}$, $(\alpha, \beta) \in \{(1, 2), (1, 1)\}$:

$$\begin{aligned}
\sum_{(k,\ell)} p_{2\mathrm{m}-2\mathrm{k},2\mathrm{n}-2\ell}\, \mathrm{f}(\mathrm{k}, \ell) &= p_{2,2}\, \mathrm{f}(\mathrm{m}-1, \mathrm{n}-1) + p_{2,0}\, \mathrm{f}(\mathrm{m}-1, \mathrm{n}) \\
&\quad + p_{2,-2}\, \mathrm{f}(\mathrm{m}-1, \mathrm{n}+1) + p_{0,2}\, \mathrm{f}(\mathrm{m}, \mathrm{n}-1) \\
&\quad + p_{0,0}\, \mathrm{f}(\mathrm{m}, \mathrm{n}) + p_{0,-2}\, \mathrm{f}(\mathrm{m}, \mathrm{n}+1) \\
&\quad + p_{-2,2}\, \mathrm{f}(\mathrm{m}+1, \mathrm{n}-1) + p_{-2,0}\, \mathrm{f}(\mathrm{m}+1, \mathrm{n}) \\
&\quad + p_{-2,-2}\, \mathrm{f}(\mathrm{m}+1, \mathrm{n}+1) \\
&= \mathrm{f}\,(\mathrm{m}, \mathrm{n}) = \mathrm{f}\left(\frac{2\mathrm{m}}{2}, \frac{2\mathrm{n}}{2}\right),
\end{aligned} \tag{3.4.27}$$

which holds true for any $f \in \Pi_3$.

(f) For $i = 2m$, and $j = 2n + 1$ with $f(x, y) = x^{\alpha}y^{\beta}$, $(\alpha, \beta) \in \{(1, 2), (1, 1)\}$:

$$\begin{aligned}
\sum_{(\mathrm{k},\ell)} p_{\mathrm{i}-2\mathrm{k},\mathrm{j}-2\ell}\, \mathrm{f}(\mathrm{k}, \ell) &= p_{2,3}\, \mathrm{f}(\mathrm{m}-1, \mathrm{n}-1) + p_{2,1}\, \mathrm{f}(\mathrm{m}-1, \mathrm{n}) \\
&\quad + p_{2,-1}\, \mathrm{f}(\mathrm{m}-1, \mathrm{n}+1) + p_{2,-3}\, \mathrm{f}(\mathrm{m}-1, \mathrm{n}+2) \\
&\quad + p_{0,3}\, \mathrm{f}(\mathrm{m}, \mathrm{n}-1) + p_{0,1}\, \mathrm{f}(\mathrm{m}, \mathrm{n}) + p_{0,-1}\, \mathrm{f}(\mathrm{m}, \mathrm{n}+1) \\
&\quad + p_{0,-3}\, \mathrm{f}(\mathrm{m}, \mathrm{n}+2) + p_{-2,3}\, \mathrm{f}(\mathrm{m}+1, \mathrm{n}-1) \\
&\quad + p_{-2,1}\, \mathrm{f}(\mathrm{m}+1, \mathrm{n}) + p_{-2,-1}\, \mathrm{f}(\mathrm{m}+1, \mathrm{n}+1) \\
&\quad + p_{-2,-3}\, \mathrm{f}(\mathrm{m}+1, \mathrm{n}+2) \\
&= -\frac{1}{16}\,[\mathrm{f}(\mathrm{m}-1, \mathrm{n}-1) + \mathrm{f}(\mathrm{m}-1, \mathrm{n}+1) \\
&\quad + \mathrm{f}(\mathrm{m}+1, \mathrm{n}) + \mathrm{f}(\mathrm{m}+1, \mathrm{n}+2)] \\
&\quad + \frac{1}{8}\,[\mathrm{f}(\mathrm{m}-1, \mathrm{n}) + \mathrm{f}(\mathrm{m}+1, \mathrm{n}+1)] \\
&\quad + \frac{1}{2}\,[\mathrm{f}(\mathrm{m}, \mathrm{n}) + \mathrm{f}(\mathrm{m}, \mathrm{n}+1)].
\end{aligned} \tag{3.4.28}$$

Now by using (3.4.28) with $\mathrm{f(x, y)} = \mathrm{x\,y}$, we have

$$\sum_{(\mathrm{k},\ell)} p_{\mathrm{i}-2\mathrm{k},\mathrm{j}-2\ell}\, \mathrm{f}(\mathrm{k}, \ell) = \mathrm{m}\left(\mathrm{n} + \frac{1}{2}\right) = \mathrm{f}\left(\mathrm{m}, \frac{\mathrm{n}+1}{2}\right),$$

and for $(\alpha, \beta) \in \{(1, 2)\}$, and using (3.4.28) with $\mathrm{f(x, y)} = \mathrm{x\,y}^2$, we have

$$\begin{aligned}
\sum_{(\mathrm{k},\ell)} p_{\mathrm{i}-2\mathrm{k},\mathrm{j}-2\ell}\, \mathrm{f}(\mathrm{k}, \ell) &= -\frac{1}{16}\left[(m-1)(n-1)^2 + (m-1)(n+1)^2 + (m+1)n^2\right. \\
&\quad \left. + (m+1)(n+2)^2\right] + \frac{1}{8}\left[(m-1)n^2 + (m+1)(n+1)^2\right] \\
&\quad + \frac{1}{2}\left[mn^2 + m(n+1)^2\right] \\
&\quad \mathrm{m}\left(\mathrm{n} + \frac{1}{2}\right)^2 = \mathrm{f}\left(\mathrm{m}, \frac{\mathrm{n}+1}{2}\right).
\end{aligned} \tag{3.4.29}$$

Thus, combining all the cases, we have established the polynomial filling property

$$\sum_{(\mathrm{k},\ell)} p_{\mathrm{i}-2\mathrm{k},\mathrm{j}-2\ell}\, \mathrm{f}(\mathrm{k}, \ell) = \mathrm{f}\left(\frac{\mathrm{i}}{2}, \frac{\mathrm{j}}{2}\right), \quad \mathrm{i}, \mathrm{j} \in \mathbb{Z},\ \mathrm{f} \in \Pi_3^2.$$

$\square$

In the next section, we derive a new interpolatory subdivision scheme, as was obtained in [vdB10], before attempting to similarly derive the Butterfly subdivision scheme directly from equation (2.6.6).

3.5 Derivation of a new interpolatory scheme

The Laurent polynomial equation (2.6.6), after normalisation,

$$\mathcal{P}^I(z_1, z_2) + \mathcal{P}^I(-z_1, z_2) + \mathcal{P}^I(z_1, -z_2) + \mathcal{P}^I(-z_1, -z_2) = 1. \tag{3.5.1}$$

may be used to construct interpolatory bivariate subdivision schemes in a direct manner.

In order to solve equation (3.5.1), we use the normalized symbol of the Courant hat function [3]

$$P(z_1, z_2) := \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1z_2}{2}\right), \tag{3.5.2}$$

which is used to define the interpolatory symbol $\mathcal{P}^I(z_1, z_2)$ as

$$\mathcal{P}^I(z_1, z_2) := P(z_1, z_2)\,\mathcal{A}(z_1, z_2), \tag{3.5.3}$$

where $\mathcal{A}(z_1, z_2)$ is an arbitrary Laurent polynomial.

Now, substituting (3.5.3) into (3.5.1) yields

$$\begin{gathered}\mathcal{P}^I(z_1, z_2) + \mathcal{P}^I(-z_1, z_2) + \mathcal{P}^I(z_1, -z_2) + \mathcal{P}^I(-z_1, -z_2) = 1 \Leftrightarrow \\ \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1z_2}{2}\right)\mathcal{A}(z_1, z_2) + \left(\frac{1-z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1-z_1z_2}{2}\right)\mathcal{A}(-z_1, z_2) + \\ \left(\frac{1+z_1}{2}\right)\left(\frac{1-z_2}{2}\right)\left(\frac{1-z_1z_2}{2}\right)\mathcal{A}(z_1, -z_2) + \left(\frac{1-z_1}{2}\right)\left(\frac{1-z_2}{2}\right)\left(\frac{1-z_1z_2}{2}\right)\mathcal{A}(-z_1, -z_2) = 1.\end{gathered} \tag{3.5.4}$$

Letting $\mathcal{A}(z_1, z_2) := f(z_1z_2)$, $u = z_1z_2$, where f is a univariate Laurent polynomial, and substituting into equation (3.5.4), we get a particular solution to the identity (3.5.4), as given by $\mathcal{A}(z_1, z_2) := \dfrac{1}{z_1z_2}$.

Assuming that the general solution $\mathrm{G}(z_1, z_2) := \mathcal{A}(z_1, z_2) + \sum_{i=1}^{k} \alpha_i\, \mathrm{H}_i(z_1, z_2)$ exists, then a

[3] A Laurent polynomial symbol $\mathrm{P}(z_1, z_2)$ is said to be normalised if P satisfies $\mathrm{P}(1,1) = 1$.

set of homogeneous solutions $\mathrm{H_i}$ are given, according to [vdB10], by

$$\left.\begin{aligned}
\mathrm{H}_1(z_1,z_2) &:= \frac{(1-z_1)\,(1+z_2)\,(1-z_1z_2)}{z_1z_2}\\
\mathrm{H}_2(z_1,z_2) &:= \frac{(1+z_1)\,(1-z_2)\,(1-z_1z_2)}{z_1z_2}\\
\mathrm{H}_3(z_1,z_2) &:= \frac{(1-z_1)\,(1-z_2)\,(1-z_1z_2)}{z_1z_2}\\
\mathrm{H}_4(z_1,z_2) &:= \frac{(1-z_1)\,(1+z_2)\,(1-z_1z_2)}{z_1^3z_2^3}\\
\mathrm{H}_5(z_1,z_2) &:= \frac{(1+z_1)\,(1-z_2)\,(1-z_1z_2)}{z_1^3z_2^3}\\
\mathrm{H}_6(z_1,z_2) &:= \frac{(1-z_1)\,(1-z_2)\,(1-z_1z_2)}{z_1^3z_2^3}
\end{aligned}\right\}, \tag{3.5.5}$$

that is, the functions $\mathrm{H} := \mathrm{H}_i$, $i = 1, 2, \ldots, 6$, satisfy the equation

$$\begin{aligned}
&P(z_1,z_2)\,\mathrm{H}(z_1,z_2) + P(-z_1,z_2)\,\mathrm{H}(-z_1,z_2)\\
&+ P(z_1,-z_2)\,\mathrm{H}(z_1,-z_2) + P(-z_1,-z_2)\,\mathrm{H}(-z_1,-z_2) = 0,
\end{aligned} \tag{3.5.6}$$

where $P(z_1, z_2)$ is the symbol as in (3.5.2).

It follows that a Laurent polynomial solution to equation (3.5.1) is given by

$$\left.\begin{aligned}
\mathrm{G}(z_1,z_2) := \mathcal{A}(z_1,z_2) + \alpha\,\mathrm{H}_1(z_1,z_2) + \beta\,\mathrm{H}_2(z_1,z_2) + \gamma\,\mathrm{H}_3(z_1,z_2)\\
+ \alpha\,\mathrm{H}_4(z_1,z_2) + \beta\,\mathrm{H}_5(z_1,z_2) - \gamma\,\mathrm{H}_6(z_1,z_2)
\end{aligned}\right\}, \tag{3.5.7}$$

for $(z_1, z_2) \in \mathbb{C}^2\backslash\{(0,0)\}$, and with α, β and γ denoting arbitrary constants. Hence the polynomial

$$\mathcal{P}^I(z_1,z_2) := \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1z_2}{2}\right)\mathrm{G}(z_1,z_2);\quad (z_1,z_2)\in\mathbb{C}^2\backslash\{(0,0)\} \tag{3.5.8}$$

satisfies equation (3.5.1), that is, $\mathcal{P}^I$ is an interpolatory mask symbol.

Observe from (3.5.8), (3.5.7), (3.5.6), together with $\mathcal{A}(z_1, z_2) = \dfrac{1}{z_1z_2}$, that the Laurent polynomial $\mathcal{P}^I$ satisfies the symmetric condition $\mathcal{P}^I(z_1, z_2) = \mathcal{P}^I(z_2, z_1)$ for $(z_1, z_2) \in \mathbb{C}^2\backslash\{(0,0)\}$.

Simplifying the Laurent polynomial $\mathcal{P}^I$, we get

$$\begin{aligned}
\mathcal{P}^I(z_1,z_2) := \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1z_2}{2}\right)\Bigg[&\frac{1}{z_1z_2} + \frac{\alpha\,(1-z_1)\,(1+z_2)\,(1-z_1z_2)}{z_1z_2}\\
&+\frac{\beta\;(1+z_1)\,(1-z_2)\,(1-z_1z_2)}{z_1z_2} + \frac{\gamma\;(1-z_1)\,(1+z_2)\,(1-z_1z_2)}{z_1z_2}\\
&+\frac{\alpha\;(1-z_1)\,(1+z_2)+(1-z_1z_2)}{z_1^3z_2^3} + \frac{\beta\;(1+z_1)\,(1-z_2)\,(1-z_1z_2)}{z_1^3z_2^3}\\
&-\frac{\gamma\;(1-z_1)\,(1-z_2)\,(1-z_1z_2)}{z_1^3z_2^3}\Bigg].
\end{aligned} \tag{3.5.9}$$

Under the preference of a refinement mask with smallest support, we set $\gamma = \alpha + \beta$, and it follows that

$$P(z_1, z_2) := \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1^{-1}z_2^{-1}}{2}\right)\Bigg[1 + 2(1 - z_1z_2)(\alpha + \beta - \alpha z_1 - \beta z_2) + 2(1 - \frac{1}{z_1z_2})(\alpha + \beta - \frac{\alpha}{z_1} - \frac{\beta}{z_2})\Bigg]. \quad (3.5.10)$$

Under the preference of a symmetric mask, we need that $\mathcal{P}^I$ must satisfy $\mathcal{P}^I(z_1, z_2) = \mathcal{P}^I(z_2, z_1)$, $(z_1, z_2) \in \mathbb{C}^2\backslash\{(0, 0)\}$, so that, from (3.5.10), we impose the further restriction that $\alpha = \beta$, after which (3.5.10) becomes

$$\begin{aligned}\mathcal{P}^I(z_1, z_2) &:= \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1^{-1}z_2^{-1}}{2}\right)\cdot[1 + 2\alpha(1 - z_1z_2)(2 - z_1 - z_2) \\ &\qquad + 2\alpha(1 - \frac{1}{z_1z_2})(2 - \frac{1}{z_1} - \frac{1}{z_2})] \\ &= \left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)\left(\frac{1+z_1^{-1}z_2^{-1}}{2}\right)[1 - w\,\mathrm{K}(z_1, z_2)],\end{aligned}$$

where K is the Laurent polynomial

$$\begin{aligned}\mathrm{K}(z_1, z_2) := {} & z_1^{-1}z_2^{-2} + z_1^{-2}z_2^{-1} - 2z_1^{-1}z_2^{-1} - z_1^{-1} - z_2^{-1} \\ & + 4 - z_1 - z_2 - 2z_1z_2 + z_1^2z_2 + z_1z_2^2,\end{aligned} \quad (3.5.11)$$

for $(z_1, z_2) \in \mathbb{C}^2\backslash\{(0, 0)\}$.

The subdivision scheme corresponding to the mask $\mathcal{P}^I$ is similar to the Butterfly subdivision scheme, as given in [DLG90, KLY07, DHL11].

We have as yet not succeeded in deriving the Butterfly subdivision scheme using the above approach . This is because (3.5.5) does not represent a full basis for the solution space of the homogeneous equation (3.5.6). This unresolved problem represents a good research project for the future. However, we have successfully verified, in Section 3.4, the interpolatory Butterfly subdivision scheme by using the algebraic approach.

Chapter 4

Convergence and smoothness analysis

In this chapter, we analyze the convergence and smoothness of the interpolatory Butterfly subdivision scheme as well as the new interpolatory scheme of Section 3.5, with some graphical illustrations of limit functions.

As was done for the univariate case in [dVC10, WW02], our approach to analyse the behaviour of a bivariate subdivision scheme is to associate a piecewise bilinear function $f_k(z_1, z_2)$ with the entries of the vector $\mathbf{c}^k$ plotted on the grid $\frac{1}{2^k}\mathbb{Z}^2$. In essence, the value of this function $f_k(z_1, z_2)$ at the grid point $\left(\frac{i}{2^k}, \frac{j}{2^k}\right)$ is simply the $(i, j)^{th}$ coefficient of the vector $\mathbf{c}^k$, that is,

$$f_k\left(\frac{i}{2^k}, \frac{j}{2^k}\right) = \mathbf{c}^k_{i,j}. \tag{4.0.1}$$

We pursue our study of convergence and smoothness of the bivariate ones by asking ourselves the following core questions, as was done for the univariate case in [dVC10].

Given a sequence of functions $f_k(z_1, z_2)$ satisfying (4.0.1),

- Does this sequence of functions $f_k(z_1, z_2)$ converge to a limit?
- If so, does the limit function satisfy a smoothness condition?

4.1 Analysis of convergence and smoothness

The following definitions are basic about boundedness and contractivity of operators in normed linear spaces.

Definition 4.1.1. *Let $(\mathcal{B}, \|\cdot\|)$ be a normed linear space and suppose $\mathbb{L}$ is a linear operator mapping $\mathcal{B}$ into itself. We say that $\mathbb{L}$ is bounded if*

$$\|\mathbb{L}\| := \sup\left\{\frac{\|\mathbb{L}f\|}{\|f\|} : f \in \mathcal{B},\ f \neq 0\right\} < \infty. \tag{4.1.1}$$

Moreover, if it holds $\|\mathbb{L}\| < 1$, then we say that $\mathbb{L}$ is contractive.

Observe in particular from (4.1.1) that

$$\|\mathbb{L}f\| \leq \|\mathbb{L}\|\|f\|, \quad f \in \mathcal{B}, \tag{4.1.2}$$

for any bounded linear operator $\mathbb{L}$ mapping a normed linear space $\mathcal{B}$ into itself.

The following results in [KLY07, DLM89] are on the smoothness and convergence of a bivariate subdivision scheme.

Theorem 4.1.2. *Let $\mathbb{S}$ be a bivariate subdivision scheme with associated mask symbol $\mathrm{P}(z_1, z_2)$. Then $\mathbb{S}$ is convergent if and only if the subdivision scheme with symbols*

$$\mathrm{P}_{\mathrm{a,b}}(z_1, z_2) = 2\frac{\mathrm{P}(z_1, z_2)}{1 + z_1^{\mathrm{a}} z_2^{\mathrm{b}}}, \quad (\mathrm{a}, \mathrm{b}) \in \{(0,1), (1,0), (1,1)\}, \tag{4.1.3}$$

are contractive.

For univariate subdivision, we have the following result from [DLM89].

Theorem 4.1.3. *[DLM89] Suppose that, for some $m \in \mathbb{N}$,*

$$\mathrm{P}(z) := 2^{-m}\left(1 + z^{-1}\right)^{m+1} r(z), \tag{4.1.4}$$

for some Laurent polynomial

$$r(z) := \sum_j \mathrm{r}_j\, z^j, \quad z \in \mathbb{C},$$

satisfying $r(1) = 1$, and that, for some $n \in \mathbb{N}$,

$$\|\mathbb{S}_r^n\|_\infty := \sup\left\{\|\mathbb{S}_r^n c\|_\infty : \|c\|_\infty \leq 1\right\} < 1, \tag{4.1.5}$$

where

$$\|c\|_\infty := \sup\left\{|c_j| : j \in \mathbb{Z}\right\},$$

and $\mathbb{S}_r^n$ is the n^{th}-iterate of the linear operator $\mathbb{S}_r$. Then the subdivision scheme $\mathbb{S}_{\boldsymbol{p}}$ is a convergent subdivision scheme of order m, that is, $[C^m SS]$ as in (2.3.3).

Equation (4.1.5) is a condition for the contractivity on r which is equivalent to

$$\lim_{n\to\infty} \|\mathbb{S}_r^n\|_\infty = 0, \tag{4.1.6}$$

with $\mathbb{S}_r$ denoting the subdivision scheme associated with the mask $\{r_j : j \in \mathbb{Z}\}$.

Definition 4.1.4. *The Laurent polynomial $r(z)$ is said to be strictly stable if (4.1.6) holds.*

The extension of the contractivity property of subdivision operators to the bivariate case is given in the following remark.

Remark 4.1.5. *According to [DLM89], the contractivity condition for assuring convergence of bivariate subdivision schemes is given by $\|\mathbb{S}_{\boldsymbol{r}}^n\|_\infty < 1$, where*

$$\|\mathbb{S}_r^n\|_\infty := \sup_{0 \leq i_1, i_2 \leq 2^n - 1} \left\{\sum_{\mathrm{j}\in\mathbb{Z}^2} \left|\mathrm{r}_{\mathrm{i}+2^n\mathrm{j}}^{[n]}\right|\right\}, \quad \mathrm{i} := (i_1, i_2) \in \mathbb{R}^2. \tag{4.1.7}$$

More precisely, the relevant two results from [DLM89] are as follows.

Theorem 4.1.6. *Suppose that*

$$\mathrm{P}(z_1, z_2) = 2^{-1}(1 + z_1^{-1})(1 + z_2^{-1})(1 + z_1^{-1}z_2^{-1})\, r(z_1, z_2), \tag{4.1.8}$$

where r *is a Laurent polynomial satisfying* $\mathrm{r}(\boldsymbol{1}) = 1$, *and the polynomials* $(1 + z_1^{-1})\, r(z_1, z_2)$ *and* $(1 + z_2^{-1})\, r(z_1, z_2)$ *are strictly stable. Then the subdivision scheme* $\mathbb{S}_{\boldsymbol{p}}$ *is* C^1SS.

Theorem 4.1.7. *If the mask symbol* $r(z)$ *is strictly stable, then* $\{z^m\, r(z) : m \in \mathbb{Z}\}$ *is strictly stable.*

By applying Theorems 4.1.6 and 4.1.7, we can now prove the following convergence result for the Butterfly subdivision scheme.

Theorem 4.1.8. *The interpolatory Butterfly subdivision scheme as given in Section 3.4, with mask symbol as in (3.4.1), converges to* $C^1(\mathbb{R}^2)$.

Proof. Using (4.1.5), together with Theorem 4.1.2, we need to check that the subdivision scheme with symbol $\mathcal{P}_w(z_1, z_2)$ as in (3.4.1) is C^1, that is, it suffices to show ([DLG90, DHL11, DLM89]) that any two of the schemes

$$2\frac{\mathcal{P}_w(z_1, z_2)}{1 + z_1},\ 2\frac{\mathcal{P}_w(z_1, z_2)}{1 + z_2},\ 2\frac{\mathcal{P}_w(z_1, z_2)}{1 + z_1z_2} \quad \text{are in } C^0.$$

We shall use the notation

$$\mathrm{P}_{1,0}(z_1, z_2) := 2\frac{\mathcal{P}_w(z_1, z_2)}{1 + z_1}; \quad \mathrm{P}_{0,1}(z_1, z_2) := 2\frac{\mathcal{P}_w(z_1, z_2)}{1 + z_2}; \quad \mathrm{P}_{1,1}(z_1, z_2) := 2\frac{\mathcal{P}_w(z_1, z_2)}{1 + z_1z_2}.$$

We shall show that

$$\mathrm{P}_{0,1}(z_1, z_2) = \frac{1}{2}(1 + z_1)(1 + z_1z_2)(1 - w\, r(z_1, z_2))(z_1z_2)^{-1} \tag{4.1.9}$$

and

$$\mathrm{P}_{1,0}(z_1, z_2) = \frac{1}{2}(1 + z_2)(1 + z_1z_2)(1 - w\, r(z_1, z_2))(z_1z_2)^{-1} \tag{4.1.10}$$

where $r(z_1, z_2)$ is as given in (3.4.2), are contractive.

By Theorem 4.1.7, it suffices to prove that $(1 + z_1)(z_1z_2)(1 - w\, r(z_1, z_2))$ is strictly stable, or, equivalently, say, $Q(z_1, z_2)$, which is obtained by applying Theorem 4.1.7 on $P_{0,1}(z_1, z_2)$ to get

$$\mathrm{Q}(z_1, z_2) := (1 + z_1)(1 - w\, r(z_1, z_2)), \tag{4.1.11}$$

is strictly stable. The other two polynomials in Theorem 4.1.6 are strictly stable by the symmetries of $r(z_1, z_2)$, that is, $r(z_1, z_2) = r(z_2, z_1) = r(z_1^{-1}, z_2^{-1})$.

Observe that, for $n = 1$,

$$\begin{aligned} \|\mathbb{S}_{\mathrm{Q}}\|_\infty := & \max_{0\le k,\ell\le 1}\left(\sum_{i,j}|\mathrm{Q}_{2i+k,2j+\ell}|\right) \\ = & \max\left(\sum_{i,j}|\mathrm{Q}_{2i,2j}|, \sum_{i,j}|\mathrm{Q}_{2i+1,2j}|, \sum_{i,j}|\mathrm{Q}_{2i,2j+1}|, \sum_{i,j}|\mathrm{Q}_{2i+1,2j+1}|\right). \end{aligned} \tag{4.1.12}$$

From the strict stability property as in Theorem 4.1.7, it is sufficient to show $\mathrm{Q}(z_1, z_2)$ is strictly stable.

Now, from (3.4.2) together with (4.1.11), we get

$$\begin{aligned} \mathrm{Q}(z_1, z_2) \;=\; & -2wz_1^3z_2 - 2wz_1^2z_2^2 + 2wz_1^2z_2 - 2wz_1z_2^2 + 4wz_1^2 + 8wz_1z_2 \\ & -2wz_1^2z_2^{-1} - 8wz_1 + 2wz_2 + 2wz_1z_2^{-1} - 2wz_1^{-1}z_2 - 8w \\ & +z_1 + 4wz_1^{-1} + 8wz_2^{-1} + 2wz_1^{-1}z_2^{-1} - 2wz_2^{-2} - 2wz_1^{-1}z_2^{-2} \\ & -2wz_1^{-2}z_2^{-1} + 1 + O(w^2). \end{aligned} \tag{4.1.13}$$

Looking at the necessary coefficients of $\mathrm{Q}(z_1, z_2)$, we see that

$$\mathrm{Q}_{2,2} = -2w; \quad \mathrm{Q}_{2,0} = 4w; \quad \mathrm{Q}_{0,0} = 1 - 8w; \quad \mathrm{Q}_{0,-2} = -2w,$$

and thus

$$\begin{aligned} \sum_{i,j}|\mathrm{Q}_{2i,2j}| &= |-2w| + |4w| + |1-8w| + |-2w| \\ &= |1-8w| + |8w|. \end{aligned} \tag{4.1.14}$$

Hence, $\|\mathbb{S}_{\mathrm{Q}}\|_\infty \ge 1$ for all values of w.

Considering $\mathbb{S}_{\mathrm{Q}}^2$, we proceed to find an interval $(0, w_0)$ such that $\|\mathbb{S}_{\mathrm{Q}}^2\|_\infty < 1,\ w \in (0, w_0)$.

Since it is very difficult to compute the exact value of w_0, we only consider the linear terms in w.

From (4.1.13), we obtain the expansion

$$\begin{aligned} \mathrm{Q}^{[2]}(z_1, z_2) = {} & \mathrm{Q}(z_1, z_2)\,\mathrm{Q}(z_1^2, z_2^2) = (1 + z_1 + z_1^2 + z_1^3)\,(1 - w\,\mathrm{r}(z_1, z_2) - w\,\mathrm{r}(z_1^2, z_2^2) + O(w^2)) \\ = {} & -2wz_1^7z_2^2 - 2wz_1^5z_2^4 - 2wz_1^6z_2^2 - 2wz_1^4z_2^4 + 2wz_1^5z_2^2 - 2wz_1^3z_2^4 - 2wz_1^5z_2 \\ & - 2wz_1^2z_2^4 + 4wz_1^5 + 2wz_1^4z_2 + 6wz_1^3z_2^2 + 8wz_1^4 + 6wz_1^3z_2 + 6wz_1^2z_2^2 \\ & - 2wz_1^5z_2^{-2} - 2wz_1^4z_2^{-2} - 16wz_1^3 + 4wz_1^2z_2 - 2wz_1^4z_2^{-2} + 2wz_1^3z_2 - 12wz_1^2 \\ & + 6wz_1z_2 + 2wz_2^2 + z_1^3 + 2wz_1^3z_2^{-2} + 6wz_1^2z_2^{-1} - 12wz_1 - 2wz_2^2z_1^{-1} + 2wz_2 \\ & + z_1^2 + 4wz_1z_2^{-1} - 2wz_2z_1^{-1} - 2wz_1^{-2}z_2^2 - 16w + z_1 + 6wz_1z_2^{-2} + 8wz_1^{-1} \\ & + 6wz_2^{-1} + 2wz_1^{-1}z_2^{-1} + 4wz_1^{-2} + 6wz_2^{-2} - 2wz_1z_2^{-4} - 2wz_1^{-2}z_2^{-1} + 2wz_1^{-2}z_2^{-2} \\ & - 2wz_2^{-4} - 2wz_1^{-1}z_2^{-4} - 2wz_1^{-3}z_2^{-2} - 2wz_1^{-2}z_2^{-4} - 2wz_1^{-4}z_2^{-2} + 1 + O(w^2). \end{aligned} \tag{4.1.15}$$

By looking at the necessary coefficients of $\mathrm{Q}^{[2]}(z_1, z_2)$, we find that $\mathrm{Q}^{[2]}_{i,j} = O(w)$ for $j \ne 0$, while $\mathrm{Q}^{[2]}_{i,0} = 1 + O(w)$, for $i = 0, 1, 2, 3$.

Thus, it is sufficient to show that, for w sufficiently small, it holds that

$$\sum_{i,j} \mathrm{Q}^{[2]}_{4i+\ell,4j} < 1, \ \ell = 0, 1, 2, 3.$$

For the case $\ell = 0$, all the non-zero coefficients are given by

$$\mathrm{Q}_{0,0} = 1 - 16w + O(w^2); \quad \mathrm{Q}_{4,4} = Q_{0,-4} = -2w + O(w^2); \quad \mathrm{Q}_{4,0} = 8w + O(w^2).$$

Hence $\sum_{i,j} |\mathrm{Q}^{[2]}_{4i,4j}| = |1 - 16w| + 12|w| + O(w^2) < 1$, for $w > 0$ sufficiently small.

For the case $\ell = 1$, all the relevant coefficients are

$$\mathrm{Q}^{[2]}_{1,0} = 1 - 12w + O(w^2); \quad \mathrm{Q}^{[2]}_{5,0} = 4w + O(w^2); \quad \mathrm{Q}^{[2]}_{5,4} = \mathrm{Q}^{[2]}_{1,-4} = -2w + O(w^2).$$

Hence, $\sum_{i,j} |\mathrm{Q}^{[2]}_{4i+1,4j}| = |1 - 12w| + 8|w| + O(w^2) < 1$, for $w > 0$ sufficiently small.

For the case $\ell = 2$, all the relevant coefficients are

$$\mathrm{Q}^{[2]}_{2,0} = 1 - 12w + O(w^2); \quad \mathrm{Q}^{[2]}_{5,4} = 4w + O(w^2); \ \mathrm{Q}^{[2]}_{2,4} = -2w + O(w^2).$$

Thus, $\sum_{i,j} |\mathrm{Q}^{[2]}_{4i+2,4j}| = |1 - 12w| + 8|w| + O(w^2) < 1$

for sufficiently small $w > 0$. That is, the case $\ell = 2$ is the same as $\ell = 1$.

For the case $\ell = 3$, we get $\sum_{i,j} |\mathrm{Q}^{[2]}_{4i+3,4j}| = |1 - 16w| + 12|w| + O(w^2) < 1$,

which is the same as the case for $\ell = 0$ for sufficiently small $w > 0$.

Hence the interval for which the tension parameter w provides the shape of the limit function of the Butterfly subdivision scheme lies in $(0, \frac{1}{12})$. The best value of w that makes the Butterfly subdivision scheme to reproduce cubic polynomials is attained when $w = \frac{1}{16}$. □

4.2 Convergence of the new interpolatory scheme

As mentioned in Section 3.5, we consider next the new interpolatory subdivision scheme, as obtained in [vdB10]. It exhibits similar features to the Butterfly subdivision scheme, and its interpolatory mask is given by

$$\left.\begin{aligned}
&p_{-3,-1} = p_{-3,-2} = p_{-2,-3} = p_{-1,-3} = p_{3,2} = p_{2,3} = p_{1,3} = p_{3,1} = -w/2;\\
&p_{1,2} = p_{2,1} = p_{-1,-2} = p_{-2,-1} = w/2;\\
&p_{0,0} = 1;\\
&p_{-1,-1} = p_{1,1} = p_{0,-1} = p_{-1,0} = p_{0,1} = p_{1,0} = 1/2;\\
&p_{1,-1} = p_{-1,1} = w;\\
&p_{i,j} = 0, \quad \text{otherwise}.
\end{aligned}\right\} \tag{4.2.1}$$

Writing equation (4.2.1) for the mask sequence $\{p_{i,j} : -3 \leq i, j \leq 3\}$ as a matrix leads to

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{w}{2} & -\frac{w}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{w}{2} \\ 0 & 0 & w & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{w}{2} \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ -\frac{w}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & w & 0 & 0 \\ -\frac{w}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{w}{2} & -\frac{w}{2} & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.2.2}$$

In Figure 4.1, the graph of the corresponding basis function ϕ_p for this subdivision scheme was obtained by means of the cascade algorithm.

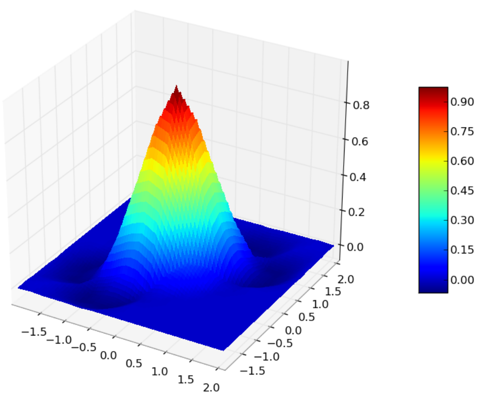

Figure 4.1: *The basis function ϕ_p for the new interpolatory scheme* $[w = \frac{1}{16}]$.

This subdivision scheme exhibits similar behaviour to the Butterfly subdivision scheme with respect to symmetry, convergence and smoothness, with, in particular:

- The new interpolatory subdivision scheme is symmetric, that is, $p_{i,j} = p_{j,i}$ for any $i, j \in \mathbb{Z}$.
- The new scheme seemingly shares the same structural pattern with the Butterfly subdivision scheme;
- The subdivision polynomial $\mathrm{P}^I(z_1, z_2)$ has $(1+z_1)$, $(1+z_2)$ and $(1+z_1z_2)$ as factors.

The corresponding bivariate Laurent polynomial is given by

$$\begin{aligned} \mathrm{P}^I(z_1, z_2) &= \frac{1}{2}(1+z_1)(1+z_2)(1+z_1^{-1}z_2^{-1})\,[1 - w\,\mathrm{r}(z_1, z_2)] \\ &= 2^{-1}(1+z_1^{-1})(1+z_2^{-1})(1+z_1^{-1}z_2^{-1})\,[(z_1z_2)\,(1 - w\,\mathrm{r}(z_1, z_2))]\,, \end{aligned} \tag{4.2.3}$$

where

$$\begin{aligned} \mathrm{r}(z_1, z_2) \;=\; & z_1^{-1}z_2^{-2} + z_1^{-2}z_2^{-1} - 2z_1^{-1}z_2^{-1} - z_1^{-1} - z_2^{-1} + 4 - z_1 - z_2 \\ & -2z_1z_2 + z_1^2z_2 + z_1z_2^2, \end{aligned} \tag{4.2.4}$$

with $\mathrm{r}(z_1, z_2) = \mathrm{r}(z_2, z_1) = \mathrm{r}(z_1^{-1}, z_2^{-1})$.

By applying Theorem 4.1.6 and 4.1.7, we now prove the following convergence result.

Theorem 4.2.1. *The new interpolatory subdivision scheme, with subdivision symbol as in (4.2.3), converges to a C^1-limit surface.*

Proof. According to Theorem 4.1.6, to show that the subdivision scheme converges to $C^1(\mathbb{R}^2)$ surface, it suffices to show that $(1+z_1)(z_1z_2)(1-w\,\mathrm{r}(z_1,z_2))$ is strictly stable, or equivalently, that

$$\mathrm{Q}(z_1,z_2) := (1+z_1)(1-w\,\mathrm{r}(z_1,z_2)) \tag{4.2.5}$$

is strictly stable. The other two polynomials in Theorem 4.1.6 are strictly stable by the symmetries of $\mathrm{r}(z_1,z_2)$. Observe that, for $n=1$,

$$\begin{aligned}\|\mathbb{S}_\mathrm{Q}\|_\infty :&= \max_{0\le k,\ell\le 1}\left(\sum_{i,j}|\mathrm{Q}_{2i+k,2j+\ell}|\right)\\ &= \max\left(\sum_{i,j}|\mathrm{Q}_{2i,2j}|,\sum_{i,j}|\mathrm{Q}_{2i+1,2j}|,\sum_{i,j}|\mathrm{Q}_{2i,2j+1}|,\sum_{i,j}|\mathrm{Q}_{2i+1,2j+1}|\right).\end{aligned} \tag{4.2.6}$$

From the strict stability property as in Theroem 4.1.7, it is sufficient to show $\mathrm{Q}(z_1,z_2)$ is strictly stable.

Now, applying (4.2.4) to equation (4.2.5), we get

$$\begin{aligned}\mathrm{Q}(z_1,z_2) \quad &= (1+z_1)(1-w\,\mathrm{r}(z_1,z_2)) + O(w^2)\\ &= -wz_1^3z_2 - wz_1^2z_2^2 + wz_1^2z_2 - wz_1z_2^2 + 3wz_1z_2 - 3wz_1\\ &\quad + wz_2 + wz_1z_2^{-1} - 3w + z_1 + wz_1^{-1} + 3wz_2^{-1} + wz_1^{-1}z_2^{-1}\\ &\quad - wz_2^{-2} - wz_1^{-1}z_2^{-2} - wz_1^{-2}z_2^{-1} + 1 + O(w^2).\end{aligned} \tag{4.2.7}$$

Looking at the necessary coefficients of $\mathrm{Q}(z_1,z_2)$, we see that

$$\mathrm{Q}_{2,2} = -w;\quad \mathrm{Q}_{2,0} = w;\quad \mathrm{Q}_{0,0} = 1-3w;\quad \mathrm{Q}_{0,-2} = -w.$$

and thus,

$$\begin{aligned}\sum_{i,j}|\mathrm{Q}_{2i,2j}| &= |-w| + |w| + |1-3w| + |-w| + O(w^2)\\ &= |1-3w| + 3|w| + O(w^2)\end{aligned} \tag{4.2.8}$$

Thus, $\|\mathbb{S}_\mathrm{Q}\|_\infty \ge 1$ for all values of w. Considering $\mathbb{S}_\mathrm{Q}^2$, we proceed to find an interval $(0,w_0)$ such that $\|\mathbb{S}_\mathrm{Q}^2\|_\infty < 1$, $w\in(0,w_0)$.

Since it is very difficult to compute the exact value of w_0, we only consider linear terms in w.

From (4.2.7), we obtain the expansion

$$\begin{aligned}\mathrm{Q}^{[2]}(z_1,z_2) &= \mathrm{Q}(z_1,z_2)\,\mathrm{Q}(z_1^2,z_2^2) = \left[(1+z_1+z_1^2+z_1^3)\left(1-w\,\mathrm{r}(z_1,z_2)-w\,\mathrm{r}(z_1^2,z_2^2)+O(w^2)\right]\right.\\ &= -wz_1^7z_2^2 - wz_1^5z_2^4 - wz_1^6z_2^2 - wz_1^4z_2^4 + wz_1^5z_2^2 - wz_1^3z_2^4 - wz_1^5z_2 - wz_1^2z_2^4 + wz_1^5\\ &\quad + wz_1^4z_2 + 2wz_1^3z_2^2 + 2wz_1^4 + 2wz_1^3z_2 + 2wz_1^2z_2^2 - 6wz_1^3 + 2wz_1^2z_2 + wz_1^3z_2^{-1}\\ &\quad - 5wz_1^2 + 3wz_1z_2 + wz_2^2 + z_1^3 + wz_1^3z_2^{-2} + 3wz_1^2z_2^{-1} - 5wz_1 + wz_2\\ &\quad + z_1^2 + 2wz_1z_2^{-1} - 6w + z_1 + 2wz_1z_2^{-2} + 2wz_1^{-1} + 2wz_2^{-1}\\ &\quad + wz_1^{-1}z_2^{-1} + wz_1^{-2} + 2wz_2^{-2} - wz_1z_2^{-4} - wz_1^{-2}z_2^{-1} + wz_1^{-2}z_2^{-2} - wz_2^{-4}\\ &\quad - wz_1^{-1}z_2^{-4} - wz_1^{-3}z_2^{-2} - wz_1^{-2}z_2^{-4} - wz_1^{-4}z_2^{-2} + 1 + O(w^2).\end{aligned} \tag{4.2.9}$$

By looking at the necessary coefficients of $\mathrm{Q}^{[2]}(z_1, z_2)$, we find that $\mathrm{Q}^{[2]}_{i,j} = O(w)$ for $j \neq 0$, while $\mathrm{Q}^{[2]}_{i,0} = 1+O(w)$, for $i = 0, 1, 2, 3$. Thus, it is sufficient to show that, for w sufficiently small,

$$\sum_{i,j} \mathrm{Q}^{[2]}_{4i+\ell,4j} < 1, \quad \ell = 0, 1, 2, 3.$$

For the case $\ell = 0$, all the non-zero coefficients are given by

$$\mathrm{Q}_{0,0} = 1 - 6w + O(w^2); \quad \mathrm{Q}_{4,4} = -w + O(w^2); \quad \mathrm{Q}_{0,-4} = -w + O(w^2); \quad \mathrm{Q}_{4,0} = 2w + O(w^2).$$

Hence,

$$\begin{aligned} \sum_{i,j} |\mathrm{Q}^{[2]}_{4i,4j}| &= |1 - 6w| + |-w| + |-w| + |2w| + O(w^2) \\ &= 4|w| + |1 - 6w| + O(w^2). \end{aligned}$$

For the case $\ell = 1$, all the relevant coefficients are

$$\mathrm{Q}_{1,0} = 1 - 5w + O(w^2); \quad \mathrm{Q}_{5,0} = -w + O(w^2); \quad \mathrm{Q}_{5,4} = -w + O(w^2); \quad \mathrm{Q}_{1,-4} = -w + O(w^2).$$

Hence, $\sum_{i,j} |\mathrm{Q}^{[2]}_{4i+1,4j}| = |1 - 5w| + 3|w| + O(w^2) < 1$ for $w > 0$ sufficiently small.

For the case $\ell = 2$, all the relevant coefficients are

$$\mathrm{Q}_{2,0} = 1 - 5w + O(w^2); \quad \mathrm{Q}_{6,4} = Q_{6,0} = \mathrm{Q}_{2,-4} = 0 + O(w^2).$$

Hence,

$$\sum_{i,j} |\mathrm{Q}^{[2]}_{4i+1,4j}| = |1 - 5w| + O(w^2) < 1$$

for $w > 0$ sufficiently small.

For the case $\ell = 3$, all the relevant coefficients are

$$\mathrm{Q}_{3,0} = |1 - 6w| + O(w^2); \ \mathrm{Q}_{3,4} = |-w| + O(w^2); \quad \mathrm{Q}_{3,-4} = |-w| + O(w^2); \quad \mathrm{Q}_{1,-4} = |-w| + O(w^2).$$

Hence, $\sum_{i,j} |\mathrm{Q}^{[2]}_{4i,4j}| = |1 - 6w| + 2|w| + O(w^2) < 1$ for $w > 0$ sufficiently small.

Thus, we have shown, according to Theorem 4.1.6, that Q is strictly stable.

Hence, the new interpolatory subdivision scheme is C^1SS for $w > 0$ sufficiently small and the interval range for the tension parameter w to get C^1 lies in $(0, \frac{1}{6})$.

As shown in Figure 4.1, as obtained by means of the cascade algorithm, the tension parameter values $w \in (0, \frac{1}{6})$ seems to produce a C^1 limit surface. $\square$

Further analysis of this subdivision scheme would be an interesting research project for the future.

Chapter 5

Conclusions

This study deals with interpolatory bivariate refinable functions, which are refinable functions that assume the value 1 at the origin and vanish at all other integer pairs [Jia95, HJ98, HJ97]. We studied characterizations of refinement masks of these refinable functions with their associated mask symbols, by taking the integer dilation matrix $A = 2\mathrm{I}_2$ into consideration.

We focused on interpolatory subdivision schemes, which are schemes that preserve initial control points at all steps of the iterative process to construct subdivision surfaces, by giving special attention to the Butterfly interpolatory subdivision scheme.

Taking the existence of interpolatory bivariate refinable functions as given, we focused on the classical $2\mathrm{I}_2$-refinable box splines, which are basis functions for subdivision analysis, as subdivision of box splines is the cornerstone of many popular multivariate subdivision schemes.

Besides using tensor products of univariate refinable functions, the link between the subdivision algorithm and the cascade algorithm was implemented to generate a non-tensor product refinable bivariate function, which in turn plays a key role in the construction of smooth surface. It is good to note that convergence of the cascade algorithm implies convergence of the corresponding subdivision scheme, which in turn is strongly bonded to the existence of the corresponding interpolatory bivariate refinable function.

We have used bivariate box splines as helpful tools for constructing subdivision surfaces in $\mathbb{R}^3$. Following the work of [CCJZ11], the mask symbol characterization of the Butterfly subdivision scheme was found to be of an algebraic nature. Using the algebraic approach to systematically construct bivariate subdivision schemes, we verified that the Butterfly interpolatory subdivision scheme emanates from the normalized box spline generators of some polynomial ideal $\mathcal{I}$ and its power $\mathcal{I}^k$. That is, the mask symbol of the Butterfly subdivision scheme can be expressed as $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}]$-combinations of normalised box spline symbols $\{\tilde{\mathbb{B}}_{\mathcal{D}}(z_1, z_2) : (z_1, z_2) \in \mathbb{C}^2 \backslash (0,0)\}$. The convergence analysis of this subdivision scheme was also shown by using contractivity of subdivision operators.

We attempted to verify the Butterfly subdivision scheme from a bivariate Laurent polynomial equation as in (3.5.1), which remains as yet unsuccessful, though we have successfully verified the Butterfly Scheme using an algebraic approach.

In particular, any attempt to obtain the Butterfly scheme as in Section 3.5 will depend on obtaining a full basis for the homogeneous equation (3.5.6) . Hence, deriving the Butterfly subdivision scheme as a general solution of the non-homogeneous bivariate equation as in (3.5.4) is an interesting open problem which will be considered in future research.

The symmetric new interpolatory subdivision scheme, as in (4.2.3) and (4.2.4), satisfies much of the same properties, like smoothness, convergence and interpolation of the Butterfly subdivision scheme. Furthermore, we have proved the convergence of this new interpolatory subdivision scheme by using contractivity property of subdivision operators. Further properties are to be studied in the future.

Appendices

Appendix A

Gröbner basis computation

```
> ring R = 0, (x,y),dp;
//Ring of characteristic 0 over field C.
> ideal I= (x2-1)*(x+1),(x2-1)*(y+1),(x+1)*(y+1)*(x+1),
(x+1)*(y+1)*(y+1),(x2-1)*(y-1),(y2-1)*(y-1),(x+1)*(y2-1),(y2-1)*(y+1);
// The ideal generated by the given polynomials
> std(I);//gives Gr\"{o}bner   basis.
_[1]=y2-1
_[2]=xy+x+y+1
_[3]=x2-1
```

Ideal membership

```
 poly f = (7/16)*xy*((1+x)^2)*((1+y)^2)*(1+xy)^2 -(1/16)*x*(1+x)*((1+y)^3)*((1+xy)^3)
 -(1/16)*y*((1+x)^3)*(1+y)*(1+xy)^3-(1/16)*((1+x)^3)*((1+y)^3)*(1+xy);
> NF(f, std(I));//normal form of an ideal I
0
//This shows that the polynomial f is contained in I if and only if NF(f, std(I));
evaluates to 0.


> ring r = 0, (x,y), dp;//Ring of characteristic 0 over field C.
> ideal I= 1-x^2, 1-y^2, (1+x)*(1+y)/4;
> poly P= (7/16)*xy*((1+x)^2)*((1+y)^2)*(1+xy)^2
-(1/16)*x*(1+x)*((1+y)^3)*((1+xy)^3) -(1/16)*y*((1+x)^3)*((1+y)^3)*(1+xy);//Polynomial f
> I=std(I);
> I;
I[1]=y2-1
I[2]=xy+x+y+1
I[3]=x2-1
> matrix C = lift (I,P);
> C;
C[1,1]=-1/16x5y4+1/4x5y3-1/16x4y4+5/8x5y2+7/16x4y3+5/8x5y+13/8x4y2+1/16x3y3
+5/8x5+9/4x4y+5/4x3y2-3/16x2y3+9/4x4+41/16x3y-1/16x2y2-1/16xy3+11/4x3+11/16x2y
-3/8xy2+17/16x2-7/16xy-1/16y2-5/16x-3/16y-1/4
C[2,1]=3
C[3,1]=5/8x3y+5/8x3+9/4x2y+9/4x2+27/8xy+27/8x+13/4y+13/4

> poly k=C[1,1]*I[1]+ C[2,1]*I[2] + C[3,1]*I[3];
```

```
> k;
-1/16x5y6+1/4x5y5-1/16x4y6+11/16x5y4+7/16x4y5+3/8x5y3+27/16x4y4+1/16x3y5+29/16x4y3
+5/4x3y4-3/16x2y5+5/8x4y2+5/2x3y3-1/16x2y4-1/16xy5+3/2x3y2+7/8x2y3-3/8xy4+3/16x3y
+9/8x2y2-3/8xy3-1/16y4+5/16x2y+1/16xy2-3/16y3-1/16x2+1/16xy-3/16y2-1/16x-1/16y
> k = P;
> k - P;
0// This assures  the coefficient polynomials are the right ones
to decompose the original poly P.
```

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