NUCLEAR REACTIONS WITH POLARIZED SPIN-$\frac{1}{2}$ BEAMS

Part I: The polarization formalism for elastic proton scattering

and

Part II: Relativistic effects on measured polarization observables

by

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"Declaration"

I the undersigned hereby declare that the work contained in this thesis is my own original work and has not previously in its entirety or in part been submitted at any University for a degree.

Date: 9 February 1990

To my parents

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Abstract

The first part of the thesis deals with the subject of polarization in nuclear reactions. The case where the incoming beam is polarized, is emphasized, and the theory is developed for the simple and illustrative cases of spin–½ on spin–½ scattering as well as spin–½ on spin zero scattering. Finally, the polarization observables (or alternatively, the Wolfenstein–parameters) are introduced, their physical meanings and the ways in which they are measured, are discussed.

In the second part we investigate the sensitivity of a complete set of spin–½ on spin zero elastic scattering observables $\left[\frac{d\sigma}{d\Omega} , P, Q \right]$ to relativistic and nonrelativistic predictions. The calculational approach is based on a simple relativistic extension of the first–order KMT optical potential within the context of a Dirac equation description. A formulation of this problem in terms of momentum space integral equations displaying an identifiable nonrelativistic sector is described and applied. The computer code WIZARD 1 is used in this respect and applied to the case of elastic scattering of polarized protons with energies of 135 MeV and 155 MeV on ^{24}Mg, ^{28}Si and ^{32}S targets. These results clearly demonstrate that the relativistic predictions are in far better agreement with the measured values than are the nonrelativistic results. We then propose the type of future measurements that can be made at the National Accelerator Centre, once the polarized–ion source has been installed. We also advise the implementations of *relativistic* calculations of the scattering observables.

Opsomming

Die eerste gedeelte van hierdie tesis handel oor die onderwerp van polarisasie in kern reaksies. Dié geval waar die inkomende bundel gepolariseerd is, word beklemtoon, en die teorie word ontwikkel vir die eenvoudige en illustratiewe gevalle van spin–½ op spin–½ verstrooiing asook spin–½ op spin zero verstrooiing. Daarna word die polarisasie waarneembares (ook bekend as die Wolfenstein parameters) ingevoer en die fisiese betekenis en metings van hierdie parameters word bespreek.

In die tweede gedeelte word 'n volledige stel waarneembares $\left[\frac{d\sigma}{d\Omega}, P, Q \right]$, vir elastiese spin–½ op spin zero verstrooiing, se sensitiwiteit tot relatiwistiese en nie–relatiwistiese berekeninge ondersoek. Die berekeninge word gebasseer op 'n eenvoudige relatiwistiese uitbreiding van die eerste orde KMT optiese potensiaal binne die raamwerk van Dirac se vergelyking. Die probleem word geformuleer in terme van momentum ruimte integraal vergelykings wat 'n mens in staat stel om die nie–relatiwistiese sektor van die probleem te identifiseer. Die rekenaar program Wizard 1 word gebruik vir hierdie doel vir die geval van elastiese verstrooiing van 'n gepolariseerde bundels protone met energieë van 135 MeV en 155 MeV op ^{24}Mg, ^{28}Si en ^{32}S teikens. Die resultate toon duidelik aan dat die relatiwistiese voorspellings in baie beter ooreenstemming is met die eksperimentele waardes as die nie–relatiwistiese geval. Ons stel voor watter tipe metings in die toekoms by die Nasionale Versneller Sentrum uitgevoer kan word, sodra die gepolariseerde ioonbron geïnstalleer word. Verder word daar aanbeveel dat *relatiwitiese* berekeninge van die verstrooiings waarneembares geïmplementeer word.

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PART I

THE POLARIZATION FORMALISM FOR ELASTIC PROTON SCATTERING

CHAPTER 1

1. Introduction

It is known from nucleon–nucleon (NN–) scattering that the interaction between two nucleons is dependent on their relative spin–orientations (spin–dependent). Therefore, a reaction between nuclei in a target and a beam of proton which all have their spins in the same direction (a polarized beam) will definitely yield more information on their interaction than in the case where the beam has protons with randomly orientated spins (an unpolarized beam).

The theory of nuclear reactions with polarized beams has been developed over many years (by Wolfenstein and others) and has been cast into a form where the polarization aspects of a reaction are expressed in terms of polarization observables, which can be measured in a nuclear reaction. The National Accelerator Centre (NAC) at Faure will, in due course, install an ion source for polarized protons (spin–$\frac{1}{2}$) and deuterons (spin–1) and embark on reactions with polarized beams. In order to support such experiments, it is the purpose of this study to start an indepth study of the polarization aspects of nuclear reactions and the way in which it yields additional information on the interaction between nucleons. In addition, it is aimed to investigate the relativistic effects on the theoretical description of the polarization observables, a phenomenon which has recently been pointed out to be of importance.

For this reason, as well as for convenience, this thesis is divided into two main parts:

PART I: The polarization formalism for elastic proton scattering, and

PART II: Relativistic effects on the measured polarization observables.

Part I commences with the historical development of polarization in nuclear reactions and its applications. It proceeds with the description of polarized beams (the density matrix representation), the scattering amplitude formalism for nuclear reactions and the incorporation of polarization effects. For the clearest illustration of the principles involved, the theory is applied to the most simple cases: that of nucleon–nucleon and that of spin–$\frac{1}{2}$ on spin zero

scattering. (The description of the polarization of deuterons with spin one is more complicated and involves rank–2 tensors; this is briefly introduced in Appendix N).

Because some historically introduced polarization terminology is ambiguous and often leads to confusion, this presentation strongly emphasizes the present definition and interpretation of polarization parameters. It leads to the definition of the polarization observables (also expressed as the well–known Wolfenstein–parameters), and a discussion of their physical meanings and the principles of their measurements. It is finally shown that for the simple case of spin–$\frac{1}{2}$ particles scattered by spin zero nuclei, the measurement of three independent scattering observables is required: the differential reaction cross section $d\sigma/d\Omega$, the polarization parameter P and the spin rotation function Q, which will be defined later.

Concerning part II, it became clear in recent years that the measured polarization observables can not be well reproduced by scattering theories which are based on only a non–relativistic (Schrödinger) microscopic description of the interaction. A very convenient formalism which enables a comparison between non–relativistic and relativistic (Dirac) calculations has been presented by Hynes, Picklesimer, Tandy and Thaler (Hy 85). It was decided to apply the theory of Part I to a practical case and calculate some polarization observables which will be within the realm of the NAC and, at the same time, investigate the sensitivity of the relativistic correction in order to reproduce experimental data. The formalism of Hynes et al., is employed and, for the numerical calculation, the computer code WIZARD1 was finally obtained from Drs. M. Hynes and A. Picklesimer of the Los Alamos National Laboratory. This code was implemented on the VAX computer system at NAC and calculations were made for the elastic scattering of polarized protons of 135 MeV and 155 MeV by the spin zero targets of ^{24}Mg, ^{28}Si and ^{32}S. Part II first discusses the relevant scattering theory and its adaptation to a relativistic (Dirac) description, and finally presents the numerical results.

Throughout the text it is assumed that the reader is familiar with the basic elements of quantum mechanical scattering theory and relativistic quantum mechanics. In order to avoid impeding the main line of discussions, much of the underlying theoretical elements and relations are dealt with in Appendices A to N.

CHAPTER 2

2. Historical Development of Polarization Studies:

It seems necessary to sketch briefly the history of the subject of polarization at this stage, and give some of the relevant references, so as to place the subject matter in perspective. For convenience, we shall consider the theoretical and experimental studies of polarization in nuclear reactions separately.

2.1 Historical development of the *Theory* of Polarization

Schwinger (Sc 46, Sc 48) was the first to point out the possible importance of polarization studies in nuclear research, as well as give definite suggestions for obtaining and detecting polarized beams. *Lepore* (Le 50) made detailed calculations based on the first of *Schwinger's* suggestions, namely that a neutron beam could be polarized and analyzed by scattering from helium.

Wolfenstein (Wo 49, Wo 52, Wo 54, Wo 56) has made extensive studies aimed at the problem of nucleon–nucleon scattering. *Blin–Stoyle* (Bl 51) first made the very important observation that polarized neutrons should be a frequent product of nuclear reactions. This meant that by proper choice of the neutron–producing reaction, a strongly polarized beam could be obtained for use in the study of another reaction, a large intensity factor thereby being gained. *Blin–Stoyle* at the same time made a real beginning on a theory of reactions with arbitrary spins involved.

Blatt and *Biedenharn* (Bl 52) were first to give a complete account of the angular distribution resulting from an unpolarized beam incident on an unpolarized target. They made use of the Racah techniques, which were quickly seen to be completely appropriate for the more complex problem involving polarized beams, when augmented by the concept of statistical tensors introduced by *Fano* (Fa 52).

The generalization was performed by *Simon* and *Welton* (Si 53a, Si 53b), so that general formulae were then available for calculating the final polarization state in terms of the initial polarization state. Simultaneously and independently, the general theory was given by *Coester* and *Jauch* (Co 53) and by *Satchler* (Sa 53).

Devons and *Goldfarb* (De 57) have given a most elegant and complete treatment of the general problem of angular correlations with polarization. Other very useful collections of results have been given by *Goldfarb* and *Rook* (Go 59) and by *Baldin*, *Goldanskii* and *Rozenthal* (Ba 61). A treatment which is primarily aimed making at many of the complex formulae easily accessible and reliable, and where all derivations have been made afresh with extensive checks, is given by *Welton* (We 63). More recently Pal (Pa 83) discusses the theory of polarization for nucleon–nucleon and spin–$\frac{1}{2}$ on spin zero scattering in a masterful way. Ohlsen [Oh 70, Oh 72(a), Oh 72(b)] discusses the latter as well as scattering involving higher spins. Both Pal and Ohlsen base their work on the original article written by Wolfenstein (Wo 56).

2.2 Historical Development of Polarization *experiments*

In the section we shall briefly comment on the historical background of polarization studies in nuclear physics. More extensive literature on this subject may be found in the proceedings of conferences dealing with polarization studies in nuclear physics (Hu 60, Hu 65, Ba 70, Gr 75, Oh 80, Ko 85).

Consider an unpolarized beam of particles incident on an unpolarized target. Usually the scattered particles are polarized. The question arises as to what causes the outgoing particles to be polarized. In general there are two sources of polarization in scattering experiments i.e.:

(i) electromagnetic spin–orbit interactions and

(ii) nuclear spin–orbit interactions

Let us consider the historical development and description of these sources of polarization.

2.2.1 Mott Scattering of Polarized Electrons (Electromagnetic spin–orbit interaction)

In 1929, as part of his work on scattering of electrons by nuclei, *Mott* (Mo 29) suggested that scattered electrons should be expected to have a preferred orientation of their magnetic moments, i.e., the scattered electrons should be partially polarized. The mechanism leading to such a partially polarized beam is described in appendix A.

The prediction that electrons scattered by nuclei are polarized was verified by a double scattering experiment by *Shull*, *Chase* and *Myers* (Sh 43) in 1943.

2.2.2 Polarization of nucleons : Nuclear Spin–orbit interactions

In nuclear physics the spin–orbit interaction involving nuclear forces is a much larger effect than that caused by electromagnetic forces. Polarization experiments usually have the purpose of studying the nuclear spin–orbit interaction.

In appendix B is shown that the large nuclear spin–orbit interaction is a natural consequence when using the Dirac equation of relativistic nuclear physics. A Foldy–Wouthuysen reduction of the Dirac equation for a nucleon moving in the scalar and vector fields $\phi_0(r)$ and $V_0(r)$ allows one to identify the effective single–particle spin–orbit interaction $\alpha_{s\cdot o}\vec{S}.\vec{L}$ (Wa 85),

where $\alpha_{s\cdot o}(r) = \dfrac{g_v\ V_0'(r) + g_s\ \phi_0'(r)}{2\ M^2 r}$.

Refer to appendix B for derivation and notation involved in this expression.

Note that V_0 and ϕ_0 tend to add up and thus provide the large nuclear spin–orbit interaction required for understanding polarization phenomena.

The spin–orbit splitting has long been considered a characteristic feature of the shell model of the nucleus. The origin of the effective shell model spin–orbit interaction has been pursued for

many years. In an article written by *Signell* (Si 70) he states: "One concludes that we are a long way from understanding the origin of that part of the nucleon–nucleon interaction which we believe produces the main part of the spin–orbit splitting".

The first mention of nuclear spin–orbit coupling was in a paper by *Inglis* (In 36), who in 1936 interpreted the lowest two states in ^{7}Li as a $p_{\frac{1}{2}} - p_{\frac{3}{2}}$ doublet. His calculated splitting of 0.2 MeV was in reasonable agreement with the observed value of 0.4 MeV. This accidental agreement led physicists to ignore evidence for strong nuclear spin–orbit coupling for many years. Later, however, it was learned that in general the theory greatly underestimated the splitting.

The first clear evidence for a large spin–orbit force in nuclei was reported from studies in which fast neutrons were scattered by helium, but here again progress was delayed because of contradictory interpretations of the results by different researchers. In spite of work done by *Landau* and *Smorodinski* (La 44) in 1944 to conclude the existence of a large spin–orbit force in nuclei, it took several years before a nuclear spin–orbit splitting of several MeV was accepted. The large nuclear splitting was rediscovered in 1949 when the shell model was introduced to explain the systematics of nuclear ground states.

Methods to detect polarization effects in nuclear reactions or scattering were proposed in the literature several years before the first experiments in 1952. One idea, advanced by *Schwinger* in 1948 (Sc 48), was to produce and detect neutron polarization using the magnetic moment interaction discussed in appendix A. For nucleons, however, we are dealing with a much smaller magnetic moment (compared to that for electrons) and it can be shown that magnetic–moment scattering of neutrons produces large polarization only at very small scattering angles ($\simeq$1%).

For fast neutrons, detection of small angle scattering is very difficult because of the large background from the direct beam, but Mott–Schwinger scattering of fast neutrons was demonstrated in 1956 at *Harwell* (Vo 56).

An even earlier proposal by *Schwinger* (Sh 46) was to detect double scattering of neutrons by helium. While this experiment was not practical with neutrons, *Heusinkveld* and *Freier* (He 52) published the results of a proton double–scattering experiment by Helium early in 1952. Not only did this experiment provide the first clear demonstration of a polarization effect, but it also confirmed that the sign of the nuclear spin–orbit coupling is opposite to that in atoms, in agreement with the shell model.

The first neutron polarization experiments were also reported in 1952 by two groups (Hu 53, Ri 53) in Switzerland. These experiments took advantage of a suggestion by *Wolfenstein* (Wo 45) in 1949, that neutrons produced in a nuclear reaction might already be polarized on account of the spin–dependence in the neutron producing reaction.

Activity in the polarization field increased rapidly. Already in 1953, the first "high–energy" polarization experiments were reported (Ox 53, Ox 54), using the 220 MeV proton beam from the Rochester cyclotron. This led to the first measurement of the polarization in pp scattering. Shortly thereafter, additional results with 300 MeV protons were published by the group at *Berkley* (Ch 54).

Today, one has only to page through a journal in nuclear physics to appreciate the importance of polarization.

I shall now stress the reasons for performing polarization experiments.

CHAPTER 3

3. Why Polarization Experiments?

The aim of this section is to illustrate why it is important to perform polarization experiments. Much of the terminology might not be understood at this stage, as this serves only to inform about the use of polarization experiments. Most points mentioned do not fall into our field of specialization – so, what follows, merely serves to enlighten.

(a) Polarization experiments are useful because they help determine

(1) the spin–dependent parts of the nuclear Hamiltonian

(2) the quantum numbers, particularly the spin quantum numbers of excited states (Ma 70).

(b) Investigations of the polarization in inelastic scattering when combined with elastic scattering can provide definite information on multipole moments and on the nature of the nucleon–nucleus interaction and the nuclear transition it induces. The need for a vibrating spin–orbit term was discovered in this fashion (Ba 70).

(c) Performing stripping reactions, using polarized deuteron beams, have been used to determine the j–values transferred in the reaction (Ad 66).

(d) A careful polarization experiment can help to determine various resonance parameters such as total width, partial width and resonance energy (Ad 66).

(e) Since spin–orbit coupling can cause polarization to occur one may say that the experiments provide rather direct evidence for the spin–orbit coupling. Thus, one has become more cautious about accepting calculations based on the assumption of central forces. It would appear therefore, that polarization experiments have helped to make the interpretation of reactions more realistic (Hu 60).

(f) Data on spin rotation– and polarization transfer observables in $(\vec{p},\vec{p})$, $(\vec{p},\vec{p}\,')$ and $(\vec{p},\vec{n})$ experiments allow one to approach an experimental decomposition of the various spin–dependent contributions to effective nucleon–nucleon scattering amplitudes, and thereby to isolate the longitudinal and transverse spin responses of nuclei (In 85, p.53).

(g) A major experiment has been undertaken at Indiana to search for charge symmetry breaking (CSB) components of the free NN interaction, by measuring left–right asymmetries in the elastic scattering of polarized neutrons by polarized protons (In 85,p.53).

(h) Calculations of the Wolfenstein parameter D and the spin correlation parameter C_{nn} demonstrate an appreciable off–shell sensitivity for these second–order observables. As more accurate measurements become feasible, second–order polarization observables will undoubtedly prove to be valuable testing ground for off–shell effects (In 85, p.69).

(i) The analyzing–power angular distributions at medium energies exhibit characteristic signatures of dynamic spin effects which allow us to determine in particular the sign, strength and energy dependence of the imaginary spin–orbit part relative to those of the corresponding real part of the optical potential (In 85, p.80).

(j) A complete set of observables would provide more stringent tests of the predictions of various microscopic models (In 85, p.83).

(k) The spin rotation parameter, Q, is essential for resolving ambiguities and imposing severe constraints on the parameterization of phenomenological potential models. The spin observables A_y (analyzing power) and Q also play a key role in establishing the importance of relativistic effects in elastic scattering. Part II of this thesis will deal with the latter point. Measurements of Q, in conjunction with $\frac{d\sigma}{d\Omega}$, the differential cross–section, and A_y will also permit the direct determination of the full scattering matrix for a spin $\frac{1}{2}$–spin 0 system (see section 4.3.5).

(l) Much of the recent emphasis in the inelastic proton scattering program at Indiana has been to investigate certain open questions (eg. prescription for spin flip and isoscalar terms) and to try to elucidate the nature of the effective nucleon–nucleon interaction by examining observables, other than cross–section and analyzing power, which are especially sensitive to the spin–dependent components of the interaction. Measurements of the spin–flip probability, and the difference between outgoing polarization and analyzing power, P–A, clearly indicate the enhanced sensitivity of these quantities to only a few spin–dependent pieces of the force (In 85, p.98/99).

(m) Measurements of polarization transfer coefficients at Indiana for inelastic scattering will examine the 1^+, T=0 transition in ^{12}C. These results for isoscalar and isovector transitions will be complementary, and should provide particularly stringent tests of the Distorted–Wave Impulse Approximation (DWIA) and the underlying NN interaction (In 85, p.100).

(n) Measurements of the transverse spin transfer in (p,n) reactions provide a unique signature for Gamow–Teller (GT) transitions which heretofore have been identified only through less direct arguments (In 85, p.109).

(o) From the measurement of analyzing power and differential cross–section one can determine the scattering phase shifts $\delta^J_\ell(E)$ (Ha 85, p.15).

(p) The ambiguity between the absorption term and the spin–orbit term in the analysis of cross–section data is resolved by the addition of analyzing power measurements (Ha 85, p.34).

(q) Measurements of tensor analyzing powers T_{21} (see appendix N) and of the linear combination $T_{22} - \sqrt{\frac{3}{2}}\, T_{20}$ are particularly useful to study tensor potentials, because to a good approximation these quantities are not affected by the spin–orbit term. In contrast, T_{20} and T_{22} are sensitive to spin–orbit and tensor potentials (Ha 85, p.38).

(r) Tensor analyzing powers have been recently measured to determine the quadrupole moment of the deuteron at low bombarding energies, when only Coulomb scattering is important (Ha 85, p.39).

(s) In the case of proton elastic scattering from ^{3}He, measurements of the spin correlation parameter are sufficient to discriminate between otherwise equally acceptable solutions in a phase shift analysis (Ha 85, p.44).

(t) Measurements of the depolarization parameter D, on targets with spin can serve to detect a spin–spin term in the nucleon optical potential. Another method to detect spin–spin effects is to look for a difference in total cross–section for parallel and anti–parallel spin (Pr 90).

(u) Measurements of complete sets of observables in elastic scattering (differential cross–section $\frac{d\sigma}{d\Omega}$, induced polarization P or analyzing power A_y, and spin rotation functions Q) have played a crucial role in arriving at a successful theoretical description of the elastic scattering amplitudes. At present impressive agreement with experiment has been obtained at energies of 200 – 500 MeV (Wa 85).

We have stressed the importance of performing polarization experiments. Perhaps we can appropriately conclude this section by quoting from Stanley Hanna's summary of the Fifth International Symposium on Polarization Phenomena in Nuclear Physics (Sa 80): "I believe this conference has greatly reinforced the idea that *all* nuclear reactions should be carried out polarized. I do not mean, of course, that there cannot be exceptions such as searching for a state or measuring an important cross section. But if one wants to know the reaction mechanism and the nuclear structure involved the polarization parameters are essential".

Next, we introduce the terminology used in the description of polarization experiments.

CHAPTER 4

4. Description of Polarized beams and Polarization Experiments for Spin–½ particles

4.1 Terminology

The polarization of an assembly (beam or target) of spin–½ particles is characterized by a direction and magnitude, namely the average value of the particles spins, the average being taken over all the particles of the assembly (see section 4.3.1.1).

For particles of spin–1, one distinguishes between *vector polarization* and *tensor polarization* of the beam. The reason for this will be discussed later. The parameters describing the two kinds of polarization will be considered at a later stage (refer to appendix N).

The term tensor polarization refers to the fact that the description of spin–1 particles involves a second rank tensor. In general, particles of spin–I are described by tensors of rank 2I so that for I>1 one would have to distinguish between rank–2 polarization parameters, rank–3 polarization parameters, etc. The parameters describing the polarization state are referred to as the *polarization parameters* of the beam or as the *spin tensor moments*. A beam is called *polarized* if any of the parameters are different from zero, no matter whether the polarization is of vector or tensor type, or both. Some authors speak of a beam as being *aligned* if it possesses any form of rank–2 tensor polarization (i.e. a beam with only even rank components), while others reserve the term for spin systems with equal populations in states m_I and $-m_I$.

With the growing practical importance of deuteron polarization experiments it has become convenient to adopt certain conventions of nomenclature and notation for the description of polarization of spin–1 particles. This thesis follows the so–called *Madison Convention* which was adopted in 1970 at the Third International Symposium on Polarization Phenomena in Madison, Wisconsin (Ba 70). Some of the recommendations are applicable to particles of any spin. One of these is a proposed shorthand notation to indicate the character of the polarization experiment. In writing a nuclear reaction A(a,b)B, an arrow is to be placed over the particles whose polarization is being observed.

Thus $^3H(\vec{d},\vec{n})^4He$ indicates that an unpolarized ^3H target is bombarded with polarized deuterons and that the polarization of the outgoing reaction is observed.

When one speaks of measuring the *polarization* of particles b in a nuclear reaction, one is referring exclusively to the process $A(a,\vec{b})B$, i.e. observation of the polarization state of particles b with an unpolarized incident beam and target. The parameters which describe the change in cross–section when the incident beam or target (but not both) is polarized are referred to as the *analyzing powers* in the reaction $A(\vec{a},b)B$. The two quantities (i.e., polarization and analyzing power) must be carefully distinguished except in some special cases (see appendix J on the polarization–asymmetry theorem).

Reactions of the type $A(\vec{a},\vec{b})B$, $\vec{A}(a,\vec{b})B$, etc. are often called *polarization transfer* experiments. The parameters which relate the spin moments of particles $\vec{b}$ to those of $\vec{a}$, for example, are referred to as the *polarization transfer coefficients*. In the case of elastic scattering the same parameters are traditionally called triple scattering parameters.

Finally, the term *spin correlation experiments* refers to reactions $\vec{A}(\vec{a},b)B$ or $A(a,\vec{b})\vec{B}$ where in the latter case, the polarization of both outgoing particles is measured for the same event in coincidence.

We now show which coordinate systems are convenient for describing polarization experiments.

4.2 Coordinate Systems

The description of the polarization of either the incoming or outgoing beam in a nuclear reaction must refer to a particular coordinate system. The Basel Convention (Hu 60) addressed itself to the special case of a reaction $A(a,\vec{b})B$ where $\vec{b}$ is a particle of spin–$\frac{1}{2}$.

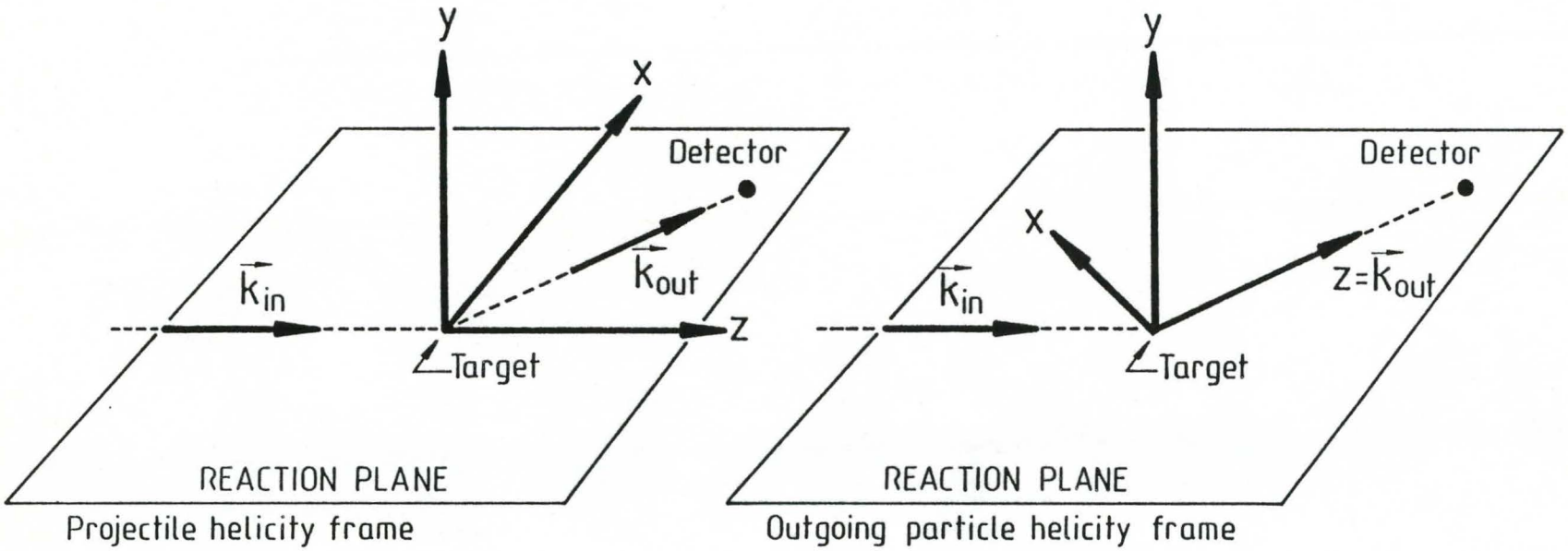

Fig. 4.1 Coordinate systems recommended by the Madison Convention. The system on the left is normally used when the incident beam is polarized, the system on the right is used to describe the polarization of the outgoing beam.

Parity conservation in strong reactions requires that the polarization vector of particle b is normal to the reaction plane $\vec{k}_{in} \times \vec{k}_{out}$, where $\vec{k}_{in}$ and $\vec{k}_{out}$ are the momentum vectors of the incoming and outgoing beams, respectively. According to Basel Convention the polarization is positive if the spin of $\vec{b}$ is predominantly in the direction $\vec{k}_{in} \times \vec{k}_{out}$.

While the opposite definition was used often prior to 1960, all recent papers have adhered to this convention. The Madison Convention specifies the following: The polarization of beams of particles should be referred to a right–handed coordinate system in which the positive z–axis is along the direction of momentum of the particles, and the positive y–axis is along $\vec{k}_{in} \times \vec{k}_{out}$ for the nuclear reaction which the polarized particles initiate, or from which they are emerging. In the latter case it should be stated whether $\vec{k}_{out}$ is in the centre–of–mass or laboratory system.

The two recommended systems are shown in figure 4.1. The one on the left would normally be used to describe the polarization of the incident beam (called the projectile helicity frame), and the one on the right to specify the polarization of the outgoing beam (called the outgoing particle helicity frame). In the latter case, in reporting the results of calculations, the z–axis is usually chosen along the outgoing momentum in the centre–of–mass system, but the laboratory system is often preferred when reporting experimental results.

Further reading about coordinate systems can be found in the following references: Ha 74, Ha 85, Oh 70, Oh 72(a), Oh 72(b).

4.3 Polarization in the scattering of a spin–½ particle on a spin zero target and also in nucleon–nucleon scattering

4.3.1 General concepts and definitions

4.3.1.1 Polarization of an ensemble of spin–½ particles

A spin–½ particle, such as a nucleon, is described by a two–component wave

function. It is explained in appendix D that the spin of a particle described by a wave function $\begin{bmatrix} a \\ b \end{bmatrix}$ "points in the direction" of the unit vector $\hat{n}$ specified by the angles (θ,ϕ), such that

$$a = \cos\left[\frac{\theta}{2}\right]e^{-i\left[\frac{\phi}{2}\right]}$$

$$b = \sin\left[\frac{\theta}{2}\right]e^{i\left[\frac{\phi}{2}\right]} \qquad 4\text{–}1$$

This statement means that the operator $(\vec{\sigma}.\hat{n})$ has the unit eigenvalue for the state $\begin{bmatrix} a \\ b \end{bmatrix}$.

A *single* spin $\frac{1}{2}$–particle, described by any spinor wavefunction, is therefore *completely* polarized in the direction $\hat{n}$ whose (θ,ϕ)–angles are given by (4–1). Since $(\vec{\sigma}.\hat{n})$ has the value unity, we conclude that $\vec{\sigma}$ has the value $\hat{n}$ for the spin state $\begin{bmatrix} a \\ b \end{bmatrix}$. Therefore, the expectation value $< \vec{\sigma} >$ for the spin state $\begin{bmatrix} a \\ b \end{bmatrix}$ can be taken to be the definition of polarization of this state because the expectation value has *magnitude* unity and *direction* $\hat{n}$.

The concept of partial polarization or zero polarization applies to a *beam* of particles. Suppose we have a beam of N spin–$\frac{1}{2}$ particles, specified by the spin states $\begin{bmatrix} a^{(r)} \\ b^{(r)} \end{bmatrix}$ with $r = 1,2,....,N$. For the r^{th} particle, the expectation value of $\vec{\sigma}$ defines the direction, say $\hat{n}^{(r)}$. In general, this direction may vary for different particles. The *polarization* of the *entire beam* is defined to be the average of $\vec{\sigma}$ over the whole beam, i.e., by

$$N^{-1}\sum_{r=1}^{N}\hat{n}^{(r)}$$

We shall denote the average of the expectation value of any quantity for the whole beam of particles by an overhead bar. Thus, the definition of the polarization of a *beam* of spin–$\frac{1}{2}$ particles is given by

$$\vec{P} = \langle \overline{\vec{\sigma}} \rangle = N^{-1} \sum_{r=1}^{N} (a^{(r)}\, b^{(r)}) \, \vec{\sigma} \begin{bmatrix} a^{(r)} \\ b^{(r)} \end{bmatrix}$$

$$= N^{-1} \sum_{r=1}^{N} \hat{n}^{(r)} \qquad 4\text{–}2$$

where the polar angles $\theta^{(r)}$, $\phi^{(r)}$ specifying the unit vector $\hat{n}^{(r)}$ are given in terms of the state amplitudes $a^{(r)}$, $b^{(r)}$ according to 4–1.

If the beam consists of a very large number of particles having the directions $\hat{n}^{(r)}$ *orientated completely at random*, then the summation in (2) can obviously be replaced by an integration over the entire range of the (θ,ϕ)–angles. The components of $\hat{n}$,

$n_x = \sin\theta \cos\phi,$

$n_y = \sin\theta \sin\phi,$

$n_z = \cos\theta,$

all give zero when integrated over

$$\int_0^{2\pi} d\phi \int_{+1}^{-1} d(\cos\theta)$$

Thus, the definition 4–2 yields *zero polarization* for a beam of spin $\frac{1}{2}$ particles having individual spin directions $\hat{n}$ completely at random in space.

Clearly, the maximum value of the magnitude of $\vec{P}$ is achieved when all the N unit vectors point in the same direction, i.e., $|\vec{P}| = 1$. We can also conclude that if the directions are not completely at random, then $|\vec{P}| < 1$. Therefore, in general, the polarization $\vec{P}$, as defined in 4–2 has a magnitude $|\vec{P}|$ less than or equal to one and a resultant direction determined by the vector addition of the N unit vectors.

4.3.1.2 Introducing the Density Matrix

A very convenient quantity for carrying out the averaging over all the particles just mentioned is the statistical density matrix first introduced by J von Neumann in 1927. We denote the density matrix by ρ, and shall precisely define it in the next paragraph.

Since our discussion is geared to polarization, the states drawing our attention are the spin states of the particles, and hence the density matrix for our purpose is a matrix in spin–space. One should, however, remember that the concept of a density matrix is broad–based and not restricted to spin states. In the definition that follows, the notation is therefore kept general.

Let a complete set of basic states for the specification of the states of the individual particles be denoted by

$|i\rangle, i = 1,2,...n.$

In terms of these states, we can specify the state $|\Psi^{(r)}\rangle$ of the r^{th} particle as:

$$|\Psi^{(r)}\rangle = \sum_{i=1}^{n} |i\rangle \langle i|\Psi^{(r)}\rangle \qquad 4\text{–}3$$

$$= \sum_{i=1}^{n} C_i^{(r)} |i\rangle \qquad 4\text{–}4$$

where

$$C_i^{(r)} \equiv \langle i \mid \Psi^{(r)}\rangle \qquad 4\text{–}5$$

The operator ρ corresponding to the density matrix is now defined by (Pa 83)

$$\rho = N^{-1} \sum_{r=1}^{N} |\Psi^{(r)}\rangle \langle\Psi^{(r)}| \qquad 4\text{–}6$$

such that its matrix element between the basic states $\langle i|$ and $|j\rangle$ becomes

$$\langle i|\rho|j\rangle = N^{-1} \sum_{r=1}^{N} \langle i|\Psi^{(r)}\rangle \langle \Psi^{(r)}|j\rangle \qquad 4\text{–}7$$

$$= N^{-1} \sum_{r=1}^{N} C_i^{(r)} C_j^{(r)*} \qquad 4\text{–}8$$

Sakurai (Sa 85) defines the density operator by

$$\rho = \sum_{i=1} \omega_i |\Psi^{(i)}\rangle \langle \Psi^{(i)}| \qquad 4\text{–}9$$

where ω_i is the fraction of particles characterized by the ket $|\Psi^{(i)}\rangle$ such that the fractional populations are constrained to satisfy the normalization condition

$$\sum_i \omega_i = 1 \qquad 4\text{–}10$$

See appendix E for clarification of the nomenclature "*density* matrix". Note that ρ is a projection operator onto the state $|\Psi^{(i)}\rangle$. Furthermore, the number of terms in the i sum of 4–10 need not coincide with the dimensionality N of the ket space; it can easily exceed N. For example, for spin $\frac{1}{2}$ systems with N=2, we may consider 40% with spin in the positive z–direction, 30% with spin in the positive x–direction, and the remaining 30% with spin in the negative y–direction.

From definitions 4–6 and 4–9 we see that the density operator contains all the physically significant information we can possibly obtain about the ensemble (an ensemble is by definition a collection of physical systems) in question.

The usefulness of the density matrix becomes apparent when we determine the ensemble average of any operator A denoting a physical observable. By the ensemble average of A we mean the average measured value of A when a large number of measurements are carried out.

By definition the expectation value of any operator A, denoting a physical observable, for the r^{th} particle is given by

$$\langle A \rangle = \frac{\langle \Psi^{(r)} | A | \Psi^{(r)} \rangle}{\langle \Psi^{(r)} | \Psi^{(r)} \rangle}$$

$$= \frac{\sum_{j}^{n} \sum_{i}^{n} \langle \Psi^{(r)} | j \rangle \langle j | A | i \rangle \langle i | \Psi^{(r)} \rangle}{\sum_{i} \langle \Psi^{(r)} | i \rangle \langle i | \Psi^{(r)}}$$

$$= \frac{\sum_{j,i}^{n} C_i^{(r)} C_j^{(r)*} \langle j | A | i \rangle}{\sum_{i} |C_i^{(r)}|^2} \qquad 4\text{–}11$$

The denominator accounts for the fact that the state $\Psi^{(r)}$ may not be normalized to unity. Also assume, however, the normalization constant necessary for this purpose to be independent of r.

Our task now is to carry out the averaging of the expectation value over all the particles in the beam. We thus obtain

$$\langle \overline{A} \rangle = N^{-1} \sum_{r=1}^{N} \frac{\langle \Psi^{(r)} | A | \Psi^{(r)} \rangle}{\langle \Psi^{(r)} | \Psi^{(r)} \rangle} \qquad \text{(per definition of the average value)}$$

$$= \frac{N^{-1} \sum_{i,j}^{n} \sum_{r=1}^{N} C_i^{(r)} C_j^{(r)*} \langle j | A | i \rangle}{N^{-1} \sum_{i} \sum_{r=1}^{N} |C_i^{(r)}|^2}$$

where the last expression is obtained from 4–11 plus the fact that the normalization of $\langle \Psi^{(r)} | \Psi^{(r)} \rangle$ in the denominator is independent of r.

$$\therefore \langle \overline{A} \rangle = \frac{\sum_{i,j}^{n} \langle i|\rho|j\rangle \langle j|A|i\rangle}{\sum_{i} \langle i|\rho|i\rangle} \quad \text{(definition 4–8 has been applied)}$$

$$= \frac{\mathrm{Tr}\ (\rho A)}{\mathrm{Tr}\ (\rho)} . \qquad 4\text{–}12$$

For example, the polarization of a beam, defined in equation 4–2 and which is relevant in this project is

$$\vec{P} = \mathrm{Tr}\ (\rho\vec{\sigma})/\mathrm{Tr}\ \rho$$

In the final step in the derivation of 4–12 "Tr" denotes the trace, i.e., the sum of the diagonal elements of the corresponding matrices. The result 4–12 is very important, and it shows that a knowledge of the n^2 matrix elements of the density matrix ρ is enough to carry out statistical averaging over all the particles of an ensemble.

In fact, the total number of independent matrix elements of ρ is even less than n^2. From 4-8, it is obvious that

$$\langle j|\rho|i\rangle = \langle i|\rho|j\rangle^{*} \qquad 4\text{–}13$$

i.e., the matrix is *Hermitean*. Thus, the n diagonal matrix elements and half of the n(n–1) nondiagonal matrix elements are really independent, so that, for carrying out a statistical average, the number of matrix elements we need to specify is

$$n + \tfrac{1}{2}n\,(n-1) = \tfrac{1}{2}n\,(n+1)$$

Furthermore, we have already assumed $\langle \Psi^{(r)} | \Psi^{(r)} \rangle$ to be independent of r; Therefore, let

$$\langle \Psi^{(r)} | \Psi^{(r)} \rangle \equiv \sum_{i=1}^{n} |C_i^{(r)}|^2 = C \qquad 4\text{–}14$$

and hence

$$\mathrm{Tr}\,\rho = \sum_{i=1}^{n} N^{-1} \sum_{r=1}^{N} |C_i^{(r)}|^2$$

$$= N^{-1} \sum_{r=1}^{N} \sum_{i=1}^{n} |C_i^{(r)}|^2$$

$$= C. \qquad 4\text{–}15$$

In particular, if all the states $\Psi^{(r)}$ are normalized to unity, then

$$\mathrm{Tr}\,\rho = 1 \qquad 4\text{–}16$$

$$\langle \overline{A} \rangle = \mathrm{Tr}\,(\rho A) \qquad 4\text{–}17$$

The subsidiary condition 4–15 or 4–16 imposes a further restriction on the n diagonal elements of ρ.

It is strongly recommended to read about the density operator as presented by Sakurai (Sa 85) – he gives several instructive examples of the density matrix.

The density matrices for polarized–, partially polarized– and unpolarized beams are discussed in appendix F.

4.3.1.2.1 Density matrices for the incident channel (ρ_{inc}) in a scattering experiment

I shall now motivate very briefly why it is important to discuss the incident–channel density matrices. Remember, our goal is to define and give meaning to the various spin observables (also called polarization observables). Now, as we shall demonstrate later on, all the spin observables can be expressed in terms of the final density matrix, ρ_{scatt},

where $\rho_{scatt} = M\,\rho_{inc}\,M^{\dagger}$ (see section 4.3.1.3)

where M is the scattering amplitude to be defined further on.

So, to be able to determine the spin observables, we need both the scattering matrix M and the initial density matrix (or incident–channel density matrix).

The discussion that follows may appear to be rather lengthy, but it is necessary to explain the meaning of the various spin observables as well as the physical principles involved in the formalism; the original article by Wolfenstein (Wo 52) discusses these observables in a much too formal way. A recipe in appendix F enables one to write down rather quickly, all the expressions for the spin observables. For convenience, refer to appendix F after completion of Chapter 4.

Having introduced the general concepts of a density matrix, we now start specializing it to the problem of spin polarization in nucleon–nucleon and spin–½ or spin zero scattering.

4.3.1.2.1.1 Spin–½ on spin zero scattering

We deal first with the simpler case of a beam of spin–½ particles scattered on a spin zero target. In this case the combined spin–space of the projectile and target is two–dimensional. In general, the dimension of combined projectile–target spin space is given by $(2S_p+1)(2S_t+1)$ where t refers to the target and p the projectile.

Hence we can expand the density matrix, ρ_{inc}, in terms of a complete set of 2×2 matrices, namely the 2×2 unit matrix 1, and the three Pauli spin matrices of appendix C. We introduce the symbol S_μ (μ=1,..., 4) to denote these four matrices.

Therefore,

$$\rho = \sum_{\mu=1}^{4} C_\mu S_\mu \qquad 4\text{–}18$$

From the properties of the Pauli matrices given in appendix C, equation (C7), we have

$$\mathrm{Tr}(S_\mu S_\nu) = 2\delta_{\mu\nu} \qquad 4\text{–}19$$

Multiply both sides of eq. 4–18 by S_ν and take the trace using the result 4–19. Then

$$C_\mu = \tfrac{1}{2}\,\mathrm{Tr}(\rho S_\mu) \qquad 4\text{–}20$$

Therefore,

$$\rho = \tfrac{1}{2} \sum_{\mu=1}^{4} \mathrm{Tr}\,(\rho S_\mu) S_\mu$$

$$= \tfrac{1}{2}\,\mathrm{Tr}(\rho)1 + \tfrac{1}{2}\,\mathrm{Tr}(\rho\vec{\sigma}).\vec{\sigma}$$

$$= \tfrac{1}{2}\,\mathrm{Tr}(\rho)\,\{1 + \langle\vec{\sigma}\rangle.\vec{\sigma}\} \qquad \text{(from eq. 4–12).}$$

$$\therefore \quad \boxed{\rho = \tfrac{1}{2}\,\mathrm{Tr}(\rho)\,\{1 + \vec{P}.\vec{\sigma}\}} \qquad \text{(from eq. 4–2)} \qquad 4\text{–}21$$

In the second step, the four operators 1 and σ_x, σ_y, σ_z for S_μ have been explicitly introduced.

4.3.1.2.1.2 Nucleon–nucleon scattering (or spin–$\frac{1}{2}$ spin–$\frac{1}{2}$ scattering)

In the case of nucleon–nucleon scattering, both the incident and target particles have spin–$\frac{1}{2}$, and hence the combined spin–space is four–dimensional. The

density matrix, ρ_{inc}, is expanded in terms of the 16 matrices:

$$1, \quad \vec{\sigma}_1, \quad \vec{\sigma}_2, \quad \vec{\sigma}_1\vec{\sigma}_2$$

where indices 1 and 2 label respectively the incident and target nucleons. Each of the vectors $\vec{\sigma}_1$, $\vec{\sigma}_2$ stands for three components, and the quantity $\vec{\sigma}_1\vec{\sigma}_2$ actually represents the nine components $\sigma_{1\alpha}\sigma_{2\beta}$, where each of α and β can be any of the components x, y, z; 1 is the four–dimensional unit matrix. The four–dimensional matrix corresponding to $\sigma_{1\alpha}$ or $\sigma_{2\beta}$ is obtained by taking the direct product of this 2×2 Pauli matrix with the 2×2 unit matrix for the target nucleon.

Any components of $\vec{\sigma}_1\vec{\sigma}_2$ are represented by the direct product of the corresponding Pauli matrices.

Now, the general expression of the density matrix, ρ_{inc}, is given by

$$\rho = \sum_{\mu=1}^{16} C_\mu S_\mu \qquad 4\text{–}22$$

where S_μ denotes any of the 16 four–dimensional matrices mentioned above.

It can easily be verified that:

$$\mathrm{Tr}\, S_\mu S_\nu = 4\delta_{\mu\nu} \qquad 4\text{–}23$$

One again, multiply both sides of eq. 4–22 by S_ν and take the trace using the result 4–23.

Then,

$$C_\mu = \tfrac{1}{4}\,\mathrm{Tr}\,(\rho\, S_\mu) \qquad 4\text{–}24$$

Therefore,

$$\rho = \tfrac{1}{4} \sum_{\mu=1}^{16} \mathrm{Tr}\,(\rho S_\mu) S_\mu \qquad 4\text{–}25$$

$$\rho = \tfrac{1}{4}\mathrm{Tr}(\rho)1 + \tfrac{1}{4}\mathrm{Tr}(\rho\vec{\sigma}_1).\vec{\sigma}_1 + \tfrac{1}{4}\mathrm{Tr}(\rho\vec{\sigma}_2).\vec{\sigma}_2 + \tfrac{1}{4}\,\mathrm{Tr}\Big(\sum_{\alpha,\beta}\rho\sigma_{1\alpha}\sigma_{2\beta}\Big)\,\sigma_{1\alpha}\sigma_{2\beta}$$

$$= \tfrac{1}{4}(\mathrm{Tr}\rho)\{1 + \vec{P}_1.\vec{\sigma}_1 + \vec{P}_2.\vec{\sigma}_2 + \sum_{\alpha,\beta} \langle\overline{\sigma_{1\alpha}\sigma_{2\beta}}\rangle\, \sigma_{1\alpha}\sigma_{2\beta}\} \qquad 4\text{–}26$$

In the second step the 16 operators for S_μ have been explicitly used. In the last line, each of the summations over α, β goes over the x–, y–, z–components, $\vec{P}_1$ and $\vec{P}_2$ are respectively the polarizations of the incident and target nucleons, and the average value of the product $\sigma_{1\alpha}\sigma_{2\beta}$ is called the *polarization correlation function.*

At this stage it is convenient to derive two very important relationships to be used in section 4.3.3 which deal with the principle of polarization experiments.

4.3.1.3 Two very important relationships

We now proceed to derive the following important relationships:

(i) $\rho_{\mathrm{scatt}} = M\rho_{\mathrm{inc}}\,M^{\dagger}$, and

(ii) $\dfrac{d\sigma}{d\Omega} = \dfrac{\mathrm{Tr}\ \rho_{\mathrm{scatt}}}{\mathrm{Tr}\ \rho_{\mathrm{inc}}}$ (Differential cross section).

In scattering formalism (Ja 70, Ta 72) the incident beam is represented by the incident wave (we take the direction of the incident beam to be the z–axis). Consider this beam as fully polarized, with spin projection m:

$$\Psi_{\mathrm{inc}} = e^{ikz}\,\chi^{s}_{m} \qquad 4\text{–}27$$

Corresponding to this incident wave, the scattered wave of spin projection m′ is described as:

$$\Psi_{scatt} = r^{-1} e^{ikr} M^{s}_{m',m}(\theta,\phi)\, \chi^{s}_{m}, \qquad 4\text{–}28$$

where M is the scattering amplitude.

In general, the scattered wave is found to consist of waves of all spin projections, and hence given by the sum

$$\Psi_{scatt} = r^{-1} e^{ikr} \sum_{m} M^{s}_{m',m}\, \chi^{s}_{m}, \qquad 4\text{–}29$$

Since our aim is to consider a more general polarized beam, we shall start with an incident wave in which the various spin states occur with *different* amplitudes a^{s}_{m}. Thus,

$$\Psi_{inc} = e^{ikz} \sum_{s,m} a^{s}_{m}\, \chi^{s}_{m} \qquad 4\text{–}30$$

The scattered wave corresponding to this incident wave is obviously obtained by multiplying eq. 4–29 by a^{s}_{m} and carrying out m and s summations. Thus,

$$\Psi_{scatt} = r^{-1} e^{ikr} \sum_{s,m} \sum_{m} M^{s}_{m',m}\, a^{s}_{m}\, \chi^{s}_{m},$$

$$= r^{-1} e^{ikr} \sum_{s,m} b^{s}_{m'}\, \chi^{s}_{m}, \qquad 4\text{–}31$$

$$\text{where } b^{s}_{m'} = \sum_{m} M^{s}_{m',m}\, a^{s}_{m} \qquad 4\text{–}32$$

The functions e^{ikz} and $r^{-1} e^{ikr}$ merely take account of the spatial dependence of the incident and scattered waves. The information on the incident and scattered density matrices is contained, according to eq. 4–8, in the coefficients a^{s}_{m} and b^{s}_{m}. For sake of simplicity, we use the compact notation i to denote the two quantum numbers (s,m) when we write, according to the definition 4–8, the density matrices for the incident and scattered state as:

$$\langle i|\rho_{inc}|j\rangle = N^{-1} \sum_{r=1}^{N} a_i^{(r)} a_j^{(r)*} \qquad 4\text{–}33$$

$$\langle i|\rho_{scatt}|j\rangle = N^{-1} \sum_{r=1}^{N} b_i^{(r)} b_j^{(r)*} \qquad 4\text{–}34$$

where N is the total number of particles and (r) labels the r^{th} particle. According to our simpler notation, the definition 4–32 can be rewritten as:

$$b_i = \sum_k M_{ik} a_k \qquad 4\text{–}35$$

Now,

$$\langle i|\rho_{scatt}|j\rangle = N^{-1} \sum_{r=1}^{N} b_i^{(r)} b_j^{(r)*}$$

$$= N^{-1} \sum_{r=1}^{N} \sum_k M_{ik} a_k^{(r)} \sum_\ell M_{j\ell}^* a_\ell^{(r)*} \quad \text{(from 4–35).}$$

$$= \sum_{k,\ell} M_{ik} \{N^{-1} \sum_{r=1}^{N} a_k^{(r)} a_\ell^{(r)*}\} (M^\dagger)_{\ell j}$$

$$= \sum_{k,\ell} M_{ik} \langle k|\rho_{inc}|\ell\rangle (M^\dagger)_{\ell j}$$

$$= \langle i|M\rho_{inc}M^\dagger|j\rangle \qquad 4\text{–}36$$

Thus, we have derived a very convenient result for later implementation, namely,

$$\boxed{\rho_{scatt} = M\rho_{inc}M^\dagger} \qquad 4\text{–}37$$

From the theory of scattering (Ja 70, Ta 72), the differential scattering cross–section may be defined as:

$$\frac{d\sigma}{d\Omega} d\Omega = \frac{|\vec{J}_{scatt}|\ r^2 d\Omega}{|\vec{J}_{inc}|} \qquad 4\text{–}38$$

where J_{scatt} and J_{inc} represent the scattered and incident probability currents respectively as $r \longrightarrow \infty$

$$\text{and } \vec{J} = \frac{\hbar}{\mu} \text{Im} \{\Psi^* \vec{\nabla} \Psi\} \qquad 4\text{–}39$$

where μ signifies the reduced mass of the system.

From eqs. 4–30 and 4–39 with $\hbar = 1$

$$|\vec{J}_{inc}| = v \sum_{s,m} |a_m^s|^2 = v \sum_i |a_i|^2 \qquad 4\text{–}40$$

where v is the incident velocity, and i the abbreviated notation for (s,m). The statistical average of the flux over the particles in the incident beam can be written as:

$$N^{-1} \sum_{r=1}^{N} v \sum_i |a_i^r|^2 = v(\text{Tr } \rho_{inc}) \quad \text{(from 4–33)} \qquad 4\text{–}41$$

Similarly, from eqs. 4–31 and 4–39

$$|\vec{J}_{scatt}| = \frac{v}{r^2} \sum_i |b_i|^2 \qquad 4\text{–}42$$

The statistical average of the outgoing flux through the area $r^2 d\Omega$ for the scattered state 4–31 is:

$$N^{-1} v d\Omega \sum_{r=1}^{N} \sum_i |b_i^r|^2 = v d\Omega \, (\text{Tr } \rho_{scatt}) \qquad 4\text{–}43$$

Therefore, the differential cross–section according to definition 4–38, is given by:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\text{Tr } \rho_{scatt}}{\text{Tr } \rho_{inc}} = \sigma(\theta)} \qquad 4\text{–}44$$

This is the second important relationship that will be of use in section 4.3.3. We now proceed to derive a general expression of the scattering amplitude, which contains much of the physics of the scattering problem.

4.3.2 General Expression of the Scattering Amplitude

The importance of obtaining expressions for the scattering amplitudes will become clear as soon the principle of polarization experiments is discussed.

Before we consider the general expression of M for nucleon–nucleon scattering, we examine a spin–½ particle scattered on a zero spin target. This case is much simpler and demonstrates the principles involved.

4.3.2.1 Spin–½ on spin zero scattering

In the case of a spin–½ particle scattering on a spin zero target, the scattering amplitude matrix M is two–dimensional (Ja 70). This means that M is expressible in terms of the four 2 × 2 matrices 1, σ_x, σ_y and σ_z. However, it has to be a scalar quantity (because its *magnitude* should not depend on the orientation of the coordinate frame), and we must therefore try to construct scalar expressions with the four operators and the kinematic variables for the scattering process. The latter are given by the incident and outgoing momentum vectors $\vec{k}_i$ and $\vec{k}_f$ in the laboratory system. Each of these is a polar vector* and, for *elastic* scattering (which we consider), both have that same *magnitude* k. The angle between them is the scattering angle θ. We can use them to construct the following different types of quantities:

$$k_i^2 = k^2,\ k_f^2 = k^2,\quad \vec{k}_i.\vec{k}_f = k^2\cos\theta \qquad \text{(scalar)};$$

$$\hat{X} = \frac{(\vec{k}_f - \vec{k}_i)}{|\vec{k}_f - \vec{k}_i|},\quad \hat{Z} = \frac{\vec{k}_i + \vec{k}_f}{|\vec{k}_i + \vec{k}_f|} \qquad \text{(polar vector)}; \qquad 4\text{–}45$$

$$\hat{y} = \hat{n} \text{ (used conventionally for unit vector normal on the scattering plane)}$$

$$\text{where } \hat{n} = \frac{(\vec{k}_i \times \vec{k}_f)}{|\vec{k}_i \times \vec{k}_f|} \qquad \text{(axial vector*)};$$

second–rank tensor.

*Vectors that change sign under a parity reflection are called *polar vectors*, if they conserve their sign, they are called *axial*–, or *pseudo vectors*.

The second–rank tensor constructed with $\vec{k}_i$ and $\vec{k}_f$ has not been explicity written down because it will not be able to combine to form a scalar quantity with any term from 1 to $\vec{\sigma}$. As there are no independent bilinear combinations of the components of a *single* spin operator $\vec{\sigma}$ [since $\sigma_\alpha^2 = 1$ and $\sigma_\alpha\sigma_\beta = i\sigma_\gamma$ (α, β, γ being a cyclic permutation of the x–, y–, z–components) it is not worthwhile to use bilinear terms in the components of $\vec{\sigma}$] we have only

1 (scalar)

$\vec{\sigma}$ (axial vector) 4–46

We are permitted to take the *scalar product* of these two matrices with the corresponding quantities in 4–45 thus forming scalar quantities. Thus the scattering amplitude is given by

$$\boxed{M = g(k^2, \cos\theta)1 + h(k^2, \cos\theta)\vec{\sigma}.\hat{n}} \qquad 4\text{–}47$$

where g and h are scalar functions of the kinematic variables k^2 and $\cos\theta$ appearing in the scalar expression of 4–45. It is worth noting that some authors multiply the amplitude h by the complex factor i.

Note that the requirement of a scalar expression of M guarantees its invariance under the rotation and reflection of the coordinate frame. Since the scattering potential has the additional property of time–reversal invariance, we should ascertain whether or not 4–47 satisfies this requirement. Under the time–reversal operation

$\vec{\sigma} \rightarrow -\vec{\sigma}$

$\vec{k}_i \rightarrow -\vec{k}_f$

$\vec{k}_f \rightarrow -\vec{k}_i$

when $t \rightarrow -t$.

$$\hat{n} \longrightarrow \frac{(\vec{k}_f \times \vec{k}_i)}{|\vec{k}_f \times \vec{k}_i|} = -\hat{n}$$

Therefore, 4–47 satisfies the time reversal invariance:

4.3.2.2 Nucleon–nucleon scattering

The foregoing considerations can be extended to the case of nucleon–nucleon scattering, except that now we have two spin operators $\vec{\sigma}_1$ and $\vec{\sigma}_2$ in order to construct the various scalar, vector and tensor quantities. The list 4–45 remains unchanged, but by $\vec{k}_i$ and $\vec{k}_f$ we now understand the *initial* and *final momenta* of the two–nucleon scattering *in the centre–of–mass coordinate system.* Further, the second–rank tensors need to be explicitly written. Once again M must be expressed as a scalar quantity such that invariance under the rotation and reflection of the coordinate frame is guaranteed. Furthermore, M must be invariant under the time reversal operation. More detail can be found in Pa 83. The general expression for M (for elastic scattering) is finally given by:

$$\begin{aligned} M = {} & g_o(k^2, \cos\theta)1 + f(k^2, \cos\theta)(\vec{\sigma}_1 - \vec{\sigma}_2).\hat{n} \\ & + h_o(k^2, \cos\theta)(\vec{\sigma}_1 + \vec{\sigma}_2).\hat{n} + h_X(k^2, \cos\theta)(\vec{\sigma}_1.\hat{X})(\vec{\sigma}_2.\hat{X}) \\ & + h_Z(k^2, \cos\theta)(\vec{\sigma}_1.\hat{Z})(\vec{\sigma}_2.\hat{Z}) + h_n(k^2, \cos\theta)(\vec{\sigma}_1.\hat{n})(\vec{\sigma}_2.\hat{n}) \end{aligned} \qquad 4\text{–}48$$

Note that for *elastic* scattering, $k_f = k_i = k$, and hence

$$\hat{X}.\hat{Z} = 0$$

$$\hat{Z} \times \hat{X} = \hat{n}$$

Thus, $\hat{X}$ and $\hat{Z}$ are perpendicular to each other, and a right–handed screw turned from $\hat{Z}$ to $\hat{X}$ goes towards the normal direction $\hat{n}$. Therefore, the direction $\hat{Z}$, $\hat{X}$ and $\hat{n}$ define a set of right–handed orthogonal coordinate axes. So far, the invariance requirements imposed on M imply that the interaction giving rise to the scattering is such that total angular momentum and parity are conserved.

Now let us consider the consequences imposed by the symmetry property of invariance under the exchange of the two nucleons. This, of course, presupposes that we are using the language of isospin to describe the two nucleons, where the nucleons are then completely equivalent in our description.

Each of the momenta $\vec{k}_i$ and $\vec{k}_f$, being the *relative momentum* of the nucleons, must change sign under an exchange of nucleons. Therefore,

$\hat{X} \rightarrow -\hat{X}$

$\hat{Z} \rightarrow -\hat{Z}$

$\hat{n} \rightarrow +\hat{n}$

and $\vec{\sigma}_1 \rightarrow \vec{\sigma}_2$

$\vec{\sigma}_2 \rightarrow \vec{\sigma}_1$

An examination of the different terms of M now reveals that the f– term changes sign under the exchange of the two nucleons, whereas all the other terms remain unchanged. We therefore discard the f–term and rewrite the general expression of M as

$$M = g_0 1 + h_0(\vec{\sigma}_1 + \vec{\sigma}_2).\hat{n} + h_X(\vec{\sigma}_1.\hat{X})(\vec{\sigma}_2.\hat{X}) + h_Z(\vec{\sigma}_1.\hat{Z})(\vec{\sigma}_2.\hat{Z}) + h_n(\vec{\sigma}_1.\hat{n})(\vec{\sigma}_2.\hat{n}) \qquad 4\text{–}49$$

where each of the coefficients g, h here is a function of k_2 and $\cos\theta$.

Finally, we try to express 4–49 in a more compact manner. Collecting the terms that are independent of $\vec{\sigma}_1$, we define their sum $g(k_2, \cos\theta)$ as

$$g(k^2, \cos\theta) = g_0(k^2, \cos\theta)1_2 + h_0\,\vec{\sigma}_2.\hat{n} \qquad 4\text{–}50$$

Similarly, the coefficients of all the terms dependent on $\vec{\sigma}_1$ are put together, and we define:

$$\vec{h}(k^2, \cos\theta) = h_o(k^2, \cos\theta)\hat{n}1_2 + h_X(k^2, \cos\theta)(\vec{\sigma}_2.\hat{X})\hat{X} + h_Z(k^2, \cos\theta)(\sigma_2.\hat{Z})\hat{Z} + h_n(k^2, \cos\theta)(\sigma_2.\hat{n})\hat{n} \qquad 4\text{–}51$$

With these definitions, M is clearly expressible as:

$$M = g(k^2, \cos\theta)1_1 + \vec{h}(k^2, \cos\theta).\vec{\sigma}_1 \qquad 4\text{–}52$$

Each of the expressions 4–50 and 4–51 is a matrix in the space of the target nucleon. The subscripts 2 and 1 on the unit matrices in 4–50 to 4–52 indicate that these are 2×2 unit matrices in the spin–space of the corresponding nucleons.

The formal similarity of the expression 4–52 and the expression 4–47 for a spin–$\frac{1}{2}$ particle scattered on a zero spin target should be noted.

4.3.3 Principles of Polarization Experiments

We are now in a position to work out the expressions that are necessary for a complete understanding of the different types of polarization measurements.

4.3.3.1 Differential Cross–Section of Polarized and Unpolarized Beams

$$\sigma(\theta) = \frac{d\sigma}{d\Omega} = \frac{Tr\ \rho_{scatt}}{Tr\ \rho_{inc}} \qquad \text{(from 4–44)}$$

$$= \frac{Tr\ (M\rho_{inc}\ M^{\dagger})}{Tr\ \rho_{inc}} \qquad \text{(from 4–37)}$$

$$= \frac{1}{4\ Tr\ \rho_{inc}} \sum_{\mu} Tr(\rho_{inc}\, S_{\mu})\, Tr(MS_{\mu}\, M^{\dagger}) \qquad \text{(from 4–25)}$$

$$= \tfrac{1}{4} \sum_{\mu} \langle S_{\mu} \rangle_{inc}\, Tr(MS_{\mu}\, M^{\dagger}) \qquad \text{(from 4–12)} \qquad 4\text{–}53$$

Making use of the 16 S_μ operators from 4–25, we derive the result

$$\sigma(\theta) = \tfrac{1}{4}\,\mathrm{Tr}(MM^\dagger) + \tfrac{1}{4}\,\vec{P}_1^{\,inc}.\mathrm{Tr}(M\vec{\sigma}_1 M^\dagger) + \tfrac{1}{4}\,\vec{P}_2^{\,inc}.\mathrm{Tr}(M\vec{\sigma}_2 M^\dagger) + \tfrac{1}{4}\sum_{\alpha,\beta} \langle\overline{\sigma_{1\alpha}\sigma_{2\beta}}\rangle\,\mathrm{Tr}(M\sigma_{1\alpha}\sigma_{2\beta}M^\dagger) \qquad 4\text{–}54$$

where $\vec{P}_1^{\,inc}$ and $\vec{P}_2^{\,inc}$ denote the polarization of the two nucleon ensembles in the initial state (i.e., the state Ψ_{inc}). Expression 4–54 is the general expression for the differential cross–section when both the incoming beam and the target are polarized. This expression can easily be specialized to the following two cases:

4.3.3.1.1 Both incoming and target nucleons unpolarized

In this case

$$\vec{P}_1^{\,inc} \text{ and } \vec{P}_2^{\,inc} = 0,\ \langle\overline{\sigma_{1\alpha}\sigma_{2\beta}}\rangle = 0$$

For this special case 4–54 becomes

$$\sigma_0 = \tfrac{1}{4}\,\mathrm{Tr}(MM^\dagger) \qquad 4\text{–}55$$

I shall now show in detail how to obtain an expression for σ_0 in terms of the matrix elements of M. This should give one an idea of how other similar expressions can be obtained for the differential cross–section. In future we shall merely mention results and refer the reader to Pa 83 for the detail involved in deriving these expressions.

We now have

$$\begin{aligned}
\sigma_0 &= \tfrac{1}{4}\,\mathrm{Tr}(MM^\dagger) \\
&= \tfrac{1}{4}\,\mathrm{Tr}\{(g1_1 + \vec{h}.\vec{\sigma}_1)(1_1 g^\dagger + \vec{\sigma}_1.\vec{h}^\dagger)\} \text{ (from 4–52 and its Hermitian conjugate)} \\
&= \tfrac{1}{4}\,\mathrm{Tr}\{g1_1 g^\dagger + g\vec{\sigma}_1.\vec{h}^\dagger + \vec{h}.\vec{\sigma}_1 g^\dagger + (\vec{h}.\vec{\sigma}_1)(\vec{\sigma}_1.\vec{h}^\dagger)\} \\
&= \tfrac{1}{4}\,\mathrm{Tr}_2\{2gg^\dagger + 0 + 0 + 2\vec{h}.\vec{h}^\dagger\} \qquad 4\text{–}56
\end{aligned}$$

In this step we have evaluated the traces of spin operators belonging to nucleon 1, using the standard results given in appendix C for the Pauli spin matrices. In the resultant expression, we still have to evaluate the trace in the space of nucleon 2, as indicated by Tr_2, because the original trace was in the four–dimensional combined spin–space of the two nucleons.

To simplify 4–56 further, we make use of 4–50 and 4–51 and recall that:

$$\vec{\sigma}_2.\hat{Z} = \sigma_{2x'}$$

$$\vec{\sigma}_2.\hat{X} = \sigma_{2y'}$$

$$\vec{\sigma}_2.\hat{n} = \sigma_{2z'}$$

Therefore, in evaluating Tr_2, we can again use the standard results given in appendix C. In this way, we obtain

$$\boxed{\sigma_0(\theta) = |g_0|^2 + 2|h|^2 + |h_X|^2 + |h_Z|^2 + |h_n|^2} \qquad 4\text{–}57$$

The right–hand side is function of k^2 and cos θ. There is no dependence on the azimuthal angle ϕ of the direction $\vec{k}_f$ of scattering. This is a consequence of the lack of polarization of the incident beam.

4.3.3.1.2 Target nucleons unpolarized; Incident nucleons polarized

Here

$$\vec{P}_2^{\,inc} = 0, \; \langle \overline{\sigma_{1\alpha}\sigma_{2\beta}} \rangle = 0$$

Expression 4–54 yields

$$\sigma(\theta) = \tfrac{1}{4}\mathrm{Tr}(MM^\dagger) + \tfrac{1}{4}\,\vec{P}_1^{\,inc}.\mathrm{Tr}(M\vec{\sigma}M^\dagger)$$

$$= \sigma_0(\theta)(1 + \vec{P}_1^{\,inc}.\vec{a}) \qquad 4\text{–}58$$

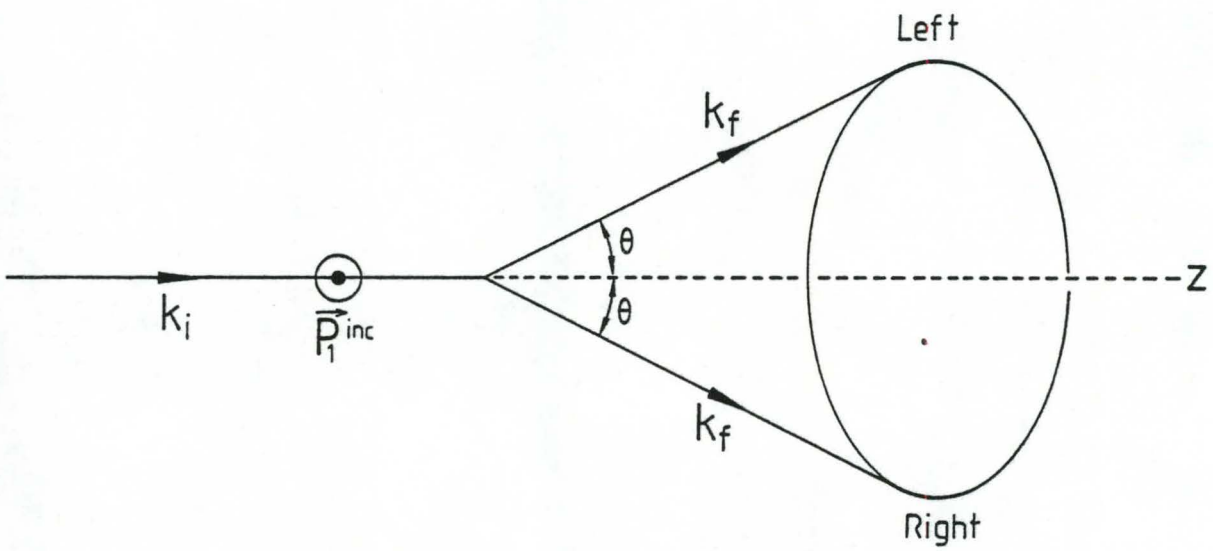

Fig. 4.2 Set-up for measuring left-right asymmetry of polarized nucleons.

where $\vec{a}$ is given by

$$\vec{a} = \frac{\mathrm{Tr}\ M\vec{\sigma}_1 M^\dagger}{\mathrm{Tr}\ MM^\dagger} \qquad 4\text{–}59$$

To simplify 4–59, we need the value of $\mathrm{Tr}(M\vec{\sigma}_1 M^\dagger)$, which is evaluated, as before, in two steps: first, Tr_1 is taken in the spin–space of nucleon 1, using M and $M^\dagger$ from 4–52; and then Tr_2 of the resultant expression is evaluated using 4–50 and 4–51. We finally get

$$\vec{a} = a\hat{n}$$

with

$$a = A_y = \frac{8\ \mathrm{Re}\,(g_0 + h_n)h_0^*}{|g_0|^2 + 2|h_0|^2 + |h_X|^2 + |h_Z|^2 + |h_n|^2} \qquad 4\text{–}60$$

Recall that 4–58 is invariant under the parity operation – in principle one could use this expression as a test of violation effects when comparing theory to experiment. The quantity a or A_y is called the *asymmetry parameter* for the present. At a later stage we shall see why A_y is also referred to as the *analyzing power* of the target. The reason for calling "a" the *asymmetry* parameter becomes obvious from the discussion that follows:

Recall that $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{k}_i$ and $\vec{k}_f$. The direction of $\vec{k}_i$ is the z–axis, and $\vec{k}_f$ makes an angle θ with this direction. Consider the very simple cases of $\vec{k}_f$ pointing "Left" (L) and "Right" (R) with respect to the incident beam direction $\vec{k}_i$, as shown in figure 4.2. Both directions $\vec{k}_f$ and $\vec{k}_i$ are assumed to be contained in the plane of the paper. However, according to the definition of $\hat{n}$, it may be directed away or towards the reader for the right and left cases respectively, thus

$$\sigma_L(\theta) = \sigma_0(\theta)\,(1 + aP_{1y}^{\mathrm{inc}}) \qquad 4\text{–}61$$

$$\sigma_R(\theta) = \sigma_0(\theta)\,(1 - aP_{1y}^{\mathrm{inc}}) \qquad 4\text{–}62$$

Therefore,

$$\frac{\sigma_L(\theta) - \sigma_R(\theta)}{\sigma_L(\theta) + \sigma_R(\theta)} = aP_{1y}^{inc} = A_y P_{1y}^{inc} = a\vec{P}_1^{inc}.\hat{n} \qquad 4\text{–}63$$

The left–hand expression is logically called the *left–right asymmetry.* This is exactly equal to a, the so–called asymmetry parameter, if $P_{1y} = 1$ (which means that the incoming beam is fully polarized perpendicular to the scattering plane). The latter experimental condition directly enables the measurement of a.

In general it is possible to rotate $\vec{k}_f$ around $\vec{k}_i$ on the surface of a cone, keeping the angle θ fixed. This amounts to taking the ϕ–angle of the direction $\vec{k}_f$ through its entire range, i.e., 0 to 2π. While rotating $\vec{k}_f$, the direction of $\hat{n}$ keeps changing continuously. Therefore, for a fixed θ, the direction $\hat{n}$ is related to the ϕ–angle of $\vec{k}_f$. It should be clear from the foregoing details that a nonvanishing value of "a" gives rise to the azimuthal asymmetry in the scattering of a polarized beam, and hence the nomenclature. Furthermore, one sees that if the scattering plane contains the direction of incident polarization, no asymmetry is observed. It is further obvious that the notation $\sigma(\theta)$ on the left–hand side of 4–58 is a little misleading because it conceals the ϕ–dependence of the quantity. We, however, choose to keep the notation simple, at the risk of some confusion.

The foregoing discussion can easily be specialized to the case of nucleon scattering on a spinless target. The expression 4–53 is obviously valid, except the summation μ now goes over the four operators 1 and σ and the factor $\frac{1}{4}$ is replaced by $\frac{1}{2}$ due to expression 4–19. Therefore, 4–54 changes to the simpler expression

$$\boxed{\sigma(\theta) = \tfrac{1}{2}\mathrm{Tr}(MM^\dagger) + \tfrac{1}{2}\,\vec{P}^{inc}.(\mathrm{Tr}M\vec{\sigma}M^\dagger)} \qquad 4\text{–}64$$

Corresponding to the discussion on nucleon–nucleon scattering, we now have

$$\boxed{\sigma_0(\theta) = \tfrac{1}{2}\mathrm{Tr}\, MM^{\dagger} = |g|^2 + |h|^2} \quad \text{(unpolarized incident nucleons)} \qquad 4\text{–}65$$

$$\boxed{\sigma(\theta) = \sigma_0(1 + a\vec{P}^{inc}.\hat{n})} \quad \text{(polarized incident nucleon beam)} \qquad 4\text{–}66$$

$$\text{with } a = \frac{2\mathrm{Re}\, gh^*}{|g|^2 + |h|^2}$$

Here g and h are referred to respectively as spin–nonflip and the spin–flip amplitudes (Ta 72). As before, the asymmetry parameter at an angle θ can be determined from the left–right azimuthal asymmetry at that angle; the expression of the asymmetry is still given by 4–63.

We now proceed to derive an expression for the polarization of the scattered nucleon beam. In doing so, we shall introduce the famous Wolfenstein parameters which are mentioned in the literature on polarization experiments. The use of the term "Wolfenstein parameters" is gradually becoming extinct and at present the term "polarization transfer parameters" is preferred in the literature.

4.3.3.2 Polarization of a Scattered Beam

Since we are interested in the polarization of the beam *after* scattering has taken place, we must use, in definition 4–12, the density matrix for the final state, i.e.,

$$\rho_{scatt} = M\rho_{inc}M^{\dagger}$$

Therefore, the expression we obtain for the polarization of the scattered beam is

$$\vec{P}_1^{scatt} = \frac{Tr\,(M\rho_{inc}\,M^\dagger\vec{\sigma}_1)}{Tr\,\rho_{scatt}} \qquad \text{(as in 4–12)}$$

$$= \frac{Tr\,\rho_{inc}}{Tr\,\rho_{scatt}} \times \frac{Tr\,(M\rho_{inc}\,M^\dagger\vec{\sigma}_1)}{Tr\,\rho_{inc}}$$

$$= [\sigma(\theta)]^{-1}\frac{Tr\,(M\rho_{inc}\,M^\dagger\vec{\sigma}_1)}{Tr\,\rho_{inc}} \qquad \text{(from eq. 4–44)}$$

$$\text{or } \sigma(\theta)\vec{P}_1^{scatt} = \frac{1}{4Tr\rho_{inc}}\sum_\mu (Tr\,\rho_{inc}\,S_\mu)(Tr\,MS_\mu\,M^\dagger\vec{\sigma}_1) \qquad \text{(from 4–25)} \qquad 4\text{–}67$$

Using the definition of $\langle\overline{S_\mu}\rangle$ in the foregoing equation, we get

$$\sigma(\theta)\vec{P}_1^{scatt} = \tfrac{1}{4}\sum_\mu \langle\overline{S_\mu}\rangle_{inc}\,Tr(MS_\mu\,M^\dagger\,\vec{\sigma}_1)$$

$$= \tfrac{1}{4}\,Tr\,MM^\dagger\,\vec{\sigma}_1 + \tfrac{1}{4}\,\vec{P}_1^{inc}.(Tr\,M\vec{\sigma}_1\,M^\dagger\,\vec{\sigma}_1) \qquad 4\text{–}68$$

In the final step, we have used the explicit forms of S_μ, and dropped the terms containing $\vec{P}_2^{inc}$ and $\langle\sigma_{1\alpha}\sigma_{2\beta}\rangle$ (see 4–26) on the understanding that we shall always consider initially *unpolarized targets.*

We now derive the following very important result for the elastic scattering of an unpolarized beam from an unpolarized target, namely, as a result of the scattering there is a net alignment of spins in the y–direction (perpendicular to scattering plane) i.e., and the unpolarized incident beam which had no preferred spin direction has been converted into a polarized scattered beam perpendicular to the scattering plane.

Consider the case of an unpolarized incident beam, i.e.,

$$\vec{P}_1^{inc} = 0$$

In this case 4–68 reduces to:

$$\sigma_0(\theta)\vec{P}_1^{scatt} = Tr(MM^\dagger\sigma_1)$$

$$\text{or} \quad \vec{P}_1^{scatt} = \frac{Tr(MM^\dagger\vec{\sigma}_1)}{Tr(MM^\dagger)} \qquad \text{(See 4–55)} \qquad 4\text{–}69$$

Now, one can verify by direct evaluation that:

$$Tr(MM^\dagger\vec{\sigma}_1) = Tr(M\vec{\sigma}_1M^\dagger) \qquad 4\text{–}70$$

Note that since the $\vec{\sigma}$ matrices do not commute with M, this is not a mathematical identity. It can be shown (Wo 52) that this relation follows from the condition on M of invariance under time reversal.

Thus, using 4–70 we see that:

$$\vec{P}_1^{scatt} = \frac{Tr(M\vec{\sigma}_1M^\dagger)}{Tr(MM^\dagger)} = \vec{a} = a\hat{n} \quad \text{(see 4–59 and 4–60)} \qquad 4\text{–}71$$

If we write

$$\boxed{\vec{P}_1^{scatt} = P\hat{n}} \qquad 4\text{–}72$$

then the magnitude P of the polarization is given by the asymmetry parameter a. The same result holds for the scattering of an unpolarized nucleon beam on a spin zero target. In this case P is equal to the asymmetry parameter given in eq. 4–66.

For a discussion on the relationship between polarization and analyzing power (also called asymmetry parameter) refer to appendix J.

We next consider the more general case described by 4–68, where the initial beam has a polarization $\vec{P}_1^{inc}$. According to 4–69, 4–55 and 4–72 we can write the first term of expression 4–68 as:

$$\tfrac{1}{4}\,\mathrm{Tr}(MM^{\dagger}\vec{\sigma}_1) = \sigma_0\vec{P}_1^{scatt} = \sigma_0 P\hat{n} \qquad 4\text{–}73$$

The second term of expression 4–68 is simplified by explicitly evaluating the trace and making use of expressions 4–50, 4–51 and 4–52.

One then obtains

$$\begin{aligned}\tfrac{1}{4}\,\vec{P}_1^{inc}.(\mathrm{Tr}\,M\vec{\sigma}_1M^{\dagger}\vec{\sigma}_1) &= (|g_0|^2 + 2|h_0|^2 + |h_n|^2 - |h_X|^2 - |h_Z|^2)(\vec{P}_1^{inc}.\hat{n})\hat{n} \\ &+ [(|g_0|^2 + |h_X|^2 - |h_Z|^2 + |h_n|^2)(\vec{P}_1^{inc}.\hat{X}) + 2\mathrm{Im}\,\{h_0^*(g_0-h_n)\}(\vec{P}_1^{inc}.\hat{Z})]\hat{X} \\ &+ \{(|g_0|^2 + |h_Z|^2 - |h_X|^2 - |h_n|^2)(\vec{P}_1^{inc}.\hat{Z}) - 2\mathrm{Im}\,\{h_0^*(g_0-h_n)\}(\vec{P}_1^{inc}.\hat{X})]\hat{Z} \qquad 4\text{–}74\end{aligned}$$

It is more convenient to consider the components of $\vec{P}_1^{inc}$ in another coordinate system other than the coordinate system defined by $\hat{X}$, $\hat{n}$ and $\hat{Z}$ in 4–45. For this reason we consider the geometry shown in figure 4.3.

$\vec{k}_i$ and $\vec{k}_f$ are the incident and outgoing momenta (each of magnitude k since we are considering elastic scattering) in the nucleon–nucleon centre–of–mass frame. The angle θ between them is the scattering angle in the centre–of–mass system. For nucleon–nucleon elastic scattering where $k_i = k_f$, one can show, by kinematical considerations (see, for example p.52 of Pa 83), that the relationship between the scattering angle θ_ℓ in the laboratory system and the same θ in the centre–of–mass system is given by

$$\theta_\ell = \frac{\theta}{2} \qquad 4\text{–}75$$

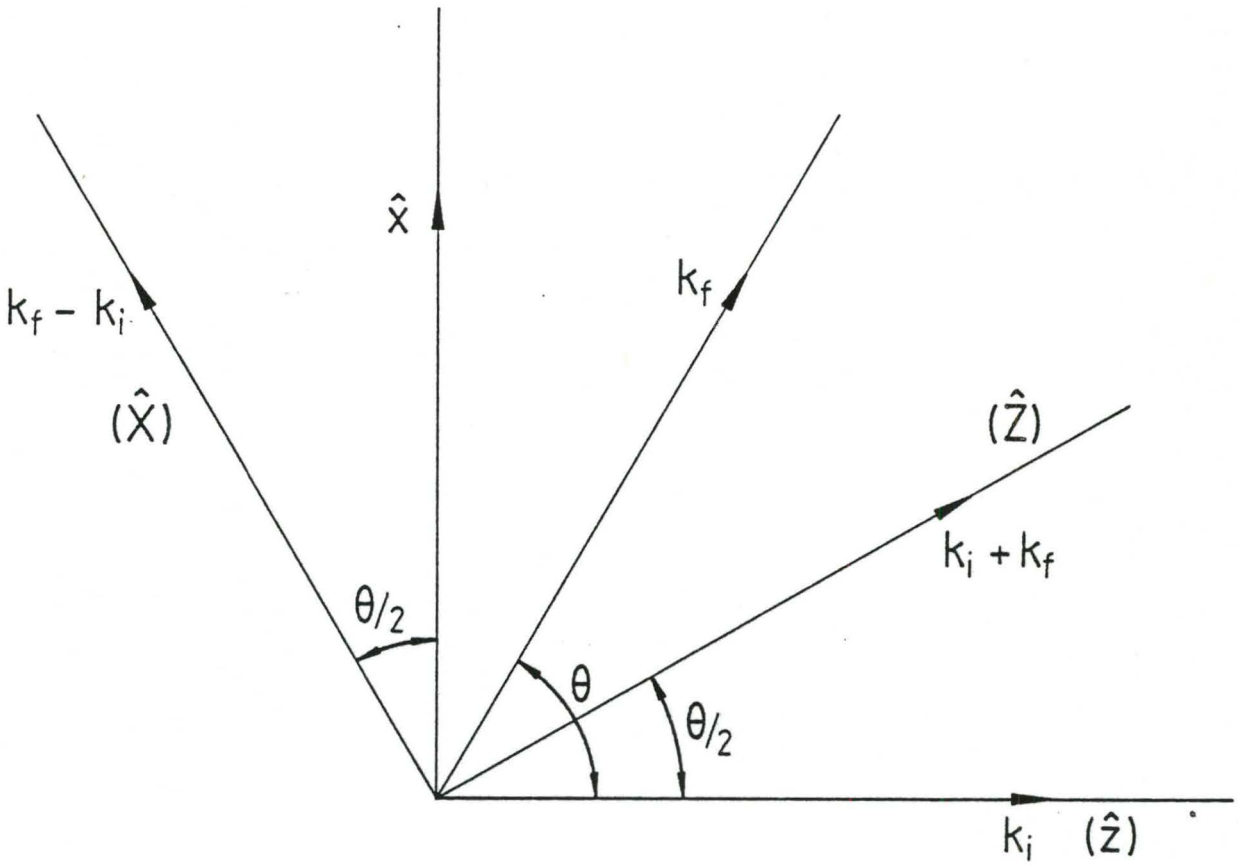

Fig. 4.3 Two possible sets of coordinate axes in the description of polarization phenomena.

Since the incident momentum $\vec{k}_i$ has the same direction in the laboratory and the centre–of–mass system, we conclude that the final momentum $(\vec{k}_f)_{lab}$ in the laboratory system points in the same direction as $\hat{Z}$ (refer to geometrical considerations in figure 4.3).

Therefore the last term in 4–74 gives the component of $\vec{P}_i^{scatt}$ in the direction of the outgoing momentum in the laboratory frame. The second term similarly gives the component of $\vec{P}_i^{scatt}$ in the scattering plane, but perpendicular to the direction of the outgoing momentum in the laboratory frame. The first term together, with 4–73 gives the component of $\vec{P}_i^{scatt}$ in a direction normal to the scattering plane. So we see that a convenient set of axes to describe the *outgoing* polarization is defined by $\hat{X}$, $\hat{n}$ and $\hat{Z}$ of 4–45.

The coefficients of the three terms in 4–74 contain the components of the *incident* polarization along the directions $\hat{X}$, $\hat{n}$ and $\hat{Z}$ However, it is more convenient to describe the components of $\vec{P}_i^{inc}$ with respect to the following three directions, indicated by their unit vectors (see figure 4.3): $\hat{z}$, $\hat{y}$, $\hat{x}$ where $\hat{z}$ is along the incident momentum in the laboratory frame (the same as the direction $\vec{k}_i$), $\hat{x}$ perpendicular to $\hat{z}$ in the scattering plane, and $\hat{y} = \hat{n}$ normal to the scattering plane. Note that the unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$ form a right–handed coordinate system, where $\hat{z}$ is a unit vector along the direction of $\vec{k}_i$.

We have just succeeded in defining the projectile helicity coordinate frame and the outgoing particle helicity frame recommended by the Madison Convention (Ba 70). The reason for defining such coordinate systems should become apparent when we describe double and triple scattering experiments. Furthermore, these coordinate systems have convenient properties with respect to time–reversal and parity operations (as we saw in our discussion in section 4.3.2).

Our aim, of expressing $\vec{P}_i^{inc}$ in terms of the projectile helicity frame, is achieved if we use the geometry of figure 4.3 to write the following:

$$\vec{P}_1^{inc}.\hat{X} = -(\vec{P}_1^{inc}.\hat{z})\sin\frac{\theta}{2} + (\vec{P}_1^{inc}.\hat{x})\cos\frac{\theta}{2}$$

$$\vec{P}_1^{inc}.\hat{Z} = +(\vec{P}_1^{inc}.\hat{z})\cos\frac{\theta}{2} + (\vec{P}_1^{inc}.\hat{x})\sin\frac{\theta}{2} \qquad 4\text{–}76$$

Substitute 4–76 in 4–74, and then substitute the resultant expression and 4–73 in 4-68.

Thus,

$$\frac{\sigma(\theta,\phi)}{\sigma_0(\theta)}\vec{P}_1^{scatt} = (P + D\vec{P}_1^{inc}.\hat{n})\hat{n} + (A'\vec{P}_1^{inc}.\hat{z} + R'\vec{P}_1^{inc}.\hat{x})\hat{Z} + (A\vec{P}_1^{inc}.\hat{z} + R\vec{P}_1^{inc}.\hat{x})\hat{X} \qquad 4\text{–}77$$

The quantities D, A, R, A′ and R′ are respectively given by:

$$\sigma_0(\theta)[1-D(\theta)] = 2(|h_X|^2 + |h_Z|^2); \qquad 4\text{–}78$$

$$\sigma_0(\theta)\,A(\theta) = -(|g_0|^2+|h_X|^2-|h_Z|^2-|h_n|^2)\sin\frac{\theta}{2} + 2\mathrm{Im}[h_0^*(g_0-h_n)]\cos\frac{\theta}{2}; \qquad 4\text{–}79$$

$$\sigma_0(\theta)\,R(\theta) = (|g_0|^2+|h_X|^2-|h_Z|^2-|h_n|^2)\cos\frac{\theta}{2} + 2\mathrm{Im}[h_0^*(g_0-h_n)]\sin\frac{\theta}{2}; \qquad 4\text{–}80$$

$$\sigma_0(\theta)\,A'(\theta) = (|g_0|^2+|h_Z|^2-|h_X|^2-|h_n|^2)\cos\frac{\theta}{2} + 2\mathrm{Im}[h_0^*(g_0-h_n)]\sin\frac{\theta}{2}; \qquad 4\text{–}81$$

$$\sigma_0(\theta)\,R'(\theta) = (|g_0|^2+|h_Z|^2-|h_X|^2-|h_n|^2)\sin\frac{\theta}{2} + 2\mathrm{Im}[h_0^*(g_0-h_n)]\cos\frac{\theta}{2}; \qquad 4\text{–}82$$

The parameter P (equal to the asymmetry parameter a) has already occurred in expression 4–69 of the polarization of a scattered beam, when the initial beam was unpolarized; it also occurred as the asymmetry in the expression 4–66 of the differential cross–section of an initially polarized beam. This parameter is usually measured in a double scattering experiment to be described in section 4.3.4. The parameters R, R′ and A, A′ are called the spin rotation parameters, while D is called the depolarization parameter. Note that the parameter A is not the same as the asymmetry parameter a. The parameters D, A, R, A′ and R′ are collectively referred to as the Wolfenstein parameters. They are measured in different layouts of triple scattering experiments to be considered in section 4.3.4. The meaning of the

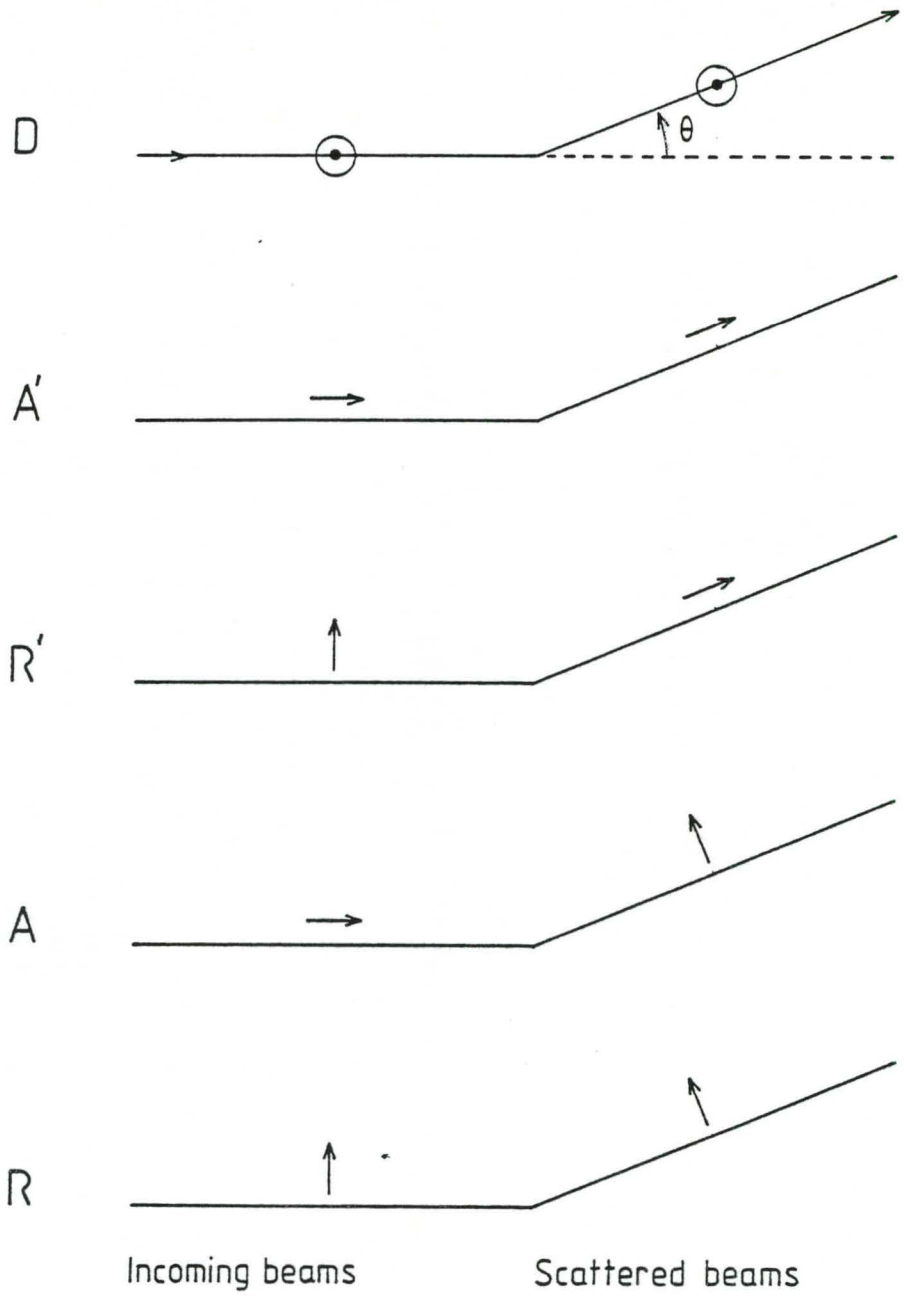

Fig. 4.4 Each diagram illustrates the polarization component in the incoming and final beams which are related by the Wolfenstein parameters indicated on the left.

Wolfenstein parameters (also called polarization transfer parameters – see section 4.3.4.6) as defined by eq. 4–77, are demonstrated pictorially by Fig. 4.4. The polarization components which are related by each of the Wolfenstein parameters are illustrated in the diagram.

Let us consider the expressions corresponding to equations 4–77 to 4–82, for the scattering of nucleons on a spin zero target. For comparative reasons we leave the projectile helicity and outgoing partical helicity frames as in the case of nucleon–nucleon scattering, despite the fact that the projectile mass and target mass differ. For this case, expression 4–77 remains the same with $\sigma_0(\theta)$ and $\sigma(\theta)$ given by expressions 4–65 and 4–66. Furthermore, to specialize 4–52 to 4–47 all we need to do is to replace the following quanties in 4–50, 4–51 and 4–52.

$$\vec{h} \rightarrow h_0\hat{n} \rightarrow h\hat{n}$$

$$\vec{\sigma}_2 \rightarrow 0$$

$$\vec{\sigma}_1 \rightarrow \vec{\sigma} \qquad 4\text{–}83$$

$$h_X, h_Z, h_n \rightarrow 0$$

$$g \rightarrow g_0 \rightarrow g$$

Having made all these substitutions into 4–52 should yield 4–47.

Expressions 4–78 to 4–82 then become

$$D(\theta) = 1; \qquad 4\text{–}84$$

$$\sigma_0(\theta)\, A(\theta) = -|g|^2 \sin\frac{\theta}{2} + 2\mathrm{Im}\,(h^* g)\cos\frac{\theta}{2}; \qquad 4\text{–}85$$

$$\sigma_0(\theta)\, R(\theta) = |g|^2 \cos\frac{\theta}{2} + 2\mathrm{Im}\,(h^* g)\sin\frac{\theta}{2}; \qquad 4\text{–}86$$

$$\sigma_0(\theta)\, A'(\theta) = |g|^2 \cos\frac{\theta}{2} + 2\mathrm{Im}\,(h^* g)\sin\frac{\theta}{2}; \qquad 4\text{–}87$$

$$\sigma_0(\theta)\, R'(\theta) = |g|^2 \sin\frac{\theta}{2} - 2\mathrm{Im}\,(h^* g)\cos\frac{\theta}{2}; \qquad 4\text{–}88$$

Note that $A' = R$ and $A = -R'$. In section 4.3.5, we shall discuss whether the various parameters are mutually independent or not. We shall also find the minimum set of experiments required for a complete and unambiguous determination

of the M matrix for the case of a nucleon scattering on a spin zero target, since this case is dealt with in Part II of this thesis.

4.3.3.3 Spin–Correlation Parameters

We now discuss the correlation of spin polarization of the two outgoing nucleons (i.e., the scattered nucleon and the recoiling target nucleon). Consider the following simple case:

$$\vec{P}_1^{\,inc} = 0$$

$$\vec{P}_2^{\,inc} = 0$$

$$\langle \overline{\sigma_{1\alpha}\sigma_{2\beta}} \rangle_{initial} = 0$$

The correlation in spin is described by the various components of $\langle \overline{\vec{\sigma}_1 \vec{\sigma}_2} \rangle_{final}$, where 2 is the target nucleon. This quantity is obtained by a quantity analogous to that for the derivation of 4–69. Since we have assumed an unpolarized target, only the analogue of the first term of 4–69 is present, and is given by:

$$\sigma_0(\theta) \langle \overline{\vec{\sigma}_1 \vec{\sigma}_2} \rangle_{final} = \tfrac{1}{4} \mathrm{Tr}(MM^{\dagger} \vec{\sigma}_1 \vec{\sigma}_2) \qquad 4\text{–}89$$

Evaluating the Tr quantity in 4–89, using expressions 4–50, 4–51 and 4–52, we obtain

$$\langle \overline{\vec{\sigma}_1 \vec{\sigma}_2} \rangle_{final} = C_{nn} \hat{n}\hat{n} + C_{XZ} \hat{X}\hat{Z} + C_{ZX} \hat{Z}\hat{X} + C_{ZZ} \hat{Z}\hat{Z} + C_{XX} \hat{X}\hat{X} \qquad 4\text{–}90$$

where

$$\sigma_0(\theta) C_{nn} = 2|h_0|^2 + 2\mathrm{Re}\,(g_0 h_n^{*} - h_Z h_X^{*}), \qquad 4\text{–}91$$

$$\sigma_0(\theta) C_{XZ} = 2\mathrm{Im}\,[h_0(h_X^{*} - h_Z^{*})] \qquad 4\text{–}92$$

$$C_{ZX} = C_{XZ} \qquad 4\text{–}93$$

$$\sigma_0(\theta) C_{ZZ} = 2\mathrm{Re}\,(g_0 h_Z^{*} - h_X h_n^{*}) \qquad 4\text{–}94$$

$$\sigma_0(\theta) C_{XX} = 2\mathrm{Re}\,(g_0 h_X^{*} - h_Z h_n^{*}) \qquad 4\text{–}95$$

In 4–90, the two vectors in each term correspond to the direction of polarization of the first and second particle. Thus, C_{nn} describes the *correlation* in their *polarizations* perpendicular to the scattering plane. If one recalls the discussion associated with figure 4.3, then as far as the polarization of the scattered particle (here particle 1) is concerned [i.e., the first vector in each term of 4–90], $\hat{Z}$ specifies a polarization parallel to its motion, and $\hat{X}$ a polarization in the scattering plane perpendicular to its direction of motion.

To investigate similar facts on the recoiling target particle (here particle 2), we must use two results associated with nucleon–nucleon scattering.

(i) the recoiling particle in the centre–of–mass system moves in a direction opposite to that of the scattered particle; and

(ii) the angle giving the direction of a particle in the laboratory system is half the same angle measured in the centre–of–mass system.

It is therefore clear from figure 4.3 that the *recoiling* particle moves in the direction $-\vec{k}_f$ in the centre–of–mass system, and $-\hat{X}$ in the laboratory system. Therefore, whenever the second vector in 4–90 is $\hat{X}$, it describes the polarization of the recoiling particle opposite to the direction of its motion in the *laboratory frame*, and whenever it is $\hat{Z}$, it describes the polarization in the scattering plane perpendicular to the direction of motion. We shall see in the following section that the polarization of a particle, along or opposite to its direction of motion is more difficult to measure than the polarization in its directions perpendicular to the motion of the particle. Therefore, the coefficients C_{nn} and C_{XZ} are easier to measure than the other coefficients in 4–90.

In the next section we discuss how the Wolfenstein parameters are measured in double and triple scattering experiments. The nomenclature used for the various parameters should also become clear.

4.3.4 Description of Double and Triple Scattering Experiments

There are four key equations which are repeatedly used in this discussion. For convenience, we mention them again:

$$\frac{\sigma_L(\theta) - \sigma_R(\theta)}{\sigma_L(\theta) + \sigma_R(\theta)} = a\vec{P}_1^{inc}.\hat{n} \longrightarrow \quad 4\text{–}63$$

$$\sigma(\theta) = \sigma_0(1 + a\vec{P}_1^{inc}.\hat{n}) \longrightarrow \text{(polarized incident nucleon beam)} \quad 4\text{–}66$$

$$\vec{P}_1^{scatt} = P\hat{n};\ P = a \longrightarrow \quad 4\text{–}72$$

$$\frac{\sigma(\theta,\phi)}{\sigma_0(\theta)}\vec{P}_1^{scatt} = (P + D\vec{P}_1^{inc}.\hat{n})\hat{n} + (A'\vec{P}_1^{inc}.z + R'\vec{P}_1^{inc}.x)\hat{Z} + (A\vec{P}_1^{inc}.\hat{z} + R\vec{P}_1^{inc}.\hat{x})\hat{X} \longrightarrow \quad 4\text{–}77$$

where 4–72 applies to an unpolarized incident beam.

At this stage it is important to explain the principle of a polarizer and analyzer (usually a C^{12} target, which is a *spin zero target*) used in double and triple scattering experiments.

4.3.4.1 Polarizer and Analyzer

Consider the scattering of an unpolarized beam on a C^{12} target (spinless). After the scattering, the beam is polarized perpendicular to the scattering plane [see 4–72 and the paragraph following it], the magnitude P_c of the polarization (C refers to C^{12}) being equal to the asymmetry parameter a_c of 4–72. The scattering is shown in figure 4.4.

The following notation will be used in the sketches to follow:

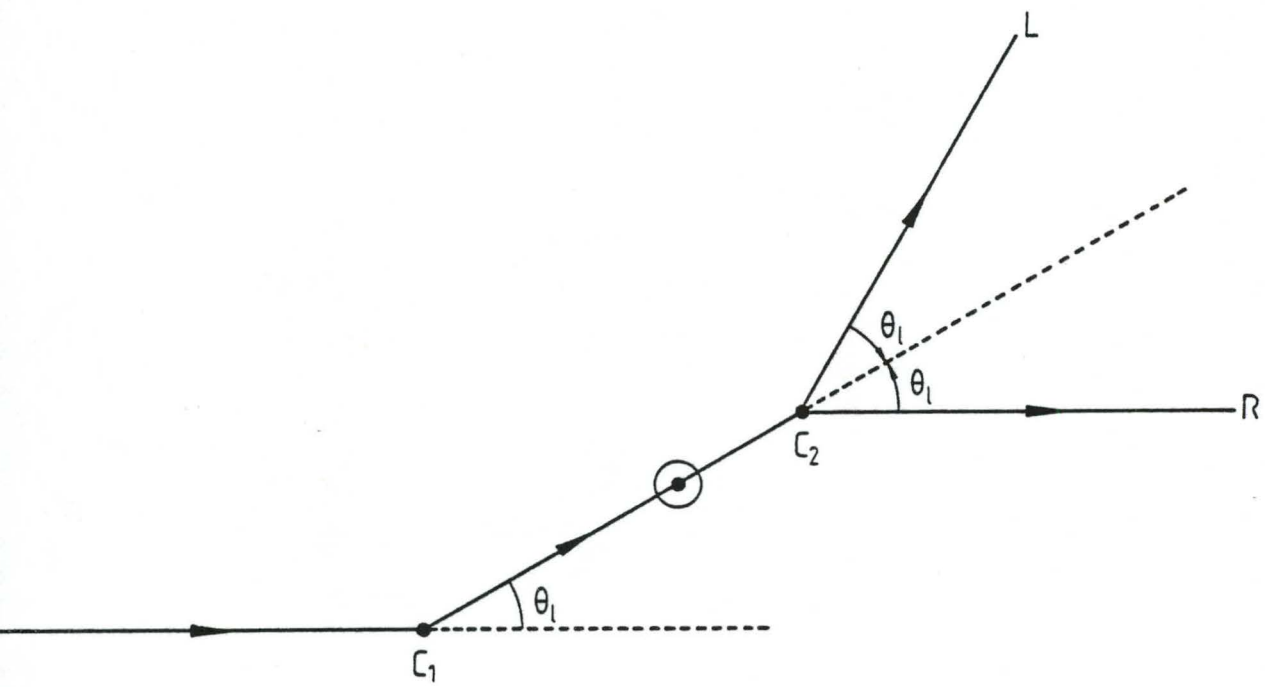

Fig. 4.5 Set-up for measuring the polarizing power of a target.

Arrows (⟶) indicate the direction of the incident and outgoing nucleons.

Polarization is either indicated by ⊙ or ⊗ where ⊙ (⊗) refers to polarization perpendicular to the plane of the figure and pointing upwards (downwards).

θ_ℓ refers to the scattering angle in the laboratory.

Subscript 1 refers to the first scattering and subscript 2 to the second.

In the discussion to follow, we use the notation:

P_c is the polarization which results from the scattering of an unpolarized beam from a carbon target;

P'_c is the polarization of the beam provided by the ion source;

P is the polarization which results from the scattering of an unpolarized nucleon beam from a target nucleon.

As already mentioned, the polarization of the outgoing beam (after first scattering) is perpendicular to the plane of figure 4.5 and points upwards. This beam is next scattered on a second C^{12} target (C_2 in figure 4.5). We are now in a position to be able to measure the left–right asymmetry of the final beam in the two directions (also in the plane of the paper) shown after C_2. This asymmetry is given by 4–63, i.e.,

$a_2 \vec{P}_1^{\,inc}.\hat{n}_2$

where $\vec{P}_1^{\,inc}$ is the polarization of the beam incident on C_2 and is equal to $P_{c1}\hat{n}_1$.

The direction of the polarization ($\hat{n}_1$) and the normal $\hat{n}_2$ to the second scattering plane both point upwards with respect to the plane of the paper, and thus $\hat{n}_1.\hat{n}_2 = 1$. Hence, we now have:

$$\frac{\sigma_L(\theta) - \sigma_R(\theta)}{\sigma_L(\theta) + \sigma_R(\theta)} = a_2 \vec{P}_1^{\,inc}.\hat{n}_2$$

$$= a_2 P_{c1}\, \hat{n}_1.\hat{n}_2$$

$$= P_{c2} P_{c1} \quad [P_{c2} = a_2: \text{ equation 4–72}]$$

Recall, that in general both P_{c1} and P_{c2} are functions of incoming beam energy and scattering angle. If we, however, take the scattering angles in both scattering

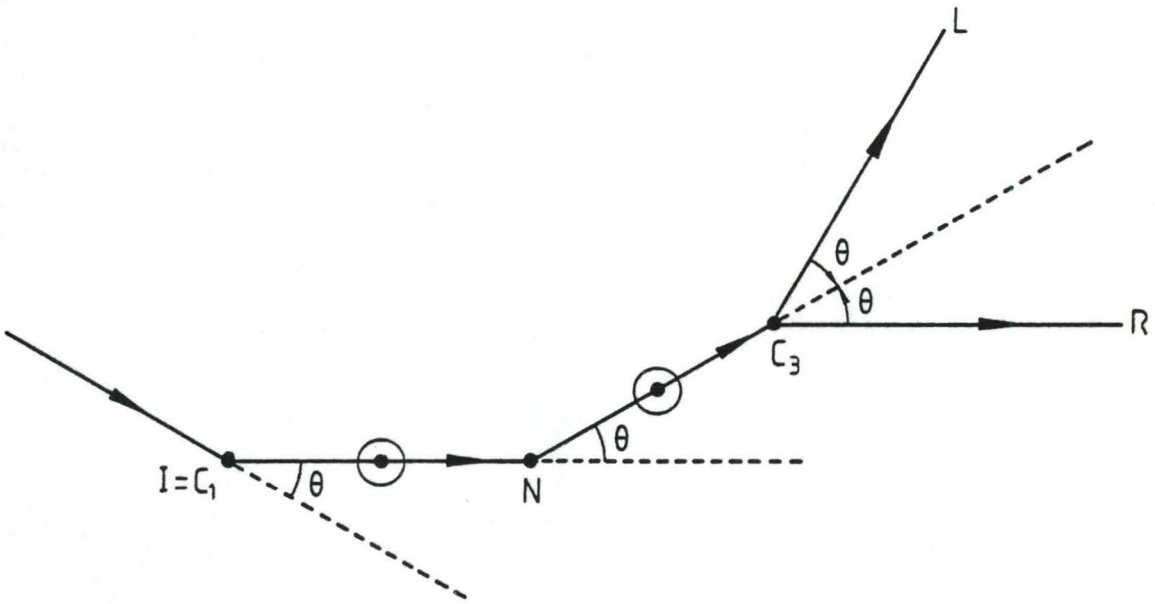

Fig. 4.6 Triple scattering set-up for measuring the depolarization parameter.

processes to be the same and we assume that the scattered particles have the same energy for both scattering process from carbon, then we may write $P_{c1}P_{c2}$ and the latter left–right asymmetry yields P_c^2, which can easily be measured in such a double scattering experiment.

In the experiment just described, the first scattering caused the polarization with *polarizing power* equal to P_{c1}, whereas *the second scattering served as the analyzer of this polarization* through its asymmetry parameter a_2. Therefore, *a* is also referred to as the *analyzing power* of the same target.

4.3.4.2 Measurement of P

In the set–up described in figure 4.5, we substitute the first scatterer by a nucleon scatterer (unpolarized nucleon target, for example, hydrogen gas). The first scattering causes a polarization of the outgoing nucleon beam equal to $P\hat{n}_1$, where P refers to nucleon–nucleon scattering and is the quantity to be measured. The second scattering on C^{12} serves the purpose of the analyzer and, according to 4–63

$$\frac{\sigma_L(\theta) - \sigma_R(\theta)}{\sigma_L(\theta) + \sigma_R(\theta)} = a_2 \vec{P}_1^{\,inc}.\hat{n}_2$$

$$= P_c P \quad \text{(refer to discussion on polarizer and analyzer).}$$

Since P_c is already known, from the previous experiment, we obtain P from this double scattering experiment.

4.3.4.3 Measurement of D

The quantities D, R, R′, A, and A′ occur in the expression 4–77 of the polarization of the outgoing beam corresponding to an *incident polarized beam.* Therefore, in this case, we need a polarized nucleon beam before the nucleon–nucleon scattering can be used. Such a polarized nucleon beam is provided by a polarized–ion source (see appendix K) rather than scattering an unpolarized nucleon beam on a ^{12}C target. With the use of a polarized–ion source the first scattering is the nucleon–nucleon scattering under investigation; the various components of the polarization of the outgoing beam are then analyzed by the second scatterer, which is a C^{12} target.

The simplest triple scattering set–up for measuring D is shown in figure 4.6. Here both planes are coincident with the plane of the paper, and the normal direction $\hat{n}$ applies to both scatterings. The nucleon scatterer is denoted by N. Let's suppose that the beam IN from the polarized ion source is polarized perpendicular to the paper such that $\vec{P}_1^{inc} = P'_c\hat{n}$ [we retain the notation P'_c, since in the past this polarization was produced as a result of the scattering of a nucleon beam by C^{12}; thus in Fig. 4.6 the notation $I = C_1$ indicates either ion source (I) which provides a beam with polarization P'_c or scattering from $^{12}C(C_1)$ to obtain a polarized beam]. Since the normal direction for the first scattering is the same as that of the incident beam, we now have only the $(\vec{P}_1^{inc}.\hat{n})$–term in 4–77. The other components in the scattering plane, namely, $\vec{P}_1^{inc}.\hat{z}$ and $\vec{P}_1^{inc}.\hat{x}$, are zero in the present set–up of the experiment. Therefore, from 4–77, the polarization of the outgoing beam NC_3 is predicted to be:

$$\frac{\sigma(\theta,\phi)}{\sigma_0(\theta)}\vec{P}_1^{scatt} = (P + D\vec{P}'_c)\hat{n} \qquad 4\text{–}96$$

in a direction normal to the scattering plane. According to 4–66 and the fact that $a = P$, we obtain

$$\sigma(\theta) = \sigma_0(\theta)(1 + PP'_c) \qquad 4\text{–}97$$

Therefore, substituting 4–97 into 4–96

$$\vec{P}_1^{scatt} = \frac{P + DP'_c}{1 + PP'_c}\hat{n} \qquad 4\text{–}98$$

The left–right asymmetry in the second scattering is therefore given by [see 4–63].

$$\frac{\sigma_L(\theta) - \sigma_R(\theta)}{\sigma_L(\theta) + \sigma_R(\theta)} = P_c\vec{P}_1^{scatt}.\hat{n}$$

$$= \frac{P + DP'_c}{1 + PP'_c} \times P_c \quad [\text{from } 4\text{–}98]$$

Since P, P_c and P'_c (from ion–source) are already known, this experiment enables one to determine D.

If the incident beam was completely polarized (i.e. $P'_c = 1$), then, according to 4–98

$$\vec{P}_1^{scatt} = \frac{P + D}{1 + P}\hat{n} \qquad 4\text{–}99$$

Therefore, the outgoing beam *can remain fully polarized* only if $D = 1$, which is the maximum value of D. For any value of $D < 1$, the outgoing beam is found to be only partially polarized. This explains the nomenclature *depolarization parameter.*

Now that we have seen the definition of D, the following meaning of D should readily be accepted, namely,

$$D = n_{\uparrow\uparrow} - n_{\uparrow\downarrow} \qquad 4\text{–}100$$

where $n_{\uparrow\uparrow}$ is the fraction of particles that maintained their polarization direction after scattering

and $n_{\uparrow\downarrow}$ represents the fraction of particles which flipped their spin in the scattering process.

Note, that according to 4–84, $D = 1$ for scattering from spinless targets. This can be ascribed to be the fact that parity conservation and angular momentum conservation have been built into the M matrix. For a target with spin, the scattered nucleon can undergo spin flip (the required angular momentum change can be taken up by reorientating the target spin), so that for a completely polarized incident beam the scattered beams need no longer be completely polarized.

There is, however, a limit to the fraction of particles $n_{\uparrow\downarrow}$ that flip their spins. This has to do with the polarization – asymmetry theorem (see Appendix J). It can be shown that the polarization–asymmetry theorem for elastic scattering is equivalent to saying that particles whose spin has been flipped are scattered symmetrically to the left and the right.

To relate the depolarization D to the asymmetry parameter a, assume a completely polarized incident beam ($P^{inc} = 1$) of which x% ($x < 50$) is scattered to the one side (say right). According to 4–63:

$$a = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = (1-x) - x = 1 - 2x \qquad 4\text{–}101$$

The maximum number of spin–flipped particles scattered equally to both sides (=2x) can in this case, according to the polarization asymmetry theorem, be considered as the maximum value of $n_{\uparrow\downarrow}$.

Thus

$$\begin{aligned} D_{min} &= (n_{\uparrow\uparrow})_{min} - (n_{\uparrow\downarrow})_{max} \\ &= (1-2x) - 2x \\ &= -1 + 2a \end{aligned}$$

in terms of a (or A_y) of 4–101. Summarizing, we thus obtain the limits of D, namely

$$-1 + 2|A_y| \leq D \leq 1 \qquad 4\text{–}102$$

4.3.4.4 Measurement of R

Figure 4.7 show the set–up for the measurement of R. The polarized–ion source (I) provides a beam of known polarization or a polarized beam is obtained from scattering at C_1. The subsequent scattering planes are crossed with respect to each other such that

$$\hat{n}_1.\hat{n}_2 = \hat{n}_2.\hat{n}_3 = 0$$

Let the first plane be horizontal. Through the direction IN (beam from ion source with polarization perpendicular and pointing out of this horizontal plane (we draw the vertical plane) in solid lines containing the second scatterer C_3. As before, the

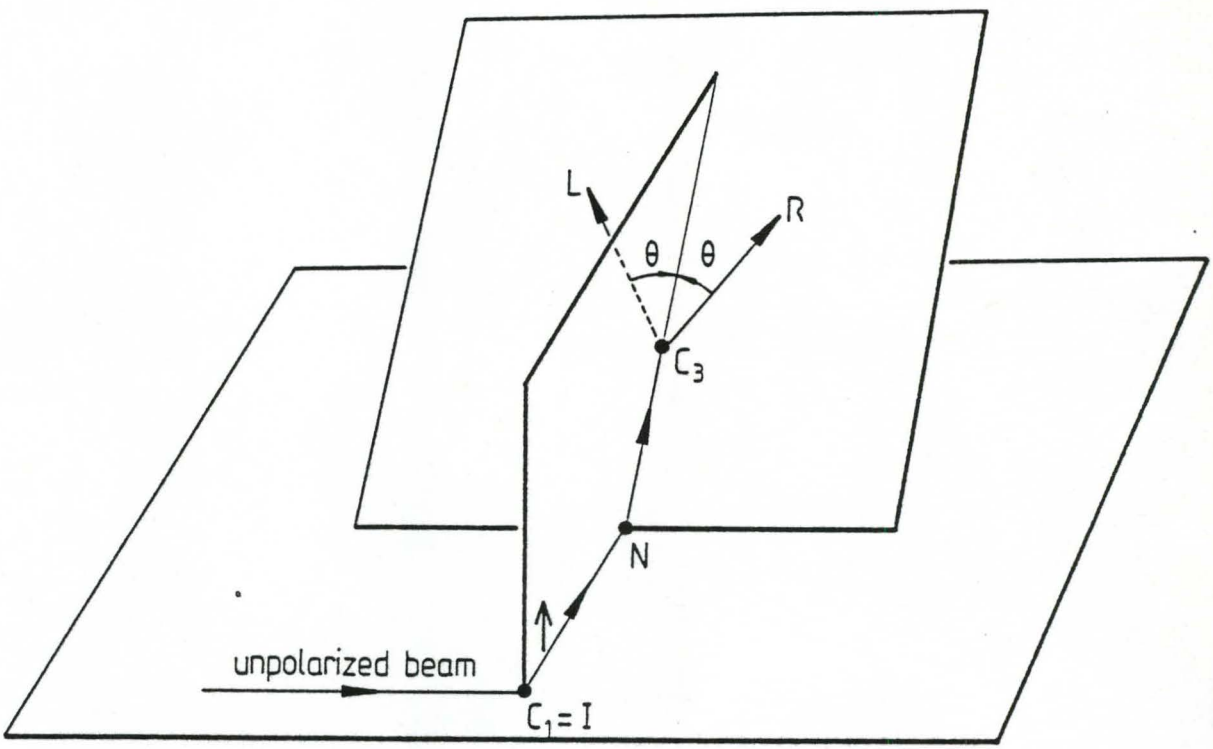

Fig. 4.7 Triple scattering set-up for measuring parameter R.

first scatterer N is a nucleon target. The third plane has been drawn through the line NC_3. The left–right asymmetry of the final scattering in this plane is the quantity measured in the experiment.

The polarization of the beam IN is given by $\vec{P}_1^{inc} = P'_c\hat{n}_1$; but $\hat{n}_1$, which is in the vertical direction, is now contained in the plane of the second scattering. The direction $\hat{n}_2$, normal to the second plane (i.e., the plane containing IN and NC_3), is horizontal, and hence $\vec{P}_1^{inc}$ does not have any normal component with respect to this plane. In 4–77, the direction $\hat{z}$ is along the incident laboratory momentum, and $\hat{x}$ is perpendicular to this direction in the scattering plane. Therefore, in this case $\vec{P}_1^{inc}$ points in the direction $\hat{x}$ for the first scattering. Furthermore, since $\hat{n}_2$ and $\hat{n}_1$ are perpendicular, equation 4–66 yields

$$\sigma(\theta) = \sigma_0(\theta)(1 + PP'_c\,\hat{n}_1.\hat{n}_2) = \sigma_0(\theta) \qquad 4\text{–}103$$

Therefore, 4–77 yields, for the polarization of the beam NC_3, the expression

$$\vec{P}_1^{scatt} = RP'_c\hat{X} + R'P'_c\hat{Z} \qquad 4\text{–}104$$

where $\hat{Z}$ is the direction of NC_3 (the outgoing direction in the laboratory) and $\hat{X}$ is *in* the vertical plane but perpendicular to the outgoing direction. Since the second *scattering plane* (containing NC_3, L and R) must pass through NC_3, it is clear that the normal to this plane, $\hat{n}_3$, is always perpendicular to $\hat{Z}$. Therefore, in the left–right asymmetry measurement in the second scattering plane, the term $\hat{Z}$ does not occur.

According to 4–63, this asymmetry is given by

$$\frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R}P_c = \vec{P}_1^{scatt}.\hat{n}_3 = P_cP_c'(R\hat{X} + R'\hat{Z}).\hat{n}_3$$

$$= P_cP_c'\,R\hat{X}.\hat{n}_3$$

$$= P_cP_c'R \qquad 4\text{–}105$$

since $\hat{X}$ is coincident with $\hat{n}_3$, i.e., $\hat{X}.\hat{n}_3 = 1$.

Thus, we are able to measure R from the measured asymmetry and known P_c' and P_c.

4.3.4.5 Measurement of A

The measurement of the parameter A (refer to figure 4.8; the unpolarized beam leading to C_1 is not shown as before. The polarized beam is provided by the ion source I.) is very similar in principle to that of R. Here we require the incident polarization, from the ion source, to be parallel to the direction of motion of the incident beam. A small arrow above the line segment IN indicates the direction of polarization. The scattered beam NC_3 and incident beam IN are assumed to be in the plane of the paper. The second scattering on C_3 is in the vertical plane, and the left–right asymmetry is measured as shown.

Since the polarization of the incident beam is along the direction $\hat{z}$, only the two $\hat{z}$–terms in 4–77 are nonvanishing, and so we get

$$\frac{\sigma(\theta,\phi)}{\sigma_0(\theta)}\vec{P}_1^{scatt} = P_c'(A\hat{X} + A'\hat{Z}) \qquad 4\text{–}106$$

But, since the direction of incident polarization continues to be perpendicular to the normal $\hat{n}$, it can be seen that $\sigma(\theta,\phi) = \sigma_0(\theta)$ (see 4–66).

Thus

$$\vec{P}_1^{scatt} = P_c'(A\hat{X} + A'\hat{Z}). \qquad 4\text{–}107$$

The left–right asymmetry measurement in the crossed plane yields

$$P_c\vec{P}_1^{scatt}.\hat{n}_3 = P_cP_c'(A\hat{X} + A'\hat{Z}).\hat{n}_3 \quad \text{(from 4–107)}$$

$$= P_cP_c'A \qquad 4\text{–}108$$

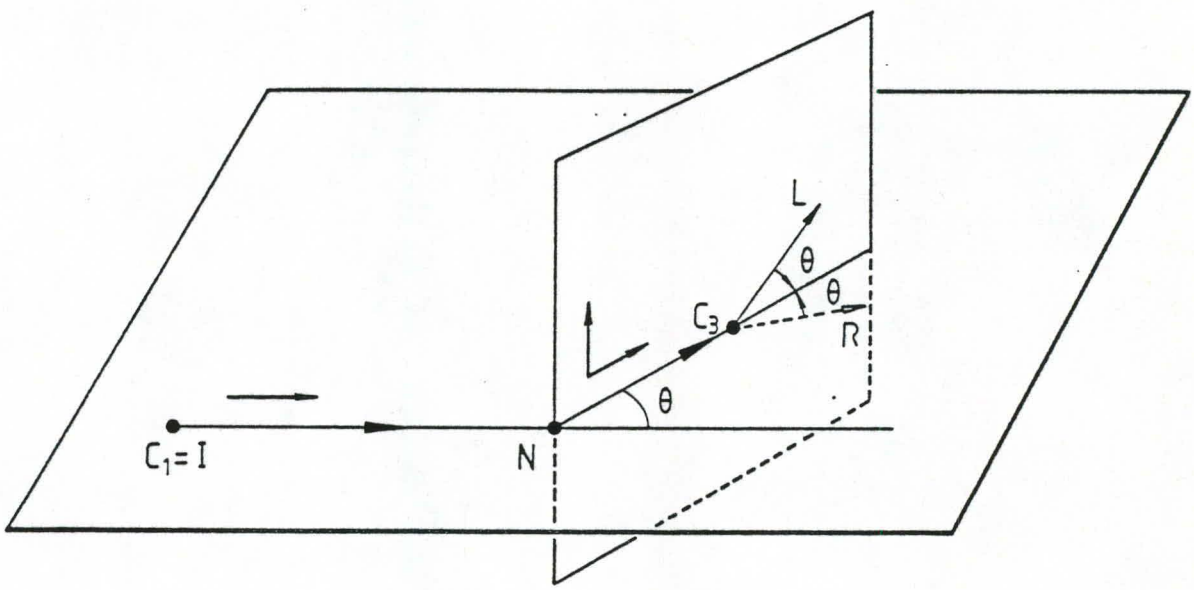

Fig. 4.8 Triple scattering set-up for measuring the parameter A.

This enables us to determine A.

Measurements of R′ and A′ require the use of magnetic fields between the first and second scatterings. Such experiments are usually not done, and for this reason we will not describe such measurements.

4.3.4.6 The meaning of the spin rotation parameters (R, R′, A, A′)

The aim of this section is to show why these Wolfenstein parameters are also called spin rotation parameters. We shall see that such terminology is only suggestive, and should not be taken literally.

Breit and McIntosh (Br 59) show that for scattering on a spinless target (simplest case):

$$\frac{\sigma(\theta,\phi)}{\sigma_0(\theta)}\,\vec{P}_1^{\,\text{scatt}} = (P + \vec{P}_1^{\,\text{inc}}.\hat{n})\hat{n} + (1-P^2)^{\frac{1}{2}}\,[\{(\vec{P}_1^{\,\text{inc}}.\hat{z})\cos(\theta+\beta) - (\vec{P}_1^{\,\text{inc}}.\hat{x})\sin(\theta+\beta)\}\hat{Z} + \{(\vec{P}_1^{\,\text{inc}}.\hat{z})\sin(\theta+\beta) + (\vec{P}_1^{\,\text{inc}}.\hat{x})\cos(\theta+\beta)\}\hat{X}] \qquad 4\text{–}109$$

Here, we merely rewrote expression (3.38) in Br 59 in terms of the notation we are familiar to using. Comparison with 4–77 shows that:

$$A' = R = (1-P^2)^{\frac{1}{2}}\cos(\theta+\beta), \qquad 4\text{–}110$$

$$A = -R' = (1-P^2)^{\frac{1}{2}}\sin(\theta+\beta), \text{ and} \qquad 4\text{–}111$$

$$D = 1$$

The relationship among these parameters is identical to those obtained using expressions 4–84 to 4–88. Similar expressions are also derived (but in less detail than Breit and McIntosh) by Wolfenstein (Wo 56); namely

$$R = A' = (1 - P^2)^{\frac{1}{2}} \cos(\beta - \theta_{lab}) \qquad 4\text{–}112$$

$$A = R' = (1 - P^2)^{\frac{1}{2}} \cos(\beta - \theta_{lab}) \qquad 4\text{–}113$$

The slight difference between expressions 4–110, 4–111 and 4–112, 4–113 is due to different definitions of the scattering angle θ. To understand the meaning of the angle β, consider only the components of $\vec{P}_1^{\,inc}$ lying in the scattering plane defined by the vectors $\vec{k}_{in}$ and $\vec{k}_{out}$. We now relate the components of the incident polarization, lying in the scattering plane, to the components of $\vec{P}_1^{\,scatt}$ in the scattering plane. From 4–109 we obtain (substitute $\theta + \beta$ for $\beta - \theta_{lab}$ to agree with Wolfenstein).

$$\begin{bmatrix} P_{1Z}^{\,scatt} \\ P_{1X}^{\,scatt} \end{bmatrix} = (1 - P^2)^{\frac{1}{2}} \begin{bmatrix} \cos(\beta - \theta_{lab}) & -\sin(\beta - \theta_{lab}) \\ \sin(\beta - \theta_{lab}) & \cos(\beta - \theta_{lab}) \end{bmatrix} \begin{bmatrix} P_{1z}^{\,inc} \\ P_{1x}^{\,inc} \end{bmatrix} \qquad (4\text{–}114)$$

From the form of the 2×2 matrix in 4–114 it is clear that β is the *rotation* angle of the in–plane component of $\vec{P}_1^{\,scatt}$ (in the outgoing particle helicity frame) with respect to the original in–plane spin direction (in the projectile helicity frame). For this reason β is called the *spin rotation angle* and since the Wolfenstein parameters R, R′, A and A′ are related to β, they are called the spin rotation parameters. Now, it is important to note that, strictly speaking, a rotation does not change the magnitude of a vector that is being rotated. But here we have a case where the magnitude of $\vec{P}_1^{\,inc}$ in the scattering plane is changed by a multiplicative factor of $(1 - P^2)^{\frac{1}{2}}$. It is for this reason that the terminology "spin rotation parameter" should not be taken literally.

For obvious reasons, Ohlsen [Oh 72(a)] refers to the Wolfenstein parameters as polarization transfer coefficients, which are denoted as follows:

$$\begin{aligned} k_x{}^{x'} &\equiv D_{SS'} \equiv R \\ k_z{}^{x'} &\equiv D_{LS'} \equiv A \\ k_z{}^{z'} &\equiv D_{LL'} \equiv A' \\ k_x{}^{z'} &\equiv D_{SL'} \equiv R' \end{aligned} \qquad 4\text{–}115$$

where the primed quantities refer to the outgoing particle helicity frame and the subscripts refer to the projectile helicity frame (i.e. $\hat{x}, \hat{y}, \hat{z} \rightarrow x', \hat{n}, z'$ or $\hat{x}, \hat{n}, \hat{z} \rightarrow \hat{X}, \hat{n}, \hat{Z}$ in our notation). The subscripts S and L refer to the "sideways" and "longitudinal" components respectively. This notation is usually used by experimentalists (Oh 80).

4.3.4.7 Measurement of the Correlation Parameters C_{nn} and C_{XZ}

The set–up for measuring C_{nn} is shown in figure 4.9. The two C^{12} scatters, C_1 and C_2, are in the path of the scattered and recoiling nucleons (respectively), which are detected *in coincidence* after the first scattering. The incident beam on N is unpolarized. According to 4–90, the polarization correlation of the beams NC_1 and NC_2 is given by

$$\langle \overline{\vec{\sigma}_1 \vec{\sigma}_2} \rangle_{final} = C_{nn}\, \hat{n}\hat{n} + C_{XZ}\, \hat{X}\hat{Z} + C_{ZX}\, \hat{Z}\hat{X} + C_{ZZ}\, \hat{Z}\hat{Z} + C_{XX}\, \hat{X}\hat{X}$$

Since the left–right asymmetry is measured after scattering on C_1 and C_2 in the same plane as the first scattering all normals point perpendicular to the plane of the paper.

Now, if $\hat{n}$ describes the *upward* normal, then in a manner analogous to 4–63, we define the left–right asymmetry of the polarization correlation as

$$P_{c1}P_{c2}\langle \overline{\vec{\sigma}_1 \vec{\sigma}_2} \rangle . \hat{n}\hat{n} = P_c^2 C_{nn} = \frac{\sigma_{1L}(\theta) - \sigma_{1R}(\theta) + \sigma_{2R}(\theta) - \sigma_{2L}(\theta)}{\sigma_{1L}(\theta) + \sigma_{1R}(\theta) + \sigma_{2L}(\theta) + \sigma_{2R}(\theta)} \qquad 4\text{–}116$$

Similar principles are involved in the measurement of C_{XZ}.

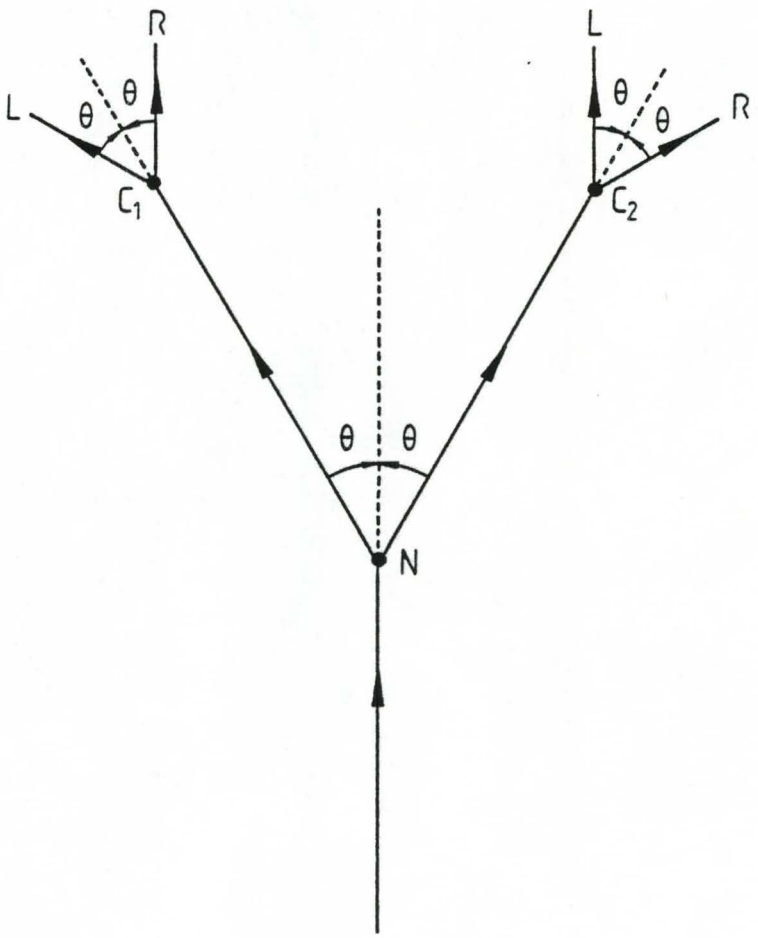

Fig. 4.9 Set-up for measuring correlation parameters C_{nn} and C_{XZ}.

4.3.5 Minimum Set of Measurements required for a Complete and Unambiguous Determination of the scattering matrix M

Breit and McIntosh (Br 59) show that for the scattering of a spin–½ particle on a spinless centre:

$$g+h = \{(|g|^2 + |h|^2)(1+P)\}^{\frac{1}{2}} e^{i(\alpha-\frac{1}{2}\beta)}, \text{ and} \qquad 4\text{–}117$$
$$g-h = \{(|g|^2 + |h|^2)(1-P)\}^{\frac{1}{2}} e^{i(\alpha+\frac{1}{2}\beta)}$$

where α and β are real constants and

$$P = a = \frac{2\mathrm{Re}\, gh^*}{|g|^2+|h|^2} \qquad \text{[see 4–66]}$$

is the polarization produced by scattering of an unpolarized beam (spin–½ beam) from a spinless target. In the previous section, it was shown that β is called the spin rotation angle because it is the angle of rotation (in the scattering plane) of any incident polarization in the scattering plane. The constant α is an overall phase factor which has no bearing whatsoever on the spin orientation and is not accessible to measurement.

From 4–117 we see that in order to determine g and h (and thus M – see 4–47), we require the measurement of three observables:

(i) the differential cross–section, $\sigma_0(\theta)$, for the scattering of an unpolarized nucleon beam from a spinless target, i.e., $\sigma_0(\theta) = |g|^2 + |h|^2$ (see 4–65).

(ii) The analyzing power, $P = A_y$, which is given by 4–72 (possibly with a sign ambiguity).

(iii) The spin rotation parameter, β (possibly with a sign ambiguity) or any quantity related to this parameter, e.g. the Wolfenstein parameters, R, R′, A, A′ (see 4–112 and 4–113). In this thesis, however, we have chosen to measure a particular spin rotation function Q (see Part II) as suggested by Glauber and Osland (Gl 79). Some attention is now devoted the discussion of the latter–mentioned observable.

Glauber and Osland (Gl 79) point out that besides efficiently completing our knowledge of the nuclear scattering amplitude, the measurement of the function Q would furnish information both about the nuclear density distribution and about parts of the fundamental proton–nucleon spin–orbit scattering amplitudes. The spin rotation function Q is defined to be related to the spin rotation angle β by:

$$Q = (1 - P^2)^{\frac{1}{2}} \sin \beta \qquad 4\text{–}118$$

where $P = A_y$ is defined above. It can be shown [using 4–112 and 4–113] that in the laboratory frame, Q can also be written as a linear combination of the Wolfenstein–rotation parameters A and R,

$$Q = A \cos \theta_{lab} + R \sin \theta_{lab} \qquad 4\text{–}119$$

where θ_{lab} is the laboratory scattering angle. Similarly, because of the symmetries of the system [see 4–112 and 4–113]

$$Q = A' \sin \theta_{lab} - R' \cos \theta_{lab}. \qquad 4\text{–}120$$

Using eq. (3.21) in Br 59, namely

$$\cos \beta = \frac{|g|^2 - |h|^2}{(|g|^2 + |h|^2)(1 - P^2)^{\frac{1}{2}}}, \quad \sin \beta = \frac{2\ \mathrm{Im}(gh^*)}{(|g|^2 + |h|^2)(1 - P^2)^{\frac{1}{2}}},$$

we can show that equation 4–118 becomes:

$$Q = \frac{2\ \mathrm{Im}(gh^*)}{|g|^2 + |h|^2}. \qquad 4\text{–}121$$

The first measurement of the spin rotation function Q was done for p–^{40}Ca elastic scattering at 500 MeV by B.Aas *et al.* (Aa 81). For an article on spin observables for elastic proton scattering, which is more directly related to calculations performed in this thesis, refer to Aa 85.

Thus, we have seen that for spin–½ on spin zero scattering the measurement of only three observables [$\sigma_0(\theta)$, A_y, Q] allows one to extract the scattering amplitude M uniquely.

Nucleon–nucleon scattering experiments, involving two particles of spin ½, are much more complicated to analyze than the scattering of a nucleon on a spinless centre, To completely describe the scattering matrix M, one in general requires eleven parameters (see expression 4–48; remember that, as before, there is an overall phase that cannot be determined $\Longrightarrow$ 12 real quantities of g and h reduce to 11 parameters) that are functions of energy and angle, instead of the previous three. The number is reduced to nine if the nucleons are identical, or if one assumes charge independence (as we have done; refer to expression 4–49). To obtain them one has to perform a number of experiments (as we have already described) to determine the scattering cross–section, polarization, Wolfenstein parameters, and, possibly the correlation coefficients.

To completely determine the nine parameters mentioned above, it is possible to choose different sets of measurements: for an unpolarized beam on an unpolarized target a theoretically possible set is: $\sigma(\theta,\phi)$, P = A_y, D, R, R′, A, A′; C_{nn}, C_{XZ}, C_{ZZ}, C_{XX}. With a polarized beam and an unpolarized target, or an unpolarized beam and a polarized target, it suffices to measure fewer quantities, and still fewer with a polarized beam and a polarized target (for a detailed discussion see Bethe and Schumacher; Sc 61).

Up until this stage we have dealt with:

(a) elastic spin–½ on spin zero scattering

(b) elastic spin–½ on spin–½ scattering .

Since this thesis only deals with spin–½ particles been scattered, we refer the reader to appendix N for the description of a spin 1 polarized (e.g. deuteron) beam.

So far, we have defined the various polarization observables. The meaning and measurement of the observables have also been discussed.

The next stage is concerned with the actual calculation of the three independent observables $[\sigma_0(\theta), A_y, Q]$ which completely describe the elastic scattering of a beam of nucleons from a spin zero target. Recall that for elastic scattering of a nucleon on a spin zero target, the scattering amplitude is of the form:

$$M = g(k^2, \cos\theta) + h(k^2, \cos\theta)\,\vec{\sigma}.\hat{n} \qquad 4\text{–}47$$

and the following observables can be measured independently:

(a) the differential cross section for an unpolarized beam:

$$\sigma_0(\theta) = |g|^2 + |h|^2, \qquad 4\text{–}65$$

(b) the analyzing power or polarization after scattering of an unpolarized beam:

$$P = A_y = \frac{2\mathrm{Re}\, gh^*}{|g|^2+|h|^2}, \qquad 4\text{–}72$$

(c) the spin rotation function Q

$$Q = \frac{2\mathrm{Im}\, gh^*}{|g|^2+|h|^2} \qquad 4\text{–}121$$

Now, in order to be able to theoretically calculate these observables, we need to employ a model which will enable us to calculate the spin–independent and spin–dependent amplitudes, g and h, respectively.

PART II

RELATIVISTIC EFFECTS ON THE MEASURED POLARIZATION OBSERVABLES

CHAPTER 5

5.1 Introduction to Part II

In due course the NAC (National Accelerator Centre), at Faure, will install an ion source capable of delivering polarized proton (spin–$\frac{1}{2}$) and polarized deuteron (spin–1) beams up to a maximum beam energy of 200 MeV. This means that for elastic spin–$\frac{1}{2}$ on spin zero scattering, experiments will be performed to determine the $\sigma_0(\theta)$, A_y and Q observables mentioned in section 4.3.5. It is our intention, therefore, to perform a study which will render us capable of *theoretically* calculating the above–mentioned observables. For this purpose, we adopt the following model:

5.2 The Model

(i) In nuclear physics, knowledge of the interaction between an incident nucleon and a target nucleus, enables one to calculate the scattering observables. The reason for this will become apparent as the discussion proceeds. We shall adopt the *optical potential* approach as a model for introducing the projectile–target nucleus interaction. In the optical potential (Fe 54, Gl 83, Sa 83, Ja 70, Fe 58, Fe 59) approach (called the optical model) the many–body problem for the elastic scattering of a projectile with a nucleus is reduced to a one–body problem with an effective interaction potential called the optical potential. The optical potential is so called because of the similarity between the transmission and the absorptive nature of the scattering process with that found in optics for the transmission and absorption of an electromagnetic wave through a medium.

The two main approaches in the study of elastic scattering phenomena have been through the use of *phenomenological* optical potentials and optical potentials based on fundamental *microscopic* ideas. The phenomenological approach (Me 81) has reaped rewards but needs to be supplemented by a refined theory. That is, since the phenomenological approach is empirical in nature, it can only give information about the gross features of the angular distributions involved in elastic scattering.

The microscopic approach, on the other hand, is concerned with the formulation of the scattering processes through the study of the interaction of the projectile with each nucleon of the target (Wa 53, Ke 59, Pi 82). Since the microscopic approach is more fundamental, many more details of elastic scattering can in principle be investigated and determined. It is the *microscopic approach* that will be used in this thesis. The specific microscopic model we use is the *multiple scattering approach*, where the information on the two–nucleon system is introduced via the t–matrix in order to solve to the problem of scattering of two nucleons.

(ii) Furthermore, we adopt the *Dirac equation*, as our basic equation of motion, with its relativistic treatment of the dynamics and its implicit incorporation of spin. A calculation based on the latter is referred to as a relativistic calculation. The question now arises as to why we consider relativistic as apposed to nonrelativistic calculations of the scattering observables. Let me explain:

When the word "relativistic" is mentioned, most students picture an object moving at a high speed, i.e., speeds comparable to the speed of light. Thus, it is thought that we need only consider relativistic calculations when the kinetic energy of the projectile is high (in the laboratory system). This is true if we wish to describe the kinematics involved correctly. Indeed, for a projectile of high energy one can employ the Schrödinger equation and include relativistic kinematics when performing *microscopic* calculations of the various polarization observables. For low energy experiments one may use nonrelativistic kinematics when solving the Schrödinger equation. Both the above–mentioned methods, however, fail to predict the spin observables, A_y and Q, correctly when one employs *microscopic* input in the calculations [Cl 83(a), Cl 83(b), Sh 83(a), Sh 83(b), Hy 85, Aa 85].

Clark [Cl 83(a), Cl 83(b)] and others [Sh 83(a), Sh 83(b), Hy 85, Aa 85], however, have used a Dirac optical potential to successfully describe the spin observables, where calculations based on the Schrödinger approach seem to be failing. Furthermore, what is

very interesting, is that it has been found (Wa 84) that relativistic effects play a very important role, *especially in low energy experiments*, since only the use of a relativistic Dirac formalism predicts the spin observables correctly when performing microscopic calculations. We shall see that the success can, amongst other things be attributed to the following aspects:

(1) The enlarged Hilbert space which includes both particle and antiparticle degrees of freedom.

(2) The increased strength (as opposed to a Schrödinger formalism) of the spin–orbit interaction which is a natural consequence of the Dirac formalism (see appendix B).

Note that when we use the word "relativistic" in this thesis, we refer to two things:

(1) The use of relativistic kinematics, AND

(2) The use of the Dirac equation as the basic equation of motion.

In this thesis we extend previous nonrelativistic calculations (Pi 84) to the case where the dynamics of the *projectile* is described by *a Dirac equation* (Hy 85, Hy 84).

(iii) Finally, we choose a *momentum–space representation* for the calculations of the optical potential and for the solution of the elastic scattering equation primarily because the off–shell NN t–matrix arising from a realistic potential model is naturally defined in this representation (Pi 84). The momentum–space method is also ideally suited to treating nonlocalities which arise in the microscopic optical potential. [Refer to section 5.2.3.2 for the meaning of the terms "non–local" and "off–shell" nucleon–nucleon t–matrix elements]. Additional advantages include the simple incorporation of relativistic kinematics and recoil.

In adopting the above mentioned model, we hope to achieve the following:

(1) Investigate the sensitivity of the observables $\sigma_0(\theta)$, A_y and Q to relativistic effects, i.e., compare relativistic and nonrelativistic calculations.

(2) Identify the nonrelativistic sector of the relativistic optical potential.

The model we adopt may be summarized as follows: We address proton–nucleus elastic scattering through the use of a *microscopic first–order optical potential* (section 5.2.1.2) based on the *multiple scattering theory* (section 5.2.1.1) of Kerman, McManus and Thaler (Ke 59)[hereafter referred to as *KMT*]. The model of *Love and Franey* (Lo 81) [section 5.2.3.3.2] will be employed for the nucleon–nucleon t–matrix as the fundamental ingredient of the optical potential calculation. In constructing the optical potential, we make use of the *optimum factorization procedure* (Pi 84) [section 5.2.2.1].

We now proceed to discuss the theory underlying the model which we adopt. Much of what is about to be discussed can be found in reference Wo 83.

5.2.1 The Theory

5.2.1.1 Multiple Scattering Theory

The methods employed in our calculations require the use of multiple scattering theory, and one of the main aspects is the development and use of an optical potential. We briefly review the concepts used in multiple scattering theory.

The scattering state that satisfies the many–body *Schrödinger** *equation* will be denoted by $|\Psi^+\rangle$, where the superscript indicates that the asymptotic behaviour is

*Our ultimate aim is to obtain the first–order optical potential in the KMT *nonrelativistic* multiple scattering theory – the reason for this will become clear in section 5.2.3.1.2.

that of an outgoing spherical scattered wave and an incoming plane wave of relative projectile–target motion multiplied by the internal ground state $|\phi_0>$ of the target nucleus. That is, the asymptotic behaviour is

$$|\Psi^+> \xrightarrow[r \to \infty]{} \{ e^{i\vec{k}_0 \cdot \vec{r}} + f(\theta) \frac{e^{ikr}}{r} \} |\phi_0> \qquad 5\text{–}1$$

where $\vec{r}$ is the position of the centre–of–mass of the projectile relative to the centre–of–mass of the target nucleus. Here, for simplicity, we refer to the scattering of a spin zero projectile from a spin zero target. For the scattering state of a projectile of non–zero spin on a spin zero target refer to section 4.3.1.3.

The wave function $e^{i\vec{k}_0 \cdot \vec{r}} |\phi_0>$ will be denoted by $|\phi>$ and obeys the equation:

$$(E - H_0)|\phi> = 0 \qquad 5\text{–}2$$

where E is the total relative energy:

$$E = \frac{\hbar^2 k_0^{\,2}}{2 \mu_{NA}} \qquad 5\text{–}3$$

and $\hbar k_0$ is the initial relative momentum

and μ_{NA} is the reduced mass. The non–interacting Hamiltonian H_0 is defined as

$$H_0 = H_T + K_0 \qquad 5\text{–}4$$

where H_T is the internal hamiltonian of the target and K_0 the relative kinetic energy operator. The total hamiltonian is

$$H = H_0 + V \qquad 5\text{–}5$$

where V is the sum of the interaction potentials between the projectile and each constituent nucleon of the target nucleus.

The Schrödinger equation for $|\Psi^+>$ is written as:

$$(E^+ - H_0)|\Psi^+> = V|\Psi^+> \qquad 5\text{–}6$$

Here $E^+ = E + i\epsilon$, where ϵ is real, positive and is meant to approach zero after the calculation. Equation 5–6 is obtained through the use of the formal scattering definition of $|\Psi^+>$ [Go 64], that is:

$$|\Psi^+(\epsilon)> = (E + i\epsilon - H_0 - V)^{-1}(E + i\epsilon - H_0)|\phi> \qquad 5\text{–}7$$

with the understanding that the physical solutions we are interested in are

$$|\Psi^+> = \lim_{\epsilon \to 0_+} |\Psi^+(\epsilon)> \qquad 5\text{–}8$$

With the introduction of the Green's function operators

$$G^+ = (E + i\epsilon - H_0 - V)^{-1} \qquad 5\text{–}9$$

and

$$G_0^+ = (E^+ - H_0)^{-1} \qquad 5\text{–}10$$

equation 5–7 can be written as

$$|\Psi^+> = G^+(E^+)\,[G_0^+(E^+)]^{-1}|\phi> \qquad 5\text{–}11$$

The relation between G^+ and G_0^+ is obtained through the use of the operator identity

$$\frac{1}{A} - \frac{1}{B} = \frac{1}{B}(B\text{–}A)\frac{1}{A} \qquad 5\text{–}12$$

Setting $A = (G_0^+)^{-1}$ and $B = (G^+)^{-1}$, we have

$$G_0^+ = G^+ - G^+VG_0^+ \qquad 5\text{–}13$$

or reversing the roles of A and B,

$$G^+ = G_0^+ + G_0^+VG^+ \qquad 5\text{–}14$$

Now, using equation 5–14, equation 5–11 can be rewritten as

$$|\Psi^+\rangle = |\phi\rangle + G_0^+(E^+)V\,|\Psi^+\rangle \qquad 5\text{–}15$$

Equation 5–15 is the *Lippmann–Schwinger equation* satisfied by the many–body state. It is a combination of the plane–wave solution of equation 5–2 (homogeneous solution, i.e., $V = 0$) and the solution for the inhomogeneous case (i.e., $V \neq 0$) represented in equation 5–6. Note at this stage that the same E appears in equations 5–2 and 5–6. This can be explained as follows: the presence of V in equation 5–6 causes the energy eigenstate to be different from a free particle state. However, if the scattering is to be elastic (as is the case we are interested in) – that is, no change in energy – we are interested in obtaining a solution to the full–Hamiltonian Schrödinger equation with the same energy eigenvalue as the non–interacting Hamiltonian (Sa 85).

It is useful at this point to define an operator which has the property that

$$\Omega^+|\phi\rangle = |\Psi^+\rangle \qquad 5\text{–}16$$

This operator, Ω^+, is known as the Möller wave operator (Go 64) and it transforms the asymptotic (plane wave) into the total scattering state.

Inserting equation 5–13 into equation 5–15, gives

$$\begin{aligned} |\Psi^+\rangle &= |\phi\rangle + G^+V\,(|\Psi^+\rangle - G_0^+V|\Psi^+\rangle) \\ &= |\phi\rangle + G^+V|\phi\rangle \qquad \text{(from equation 5–15)} \\ &= (1 + G^+V)|\phi\rangle \qquad 5\text{–}17 \end{aligned}$$

Comparing equations 5–16 and 5–17, we can identify the Möller operator as being

$$\Omega^+ = 1 + G^+V \qquad 5\text{–}18$$

In scattering theory the transition operator, T, is defined as

$$T|\phi> = V|\Psi^+> \qquad 5\text{–}19$$

Using definition 5–16, we can write

$$T|\phi> = V\Omega^+|\phi>$$

Hence

$$T = V\Omega^+ \qquad 5\text{–}20$$

and thus

$$T = V + VG^+V \quad \text{(from equation 5–18)} \qquad 5\text{–}21$$

$$= V + VG_0^+V + VG_0^+ VG^+V \quad \text{(from equation 5–14)}$$

$$= V + VG_0^+V\,(1 + G^+V)$$

$$= V + VG_0^+T \quad \text{(from equations 5–18 and 5–20)} \qquad 5\text{–}22$$

Since we deal always with outgoing scattered wave boundary conditions, we henceforth omit the superscript and write

$$\boxed{T = V + VG_0T} \qquad 5\text{–}23$$

which is the *Lippmann–Schwinger equation* for the transition operator.

With the substitution:

$$V = \sum_{i=1}^{A} v_{oi}, \qquad 5\text{–}24$$

where v_{oi} is the interaction between the projectile and the i^{th} nucleon of the target nucleus (which has A nucleons : mass number), equation 5–23 in principle contains all the information for describing the elastic scattering problem. Its many–body nature (due to presence of H_T and G_0), however, necessitates the introduction of approximation methods.

The total wave function $|\Psi>$ may be expanded in terms of a complete set of internal target states $|\phi_i>$ as

$$|\Psi> = \sum_i \Psi_i|\phi_i> \qquad 5\text{–}25$$

In order to define the optical potential, the projection operator technique of Feshbach (Fe 58) is used where the projection operators P and Q, respectively, project on and off the channel of interest. Hence, if ϕ_0 corresponds to the ground state of the target, we can use projection operators to distinguish the ground state from the excited states of the target nucleus by defining

$$P = |\phi_0><\phi_0|, \; Q = \sum_{i\neq 0} |\phi_i><\phi_i| \qquad 5\text{–}26$$

where

$$P + Q = 1 \qquad 5\text{–}27$$

Now we can express G_0 as

$$G_0 = PG_0 + QG_0 \qquad 5\text{–}28$$

Using the result in equation 5–23, we obtain:

$$T= V + VPG_0T + VQG_0T, \qquad 5\text{–}29$$

which can be rewritten as:

$$(1 - VQG_0)T = V + VPG_0T$$

$$\text{or} \quad T = (1 - VQG_0)^{-1}V + (1 - VQG_0)^{-1}VPG_0T \qquad 5\text{–}30$$

If we define:

$$U = (1 - VQG_0)^{-1}V, \qquad 5\text{–}31$$

equation 5–30 can be expressed in the following form:

$$\boxed{T = U + UPG_0T} \qquad 5\text{–}32$$

The expression shown in equation 5–31 can be rearranged, resulting in:

$$\boxed{U = V + VQG_0U} \qquad 5\text{–}33$$

We see that in equation 5–32 all the intermediate states are restricted by P to be the nuclear ground state only. There is, however, a price to pay for the truncation (i.e., considering elastic scattering in the intermediate states by virtue of the P operator). The price we pay is the appearance of an effective optical potential, U, which incorporates all intermediate nuclear excitations. Note that the pair of equations 5–32 and 5–33 is completely equivalent to equation 5–23.

The rationale for the *optical potential method* is that approximate calculations of U, when employed to obtain the full solution to equation 5–32 should be much more valuable and meaningful than approximations to equation 5–23. The elastic channel matrix elements of equation 5–32 produces a one–body equation which can be solved.

The task now is to express U in terms of the solution to the two–nucleon problem. With the aid of equation 5–24, equation 5–33 becomes:

$$U = \sum_i U^i = \sum_i v_{oi} + \sum_i v_{oi} Q G_0 U, \qquad 5\text{–}34$$

$$U^i = v_{oi} + v_{oi}\, Q G_0 \sum_j U^j, \qquad 5\text{–}35$$

$$U^i = v_{oi} + v_{oi}\, Q G_0 U^i + v_{oi}\, Q G_0 \sum_{j \neq i} U^j \qquad 5\text{–}36$$

Rearranging, we have

$$(1 - v_{oi}\, Q G_0) U^i = v_{oi} + v_{oi}\, Q G_0 \sum_{j \neq i} U^j \qquad 5\text{–}37$$

hence

$$U^i = \tau_{oi} + \tau_{oi}\, Q G_0 \sum_{j \neq i} U^j \qquad 5\text{–}38$$

where we have introduced

$$\tau_{oi} = (1 - v_{oi}\, Q G_0)^{-1} v_{oi} \qquad 5\text{–}39$$

From equation 5–38, 5–34 becomes the following series:

$$U = \sum_i U^i = \sum_i^A \tau_{oi} + \sum_{j \neq i} \tau_{oi}\, Q G_0\, \tau_{oj} + \sum_{i \neq j \neq k} \tau_{oi}\, Q G_0\, \tau_{oj}\, Q G_0\, \tau_{ok} + \ldots \qquad 5\text{–}40$$

This is known as the Watson [Wa 57] multiple scattering series for the optical potential. Equation 5–40 is a series development in terms of a well–behaved (nonsingular) two–body scattering operator τ (Gl 83,p.83). Watson's series represents the successive scattering of the projectile from different nucleons. The first term represents the complete scattering from any single nucleon in the target, summed over all target nucleons. The second term represents the complete scattering of the incident nucleon by any two different bound nucleons: first by nucleon j, its propagation through an excited intermediate state of the nucleus until it scatters

from nucleon i and then emerges, and such events are *summed over all pairs.* The third term represents the complete scattering from three different bound nucleons, and so on.

Now, since we are interested in calculating elastic scattering, that is, events in which the nucleus is in its ground state initially and finally, we take the ground state expectation value of both sides of equation 5–32. This yields

$$\langle\phi_0|T|\phi_0\rangle = \langle\phi_0|U|\phi_0\rangle + \langle\phi_0|U|\phi_0\rangle\, \Gamma_0\, \langle\phi_0|T|\phi_0\rangle \qquad 5\text{–}41$$

which can be more compactly written as

$$T_{00} = U_{00} + U_{00}\, \Gamma_0\, T_{00}, \qquad 5\text{–}42$$

where

$$U_{00} = \langle\phi_0|\sum_i \tau_{0i}|\phi_0\rangle + \ldots \qquad \text{[from equation 5–40]}$$

$$\text{and } \Gamma_0 = \frac{1}{E(k_0) + i\epsilon - K_0} \quad . \qquad 5\text{–}43$$

Note that we have taken the energy of the ground state of the target nucleus to be zero, i.e., $H_T|\phi_0\rangle = 0|\phi_0\rangle$.

It is important to state at this point, that in this thesis we adopt the *single scattering approximation* for the optical potential. This means that we only consider the first term of the Watson multiple scattering series (eq. 5–40) of the optical potential. This first term of the Watson multiple scattering series is associated with the multiple scattering from the (A) individual target nucleons, one at a time, where the nucleus remains in its ground state at the conclusion of each scattering. This term is referred to as the first–order optical potential. We will consider this term throughout the work presented here as the only term of the Watson series which is of importance. In fact, the contributions from double, triple and higher order scattering have been

shown to be small (Bo 77, Ul 74, Ra 79). Recall that the second term of the Watson multiple scattering series is associated with the multiple scattering from $[\frac{1}{2} A(A-1)]$ pairs of particles, with the nucleus again remaining in its ground state at the conclusion of each scattering. In general these higher–order terms correspond to the correlation effects of the nucleons. For example, the second–order term of the optical potential can be shown to be proportional to the two–nucleon correlation function of the nucleus (Er 77). Hence, the use of the single scattering approximation is equivalent to neglecting nucleon–nucleon correlations, and as others (Pi 84) have shown, is a reasonably good assumption for elastic scattering calculations.

Also, we assume that the *target nucleons are indistinguishable* and consequently we write

$$\langle\phi_0|\sum_{i=1}^{A}\tau_{0i}|\phi_0\rangle = A\langle\phi_0|\tau|\phi_0\rangle = A\tau_{00} \qquad 5\text{–}44$$

where A is the number of nucleons in the target nucleus, and τ is any one of the τ_{0i}. Note that A occurs as a factor because of the identity of the target nucleons.

Thus, adopting the single scattering approximation and at the same time respecting the identity of the target nucleons, equation 5–42 becomes:

$$\boxed{T_{00} = A\tau_{00} + A\tau_{00}\Gamma_0 T_{00},} \qquad 5\text{–}45$$

The τ which is to be used in equation 5–45 is not easily calculated because of the projection operator Q and the many–body nature of the Green's function in equation 5–39. We shall now show how the complications caused by the projection operator Q can be avoided. First equation 5–39 can be expressed as:

$$\tau_{0i} = v_{0i} + v_{0i} QG_0 \tau_{0i} \qquad 5\text{–}46$$

With equation 5–27, we now write eq. 5–46:

$$\tau_{oi} = v_{oi} + v_{oi}(1-P)\,G_0\tau_{oi}, \qquad 5\text{–}47$$

$$\tau_{oi} = v_{oi} + v_{oi}G_0\tau_{oi} - v_{oi}PG_0\tau_{oi}, \qquad 5\text{–}48$$

$$(1 - v_{oi}G_0)\tau_{oi} = v_{oi} - v_{oi}\,PG_0\,\tau_{oi}, \qquad 5\text{–}49$$

$$\tau_{oi} = \tilde{t}_{oi} - \tilde{t}_{oi}\,PG_0\tau_{oi} \qquad 5\text{–}50$$

where we have introduced the two–body scattering operator $\tilde{t}_{oi}$ for an incident nucleon striking a bound target nucleon i in a nucleus with A nucleons,

$$\tilde{t}_{oi} = (1 - v_{oi}G_0)^{-1}\,v_{oi}, \text{ or} \qquad 5\text{–}51$$

$$\tilde{t}_{oi} = v_{oi} + v_{oi}G_0\,\tilde{t}_{oi} \qquad 5\text{–}52$$

which is the solution that would be obtained if the intermediate states were not restricted to excited states of the target. If the operator $\tilde{t}_{oi}$ was available, then the required elastic channel matrix elements of τ_{oi} can easily be calculated from equation 5–50 which becomes a one–body equation.

The KMT method that we describe in the next section takes advantage of this by combining equations 5–52 and 5–45 to describe the optical potential solely in terms of $\tilde{t}_{oi}$. A frequently used approximation for $\tilde{t}_{oi}$ of equation 5–52 is t^F_{oi}, the free–particle transition operator. The free t–matrix equation has the following form:

$$t^F_{oi} = v_{oi} + v_{oi}g_0t^F_{oi} \qquad 5\text{–}53$$

where g_0 is the two–body Green's function:

$$g_0 = \frac{1}{E_{NN} - K_0 - K_1}\ . \qquad 5\text{–}45$$

Here E_{NN} is the nucleon–nucleon relative energy, K_0 and K_1 are the kinetic energy operators of the two nucleons. From equation 5–53 and 5–54, we can derive the relation between the required $\tilde{t}_{oi}$ and the available t^F_{oi} as:

$$\tilde{t}_{oi} = v_{oi} + v_{oi}g_0\tilde{t}_{oi} + v_{oi}(G_0 - g_0)\tilde{t}_{oi},$$

$$(1 - v_{oi}g_0)\tilde{t}_{oi} = v_{oi} + v_{oi}(G_0 - g_0)\tilde{t}_{oi},$$

$$\tilde{t}_{oi} = (1 - v_{oi}g_0)^{-1} v_{oi}\{1 + (G_0 - g_0)\tilde{t}_{oi}\},$$

$$\tilde{t}_{oi} = t^F_{oi} + t^F_{oi}(G_0 - g_0)\tilde{t}_{oi} \qquad 5\text{–}55$$

This means that if the difference between G_0 and g_0 is not too great so that $\tilde{t}_{oi} \approx t^F_{oi}$, then the scattering amplitude is given approximately by the free–nucleon amplitude. The replacement of $\tilde{t}_{oi} \approx t^F_{oi}$, the free–particle transition operator, is known as the impulse approximation.

In this thesis we adopt the *impulse approximation* and at a later stage (section 5.2.3.3.2) we shall say more about the *Love and Franey model* (Lo 81) we employ for the calculation of the free nucleon–nucleon t–matrix.

In the impulse approximation, the projectile and target nucleon interact as if they were free particles. Now equation 5–50 can be written as:

$$\tau \simeq t^F - t^F PG_0\tau \qquad 5\text{–}56$$

where $\tau = \tau_{oi}$, and $\tilde{t}_{oi} = t^F$. In this approximation one neglects the effects of the nuclear medium on the nucleon–nucleus scattering. Corrections to the impulse approximation could include such effects as nuclear binding and the restrictions from the Pauli principle.

5.2.1.2 The Kerman–McManus–Thaler (KMT) method

The close connection between the elastic scattering optical potential and the free nucleon–nucleon t–matrix was first demonstrated by Kerman, McManus and Thaler (Ke 59). We can easily obtain the KMT results by starting with equation 5–32 and making use of equation 5–44:

$$T = A\tau + A\tau PG_0T \qquad 5\text{–}57$$

With the use of equation 5–50, which can be expressed as

$$\tau = \tilde{t} - \tilde{t}\, PG_0\tau, \qquad 5\text{–}58$$

we solve for $\tilde{t}$ to get:

$$\tilde{t} = (1 + \tilde{t}PG_0)\tau, \qquad 5\text{–}59$$

and consequently

$$\tau = (1 + \tilde{t}PG_0)^{-1}\,\tilde{t} \qquad 5\text{–}60$$

When equation 5–60 is substituted into equation 5–57, we obtain:

$$(1 + \tilde{t}PG_0)T = A\tilde{t} + A\tilde{t}PG_0T,$$

$$T = A\tilde{t} + (A-1)\tilde{t}\, PG_0T \qquad 5\text{–}61$$

Now if we define an auxiliary transition operator T' as

$$T' = \frac{A-1}{A}T, \qquad 5\text{–}62$$

we can express equation 5–61 as

$$T' = (A-1)\tilde{t} + (A-1)\tilde{t}\, PG_0T'. \qquad 5\text{–}63$$

Equation 5–63 is the KMT result, where T' is not the physical T–matrix but the latter can be obtained from T' through relation 5–62. The equation 5–63 can be written as a Lippmann–Schwinger equation

$$\boxed{T' = U' + U'PG_0T'} \qquad 5\text{–}64$$

thus enabling us to identify the KMT first–order pseudo–optical potential:

$$\boxed{U' = (A-1)\bar{t}} \qquad 5\text{–}65$$

In the impulse approximation, we write:

$$\boxed{U' = (A-1)t^F} \qquad 5\text{–}66$$

Note that we have already employed the fact that the target nucleons are indistinguishable [see eq. 5–44]. So far we have not addressed the identity of the projectile in the description of the theory underlying our approach. We adopt the approach of Picklesimer and Thaler (Pi 81), who show that it is possible to develop a multiple scattering expansion in the case where the projectile is identical with the target nucleons by distributing, in an exact way, the Pauli exchanges involving the projectile amongst all terms of the multiple scattering expansion so that in each term the identity of all the "active" particles experiencing the residual interaction is respected. For the first order multiple scattering term, this formulation leads to the same expression as given by equation 5–65 for the auxiliary KMT optical potential, with the requirement that $\bar{t}_{oi}$ be antisymmetrized with respect to exchange of the projectile with just particle "i" of the target i.e.:

$\bar{t}_{oi}$ is replaced by $\bar{t}_{oi}(1 - P_{oi})$ (see section 5.2.4.3.2.1)

where P_{oi} is the operator that interchanges all coordinates (spin–, isospin– and space coordinates) of the "active" particles o (the projectile) and i (the target nucleon interacting with the projectile). For a complete and detailed study of antisymmetry, see reference Pi 81. Also refer to Gl 83.

5.2.2 Approximations to the KMT First–Order Optical Potential

We have seen from eq. 5–66 that the KMT nonrelativistic first–order optical potential for elastic scattering can be expressed as

$$U' = (A-1) \langle \phi_0 | t | \phi_0 \rangle \qquad 5\text{–}67$$

where t is the free nucleon–nucleon antisymmetrized t–matrix (mentioned in the last paragraph of the previous section) and $|\phi_0\rangle$ is the target ground state normalized to unity.

To avoid confusion in the remainder of this thesis, always bear in mind that our model retains a nonrelativistic treatment of the target ground state, and considers the effects of negative energy intermediate states and propagation for the projectile alone i.e., the projectile is regarded as a Dirac particle.

In this section we wish to derive explicit expressions for the first order optical potential which takes into account both off–shell and non–local effects, as well as recoil effects. To do this we adopt the optimum factorization approach of Ernst and Weiss (Er 82, Er 83; also see Pi 84).

5.2.2.1 Optimum Factorization

We start by considering the matrix elements of U′ in the momentum space of the projectile (see appendix H).

$$\langle \vec{k}' | U' | \vec{k} \rangle = U'(\vec{k}', \vec{k}) = (A-1) \int d^3p \sum_{s,i} \langle \vec{k}'; \vec{p} - \tfrac{1}{2}\vec{q}, s,i | t | \vec{k}; \vec{p} + \tfrac{1}{2}\vec{q}, s,i \rangle$$

$$\times \rho^{s,i}_{int} \left[\vec{p} - \frac{A-1}{A}\frac{\vec{q}}{2} + \frac{\vec{K}}{A}; \vec{p} + \frac{A-1}{A}\frac{\vec{q}}{2} + \frac{\vec{K}}{A} \right] \qquad 5\text{–}68$$

where $\vec{k}$ and $\vec{k}'$ refer to respectively to the initial and final momentum of the projectile in the nucleon–nucleus centre–of–mass system. The initial momentum of the struck nucleon is represented by $\vec{p} + \frac{1}{2}\vec{q}$ (in N–Nucleus c.m.–system). The initial (final) momentum of the struck nucleon relative to the A–nucleus centre–of–mass system (c.m.–system) is given by

$$\vec{p} + (-)\frac{A-1}{A}\frac{\vec{q}}{2} + \frac{\vec{K}}{A}$$

where

$\vec{K} = \frac{1}{2}(\vec{k}' + \vec{k})$, and

$\vec{q} = \vec{k}' - \vec{k}$ (momentum transferred to the projectile nucleon).

Note that the momentum $\vec{p}$ is an integration variable. In equation 5–68, s and i are the spin and isospin projections of the *struck* nucleon which must be conserved for elastic scattering; $U'(\vec{k}',\vec{k})$ and the matrix element of t, are operators in the spin and isospin space of the projectile nucleon.

The quantity $\rho^{s,i}_{int}(\vec{m}',\vec{m})$,

where $\vec{m}' = \vec{p} - \frac{A-1}{A}\frac{\vec{q}}{2} + \frac{\vec{K}}{A}$ and

$$\vec{m} = \vec{p} + \frac{A-1}{A}\frac{\vec{q}}{2} + \frac{\vec{K}}{A},$$

is the one–nucleon density matrix of the target corresponding to a change in the intrinsic momentum from $\vec{m}$ to $\vec{m}'$ for a nucleon of spin and isospin projections s,i. By intrinsic momentum we mean the momentum of a nucleon relative to the centre–of–mass of the A nucleons constituting the nucleus.

To simplify matters, we assume that we are dealing with a *spin–saturated nucleus* (eg. ^{16}O, ^{40}Ca, ^{24}Mg, ^{28}Si, ^{32}S). In such a nucleus, the probability of finding a spin–up nucleon with specified momentum is exactly the same as the probability of finding a spin–down nucleon with the same momentum, i.e., $\rho_{int}^{s,i}(\vec{m}';\vec{m}) = \rho_{int}^{s',i}(\vec{m}';\vec{m}) = \rho_{int}^{i}(\vec{m}';\vec{m})$. In this circumstance, the spin trace will eliminate those components of the t–matrix which depend linearly on the spin of the *struck* nucleon (see Gl 83, p.193 and equations 5–162 to 5–166). The eliminated terms are the tensor term, and the $\vec{\sigma}_1.\vec{\sigma}_2$ and $(\vec{\sigma}_1.\vec{\sigma}_2)(\vec{\tau}_1.\vec{\tau}_2)$ part of the central term. The remaining terms are the spin–independent central term and part of the spin–orbit term for each of the pp and np t–matrices. We denote these "reduced" t–matrices by t'_α and the corresponding density matrix by ρ_{int}^{α} where $\alpha = n,p$. Thus for a proton projectile, equation 5–68 may be written as an average, namely

$$U'(\vec{k}',\vec{k}) = \frac{A-1}{A} \sum_{\alpha=n,p} \int d^3p \langle \vec{k}';\vec{p}-\tfrac{1}{2}\vec{q}\,|t'_\alpha|\,\vec{k};\vec{p}+\tfrac{1}{2}\vec{q}\rangle \times \rho_{int}^{\alpha}(\vec{m}';\vec{m}) \qquad 5\text{–}69$$

where the density matrices are normalized to N for neutrons and Z for protons. It is convenient to make a change of integration variable from $\vec{p}$ to $\vec{P} = \vec{p} + \frac{\vec{K}}{A}$, so that equation 5–69 becomes

$$U'(\vec{k}',\vec{k}) = \frac{A-1}{A} \sum_{\alpha=n,p} \int d^3P \langle \vec{k}';\vec{P}-\frac{\vec{q}}{2}-\frac{\vec{K}}{A}\,|t'_\alpha|\,\vec{k};\vec{P}+\frac{\vec{q}}{2}-\frac{\vec{K}}{A}\rangle \times$$

$$\rho_{int}^{\alpha}\left[\vec{P}-\gamma\frac{\vec{q}}{2};\vec{P}+\gamma\frac{\vec{q}}{2}\right] \qquad 5\text{–}70$$

where $\gamma = \frac{A-1}{A}$. We note that the density matrix is related to the momentum–space density profile $\rho^{\alpha}(q)$ of the nucleus by

$$\rho^{\alpha}(q) \equiv \rho_{int}^{\alpha}\left(\frac{A-1}{A}q\right)$$

$$= \int d^3P\, \rho_{int}^{\alpha}\left[\vec{P}-\frac{A-1}{A}\frac{\vec{q}}{2};\vec{P}+\frac{A-1}{A}\frac{\vec{q}}{2}\right], \qquad 5\text{–}71$$

where $\rho^{\alpha}(q)$ measures the ability of the nucleus to absorb momentum $\vec{q}$ and still remain in the ground state and where $\rho^{\alpha}(q)$ is the Fourier transform of the position space point neutron or point proton density profile defined in a system of coordinates with the origin at the centre–of–mass of the nucleus. The normalization is such that $\rho^{\alpha}(q=0)$ = N or Z. We deal with $\rho^{\alpha}(q)$, since for protons it can be obtained from the nuclear charge density measured by electron scattering after the intrinsic proton charge form factor is divided out. We will take the point neutron density to be equal to the point proton density for N=Z nuclei. This is expected to be a good approximation for N=Z nuclei. It is perhaps relevant at this stage to mention that in our final calculations the major source of uncertainty is the lack of constraint on the nuclear density. Lack of knowledge of the large momentum transfer components of the density is the principle question that needs to be addressed by any future measurements of scattering observables in an extended angular range.

In our expression for the N–nucleus first order optical potential the free t–matrix describes the NN collision as seen in the N–nucleus centre–of–mass system. In the following section we express the collision matrix in terms of the usual collision matrix defined in the NN centre–of–mass system, which is directly related to the phase shifts obtained from performing experiments.

To relate the t–matrix of equation 5–70, which is in the nucleon–nucleus system, to the corresponding t–matrix in the nucleon–nucleon system, we need to introduce the Möller transformation factor η (Go 64, Mo 45, Wo 83)

such that

$$\eta = \frac{E_0^{\,2}}{E_1\, E_2} \qquad 5\text{–}72$$

where

E_0 is the total energy of the incident nucleon in the nucleon–nucleon centre–of–mass system, E_1 is the total energy of the incident nucleon in the nucleon–nucleus centre–of–mass system, and E_2 the total energy of the target nucleon in the nucleon–nucleus centre–of–mass system.

We then have

$$\boxed{<\vec{k}';\vec{P}-\frac{\vec{q}}{2}-\frac{\vec{K}}{A}|t'_\alpha|\vec{k};\vec{P}+\frac{\vec{q}}{2}-\frac{\vec{K}}{A}>_{NA}=\eta(\vec{P},\vec{q}\,\vec{K})<\vec{\kappa}'|t'_\alpha|\vec{\kappa}>_{NN}} \qquad 5\text{–}73$$

where the initial nucleon and final nucleon momentum $\vec{\kappa}$ and $\vec{\kappa}'$ in the NN centre–of–mass system are given by the definitions

$$\vec{\kappa} = \tfrac{1}{2}\{\vec{k}-\left[\vec{P}+\frac{\vec{q}}{2}-\frac{\vec{K}}{A}\right]\} \qquad 5\text{–}74$$

and

$$\vec{\kappa}' = \tfrac{1}{2}\{\vec{k}'-\left[\vec{P}-\frac{\vec{q}}{2}+\frac{\vec{K}}{A}\right]\} \qquad 5\text{–}75$$

The Möller factor η is given more specifically, by

$$\eta(\vec{P},\vec{q},\vec{K})=\left\{\frac{E_N(\vec{\kappa}')E_N(-\vec{\kappa}')E_N(\vec{\kappa})E_N(-\vec{\kappa})}{E_N(\vec{k})E_N(\vec{P}-\frac{\vec{q}}{2}-\frac{\vec{K}}{A})E_N(\vec{k})E_N(\vec{P}+\frac{\vec{q}}{2}-\frac{\vec{K}}{A})}\right\}^{\frac{1}{2}} \qquad 5\text{-}76$$

where $E_N(\vec{k})$ is the relativistic energy of a nucleon of momentum $\vec{k}$. The Möller factor is chosen such that our theory is consistent with the special theory of relativity and because it imposes the Lorentz invariance of flux.

The NN t–matrix on the right–hand side of equation 5–73 can be written in the form

$$<\vec{\kappa}'|t'_{\alpha}|\vec{\kappa}>_{NN} = t'_{\alpha}(\vec{q},2\vec{\chi}) \qquad 5\text{–}77$$

where

$$\vec{q} = \vec{\kappa}' - \vec{\kappa} = \vec{k}' - \vec{k} \qquad 5\text{–}78$$

and

$$\vec{\chi}(\vec{P}) = \tfrac{1}{2}(\vec{\kappa}' + \vec{\kappa}) = \tfrac{1}{2}\left[\frac{A+1}{A}\vec{K} - \vec{P}\right] \qquad 5\text{–}79$$

We write expression 5–77 since the Love and Franey model (Wo 83) of the NN t–matrix that we employ is specified in terms of $\vec{q}$ and $2\vec{\chi}$ [see section 5.2.4.3.2.1]. With these notations 5–70 becomes

$$\boxed{U'(\vec{k}',\vec{k}) = \frac{A-1}{A}\sum_{\alpha=n,p}\int d^3P\ \eta(\vec{P},\vec{q},\vec{K})t'_{\alpha}[\vec{q},2\vec{\chi}(\vec{P})]\times \rho^{\alpha}_{int}\left[\vec{P}-\gamma\frac{\vec{q}}{2};\ \vec{P}+\gamma\frac{\vec{q}}{2}\right]} \qquad 5\text{–}80$$

This "full–folding" expression for the first–order KMT optical potential is quite difficult to compute. In this thesis we employ the optimum factorization approximation for equation 5–80. By factorization we mean that since the t–matrix in equation 5–80 is more slowly varying with $\vec{P}$ than is ρ_{int} (because the size of the nucleus is significantly larger than the range of the nucleon–nucleon interaction), we estimate the result by evaluating $\eta t'_{\alpha}$ at a fixed value of $\vec{P}$ and remove it from the integral. In this way, we systematically include off–shell, non–local and recoil effects into the optical potential. The optimum value of $\vec{P}$ is found by expanding $\eta t'_{\alpha}$ in a Taylor series in $\vec{P}$ about a fixed value $\vec{P}_0$ chosen such that the second term in the Taylor series is zero.

The expansion

$$\eta(\vec{P})t'_{\alpha}(\vec{P}) = \eta(\vec{P}_0)t'_{\alpha}(\vec{P}_0) + (\vec{P}-\vec{P}_0).\partial_{\vec{P}_0}\eta(\vec{P}_0)t'_{\alpha}(\vec{P}_0) + \ldots \qquad 5\text{–}81$$

when used in equation 5–80 gives:

$$U'(\vec{k}',\vec{k}) = \frac{A-1}{A}\,\eta(\vec{P}_0,\vec{q},\vec{K}) \sum_{\alpha=n,p} t'_\alpha[\vec{q},\, 2\mathcal{K}(\vec{P}_0)] \times \rho^\alpha(q) + \Delta U' + \ldots \qquad 5\text{–}82$$

where

$$\Delta U' = \frac{A-1}{A} \sum_{\alpha=n,p} [\partial_{\vec{P}_0}\eta(\vec{P}_0)t'_\alpha(\vec{P}_0)] \times \int d^3P\,(\vec{P}-\vec{P}_0)\rho^\alpha_{int}\left[P-\gamma\frac{\vec{q}}{2};P+\gamma\frac{\vec{q}}{2}\right] \qquad 5\text{–}83$$

and $\rho^\alpha(q)$ is given by eq. 5–71.

Now, the time–reversal invariance property of the ground–state density matrix for even–even nuclei $\left[\vec{J} = \sum_{i=1}^{A}(\vec{\ell}_i + s_i) = 0\right]$ leads to

$$\int d^3\vec{P}\;\vec{P}\;\rho^\alpha_{int}\left[\vec{P}-\gamma\frac{\vec{q}}{2};\;\vec{P}+\gamma\frac{\vec{q}}{A}\right] = 0 \qquad 5\text{–}84$$

so that $\Delta U'$ is zero if we choose $\vec{P}_0 = 0$. Thus the optimally factorized optical potential is

$$\boxed{U'(\vec{k}',\vec{k}) = \frac{A-1}{A}\,\eta(\vec{q},\vec{K}) \sum_{\alpha=n,p} t'_\alpha\left[\vec{q},\,\frac{A-1}{A}\vec{K}\right]\rho^\alpha(q)} \qquad 5\text{–}85$$

where $\eta(\vec{q},\vec{K})$ shall be used to stand for $\eta(\vec{P}=0,\,\vec{q},\,\vec{K})$.

Referring back to equation 5–73, we see that the optimum choice for factorization, $\vec{P}=0$, selects the initial momentum of the struck nucleon to be $\frac{\vec{q}}{2}-\frac{\vec{K}}{A}$ and the final momentum to be $-\frac{\vec{q}}{2}-\frac{\vec{K}}{A}$ in the nucleon–nucleus centre–of–mass system. That this choice physically makes sense can be seen in the limit of a single nucleon for the target (A), where these momenta become $-\vec{k}$ and $-\vec{k}'$, which are the correct values of NN scattering. Also, in this limit, $\eta \to 1$ and $\rho^\alpha(q) \to 1$. Note that in general, the *recoil* effects are included through the kinematic effects employed in equation 5–85.

The optical potential obtained thus far is an operator in the spin–space of the projectile. To make the spin dependence explicit we write the t–matrix t'_α (which has been averaged over the spin of the struck nucleon) in the form (see section 5.2.3.3.2).

$$t'_\alpha(\vec{q},2K) = t^C_\alpha(q,2K) + \tfrac{1}{2}\,\vec{\sigma}.\vec{q} \times K\, t^{LS}_\alpha(\vec{q}.2K) \qquad 5\text{–}86$$

The first term corresponds to the central spin–independent contribution, while the second term corresponds to the spin–orbit contribution. In the latter term the usual total Pauli operator of the NN system is replaced by just the projectile's Pauli spin operator, the other having been eliminated by the trace over the spin of the struck nucleon. Substitution of equation 5–86 (bearing in mind that $\vec{P} = 0$) into equation 5–85 gives the optical potential as

$$U'(\vec{k}',\vec{k}) = U'^{C}(\vec{k}',\vec{k}) + \frac{i}{2}\,\vec{\sigma}.\vec{q} \times \vec{K}\, U'^{LS}(\vec{k}',\vec{k}) \qquad 5\text{–}87$$

where the central term is given by

$$U'^{C}(\vec{k}',\vec{k}) = \frac{A-1}{A}\,\eta(\vec{q},\vec{K}) \sum_{\alpha=n,p} t^c_\alpha\left[\vec{q}, \frac{A+1}{A}\vec{K}\right]\rho^\alpha(q) \qquad 5\text{–}88$$

and the spin–orbit term is given by

$$U'^{LS}(\vec{k}',\vec{k}) = \frac{A-1}{A}\,\eta(\vec{q},\vec{K})\left\{\frac{A+1}{2A}\right\}\sum_{\alpha=n,p} t^{LS}_\alpha\left[\vec{q}, \frac{A+1}{A}\vec{K}\right]\rho^\alpha(q) \qquad 5\text{–}89$$

This completes our optimum factorization results for the first–order optical potential.

The partial wave components of $U'^{C}(\vec{k}',\vec{k})$ and $U'^{LS}(\vec{k}',\vec{k})$ can be calculated in terms of NN t–matrix components and nuclear densities. For example, the projection can be performed numerically by evaluating the integral (Pi 84, Wo 83).

$$U'^{C}_{L}(k',k) = \tfrac{1}{2}\int_{-1}^{+1} dx P_L(x)\, U'^{C}(\vec{k}',\vec{k}), \qquad 5\text{–}90$$

$$= \tfrac{1}{2}\int_{-1}^{+1} dx P_L(x) \frac{A-1}{A}\, \eta[\vec{q}(x),\vec{K}(x)] \sum_{\alpha=n,p} t^{c}_{\alpha}\left[\vec{q}, \frac{A+1}{A}\vec{K}(\vec{x})\right] \rho^{\alpha}[\vec{q}(x)] \qquad 5\text{–}91$$

where $q^2(x) = k'^2 + k^2 - 2k'kx$, $K^2(x) = \tfrac{1}{4}(k'^2 + k^2 + 2k'kx)$,

$\vec{q}.\vec{K} = \tfrac{1}{2}(k'^2 - k^2)$, and $x = \cos\theta = \hat{k}'.\hat{k}$

Alternatively, in terms of the partial wave components of t^{C}_{α}, t^{LS}_{α}, and ρ^{α} defined with the same convention as equation 5–90, the result can be expressed as

$$U'^{C}_{L}(k',k) = \frac{A-1}{A}\,\eta \sum_{\ell,\ell'} \sum_{\alpha=n,p} \frac{(2\ell+1)(2\ell'+1)}{(2L+1)} \langle \ell 0;\ell' 0 | L0\rangle^2\, t^{c}_{\alpha,\ell}(k',k)\rho_{\ell'}(k',k) \qquad 5\text{–}92$$

Analogous results hold for $U'^{LS}_{L}(k',k)$.

5.2.2.2 Off–Shell and Non–Local effects

We note that the optical potential is non–local (i.e., not simply a function of $|\vec{k}'-\vec{k}|$) and involves off–shell values of the nucleon–nucleon t–matrix. To clarify the latter point, we note that no mention has yet been made of a prescription for determining the energy at which the t–matrix is to be evaluated. For the optimum factorization calculations performed in this thesis, we employ the prescription energy of choosing the nucleon–nucleon centre–of–mass kinetic energy of the t–matrix to be one–half the beam kinetic energy. This corresponds to assuming that the struck nucleon is always at rest prior to the collision. We have seen that in the optimum factorization treatment, the initial momentum of the struck nucleon is actually $\frac{\vec{q}}{2} - \frac{\vec{K}}{A}$; therefore this prescription is quite reasonable for forward angle scattering ($q = 0$) from nuclei with large A.

In actual fact:

$$t_{\alpha}^{C,LS} = t_{\alpha}^{C,LS}\left[\epsilon;\ \vec{q},\ \frac{A+1}{A}\vec{K}\right]$$

where ϵ is the NN centre–of–mass frame kinetic energy. The off–shell effects enter in equation 5–88 and 5–89 because the three arguments of t_{α}^{c} and t_{α}^{LS} are completely independent. Inspection of equation 5–73 to 5–76 shows that the imposition of the two–body, on shell constraint $|\vec{\kappa}'| = |\vec{\kappa}|$ and $\epsilon = E_N(\vec{\kappa}) + E_N(-\vec{\kappa})$ in terms of the variables $\vec{q}$ and $\vec{K}$ leads to

$$q.K = 0 \qquad 5\text{–}93$$

$$q^2\left[\frac{A-1}{A}K\right]^2 = 4\kappa^2 \qquad 5\text{–}94$$

where $\vec{\kappa}$ is the on–shell nucleon momentum in the zero momentum frame of the NN system and is calculated from the prescribed value of ϵ. Thus, under the on–shell constraint, the second argument $\left[\frac{A+1}{A}\right]\vec{K}$ of the t–matrices in equation 5–88 and 5–89 is completely determined by ϵ and $\vec{q}$ and the optical potential becomes local, apart from the nonlocality of the Möller factor $\eta(\vec{q},\vec{K})$. This nonlocality is found to be quite negligible (Pi 84). In the calculations employing the optimally factorized optical potential we shall make use of the Love and Franey model (Lo 81) of t_{NN} in which the t_{α}^{c} and t_{α}^{LS} can be obtained as functions of the three independent variables as required in equations 5–88 and 5–89. Wolfe (Wo 83) investigates whether off–shell and nonlocal dependence of the t–matrix is important in calculations of the scattering observables. This thesis is concerned with the calculation of the scattering observables taking into account the off–shell and nonlocal dependences discussed above. Note that the origin of the above–mentioned nonlocal and off–shell effects was due to the fact that the NN t–matrix is antisymmetrized (see section 5.2.4.3.2) and because target recoil is included i.e., we have taken the presence of a nuclear medium into account in this way.

5.2.3 Relativistic (Dirac equation) effects in microscopic elastic scattering calculations

Up until now we have described the nonrelativistic (Schrödinger equation) impulse approximation (NRIA) to the optical potential for elastic proton scattering from nuclei. Wolfe (Wo 83) and Picklesimer (Pi 84) investigated the influence of off–shell and nonlocal effects and the related ambiguities associated with approximation treatments of the nuclear matrix element of the two–body scattering operators. It was found that the scattering observables at forward angles are not affected significantly by these sources of ambiguities in current approximation methods for implementing the NRIA.

Although the NRIA yields qualitatively adequate theoretical predictions, especially at several hundred MeV, these predictions do not provide completely satisfactory descriptions of the high precision data currently available. In particular, some details of the spin observables are poorly given by the NRIA and even the descriptions of differential cross sections are not completely satisfactory when measured against the standards set by the data. Moreover, the successes and failures of the NRIA do not generally appear to follow any systematic dependence upon projectile energy or momentum transfer which might be attributed to higher order effects. Thus one cannot appeal to what have become the standard sources of error for an explanation of the inadequacy of the NRIA. It appears that there may be an omission of a fundamentally important process in the theoretical approach itself. Calculations by Shepard (Sh 83) and Clark (Cl 83) strongly suggest that an approach within the context of a Dirac wave equation may go a long way towards a resolution of this problem. In this thesis we discuss a relativistic approach of this type and present calculations based on a formulation which keeps the relation with the nonrelativistic description in focus. The previous nonrelativistic calculations (Pi 84) are extended to the case where the dynamics of the *projectile* are described by a *Dirac equation* (Hy 85) and the target ground state retains a nonrelativistic (Schrödinger equation) treatment. The calculation is based on work done by Hynes, Picklesimer, Tandy and Thaler (Hy 85) and by Wolfe (Wo 83, Pi 84).

In section 5.2.3.1.1 the Dirac equation in the presence of an external interaction is cast into a form of a integral equation where the operators couple positive and negative energy plane–wave states for the projectile. After defining our notation and the method we employ for handling the Dirac equation, we turn to the question of the microscopic content of the nucleon–nucleus interaction (see section 5.2.3.1.2). The ansatz employed to obtain this from the first–order nonrelativistic optical potential is then described. The partial–wave decomposition of the positive and negative energy sectors of the optical potential is given in section 5.2.3.2 as is the partial wave form of the Dirac integral equation. The method for solving the coupled integral equations is discussed in section 5.2.3.3. The numerical results, conclusions and furture proposals are presented and discussed in Chapter 6.

5.2.3.1 Theoretical framework

5.2.3.1.1 Dirac equation with an external field

The differential form of the Dirac equation for the scattering of a spin–$\frac{1}{2}$ projectile of mass m from an external central field U is (we choose $\hbar = c = 1$)

$$\boxed{(\not{p} - m - U)|\Psi> = 0}\,, \qquad 5\text{–}95$$

where $\not{p} = \gamma^{\mu} p_{\mu} = \gamma_{\mu} p^{\mu}$ (pronounced "p slash")

$p^{\mu} = (E, -i\vec{\nabla})$, $p_{\mu} = (E, i\vec{\nabla})$

$\mu = 0, 1, 2, 3,$ and the

γ^{μ} represent the Dirac matrices

The notation for four vectors and the gamma matrices is that of Bjorken and Drell (Bj 64) – also refer to appendix B for an explanation of notation.

Now the positive energy free state with momentum $\vec{k}$ and rest frame spin projection s satisfies

$$(\not{p} - m)|\vec{k},s(+)> = 0 \qquad 5\text{–}96$$

Refer to appendix I for coordinate space forms of $|\vec{k},s(\pm)>$ as well as for the orthonomality and completeness relations for the basis states.

From equations 5–95 and 5–96, the integral equation equivalent of equation 5–95 is

$$|\Psi> = |\vec{k},s(+)> + \frac{1}{\not{p} - m + i\delta} U|\Psi> \qquad 5\text{–}97$$

which is essentially the Lippmann–Schwinger equation 5–15 and implements the outgoing spherical wave boundary conditions with the limit $\delta \longrightarrow 0+$. Subsequently, we shall not explicitly display the $i\delta$ term in the Dirac propagator. With a transition operator defined as

$$T|\vec{k},s(+)> = U|\Psi>, \qquad 5\text{–}98$$

(compare with equation 5–19)

equation 5–97 leads to the operator integral equation

$$\boxed{T = U + U\frac{1}{\not{p} - m}T} \qquad 5\text{–}99$$

(compare with equation 5–23)

which is completely equivalent to the differential form of the Dirac equation 5–95. In order to define transition probabilities that are Lorentz scalar quantities, we introduce an operator $\tilde{T}$ such that

$$\tilde{T} = \gamma^o T \qquad 5\text{–}100$$

The transition matrix elements are given by

$$\begin{aligned} T^{ab}_{s's}(\vec{k}',\vec{k}) &= <\vec{k}',s'(a)|\tilde{T}|\vec{k},s(b)> \\ &= \overline{<\vec{k}',s'(a)|T|\vec{k},s(b)>} \\ &= <\chi_{s'}|T^{ab}(\vec{k}',\vec{k})|\chi_s> \end{aligned} \qquad 5\text{–}101$$

where

$$T^{ab}(\vec{k}',\vec{k}) = \langle\overline{\vec{k}',a}|T|\vec{k},b\rangle \qquad 5\text{–}102$$

since

$$|\vec{k},s(\pm)\rangle \equiv |\vec{k},\pm\rangle|\chi_s\rangle$$

and

$$\langle\overline{\vec{k},(\pm)}| = \langle\vec{k},\pm|\gamma^0$$

and where a and b can each be either + or −. $|\chi_s\rangle$ is a usual Pauli two–component spinor. The primed quantities refer to the projectile nucleon after the incident nucleon (unprimed quantities) has been scattered elastically by the nucleus.

To make the integral equation 5–99 more explicit, the Dirac propagator can be expanded as (see appendix L)

$$(p-m)^{-1} = \int d^3k \left\{ \frac{|\vec{k},+\rangle\langle\vec{k},+|}{E - E_k} + \frac{|\vec{k},-\rangle\langle\vec{k},-|}{E + E_k} \right\} \qquad 5\text{–}103$$

where $E_k = k^2 + m^2$

This expansion implies that the projectile can propagate firstly as a particle (in $|\vec{k},+\rangle$ states) and then as a virtual antiparticle (in $|\vec{k},-\rangle$ states). When equation 5–103 is employed in equation 5–99, and matrix elements are taken with respect to the same complete basis, the resulting coupled pair of integral equations can be written in the form:

$$T^{++}(\vec{k}',\vec{k}) = U^{++}(\vec{k}',\vec{k}) + \int d^3k''\, U^{++}(\vec{k}',k'')\frac{1}{E - E_{k''}}T^{++}(\vec{k}'',\vec{k}) + \int d^3k''\, U^{+-}(\vec{k}',k'')\frac{1}{E - E_{k''}}T^{-+}(\vec{k}'',\vec{k}), \qquad 5\text{–}104$$

$$T^{-+}(\vec{k}',\vec{k}) = U^{-+}(\vec{k}',\vec{k}) + \int d^3k''\, U^{-+}(\vec{k}',k'')\frac{1}{E - E_{k''}}T^{++}(\vec{k}'',\vec{k}) + \int d^3k''\, U^{--}(\vec{k}',k'')\frac{1}{E - E_{k''}}T^{-+}(\vec{k}'',\vec{k}), \qquad 5\text{–}105$$

where k',k and k'' are in units of fm^{-1} and E is in MeV. Hence $T(\vec{k}',\vec{k})$ is in MeV.fm^3. The remainder of this thesis concerns understanding and solving these coupled integral equations. It will be shown at a later stage (equations 5–152 and 5–153) that the reason for wanting to solve these equations is because the scattering coefficients $g(\theta)$ and $h(\theta)$ [expression 4–40] can be expressed in terms of these solutions. The projected interactions are:

$$U^{ab}(\vec{k}',\vec{k}) = \overline{<\vec{k}',a|U|\vec{k},b>}, \qquad 5\text{–}106$$

where a and b can stand for either of the labels + or −. The various quantities $T(\vec{k}',\vec{k})$ and $U(\vec{k}',\vec{k})$ in equation 5–104 and 5–105 are operators in Pauli spin space (2 × 2 matrices) and the pair of coupled equations 5–104 and 5–105 are just another version of the four–component Dirac equation, equation 5–95.

Now, if we were to truncate the formalism so that only particle degrees of freedom were allowed, then the equations that determine T^{++}, namely equation 5–104 and 5–105, become simply the usual Lippmann–Schwinger equation (with relativistic kinematics), viz.,

$$T^{++}(\vec{k}',\vec{k}) = U^{++}(\vec{k}',\vec{k}) + \int d^3k'' \frac{U^{++}(\vec{k}',\vec{k}'')\, T^{++}(\vec{k}'',\vec{k})}{E - E_{k''}} \qquad 5\text{–}107$$

It is therefore natural to interpret T^{++} as the "nonrelativistic" component of the transition operator. We shall make use of this shortly. The other components, namely T^{+-} and T^{-+} are purely relativistic components. Thus, the essential relativistic feature of the Dirac equation is the enlarged Hilbert space due to the appearance of the antiparticle degree of freedom, on an equal footing with the particle degree of freedom.

5.2.3.1.2 Microscopic Approach

We recall from equation 5–67 that the first–order optical potential in the Kerman–McManus–Thaler (KMT) nonrelativistic (NR) multiple scattering theory is

$$U'_{NR}(\vec{k}',\vec{k}) = (A-1)\langle \vec{k}';\phi_0|t|\phi_0;\ \vec{k}\rangle \qquad 5\text{–}108$$

where $|\vec{k}\rangle$ is a Schrödinger plane wave and $|\phi_0\rangle$ is the target ground state, normalized to unity. Recall that the KMT presentation was motivated by the desire to obtain a scattering theory which explicitly incorporates the target indistinguishability of the nucleons at all stages of the scattering process and which defines the projectile nucleus optical potential in terms of elementary two–body operators, where occupation of the entire Hilbert space in intermediate scattering states is allowed.

For the purposes of our calculation, we do not follow a field theoretical approach. Instead, we follow the procedure adopted by Hynes, Picklesimer, Tandy and Thaler (Hy 85) which seeks an extension of the nonrelativistic first order mechanism for use in the Dirac equation. It will be shown that a very simple ansatz for the relativistic extension to the negative energy sector can reveal significant insight into the important features of the Dirac description of elastic scattering. The *ansatz* we employ is to make the identification

$$\boxed{U^{++}(\vec{k}',\vec{k}) = U'_{NR}(\vec{k}',\vec{k})} \qquad 5\text{–}109$$

and hope to specify the other relativistic components (U^{++}, U^{-+}, U^{--}) of the Dirac optical potential.

An optimum factorization treatment of the matrix element in equation 5–108 yields (refer to section 5.2.2.1 for a discussion of the optimum factorization procedure) equations 5–87 to 5–89, namely:

$$\boxed{U^{++}(\vec{k}',\vec{k}) = U'^{C}(\vec{k}',\vec{k}) + \frac{i}{2}\,\vec{\sigma}.\vec{k}' \times \vec{k}\ U'^{LS}(\vec{k}',\vec{k})} \qquad 5\text{–}110$$

where the central term is given by

$$U'^{c}(\vec{k}',\vec{k}) = \frac{A-1}{A}\eta(\vec{k}',\vec{k}) \sum_{\alpha=n,p} t^{c}_{\alpha}(\vec{q}.\vec{K})\rho^{\alpha}(q) \qquad 5\text{–}111$$

and the spin–orbit is given by

$$U'^{LS}(\vec{k}',\vec{k}) = \frac{A-1}{A}\eta(\vec{k}',\vec{k})\left\{\frac{A+1}{2A}\right\} \sum_{\alpha=n,p} t^{LS}_{\alpha}(\vec{q}.\vec{K})\rho^{\alpha}(q) \qquad 5\text{–}112$$

where the meaning of the symbols has already been discussed.

We now change to a Dirac representation by writing:

$$U^{++}(\vec{k}',\vec{k}) = \langle\overline{\vec{k}',+}|S+\gamma^{o}V|\vec{k},+\rangle \qquad 5\text{–}113$$

where S is a Lorentz scalar, and V is the timelike component of a Lorentz four–vector (see appendix B) which together describe the N–nucleus interaction.

The *ansatz* we employ is that the S and V derived from the *forced equality* of the representation in equation 5–110 and 5–113 is to be used to calculate the extensions U^{+-}, U^{-+} and U^{--}. That is, we take:

$$U^{ab}(\vec{k}',\vec{k}) = \langle\overline{\vec{k}',a}|S+\gamma^{o}V|\vec{k},b\rangle \qquad 5\text{–}114$$

where again a and b can each be either + or −. Equations 5–110 to 5–114 define the Dirac optical potential which is employed in this thesis together with the integral equations 5–104 and 5–105 to calculate scattering observables. The extension ansatz embodied by equation 5–113 and 5–114 does not rely upon detailed knowledge of the microscopic content of U'^{c} and U'^{LS}. The expression for U'^{c} and U'^{LS} in terms of NN scattering operators and nuclear densities will vary with the type of microscopic approach. We have chosen to employ expressions for U'^{c} and U'^{LS} that come directly from a nonrelativistic approach. Note that at this level of

treatment there is no fundamental reason for choosing the Lorentz structure $S + \gamma^o V$ in equation 5–113. There are many other Lorentz structures with different momentum and gamma matrix dependence that could be used in equation 5–113 while still reproducing the form given in equation 5–110. For example, a Lorentz tensor could be used in place of either the scalar or vector (Hy 85). The results for U^{+-}, U^{++} and U^{--} would then, in general, be different. We choose $S + \gamma^o V$ for simplicity and for comparison with both phenomenological analysis of scattering (Ar 79) and mean–field treatment of nuclear ground states (Wa 85).

5.2.3.2 Partial Wave Projections of the Interactions and the Integral Equations

The Dirac momentum–space integral equations [equations 5–104 and 5–105] are to be solved separately for each angular momentum state. Here we outline the expansion of each of these quantities in Pauli spinor–spherical harmonics, and derive the resulting projected form of the integral equations. We first give the explicit forms for the four components U^{++}, U^{+-}, U^{-+} and U^{--} of the Dirac optical potential employed here. From equations (I2) and (I3), with the notation $U = S+\gamma^o V$, we find.

$$U^{++}(\vec{k}',\vec{k}) = \langle \overline{\vec{k}',+} | S + \gamma^o V | \vec{k},+ \rangle$$

$$= N_{k'} N_k \left\{ (V + S) + \frac{(V - S)}{\epsilon_{k'} \epsilon_k} (\vec{k}'.\vec{k} + i\vec{\sigma}.\vec{k}' \times \vec{k}) \right\} \qquad 5\text{–}115$$

where

$$\epsilon_k = E_k + m = (k^2 + m^2)^{\frac{1}{2}} + m \qquad 5\text{–}116$$

and from equation (I3) the normalization constant of the Dirac spinors is

$$N_k = \left[\frac{\epsilon_k}{2E_k} \right]^{\frac{1}{2}} . \qquad 5\text{–}117$$

For convenience we denote $V(\vec{k}',\vec{k})$ and $S(\vec{k}',\vec{k})$ by simply V and S. Note the similarity between equation 5–110 and 5–115 which we will use in due course. In a similar manner, equation 5–114 also yields

$$U^{+-}(\vec{k}',\vec{k}) = \overline{<\vec{k}',+|S+\gamma^{o}V|\vec{k},->}$$

$$= N_{k'}N_{k}\left\{\frac{\vec{\sigma}.\vec{k}'}{\epsilon_{k'}}(V-S)-(V+S)\frac{\vec{\sigma}.\vec{k}}{\epsilon_{k}}\right\} \qquad 5\text{–}118$$

$$U^{-+}(\vec{k}',\vec{k}) = \overline{<\vec{k}',-|S+\gamma^{o}V|\vec{k},+>}$$

$$= N_{k'}N_{k}\left\{(V-S)\frac{\vec{\sigma}.\vec{k}}{\epsilon_{k}}-\frac{\vec{\sigma}.\vec{k}'}{\epsilon_{k'}}(V+S)\right\} \qquad 5\text{–}119$$

and

$$U^{--}(\vec{k}',\vec{k}) = \overline{<\vec{k}',-|S+\gamma^{o}V|\vec{k},->}$$

$$= N_{k'}N_{k}\left\{(V-S)+\frac{(V+S)}{\epsilon_{k'}\epsilon_{k}}(\vec{k}'.\vec{k}+i\vec{\sigma}.\vec{k}'\times\vec{k})\right\} \qquad 5.120$$

It is useful to define auxiliary quantities D and F such that

$$D(\vec{k}',\vec{k}) = N_{k'}N_{k}\{V(\vec{k}',\vec{k})-S(\vec{k}',\vec{k})\} \qquad 5\text{–}121$$

and

$$F(\vec{k}',\vec{k}) = N_{k'}N_{k}\{V(\vec{k}',\vec{k})+S(\vec{k}',\vec{k})\} \qquad 5\text{–}122$$

since these are the only combinations that appear. The full Dirac optical potential can then be expressed as

$$U^{++}(\vec{k}',\vec{k}) = F(\vec{k}',\vec{k})+\frac{D(\vec{k}',\vec{k})}{\epsilon_{k'}\epsilon_{k}}(\vec{k}'.\vec{k}+i\vec{\sigma}.\vec{k}'\times\vec{k}), \qquad 5\text{–}123$$

$$U^{+-}(\vec{k}',\vec{k}) = \frac{\vec{\sigma}.\vec{k}'}{\epsilon_{k'}}D(\vec{k}',\vec{k})-F(\vec{k}',\vec{k})\frac{\vec{\sigma}.\vec{k}}{\epsilon_{k}} \qquad 5\text{–}124$$

$$U^{-+}(\vec{k}',\vec{k}) = D(\vec{k}',\vec{k})\frac{\vec{\sigma}.\vec{k}}{\epsilon_{k}}-\frac{\vec{\sigma}.\vec{k}'}{\epsilon_{k'}}F(\vec{k}',\vec{k}) \qquad 5\text{–}125$$

and

$$\bar{U}(\vec{k}',\vec{k}) = D(\vec{k}',\vec{k}) + \frac{F(\vec{k}',\vec{k})}{\epsilon_{k'}\,\epsilon_k}.(\vec{k}'.\vec{k} + i\vec{\sigma}.\vec{k}' \times \vec{k}) \qquad 5\text{–}126$$

Given the nonrelativistic optical potential in the form of equation 5–110, our ansatz for determining V and S can now be expressed as [by the forced equality of equations 5–110 and 5–123]

$$\boxed{D(\vec{k}',\vec{k}) = \frac{\epsilon_{k'}\,\epsilon_k}{2}\,U'^{LS}(\vec{k}',\vec{k})} \qquad 5\text{–}127$$

and

$$\boxed{F(\vec{k}',\vec{k}) = U'^{C}(\vec{k}',\vec{k}) - \tfrac{1}{2}\,\vec{k}'.\vec{k}\;U^{LS}(\vec{k}',\vec{k})} \qquad 5\text{–}128$$

Thus, we can now express the full relativistic potential in terms of the nonrelativistic optical potentials, U'^{C} and U'^{LS}. Recall that our aim in this section is to obtain and solve the integral equations 5–104 and 5–105 in partial wave form.

The angular momentum expansions for the various quantities to be considered here are based on the following (see appendix M):

(1) The assumption that the nucleon–nucleus interaction is rotationally invariant.

(2) The Wigner–Eckart theorem (Sc 83).

(3) The expansion of an abstract operator $\hat{\Omega}$, namely,

$$\hat{\Omega} = I\hat{\Omega}I = \sum_{n,m} |n\rangle \langle n|\hat{\Omega}|m\rangle \langle m|$$

where I is a unit operator and n and m label a complete set of basis states.

These considerations will also determine which quantum numbers are to be used as labels in the various expansions. The angular momentum expansion for $F(\vec{k}',\vec{k})$ is (see appendix M)

$$F(\vec{k}',\vec{k}) = 4\pi \sum_{JLM} \mathcal{Y}^{M}_{JL}(\hat{k}')\, F_L(k',k)\, \mathcal{Y}^{M\dagger}_{JL}(\hat{k}) \qquad 5\text{–}129$$

where $\mathcal{Y}^{M}_{JL}(\hat{k})$ is the standard Pauli spinor–spherical harmonic defined by

$$\mathcal{Y}^{M}_{JL}(\hat{k}) = \sum_{m_L s} Y_L^{m_L}(\hat{k})\chi_s \langle Lm_L; \tfrac{1}{2}s|JM\rangle \qquad 5\text{–}130$$

in terms of normal spherical harmonics $Y_L^{m_L}(\hat{k})$ and Clebsch–Gordon coefficients. Expansions similar to equation 5–129 also exist for D, U^c and U^{LS}. By making use of these expansions, and a recurrence relation for the Legendre polynomials, we get (Pi 84, Hy 85) the angular momentum projected forms of equations 5–127 and 5–128, namely

$$D_L(k',k) = \frac{\epsilon_{k'}\epsilon_k}{2} U'^{LS}_L(k',k) \qquad 5\text{–}131$$

and

$$F_L(k',k) = U'^{C}_L(k',k) - \frac{k'k}{2(2L+1)} \times \left\{ (L+1)U'^{LS}_{L+1}(k',k) + LU'^{LS}_{L-1} \right\}. \qquad 5\text{–}132$$

We now require the angular momentum expansions of the Dirac optical potentials, U^{++}, U^{+-}, U^{-+} and U^{--}. There are only two types of expansions required because when the roles of F and D are reversed, U^{++} becomes U^{--}, and when the roles of F and –D are reversed, U^{+-} becomes U^{-+}. The following angular momentum expansions are derived (see appendix M, Pi 84, Hy 85)

$$U^{++}(\vec{k}',\vec{k}) = 4\pi \sum_{JLM} \mathcal{Y}^{M}_{JL}(\hat{k}')\, U^{++}_{JL}(k',k)\, \mathcal{Y}^{M\dagger}_{JL}(\hat{k}), \qquad 5\text{–}133$$

with

$$U^{++}_{JL}(k',k) = F_L(k',k) + \frac{k'k}{\epsilon_{k'}\epsilon_k} D_{\bar{L}}(k',k) \qquad 5\text{–}134$$

where $\bar{L} = 2J - 1$;

$$4\pi \sum_{JLM} \mathcal{Y}^{M}_{JL}(\hat{k}')\, U^{+-}_{JL}(k',k)\, \mathcal{Y}^{M\dagger}_{J\bar{L}}(\hat{k}), \qquad 5\text{–}135$$

where

$$U^{+-}_{JL}(k',k) = F_L(k',k)\frac{k}{\epsilon_k} - \frac{k'}{\epsilon_{k'}} D_{\bar{L}}(k',k); \qquad 5\text{–}136$$

$$U^{-+}(\vec{k}',\vec{k}) = 4\pi \sum_{JLM} \mathcal{Y}^{M}_{J\bar{L}}(\hat{k}')\, U^{-+}_{J\bar{L}}(k',k)\, \mathcal{Y}^{M\dagger}_{JL}(\hat{k}), \qquad 5\text{–}137$$

where

$$U_{JL}^{-+}(k',k) = \frac{k'}{\epsilon_{k'}} F_L(k',k) - D_{\bar{L}}(k',k) \frac{k}{\epsilon_k} \qquad 5\text{–}138$$

From equations 5–123, 5–126 and 5–134, an interchange of the roles of F and D allows the expansion of $U^{--}(\vec{k}',\vec{k})$ to be written as

$$U_{JL}^{--}(k',k) = D_{\bar{L}}(k',k) + \frac{k'k}{\epsilon_{k'}\epsilon_k} F_L(k',k) \qquad 5\text{–}139$$

The above–mentioned angular momentum expansion of the optical potentials, serves to define similar expansions for the T–matrix elements of equations 5–104 and 5–105. This leads to an angular momentum quantized form of the important integral equations 5–104 and 5–105.

$$T^{++}(\vec{k}',\vec{k}) = 4\pi \sum_{JLM} \mathcal{Y}^{M}_{J\,L}(\hat{k}')\, T_{JL}^{++}(k',k)\, \mathcal{Y}^{M\dagger}_{J\,L}(\hat{k}) \qquad 5\text{–}140$$

and

$$T^{-+}(\vec{k}',\vec{k}) = 4\pi \sum_{JLM} \mathcal{Y}^{M}_{J\,\bar{L}}(\hat{k}')\, T_{JL}^{-+}(k',k)\, \mathcal{Y}^{M\dagger}_{J\,L}(\hat{k}). \qquad 5\text{–}141$$

The integral equations in partial wave form are

$$\boxed{T_{JL}^{++}(k',k) = U_{JL}^{++}(k',k) + 4\pi \sum_{\alpha=+,-} \int dk''k''^2\, U_{JL}^{+\alpha}(k'k'') \frac{1}{E - \lambda_a E_{k''}} T_{JL}^{\alpha+}(k'',k)} \qquad 5\text{–}142$$

$$\boxed{T_{JL}^{-+}(k',k) = U_{JL}^{-+}(k',k) + 4\pi \sum_{\alpha=+,-} \int dk''k''^2\, U_{JL}^{-\alpha}(k'k'') \frac{1}{E - \lambda_a E_{k''}} T_{JL}^{\alpha+}(k'',k)} \qquad 5\text{–}143$$

where $\lambda_+ = +1$, and $\lambda_- = -1$. We use relativistic kinematics such that

$$E = E(k) = E_1(k) + E_2(k) \qquad 5\text{–}144$$

where the subscripts 1 and 2 refer to the two colliding particles, and

$$E_i^2(k) = (\hbar c k)^2 + m_i^2 \qquad 5\text{–}145$$

with the momentum expressed in fm^{-1}.

At this stage I would like to express the reason for wanting to solve equations 5–142 and 5–143.

First of all we state that it is convenient to use units in which momentum is in MeV.

We therefore use:

$$\hat{k} = (\hbar c)k \qquad 5\text{–}146$$

and define

$$\frac{T_{JL}}{(\hbar c)^3} = \hat{T}_{JL} \qquad 5\text{–}147$$

$$\frac{U_{JL}}{(\hbar c)^3} = \hat{U}_{JL} \qquad 5\text{–}148$$

$$E_i^2(\hat{k}) = \hat{k}^2 + m_i^2 \qquad 5\text{–}149$$

where $\hat{T}_{LJ}$, $\hat{U}_{LJ}$ and E_i^{-2} are now in units of MeV^{-2}.

Furthermore, if we multiply both $\hat{T}_{LJ}$ and $\hat{U}_{LJ}$ by a factor $2\pi^2$, the partial wave of the coupled integral equations become:

$$T_{JL}^{++}(k',k) = U_{JL}^{++}(k',k) + \frac{2}{\pi}\sum_{\alpha=+,-}\int dk'' k''^2\, U_{JL}^{+\alpha}(k',k'')\frac{1}{E-\lambda_\alpha E_{k''}}T_{JL}^{\alpha+}(k'',k) \qquad 5\text{–}150$$

$$T_{JL}^{-+}(k',k) = U_{JL}^{-+}(k',k) + \frac{2}{\pi}\sum_{\alpha=+,-}\int dk'' k''^2\, U_{JL}^{-\alpha}(k',k'')\frac{1}{E-\lambda_\alpha E_{k''}}T_{JL}^{\alpha+}(k'',k) \qquad 5\text{–}151$$

The scattering amplitude $M(\theta)$, in units of fermis, can be directly related to the t–matrix T_{JL} through the partial wave expansion (Wo 83, Go 64):

$$\boxed{g(\theta) = \frac{-\rho(E)}{k_0}\sum_{L=0}^{\infty}\{(L+1)T_L^+ + L\,T_L^-\}\,P_L(\cos\theta)}\,, \qquad 5\text{–}152$$

$$\boxed{h(\theta) = \frac{-\rho(E)}{k_0}\sum_{L=1}^{\infty}(T_L^+ - T_L^-)\,P_L^1(\cos\theta)}\,, \qquad 5\text{–}153$$

where the superscripts $(\pm)$ on T_L refer to the total angular momentum, which has values $J = L \pm \frac{1}{2}$, and

$$T_{JL} = T_{JL}^{++} + T_{JL}^{+-} + T_{JL}^{-+} \; T_{JL}^{--},$$

$$M(\theta) = g(\theta) + h(\theta)\vec{\sigma}.\hat{n}, \qquad 5\text{–}154$$

$\hat{n}$ is a unit vector in the direction $(\vec{k}' \times \vec{k})$,

k_0 is the on–shell momentum in fm^{-1},

$$\rho(E) = 2k_0 \left\{ \frac{E_N(k_0)E_A(k_0)}{E_N(k_0) + E_A(k_0)} \right\} \quad \text{in } (MeV)^2 \text{ units} \qquad 5\text{–}155$$

where the subscripts N and A refer to the projectile and target respectively.

Therefore, we see that once we have determined T_{JL}^{++}, T_{JL}^{+-}, T_{JL}^{-+} and T_{JL}^{--}, then in principle we can determine the scattering amplitudes $g(\theta)$ and $h(\theta)$ and hence also the observables $\frac{d\sigma}{d\Omega}$, Ay and Q.

If we include the Coulomb potential, in addition to the nuclear potential, which affects the scattering of charged particles, the scattering amplitudes $g(\theta)$ and $h(\theta)$ are found to be

$$\boxed{g(\theta) = f_{coul}(\theta) - \frac{\rho(E)}{k_0} \sum_{L=0}^{\infty} e^{2i\sigma_L} \{(L+1)T_L^+ + L\,T_L^-\} P_L(\cos\theta)}, \qquad 5\text{–}156$$

$$\boxed{h(\theta) = -\frac{\rho(E)}{k_0} \sum_{L=1}^{\infty} e^{2i\sigma_L} (T_L^+ - T_L^-) P_L^1(\cos\theta)} \qquad 5\text{–}157$$

where $f_{coul}(\theta)$ is the Coulomb scattering amplitude and σ_L is a pure Coulomb scattering phase shift (Wo 83, Go 64).

So now we see where the partial wave forms of the coupled integral equations [equations 5–104 and 5–105] fit into the calculations of the scattering observables.

The procedure leading to the scattering observables may be summarized schematically as follows:

Input:	$\rho^{\alpha}, t^{c}_{\alpha}, t^{LS}_{\alpha}$	[equation 5–9]
	↓	
	U'^{c}_{L}, U'^{LS}_{L}	[equation 5–9]
	↓	
	F_L, D_L	[equations 5–131 and 5–132]
	↓	
	$U^{++}_{JL}, U^{+-}_{JL}, U^{-+}_{JL}, U^{--}_{JL}$	[equations 5–134, 5–136, 5–137 and 5–139]
	↓	
	$T^{++}_{JL}, T^{+-}_{JL}, T^{-+}_{JL}, T^{--}_{JL}$	[equations 5–142 and 5–143]
	↓	
	$g(\theta), h(\theta)$	[equations 5–156 and 5–157]
	↓	
Scattering observables	$\frac{d\sigma}{d\Omega}$, Ay, Q	[equations 4–65, 4–72, 4–12]

5.2.3.3 Solving the Coupled Integral Equations for Elastic Scattering

This section will be brief. Details can be found in Wo 83.

Equations 5–150 and 5–151 are in the form of the Lippmann–Schwinger equation which is to be solved using the *computer code WIZARD 1.*

From equations 5–150 and 5–151 we see that in order to solve the coupled integral equations we require $U_L'^{c}(k',k)$ and $U_L'^{LS}(k',k)$. These quantities are, in turn, expressed in terms of t_α^c, t_α^{LS} and ρ^α [see equations 5–91 and 5–92]. We now proceed to discuss these latter quantities very briefly.

5.2.3.3.1 Fourier transform of position space nuclear density profile

In the incorporation of the density into the optical potential, separate proton and neutron densities are calculated. The particular form of the density used for calculations in this thesis is the *three–parameter Fermi density*. The density has the following form in coordinate space

$$\rho(r) = \rho_0 \left[1 + w \left[\frac{r}{c} \right]^2 \right] \left[1 + e^{(r-c)/t} \right]^{-1} \qquad 5\text{–}158$$

where

$$\rho_0 = \frac{3N}{4\pi(c^3 + \pi^2 t^2 c)} \left\{ 1 + \frac{3w}{5} \left[1 + \frac{7}{3} \left[\frac{\pi t}{c} \right]^2 \right] \right\}^{-1} \qquad 5\text{–}159$$

Here, N is either the number of protons or neutrons, and c, t, and w are the density distribution parameters (c is the radius and w is the surface thickness of the distribution and t is some type of diffuseness parameter). The proton density parameters that are used for calculations are chosen from those appropriate from electron scattering (De 74). The neutron parameters are taken to be the same as the proton parameters. The constant ρ_0 normalizes the distribution to the number of particles N, that is, $\rho(q=0) = N$.

In momentum space, the Fourier transform of $\rho(r)$ is (Wo 83)

$$\begin{aligned} \rho(q) &= \int d\vec{r}\, e^{i\vec{q}\cdot\vec{r}}\, \rho(r) \\ &= \frac{4\pi}{q} \int_0^\infty \rho(r) \sin(qr)\, r dr \\ &= \frac{4\pi}{q} \rho_0 \left\{ \phi(q) - \frac{w}{c^2} \phi'(q) \right\} \end{aligned} \qquad 5\text{–}160$$

where

$$\phi(q) = \pi c\left\{\frac{-\cos(qc)}{\sin h(qt\pi)} + \frac{\pi t}{c}\frac{\sin(qc)\ \cot\ h(qt\pi)}{\sin h(qt\pi)} - \frac{2t}{c}\sum_{n=1}^{\infty}(-b)^n\frac{nq\,t}{[(qt)^2 + n^2]^2}\right\} \quad 5\text{–}161$$

It should be noted that $b = e^{-c/t}$ and hence the sum in equation 5–161 converges very rapidly.

5.2.3.3.2 The Love and Franey Model for the Free Nucleon–Nucleon t–matrix

The Love and Franey (LF) t–matrix (Lo 81) is written as a sum of three terms, namely, the central term, the spin–orbit term and the tensor term. Each term corresponds to a particular spin dependence known to be important in describing nucleon–nucleon scattering. We can write the LF t–matrix as (Wo 83, Gl 83).

$$t_{LF}(t,\vec{k}) = t_c(r) + t_{LS}(r)\vec{L}.\vec{S} + t_T(r)S_{12} \quad 5\text{–}162$$

with

$$\vec{L} = \vec{r} \times \vec{k}\ ,\quad \vec{r} = \vec{r}_1 - \vec{r}_2,\quad \vec{k} = \tfrac{1}{2}(\vec{k}_1 - \vec{k}_2)$$

$$S = \tfrac{1}{2}\vec{\sigma},\quad \vec{\sigma} = \vec{\sigma}_1 + \vec{\sigma}_2$$

The subscripts 1 and 2 on the spin matrices correspond to the projectile and target nucleon respectively.

Furthermore, the central contribution,can be further decomposed as

$$t_c(r) = t_o^c(r) + t_\sigma^c(r)\vec{\sigma}_1.\vec{\sigma}_2 + t_\tau^c(r)\vec{\tau}_1.\vec{\tau}_2 + t_{\sigma\tau}^c(r)(\vec{\sigma}_1.\vec{\sigma}_2)(\vec{\tau}_1.\vec{\tau}_2) \quad 5\text{–}163$$

where $\vec{\sigma}_i$ and $\vec{\tau}_i$ are the spin and isospin operators for nucleon i. The various coefficients t_o^c, t_σ^c, t_τ^c, and $t_{\sigma\tau}^c$ depend on the coordinates of the two nucleons and can be expressed in terms of combinations of singlet–even, singlet–odd, triplet–even and triplet–odd components for each nucleon–nucleon channel where even or odd refer to the parity of the two–nucleon state (Gl 83, Wo 83). Here $\vec{L}.\vec{S}$ is the usual spin–orbit operator in dimensionless units. The spin–orbit contribution to equation 5–162, $t_{LS}(r)$, is written as

$$t_{LS}(r) = t_o^{LS}(r) + t_\tau^{LS}(r)\,\vec{\tau}_1.\vec{\tau}_2\ . \quad 5\text{–}164$$

Finally $t_T S_{12}$ is the tensor contribution, where t_T is written as

$$t_T(r) = t_0^T(r) + t_\tau^T(r)\, \vec{\tau}_1.\vec{\tau}_2 \qquad 5\text{–}165$$

and S_{12} is the tensor operator defined as

$$S_{12} = \frac{3}{r^2}(\vec{\sigma}_1.\vec{r})(\vec{\sigma}_2.\vec{r}) - \vec{\sigma}_1.\vec{\sigma}_2 \quad . \qquad 5\text{–}166$$

Recall (section 5.2.3.1) that for elastic scattering from a $J = 0$ spin–saturated nucleus, the tensor term vanishes from integration over the spin coordinates of the target nucleon; thus only central and spin–orbit terms in equation 5–162 will be needed in forming the optical potential.

In the LF model, each of the terms in equation 5–162 is written as a combination of a forward–angle part (direct) and a backward–angle part (space exchange) (Wo 83, Gl 83). That is, the *fully antisymmetrized t–matrix* (see section 5.2.3.3.2.2) is written as

$$t_{TS}(\vec{q},\vec{\kappa}) = t_{TS}^D(\vec{q}) + (-)^{T+S+1}\, t_{TS}^E(2\vec{\kappa}) \qquad 5\text{–}167$$

where the superscripts D and E refer to the direct and exchange contributions, respectively. The functional form of t_{TS}^E is the same as for t_{TS}^D, where $\vec{q}$ and $\vec{\kappa}$ have the definitions

$$\vec{q} = \vec{\kappa}' - \vec{\kappa}; \quad \vec{\kappa} = \tfrac{1}{2}(\vec{k}' + \vec{k}) \qquad 5\text{–}168$$

in terms of the initial and final momenta of the two nucleons in the nucleon–nucleon centre–of–mass system. The subscripts T and S are the isospin and spin quantum numbers, and the factor $(-)^{T+S+1}$ causes the direct and exchange terms to be added or subtracted, depending on the nucleon–nucleon spin–isospin channel. For computational simplicity and to reflect the exchange of various mesons, the direct and and exchange terms are expressed as a sum of

Yukawa form factors with various strengths and ranges so that the complete t–matrix when evaluated on–shell reproduces nucleon–nucleon scattering data within an energy range of 100 to 800 MeV. The LF model is described by a set of strengths and ranges for a number of representative beam energies to cover the above range (Lo 81). Hence, the LF t–matrix has an energy dependence built into it. We are now going to investigate the structure of the LF t–matrix elements.

5.2.3.3.2.1 The Central and Spin–Orbit contributions to the LF t–matrix

Amongst other things, in this section we will illustrate that the direct terms for both central and spin–orbit contributions to the t–matrix are local, whereas the exchange terms are both non–local. Furthermore, it is also shown why

$t = t(\vec{q}, 2\vec{k})$ [refer to equation 5.77].

5.2.3.3.2.2 The Central Contribution

For illustrative purposes consider a single Yukawa function of unit strength, such that

$$t_c^D(r) = \frac{e^{-\mu r}}{\mu r} \qquad 5\text{–}169$$

where μ^{-1} is the range of a particular meson exchange in position space. The extension of this to the case of a sum of Yukawa terms with different complex strengths, as required by the LF model, is obvious.

The momentum space matrix elements of t_c^D are then

$$\langle \vec{\kappa}' | t_c^D | \vec{\kappa} \rangle = \int \langle \vec{\kappa}' | \vec{r}' \rangle \langle \vec{r}' | t_c^D | \vec{r} \rangle \langle \vec{r} | \vec{\kappa} \rangle \, d\vec{r}' \, d\vec{r} \qquad 5\text{–}170$$

where

$$\langle \vec{r}' | t_c^D | \vec{r} \rangle = \delta(\vec{r}' - \vec{r}) t_c^D(r) \qquad 5\text{–}172$$

and

$$\langle \vec{r} \,|\, \vec{\kappa} \rangle \quad = \frac{1}{(2\pi)^{3/2}} \, e^{i \vec{\kappa} . \vec{r}}$$

Evaluating equation 5–170 using equations 5–169, 5–171 and 5–172 gives

$$\boxed{\langle \vec{\kappa}' | t_c^D | \vec{\kappa} \rangle = \frac{1}{2\pi^2} \left[\frac{\frac{1}{\mu}}{\mu^2 + q^2} \right] = t_c^D(q)} \qquad 5\text{–}173$$

where

$$q^2 = \kappa'^2 + \kappa^2 - 2\kappa' \kappa \cos \theta \qquad 5\text{–}174$$

We now look at the space exchange term for the central contribution to the t–matrix. Recall that the full antisymmetrized $\hat{t}$–matrix is given by

$$\langle \vec{\kappa}' | \hat{t} | \vec{\kappa} \rangle = \langle \vec{\kappa}' | t | \vec{\kappa} \rangle (1 - P_{oi}) \qquad 5\text{–}175$$

where P_{oi} is the operator that interchanges all coordinates of particles o and i. This operator may be written as:

$$P_{oi} = P^x \sum_{S,T} (-)^{S+T} P_S^\sigma P_T^\tau \qquad 5\text{–}176$$

where P^x exchanges spatial coordinates, P_S^σ is the spin–singlet (S=0) or –triplet (S=1) projection operator, with an analogous definition of P_T^τ. Therefore, if we separate t into pieces that act on singlet or triplet states in spin and isospin,

$$\langle \vec{\kappa}' | t | \vec{\kappa} \rangle = \sum_{S,T} \langle \vec{\kappa}' | t^{TS} | \vec{\kappa} \rangle P_S^\sigma P_T^\tau \qquad 5\text{–}177$$

then

$$\langle \vec{\kappa}' | \hat{t} | \vec{\kappa} \rangle = \sum_{S,T} \langle \vec{\kappa}' | t^{TS} | \vec{\kappa} \rangle [1 + (-)^{S+T+1} P_x] P_S^\sigma P_T^\tau \; . \qquad 5\text{–}178$$

If we define t^D and t^E by

$$\langle \vec{\kappa}' | t^D | \vec{\kappa} \rangle = \sum_{S,T} \langle \vec{\kappa}' | t^{TS} | \vec{\kappa} \rangle P_S^\sigma P_T^\tau , \qquad 5\text{–}179$$

$$\langle \vec{\kappa}' | t^E | \vec{\kappa} \rangle = - \sum_{S,T} (-)^{S+T} \langle \vec{\kappa}' | t^{TS} | \vec{\kappa} \rangle P_S^\sigma P_T^\tau \qquad 5\text{–}180$$

Then

$$\langle \vec{\kappa}' | \hat{t} | \vec{\kappa} \rangle = \langle \vec{\kappa}' | t^D | \vec{\kappa} \rangle + \langle \vec{\kappa}' | t^E | \vec{\kappa} \rangle P^x \quad . \qquad 5\text{–}181$$

For a two–particle spin–isospin state $|S,T\rangle$, the projection operators P_S^σ and P_T^τ simply select out the relevant part of $\hat{t}$.

We now consider the momentum space matrix elements of the exchange term, namely

$$\langle \vec{\kappa}' | t_c^E | \vec{\kappa} \rangle P^x = \int \langle \vec{\kappa}' | \vec{r}\,' \rangle \langle \vec{r}\,' | t_c^E | \vec{r} \rangle P^x \langle \vec{r} | \vec{\kappa} \rangle \, d\vec{r}\,' d\vec{r}$$

$$= \frac{1}{(2\pi)^3} \int e^{-i\vec{\kappa}.\vec{r}} \; e^{-i\vec{\kappa}.\vec{r}} \; t_c^E(r) d\vec{r}$$

$$= \frac{1}{(2\pi)^3} \int e^{-i2\vec{\kappa}.\vec{r}} \; \frac{e^{-\mu r}}{\mu r} d\vec{r}$$

$$= \frac{1}{(2\pi)^3} \left\{ \frac{\frac{1}{\mu}}{\mu^2 + (2\kappa)^2} \right\} \qquad 5\text{–}182$$

where we have used equations 5–168, 5–171, 5–172 and the fact that

$$P^x e^{i\vec{\kappa}.\vec{r}} = e^{+i\vec{\kappa}.(-\vec{r})} \qquad 5\text{–}183$$

and

$$t_c^E(r) = \frac{e^{-\mu r}}{\mu r} . \qquad 5\text{–}184$$

Hence

$$\boxed{\langle \vec{\kappa}' | t_c^E | \vec{\kappa} \rangle P^x = \frac{1}{(2\pi)^2} \left\{ \frac{\frac{1}{\mu}}{\mu^2 + (2K)^2} \right\} = t_c^E(2K)} \qquad 5\text{–}185$$

5.2.3.3.2.3 The Spin–orbit Contribution

Here we have an additional complication in the form of the spin–orbit operator $\vec{L}.\vec{S}$. The matrix element can be written as

$$\langle \vec{\kappa}' | t_{LS}^D \vec{L}.\vec{S} | \vec{\kappa} \rangle = \frac{1}{(2\pi)^3} \int d\vec{r}\, e^{-i\vec{\kappa}'.\vec{r}}\, t_{LS}^D(r) \vec{r} \times \frac{1}{i} \vec{\nabla}_r.\vec{S}\, e^{i\vec{\kappa}.\vec{r}} \qquad 5\text{–}186$$

where $S = \frac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2)$ and we consider a single unit strength Yukawa function. We have also defined $\vec{L}$ accordinging to its usual meaning for angular momentum in dimensionless units

$$\vec{L} = \vec{r} \times \frac{1}{i} \vec{\nabla}_r$$

Equation 5–186 can be further manipulated to yield

$$\begin{aligned} \langle \vec{\kappa}' | t_{LS}^D \vec{L}.\vec{S} | \vec{\kappa} \rangle &= \frac{1}{(2\pi)^3} \int d\vec{r}\, e^{-i\vec{q}.\vec{r}}\, t_{LS}^D(r)(\vec{r} \times \vec{\kappa}.\vec{S}) \\ &= \frac{i}{(2\pi)^3} \int d\vec{r}\, (\vec{\nabla}_q \times \vec{\kappa}.\vec{S})\, e^{-i\vec{q}.\vec{r}}\, t_{LS}^D(r) \end{aligned} \qquad 5\text{–}187$$

where we have used

$$\vec{r}\, e^{-i\vec{q}.\vec{r}} = i\vec{\nabla}_q e^{-i\vec{q}.\vec{r}}$$

We can now write

$$\langle \vec{\kappa}' | t^D_{LS}\, \vec{L}.\vec{S} | \vec{\kappa} \rangle = \frac{i}{(2\pi)^3} (\vec{\nabla}_q \times \vec{\kappa}.\vec{S}) \int d\vec{r}\; e^{-i\vec{q}.\vec{r}}\, t^D_{LS}(r) \qquad 5\text{–}188$$

$$= i(\vec{\nabla}_q \times \vec{\kappa}.\vec{S})\, t^D_{LS}(q) \,. \qquad 5\text{–}189$$

Now we use

$$\vec{\nabla}_q\, t^D_{LS}(q) = \hat{q}\, \frac{\partial}{\partial q}\, t^D_{LS}(q)$$

where $\hat{q}$ is a unit vector.

We can now rewrite equation 5–189 as

$$\langle \vec{\kappa}' | t^D_{LS}\, \vec{L}.\vec{S} | \vec{\kappa} \rangle = i(\vec{q} \times \vec{\kappa}.\vec{S})\, \frac{1}{q} \frac{\partial}{\partial q}\, t^D_{LS}(q) \qquad 5\text{–}190$$

Since $\vec{q} \times \vec{\kappa} = \vec{\kappa}' \times \vec{\kappa} = \vec{q} \times \frac{1}{2}(\vec{\kappa}' + \vec{\kappa}) = \vec{q} \times \vec{K}$, we can write

$$\langle \vec{\kappa}' | t^D_{LS}\, \vec{L}.\vec{S} | \vec{\kappa} \rangle = \frac{i}{2} (\vec{q} \times \vec{\kappa}).(\vec{\sigma}_1 + \vec{\sigma}_2)\, \frac{1}{q} \frac{\partial}{\partial q}\, t^D_{LS}(q)$$

which shows that the *spin–orbit* operator $\vec{L}.\vec{S}$ causes the otherwise local form of t^D_{LS} to be *non–local*. Now, since we have assumed a single Yukawa of unit strength for t^D_{LS}, we find that

$$\frac{1}{q} \frac{\partial}{\partial q}\, t^D_{LS}(q) = \frac{1}{q} \frac{\partial}{\partial q} \left\{ \frac{1}{2\pi^2} \left[\frac{\frac{1}{\mu}}{\mu^2 + q^2} \right] \right\}, \qquad 5\text{–}191$$

$$= \frac{1}{2\pi^2} \frac{-\frac{2}{\mu}}{(\mu^2 + q^2)^2} \,.$$

Hence,

$$\boxed{\langle \vec{\kappa}' | t^D_{LS}\, \vec{L}.\vec{S} | \vec{\kappa} \rangle = \frac{i}{2} (\vec{q} \times \vec{K}).(\vec{\sigma}_1 + \vec{\sigma}_2) \times \frac{(-2/\mu)}{2\pi^2(\mu^2+q^2)^2} = t^D_{LS}(q)} \qquad 5\text{–}192$$

Similarly, the exchange term of the spin–orbit contribution is found to be

$$\boxed{<\vec{\kappa}'|t^{E}_{LS}\ \vec{L}.\vec{S}\ |\vec{\kappa}> = \frac{i}{2}(\vec{q}\times\vec{\mathcal{K}}).(\vec{\sigma}_1+\vec{\sigma}_2)\times\frac{(-2/\mu)}{2\pi^2(\mu^2+(2\mathcal{K})^2]^2} = t^{E}_{LS}(2K)} \qquad 5\text{–}193$$

To summarize, the central and spin–orbit contributions to the t–matrix can be written in terms of direct and exchange parts as

$$\boxed{t^{C}_{TS}(q,2\mathcal{K}) = \sum_i t^{C}_{i,TS}[F_1(q,\mu_i) + (-)^{T+S+1}\,F_i(2\mathcal{K},\mu_i)]} \qquad 5\text{–}194$$

Set $w_{LS}(q) = \frac{1}{q}\frac{\partial}{\partial q}t_{LS}(q)$, then

$$\boxed{w^{LS}_{TS}(q,2\mathcal{K}) = \sum_i w^{LS}_{i,TS}[G_i(q,\mu_i) + (-)^{T+S+1}\,G_i(2\mathcal{K},\mu_i)]} \qquad 5\text{–}195$$

where

$$F_i(q,\mu_i) = \frac{1}{(2\pi)^3}\frac{4\pi/\mu_i}{(q^2+\mu_i^2)};\quad G_i(q,\mu_i) = \frac{1}{(2\pi)^3}\frac{-8\pi/\mu_i}{[q^2+\mu_i^2]^2} \qquad 5\text{–}196$$

and the t_i, w_i are complex t–matrix strengths in MeV units. We can now write the t–matrix as

$$\boxed{t_{TS}(q,2\mathcal{K}) = t^{C}_{TS}(q,2\mathcal{K}) + \frac{i}{2}(\vec{\sigma}_1+\vec{\sigma}_2).\vec{q}\times\vec{\mathcal{K}}\ w^{LS}_{TS}(q,2\mathcal{K})} \qquad 5\text{–}197$$

where from equations 5–194 and 5–195 we see that the direct terms for both the central and spin–orbit contributions to the t–matrix are local, whereas the exchange terms are both non–local.

Thus, we have seen how to treat the t–matrix in terms of appropriate spin and isospin combinations and how to incorporate separate proton and neutron densities into calculations of the optical potential. The detail concerning the numerical method of solving the integral equations can be found in Wolfe's thesis (Wo 83). We have implemented the program Wizard 1 successfully on the VAX at NAC to perform the calculations of the differential cross section, analyzing power and spin rotation function. We shall now present the results of such calculations for ^{24}Mg, ^{28}Si and ^{32}S targets for a beam of incident protons at energies 135 and 155 MeV.

CHAPTER 6

6. Results, Conclusions and Future Proposals

Some representative calculations for the elastic scattering of protons from ^{24}Mg, ^{28}Si and ^{32}S are presented in this chapter.

These target nuclei were chosen firstly because they are zero–spin, spin saturated, N = Z nuclei, for which the proton and neutron distributions may be considered almost identical [see section 5.2.3.1]. Secondly, the mass numbers A, are of such a magnitude that the optimum factorization approximation should be quite reasonable for forward scattering (q = 0) [see section 5.2.2.1]. Thirdly, these are very probable candidates as target nuclei to study the scattering of polarized protons at the National Accelerator Centre (NAC) [Co 89].

We now briefly recap the content of our calculations: the calculations employ the optimum factorization procedure (see section 5.2.3.1) for the first–order KMT optical potential (see section 5.2.1.2); the Love and Franey model (see section 5.2.4.3.2) is used for the t–matrix; the nuclear densities employed are obtained from three–parameter Fermi representations of the nuclear charge form factor tabulated in ref. De 74 (see section 5.2.4.3.1). The point–proton density is obtained by dividing out the intrinsic proton charge form factor from the nuclear charge form factor. The point–neutron density was assumed to be equal to the point–proton density.

We show calculations for the differential cross section $\frac{d\sigma}{d\Omega}$, the analyzing power A_y (or polarization P), and the spin rotation function Q (see section 4.3.5) for the angular region extending from 5° to 55°. We consider 135 MeV and 155 MeV incident proton kinetic energies, mainly because these energies fall into the energy range of the cyclotron at the National Accelerator Centre (NAC) and because published data is available for $\frac{d\sigma}{d\Omega}$ and P at these energies. Unfortunately no measurements have been made of the spin rotation function Q at these energies.

Our major aim is to use the computer program WIZARD 1 in order to test the sensitivity of the $\frac{d\sigma}{d\Omega}$, P and Q observables to relativistic and nonrelativistic calculations. The relativistic results are calculated according to the Dirac–based description discussed in section 5.2.4. The nonrelativistic results are obtained by simply setting U^{+-}_{JL} to zero in equations 5–142 and 5–143, thereby removing coupling to negative energy states and reverting, in effect, to equation 5–107.

Our calculated results for the nuclei ^{24}Mg, ^{28}Si and ^{32}S, each at incident proton energies of 135 MeV and 155 Mev are presented in figs. 5.1 to 5.5, together with the existing measured data. Other possible candidates for targets are ^{20}Ne and ^{36}Ar, but the absence of density parameters and measured data in the 100 MeV to 200 MeV region disqualifies their use. We first devote our attention to fig. 5.1 which concerns the scattering of 135 MeV protons from ^{24}Mg. This energy is too low to expect a priori that a free impulse approximation, to the first–order optical potential, will yield good agreement with the data. Nevertheless, the comparison of relativistic and nonrelativistic predictions can be most informative.

Unfortunately there are no published data available for $\frac{d\sigma}{d\Omega}$ at 135 MeV. However, we see that the relativistic and nonrelativistic predictions follow the same general trend. If we now consider the polarization P, we observe the deficiencies of the nonrelativistic calculations. The nonrelativistic (dotted) curve rises beyond 20°, showing a peak where the data shows a definite minimum. Again, at 45° the nonrelativistic curve has a broad shoulder where the data shows a definite decline. The relativistic (solid) curve, on the other hand, has none of these unsatisfactory features. There is indeed a dip in the 23° angular region. Were it not for the fact that the relativistic calculation shows deeper minima than the data, a fairly accurate description of the data would be evident. A glance at the curves for Q shows that the qualitative difference between the two curves for P in the 15° to 30° region is reflected in the large qualitative difference in the same region for Q.

Next we consider fig. 5.2 which deals with the elastic scattering of 155 MeV protons from ^{24}Mg. For $\frac{d\sigma}{d\Omega}$, both relativistic (solid) and nonrelativistic (dotted) curves follow the same trend as the

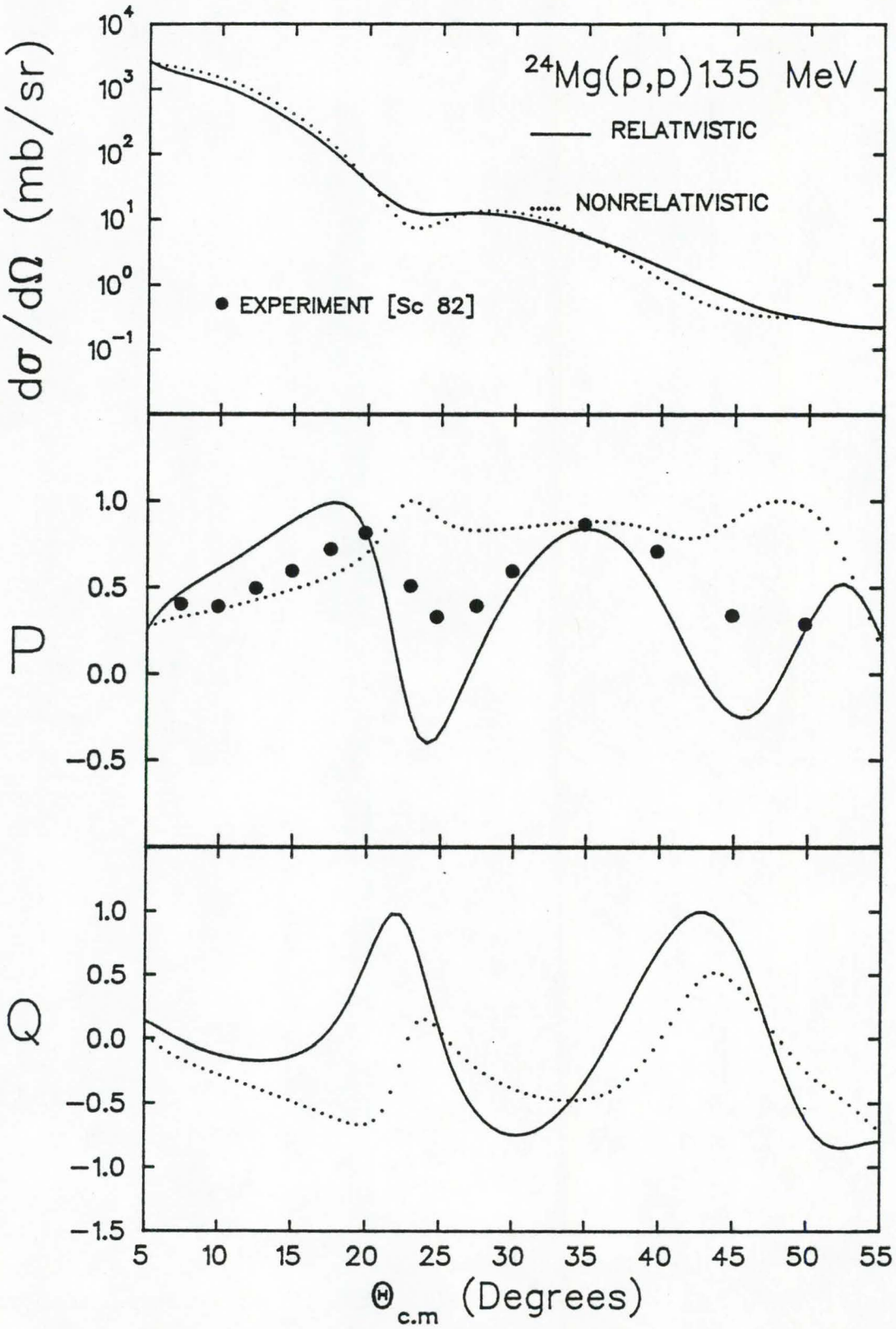

Fig. 5.1 Differential cross section, analyzing power (or polarization), and spin rotation function for 135 MeV protons scattered from ^{24}Mg. The solid and dashed lines represent respectively the relativistic and nonrelativistic calculations described in this thesis. The data are from ref. Sc 82.

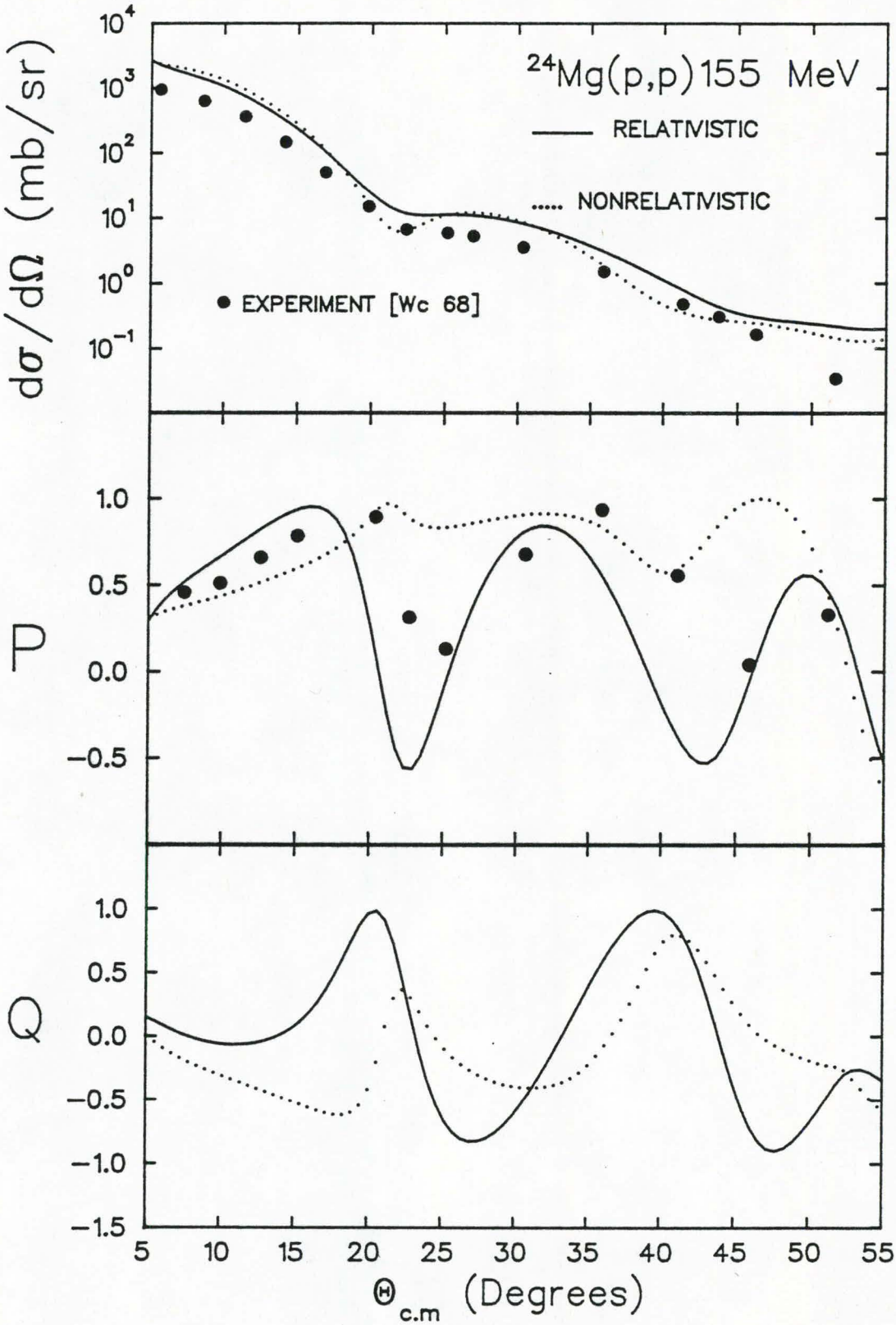

Fig. 5.2 Same as for Fig. 5.1, except the energy is 155 MeV and the data are from ref. Wc 68.

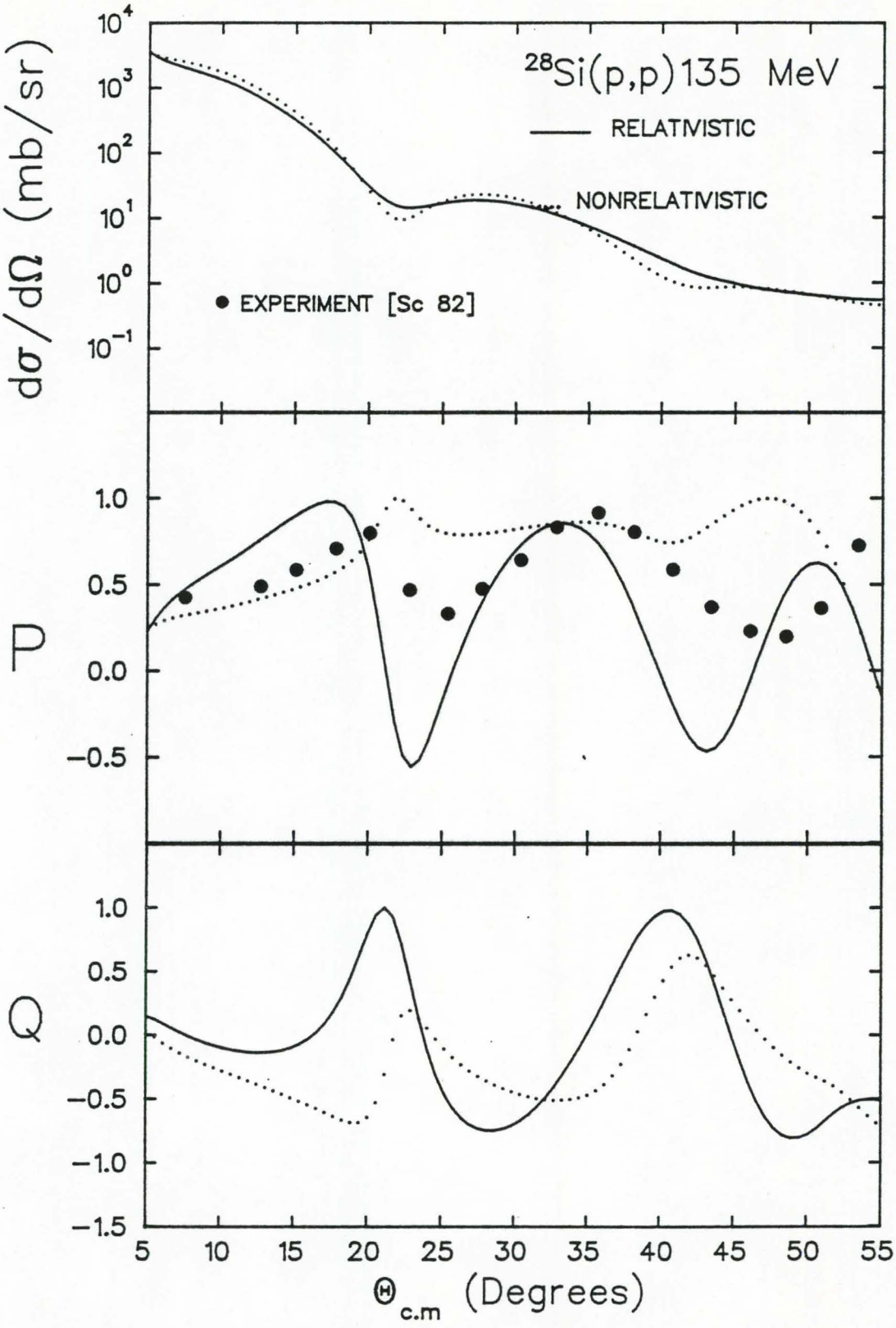

Fig. 5.3 Same as for Fig. 5.1, except for ^{28}Si.

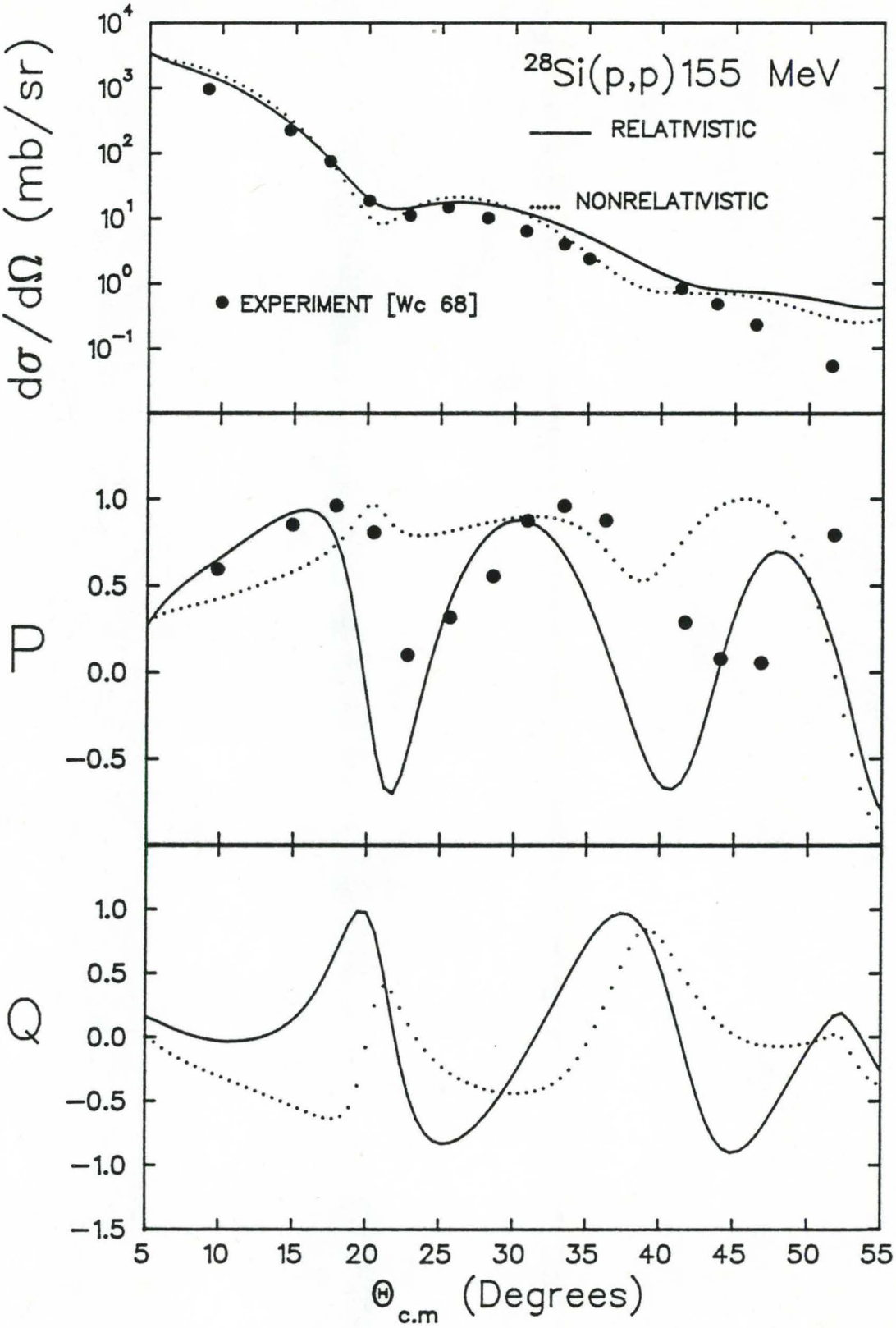

Fig. 5.4 Same as for Fig. 5.1, except for ^{28}Si at an energy of 155 MeV, and the data are from ref. Wc 68.

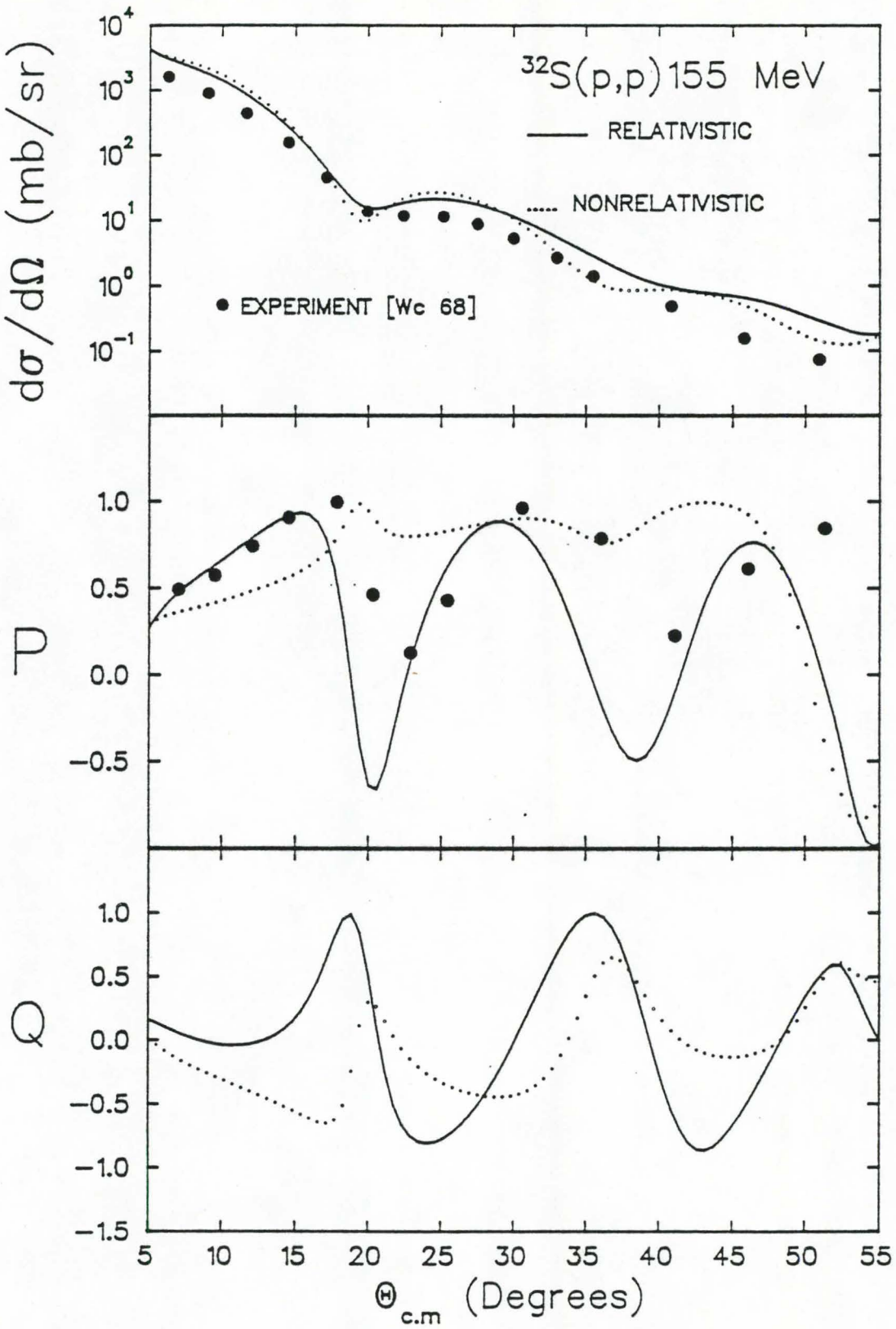

Fig. 5.5 Same as for Fig. 5.1, except for ^{32}S at an energy of 155 MeV, and the data are from ref. Wc 68.

trend as the data. The relativistic curve gives a diffraction pattern which is more spread out and whose diffraction minima are shallower than the nonrelativistic curve. It is difficult to judge which prediction is in better agreement with the data. For the polarization, the relativistic calculations, once again, provides an improved fit by deepening and shifting of the minima. The same comment, as for the 135 MeV case applies here for the spin rotation function.

Similar results apply to ^{28}Si and ^{32}S (see figures 5.3 and 5.5).

We are now in a position to make a few conclusions. It is perhaps relevant to mention that Hynes (Hy 85) reports that for ^{40}Ca and ^{16}O at 500 MeV the relativistic predictions for $\frac{d\sigma}{d\Omega}$, P and Q are in far better agreement with the data than the nonrelativistic predictions. Overall, we observe that the relativistic calculations are in much better agreement with the experimental data than are the nonrelativistic predictions. One would probably obtain better relativistic results if

(a) one were to perform the full–folding integral (see eq. 5–80) instead of adopting the optimum factorization procedure,

(b) a higher–order KMT optical potential was used,

(c) higher incident proton energies were investigated and

(d) more realistic densities were used, especially since ^{24}Mg, ^{28}Si and ^{32}S are known to be deformed nuclei.

Finally, I would like to suggest that once a polarized ion source has been installed at NAC, thus providing polarized proton beams up to a maximum beam energy of 200 MeV, it is important to use a relativistic Dirac–based formalism when performing microscopic calculations. Any microscopic Schrödinger–based calculation fails to reproduce P correctly and due to the mutual association of all the polarization observables, one would expect the same behaviour for Q. As an initial experiment it would be a good idea to measure the scattering observables for any one of the nuclei ^{24}Mg, ^{28}Si and ^{32}S at incident proton energies of 135 MeV and 155 MeV.

Appendix A

Polarization by electromagnetic interactions (Ha 85, Ba 67)

The first interest in particle polarization phenomena was the result of *Mott's* (Mo 29) work on the scattering of electrons by nuclei. *Mott* pointed out that the magnetic moments of the scattered electrons would have a preferred orientation i.e., the scattered beam would be partially polarized. Figure (A1) illustrates the mechanism which causes this effect.

Electrons which pass near a nucleus of charge + Ze are attracted by the Coulomb force towards the nucleus. In addition to the electrostatic force, there will be an interaction between the magnetic moment ($\vec{\mu}$) of the moving electron and the charge of the nucleus. This is most easily visualized in the frame of reference in which the electron is at rest. In this frame of reference, the moving nucleus appears as a current which produces a cylindrical magnetic field $\vec{B}$ at the electron. Since this field is inhomogeneous, it exerts a force on the magnetic moment of the electron which is in the same or opposite direction as the electrostatic force, depending on the orientation of the magnetic moment. Thus, since spin–up electrons of a given impact parameter are deflected differently from spin–down electrons, the scattered electrons are partially polarized. This additional force is therefore called the spin–orbit force.

To illustrate the above argument consider the following example together with figure (A1): Consider the case where the electron's magnetic moment ($\vec{\mu}$) points down (electron e_1 on fig. A1) and the projectile electron follows an incident path as indicated by A. In this case both $\vec{\mu}$ and $\vec{B}$ point down, which implies that their interaction energy, given by

$$V = -\vec{\mu}.\vec{B},$$

becomes

$$V = -\mu B$$

As $\vec{B}$ is inhomogeneous ($B \propto r^{-2}$) the electron will experience a force (the spin–orbit force) to a region where V is lower ($\vec{F} = -\vec{\nabla} V$). For electron e_1 this force will be directed towards larger

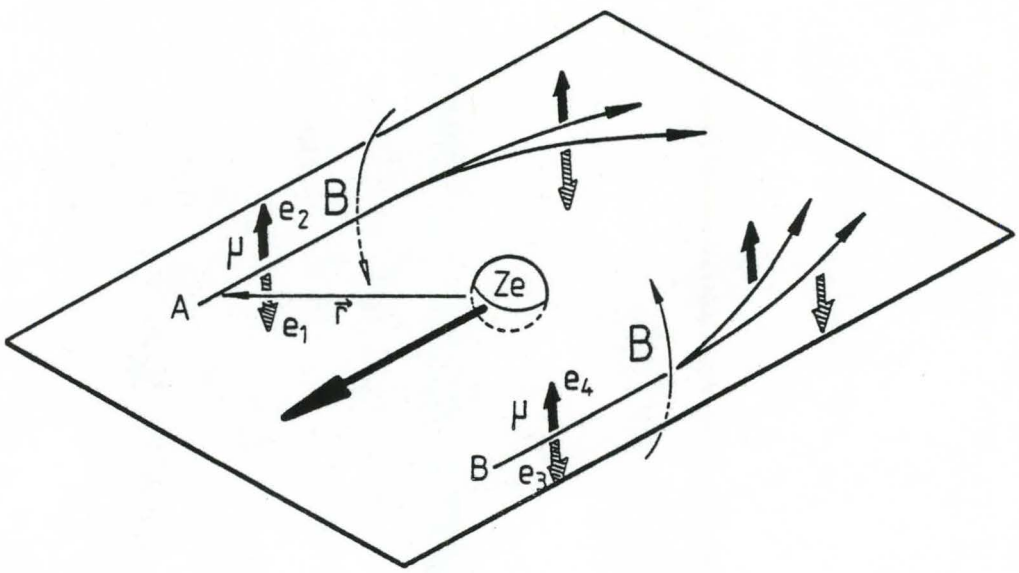

Fig. (A1) Polarization by electronmagnetic interactions

$\vec{B}$, nearer to the nucleus, i.e. towards the right. For electron e_2 [see figure (A1)] with $\vec{\mu}$ pointing up however, the spin orbit force will, by similar reasoning, be to the left. For two electrons which pass on the other side (path B on sketch) of the nucleus with $\vec{\mu}$ pointing down and up (electron e_3 and e_4 respectively) the spin orbit force will be similarly to the right and the left respectively, therefore the same two directions of the spin–orbit forces are experienced if a whole electron beam is scattered by a target containing many atoms at random. As a consequence, among the electrons scattered in a given direction the number with magnetic moments pointing up or down will not be equal, and the same applies to their spins. A beam of this kind is said to be partially polarized.

It was pointed out by Schwinger (Sc 46, Sc 48) that the same mechanism should be effective for nucleons since they also have a magnetic moment. The electromagnetic interaction involving the spin and the orbital motion of a particle is an example of what is usually called a spin–orbit interaction.

This spin–orbit interaction has long been observed in atoms where it agrees with the order of magnitude expected. We can get an idea of the order of magnitude of the strength of the nuclear spin–orbit interaction by assuming it to be of electromagnetic origin and evaluating $(\vec{\mu}.\vec{B})$ where $\vec{\mu}$ is the nucleon's magnetic moment and $\vec{B}$ is the magnetic field experienced, in its rest frame, by a nucleon moving in a nucleus. Calculation shows that $\vec{\mu}.\vec{B}$ is at most of order 10^3 eV. This is much too small to explain the experimental findings (Ta 63). In nuclear spectra the separation between the two components of a doublet is observed to be of the order 1 to 6 MeV. The spin–orbit interaction in nuclei cannot therefore be of electromagnetic origin, since, as we saw above, these energies are expected to be at most of the order 10^3 eV.

Appendix B

The Spin–Orbit interaction resulting from a Foldy–Wouthuysen reduction of the Dirac equation for a nucleon moving in a scalar and a vector field

For the purpose of the Foldy–Wouthuysen transformation and for relativistic calculations to appear later in this thesis, we require the following Dirac equation,

$$[\, i\, \gamma_\mu \partial^\mu - g_v\, \gamma^0 V_0 - (M - g_s\, \phi_0)]\, \Psi = 0$$

where ϕ_0 and V_0 are scalar and vector fields respectively.

I We shall now attempt to show the derivation of this equation as well as the meaning of the various symbols. For more detail refer to reference Wa 85. Empirical observation (Er 74, Ho 81(a), Mc 83) shows that there are large Lorentz scalar and four–vector components in the N–N interaction. These must, of course, be reproduced in any relativistic theory of nuclear structure, and the simplest way to do this is through the exchange of scalar and vector mesons. We assume that the neutral scalar meson couples to the scalar density of baryons through $g_s\overline{\Psi}\Psi\phi$ and that the neutral vector meson couples to the conserved baryon current through $g_v\overline{\Psi}\gamma_\mu\Psi V^\mu$.

Our model contains the following fields:

Field	Description	Particles	Mass
Ψ	Baryon	p, n, ...	M
ϕ	Neutral scalar meson	σ	m_s
V_μ	Neutral vector meson	ω	m_v

The Lagrangian density for the present model is:

$$\mathscr{L} = \overline{\Psi}(i\gamma_\mu \partial^\mu - M)\Psi + \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m_s^2 \phi^2)$$

$$+ (\frac{-1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_v^2 V_\mu V^\mu) - \overline{\Psi} \gamma_\mu g_v V^\mu \Psi + \overline{\Psi} g_s \phi \Psi + \delta\mathscr{L}$$

where γ_μ ($\mu = 0,1,2,3$) are the 4 Dirac gamma matrices

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}$$

$$\overline{\Psi} \equiv \Psi^\dagger \gamma^0$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left[\frac{\partial}{\partial x^0}, -\vec{\nabla} \right]$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left[\frac{\partial}{\partial x^0}, \vec{\nabla} \right]$$

$$x^\mu \equiv (t, \vec{x})$$

$$x_\alpha = g_{\alpha\beta} x^\beta, \quad g_{\alpha\beta} = g^{\alpha\beta} = g_{\beta\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{(called the metric tensor)}$$

$$x^\alpha = g^{\alpha\beta} x_\beta$$

$$F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu$$

$$\partial^\mu V_\mu = \partial_\mu V^\mu = \frac{\partial V^0}{\partial x^0} + \vec{\nabla}.\vec{V}$$

$$\partial_\mu \partial^\mu \equiv \frac{\partial^2}{\partial x^{0 2}} - \vec{\nabla}^2$$

$$V_\mu V^\mu = V.V = V^0 V^0 - \vec{V}.\vec{V}$$

For future conveniance we also define $\vec{\alpha}$ and β such that

$$\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}, \gamma^0 = \beta \text{ and } \vec{\gamma} = \beta\vec{\alpha}.$$

The term $\delta\mathscr{L}$ contains renormalization counterterms required for the quantum field theory; for the present discussion this term will not be needed.

Lagrange's equations

$$\frac{\partial}{\partial x^{\mu}}\left[\frac{\partial \mathscr{L}}{\partial(\partial q_i/dx^{\mu})}\right] - \frac{\partial \mathscr{L}}{\partial q_i} = 0 \tag{B1}$$

where q_i is one of the generalized coordinates, yield the field equations

$$q_i \equiv \phi \Rightarrow (\partial_{\mu}\partial^{\mu} + m_s^2)\,\phi = g_s\,\overline{\Psi}\,\Psi \tag{B2}$$

$$q_i \equiv V^{\mu} \Rightarrow \partial_{\mu}F^{\mu\nu} + m_v^2\,V^{\nu} = g_v\,\overline{\Psi}\,\gamma^{\nu}\Psi \tag{B3}$$

$$q_i \equiv \overline{\Psi} \Rightarrow [\gamma_{\mu}(i\partial^{\mu} - g_v\,V_{\mu}) - (M - g_s\phi)]\Psi = 0 \tag{B4}$$

Equation (B2) is simply the Klein–Gordon equation with a scalar source. Equation (B3) looks like QED for particles with mass with the conserved baryon current rather than the conserved electromagnetic current as source. Equation (B4) is the Dirac equation with the scalar and vector fields introduced in a minimal fashion.

Equations (B2) – (B4) are nonlinear quantum field equations, and their exact solutions are very complicated. We have made little progress by writing down these equations without a suitable method for solving them. Fortunately, there is an approximate solution that should become increasingly valid as the nuclear density increases (called the Mean–Field Theory approximation): Consider a uniform system of B baryons in a box of volume V. As the baryon density increases, so do the source terms of the right–hand sides of equation (B2) and (B3). When the source terms are large, the meson field operators can be replaced by their expectation values, which are classical fields:

$$\phi \rightarrow <\phi> \equiv \phi_0$$

$$V_{\mu} \rightarrow <V_{\mu}> \equiv \delta_{\mu 0}\,V_0 \tag{B5}$$

This assumption means that for a dense medium it does not really matter which specific baryon interacts, via exchange of mesons, with a nucleon. A certain *average* number of mesons interact with the nucleon as it travels through the nuclear medium i.e., the nucleon interacts with an average meson field.

For a static uniform system, quantities ϕ_s and V_0 are independent of x_μ and constant. Rotational invariance implies that the expectation value $< \vec{V} >$ vanishes.

The meson field equations (B2) and (B3) can be solved immediately for constants ϕ_0 and V_0 to give

$$\phi_0 = \frac{g_s}{m_s^2} < \bar{\Psi} \Psi > \equiv \frac{g_s}{m_s^2} \rho_S \tag{B6}$$

$$V_0 = \frac{g_v}{m_v^2} < \Psi^\dagger \Psi > \equiv \frac{g_v}{m_v^2} \rho_B \tag{B7}$$

where ρ_S is the scalar density and ρ_B is the baryon density.

When the classical meson field of (B6) and (B7) are substituted into (B4) for the Dirac field, that equation is linear,

$$\boxed{[i\gamma_\mu \partial^\mu - g_v \gamma^0 V_0 - (M - g_s \phi_0)] \Psi = 0} \tag{B9}$$

and may be solved directly.

Spatially inhomogeneous systems can be accommodated, for example if we allow spherically symmetric spatial variations in the mean fields $\phi_0(|\vec{x}|)$ and $V_0(|\vec{x}|)$. Also note that nucleons, although not fundamental Dirac particles, can usually be treated as such to good approximation.

II Now that we have obtained (B9), we are in a position to perform a Foldy–Wouthuysen reduction of the Dirac equation (B9) for a nucleon moving in the scalar and vector fields $\phi_0(r)$ and $V_0(r)$, where $r = |\vec{x}|$.

We first have to define even and odd matrix operators.

Operators of the form $\begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix}$ are called odd (a_1, b_1, are 2×2 matrices), i.e., they mix the "large" and "small" components of a Dirac 4–spinor; For example, $\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}$ is an odd operator, since $\begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} \vec{\sigma} & \Psi_2 \\ \sigma & \Psi_1 \end{bmatrix}$. In general we denote an odd operator by the symbol O.

An even operator, on the other hand, is of the form $\begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}$ where a_2 and b_2 are 2×2 matrices, i.e., it does not mix the "large" and "small" components of a Dirac 4–spinor; For example, β is an even operator, since $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} \Psi_1 \\ -\Psi_2 \end{bmatrix}$. In general we denote an even operator by E.

A Foldy–Wouthuysen (F.W.) reduction is a transformation, designed to put the Dirac Hamiltonian into a form that does not contain any odd operator. Thus, the Foldy–Wouthuysen. It thus reduces the Dirac equation to two uncoupled equations for positive and negative energy solutions. The Dirac equation associated with positive energy solutions then allows one to find the proper non–relativistic Hamiltonian. We shall briefly state the idea behind the F.W. transformation. The theory underlying the F.W. transformation can be found in references Bj 64, Fo 50, Ac 88, Ho 81.

For a particle undergoing interactions, the Dirac Hamiltonian can be written as:

$$H = \beta M + O + E \tag{B10}$$

Now it is necessary to make an infinite number of Foldy–Wouthuysen transformations, each of which successively removes odd terms to one order in $\left[\frac{1}{M}\right]$ from H. After three such transformations, H will contain only even operators in $\left[\frac{1}{M}\right]^0$, $\left[\frac{1}{M}\right]^1$ and $\left[\frac{1}{M}\right]^2$, but higher orders will still contain even and odd operators.

For each of the successive transformations

$$S = -\left[\frac{i}{2M}\right] \beta \times (\text{odd terms in Hamiltonian of lowest order in } \frac{1}{M}).$$

where each transformation is given by $\Psi' = e^{iS}\Psi$.

The result for the first three transformations is

$$\boxed{H''' = \beta(M + \frac{O^2}{2M} - \frac{O^4}{8M^3}) + E - \frac{1}{8M^2}[O,[O, E]] - \frac{i}{8M^2}[O, \dot{O}]} \tag{B11}$$

Now, the Dirac equation we are interested in is:

$$\boxed{[i\gamma_\mu \partial^\mu - g_v \gamma^0 V_0 - (M - g_s \phi_0)]\ \Psi = 0} \qquad \text{[from (B9)]}$$

or, since $\beta = \gamma^0$

$$\left[i\beta \frac{\partial}{\partial t} + i\vec{\gamma}.\vec{\nabla} - g_v \beta V_0 - M + g_s \phi\right] \Psi = 0$$

Multiply by β (remember $\vec{\alpha} = \beta\vec{\gamma}$, $\beta^2 = 1$)

$$[i\vec{\alpha}.\vec{\nabla} - g_v V_0 - \beta(M - g_s \phi_0)]\ \Psi = -i \frac{\partial}{\partial t} \Psi$$

and we are able to identify

$$H = \vec{\alpha}.\vec{p} + g_v V_0 + \beta(M - g_s\phi_0)$$

$$\equiv \vec{\alpha}.\vec{p} + V + \beta M - \beta S$$

where $V \equiv g_v V_0$

and $S \equiv g_s\phi_0$

Write,

$$H = \beta M + E + O \tag{B12}$$

where $E = V - \beta S$

$$O = \vec{\alpha}.\vec{p}$$

A F.W. transformation of eq. (B12) will yield (B11). Now let us proceed to evaluate all the terms in eq. (B11). We shall divide the procedure into seven steps.

(i) $$O^2 = (\vec{\alpha}.\vec{p})^2 = p^2 1_4 , \tag{B13}$$

where 1_4 is a 4×4 unit matrix, and we made use of the identity:

$$(\vec{\sigma}.\vec{a})(\vec{\sigma}.\vec{b}) = \vec{a}.\vec{b} + i\vec{\sigma}.(\vec{a} \times \vec{b}) \tag{B14}$$

(ii) $$O^4 = p^4 1_4 \tag{B15}$$

(iii) Let us adopt the following notation:

$\vec{\nabla}$ operates on everything standing on its right hand side.

$\vec{\partial}$ operates on objects only standing directly to its right.

Now:

$$\begin{aligned}
[O,E] &= [\vec{\alpha}.\vec{p}, V - \beta S] \\
&= -i[\vec{\alpha}.\vec{\nabla}, V] + i[\vec{\alpha}.\vec{\nabla}, \beta S] \\
&= -i(\vec{\alpha}.\vec{\nabla}\, V) + iV\,\vec{\alpha}.\vec{\nabla} + i\vec{\alpha}\,\beta.\vec{\nabla} S - i\beta S\,\vec{\alpha}.\vec{\nabla} \\
&= -i\,\vec{\alpha}.\vec{\partial}\,\vec{\nabla} + i\,\vec{\alpha}\beta.\vec{\nabla} S - i\beta S\,\vec{\alpha}.\vec{\nabla}
\end{aligned} \tag{B16}$$

Using (B16) we get:

$$\text{(iv)}\quad [O,[O,E]]\Psi = \overbrace{-(\vec{\alpha}.\vec{\nabla})(\vec{\alpha}.\vec{\partial}V)\Psi}^{(1)} + \overbrace{(\vec{\alpha}.\vec{\nabla})(\vec{\alpha}\vec{\beta}.\vec{\nabla}S)}^{(2)}$$
$$\overbrace{-(\vec{\alpha}.\vec{\nabla})\ S\beta\ (\vec{\alpha}.\vec{\partial}\Psi)}^{(3)} + \overbrace{(\vec{\alpha}.\vec{\partial}V)(\vec{\alpha}.\vec{\partial}\Psi)}^{(4)}$$
$$\overbrace{-(\vec{\alpha}\beta.\vec{\nabla}S)(\vec{\alpha}.\vec{\partial}\Psi)}^{(5)} + \beta S\overbrace{(\vec{\alpha}.\vec{\nabla})(\vec{\alpha}.\vec{\partial}\Psi)}^{(6)} \qquad \text{(B17)}$$

where (1) $\equiv -(\vec{\alpha}.\vec{\nabla})(\vec{\alpha}.\vec{\partial}V)\Psi$

$$= -\begin{bmatrix} (\vec{\sigma}.\vec{\nabla})(\vec{\sigma}.\vec{\partial}V) & 0 \\ 0 & (\vec{\sigma}.\vec{\nabla})(\vec{\sigma}.\vec{\partial}V) \end{bmatrix}\Psi$$

Use the identity, (B14),

$$\text{then (1)} \equiv -\begin{bmatrix} \vec{\nabla}.\vec{\partial}V + i\vec{\sigma}.(\vec{\nabla}\times\vec{\partial}V) & 0 \\ 0 & \vec{\nabla}.\vec{\partial}V + i\vec{\sigma}.(\vec{\nabla}\times\vec{\partial}V) \end{bmatrix}\Psi$$

$$= -(\vec{\partial}^2V)\Psi - \vec{\partial}V.\vec{\partial}\Psi - i\vec{\Sigma}.\{(\underbrace{\vec{\partial}\times\vec{\partial}V}_{=0})\Psi + \vec{\partial}\Psi\times\vec{\partial}V\}$$

where $\vec{\Sigma} \equiv \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}$ (spin operator)

Similarly (4) $\equiv (\vec{\partial}V.\vec{\partial}\Psi) + i\vec{\Sigma}.(\vec{\partial}V\times\vec{\partial}\Psi)$

and therefore

$$(1)+(4) = -(\partial^2V)\Psi + 2i\vec{\Sigma}.(\vec{\partial}V\times\partial\Psi)$$

$$= -(\partial^2V)\Psi - 2\vec{\Sigma}.(\vec{\partial}V\times\vec{p}\Psi)$$

Furthermore

$$(2)+(3)+(5)+(6) = \overbrace{\beta\,(\vec{\alpha}.\vec{\nabla})[(\vec{\alpha}.\vec{\partial}S)\Psi + S\vec{\alpha}.\vec{\partial}\Psi)]}^{(2)} \quad \overbrace{+2\beta\,(\vec{\alpha}.\vec{\nabla})S(\vec{\alpha}.\vec{\partial}\Psi)}^{(3)+(5)}$$

$$\overbrace{+S\beta\,(\vec{\alpha}.\vec{\nabla})(\vec{\alpha}.\vec{\partial}\Psi)}^{(6)}$$

$$= \overbrace{\beta(\vec{\alpha}.\vec{\nabla})(\vec{\alpha}.\vec{\partial}S)\Psi + \beta(\vec{\alpha}.\vec{\partial}S)(\alpha.\partial\Psi) + \beta S(\vec{\alpha}.\vec{\nabla})(\vec{\alpha}.\vec{\partial}\Psi)}^{(2)}$$

$$\overbrace{+2\beta(\vec{\alpha}.\vec{\partial}S)(\vec{\alpha}.\vec{\partial}\Psi) + 2\beta S(\vec{\alpha}.\vec{\nabla})(\vec{\alpha}.\vec{\partial}\Psi)}^{(3)+(5)} \overbrace{+S\beta(\vec{\alpha}.\vec{\nabla})(\vec{\alpha}.\vec{\partial}\Psi)}^{(6)}$$

$$= \beta(\vec{\alpha}.\vec{\nabla})(\vec{\alpha}.\vec{\partial}S) + 3\beta(\vec{\alpha}.\vec{\partial}S)(\vec{\alpha}.\vec{\partial}\Psi) + 4\beta S(\vec{\alpha}.\vec{\nabla})(\vec{\alpha}.\vec{\partial}\Psi)$$

$$= \beta[\vec{\nabla}.\vec{\partial}S + i\vec{\Sigma}.(\vec{\nabla}\times\vec{\partial}S)] + 3\beta[\vec{\partial}S.\vec{\partial}\Psi + i\vec{\Sigma}.(\vec{\partial}S\times\vec{\partial}\Psi)]$$

$$+ 4\beta S[\vec{\nabla}.\vec{\partial}\Psi + i\,\vec{\Sigma}.\underbrace{(\vec{\nabla}\times\vec{\partial}\Psi)}_{=0}] \text{ (using (B14)}$$

$$= \beta[\vec{\partial}^2S + \vec{\partial}S.\vec{\partial}\Psi + i\,\vec{\Sigma}.(\underbrace{\vec{\partial}\times\vec{\partial}S}_{=0}\Psi + \vec{\partial}\Psi\times\vec{\partial}S) + 3\vec{\partial}S.\vec{\partial}\Psi$$

$$+ 3i\vec{\Sigma}.(\vec{\partial}S\times\vec{\partial}\Psi) + 4S\,\vec{\nabla}.\vec{\partial}\Psi]$$

$$= \beta[\vec{\partial}^2S + 4(\vec{\partial}S.\vec{\partial}\Psi + S\vec{\nabla}.\vec{\partial}\Psi + 2i\,\vec{\Sigma}.(\vec{\partial}S\times\vec{\partial}\Psi)]$$

$$= \beta[\vec{\partial}^2S - 4(\vec{p}\,S).\vec{p} - 2\vec{\Sigma}.(\vec{\partial}S\times\vec{p})]$$

Thus $[O[O,E]] = -\vec{\sigma}^2 V - 2\vec{\Sigma}.(\vec{\sigma} V \times \vec{p}) + \beta[\vec{\sigma}^2 S - 4(\vec{p} S).\vec{p} - 2\vec{\Sigma}.(\vec{\sigma} S \times \vec{p})]$ (B18)

(v) Now, $\dot{O} = 0$

Therefore $[O,\dot{O}] = 0$

(vi) Finally, we have $\left[\text{to order } \frac{1}{M^2}\right]$

$$H''' = \beta\left[M + \frac{p^2}{2M} - \frac{p^4}{8M^3}\right] + V - \beta S - \frac{1}{8M^2}\{\beta\vec{\sigma}^2 S - \vec{\sigma}^2 V - \beta 4(\vec{p} S).\vec{p} - 2\vec{\Sigma}.[\vec{\sigma}(V + \beta S) \times \vec{p}]\} \quad \text{(B19)}$$

In the nonrelativistic limit, we only use the blocked sections of β and $\vec{\Sigma}$, namely

$$\beta = \begin{bmatrix} \boxed{1} & 0 \\ 0 & -1 \end{bmatrix}, \quad \vec{\Sigma} = \begin{bmatrix} \boxed{\vec{\sigma}} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}$$

Hence, in the nonrelativistic limit we have

$$\boxed{H''' = M + \frac{p^2}{2M} - \frac{p^4}{8M^3} + V - S + \frac{1}{8M^2}\vec{\sigma}^2(V - S) + \frac{1}{2M^2}\vec{p} S.\vec{p} + \alpha(r)\,\vec{\mathcal{L}}\vec{S}} \quad \text{(B20)}$$

where $\vec{L} = \vec{r} \times \vec{p}$ and $\vec{S} = \frac{\vec{\sigma}}{2}$

If we assume that $S = S(r)$ and $V = V(r)$, then the term associated with spin–orbit coupling is given by:

$$\boxed{\alpha(r)\,\vec{\mathcal{L}}\vec{S} = \frac{1}{4M^2 r}\vec{\sigma}.\left[\frac{\partial V}{\partial r} + \frac{\partial S}{\partial r}\right]\vec{r} \times \vec{p}} \quad \text{(B21)}$$

So, we have succeeded in showing the following:

A Foldy–Wouthuysen reduction of the Dirac equation for a nucleon moving in the scalar and vector fields $\phi_0(r)$ and $V_0(r)$ allows us to identify the effective single–particle spin–orbit interaction $\alpha_{so}(r)\,\vec{\mathcal{L}}\vec{S}$, where

$$\alpha_{so}(r) = \frac{g_v\, V_0'(r) + g_s\, \phi_0'(r)}{2M^2 r} \qquad \text{(B22)}$$

Appendix C:

The Pauli Spin Matrices

The derivation of the Pauli spin matrices can be found in any standard text on angular momentum (see, for example Ro 57).

Spin angular momentum is a special case of angular momentum (with $j = \frac{1}{2}$) for which we use the symbol $\vec{S}$ instead of $\vec{J}$. The Pauli spin operator $\vec{\sigma}$ is defined by

$$\vec{\sigma} = 2\vec{S} \tag{C1}$$

The matrices for the components of $\vec{\sigma}$ are given by

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{C2}$$

Some Properties of the Pauli Spin Matrices:

With i, j, k as indices for cyclic coordinates x, y, z,

$$\sigma_i\sigma_j = i\sigma_k, \tag{C3}$$

$$\sigma_i^2 = 1 \text{ (unit matrix)}, \tag{C4}$$

$$[\sigma_i,\sigma_j] = 2i\sigma_k, \tag{C5}$$

the well–known angular momentum commutator

These lead to some concise expressions for the traces (Tr) of the Pauli matrices and their products, which considerably simplify the derivation of the polarization expressions in Chapter 4, namely

$$\mathrm{Tr}\ \sigma_i = 0 \tag{C6}$$

$$\mathrm{Tr}\ (\sigma_i\sigma_j) = 2\,\delta_{ij} \tag{C7}$$

$$\mathrm{Tr}\ (\sigma_i\sigma_j\sigma_k) = 2i\,\epsilon_{ijk} \tag{C8}$$

where ϵ_{ijk} is the well known Levi–Cevita symbol, defined as

$$\epsilon_{ijk} = \begin{cases} +1 \text{ for cyclic ijk} \\ -1 \text{ for anticyclic ijk} \\ 0 \text{ otherwise.} \end{cases} \tag{C9}$$

Appendix D:

Arbitrary Spin State:

In the two–dimensional representation used for the Pauli spin matrices the basis vectors are the two eigenstates

$\chi^{\frac{1}{2}}_{m}$ with $m = \pm \frac{1}{2}$

and are thus represented by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. σ_z has the value +1 and –1 for these two respective states; therefore, these two states correspond to spin projection along the z–axis and along the direction opposite to the z–axis, respectively.

It will now be seen that an arbitrary spin state

$$\chi = \begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{with } (a^* \; b^*) \begin{bmatrix} a \\ b \end{bmatrix} = |a|^2 + |b|^2 = 1 \text{ (normalization condition)} \tag{D1}$$

has spin projection in a direction (θ,ϕ) in space, determined in a particular way by the values of a and b. Let $\hat{n}$ be the unit vector in the direction (θ,ϕ) such that

$$n_x = \sin\theta\cos\phi, \quad n_y = \sin\theta\sin\phi, \quad n_z = \cos\theta \tag{D2}$$

By our assertion, $(\vec{\sigma}.\hat{n})$ has the eigenvalue unity for the state $\begin{bmatrix} a \\ b \end{bmatrix}$, that is,

$$(\vec{\sigma}.\hat{n}) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{D3}$$

It is a straight forward exercise to obtain, from this equation, with the help of (C2) and the Pauli spin matrices [eq. (C2)], the result

$$\frac{a}{b} = e^{-i\phi} \cot\left[\frac{\theta}{2}\right] \tag{D4}$$

Using (D1), we get

$$a = e^{-i\xi} \cos\left[\frac{\theta}{2}\right], \quad b = e^{-i(\xi - \phi)} \sin\left[\frac{\theta}{2}\right]$$

where ξ is an arbitrary phase. For convenience, we choose $\xi = \left[\frac{\phi}{2}\right]$, and then

$$a = \cos\left[\frac{\theta}{2}\right] e^{-i(\phi/2)}, \quad b = \sin\left[\frac{\theta}{2}\right] e^{i(\phi/2)}$$

Finally, it is worth clarifying the phrase that the spin "points in a direction" (see section 4.3.1.1) specified by θ, ϕ in space: Quantum mechanically the measurement of the spin of a nucleon in a given direction is (by virtue of the Heisenberg uncertainty relation) not allowed; one can only measure its magnitude ($\sqrt{\frac{3}{4}}\,\hbar$) and its *projection* with respect to a certain quantization axis specified by θ and ϕ as either $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$. The term "pointing in a direction" means, that if this θ, ϕ direction has been chosen as the measuring axis, only a projection of $+\frac{1}{2}\hbar$ will be measured.

Appendix E:

The name "*density*" matrix

The question arises as to why ρ is called the *density* operator. In what follows, we give two reasons as a motivation for this nomenclature.

I Time evolution of ensembles

How does the density operator ρ changes as a function of time?

Let us suppose that at some time t_o the density operator is given by

$$\rho(t_o) = \sum_i w_i |\Psi^{(i)}\rangle \langle\Psi^{(i)}| \tag{E1}$$

If the ensemble is left undisturbed, we cannot change the fractional population w_i. So the change in ρ is governed solely by the time evolution of the state ket $|\Psi^{(i)}\rangle$:

$$|\Psi^{(i)}\rangle \text{ at } t_o \longrightarrow |\Psi^{(i)}, t_o, t\rangle \tag{E2}$$

From the fact that $|\Psi^{(i)}, t_o, t\rangle$ satisfies the Schrödinger equation, we obtain

$$\begin{aligned} i\hbar \frac{d\rho}{dt} &= \sum_i w_i \{H|\Psi^{(i)}, t_o, t\rangle \langle\Psi^{(i)}, t_o, t| - |\Psi^{(i)}, t_o, t\rangle \langle\Psi^{(i)}, t_o, t|H\} \\ &= -[\rho, H] \end{aligned} \tag{E3}$$

This looks like the Heisenberg equation of motion except that the sign is wrong! This is not disturbing because ρ is not a dynamic observable in the Heisenberg picture. On the contrary, ρ is built up of Schrödinger–picture state kets and state bras which evolve in time according to the Schrödinger equation.

(E3) can be regarded as the quantum–mechanical analogue of Liouville's theorem in classical statistical mechanics,

$$\frac{d\rho}{dt}_{classical} = -[\rho_{classical}, H_{classical}] \tag{E4}$$

where [ ] stands for Poisson brackets and $\rho_{classical}$ stands for the *density* of representative points in phase space. Thus, the name *density* operator for ρ appearing in (E3) is appropriate.

II Continuum case

So far we have considered base kets that are labelled by eigenvalues of some observable, for example $\langle i|\rho|j\rangle$. The concept of density matrix can, however, be generalized to cases where the base kets used are labelled by continuous eigenvalues. In particular, let us consider the ket space spanned by the position eigenkets $|x'\rangle$.

The density matrix elements here are actually functions of x' and x'', namely,

$$\langle x''|\rho|x'\rangle = \langle x''|\sum_i w_i|\Psi^{(i)}\rangle\langle\Psi^{(i)}|x'\rangle$$

$$= \sum_i w_i \Psi_i(x'')\Psi_i^*(x') \tag{E5}$$

where Ψ_i is the wave function corresponding to the state ket $|\Psi^{(i)}\rangle$. Notice that the diagonal element (i.e., $x' = x''$) of this is just the weighted sum of *probability densities*. Once again, we see that the term density matrix for ρ is indeed appropriate.

Appendix F:

Density matrices for polarized–, partially polarized– and unpolarized beams

Consider the density operator defined as

$$\rho = \sum_{i=1} w_i |\Psi^{(i)}\rangle \langle\Psi^{(i)}| \quad \text{(see equation (4–9))}$$

I Pure Ensemble (Polarized beam)

A pure ensemble by definition is a collection of physical systems such that every member is characterized by the same ket $|\Psi\rangle$. A pure ensemble is specified by $w_i=1$ for some $|\Psi^{(i)}\rangle$ — with i=n for instance — and $w_i=0$ for all other conceivable state kets; so the density operator is written as

$$\rho = |\Psi^{(n)}\rangle \langle \Psi^{(n)}| \tag{F1}$$

with no summation.

Clearly, the density operator for a pure ensemble is idempotent, that is,

$$\rho^2 = \rho. \tag{F2}$$

Thus, for a pure ensemble only, we have

$$\mathrm{Tr}(\rho^2) = 1$$

in addition to eq. 4–16.

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in addition to eq. 4–16.

We shall now derive the form for the diagonalized density matrix of a pure ensemble. Let us diagonalize ρ such that

$$\rho \equiv \begin{bmatrix} \rho_1 & & & & \\ & \rho_2 & & & \\ & & \rho_3 & & \\ & & & \ddots & \\ & & & & \rho_n \end{bmatrix}.$$

Then

$$\rho^2 \equiv \begin{bmatrix} \rho_1^2 & & & & \\ & \rho_2^2 & & & \\ & & \rho_3^2 & & \\ & & & \ddots & \\ & & & & \rho_n^2 \end{bmatrix}.$$

Using (F2), we see that for a pure ensemble

$$\begin{bmatrix} \rho_1^2 & & & & \\ & \rho_2^2 & & & \\ & & \rho_3^2 & & \\ & & & \ddots & \\ & & & & \rho_n^2 \end{bmatrix} = \begin{bmatrix} \rho_1 & & & & \\ & \rho_2 & & & \\ & & \rho_3 & & \\ & & & \ddots & \\ & & & & \rho_n \end{bmatrix}.$$

Now, since the matrix elements are merely numbers it follows that

$\rho_i = 1$ or 0

But, from eq. 4–16

$\mathrm{Tr}\rho = 1.$

This means that only one of the diagonal elements is one, while all the others are zero. Therefore, when diagonalized, the density matrix for a pure ensemble must look like

$$\rho \equiv \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix} \quad \text{(diagonal form).} \tag{F3}$$

II $\underline{\text{Tr}(\rho^2) \text{ for pure– and mixed ensembles}}$

Our following, immediate aim, is to show that $\text{Tr}(\rho^2)$ is maximal when the ensemble is pure and that for a mixed ensemble $\text{Tr}(\rho^2)$ is a positive number less than one.

We start by proving the important property that ρ is positive semi–definite which means that all its eigenvalues are non–negative. For ρ diagonal the eigenvalues are merely the diagonal matrix elements. The diagonal matrix elements of ρ are given by

$$\langle i|\rho|i\rangle = \sum_r w_r \langle i|\Psi^{(r)}\rangle\langle\Psi^{(r)}|i\rangle \quad \text{(from eq. (4–9))}$$

$$= \sum_r w_r |\langle i|\Psi^{(r)}\rangle|^2$$

But, per definition

$$\left.\begin{matrix} |\langle i|\Psi^{(r)}\rangle|^2 \\ w_i \end{matrix}\right\} \geq 0$$

Therefore

$$\langle i|\rho|i\rangle \geq 0 \qquad \text{(F4)}$$

Hence ρ is positive semi–definite.

Combining the results from eq. (4–16) and (F4), we conclude that

$$0 \leq \langle i|\rho|i\rangle \leq 1. \qquad \text{(F5)}$$

We now proceed to prove what we originally set out to do. We write a general diagonalized density matrix (consider a 3×3 matrix for simplicity) as:

$$\rho \equiv \begin{bmatrix} a & & \\ & b & \\ & & 1-a-b \end{bmatrix} \tag{F6}$$

where $0 \leq a \leq 1$
$0 \leq b \leq 1$
$0 \leq (1-a-b) \leq 1$ } [from (F5)]

and $\quad \mathrm{Tr}\,\rho = 1.$

Firstly, consider the diagonalized density matrix for a mixed ensemble i.e.,

$$0 < a < 1 \qquad 0 < b < 1 \tag{F7}$$

For this case, it can be correctly stated that:

$$0 < |a|^2 < a \quad \text{and} \quad 0 < |b|^2 < b. \tag{F8}$$

From (E6) it follows that

$$\rho^2 \equiv \begin{bmatrix} a^2 & & \\ & b^2 & \\ & & (1-a-b)^2 \end{bmatrix}$$

and $\mathrm{Tr}(\rho^2) \quad = a^2 + b^2 + (1-a-b)^2$

$< a + b + (1-a-b) = 1$ [Eq. from (F7) and (F8)].

Thus, for a mixed ensemble $\mathrm{Tr}(\rho^2)$ is a positive number less than one.

For a pure ensemble, it can be seen from (F3) that $\mathrm{Tr}(\rho^2) = 1$.

Thus, we have succeeded in showing that $Tr(\rho^2)$ is maximal when the ensemble is pure and for a mixed ensemble $Tr(\rho^2)$ is a positive number less than one.

III Unpolarized beam

We now proceed to discuss the density matrix for an unpolarized beam. If we consider for our system a particle of spin S then with respect to any orthonormal basis, ρ becomes $(2S + 1)$–dimensional. At this stage, we define an unpolarized assembly of particles as one in which each of the $(2S + 1)$ states of any *orthonormal basis*, $\{|m\rangle\}$, is equally populated. The density matrix describing this situation is just [in eq. (4–9) $|\Psi^{(r)}\rangle \longrightarrow |m\rangle$ and $w_r = \frac{1}{2S + 1}$]

$$\rho = \sum_m \frac{1}{2S + 1} |m\rangle\langle m|$$

$$= \frac{1}{2S + 1} I \qquad \text{(F10)}$$

where I is the $(2S + 1)$ dimensional unit matrix. That is, the density matrix for an unpolarized system is a multiple of the unit matrix.

It is satisfactory that the result (F10) is independent of the orthonormal basis used, since we certainly want our definition of an unpolarized system to be basis–independent (This means, for example, that an unpolarized beam of electrons can be regarded as being 50% spin up and 50% spin down, or alternatively as 50% spin right and 50% spin left; either view gives the same density matrix and therefore corresponds to the same experimental situation).

For any typical experimental system (eq. a beam of particles emerging from a cyclotron) a much more natural definition would be that an unpolarized system is one whose spins are randomly orientated. We shall now explain why definition (F10) and the latter more "natural" definition are equivalent.

The characteristic feature of a system of randomly orientated spins is that the measurement of any spin observable should be unaffected by rotation of all spins. This means that the corresponding density matrix satisfies:

$$\rho = R^{S\dagger} \rho R^S \quad \text{or} \quad [\rho, R^S] = 0.$$

(In appendix G it is shown that all the various spin observable can be expressed in terms of the density matrix).

where R^S is the $(2S + 1)$–dimensional rotation matrix for any rotation. Now it is a well–known result from the theory of the rotation group (Schur's Lemma: Ha 62) that the only $(2S + 1)$–dimensional matrices that commute with all of the rotation matrices are multiples of the unit matrix. Thus,

$$\rho = cI$$

and, taking into account the normalization (eq. 4–16) we see that ρ has precisely the form (F10). That is, our two definitions of an unpolarized system lead to the same density matrix and are therefore completely equivalent.

To summarize, the density matrix for a beam of particles of spin S is a $(2S + 1)$–dimensional Hermitian matrix ρ, usually normalized to satisfy

$$\text{Tr}\, \rho = 1.$$

The density matrix appropriate to a completely polarized beam of particles is characterized by the property

$$\rho^2 = \rho;$$

that for a completely unpolarized beam of particles has the unique form

$$\rho = \frac{1}{2S + 1} I;$$

and any ρ which has neither of these properties corresponds to a state of partial polarization. Thus, instead of casual worded definitions for the various degrees of polarization, we now have a more formal way of expressing the various degrees of polarization in terms of density matrices.

Appendix G:

Recipe for determining the spin observables for any scattering experiment of the general form $\vec{A} + \vec{B} \longrightarrow \vec{C} + \vec{D}$:

The recipe is be presented in point form:

(1) Determine the dimensionality of the initial density matrix. This is given by $(2S_p + 1) \times (2S_t + 1)$ where S_p and S_t refer respectively to the spin of the projectile and the target particles.

(2) Expand the initial density matrix in terms of a complete set of basis matrices of correct dimensionality.

(3) Determine the expansion coefficients.

We now have a suitable expression for the initial density matrix. At the stage we are in a position to determine the expressions for the observables.

(4) The differential cross–section:
$$\frac{d\sigma}{d\Omega} = \mathrm{Tr}\,\rho^f \quad \left[\mathrm{Tr}\,\rho^i = 1 \;\; \begin{matrix} f \Rightarrow \text{final} \\ i \Rightarrow \text{initial} \end{matrix}\right] \qquad \text{(G1)}$$
$$= \mathrm{Tr}\,(M\rho^i M^\dagger)$$

(5) To obtain the polarization quantities of the outgoing particles in terms of the polarization of the incident particles, one only has to multiply the expectation values of the spin matrices describing the spin of the scattered particles by the differential cross section.

(6) Determining the form of the scattering matrix (M): First of all, it is necessary to determine the dimensionality of M. This is done by looking at what type of matrix connects the outgoing–particle spinor χ^f to the incoming–particle spinor χ^i, i.e., $\chi^f = M\chi^i$.

Once this is done, it is possible to expand M in terms of a complete set of basis matrices. Parity and time–reversal invariance arguments serve to simplify this expression. Generally, the expansion coefficients are functions of the kinematics of the experiment, namely, energy and scattering angle.

As an illustration of the above procedure, consider the following case of elastic scattering (for another example see p.20 of Oh 70).

$$\underbrace{\vec{\tfrac{1}{2}} + 0}_{\rho^i} \longrightarrow \underbrace{\vec{\tfrac{1}{2}} + 0}_{\rho^f}$$

$$\chi^i \qquad\qquad \chi^f$$

(i) Dimensionality of ρ^i is 2.

(ii) Consequently we can conveniently expand in terms of the following 2 × 2 basis matrices: 1 (2 × 2 unit matrix), σ_x, σ_y, and σ_z. In general we denote these four matrices by S_μ where $\mu = 0, 1, 2, 3$.

$$\text{Therefore } \rho^i = \sum_{\mu=0}^{3} a_\mu S_\mu \qquad \text{(G2)}$$

(iii) Now

$$\text{Tr } S_\mu \sigma_\nu = 2\delta_{\mu\nu} \quad \text{(orthonormality condition)} \qquad \text{(G3)}$$

Multiply both sides of (G2) by S_ν and take the trace using the result (G3).

Then

$$\text{Tr}(\rho^i S_\nu) = 2a_\nu$$

$$\therefore a_\nu = \tfrac{1}{2}\text{Tr}(\rho^i S_\nu)$$

$$= \tfrac{1}{2}\langle \overline{S_\nu} \rangle \quad \text{(from equation 4–17).}$$

$$\text{Hence } \rho^i = \tfrac{1}{2}\left(1 + \sum_{k=1}^{3} \langle \overline{S_k} \rangle S_k\right) \qquad \text{(G4)}$$

(iv) $$\frac{d\sigma}{d\Omega} = \mathrm{Tr}(M\rho^i M^\dagger)$$

$$= \tfrac{1}{2}\,\mathrm{Tr}[M\,(1 + \sum_{k=1}^{3} \langle\overline{S_k}\rangle S_k) M^\dagger] \quad \text{[from (G3)]}$$

$$= \tfrac{1}{2}\mathrm{Tr}\left\{ MM^\dagger \left[1 + \frac{\sum_{k=1}^{3} \langle\overline{S_k}\rangle\, (M S_k M^\dagger)}{\mathrm{Tr}(MM^\dagger)} \right] \right\} \tag{G5}$$

$$= \left[\frac{d\sigma}{d\Omega}\right]_{\text{unpol.}} (1 + \Sigma \langle\overline{S_k}\rangle A_k(\theta)) \tag{G6}$$

where $\left[\frac{d\sigma}{d\Omega}\right]_{\text{unpol.}}$ is the differential cross–section for the scattering of an unpolarized beam and $A_k(\theta) = \dfrac{\mathrm{Tr}(M S_k M^\dagger)}{\mathrm{Tr}(MM^\dagger)}$ where $A_k(\theta)$ is the analyzing power of the reaction for the k^{th} initial polarization component ($k = 1, 2, 3$).

(v) To consider the polarization quantities of the outgoing particles, it is convenient to define a renormalized final–density matrix $(\rho^f)'$ by the relation

$$(\rho^f)' = \frac{\rho^f}{\mathrm{Tr}\ \rho^f}$$

such that $\mathrm{Tr}(\rho^f)' = 1$

and $\langle\overline{S_j'}\rangle = \mathrm{Tr}\{(\rho^f)' S_j\} \qquad j = 1, 2, 3.$

Now, we see that using (G5)

$$\langle\overline{S_j'}\rangle \left[\frac{d\sigma}{d\Omega}\right] = \left[\frac{d\sigma}{d\Omega}\right]_{\text{unpol.}} \left\{ \frac{\mathrm{Tr}(MM^\dagger S_j')}{\mathrm{Tr}(MM^\dagger)} + \sum_{k=1}^{3} \langle\overline{S_k}\rangle \frac{\mathrm{Tr}(M S_k M^\dagger S_j')}{\mathrm{Tr}(MM^\dagger)} \right\} \begin{matrix}[\text{from } (G1) \\ \text{and } (G5)]\end{matrix}$$

$$= \left[\frac{d\sigma}{d\Omega}\right]_{\text{unpol.}} \{P_k'(\theta) + \sum_{k=1}^{3} \langle\sigma_k\rangle K_k^{j\,\prime}\} \tag{G7}$$

where $P_k' = \dfrac{\mathrm{Tr}(MM^\dagger S_k')}{\mathrm{Tr}(MM^\dagger)}$

$$\text{and } K_k^{j\prime} = \frac{\mathrm{Tr}(M S_k M^\dagger S_j')}{\mathrm{Tr}(MM^\dagger)}$$

Note that the prime on S_j' refers to the outgoing nucleons.

$P_k'(\theta)$ is the k^{th} component of polarization produced by an unpolarized beam, and $K_k^{j\prime}(\theta)$ is the polarization transfer coefficient that relates the k^{th} initial polarization component to the j^{th} final polarization component.

Appendix H:

Derivation of the "$t\rho$" form (i.e., expression 5–68) of expression 5–57

According to equation 5–67, the matrix elements of the KMT nonrelativistic first–order optical potential in the momentum space of the projectile are given by

$$<\vec{k}'|U'|\vec{k}> \;= U'(\vec{k}',\vec{k}) \;= (A-1)<\vec{k}';\phi_o|t|\phi_o;\vec{k}>. \tag{H1}$$

Now, the exact ground state wave vector $|\phi_o>$ can be written as

$$|\phi_o> = \sum_{s_1\ldots s_A} \sum_{i_1\ldots i_A} \int d\vec{k}_1 \ldots \int d\vec{k}_A \, |\vec{k}_1,s_1,i_1;\ldots;\vec{k}_A,s_A,i_A><\vec{k}_1,s_1,i_1;\ldots;\vec{k}_A,s_A,i_A|\phi_o> \tag{H2}$$

where k_i, s_i, i_i respectively refer to the momentum, spin– and isospin projection of the i^{th} target nucleon and we have made use of the completeness of basis states. Since our final expression requires matrix elements of the form

$$<\vec{k}';\vec{k}'_1,s'_1,i'_1|t|\vec{k};\vec{k}_1,s_1,i_1> \tag{H3}$$

it is convenient to write (H2) as

$$|\phi_o> = \sum_{s_1,i_1} \int d\vec{k}_1 |\vec{k}_1,s_1,i_1> \Big\{ \sum_{s_2\ldots s_A} \sum_{i_2\ldots i_A} \int d\vec{k}_2 \ldots \int d\vec{k}_A \, |\vec{k}_2,s_2,i_2;\ldots;\vec{k}_A,s_A,i_A> \times$$

$$<\vec{k}_2;s_2;i_2;\ldots;\vec{k}_A,s_A,i_A|\phi_o> \Big\} \tag{H4}$$

Note, that in this discussion we have assumed that the A target nucleons are identical and, hence indistinguishable. This means that any index number (other than 1) could have been used in eq. (H3).

Using (H4), we may write (H1) as

$$U'(\vec{k}',\vec{k}) = (A-1)\sum_{s_1',s_1}\sum_{i_1',i_1}\int d\vec{k}'_1\int d\vec{k}_1 <\vec{k}';\vec{k}'_1,s'_1,i'_1|t|\vec{k};\vec{k}_1,s_1,i_1> \times$$

$$\sum_{\substack{s_2'\ldots s_A' \\ s_2\ldots s_A}}\sum_{\substack{i_2'\ldots i_A' \\ i_2\ldots i_A}}\int d\vec{k}'_2\ldots\int d\vec{k}'_A\int d\vec{k}_2\ldots\int d\vec{k}_A \times$$

$$<\vec{k}'_2,s'_2,i'_2;\ldots\vec{k}'_A,s'_A,i'_A|\vec{k}_2;s_2;i_2;\ldots;\vec{k}_A,s_A,i_A> \times$$

$$<\vec{k}_1,s_1,i_1;\ldots\vec{k}_A,s_A,i_A|\phi_0><\phi_0|\vec{k}'_1,s'_1,i'_1;\ldots;\vec{k}'_A,s'_A,i'_A> \qquad \text{(H5)}$$

If we make use of the orthonormality condition

$$<\vec{k}'_2;s'_2;i'_2;\ldots;\vec{k}'_A,s'_A,i'_A|\vec{k}_2;s_2;i_2;\ldots;\vec{k}_A,s_A,i_A>$$

$$= \delta(\vec{k}'_2-\vec{k}_2)\times\ldots\times\delta(\vec{k}'_A-\vec{k}_A)\times\delta_{s'_2s_2}\times\ldots\times\delta_{s'_As_A}\times\delta_{i'_2i_2}\times\ldots\times\delta_{i'_Ai_A} \qquad \text{(H6)}$$

(H5) becomes:

$$U'(\vec{k}',\vec{k}) = (A-1)\sum_{s_1',s_1}\sum_{i_1'i_1}\int d\vec{k}'_1\int d\vec{k}_1<\vec{k}';\vec{k}'_1,s'_1,i'_1|t|\vec{k};\vec{k}_1,s_1,i_1> \times$$

$$\sum_{s_2\ldots s_A}\sum_{i_2\ldots i_A}\int d\vec{k}_2\ldots\int d\vec{k}_A<\vec{k}'_1,s'_1,i'_1;\vec{k}_2,s_2,i_2;\ldots;\vec{k}_A,s_A,i_A|\phi_0> \times$$

$$<\phi_0|\vec{k}_1,s_1,i_1;\vec{k}_2,s_2,i_2;\ldots;\vec{k}_A,s_A,i_A> \qquad \text{(H7)}$$

We assume that the operator t obeys the conservation of total momentum of the two nucleons involved in the collision; in the present formalism:

$$\vec{k}' + \vec{k}'_1 = \vec{k} + \vec{k}_1$$

In this respect, we define $\vec{q}$ as the momentum transferred to the scattered nucleon, i.e.

$$\vec{q} = \vec{k}' - \vec{k} = \vec{k}_1 - \vec{k}'_1$$

Furthermore, for elastic scattering the spin and isospin projections of the struck nucleon must be conserved. These considerations allow one to write (H7) as:

$$U'(\vec{k}',\vec{k}) = (A-1)\int d\vec{k}_1 \sum_{s_1,i_1} <\vec{k}';\vec{k}_1-\vec{q},s_1,i_1|t|\vec{k};\vec{k}_1,s_1,i_1> \rho_{int}^{s_1,i_1}(\vec{k}_1-\vec{q};k_1) \qquad (H8)$$

where

$$\rho_{int}^{s_1,i_1}(\vec{k}_1-q,\vec{k}_1) = \sum_{s_2\ldots s_A} \sum_{i_2\ldots i_A} \int d\vec{k}_2 \ldots \int d\vec{k}_A <\vec{k}_1-\vec{q},s_1,i_1;\ \vec{k}_2,s_2,i_2\ldots\vec{k}_A,s_A,i_A|\phi_o> \times$$
$$<\phi_o|\vec{k}_1;s_1;i_1;\vec{k}_2,s_2,i_2;\ldots;\vec{k}_A,s_A,i_A> \qquad (H9)$$

Now, since we wish to maintain time–reversal invariance, it is convenient to make a change in the integration variable $\vec{k}_1$, to another integration variable $\vec{p}$ such that the t–matrix is symmetric in $\vec{k}$ and $\vec{k}'$. Such a transformation is brought about if $\vec{p}$ is defined as

$$\vec{p} = \vec{k}_1 - \tfrac{1}{2}\vec{q} \qquad (H10)$$

Furthermore, to bring (H8) into relation with expression 5–57, $\vec{k}_1$ and $\vec{k}'_1$ need to be expressed relative to the A nucleus c.m. system. If this is done and we also take the transformation (H10) into consideration, then (H8) becomes

$$U'(\vec{k}',\vec{k}) = (A-1)\int d^3p \sum_{s_1,i_1} <\vec{k}';\vec{p}-\tfrac{1}{2}q,s_1,i_1|t|\vec{k};\ \vec{p}+\tfrac{1}{2}q,s_1,i_1> \rho_{int}^{s_1,i_1}[\vec{m}(\vec{p},\vec{q};\vec{K});$$
$$\vec{m}(\vec{p},\vec{q},\vec{K})] \qquad (H11)$$

where $\vec{m}$ is defined in section 5.2.3.1.

If we rename s_1 and i_1 to be s and i respectively then (H11) is exactly the same as expression 5–68.

Appendix I:

Coordinate Space forms of $|\vec{k},s(\pm)>$ as well as the Orthonormality and Completeness relations for these basis states

As was already mentioned, the positive energy free state with momentum $\vec{k}$ and rest frame spin projection s satisfies

$$(\not{p}-m)|\vec{k},s(+)> = 0 \tag{I1}$$

The coordinate space form of this state is

$$<\vec{r}|\vec{k},s(+)> = \frac{e^{i\vec{k}.\vec{r}}}{(2\pi)^{3/2}} \underbrace{u(\vec{k},+)|\chi_s>}_{\substack{\text{Kronecker}\\ \text{product}}}, \tag{I2}$$

where $|\chi_s>$ is a Pauli (two–component) spinor and $u(\vec{k},+)$ is a Dirac (four component) spinor given by

$$u(\vec{k},+) = \left[\frac{E_k + m}{2E_k}\right]^{\frac{1}{2}} \begin{pmatrix} 1 \\ \frac{\vec{\sigma}.\vec{k}}{E_k + m} \end{pmatrix} \tag{I3}$$

In equation (I3), $E_k^2 = k^2 + m^2$, $\vec{\sigma}$ is the usual 2×2 Pauli spin matrix, and 1 is the 2×2 unit matrix. The corresponding negative energy solution of equation (I1), with energy $-E_k$, is

$$<\vec{r}|\vec{k},s(-)> = \frac{e^{-i\vec{k}.\vec{r}}}{(2\pi)^{3/2}} u(\vec{k},-)|\chi_s>, \tag{I4}$$

where

$$u(\vec{k},-) = \left[\frac{E_k + m}{2E_k}\right]^{\frac{1}{2}} \begin{pmatrix} \frac{-\vec{\sigma}.\vec{k}}{E_k + m} \\ 1 \end{pmatrix} \tag{I5}$$

Note that the normalization constant $\left[\frac{E_k + m}{2E_k}\right]^{\frac{1}{2}}$ is chosen such that it is a Lorentz invariant quantity (Bj 64). The positive and negative energy free states relate to the particle and antiparticle degrees of freedom, respectively. The orthonormality relations for these basis states are

$$<\vec{k}',s'(\pm)|\vec{k},s(\pm)> = \delta_{s's}\,\delta(\vec{k}'-\vec{k}) \tag{I6}$$

and

$$<\vec{k}',s'(\pm)|\vec{k},s(\mp)> = 0 \tag{I7}$$

Note that the adjoint state vectors in equations (I6) and (I7) are the Hermitian adjoints, viz.

$$<\vec{k}',s'(\pm)|\vec{r}> = <\chi_s|u(\vec{k},\pm)^\dagger \frac{e^{-i\vec{k}.\vec{r}}}{(2\pi)^{3/2}}, \tag{I8}$$

rather than the Dirac adjoints which are

$$\overline{<\vec{k}',s'(\pm)|} = <\vec{k}',s'(\pm)|\gamma^0 \tag{I9}$$

so as to ensure that scalar, vector and tensor quantities behave correctly when transformed. In terms of the basis states already introduced, the completeness relation is

$$\sum_s \int d^3k \{|\vec{k},s(+)><\vec{k},s(+)| + |\vec{k},s(-)><\vec{k},s(-)|\} = 1 \ . \tag{I10}$$

Appendix J:

The Polarization–Asymmetry Theorem:

One does not need to learn any complicated theory to convince oneself that polarization P_y and analyzing power A_y in a reaction are separate, independent quantities. This is obvious for reactions where the emitted particle might be an alpha particle which has no spin and hence can have no polarization, but where it is still quite possible for an asymmetry to occur when the reaction is induced by polarized nucleons (Ha 85, Ba).

However, even in this case, an important non–trivial relationship is found if we compare the polarization P_y in a reaction to the analyzing power in the inverse reaction:

$P_y = A_y$ (inverse reaction).

This so–called polarization–asymmetry theorem can be derived from time–reversal invariance (Sa 58, Bi 59, Sa 83), although it was pointed out by Biedenharn (Bi 59) that a symmetric somewhat weaker condition called reciprocity (which assumes that the S–matrix is symmetric) already is sufficient.

The theory says that

$$P_y \text{ in } A(a,\vec{b})B = A_y \text{ in } B(\vec{b},a)A \qquad \text{(J1)}$$

Often, the inverse reaction cannot be investigated experimentally, because the final nucleus of the reaction is unstable. Only if one is interested in a reaction which leads to the ground state of a stable nucleus can the inverse reaction be carried out. Elastic scattering leads back to the ground state of the target nucleus, and thus it is it's own inverse. Thus, for elastic scattering

$$P_y = A_y \quad \text{(elastic scattering)}. \qquad \text{(J2)}$$

Appendix K:

The Advent of Polarized–Ion Sources

Originally, the only available means to produce energetic polarized particles was to bombard a target with an unpolarized beam, and to use the fact that the spin–dependence of nuclear forces in general caused the outgoing particles from the reaction to be polarized to some extent. It was found that for certain scattering angles quite high polarization of the scattered protons was obtained by proton elastic scattering from carbon.

However, the intensity of the scattered beam was low, the scattered beam could not easily be focused into a well formed, directed beam, the scattered beam had a considerable spread in energy on account of the relatively thick targets that were required for reason of intensity, and the beam energy could not be varied at will without also changing the polarization. Nevertheless, until about twenty years ago, double–scattering experiments were common. The first scattering served to polarize the protons, and the second served to study polarization effects in the reaction of interest (refer to the description of double and triple scattering experiments in section 4.3.4).

The development of polarized–ion sources constituted a major advance in the study of polarization effects in nuclear reactions. Not only did these sources produce vastly improved intensities of polarized particles, but the polarized beam is much more superior, to a scattered beam, in energy definition. Also, the polarization is large and constant, it can be reversed easily at the ion source and, for deuterons, one can obtain at will either vector or tensor polarization. For further reading on ion–sources refer to Ba 60 and Ka 65.

Appendix L:

Expansion of the Dirac Propagator $(p\!\!\!/-m)^{-1}$

Now

$$(p\!\!\!/-m) = (p_\mu \gamma^\mu - m) \quad \text{(refer to appendix B)}$$
$$= (\gamma^o E - \vec{\gamma}.\vec{p} - m) \tag{L1}$$

Therefore

$$\gamma^o(p\!\!\!/-m) = (E1 - \vec{\alpha}.\vec{p} - \gamma^o m) \qquad (\vec{\alpha} = \gamma^o\vec{\gamma},\ \gamma^o\gamma^o = 1) \tag{L2}$$

We can expand the following operator as

$$(\vec{\alpha}.\vec{p} + \gamma^o m) = \int d^3k \sum_s \{|\vec{k},s(+)><\vec{k},s(+)|.E_k + |\vec{k},s(-)><\vec{k},s(-)|.(-E_k)\} \tag{L3}$$

where E_k and $(-E_k)$ are positive and negative energy eigenvalues, respectively. Here we have made use of the completeness relation (I10). Thus, we can write (L2) as

$$(E_1 - \vec{\alpha}.\vec{p} - \gamma^o m) = \int d^3k \sum_s \{|\vec{k},s(+)><\vec{k},s(+)|(E-E_k) + |\vec{k},s(-)><\vec{k},s(-)|(E+E_k)\} \tag{L3}$$

$$= \gamma^o(p\!\!\!/-m) \tag{L4}$$

At this stage we have the expansion for $\gamma^o(p\!\!\!/-m)$. We require, however, the expansion for $(p\!\!\!/-m)^{-1}\gamma^o$, which is inverse of (L2). Thus, the expansion for $(p\!\!\!/-m)^{-1}\gamma^o$ must be such that

$$(p\!\!\!/-m)^{-1}\gamma^o\gamma^o(p\!\!\!/-m) = 1 \tag{L5}$$

If we assume that the expansion for the Dirac propagator is of the form

$$(p\!\!\!/-m)^{-1}\gamma^o = \int d^3k \sum_s \{A'_k|\vec{k},s'(+)><\vec{k}',s'(+)| + B'_k|\vec{k}',s'(-)><\vec{k}',s'(-)|\} \tag{L6}$$

then an expansion satisfying (L5) would be

$$(\not{p}-m)^{-1}\gamma^0 = \int d^3k \sum_{s'} \left\{ \frac{|\vec{k}',s'(+)><\vec{k}',s'(+)|}{E - E_{k'}} + \frac{|\vec{k}',s'(-)><\vec{k}',s'(-)|}{E + E_{k'}} \right\} \tag{L7}$$

If we define

$$<\vec{k},s(\pm)| \equiv <\vec{k},\pm|<\chi_s|$$

and $\overline{<\vec{k},\pm|} = <\vec{k},\pm|\gamma^0$

and make use of the fact that

$$|\chi_s><\chi_s| = 1$$

then

$$\boxed{(\not{p}-m)^{-1} = \int d^3k \left\{ \frac{|\vec{k},s(+)><\vec{k},s(+)|}{E - E_k} + \frac{|\vec{k},s(-)><\vec{k},s(-)|}{E + E_k} \right\}} \tag{L8}$$

Appendix M:

Angular Momentum Expansion of the Interactions

The angular momentum expansions for the various quantities to be considered here are based on the following considerations:

(a) The expansion of any abstract operator Ω.

(b) The Wigner–Eckart theorem.

(c) The assumption that the nucleon–nucleus interaction is rotationally invariant.

These considerations will also determine which quantum numbers label the various expansion coefficients. For convenience we work in abstract Hilbert space; the position space interpretation follows analogously.

1. Start by considering the abstract identity operator I,

$$I = \sum_n |n\rangle\langle n| \tag{M1}$$

where the n's refer to a complete set of basis states. Consequently, one is able to expand any abstract operator Ω as

$$\Omega = I\Omega I = \sum_{n,m} |n\rangle\langle n|\Omega|m\rangle\langle m| \tag{M2}$$

2. At this stage it would be convenient to quote two results that will be of use to us (Sc 83), namely

The Wigner–Eckart Theorem applied on a tensor operator T(kq), yields

$$\boxed{\langle j_2 m_2|T(kq)|j_1 m_1\rangle = (2j_2+1)^{-\frac{1}{2}}\langle j_2\|T(k)\|j_1\rangle\langle j_1 m_1\, kq|j_2 m_2\rangle} \tag{M3}$$

where the reduced matrix element is independent of m_1 and m_2.

If $T(k_1q_1)$ and $T(k_2q_2)$ are tensor operators which act on different parts of the coupled system, then the coupled tensor operator $T(k_1k_2)kq$ may be defined as

$$T(k_1k_2)kq = \sum_{q_1q_2} <k_1q_1\, k_2q_2|kq>\, T^{k_1}_{q_1} T^{k_2}_{q_2} \qquad \text{(M4)}$$

If one makes use of (M3), (M4) and the definition of the 9j–symbol, the following result is derived

$$\boxed{<(j_1j_2)j\|T(k_1k_2)k\|(j_1'j_2')j'> = \begin{Bmatrix} k_1 & k_2 & k \\ j_1 & j_2 & j \\ j' & j' & j' \end{Bmatrix} \times [(2j'+1)\ (2j+1)\ (2k+1)]^{\frac{1}{2}} <j_1\|T_1(k_1)\|j_1'> <j_2\|T_2(k_2)\|j_2'>} \qquad \text{(M5)}$$

I would now like to explain how we apply the above–mentioned results to obtain the desired partial wave expansions.

2.1 As an illustration let us consider the quantity in eq. 5–110,

$$\vec{\sigma}.(\vec{k}' \times \vec{k})U'^{LS}(\vec{k}',\vec{k}) \qquad \text{(M6)}$$

which, per assumption, is rotationally invariant and hence is a scalar quantity [T(k=0, q=0)]. At this stage we identify the following two tensor operators,

$$T(k_1 = 1, q_1) = \vec{\sigma} \quad \text{and} \quad T(k_2 = 1, q_2) = (\vec{k}' \times \vec{k})U'^{LS} \qquad \text{(M7)}$$

Now, in the coupled representation, application of (M2) leads to

$$\vec{\sigma}.(\vec{k}' \times \vec{k})\, U'^{LS}(\vec{k}',\vec{k}) =$$

$$\sum_{\substack{J\ ,L\ ,M \\ J',L',M'}} |(L\tfrac{1}{2})JM> \underbrace{<(L\tfrac{1}{2})JM|\,\vec{\sigma}.(\vec{k}'\times\vec{k})\ U'^{LS}(\vec{k}',\vec{k})\,|(L'\tfrac{1}{2})J'M'>}_{A} <(L'\tfrac{1}{2})J'M'|$$

Application of the Wigner–Eckart theorem (M3) as well as (M5) to the matrix element marked A yields

$$<(L\tfrac{1}{2})JM|\,\vec{\sigma}.(\vec{k}'\times\vec{k})\,U'^{LS}(\vec{k}',\vec{k})\,|(L'\tfrac{1}{2})J'M'>$$

$$= (2J+1)^{-\frac{1}{2}}\,<(L\tfrac{1}{2})J\|\vec{\sigma}.(\vec{k}'\times\vec{k})\,U'^{LS}(\vec{k}',\vec{k})\|(L'\tfrac{1}{2})J'>\,<J'M'00|JM>$$

$$= \begin{Bmatrix} 1 & 1 & 0 \\ L & \frac{1}{2} & J \\ L' & \frac{1}{2} & J' \end{Bmatrix} \times [(2J+1)\,(2J'+1)]^{\frac{1}{2}}\,(2J+1)^{-\frac{1}{2}} \times <\tfrac{1}{2}\|\vec{\sigma}\|\tfrac{1}{2}> \times$$
$$<L\|(\vec{k}'\times\vec{k})\,U'^{LS}(\vec{k}',\vec{k})\|L'> \times <J'M'00|JM> \qquad \text{(M9)}$$

Furthermore

$$<J'M'00|JM> \;=\; \delta_J^{J'}\,\delta_M^{M'} \qquad \text{(M10)}$$

Therefore

$$<(L\tfrac{1}{2})JM|\,\vec{\sigma}.(\vec{k}'\times\vec{k})\,U'^{LS}(\vec{k}',\vec{k})\,|(L'\tfrac{1}{2})J'M'> \;=\; F^{J,L,L'}(k',k)$$

$$= \begin{Bmatrix} 1 & 1 & 0 \\ L & \frac{1}{2} & J \\ L' & \frac{1}{2} & J \end{Bmatrix} \times (2J+1)^{\frac{1}{2}} \times <\tfrac{1}{2}\|\vec{\sigma}\|\tfrac{1}{2}>\,<L\|(\vec{k}'\times\vec{k})\,U'^{LS}(\vec{k}',\vec{k})\|L'>\,\delta_J^{J'}\,\delta_M^{M'} \qquad \text{(M11)}$$

where $<L\|(\vec{k}'\times\vec{k})\,U'^{LS}(\vec{k}',\vec{k})\|L'>$ is independent of the orientation of the system.

Inserting (M11) into (M8) gives

$$\vec{\sigma}.(\vec{k}'\times\vec{k})\,U'^{LS}(\vec{k}',\vec{k}) = \sum_{J,L,L',M} |(L\tfrac{1}{2})JM> \begin{Bmatrix} 1 & 1 & 0 \\ L & \frac{1}{2} & J \\ L' & \frac{1}{2} & J \end{Bmatrix} \times (2J+1)^{\frac{1}{2}} \times <\tfrac{1}{2}\|\vec{\sigma}\|\tfrac{1}{2}> \times$$
$$<L\|(\vec{k}'\times\vec{k})\,U'^{LS}(\vec{k}',\vec{k})\|L'>\,<(L'\tfrac{1}{2})JM| \qquad \text{(M12)}$$

In position space, the result analogous to (M12) is

$$\vec{\sigma}.(\vec{k}'\times\vec{k})\,U'^{LS}(\vec{k}',\vec{k}) = \sum_{J,L,L',M} \mathcal{Y}_{JL}^{M}(\hat{k}')\,F^{J,L',L}(\vec{k}',\vec{k})\,\mathcal{Y}_{JL'}^{M\dagger}(\hat{k}) \qquad \text{(M13)}$$

where $\mathcal{Y}^{M}_{JL}(\hat{k})$ is defined in eq. 5–130.

2.1 As a second example, consider the angular momentum expansion of $U'^{LS}(\vec{k}',\vec{k})$ in equation 5–110.

We write $U'^{LS}(\vec{k}',\vec{k}) = 1_S . U'^{LS}(\vec{k}',\vec{k})$

where 1_S is the identity operator in spin space. Since we are considering a rotationally invariant interaction, $U'^{LS}(\vec{k}',\vec{k})$ must be a scalar operator, thus we identify the following two tensor operators:

$$T(k_1 = 0, q_1 = 0) = 1_S \quad \text{and} \quad T(k_2 = 0, q_2 = 0) = U'^{LS}(\vec{k}',\vec{k}).$$

In a manner similar to the previous example, we find that, in position space

$$U'^{LS}(\vec{k}',\vec{k}) = \sum_{J,L,M} \mathcal{Y}^{M}_{JL}(\hat{k}')\, U'^{LS}_{L}(\vec{k}',\vec{k})\, \mathcal{Y}^{M\dagger}_{JL}(\hat{k}). \qquad \text{(M14)}$$

In this way, one can derive all the angular momentum expansions used in section 5.2.3.2 of this thesis.

Appendix N:

Description of a beam of spin–1 particles

It has been shown in Chapter 4 that the three components of $\langle \vec{P} \rangle$, given by the expression appearing after equation 4–12, provide a *complete* description of the polarization of a beam of spin–½ particles. A beam of spin–1 particles (e.g. deuterons) will now be briefly discussed and it will be seen that more parameters are now required to fully specify the polarization of such a beam. The generalization to higher spins will become obvious.

For a beam of spin–1 particles the measurement of the z–component of the spin in units of $\hbar$ (z being along the quantization axis) yields +1, 0 or –1. This leads to three possible spin functions, and their eigenvalue equations for the spin operator S_z are:

$$\left.\begin{aligned} S_z\chi_+ &= +1\chi_+ , \\ S_z\chi_0 &= 0\chi_0 \text{ , and} \\ S_z\chi_- &= -1\chi_- \end{aligned}\right\} \qquad \text{(N1)}$$

All three spin functions also, of course, obey the eigenvalue equation:

$$S^2\chi = S(S+1)\chi \quad \text{with } S = 1 \qquad \text{(N2)}$$

With the three spin functions in spin vector form:

$$\chi_+ = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \chi_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and} \qquad \chi_- = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{(N3)}$$

S_z has obviously the matrix representation:

$$S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{(N4)}$$

In order to obey the well–known commutation relation for the spin component operators, as well as (N2), the other two components are:

$$\left.\begin{aligned} S_x &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and} \\ S_y &= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \end{aligned}\right\} \qquad \text{(N5)}$$

A general spin–1 function can be expanded in terms of the complete set of spin functions (N3) as:

$$\chi = a\chi_+ + b\chi_0 + c\chi_- \qquad \text{(N6)}$$

A beam of spin–1 particles can generally be described by a 3 × 3 density matrix, similar to that for the spin–½ particle beam in a 2 × 2 space (section 4.3.1.2). Such a density matrix with its nine elements can be expanded in terms of a complete set of nine basis matrices τ_ν, normalized such that

$$\mathrm{Tr}(\tau_\mu \tau_\nu) = 3\,\delta_{\mu\nu} \qquad \text{(N7)}$$

[Compare this with the case of a spin–½ beam, section 4.3.1.2.1.1. In that case the 2 × 2 density matrix could be expanded in terms of the four matrices σ_x, σ_y, σ_z and the unit matrix I, which obey a similar trace relation, equation 4–19]

The nine cartesian basis matrices for the expansion are respectively:

- the 3 × 3 unit matrix I,
- $P = S_i$ with i = x,y,z [equations (N4) and (N5)],
- as well as the traceless tensor operators

$$P_{ij} = \frac{3}{2}(S_iS_j + S_jS_i) - 2\,\delta_{ij} \qquad \text{(N8)}$$

with i,j = x,y,z.

The P_{ij}'s are tensor operators and five independent matrices, as we shall now explain: although their are altogether nine matrices P_{ij}, they are clearly symmetric which leaves only the three P_{ij} $(i \neq j)$ independent. In addition the eigenvalue equation (N2) leads to the constraint

$$P_{xx} + P_{yy} + P_{zz} = 0 \tag{N9}$$

which finally leaves only five independent matrices P_{ij}.

It is worthwhile to give a few examples of the P_{ij} matrices of (N8) explicitly:

$$\left.\begin{aligned} P_{xx} &= 3S_z^2 - 2I \\ P_{zz} &= \frac{3}{2}(S_xS_z + S_zS_x) \end{aligned}\right\} \tag{N10}$$

The P_i and P_{ij} operators are *cartesian* components of the spin operator. Because of their convenient transformation properties in a coordinate transformation, it is more convenient to express them as *spherical tensor* components. In addition, the tensor form leads directly to a generalization to higher spins values. The spherical tensor components τ_{kq} for the different ranks $k = 0,1,...2S$ (S=1 presently), with the component index $q = k,k-1,...-k$ for each k, are expressed in terms of the cartesian components (N4), (N5) and (N8) as*:

$$\left.\begin{aligned} \tau_{00} &= 1 \\ \tau_{10} &= \sqrt{\frac{3}{2}}\, S_z, \\ \tau_{11} &= -\frac{\sqrt{3}}{2}(S_x + iS_y) \end{aligned}\right\} \tag{N12}$$

$$\left.\begin{aligned} \tau_{20} &= \sqrt{\frac{1}{2}}\,(S_z^2 - 2) \\ \tau_{21} &= -\sqrt{\frac{3}{2}}\,[(S_x + iS_y)S_z + S_z(S_x + iS_y)] \\ \tau_{22} &= -\sqrt{\frac{3}{2}}\,(S_x + iS_y)^2 \end{aligned}\right\} \tag{N13}$$

*Note that all the notations used in this appendix are in agreement with the Madison Convention (Ba 70).

with $\tau_{k-q} = (-)^q \tau^{\dagger}_{kq}$ (N14)

Expressions (N11), (N12) and (N13) describe the scalar, vector and tensor operators respectively, and have the advantage that in a co–ordinate transformation, these different rank tensors transform separately, as is well known.

The trace relation (N7) becomes in terms of these spherical tensors:

$$\mathrm{Tr}(\tau_{kq}\,\tau^{\dagger}_{k',q'}) = 3\,\delta_{kk'}\,\delta_{qq'} \qquad \text{(N15)}$$

which enables the expansion of the density matrix of a beam as:

$$\rho = \sum_{kq} \mathrm{Tr}(\delta\tau_{kq})\tau^{\dagger}_{kq}\,, \qquad \text{(N16)}$$

a form similar to that of (4–25).

The expectation values of the polarization parameters are analogous to 4–17:
in Cartesian form:

$$\left.\begin{aligned} p_i &= \langle P_i \rangle = \mathrm{Tr}(\rho P_i) \\ p_{ij} &= \langle P_{ij} \rangle = \mathrm{Tr}(\rho P_{ij}) \end{aligned}\right\} \qquad \text{(N17)}$$

and in spherical tensor form:

$$t_{kq} = \langle \tau_{kq} \rangle = \mathrm{Tr}\,(\rho\tau_{kq}) \qquad \text{(N18)}$$

The full extent of spin–1 polarization can best be illustrated by considering the polarization of a spin–1 beam from a polarized ion source which has axial symmetry (because of the magnetic field present in the ion–source). If, for convenience, we take the axis of quantization (z–axis) along the magnetic field direction, then an axially symmetric polarization state can be represented by a mixture of a fraction n_+ of particles with spin pointing along the +z–axis, a fraction n_- of particles with spin along the –z–axis and a fraction n_0 of particles with spins uniformly distributed in a plane perpendicular to the z–axis. From equation 4–9, the density matrix for this system is

$$\rho = n_+ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} (1\,0\,0) + n_0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} (0\,1\,0) + n_- \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (0\,0\,1)$$

$$= \begin{bmatrix} n_+ & 0 & 0 \\ 0 & n_0 & 0 \\ 0 & 0 & n_- \end{bmatrix} \qquad \text{(N19)}$$

Thus an axially symmetric system yields a diagonal density matrix and the non–zero expectation values of polarization observables (N4), (N5) and (N8) follow easily from (N17) with (N20) substituted:

$$\langle P_z \rangle = \mathrm{Tr}(\rho P_z) = n_+ - n_-, \text{ and} \qquad \text{(N20)}$$

$$\left.\begin{aligned} \langle P_{zz} \rangle = \mathrm{Tr}(\rho P_{zz}) &= n_+ + n_- - 2n_0 \\ &= 1 - 3n_0 \end{aligned}\right\} \qquad \text{(N21)}$$

where we made use of the fact that $n_+ + n_- + n_0 = 1$, i.e.,

$\mathrm{Tr}(\rho) = 1$ (see equation 4–16)

The zero values of all the independent $\langle P_i \rangle$'s and $\langle P_{ij} \rangle$'s which contain x and y components, illustrate the axial symmetry. [$\langle P_{xx} \rangle = \langle P_{yy} \rangle$ are nonzero as they depend on $\langle P_{zz} \rangle$, according to the constraint (N9).]

The results (N20) and (N21) illustrate that $\langle P_z \rangle$ provides a measure of only the polarization of the beam between the states with $S_z = +1$ and -1, in a similar way as P measures the polarization between $P_z = +\frac{1}{2}$ and $-\frac{1}{2}$ states for a spin–$\frac{1}{2}$ beam (Chapter 4). $\langle P_z \rangle$ is called the vector polarization. On the other hand, $\langle P_{zz} \rangle$ provides a measure of the deviation of n_0 from its value for a completely unpolarized beam. It is called the tensor polarization.

The different types of polarization are illustrated by the following four extreme examples: applied to (N20) and (N21):

(i) $n_+ = n_- = n_0$: $\langle P_z \rangle = 0$, $\langle P_{zz} \rangle = 0$ (completely unpolarized beam)

(ii) $n_0 = \frac{1}{3}$: $\langle P_z \rangle = n_+ - n_-$, $\langle P_{zz} \rangle = 0$ (pure vector polarization)

(iii) $n_+ = n_-$, $n_0 = 0$: $\langle P_z \rangle = 0$, $\langle P_{zz} \rangle = +1$ (pure tensor polarization)

(iv) $n_+ = n_- = 0$: $\langle P_z \rangle = 0$, $\langle P_{zz} \rangle = -2$ (pure tensor polarization)

The latter two cases illustrate the two extreme values of tensor polarization.

Note from equations (N12) and (N13) that the spherical tensors τ_{10} and τ_{20} depend on only P_z and P_{zz} respectively, and thus provide the same physical content in terms of beam polarization.

With this description of a spin–1 beam, it is clear that beams of higher spin values can similarly be described in terms of higher order spherical tensors.

All the equations of Chapter 4 concerning the spin change in a nuclear reaction, apply identically to cases of higher particle spins; only the dimensions of the initial and final density matrices (see appendix G) and therefore of the scattering matrix M are changed.

For more details on spin–1 and higher spin reactions refer to La 55, Sa 60, Go 58, Oh 70, Oh 72(a), Oh 72(b) and for a review of the history and a complete set of reference refer to Ba 70.

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