Liquidity Risk and No Arbitrage

by

Laila El Ghandour

Thesis presented in partial fulfilment of the
academic requirements for the degree of
Master of Science
at the Stellenbosch University

Supervisor: Dr. Peter Ouwehand
(**University of Stellenbosch**)

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February 10, 2013

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Laila **El GHANDOUR** Date

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Abstract

In modern theory of finance, the so-called First and Second Fundamental Theorems of Asset Pricing play an important role in pricing options with no-arbitrage. These theorems gives a necessary and sufficient conditions for a market to have no-arbitrage and for a market to be complete. An early version of the First Fundamental Theorem of Asset Pricing was proven by Harrison and Kreps [30] in the case of a finite probability space. A more general version was proven by Harrison and Pliska [31] in the case of a finite probability space and discrete time. In the case of continuous time, Delbaen and Schachermayer [19] introduced a more general concept of no-arbitrage called "No-Free Lunch With Vanishing Risk" (NFLVR), and showed that for a locally-bounded semimartingale price process NFLVR is essentially equivalent to the existence of an equivalent local martingale measure.

The goal of this thesis is to review the theory of arbitrage pricing and the extension of this theory to include liquidity risk. At the current time, liquidity risk is a key challenge faced by investors. Consequently there is a need to develop more realistic pricing models that include liquidity risk. We present an approach to liquidity risk by Çetin, Jarrow and Protter [10]. In to this approach the liquidity risk is embedded into the classical theory of arbitrage pricing by having investors act as price takers, and assuming the existence of a supply curve where prices depend on trade size. This framework assumes that the quantity impact on the price transacted is momentary. Using trading strategies that are both continuous and of finite variation allows one to avoid liquidity costs. Therefore, the First and Second Fundamental Theorems of Asset Pricing and the Black-Scholes model can be extended.

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Opsomming

In moderne finansiële teorie speel die sogenaamde Eerste en Tweede Fundamentele Stellings van Bateprysbepaling 'n belangrike rol in die prysbepaling van opsies in arbitrage-vrye markte. Hierdie stellings gee nodig en voldoende voorwaardes vir 'n mark om vry van arbitrage te wees, en om volledig te wees. 'n Vroeë weergawe van die Eerste Fundamentele Stelling was deur Harrison en Kreps [30] bewys in die geval van 'n eindige waarskynlikheidsruimte. 'n Meer algemene weergawe was daarna gepubliseer deur Harrison en Pliska [31] in die geval van 'n eindige waarskynlikheidsruimte en diskrete tyd. In die geval van kontinue tyd het Delbaen en Schachermayer [19] 'n meer algemene konsep van arbitragevryheid ingelei, naamlik "No–Free–Lunch–With–Vanishing–Risk" (NFLVR), en aangetoon dat vir lokaalbegrensde semimartingaalprysprosesse NFLVR min of meer ekwivalent is aan die bestaan van 'n lokaal martingaalmaat.

Die doel van hierdie tesis is om 'n oorsig te gee van beide klassieke arbitrageprysteorie, en 'n uitbreiding daarvan wat likideit in ag neem. Hedendaags is likiditeitsrisiko 'n vooraanstaande uitdaging wat beleggers die hoof moet bied. Gevolglik is dit noodsaaklik om meer realistiese modelle van prysbepaling wat ook likiditeitsrisiko insluit te ontwikkel. Ons bespreek die benadering van Çetin, Jarrow en Protter [10], waar likiditeitsrisiko in die klassieke arbitrageprysteorie ingesluit word deur die bestaan van 'n aanbodkromme aan te neem, waar pryse afhanklik is van handelsgrootte. In hierdie raamwerk word aangeneem dat die impak op die transaksieprys slegs tydelik is. Deur gebruik te maak van handelingsstrategieë wat beide kontinu en van eindige variasie is, is dit dan moontlik om likiditeitskoste te vermy. Die Eerste en Tweede Fundamentele Stellings van Bateprysbepaling en die Black–Scholes model kan dus uitgebrei word om likiditeitsrisiko in te sluit.

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Dedication

To my parents, sisters and brothers.

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Acknowledgements

First, I would like to say thank you to my supervisor, Dr. Peter Ouwehand, for allowing me to work under his able supervision and for constantly criticizing my work constructively. The finished work would not have been achieved without his professional assistance. Second, I would like to thank the African Institute for Mathematical Sciences (AIMS) and the Department of Mathematical Sciences at the University of Stellenbosch for supporting me financially and personally.

Finally, to my roots, I would like express my profound love to my dearest mother, father, sisters and brothers. Thank you for being there for me when I wanted you. Allah bless you all.

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Chapter 1

Introduction

1.1 Motivation

In 1900, more than a century ago, Louis Bachelier developed, in his thesis under Poincaré, a crucial result in the field of mathematical finance. His thesis "Theory of Speculation", studied a model of the securities price fluctuation using Brownian motion five years before Einstein's paper of 1905 [23]. Bachelier's goal was to develop theoretical values for various types of options. However, his work was ignored and forgotten for some time. More than 50 years later, Samuelson [62], modelled the stock market using geometric Brownian motion instead of Bachelier's ordinary Brownian motion to eliminate the possibility of negative stock price. The use of geometric Brownian motion to model stock prices was developed between 1950 and 1970.

In the early 1970s, Fisher Black, Myron Scholes and Robert Merton presented a ground-breaking discovery in pricing financial instruments by developing what has become known as the Black-Scholes model. The Black-Scholes model displays the importance of mathematics in the field of finance. The famous Black-Scholes paper [5] provides a closed formula for European call and put options assuming that stock prices follow a geometric Brownian motion. This formula was obtained solving by a PDE called Black-Scoles PDE, and can be interpreted as the expectation of the discounted payoff of the options under a risk-neutral measure. This work was done using Itô stochastic calculus and the Markov property of diffusions. On the other hand, the theory of stochastic integration for general semimartingales

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was developed independently in the 1970s and 1980s. This theory of stochastic integral was developed originally by Itô's, then continued by Kunita-Watanaba, and thereafter mainly Meyer and the Strasbourg School.

Harrison and Kreps [30] in 1979 and Harrison and Pliska [31] in 1981, combined the advanced theory of stochastic integration and the work of Black and Scholes, to prove what have become known as the First and Second Fundamental Theorems of Asset Pricing respectively. These two theorems provide the connection between no-arbitrage and the martingale theory in a frictionless market with continuous trading. The first fundamental theorem states that a mathematical model of a discounted financial security S, is free of arbitrage if, and only if, it is a martingale under an equivalent martingale measure, while the second fundamental theorem relates the market-completeness to the uniqueness of an equivalent martingale measure. However, it turns out that only in the simplest case of competitive markets, markets without friction, with a finite number of trading dates, and finite number of states of the world and finite number of assets, such results are true.

What will happen in a more general setting?

Let us start with the case of an infinite number of trading dates and infinite number of states of the world. In this case we need to generalize the concept of arbitrage to free lunch (Kreps [49]) or free lunch with vanishing risk (Delbaen and Schachermayer [19]). It has been proven that a locally-bounded process S satisfies no free lunch with vanishing risk if, and only if, there exists a probability measure $\mathbb{Q}$ (Definition 2.1.5) such that S is a local martingale. Furthermore, such a $\mathbb{Q}$ is unique if, and only if, the market is complete.

The goal of this thesis is to study what happens when we weaken the assumptions of competitive and frictionless markets. A frictionless market is one where there are no restrictions on trade, no transaction costs and no bid and ask spreads; a market where one can trade infinitesimal amounts of a security as long/short as one wants. A competitive market is one in which there is no quantity impact on the price received or paid for a trade. In reality, these assumptions are not true. However, both assumptions are needed for arbitrage pricing theory and the Black-Scholes model. Classical arbitrage pricing theory has been successfully used by practitioners for over 20 years. These questions arise then: Do

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we really need to modify this theory? How will this theory change if these two assumptions mentioned above are relaxed?
The answers to these questions has been the subject of many papers from the 1990s to the present, (see Jarrow [34], [35], Çetin , Jarrow, Protter [10], Çetin, Jarrow, Protter and Warachka [11], Çetin, Soner and Touzi [13], Roger and Singh [58], Gökay and Soner [28] and Roch [57]). Weakening these assumptions generates what it called "liquidity risk".

1.2 Liquidity Risk

Liquidity can be divided into two types: *funding liquidity* and *market liquidity*. *Funding liquidity* is known as the ability to resolve agreements immediately, for example a bank has good funding liquidity if it has enough available funding. *Market liquidity* is the ease of which securities can be bought or sold in the market without affecting the asset's price, in other words, it is the facility to trade an asset immediately, with low cost, and with negligible impact on its price [54].

Given these definitions, liquidity risk can be described as follows: *funding liquidity risk* is the risk from the possibility that a trader cannot fund his position and he will be obliged to unwind. *Market liquidity risk* is the risk that the market liquidity gets worse when one needs to relax a position. One can see the funding liquidity risk when depositors to a bank may extract their funds and the bank is not able to borrow from other banks. Market liquidity risk can be seen when a leveraged hedge fund is denied the ability to borrow from their bank, and as a result is obliged to sell their securities [54].

Liquidity generally varies over time and across markets, and since 2007 a number of markets have experienced extreme liquidity risk [54]. The utmost form of market liquidity risk is when traders go out of business (which means no bids), while funding liquidity risk is when banks are short on capital, so they need to lower their trading that requires capital, and also lower the amount of capital they lend to other traders. When banks cannot fund themselves, they cannot fund their clients either. Funding liquidity and market liquidity are related to each other in "liquidity spirals" where less funding leads to less trading; this reduces market liquidity, growing margins and constriction risk management, thus worsening funding [54].

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1.3 Liquidity Risk and Option Pricing

Before the recent global financial crisis, liquidity risk was not of interest to everyone and financial models consistently failed to include liquidity risk. Our interest is the inclusion of liquidity risk in classical arbitrage pricing theory. The papers by Jarrow and Turnbull [43] and Jarrow [35] were first to incorporate liquidity risk into arbitrage pricing theory using the convenience yield approach. However, this approach omits the impact of different trade sizes on the prices. Then, the inclusion of liquidity risk into arbitrage pricing theory that integrates the impact of differing trade sizes on the price was proposed by Çetin, Jarrow, Protter [10]. They hypothesized a stochastic supply curve for the stock price that depends on the trade size, such that traders act as price takers. By adding the assumption that investors face a twice-continuously differentiable price/quantity schedule, the first and second fundamental theorems of asset pricing were extended. In this framework, it was shown that markets do not allow arbitrage if, and only if, there exists an equivalent martingale measure. Further the market is approximately complete if there exists a unique martingale measure. Indeed, the key was that trading strategies that are both continuous and of finite variation can approximate arbitrary-predictable trading strategies in the L^2 sense, and deflect all liquidity costs. Therefore, the arbitrage-free price of any derivative is given by the same price as in the classical economy with no liquidity costs, which is equal to the expected value of its payoff under the risk-neutral measure. Nevertheless, the classical hedge will not be useful to replicate the option in this model with illiquidities, instead a continuous and finite variation approximation will be used. On the other hand, it was shown that the liquidity premium for options in discrete-time is non-zero. Also, the self-financing condition in continuous time is defined to be the self-financing condition in discrete time when the time step tends to zero. Therefore, "one naturally wonders what happens to the liquidity premium when one passes to the continuous-time limit as it is shown by [10] that the pricing formulas for the contingent claims in their model coincide with those in the frictionless markets" [13]. Actually, Çetin, Soner and Touzi [13] report this situation as being a paradox. Their paper resolves this paradox by defining an appropriate set of admissible strategies so that the cost of liquidity will exist in a non-negligible way in the continuous time limit.

Resolving this paradox was also the main interest of other researchers, such as Roch [57].

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This paper combines both notions of liquidity risk in [10] and [13] by adding the impact of large traders on prices. Inspired by the model of Çetin, Jarrow and Protter, Roch [57] assumed the existence of a linear supply curve and an impact of trades on prices. Self-financing trading strategies in which the profit is somewhat affected by the level of liquidity is characterised. The problem of replicating contingent claims was solved using variance swaps which turned out to be the simplest hedging tools in this context.

The study of financial bubbles is a very exciting topic in the economy, this topic was also investigated using liquidity risk modelling. Indeed, it was shown that there is a connection between asset price bubbles and liquidity risk. This connection was studied by Jarrow, Protter and Roch [40]; they developed a liquidity based model for financial asset price bubbles using the model by Roch [57]. It was claimed by [40] that the quantity impact of trading activities on the fundamental price process is what generated financial bubbles.

The outline of this thesis is as follows:

Chapter 2 will tackle the classical theory of arbitrage pricing, we discuss the relation between the theory of arbitrage and the martingale measure as given by the first and second fundamental theorems of asset pricing. We start with the case of finite discrete space, then move to a more general space. We finish the chapter with a section on a very popular model in finance, the Black-Scholes model.

Chapter 3 will discuss the main topic of this thesis, how the classical theory of arbitrage presented in Chapter 2 will change when assuming a market with liquidity risk. We first begin with the model of Çetin, Jarrow and Protter, which will be called CJP model, we give the extended First and Second Fundamental Theorems of Asset pricing, a brief summary of the Roch [57] model, the extended Black-Scholes model for a market with liquidity risk, and an example of a linear supply curve.

For convenience of reference, Appendix A present results in stochastic integration and semimartingales used all through the thesis. Appendix B contain tools used in Chapter 2 to prove the first and second fundamental theorems of asset pricing. Appendix C contain results used in Chapter 3. We then conclude the thesis in Chapter 4.

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Chapter 2

Classical Arbitrage Theory

An *arbitrage opportunity* has been defined in the literature (for example in [21]) as the possibility of making a profit in a financial market without taking any risk. An equilibrium model of a financial market should exclude such arbitrage opportunities. The reason is that, if such opportunities of riskless profit exist, all the traders would try to collect them. Then prices would move in response to an imbalance between supply and demand. Therefore, no such opportunity should be possible in a equilibrium market. Advanced tools from the theory of stochastic processes and functional analysis turn out to be needed in order to mathematically characterize the notion of an arbitrage-free market.

In order to present an extension of the classical theory of arbitrage pricing in the next chapter, we now examine the classical arbitrage theory in detail, from a toy example to discrete time case to a continuous time case.

In a financial market, the up and down movements of market variables (for example stocks, bonds, currencies, interest rates), is what create risks. *Options* [1] are an example of derivatives that are in the financial market to transfer risks from one party to another.

Let us first explain the idea of arbitrage with a simple example in which we assume interest rates are zero. Suppose $S = (S_t)_{0 \leq t \leq T}$ represents the price process of a risky asset. We know its value today $S_0 = 1$, but we don't know its value tomorrow. To model this uncertainty stochastically, we define a sample space Ω containing two states of the stock,

[1] Options are contracts that gives the holder the right but not the obligation to buy,"call option", or sell "put option" a type of security.

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$\Omega = \{up, down\}$, we define S_T to be the stock price at a future time T which is unknown. We have $S_T(\omega) = 2$ for $\omega = up$ or, $S_T(\omega) = 0.5$ for $\omega = down$, with $\mathbb{P}(up) = \mathbb{P}(down) = 0.5$. In order to reduce the risk, one can use, for example, a call option where the buyer of the option has the right but not obligation to buy one stock at time T at a agreed price K fixed at time $t = 0$. Then the payoff at time T for a call option with a strike price $K = 1$ will be

$$C_T = (S_T - K)^+. \tag{2.1}$$

Therefore, if the stock gains value the option is worth $C_T = 1$; else it is worth $C_T = 0$. The question now is what is the price C_0 of the option at time $t = 0$?

A common guess is that because the option pays 1 or 0 with a probability of a half-half, it should be worth

$$0.5 \times 1 + 0.5 \times 0 = 0.5$$

at time 0 i.e. $C_0 = \mathbb{E}[C_T]$. This guess is wrong, and it is necessary to understand the reason. Note that we start with a risky asset $S_0 = 1$ and bond $B_0 = 1$, so we have $\mathbb{E}[S_T] = 1.25 > S_0 = 1$. Thus in average the stock has a better performance than the bond. This means that this method is not applicable for pricing stocks and therefore there is no reason to believe that it is applicable to pricing options either.

Let us use a different approach to price the option. One can buy a portfolio V consisting of s shares and b in cash with the property that at time T, regardless of the movement of the stock market, the portfolio has the same value as the option. Thus we have the following system of linear equations to solve for s and b:

$$\begin{cases} 2s + b = 1 \\ 0.5s + b = 0 \end{cases} \Leftrightarrow \begin{cases} s = 2/3 \\ b = -1/3 \end{cases}.$$

Then the portfolio value is known at time 0, $V_0 = \frac{2}{3}S_0 - \frac{1}{3}B_0 = \frac{1}{3}$. We know that the portfolio replicates the option i.e. $C_T = V_T$, we claim that then we must have $C_0 = V_0 = \frac{1}{3}$. For now, suppose that $C_0 = 0.5$ as we proposed above, and that we can buy the replication portfolio and sell an option, and keep the difference. At time t our portfolio is worth

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exactly what the holder of the option owes. By definition of this portfolio, we made money without taking any risk. This is called an arbitrage opportunity in the market, and it is an important assumption that such opportunities should not exist.

This example shows that the arbitrage-free price of the option C is not $\mathbb{E}[C]$, which means the probability $\mathbb{P}$ is not useful at all. Instead, it is $\mathbb{E}_{\mathbb{Q}}[C]$, such that $\mathbb{Q}$ is a new probability measure, in this case defined by $\mathbb{Q}(\omega = up) = \frac{1}{3}$ and $\mathbb{Q}(\omega = down) = \frac{2}{3}$, and is the unique solution satisfying $\mathbb{E}_{\mathbb{Q}}[S_T] = S_0 = 1$. Thus, the price process for the stock is a $\mathbb{Q}$-martingale and $\mathbb{Q}$ is called a martingale measure for the stock price process. Moreover, If S is a martingale under $\mathbb{Q}$, and $\mathbb{Q}$ is unique, then every contingent claim C with the underlying stock S, can be replicated, as we will see in the next section.

This chapter's main goal is to show the connection between arbitrage and the existence of a equivalent martingale measure, namely, a measure that is equivalent to the original one under which discounted price processes have the martingale property. The above connection is given by the so-called fundamental theorem of asset pricing.

The First Fundamental Theorem of Asset Pricing was proven in the case of discrete-state-space, multi-period discrete time by Harrison and Kreps [30]. In this case, it was proven that for a stochastic process S_t, the existence of an equivalent martingale measure is basically equivalent to the absence of arbitrage opportunities. The separating hyperplane theorem in finite-dimensional Euclidean spaces was crucial in the proof. However, a more general version of the theorem was proven by Harrison and Pliska [31], in the case of finite Ω and finite discrete time. Although, the results in this paper were attractive, one has to consider the severity of the restriction to finite Ω. When we assume infinitely many trading dates, an infinite number of assets or the presence of market frictions, a simple version of a fundamental theorem of asset pricing has not yet been proven, thus the absence of arbitrage is not sufficient to construct martingale measures under which the price process is a martingale. Kreps [50] replaces the condition of no-arbitrage by the notion of "free lunch". The first to prove the First Fundamental Theorem of Asset Pricing for a more general semimartingale model is the paper by Delbaen and Schachermayer [19], introducing the condition of "No free lunch with vanishing risk". This condition is the most effective in obtaining a general version of the fundamental theorem of asset pricing, and is economically acceptable and mathematically convenient. The next important result

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on which modern finance theory is based is what it is called the Second Fundamental Theorem of Asset Pricing. This theorem gives the relation between the notion of market completeness and the uniqueness of the equivalent martingale measure. We recall that a well-known application of the theory of asset pricing by arbitrage is the Black-Scholes formula, which is still widely used by practitioners and will be discussed in Section 2.2.3.

2.1 The Discrete Time Model

This section introduces a discrete model of a financial market. We start by defining self-financing trading strategies. Then we discuss the First Fundamental Theorem of Asset Pricing and market completeness.

We assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω has infinitely many elements. Also we have a fixed time $T \geq 0$, and a filtration $(\mathcal{F}_t)_{t=0}^T$ on Ω.

A financial market model is a $d+1$-dimensional stochastic process $\tilde{S} = (\tilde{S}_t^0, \tilde{S}_t^1, \cdots, \tilde{S}_t^d)_{t=0}^T$ based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$, The risk-free bank account process $\tilde{S}_t^0$ is positive a.s, and is the process used as a numéraire, that compares money between 0 and $t > 0$. In this case $\tilde{S}$ is assumed to be adapted which means $\tilde{S}_t$ is $\mathcal{F}_t$-measurable to the filtration $(\mathcal{F}_t)_{t=0}^T$, in other words, at time t, the past and the current prices are known for each security.

We also need to define a *trading strategy*, which is a process $(\tilde{X}_t)_{t=1}^T = (\tilde{X}_t^0, \tilde{X}_t^1, \cdots, \tilde{X}_t^d)_{t=1}^T$ with values in $\mathbb{R}^d$. This process is predictable, that is, $\tilde{X}_t$ is $\mathcal{F}_{t-1}$-measurable for $t = 1, \cdots, T$. The component $\tilde{X}_t^d$ represents the number of securities $\tilde{S}_t^d$ held by the investor between times $t-1$ and t. The vector $\tilde{X}_t$ (also called a portfolio) is what the investor holds between times $t-1$ and t, the decision of holding $\tilde{X}_t$ made at time $t-1$, which explains why we need $\tilde{X}_t$ to be predictable.

Given a security $\tilde{S}_t$ and a portfolio $\tilde{X}_t$, the inner product

$$\tilde{V}_0 = \sum_{i=0}^{d} \tilde{X}_1^i \tilde{S}_0^i \quad \tilde{V}_t = \sum_{i=0}^{d} \tilde{X}_t^i \tilde{S}_t^i, \ \ t = 1, \cdots, T,$$

is called the market value of the portfolio $\tilde{X}_t$.

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The trading strategies that we interested to study are those that do not allow funds to be added or withdrawn from the value of the portfolio. These strategies have the special name *"self-financing trading strategies"*. Mathematically:

Definition 2.1.1. *[21] A trading strategy $\tilde{X}_t$ is self-financing if it satisfies the following equation:*

$$\tilde{X}_{t+1}\tilde{S}_t = \tilde{X}_t\tilde{S}_t \ \ for \ \ t = 1, \cdots, T-1. \tag{2.2}$$

We can interpret this as follows; by changing the portfolio form $\tilde{X}_{t-1}$ to $\tilde{X}_t$, there is no input or outflow of the money.

Let us define the discounted price process $S_t = (S_t^1, \cdots, S_t^d)$ which is an adapted process with values in $\mathbb{R}^d$ such that

$$S_t^i = \frac{\tilde{S}_t^i}{\tilde{S}_t^0}, \text{ for } i = 1, \cdots, d.$$

We can observe that there is no need to include coordinate 0, since we have $S_t^0 = \dfrac{\tilde{S}_0^t}{\tilde{S}_0^t} = 1$.

We then define the predictable trading strategy $(X_t)_{t=1}^T = (X_t^1, \cdots, X_t^d)_{t=1}^T$ with values in $\mathbb{R}^d$ obtained by discarding the $0'th$ coordinate of the $\mathbb{R}^{d+1}$-valued process $(\tilde{X})_{t=1}^T = (\tilde{X}_t^0, \tilde{X}_t^1, \cdots, \tilde{X}_t^d)_{t=1}^T$ so that

$$X_t^i = \tilde{X}_t^i, \text{ for } i = 1, \cdots, d.$$

An important result that should be mentioned here is that for a $\mathbb{R}^d$-valued trading strategy X_t, a unique $\mathbb{R}^{d+1}$-valued self-financing trading strategy $\tilde{X}$ can be constructed such that $\tilde{X}_t^i = X_t^i$ for all $i = 1, \cdots, d$ and $t = 1, \cdots, T$, with the assumption $\tilde{X}_1^0 = 0$, and finding $\tilde{X}_t^0$ inductively using equation 2.2. Another observation is that we can define a discounted market value V_t as follows:

$$V_t = \frac{\tilde{V}_t}{\tilde{S}_t^0}, \ t = 0, 1, \cdots, T.$$

This discounted market value is independent of the process $\tilde{X}_t^0$, as we can justify as follows: we have $\tilde{S}_0^0 = 1$ and $X_1^0 = 0$, then the market value at time 0 is $V_0 = \tilde{V}_0 = \tilde{X}_1^1\tilde{S}_1^1 + \cdots + \tilde{X}_1^d\tilde{S}_1^d$.

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Using the self-financing condition $\tilde{X}_t \tilde{S}_t = \tilde{X}_{t+1} \tilde{S}_t$, the change in V_t is

$$\begin{aligned} V_{t+1} - V_t &= \frac{\tilde{V}_{t+1}}{\tilde{S}^0_{t+1}} - \frac{\tilde{V}_t}{\tilde{S}^0_t} \\ &= \frac{1}{\tilde{S}^0_{t+1}} \sum_{i=0}^{d} \tilde{X}^i_{t+1} \tilde{S}^i_{t+1} - \frac{1}{\tilde{S}^0_t} \sum_{i=0}^{d} \tilde{X}^i_{t+1} \tilde{S}^i_t \\ &= \sum_{i=0}^{d} X^i_{t+1} \left(S^i_{t+1} - S^i_t \right) = \sum_{i=1}^{d} X^i_{t+1} \Delta S^i_{t+1}. \end{aligned}$$

We showed that V_t is independent of $\tilde{X}^0_t$. Then we can conclude that as long as we are interested in discounted portfolio values only, there is no loss of information when we consider the $\mathbb{R}^d$-valued strategy X_t instead of the $\mathbb{R}^{d+1}$-valued strategy $\tilde{X}_t$.

We then define the "stochastic integral" $X \bullet S$ as the $\mathbb{R}$-valued process $((X \bullet S)_t)_{t=0}^T$ given by

$$(X \bullet S)_t = \sum_{n=1}^{t} X_n \Delta S_n \quad t = 0, \cdots, T,$$

where $V_0 = (X \bullet S)_0 = 0$ is the initial investment, then the self-financing condition implies

$$V_t = V_0 + \sum_{n=1}^{t} X_n \Delta S_n = (X \bullet S)_t, \quad t = 0, \cdots, T. \tag{2.3}$$

2.1.1 The First Fundamental Theorem of Asset Pricing

To formulate the proof of the First Fundamental Theorem of Asset Pricing (FFTAP), we need to introduce some mathematical concepts. In this section on discrete time models we will restricted to a finite probability space $\Omega = \{\omega_1, \cdots, \omega_N\}$ and a probability measure $\mathbb{P}$ such that $\mathbb{P}[\omega_n] = p_n > 0$ for $n = 1, \cdots, N$, where N denotes the number of the states of the world. Let $\mathcal{H}$ be the set of predictable processes X_t with values in $\mathbb{R}^d$. Let $L^0(\Omega, \mathcal{F}, \mathbb{P})$ be the space of measurable functions on Ω, and $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is the set of bounded functions on Ω. Let us mention here that in the finite case we simply have $L^0(\Omega, \mathcal{F}, \mathbb{P}) = L^\infty(\Omega, \mathcal{F}, \mathbb{P}) = \mathbb{R}^N$. We also would like to introduce the positive (resp

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negative) orthant of $L^0(\Omega, \mathcal{F}, \mathbb{P})$ as follows:

$$L^0_+(\Omega, \mathcal{F}, \mathbb{P}) = \left\{ f \in L^0(\Omega, \mathcal{F}, \mathbb{P}), f \geq 0 \right\} \tag{2.4}$$

$$L^0_-(\Omega, \mathcal{F}, \mathbb{P}) = \left\{ f \in L^0(\Omega, \mathcal{F}, \mathbb{P}), f \leq 0 \right\}. \tag{2.5}$$

Note that we will write the notation $L^0(\Omega, \mathcal{F}, \mathbb{P})$ and $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, even if we know that these spaces are simply $\mathbb{R}^N$, to indicate which function spaces we shall face when we move to a continuous time model on the next section, and also to know if an element of $\mathbb{R}^N$ is a contingent claim [2] or an element of the price vector [21].

Definition 2.1.2. *[21](attainable) We call the subspace $\mathcal{K} \subset L^0(\Omega, \mathcal{F}, \mathbb{P})$ defined by*

$$\mathcal{K} = \{(X \bullet S)_T \mid X \textit{ is predictable } \}, \tag{2.6}$$

$\mathcal{K}$ is a vector space of all possible replicable pay-offs of self-financing portfolios with zero-initial value, i.e. the set of contingent claims attainable at price 0. We call the convex cone $\mathcal{C} \subset L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, defined by

$$\mathcal{C} = \{g \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \mid \textit{there exists } f \in \mathcal{K} \textit{ with} f \geq g\}, \tag{2.7}$$

the set of contingent claims super-replicable at price 0, we say "super-replicable" since such claims are dominated by attainable claims.

The affine space $\mathcal{K}_a = a + \mathcal{K}$ is the set of contingent claims attainable at price a , for $a \in \mathbb{R}$, and is obtained by moving $\mathcal{K}$ by the constant function a. An economical interpretation, is that these are precisely the contingent claims that a trader may replicate with an initial investment of a by following some predictable trading strategy X. Economically, a contingent claim $g \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is super-replicable at price 0, if it is possible to achieve g with 0 net investment by following predictable trading strategies X. As a result, we arrive at some contingent claim f and if needed, we "throw away money" [21] to arrive at g.

The definition of *arbitrage opportunity* can be given as follows: it is a trading strategy X with 0 initial value, the resulting contingent claim $f = (X \bullet S)_T$ is non-negative and not identically equal to zero a.s. It is obvious that every arbitrageur would like to have such an opportunity. Then an equilibrium financial market should not allow for arbitrage opportunities.

[2]Contingent claim is a random variable represent the pay-off at time T from seller to buyer.

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Definition 2.1.3. *[21] A financial market S satisfies the No-arbitrage condition (NA) if*

$$\mathcal{K} \cap L^0_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}\,, \tag{2.8}$$

or equivalently

$$\mathcal{C} \cap L^0_+(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}\,. \tag{2.9}$$

Proposition 2.1.4. *[21] The condition (NA) for S implies that*

$$\mathcal{C} \cap (-\mathcal{C}) = \mathcal{K}.$$

Proof. We have $\mathcal{K} \subset \mathcal{C}$ and since $\mathcal{K}$ is a vector space, then also $\mathcal{K} \subset -\mathcal{C}$, thus $\mathcal{K} \subset \mathcal{C} \cap -\mathcal{C}$. Conversely suppose $h \in \mathcal{C} \cap (-\mathcal{C})$, then we can write $h = f_1 - g_1 = f_2 + g_2$ for some elements $f_1, f_2 \in \mathcal{K}$ and $g_1, g_2 \in L^0_+$. We have $f_1 - g_1 = f_2 + g_2$. Hence, $f_1 - f_2 \in K \cap L^0_+ = \{0\}$ (by the NA condition). It follows that $f_1 = f_2$ $g_1 = g_2 = 0$. Thus, $h = f_1 = f_2 \in \mathcal{K}$. □

Definition 2.1.5. *A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is called an equivalent martingale measure for S if, and only, if, $\mathbb{P} \equiv \mathbb{Q}$ and S is a martingale measure under $\mathbb{Q}$, i.e.,*

$$\mathbb{E}_{\mathbb{Q}}\left[S_{t+1} \mid \mathcal{F}_t\right] = S_t \;\; for\ all\ t = 0, \cdots, T-1.$$

Let $\mathcal{M}^e(S)$ denote the set of equivalent martingale measures for S.

Lemma 2.1.6. *[21] For a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, the following are equivalent:*

i) S is martingale under $\mathbb{Q}$.

ii) $\mathbb{E}_{\mathbb{Q}}[f] = 0$, for all $f \in \mathcal{K}$.

iii) $\mathbb{E}_{\mathbb{Q}}[g] \leq 0$, for all $g \in \mathcal{C}$.

Proof. [21] Let us start by proving i)⇒ ii): Suppose we have a $\mathbb{Q}$-martingale S and a trading strategy X, then

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[X_t \Delta S_t] &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\sum_{i=1}^{d} X^i_t (S^i_t - S^i_{t-1}) \mid \mathcal{F}_{t-1}\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\sum_{i=1}^{d} X^i_t \mathbb{E}_{\mathbb{Q}}[S^i_t - S^i_{t-1} \mid \mathcal{F}_{t-1}]\right] \;\; (X_t \;\; is\ \mathcal{F}_{t-1} - measurable). \\
&= 0 \;\; (S \;\; is \;\; \mathbb{Q} - martingale).
\end{aligned}$$

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We have

$$\mathbb{E}_{\mathbb{Q}}\left[(X \bullet S)_t \mid \mathcal{F}_{t-1}\right] = \mathbb{E}_{\mathbb{Q}}\left[\sum_{u=1}^{t-1} X_u \Delta S_u + X_t \Delta S_t \mid \mathcal{F}_{t-1}\right] = (X \bullet S)_{t-1},$$

which means $(X \bullet S)_t$ is also $\mathbb{Q}$-martingale. In particular, $\mathbb{E}_{\mathbb{Q}}[(X \bullet S)_T] = (X \bullet S)_0 = 0$.

For ii) $\Rightarrow$ i), we consider a $\mathcal{F}_{t-1}$-measurable set A and the strategy $X(\omega, s) = 1_A(\omega)1_{(t-1,t]}(s)$. We have $(X \bullet S)_T = 1_A(S_t - S_{t-1})$ and so $\mathbb{E}_{\mathbb{Q}}\left[1_A(S_t - S_{t-1})\right] = 0 \in \mathbb{R}^d$ and

$$\mathbb{E}_{\mathbb{Q}}\left[S_t \mid \mathcal{F}_{t-1}\right] = S_{t-1} \text{ for } t = 1, \cdots, T.$$

For ii) $\Leftrightarrow$ iii), if we have $f \in \mathcal{K}$ is equivalent to say $g \in \mathcal{C}$ such that $g \leq f$, then $\mathbb{E}_{\mathbb{Q}}[g] \leq \mathbb{E}_{\mathbb{Q}}[f] = 0$. $\square$

Before we present the FFTAP, we introduce some concepts and important results from convex analysis that are required to prove this theorem. These definitions and results are from [33].

Let us assume a linear topological space T, which in this section is simply $T = \mathbb{R}^N$.

- A set $A \subseteq T$ is said to be convex if whenever $x, y \in A$, then $[x, y] \subset A$ where $[x, y] = \{\alpha x + (1-\alpha)y,\ 0 \leq \alpha \leq 1\}$.
- If we have a family of convex sets, their intersection is again a convex set. Then for any convex $A \subseteq T$, we can define the convex hull of A to be the intersection of all convex sets in T that contain A, and the smallest convex set contains A.
- A hyperplane in $\mathbb{R}^N$ is the set of all the points where $L(x) = a$ for some linear functional L, with $a \in \mathbb{R}$.
- Two sets A and B in T is said to be separated by a hyperplane if there exists a continuous linear functional L such that for $a \in \mathbb{R}$ we have

$$L(x) \leq a,\ for\ x \in A \tag{2.10}$$

$$L(x) \geq a,\ for\ x \in B. \tag{2.11}$$

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- One of the nicest results in functional analysis says that two convex disjoint sets in a topological vector space can be separated, this result is known as the Separation Theorem:

Theorem 2.1.7. *[63] Separation Theorem: Assume A and B are disjoint, non-empty and convex subsets of a topological vector space E. Then,*

1. *If A is open, there exist $L \in E^*$ and $\alpha \in \mathbb{R}$ such that*
$$L(x) < \alpha \leq L(y). \tag{2.12}$$

2. *If E is locally convex, A is compact and B is closed, there exist $L \in E^*$ and $\alpha,\ \beta \in \mathbb{R}$ such that*
$$L(x) < \alpha < \beta < L(y), \tag{2.13}$$
for every $x \in A$ and $b \in B$.

Now we have the tools needed to prove the following theorem.

Theorem 2.1.8. *[31] A financial market S modelled on a finite stochastic space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ satisfies the no-arbitrage property if, and only if, $\mathcal{M}^e(S) \neq \emptyset$.*

Proof. [31] Suppose, first, that there is $\mathbb{Q} \in \mathcal{M}^e(S)$ by applying the lemma 2.1.6; we have
$$E_{\mathbb{Q}}\left[g\right] \leq 0 \text{ for } g \in \mathcal{C}. \tag{2.14}$$
On the other hand, suppose that there is $g \in \mathcal{C} \cap L_+^\infty$, $g \neq 0$; using the fact that $\mathbb{Q} \equiv \mathbb{P}$ we have
$$E_{\mathbb{Q}}\left[g\right] > 0, \text{ contradiction.} \tag{2.15}$$
The converse is the most important part of the theorem. It connects the no-arbitrage with the martingale theory. To prove the converse we use Theorem 2.1.7 for the existence of an equivalent martingale measure.

We want to prove the existence giving that there is no-arbitrage, by Definition 2.1.3 we have $\mathcal{K} \cap L_+^\infty = \{0\}$ which means that $\mathcal{K}$ and $L_+^\infty \setminus \{0\}$ are disjoint and also convex. But this is not enough for us in order to use Theorem 2.1.7, and have a strict separation between $\mathcal{K}$ and $L_+^\infty \setminus \{0\}$, because we need the compactness of one of them. Since we are working in a finite probability space we can consider the convex hull of the unit vectors $(1_{\{\omega_n\}})_{n=1}^N$ in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ i.e.
$$P := \left\{ \sum_{n=1}^N \mu_n 1_{\{\omega_n\}} \mid \mu_n \geq 0, \sum_{n=1}^N \mu_n = 1 \right\}. \tag{2.16}$$

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The set P is a convex, compact subset of L^∞_+. By the (NA) assumption we have $\mathcal{K} \cap L^\infty_+ = \{0\}$.

Then P is a compact subset of $L^\infty_+(\Omega, \mathcal{F}, \mathbb{P})$, and P, $\mathcal{K}$ are disjoint. Therefore, we can apply Theorem 2.1.7 to separate the convex compact set P from $\mathcal{K}$, by a linear functional $L \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* = L^1(\Omega, \mathcal{F}, \mathbb{P})$.

$$L(f) \leq \alpha < \beta \leq L(g) \text{ for } f \in \mathcal{K}, h \in P \textit{ and } \alpha,\ \beta \in \mathbb{R}. \tag{2.17}$$

$\mathcal{K}$ is a linear space, then we have $\forall \lambda > 0$, $\lambda f \in \mathcal{K}$ and $-\lambda f \in \mathcal{K}$ for all $f \in \mathcal{K}$

$$L(\lambda f) = \lambda L(f) \leq \alpha \tag{2.18}$$
$$L(-\lambda f) = -\lambda L(f) \leq \alpha. \tag{2.19}$$

We showed that $\alpha \geq 0$ and we can replace α by 0. Therefore, $\beta > 0$. To finish the proof, let us define $L_i = L(1_{\{\omega_i\}}) \geq \beta > 0$. By normalizing L we have $\mathbb{Q}(\{\omega\}) = (\frac{L_1}{\sum_{i=1}^N L_i}, \cdots, \frac{L_N}{\sum_{i=1}^N L_i}) > 0$, as $\mathbb{Q}$ is strictly positive on each $1_{\{\omega_i\}}$. Therefore, we have found a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ such that $\mathbb{E}_\mathbb{Q}[f] \leq 0$, using again the linearity of $\mathcal{K}$ we get $\mathbb{E}_\mathbb{Q}[f] = 0$. By Lemma 2.1.6 we found an equivalent martingale measure $\mathbb{Q}$ for the process S. □

2.1.2 Market Completeness

Now we assume that the market is arbitrage-free and that there exists a equivalent martingale measure. Here we want to investigate when we can say that a market is complete.

Definition 2.1.9. *[21] A contingent claim C on $(\Omega, \mathcal{F}, \mathbb{P})$ is attainable if there exists a predictable strategy X such that $C = a + (X \bullet S)_T$ for some $a \in \mathbb{R}$.*

An attainable contingent claim can be seen as a random pay-off at time T that can be achieved by following a self-financing strategy involving some initial investment a. Now the definition of a complete market can be given.

Definition 2.1.10. *[21] A market is said to be complete if every contingent claim is attainable.*

We should mention that the results we give below are from [8], and we are restricted to the case of finite space.

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- The polar $\mathcal{C}^0$ for $\mathcal{C} \subseteq L^0_+(\Omega, \mathcal{F}, \mathbb{P})$ is given by

$$\mathcal{C}^0 = \{g \in L^0_+ : \mathbb{E}[fg] \leq 1, \ \ \forall f \in \mathcal{C}\} \tag{2.20}$$

- The bipolar $\mathcal{C}^{00}$ of $\mathcal{C} \subseteq L^0_+(\Omega, \mathcal{F}, \mathbb{P})$ is given by

$$\mathcal{C}^{00} = \left\{f \in L^0_+ : \mathbb{E}[fg] \leq 1, \ \ \forall g \in \mathcal{C}^0\right\}. \tag{2.21}$$

Theorem 2.1.11. *[8] Bipolar Theorem : The bipolar $\mathcal{C}^{00}$ is the smallest closed convex set in $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$ containing $\mathcal{C}$.*

Remark 2.1.12. *If $\mathcal{C}$ is closed under multiplication by positive scalars, the polar $\mathcal{C}^0$ may be written as*

$$\mathcal{C}^0 = \{g \in L^0_+ : \mathbb{E}[fg] \leq 0, \ \ \forall f \in \mathcal{C}\}. \tag{2.22}$$

Definition 2.1.13. *We denote $\mathcal{M}^a(S)$ the set of absolutely continuous martingale measures for S, i.e the set of all probability measures $\mathbb{Q}$ that are absolutely continuous with respect to $\mathbb{P}$ such that S is martingale under $\mathbb{Q}$.*

We recall that in this setting of finite probability space Ω with $\mathbb{P}[\omega] > 0$ for each $\omega \in \Omega$, we have $\mathbb{Q} \equiv \mathbb{P}$ if, and only if, $\mathbb{Q}[\omega] > 0$, for each $\omega \in \Omega$. The measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is identified by its Radon-Nikodým derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

The cone generated by $\mathcal{M}^a(S)$ is defined as follows:

$$cone(\mathcal{M}^a(S)) = \left\{f = \lambda\frac{d\mathbb{Q}}{d\mathbb{P}} : \ \lambda \geq 0 \ and \ \mathbb{Q} \in \mathcal{M}^a(S)\right\}. \tag{2.23}$$

Proposition 2.1.14. *[19] Assume S satisfies no-arbitrage and the convex cone $\mathcal{C}$ is closed. The polar $\mathcal{C}^0$ is equal to cone $(\mathcal{M}^a(S))$, the cone generated by $\mathcal{M}^a(S)$, and $\mathcal{M}^e(S)$ is dense in $\mathcal{M}^a(S)$. Hence the following assertions are equivalent for an element $f \in L^\infty(\Omega, \mathcal{F}, \mathbb{C})$:*

1. *$f \in \mathcal{C}$,*

2. *$\mathbb{E}_\mathbb{Q}[f] \leq 0, \ for \ \ all\, \mathbb{Q} \in \mathcal{M}^a(S)$,*

3. *$\mathbb{E}_\mathbb{Q}[f] \leq 0, \ for \ \ all\, \mathbb{Q} \in \mathcal{M}^e(S)$.*

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Proof. Let $\mathbb{Q} \in \mathcal{M}^a(S)$, and let $\lambda > 0$. Let $f = \lambda \frac{d\mathbb{Q}}{d\mathbb{P}} \in cone(\mathcal{M}^a(S))$, then we have

$$\mathbb{E}[fg] = \mathbb{E}\left[\lambda \frac{d\mathbb{Q}}{d\mathbb{P}} g\right] \tag{2.24}$$

$$= \lambda \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} g\right] \tag{2.25}$$

$$= \lambda \mathbb{E}_{\mathbb{Q}}[g] \leq 0, \ \forall g \in \mathcal{C} \text{ [Lemma 2.1.6].} \tag{2.26}$$

Thus $cone(\mathcal{M}^a(S)) \subset \mathcal{C}^0$. Conversely, we have $L^\infty_-(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathcal{C}$ and $\mathcal{C}^0 \subseteq L^1_+(\Omega, \mathcal{F}, \mathbb{P})$, then if $f \in \mathcal{C}^0 \subset L^1_+$, we can write $f = \lambda \frac{d\mathbb{Q}}{d\mathbb{P}}$, for $\lambda \geq 0$ and probability measure $\mathbb{Q}$. Let us show that $\mathbb{Q} \in \mathcal{M}^a(S)$, we have $0 \geq \mathbb{E}[fg] = \mathbb{E}[\lambda \frac{d\mathbb{Q}}{d\mathbb{P}} g] = \lambda \mathbb{E}_{\mathbb{Q}}[g]$, $\forall g \in \mathcal{C}$, by lemma 2.1.6, we have $\mathbb{Q} \in \mathcal{M}^a(S)$. Then $\mathcal{C}^0 \subset cone(\mathcal{M}^a(S))$. Thus we proved that $\mathcal{C}^0 = cone(\mathcal{M}^a(S))$. Hence 1) and 2) are equivalent from the bipolar Theorem 2.1.11.

To prove the density between $\mathcal{M}^e(S)$ and $\mathcal{M}^a(S)$, we have S satisfy no-arbitrage then from Theorem 2.1.8, we have the existence of at least one $\mathbb{Q}^* \in \mathcal{M}^e(S)$. Then for all $\mathbb{Q} \in \mathcal{M}^a(S)$ and $0 \leq \upsilon \leq 1$, we have $\upsilon \mathbb{Q}^* + (1-\upsilon)\mathbb{Q} \in \mathcal{M}^e(S)$, since $(1-\upsilon)\mathbb{Q}$ is absolutely continuous with respect to $\mathcal{M}^e(S)$. We showed that any neighbourhood of $\mathbb{Q} \in \mathcal{M}^a(S)$ contains at least one element of $\mathcal{M}^e(S)$, which implies the density of $\mathcal{M}^e(S)$ in $\mathcal{M}^a(S)$. Then 2) and 3) are equivalent because of the density of $\mathcal{M}^e(S)$ and $\mathcal{M}^a(S)$. □

Proposition 2.1.15. *[21] Assume S satisfies no-arbitrage. Then for $f \in L^\infty$, the following assertions are equivalent:*

1. *$f = (X \bullet S)_T \in \mathcal{K}$ for $X \in \mathcal{H}$,*
2. *$\forall\, \mathbb{Q} \in \mathcal{M}^e(S)$, we have $\mathbb{E}_{\mathbb{Q}}[f] = 0$,*
3. *$\forall\, \mathbb{Q} \in \mathcal{M}^a(S)$, we have $\mathbb{E}_{\mathbb{Q}}[f] = 0$.*

Proof. [21] By Proposition 2.1.4, we have that $f \in \mathcal{K}$ if, and only if, $f \in \mathcal{C} \cap (-\mathcal{C})$. Hence by Proposition 2.1.14, and since $f \in \mathcal{C}$ and $f \in (-\mathcal{C})$, we have the equivalence. □

Corollary 2.1.16. *[21] Suppose there are no-arbitrage opportunities.*

1. *The model is complete if, and only if, there exists a unique equivalent martingale measure.*
2. *In case of completeness the representation $g = a + g_0$ with $a \in \mathbb{R}$ and $g_0 \in \mathcal{K}$ of claim $g \in L^\infty$ is unique. In this case $a = \mathbb{E}_{\mathbb{Q}}[f]$, the stochastic process $X \bullet S$ is unique and we have*

$$\mathbb{E}_{\mathbb{Q}}[f \mid \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[f] + (X \bullet S)_t, \ t = 0, \cdots, T. \tag{2.27}$$

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Proof. [21] 1. Suppose that there exists a unique martingale measure i.e $\mathcal{M}^e(S) = \{\mathbb{Q}\}$ and take $f \in L^\infty$. By Proposition 2.1.15 we have $f - \mathbb{E}_{\mathbb{Q}}[f] \in \mathcal{K}$, hence f is attainable. Conversely, assume that there is $\mathbb{Q}_1 \neq \mathbb{Q}_2$ in $\mathcal{M}^e(S)$. Then there exists an $f \in L^\infty$ such that $\mathbb{E}_{\mathbb{Q}_1}[f] \neq \mathbb{E}_{\mathbb{Q}_2}[f]$. If the market is complete, and this f is attainable, there would exist $a \in \mathbb{R}$ such that $f - a \in \mathcal{K}$. Using Proposition 2.1.15 this would imply that $\mathbb{E}_{\mathbb{Q}_1}[f] = a = \mathbb{E}_{\mathbb{Q}_2}[f]$, a contradiction.

2. We start by showing the uniqueness of the constant $a \in \mathbb{R}$; assume there are two representations $g = a_1 + g_0^1$, and $g = a_2 + g_0^2$ with $a_1 \neq a_2$, $g_0^1 = (X^1 \bullet S)_T$ and $g_0^2 = (X^2 \bullet S)_T$. If we assume that $a_1 > a_2$, then an arbitrage opportunity could be possible by considering the trading strategy $X_2 - X_1$. Indeed, we have $a_1 - a_2 = ((X^2 - X^1) \bullet S)_T$, which means the trading strategy $X^2 - X^1$ creates a strictly positive profit at time T, and that contradicts the no-arbitrage condition. □

2.2 The Continuous-Time Model

In this section we present the theory of arbitrage in continuous time. Most of the results in this section are from the paper by Delbaen and Schachermayer [19]. As in the case of discrete time models, we will define trading strategies, investigate a version of the First Fundamental Theorem of Asset Pricing, and finally study market completeness and the Second Fundamental Theorem of Asset Pricing.

In the case of finite number of trading dates presented previously, trading strategies are defined to be a linear combinations of buy and hold strategies. Mathematically, such buy and hold strategies are given by function $X = H1_{]T_1,T_2]}$, where $T_1 < T_2$ are finite stopping times, H is $\mathcal{F}_{T_1}$-measurable, which means in economics, the following: buy $H(\omega)$ units of financial security at time $T_1(\omega)$, wait until time $T_2(\omega)$ and sell them. A sum of such strategies is called a simple integrand. Then by using the simple integrand X, the capital gain, defined by the stochastic integral $(X \bullet S) = \int X dS$, is well-defined for adapted processes S.

In the case of continuous time and continuous trading, it turns out that more general integrands or trading strategies need to be considered. More precisely, we need to consider predictable trading strategies. Then a problem arises, when we leave the framework of simple integrands to more general trading strategies, the stochastic integral $X \bullet S$ has to exist. As mentioned in Delbaen and Schachermayer [19], the process S has to be a

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semimartingale for $X \bullet S$ to be well-defined for general integrands. Actually, the Bichteler-Dellacherie theorem A.2.12 give us the equivalence between S being a semimartingale and S as a sum of a local martingale process and a finite variation process, which makes semimartingales called "good integrators"[3]. In fact, semimartingales provide a suitable framework for the discussion of general concepts of financial theory such as arbitrage and hedging problems.

As usual we start with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t\geq 0}$ on Ω. We consider a model of a financial market formed from a bank account with a price process Y_t, and a discounted risky asset price with values in $\mathbb{R}^d$, denoted by $S = (S_t)_{t\geq 0}$, based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, which we assume to be semimartingale. That is,

$$S_t = S_0 + M_t + A_t,$$

with M as a local martingale and A an adapted càdlàg process of finite variation.

We assume a frictionless financial market where there is no bid-ask spread [4], and no transaction costs. In other words, we assume a completely liquid market, where one can buy and sell unlimited quantities on the market without changing the financial securities prices. Also we assume that the stock prices are discounted.

From now onwards we give a trading strategy $((X, Y)_{t\geq 0}$ which presents, respectively, the quantity of the asset held by the investor and money market accounts. Let us assume that a trading occurs at time t and $t+dt$, but not in between, which means the holding of stock X and the money market account Y stay constant between t and $t+dt$. Therefore, if there is no incoming or outgoing cashflow, and if a trading occurs at $t+dt$, it has to be done only with available funds i.e. $Y_t + X_t S_{t+dt} = Y_{t+dt} + X_{t+dt} S_{t+dt}$. The intuition behind this is as follows:

$$Y_{t+dt} - Y_t = -\left(X_{t+dt} - X_t\right) S_{t+dt}.$$

[3] Appendix A contain definitions and results from stochastic integration theory needed in this thesis.
[4] The selling price is the same as the buying price of all securities.

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Then in continuous time we have:

$$\begin{aligned} Y_{t+dt} - Y_t &= -\left(X_{t+dt} - X_t\right)\left(S_{t+dt} - S_t\right) - (X_{t+dt} - X_t)S_t \\ dY_t &= -S_{t^-}\, dX_t - d\left[S, X\right]_t \end{aligned}$$

Using integration by parts,

$$dY_t = d(X_{t^-}S_t) - X_t S_t$$

Therefore, the following is the the formal definition of the self-financing trading strategies.

Definition 2.2.1. *A trading strategy (X, Y) is self-financing if X_t and Y_t are predictable processes, with $X_0 = 0$, then we have*

$$Y_t + X_t S_t = Y_0 + X_0 S_0 + \int_0^t X_s dS_s. \tag{2.28}$$

Note that X is *predictable* so we can replace X_{t^-} by X_t. The equation 2.28 implies that $Y_t + X_t S_t$ is càdlàg, it also shows that it is reasonable to consider processes X predictable, because X is the investor's holding at time t, and this is based on information obtained at time strictly before t, but not t itself.

As in the discrete time model, a continuous time financial model is arbitrage-free if there exists an equivalent probability measure $\mathbb{Q}$ such that S is a $\mathbb{Q}$-local martingale. This will be investigated in the next section.

2.2.1 The First Fundamental Theorem of Asset Pricing

This section presents an outline proof of the FFTAP for general models S of financial markets in continuous time, and for general trading strategies X.

To define an appropriate class of trading strategies, as mentioned by Delbaen and Schachermayer [19], one has to restrict the choice of the integrands X to make sure that the process $X \bullet S$ exists. Besides the qualitative restriction coming from the theory of stochastic integration, one has to avoid problems that arise from so-called doubling strategies. An

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example of such a strategy is the coin-betting game. The player starts with 1\$, we draw the coin, if heads come out s/he wins 2 times his/her bets, and s/he stops. If tails come up s/he loses his/her bets, his/her next bet is 2\$, then s/he continues doubling his/her bet until s/he wins. This strategy leads to a certain gain of 1\$ without risk. Now there is no limit to how much money the player will have to pay before winning 1\$. Unfortunately, no one has such infinite resources to play such a game. Mathematically, this problem can be avoided by acquiring trading strategies that are bounded below by a constant. So, we have the following definition:

Definition 2.2.2. *[19] Let α be a positive real number, and suppose $S = (S_t)_{t\geq 0}$ a $\mathbb{R}^d$-valued semi-martingale. An $\mathbb{R}^d$-valued predictable process $X = (X_t)_{t\geq 0}$ is called an α-admissible integrand for the semi-martingale S, if*

a) X is S-integrable, which means the stochastic integral $X \bullet S = ((X \bullet S)_t)_{t\geq 0}$ is well-defined and that $\lim_{t\to+\infty} (X \bullet S)_t = (X \bullet S)_\infty$ exists,

b) $X_0 = 0$, and

$$(X \bullet S)_t \geq -\alpha \quad \textit{for all} \quad t \geq 0.$$

We say X is admissible if it is admissible for some $\alpha \geq 0$.

The following definitions from functional analysis are needed in this chapter:

Let $\mathbb{R}$ be the space of real numbers, and E a topological vector space over $\mathbb{R}$ which is locally convex. We need to define a different topology on E using the continuous dual space E^*. The dual space contains all linear functions from E into $\mathbb{R}$ which are continuous with respect to the given topology [5]. The weak topology on E, denoted by $\sigma(E, E^*)$, is the weakest topology such that each element of E^* is a continuous function. An important observation to make is that every point $x \in E$ induces a linear functional f_x on E^*, defined by $f_x\Lambda = \Lambda x$ and that $\{f_x : x \in X\}$ separates the points on E^*. The weak* topology of E^*, denoted by $\sigma(E^*, E)$, is the weakest topology making all the maps $\Lambda \to \Lambda x$ continuous [60].

In the infinite dimensional setting we work with the following spaces: we recall the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ as presented in the previous section. The space $L^0(\Omega, \mathcal{F}, \mathbb{P})$

[5] If the topology on X is induced by a norm $\|\cdot\|$, a linear functional Λ belongs to E^* if and only if the unit ball in E is mapped into a bounded subset of $\mathbb{R}$. The norm of Λ will be defined by $\| \Lambda \| = \sup_{\|x\|\leq 1} | \Lambda(x) |$ and we have the relation $| \Lambda(x) | \leq \| \Lambda \| \| x \|$ for every $x \in E$ [60].

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denotes the vector space of all real-valued $\mathcal{F}$ measurable functions defined on Ω, where as usual, two functions equal almost surely, are identified. The space L^0 is endowed with the topology of convergence in probability. It is a complete metrisable topological vector space, a Fréchet space. The space is not locally convex and cannot be given an equivalent norm, in general, there are no non-trivial continuous linear functions from L^0 to $\mathbb{R}$[20]. The space $E = L^1(\Omega, \mathcal{F}, \mathbb{P})$ is the Banach space of all integrable $\mathcal{F}$-measurable functions. The dual space is identified with $E^* = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ the space of equivalence classes of bounded measurable functions on $(\Omega, \mathcal{F}, \mathbb{P})$ where, two functions equal almost surely, are identified. Applying the Separation Theorem 2.1.7 in the spaces L^∞, poses the problem that the dual space of L^∞ is not L^1. In order to obtain a duality between L^1 and L^∞, we consider the weak* topologies instead of the norm topologies i.e. $\sigma(L^\infty, L^1)$, and work with sets that are weak* closed ($\sigma(L^\infty, L^1)$-closed).

We also define the following sets:

$$\mathcal{K}_0 = \left\{ (X \bullet S)_\infty = \int_0^\infty X_s dS_s \mid X \text{ admissible and } (X \bullet S)_\infty = \lim_{t \to \infty} (X \bullet S)_t \text{ exist a.s} \right\}$$

which forms a convex cone of function in L^0. $\mathcal{K}_0$ is the set of contingent claims, i.e. pay-off functions, available at price 0 pursuing admissible trading strategies. The convex cone $\mathcal{C}_0$ in L^∞ defined by

$$\mathcal{C}_0 = \{ g \in L^\infty \mid g \le f \text{ for some } f \in \mathcal{K}_0 \},$$

is the cone of functions dominated by an element of $\mathcal{K}_0$ i.e. $\mathcal{C}_0 = \mathcal{K}_0 - L^0_+$.
$\mathcal{C}_0$ is the set of contingent claims super-replicable at zero costs. Then, we define the following intersections with the space of bounded functions L^∞,

$$\mathcal{K} = \mathcal{K}_0 \cap L^\infty$$
$$\mathcal{C} = \mathcal{C}_0 \cap L^\infty.$$

The convex cone $\mathcal{K}$ contains all bounded contingent claims that are available at price 0 pursuing admissible trading strategies.

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The cone convex $\mathcal{C}$ contains all bounded contingent claims of L^{∞} that are super-replicable at zero cost.

In order to formulate the idea that it is impossible to make something out of nothing, the following properties for a semimartingale S should be given:

No Arbitrage: The process S satisfies "no-arbitrage" condition if

$$\mathcal{C} \cap L_{+}^{\infty} = \{0\}\,. \tag{2.29}$$

Therefore, the existence of an arbitrage opportunity is the same thing as the existence of an admissible trading strategy having zero- initial cost and the resulting contingent claim is non-negative and not identically equal to zero.

A "free lunch with vanishing risk" is a self-financing trading strategy that can be approximated by a sequence of admissible self-financing strategies that converge to an arbitrage strategy. The condition, "No free lunch with vanishing risk", is a generalisation of the condition of "no- arbitrage". If a free lunch with vanishing risk is satisfied then we have $f_0 \in \overline{\mathcal{C}} \cap L_{+}^{\infty}$. Then there exists a sequence $(f_n)_{n\geq 1}$ of elements in $\mathcal{C}$, such that $\parallel f_n - f_0 \parallel_{L^{\infty}} \to 0$ i.e. a uniform approximation of the claim f_0. If $f_0 \geq 0$, we have $f_n - f_0 \geq \frac{-1}{n}$, also we have $f_n \geq \frac{-1}{n}$, since $\mathcal{C}$ is defined using admissible trading strategies. Economically this means that the possible losses f_n converge uniformly to 0, then the risk vanishes. Therefore, we have the following definition:

No Free Lunch with Vanishing Risk: The process S satisfies the condition of (NFLVR) if

$$\overline{\mathcal{C}} \cap L_{+}^{\infty} = \{0\}\,.$$

where $\overline{\mathcal{C}}$ denotes the closure of $\mathcal{C}$ under the norm topology of L^{∞}.

To have a better understanding of the property of "No free lunch with vanishing risk", we give the following proposition:

Proposition 2.2.3. *[19] If a semi-martingale S fails the property (NFLVR) then, one of these sentences is true,*

1. *S fails NA or*

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2. *There exists a sequence* $f_n = \int_0^\infty X^n dS \in \mathcal{K}_0$, *where* X^n *is a* $\frac{1}{n}$*-admissible trading strategy, such that* $f_n \to f_0$ *in probability, with* f_0 *not identically 0.*

Before we give the proof of this proposition, we introduce the relation between (NFLVR) and a boundedness property in L^0. We recall that a subset A in a topological vector space X, said to be bounded if for each 0-neighbourhood U in X, there exists $\lambda > 0$ such that $A \subset \lambda U$ (i.e. $A \subseteq X$ is bounded iff it is absorbed by every 0-neighbourhood of X)[64]. In our setting, since we are working on the topological vector space L^0, the notion of boundedness in L^0 is given by the following definition:

Definition 2.2.4. *[8] A subset* $A \subset L^0(\Omega, \mathcal{F}, \mathbb{P})$ *is bounded in probability if, for* $\varepsilon > 0$, *there is* $\alpha > 0$ *such that*

$$\mathbb{P}\left[f > \alpha\right] < \varepsilon, \quad for \quad all\, f \in A. \tag{2.30}$$

Then we have the following lemma:

Lemma 2.2.5. *[19] If* S *a semi-martingale which satisfies (NFLVR) then the set*

$$\mathcal{K}_1 = \{(X \bullet S)_\infty \mid X \text{ } is \text{ } 1 - admissible \text{ } and \text{ } of \text{ } bounded \text{ } support\},$$

is bounded in L^0 *(i.e in probability).*

Proof. [19] We recall that saying X is 1-admissible, means X is S-integrable and $(X \bullet S)_t \geq -1$ and saying X of bounded support is that X is 0 outside $[0, T]$, for some $T < \infty$. The proof will proceed with contradiction by supposing that $\mathcal{K}_1$ is not bounded in L^0. That means there exists a sequence X^n of 1-admissible integrands of bounded support and the existence of $\varepsilon > 0$ such that $\mathbb{P}\left[(X^n \bullet S)_\infty \geq n\right] > \varepsilon$. Let us take the sequence $f_n = \left(\frac{1}{n}(X^n \bullet S)_\infty\right) \wedge 1$ which is in $\mathcal{C}$, $\mathbb{P}\left[f_n = 1\right] > \varepsilon > 0$ and $\| f_n^- \|_\infty \leq \frac{1}{n}$. The set of all convex combinations of the sequences $(f_n)_{n\geq 1}$ in $\mathcal{C}$ is denoted by $conv(f_n, f_{n+1}, \cdots)$, then Lemma B.1.1 provides a sequence $h_n \in conv(f_n, f_{n+1}, \cdots)$ converging almost surely to $h \geq 0$, we have $\mathbb{E}[h] \geq \varepsilon$ and then $\mathbb{P}\left[h > 0\right] = \beta \geq \varepsilon > 0$. So we have h_n converge to h a.s and then using Egorov's Theorem B.1.2, there exists Γ such that $\mathbb{P}\left[\Gamma > 0\right] > \beta$ and $\lim_n \| h_n I_\Gamma - h I_\Gamma \|_\infty = 0$ (i.e. $h_n \to h$ uniformly on the set Γ), the function $\min(h_n, 1_\Gamma)$ is in $\mathcal{C}$ and $\min(h_n, 1_\Gamma) \to h I_\Gamma$ in L^∞. The fact that $\mathbb{P}\left[h I_\Gamma > 0\right] \geq \beta > 0$ contradicts the (NFLVR) assumption. Hence $\mathcal{K}_1$ is bounded in L_0. $\square$

Now we proceed with the proof of the Proposition 2.2.3.

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Proof. [19] If such a sequence in the proposition (condition 2) exists, then we will have a sequence of 1-admissible strategies $(nX^n)_{n\geq 1}$ and the set $\{n(X^n \bullet S)_\infty : n \geq 1\}$ in $\mathcal{K}_0$, the set is unbounded in L^0 and the limit $(X^n \bullet S)_\infty$ exist. That contradicts the Lemma 2.2.5. Hence the existence of the sequences in the proposition flouts (NFLVR).

For the converse of Proposition 2.2.3, we assume that S fails (NFLVR) and satisfies no-arbitrage condition and we proof that condition 2 in Proposition 2.2.3 is true. Since S fails the property of (NFLVR) i.e. $\overline{\mathcal{C}} \cap L^\infty \neq \{0\}$, we have a sequence $(h_n)_{n\geq 1} \in \mathcal{C}$ such that $h_n \to h_0$ in L^∞ with the existence of $\varepsilon > 0$ such that $\mathbb{P}\left[h_0 > \varepsilon\right] > \varepsilon$ and the possible losses $\| h_n^- \|_\infty$ tends to 0. Passing to a subsequence, we may take $\| h_n^- \|_\infty \leq \frac{1}{n}$. Then for each n we have $h_n \leq g_n$, where $g_n = (H^n \bullet S)_\infty \in \mathcal{K}_0$ and $\| g_n^- \|_\infty \leq \| h_n^- \|_\infty \leq \frac{1}{n}$, then since S satisfies no-arbitrage condition and by Proposition B.1.4 we have H^n is $\frac{1}{n}$-admissible. By Lemma B.1.1 we may replace the sequence g_n by $f_n \in conv(g_n, g_{n+1}, \cdots)$ such that $f_n \to f_0$ in probability. Suppose X^n is a convex combination of the integrand $(H^k)_{k\geq n}$. H^n is still $\frac{1}{n}$-admissible and $f_n^- \to 0$ in L^∞. We have $\| h_n - h_0 \|_\infty \leq \varepsilon/2$ for n large enough, then $\mathbb{P}\left[g_n > 0\right] \geq \mathbb{P}\left[h_n > \varepsilon/2\right] > \varepsilon/2$. The result 2 in Lemma B.1.1 gives us $\mathbb{P}\left[f_0 > 0\right] > 0$.

□

Theorem 2.2.6. *[20] The process S satisfies the (NFLVR) condition if, and only if, it satisfies*

1. *No-arbitrage*
2. *$\mathcal{K}_1$ is bounded in the space L^0.*

Proof. [19] If S satisfies (NFLVR), we have $\overline{\mathcal{C}} \cap L_+^\infty = \{0\}$, and since $\mathcal{C} \subset \overline{\mathcal{C}}$ then we have $\mathcal{C} \cap L_+^\infty = \{0\}$ which is equivalent to $\mathcal{K} \cap L_+^\infty = \{0\}$. Hence there is no-arbitrage. and by Lemma 2.2.5, $\mathcal{K}_1$ is bounded in L^0.

For the converse, we use the proof of the Proposition B.1.3 which gives us the set

$$\left\{ \sup_{0\leq t}(X \bullet S)_t \mid \ X\ 1-admissible \right\}, \tag{2.31}$$

that is also bounded in L^0. If a sequence $(h_n)_{n\geq 1} \in \mathcal{K}_0$, such that $\| h_n^- \|_\infty \to 0$, then applying the NA condition and Proposition B.1.4 we have $h_n = (X^n \bullet S)_\infty$ where X^n is ε_n-admissible with $\varepsilon_n = \| h_n^- \|_\infty$. To make the sequence $\frac{1}{\varepsilon_n} h_n$ bounded, we should have $h_n \to 0$ in probability. Therefore, from Corollary B.1.5, there is (NFLVR). □

The boundedness of the set $\mathcal{K}_1$ has the following economic interpretation given by [19]: For outcomes that have a maximal loss bounded by 1, the profit is bounded in probability, this

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means that the probability of making a huge profit can be estimated from above, uniformly, over all such results.

Let us first recall the definition of Fatou convergence and Fatou closedness.

Definition 2.2.7. *[63] Let $(f^n)_{n\geq 1}$ be a sequence defined on the cone of $\mathbb{R}\cup\{+\infty\}$-valued $\mathcal{F}$-measurable functions $F(\Omega,\mathcal{F},\mathbb{P})$. The sequence $(f^n)_{n\geq 1}$ is Fatou-convergent to $f_0 \in F(\Omega,\mathcal{F},\mathbb{P})$, if $(f^n)_{n\geq 1}$ is uniformly bounded from below (i.e. $\exists\, M \in \mathbb{R}_+$ such that $f_n(\omega) \geq -M$, $\mathbb{P}$-almost surely) and*

$$f_0 = \lim_{n\to\infty} f_n(\omega),\ \mathbb{P}-almost\ surely. \tag{2.32}$$

A set D of random variables is said to be Fatou-closed if, whenever $(f_n)_{n\geq 1}$ is a sequence in D which Fatou converges to some function f_0, then $f_0 \in D$ also.

Theorem 2.2.8. *[19] If S is a bounded semi-martingale satisfying (NFLVR), then*

1. *$\mathcal{C}_0$ is Fatou-closed and hence,*
2. *$\mathcal{C} = \mathcal{C}_0 \cap L^\infty$ is $\sigma(L^\infty, L^1) - closed$.*

Actually, the first part of this theorem is a crucial result to prove the FFTAP that we will mention later. In the paper by Delbaen and Schachermayer [19], the proof that $\mathcal{C}_0$ is Fatou-closed takes more that ten pages, and as the authors mentioned, the proof is very technical. In this dissertation we only give a sketch of this proof.

To prove that $\mathcal{C}_0$ is Fatou-closed, it suffices to consider a sequence $(f_n)_{n\geq 1} \in \mathcal{C}_0$ such that $f_n \geq -1$ and $\lim\limits_{n\to\infty} f_n = f_0$. The job is to prove that $f_0 \in \mathcal{C}_0$. In order to prove that, one has to find an element $f \in \mathcal{K}_0$ such that $f_0 \leq f$. As a first move let us consider the following set:

$$D = \{h : \text{there is a sequence of 1-admissible integrand } H^n \text{ such that} (H^n \bullet S)_\infty \to h, h \geq f_0\}.$$

We have to show that this set is non-empty and has a maximal element f.

To prove that $D \neq \emptyset$, we take $g_n \in \mathcal{K}_0$, such that $g_n \geq f_n$. Applying Lemma B.1.1, there exists a sequence $h_n \in$ *conv* $\{f_n, f_{n+1}, \ldots\} \subset \mathcal{K}_1$ that converges almost surely to h_0. Since $h_0 \geq f_0$, then $h_0 \in D$. Thus $D \neq \emptyset$.

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We have $D = \overline{\mathcal{K}}_1 \cap [f_0, \infty[$ with $[f_0, \infty[= \{h \in L^0, h \geq f_0\}$ and $\overline{\mathcal{K}}_1$ is the closure of $\mathcal{K}_1$ defined as follows:

$$\overline{\mathcal{K}}_1 = \{h, \text{there is a sequence of 1-admissible such that } (H^n \bullet S)_\infty \to h\}. \qquad (2.33)$$

We also have $D \subset \overline{\mathcal{K}}_1$, so D is closed. We recall the fact that S satisfies (NFLVR). Then by Theorem 2.2.6 we have $\mathcal{K}_1$ bounded in L^0. Thus D is bounded in L^0. The set D is bounded closed on L^0. Using a well known result that says a non-empty closed bounded subset of L^0 has a maximal element, we conclude that D has an maximal element f.

Now we have f a maximal element of D, let $h_n = (X^n \bullet S)_\infty$ where X^n is 1-admissible strategy, and $h_n \to f$ almost surely. The last step to prove the first part of Theorem 2.2.8 is to prove that $f \in \mathcal{K}_0$.

Let us prove the second assertion in Theorem 2.2.8. Given that $\mathcal{C}_0$ is Fatou-closed, we take a sequence $(f_n)_{n\geq 0}$ in $\mathcal{C}$, uniformly bounded in absolute value by 1, such that $f_n \to f$ almost surely. Since $\mathcal{C}_0$ is Fatou-closed, $f \in \mathcal{C}_0$ and hence also $f \in \mathcal{C}$. Thus $\mathcal{C} = \mathcal{C}_0 \cap L^\infty$ is Fatou-closed for the topology $\sigma(L^\infty, L^1)$ by Theorem B.1.9.

Defining (NA) and (NFLVR) conditions, we can establish the link between absence of arbitrage and the semi-martingale property, known as The Fundamental Theorem of Asset Pricing.

Theorem 2.2.9. *[19] FFTAP Let $S = (S_t)_{0\leq t\leq T}$ be a bounded semi-martingale. Then S satisfies (NFLVR) if, and only if, there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which S is a local-martingale.*

Proof. Suppose we have the (NFLVR) condition. Since S satisfies the no-arbitrage, we have $\mathcal{C} \cap L^\infty_+ = \{0\}$ i.e. $\mathcal{C}$ and $L^\infty_+ \setminus \{0\}$ are disjoint. Using Theorem 2.2.8, we have that $\mathcal{C}$ is weak*-closed in L^∞ (that is closed in $\sigma(L^1, L^\infty)$). By Kreps-Yan Separation Theorem B.1.10, there exist a equivalent martingale measure $\mathbb{Q}$ such that $\mathbb{E}_\mathbb{Q}[f] \leq 0$, $\forall\, f \in \mathcal{C}$. The boundedness of S gives that for each $s < t$, $B \in \mathcal{F}_s$, $\alpha \in \mathbb{R}$, we have $\alpha(S_t - S_s)1_B \in \mathcal{C}$. Hence, $\mathbb{E}_\mathbb{Q}[(S_t - S_s)] = 0$ and $\mathbb{Q}$ is a martingale measure for S.

For the converse, note that no-free lunch with vanishing risk remains unchanged with an equivalent probability, so we may suppose that under $\mathbb{P}$, S is a martingale. If X is admissible, then $(\int_0^t X_s dS_s)_{t\geq 0}$ is a local martingale, bounded below, hence it is a supermartingale. Since $\mathbb{E}[X_0 S_0] = 0$, we have, as well, $\mathbb{E}\left[\int_0^\infty X_s dS_s\right] \geq \mathbb{E}[X_0 S_0] = 0$. That is, for every function $f \in \mathcal{C}$, we have $\mathbb{E}[f] \geq 0$. Therefore it is true as well for $f \in \overline{\mathcal{C}}$, the closure of $\mathcal{C}$ in L^∞. Thus we have $\overline{\mathcal{C}} \cap L^\infty_+ = \{0\}$. ∎

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2.2.2 Second Fundamental Theorem of Asset Pricing

Let us assume a discounted asset price process S with values in $\mathbb{R}^d$. We recall that the process (X, Y) represents respectively, the quantity of asset hold and the money market account. We assume a zero interest rate. A contingent claim is a random variable $C \in \mathcal{F}_T$, that can be seen as a random payoff at time T. The main goal is to show that there exists a trading strategy (X, Y) that can be used to replicate C at time T or at least come as close as possible in an approximate sense.

Definition 2.2.10. *A contingent claim $C \in \mathcal{F}_T$ is attainable if there exists an admissible self-financing strategy (X, Y) such that*

$$C = X_0 S_0 + Y_0 + \int_0^T X_s dS_s. \tag{2.34}$$

Note that the stochastic integral preserves the martingale property in the case when $X \in \mathbb{L}$, (the set of adapted càglàd processes). If $\mathbb{Q}$ is any equivalent martingale measure such that S is a martingale, and C has finite expectation under $\mathbb{Q}$, then

$$\mathbb{E}_{\mathbb{Q}}[C] = \mathbb{E}_{\mathbb{Q}}[X_0 S_0 + Y_0] + \mathbb{E}_{\mathbb{Q}}\left[\int_0^T X_s dS_s\right] \tag{2.35}$$

$$= \mathbb{E}_{\mathbb{Q}}[X_0 S_0 + Y_0]. \tag{2.36}$$

Theorem 2.2.11. *[10] Let C be an attainable contingent claim such that there exists an equivalent martingale measure $\mathbb{Q}$ with $C \in \mathcal{L}_{\mathbb{Q}}(S)$[6]. Then there exists a unique no- arbitrage price of C and it is $\mathbb{E}_{\mathbb{Q}}[C]$.*

Proof. [55] We show first that $\mathbb{E}_{\mathbb{Q}}[C]$ does not change under any equivalent measure. Let us assume the existence of two equivalent martingale measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$ such that

$$\mathbb{E}_{\mathbb{Q}_i}[C] = \mathbb{E}_{\mathbb{Q}_i}[X_0 S_0 + Y_0] + \mathbb{E}_{\mathbb{Q}_i}\left[\int_0^T X_s dS_s\right], \; i = 1, 2. \tag{2.37}$$

We know that $\mathbb{E}_{\mathbb{Q}_i}\left[\int_0^T X_s dS_s\right] = 0$, because S is martingale under $\mathbb{Q}_1$ and $\mathbb{Q}_2$. Thus $\mathbb{E}_{\mathbb{Q}_i}[X_0 S_0 + Y_0] = X_0 S_0 + Y_0$, since X_0, S_0 and Y_0 are known at time 0, and assumed to be constant.

Therefore the fair price of C is $\mathbb{E}_{\mathbb{Q}}[C]$ by the law of one price[7]. □

[6] The class of strategies (X, Y) such that $(\int_0^t X_s^2 d[S, S]_s)^{1/2}$ is locally integrable.

[7] Two assets that guaranteed to have the same value at time $t = T$ must have the same value at $t = 0$.

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Note that a market model is said to be complete if for any $C \in L^1(\mathcal{F}_T, d\mathbb{Q})$, there exists an admissible self-financing strategy satisfying 2.34, with $(\int_0^t X_s dS_s)_{t\geq 0}$ is uniformly integrable. Thus, a complete market is one for which every claim is attainable. One can relate the definition of market completeness to the predictable representation property for martingales [8]. Indeed, only a few martingales have this property, for example Brownian motion, the Compensated Poisson process, and the Azéma martingale [55].

Unfortunately, most of the models are therefore not complete, and most of the practitioners believe that the real financial world is not complete [55]. Then we have the following result:

Theorem 2.2.12. *[55] S has a unique local martingale measure $\mathbb{Q}$ only if the market is complete.*

This theorem is a consequence of Dellacherie's approach to martingale representation: if there is a unique probability under which the process S is a local martingale, then S must have the martingale representation property. Therefore the market is complete. This theory was well studied by Jacod and Yor, we refer to Appendix B.2 and for more detail we refer to [56, Chapter IV].

The result which is known to have a unique martingale measure $\mathbb{Q}$, is one-dimensional Brownian motion (which implies that every contingent claim is attainable in the Black-Scholes model). But, as in Jarrow et al. [38], there are examples where we can have complete markets without the uniqueness of the equivalent martingale measure. However, when we consider models with continuous processes, the situation is simpler. The next theorem, called The Second Fundamental Theorem of Asset Pricing, will be proved in the L^2 sense and with continuous process S.

Theorem 2.2.13. *[55] If the process S has continuous paths, then the market is complete if, and only if, there is a unique $\mathbb{Q}$ such that S is an L^2 ($\mathbb{Q}$)-martingale.*

Proof. [55] If $\mathbb{Q}$ is unique i.e $\mathcal{M}^2(S) = \{\mathbb{Q}\}$, then $\mathbb{Q}$ is an extremal point of $\mathcal{M}^2(S)$ (Definition B.2.1), and by Theorem B.2.5, S has the predictable representation property. Thus the market is complete. Suppose $\mathbb{Q}$ is not unique but the market is complete, then by Theorem B.2.3, $\mathbb{Q}$ is still an extremal point in the space of probability measure making S an L^2 martingale. Suppose there exists another extremal probability measure $\mathbb{Q}^*$. Let

[8] We say that a martingale S have the predictable representation property if for any $C \in L^2(\mathcal{F}_T)$ satisfy $C = \mathbb{E}[C] + \int_0^T X_s dS_s$ for some predictable $X \in \mathcal{L}(S)$.

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$L_\infty = \frac{d\mathbb{Q}^*}{d\mathbb{Q}}$ and $L_t = \mathbb{E}\left[\frac{d\mathbb{Q}^*}{d\mathbb{Q}} \mid \mathcal{F}_t\right]$, with $L_0 = 1$. Let $T_n = \inf\{t > 0 : \mid L_t \mid \geq n]$, $L_t^n = L_{t\wedge T_n}$ is bounded and by Theorem B.2.5, L is continuous. For bounded $C \in \mathcal{F}_s$, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^*}\left[S_{t\wedge T_n} C\right] &= \mathbb{E}_{\mathbb{Q}}\left[S_{t\wedge T_n} L_t^n C\right] \\ &= \mathbb{E}_{\mathbb{Q}^*}\left[S_{s\wedge T_n} C\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[S_{s\wedge T_n} L_s^n C\right].\end{aligned}$$

Thus SL^n is a martingale, hence L^n is a bounded $\mathbb{Q}$-martingale strongly orthogonal to S (Definition B.2.2). Thus L^n is null by Theorem B.2.3, and since L^n is constant we conclude that $L_\infty = \frac{d\mathbb{Q}^*}{d\mathbb{Q}} \equiv 1$ and thus $\mathbb{Q}^* = \mathbb{Q}$. □

2.2.3 Example: Black-Scholes Model

In the early 1970s, the pricing of stock options had a breakthrough. This breakthrough led to what has become known as the Black-Scholes model. The model has a huge influence on the way that traders price and hedge options. It has also been the key to the development and success of financial engineering in the 1980s and 1990s.

The model is based on the following assumptions: the price of the underlying asset follows a geometric Brownian motion with constant drift and volatility, and also no dividends are paid.

The Black-Scholes model assumed two assets in the market, the bank account (or Bond) and the stock. Their prices increments are, respectively,

$$d\tilde{B}_t = r\tilde{B}_t\, dt \tag{2.38}$$

$$d\tilde{S}_t = \mu\, \tilde{S}_t\, dt + \sigma\, \tilde{S}_t\, dw_t, \tag{2.39}$$

where w is a standard Brownian motion with natural base $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, r is the riskless rate of interest and μ is the drift of the stock price. These two formulas can be solved, respectively, as follows:

$$\tilde{B}_t = e^{rt} \tag{2.40}$$

$$\tilde{S}_t = S_0 e^{\sigma w_t + (\mu - \frac{\sigma^2}{2})t} \quad , \quad 0 \leq t \leq T \tag{2.41}$$

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In our setting we use the bank account as a numéraire, then we have

$$B_t = \frac{\tilde{B}_t}{\tilde{B}_t} = 1$$
$$S_t = \frac{\tilde{S}_t}{\tilde{B}_t} = S_0 e^{\sigma w_t + (\mu - r - \frac{\sigma^2}{2})t},$$

where $\mu - r$ is called the excess return. Note that the process

$$S_t = S_0 e^{\sigma w_t + (\mu - r - \frac{\sigma^2}{2})t} \ , \ 0 \leq t \leq T$$

is usually not martingale under $\mathbb{P}$.

The unique martingale measure $\mathbb{Q}$ for S which is absolutely $\mathbb{P}$-continuous, is given by Girsanov's Theorem B.2.7.

Example: European call option

Let us price and hedge the contingent claim $C(S_T, T) = (S_T - Ke^{-rT})_+$, which is the discounted pay-off function of the European call option with time to maturity T and a agreed price K.

$(w_t + \nu t)_{t=0}^{\infty}$ with $\nu = \mu - r$ is a standard Brownian motion under $\mathbb{Q}$, following some calculations in [19] we may have.

$$C(S_0, T) = \mathbb{E}_{\mathbb{Q}}[C(S_T, T)] = \mathbb{E}_{\mathbb{Q}}\left[\left(S_0 e^{\sigma(w_T + \nu T) - \frac{\sigma^2}{2}T} - Ke^{-rT}\right)_+\right] \tag{2.42}$$
$$= S_0 \mathbb{E}_{\mathbb{Q}}\left[e^{\sigma\sqrt{T} - \frac{\sigma^2 T}{2}} \chi_{\{S_T \geq K\}}\right] - Ke^{-eT}\mathbb{Q}\left[S_T \geq K\right] \tag{2.43}$$

After an elementary calculation (see [19]), this yields the famous Black-Scholes formula:

$$C(S_0, T) = S_0 \phi\left(\frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \tag{2.44}$$
$$- Ke^{-rT}\phi\left(\frac{\ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right), \tag{2.45}$$

where ϕ is the cumulative standard normal distribution function, and by the same token, for $0 \leq t \leq T$ and $S_t > 0$,

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$$
\begin{aligned}
C(S_t, T-t) = S_0\phi\left(\frac{\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\
- Ke^{-rT}\phi\left(\frac{\ln(\frac{S_t}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned}
$$

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Chapter 3

Liquidity Risk and Arbitrage Pricing Theory

The classical theory of arbitrage pricing presented previously, employs unrealistic assumptions: Firstly, the investor can buy/sell an unlimited quantity of a security without changing the price (*Competitive market*). Secondly, there are no restrictions on trade and no transaction costs (*Frictionless market*). Weakening the assumptions of frictionless and competitive markets introduces liquidity risk. *Liquidity risk* is the risk owing to the difficulty of selling/buying an asset. From another point of view, liquidity risk is the "volatility" in asset prices when trading or hedging financial securities due to the size of the transaction itself. Specifically, when markets are tranquil and transaction sizes are small, liquidity risk is small. When markets are in crisis or the transaction sizes are considerable, liquidity risk is large.

Now the question to ask is: How has the classical financial modelling been modified to include liquidity risk?

A large number of authors have developed models that include liquidity risk. In the literature, those models are divided into two classes: The first contains models that assume the price of a stock to be dependent on the size of the transaction and the depth of the order book. The second category consists of models which assume that large trader activity affects prices in addition to the assumptions of the first category.

A model in the first category is proposed by Çetin, Jarrow and Protter [10]. To include

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liquidity risk in arbitrage theory, they hypothesize the existence of a *supply curve* $S(t, x)$, with x denote the size of the trade. Several assumptions were made on this supply curve to embed liquidity risk into the classical theory. In this setting, using strategies with infinite quadratic variation will incur infinite liquidity costs. To avoid these liquidity costs, trading strategies must be continuous and of finite variation. The quantity impact of trades on prices is then negligible. This yields the ability to apply the classical arbitrage theory to price the options in the presence of liquidity risk. Thus, the First and the Second Fundamental Theorems can be extended due to the fact that trading strategies that are both continuous and of finite variation can approximate (in an L^2 sense) arbitrary-predictable trading strategies. As a result, the arbitrage-free price of any derivative is shown to be the same price as in the classical economy with no liquidity cost.

A model in the second category was studied by Roch [57]. The paper was based on the results of Çetin, Jarrow and Protter [10], and assumes the same supply curve $S(t, x)$, except that quantity impacts of trades on prices are not negligible, but there is an impact of trades on prices coming from the activity of large traders. Also, the supply curve is linear (based on some interesting empirical studies done by Blais [6]). The main observation in [57] is that the magnitude of price impacts is related to the level of liquidity of the asset. A new characterization of self-financing trading strategy is derived, and a sufficient condition for no-arbitrage is given. A non-linear contingent claim like a put or call option on a stock failed in this model, and variance swaps were the simplest way to complete the market.

In this thesis we aim to give a detailed exposition of the model of Çetin, Jarrow and Protter [10]. We will look at the assumptions made on the supply curve, how the self-financing trading strategy was derived, and an extension of the first and second theorems of asset pricing under liquidity risk. Also we will give a summary of the model by Roch [57], and some examples of models that include liquidity risk.

3.1 Çetin, Jarrow, Protter Model

In this section we introduce an extension of the classical arbitrage theory to include liquidity risk. Given some assumptions about the supply curve, we derive a self-financing trading

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strategy condition and an extension of the First and Second Fundamental Theorems of Asset Pricing.

3.1.1 Supply Curve and Trading Strategies

As usual, we start with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ which satisfies Definition A.1.2, with $\mathbb{P}$ as the empirical probability measure. Let us consider a security in a market and call it a stock[1]. Consider also a money market account that uses the spot rate of interest as return. The assumption that the spot rate of interest is zero gives an initial value for the money at all times.

The first assumption made in order to include liquidity risk in the classical theory of arbitrage is that there exists a supply curve for a security's price as a function of the transaction size. Actually, in the classical theory the supply curve was assumed to be horizontal which means there is only one price for any order size. The supply curve in this new approach, denoted by $S(t, x)$, is the stock price paid or received per share at time t for a trade of size x. The stock is purchased if $x > 0$, and is sold if $x < 0$. The zero trade $x = 0$ represents the marginal trade [2], and we can see the price $S(t, 0)$ as the price in the classical theory.

In this framework, the larger the purchase order, the higher the average price paid per share. Thus, the supply curve is an increasing function of x. Another assumption in this approach is that the supply curve is C^2 in x. This means that investors face a twice-continuously differentiable price/quantity schedule, instead of a single price for all shares traded. Indeed, with the fact that the quantity impact of trades on market prices is assumed to be negligible, and under C^2 supply curve with continuous trading strategies, all liquidity costs can be avoided when trading in the stock. This will be the key to extend the first and the second fundamental theorems of asset pricing to a model of an economy with liquidity risk.

Definition 3.1.1. *[10] The supply curve $S(t, x)$ is the price per unit when x unit is bought or sold, and it has the following properties:*

[1]The model applies also to bonds, commodities, foreign currencies, etc.
[2]Purchase or sale of infinitesimal quantities.

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1. *$S(t,x,.)$ is $\mathcal{F}_t$- measurable and non-negative.*
2. *for λ-almost every t, the supply curve $x \to S(t,x,\omega)$ is $\mathbb{P}$-almost surely increasing, where λ is the Lebesgue measure.*
3. *S is C^2 in x, and $\partial S(t,x)/\partial x$, $\partial^2 S(t,x)/\partial x^2$ are continuous in t.*
4. *$S(.,0)$ is a semi-martingale.*
5. *$S(.,x)$ has a continuous sample path for all values of x.*

Example: A simple example of a supply curve is if $S(t,x) \equiv f(t,D_t,x)$, where D_t is an n-dimensional continuous adapted semi-martingale, and $f : \mathbb{R}^{n+2} \to \mathbb{R}^+$ is a Borel-measurable function increasing in x and sufficiently smooth. This non-negative function f can represent a supply curve produced by a market equilibrium process in a complex and dynamic financial economy. D_t represents the uncertainty in the economy.

For example, in the Black-Scholes model, $S(t,0)$ follows a geometric Brownian motion. The difference here is that when studying liquidity risk, the supply curve and its stochastic changeability across time need to be modelled. For example, we consider the following form of the supply curve [3]:

$$S(t,x) = e^{h(t,D_t,x)} S(t,0), \tag{3.1}$$

where h is a Borel-measurable function, which is C^1 in t, C^2 in all its other arguments, with $h(t,D_t,0) = 0$, $h(t,D_t,x) > 0$ if $x > 0$ and $h(t,D_t,x) < 0$, if $x < 0$. This supply curve is stochastic. It varies randomly through time, its shape changing, although always remaining upward sloping. When markets are calm, the shape will be more horizontal, or else when markets are hectic, the shape will be more upward tilting [37].

Definition 3.1.2. *[10] A trading strategy is a triplet $((X_t, Y_t : t \in [0,T]), \tau)$ where X_t represents the number of the stock holding at time t and Y_t represents the money market account position at time t. τ is the liquidation time of the stock position and has the following restrictions:*

(a) *The processes X_t and Y_t are predictable and optional, respectively, with $X_{0^-} \equiv Y_{0^-} \equiv 0$.*

[3] In section 3.4 we will give an example of a linear supply curve based on the empirical studies by Blais [6]

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(b) *τ is a predictable $(\mathcal{F}_t, 0 \leq t \leq T)$-stopping time* [4] *with $\tau \leq T$ and $X = H1_{[0,\tau)}$ for some predictable process $H(t,\omega)$. It follows that $X_T = 0$.*

Let us recall that in the classical theory, a trading strategy must be a predictable process with $X_0 = 0$. The trader's holding position X_t is predictable because it is based on information obtained at a time strictly before t and not t itself. The money market account Y_t has to be optional in order to define the self-financing condition, and for the portfolio to be càdlàg [55]. In the case with liquidity risk we have trading strategies that are predictable with $X_{0^-} = 0$, this convention is needed in order to define the quadratic variation of X, we did not need that in the classical case. Also at time T, the stock position returns to zero units, in this case the stock position can be liquidated prior to time T at a predictable stopping time τ. After the liquidation time, the stock position is zero, i.e. $X_t = 0$ for $t \geq \tau$.

In the same way as the classical theory, we are interested in self-financing trading strategies i.e. trading strategies that generate no cash flows for all times $t \in [0,T]$. This means that a purchase or sale of the stock must be financed by borrowing or investing in the money market account. Therefore Y is uniquely determined by (X,τ). Let us explain the intuition behind the self-financing condition, but now by using the restriction on the supply curve and trading strategies in the case of liquidity risk. Then we have the following:

$$\begin{aligned} dY_t =& - S(t+dt, dX_t)dX_t && (3.2)\\ =& - [S(t+dt,0) - S(t,0)]\, dX_t - S(t,0)dX_t && (3.3)\\ & - [S(t+dt,dX_t) - S(t+dt,0)]\, dX_t. && (3.4) \end{aligned}$$

In the classical case, the two first terms become

$$\begin{aligned} - [S(t+dt,0) - S(t,0)]\, dX_t - S(t,0)dX_t &= -d[X^c, S]_t - S(t,0)dX_t \\ &= X_{t^-} dS(t,0) \text{ (Integration by part)}, \end{aligned}$$

with X^c denote the continuous part of X. Using the fact that $dX_t = dX_t^c + \Delta X$, the last

[4] See Definition A.2.14 for the definition of stopping time.

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term becomes

$$[S(t+dt,dX_t) - S(t+dt,0)]\,dX_t = [S(t+dt,dX_t^c) - S(t+dt,0)]\,dX_t^c + [S(t,\Delta X_t) - S(t,0)]\,\Delta X_t.$$

Using Itô's formula and ignoring the higher-order terms we have,

$$-\,[S(t+dt,dX_t^c) - S(t+dt,0)]\,dX_t^c = -\frac{\partial S}{\partial x}(t,0)d[X,X]_t^c.$$

Combining the results together the equation 3.2 become:

$$dY_t = X_{t^-}dS(t,0) - \Delta X_t\,[S(t,\Delta X_t) - S(t,0)] - \frac{\partial S}{\partial x}(t,0)d[X,X]_t^c.$$

First we recall from the literature (for example Protter [55]) that for a càdlàg, adapted process X which is not a semimartingale, the quadratic variation can still have a mining using limit in probability, then we can write

$$[X,X]_t = \lim_{n\to\infty} \sum_{t_i\in\pi^n\in[0,t]} (X_{t_{i+1}} - X_{t_i})^2,$$

where $\pi^n[0,t]$ is a sequence of finite partitions of $[0,t]$ with $\lim\limits_{n\to+\infty} \mathrm{mesh}(\pi^n) = 0$. This limit always exists if X is a semimartingale [55]. Then its quadratic variation is equal to the sum of its continuous paths and the squares of the jumps of the process X, thus

$$[X,X]_t = [X,X]_t^c + \sum_{0\le s\le t} (\Delta X_s)^2.$$

Therefore, the following definition refer to the *self-financing strategies* in the case of liquidity risk.

Definition 3.1.3. *[10]*
A self-financing trading strategy is a trading strategy $((X_t, Y_t : t\in[0,T]),\tau)$ *where,*

(a) X_t *is càdlàg if* $\partial S(t,0)/\partial x = 0$ *for all* t*, and* X_t *is càdlàg with finite quadratic variation* ($[X,X]_T < \infty$) *otherwise.*

(b) $Y_0 = -X_0 S(0,X_0)$ *and*

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(c)

$$\begin{aligned} Y_t + X_t S(t,0) &= \int_0^t X_{u^-} dS(u,0) \\ &\quad - \sum_{0\le u\le t} \Delta X_u \left[S(u,\Delta X_u) - S(u,0)\right] - \int_0^t \frac{\partial S}{\partial x}(u,0) d[X,X]^c_u. \end{aligned} \tag{3.5}$$

Now with the assumption that X_t is càdlàg and of finite variation, we simply have $[X,X]^c_t = 0$, which makes the Equation 3.5 well defined. If X_t is only càdlàg, then for the Equation 3.5 to be well-defined, this $\frac{\partial S}{\partial x}(u,0) = 0$ should hold. Therefore, condition (a) in the Definition 3.1.3 gives us the class of trading strategies possible for the Equation 3.5 to be well-defined. Also, since X_t is càdlàg, it is right- continuous and not in general predictable, thus in the stochastic integral we need to use the left- continuous version of X which is X_{t^-} instead of using X_t as in the classical case. Here is a example of trading strategy that is applied in the classical case, while not permitted in the case of liquidity risk.

Example: $X_t = 1_{[S(t,0)>K]}$, with $S(t,0)$ is a Brownian motion and K a positive constant. The quadratic variation of X_t is not defined:

$$[X,X]_t = \lim_{n\to\infty} \sum_{t_i\in\pi^n\in[0,t]} (1_{[S(t_{i+1},0)>K]} - 1_{[S(t_i,0)>K]})^2 = \infty. \tag{3.6}$$

Then Y_t in Equation 3.5 is not defined either.

The condition $Y_0 = -X_0 S(0,X_0)$ implies that at time 0 the strategy requires zero initial investment but when we will study complete markets we will use $Y_0 + X_0 S(0,X_0) \neq 0$ instead. The first term on the right-hand side of the expression 3.5 is the classical self-financing condition when the supply curve is horizontal, since the last two terms in the expression do not exist in the classical theory. These last two terms on the right-hand side are, respectively, the price impact costs of discrete changes in share holdings, and the price impact costs of continuous changes in the share holdings. These terms represent the impact of illiquidity, both are negative.

Let us give a limiting explanation to Equation 3.5 as in [10].

Consider a fixed time t and a sequence of random partitions (τ_n) of $[0,t]$ tending to identity,

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i.e. each τ_n is a finite increasing sequence of stopping times covering the interval $[0,t]$, and the mesh of τ_n tends to zero as $n \to \infty$. That is, $\tau_n : 0 = T_0^n \leq T_0^n \leq \cdots \leq T_{k_n}^n = t$ and $\lim_{n\to\infty} \sup_k \mid T_{k+1}^n - T_k^n \mid = 0$ almost surely. The self-financing condition between two successive trading times t_1 and t_2 is given by

$$Y_{t_2} - Y_{t_1} = -\left(X_{t_2} - X_{t_1}\right)\left[S(t_2, X_{t_2} - X_{t_1})\right].$$

Using the fact that $Y_t = Y_0 + \sum_{k\geq 1}(Y_{T_k^n} - Y_{T_{k-1}^n})$, we can define Y_t as a limit whenever it exists as follows:

$$Y_t = Y_0 - \lim_{n\to\infty} \sum_{k\geq 1} (X_{T_k^n} - X_{T_{k-1}^n}) S(T_k^n, X_{T_k^n} - X_{T_{k-1}^n}) \ \text{ for all } \ n. \tag{3.7}$$

Now we have to prove that the Expression 3.7 holds if, and only if, Expression 3.5 does.

Proof.

$$\begin{aligned}
Y_t &= Y_0 - \lim_{n\to\infty} \sum_{k\geq 1} \left(X_{T_k^n} - X_{T_{k-1}^n}\right) S\left(T_k^n, (X_{T_k^n} - X_{T_{k-1}^n})\right) \\
&= -X(0)S(0, X_0) \\
&- \lim_{n\to\infty} \sum_{k\geq 1} \left(X_{T_k^n} - X_{T_{k-1}^n}\right) \left[S(T_k^n, (X_{T_k^n} - X_{T_{k-1}^n})) - S(T_k^n, 0)\right] \\
&- \lim_{n\to\infty} \sum_{k\geq 1} \left(X_{T_k^n} - X_{T_{k-1}^n}\right) S(T_k^n, 0).
\end{aligned}$$

We know from the classical case that the last sum converges to $-X_0 S(0,0) + X_t S(t,0) - \int_0^t X_{u-} dS(u,0)$. For any $t > 0$, we have $\sum_{0\leq s\leq t} (\Delta X_s)^2 \leq [X,X]_t < \infty$ almost surely. Since $\sum_{0\leq s\leq t} (\Delta X_s)^2$ is convergent almost surely, then we can partition the jumps of X on $(0,t]$ into $A = A(\epsilon, t)$ a set of jumps of X that has almost surely a finite number of times s (we can see A as the number of times when we have "large" jumps), and $B = B(\epsilon, t)$ such that $\sum_{s\in B} (\Delta X_s)^2 \leq \epsilon^2$ (we can see B as the number of times when we have "small"

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jumps), where A and B are disjoint and $A \cup B$ exhaust the jumps of X on $(0,t]$. Thus,

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_{k\geq 1} \left(X_{T_k^n} - X_{T_{k-1}^n}\right) \left[S(T_k^n, (X_{T_k^n} - X_{T_{k-1}^n})) - S(T_k^n, 0)\right] \\
&= \lim_{n\to\infty} \sum_{k,A} \left(X_{T_k^n} - X_{T_{k-1}^n}\right) \left[S(T_k^n, (X_{T_k^n} - X_{T_{k-1}^n})) - S(T_k^n, 0)\right] \\
&+ \lim_{n\to\infty} \sum_{k,B} \left(X_{T_k^n} - X_{T_{k-1}^n}\right) \left[S(T_k^n, (X_{T_k^n} - X_{T_{k-1}^n})) - S(T_k^n, 0)\right],
\end{aligned}
$$

where $\sum_{k,A}$ denotes $\sum_{k\geq 1} 1_{[A\cap(T_{k-1}^n, T_k^n]\neq\emptyset]}$, and $\sum_{k,B}$ denotes $\sum_{k\geq 1} 1_{[B\cap(T_{k-1}^n, T_k^n]=\emptyset]}$. Since A has only finitely many elements, ω by ω, then we have

$$
\lim_{n\to\infty} \sum_{k,A} \left(X_{T_k^n} - X_{T_{k-1}^n}\right) \left[S(T_k^n, (X_{T_k^n} - X_{T_{k-1}^n})) - S(T_k^n, 0)\right] = \sum_{u\in A} [S(u, \Delta X_u) - S(u,0)]\Delta X_u. \tag{3.8}
$$

To deduce the second limit we need to recall Taylor's formula up to the first order

$$
f(y) - f(x) = f^{'}(x)(y-x) + R(x,y), \tag{3.9}
$$

such that $\mid R(x,y) \mid \leq r(\mid y-x \mid)(y-x)^2$,with $r : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function with $\lim_{u\downarrow 0} r(u) = 0$. This is true for f twice- continuously differentiable, defined on a compact.

Applying Taylor's formula to each $S(T_k^n, .)$, the second limit becomes

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_{k,B} \frac{\partial S}{\partial x}(T_k^n, 0) \left(X_{T_k^n} - X_{T_{k-1}^n}\right)^2 \\
&\underbrace{+ \lim_{n\to\infty} \sum_{k,B} \left(X_{T_k^n} - X_{T_{k-1}^n}\right) R\left(T_k^n, \mid X_{T_k^n} - X_{T_{k-1}^n} \mid\right)}_{Z} \\
&= \lim_{n\to\infty} \sum_{k\geq 1} \frac{\partial S}{\partial x}(T_k^n, 0) \left(X_{T_k^n} - X_{T_{k-1}^n}\right)^2 \\
&- \lim_{n\to\infty} \sum_{k,A} \frac{\partial S}{\partial x}(T_k^n, 0) \left(X_{T_k^n} - X_{T_{k-1}^n}\right)^2 + Z \\
&= \lim_{n\to\infty} \sum_{k\geq 1} \frac{\partial S}{\partial x}(T_{k-1}^n, 0) \left(X_{T_k^n} - X_{T_{k-1}^n}\right)^2 \\
&- \lim_{n\to\infty} \sum_{k,A} \frac{\partial S}{\partial x}(T_k^n, 0) \left(X_{T_k^n} - X_{T_{k-1}^n}\right)^2 \\
&+ \lim_{n\to\infty} \sum_{k\geq 1} \left[\frac{\partial S}{\partial x}(T_k^n, 0) - \frac{\partial S}{\partial x}(T_{k-1}^n, 0)\right] \left(X_{T_k^n} - X_{T_{k-1}^n}\right)^2 + Z.
\end{aligned} \tag{3.10}
$$

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In our case we assume that $\frac{\partial S}{\partial x}(.,0)$ is continuous. Then using the fact that X has a finite quadratic variation and $\frac{\partial S}{\partial x}(.,0)$ is uniformly continuous over the compact domain $[0,T]$, the third limit is zero. By Theorem A.6.2, the two first limits converge to

$$\int_0^t \frac{\partial S}{\partial x}(u,0)d[X,X]_u - \sum_{u\in A}\frac{\partial S}{\partial x}(u,0)(\Delta X_u)^2. \tag{3.11}$$

Now we will show as ϵ tends to 0, the term Z in expression 3.10 vanishes. We assumed first that X is bounded by k. Since $\frac{\partial^2 S}{\partial^2 x}(t,x)$ is continuous on the interval $[-K,K]$, it is bounded on that interval, thus $\mid \frac{\partial^2 S}{\partial^2 x}(t,x)\mid < K < \infty$ uniformly in x and t,

$$\begin{aligned}
&\left|R\left(T_k^n, |X_{T_k^n} - X_{T_{k-1}^n}|\right)\right| && (3.12)\\
&\leq \sup_{0\leq|x|\leq|X_{T_k^n}-X_{T_{k-1}^n}|}\left|\frac{\partial S}{\partial x}(T_k^n,x) - \frac{\partial S}{\partial x}(T_k^n,0)\right|\left|(X_{T_k^n} - X_{T_{k-1}^n})\right| && (3.13)\\
&\leq \sup_{0\leq|y|\leq|x|\leq|X_{T_k^n}-X_{T_{k-1}^n}|}\left|\frac{\partial^2 S}{\partial^2 x}(T_k^n,y)x\left(X_{T_k^n} - X_{T_{k-1}^n}\right)\right| && (3.14)\\
&\leq K\left(X_{T_k^n} - X_{T_{k-1}^n}\right)\left(X_{T_k^n} - X_{T_{k-1}^n}\right), && (3.15)
\end{aligned}$$

where the second inequality follows from Mean Value Theorem. Therefore,

$$\begin{aligned}
&\mid Z\mid \leq K\lim_{n\to\infty}\sum_{k,B}\left(\left|X_{T_k^n} - X_{T_{k-1}^n}\right|\right)^3 && (3.16)\\
&\leq K\lim_{n\to\infty}\sup_{k,B}\mid X_{T_k^n} - X_{T_{k-1}^n}\mid\sum_k\left(\left|X_{T_k^n} - X_{T_{k-1}^n}\right|\right)^2 && (3.17)\\
&\leq K\epsilon[X,X]_t. && (3.18)
\end{aligned}$$

Note that ϵ could be chosen arbitrarily small and X has a finite quadratic variation process. Also the sums $\sum_k\left(\left|X_{T_k^n} - X_{T_{k-1}^n}\right|\right)^2$ converges to $[X,X]_t$. Moreover, because all summands are positive, as $\epsilon\to 0$, the expression 3.8 converges to

$$\sum_{0\leq u\leq t}[S(u,\Delta X_u) - S(u,0)]\Delta X_u, \tag{3.19}$$

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and 3.11 converges to

$$\int_0^t \frac{\partial S}{\partial x}(u,0)d[X,X]_u - \sum_{0<u\le t} \frac{\partial S}{\partial x}(u,0)(\Delta X_u)^2 \tag{3.20}$$

$$= \int_0^t \frac{\partial S}{\partial x}(u,0)d[X,X]_u^c. \tag{3.21}$$

To obtain the results in the general case when $\mid \frac{\partial^2 S}{\partial^2 x}(t,x) \mid > k$, let $V_k^x = \inf\{t > 0 : \frac{\partial^2 S}{\partial^2 x}(t,x) > k\}$. Then we have

$$\begin{aligned} S(t,x) &= S(t,x)1_{[0,V_k^x)} + S(t,x)1_{[V_k^x,+\infty)} \\ &= \tilde{S}(t,x) + S(t,x)1_{[V_k^x,+\infty)}. \end{aligned}$$

We have $\tilde{S}(t,x) = S(t,x)1_{[0,V_k^x)}$ as a semimartingale, since is a product of two semimartingales. By the previous case, Equation 3.5 holds for $\tilde{S}(t,x)$,

$$\begin{aligned} Y_t + X_t\tilde{S}(t,0) &= \int_0^t X_{u^-} d\tilde{S}(u,0) \\ &- \sum_{0\le u\le t} \Delta X_u \left[\tilde{S}(u,\Delta X_u) - \tilde{S}(u,0)\right] - \int_0^t \frac{\partial \tilde{S}}{\partial x}(u,0)d[X,X]_u^c. \end{aligned} \tag{3.22}$$

For k large enough we have $\tilde{S}(t,x) = S(t,x)$, hence the equation holds for $S(t,x)$ as well (see the proof of Itô Theorem A.7.1). □

3.1.2 Liquidity Cost from the Marked-to-Market Value of a Self-Financing Trading Strategy

In the classical theory, when we having a money market account and a stock, we have one value for a portfolio only. However, when price depends on the size of the trade, there exists a non-unique value of a portfolio before liquidation. In fact, at least three definitions of the portfolio value exist: The immediate liquidation value if $X_t > 0$ then $Y_t + X_tS(t,-X_t)$, the accumulated cost of forming the portfolio (Y_t), the portfolio defined by using the price from the marginal trade (zero trade size) $(Y_t + X_tS(t,0))$. The expression $(Y_t + X_tS(t,0))$ is called the marked-to-market value of the self-financing trading strategy (X,Y,τ), and it corresponds to the value of the portfolio in the classical model. We can remark that at the liquidation time τ, the valuations gives the portfolio the same value due to the fact that $X_\tau = 0$.

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Therefore, the liquidity cost of trading strategies in a marked-to-market value is presented as the difference between the accumulated gains/losses of the portfolio, obtained as if all trades are executed at the standard price $S(t,0)$, and the marked-to-market value of the portfolio. Thus, the liquidity cost of a self-financing trading strategy (X, Y, τ) is given as follows:

$$L_t = \int_0^t X_{u^-} dS(u,0) - [Y_t + X_t S(t,0)], \tag{3.23}$$

which leads to the following Lemma:

Lemma 3.1.4. *[10]*

$$\begin{aligned} L_t &= \sum_{0 \le u \le t} \Delta X_u \left[S(u, \Delta X_u) - S(u,0) \right] \\ &+ \int_0^t \frac{\partial S}{\partial x}(u,0) d\left[X, X\right]_u^c \\ &\ge 0, \end{aligned} \tag{3.24}$$

where $L_{0^-} = 0$, $L_0 = X_0 \left[S(0, X_0) - S(0,0)\right]$ *and* L_t *is non-decreasing in* t.

Proof. [10] The first equality is obtained from the definition of a self-financing trading strategy i.e.

$$\begin{aligned} L_t &= \int_0^t X_{u^-} dS(u,0) - [Y_t + X_t S(t,0)] \\ &= \int_0^t X_{u^-} dS(u,0) - \int_0^t X_{u^-} dS(u,0) + \sum_{0 \le u \le t} \Delta X_u \left[S(u, \Delta X_u) - S(u,0)\right] \\ &+ \int_0^t \frac{\partial S}{\partial x}(u,0) d\left[X, X\right]_u^c \\ &= \sum_{0 \le u \le t} \Delta X_u \left[S(u, \Delta X_u) - S(u,0)\right] + \int_0^t \frac{\partial S}{\partial x}(u,0) d\left[X, X\right]_u^c. \end{aligned}$$

Using the fact that $S(u,x)$ is increasing in x ($x \le y$ implies $S(t,x) \le S(t,y)$), we have $L_{0^-} = 0$, $L_0 = (X_0 - X_{0^-})\left[S(0, X_0) - S(0,0)\right] = X_0 \left[S(0, X_0) - S(0,0)\right] \ge 0$ and L_t is non-decreasing in t, then we conclude that $L_t \ge 0$ for all t.

□

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We remark that the liquidity cost is non-negative and non-decreasing in t. It has two components. The first term presents the discontinuous changes in the share holdings. The second term is the continuous component. If we assume X is of bounded variation, we have $[X,X]^c_u = 0$, and when X is continuous, we have $\Delta X_u = 0$. Thus, if we assume that X is both continuous and of bounded variation, the first term in the liquidity costs Equation 3.23 equals its value at zero L_0. Thus the liquidity cost is $L_t = X_0 \left[S(0, X_0) - S(0,0)\right]$. Indeed, that is the key to extend the First and Second Fundamental Theorems of Asset Pricing in a market with liquidity risk.

3.1.3 The First Fundamental Theorem of Asset Pricing under Liquidity Risk

This section presents an extension of the FFTAP to include liquidity risk. Due to the restrictions on the supply curve and trading strategies made in the approach by Çetin et al. [10], we are able to apply the classical theory in a economy with liquidity risk. Thus, with a supply curve that is continuous and twice differentiable at the origin (C^2 assumption), and using a continuous and finite variation trading strategy, the first and second theorems of asset-pricing are extended.

Recall that in the classical theory, an arbitrage opportunity is a portfolio that starts with zero value, it has no intermediate cash flows, and the portfolio is liquidated at some future date T with a non-negative value with probability one, and a strictly positive value with positive probability. The only change in the liquidity risk model from the classical definition is the use of the liquidation value Y_T, and not the marked-to-market value of the portfolio at time T. Thus an arbitrage opportunity appears if there exists a self-financing trading strategy (X, Y, τ) such that $\mathbb{P}(Y_T \geq 0) = 1$ and $\mathbb{P}(Y_T > 0) > 0$ [5].

Let us explain what happens at time $t = 0$. We recall that S_t is a semimartingale with the following decomposition: $S_t = S_0 + M_t + A_t$ with $M_0 = A_0 = 0$. In the previous chapter we assumed that X_t is predictable, thus $\int X_s dS_s = X_0 \Delta S_0 + \int M_s \, dS_s + \int X_s \, dA_s$, the initial term $X_0 \Delta S_0$ is 0, since $X_0 = 0$. In the current chapter, the trading strategy X is càdlàg and adapted, thus we have to take the left-continuous version of X and then we

[5] These equations are still hold under a martingale measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$.

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have $\int X_{s^-} dS_s = X_{0^-}\Delta S_0 + \int X_{s^-}\ dM_s + \int X_{s^-}\ dA_s$, the initial term $X_{0^-}\Delta S_0$ is 0, since we have $X_{0^-} = 0$.

Notation: Let $(X_- \bullet S)_t = \int_0^t X_{u^-} dS(u,0)$, and for $\alpha \geq 0$, let

$$\Theta_\alpha = \{\text{ s.f.t.s } (X,Y,\tau) | (X_- \bullet S)_t \geq -\alpha \text{ for all } t \text{ almost surely}\}.$$

Given an $\alpha \geq 0$, a s.f.t.s. (X,Y,τ) is said to be α-admissible if $(X,Y,\tau) \in \Theta_\alpha$. A self-financing trading strategy is admissible if it is α-admissible for some α.

Lemma 3.1.5. *[10] If there exists a probability measure $\mathbb{Q} \equiv \mathbb{P}$ such that $S(.,0)$ is a $\mathbb{Q}$-local martingale, and if $(X,Y,\tau) \in \Theta_\alpha$ for some α, then $Y_t + X_t S(t,0)$ is a $\mathbb{Q}$-supermartingale.*

Proof. We have $S(t,0)$ as $\mathbb{Q}$-local martingale under the $\mathbb{Q}$-martingale measure, and since local martingales are reserved under stochastic integration, then $(X_- \bullet S)_t$ is a $\mathbb{Q}$-local martingale. Since (X,Y,τ) is admissible, by Theorem A.5.12 we have $(X_- \bullet S)_t$ is a supermartingale. On the other hand, we have $Y_t + X_t S(t,0) = (X_- \bullet S)_t - L_t$, then by Lemma 3.1.4 we have that L_t as non-negative and non-decreasing. We conclude using Optional Decomposition Theorem A.5.11, that the process $Y_t + X_t S(t,0)$ is also a $\mathbb{Q}$-supermartingale. $\square$

To guarantee that a market with liquidity risk is arbitrage-free, one only needs to consider the classical case, and study the properties of the marginal stock price process $S(t,0)$. Since, the portfolio in this market differs from the classical portfolio only by subtraction of non-negative liquidity cost, then if the classical portfolio admits no-arbitrage, the actual portfolio cannot admit arbitrage either. Hence we have the following result:

Theorem 3.1.6. *[10] (No-arbitrage condition). If there exists a probability measure $\mathbb{Q} \equiv \mathbb{P}$ such that $S(.,0)$ is a $\mathbb{Q}$-local martingale, then there is no-arbitrage for $(X,Y,\tau) \in \Theta_\alpha$ for any α.*

Proof. Using the fact that $(Y_t + X_t S(t,0))$ is a $\mathbb{Q}$-supermartingale, together with the definition, of the liquidation time that says $Y_\tau + X_\tau S(\tau,0) = Y_\tau$. This gives for a self-financing trading strategy, $\mathbb{E}_\mathbb{Q}[Y_\tau] = \mathbb{E}_\mathbb{Q}[Y_\tau + X_\tau S(\tau,0)] \leq 0$. But, by the definition of an arbitrage opportunity, we have $\mathbb{E}_\mathbb{Q}[Y_\tau] > 0$ which leads to the conclusion that there are no-arbitrage opportunities in this market with liquidity risk. $\square$

In the same way as in the previous chapter, we can investigate the existence of a equivalent local martingale measure. We start by giving the definition of free-lunch with vanishing risk derived from Proposition 2.2.3.

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Definition 3.1.7. *[10] A free lunch with vanishing risk (FLVR) is either:*

1. *an admissable self-financing trading strategy, that is, an arbitrage opportunity, or*
2. *a sequence of ϵ_n-admissible self-financing trading strategy $(X^n, Y^n, \tau^n_{n\geq 1})$ and a non-negative $\mathcal{F}_T$-measurable random variable, f_0, not identically 0 such that $\epsilon_n \to 0$ and $Y^n_T \to f_0$ in probability.*

The fictitious economy is defined as an economy with liquidity risk with the assumption that $S(t,x) = S(t,0)$. This assumption is needed to state the FFTAP. This fictitious economy is simply the economy we discussed in the previous chapter where the self-financing trading strategy (X, Y^0, τ) satisfies the classical condition with $X_0 = 0$, $Y^0_t = \int_0^t X_{u^-} dS(u,0) - X_t S(t,0)$, and X is a general $S(.,0)$ integrable predictable process. Therefore, the definitions of arbitrage opportunity, admissible trading strategies and no free-lunch with vanishing risk in this fictitious economy coincide with those in the classical economy as in Chapter 2. Now we can state the extended FFTAP to include liquidity risk.

Theorem 3.1.8. *[10] Suppose there are no-arbitrage opportunities in the fictitious economy. There is (NFLVR) if, and only if, there exists a probability measure $\mathbb{Q} \equiv \mathbb{P}$ such that $S(.,0)$ is a $\mathbb{Q}$-local martingale.*

The proof of this theorem will be the subject of the rest of this section. First we consider a fictitious market where all trades are executed at the marginal stock price. In this fictitious economy, Theorem 3.1.8 coincides with Theorem 2.2.9. Second, we show that the theorem in this fictitious market is sufficient to obtain the results in this economy with liquidity risk.

We recall that in the classical theory of arbitrage in Chapter 2, the trading strategies start with $X_0 = 0$, whereas in this theory the trading strategies start with $X_{0^-} = 0$. In the proof as [10] suggested and without loss of generality, we restrict ourselves to predictable processes with $X_0 = 0$, and that to make use of the classical theory, we choose Y^0 such that $Y^0_0 = -X_0 S(0,0)$ and $X_t S(t,0) + Y^0_t = X_0 S(0,0) + Y^0_0 + \int_{0+}^T X_u dS(u,0) = \int_{0+}^T X_u dS(u,0)$. Define $\widehat{X} = 1_{(0,T]} X$. The trading strategy $\widehat{X}$ is predictable, $\widehat{X}_0 = 0$. Then $\int_{0+}^T X_u dS(u,0) = \int_0^T 1_{(0,T]} X_u dS(u,0) = \int_0^T \widehat{X}_u dS(u,0)$. Hence we can prove the Theorem 3.1.8 for $\widehat{X}$.

In the fictitious market we have the following corollary, that gives us a necessary condition for the (NFLVR).

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Corollary 3.1.9. *Assume there is no-arbitrage in the fictitious market. Given Definition 3.1.1, if there's free-lunch with vanishing risk in the fictitious market, there exists a sequence of ϵ_n-admissible trading strategies X^n, continuous and of finite variation random variable f_0, not identically zero such that $\epsilon_n \to 0$ and $(X^n \bullet S)_T \to f_0$ in probability.*

Section C.1 in the Appendix presents the following result: given $S(.,0)$ has only totally inaccessible jumps, the stochastic integral of a predictable, continuous process with finite variation can be approximated with continuous and finite variation integrand. Using this result together with Definition 3.1.7 the above corollary is obtained.

Now consider an illiquid market, with a self-financing trading strategy (X, Y, τ) satisfied the Equation 3.5. If we assume X is continuous and of finite variation by Lemma 3.1.4 we have $L_t = L_0 = X_0 [S(0, X_0) - S(0,0)]$. Moreover, according to the observation we made above we have $X_0 = 0$. Thus $Y_t = (X \bullet S)_t - X_t S(t,0)$, and at time T we have $Y_T = (X \bullet S)_T$ since $X_T = 0$. This is the same value of the self-financing trading strategy in the classical theory.

The following two lemmas are crucial in the prove of the extended FFTAP 3.1.8.

Lemma 3.1.10. *[10] Let X be an α-admissible trading strategy which is continuous and of finite variation in the fictitious market. Then there is a sequence of $(\alpha + \epsilon_n)$-admissible trading strategies, in the illiquid market $(H^n, Y^n, \tau^n)_{n \geq 1}$ of finite variation and continuous on $[0, \tau^n]$, such that $H^n_T = 0$, Y^n_T tends to $(X \bullet S)_T$, in probability, and $\epsilon_n \to 0$.*

Proof. [10] Consider $T_n = T - \frac{1}{n}$ and construct the following sequence:

$$f_n(t) = 1_{[T_n \leq t \leq T_{n+1}]} \frac{X_{T_n}}{T_n - T_{n+1}} (t - T_{n+1})$$

such that $f_n(T_n) = X_{T_n}$ and $f_n(T_{n+1}) = 0$, we have $f_n(t) \to 0$ almost surely $\forall\ t$. Then, consider $X^n_t = X_t 1_{[t < T_n]} + f_n(t)$. X^n is continuous and of finite variation by this given definition. Since T is a fixed time, X^n is predictable. Using integration by parts we have

$$X^n \bullet S = X^n S - \int S dX^n.$$

By the definition of X^n_t given above we have $X^n_t \to X_t$ uniformly on compacts in probability. Therefore, $X^n S \to XS$ and $\int S dX^n \to \int S dX$ in ucp on $[0, T]$. Hence, $X^n \bullet S \to X \bullet S$ in ucp on $[0, T]$ since $X \bullet S = XS - \int S dX$.

Let us consider a sequence $(\epsilon_n)_{n \geq 1}$ of positive real numbers that converge to 0 such that $\sum_n \epsilon_n < \infty$. Let $\tau^n = \inf \{t > 0 : (X^n \bullet S)_t < -\alpha - \epsilon_n\} \wedge T$. τ^n is a hitting time, and

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$S(.,x)$ is continuous process, so τ^n is a predictable stopping time. Since $X^n \bullet S$ converges to $X \bullet S$ uniformly on compacts in probability we have

$$\mathbb{P}\left[\sup_{0\le t\le T} \mid (X^n \bullet S)_t - (X \bullet S)_t \mid\ge \epsilon_n\right] \le \epsilon_n.$$

Then, $\mathbb{P}\left[\tau^n < T\right] \le \epsilon_n$ which means $\tau^n \to T$ in probability. Furthermore, strictly before time T_n we have $f_n(t) = 0$ and at time T_n, $f_n(T_n) = X_{T_n}$, then $X^n = X$ up to time T_n, which means $\tau^n \ge T_n$. Now we will consider the sequence of trading strategies $(H^n, \tau^n)_{n\ge1}$ such that $H^n = X^n 1_{[0,\tau_n)}$. We know that $(H^n \bullet S)_t \ge -\alpha - \epsilon_n$, for all $t \in [0, \tau^n]$ (we take closed interval in τ_n because $H^n_{\tau_n} = 0$ for all n). Therefore, $(H^n, \tau^n)_{n\le1}$ is a sequence of $(\alpha + \epsilon_n)$-admissible trading strategies.

Now we have to prove that Y^n_T tends to $(X \bullet S)_T$, in probability, and $\epsilon_n \to 0$. We have that the value of the portfolio at liquidation for each trading strategy with the assumption that H^n is of finite variation and jumps only at τ^n for each n, by the continuity of X^n, is given by

$$Y^n_{\tau_n} = X^n(\tau^n)\left[S(\tau^n, -X^n(\tau^n)) - S(\tau^n, 0)\right] + (X^n \bullet S)_{\tau^n}.$$

Because, $\sum_n \mathbb{P}(\tau^n < T) \le \sum_n \varepsilon_n < \infty$, the first Borel-cantelli lemma tells us that for almost all $\omega \in \Omega$, there are only a finite number of natural numbers n such that $\tau^n < T$ (i.e. $\mathbb{P}\left[\tau^n < T\ i.o\right] = 0$). This implies $X^n(\tau^n) = X^n(T) = 0$, with probability 1, for all but at most finitely many n, together with $\tau^n \to T$ in probability, we have, $Y^n_T \to (X \bullet S)_T$. □

Lemma 3.1.11. *[10] Assume there are no-arbitrage opportunities in the fictitious economy. There is NFLVR in the fictitious market if, and only if, there is NFLVR in the illiquid market.*

Proof. [10] Assume there is NFLVR in the classical theory "fictitious economy". Then, given any self financing trading strategy (X, Y, τ) in the illiquid market, we have $Y_\tau = (X \bullet S)_\tau - X_\tau S(t, 0) \le (X \bullet S)_\tau$ since $X_\tau = 0$. Using the property of NFLVR in the fictitious economy (Proposition 2.2.3), there exists a sequence X^n of continuous and finite variation process such that $(X^n \bullet S)_\tau \to 0$. Then there exists an ϵ_n-admissible trading strategy $(H^n, Y^n, \tau^n)_{n\ge1}$ such that $Y^n_\tau \le (X^n \bullet S)_\tau$ and $Y^n_\tau \to 0$, hence there exists NFLVR in the illiquid market.

Contrariwise, suppose there is FLVR in the fictitious market. By Corollary 3.1.9 there exists a sequence $(X^n)_{n\ge1}$ of continuous and finite variation process with, and ϵ_n-admissible trading strategies such that $(X^n \bullet S)_T \to f_0$ in probability, with $\epsilon_n \to 0$. Applying Lemma 3.1.10, there exists a sequence of α_n-admissible trading strategies, $(H^n, Y^n, \tau^n)_{n\ge1}$, where $\alpha_n \to 0$ in the illiquid market such that $Y^n_{\tau_n} \to f_0$ in probability, with f_0 a non-negative $\mathcal{F}_T$ measurable random variable that is not 0, and that gives a free lunch with vanishing risk in the illiquid market. □

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Proof. (Theorem 3.1.8) [10] By Theorem 2.2.9 in Chapter 2, there is NFLVR in the fictitious market if, and only if, there exists an equivalent martingale measure $\mathbb{Q} \equiv \mathbb{P}$ such that $S(.,0)$ is a $\mathbb{Q}$-local martingale. Using Lemma 3.1.11, we have that NFLVR in the illiquid market is equivalent to NFLVR in the fictitious market, which is equivalent to the existence of a martingale measure. □

Let us summarize: In the classical theory presented on the previous chapter, we assumed a market that is perfectly liquid with predictable trading strategies, and we showed that the existence of NFLVR in this market is equivalent to the existence of an equivalent martingale measure. In this Chapter we considered a market with liquidity risk, and we derived a self-financing condition with trading strategies that are càdlàg and a supply curve that is C^2 continuous, we observed additional terms which refer to the liquidity costs. The implication that Çetin, Jarrow and Protter [10] made is that the use of trading strategies that are continuous and of finite variation eliminate all liquidity costs. Thus we can apply the classical theory in a liquid market to extend the First Fundamental Theorem of Asset Pricing in a illiquid market.

3.1.4 The Second Fundamental Theorem of Asset Pricing under Liquidity Risk

Now that we know that for a market with liquidity risk to be arbitrage-free, an equivalent martingale measure should exist, the question to ask is, is this martingale measure unique? This is what this section will investigate.

In the classical theory, the Second Fundamental Theorem of Asset Pricing (Theorem 2.2.13) connects the uniqueness of the equivalent martingale measure to the completeness of markets. In an economy with liquidity risk we have the same result using the marginal price $S(t,0)$ as the classical case. The small difference is that the uniqueness of the equivalent measure gives us an approximately complete market. The definition of approximately complete as [10] defines it is as follows: given any random variable, there exists a self-financing trading strategy whose liquidation valued is arbitrarily close (in an L^2 sense) to the given random variable. Indeed, by using trading strategies that involve quick trading of small quantities, we can apply the classical result in the analysis of liquidity risk. To study an economy with illiquidities, the Second Fundamental Theorem of Asset Pricing can be

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extended by assuming that there exists an equivalent local martingale measure $\mathbb{Q}$ so that the market is arbitrage-free and there is no free-lunch with vanishing risk (NFLVR). A modification in the definition of a s.f.t.s (X, Y, τ) will be used in this section. In particular, a s.f.t.s (X, Y, τ) will satisfy Definition 3.1.3 without condition (b) [6]. That is, a s.f.t.s which allows for a non-zero investment at time 0.

Recall that the set $\mathcal{H}^2_{\mathbb{Q}}$ is the space of semimartingales with respect to the equivalent local martingale measure $\mathbb{Q}$, with finite $\mathcal{H}^2$ norm (Definition A.2.15). Assume that $S(.,0) \in \mathcal{H}^2_{\mathbb{Q}}$, then the stochastic integral $X \bullet S(.,0) \in \mathcal{H}^2_{\mathbb{Q}}$. So, there is no need for $X \bullet S(.,0)$ to be uniformly bounded from below since $\mathcal{H}^2_{\mathbb{Q}}$ has finite $\mathcal{H}^2$ norm. This integrability condition is needed to study contingent claims [10].

3.1.4.1 Contingent Claims

A contingent claim is any $\mathcal{F}_T$-measurable random variable C with $\mathbb{E}^{\mathbb{Q}}[C^2] < \infty$. This is considered at a time T, prior to the liquidation of position. If the contingent claim's payoff depends on the stock price at time T, then the dependence on the contingent claim's payoff must be made explicit, or else the contingent claim's payoff is not well-defined. Çetin et al. [10] give the following example to explain this matter.

Examples of Contingent Claims

A well-defined contingent claim can be given by an European call option on the stock with a strike price of K and maturity $T_0 \leq T$. To give the modified boundary condition for the option using the supply curve for the stock, two types of settlement that the option contract could have are: *cash settlement* and *physical settlement*.

Suppose first an option contract that requires a *cash settlement*, the buyer of this option receives cash at maturity if the option ends up in the money. The synthetic option position has to be liquidated prior to time T_0, in order to match the cash settlement. The stock position is liquidated in the case when the synthetic option is liquidated, and then the position in the stock is zero at time T_0. Selling the stock implies that the price received

[6] $(Y_0 \neq -X_0 S(0, X_0))$ i.e. we do not required that the initial value $(Y_0 + X_0 S(0, X_0))$ is zero

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depends on the number of shares sold. Thus the boundary condition for the option will be $C \equiv max\left[S(T_0, -1) - K, 0\right]$, where $(\Delta X_{T_0} = -1)$. The stock can be liquidated prior to time T_0 using a continuous and finite variation process. This alternative strategy can be used to avoid liquidity cost at time T_0. Then the boundary condition in this case is $C \equiv \max[S(T_0, 0) - K, 0]$ where $\Delta X_{T_0} = 0$. Because liquidation occurs right before T_0, the option payoff can only be approximately obtained to a given level of accuracy.

In the case when the option contract required a *physical settlement* the synthetic option position should match the underlying asset in the physical delivery. This means that the option contract obligates the seller to deliver the stock shares. In order to match the physical delivery, the stock position in the synthetic option is not sold. However, the model requires the stock position to be liquidated at time T_0. To approximate physical delivery the boundary condition used is $C \equiv \max[S(T_0, 0) - K, 0]$ where $\Delta X_{T_0} = 0$. Then there is no liquidity cost.

Note that trading in options in the case of physical delivery is an expansion of the economy which gives a possibility to avoid liquidity costs at time T.

3.1.4.2 Market Completeness

Definition 3.1.12. *A market model is complete if given any contingent claim C, there exist a self-financing trading strategy (X, Y, τ), with $\mathbb{E}_{\mathbb{Q}}\left[\int_0^T X_u^2 d[S(u,0), S(u,0)]\right] < \infty$* [7] *such that $Y_T = C$.*

Let us consider a contingent claims C in $L^2(d\mathbb{Q})$ where there exists a non-zero initial investment self-financing trading strategy (X, Y, τ) such that $C = c + \int_0^T X_u dS(u, 0)$ where, $c \in \mathbb{R}$ and $\mathbb{E}_{\mathbb{Q}}\left[\int_0^T X_u^2[S(u,0), S(u,0)]\right] < \infty$. Since $X_0 S(0,0)$ is the initial value of the $\mathbb{Q}$-martingale $\int_0^T X_u dS(u, 0)$, then $\mathbb{E}_{\mathbb{Q}}\left[C\right] = c + X_0 S(0,0)$. In this case a long position in the contingent claim C is attainable if there is no liquidity cost. However, by lemma 3.1.4, the liquidity costs in trading this stock position are

$$L_t = \sum_{0 \le u \le t} \Delta X_u \left[S(u, \Delta X_u) - S(u, 0)\right] + \int_0^T \frac{\partial S}{\partial x}(u, 0) d\left[X, X\right]_u^c \ge 0. \tag{3.25}$$

[7] Since X is càdlàg and not predictable in general, we need to require $X \bullet S(., 0)$ to be locally square integrable.

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The self financing condition is given by Definition 3.1.3 as follows:

$$Y_T = Y_0 + X_0 S_0 + \int_0^T X_{u^-} dS(u,0) - X_T S_T - L_T + L_0,$$

and since S_u is assumed continuous, and $X_0 S_0 = X_0 \Delta S_0 = 0$, we have

$$X_0 S_0 + \int_0^T X_{u^-} dS(u,0) = \int_0^T X_u dS(u,0).$$

Using the fact that $C = c + \int_0^T X_u dS(u,0)$, liquidation time criteria $X_T = 0$ and $Y_0 = c$ leads to the following equation:

$$Y_T = C - (L_T - L_0) \leq C.$$

The trading strategy sub-replicates a long position in this contingent claim's payoffs. In the same way, a short position in this contingent claim is given by

$$Y_T^* = -C - (L_T^* - L_0^*) \leq -C,$$

where Y^* is the value of the money market account and L^* is the liquidity cost associated with the non-zero initial investment s.f.t.s $-X$. Analogously taking $Y_0^* = -C$, we get

$$-Y_T^* = C + (L_T^* - L_0^*) \geq C.$$

The selling out (liquidation) value of the trading strategies (buy and sell) provide a lower and upper bound on obtaining the payoffs of the contingent claim.

Now let us go back to the liquidity costs equation 3.25, if we assume $\frac{\partial S}{\partial x}(.,0) \equiv 0$, then we get $L_t = \sum_{0 \leq u \leq t} \Delta X_u \left[S(u, \Delta X_u) - S(u,0)\right]$. If X is chosen to be a continuous trading strategy, then $\Delta X_u = X_0$ for $u > 0$ and we have $L_t = X_0 \left[S(0, X_0) - S(0,0))\right] = L_0$. Thus all claims C where there exist a s.f.t.s $(X, Y.\tau)$, such that $C = c + \int_0^T X_u dS(u,0)$ can be replicated if X is continuous. For example, if $S(.,0)$ is a geometric Brownian Motion, then a call option can be replicated while the Black-Scholes hedge is a continuous s.f.t.s.

If $\frac{\partial S}{\partial x}(.,0) \geq 0$, and X is a trading strategy that is of finite variation, we have $[X,X]^c = 0$, which gives $L_t = \sum_{0 \leq u \leq t} \Delta X_u \left[S(u, \Delta X_u) - S(u,0)\right]$. Then if we choose X to be continuous we have $L = L_0$. Therefore all claims C where there exists a s.f.t.s. (X, Y, τ), such that

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$C = c + \int_0^T X_u dS(u,0)$ can be replicated, with X continuous of finite variation.
Thus it can be concluded that if we can approximate X using a finite and continuous trading strategy, in a limiting sense it is the way to avoid all the liquidity costs in the replication strategy. Then the following result shows how we could approximate with a continuous and finite self-financing trading strategy.

Lemma 3.1.13. *[10] Given a predictable X with $\int_0^T X_u dS(u,0) \in L^2(d\mathbb{Q})$, there exists a sequence $X^n_{n\geq 1}$ with X^n continuous and with finite variation paths, such that $\int_0^T X^n_u dS(u,0) \in L^2(d\mathbb{Q})$. $X^n_0 = X_0$ for all n and*

$$\int_0^T X^n_u dS(u,0) \to \int_0^T X_u dS(u,0) \text{ as } n \to \infty \text{ in } L^2(d\mathbb{Q}). \tag{3.26}$$

Proof. [10] For a given $H \in \mathbb{L}$ (Definition A.2.1), with $H_0 \neq 0$, we can define H^n by $H^n_t = n \int_{t-\frac{1}{n}}^t H_u(\omega)du$, for all $t \geq 0$, H is the pointwise limit of the processes H^n that are continuous and of finite variations. Restricting ourselves to the set of processes that are bounded and continuous, with finite variation paths on compacts time sets, Theorem A.5.3 will be still true. That, for $X \in \mathcal{H}^2$, and $H \in b\mathcal{P}$, there exists a bounded continuous of finite variation process J such that $d_X(H,J) < \varepsilon$, for any $\varepsilon > 0$.

Assume a predictable process X with $X_0 = 0$ and $\mathbb{E}_{\mathbb{Q}}\left[\int_0^T X_u d[S(u,0), S(u,0)]_u\right] \leq \infty$. We have $X \bullet S = \lim_{k\to\infty} \overline{X}^k \bullet S$, with the convergence in $\mathcal{H}^2$ and $\overline{X}^k = X 1_{\{|X|\leq k\}}$. Theorem A.5.10 gives an important result in stochastic integration, that we can approximate a stochastic integral for predictable processes that are not necessarily bounded with processes in $b\mathcal{P}$. Since the set of processes that are bounded, continuous and of FV is dense in $b\mathcal{P}$, then we can say that there exists a sequence of continuous and bounded processes of finite variation, $(X^n)_{n\geq 1}$, such that $\mathbb{E}_{\mathbb{Q}}\left[\int_0^T (X^n_u)^2 d[S(u,0), S(u,0)]\right] < \infty$. $X^n_0 = 0$, for all n and $X^n \bullet S$ is a Cauchy sequence in $\mathcal{H}^2$. Since $\mathcal{H}^2$ is a Banach space, then we have $\int_0^T X^n_u dS(u,0) \longrightarrow \int_0^T X_u dS(u,0)$ with convergence in $L^2(d\mathbb{Q})$.

□

Lemma 3.1.14. *[10] Let $C = c + \int_0^T X_u dS(u,0)$ in $L^2(d\mathbb{Q})$ for X predictable such that $\mathbb{E}_{\mathbb{Q}}\left[\int_0^T X_u^2 d[S(u,0), S(u,0)]_u\right] < \infty$. Then there exists a sequence of s.f.t.s $(X^n, Y^n, \tau^n)_{n\geq 1}$ with X^n bounded, continuous and of finite variation such that*
$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T (X^n_u)^2 d[S(u,0), S(u,0)]_u\right] < \infty$.

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$X_0^n = 0,\ X_T^n = 0,\ Y_0^n = \mathbb{E}_{\mathbb{Q}}[C]$ *for all* n *and*

$$Y_T^n = Y_0^n + X_0^n S(0, X_0^n) + \int_0^T X_{u-}^n dS(u,0) - X_T^n S(T,0) - L_T^n$$
$$\longrightarrow C = c + \int_0^T X_u dS(u,0) \ \ in\ \ L^2(d\mathbb{Q}),$$

Proof. [10]

We choose X^n as with the previous Lemma, then X^n is continuous of FV. Note that $X_T^n = 0$ and $\tau^n = T$ for all n (see Theorem C.1.3 and Corollary C.1.4) . Let Y^n satisfy expression 3.5 with $Y_0^n = c = Y_0$ for all n, then (X^n, Y^n, τ^n) is a s.f.t.s. By Expression 3.5,

$$Y_T^n = Y_0^n + X_0^n S(0, X_0^n) + \int_0^T X_{u-}^n ds_u - X_T^n S(T,0) - L_T^n.$$

Given $X_T^n = 0$, and the fact that $X_0^n S(0,0) + \int_0^T X_{u-}^n dS(u,0) = \int_0^T X_u^n dS(u,0)$, we have $Y_T^n = Y_0^n + \int_0^T X_u^n dS(u,0) - L_T^n + L_0$[8]. Using $Y_0^n = c$ and the fact that $L_T^n = L_0$, since X^n is continuous and of finite variation, gives $Y_T^n = c + \int_0^T X_u^n dS(u,0)$. By the previous Lemma we have $Y_T^n \to c + \int_0^T X_u dS(u,0)$ in $L^2(d\mathbb{Q})$. □

Now we are able to give the definition of a complete market in the case of liquidity risk, also an extension of the Second Fundamental Theorem of Asset Pricing.

Definition 3.1.15. *[10] A market is said to be approximately complete if given any contingent claim C, there exists a sequence of non-zero initial investment s.f.t.s.* (X^n, Y^n, τ^n) *with* $\mathbb{E}_{\mathbb{Q}}\left[\int_0^T (X_u^n)^2 d[S(u,0), S(u,0)]_u\right] < \infty$*, such that* $Y_T^n \longrightarrow C$ *as* $n \longrightarrow \infty$ *in* $L^2(d\mathbb{Q})$*.*

Theorem 3.1.16. *(Second Fundamental Theorem)[10] Suppose there exists a unique probability measure $\mathbb{Q} \equiv \mathbb{P}$ such that $S(.,0)$ is a martingale under $\mathbb{Q}$. Then the market is approximately complete.*

Proof. [10] The proof of this theorem will proceed in the same way as the proof of the FFTAP. Thus, hypothesizing $S(.,x) = S(.,0)$ leads to a fictitious economy with a self-financing trading strategy coinciding with the self-financing trading strategy in the classical case i.e. $Y_t + X_t S(t,0) = Y_0 + X_0 S(0,0) + \int_0^t X_{u-} dS(u,0)$. Then Theorem 2.2.12 ensures that this fictitious market is complete if, and only if, $\mathbb{Q}$ is unique. Hence there exists a predictable X such that $C = c + \int_0^T X_u dS(u,0)$ with $\mathbb{E}^{\mathbb{Q}}\left[\int_0^T X_u^2 d[S(u,0), S(u,0)]_u\right] < \infty$. By applying Lemma 3.1.14, there exists a sequence of non-zero initial investment s.f.t.s.

[8] We recall that $L^0 = X_0^n S(0, X_0^n) - X_0^n S(0,0)$.

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(X^n, Y^n, τ^n) with X^n continuous and of finite variation, $X_0^n = X_0$, $Y_0^n = c$ for all n so that $Y_T^n = Y_0 + X_0 S(0, X_0) - L_T + \int_0^T X_{u^-}^n dS(u,0) \to c + \int_0^T X_{u^-} dS(u,0)$ in $L^2(d\mathbb{Q})$. Thus the market is approximately complete. □

The sequence (X^n, Y^n, τ^n) is given by the theorem called an approximating trading strategy sequence for the contingent C.

Theorem 3.1.17. *Let C be a contingent claim and $(X^n, Y^n, \tau^n)_{n\geq 1}$ be a sequence of approximating strategies. Then, $\lim_{n\to\infty} Y_0^n + X_0^n S(0, X_0^n)$ is independent of the choice of approximating trading strategies.*

Proof. [9] Suppose we have two approximating trading strategies $(X^n, Y^n, \tau^n)_{n\geq 1}$ and $(\overline{X}^n, \overline{Y}^n, \tau^n)_{n\geq 1}$; we have to prove that $\lim_{n\to+\infty} Y_0^n + X_0^n S(0, X_0^n) = \lim_{n\to+\infty} \overline{Y}_0^n + \overline{X}_0^n S(0, X_0^n)$. We have $\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}} \left[Y_T^n - \overline{Y}_T^n\right]^2 = 0$. Then

$$\mathbb{E}_{\mathbb{Q}} \left[Y_T^n - \overline{Y}_T^n\right]^2 = \mathbb{E}_{\mathbb{Q}} \left[Y_0^n + X_0^n S(0, X_0^n) + \int_0^T X_u^n dS(u,0) \right. \tag{3.27}$$

$$\left. - \left(\overline{Y}_0^n + \overline{X}_0^n S(0, X_0^n) + \int_0^T \overline{X}_u^n dS(u,0)\right)\right]^2 \tag{3.28}$$

$$= \left(Y_0^n + X_0^n S_0 - \left(\overline{Y}_0^n + \overline{X}_0^n S_0\right)\right)^2 \tag{3.29}$$

$$+ \mathbb{E}_{\mathbb{Q}} \left[\int_0^T \left(X_u^n - \overline{X}_u^n\right)^2 d\langle S(u,0), S(u,0)\rangle_u\right] \tag{3.30}$$

$$+ 2\mathbb{E}_{\mathbb{Q}} \left[\left(Y_0^n + X_0^n S_0 - \left(\overline{Y}_0^n + \overline{X}_0^n S_0\right)\right) \int_0^T (X_u^n - \overline{X}_u^n) dS(u,0)\right]. \tag{3.31}$$

Let us recall that $\langle S(.,0), S(.,0)\rangle$ is the compensator of $[S(.,0), S(.,0)]$, if $S(.,0)$ is a continuous semimartingale then $[S(.,0), S(.,0)]$ is continuous and also predictable which gives $[S(.,0), S(.,0)] = \langle S(.,0), S(.,0)\rangle$. We have $(X^n - \overline{X}^n) \bullet S$ as a $\mathbb{Q}$-martingale, Expression 3.31 is 0. Then,

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}} \left[Y_T^n - \overline{Y}_T^n\right]^2 = 0 \Longleftrightarrow \lim_{n\to+\infty} \left(Y_0^n + X_0^n S_0 - \left(\overline{Y}_0^n + \overline{X}_0^n S_0\right)\right)^2 = 0. \tag{3.32}$$

and

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}} \left[\int_{0+}^T \left(X_u^n - \overline{X}_u^n\right)^2 d\langle S(.,0), S(.,0)\rangle_u\right] = 0. \tag{3.33}$$

From Equation 3.33 we have $\lim_{n\to\infty} (X^n - \overline{X}^n) = 0$, $d\mathbb{Q} \times d\langle S(.,0), S(.,0)\rangle$ almost everywhere. Thus, $\lim_{n\to\infty} (X_0^n - \overline{X}_0^n) = 0$. Also from the Equation 3.32 we have $\lim_{n\to\infty} (Y_0^n - \overline{Y}_0^n) = 0$. Hence,

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$$\lim_{n\to+\infty} Y_0^n + X_0^n S(0, X_0^n) = \lim_{n\to+\infty} \overline{Y}_0^n + \overline{X}_0^n S(0, X_0^n). \qquad \square$$

Corollary 3.1.18. *[10] Assume there exists a unique probability martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which $S(.,0)$ is a local martingale. Then the valuation of any contingent claim C is $\mathbb{E}_{\mathbb{Q}}[C]$.*

Proof. [10] Given an approximating sequence $(X^n, Y^n, \tau^n)_{n\geq 1}$ for C, we have $\mathbb{E}_{\mathbb{Q}}[Y_T^n - C]^2 \to 0$, which gives $\mathbb{E}_{\mathbb{Q}}[Y_T^n - C] \to 0$. On the other hand,

$$\mathbb{E}_{\mathbb{Q}}[Y_T^n] = \mathbb{E}_{\mathbb{Q}}\left[Y_0^n + X_0^n S(0, X_0^n) + \int_0^T X_{u^-}^n dS(u,0) - L_T^n\right] \tag{3.34}$$

$$= Y_0^n + X_0^n S(0, X_0^n) - \mathbb{E}_{\mathbb{Q}}[L_T^n], \tag{3.35}$$

since $\mathbb{E}_{\mathbb{Q}}[(X_u^n)^2 d[S(u,0), S(u,0)]_u] < \infty \ \forall\, n$, and $\int_0^T X_{u^-}^n dS(u,0)$ is a martingale under $\mathbb{Q}$. Thus $Y_0^n + X_0^n S(0, X_0) - \mathbb{E}_{\mathbb{Q}}[L_T^n] - \mathbb{E}_{\mathbb{Q}}[C] \to 0$, that gives $\lim_{n\to\infty} Y_0^n + X_0^n S(0, X_0^n) \geq \mathbb{E}_{\mathbb{Q}}[C]$, since $L^n \geq 0$. However by Lemma 3.1.14, there exists $(\bar{X}^n, \bar{Y}^n, \bar{\tau}^n)_{n\geq 1}$, an approximating sequence such that $\bar{L}^n = 0$, for all n. Hence $\lim_{n\to\infty} Y_0^n + X_0^n S(0, X_0^n) = \mathbb{E}_{\mathbb{Q}}[C]$, $\square$

This corollary implies that given a unique martingale measure for $S(.,0)$, then the standard contingent claim valuation formulae are valid, except for the initial liquidity cost of constructing the replicating portfolio. Nevertheless, after its initial construction, only an approximate hedge can be obtained.

To summarise: Using the fact that a bounded continuous process with finite variation can approximate an arbitrary predictable process in the L^2 sense, we are able to apply the classical arbitrage theory and extend the Second Fundamental Theorem of Asset Pricing.

The paper by Çetin, Jarrow and Protter [10] did not neglect the case where we have a discontinuous supply curve. The previous economy has been extended in the case of a discontinuous simple path of the supply curve. The continuity assumption replaced by $\partial S(.,0)/\partial x$ has a finite quadratic variation; that is, for $[\partial S(.,0)/\partial x, [X,X]]$ to be well-defined. As a result, the self-financing trading strategy definition will include possible jumps in the first partial derivatives of the supply curve. A sufficient condition for the no arbitrage is obtained, and the market is approximately complete if there exists a probability measure $\mathbb{Q} \equiv \mathbb{P}$ such that the discontinuous supply curve $S(.,0)$ is a $\mathbb{Q}$-local martingale.

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3.1.5 Weaknesses of Çetin, Jarrow and Protter Model

The CJP model assumes trading strategies that are continuous and of finite variation, in order to avoid all liquidity costs theoretically. In practice, liquidity risk does exist, we cannot avoid it, and using continuous trading strategies are impossible. To make liquidity risk unavoidable in theory as well, a continuous trading strategy should be excluded. The previous theory in [10] was modified by Çetin, Jarrow, Protter and Warachka [11] considering only discrete trading strategies. Optimal hedging trading strategies that super-replicate an option were derived by solving a dynamics program. Empirical studies have proved that the Black-Scholes hedge and the super-replication costs to an option are economically significant. This study was based on the TAQ database of five well-known firms trading on the NYSE over the period 1995 to 1998.

Another assumption that the CJP model uses, is that the impact of the trade size on the price process is temporary. The model neglects the case in which a large trader buys and sells large quantities of assets affecting the prices in a non-negligible way. Including this assumption is what is known as "large trader" models, which will be summarized in the next section.

3.2 Liquidity Risk and Price Impact

The recent paper by Roch [57] presents a model that captures liquidity risk and price impact that the model by [10] did not take into account. The idea of the paper is that instead of modelling the price process directly, one models the dynamics of the supply and demand for the asset and its impact on the execution costs [27], that is, considers a risky asset traded through a limit order book [9]. Let us consider that a supply curve that the hedger would expect to observe if he did not trade, will be denoted by S and called the unaffected supply curve. S represents the limit order book coming from all trader's limit and market order. Assuming that the hedger's trade has a lasting impact on prices, the actual observed supply curve denoted by S^0 will then be the impact on prices due to the hedger's trade added to S.

[9]A limit order is an order to buy or to sell the stock at a specific price which is not immediately executed.

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The unaffected supply curve $S(t,x)$ in this model is assumed to be linear based on some empirical studies done by Blais [6], Blais and Protter [7] on a large data set of stocks in the year 2003, and it is given by the following linear structure:

$$S(t,x) = S_t + M_t x \ \text{ for } \ x \in \mathbb{R}, \tag{3.36}$$

where $(M_t)_{t\geq 0}$, $(S_t)_{t\geq 0}$ are positive continuous semimartingales. S_t is called the marginal price, it is the price per share for a purchase or sale of an infinitesimal quantity of shares $(x = 0)$. We can assume S_t to be a geometric Brownian motion, for example.

Using Equation 3.36, it was shown that the limit order book has a constant density at time t given by $\frac{1}{2M_t}$. The effective impact on price of a trade of size ΔX_t is to shift the quoted price to $S_t^0 + 2\lambda M_t \Delta X_t$. The observed marginal price is obtained by adding the unaffected price to the impact on prices as follows: [10]

$$S_{t+}^0 = S_t + 2\lambda \int_0^t M_{u^-} dX_u + 2\lambda \int_0^t d[M,X]_u, \tag{3.37}$$

where X denotes the stock position that is a semimartingale, and S_{t+}^0 is the observed price after time t. The parameter λ denotes the resilience of the market taking value on the interval $[0,1]$.

The money market account Y and the position X in the stock satisfy the following as in [57] (Proposition 2.2):

$$\begin{aligned} Y_t + X_t(S_{t+}^0 - \lambda M_t X_t) = Y_{t_{0^-}} + X_{t_{0^-}}(S_{t_0}^0 - \lambda M_{t_0} X_{t_{0^-}}) + \int_{t_0}^t X_{u^-} dS_u \\ - \lambda \int_{t_0}^t X_{u^-}^2 dM_u - \int_{t_0}^t (1-\lambda) M_{u^-} d[X,X]_u. \end{aligned} \tag{3.38}$$

We can observe that in the case when $\lambda = 0$ we get a linear version of the CJP model. The integral with respect to M is related to the impact of trading. Indeed, when $\lambda = 0$, the limit order book is refilled after a market order, as in the CJP model. In contrast, when $\lambda = 1$, the impact of trading is in its fullest[27].

When taking into account the trades impact, the Equation 3.38 has an integral with respect

[10] S_t^0 will not be càdlàg in general, then we use the right limit version S_{t+}^0.

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to M. Then, since the integrand $(-\lambda X_{t^-}^2)$ is negative, and to make the profit coming from this integral negative on average, the process M has to be submartingale under the risk neutral measure. The Theorem 2.5 in [57] states that if there exists a martingale measure $\mathbb{Q} \equiv \mathbb{P}$ such that S is a $\mathbb{Q}$-martingale and M is a $\mathbb{Q}$-local submartigale, then there is no-arbitrage in a market with liquidity risk and price impacts.

Pricing options in the presence of liquidity risk has become the focus of many research articles ([13], [12], [59], [58], [28]) that is, after the work by Çetin, Jarrow and Protter [10]. Most of them try to solve the problem of replication in the presence of illiquidities using more general trading strategies rather than only trading strategies that are continuous and of finite variation. Others try to explain the phenomena of formation and bursting of financial bubbles on the market using the model by Roch [57]. The paper by Jarrow, Protter and Roch [40] presents more detail in this context.

3.3 Example: Black-Scholes Model under Liquidity Risk

It is well known that most of the models and techniques used by analysts today are rooted in the model of Black and Scholes presented earlier. That is why an extension of this model to include illiquidity is needed. The supply curve is given by the following equation:

$$S(t,x) = e^{\alpha x} S(t,0) \text{ with } \alpha > 0 \tag{3.39}$$

$$S(t,0) \equiv \frac{s_t}{e^{rt}} = \frac{s_0 e^{\mu t + \sigma W_t}}{e^{rt}}, \tag{3.40}$$

in which μ, σ are constant and W_t is a standard Brownian motion. The marginal stock price indicated in equation 3.40 follows a geometric Brownian motion and is normalized by the money market account's value. The extended Black-Scholes economy's supply curve is formed in equation 3.39. Note that the choice of the function $e^{\alpha x}$ was chosen for simplicity and is easily generalized. This choice of the supply curve, as a first estimate, happens to be consistent with the data in Çetin, Jarrow, Protter and Warachka [11].

The supply curve satisfies Definition 3.1.1, applying the extended first and second arbitrage theorems pricing theorems 3.1.8 and 3.1.16, there exists a unique martingale measure for $S(t,0) = s_t$; then the market is arbitrage-free and is approximately complete.

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Now in the same way as in the classical case, one can price a European call option. Only now we use the supply curve in equation 3.39. We can show that in a market which is approximately complete, the cost of an option is equal to its discounted expected payoff with the expectation being taken under the martingale measure.

We consider a European call option with price K and maturity date T on this stock with physical delivery. Given physical delivery, the payoff to the option at time T is $C_T = max\left[S(T,0) - Ke^{-rT}, 0\right]$. Under this structure, by Corollary 3.1.18, the value of a long position in the option is

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[C_T\right] &= \mathbb{E}_{\mathbb{Q}}\left[max\left[S(T,0) - Ke^{-rT}, 0\right]\right] + L_0 \\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[max\left[s_T - K, 0\right]\right] + L_0,\end{aligned}$$

where $L_0 = X_0\left[S(0,X_0) - S(0,0)\right]$. The analytical value for the expectation value of the Black-Scholes is

$$\mathbb{E}_{\mathbb{Q}}\left[C_T\right] = s_0 N(h_0) - Ke^{-rT}N(h_0 - \sigma\sqrt{T}),$$

where $N(.)$ is the density function for the normal distribution

$$h_t = \frac{\log(s_t) - \log K + r(T-t)}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t}.$$

After pricing the option, the second step is replicating the options. It is well known that in the classical case the replicating strategy is given by

$$X_t = \frac{\partial C_t}{\partial S(t,0)} = N(h_t). \tag{3.41}$$

When we have an upward sloping supply curve, the procedure for determining the replicating portfolio is the same. The difference is that the classical replicating strategy is continuous, but not of finite variation. Then a modified replicating strategy will need to be used to reduce price impact costs.

In fact, using this strategy, a liquidity cost of the Black-Scholes hedge can be calculated

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(see [11]);

$$\begin{aligned}
L_T &= X_0\left(S(0,X_0)-S(0,0)\right)+\int_0^T \alpha s_u d[N(h),N(h)]_u^c \\
&= X_0\left(S(0,X_0)-S(0,0)\right)+\int_0^T \alpha s_u (N'(h_u))^2 d[h,h]_u^c \\
&= X_0\left(S(0,X_0)-S(0,0)\right)+\int_0^T \alpha s_u (N'(h_u))^2 \frac{1}{s_u^2\sigma^2(T-u)} d[s,s]_u \\
&= X_0\left(S(0,X_0)-S(0,0)\right)+\int_0^T \frac{\alpha (N'(h_u))^2}{\sigma^2 s_u (T-u)}\sigma^2 s_u^2 du \\
&= X_0\left(S(0,X_0)-S(0,0)\right)+\int_0^T \frac{\alpha (N'(h_u))^2}{T-u} s_u du.
\end{aligned}$$

The value at time 0 of the Black-Scholes hedge is

$$X_0(S(0,X_0)-S(0,0))+\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \frac{\alpha(N'(h_u))^2 s_u}{T-u}du\right]=\infty.$$

This hedging strategy will yield a portfolio whose initial cost of construction is infinite. Instead, the self financing trading strategy, that is continuous and of finite variation, that approximates the call option's payoff as given by [10], is as follows:

$$X_t^n = \begin{cases} n1_{\left[\frac{1}{n},T-\frac{1}{n}\right]}(t)\int_{(t-\frac{1}{n})^+}^t N(h(u))du & \text{if } 0\le t\le T-\frac{1}{n} \\ n\left(TX^n_{(T-\frac{1}{n})}-tX^n_{(T-\frac{1}{n})}\right) & \text{if } T-\frac{1}{n}\le t\le T \end{cases}$$

This strategy starts with $X_0^n=0$. Given these expressions, together with the use of a limited trading strategy, we can obtain the call options price in time T as $Y_T^n = Y_0^n + \int_0^T X_{u^-}^n dS_u \longrightarrow C_T = \max\left[S(T,0)-K,0\right]$ in $L^2(\mathbb{Q})$.

The above trading strategy is a version of the Black-Scholes hedging strategy with more smoothness which incurs no liquidity costs [10].

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3.4 Example of Linear Supply Curves

Now we will discuss an example of a supply curve based on the results of [6] and [7]. Recall that in the approach of Çetin, Jarrow and Protter [10], the classical theory with unlimited liquidity is embedded in what we discussed previously. Thus the standard price $S_t = S(t,0)$ can be obtained by reducing the supply curve $x \to S(t,x)$ to $x \to S(t,0)$. We can have the following linear form of the supply curve [7]

$$x \to S(t,x) = M_t x + S(t,0),$$

with M_t as a stochastic process with continuous paths. Where $M_t = 0$, we simply have the classical case.

Theorem 3.4.1. *[39] A liquid stock with linear supply curve*

$$x \to S(t,x) = M_t x + S(t,0),$$

with X, is a càdlàg trading strategy with finite quadratic variation, the value in the money market account for a self-financing trading strategy is given by

$$Y_t = -X_t S(t,0) + \int_0^t X_{u^-} dS(u,0) - \int_0^t M_u d[X,X]_u,$$

with the possibility of the quadratic differential term to have jumps.

A new problem arises for non-liquid stocks. The C^2 hypothesis on the supply curve Definition 3.1.1 no longer holds and the supply curve is jump linear and has one jump that can be described as the bid-ask spread. Since, Çetin, Jarrow and Protter [10] used the C^2 hypothesis only in the derivation of the self-financing trading strategy, then it can be eliminated in the jump-linear case.

The non-continuity property of the supply curve gives the following form of the bid ask spread $\gamma(t) = S(t,0) - S(t,0^-)$, where $S(t,0^-)$ is the marginal ask, while $S(t,0)$ is the marginal bid. Let us assume that $\Lambda = \{(s,\omega) : \Delta X_s(\omega) < 0\}$, and the supply curve has the following jump linear form:

$$S(t,x) = \begin{cases} \beta(t)x + S(t,0) & x \geq 0 \\ \alpha(t)x + S(t,0^-) & x < 0 \end{cases} \tag{3.42}$$

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Using this supply curve form, we have the following theorem:

Theorem 3.4.2. *[39] The value in the money market account for a an illiquid stock with a jump linear supply curve as in the Equation 3.42, and a càdlàg with finite variation trading strategy X, is given as follows:*

$$Y_t = -X_t S(t,0) + \int_0^t X_{u^-} dS(u,0) - \int_0^t \beta(u) 1_{\Lambda^c}(u) + \alpha(u) 1_{\Lambda}(u) d[X,X]_u - \int_0^t 1_{\Lambda}(u) d[\gamma, X]_u,$$

with $\gamma(t) = S(t,0) - S(t,0^-)$ as the bid-ask spread.

Proof. see [39]. □

From the Theorem, we can define the liquidity cost as follows:

$$L_t = -\int_0^t \beta(u) 1_{\Lambda^c}(u) + \alpha(u) 1_{\Lambda}(u) d[X,X]_u - \int_0^t 1_{\Lambda}(u) d[\gamma, X]_u.$$

This liquidity cost generated by the bid-ask spread in the Equation 3.42 is not necessarily infinite [39].

The study by Blais [6], Blais and Protter [7] was based on the analysis of the book data provided to them by Robert Ferstenberg of Morgan Stanley. Using the book data was more accurate than using the tick data as in the study by Çetin, Jarrow, Protter and Warachka [11]. The issue with the tick data is given where a trade is not clear if it is a buy or a sell. Although, the well-known algorithm to distinguish between buys and sells, Lee and Ready algorithm [52], did not give an accurate results.

The conclusion from these studies is that a supply curve definitively exists. Moreover, it has been shown that the supply curve has a linear structure for highly liquid stocks, and a jump-linear structure for illiquid stocks. Furthermore, the model by Çetin, Jarrow and Protter [10] has been criticized since the model claims that the price of a European call option in a market with liquidity risk is the same as the price in a perfect market without a liquidity assumption. Indeed, the option price has a linear supply curve that is not horizontal, and the issue of liquidity does exist for options as well.

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3.5 Liquidity Risk and Financial Bubbles

In this section, we give a short review on a very large topic of interest in financial economics, financial bubbles. The word "bubble" refers to the image of an object growing little by little until it finally pops. From financial crises history, a bubble is an upward price movement over an extended range that then crashes. In other words, a bubble is a situation in which prices for stock rise far above their actual value. Usually, bubbles occur when traders believe that demand for stock will continue to rise or that stock will become profitable in short order.

There are many famous episodes in financial history of bubbles in prices, here we mention two of them: the Dot.com bubbles in the mid-1990s, when the stock market increased on technology and Internet stocks, and the stock prices went sky hight ending with the crash in the 2000. And, the stock market bubble peaking in October 2007 [66].

Recently, the issue of modelling financial bubbles has attracted the attention of several authors: we start with Cox and Hobson [18], their paper studied the martingale approach to model price bubbles by assuming that the stock market price is exogenous and the fundamental price [11] is endogenous. The characterisation theorem for bubbles under a standard no-arbitrage framework was studied by Jarrow, Protter and Shimbo [41] only in a complete market. In addition, Jarrow, Protter and Shimbo [42] dealt with a case of an incomplete market by studying a continuous time model using the local martingale approach and proposing a new theory of bubbles birth. Finally, they investigated the pricing of derivatives when asset price bubbles exist in the market.

The main purpose of the paper by Protter and Roch [40] was to link the theory of liquidity risk with asset price bubbles. Precisely, the liquidity risk model of Roch [57] was used to analyse the birth and bursting of bubbles. Indeed, asset price bubbles and market trading activity are related. Since, the trading activities affect prices, it causes the existence of liquidity risk. Then, this quantity impact of trading activity deviates the market price from its fundamental value, which is what it called, price bubbles. To model price bubbles, a specific martingale approach was considered; it was assumed, and in contrast to the previous studies in modelling bubbles, that the asset's fundamental price is exogenous and

[11] The expected future cash flows with respect to a martingale measure

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asset price bubbles are endogenous which is generated by market trading activity including the volume of market order, resiliency parameters, and levels of liquidity in the market. The fundamental price process will be the market process coming from the trading activity against the limit order book under a normal market with normal conditions. Now, when the resiliency of the limit order book is weak, there is a divergence from this fundamental market price process, and this divergence creates price bubbles in this model. The price bubble burst, when the resiliency is restored.

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Chapter 4

Conclusion

Many practitioners accept the Black-Scholes model and classical arbitrage pricing theory as it gave reasonably good results and was easy to apply. Nevertheless, this theory assumes a competitive and frictionless market, which is not actually true in the real world. For this reason it is necessary to extend the theory to a more general setting. Once the market is not competitive and is not frictionless, there is a possibility of liquidity risk in the market.

In this dissertation, we reviewed a model that captures liquidity risk and studies the arbitrage theory in the presence of liquidity risk: The CJP model that assumes an upward sloping stochastic supply curve for the stock price, instead of a horizontal supply curve as in the classical theory. A self-financing trading strategy condition is derived using this supply curve to give a value of the portfolio as the classical value minus liquidity costs. The weakness of this model lies in the result that liquidity risk can be avoided by using trading strategies that are both continuous and of finite variation. This assumption of continuity and finite variation trading strategies was the key to extend the First and the Second Fundamental Theorems of Asset Pricing. We also summarized the model by Roch that takes the CJP model further and assumes trading impacts on prices, and a linear supply curve, based on some recent empirical studies. An interesting example of a linear supply curve for liquid and illiquid stocks was derived from the book order data, and shows that the supply curve considered in the theory actually exists in practice as well. Indeed, liquidity risk is an issue in the market that cannot be avoided, and using continuous trading strategies is impossible in practice. However, trading with high frequency in a small amount gives small liquidity charges. In the end we bring to light a very large topic in

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mathematical finance, financial bubbles. The topic is much larger than the subject of this thesis, but we noted that the analysis of financial bubbles was studied using liquidity risk models. Indeed, the model by Roch was used to explain the formation and bursting of financial bubbles on the market.

Future work, which could extend what we have discussed in this thesis, is to study the dependence structure between supply curves for stocks and options as well. Another point of interest is to develop more accurate, fast and applicable numerical methods to simulate some of the complicated stochastic integrals we have used in this thesis, so it will be easy for practitioners to use more complicated models based on a more complicated stochastic analysis.

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Appendix A

Stochastic Integrals and Semimartingales

The main goal of this chapter is to present some basics results on stochastic integration and semimartingale processes which have been used throughout this thesis. The main reference for this chapter is Protter [56].

A.1 Preliminaries

A.1.1 Basic Definition and Notation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq\infty}, \mathbb{P})$ be a complete probability space. In addition, we are given a filtration, $(\mathcal{F}_t)_{0\leq t\leq\infty}$. By a filtration we mean a family of σ-algebras, $(\mathcal{F}_t)_{0\leq t\leq\infty}$, that is increasing, i.e. $\mathcal{F}_s \subset \mathcal{F}_t$ if, $s < t$.

Definition A.1.1. *A map $T : \Omega \longrightarrow [0, \infty]$ is called a random time, if T is a random variable. T is called a stopping time (w.r.t the filtration $(\mathcal{F}_t)_{0\leq t\leq\infty}$, or an $\mathcal{F}_t$-stopping time) if, for all $t \geq 0$, the set $\{T \leq t\} \in \mathcal{F}_t$.*

Definition A.1.2. *A filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq\infty}, \mathbb{P})$ is said to satisfy the usual hypotheses if*

- *$\mathcal{F}_0$ contains all the $\mathbb{P}$-null of $\mathcal{F}$;*
- *$\mathcal{F}_t = \cap_{u>t}\mathcal{F}_u$, all t, $0 \leq t \leq \infty$; i.e the filtration $\mathcal{F}_t$ is right-continuous.*

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Definition A.1.3. *A stochastic process X on $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of $\mathbb{R}$-valued of $\mathbb{R}^d$-valued random variables $(X_t)_{0 \le t \le \infty}$. The process X is called adapted if $X_t \in \mathcal{F}_t$ (that is, X_t is $\mathcal{F}_t$ measurable) for each t.*

Definition A.1.4. *[56] A stochastic process X is said to be a càdlàg process if it almost surely has sample paths which are right-continuous, with left limits. Also, a stochastic process X is a càglàd process if it almost surely has sample paths which are left-continuous, with right limits.(The words càdlàg and càglàd are abbreviations from the French for "continu à droite, limites a gauche" and "continu à gauche, limits à droite", respectively).*

Theorem A.1.5. *Let S, T be stopping times. Then the following are stopping times:*

1. $S \wedge T = min(S, T)$.
2. $S \vee T = max(S, T)$.
3. $S + T$.
4. αS, *where* $\alpha > 1$.

A.1.2 Martingales

Martingale theory plays a very interesting and useful role in studying stochastic processes. The word "martingale" has been known in 18th-century in France as a class of betting strategies. Recall we can only give a definition of a martingale if we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with an information flow, $\mathcal{F}_t$.

Definition A.1.6. ***Martingale**: A càdlàg process, $(X_t)_{t \in [0,T]}$, is said to be a martingale if X is adapted to $\mathcal{F}_t$, and $\mathbb{E}[| X_t |]$ is finite for any $t \in [0, T]$ and,*

$$\mathbb{E}[X_s \mid \mathcal{F}_t] = X_t \quad \forall s > t. \tag{A.1}$$

A Supermartingale is defined similarly only in that A.1 is replaced by

$$\mathbb{E}[X_s \mid \mathcal{F}_t] \le X_t \quad \forall s > t.$$

A Submartingale is defined with A.1 replaced by

$$\mathbb{E}[X_s \mid \mathcal{F}_t] \ge X_t \quad \forall s > t.$$

In other words, the best prediction of a martingale's future value is its present value. Note that martingales are only defined on $[0, \infty[$, that is, for finite t and not $t = \infty$. It is often

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possible to extend the definition to $t = \infty$. The fundamental property of a martingale is what it called "Doob's Optional Sampling Theorem".

Definition A.1.7. *A martingale X is said to be closed by a random variable Y if* $\mathbb{E}[| Y |] < \infty$ *and* $X_t = \mathbb{E}[Y \mid \mathcal{F}_t]$, $0 \leq t < \infty$.

Theorem A.1.8. *Let X be a right-continuous sub-martingale with* $\sup_{t\geq 0} \mathbb{E}[| X_t |] < \infty$. *Then* $X_\infty = \lim_{t\to\infty} X_t$ *exists almost surely and* $X_\infty \in L^1$.

Definition A.1.9. *A collection of random variables $(U_\alpha)_{\alpha\in A}$ is uniformly integrable if*

$$\lim_{n\to\infty} \sup_{\alpha\in A} \mathbb{E}\left[1_{|U_\alpha|>n} U_\alpha\right] = 0. \tag{A.2}$$

Theorem A.1.10. *:[56]* ***Doob's Optional Sampling Theorem****. Assume a right continuous martingale X, which is closed by a random variable X_∞, and two stopping times S and T such that $S \leq T$. Then X_S and X_T are integrable and*

$$X_S = \mathbb{E}[X_T \mid \mathcal{F}_S] \quad \textit{almost} \quad \textit{surely.}$$

Definition A.1.11. *[56] Suppose X is a stochastic process and T a random time. We say that X^T is a stopped process at T if $X_t^T = X_{t\wedge T}$.*

Note that if X is adapted and càdlàg, and if T is a stopping time, then

$$X_t^T = X_{t\wedge T} = X_t 1_{t<T} + X_T 1_{t\geq T},$$

is also adapted. Furthermore, a martingale stopped at a stopping time is also a martingale, and we have the following theorem.

Theorem A.1.12. *[56] Assume a uniformly integrable right continuous martingale X, and a stopping time T. Then $X^T = (X_{t\wedge T})_{0\leq t<\infty}$ is also a uniformly integrable right continuous martingale.*

Proof. [56]

Since X is right-continuous and T is a stopping time, X^T is right-continuous. Then by Theorem A.1.10, we have

$$\begin{aligned} X_{t\wedge T} &= \mathbb{E}[X_T \mid \mathcal{F}_{t\wedge T}] \\ &= \mathbb{E}\left[X_T 1_{\{T<t\}} + X_T 1_{\{T\geq t\}} \mid \mathcal{F}_{t\wedge T}\right] \\ &= X_T 1_{\{T<t\}} + \mathbb{E}\left[X_T 1_{\{T\geq t\}} \mid \mathcal{F}_{t\wedge T}\right]. \end{aligned}$$

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Since $H \in \mathcal{F}_t$, we have $H1_{\{T \geq t\}} \in \mathcal{F}_T$. Thus

$$= X_T 1_{\{T<t\}} + \mathbb{E}\left[X_T \mid \mathcal{F}_t\right] 1_{\{T \geq t\}}.$$

Therefore,

$$\begin{aligned} X_{t \wedge T} &= X_T 1_{\{T<t\}} + \mathbb{E}\left[X_T \mid \mathcal{F}_t\right] 1_{\{T \geq t\}} \\ &= \mathbb{E}\left[X_T \mid \mathcal{F}_t\right], \end{aligned}$$

since $X_T 1_{\{T<t\}}$ is $\mathcal{F}_t$ measurable. Thus, X^T is a uniformly integrable $\mathcal{F}_t$ martingale. □

The next theorem is an elementary result.

Theorem A.1.13. *[56](Jensen's Inequality) If φ is a convex function, and X is an integrable random variable, then for any σ-algebra $\mathcal{G}$, we have $\varphi(\mathbb{E}\left[X \mid \mathcal{G}\right]) \leq \mathbb{E}\left[\varphi(X) \mid \mathcal{G}\right]$.*

Definition A.1.14. *A martingale X with $X_0 = 0$ and $\mathbb{E}\left[X_t^2\right] < \infty$ for each $t > 0$, is called a square integrable martingale. If, in addition $\mathbb{E}\left[X_\infty^2\right] < \infty$, so X is called an L^2 martingale.*

Corollary A.1.15. *[56] Any L^2 martingale is also a square integrable martingale.*

Proof. [56] Let X be an L^2-martingale. Then $\mathbb{E}\left[X_\infty^2\right] < \infty$, by Theorem A.1.13, we have

$$\begin{aligned} \mathbb{E}\left[X_t^2\right] &= \mathbb{E}\left[\left(\mathbb{E}\left[X_\infty \mid \mathcal{F}_t\right]\right)^2\right] && \text{(A.3)} \\ &\leq \mathbb{E}\left[\mathbb{E}\left[X_\infty^2 \mid \mathcal{F}_t\right]\right] && \text{(A.4)} \\ &= \mathbb{E}\left[X_\infty^2\right] < \infty, && \text{(A.5)} \end{aligned}$$

which gives X_t as a square integrable martingale. □

A.1.3 Local Martingale

Definition A.1.16. *An adapted càdlàg X is a local martingale if there exists a sequence of increasing stopping times, T_n, with $\lim_{n \to +\infty} T_n = \infty$ almost surely, such that $X_{t \wedge T_n} 1_{T_n > 0}$ is a uniformly integrable martingale for each n. This stopping times sequence T_n are called an fundamental sequence.*

Definition A.1.17. *A stopping time T reduces a process M if M^T is a uniformly integrable martingale.*

Theorem A.1.18. *Let M, N be local martingales and let S and T be stopping times.*

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1. *If T reduces M and $S \leq T$ a.s, then S reduces M.*
2. *The sum $M + N$ is also a local martingale.*
3. *If S, T both reduce M, then $S \vee T$ also reduces M.*
4. *The processes M^T, $M^T 1_{T>0}$ are local martingales.*
5. *Let X be a càdlàg process and let T_n be a sequence of stopping times increasing to ∞ a.s such that for each n, $X^{T_n} 1_{T_n > 0}$ is a local martingale. Therefore, X is a local martingale.*

Proof. see [56] □

Example [56]: Suppose $(A_n)_{n\geq 1}$ is a measurable partition of Ω with $\mathbb{A}_n = 2^n$ for all n. Suppose $(Z_n)_{n\geq 1}$ is a sequence of random variables independent of the A_n and such that $\mathbb{P}[Z_n = -2^n] = \frac{1}{2}$. Define $\mathcal{F}_t = \sigma(A_n : n \geq 1)$ for all $0 \leq t < 1$ and $\mathcal{F}_t = \sigma(A_n, Z_n : n \geq 1)$ for all $t > 1$, completed with respect to $\mathbb{P}$. We assume $Y_n = \sum_{1\leq k\leq n} Z_k 1_{A_n}$ and $T_n = \infty 1_{\{\cup_{1\leq k\leq n} A_k\}}$. Let

$$X_t = \begin{cases} 0 & for\ t < 1 \\ Y_\infty & for\ t \geq 1, \end{cases}$$

if $(T_n)_{n\geq 1}$ is a sequence of stopping times for X, and

$$X_t^{T_n} = \begin{cases} 0 & for\ t < 1 \\ Y_n & for\ t \geq 1. \end{cases}$$

Since Y_n is bounded for each n, X^{T_n} is a uniformly integrable martingale. However, X is not a martingale because $X_1 = Y_\infty$ is not in L^1.

A.2 Semimartingales

Semimartingale processes are those processes that can be decomposed into a local martingale and an adapted finite variation process. Those processes are called "good integrators" and form a large class of processes with respect to which the Itô integral is well-defined.

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Definition A.2.1. *[56] Let $\mathbb{L}$ denote the space of adapted processes with càglàd paths and denote $b\mathbb{L}$ processes in $\mathbb{L}$ with bounded paths.*

Definition A.2.2. *[16] A process X is said to be simple predictable if X can be presented as follows:*

$$X_t = X_0 1_0(t) + \sum_{i=0}^{n} X_i 1_{]T_i,T_{i+1}]}(t), \tag{A.6}$$

where $0 = T_1 \leq \ldots \leq T_{n+1} < \infty$ is a finite sequence of stopping times, $X_i \in \mathcal{F}_{T_i}$ with $\mid X_i \mid < \infty$, almost surely, $0 \leq i \leq n$. (The collection of simple predictable processes denoted by $\mathbb{S}$),

Definition A.2.3. *The space $L^0 = L^0(\Omega, \mathcal{F}, \mathbb{P}) = \{X \in \mathcal{F} : X \text{ is finite-valued, almost surely}\}$ with the topology induced by convergence in probability under $\mathbb{P}$.*

Definition A.2.4. *[16] **Semimartingale** A adapted càdlàg process S is called a semimartingale if the stochastic integral of the simple predictable processes with respect to $\mathbb{S}$,*

$$\mathbb{S} \longrightarrow L^0$$

$$X = X_0 1_0 + \sum_{i=0}^{n} X_i 1_{]T_i,T_{i+1}]} \longrightarrow \int_0^T X dS = X_0 S_0 + \sum_{i=0}^{n} X_i (S_{T_{i+1}} - S_{T_i}),$$

verifies the following continuity property: for every X^n, $X \in \mathbb{S}$ if

$$\sup_{(t,\omega)\in[0,T]\times\Omega} \mid X_t^n(\omega) - X_t(\omega) \mid \longrightarrow 0 \ , \ n \to \infty \ \text{then} \ \int_0^T X^n dS \longrightarrow \int_0^T X dS.$$

If this continuity property does not hold, as [16] interpreted, and if a security is modelled by (S_t), then a little error in the composition of a strategy can cause a large change in the portfolio value. Therefore, it has been claimed that it is better, in models of continuous time trading, to use special stochastic processes, those that called semimartingales, or else, our model may gives results which are difficult to use or interpret.

Theorem A.2.5. *[56] i) The set of semimartingales is a vector space.*
ii) If a measure $\mathbb{Q}$ is equivalent to a measure $\mathbb{P}$, then a semimartigale under $\mathbb{P}$ is also a semimartingale under $\mathbb{Q}$.

Proof. i) The sum and scalar of multiples of semimartingales is a semimartingle (sums and scalars of continuous operation are continuous). ii) $\mathbb{P}$-probability convergence implies $\mathbb{Q}$-probability convergence. $\square$

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Semimartingale and Decomposable processes

Historically the stochastic integral was first proposed for Brownian motion, for continuous martingales, and then for square integrable martingales, and finally for decomposable processes, which are processes that can be presented as the sum of a locally square integrable local martingale and adapted, càdlàg processes with paths of finite variation on compacts.

Definition A.2.6. *An adapted càdlàg process A is a finite variation process (FV) if almost surely the paths of A are of finite variation on each compact interval of $[0, \infty]$. We write $\int_0^\infty \mid dA_s \mid$ or $\mid A \mid_\infty$ for the random variable that is, the total variation of the paths of A.*

Definition A.2.7. *An adapted process X is decomposable if there exist processes N, A such that*

$$X_T = X_0 + N_t + A_t,$$

with $N_0 = A_0 = 0$, where N a locally square integrable local martingale and A is a finite variation process.

Definition A.2.8. *An adapted càdlàg process Y is a classical semimartingale if there exist processes N, B with $N_0 = B_0 = 0$ such that*

$$Y_t = Y_0 + N_t + B_t, \tag{A.7}$$

where N is a local martingale and B is a finite variation process.

Theorem A.2.9. *[56] If X is an adapted càdlàg process, the following are equivalent:*

1. *X is a semimartingale.*
2. *X is decomposable.*
3. *Given $\beta > 0$, there exist M, A with $M_0 = A_0 = 0$, and M is a local martingale that has jumps bounded by β, A an finite variation process, such that $X_t = X_0 + M_t + A_t$.*
4. *X is a classical semimartingale.*

Definition A.2.10. *Suppose X is a semimartingale. If X has a decomposition $X_t = X_0 + M_t + A_t$ with $M_0 = A_0 = 0$, M a local martingale, A of finite variation, and A is predictable, then X is said to be a special semimartingale.*

Theorem A.2.11. *Let X be a semimartingale. If X has a decomposition $X_t = X_0 + M_t + A_t$ with M as a local martingale and A a predictable measurable finite variation process, $M_0 = A_0 = 0$, then such a decomposition is unique.*

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Theorem A.2.12. *[56] (Bichteler-Dellacherie Theorem) An adapted, càdlàg process X is a semimartingale if, and only if, it is a classical semimartingale. That is, X is a semimartingale if, and only if, can be written as $X = M + A$, where M is a local martingale and A is of finite variation.*

Let us recall some definitions that are needed in the next section.

Definition A.2.13. *The predictable σ-algebra $\mathcal{P}$ on $\mathbb{R}_+ \times \Omega$ is the smallest σ-algebra making all processes in $\mathbb{L}$ measurable. That is, $\mathcal{P} = \sigma\{H : H \in \mathbb{L}\}$.*

Definition A.2.14. *A stopping time T is predictable if there exists a sequence of stopping times $(T_n)_{n\geq 1}$ such that T_n is increasing, $T_n < T$ on $\{T > 0\}$, all n, and $\lim_{n\to\infty} T_n = T$, almost surely.*

Definition A.2.15. *Let X be a special semimartingale with canonical decomposition $X = N + A$. The $\mathcal{H}^2$ norm of X is defined as*

$$\| X \|_{\mathcal{H}^2} = \| [N,N]_\infty^{1/2} \|_{L^2} + \| \int_0^\infty | dA_s | \|_{L^2} .$$

The space of semimartingale $\mathcal{H}^2$ consists of all special semimartingales with finite $\mathcal{H}^2$ norm.

Theorem A.2.16. *The space of an $\mathcal{H}^2$ semimartingale is a Banach space.*

A.3 Examples of Semimartingales

We will see in this section that many common processes are semimartingales. For example, the Poisson process, Brownian motion, and the more general processes as Lévy processes are semimartingales.

Theorem A.3.1. *[56] An adapted process with càdlàg paths of finite variation on compacts (respectively, of finite total variation) is a semimartingale (respectively, a total semimartingale),*

Proof. [56] It suffices to observe that $| \int_0^T HdS | \leq \| H \|_u \int_0^\infty | dS_s |$, with $\int_0^\infty | dS_s |$ denotes the Lebesgue-Stieltjes total variation and $\| H \|_u = sup_{(t,\omega)} | H(t,\omega) |$. □

Theorem A.3.2. *[56] Each L^2 martingale with càdlàg paths is a semimartingale.*

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Proof. [56] Assume X as an L^2 martingale with $X_0 = 0$, and $H \in \mathbb{S}$ using Doob's optional sampling Theorem and the L^2 orthogonality of the increments of L^2 martingale, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int HdX\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=0}^{n} H_i(X_{T_{i+1}} - X_{T_i})\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{n} H_i^2(X_{T_{i+1}} - X_{T_i})^2\right] \quad X \text{ is a Martingale} \\
&\leq \| H \|_u^2 \, \mathbb{E}\left[\sum_{i=0}^{n} \left(X_{T_{i+1}} - X_{T_i}\right)^2\right] \\
&= \| H \|_u^2 \, \mathbb{E}\left[X_{T_{n+1}}^2\right] \, X \text{ is a Martingale} \\
&\leq \| H \|_u^2 \, \mathbb{E}\left[X_\infty\right] \text{ Jensen's inequality .}
\end{aligned}$$

□

Corollary A.3.3. *[56] The Wiener process (that is Brownian motion) is a semimartingale.*

Proof. [56] The Wiener process B_t is a martingale with continuous paths if B_0 is integrable. Then it is a continuous local martingale. □

Theorem A.3.4. *[56] A decomposable process is a semimartingale.*

Proof. [56] If $S_t = S_0 + M_t + A_t$ a decomposition of S, then M is a semimartingale. And A is a semimartingale by Theorem A.3.1. Since semimartingales form a vector space, the summation of two semimartigales is a semimartingale. □

Corollary A.3.5. *A Lévy process is a semimartingale.*

Proof. We know that a Lévy process is decomposable, see [56]. Then, by Theorem A.3.4, we have the result. □

A.4 Stochastic Integrals and Semimartingales

In the previous section we defined semimartingales as adapted, càdlàg processes that were assumed to be "good integrators" on the *simple predictable* processes. A particularly nice class of adapted processes for our purposes is the class of adapted processes with left-continuous paths that have right limits.

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Definition A.4.1. *[56] A sequence of processes $(H^n)_{n\geq 1}$ converges to a process H uniformly on compacts in probability (ucp) if, for each $t > 0$, $\sup_{0\leq s\leq t} \mid H^n_s - H_s \mid$ converges to 0 in probability.*

We write $\mathbb{D}_{ucp}$, $\mathbb{L}_{ucp}$, $\mathbb{S}_{ucp}$ to denote the respective spaces endowed with the ucp topology. Note that $\mathbb{D}_{ucp}$ is a metrizable space and there exists a compatible metric such that $\mathbb{D}_{ucp}$ is complete.

Theorem A.4.2. *[56] The space $\mathbb{S}$ is dense in $\mathbb{L}$ under the ucp topology.*

Definition A.4.3. *[56] For $H \in \mathbb{S}$, and X a càdlàg process, a linear mapping is defined as follows:*

$$J_X : \mathbb{S} \longrightarrow \mathbb{D} \text{ by } J_X(H) = H_0X_0 + \sum_{i=1}^{n} H_i(X^{T_{i+1}} - X^{T_i}),$$

for $H \in \mathbb{S}$ with the standard representation.

Theorem A.4.4. *[56] Let X be a semimartingale. Then the mapping $J_X : \mathbb{S}_{ucp} \longrightarrow \mathbb{D}_{ucp}$ is continuous.*

This operator is continuous on $\mathbb{S}_{ucp}$ but also $\mathbb{S}_{ucp}$ is dense in $\mathbb{L}_{ucp}$. Hence, the linear integration operator J_X can be extended from $\mathbb{S}$ to $\mathbb{L}$ by continuity because $\mathbb{D}_{ucp}$ is a complete metric space.

Definition A.4.5. *Let X be a semimartingale. We can obtain the continuous linear mapping $J_X : \mathbb{L}_{ucp} \longrightarrow \mathbb{D}_{ucp}$ as the extension of $J_X : \mathbb{S} \longrightarrow \mathbb{D}$ and it is called the stochastic integral. We have the following notations for X, a semimartingale, and $H \in \mathbb{L}$:*

$$J_X(H) = H \bullet X = \int H_s dX_s.$$

From now, X will denotes a semimartingale and H will denotes an element of $\mathbb{L}$.

Theorem A.4.6. *[56] The stochastic integral process $Y = H \bullet X$ is itself a semimartingale, and for $G \in \mathbb{L}$ we have*

$$\int G_s dY_s = \int G_s H_s dX_s.$$

Theorem A.4.6 shows that the property of being a semimartingale is preserved by stochastic integration. Also there is a result stating that if the semimartingale X is an finite variation process, then the stochastic integral coincides with the Lebesgue-Stieltjes integral, and

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by the theory of Lebesgue-Stieltjes integration, we can say that the stochastic integral is an finite variation process as well. That is, the property of being a finite variation process is preserved by stochastic integration for integrands in $\mathbb{L}$. A question may well be asked if other properties are preserved by stochastic integration, in particular, whether the stochastic integrals for martingales and local martingales still martingales are indeed preserved by stochastic integration, but we are not yet able to easily prove it [56]. Instead it has been shown that locally-square integrable local martingales are preserved by stochastic integration for integrands in $\mathbb{L}$.

Theorem A.4.7. *[56] If X a locally-square integrable martingale, and $H \in \mathbb{L}$, then the stochastic integral $H \bullet X$ is also a locally-square integrable local martingale.*

Proof. see [56]. □

Theorem A.4.8. *[56] Let X be a semimartingale. Suppose that $(\sigma_n)_{n\geq 1}$ is a sequence of random partition, $\sigma_n : 0 = T_0^n \leq T_1^n \leq \cdots \leq T_{k_n}^n$, where the T_k^n are stopping times such that $\lim_{n\to\infty} T_{k_n}^n = \infty$, almost surely, and $\sup_k \mid T_{k+1}^n - T_k^n \mid$ converges to 0, almost surely. If $H \in \mathbb{L}$, then*

$$\sum_k H_{T_k^n}\left(X^{T_{k+1}^n} - X^{T_k^n}\right) \to \int H_- dX.$$

Example: Let $M_t = N_t - \lambda t$ be a compensated Poisson process and assume $H = 1_{[0,T_1)}$, with T_1 as the first jump time of the Poisson process. Then $\int_0^t H_s dM_s = -\lambda(t \wedge T_1)$, which is not a local martingale. That is why our integrand has to be from $\mathbb{L}$.

A.5 Stochastic Integrals with Respect to Predictable Processes

In the previous section, we discussed stochastic integrals with respect to elements of $\mathbb{L}$. This set of adapted processes that are left continuous with right limits is enough for some proofs in stochastic integration such as the Itô's formula. However, this set is not rich enough for some other proves such as martingale representation theorems, and also many formulas with semimartingales local time. Thus, we need to define a large space of integrands for these uses.

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Definition A.5.1. *The predictable σ-algebra $\mathcal{P}$ on $\mathbb{R}_+ \times \Omega$ is the smallest σ-algebra making all processes in $\mathbb{L}$ measurable. That is, $\mathcal{P} = \sigma\{H : H \in \mathbb{L}\}$. Then $b\mathcal{P}$ will be the set of bounded processes that are $\mathcal{P}$-measurable.*

Definition A.5.2. *Let $X \in \mathcal{H}^2$ with $X = N + A$ as its canonical decomposition, suppose $H,\ J \in b\mathcal{P}$. We define $d_X(H, J)$ by*

$$d_X(H,J) = \parallel \left(\int_0^\infty (H_s - J - s)^2 d[N,N]_s\right)^{1/2} \parallel_{L^2} + \parallel \int_0^\infty \mid H_s - J_s \mid\mid dA_s \mid\parallel_{L^2}. \quad (A.8)$$

Theorem A.5.3. *[56] Let $X \in \mathcal{H}^2$, the space $b\mathbb{L}$ is dense in $b\mathcal{P}$ under $d_X(.,.)$.*

Proof. [56] Define

$$\mathcal{L} = \{H \in b\mathcal{P} : \text{ for all } \varepsilon > 0 \text{ there is } J \in b\mathbb{L} \text{ such that } d_X(H,J) < \varepsilon\}, \quad (A.9)$$

we have $b\mathbb{L}$ as a multiplicative class and $\mathcal{L}$ as a monotone vector space containing $b\mathbb{L}$. If $H^n \in \mathcal{L}$ and $H^n \uparrow H$ with H-bounded, then $H \in b\mathcal{P}$. By the dominated converge theorem, $d_X(H^n, H) \to 0$. Let $\varepsilon > 0$ and pick n_0 such that $d_X(H^{n_0}, H) < \varepsilon/2$, and pick $J \in b\mathbb{L}$ such that $d_X(H^{n_0}, J) < \varepsilon/2$. Then $d_X(H, J) < \varepsilon$, so $H \in \mathcal{L}$. Therefore, $\mathcal{L}$ is a monotone vector space. By Monotone class theorem, $b\mathcal{P} = b\sigma(b\mathbb{L}) \subseteq \mathcal{L}$. $\square$

Theorem A.5.4. *[56] Let $X \in \mathcal{H}^2$ and $H \in b\mathcal{P}$. Suppose $H^n \in b\mathbb{L}$ are two sequences such that $\lim_n d_X(H^n, H) = \lim_m d_X(J^m, H) = 0$. Then $H^n \bullet X$ and $J^m \bullet X$ tend to the same limit in $\mathcal{H}^2$.*

Definition A.5.5. *Let X be a semimartingale in $\mathcal{H}^2$ and let $H \in b\mathcal{P}$. Let $H^n \in b\mathbb{L}$ be such that $\lim\limits_{n\to\infty} d_X(H^n, H) = 0$. The stochastic integral $H \bullet X$ is the unique semimartingale $Y \in \mathcal{H}^2$ such that $\lim\limits_{n\to\infty} H^n \bullet X = Y$. We write $H \bullet X = (\int_0^t H_s dX_s)_{t\geq 0}$.*

Theorem A.5.6. *[56] Let X be a semimartingale in $\mathcal{H}^2$. Then*

$$\mathbb{E}\left[(sup_t \mid X_t \mid)^2\right] \leq 8 \parallel X \parallel^2_{\mathcal{H}^2}.$$

Proof. Suppose that $X = \overline{N} + \overline{A}$, and $X^* = sup_t \mid X_t \mid$. Then

$$X^* \leq \overline{N^*} + \int_0^\infty \mid \overline{dA}_s \mid. \quad (A.10)$$

By Doob's maximal inequality

$$\mathbb{E}\left[(N^*)^2\right] \leq 4\mathbb{E}\left[\overline{N}^2_\infty\right] = 4\mathbb{E}\left[[\overline{N}, \overline{N}]_\infty\right]. \quad (A.11)$$

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Using the fact that $(X^*)^2 \leq 2(\overline{N}^*_\infty)^2 + 2(|\ \overline{A}\ |_\infty)^2$, we have

$$\mathbb{E}\left[(X^*)^2\right] \leq 2\mathbb{E}\left[(\overline{N}^*)^2\right] + 2\mathbb{E}\left[\int_0^\infty (|\ \overline{dA}_s\ |)^2\right] \tag{A.12}$$

$$\leq 8\mathbb{E}\left[[\overline{N},\overline{N}]_\infty\right] + 2 \parallel \int_0^\infty |\ \overline{dA}_s\ | \|_{L^2}^2 \tag{A.13}$$

$$\leq 8 \parallel X \parallel_{\mathcal{H}^2}^2 . \tag{A.14}$$

□

Corollary A.5.7. *If $X^n \to X$ in $\mathcal{H}^2$ then there is a subsequent $(X^{n_k})_{k\geq 1}$ such that $(X^{n_k} - X)^*_\infty$, almost surely.*

Theorem A.5.8. *[56] Let X, $Y \in \mathcal{H}^2$ and $H, K \in b\mathcal{P}$.*
i) $(H + K) \bullet X = H \bullet X + K \bullet X$.
ii) $H \bullet (X + Y) = H \bullet X + H \bullet Y$.
iii) If T is a stopping time then $(H \bullet X)^T = (H \bullet 1_{[0,T]}) \bullet X = H \bullet (X^T)$.
iv) $\Delta(H \bullet X) = H\delta X$.
v) If T is a stopping time then $H \bullet (X^{T^-}) = (H \bullet X)^{T^-}$.
vi) If X has finite variation then $H \bullet X$ coincides with the Lebesgue-Stieltjes integral.
vii) $H \bullet (K \bullet X) = (HK) \bullet X$.
viii) If X is a local martingale the $H \bullet X$ is an L^2-martingale.
ix) $[H \bullet X, K \bullet Y] = (HK) \bullet [X, Y]$.

Definition A.5.9. *Let $X \in \mathcal{H}^2$ with the canonical decomposition $X = \overline{N} + \overline{A}$. $H \in \mathcal{P}$ is said to be $(\mathcal{H}^2, X)$ integrable if*

$$\mathbb{E}\left[\int_0^\infty H_s^2 d\left[\overline{N},\overline{N}\right]_s\right] + \mathbb{E}\left[\left(\int_0^\infty |\ H_s\ |\ |\ \overline{dA}_s\ |\right)^2\right] < \infty. \tag{A.15}$$

Theorem A.5.10. *[56] Let X be a semimartingale and let $H \in \mathcal{P}$ be $(\mathcal{H}^2, X)$ integrable. Let $H^n = H1_{\{|H|\leq n\}} \in b\mathcal{P}$. Then $H^n \bullet X$ is a Cauchy sequence in $\mathcal{H}^2$.*

Proof. $H^n = H1_{|H|\leq n}$, $H^n \in b\mathcal{P}$, so $H^n \bullet X$ is well defined.

$$\parallel H^n \bullet X - H^m \bullet X \parallel_{\mathcal{H}^2} = d_X(H^n, H^m) \tag{A.16}$$

$$= \parallel ((H^n \bullet H^m)^2 \bullet [N])^{1/2} \parallel_{L^2} \tag{A.17}$$

$$+ \parallel |\ (H^n - H^m)^2 \bullet A\ |_\infty \|_{L^2} \to 0 \tag{A.18}$$

by the dominated convergence theorem. □

Theorem A.5.11. *Optional Decomposition Theorem [48]; let $(V_t)_{t\geq 0}$ be a positive process. Then V is a supermartingale if, and only if, there exists an X-integrable predictable process*

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$H = (H^i)_{1\leq i\leq d}$ and an adapted increasing process C such that

$$V_t = V_0 + (H \bullet X)_t - C_t, \ t \geq 0. \tag{A.19}$$

Proof. [48] □

Theorem A.5.12. *[21] If M is a local martingale and if X is an admissible integrand for M, then $X \bullet M$ is a local martingale. Therefore, $X \bullet M$ is a supermartingale.*

Proof. [25] □

A.6 Quadratic Variation of Semimartingales

The quadratic variation process (or the bracket process) of a semimartingale, is a simple object that plays a fundamental role in stochastic analysis.

Definition A.6.1. *[56] If X, Y be semimartingales, then the quadratic variation process of X, denoted $[X,X] = ([X,X]_t)_{t\geq 0}$, is defined by*

$$[X,X] = X^2 - 2\int X_- dX,$$

with $X_{0^-} = 0$. The quadratic covariation of X, Y, also known by the name of the bracket process of X, Y, is defined by

$$[X,Y] = XY - \int X_- dY - \int Y_- dX.$$

The operator $(X,Y) \to [X,Y]$ is bilinear and symmetric. Therefore the following polarization identity is defined:

$$[X,Y] = \frac{1}{2}\left([X+Y, X+Y] - [X,X] - [Y,Y]\right).$$

Theorem A.6.2. *[56] The quadratic variation process of X is a càdlàg increasing, adapted process. Also we have, $[X,X]_0 = X_0^2$ and $\Delta[X,X] = (\Delta X)^2$. If σ_n is a sequence of random partitions tending to the identity, we have the following:*

$$X_0^2 + \sum_i (X_{T_{i+1}^n} - X_{T_i^n})^2 \to [X,X], \tag{A.20}$$

with convergence in ucp, where σ_n is the sequence $0 = T_0^n \leq T_1^n \leq \cdots \leq T_i^n \leq \cdots \leq T_{k_n}^n$, and T_i^n are stopping times.

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Proof. [56] Recall that $X_{0^-} = 0$, so $\int_0^0 X_{s^-} dX_s = 0$ and $[X]_0 = (X_0)^2$. For any $t > 0$,

$$\begin{aligned}(\Delta X_t)^2 = (X_t - X_{t^-})^2 &= X_t^2 + X_{t^-}^2 - 2X_t X_{t^-} \\ &= X_t^2 - X_{t^-}^2 - 2X_{t^-}\Delta X_t \\ &= (\Delta X^2)_t - 2\Delta(X_- \bullet X)_t \\ &= \Delta(X^2 - 2(X_- \bullet X)_t)_t = \Delta[X, X]_t.\end{aligned}$$

Now let us prove A.20, without loss of generality we replace $\tilde{X} = X - X_0$ with $X_0 = 0$. Let $R_n = \sup_i T_i^n < \infty$ and $\lim_n R_n = \infty$, almost surely. Thus

$$(X^2)^{R_n} = \sum_i \left[(X^2)^{T_{i+1}^n} - (X^2)^{T_i^n}\right] \to X^2, \tag{A.21}$$

with convergence in ucp. On the other hand, we have by Theorem A.4.8

$$\sum_i X_{T_i^n}(X^{T_{i+1}^n} - X^{T_i^n}) \to \int X_- dX, \tag{A.22}$$

with convergence in ucp. Thus

$$\sum_i \left((X^2)^{T_{i+1}^n} - (X^2)^{T_i^n} - 2X^{T_i^n}(X^{T_{i+1}^n} - X^{T_i^n})\right) = \sum_i \left(X^{T_{i+1}^n} - X^{T_i^n}\right)^2 \tag{A.23}$$

$$\to X^2 - 2\int X_- dX = [X, X], \tag{A.24}$$

with convergence in ucp. Also, $[X, X]$ is non-decreasing, since the approximation sum in A.21 includes more terms that are non-negatives. □

Remark A.6.3. *[56] The bracket process $[X, Y]$ of two semimartingales has paths of finite variation on compacts, and it is also a semimartingale. We also have $\Delta[X, Y] = \Delta X \Delta Y$. Since the process $[X, X]$ is paths that is non-decreasing and right-continuous, and $\Delta[X, X]_t = (\Delta X_t)^2$ for all $t \geq 0$ (with $X_{0^-} = 0$), we can break $[X, Y]$ path-by-path into its continuous part and its pure jump part.*

Definition A.6.4. *For a semimartingale X, the process $[X, X]^c$ denotes the path-by-path continuous part of $[X, X]$. We can then write*

$$[X, X]_t = [X, X]_t^c + X_0^2 + \sum_{0<s\leq t} (\Delta X_s)^2 = [X, X]_t^c + \sum_{0<s\leq t} (\Delta X_s)^2.$$

Observe that $[X, X]_0^c = 0$. Analogously, $[X, Y]^c$ denotes the path-by-path continuous part of $[X, Y]$.

Theorem A.6.5. *[56] properties of the quadratic variation process. Let X and Y be two*

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semimartingales, and let H, $K \in \mathbb{L}$. Then

$$\left[\int HdX, \int KdY\right]_t = \int_0^t H_s K_s d\,[X,Y]_s\,,$$

and, in particular,

$$\left[\int HdX, \int HdX\right]_t = \int_0^t H_s^2 d\,[X,X]_s\,.$$

Definition A.6.6. *Let X be a semimartingale. The continuous part of quadratic variation, $[X,X]^c$, is defined by*

$$[X,X]_t = [X,X]_t^c + X_0^2 + \sum_{0<s\leq t} (\Delta X_s)^2.$$

We say that X is a quadratic pure-jump semimartingale if we have $[X,X]^c = 0$.

Theorem A.6.7. *[56] If X is an adapted, càdlàg process that have a finite variation path, we say that X is a quadratic pure-jump semimartingale.*

Proof. [56] The Lebesgue-Stieltjes integration by part formula is as follows:

$$X^2 = \int X_- dX + \int XdX,$$

and, by definition of $[X,X]$, we have

$$X^2 = 2\int X_- dX + [X,X].$$

And

$$\int XdX = \int (X_- + \Delta X)dX = \int X_- dX + \sum (\Delta X)^2.$$

Then $[X,X] = \sum(\Delta X)^2$. Therefore X is a quadrartic pure-jump semimartingale. □

Examples of Quadratic Variation processes

Example 1: Suppose $B_t = \sigma\omega_t$ where ω is a standard Wiener process. Then $[B,B]_t = \sigma^2 t$. We see that here the quadratic variation process is equal to the variance of B_t. This reminds us of the term "realized volatility" that is used to denote the empirically calculated quadratic variation of returns. The Brownian motion case is unique, in general, this is a stochastic process.

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Example 2: Let N_t be a counting process with jump times T_i, and jump sizes Z_i, where $Z_i \in \mathcal{F}_t$. We have

$$X_t = \sum_{i=1}^{\infty} Z_i 1_{\{i \leq N_t\}},$$

which implies

$$[X, X]_t = \sum_{i=1}^{\infty} \mid Z_i \mid^2 1_{\{i \leq N_t\}} = \sum_{0 \leq s \leq t} \mid \Delta X_s \mid^2 .$$

A.7 Itô's Formula

We know that, if $f : \mathbb{R} \longrightarrow \mathbb{R}$, $g : [0, T] \longrightarrow \mathbb{R}$ are smooth (C^1) functions, then the change of variables formula for smooth functions is as follows:

$$f(g(t)) - f(g(0)) = \int_0^t f^{'}(g(s))g^{'}(s)ds \tag{A.25}$$

$$= \int_0^t f^{'}(g(s))dg(s). \tag{A.26}$$

Applying this to $f(x) = x^2$, we get

$$g(t)^2 - g(0)^2 = 2 \int_0^2 g(s)dg(s).$$

However, when X is a semimartingale we have that

$$X_t^2 - X_0^2 = 2 \int_0^t X_{s^-} dX_s + [X, X]_t ,$$

where $[X, X] \neq 0$. Therefore, stochastic integrals with respect to semimartingales, seem to not be following the usual change of variable formulae for smooth functions. The aim is to give formulae analogous to A.25 for $f(t, X_t)$ when f is a smooth function and X is a semimartingale with jumps.

The Itô's formula for Brownian integrals is well-known, that if f is a C^2 function and $X_t = \int_0^t \sigma_s dW_s$, then

$$f(X_t) = f(0) + \int_0^t f^{'}(X_s)\sigma_s dW_s + \int_0^t \frac{1}{2}\sigma_s^2 f^{"}(X_s)ds.$$

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If X is a Lévy process, then $Y_t = f(t, X_t)$ is not a Lévy process anymore: However, it can be expressed in terms of stochastic integrals so it is still a semimartingale. Therefore, if (Y_t) is a random process driven by the Lévy process (X_t), in order to consider quantities like $f(t, X_t)$, we need to have a change of variable formulae for discontinuous semimartingales such as (Y_t).

Let (X) be a semimartingale with a quadratic variation process $[X, X]$. Since the quadratic variation is an increasing process, it can be split into both jump part and a continuous part. The continuous part will be written as $[X, X]^c$.

Theorem A.7.1. *[56] Itô's formula for semimartingales. Let $(X_t)_{t\geq 0}$ be a semimartingale. For any $C^{1,2}$ function, $f : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$.*

$$f(X_t) - f(X_0) = \int_0^t \frac{\partial f}{\partial x}(X_{s^-})dX_s \tag{A.27}$$

$$+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial^2 x}(X_{s^-})d\,[X, X]_s^c \tag{A.28}$$

$$+ \sum_{0\leq s\leq t}\left[f(X_s) - f(X_{s^-}) - \Delta X_s \frac{\partial f}{\partial x}(X_{s^-})\right]. \tag{A.29}$$

Proof. This theorem was proved in two cases, when X is a continuous process and when it is a general process. The proof make use of the known Taylor's theorem which says:

$$f(y) = f(x) + f'(x)(y - x) + \frac{1}{2}f''(x)(y - x)^2 + R(x, y), \tag{A.30}$$

where $R(x, y) \leq r \mid y - x \mid (y - x)^2$, with $r : \mathbb{R}_+ \to \mathbb{R}_+$ as an increasing function and $\lim_{s\to 0} r(s) = 0$. First, we consider X as a continuous semimartingale. For a fix $t > 0$, consider a sequence of partition, of the interval $[0, t]$: $\sigma_n = (0 = T_0^n \leq T_1^n \leq \cdots \leq T_{k_n}^n = t)$. Then we have

$$f(X_t) - f(X_0) = \sum_{i=0}^{k_n}\left[f(X_{T_{i+1}^n}) - f(X_{T_i^n})\right] \tag{A.31}$$

$$= \sum_i f'(X_{T_i^n})(X_{T_{i+1}^n} - X_{T_i^n}) + \frac{1}{2}\sum_i f''(X_{T_i^n})(X_{T_{i+1}^n} - X_{T_{i+1}^n})^2 \tag{A.32}$$

$$+ \sum_i R(X_{T_i^n}, X_{T_{i+1}^n}). \tag{A.33}$$

The two sums in the Equation A.32 converge in probability, respectively, to $\int_0^t f'(X_{s^-})dX_s$

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and $\frac{1}{2}\int_0^t f''(X_s)d[X,X]_s$ (Theorem A.4.8 and Theorem A.6.2). For the last sum we have

$$|\sum_i R(X_{T_i^n}, X_{T_i^n})| \leq \sup_i r \mid X_{T_{i+1}^n} - X_{T_i^n} \mid \sum_i (X_{T_{i+1}^n} - X_{T_i^n})^2, \tag{A.34}$$

we have $\sum_i (X_{T_{i+1}^n} - X_{T_i^n})^2 \to [X,X]_t$ in probability, also the path $s \to X_s(\omega)$ is uniformly continuous on $[0,t]$, and since $\lim\limits_{n\to+\infty} \sup_i \mid T_{i+1}^n - T_i^n \mid = 0$, we conclude that $\lim\limits_{n\to+\infty} \sum_i R(X_{T_i^n}, X_{T_i^n}) = 0$. Thus Equation A.27 holds in the case of X continuous. The continuity assumption eliminates the dependence of the exceptional set on t.

Now the general case needs a closer analysis. For $t > 0$, $\sum_{0<s\leq t}(\Delta X_s)^2 \leq [X,X]_t < \infty$ almost surely, then $\sum_{0<s\leq t}(\Delta X_s)^2$ converges, so we can partition the jumps of X to A a subset of $\mathbb{R}_+ \times \Omega$ such that $\sum_{s\in A}(\Delta X_s)^2 \leq \varepsilon^2$, for a given $\varepsilon > 0$, and $B = \{(s,\omega) : (\Delta X_s)^2 > 0, (s,\omega) \textit{ not in } A\}$. Then the Equation can be rewritten as follows:

$$f(X_t) - f(X_0) = \sum_{i=0}^{k_n} \left[f(X_{T_{i+1}^n}) - f(X_{T_i^n}) \right] \tag{A.35}$$

$$= \sum_i f'(X_{T_i^n})(X_{T_{i+1}^n} - X_{T_i^n}) + \frac{1}{2}\sum_i f''(X_{T_i^n})(X_{T_{i+1}^n} - X_{T_i^n})^2 \tag{A.36}$$

$$+ \sum_i 1_{\{B\cap]T_i^n, T_{i+1}^n] \neq \emptyset\}} [f(X_{T_{i+1}^n}) - f(X_{T_i^n}) - f'(X_{T_i^n})(X_{T_{i+1}^n} - X_{T_i^n}) \tag{A.37}$$

$$- \frac{1}{2} f''(X_{T_{i+1}^n})(X_{T_{i+1}^n} - X_{T_i^n})^2] \tag{A.38}$$

$$+ \sum_i 1_{\{B\cap]T_i^n, T_{i+1}^n] = \emptyset\}} R(X_{T_i^n}, X_{T_{i+1}^n}). \tag{A.39}$$

As the continuous case, the two sums in Equation A.36 converge respectively to $\int_0^t f'(X_{s^-})dX_s$ and $\frac{1}{2}\int_0^t f''(X_s)d[X,X]_s$. The sum in the Equation A.37 converge to

$$\sum_{s\in B, |\Delta X_s|>0} \left[f(X_s) - f(X_{s^-}) - f(X_{s^-})\Delta X_s - \frac{1}{2} f''(X_{s^-})(\Delta X_s)^2 \right]. \tag{A.40}$$

Now let us prove that the sum in the Equation A.39 converges to 0. We have $\lim\limits_n \sum_i \mid T_{i+1}^n - T_i^n \mid = 0$ for n large enough, $\mid X_{T_i^n} - X_{T_{i+1}^n} \mid \leq 2\varepsilon$ when $B \cap]T_i^n, T_{i+1}^n] = \emptyset$. On the other hand, $R(x,y) \leq r \mid y - x \mid (y-x)^2$. Then

$$\sum_i 1_{\{B\cap]T_i^n, T_{i+1}^n] = \emptyset\}} R(X_{T_i^n}, X_{T_{i+1}^n}) \leq r(2\varepsilon) \sum_i (X_{T_i^n} - X_{T_{i+1}^n})^2 \tag{A.41}$$

$$\leq r(2\varepsilon)[X,X]_t, \tag{A.42}$$

as $\varepsilon \to 0$, $r(2\varepsilon) \to 0$, then Equation A.39 converges to 0 as well. Moreover, A.40 converges

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as $\varepsilon \to 0$ to

$$\sum_{0<s\leq t} \left[f(X_s) - f(X_{s^-}) - f(X_{s^-})\Delta X_s - \frac{1}{2} f''(X_{s^-})(\Delta X_s)^2 \right]. \tag{A.43}$$

To finish the proof we have to show that A.43 is absolutely convergent.

Let $V_k = \inf\{t > 0 :\mid X_k \mid\geq k\}$, $X_0 = 0$, the Equation A.27 holds for $X1_{[0,T_k[}$, since it is a product of semimartingales which is a semimartingale, then it suffices to consider semimartingales on the interval $[-k, k]$. Restricting f on the interval $[-k, k]$, we have $\mid f(y) - f(x) - (y-x)f'(y-x) \mid\leq c(y-x)^2$. Then we have:

$$\sum_{0<s\leq t} [f(X_s) - f(X_{s^-}) - f(X_{s^-})\Delta X_s] \leq c_1 \sum_{0<s\leq t} (\Delta X_s)^2 \tag{A.44}$$

$$\leq c_1 [X, X]_t < \infty. \tag{A.45}$$

and

$$\sum_{0<s\leq t} [f''(X_{s^-})(\Delta X_s)^2] \leq c_2 \sum_{0<s\leq t} (\Delta X_s)^2 \tag{A.46}$$

$$\leq c_2 [X, X]_t \leq \infty, \tag{A.47}$$

which proves that A.43 is absolutely convergent. □

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Appendix B

Arbitrage Pricing Theory

B.1 Tools Needed to Prove The First Fundamental Theorem of Asset Pricing in General Space

Lemma B.1.1. *[19] Let ψ_n be a sequence of non-negative measurable functions. There exists a sequence $\varphi_n \in convex(\psi_n, \psi_{n+1}, \cdots)$, such that $\varphi_n \to \varphi$, almost surely.*

1. *If $convex(\psi_n, \psi_{n+1}, \cdots)$ is bounded in L^0, then $\varphi < \infty$, almost surely.*
2. *If $\mathbb{P}[\psi_n \geq \alpha] \geq \alpha$ for all n, then $\mathbb{P}[\varphi > 0] > 0$.*

Proof. [19] The first step in the proof is to show that there exists φ_n converge to φ in probability. Since we are working on the compact metric space $[0, \infty]$ which is complete, we will use the well-known result that every Cauchy sequence converges in a complete space. We recall that a sequence g_n in $[0, \infty]$ is a Cauchy sequence iff for each ε, there is δ so that for all $n, m > \delta$ we have $\mid g_n - g_m \mid \leq \varepsilon$ or $\min(g_n, g_m) \geq \varepsilon^{-1}$. To choose such a sequence φ_n, Delbaen and Schachermayer [19] suggest the following:

- Assume a function $u : \mathbb{R}_+ \cup \{+\infty\} \to [0, 1]$ such that $u(x) = 1 - e^{-x}$.
- Define a sequence $s_n = \sup\{\mathbb{E}[u(\varphi)] \mid \varphi \in conv\{\psi_n, \psi_{n+1}, \cdots\}\}$.
- Choose $\varphi_n \in conv\{\psi_n, \psi_{n+1}, \cdots\}$ such that $\mathbb{E}[u(\varphi)] > s_n - \frac{1}{n}$.

The sequence s_n decreases to $s_0 \geq 0$ and $\lim\limits_{n\to\infty} \mathbb{E}[u(\varphi_n)] = s_0$. The function u is concave which means $\forall x, y \in [0, \infty]$ and $\forall\ 0 \leq \alpha \leq 1$, and we have $u(\alpha x + (1 - \alpha)y) \geq \alpha u(x) +$

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$(1-\alpha)u(y)$. The function u has also the following property that will be needed, for $\alpha > 0$ there is $\beta > 0$, so that $\mid x - y \mid > 0$ and $\min(x, y) \leq \frac{1}{\alpha}$, implies $u(\frac{x+y}{2}) > \frac{1}{2}(u(x) + u(y)) + \beta$.

To proof that φ_n is a Cauchy sequence in probability, we have to show the following:

$$\lim_{n,m\to\infty} \mathbb{P}\left[\mid \varphi_n - \varphi_m \mid > \alpha \text{ and } \min(\varphi_n, \varphi_m) \leq \frac{1}{\alpha}\right] = 0$$

.

Using the property of u as mentioned above, we have that for $\alpha > 0$, there exists β such that

$$u(\frac{1}{2}(\varphi_n + \varphi_m)) \geq \frac{1}{2}(u(\varphi_n) + \varphi_m) + \beta I_{\{|\varphi_n - \varphi_m| > \alpha \text{ and } \min(\varphi_n, \varphi_m) < \frac{1}{\alpha}\}} \tag{B.1}$$

$$\mathbb{E}\left[u(\frac{1}{2}(\varphi_n + \varphi_m))\right] \geq \frac{1}{2}\mathbb{E}[u(\varphi_n)] + \frac{1}{2}\mathbb{E}[u(\varphi_m)] + \mathbb{E}[\beta I_{\{|\varphi_n - \varphi_m| > \alpha \text{ and } \min(\varphi_n, \varphi_m) < \frac{1}{\alpha}\}}] \tag{B.2}$$

$$\geq \frac{1}{2}\mathbb{E}[u(\varphi_n)] + \frac{1}{2}\mathbb{E}[u(\varphi_m)] + \tag{B.3}$$

$$\beta\mathbb{P}\left[\mid \varphi_n - \varphi_m \mid > \alpha \text{ and } \min(\varphi_n, \varphi_m) < \frac{1}{\alpha}\right]. \tag{B.4}$$

It follows that

$$\beta\mathbb{P}\left[\mid \varphi_n - \varphi_m \mid > \alpha \text{ and } \min(\varphi_n, \varphi_m) < \frac{1}{\alpha}\right] \tag{B.5}$$

$$\leq \mathbb{E}\left[u(\frac{1}{2}(\varphi_n + \varphi_m))\right] - (\frac{1}{2}\mathbb{E}[u(\varphi_n)] + \frac{1}{2}\mathbb{E}[u(\varphi_m)]) \tag{B.6}$$

$$\leq 0 \text{ (by the concavity of u and linearity of the expectation).} \tag{B.7}$$

Therefore, the sequence φ_n is a Cauchy sequence in probability, then it converges to a function g in probability.

1. If $conv\{\psi_n, \psi_{n+1}, \ldots\}$ is bounded in L_0, which means $\forall\ \varepsilon > 0$, there is N so that $\mathbb{P}[h > N] < \varepsilon$ for all $h \in conv\{\psi_n, \psi_{n+1}, \ldots\}$. Then for $\varphi_n \in conv\{\psi_n, \psi_{n+1}, \ldots\}$, we have $\mathbb{P}[\varphi_n > N] < \varepsilon$. Hence $\mathbb{P}[\varphi > N] \leq \varepsilon$, therefore φ is finite, almost surely.

2. If $\mathbb{P}[\psi_n \geq \alpha] \geq \alpha$, we have that $\varphi_n \to \varphi$, and $u(\varphi_n) \to u(\varphi)$. Using the bounded convergence theorem that states that if $X_n \to X$ and X_n bounded, then $\mathbb{E}[X_n] \to \mathbb{E}[X]$. So we have $\mathbb{E}[u(\varphi_n)] \to \mathbb{E}[u(\varphi)]$. Since $\mathbb{E}[\varphi_n] > 0$, then $\mathbb{E}[u(\varphi)] > 0$. Hence $\mathbb{P}[\varphi > 0] > 0$.

$\square$

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Theorem B.1.2. ***Egorov's Theorem***[46][Exercise 6.1.3] *Suppose a finite measure space* (Ω, Σ, μ) *and a sequence of* Σ *measurable function* f_n *converges almost everywhere to* f*. Then, for every* $\varepsilon > 0$*, there is a set* $\Gamma \in \Sigma$ *with* $\mu(\Omega \setminus \Gamma) < \varepsilon$ *and* $\sup_{\omega \in \Gamma} \mid f_n(\omega) - f(\omega) \mid \to 0$ *as* $n \to \infty$ *(i.e.* f_n*, almost uniformly, converges to* f*).*

Proposition B.1.3. *[19] If* S*, a semimartingale satisfies NFLVR, then for each admissible* X*, the function* $(X \bullet S)^* = \sup_{0 \le t} \mid (X \bullet S)_t \mid$ *is finite almost everywhere and the set* $\{(X \bullet S)^* \mid X \ 1-admissible\}$ *is bounded in* L^0*.*

Proof. [19] We proceed by contradiction, assume that the set is not bounded, then we have a sequence of 1-admissible sequence X^n, stopping time T_n and $\alpha > 0$, such that $\mathbb{P}[T_n < \infty] > \alpha > 0$ and $(X^n \bullet S)_{T_n} > n$ on $\{T_n < \infty\}$. Taking t_n to be large enough for each n, we have $\mathbb{P}[T_n \le t_n] > \alpha$ and taking $K^n = X^n 1_{[0, min(T_n, t_n)]}$, gives K^n is of bounded support, and

$$\mathbb{P}[(K^n \bullet S)_\infty > n] > \alpha > 0. \tag{B.8}$$

Then we have that S satisfied NFLVR and K^n 1-admissible and of bounded support, these together with Equation B.8 contradicts Lemma 2.2.5. □

Proposition B.1.4. *[19] If* S *is a semimartingale satisfying NA, then for every admissible integrand* X*, such that* $(X \bullet S)_\infty = \lim_{t \to \infty} (X \bullet S)_t$ *exists, we have for each* $t \in \mathbb{R}_+$*,*

$$\| (X \bullet S)_t^- \|_\infty \le \| (X \bullet S)_\infty^- \|_\infty . \tag{B.9}$$

Proof. Assume the opposite, that we have $\| (X \bullet S)_t^- \|_\infty > \| (X \bullet S)_\infty^- \|_\infty$, then the set is constructed as follows:

$$D = \left\{ (X \bullet S)_t < - \| (X \bullet S)_\infty^- \|_\infty \right\}.$$

$D \in \mathcal{F}_t$ and $K = 1_D 1_{]t, \infty[}$, That contradicts NA. □

Corollary B.1.5. *[19] The semimartingale* S *satisfies the condition NFLVR if, and only if, for a sequence* $(h_n)_{n \ge 1} \in K_0$*, the condition* $\| g_n^- \| \to 0$ *implies that* g_n *tends to 0 in probability.*

Theorem B.1.6. *[63] Krein Smulian Theorem. Suppose* E *a Frèchet space and* E^* *its dual. We say that a convex* $\mathcal{C}$ *in* E^* *is* $\sigma(E^*, E)$ *closed if, and only if, for each balanced, convex* $\sigma(E^*, E)$ *closed set* M *of* E^**,* $\mathcal{C} \cap M$ *is* $\sigma(E^*, E)$ *closed.*

Free Lunch

The notion of "Free Lunch" was first introduced by Kreps [50] as a generalized form of an arbitrage. Kreps claimed that the no-arbitrage notion is too weak to imply the existence

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of an equivalent martingale measure in general, and suggest to allow for a passage to limit as given by the following definition:

Definition B.1.7. *[50] A stochastic process S admit a free lunch (FL), if there exist a random variable $f \in L^{\infty}_{+}(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}[f > 0] > 0$ and a net $(f_\alpha)_{\alpha \in I} = (g_\alpha - h_\alpha)_{\alpha \in I}$ such that $g_\alpha = \int_0^T X_t^\alpha dS_t$, for some admissible trading strategies X^α, $h_\alpha \geq 0$ and $(f_\alpha)_{\alpha \in I}$ converges to f in the weak* topology of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$.*

Kreps gave the following topological notion of this concept:

Let S be a bounded process, and let $\mathcal{K}^s = \{(X \bullet S)_\infty \mid X \text{ simple, admissible}\}$ be the set of outcomes with respect to bounded simple integrands, and

$$\mathcal{C}^s = \mathcal{K}^s - L^{\infty}_{+} = \{f - k \mid f \in \mathcal{K}^s, f \in L^\infty, k \geq 0\}.$$

Definition B.1.8. *[19] S satisfies the NFL condition if the closure $\overline{\mathcal{C}^s}$ of $\mathcal{C}^s$, with respect to the weak * topology of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, satisfies*

$$\overline{\mathcal{C}^s} \cap L^{\infty}_{+}(\Omega, \mathcal{F}, \mathbb{P}) = \{0\}.$$

The following result is crucial in order to prove Theorem 2.2.8. The proof of the below theorem is based on the Krein Smulian Theorem.

Theorem B.1.9. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If $\mathcal{C}$ is a convex cone of L^∞, then $\mathcal{C}$ is weak* closed if, and only if, for each sequence $(f_n)_{n \geq 1} \in \mathcal{C}$, that is uniformly bounded by 1 and $f_n \to f_0$ in probability, we have $f_0 \in \mathcal{C}$.*

Proof. $\Rightarrow$ using a result of the Krein Smulian Theorem, which states that for a Banach space X, and a convex subset of $\mathcal{C}$ in X^*, $\mathcal{C}$ is weak* closed if, and only if, $\mathcal{C} \cap B_r(X^*)$ $r > 0$ is weak* closed, with $B_r(X^*) = \{f \in X^* \mid \|f\| \leq r\}$. In our case, we have $X^* = L^\infty$. Then if we assume a Cauchy sequence $f_n \in \mathcal{C} \cap B_1(L^\infty)$ with respect to convergence in probability, there exists a subsequence converge, almost surely to $f \in B_1(L^\infty)$. Then to show that $f \in \mathcal{C}$, we make use of the convergence dominated theorem that give us $\lim\limits_{n \to \infty} \int f_n g d\mathbb{P} = \int f g d\mathbb{P}$ for $g \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Thus, f_n converges to f in the weak* topology, then $f \in \mathcal{C}$.

Theorem B.1.10. *[19]**Kreps-Yan Separation Theorem** A locally bounded stochastic process S satisfies NFL if, and only if, there exists an equivalent martingale measure, i.e., (NFL $\Leftrightarrow$ EMM).*

Lemma B.1.11. *[19] Let $\mathbb{Q}$ be a probability measure on $\mathcal{F}$ which is absolutely continuous with respect to $\mathbb{P}$. A locally bounded stochastic process S is a local martingale under a*

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probability measure $\mathbb{Q}$ if, and only if, $\mathbb{E}_{\mathbb{Q}}\left[(X \bullet S)_\infty\right] = 0$ for each admissible simple trading strategy X.

Proof. Let $(\tau_n)_{n=1}^{\infty}$ be a sequence of finitely valued stopping times increasing $\mathbb{P}$ almost surely, to infinity such that each S^{τ_n} is bounded. Assume that $\mathbb{E}_{\mathbb{Q}}\left[(X \bullet S)_\infty\right] = 0$ is true for each simple admissible integrand; we will have to show that each S^{τ_n} is a $\mathbb{Q}$-martingale i.e. for each $n \geq 1$ and each pair of stopping times $0 \leq \sigma_1 \leq \sigma_2 \leq \tau_n$, we have $\mathbb{E}_{\mathbb{Q}}\left[S_{\sigma_2} \mid \mathbb{F}_{\sigma_1} = S_{\sigma_1}\right]$. This is equivalent to proving that for each $\mathbb{R}^d$-valued $\mathcal{F}_{\sigma_1}$-measurable bounded function h, we have,

$$\mathbb{E}_{\mathbb{Q}}\left[(h, S_{\sigma_2} - S_{\sigma_1})\right] = 0,$$

which is true by the assumption $\mathbb{E}_{\mathbb{Q}}\left[(X \bullet S)_\infty\right] = 0$. Therefore, S is a local martingale. For the converse implication, we have that the local $\mathbb{Q}$-martingale property of S implies $\mathbb{E}_{\mathbb{Q}}\left[(X \bullet S)_\infty\right] = 0$ for each admissible integrand (by Lemma 2.1.6). □

Proof of Theorem B.1.10

Proof. [63] Suppose that there exists an equivalent martingale measure $\mathbb{Q}$, with $h = \frac{d\mathbb{Q}}{d\mathbb{P}}$ as its Radon-Nikoým derivative. From the definition of martingale it follows that

$$\mathbb{E}_{\mathbb{Q}}\left[f\right] = \langle f, h\rangle = \int f(\omega)\ h(\omega)d\mathbb{P} = 0 \ for\ each\ f \in \mathcal{K}^s, \tag{B.10}$$

and therefore,

$$\mathbb{E}_{\mathbb{Q}}\left[f\right] = \langle f, h\rangle = \int f(\omega)\ h(\omega)d\mathbb{P} \leq 0 \ for\ each\ f \in \mathcal{C}^s. \tag{B.11}$$

The weak* continuity of h, gives $\mathbb{E}_{\mathbb{Q}}\left[f\right] \leq 0$ for $f \in \overline{\mathcal{C}}^s$. On the other hand, for $f \in L^\infty_+$, $f \neq 0$ we have

$$\mathbb{E}_{\mathbb{Q}}\left[f\right] = \langle f, h\rangle > 0, \tag{B.12}$$

which applies that $\overline{\mathcal{C}}^s$ is disjoint from $L^\infty_+ \setminus \{0\}$. Thus there is No Free-lunch.

Now, assume that S satisfies the condition of Free-lunch. The first step is to apply the Separation Theorem 2.1.7 to the weak*-closed convex set $\overline{\mathcal{C}}$ and the compact set $\{f\}$, then there exist $g \in L^1$ and $\alpha < \beta$ such that $g \mid_{\overline{\mathcal{C}}} \leq \alpha$ and $\langle f, g\rangle > \beta$. Since $0 \in \mathcal{C}$, we have $\alpha \geq 0$, since $\overline{\mathcal{C}}$ is a cone, we have that g is zero or negative on $\overline{\mathcal{C}}$ and non-negative on L^∞_+, that is, $g \in L^\infty_+$.

The second step is considers a set $\mathcal{G}$ of all $g \in L^1_+$, $g \leq 0$ on $\mathcal{C}$. We have $0 \in \mathcal{G}$ by step 1, then $\mathcal{G} \neq 0$. Let $\mathcal{S}$ be the family of subset of Ω formed by the supports $\{g > 0\}$ of the

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elements $g \in \mathcal{G}$. For a sequence $(g_n)_{n=1}^{\infty} \in \mathcal{G}$, there exist strictly positive scalars $(\alpha_n)_{n=1}^{\infty}$, such that $\sum_{n=1}^{\infty} \alpha_n g_n \in \mathcal{G}$ i.e $\mathcal{S}$ is closed under countable unions. Therefore, there is $g_0 \in \mathcal{G}$ such that, for $\{g > 0\}$, we have $\mathbb{P}[\{g_0 > 0\}] = \sup\{\mathbb{P}[\{g > 0\}] \mid g \in \mathcal{G}\}$. We can conclude that g_0 is strictly positive, almost surely, that is, $\mathbb{P}[\{g_0 > 0\}] = 1$. To see this, we suppose that $\mathbb{P}[\{g_0 > 0\}] < 1$, applying step 1 to the function $f = 1_{\{g_0=0\}}$, to find $g_1 \in \mathcal{G}$ with

$$\begin{aligned}\mathbb{E}[fg_1] &= \int_{\Omega} 1_{\{g_0=0\}}(\omega) g_1(\omega) d\mathbb{P}(\omega)\\ &= \int_{\{g_0=0\}} g_1(\omega) d\mathbb{P}(\omega) > 0.\end{aligned}$$

Therefore, we have $g_0 + g_1 \in \mathcal{G}$ whose support has a $\mathbb{P}$-measure strictly bigger than $\mathbb{P}[g_0 > 0]$, a contradiction. Normalise g_0 i.e. $\| g_0 \|_1 = 1$. Let $\mathbb{Q}$ be the measure on $\mathcal{F}$ with Radon-Nikodým derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} = g_0$. Then from Lemma B.1.11, we conclude that $\mathbb{Q}$ is a local-martingale measure for S, so that $\mathcal{M}^e(S) \neq \emptyset$. □

□

B.2 Tools Needed to Prove The Second Fundamental Theorem of Asset Pricing

The space M^2 of L^2-martingales contains all martingales M such that $\sup_t \mathbb{E}[M_t^2] < \infty$ and $M_0 = 0$, almost surely. For $\mathcal{A}$ subset of M^2, $\mathcal{M}^2(\mathcal{A})$ is the set of all M^2 martingale measures for $\mathcal{A}$. Let $\mathcal{A}$ be a subset of M^2, then the stable subspace generated by $\mathcal{A}$, denoted by $\mathcal{S}(\mathcal{A})$ is the intersection of all closed, stable subspaces containing $\mathcal{A}$. (see page 178, 179 and 182 in Protter [56]).

Definition B.2.1. *[56] A measure $\mathbb{Q} \in \mathcal{M}^2(\mathcal{A})$ is an extremal point of $\mathcal{M}^2(\mathcal{A})$ if whenever $\mathbb{Q} = \lambda R + (1-\lambda)S$ with R, $S \in \mathcal{M}^2(\mathcal{A})$, $0 \leq \lambda \leq 1$, then $\lambda = 0$ or 1.*

Definition B.2.2. *[56] If N, M are two martingales, then we say that N and M are strongly orthogonal if the product $L = NM$ is martingale.*

Theorem B.2.3. *[56] Let $\mathcal{A} \subset M^2$. If $\mathcal{S}(\mathcal{A}) = M^2$, then P is an extremal point of $\mathcal{M}^2(\mathcal{A})$.*

Proof. [56] We assume that P is not extremal, then we show that $\mathcal{S}(\mathcal{A}) \neq M^2$. Using the hypothesis that P is not extremal and Definition B.2.1, there exists Q, $R \in \mathcal{M}^2(\mathcal{A})$ with $Q \neq R$, such that $P = \lambda Q + (1-\lambda)R$, $0 < \lambda < 1$. Let $L_\infty = \frac{dQ}{dP}$ and $L_t = \mathbb{E}\left[\frac{dQ}{dP} \mid \mathcal{F}_t\right]$.

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Then we have,

$$
\begin{aligned}
1 = \frac{dP}{dP} &= \lambda \frac{dQ}{dP} + (1-\lambda)\frac{dR}{dP} \\
&= \lambda L_\infty + (1-\lambda)\frac{dR}{dP} \geq \lambda L_\infty \text{ almost } \quad \text{sure.} \\
&\Leftrightarrow L_\infty \leq \frac{1}{\lambda}, \text{ almost surely.}
\end{aligned}
$$

Since $Q = P$ on $\mathcal{F}_0$, L is a bounded martingale with $L_0 = 1$. Thus, the martingale $L - L_0 \in M^2(P)$ is non-constant. Nevertheless, if $X \in \mathcal{A}$ and $H \in b\mathcal{F}_s$, then X is a martingale, and for $s < t$,

$$
\begin{aligned}
\mathbb{E}_P\left[X_t L_t H\right] &= \mathbb{E}_P\left[X_t \mathbb{E}\left[\frac{dQ}{dP} \mid \mathcal{F}_s\right] H\right] \\
&= \mathbb{E}_P\left[X_t \frac{dQ}{dP} H\right] \\
&= \mathbb{E}\left[X_t L_\infty H\right] \\
&= \mathbb{E}_Q\left[X_t H\right] = \mathbb{E}_Q\left[X_s H\right] \\
&= \mathbb{E}_P\left[X_s L_\infty H\right] = \mathbb{E}_P\left[X_s L_s H\right],
\end{aligned}
$$

and XL is a P martingale. Thus $X(L - L_0)$ is a P martingale, and $L - L_0 \in M^2$, and it is strongly orthogonal to $\mathcal{A}$. Using Theorem 37, page 181 in Protter [56], we have $\mathcal{S}(\mathcal{A}) \neq M^2$. □

Theorem B.2.4. *[56] Let $\mathcal{A} \subset M^2$. If P is an extremal point of $\mathcal{M}^2(\mathcal{A})$, then every bounded P martingale strongly orthogonal to $\mathcal{A}$ is null.*

Proof. [56] Let L be a bounded non-constant martingale strongly orthogonal to $\mathcal{A}$. Let c be bounded for $\mid L \mid$, and set

$$
dQ = (1 - \frac{L_\infty}{2c})dP \text{ and } dR = (1 + \frac{L_\infty}{2c})dP.
$$

We have $Q,\ R \in \mathcal{M}^2(\mathcal{A})$, and $P = \frac{1}{2}Q + \frac{1}{2}R$ is a decomposition that shows that P is not extremal, that is a contradiction. □

Theorem B.2.5. *[56] If $\mathcal{A} = \{M^1, \cdots, M^2\} \subset M^2$, with M^i continuous and M^i, M^j strongly orthogonal for $i \neq j$. And if P is an extremal point of $\mathcal{M}^2(\mathcal{A})$, then we have the following:*

1. *Every stopping time is accessible;*

2. *Every bounded martingale is continuous;*

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3. *Every uniformly integrable martingale is continuous; and*

4. $\mathcal{A}$ *has the predictable representations property.*

Proof. [56] □

Theorem B.2.6. *[56]**Jacod-Yor Theorem on Martingale Representation**. Let $\mathcal{A}$ be a subset of $\mathcal{H}^2$ containing constant martingale. Then $\mathcal{S}(\mathcal{A})$, the stable subspace of stochastic integral generated by $\mathcal{A}$, equals $\mathcal{H}^2$ if, and only if, the probability measure $\mathbb{P}$ is an extremal point of $\mathcal{M}^2(\mathcal{A})$, the space of probability measures making all elements of $\mathcal{A}$ square integrable martingales.*

Proof. [56] □

Girsanov Theorem for One Dimensional Brownian Motion

A probability measure $\mathbb{Q}$ defined on a probability space $(\Omega, \mathcal{F})$ is absolutely continuous with respect to $\mathbb{P}$, if any $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$, then $\mathbb{Q}(A) = 0$ as well, and it is denoted by $\mathbb{Q} << \mathbb{P}$. It is well-known in probability theory that $\mathbb{Q} << \mathbb{P}$ if, and only if, there exists a non-negative random variable Z such that $\mathbb{Q}(A) = \int_A Z d\mathbb{P}$ for all $A \in \mathcal{F}$, with Z called the Radon-Nikodým derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ given as $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. Also, if we have $\mathbb{Q} << \mathbb{P}$ and $\mathbb{P} << \mathbb{Q}$, then we say that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, and denoted by $\mathbb{P} \equiv \mathbb{Q}$.

Theorem B.2.7. *[23] Let w_t be a Brownian motion on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, and X_t be an adapted process satisfying $\mathbb{E}\left[exp(\frac{1}{2}\int_0^T X_t^2 dt)\right] < \infty$(Novikov condition). For $0 \leq t \leq T$, define*

$$
\begin{aligned}
Z_t &= exp\left(-\frac{1}{2}\int_0^t X_u dw_u - \int_0^t X_u^2 du\right) \\
\widehat{w}_t &= w_t + \int_0^t X_u du
\end{aligned}
$$

and a measure $\mathbb{Q}$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_t$. Then Z_t is a martingale measure under $\mathbb{P}$, and $\widehat{w}_t$ is a standard Brownian motion under $\mathbb{Q}$.

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Appendix C

Liquidity Risk and Arbitrage Pricing Theory

C.1 Approximating Stochastic Integrals with Continuous and of Finite Variation Integrands

Lemma C.1.1. *[10] Let X be special semimartingale with the canonical decomposition $X = \overline{N} + \overline{A}$, where N is a local martingale and A is predictable. Suppose S has inaccessible jumps, then A is continuous.*

Definition C.1.2. *Let $X \in \mathcal{H}^2$ with $X = \overline{N} + \overline{A}$ its canonical decomposition, and let $H, J \in b\mathcal{P}$. We define $d_X(H,J)$ as follows:*

$$d_X(H,J) = \| \left(\int_0^T (H_u - H_u^\epsilon)^2 d[\bar{N},\bar{N}]_u \right)^{1/2} \|_{L^2} + \| \int_0^T \mid H_u - H_u^\epsilon \mid\mid d\bar{A}_u \mid \|_{L^2}. \quad \text{(C.1)}$$

Theorem C.1.3. *[10] For $\epsilon > 0$, and continuous and finite variation processes H, there exists H^ϵ bounded continuous of finite variation, with $H_T^\epsilon = 0$ such that $d_X(H, H^\epsilon) < \epsilon$, with $X \in \mathcal{H}^2$ and $X = \overline{N} + \overline{A}$.*

Corollary C.1.4. *[10] Let $\epsilon > 0$, for any continuous bounded of finite variation process H, there exists a continuous bounded of finite variation process H^ϵ, with $H_T^\epsilon = 0$ such that $\| H \bullet S - H^\epsilon \bullet S \|_{L^2} < \epsilon$.*

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C.2 Change of Numéraire

Suppose $(S(.,x), B)$, where $S(.,x)$ is the stock price process per share of size x and B is the bond price that is strictly positive and semimartingale. Let us write $\bar{S}(.,x) = \frac{S(.,x)}{B}$ which represents the discounted price process. The self-financing condition for the economy $(\bar{S}(.,x), 1)$ is shown to be as follows:

$$\bar{Y}_t = \int_0^t \bar{X}_{u^-} d\bar{S}(u,0) - \bar{X}_t \bar{S}(t,0) - \bar{L}_t^1 - \overline{L}_t^2 \text{ for } 0 \leq t \leq \bar{\tau}^-, \tag{C.2}$$

with $\bar{L}_t^1 = \sum_{0 \leq u \leq t} \Delta X_u \left[\bar{S}(u, \Delta X_u) - \bar{S}(u,0)\right]$ and $\bar{L}_t^2 = \int_0^t \frac{\partial \bar{S}}{\partial x}(u^-, 0) d[X,X]_u^c$. The following theorem is the Numéraire Invariance Theorem.

Theorem C.2.1. *[10] Let X be a predictable càdlàg process with finite quadratic variation, and τ be a predictable stopping time. Then (X, Y, τ) is a s.f.t.s for $(S(.,x), B)$ if, and only if, (X, Y, τ) is a s.f.t.s for $(\frac{S(.,x)}{B}, 1)$.*

Proof. [10] (X, Y, τ) is a self-financing strategy for $(S(.,x), B)$ if Y satisfies

$$B_t Y_t = (X_-.s)_t - X_t S(.,0) - L_t^1 - L_t^2 + (Y_-.B)_t \text{ for } 0 \leq t \leq \tau \tag{C.3}$$

and $(X, \bar{Y}, \tau)$ is self-financing strategy for $(\bar{S}(.,x), 1)$ if $\bar{Y}$ satisfies C.2. We need to show that $Y = \bar{Y}$ because, Y and $\bar{Y}$ are uniquely determined by X and τ. Note that L^1, L^2 and

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$\bar{L}^2$ are semimartingales of finite variation. Hence,

$$\begin{aligned}
\frac{1}{B}L^1 &= L^1_-.\frac{1}{B} + \frac{1}{B}.L^1\\
&= L^1_-\frac{1}{B} + \sum_{0\le u\le t}\Delta X_u\left[S(u,\Delta X_u) - S(u,0)\right]\frac{1}{B_u}\\
&= L^1_-.\frac{1}{B} + \bar{L}^1\\
\frac{1}{B}L^2 &= L^2.\frac{1}{B} + \frac{1}{B}.L^2 = L^2.\frac{1}{B} + \frac{\partial\bar{S}}{\partial x}(.,0).[X,X]^c\\
&= L^2.\frac{1}{B} + \bar{L}^2\\
\frac{1}{B}(Y_-.B) &= \frac{1}{B_-}(Y_-.B) + (Y_-.B)_-.\frac{1}{B} + \left[\frac{1}{B},Y_-.B\right]\\
&= \frac{1}{B_-}Y_-.B + (Y_-.B)_-.\frac{1}{B} + \left[\frac{1}{B},Y_-.B\right]\\
\frac{1}{B}(X_-.S(.,0)) &= \frac{1}{B_-}X_-.S(.,0) + (X_-.S(.,0))_-.\frac{1}{B} + \left[\frac{1}{B},X_-.S(.,0)\right]\\
&= X_-.(\frac{1}{B_-}.S(.,0)) + (X_-.S(.,0))_-.\frac{1}{B} + \left[\frac{1}{B},X_-.S(.,0)\right]\\
&= X_-\bar{S}(.,0) - X_-\bar{S}_-.\frac{1}{B} - X_-.\left[\frac{1}{B},S(.,0)\right]\\
&+ (X_-.S(.,0))_-.\frac{1}{B} + \left[\frac{1}{B},X_-.S(.,0)\right]\\
&= X_-.\bar{S}(.,0) - X_-\bar{S}(.,0)_-.\frac{1}{B} - \left[\frac{1}{B},X_-.S(.,0)\right]\\
&+ (X_-.S(.,0))_-.\frac{1}{B} + \left[\frac{1}{B},X_-.S(.,0)\right]\\
&= X_-.\bar{S}(.,0) - X_-S(.,0)_-.\frac{1}{B} + (X_-.S(.,0))_-.\frac{1}{B}.
\end{aligned}$$

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Rearranging the above expressions and dividing expression C.3 by B we get

$$
\begin{aligned}
Y =& X_{-}.\bar{S} - X\bar{S} - \bar{L}^1 - \bar{L}^2 \\
& \big((X_{-}.S(.,0))_{-} - X_{-}S(.,0)_{-} - L^1_{-} - L^2 + (Y_{-}.B)_{-}\big) .\frac{1}{B} \\
& + \frac{1}{B_{-}} Y_{-}.B + \left[\frac{1}{B}, Y_{-}.B\right] \\
& = \bar{Y} + B_{-}Y_{-}.\frac{1}{B} + \frac{1}{B_{-}} Y_{-}.B + \left[\frac{1}{B}, Y_{-}.B\right] \\
& = \bar{Y} + Y_{-}.\left(B_{-}.\frac{1}{B} + \frac{1}{B_{-}}.B + \left[\frac{1}{B}, B\right]\right) \\
& \bar{Y} + Y_{-}.(1) \\
& = \bar{Y}.
\end{aligned}
$$

$\square$

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