

BAYESIAN HIGHEST POSTERIOR DENSITY INTERVALS FOR THE AVAILABILITY OF A SYSTEM WITH A 'REST-PERIOD' FOR THE REPAIR FACILITY

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ABSTRACT

In this paper Bayesian estimation for the steady state availability of a one-unit system with a rest-period for the repair facility is studied. The assumption is that the repair facility takes rest with probability p after each repair completion and the facility does not take the same with probability $(1 - p)$. The prior information is assumed to be vague and the Jeffreys' prior is used for the unknown parameters in the system. Gibbs sampling is used to derive the posterior distribution of the availability and subsequently the highest posterior density (HPD) intervals. A numerical example illustrates these results.

OPSOMMING

In hierdie artikel word die Bayes-beraming van die ewewigstoestandbeskikbaarheid van 'n stelsel wat afwisselend gebruik word, voorgestel. Daar word veronderstel dat die herstelfasiliteit na voltooiing van elke herstel óf 'n rustydperk binnegaan of nie. Die rustydperk sal geneem word met waarskynlikheid p en die waarskynlikheid dat daar nie 'n rustydperk geneem word nie, is $(1 - p)$. Jeffrey se a priori-verdeling word vir die onbekende parameters in die stelsel aanvaar. Gibbs-steekproefneming word gebruik om die a posteriori-verdeling van die beskikbaarheid en daarna die hoogste a posteriori-digtheidsintervalle (HPD) af te lei. 'n Numeriese voorbeeld illustreer hierdie resultate.

1. INTRODUCTION

To evaluate the effectiveness of a system, several concepts have been introduced: maintainability, serviceability, reparability and availability etc. (Kapur & Lamberson, 1977). Within availability itself, several measures have been proposed. In the past, nine different kinds of availability have been defined (Brender, 1968a, b). The survey of different approaches of availability is available in Lie. et al (1977), and Kumar and Agarwal (1980). Pointwise availability $A(t)$ is, undoubtedly, the most important of these since it gives the probability that the system is functioning at time t (Klassen & Van Peppen, 1989). Its limit value, when it exists, is called the steady-state or asymptotic availability (Pham-Gia & Turkkan, 1999). A_{∞} is defined as the expected fraction of time that the system operates satisfactorily in the long run.

Most of the models studied in the literature have the assumption that the repair facility is continuously available to attend the repair of the failed units. But it is reasonable to expect that a 'rest-period' might be needed to get the repair facility ready for the next repair could be taken up (Subramanian & Sarma, 1981). This rest-period usually starts after each repair completion. In this paper it is assumed that the 'rest-period' is taken with probability p , and will continue with the repairs of the unit with probability $(1 - p)$, depending on the amount of time spent by the repair facility.

The Bayesian framework for statistics and decision theory (Bernardo & Smith, 2000) offers great opportunities for applications in reliability problems (Martz & Waller, 1982), because of the possibility to take expert knowledge into account through the prior distribution for an assumed parametric model. In this paper the steady state availability of a one unit system as studied by Yadavalli et al (2001) is considered from a *Bayesian* viewpoint with emphasis on the use of a Jeffreys' prior distribution for the parameters.

Section 2 gives the necessary notation, a brief description of the model and maximum likelihood estimators of the steady state availability as derived by Yadavalli et al (2001). The Bayesian approach to this problem is introduced in section 3, followed by a numerical example illustrating the posterior analysis using the Gibbs sampling method.

2. ASSUMPTIONS AND SYSTEM DESCRIPTION

The following characteristics and assumptions describe the model:

- (1) The system consists of a unit and a single repair facility.
- (2) The lifetime and repair time of the unit are exponentially distributed with parameters λ and μ respectively.
- (3) The unit is as good as new after each repair.
- (4) Switch is perfect and switchover is instantaneous.
- (5) The repair facility is not available for a random time, called *rest-period*, which is distributed exponentially with parameter d . The rest-period (D) is a Bernoulli random variable, which is defined as follows:

$$D = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } (1-p) \end{cases}$$

The states 1 and 0 represent respectively the realisation or not of the 'rest-period'.

Let the stochastic process $\{W(t), t \geq 0\}$ with state space $\{0, 1, 2, 3\}$ describe the behaviour of the system at time t . The system transitions for the model are given below.

Table 1: System transitions

State number	State of the	
	Unit	repair facility
0	operable	available
1	under repair	available
2	operable	not available
3	waiting for repair	not available

The steady state probabilities can be determined by using the principle of flow-balance (Ravindran et al, 1987). The steady state availability is obtained in Yadavalli et al (2001), namely

$$A_{\infty} = \frac{\mu d(\lambda + d)}{d^2 \lambda + d^2 \mu + \lambda^2 d + \lambda d \mu + \lambda^2 \mu p} \quad (1)$$

Let X_1, X_2, \dots, X_n and Y_1, Y_2, \dots, Y_n be random samples of size n , each drawn from different exponential populations with failure and repair rates of the unit λ and μ respectively. Also, let Z_1, Z_2, \dots, Z_n be a random sample of size n , drawn from a different exponential population with parameter d . The Maximum Likelihood Estimator (MLE) of A_{∞} is given by (see Yadavalli et al, 2001).

$$\hat{A}_{\infty} = \frac{\bar{X}(\bar{X} + \bar{Z})}{(\bar{X}^2 + \bar{X}\bar{Y} + p\bar{Z}^2 + \bar{Y}\bar{Z} + \bar{X}\bar{Z})} \quad (2)$$

where $\bar{X} = \frac{1}{n} \sum_{j=1}^n x_j$, $\bar{Y} = \frac{1}{n} \sum_{j=1}^n y_j$ and $\bar{Z} = \frac{1}{n} \sum_{j=1}^n z_j$.

Let $\theta = (\theta_1, \theta_2, \theta_3)$ with

$$\theta_1 = \frac{1}{\lambda}, \theta_2 = \frac{1}{\mu}, \theta_3 = \frac{1}{d}. \quad (3)$$

The steady state availability given in (3) reduces to

$$A_{\infty} = \frac{\theta_1(\theta_1 + \theta_3)}{\theta_1\theta_2 + \theta_1^2 + \theta_2\theta_3 + \theta_1\theta_3 + \rho\theta_3^2} \quad (4)$$

The $(1 - \alpha)$ % asymptotic confidence limits for A_{∞} are given by

$$\hat{A}_{\infty} \pm k_{\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{n}} \quad (5)$$

where $\hat{\sigma}^2$ is a consistent estimator for $\sigma^2(\hat{\theta}) = \sum_{j=1}^3 \left(\frac{\partial A_{\infty}}{\partial \theta_j} \right)^2 \theta_j^2$, and $k_{\frac{\alpha}{2}}$ is obtained from the standardised normal tables.

3. BAYESIAN INFERENCE OF A_{∞}

Based on the attributed data, the likelihood function is given by

$$\begin{aligned} L(\lambda, \mu, d | T_1, T_2, T_3) &= \prod_{j=1}^n \lambda e^{-\lambda x_j} \cdot \prod_{j=1}^n \mu e^{-\mu y_j} \cdot \prod_{j=1}^n d e^{-d z_j} \\ &= (\lambda \mu d)^n e^{-(\lambda T_1 + \mu T_2 + d T_3)} \end{aligned} \quad (6)$$

where

$$\left(T_1 = \sum_{j=1}^n x_j, \quad T_2 = \sum_{j=1}^n y_j, \quad T_3 = \sum_{j=1}^n z_j \right)$$

is sufficient for (λ, μ, d) .

Representing (6) in terms of corresponding mean life time, mean repair time, mean rest-period time, (6) results in (see (3))

$$L(\theta_1, \theta_2, \theta_3 | T_1, T_2, T_3) = (\theta_1 \theta_2 \theta_3)^{-n} e^{-\left(\frac{1}{\theta_1} T_1 + \frac{1}{\theta_2} T_2 + \frac{1}{\theta_3} T_3 \right)} \quad (7)$$

The Jeffreys' prior distribution for $(\theta_1, \theta_2, \theta_3)$ (see Box & Tiao, 1992) is given by

$$g(\theta_1, \theta_2, \theta_3) \propto \frac{1}{(\theta_1 \theta_2 \theta_3)}, \quad \theta_j > 0 (j = 1, 2, 3) \quad (8)$$

The joint posterior distribution, according to Bayes' theorem (using (7) and (8)) is defined by

$$g(\theta_1, \theta_2, \theta_3 | T_1, T_2, T_3) \propto (\theta_1 \theta_2 \theta_3)^{-n-1} e^{-\left(\frac{1}{\theta_1 T_1} + \frac{1}{\theta_2 T_2} + \frac{1}{\theta_3 T_3}\right)} \quad (9)$$

From (9), the joint posterior of $(\theta_1, A_\infty, \theta_3)$ is given by

$$g(\theta_1, A_\infty, \theta_3 | T_1, T_2, T_3) = g\left(\theta_1, \theta_1 \left(\frac{1}{A_\infty} - A_\infty\right) - \frac{p\theta_3^2}{(\theta_1 + \theta_3)}, \theta_3 | T_1, T_2, T_3\right) \times |J| \quad (10)$$

with J the Jacobian of the transformation from $(\theta_1, \theta_2, \theta_3)$ to $(\theta_1, A_\infty, \theta_3)$.

For the joint posterior distribution (10) the marginal posterior distribution for A_∞ is obtained by using the Gibbs sampling method (Gelman et al, 1995). The Gibbs sampler allows the generation of a sample from the following full conditional distributions

$$\begin{aligned} g(\theta_1 | data, \theta_2, \theta_3) \\ g(\theta_2 | data, \theta_1, \theta_3) \\ g(\theta_3 | data, \theta_1, \theta_2) \end{aligned} \quad (11)$$

These conditional distributions are used in the iterative scheme to generate samples for each of the three marginal distributions and subsequently of A_∞ using (4).

4. NUMERICAL ILLUSTRATION

To illustrate the results in section 3, exponentially distributed samples were simulated for the three variables in the system. Table 2 gives the sample information for sample sizes of 40 and 200, respectively.

Table 2 Sample summary

	$n = 40$			$n = 200$		
	X	Y	Z	X	Y	Z
Mean	636.05	193.84	41.34	505.59	192.41	50.59
st. dev.	873.26	177.64	44.48	481.85	172.19	54.74
sum	25442.17	77.62	1653.69	101119.00	38481.16	10117.97

The full conditionals (10) are simulated with WINBUGS software, using the Gibbs sampling method. Initial burn-in samples of a 1000 were first simulated for the model, which was followed by 10 000 iterations of this procedure to simulate the marginal distribution of A_∞ . Table 3 and 4 show the posterior mean, median and the $(1 - \alpha)$ 100% HPD intervals for A_∞ , for sample with sizes 40 and 200, respectively. In both the tables the estimates were obtained for different values of the parameter p .

Table 3 Posterior mean, median and HPD intervals for A_∞ ($n = 40$)

p	post. Mean	St. dev.	median	95% HPD	90% HPD	80% HPD
0.00	0.7641	0.0401	0.7661	(0.6787; 0.8368)	(0.6938; 0.8268)	(0.7105; 0.8148)
0.10	0.7639	0.0407	0.7659	(0.6783; 0.8367)	(0.6935; 0.8266)	(0.7103; 0.8145)
0.33	0.7633	0.0406	0.7653	(0.6777; 0.8362)	(0.6931; 0.8260)	(0.7096; 0.8138)
0.50	0.7629	0.0407	0.7650	(0.6773; 0.8359)	(0.6927; 0.8257)	(0.7092; 0.8135)
0.67	0.7625	0.0407	0.7646	(0.6766; 0.8356)	(0.6921; 0.8252)	(0.7087; 0.8132)
0.90	0.7619	0.0409	0.7639	(0.6759; 0.8352)	(0.6931; 0.8248)	(0.7081; 0.8128)
1.00	0.7616	0.0409	0.7637	(0.6757; 0.8351)	(0.6910; 0.8247)	(0.7078; 0.8126)

Table 4 Posterior mean, median and HPD intervals for A_∞ ($n = 200$)

P	post. Mean	St. dev.	median	95% HPD	90% HPD	80% HPD
0.00	0.7235	0.0200	0.7238	(0.6840; 0.7615)	(0.6897; 0.7560)	(0.6974; 0.7489)
0.10	0.7230	0.0200	0.7233	(0.6836; 0.7611)	(0.6892; 0.7555)	(0.6969; 0.7485)
0.33	0.7219	0.0201	0.7222	(0.6823; 0.7602)	(0.6880; 0.7545)	(0.6956; 0.7474)
0.50	0.7211	0.0201	0.7214	(0.6814; 0.7595)	(0.6872; 0.7537)	(0.6948; 0.7466)
0.67	0.7203	0.0202	0.7205	(0.6805; 0.7589)	(0.6863; 0.7530)	(0.6940; 0.7459)
0.90	0.7192	0.0202	0.7194	(0.6794; 0.7580)	(0.6851; 0.7520)	(0.6927; 0.7448)
1.00	0.7187	0.0202	0.7189	(0.6789; 0.7576)	(0.6846; 0.7516)	(0.6922; 0.7444)

From these tables it is evident that when the parameter p decreases, the estimates of A_∞ increases. It is also evident that the influence of the parameter p is minimal in this illustration. The differences of the largest and the smallest estimates of A_∞ are all less than 0.005. Figures 2 and 3 shows the posterior distribution for A_∞ , with $p = 0.33$, for the samples of size 40 and 200, respectively.

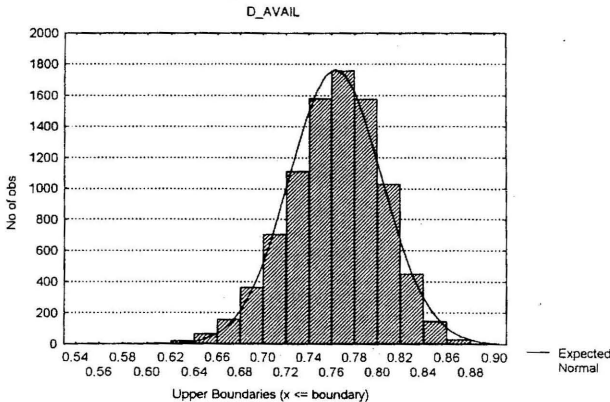


Figure 2 Posterior distribution for A_∞ ($n = 40$)

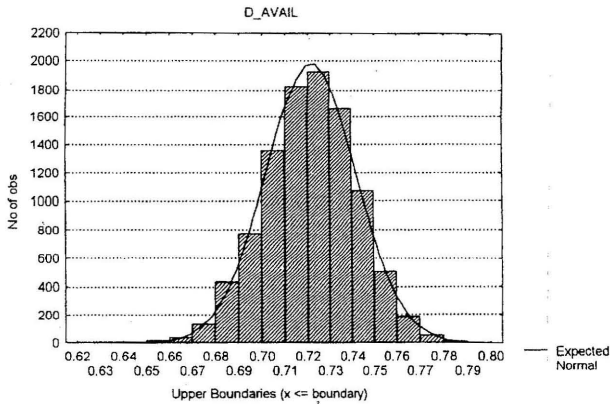


Figure 3 Posterior distribution for A_∞ ($n = 200$)

The posterior distribution for A_∞ will vary most when large rest times are observed in a system (see term $p\theta_3^2$ in expression (4) for A_∞). It is clear in this case for large probability p , that the estimates for A_∞ will dramatically decrease, relative to smaller failure and repair times.

5. REFERENCES

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