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Pricing Multi-Asset Options with Lévy Copulas

by

Jean Claude Dushimimana

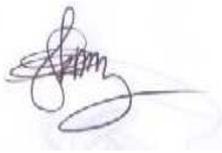
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Supervisor: Dr. Peter Ouwehand

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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and has not previously, in its entirety or in part, been submitted at any university for a degree.



Jean Claude Dushimimana

3 June 2010

Abstract

In this thesis, we propose to use Lévy processes to model the dynamics of asset prices. In the first part, we deal with single asset options and model the log stock prices with a Lévy process. We employ pure jump Lévy processes of infinite activity, in particular variance gamma and CGMY processes. We fit the log-returns of six stocks to variance gamma and CGMY distributions and check the goodness of fit using statistical tests. It is observed that the variance gamma and the CGMY distributions fit the financial market data much better than the normal distribution. Calibration shows that at given maturity time the two models fit into the option prices very well.

In the second part, we investigate the effect of dependence structure to multivariate option pricing. We use the new concept of Lévy copula introduced in the literature by Tankov [40]. Lévy copulas allow us to separate the dependence structure from the behavior of the marginal components. We consider bivariate variance gamma and bivariate CGMY models. To model the dependence structure between underlying assets we use the Clayton Lévy copula. The empirical results on six stocks indicate a strong dependence between two different stock prices. Subsequently, we compute bivariate option prices taking into account the dependence structure. It is observed that option prices are highly sensitive to the dependence structure between underlying assets, and neglecting tail dependence will lead to errors in option pricing.

Opsomming

In hierdie proefskrif word Lévy prosesse voorgestel om die bewegings van batepryse te modelleer. Lévy prosesse besit die vermoë om die risiko van spronge in ag te neem, asook om die implisiete volatilitate, wat in finansiële opsie pryse voorkom, te reproduseer. Ons gebruik suiwer-sprong Lévy prosesse met oneindige aktiwiteit, in besonder die gamma-variensie (Eng. *variance gamma*) en CGMY-prosesse. Ons pas die log-opbrengste van ses aandele op die gamma-variensie en CGMY distribusies, en kontroleer die resultate met behulp van statistiese pasgehaltetoetse. Die resultate bevestig dat die gamma-variensie en CGMY modelle die finansiële data beter pas as die normaalverdeling.. Kalibrasie toon ook aan dat vir 'n gegewe verstryktyd die twee modelle ook die opsiepryse goed pas.

Ons ondersoek daarna die gebruik van Lévy prosesse vir opsies op meervoudige bates. Ons gebruik die nuwe konsep van Lévy copulas, wat deur Tankov[40] ingelei is. Lévy copulas laat toe om die onderlinge afhanklikheid tussen bateprysspronge te skei van die randkomponente. Ons bespreek daarna die simulاسie van meerveranderlike Lévy prosesse met behulp van Lévy copulas. Daarna bepaal ons die pryse van opsies op meervoudige bates in multi-dimensionele eksponensieële Lévy modelle met behulp van Monte Carlo-metodes. Ons beskou die tweeveranderlike gamma-variensie en – CGMY modelle en modelleer die afhanklikheidsstruktuur tussen onderleggende bates met 'n Lévy Clayton copula. Daarna bereken ons tweeveranderlike opsiepryse. Kalibrasie toon aan dat hierdie opsiepryse baie sensitief is vir die afhanlikheidsstruktuur, en dat prysbepaling foutief is as die afhanklikheid tussen die sterte van die onderleggende verdelings verontagsaam word.

Dedication

To all my brothers and sisters

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Chapter 1

Introduction

While the Black-Scholes model and diffusion models constitute the main framework for derivatives pricing, they are inconsistent with market data, typically in relation to the implied volatility and the dynamics of the asset price process. The dynamics of asset prices exhibit jumps of different sizes with small jumps occurring more frequently than large jumps, leading both to asymmetries and fat tails in the asset returns [66]. Although stochastic volatility models and non-linear Markovian diffusion models yield non-normal returns, they share with the Brownian motion the continuity property, which amounts to neglecting jumps in the asset price process. These observations have led practitioners to increasingly adopt alternative processes for describing these returns.

Lévy processes are found to describe the observed reality of the financial market data in a more accurate way, both in the real world and in the risk-neutral world. A process X is a Lévy process if it has (almost surely) right-continuous paths and its increments are independent and time-homogeneous. Among them the jump-diffusion by Merton [51], the normal inverse Gaussian (NIG) by Barndorff-Nielsen [4], are mostly used. In recent years pure jump processes of infinite activity (that is, with infinitely many jumps in any finite time interval) such as the variance gamma process by Madan and Seneta [48], the CGMY process by Carr et al. [16] have been explored.

The use of processes with stationary independent increments to model the asset prices can be economically explained by the fact that time series data of financial asset returns should be stationary and any shocks which occur should be independent. Moreover, such processes,

when they have independent and identically distributed increments, are characterized by their Lévy densities that count the arrival rate of jumps of different sizes [47].

Another important feature of Lévy processes is the structure of their distributions—*infinitely divisible distributions*. For every infinitely divisible distribution there is an associated Lévy process. This is to be compared with the motivation for modeling stock returns by the Gaussian distribution, namely that this distribution is a limiting distribution of sums of n independent random variables which may be viewed as representing the effect of various shocks in the economy [29].

In this thesis, we wish to extend the univariate valuation procedure to the multivariate case. Multi-asset options have experienced a significant development in the last decade, following the increased popularity of structured equity products such as bonds or insurance policies, which typically embed multi-asset contingent claims. Despite the growing offer of multi-asset equity derivatives on OTC markets, pricing these products is a burdensome task [8]. The key point in pricing multivariate financial derivatives is the determination of the dependence between underlying assets. It is not enough to know the univariate marginal distribution of each of the underlying assets.

Many authors use a generalized Black-Scholes model to price multivariate options ([14], [18], [39], [69]). In the generalized Black-Scholes model, the dynamics of the asset returns is modeled by multidimensional Brownian motion and the distribution of log-returns is multivariate normal. It is well known that, for multivariate normal distributions, the dependence between components is characterized by the correlation matrix. Hence, in a generalized Black-Scholes model one has to estimate the correlation matrix in the determination of the dependence between components.

However, linear correlation is not a satisfactory measure of dependence in many multivariate models because of a number of reasons: Firstly, linear correlation requires the variance of the returns to be finite; otherwise it is not defined. This causes problems when working with heavy-tailed distributions. Secondly, linear correlation assumes that the marginals and the joint distributions be normal. For normal distributions, zero correlation is equivalent to independence, but it is not the case for other distributions. In the real world market, the distribution of returns is not normal. Thirdly, linear correlation is not invariant under non-linear strictly increasing transformations, implying that the returns might be uncor-

related whereas the prices are correlated or vice-versa. Embrechts et al. [27] explain more about the deficiency of using linear correlation to analyze the dependence.

A more convenient method of characterizing the dependence of the distribution of returns is to use distributional copula functions. Distributional copulas are functions that join or couple univariate marginal distributions to form a multivariate distribution function. The principal advantage of distributional copulas is that they allow us to separate the dependence structure from the marginal distributions completely. Among them, the Gaussian, student-t, Clayton copulas, etc are widely used in pricing structured products in the credit market and the equity market. With the copula method, the nature of dependence that can be modeled is more general than linear correlation and the dependence of extreme events can be considered. A number of authors have used the copula method to price credit derivative products, CDO and other multi-asset products ([53], [18], [78], [45]).

Tankov [73] proposes to use Lévy copula functions to model dependence between Lévy processes. Lévy copulas allow one to construct multidimensional Lévy processes and to characterize their dependence structure. This technique is a generalization of copulas for random variables to Lévy processes. His idea is to replace the role of a probability measure by a Lévy measure and that of distribution functions by tail integrals. Hence, Lévy copulas connect marginal Lévy measures to build joint Lévy measure. The benefit of using Lévy copula is that the resulting processes are Lévy processes. However, this method suffers a certain number of drawbacks. Lévy copula functions depend on time t of the Lévy process X_t . Secondly, modeling dependence of multidimensional Lévy processes by Lévy copulas, it is unclear which Lévy copula constructs a Lévy process. In general, the infinite divisibility property is not invariant under Lévy copula setting.

1.1 The Structure of the Thesis

In the next chapter, we review the fundamental concepts of Lévy processes and their properties. We introduce a new measure that allows us to take into account the jumps of a stochastic process. A class of Lévy processes (subordinated Lévy processes) which is highly used in finance is defined.

In chapter 3, we introduce Lévy processes into derivative pricing. We construct exponential models and discuss their tractability. We concentrate on pure jump processes with infinite activity since the infinite activity property allows Lévy processes to capture frequently small and rare large jumps which eliminates the need for the diffusion component. Examples of Lévy processes such as variance gamma and CGMY processes are thoroughly discussed, and we construct exponential model based on these processes.

In chapter 4, we introduce Lévy processes to option pricing and describe a method due to Carr and Madan [15] for pricing European options in exponential Lévy models by Fourier transform. Application of variance gamma and CGMY processes to option data is discussed. We show that the variance gamma and CGMY distributions describe the observed behavior of the asset returns and, at a given maturity time the two models fit into the option prices very well.

Chapter 5 deals with simulation of one-dimensional Lévy processes such as variance gamma and CGMY processes. We consider the case of variance gamma and CGMY processes defined as time-changed Brownian motion. The benefit of viewing a Lévy process as time-changed Brownian motion with respect to simulation is that one avoids dealing directly with the Lévy density which might be difficult to sample from. We show that when the variance gamma and the CGMY parameters are estimated from the time series, the sample path looks like that of the stock prices. Simulation by series representation is also reviewed.

In chapter 6, we review the notion of distributional copula functions and Lévy copulas. Distributional copulas were developed in order to construct multivariate distribution functions and to study their dependence structure. Sklar's theorem is stated and some important properties of distributional copulas are discussed.

The notion of Lévy copulas is then discussed in detail. Lévy copulas were first introduced in the literature in [73] in order to build and model the dependence structure of multidimensional Lévy processes. Lévy copulas are like distributional copulas but they are defined on a different domain. This is due to the fact that Lévy measures are not necessarily finite: They may have a non-integrable singularity at zero. A version of Sklar's theorem for Lévy copulas is stated and theorems parallel to those for distributional copulas are exhibited. The construction of parametric families of Lévy copulas that are tractable in mathematical finance is discussed.

Simulation of d -dimensional Lévy processes when the dependence structure is given by a Lévy copula is discussed with an illustration to two-dimensional variance gamma and CGMY processes. We show that when the dependence structure is modeled by a Lévy copula, the processes may jump in the same directions or opposite directions depending on whether there is strong dependence or weak dependence between the components.

Chapter 7, discusses application of Lévy copula to multi-asset option pricing. We construct a two-dimensional exponential Lévy model with variance gamma and CGMY margins with dependence structure given by the Clayton Lévy copula. Statistical inference on datasets is investigated and we compute the price of the options such as rainbow options and option on the weighted average between underlying assets with dependence structure given by a Lévy copula. We show that neglecting the tail dependence leads to an error in option pricing.

In the first part of this thesis, we study the capability of Lévy processes to model the dynamics of asset prices and to capture the smile/skew observed from the financial market data. We consider different stocks and show that Lévy processes are tractable from both the statistical point of view and for the calculation of option prices. We consider examples of pure jump processes of infinite activity. The infinite activity property enables a pure jump process to capture both frequently small and rare large moves/jumps, which eliminates the need for a diffusion component. Moreover, it has been argued [79], [29] that such models give a more realistic description of the price process at various time scale. The specific Lévy processes we are considering are the variance gamma and the CGMY processes.

We first fit the log-returns data to the variance gamma and the CGMY distributions and show that they capture the tails very well compared to normal distribution. We then use different statistical tests to assess the goodness of fit for the variance gamma and CGMY distributions. To calibrate the risk-neutral parameters, we use the observed option prices. We fit variance gamma and CGMY models to option prices and show that, at a given maturity time, the variance gamma and CGMY models fit the market prices very well.

We also investigate the effect of dependence structure to multi-asset option pricing. We suggest a method which is based on the new concept of Lévy copulas (see [40] and references therein). Lévy copulas allow to construct multidimensional Lévy processes and to characterize the possible dependence structure between the components. By Sklar's theorem

for Lévy processes, one can then construct multivariate exponential Lévy model by taking one-dimensional Lévy processes and one Lévy copula, possibly from a parametric Lévy copula. We follow this approach in chapter 7 to construct a two-dimensional exponential Lévy model with variance gamma and CGMY margins.

We select the Clayton Lévy copula from the family of Archimedean Lévy copulas to model the dependence between underlying assets. We consider the same set of data and estimate the historical dependence between underlying assets. We first calibrate the marginal risk-neutral parameters through the market data using the FFT method and the copula's parameters are estimated using maximum likelihood estimation. We consider popular multivariate options such as option on the weighted average and rainbow options. Because analytical formulas for option pricing are not available for most Lévy processes, we use Monte Carlo simulation method. As plain Monte Carlo brings an error in the option pricing, we apply a variance reduction scheme which uses the technique of control variates to reduce the error and to speed up the computations. From the results, we conclude that, apart from the linear correlation, the option prices are sensible to the dependence structure beyond the linear correlation. In all cases considered, we see that neglecting tail dependence lead to an error in option prices.

1.2 Literature Review

Due to the increase in popularity of multi-asset options in recent year, researchers have put their attention on multivariate models. In order to price multivariate options, one needs to take into account the dependence structure between various underlying assets. Various authors [19], [20], [26], [58], [59], [60] use the distributional copula method to price multi-assets products. With distributional copulas method, one can price multi-asset options with the information stemming from the marginals ones. Although all these works give reliable methods for pricing multi-asset options, none of them is in the framework of Lévy processes.

Modeling the dependence structure between underlying asset when the marginals are modeled by Lévy processes is desirable. If one is interested in one fixed time point say $t_1 = 1$ year, the Lévy process is simply a vector of static random variables. It is well known that

the dependence structure of a multidimensional random variable can be disentangled from its marginals using a distributional copula [38], [54]. Therefore, choosing a suitable distributional copula at t_1 , one can price multi-asset option consistently. This is the approach followed by Luciano and Schoutens to price multivariate options in equity and credit risk [45], [44], [78].

However, switching on time-dependence in the marginals, we can no longer model the dependence of the multidimensional process using distributional copula. The reason is that the resulting process will in general not be a Lévy process. The infinite divisibility of a probability distribution is not invariant under the distributional copula setting. Tankov [71] proposes to use Lévy copula method. The concept of Lévy copulas was introduced in the literature in order to characterize the dependence structure between components of multidimensional Lévy processes and the pricing of multi-asset option. They can be used to separate the dependence from the behavior of the components of the multivariate Lévy process.

By Lévy-Itô decomposition theorem, any Lévy process can be written as a sum of Brownian motion with drift and a pure jump part J with a Lévy measure ν . Since the Brownian motion part and the pure jump part are independent, and the dependence of the Brownian motion is completely characterized by the covariance matrix, we can focus on the pure jump process part that must be studied using the Lévy measure. Moreover, as the laws of the components of a multidimensional Lévy process are specified by their Lévy measures, it is convenient to define Lévy copulas with respect to tail integrals of Lévy measures rather than the distribution functions. Tail integrals play the role of distribution functions while Lévy measures play the role of probability measures for random variables.

Positive Lévy copulas are defined in [71] and [72]. As the whole concept of distributional copula functions is based on Sklar's theorem, a version of Sklar's theorem for Lévy processes is given. This theorem states that, a d -dimensional Lévy process can be constructed by taking d one-dimensional Lévy processes and couple them via an arbitrary Lévy copula. Conversely, any Lévy copula taking d one-dimensional Lévy processes as arguments construct a d -dimensional Lévy process. Parametric families of Lévy copula are constructed and theorems parallel to those for distributional copula are given. Two important theorems for simulating Lévy processes when the dependence structure is given by Lévy copula are

given. Application of Lévy copula in insurance as well as in finance is discussed.

General Lévy copulas are discussed in detail in [73] and [40]. Sklar's theorem for general Lévy processes and various theorems parallel to those for distributional copula are given. Parametric families of Lévy copulas are constructed and proof of limit theorem, which indicates how to obtain a Lévy copula of a multidimensional Lévy process X from distributional copula of the random variable X_t for fixed small time t is given. Simulation of multidimensional Lévy processes when the dependence structure is given by Lévy copula which is based on series representation of Rosiński [61] is discussed in detail in [73] and [74]. Applications of Lévy copulas to multi-asset options are also discussed.

Cont and Tankov [22] discuss Lévy processes in detail and some examples of jump processes which are tractable in finance are given. Different approach of option pricing and hedging are discussed in the case of one-dimension as well as multi-dimension. Options such as European, American, Forward and Barrier in one-dimensional exponential Lévy model are priced using the Fourier transform method. Multidimensional Lévy processes were constructed using Lévy copula functions. Positive Lévy copulas as well as Lévy copulas in general are discussed and several parametric families of Lévy copulas are used to construct multidimensional Lévy processes. Algorithms for simulating various Lévy processes using series representation are discussed. Basket options on two assets are priced using Monte Carlo simulation method.

Martin [34] studies the application of Lévy copula to the Danish fire insurance. A two one-dimensional compound Poisson processes (X_t, Y_t) model where X_t is the claims due to buildings up to time t and Y_t is the claims due to damage to furniture and personal property is constructed. The processes X_t and Y_t are assumed to have the same parameters and the estimated margins are compound Poisson processes of the same intensity. The dependence structure is modeled by the Clayton Lévy copula.

Chen [17] introduces a new method of *Discretely Sampled Process with pre-specified Marginals and pre-specified Dependence* (DSPMD) which allows to study the statistical inference of the Lévy copula. This method consists in pre-specifying the marginals and couple them using the joint law of pre-specified joint process. He proves that if the pre-specified marginals and pre-specified joint processes are Lévy processes, the DSPMD converges to a Lévy process under certain technical conditions. The DSPMD uses the copula structure on the ran-

dom variables level so that one can have access to its statistical properties. This method is applied to variance gamma process and a closed form of the copula function is obtained. He argues that, variance gamma copula is very competitive against other popular copula such Clayton, student-t, for modeling dependence of equity names. He introduces a new method for simulating Lévy processes which is also based on series representation. He argues that the simulation of multidimensional Lévy processes when the dependence structure is captured by Lévy copula of Tankov has bias: The loss of jump mass when the dependence level is low and the numerical complexity in high dimension since the Tankov's algorithm is based on conditional probability that needs to be computed recursively.

Chapter 2

Mathematical Background

One of the main issues in finance is to quantify the risk associated with a financial asset or portfolio asset. The risks in a financial asset are associated with the non-smoothness of the trajectory of the market prices and this is one crucial aspect of empirical data that one would like a mathematical model to reproduce. It is therefore reasonable to model the dynamics of the stock prices with discontinuous processes—namely *Lévy processes*. Lévy processes are processes with stationary independent increments. They play a central role in several fields of science such as physics, engineering, economics, and of course mathematical finance.

As mentioned in the introduction, Lévy processes were introduced as an alternative model of asset returns. This is because the paths of asset prices display discontinuities and the normality assumption of the log-returns is weak. Moreover, the volatility is not constant as suggested by the Black-Scholes model. In this chapter, we give a short introduction to the mathematics of Lévy processes. For additional details on Lévy processes we refer the reader to [7], [65], [66]. Unless otherwise mentioned, all proofs can be found in [65].

2.1 Definition of Lévy Processes

Definition 2.1.1. *A stochastic process $(X_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in \mathbb{R}^d is called a Lévy process if the following conditions are satisfied [22]:*

- (i) $X_0 = 0$ \mathbb{P} a.s.

(ii) *Independent increment: for every increasing sequence of times t_0, \dots, t_n , the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.*

(iii) *Stationary increments: the law of $X_{t+h} - X_t$ does not depend on t .*

(iv) *Stochastic continuity:*

$$\text{for all } \epsilon > 0, \lim_{h \rightarrow 0} \mathbb{P}[|X_{t+h} - X_t| \geq \epsilon] = 0.$$

Note that the last condition does not imply that the paths of Lévy processes are continuous. It only requires that for a given time t the probability of seeing a jump at t is zero; that is, jumps occur at random times.

A Lévy process can be seen as a random walk in continuous time with jumps occurring at random times. It is well known (see [57] chapter 4) that Lévy processes have a version with càdlàg paths, that is, paths which are right continuous and have limits from the left. A stochastic process which is càdlàg has two important path properties: the total number of jumps is at most countable, and the number of jumps whose size is bigger (in absolute value) than an arbitrary $\epsilon > 0$ is finite [22].

The primary tool in the analysis of distribution of Lévy processes is their characteristic function, or Fourier transform. The properties of the characteristic function make it to be tractable. For any random variable X , its characteristic function always exists, it is continuous, and it determines X uniquely. The characteristic function of a random variable X is defined by

$$\phi_X(u) = \mathbb{E}[\exp(iu \cdot X)] = \int_{-\infty}^{\infty} \exp(iux) dF(x), \quad (2.1)$$

where $F(x)$ is the distribution function of X defined by $F(x) = \mathbb{P}[X \leq x]$ and i is the imaginary number ($i^2 = -1$). If the distribution of the random variable X is continuous with density function $f_X(x)$, (2.1) becomes

$$\phi_X(u) = \mathbb{E}[\exp(iu \cdot X)] = \int_{-\infty}^{\infty} e^{iux} f_X(x) dx. \quad (2.2)$$

Moreover, for independent random variables X, Y ,

$$\phi_{X+Y}(u) = \phi_X(u)\phi_Y(u).$$

Definition 2.1.2. *Suppose that $\phi(u)$ is the characteristic function of a distribution X . If for any $n \in \mathbb{N}$, $\phi(u)$ is also the n th power of a characteristic function, we say that the distribution is infinitely divisible.*

In terms of X this means that one could write for any n :

$$X = Y_1^{(n)} + \dots + Y_n^{(n)},$$

where $Y_i^{(n)}$, $i = 1, \dots, n$, are i.i.d. random variables, all following a law with characteristic function $\phi(u)^{1/n}$.

A simple example of infinitely divisible distribution is the normal distribution. If $X \sim N(\mu, \sigma^2)$, then one can write $X = \sum_{i=1}^{n-1} Y_i$ where Y_i are i.i.d. with law $N(\mu/n, \sigma^2/n)$. Other examples are the gamma distribution, the Poisson distribution, the exponential distribution, the compound Poisson distribution, the Cauchy distribution, the α -stable distribution, and the Poisson distribution. A random variable having any of these distribution can be written as a sum of n i.i.d. parts having the same distribution but with modified parameters. The counter examples are the uniform distribution and the binomial distribution.

Now, consider a Lévy process $(X_t)_{t \geq 0}$. Using the fact that, for any $n \in \mathbb{N}$, and $t > 0$,

$$X_t = X_{\frac{t}{n}} + (X_{\frac{2t}{n}} - X_{\frac{t}{n}}) + \dots + (X_t - X_{\frac{(n-1)t}{n}}),$$

together with the stationarity and the independence of the increments, it follows that the law of $(X_t)_{t \geq 0}$ is infinitely divisible. The following proposition defines the characteristic function of a Lévy process.

Proposition 2.1.1. *(The characteristic function of a Lévy process).*

Let $(X_t)_{t \geq 0}$ be a Lévy process on \mathbb{R}^d . Then, there exists a continuous function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ known as the characteristic exponent of X , such that:

$$\mathbb{E}[e^{iu \cdot X_t}] = e^{t\psi(u)}, \quad u \in \mathbb{R}^d, \quad (2.3)$$

Proof. Define the characteristic function of X_t by

$$\phi_{X_t}(u) = \mathbb{E}[e^{iu \cdot X_t}], \quad u \in \mathbb{R}^d.$$

For $t > s$, by writing $X_{t+s} = X_s + (X_{t+s} - X_s)$, and using the fact that $X_{t+s} - X_s$ is independent of X_s , we obtain that $t \mapsto \phi_t(u)$ is a multiplicative function.

$$\begin{aligned} \phi_{X_{t+s}}(u) &= \phi_{X_s}(u) \phi_{X_{t+s}-X_s}(u) \\ &= \phi_{X_t}(u) \phi_{X_s}(u). \end{aligned}$$

The stochastic continuity of $t \mapsto X_t$ implies in particular that $X_t \rightarrow X_s$ in distribution when $s \rightarrow t$. Because of the convergence in distribution of X_t , it follows that $\phi_{X_s}(u) \rightarrow \phi_{X_t}(u)$ when $s \rightarrow t$ so $t \mapsto \phi_{X_t}(u)$ is continuous in t . Combining this with the above multiplicative property, it follows that $t \mapsto \phi_{X_t}(u)$ is an exponential function [22]. \square

The property of infinite divisibility gives rise to a great convenience to study the Lévy process X_t , namely one only needs to look at X_1 in order to investigate the distributional properties of X_t for any finite time t .

There is a one-to-one relationship between Lévy processes and infinitely divisible distributions. This relationship relies on the characterization of the characteristic function of an infinitely divisible distribution by the Lévy-Khintchine formula and the expression of the characteristic function of Lévy processes in terms of their characteristic triplet (A, ν, γ) . Moreover, Lévy processes are uniquely determined by their characteristic triplets and their characteristic exponent has a special representation which we discuss below.

Theorem 2.1.1. (*Lévy-Khintchine representation*).

Let $(X_t)_{t \geq 0}$ be a Lévy process on \mathbb{R}^d . There exists (A, ν, γ) called characteristic triplet with A a symmetric nonnegative definite $d \times d$ matrix, a vector $\gamma \in \mathbb{R}^d$ and ν a Lévy measure (see definition (2.2.3)), that is, a positive measure on $\mathbb{R}^d \setminus \{0\}$, satisfying

$$\int_{\mathbb{R}^d \setminus \{0\}} (|x|^2 \wedge 1) \nu(dx) < \infty.$$

such that the characteristic function of $(X_t)_{t \geq 0}$ is defined by

$$\mathbb{E}[e^{iu \cdot X_t}] = e^{t\psi(u)}, \quad u \in \mathbb{R}^d. \quad (2.4)$$

The characteristic exponent $\psi(u)$ is given by

$$\psi(z) = -\frac{1}{2}u \cdot Au + i\gamma \cdot u + \int_{\mathbb{R}^d} (e^{iu \cdot x} - 1 - iu \cdot x 1_{|x| \leq 1}) \nu(dx).$$

The Lévy measure determines the frequency and the size of jumps of the Lévy process. Jumps larger than some arbitrary ϵ must be truncated. More precisely, for every bounded measurable function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying $g(x) = 1 + o(|x|)$ as $x \rightarrow 0$ and $g(x) = O(1/|x|)$ as $x \rightarrow \infty$, the representation above can be written as

$$\psi(u) = -\frac{1}{2}u \cdot Au + i\gamma^g \cdot u + \int_{\mathbb{R}^d} (e^{iu \cdot x} - 1 - iu \cdot x g(x)) \nu(dx),$$

where $\gamma^g = \gamma + \int_{\mathbb{R}^d} x(g(x) - 1_{|x| \leq 1}) \nu(dx)$. The triplet (A, ν, γ^g) is called the characteristic triplet of $(X_t)_{t \geq 0}$ with respect to the truncation function g . If $\int_{\mathbb{R}^d} (|x| \wedge 1) \nu(dx) < \infty$, one may take $g = 0$ and the Lévy-Khintchine representation becomes

$$\psi(u) = -\frac{1}{2}u \cdot Au + i\gamma_0 \cdot u + \int_{\mathbb{R}^d} (e^{iu \cdot x} - 1) \nu(dx),$$

The vector γ_0 is in this case called *drift* of the Lévy process X .

2.2 Lévy Measure and Path Properties

In general, the sample path of a Lévy process is not continuous. Understanding the jump structure of a Lévy process is equivalent to the knowledge of the path behavior of a Lévy process. In the following, we discuss various measures that are associated with Lévy processes. They are useful in the Lévy-Itô decomposition theorem (2.2.1).

Definition 2.2.1. (*Random measure*). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let (E, \mathcal{B}) be a measurable space. A map $M : \mathcal{B} \times \Omega \mapsto \mathbb{R}$ is called a random measure on (E, \mathcal{B}) if and only if

- (i) For each $B \in \mathcal{B}$, the map $\omega \mapsto M(B, \omega)$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.
- (ii) For almost every $\omega \in \Omega$, the map $B \mapsto M(B, \omega)$ is a measure on (E, \mathcal{B}) .
- (iii) There exists a partition $B_1, B_2, \dots \in \mathcal{B}$ of E such that $M(B_k) < \infty$ almost surely for all $k \in \mathbb{N}$.

A random measure M on (E, \mathcal{B}) is said to have independent increments if and only if $M(B_1), \dots, M(B_n)$ are independent random variables whenever B_1, \dots, B_n are mutually disjoint members of \mathcal{B} . A random measure M on (E, \mathcal{B}) is called a *point process* if and only if M is a $\bar{\mathbb{Z}}_+$ -valued (including ∞). A Poisson random measure with intensity measure μ is a point process M with independent increments such that for every $B \in \mathcal{B}$, $M(B)$ is a Poisson random variable with mean $\mu(B)$. Here μ is a measure on (E, \mathcal{B}) , that is,

$$\mathbb{P}[M(B) = k] = e^{-\mu(B)} \frac{\mu(B)^k}{k!} \quad \text{for all } k \in \bar{\mathbb{Z}}_+. \quad (2.5)$$

Let $H = (0, \infty) \times \mathbb{R}^d \setminus \{0\}$. Every Lévy process X has a Poisson random measure J_X on $(H, \mathcal{B}(H))$, known as jump measure associated with it. The jump measure is defined by

Definition 2.2.2. (*Jump measure*). Let $(X_t)_{t \geq 0}$ be a Lévy process on $(\Omega, \mathcal{F}_t, \mathbb{P})$ and $B \in \mathcal{B}(B)$. For every $\omega \in \Omega$, the jump measure J_X of the process X_t is defined by

$$J_X(\omega, A) = \#\{t : (t, \Delta X_t) \in A\}, \quad (2.6)$$

J_X is just a counting measure. In particular, if $A = ((0, t] \times B)$, where $B \in \mathcal{B}(\mathbb{R}^d)$ is bounded away from zero, then $J_X((0, t] \times B)$ counts the number of jumps of X_t between times 0 and t with jump size in B .

Another very important measure in the setting of Lévy processes is the Lévy measure ν . The Lévy measure determines the frequency and size of jumps of a Lévy process X . It is a positive measure on \mathbb{R}^d satisfying the following conditions

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (1 \wedge x^2) \nu(dx) < \infty.$$

That means, a Lévy measure has no mass at the origin, but singularities (that is, $\nu(B)$ may be $+\infty$) can occur around the origin. For a Lévy process X_t , because it is càdlàg, it is possible to define the jump process $\Delta X_t = X_t - X_{t-}$. It is quite possible for the sum $\sum \Delta X_t$ to be infinite but for a bounded time interval and again because of the càdlàg property of X_t there can be only finitely many jumps whose amplitude exceed a certain (strictly positive) size.

Let $B \in \mathcal{B}(\mathbb{R}^d)$ be bounded away from zero (that $0 \notin \bar{B}$, where \bar{B} is the closure of B .) For any such B , the sum

$$\sum_{s \leq t} \Delta X_s 1_{\Delta X_s \in B}$$

will have only finitely many non zero terms and a strictly increasing sequence of stopping times $(\tau_n^B)_{n \in \mathbb{N}}$ can be introduced as follows

$$\tau_0^B = 0 \quad \tau_{n+1}^B = \inf\{t > \tau_n^B : \Delta X_t \in B\}.$$

Thus $(\tau_n^B)_{n \geq 1}$ enumerate the jump times in B [57]. We define the associated counting process $N_t(B)$ by

$$N_t(B) = \sum_{n=1}^{\infty} 1_{\{\tau_n^B \leq t\}} = \sum_{0 < s < t} 1_B(\Delta X_s).$$

Note that $N_t(B)$ is a Poisson process with intensity $\nu(B)$.

Let $\nu(B)$ be the parameter of $N_t(B)$, that is,

$$\nu(B) = \mathbb{E}[N_1(B)], \tag{2.7}$$

is the expected number of jumps of $N_t(B)$ per unit time. Note that $\nu(B)$ also gives the expected number of jumps of X which belong to B per unit time. One can easily verify that ν satisfies the conditions of a measure on $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$. Moreover, since $\nu(B) = \mathbb{E}[N_1(B)]$, the monotonic converge theorem implies that ν is also a measure.

Definition 2.2.3. (*Lévy measure*). Let $(X_t)_{t \geq 0}$ be a Lévy process on \mathbb{R}^d . The measure ν on \mathbb{R}^d defined by :

$$\nu(B) = \mathbb{E} [\#t \in [0, 1] : \Delta X_t \neq 0, \Delta X_t \in B], \quad B \in \mathcal{B}(\mathbb{R}^d),$$

is called the Lévy measure of X : $\nu(B)$ is the expected number, per unit time of jumps whose size belongs to B .

For example, the characteristic function of the compound Poisson process has the following representation

$$\mathbb{E} (\exp(iu \cdot X_t)) = \exp \left\{ t\lambda \int_{\mathbb{R}^d} (e^{iu \cdot x} - 1) f(dx) \right\}, \quad \text{for all } u \in \mathbb{R}^d, \quad (2.8)$$

where λ denotes the jump intensity and f the jump size distribution. If we introduce a new measure $\nu(A) = \lambda f(A)$, (2.8) can be written as

$$\mathbb{E} (e^{iu \cdot X_t}) = \exp \left\{ t \int_{\mathbb{R}^d} (e^{iu \cdot x} - 1) \nu(dx), \right\}, \quad \text{for all } u \in \mathbb{R}^d.$$

ν is called the *Lévy measure* of the process $(X_t)_{t \geq 0}$. It is a positive measure on \mathbb{R} but not a probability measure since $\int \nu(dx) = \lambda \neq 1$.

From the Lévy-Khintchine formula, we see that any Lévy process can be decomposed into two independent components: the continuous part described by a Brownian motion and a pure jump part described by a Poisson process. One could further decompose the jump part into two parts: one part describing large jumps and the other describing the compensated small jumps. This decomposition is known as Lévy-Itô decomposition as it is defined in the following theorem

Theorem 2.2.1. (*Lévy-Itô decomposition*).

Let $(X_t)_{t \geq 0}$ be a Lévy process on \mathbb{R}^d with a Lévy measure ν given by definition (2.2.3). Then

- ν is a random measure on $\mathbb{R}^d \setminus \{0\}$ and verifies

$$\int_{|x| \leq 1} |x|^2 \nu(dx) < \infty \quad \int_{|x| \geq 1} \nu(dx) < \infty.$$

- The jump measure of X , denoted by J_X , is a Poisson random measure on $[0, \infty[\times \mathbb{R}^d$ with intensity measure $\nu(dx)dt$.

- There exist a vector γ and a d -dimensional Brownian motion $(B_t)_{t \geq 0}$ with covariance matrix A , such that

$$X_t = \gamma t + B_t + X_t^l + \lim_{\epsilon \downarrow 0} \tilde{X}_t^\epsilon, \quad (2.9)$$

with

$$\begin{aligned} X_t^l &= \int_{|x| \geq 1, s \in [0, t]} x J_X(ds \times dx). \\ \tilde{X}_t^\epsilon &= \int_{\epsilon \leq |x| < 1, s \in [0, t]} x \{J_X(ds \times dx) - \nu(dx) ds\} \\ &= \int_{\epsilon \leq |x| < 1, s \in [0, t]} x \tilde{J}_X(ds \times dx). \end{aligned}$$

The terms in (2.9) are independent and the convergence in the last term is almost sure and uniform in t on $[0, T]$.

The first two terms in (2.9) are none other than Brownian motion with drift γ and these are the continuous terms of the Lévy process. The last two terms are the discontinuous processes incorporating jumps of X_t described by the Lévy measure ν . The condition $\int_{|x| \geq 1} \nu(dx) < \infty$ implies that the number of jumps of X_t over each finite time interval with absolute value large than 1 is finite. So the sum

$$X_t^l = \sum_{\substack{0 \leq s \leq t, \\ |\Delta X_s| \geq 1}} \Delta X_s$$

contains almost surely a finite number of terms and thus X_t^l is a compound Poisson process. There is nothing special about the value 1; it can be replaced with any $\epsilon > 0$ and the resulting Lévy process X_t^ϵ is again a well defined Lévy process. Contrarily to compound Poisson case, the sum of the small jumps $\int_{|x| \leq 1} x \tilde{J}_X(ds \times dx)$ may be infinite. However, it turns out that the compensated integral (cf. section 2.6.2 in [22])

$$\int_{|x| \leq 1} x \tilde{J}_X(ds \times dx) = \int_{|x| \leq 1} x [J_X(ds \times dx) - \nu(dx) ds],$$

is guaranteed to be finite: $\int_{|x| \leq 1} x \nu(dx) ds$ is roughly, the expected sum of small jumps by the time t and subtracting the expected sum from the actual sum leaves us with something finite.

Note that in equation (2.9) above, not all terms are martingale. Only the B_t term and the compensated term are martingales. The Lévy-Itô decomposition theorem has an important

implication which is useful both in theory and in practice such as simulation of Lévy processes [22]: Every Lévy process can be approximated with arbitrary precision by a jump-diffusion process, that is, by the sum of Brownian motion with drift and a compound Poisson process. Indeed, every Lévy process is a combination of a Brownian motion with drift and a possibly infinite sum of independent compound Poisson processes.

The Lévy measure of a Lévy process X is responsible for the path properties such as activity and variation of the Lévy process. These properties are deduced from the characteristic triplet (A, ν, γ) of the Lévy process and the Lévy-Itô decomposition [65]. If $\nu(\mathbb{R}) < \infty$ then almost all paths of X_t have a finite number of jumps on every compact interval. The Lévy process is said to be of finite activity. If $\nu(\mathbb{R}) = \infty$, then almost all paths of X_t have an infinite number of jumps on every non-degenerate compact interval. In this case, the Lévy process is of infinite activity. The Lévy process X_t is of finite variation if $A = 0$ and $\int_{|x| \leq 1} |x| \nu(dx) < \infty$. In this case, the Lévy process is a pure jump process and its characteristic exponent has a simple form

$$\psi(u) = iu \cdot \gamma' + \int_{\mathbb{R}^d \setminus \{0\}} (e^{iu \cdot x} - 1) \nu(dx),$$

where γ' is the new drift. If $A \neq 0$ or $\int_{|x| \leq 1} |x| \nu(dx) = \infty$, the Lévy process is of infinite variation.

2.3 Subordinators

Subordinators are increasing Lévy processes. They can be used to build new Lévy processes by time changing another. Subordination, or time-change of a Lévy process with a subordinator is a very important technique in building financial models based on Lévy processes. Examples in this setting are the variance gamma [49], the normal inverse Gaussian (NIG) [4], the CGMY [16], and the generalized hyperbolic (GH) [6] models.

The concept of time-changed Brownian motion has a strong economic intuition. It is clear that the market does not evolve equally all the time, sometimes the trading activity is very intensive, while other times the market is quiet and the trading activity is slow. It is therefore reasonable to measure the time scale of the market by a random business time rather than the calendar time.

Proposition 2.3.1. *Let $(X_t)_{t \geq 0}$ be a Lévy process on \mathbb{R} . The following conditions are equivalent:*

- (i) $X_t \geq 0$ a.s for some $t > 0$.
- (ii) $X_t \geq 0$ a.s for every $t > 0$.
- (iii) The sample paths of (X_t) are almost surely nondecreasing: $t \geq s$ implies $X_t \geq X_s$ a.s.
- (iv) The characteristic triplet (A, ν, γ) of (X_t) satisfies

$$A = 0, \quad \nu([-\infty, 0]) = 0, \quad \int_0^\infty (x \wedge 1) \nu(dx) < \infty, \quad \text{and } \gamma_0 \geq 0,$$

that is (X_t) has no diffusion component, only positive jumps and positive drift.

Since a subordinator S_t is a positive random variable for all t , it is conveniently described using Laplace transform rather than Fourier transform. If $(0, \rho, \gamma_0)$ is the characteristic triplet of S_t , then its moment generating function is defined as

$$\mathbb{E}(e^{uS_t}) = e^{tL(u)} \quad \text{for all } u \geq 0, \quad (2.10)$$

where

$$L(u) = \gamma_0 u + \int_0^\infty (e^{ux} - 1) \rho(dx). \quad (2.11)$$

$L(u)$ is called the *Laplace exponent* of S . The following theorem whose proof is given in [22] shows that S can be interpreted as a time deformation and is used to time change other Lévy processes.

Theorem 2.3.1. *Subordination of Lévy process.*

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(X_t)_{t \geq 0}$ be a Lévy process on \mathbb{R}^d with characteristic exponent ψ and triplet (A, ν, γ) and let $(S_t)_{t \geq 0}$ be a subordinator with Laplace exponent $L(u)$, and triplet $(0, \rho, \gamma_0)$. Then the process $(Y_t)_{t \geq 0}$ defined for each $\omega \in \Omega$ by $Y(t, \omega) = X(S(t, \omega), \omega)$ is a Lévy process. Its characteristic function is

$$\mathbb{E}(e^{iuY_t}) = e^{tL(\psi(u))},$$

that is, the characteristic exponent of Y is obtained by composition of the Laplace exponent of S with the characteristic exponent of X . The triplet (A^Y, ν^Y, γ^Y) of Y is given by

$$\begin{aligned} A^Y &= \gamma_0 A, \\ \nu^Y(B) &= \gamma_0 \nu(B) + \int_0^\infty P_s^X \rho(ds) \quad \text{for all } B \in \mathcal{B}(\mathbb{R}^d), \\ \gamma^Y &= \gamma_0 \gamma + \int_0^\infty \rho(ds) \int_{|x| \leq 1} x P_s^X(dx), \end{aligned}$$

where P_s^X is the probability distribution of X . $(Y_t)_{t \geq 0}$ is said to be subordinated to the process $(X_t)_{t \geq 0}$.

A Lévy process is a stochastic process with stationary and independent increments. Its law is completely specified by its characteristic triplet (A, ν, γ) . The characteristic function of a Lévy process is infinitely divisible and can be computed from the characteristic triplet. Every Lévy process can be decomposed into a continuous part described by a Brownian motion and a pure jump part described by the Lévy measure ν . The frequency and the jump-sizes are determined by the Lévy measure ν . A positive increasing Lévy process is called a subordinator and can be used to construct new Lévy processes by time-changing others.

Chapter 3

Lévy Processes in Finance

In this chapter, we introduce Lévy processes into derivative pricing and discuss their tractability. We concentrate on pure jump Lévy processes with infinite activity as these processes were found capable to describe the observed behavior of the financial market data. We follow [22] to represent exponential Lévy models.

3.1 Problems with the Black-Scholes Models

A well-known stochastic process that is used to model the stock price is Brownian motion. Brownian motion has been used since the beginning of modern mathematical finance when Louis Bachelier [3] proposed to model the price of an asset at the Paris Bourse as

$$S_t = S_0 + \sigma W_t, \quad (3.1)$$

where $\sigma > 0$ is a parameter and $(W_t)_{t \geq 0}$ is a standard Brownian motion. The main drawback of the Bachelier model is that the price of an asset may be negative.

In their seminal paper [9], Black and Scholes made a breakthrough in the pricing of stock options by developing what is known as Black-Scholes model (Samuelson [67] first modeled stock price dynamics using a geometric Brownian motion). They modeled the stock price as the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t). \quad (3.2)$$

Equation (3.2) has a unique solution

$$S_t = S_0 \exp \left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right), \quad (3.3)$$

which is the functional of Brownian motion called Geometric Brownian motion. The log-returns produced by the geometric Brownian motion are normally distributed $N(\mu - \sigma^2/2, \sigma^2)$ which is far from being realistic for most time series of financial data. In the real world, the asset price processes have jumps or spikes and the empirical distribution of the asset returns exhibits fat tails and skewness behavior that deviates from normality.

Figure (3.1) depicts the evolution of the logarithm of the stock price of Intel corporation (INTC) over the period January 3rd 2000 to December 30th 2005, and the empirical and fitted normal density over the same period. From the left graph of figure (3.1), one can see at least two points where the price moved by 10\$ within the period of one day. Prices' moves like these need to be taken into account and the Brownian motion assumption in the Black-Scholes model can not deal with these moves. The right graph of figure (3.1) compares the empirical density and that of fitted normal for INTC over the same period. One observes that the distribution of the asset returns has a sharp peak and the tails are heavier than that of a normal distribution. Therefore, traders need models that account for jumps and with the right distributions. For more on the drawbacks of the Brownian motion to model the stock prices see the first chapter of [22].

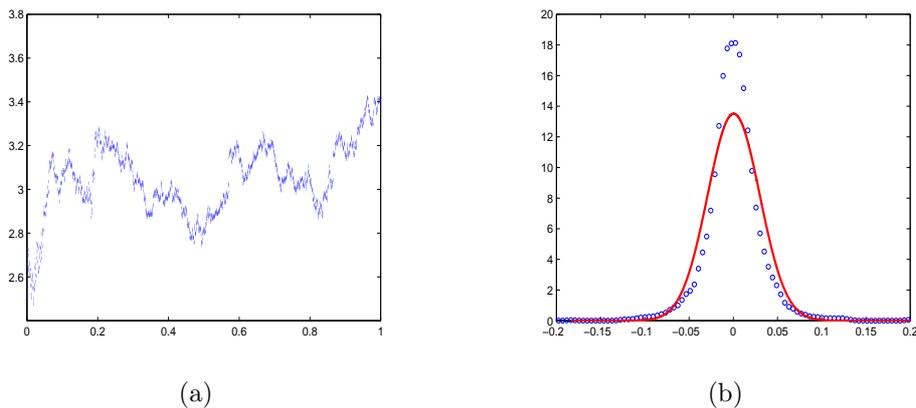


FIG. 3.1. On the left: The evolution of the logarithm of the stock price of INTC over the period January 3rd 2000 to December 30th 2005. On the right: The empirical density compared to that of fitted normal density.

3.2 Exponential Lévy Models

Exponential Lévy models are obtained by exponentiating a Lévy process; more precisely, the risk-neutral dynamics of the stock prices is given by

$$S_t = \exp(rt + X_t), \quad 0 \leq t \leq T, \quad (3.4)$$

where X_t is a Lévy process on $(\Omega, \mathcal{F}, \mathbb{Q})$ with characteristic triplet (σ, ν, γ) , r is the constant continuously compounded interest rate, and T is the fixed horizon date for all market activities.

By the first fundamental theorem of asset pricing, there is no arbitrage opportunity (to be precise, there is No Free Lunch with Vanishing Risk), if there exists a probability measure \mathbb{Q} called the *risk-neutral measure*, equivalent to \mathbb{P} , such that the discounted price process is a martingale [24].

For exponential Lévy model (3.4), the absence of arbitrage imposes that $\tilde{S}_t = e^{-rt}S_t = \exp(X_t)$, which is equivalent to the following condition on the triplet (σ, ν, γ) [22]:

$$\int_{|x|>1} \nu(dx)e^x < \infty \quad \text{and} \quad (3.5)$$

$$\gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^x - 1 - x1_{|x|\leq 1})\nu(dx) = 0. \quad (3.6)$$

$(X_t)_{t \geq 0}$ is then a Lévy process such that $\mathbb{E}[e^{X_t}] = 1$ for all t .

According to Cont and Tankov (cf. proposition 9.9 [22]), if the trajectories of the Lévy process X are not almost surely increasing nor almost surely decreasing, then the exponential Lévy model (3.4) is arbitrage-free, that is, there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $e^{-rt}S_t$ is a \mathbb{Q} -martingale.

If one would start with the stochastic integral (3.2), by replacing $rt + \sigma W_t$ by a Lévy process Z_t , we obtain the stochastic equation

$$dS_t = S_{t-}(rdt + dZ_t), \quad (3.7)$$

where S_{t-} denotes the limit from the left. The discounted stock price process $S_t e^{-rt}$ is then a martingale if and only if the Lévy process Z_t is a martingale and this is the case

if $\mathbb{E}[Z_1] = 0$. Cont and Tankov [22] proved in proposition 8.22 that the two constructions lead to the same class of processes. In our construction of exponential Lévy model, we will consider the first approach.

3.3 Examples of Lévy Processes

In this section, we discuss some of the pure jump processes of infinite activity (that is, there is an infinite number of jumps in each time interval) such as the variance gamma and the CGMY processes. The most frequently used method to generate infinite activity Lévy processes is through subordinating a Brownian motion with an independent increasing Lévy process (a process called *subordinator*). This means, a Brownian motion (possibly with a drift) is evaluated at a new stochastic time scale which is given by an independent increasing process. This time scale has a financial interpretation of "business time" [30]. The interesting feature of Lévy processes generated by time-changing a Brownian motion by a subordinator is that, the Lévy-Itô decomposition of this kind of Lévy processes does not necessarily contains a Brownian motion part leading to a purely discontinuous Lévy process [22].

Modeling asset prices by purely discontinuous but infinite activity Lévy process is justified by the argument that the jump structure of these processes is rich enough to capture both frequently small jumps and rare large jumps which amount to eliminate the need of the diffusion component (see [16], [29]).

3.3.1 The Gamma Process

An \mathbb{R}_+ -valued random variable X is said to have a gamma distribution with parameters $\mu > 0$ and $\lambda > 0$ if it has density function given by [22]

$$f_X(x, \mu, \lambda) = \frac{\lambda^\mu x^{\mu-1}}{\Gamma(\mu)} e^{-\lambda x}, \quad x > 0, \quad (3.8)$$

where $\Gamma(\mu)$ is the gamma function. The mean of X is μ/λ and its variance is given by μ/λ^2 . When μ is an integer, then X has the same distribution as the sum of μ independent exponentially distributed random variables with parameters λ . The density function has

a heavy-right tail and its kurtosis is greater than that of the normal. These properties can be clearly observed from figure (3.2) where the gamma density is compared with the normal density.

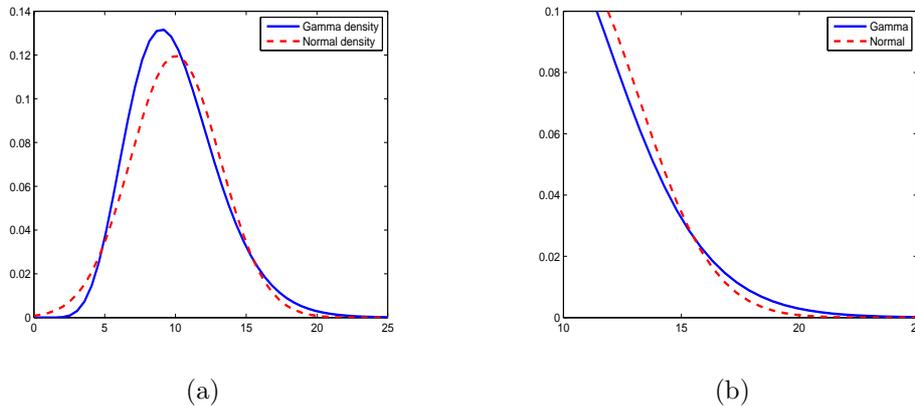


FIG. 3.2. On the left: Comparison of the gamma density function and that of the normal with the same mean and the same variance. On the right: Comparison of the right tail of the gamma distribution and that of the normal.

The characteristic function of the gamma distribution is given by

$$\phi_{\gamma}(u, \mu, \lambda) = (1 - iu/\lambda)^{-\mu}, \quad (3.9)$$

which is infinitely divisible and thus a Lévy process associated to it can be defined (cf. theorem 2.1.1). The Lévy process associated to the gamma distribution is the gamma process.

The gamma process $\gamma_t = (\gamma_t)_{t \geq 0}$ is a stochastic process whose increments $\gamma_{t+h} - \gamma_t$ over non-overlapping intervals follow for all $0 < t < t+h < T$ a gamma distribution. Time will enter only in the parameter μ so that γ_t is distributed as $\gamma(\mu t, \lambda)$.

For the reason of normalization, we will work with a gamma process such that $\mathbb{E}[\gamma_t] = t$, which in terms of the parameters imply that $\mu = \lambda$. Denote the common quantity $\mu^{-1} = \lambda^{-1}$ by α . Then $\mathbb{E}[\gamma_t] = t$ and $\text{var}[\gamma_t] = \alpha t$. Therefore the average random time change in t units of calendar time is t whereas its variance is proportional to t .

The gamma process is a pure jump process which satisfies the condition of subordinator (cf. proposition 2.3.1). It can therefore be used to build new processes. An example in this setting is the variance gamma process (see section 3.3.2) obtain by time-changing a

Brownian motion by a gamma process. The moment generating function of the gamma process is given by

$$\phi_\gamma(u, \mu, \lambda) = (1 - u/\lambda)^{-\mu}. \quad (3.10)$$

By using (2.10), it follows that $l(u) = -\mu \ln(1 - u/\lambda)$ and then we have

$$\begin{aligned} -\mu \ln(1 - u/\lambda) &= -\mu \int_0^u \frac{1}{\lambda - y} dy \\ &= -\mu \int_0^u \int_0^\infty e^{-\lambda x + yx} dx dy \\ &= -\mu \int_0^\infty e^{-\lambda x} \int_0^u e^{xy} dx dy \\ &= \mu \int_0^\infty e^{-\lambda x} \frac{1}{x} (e^{ux} - 1) dx \quad (\text{since } u \leq 0) \\ &= \int_0^\infty (e^{ux} - 1) \frac{\mu e^{-\lambda x}}{x} dx. \end{aligned} \quad (3.11)$$

It follows from the last equality of (3.11) that the Lévy measure of the gamma process is given by

$$\nu(dx) = \frac{\mu e^{-\lambda x}}{x} dx, \quad x > 0. \quad (3.12)$$

The Lévy measure of the gamma process satisfies $\int_0^\infty (x \wedge 1) \nu(dx) < \infty$, but $\int_0^\infty \nu(x) dx = \infty$, meaning that the arrival rate of jumps (with small jumps occurring more often than large jumps) in each finite time interval is infinite. Therefore, the gamma process is of finite variation but of infinite activity. The characteristic function of the gamma process is given by [22]

$$\phi_{\gamma_t}(u, \mu, \lambda) = (1 - iu/\lambda)^{-\mu t}, \quad (3.13)$$

and its Lebesgue density function is given by

$$p_t(x) = \frac{\lambda^{\mu t}}{\Gamma(\mu t)} x^{\mu t - 1} e^{-\lambda x}. \quad (3.14)$$

3.3.2 The Variance Gamma Process

Another example of pure jump processes of infinite activity we consider is the variance gamma process. This process was introduced in the literature by Madan and Seneta [48] for the symmetric case (that is, $\beta = 0$ in (3.15) below) when they considered a Brownian motion without drift time-changed by a gamma process. Madan and Milne [46] investigated

equilibrium option pricing for a symmetric variance gamma process in a representative agent model with a constant relative risk aversion utility function. The resulting risk-neutral process obtained is similar to the more general (asymmetric case) variance gamma process studied by Madan, Carr and Chang [49].

The variance gamma process was originally derived by evaluating Brownian motion at random time given by the gamma process. The modeling of the business time by a stochastic process whose increments follow a gamma distribution is motivated by the lack of memory property that is possessed by exponential distribution; that is, what happens today does not depend on what happened in the past. Mathematically speaking, if X is a random variable with exponential distribution, then

$$\mathbb{P}[X > s + t | X > s] = \mathbb{P}[X > t].$$

Note that exponential distribution is the only distribution with this property, which motivate its use for modeling the inter-arrival times of objects. Moreover, the exponential distribution is a special case of the gamma distribution. Thus, it is convenient to model the jump times by a gamma process which can also be interpreted as a model for arrival of information.

Since its introduction, variance gamma process has shown in a vast literature ([66], [29], [37], [47]) a great ability to describe asset returns in the univariate contest. The attractive feature of this process is that the log-normal density and the Black-Scholes formula are special case, making this model an extension of the standard financial modeling paradigm [49]. Besides the volatility, in these processes feature two additional parameters allowing to control skewness and kurtosis of the distribution.

Now given a Brownian motion $B(t; \theta, \sigma)$ with drift θ and volatility σ and a gamma process $\gamma(t; 1, \alpha)$ with mean rate unit and variance rate α , the variance gamma process is defined by

$$X(t; \sigma, \alpha, \theta) = \beta\gamma(t; 1, \alpha) + \sigma B(\gamma(t; 1, \alpha)). \quad (3.15)$$

The variance gamma is a 3-parameters (σ, α, θ) process, where θ controls over the skewness and α the kurtosis. When $\theta = 0$, the Lévy density is symmetric and the skewness is zero. Negative values of θ generate negative skewness.

The variance gamma distribution can be thought of as a mixture of normal distribution,

where the mixing weights density is given by the Gamma distribution of the subordinator. The density function of the variance gamma process can be obtained by first conditioning on the realization of the gamma process as a normal density function and then integrating out the density of the gamma process (3.14). This gives f_X of $X(t)$ as

$$f_{X(t)}(X) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi x}} \exp\left(-\frac{(X-\theta)^2}{2\sigma^2 x}\right) \frac{x^{\frac{t}{\alpha}-1} \exp(-\frac{x}{\alpha})}{\alpha^{\frac{t}{\alpha}} \Gamma(\frac{t}{\alpha})} dx. \quad (3.16)$$

The above integral converges and the probability density function of the variance gamma process is given by (see appendix A.1 for a proof)

$$f_X(x) = \sqrt{\frac{2}{\pi}} \frac{\exp(\frac{\theta x}{\sigma^2})}{\alpha^{t/\alpha} \Gamma(t/\alpha) \sigma} \left(\frac{x^2}{\frac{2\sigma^2}{\alpha} + \theta^2}\right)^{\frac{\xi}{2}} K_\varphi(\xi). \quad (3.17)$$

Here $K_\varphi(\cdot)$ is the modified Bessel function of the third kind with index ξ given by

$$K_\varphi(\xi) = \frac{1}{2} \int_0^\infty t^{\varphi-1} \exp\left[-\frac{\xi}{2} \left(\frac{1}{t} - t\right)\right] dt.$$

The characteristic function of the variance gamma process can be obtained by conditioning on the gamma time and using the fact that the conditional random variable is Gaussian. Then, we apply the Laplace transform to get the unconditional characteristic function.

$$\begin{aligned} \phi_{VG}(u, t) &= \mathbb{E}[e^{iuX_t}] \\ &= \mathbb{E}[\mathbb{E}[e^{iuX_t} | \gamma_t = z]] \\ &= \mathbb{E}[\mathbb{E}[e^{iu(\beta z + \sigma B_z)} | \gamma_t = z]] \\ &= \mathbb{E}[e^{iu\beta z} \mathbb{E}[e^{iu\sigma B_z} | \gamma_t = z]] \\ &= \mathbb{E}[e^{iu\beta z} e^{-\frac{u^2 \sigma^2 z}{2}} | \gamma_t = z] \\ &= \mathbb{E}[e^{(iu\beta - \frac{u^2 \sigma^2}{2})\gamma_t}] \\ &= \left(1 - \alpha u \beta + \frac{\alpha u^2 \sigma^2}{2}\right)^{-\frac{t}{\alpha}}, \end{aligned}$$

where the last equality follows from the definition of the moment generating function (3.10) of the gamma process. The characteristic exponent of the variance gamma process is then given by

$$\psi_{VG}(u, t) = -\frac{t}{\alpha} \left(1 - i\theta\alpha u + \frac{\alpha u^2 \sigma^2}{2}\right). \quad (3.18)$$

The moments of the variance gamma process are given by [49]

$$\begin{aligned}
\text{mean} &= \theta t, \\
\text{variance} &= (\theta^2 \alpha + \sigma^2) t, \\
\text{Skewness} &= \frac{(2\theta^3 \alpha^2 + 3\sigma^2 \alpha \theta) t}{((\theta^2 \alpha + \sigma^2) t)^{\frac{3}{2}}}, \\
\text{Kurtosis} &= \frac{(3\alpha \sigma^4 + 12\theta^2 \sigma^2 \alpha^2 + 6\theta^4 \alpha^3) t + (3\sigma^4 + 6\sigma^2 \theta^2 \alpha + 3\theta^4 \alpha^2) t^2}{((\theta^2 \alpha + \sigma^2) t)^2}.
\end{aligned}$$

Carr, Madan and Chang [49] also showed that the variance gamma process can be expressed as the difference of two independent gamma process $\gamma_+(t)$ and $\gamma_-(t)$ which may have different mean and variance.

$$X(t; \sigma, \alpha, \theta) = \gamma_+(t; \eta_+, \delta_+) - \gamma_-(t; \eta_-, \delta_-), \quad (3.19)$$

This representation is obtained by writing the characteristic function of the variance gamma process as a product of two characteristic functions and noticing that they are the characteristic function of the gamma processes. The relationship between the parameters (3.19) and the original parameters is given by

$$\eta_+ = \frac{1}{2} \sqrt{\theta^2 + \frac{2\sigma^2}{\alpha}} + \frac{\theta}{2}, \quad (3.20)$$

$$\eta_- = \frac{1}{2} \sqrt{\theta^2 + \frac{2\sigma^2}{\alpha}} - \frac{\theta}{2}, \quad (3.21)$$

$$\delta_+ = \alpha \eta_+^2, \quad (3.22)$$

$$\delta_- = \alpha \eta_-^2. \quad (3.23)$$

The Lévy density of the variance gamma process viewed as the difference of two gamma processes has a simple form which allows to see easily the property of an infinite arrival rate of price jumps from the gamma processes. This Lévy density is obtained by employing the Lévy density of the gamma process (3.12) as

$$\nu_{VG}(x) = \begin{cases} \frac{\eta_-^2}{\delta_-} \frac{\exp(-\frac{\eta_-}{\delta_-} |x|)}{|x|}, & x < 0 \\ \frac{\eta_+^2}{\delta_+} \frac{\exp(-\frac{\eta_+}{\delta_+} x)}{x}, & x > 0. \end{cases} \quad (3.24)$$

Once we have this expression of the Lévy density, we can easily calculate the Lévy density in terms of the original parameters.

$$\nu_{VG}(x) = \frac{\exp(\theta x/\sigma^2)}{\alpha|x|} \exp\left(\frac{\sqrt{\frac{2}{\alpha} + \frac{\theta^2}{\sigma^2}}}{\sigma}|x|\right). \quad (3.25)$$

The variance gamma has paths of finite variation since $\int_{\{|x|\leq 1\}} |x|\nu(dx) < \infty$. Furthermore, the asymptotic behavior $\nu(dx) \sim \frac{1}{|x|}dx$, $x \rightarrow 0$, yields the infinite activity of the process.

The representation of the variance gamma process makes its simulation easier since the distribution of increments is known. The variance gamma process can be simulated as a Brownian motion sampled by a random time given by gamma random variable. One can also use representation (3.19). Figure (5.1) depicts the sample path of the variance gamma process represented as a time-changed Brownian motion. On the left, the parameters were chosen randomly and are given by $\theta = 0.05$, $\alpha = 0.042$, $\sigma = 0.25$. On the right, the parameters were estimated from the historical asset returns of INTC over the period January 3rd 2000 to December 30th 2005 and are given by $\theta = 0.001$, $\alpha = 0.45$, $\sigma = 0.0270$. Notice the difference between the trajectories. While the right graph looks like the path of asset prices (with reasonable jumps), the left graph has very big jumps. Hence calibrated Lévy processes look like stock prices except that they can be negative.

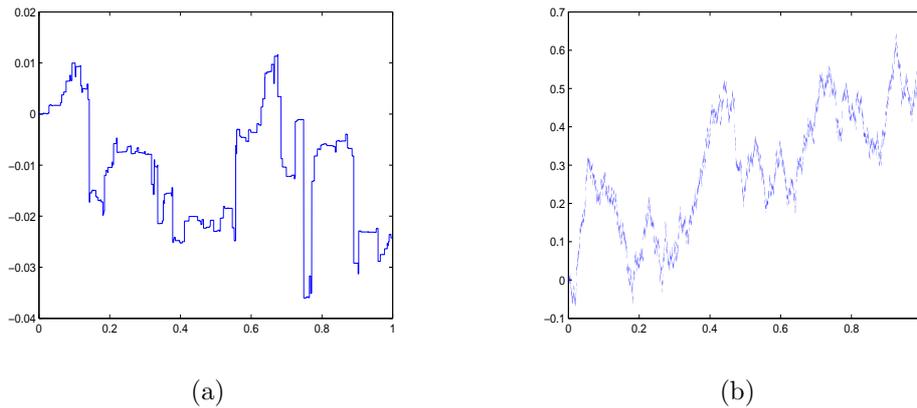


FIG. 3.3. On the left: The discretized trajectory of the variance gamma process with parameters $\theta = 0.05$, $\alpha = 0.042$, $\sigma = 0.25$. On the right: The trajectory of the variance gamma process with parameters $\theta = 0.001$, $\alpha = 0.45$, $\sigma = 0.0270$ estimated from the historical returns data of INTC from January 3rd 2000 to December 30th 2005.

3.3.3 Stable Lévy Processes

In this subsection, we look at a family of Lévy processes called *stable processes*. Stable processes are Lévy processes associated to the family of infinitely divisible distribution known as stable distribution.

Definition 3.3.1. A random variable X on \mathbb{R}^d is said to have a stable distribution if, for every $a > 0$ there exist $b(a) > 0$, and $c(a) \in \mathbb{R}^d$ such that

$$\Phi_X(z)^a = \Phi_X(zb(a))e^{iu \cdot z}, \quad \forall z \in \mathbb{R}^d. \quad (3.26)$$

It is said to have a strictly stable distribution if

$$\Phi_X(z)^a = \Phi_X(zb(a)), \quad \forall z \in \mathbb{R}^d. \quad (3.27)$$

For every stable distribution, there exist a constant $\alpha \in [0, 2]$ such that in equation (3.26), $b = a^{\frac{1}{\alpha}}$. This constant is called the *index of stability* and stable distribution with index of stability α are referred to α -stable distributions [22].

A selfsimilar Lévy process has strictly stable distribution at all times. Such processes are also called *strictly stable Lévy processes*. A strictly α -stable Lévy process satisfies

$$\left(\frac{X_{at}}{a^{\frac{1}{\alpha}}} \right)_{t \geq 0} \stackrel{d}{=} (X_t)_{t \geq 0}, \quad \text{for all } t > 0. \quad (3.28)$$

In general, α -stable Lévy process satisfies relation (3.28) up to a translation:

$$\text{for all } t > 0, \text{ there exists } C \in \mathbb{R}^d : (X_{at})_{t \geq 0} \stackrel{d}{=} \left(a^{\frac{1}{\alpha}} X_t + C(t) \right)_{t \geq 0}.$$

Proposition 3.3.1. *Stable distribution and Lévy processes.*

A distribution on \mathbb{R}^d is α -stable with $0 < \alpha < 2$ if and only if it is infinitely divisible with characteristic triplet $(0, \nu, \gamma)$ and there exists a finite measure λ on S , a unit sphere of \mathbb{R}^d , such that

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{dr}{r^{1+\alpha}}. \quad (3.29)$$

A distribution on \mathbb{R}^d is α -stable with $\alpha = 2$ if and only if it is Gaussian.

If X is a real-valued α -stable variable with $0 < \alpha < 2$, then its Lévy measure is given by

$$\nu(x) = \frac{A}{|x|^{\alpha+1}} 1_{x>0} + \frac{B}{|x|^{\alpha+1}} 1_{x<0}, \quad (3.30)$$

where A and B are positive constant.

The characteristic function at time 1 of a real-valued stable random variable X is given by

$$\begin{aligned}\Psi_X(u) &= \exp \left\{ -\sigma^\alpha |u|^\alpha \left(1 - i\beta \operatorname{sgn} u \tan \frac{\pi\alpha}{2} \right) + i\mu u \right\}, \quad \text{if } \alpha \neq 1, \\ \Psi_X(u) &= \exp \left\{ -\sigma |u| \left(1 + i\beta \frac{2}{\pi} \operatorname{sgn} u \log |u| \right) + i\mu u \right\}, \quad \text{if } \alpha = 1,\end{aligned}\quad (3.31)$$

where $\alpha \in [0, 2]$, $\sigma \geq 0$, $\beta \in [-1, 1]$, $\mu \in \mathbb{R}$ and $\operatorname{sgn} u$ is the sign of u defined by

$$\operatorname{sgn} u = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

A stable distribution in this parametrization is often denoted by $S_\alpha(\sigma, \beta, \nu)$. In this representation, σ is the scale parameter¹, μ is the shift parameter², α determines the shape of the distribution and β its skewness. When $\beta = 0$ and $\mu = 0$, X is said to have a symmetric stable distribution and the characteristic function is given by

$$\Phi_X(z) = \exp(-\sigma^\alpha |z|^\alpha).$$

The probability density of α -stable distribution is not known in closed form except for the following three cases:

- The Gaussian distribution $S_2(\sigma, 0, \mu)$ with density

$$\frac{1}{2\sigma\sqrt{\pi}} e^{-\frac{(x-\mu)^2}{4\sigma^2}}.$$

- The Cauchy distribution $S_1(\sigma, 0, \mu)$ with density

$$\frac{\sigma}{\pi(x-\mu)^2 + \sigma^2}.$$

- The Lévy distribution $S_{\frac{1}{2}}(\sigma, 1, \mu)$ with density

$$\left(\frac{\sigma}{2\pi}\right)^{\frac{1}{2}} \frac{1}{(x-\mu)^{\frac{3}{2}}} \exp\left\{-\frac{\sigma}{2(x-\mu)}\right\} 1_{x>\mu}.$$

The Gaussian and the Cauchy distributions are symmetric around their mean, while the Lévy distribution is centered on (μ, ∞) .

¹Note that it has nothing to do with the Gaussian component if $\alpha < 2$.

²When $\alpha \neq 1$ this is not true [22].

3.3.4 Tempered Stable Process

The tempered stable processes are obtained by multiplying the Lévy measure of the stable processes with a decreasing exponential function on each half real axis. After exponential softening, the small jumps keep their stable-like behavior while the big jumps become much less violent. The tempered stable processes are Lévy processes with no Gaussian components and their Lévy measures have a density of the form [22]

$$\nu_{TS}(x) = \frac{c_-}{|x|^{1+\alpha_-}} e^{\lambda_-|x|} 1_{x<0} + \frac{c_+}{|x|^{1+\alpha_+}} e^{\lambda_+|x|} 1_{x>0}, \quad (3.32)$$

where the parameters satisfy $c_{\pm} > 0$, $\lambda_{\pm} > 0$, and $\alpha_{\pm} < 2$.

The tempered stable process was first proposed by Koponen [41] under the name of *truncated Lévy flight*, and Shiryaev [68] remarked that it was misleading, and it was then replaced by *KoBoL* (see also [11], [12]). The tempered stable processes were studied in [22], [52], [62], [64] and they were used in financial modeling by [56], [70], [80].

Unlike stable processes which are defined for $\alpha_{\pm} > 0$, tempered stable processes are defined for any values of α_{\pm} less than 2. Different subclass of tempered stable processes are obtained by imposing different conditions on the parameters c_- , c_+ , α_- , and α_+ . The tempered stable process reduces to

- compound Poisson type if $\alpha_+ < 0$ and $\alpha_- < 0$,
- has trajectory of finite variation if $\alpha_+ < 1$ and $\alpha_- < 1$,
- is a subordinator if $c_- = 0$, $\alpha_+ < 1$, and the drift parameter is positive.

The tempered stable process can be represented as time-changed Brownian motion possibly with drift (cf. proposition 4.1 in [22]) if and only if $c_- = c_+$ and $\alpha_- = \alpha_+ \geq -1$. The case $c_- = c_+$ and $\alpha_- = \alpha_+$ was studied in Madan et al. [16] (see subsection 3.3.5 below) under the name of CGMY process with Lévy measure (3.34). The limiting case $\alpha_- = \alpha_+ = 0$ correspond to an infinite activity process. If in addition $c_- = c_+$, we obtain the variance gamma process discussed in subsection 3.3.2.

Because of the exponential tempering in the case of tempered stable process big jumps need not be truncated and one can use the truncation function $h(x) = x$. This gives the

following version of the Lévy-Khintchine formula

$$\mathbb{E}[e^{iuX_t}] = \exp t \left(iu\gamma_c + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux)\nu(dx) \right).$$

This form can be used because of exponential decay of the tails of Lévy measure and $\mathbb{E}(X_t) = \gamma_c t$. The characteristic function is computed by first considering the positive half of the Lévy measure and by supposing that $\alpha_{\pm} \neq 1$ and $\alpha_{\pm} \neq 0$.

$$\begin{aligned} \int_0^{\infty} (e^{iux} - 1 - iux) \frac{e^{-\lambda x}}{x^{1+\alpha}} dx &= \sum_{n=2}^{\infty} \frac{(iu)^n}{n!} \int_0^{\infty} x^{n-1-\alpha} e^{-\lambda x} dx = \sum_{n=2}^{\infty} \frac{(iu)^n}{n!} \lambda^{\alpha-n} \Gamma(n-\alpha) \\ &= \lambda^{\alpha} \Gamma(2-\alpha) \left\{ \frac{1}{2!} \left(\frac{i u}{\lambda} \right)^2 + \frac{2-\alpha}{3!} \left(\frac{i u}{\lambda} \right)^3 \right. \\ &\quad \left. + \frac{(2-\alpha)(3-\alpha)}{4!} \left(\frac{i u}{\lambda} \right)^4 + \dots \right\}. \end{aligned}$$

The expression in braces resembles the well-known power series

$$(1+x)^{\mu} = 1 + \mu x + \mu(\mu-1) \frac{x^2}{2!} + \dots$$

Comparing the two series we conclude that

$$\int_0^{\infty} (e^{iux} - 1 - iux) \frac{e^{-\lambda x}}{x^{1+\alpha}} dx = \lambda^{\alpha} \Gamma(-\alpha) \left\{ \left(1 - \frac{i u}{\lambda} \right)^{\alpha} - 1 + \frac{i u}{\lambda} \right\}. \quad (3.33)$$

The interchange of sum and integral and the convergence of power series used to obtain (3.33) can be justified if $|u| < \alpha$ but the resulting formula is extended via analytic continuation to other values of u such that $\mathcal{F}(u) > -1$ [22]. The power in (3.33) is computed by choosing a branch of z^{α} that is continuous in the upper half plane and maps positive half-line into positive half-line.

The characteristic function of the tempered stable process is given by

$$\phi_{X_t}(u) = \exp(i\psi(u)),$$

where the characteristic exponent ψ is given by

$$\begin{aligned} \psi(u) &= iu\gamma_c + \Gamma(-\alpha_+) \lambda_+^{\alpha_+} c_+ \left\{ \left(1 - \frac{i u}{\alpha_+} \right)^{\alpha_+} - 1 + \frac{i u \alpha_+}{\lambda_+} \right\} \\ &\quad + \Gamma(-\alpha_-) \lambda_-^{\alpha_-} \left\{ \left(1 + \frac{i u}{\lambda_-} \right)^{\alpha_-} - 1 - \frac{i u \alpha_-}{\lambda_-} \right\}. \end{aligned}$$

If $\alpha_+ = \alpha_- = 1$,

$$\begin{aligned}\psi(u) = & iu(\gamma_c + c_+ - c_-) + c_+(\lambda_+ - iu) \log\left(1 - \frac{iu}{\lambda_+}\right) \\ & + c_-(\lambda_- + iu) \log\left(1 + \frac{iu}{\lambda_-}\right),\end{aligned}$$

and, if $\alpha_+ = \alpha_- = 0$,

$$\begin{aligned}\psi(u) = & iu\gamma_c - c_+ \left\{ \frac{iu}{\lambda_+} + \log\left(1 - \frac{iu}{\lambda_+}\right) \right\} \\ & - c_- \left\{ -\frac{iu}{\lambda_-} + \log\left(1 + \frac{iu}{\lambda_-}\right) \right\}\end{aligned}$$

Unlike stable processes, tempered stable processes have all moments finite, including exponential moments of some order. The first moments are obtained by taking the derivatives of the characteristic exponent ψ (cf. proposition 3.13 in [22]). This gives us

$$\begin{aligned}\mu = \mathbb{E}[X_t] &= t\gamma_c, \\ V = \text{var}(X_t) &= t\Gamma(2 - \alpha_+)c_+\lambda_+^{\alpha_+-2} + t\Gamma(2 - \alpha_-)c_-\lambda_-^{\alpha_--2}, \\ S &= t\Gamma(3 - \alpha_+)c_+\lambda_+^{\alpha_+-3} - t\Gamma(3 - \alpha_-)c_-\lambda_-^{\alpha_--3}, \\ K &= t\Gamma(4 - \alpha_+)c_+\lambda_+^{\alpha_+-4} + t\Gamma(4 - \alpha_-)c_-\lambda_-^{\alpha_--4}.\end{aligned}$$

3.3.5 The CGMY Process

The CGMY model originates from Madan et al. [16] in order to consider a continuous time model which allows diffusions and jumps of both finite and infinite activity. This model improves and is a generalization of the variance gamma model studied by Madan and Seneta [48], Madan, Carr and Chang [49], and hyperbolic model by Eberlein, Keller, and Prause [25] where pure jump processes with infinite activity are considered.

As the variance gamma process is of infinite activity but finite variation, it is extended by adding a parameter that allows to control the fine structure of asset return distribution. The CGMY Lévy density is given by [16]

$$\nu_{CGMY}(x) = \begin{cases} C \frac{\exp(-G|x|)}{|x|^{1+Y}} & \text{for } x < 0 \\ C \frac{\exp(-M|x|)}{|x|^{1+Y}} & \text{for } x > 0, \end{cases} \quad (3.34)$$

where $C > 0$, $G \geq 0$, $M \geq 0$, $Y < 2$. The parameter Y allows us to gain flexibility in describing the fine structure of the stochastic process. For $Y < 0$, the Lévy process is of finite activity; for $0 \leq Y < 1$, it is of infinite activity but finite variation; for $1 \leq Y < 2$, the Lévy process is of infinite activity and infinite variation.

Choosing $Y = 0$ gives the variance gamma process with parameters

$$\begin{aligned} C &= \frac{1}{\alpha}, \\ G &= \frac{1}{\eta_-}, \\ M &= \frac{1}{\eta_+}. \end{aligned} \tag{3.35}$$

When $G = M$, the Lévy measure becomes that of the symmetric variance gamma process with C providing the control over the kurtosis [49]. When $G < M$, there are more negative jumps and the left tail of the distribution of $X(t)$ is heavier than the right tail, which is consistent with the risk-neutral distribution implied from the option prices [16]. The characteristic function of the CGMY process is given by (see appendix A.2 for a proof)

$$\mathbb{E} [\exp(iuX_{CGMY}(t))] = \exp(tC\Gamma(-Y) \{(M - iu)^Y - M^Y + (G + iu)^Y - G^Y\}),$$

which is infinitely divisible.

The moments of the CGMY process are given by [16]

$$\begin{aligned} \text{variance} &= C\Gamma(2 - Y) \left[\frac{1}{M^{2-Y}} + \frac{1}{G^{2-Y}} \right], \\ \text{skewness} &= \frac{C\Gamma(3 - Y) \left[\frac{1}{M^{3-Y}} + \frac{1}{G^{3-Y}} \right]}{(\text{variance})^{3/2}}, \\ \text{kurtosis} &= 3 + \frac{C\Gamma(4 - Y) \left[\frac{1}{M^{4-Y}} + \frac{1}{G^{4-Y}} \right]}{(\text{variance})^2}. \end{aligned}$$

Madan and Yor [79] showed that the CGMY process can be written as a Brownian motion time-changed by a one sided $\frac{Y}{2}$ -stable process. This representation allows us to simulate the CGMY process using the rejection method as given in [79]. This method requires to write the CGMY process as a subordinated Brownian motion and simulate the subordinator by rejection method (see detailed algorithm in chapter 5). Figure (3.4) depicts the trajectories of the CGMY process. On the left, the parameters were chosen randomly and are given by

$C = 0.7$, $G = 4.5$, $M = 3$, $Y = 0.5$; On the right, the parameters were estimated from asset returns of INTC over the period January 3rd 2000 to December 30th 2005 and are given by $C = 0.4815$, $G = 6.4331$, $M = 6.4386$, $Y = 0.2011$. Similarly to the variance gamma process, the same conclusion can be made for the CGMY process.

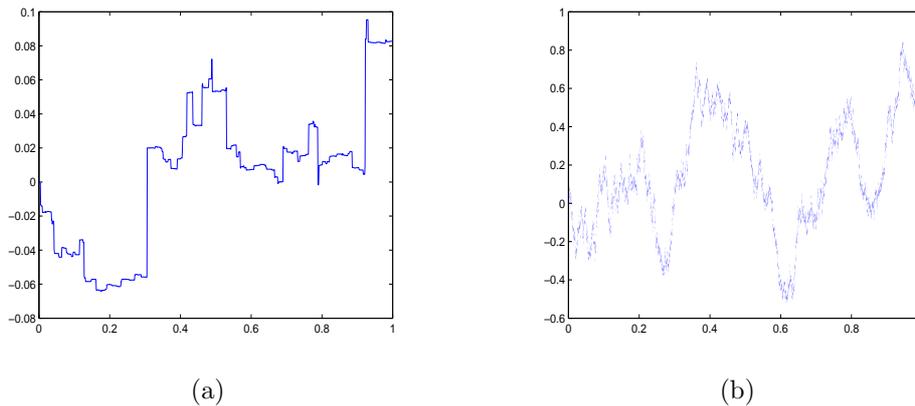


FIG. 3.4. Typical trajectories of the CGMY process. On the left the parameters are $C = 0.7$, $G = 4.5$, $M = 3$, $Y = 0.5$ and were chosen randomly. On the right, $C = 0.4815$, $G = 6.4331$, $M = 6.4386$, $Y = 0.2011$ and were estimated from the asset returns of INTC from January 3rd 2000 to December 30th 2005.

3.4 Examples of Exponential Lévy Models

In this section, we give examples of exponential Lévy models where the Lévy process is a pure jump process of infinite activities. In these models, the jump structure is already rich enough to give a nontrivial small time behavior [16] which eliminate the need of the Brownian motion component and it has been argued [16], [29] that such models give a more realistic description of the asset prices. Moreover, these models are analytically tractable compared to jump-diffusion models, as many of them can be constructed via Brownian subordination. We will use the name of the Lévy process to denote the corresponding exponential Lévy model.

3.4.1 Variance Gamma Model

The variance gamma model corresponds to the exponential Lévy model obtained by exponentiating the variance gamma process, i.e, the Lévy process in (3.4) is the variance gamma process.

The dynamics of the variance gamma stock price process under the risk-neutral measure is given by

$$S(t) = S(0) \exp((r + \omega)t + X(t, \sigma, \alpha, \theta)), \quad (3.36)$$

where r is the continuously compounded interest rate and ω is used to make the stock price process a martingale. To have $\mathbb{E}[S_t] = S_0 \exp(rt)$, we have to set

$$\omega = \frac{1}{\alpha} \log(1 - \theta\alpha - \sigma^2\alpha/2), \quad (3.37)$$

that is,

$$\begin{aligned} \mathbb{E}[S_t] &= S_0 \exp((r + \omega)t) \mathbb{E}[\exp(X_t)] \\ &= S_0 \exp((r + \omega)t) \exp\left(-\left(\frac{1}{\alpha} \log\left(1 - \theta\alpha + \frac{1}{2}\sigma^2\alpha\right)\right)t\right) \\ &= S_0 \exp\left(rt + \frac{1}{\alpha} \log\left(1 - \theta\alpha - \frac{1}{2}\sigma^2\alpha\right)t\right) \exp\left(-\left(\frac{1}{\alpha} \log\left(1 - \theta\alpha - \frac{1}{2}\sigma^2\alpha\right)\right)t\right) \\ &= S_0 \exp(rt). \end{aligned}$$

The second equality holds since $\mathbb{E}[\exp(X_t)] = \exp(\psi(-i)t)$ and $\psi(t)$ is given by (3.18).

The density function of the log-stock price process is known in closed form and is expressed in terms of the modified Bessel function of the second type [49].

Theorem 3.4.1. *The density for the log price relative $z = \ln S(t)/S(0)$ when prices follow the risk neutral variance gamma process is given by*

$$g(z) = \frac{2 \exp(\theta x/\sigma^2)}{\alpha^{t/\alpha} \sqrt{2\pi} \Gamma(t/\alpha)} \left(\frac{x^2}{2\sigma^2/\alpha + \theta^2}\right)^{\frac{t}{2\alpha} - \frac{1}{4}} \times K_{\frac{t}{\alpha} - \frac{1}{2}}\left(\frac{1}{\sigma^2} \sqrt{x^2(2\sigma^2/\alpha + \theta^2)}\right), \quad (3.38)$$

where K is the modified Bessel function of the second type and

$$x = z - rt - \frac{t}{\alpha} \ln(1 - \theta\alpha - \sigma^2\alpha/2).$$

The characteristic function of the log price is given by

$$\phi_{\ln(S_t)}(u) = \exp[iu(\ln(S_0) + (r + \omega)t)](1 - i\theta\alpha u + \frac{1}{2}\sigma^2 u^2 \alpha)^{-t/\alpha}. \quad (3.39)$$

Madan, Carr and Chang [49] also give the analytical formula for European call option price on a stock, but we will use in chapter 4 the risk-neutral characteristic function to calculate the variance gamma option price by Fast Fourier Transform (FFT) method.

3.4.2 The CGMY Model

Similarly to the variance gamma model, the CGMY model is obtained by exponentiating the CGMY process. The CGMY model was used to investigate the level of activity and variation of the continuous time process underlying the stock returns in equity and index [16], [29]. The dynamics of the asset price process under the risk-neutral measure is given by

$$S(t) = S(0) \exp((r + \omega)t + X(t)), \quad (3.40)$$

where r is the continuously compounded interest rate, $X(t)$ is the CGMY process, and ω is a parameter used to make the risk-neutral stock price process a martingale. To have $\mathbb{E}[S_t] = S_0 \exp(rt)$, we have to set

$$\omega = -C\Gamma(-Y)((M - 1)^Y - M^Y + (G + 1)^Y - G^Y). \quad (3.41)$$

The risk-neutral characteristic function of the log price process is given by

$$\phi_{\ln(S)}(u, t) = \exp(iu(\ln(S(0)) + (r + \omega)t)) \phi(u, C, G, M, Y). \quad (3.42)$$

The explicit formula for option price does not exist because analytical form of the density function is not known. Therefore, we will use equation (3.42) to evaluate the CGMY model by FFT method in chapter 4.

In conclusion, exponential models are tractable because they allow to capture the statistical properties and the jump behavior observed from the market data. When the Levy process in exponential is a pure jump process with infinite activity, it allows to capture both frequently small and rare large jumps which eliminates the need of the diffusion component. Such models give more realistic description of the price process at various time scale.

Chapter 4

Numerical Implementation

In this chapter, we apply exponential Lévy models to financial market data. We showed in section 3.1 that the normal distribution is a very poor model to fit log returns of financial assets. To achieve a better fit we replace the normal distribution by the variance gamma and the CGMY distributions. We introduce stock price models based on the variance gamma and the CGMY processes in order to price financial derivatives. We follow the Carr and Madan [15] Fast Fourier Transform (FFT) technique to price the options.

4.1 Pricing European Call Options via the FFT Method

One of the main problems in mathematics of finance is the pricing and hedging of a contingent claim. In exponential Lévy models, explicit formulas for option pricing are not available in general because the probability density function is not known in closed form. However, the availability of the characteristic function allows to price the options using the Fourier transform methods. Carr and Madan[15] provide us a pricing method called Fast Fourier Transform (FFT), which is based on the characteristic function of the process under consideration. As for hedging, perfect hedging is impossible due to the jumps in the prices [23], [22]. However, market participants can hedge the options by minimizing the risk associated to the jumps. Because of the time constraint we only discuss the pricing problem in this dissertation.

4.1.1 The Fourier Transform of Option Price

Denote k the log of the strike price K , and $C_T(k)$ the price of a European call option with maturity T . Let the risk-neutral density of the log price s_T be $q_T(s)$. The characteristic function of this density is defined by

$$\Phi_T(u) = \int_{-\infty}^{\infty} e^{ius} q_T(s) ds. \quad (4.1)$$

Writing $C_T(k)$ in terms of the risk-neutral density $q_T(s)$ we get

$$C_T(k) = \int_k^{\infty} e^{-rT} (e^s - e^k) q_T(s) ds. \quad (4.2)$$

The function C_T is not square integrable because $C_T \rightarrow 0$ as $K \rightarrow -\infty$. Now, consider its modification

$$c_T(k) = \exp(\alpha k) C_T(k). \quad (4.3)$$

which is square integrable for a suitable $\alpha > 0$. The choice of α depends on the model for S_t . The Fourier transform of $c_T(k)$ is then defined as

$$\Psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} c_T(k) dk. \quad (4.4)$$

One would like to express $\Psi_T(v)$ in terms of the characteristic function Φ_T and then obtain call price numerically using the inverse transform.

$$C_T(k) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_T(v) dv = \frac{\exp(-\alpha k)}{\pi} \int_0^{\infty} e^{-ivk} \psi_T(v) dv. \quad (4.5)$$

From (4.2) and (4.3) we can write the expression for $\psi_T(v)$ as

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} \int_k^{\infty} e^{\alpha k} (e^s - e^k) q_T(s) ds dk. \quad (4.6)$$

Applying the interchange of order of integration we get

$$\psi_T(v) = \frac{e^{-rT} \phi_T(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}. \quad (4.7)$$

The call option price can then be obtained by substituting (4.7) into (4.5).

4.1.2 Fast Fourier Transform of Option Price

Fast Fourier Transform (FFT) is an efficient algorithm to compute the sum

$$\omega(k) = \sum_{j=1}^N e^{-i\frac{2\pi}{N}(j-1)(k-1)} x(j) \quad \text{for } j = 1, \dots, N, \quad (4.8)$$

where N is a power of 2. FFT is a commonly employed to compute discrete approximation technique of Fourier transform used to reduce the computational complexity. We want to approximate relation (4.5) using discrete Fourier transform as in (4.8). Using the trapezoid rule, (4.5) can be written as

$$C_T(k) \approx \frac{\exp(-\alpha k)}{\pi} \sum_{j=1}^N e^{-iv_j k} \psi_T(v_j) \eta, \quad (4.9)$$

where $v_j = \eta(j-1)$. The effective upper limit of the integration is now

$$a = N\eta.$$

Choosing a regular spacing λ , and set a bound on the log strike to range between $-b$ and b , we get N values of k given by

$$k_u = -b + \lambda(u-1) \quad \text{for } u = 1, \dots, N. \quad (4.10)$$

where

$$b = \frac{1}{2}N\lambda.$$

Our formula (4.9) can now be written as

$$C_T(k) \approx \frac{\exp(-\alpha k)}{\pi} \sum_{j=1}^N e^{-i\lambda\eta(j-1)(u-1)} e^{ibv_j} \psi_T(v_j) \eta. \quad (4.11)$$

The fast Fourier transform is applied by setting

$$\lambda\eta = \frac{2\pi}{N}. \quad (4.12)$$

Choosing small value of η , we get a fine integration but with few strikes lying in the desired region near the stock price. Carr and Madan [15] suggest to use Simpson's weighting rule to obtain an accurate integration with large η . Then our option price formula becomes

$$C_T(k) \approx \frac{\exp(-\alpha k)}{\pi} \sum_{j=1}^N e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{ibv_j} \psi_T(v_j) \frac{\eta}{3} [3 + (-1)^j - \delta_{j-1}], \quad (4.13)$$

where δ_n is the Kronecker delta function that is unit for $n = 0$ and zero otherwise.

4.2 Datasets and Parameter Estimation

We work on a dataset drawn from CRSP and contain the daily data on the six component companies of S&P 500 with ticker names INTC, IBM, AMZN, DELL, FDX, and ABC. The data were downloaded from yahoo finance and cover the period from January 3rd 2000 to December 26th 2008. This resulted in 2254 observations in each time series data. The log-returns was calculated using the adjusted daily closing prices. We then evaluate the likelihood of observing these data on the assumption that the statistical processes are parametrized by the variance gamma and the CGMY model with parameters (σ, α, θ) and (C, G, M, Y) respectively.

Maximum likelihood requires to have the density function of the process under consideration. The density function of the variance gamma can be found analytically or by FFT. As for the CGMY process, there is no analytical form of the density function. However, the analytical form of the characteristic function for both processes is available. We use the FFT algorithm to convert the characteristic function into the density function. To obtain numerical density values at fine grid of realization, a large number N is needed for the FFT. We used $N = 16384$ to obtain a return spacing of 0.00153398. We then map the actual data to the grids by grouping the actual realizations into different bins that match the grids of the FFT and assign the same likelihood for realizations within the same bin [43].

To obtain more accurate results from the MLE, an efficient set of initial parameters is required. For the variance gamma, we recall the four moments given by

$$\begin{aligned}\mu &= \theta t, \\ V &= (\theta^2 \alpha + \sigma^2) t, \\ S &= \frac{(2\theta^3 \alpha^2 + 3\sigma^2 \alpha \theta) t}{((\theta^2 \alpha + \sigma^2) t)^{\frac{3}{2}}}, \\ K &= \frac{(3\alpha \sigma^4 + 12\theta^2 \sigma^2 \alpha^2 + 6\theta^4 \alpha^3) t + (3\sigma^4 + 6\sigma^2 \theta^2 \alpha + 3\theta^4 \alpha^2) t^2}{((\theta^2 \alpha + \sigma^2) t)^2}.\end{aligned}$$

Now assume θ to be small, thus ignoring $\theta^2, \theta^3, \theta^4$, we obtain

$$S = \frac{3\alpha\theta}{\sigma t}.$$

$$K = 3\left(1 + \frac{\alpha}{t}\right).$$

Solving for α, σ, θ , we obtain

$$\begin{aligned}\sigma &= \sqrt{\frac{V}{t}}, \\ \alpha &= \left(\frac{K}{3} - 1\right)t, \\ \theta &= \frac{S\sigma t}{3\alpha}.\end{aligned}\tag{4.14}$$

Then, the initial guess for the parameters is given by equation (4.14). We report the estimated variance gamma parameters for the six names in table (4.1).

Ticker	θ	α	σ
INTC	0.0010	0.6331	0.0285
IBM	0.0012	0.4878	0.0263
AMZN	-0.0012	0.6011	0.0570
DELL	0.0013	0.5602	0.0380
FDX	-0.0024	0.6234	0.0220
ABC	0.0005	0.5370	0.0473

TABLE. 4.1. VG parameters estimation on the six names.

As for the CGMY process, we recognize the relationship between the CGMY parameters with that of the variance gamma process with parameters (σ, α, θ) given by (see [16])

$$C = \frac{1}{\alpha},\tag{4.15}$$

$$G = \left(\sqrt{\frac{\theta^2\alpha^2}{4} + \frac{\sigma^2\alpha}{2}} - \frac{\theta\alpha}{2}\right)^{-1},\tag{4.16}$$

$$M = \left(\sqrt{\frac{\theta^2\alpha^2}{4} + \frac{\sigma^2\alpha}{2}} + \frac{\theta\alpha}{2}\right)^{-1}.\tag{4.17}$$

The initial parameters are obtained by substituting the equalities in (4.14) into equations (4.15), (4.16), (4.17). We fixed Y to 0.5 and the estimated parameters are reported in table (4.2).

We also plot the empirical density of the log-returns of the six names, that is, a Gaussian kernel density estimation and compare with the density of the fitted normal distribution,

Ticker	C	G	M	Y
INTC	0.3810	6.9081	5.6916	0.30186
IBM	0.4999	6.0057	6.7464	0.2006
AMZN	0.3057	4.0150	4.5385	0.2796
DELL	0.4025	6.4015	6.2620	0.2402
FDX	0.4416	8.1074	7.3565	0.3200
ABC	0.3331	4.6442	4.0511	0.4050

TABLE. 4.2. CGMY parameters estimation on the six names.

variance gamma distribution, and CGMY distribution. In figure (4.1) we plot the Gaussian kernel density estimator (dots line), the fitted normal density (red line), and the fitted CGMY density (green line). In figure (4.2) the fitted variance gamma density (green line) is compared with the Gaussian kernel density (dots line) and the fitted normal density (red line).

From the figures, it is clear that the empirical distribution puts more mass around zero (it is leptokurtic). This tells us that, for most of the time, stock prices do not move so much. The variance gamma and the CGMY distribution capture this effect much better than the normal distribution.

4.3 Goodness of Fit

In order to assess the goodness of fit of the variance gamma and the CGMY distributions to the option data, we use the quantile-quantile (QQ)-plot. The QQ-plot of an ordered sample $y = y_1 \leq y_2 \leq \dots \leq y_n$ provides a way of visually assessing the fit of a distribution to the sample data. A QQ-plot of a sample of n points, plots for every $j = 1, \dots, n$ the empirical $((j - \frac{1}{2})/n)$ -quantile of the data against the $((j - \frac{1}{2})/n)$ -quantile of the fitted distribution. If the plotted points lie roughly on the line $y = x$, then the compared distribution fits the data well. In order to compute the quantiles, we need the inverse of the distribution function $F(x)$. We numerically calculate the distribution function by FFT as follows:

Theorem 4.3.1. *Let X be a random variable with characteristic function $\phi(x)$ and let $e^{-\alpha x}$ be the dampening factor. X has density function $f(x)$ and distribution function $F(x)$.*

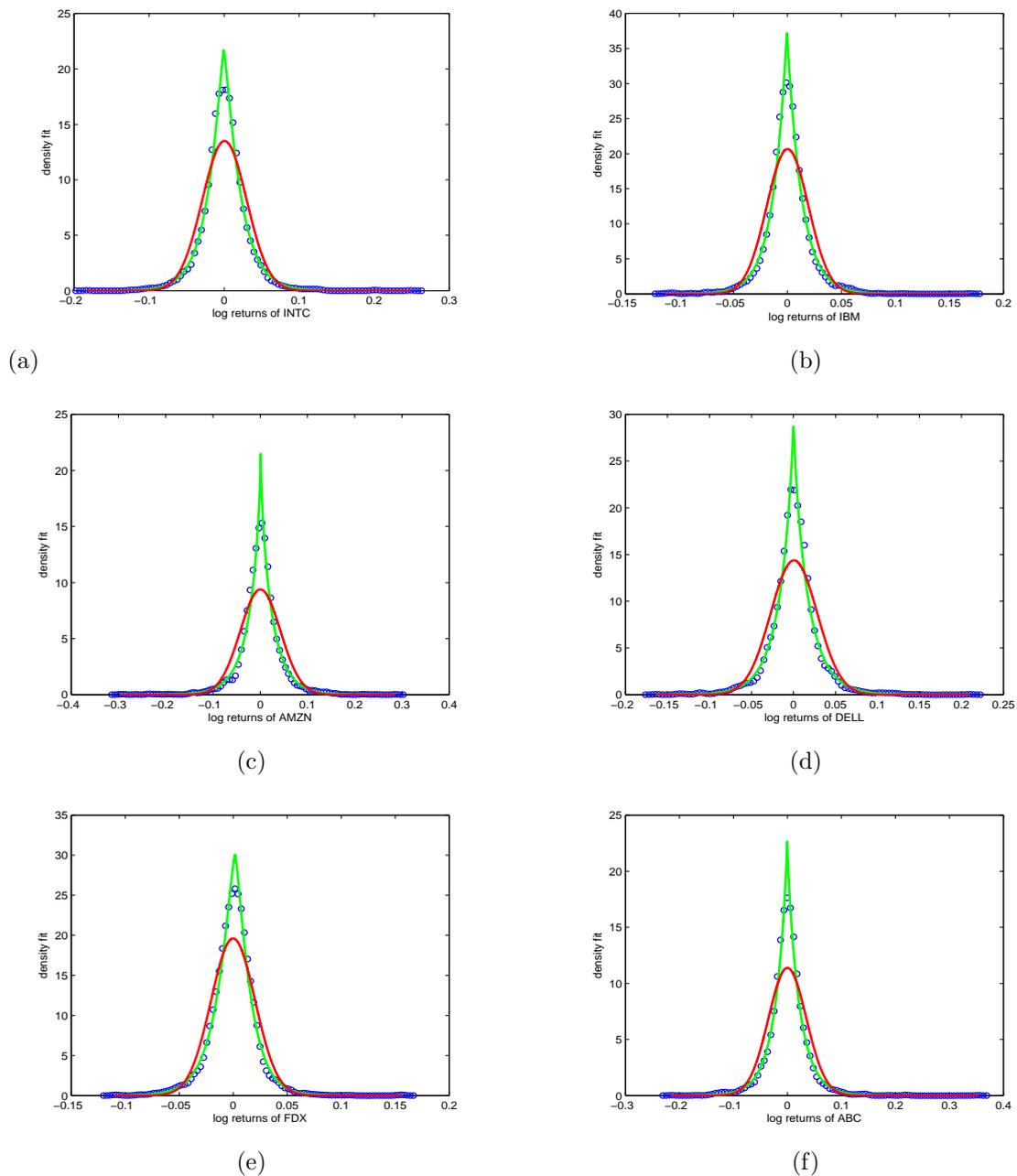


FIG. 4.1. The empirical density (dots line), fitted normal density (red line), and the fitted CGMY density (green line).

Then, the c.d.f. function $F(x)$ is given by:

$$F(x) = \frac{e^{\alpha x}}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \frac{\phi(u + i\alpha)}{\alpha - iu} du.$$

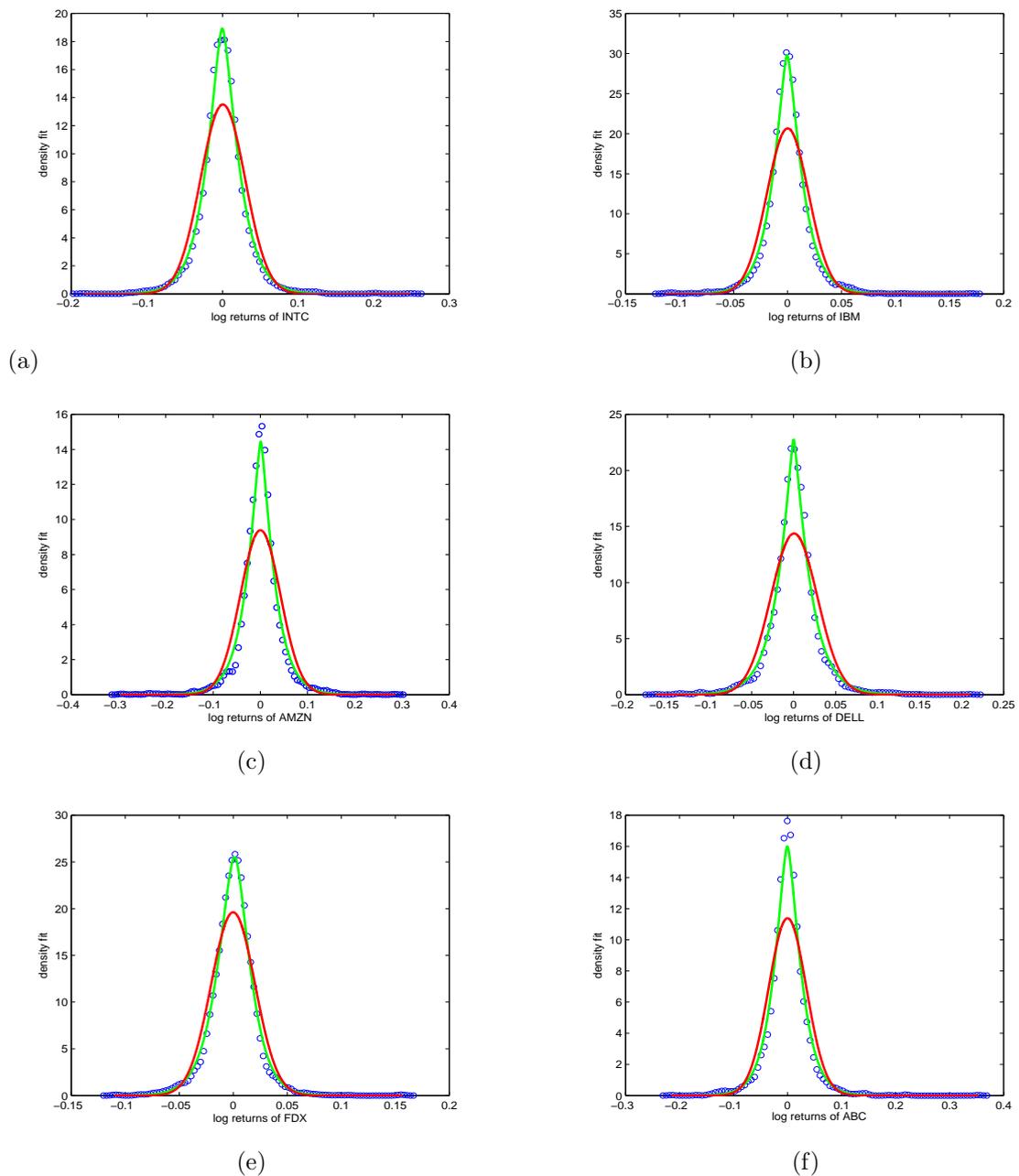


FIG. 4.2. The empirical density (dots line), fitted normal density (red line), and the fitted variance gamma density (green line).

Proof. For the proof of this theorem see appendix A.3. □

Figures (4.3), (4.4), and (4.5) depict the QQ-plot of the variance gamma distribution and that of the CGMY distribution compared with the normal distribution respectively.

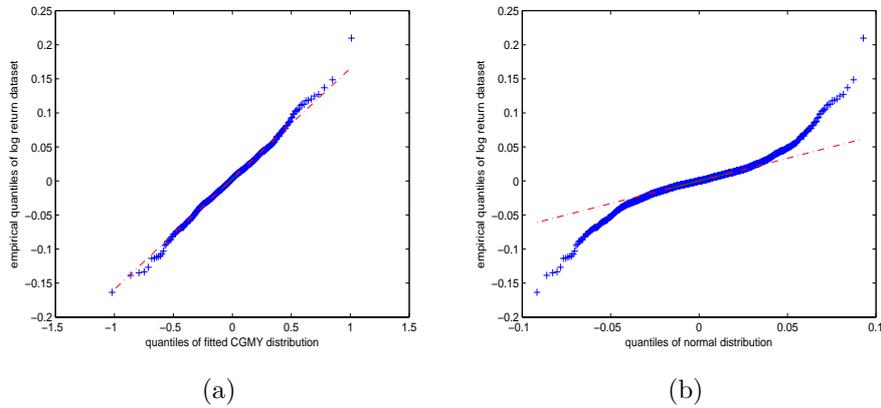


FIG. 4.3. On the left: The empirical quantiles compared to those for the VG distribution on DELL. On the right: The empirical quantiles compared to those for the normal distribution on DELL.

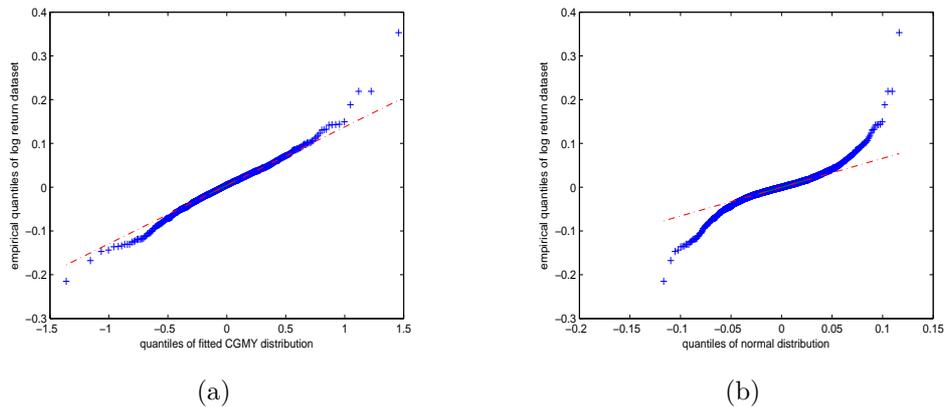


FIG. 4.4. On the left: The empirical quantiles compared to those for the CGMY distribution on INTC. On the right: The empirical quantiles compared to those for the normal distribution on INTC.

For the model based on the normal distribution, the deviation from the straight line is clearly seen (right panel). The fact that there is a strong deviation from the line $y = x$ in the small quantiles and the large quantiles shows that the empirical distribution has fatter tails than the normal distribution. This problem almost completely disappears when we use the variance gamma distribution or the CGMY distribution to fit the data (left panel). They show that the vast improvement is achieved by replacing the class of normal distributions with the class of the variance gamma or the CGMY distributions.

We also use the Kolmogorov-Smirnov and the Anderson-Darling [1] statistic goodness of

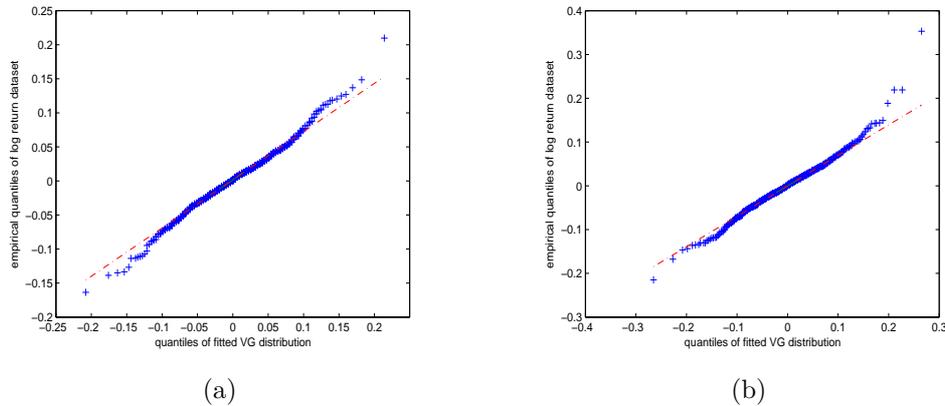


FIG. 4.5. On the left: The empirical quantiles compared to those for the variance gamma distribution on DELL. On the right: The empirical quantiles compared to those for the variance gamma distribution on INTC.

fit tests to test the null hypothesis that the log-returns are normally distributed. The Kolmogorov-Smirnov test uses the Kolmogorov-Smirnov distance of the empirical distribution F_{emp} and the fitted distribution F_{fit} to test whether x was sampled from the distribution F_{fit} . It rejects the hypothesis if the distance is too large.

The Kolmogorov-Smirnov distance (K-S) is given by

$$\text{K-S} = \max_{x \in \mathbb{R}} | F_{emp}(x) - F_{fit}(x) |,$$

while the Anderson-Darling statistic (A-D) is given by

$$\text{A-D} = \max_{x \in \mathbb{R}} \frac{| F_{emp}(x) - F_{fit}(x) |}{\sqrt{F_{fit}(x)(1 - F_{fit}(x))}}.$$

The compared values of the K-S and A-D for normal distribution, the variance gamma distribution, and the CGMY distribution are reported in table (4.3). As you can see, the variance gamma and the CGMY distributions perform better than the normal distribution.

We used the estimated parameters to compare the sample path of the asset prices under the Black-Scholes model, variance gamma model, and CGMY model for IBM. We first plotted the evolution of the logarithm of the stock price of IBM over the period January 3rd 2000 to December 26th 2008 (the left graph of figure (4.6)). From the graph, one can see that the asset prices have jumps of different sizes with small jumps occurring more often than large jumps. In the right graph of figure (4.6), the asset prices are simulated

Ticker	<i>Normal</i>		<i>CGMY</i>		<i>VG</i>	
	KS	DA	KS	DA	KS	DA
INTC	0.2785	0.6071	0.0708	0.1362	0.0216	0.1028
IBM	0.3377	0.7082	0.0795	0.1466	0.0193	0.0883
AMZN	0.2505	0.2848	0.0927	0.2116	0.0286	0.1042
DELL	0.2974	0.7232	0.0905	0.1289	0.0195	0.0922
FDX	0.3469	0.6077	0.0598	0.1456	0.0339	0.1250
ABC	0.3448	0.6207	0.0818	0.4159	0.0261	0.1069

TABLE. 4.3. Kolmogorov-Smirnov distance and Anderson-Darling statistic.

under the Brownian motion assumption in the Black-Scholes model. One can clearly see that the sample path looks like the graph of a continuous function. This means that, in the Black-Scholes framework the asset prices do not fluctuate too much. In the setting of variance gamma (left panel of figure (4.7)) and CGMY processes (right panel of figure (4.7)), the asset prices present jumps with small jumps occurring more often than large jumps reflecting what we observed from the evolution of the market prices.

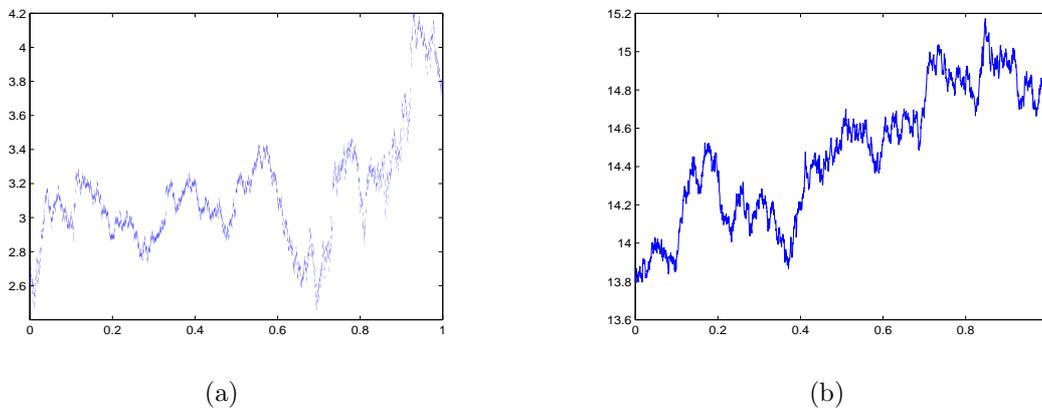


FIG. 4.6. On the left: The evolution of the logarithm of the stock price of IBM over the period January 3rd 2000 to December 26th 2008. On the right: The sample path of the asset prices of IBM with parameters fitted by MLE for the Black-Scholes model over the same period.

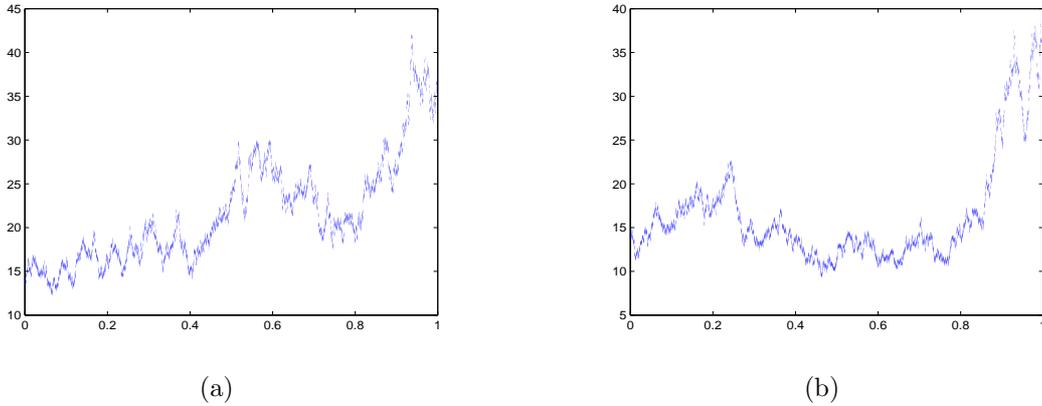


FIG. 4.7. On the left: The sample path of the asset prices of IBM with parameters fitted by MLE for the variance gamma model over the period January 3rd 2000 to December 26th 2008.. On the right: The sample path of the asset prices of IBM with parameters fitted by MLE for the CGMY model on IBM over the same period.

4.4 Calibration Procedure

The exponential Lévy models belong to the incomplete market models. There is no a unique equivalent martingale measure under which the discounted asset price process is a martingale. In a market, where options are traded on exchange, prices are available and can be used as source of information for selecting \mathbb{Q} .

In order to calibrate the risk-neutral parameters, we use the observed option prices from the market. One of the most popular calibration methods is the method of least squares. The idea is, given the observed market prices $(C_i)_{i=1}^N$, at time $t = 0$ with different strikes K_i and maturities T_i , and the risk-neutral model prices C^θ , find the parameters value θ that minimize the sum of the quadratic deviation between these prices, that is,

$$\phi^* = \arg \min \sum_{i=1}^N |C^\theta(T_i, K_i) - C_i|^2, \quad (4.18)$$

and the optimization being done by the gradient-based method. Different statistics (see Schoutens [66]) can be used to measure the quality of fit. To our knowledge no one has mentioned the existence of a better statistic measure than others. We choose the root mean square error (rmse) given by

$$rmse = \sqrt{\sum_{\text{derivatives}} \frac{(\text{Market price} - \text{Model price})^2}{\text{Number of derivatives}}}. \quad (4.19)$$

4.4.1 Calibration Results

We fitted the model on a data set of vanilla options on INTC, IBM, AMZN, DELL, FDX, and ABC taken at July 9th 2009. The option data were downloaded from yahoo finance and the maturity time is one month. The risk-neutral parameters for the variance gamma and the CGMY models are reported in table (4.4) and (4.5) respectively. Different set of parameters for the same model were obtained because each stock has its own set of strike prices and option prices.

Ticker	θ	α	σ
INTC	0.0010	0.2548	0.0200
IBM	0.0105	0.3625	0.0253
AMZN	0.0002	0.4927	0.0362
DELL	-0.0053	0.1650	0.0192
FDX	0.0009	0.1288	0.0340
ABC	0.0016	0.7070	0.0173

TABLE. 4.4. Calibrated parameters for VG model.

Ticker	C	G	M	Y
INTC	0.0551	4.1050	6.5521	0.2609
IBM	0.0683	1.1187	1.4567	0.0136
AMZN	0.4118	1.1482	2.5430	0.2697
DELL	0.0012	1.1318	2.5572	0.4743
FDX	1.5602	8.1049	9.5521	0.0007
ABC	0.0561	4.1050	6.5521	0.2610

TABLE. 4.5. Calibrated parameters for CGMY model .

Figures (4.8) and (4.9) depicts the market prices and model prices computed using the method described in subsection 4.1. As you can see, the variance gamma and the CGMY model fit the market prices at the given maturity very well.

We also plotted the implied volatility surface for the variance gamma and the CGMY models. Figure (4.10) depicts the market implied volatility surface (on the left), and the implied volatility surface as a function of strike price and maturity for the variance gamma models. In figure (4.11), we plotted the profile of the implied volatility surface as a function of strike price and maturity in the CGMY model. The empirical observations of the implied

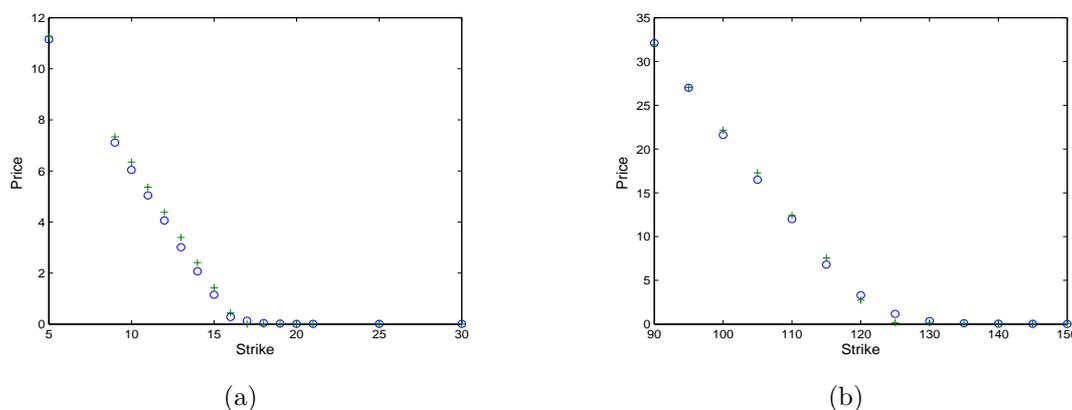


FIG. 4.8. The VG model calibration. Plus signs denote model price, o signs market prices.

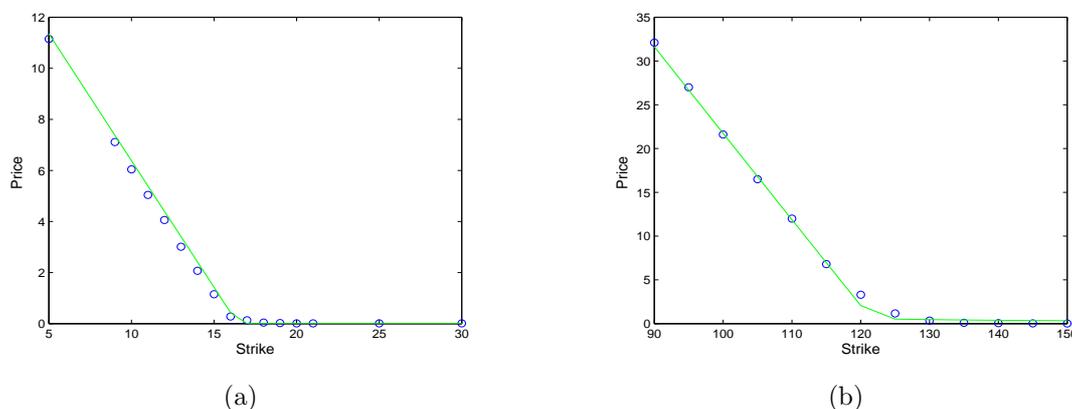


FIG. 4.9. The CGMY model calibration. Solid line denote model prices, o signs market prices.

volatility surface show that it flattens out as the maturity increases. As you can see from implied volatility in the variance gamma and the CGMY models, the implied volatility surface of the skew flattens with the maturity.

In conclusion, the variance gamma and the CGMY models allow us to take into account the jump risks observed from the financial market data. The calibrated parameters allow us to reproduce the sample path of the stock prices. Fitted to the market data, the variance gamma and CGMY distributions capture the kurtosis and skewness observed from the empirical distribution of the time series data than the normal distribution suggested in the Black-Scholes model. The calibration results to the option data indicated that, at given maturity time, the variance gamma model and the CGMY model fit the option data very

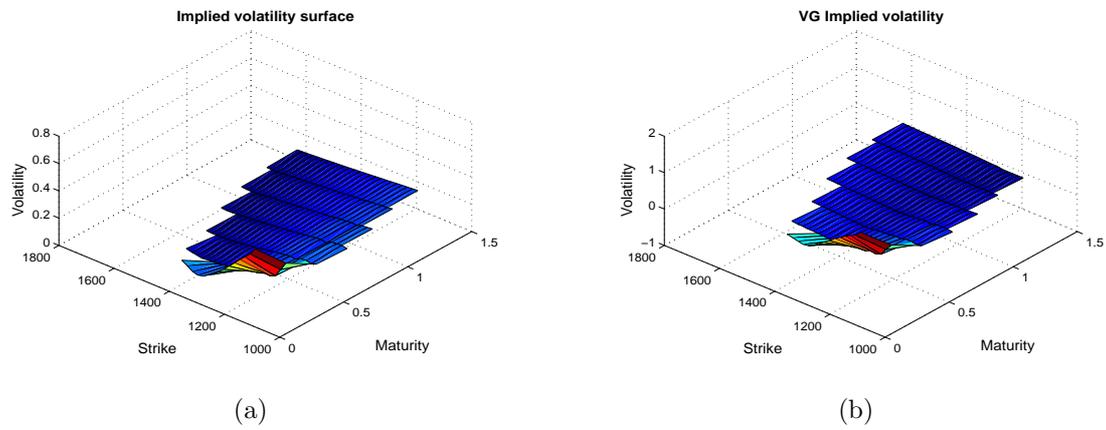


FIG. 4.10. On the left: The market implied volatility surface. On the right: Implied volatility surface as a function of strike price and maturity for VG model.

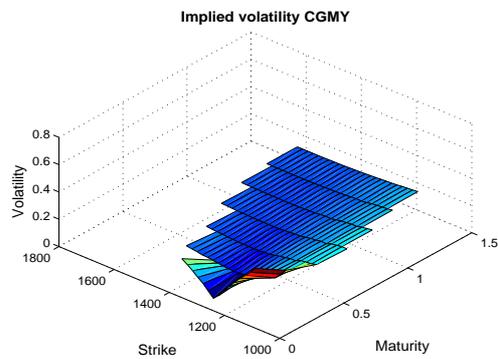


FIG. 4.11. Implied volatility surface as a function of strike price and maturity for CGMY model.

well.

Chapter 5

Estimation and Simulation of Lévy Processes

Pricing options and other financial derivatives is one of the main issues in financial mathematics. For some financial derivatives, analytical pricing solutions are not available and one has to rely on numerical methods. The most used numerical method is the Monte Carlo simulation. Monte Carlo simulation has an advantage of being flexible compare to other numerical method. Moreover, it serves as the only method of simulation in higher dimension and can be used to price derivatives with complicated structures [81]. Good sources for introduction to Monte Carlo techniques are Glasserman [32] and Gentle [31]. One can also use partial integral-differential equation (PIDE) methods, but these methods become infeasible as the dimensions of the problem grows [22].

For the valuation of exotic options such as Asian options, Lookback options, Barrier options amongst other (where the payoff depends explicitly on the values of the underlying assets at multiple dates), the knowledge of the entire asset path is required. This requires to be able to simulate the entire asset path, meaning that one should be able to simulate the chosen process to model the dynamics of the asset prices. In the setting of exponential Lévy models, one should be able to simulate Lévy processes.

The simulation of Lévy processes depends on the type of process you want to simulate. The compound Poisson process is the only Lévy process with piecewise constant sample path (cf. proposition 3.3 in [22]) and it is of finite variation. Its sample paths can be

simulated exactly without any discretization error by just simulating a finite number of jump times and jump sizes [22]. If the law of increments of the Lévy process is known, one simulates the trajectory at discrete times without any approximation. When the law of increments of the Lévy process is not known explicitly, one can approximate the Lévy process by a compound Poisson process with jump distribution proportional to $\nu^\epsilon(x) = \nu(x)1_{|x|>\epsilon}$, $\epsilon > 0$. This approximation consists in truncating the jumps smaller than ϵ . However, this approximation converges slowly when the jumps of the Lévy process are highly concentrated around zero [22]. When the precision of the Poisson approximation is low, one can improve it by re-normalizing the small jumps properly. In this case they behave like a Brownian motion [2].

Another method is to simulate Lévy processes by series representations [61]. Series representations can be thought as a more convenient way of approximating a Lévy process as a compound Poisson process. In order for the series to converge it is necessary that the magnitude of the jumps decreases as the time increases.

If the Lévy process is of infinite activity, the Lévy process can be simulated using random walk approximations [63]. Let $(X(t))_{0 \leq t < T}$ be a Lévy process with $\sigma = 0$ and an infinite Lévy measure ν . For $n \geq 1$, put $h = \frac{T}{n}$ and generate the increments $\Delta_j^h X = X(jh) - X((j-1)h)$ as i.i.d. random variables with distribution $P_h(\cdot) = \mathbb{P}[X(h) \in \cdot]$, $j = 1, \dots, n-1$. Then, the process $(X^h(t))_{0 \leq t < T}$ defined by

$$X^h(t) = \begin{cases} 0 & \text{if } 0 \leq t < h \\ \Delta_1^h X + \dots + \Delta_j^h X & \text{if } jh \leq t < (j+1)h. \end{cases}$$

is a random walk approximation to $(X(t))_{0 \leq t < T}$.

Random walk approximation requires one to be able to simulate $X(h)$. The disadvantage of this method is that one cannot precisely identify the location and the magnitude of the large jumps [63]. Another complication is that the simulation of $X(h)$ may be computationally expensive.

In this chapter, we will focus on the simulation algorithm for the variance gamma process and the CGMY process. For simplicity, we consider the case of variance gamma and CGMY processes defined as time-changed Brownian motion. The benefit of viewing a Lévy process as time-changed Brownian motion with respect to simulation is that one

avoids dealing directly with the Lévy density which might be difficult to sample from [81]. Other algorithms for simulating variance gamma process (defined as the difference of two gamma processes) see [81], [78] and for the CGMY process see [81]. For more on different methods for simulating Lévy processes we refer the reader to the book by Cont and Tankov [22].

5.1 Simulation of Variance Gamma Process

Variance gamma process was originally defined as a Brownian motion time-changed by a gamma process, that is,

$$X(t; \sigma, \alpha, \theta) = \theta\gamma(t; 1, \alpha) + \sigma B(\gamma(t; 1, \alpha)), \quad (5.1)$$

where $\gamma(t; 1, \alpha)$ is the gamma process and $B(t)$ is a Brownian motion. This representation allows us to simulate the variance gamma process as a Brownian motion sampled by a random time given by gamma random variate. We start by giving algorithm for simulating gamma variables. There are many algorithm for generating gamma random variable (see for example [22], [32]). We only consider two methods: the Jöhnk's generator (5.1.1) which should be used if $\alpha \leq 1$ (this method is the most used in application) and the Best's generator (5.1.2) for which $\alpha > 1$.

Algorithm 5.1.1. *Jöhnk's generator of gamma variables $\alpha \leq 1$.*

- *REPEAT*
 - *Generate i.i.d. uniform $[0, 1]$ random variables U, V .*
 - *Set $X = U^{1/\alpha}$, $Y = V^{1/(1-\alpha)}$.*
- *UNTIL $X + Y \leq 1$*
- *Generate an exponential random variable E .*
- *Return $\frac{XE}{X+Y}$.*

Algorithm 5.1.2. *Best's generator of gamma variables $\alpha > 1$.*

- *Set $b = \alpha - 1$, $c = 3\alpha - 4$.*
- *REPEAT*

- Generate *i.i.d.* uniform $[0, 1]$ random variables U, V .
- Set $W = U(1 - U)$, $Y = \sqrt{\frac{c}{W}}(U - \frac{1}{2})$, $X = b + Y$.
- If $X < 0$ go to *REPEAT*
- Set $Z = 64W^3V^3$.
- *UNTIL* $\log(Z) \leq 2(b \log(\frac{X}{b}) - Y)$
- *Return* X

To simulate the variance gamma process expressed as time-changed Brownian motion we do as shown in algorithm (5.1.3). Recall that the two processes γ_t (the gamma process) and B_t (the Brownian motion) are independent.

Algorithm 5.1.3. *Simulation of $X_t \sim VG(t; \sigma, \alpha, \theta)$.*

- *Simulate n independent gamma variables $\Delta S_1, \dots, \Delta S_n$, with parameter $\frac{\Delta t}{k}$, where $\Delta t = t_i - t_{i-1}$ using algorithm (5.1.2) or (5.1.1).*
- *Simulate independent and identically distributed $N(0, 1)$ random variables N_1, \dots, N_n . Set $\Delta X_i = \sigma N_i \sqrt{\Delta S_i} + \theta \Delta S_i$.*

The discretized trajectory is given by

$$X(t_i) = \sum_{k=1}^i \Delta X_k.$$

In the above algorithm, the process is simulated sequentially whereby the next value of the process is calculated from the previous one. The process is assumed to start at zero and nothing is known beyond the current time t_i . The value of the process at time t_{i+1} is then calculated using its value at time t_i .

Figure (5.1) depicts a typical trajectory of the variance gamma process defined as time-changed Brownian motion. In our simulation procedure, the drift term is zero and the parameters of the variance gamma process are $\theta = 0.01$, $\alpha = 0.05$, $\sigma = 0.3$.

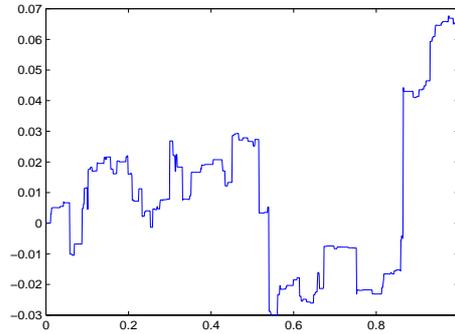


FIG. 5.1. A discretized trajectory of the variance gamma process.

5.2 Simulation of CGMY Process

The expression of the CGMY process as a time-changed Brownian motion was discovered by Madan and Yor in [79]. They showed that the CGMY process can be written as a Brownian motion time-changed by one sided $\frac{Y}{2}$ -stable process.

$$X_{CGMY}(t) = \theta Y(t) + W(Y(t)), \quad (5.2)$$

where $Y(t)$ is one sided $\frac{Y}{2}$ -stable subordinator and $W(t)$ is a Brownian motion. Within this representation, the CGMY process is simulated using the rejection method (see [61], [63], [56]). This amounts to accept every jump y_i of the stable subordinator for which $f(y_i)$ (this function is defined below) is greater than an independent uniform random variable on $[0, 1]$. The simulation algorithm is given by

Algorithm 5.2.1. *Simulation of $X_t \sim CGMY(t; C, G, M, Y)$ process.*

1. Define the time step t to be $t = C$. Then, let $A = \frac{G-M}{2}$, $B = \frac{G+M}{2}$.
2. Truncate jumps small than a small value ϵ replacing them by their expected value at a rate of

$$\begin{aligned} d &= \int_0^\epsilon y \frac{1}{y^{\frac{Y}{2}+1}} dy \\ &= \frac{\epsilon^{1-\frac{Y}{2}}}{Y \epsilon^{\frac{Y}{2}}}. \end{aligned}$$

3. The arrival rate of jumps bigger than ϵ is

$$\begin{aligned}\lambda &= \int_{\epsilon}^{\infty} \frac{1}{y^{\frac{Y}{2}+1}} dy \\ &= \frac{2}{Y\epsilon^{\frac{Y}{2}}}.\end{aligned}$$

4. The interval jump times are exponential and are simulated by

$$t_i = -\frac{1}{\lambda} \log(1 - u_{2i}),$$

for an independent uniform sequence u_{2i} .

5. Compute the actual jump times by

$$\Gamma_j = \sum_{i=1}^j t_i.$$

6. Generate jumps y_i at t_i given by $y_i = \frac{\epsilon}{(1-u_{1j})^{\frac{2}{Y}}}$, where u_{1j} is an independent uniform sequence (Inversion of the normalized Lévy measure).

7. Simulate the stable subordinator $S(t)$ by

$$S(t) = dt + \sum_{j=1}^{\infty} y_j 1_{\Gamma_j < t}.$$

8. Simulate the CGMY subordinator by

$$H(t) = dt + \sum_{j=1}^{\infty} y_j 1_{h(y_j) > u_{3j}},$$

where u_{3j} is another independent uniform sequence. The calculation of $h(y_j)$ is presented below.

9. Return $X_t = AH(t) + \sqrt{H(t)}W$, where W is a standard normal random variable.

Calculation of truncation function $h(y)$

$$1. h(y) = e^{-\frac{B^2 y}{2}} \frac{\Gamma(\frac{Y+1}{2})}{\Gamma(Y)\Gamma(\frac{1}{2})} 2^Y \left(\frac{B^2 y}{2}\right)^{\frac{Y}{2}} I(Y, B^2 y, \frac{B^2 y}{2}).$$

$$2. I(Y, 2\lambda, \lambda) = \frac{H_Y(\sqrt{2\lambda})\Gamma(Y)}{(2\lambda)^{\frac{Y}{2}}}, \text{ where } H_{\alpha}(\cdot) \text{ is the Hermite function.}$$

3. The Hermite function is explicitly known in terms of confluent Hypergeometric function ${}_1F_1$, (see [33]) where

$$H_\alpha(z) = 2^{\alpha/2} \left[\frac{1}{\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{1}{2})} {}_1F_1\left(\frac{-\alpha}{2}, \frac{1}{2}, \frac{z^2}{2}\right) - \frac{z}{\sqrt{2}\Gamma(\frac{-\alpha}{2})\Gamma(\frac{3}{2})} {}_1F_1\left(\frac{1-\alpha}{2}, \frac{3}{2}, \frac{z^2}{2}\right) \right].$$

The Rosiński rejection method supposes the existence of two Lévy measures \mathbb{Q} and \mathbb{Q}_0 with the property that

$$\frac{d\mathbb{Q}}{d\mathbb{Q}_0} \leq 1,$$

and a sequence of uniform random variables W_i such that one simulates the paths of \mathbb{Q} from those of \mathbb{Q}_0 by only accepting all jumps J_i^0 for which

$$\frac{d\mathbb{Q}}{d\mathbb{Q}_0}(J_i^0) \geq W_i. \quad (5.3)$$

The key to this method is to find an easy way to generate X_0 of \mathbb{Q}_0 from which only a small finite number of jumps must be removed to get the jumps of X .

For the CGMY process, the two Lévy measure correspond to [79]

$$\nu_0(dx) = \frac{Kdy}{y^{\frac{Y}{2}+1}},$$

and

$$\nu_1(dy) = \nu_0(dy)e^{-\frac{B^2y}{2}}\mathbb{E}[\exp(-yZ)],$$

where $Z = \frac{B^2}{2} \frac{\gamma_{Y/2}}{\gamma_{1/2}}$ and γ_a is the gamma variate of parameter a . One can then easily prove that

$$\frac{d\nu_1}{d\nu_0} = \mathbb{E}[\exp(-yZ)] < 1,$$

so we must reject all jumps in the paths of ν_0 for which

$$\mathbb{E}[\exp(-yZ)] > W_i,$$

where W_i is defined as above.

A typical trajectory of the CGMY process simulated using algorithm (5.2.1) is depicted in figure (5.2). The parameter's values are $C = 5$, $G = 10$, $M = 10$, $Y = 0.5$.

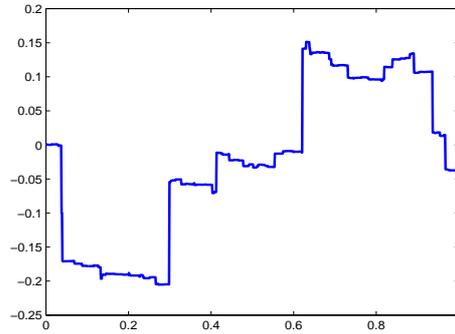


FIG. 5.2. A typical trajectory of the CGMY process defined as time-changed Brownian motion.

5.3 Series Representation

Series representation is a more convenient way of approximating a Lévy process as a compound Poisson process. For a series to converge, it is necessary that the magnitude of the jumps decreases as the time increases. The following theorem from [22] tells us how to construct such series. The original representation theorem is due to Rosiński [61].

Theorem 5.3.1. *Let $\{V_i\}_{i \geq 1}$ be independent identically distributed sequence of random elements in measurable space S . Assume that $\{V_i\}_{i \geq 1}$ is independent of the sequence $\{\Gamma_i\}_{i \geq 1}$ of jumping times of a standard Poisson process. Let $\{U_i\}_{i \geq 1}$ be sequence of independent random variables, uniformly distributed on $[0, 1]$ and independent from $\{V_i\}_{i \geq 1}$ and $\{\Gamma_i\}_{i \geq 1}$. Let*

$$H : (0, \infty) \times S \rightarrow \mathbb{R}^d$$

be a measurable function. We define a measure on \mathbb{R}^d by

$$\sigma(r, B) = \mathbb{P}(H(r, V_i) \in B), \quad B \in \mathcal{B}(\mathbb{R}^d), \quad (5.4)$$

$$\nu(B) = \int_0^\infty \sigma(r, B) dr.$$

Put

$$A(s) = \int_0^\infty \int_{|x| \leq 1} x \sigma(r, dx) dr, \quad s \geq 0.$$

(i) If ν is a Lévy measure on \mathbb{R}^d , that is,

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty,$$

and the limit $\gamma = \lim_{s \rightarrow \infty} A(s)$ exists in \mathbb{R}^d then the series

$$\sum_{i=1}^{\infty} H(\Gamma_i, V_i) 1_{U_i \leq 1}$$

converges almost surely and uniformly on $t \in [0, 1]$ to a Lévy process with characteristic triplet $(0, \gamma, \nu)$, that is, with characteristic function

$$\Phi_t(u) = \exp\left(t\left[iu\gamma + \int_{\mathbb{R}^d} (e^{iux} - 1 - iux1_{|x| \leq 1})\nu(dx)\right]\right).$$

(ii) If ν is a Lévy measure on \mathbb{R}^d and for each $v \in S$ the function

$$r \rightarrow |H(r, v)| \quad \text{is nonincreasing,} \quad (5.5)$$

then

$$\sum_{i=1}^{\infty} (H(\Gamma_i, V_i) 1_{U_i \leq t} - tc_i)$$

converges almost surely and uniformly on $t \in [0, 1]$ to a Lévy process with characteristic triplet $(0, 0, \nu)$. Here c_i are deterministic constants given by $c_i = A(i) - A(i-1)$.

Proof. The proof is given in appendix A.6. □

Cont and Tankov [22] (cf. remark 6.6) show that the truncated series

$$X_t^\tau = \sum_{K: \Gamma_K \leq \tau} (H(\Gamma_K, V_K) 1_{U_K \leq t} - t(A(K) - A(K-1))),$$

is a compound Poisson process with characteristic triplet $(0, 0, \nu_\tau)$, where $\nu_\tau(A) = \int_0^\tau \sigma(r, A) dr$. The advantage of series representation is that one is always working with the same Lévy measure instead of working with different measures of compound Poisson types. However, some series may converge slowly but with an increasing computational speed, a slow convergence may not be an issue of practical importance for some applications. More on simulation by series representation see [61], [63], and for series representation of tempered stable processes see [21], [64].

To summarize, in this chapter, we looked at the simulation of infinite activity Lévy process, in particular the variance gamma and the CGMY processes. We provided the simulation algorithms when the Lévy processes are defined as time-changed Brownian motion since viewing a Lévy process as time-changed Brownian motion with respect to simulation allows

one to avoid dealing with Lévy density which may be difficult to sample from. We also stated a theorem by [61] that allows to represent a Lévy process as series with regard to its simulation.

Chapter 6

Dependence Concepts and Lévy Processes

Over the last decades, researchers have put much emphasis on the valuation of options written on more than one underlying assets. Interest is due to the increase in popularity of these derivatives in financial markets. Contracts such as spread options can now be bought on the organized exchange while options on the minimum or maximum of two or more assets are quoted over-the-counter (OTC) [19]. The key point in evaluating multivariate options is the determination of the dependence structure between the underlying assets.

In this chapter, we discuss a powerful technique that is used to model the dependence structure between assets. This technique is the *Lévy copula* functions, a concept introduced recently by Tankov [71] in order to construct multidimensional Lévy processes and to model the dependence structure between components. The advantage of modeling dependence via Lévy copula is that the resulting probability law is automatically infinitely divisible. Moreover, Lévy copula functions allow to cover the whole range of dependence from independent to complete dependence. Applied to multi-asset options pricing, Lévy copulas allow one to price multivariate options with the information stemming from the marginals.

6.1 Dependence and Independence of Lévy Processes

Let $X = (X_t)_{t \geq 0}$ be a d -dimensional Lévy process. From the Lévy-Itô decomposition, we know that the distribution of a Lévy process is uniquely determined by the characteristic triplet (A, ν, γ) where A is the covariance matrix of the Brownian motion, γ is the drift parameter and ν is the Lévy measure determining the frequency and size of the jumps. It is worth defining criteria under which the components of a d -dimensional Lévy process X_t are independent or dependent. Since the continuous part and the pure jump part of a Lévy process are independent (see theorem 2.2.1), the dependence structure of the continuous part and the discontinuous part can be considered separately. The continuous martingale parts are independent if the covariance matrix is diagonal. It therefore remains to characterize the dependence structure of the pure jump parts that must be studied using Lévy measures. Therefore, from now on the Lévy processes that we consider have no continuous part, that is, $A = 0$. We start by defining the margins of a d -dimensional Lévy process $X := \{X_t^i\}_{t \geq 0}^{i=1, \dots, d}$.

Definition 6.1.1. *Let $I \subset \{1, \dots, d\}$ nonempty. The I -margin of X is the process $X^I := \{X_t^i\}_{t \geq 0}^{i \in I}$.*

The I -margins of the Lévy process are just the projection of the Lévy process X to the axis of $\mathbb{R}^{|I|}$, that is, $\Pi^I : \mathbb{R}^d \rightarrow \mathbb{R}^{|I|} : x \mapsto x^I$. Then the I -margins are Lévy process X^I such that $X_t^I = \Pi^I \cdot X_t$.

The Lévy measure of X^I depends only on the Lévy measure of X and can be computed as follows

Lemma 6.1.1. *Marginal Lévy measure.*

Let $I \subset \{1, \dots, d\}$ nonempty. Then the Lévy process X^I has Lévy measure ν^I given by

$$\nu^I(B) = \nu(\{x \in \mathbb{R}^d : (x_i)_{i \in I} \in B\}), \quad \forall B \in \mathcal{B}(\mathbb{R}^{|I|} \setminus \{0\}).$$

The important fact of the above lemma is that the margins of the Lévy measure can be computed exactly in the same way as the margins of a probability measure.

Proof. To prove the above lemma, recall that $\nu(B) = \mathbb{E}[N_1(B)]$ is the expected number, per unit time of jumps whose size belong to B . By definition (6.1.1), the marginal Lévy measure are then given by

$$\nu^I(B) = \nu \circ (\Pi^I)^{-1}(B).$$

□

We proceed by defining the independence between the components of an \mathbb{R}^d -valued Lévy process. The independence between components of a Lévy process is very much a Lévy measure property.

Proposition 6.1.1. *The components X_1, \dots, X_d of an \mathbb{R}^d -valued Lévy process X are independent if and only if their continuous martingale parts are independent and the Lévy measure ν is supported by the coordinate axis. The Lévy measure ν is then given by*

$$\nu(B) = \sum_{i=1}^d \nu_i(B_i), \quad B \in \mathcal{B}(\mathbb{R}^d), \quad (6.1)$$

where for every i , ν_i denotes the Lévy measure of X_i and

$$B_i = \{x \in \mathbb{R} : (\underbrace{0, \dots, 0}_{i-1 \text{ times}}, x, 0, \dots, 0) \in B\}.$$

Proposition 6.1.1 says that the components of a d -dimensional Lévy process are independent if and only if they never jump at the same time almost surely.

Proof. Let X_t be an \mathbb{R}^d -valued Lévy process with components X_t^1, \dots, X_t^d all independent from each other. By Lévy-Khintchine formula, we have

$$\begin{aligned} \mathbb{E}[e^{i\langle u, X_t \rangle}] &= \mathbb{E} \left[e^{i(u_1 X_t^1 + \dots + u_d X_t^d)} \right] \\ &= \exp \left(t \sum_{k=1}^d \left(i\gamma_k u_k + \int_{\mathbb{R} \setminus \{0\}} (e^{iu_k x_k} - 1 - iu_k x_k 1_{|x_k| \leq 1}) \nu_k(dx_k) \right) \right) \\ &= \prod_{k=1}^d \mathbb{E} \left[e^{iu_k X_t^k} \right]. \end{aligned}$$

Let $\tilde{\nu}$ be a measure on \mathbb{R}^d defined as $\tilde{\nu}(B) = \sum_{i=1}^d \nu_i(B_i)$, where ν_i is the Lévy measure of X^i and B_i is as above. By proposition 6.1.1, these measures coincide with the margins of $\tilde{\nu}$: $\tilde{\nu}_i = \nu_i$ for all i . By Lévy-Khintchine formula and the independence between components of X_t , we have

$$\mathbb{E}[e^{i\langle u, X_t \rangle}] = \exp \left(t \{ i\langle \gamma, u \rangle + \int_{\mathbb{R}^d \setminus \{0\}} (e^{i\langle u, x \rangle} - 1 - i\langle u, x \rangle 1_{|x| \leq 1}) \tilde{\nu}(dx) \} \right).$$

Using the uniqueness of the Lévy-Khintchine representation, we conclude that $\tilde{\nu}$ is a Lévy measure. □

Contrary to independence, complete dependence among components of an \mathbb{R}^d -value Lévy process is not a Lévy measure property in general. However, the components of an \mathbb{R}^d -valued Lévy process need not to be completely dependent for fixed time t ; they can be completely dependent as processes as shown by the following example [71].

Example 6.1.1. Let X_t be a pure jump process and Y_t a process constructed from the jumps of X_t : $Y_t = \sum_{s \leq t} \Delta X_s^3$. From the dynamics point of view X_t and Y_t are completely dependent, because the trajectory of the one can be reconstructed from the trajectory of the other. However, the copula of X_t and Y_t is not of complete dependence because Y_t is not a deterministic function of X_t .

Using this example, Tankov argues that the important dependence concept for Lévy processes is the dependence of jumps that should be studied using Lévy measure, because the knowledge of jumps dependence allows one to characterize the dynamic structure of the Lévy processes. We will see in subsection 6.3.2 that the dependence structure of the jumps of Lévy processes can be described by Lévy copulas since Lévy measures are completely characterized by the knowledge of Lévy copula and the margins. The following definition is needed in the sequel.

$$K = \{x \in \mathbb{R}^d : \text{sgn}x_1 = \dots = \text{sgn}x_d\}, \quad (6.2)$$

where

$$\text{sgn}x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Note that (6.2) is just the union of positive and negative orthant. To characterize complete dependence we first need the following definition.

Definition 6.1.2. A subset S of \mathbb{R}^d is ordered if, for any two vectors $u, v \in S$, either $u_k \leq v_k$, $k = 1, \dots, d$ or $u_k \geq v_k$, $k = 1, \dots, d$. Similarly, S is called strictly ordered if, for any two different vectors $u, v \in S$, either $u_k < v_k$, $k = 1, \dots, d$ or $u_k > v_k$, $k = 1, \dots, d$.

Definition 6.1.2 implies that if a set $S \subset \mathbb{R}^d$ is strictly ordered its elements can be completely determined by one coordinate only. For random variables Y_1, \dots, Y_d to be complete dependent or comonotonic, there must be a strictly ordered set $S \subset \mathbb{R}^d$ such that $(Y_1, \dots, Y_d) \in S$ with probability one. As for Lévy processes, we need to characterize complete dependence for any time $t \geq 0$.

Definition 6.1.3. *Let X be an \mathbb{R}^d -valued Lévy process. Its jumps are said to be complete dependent or comonotonic if there exists a strictly ordered subset $S \subset K$ such that $\Delta X_t = X_t - X_{t-} \in S, t \in \mathbb{R}_+$.*

This definition entails that, if the components of an \mathbb{R}^d -valued Lévy process X are completely dependent, the jumps of the all components can be determined from the jumps of the one components. This means that, knowing the trajectory of the one component is equivalent to knowing the trajectories of the others. Condition $\Delta X_t \in K$ implies that, if the components of an \mathbb{R}^d -valued Lévy process are completely dependent, they always jump in the same direction.

6.2 Distributional Copulas

Attempting to produce a suitable model of financial quantities means that it is important and relevant to more accurately understand the market co-movement and the dependence structure between underlying assets. The traditional approach relies on the normal distribution to model this dependence. In this setting the linear correlations suffice to fully characterize the dependence structure between the underlying assets. However the evidence against the Gaussian model cast doubts on the relevance of the correlation coefficient as an adequate measure of dependence.

Let us consider a simple example. Let X be a standard normal random variable $X \sim N(0, 1)$, and $Y = X^3$ so that Y is a deterministic function of X . Since the two random variables contain the same set of information, they should have a maximum dependence. Let us compute the linear correlation between X and Y . It is given by

$$\begin{aligned} \rho_{XY} &= \frac{\text{Cov}(X, Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}} \\ &= \frac{\mathbb{E}[X \cdot X^3] - \mathbb{E}[X]\mathbb{E}[X^3]}{\sqrt{\text{var}(X)}\sqrt{\text{var}(X^3)}} \\ &= \frac{3}{\sqrt{14}} < 1. \end{aligned}$$

One can see that even in this simple example the linear correlation is unable to explain the dependence structure between the two random variables (see [27], [28] for more on the

pitfalls of linear correlation as a measure of dependence). Nevertheless, Gaussian models are computational tractable and enable the development of sophisticated models for asset valuation.

Recently, there has been tendency to move away from the Gaussian assumption in financial context and move towards models that allow for extremal behavior. This has led to the introduction of copulas as a possible solution of modeling dependence structure between underlying assets. Copulas are functions that link together univariate distribution functions to form a multivariate distribution function. We follow Nielsen and Lindner [5] and call copulas for random variables distributional copulas. The role of distributional copula is to disentangle the dependence structure of the random vector from its marginals. Since each marginal distribution F_i contains information about the individual variable X_i , while the joint distribution F contains all univariate and multivariate information, it is clear that the information contained in the distributional copula must be all of the dependence information between the X_i 's. We refer the reader interested in a more detailed treatment for distributional copulas to [26], [38], [54]. For applications of distributional copula to finance see [20], [45], [50], [18], [53], [78].

Before we give the mathematical definition of distributional copula, we review the notion of volume functions and increasing functions of several random variables .

Let $\mathbb{R} = (-\infty, \infty)$ denote the ordinary real line and $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ denote the extended real line. For $a, b \in \bar{\mathbb{R}}^d$ we write $a \leq b$ if $a_k \leq b_k$, $k = 1, \dots, d$. A d -box $B = (a, b]$ is defined as the set $(a, b] = (a_1, b_1] \times \dots \times (a_d, b_d]$.

Definition 6.2.1. (*F-volume*). Let $F : S \rightarrow \bar{\mathbb{R}}$ for some subset $S \subset \bar{\mathbb{R}}^d$. For $a, b \in S$ with $a \leq b$ and $(a, b] \in S$, the F -volume of $(a, b]$ is define as

$$V_F((a, b]) = \sum_{u \in \{a_1, b_1\} \times \dots \times \{a_d, b_d\}} (-1)^{N(u)} F(u), \quad (6.3)$$

where $N(u) = \#\{k : u_k = a_k\}$, that is, the sum of the signed values of F over the vertices of $(a, b]$.

Example 6.2.1. The distribution function $F(x_1, \dots, x_d)$ of the random vector X on \mathbb{R}^d is an F -volume function. Indeed, if (X_1, \dots, X_d) is a random vector with distribution function

$$F(x_1, \dots, x_d) = \mathbb{P}[X_1 \leq x_1, \dots, X_d \leq x_d], \quad (6.4)$$

then

$$V_F((a, b]) = \mathbb{P}[X \in (a, b]],$$

for every $a, b \in \bar{\mathbb{R}}^d$ with $a \leq b$.

In the following, we define the notion of d -increasing function. This notion can be understood as the nonnegativity of the assigned probability to the d -box B .

Definition 6.2.2. (*d -increasing function*).

A function $F : S \rightarrow \bar{\mathbb{R}}$ for some set $S \subset \bar{\mathbb{R}}^d$ is called d -increasing if $V_F((a, b]) \geq 0$ for all $a, b \in S$ with $a \leq b$ and $(a, b] \subset S$.

In dependence modeling, the margins of a multivariate distribution play an important role. They are defined as follows

Definition 6.2.3. Let $F : \bar{\mathbb{R}}^d \rightarrow \bar{\mathbb{R}}$ be a d -increasing function which satisfies $F(u_1, \dots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \dots, d\}$. Furthermore let $I \subset \{1, \dots, d\}$ be a nonempty index set and denote with $I^c = \{1, \dots, d\} \setminus I$ its complement. Then, the I -margin of F is the function $F^I : \bar{\mathbb{R}}^{|I|} \rightarrow \bar{\mathbb{R}}$ such that

$$F^I((u_i)_{i \in I}) = \lim_{a \rightarrow \infty} \sum_{(u_j)_{j \in I^c} \in \{-a, \infty\}^{I^c}} F(u_1, \dots, u_d) \prod_{j \in I^c} \text{sgn}(u_j). \quad (6.5)$$

The distribution function defined in (6.4) is d -increasing. The margins of F are the marginal distribution functions of the random vector X on $\bar{\mathbb{R}}^d$ defined by $F_1(x) = F(x, \infty, \dots, \infty) = \mathbb{P}[X_1 \leq x]$.

The following property of increasing function is useful in the sequel.

Lemma 6.2.1. Let $S_k \subset \bar{\mathbb{R}}$ for $k = 1, \dots, d$ and let $F : S_1 \times \dots \times S_d \rightarrow \bar{\mathbb{R}}$ be a volume function. Let (x_1, \dots, x_d) and (y_1, \dots, y_d) be any point in $\text{Dom}F$. Then

$$|F(x_1, \dots, x_d) - F(y_1, \dots, y_d)| \leq \sum_{k=1}^d |F_k(x_k) - F_k(y_k)|. \quad (6.6)$$

The definition of distributional copula given above is intuitive but not mathematically rigid. We proceed in this section by giving the mathematical definition of the distributional copula

Definition 6.2.4. A d -dimensional copula is a function C with domain $[0, 1]^d$ such that

1. C is grounded and d -increasing.
2. C has margins C_k , $k = 1, \dots, d$, which satisfy $C_k(u) = u$ for all u in $[0, 1]$.

It is clear from the definition that a distributional copula is a multivariate distribution with support in $[0, 1]^d$, and with uniform margins. The main feature of the distributional copula is that one can take any arbitrary marginal distribution functions and joint them via a distributional copula to form a multivariate distribution function. This argument is build on the fundamental findings known as Sklar's theorem. This theorem states that, any joint probability distribution function can be written in terms of distributional copula function taking the marginal distribution functions as arguments, and conversely, any distributional copula function taking marginal distribution functions as arguments generates a joint distribution function.

Theorem 6.2.1. *Sklar's theorem.*

Let F be a d -dimensional distribution function with margins F_1, \dots, F_d . Then there exists a d -dimensional copula C such that for all x in $\bar{\mathbb{R}}^d$,

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)). \quad (6.7)$$

If F_1, F_2, \dots, F_d are all continuous, then C is unique; otherwise, C is uniquely determined on $\text{Ran}F_1 \times \dots \times \text{Ran}F_d$. Conversely, if C is a d -copula and F_1, \dots, F_d are distribution functions, then the function F defined by (6.7) is a d -dimensional distribution function with margins F_1, \dots, F_d .

This result shows that the distributional copula captures completely the dependence structure between the components of a random vector X on \mathbb{R}^d , independent of the shape of the marginal distributions.

The copula approach has great interest to finance in terms of its practical and theoretical implications. From the practical point of view, they provide a better understanding and quantification of the interactions between assets by determining the diverse dependence structure between various sources of risk [50]. As for the theoretical point of view, the dependence structure between assets can be seen as a mechanism that governs financial markets. In particular, the dependence between assets is in part the result of the interactions between the agents investing in the stock markets. Not only investors are responsible for individual asset variations and fluctuation but their choice of buying or selling one security rather than another creates dependence between assets [50].

Distributional copulas can be used to define upper and lower tail dependence, which intuitively expresses the probability of a random variable taking extrem values conditional on another random variable has taken the extrem values. Given a distribution copula C the upper tail dependence is define by

$$\lambda_U = \lim_{u \rightarrow 1^-} \frac{C(u, u)}{u},$$

and lower tail dependence is defined by

$$\lambda_L = \lim_{u \rightarrow 0^+} \frac{C(u, u) + 1 - 2u}{1 - u}.$$

The lower tail dependence is particularly interesting for prices and returns, since it corresponds to concordance in market clashes.

In the following we give an intuitive grasp of the use of distributional copulas to pricing multivariate contingent claims. Consider a market with two risky assets S_1 and S_2 and a risk-free asset B . Let us consider a bivariate rainbow option whose payoff is given by

$$P(S_1(T), S_2(T)) = \max[\min(S_1(T), S_2(T)) - K, 0].$$

The price of a call option on the minimum of the two underlying assets can easily understood by following the idea of Breeden and Litzenberger [13]. The basic concept in [13] stem from the martingale representation of option prices. For example, the price of European call option can be computed by

$$C(S, t; K, T) = \exp(-r(T - t)) \mathbb{E}^{\mathbb{Q}}[\max(S(T) - K, 0)], \quad (6.8)$$

where \mathbb{Q} is the risk-neutral measure. By computing the derivative of the pricing function with respect to the strike K , we obtain

$$\frac{\partial C(S, t; K, T)}{\partial K} = -\exp(-r(T - t))(1 - \mathbb{Q}(K|\mathcal{F}_t)), \quad (6.9)$$

where $\mathbb{Q}(K|\mathcal{F}_t)$ is the conditional distribution function under the risk-neutral measure. Defining $\bar{\mathbb{Q}}(K|\mathcal{F}_t) = 1 - \mathbb{Q}(K|\mathcal{F}_t)$, that is, $\mathbb{Q}[S(T) > K]$, we may write (6.9) as

$$\frac{\partial C(S, t; K, T)}{\partial K} = -\exp(-r(T - t))\bar{\mathbb{Q}}(K|\mathcal{F}_t). \quad (6.10)$$

The price of the call option is then obtained by integrating (6.10) from K to infinity.

$$C(S, t; K, T) = \exp(-r(T - t)) \int_K^\infty \bar{\mathbb{Q}}(u|\mathcal{F}_t) du. \quad (6.11)$$

Let us come back to our problem of pricing bivariate options. Applying the same probability distribution technique, with $f(S_1, S_2) = \min(S_1, S_2)$, we may write the call option price by

$$C(S_1, S_2, t; K, T) = B(t, T) \int_K^\infty \mathbb{Q}[\min(S_1(T), S_2(T)) > u | \mathcal{F}_t] du, \quad (6.12)$$

where the probability is computed under the risk-neutral measure.

Now consider the joint survival probability for any threshold $u : \mathbb{Q}[S_1(T) > u, S_2(T) > u | \mathcal{F}_t]$. Saying that at time T , both prices are greater than u is equivalent to saying that the lower of the two is above u . So, the price of the option becomes

$$\begin{aligned} C(S_1, S_2, t; K, T) &= B(t, T) \int_K^\infty \mathbb{Q}[S_1(T) > u, S_2(T) > u | \mathcal{F}_t] du \\ &= B(t, T) \int_K^\infty \bar{\mathbb{Q}}(u, u | \mathcal{F}_t) du, \end{aligned}$$

where $\bar{\mathbb{Q}}(u, u)$ is the joint survival distribution function under the risk-neutral measure. By Sklar's theorem (6.2.1), we can now write the price of the bivariate call option by

$$C(S_1, S_2, t; K, T) = B(t, T) \int_K^\infty C(\bar{\mathbb{Q}}_1(u), \bar{\mathbb{Q}}_1(u) | \mathcal{F}_t) du, \quad (6.13)$$

where $\bar{\mathbb{Q}}_1$ and $\bar{\mathbb{Q}}_2$ are marginal survival distributions under the risk-neutral measure. In this way, we are able to separate the marginal distributions from the dependence structure, which is represented by the distributional copula C . Different types of dependence structure between the underlying assets can be modeled by taking different types of dependence for the distributional copula C .

6.2.1 Shortcomings of Distributional Copulas for Lévy Processes

Recall that distributional copulas are functions that join univariate distribution functions to form a multivariate distribution function. For Lévy processes, one would expect to use the same principle in the construction of multidimensional Lévy processes. Since for one fixed time t , the law of X_t determines the law of the whole d -dimensional Lévy process $(X_t)_{t \geq 0}$, the dependence structure of the multidimensional Lévy process can be parametrized by a distributional copula C_t of the random vector X_t for some $t > 0$. However, this method has a number of drawbacks [73]:

- (i) For some $s \neq t$, the copula C_s cannot in general be computed from C_t because it also depends on the margins. Therefore, the copula C_t depends on t .
- (ii) Choosing a d -dimensional Lévy process and defining their dependence structure through a copula is not guaranteed to preserve infinite divisibility and thus produce a d -dimensional Lévy process.
- (iii) Since the laws of components of a multidimensional Lévy process are specified via their Lévy measures, it is not convenient to model dependence using the copula of probability distribution.

An example of a time-dependent copula for Lévy processes is given in [72]. The following example [76] illustrates the second statement.

Example 6.2.2. Let B_t^1 be a Brownian motion and B_t^2 a process such that $B_t^2 = ZB_t^1$. Z is a random variable independent from B_t^1 such that $\mathbb{P}[Z = 1] = \mathbb{P}[Z = -1] = \frac{1}{2}$. Note that B_t^2 is also a Brownian motion.

Let $0 < s < t$. Then,

$$\begin{aligned}
 \mathbb{P}[B_s(2) < x, B_t(2) - B_s(2) < y] &= \mathbb{P}[Z = 1][B_s(1) < x, B_t(1) - B_s(1) < y] \\
 &\quad + \mathbb{P}[Z = -1]\mathbb{P}[B_s(1) < x, -(B_t(1) - B_s(1)) < y] \\
 &= \frac{1}{2} (\mathbb{P}[B_s(1) < x]\mathbb{P}[B_t(1) - B_s(1) < y]) \\
 &\quad + \frac{1}{2} (\mathbb{P}[-B_s(1) < x]\mathbb{P}[-(B_t(1) - B_s(1)) < y]) \\
 &= \mathbb{P}[B_s(1) < x]\mathbb{P}[B_t(1) - B_s(1) < y] \\
 &= \mathbb{P}[B_s^1 < x]\mathbb{P}[B_{t-s}^1 < y].
 \end{aligned}$$

For $s = t$ and $x = \infty$, we have the stationarity of B_t^2 , and with $s = 0$, $x = \infty$, this shows that B_t^2 is normal. But

$$\begin{aligned}
 \mathbb{P}[B_t(2) - B_s(2) < y] &= \mathbb{P}[B_{t-s}(2) < y] \\
 &= \mathbb{P}[B_{t-s}(1) < y] \\
 &= \mathbb{P}[B_t(1) - B_s(1) < y].
 \end{aligned}$$

Then $\mathbb{P}[B_s(2) < x, B_t(2) - B_s(2) < y] = \mathbb{P}[B_s(2) < x]\mathbb{P}[B_t(2) - B_s(2) < y]$, which proves the independent increments property. Thus $B = (B^1, B^2)$ is a vector of Lévy processes.

The characteristic function of $B = (B^1, B^2)$ is given by

$$\psi_{B_t}(u) = \frac{1}{2} \left(e^{\frac{u_1^2 + u_2^2}{2}t} \right) (e^{u_1 u_2 t} + e^{-u_1 u_2 t}).$$

We want to characterize the dependence structure of the process $B = (B^1, B^2)$ using a distributional copula function. The joint cumulative distribution function of $B = (B^1, B^2)$ is given by

$$\begin{aligned} F_B(x) &= \mathbb{P}[B^1 \leq x, B^2 \leq y] \\ &= \frac{1}{2} (\mathbb{P}[B^1 \leq \min(x, y)] + \mathbb{P}[-y \leq B^1 \leq x]) \\ &= \frac{1}{2} (\min(\mathbb{P}[B^1 \leq x], \mathbb{P}[B^1 \leq y]) + \mathbb{P}[B^1 \leq x] - \mathbb{P}[B^1 < -y]) \\ &= \frac{1}{2} (\min(F(x), F(y)) + F(x) - 1 + F(y)). \end{aligned}$$

Therefore, the distributional copula of $B = (B^1, B^2)$ at time $t = 1$ is given by $C_1(u, v) = \frac{1}{2} (\min(u, v) + u + v - 1)$.

However $B = (B^1, B^2)$ to be a Lévy process, its characteristic function must be infinitely divisible. Hence,

$$\begin{aligned} \psi_{B_t}(u) &= (\psi_{B_{t/2}}(u))^2 \\ &= \left(e^{-\frac{u_1^2 + u_2^2}{2}} \right)^t \frac{1}{4} (e^{tu_1 u_2} + e^{-tu_1 u_2} + 2). \end{aligned}$$

which implies that $\psi_{B_t}(u)$ to be infinitely divisible

$$e^{u_1 u_2 t} + e^{-u_1 u_2 t} = 2, \quad \forall u = (u_1, u_2)^T \in \mathbb{R}^2,$$

which is clearly not the case.

6.3 Dependence Structure of Lévy Processes

It is well known that models based on the generalized Black-Scholes model (e.g. geometric Brownian motion) have a number of drawbacks especially when it comes to the description of the statistical properties of the distribution of asset returns and the dynamics of asset prices. In real world, the sample path of the asset price process presents discontinuities, a feature which is not observed in the diffusion models. In a diffusion model, the sample path of asset price process is always continuous with time. It is therefore of great interest to introduce jumps in the construction of multidimensional model taking also into account the dependence structure between components.

In the context of subordinated Lévy processes, it may be argued that extension to multivariate models can be done in a simple manner: time-change a multivariate Brownian

motion by a univariate subordinator. However, in doing so, one is limited to a small range of dependence which excludes independence (all processes are subordinated by the same subordinator) and all components must follow the same parametric model [73]. To clarify this argument, we consider the following example [73].

Example 6.3.1. Suppose that the stock price processes S_t^1 and S_t^2 are modeled by

$$\begin{aligned} S_t^1 &= \exp(X_t^1) & X_t^1 &= B^1(Z_t) + \mu_1 Z_t, \\ S_t^2 &= \exp(X_t^2) & X_t^2 &= B^2(Z_t) + \mu_2 Z_t, \end{aligned}$$

where B_t^1 and B_t^2 are Brownian motions with variance σ_1^2 and σ_2^2 respectively and correlation ρ , and Z_t is an increasing stochastic process (a subordinator). The correlation of returns $\rho(X_t^1, X_t^2)$ is computed by conditioning with respect to Z_t . Then (see the appendix (A.4))

$$\rho(X_t^1, X_t^2) = \frac{\sigma_1 \sigma_2 \rho \mathbb{E}[Z_t] + \mu_1 \mu_2 \text{Var}[Z_t]}{(\sigma_1^2 \mathbb{E}[Z_t] + \mu_1^2 \text{var}[Z_t])^{1/2} (\sigma_2^2 \mathbb{E}[Z_t] + \mu_2^2 \text{var}[Z_t])^{1/2}}. \quad (6.14)$$

Tankov [73] argues that, even in symmetric case (when $\mu_1 = \mu_2 = 0$), and the Brownian motions are independent ($\rho = 0$), the two stock are decorrelated but not independent, since the processes are subordinated by the same subordinator. As a result, the large jumps in the two stocks tend to arrive at the same time, which means the absolute value of returns are correlated. When $\mu_1 = \mu_2 = 0$ and $\rho = 0$, the covariance of the squares of returns is [73]

$$\text{Cov}((X_t^1)^2, (X_t^2)^2) = \text{Cov}((B^1(Z_t))^2, (B^2(Z_t))^2) \quad (6.15)$$

$$= \mathbb{E}[(B^1(Z_t))^2 (B^2(Z_t))^2 | Z_t] \quad (6.16)$$

$$= \sigma_1 \sigma_2 \text{Var}[Z_t], \quad (6.17)$$

which means that squares of returns are correlated if Z_t is not deterministic.

Another way of constructing multivariate model with jumps, is to incorporate several finite shocks into a multivariate diffusion model which are in the form of Poisson process N_t [42]. This leads us to model the log-price processes by

$$X_t^i = \mu^i t + B_t^i + \sum_{j=1}^{N_t} Y_j^i, \quad i = 1, \dots, d, \quad (6.18)$$

where $B_t = (B_t^1, \dots, B_t^d)$ is a d -dimensional Brownian motion and $\{Y_j\}_{j=1}^{\infty}$ are i.i.d. d -dimensional random vectors which determine the jump size in individual assets. Here we only have one driving Poisson shocks that affect the entire market. If we consider several

independent shocks that affect individual companies or sectors, we need to introduce several driving Poisson processes in the model, which leads us to the following form of the log-price processes [73]

$$X_t^i \mu^i t + B_t^i + \sum_{k=1}^M \sum_{j=1}^{N_t^k} Y_{jk}^i, \quad i = 1, \dots, d, \quad (6.19)$$

where N_t^1, \dots, N_t^M are Poisson processes driving M independent shocks and Y_{jk}^i is the size of jump in the i -th component after j -th shock of type k . To construct a parametric model, one needs to determine the distribution of the jump size and to characterize the dependence structure between them.

In general, when constructing multidimensional models with jumps, the following are important [73]:

- (i) The model should be flexible enough to allow any one-dimensional Lévy process with any jump size distribution.
- (ii) It should be possible to characterize all the dependence structure, from complete dependence to complete independence with smooth transition between the two.
- (iii) Since in characterizing the dependence structure between components one may not have enough information, the model should allow to characterize the dependence in a parametric way.

As a convenient method to deal with multivariate models with jumps, Tankov suggests to use Lévy copula functions. Lévy copulas offer a versatile modeling approach as they enable one to separate the dependence structure and the marginal aspects of a multidimensional Lévy process. With the Lévy copula framework, one handles the problem of modeling jumps dependence and responds to the need of flexible functions to couple the one-dimensional Lévy models.

The derivations in this section follow from the pioneering thesis by Tankov [73] and the paper on Lévy copulas for multidimensional Lévy process by Kallsen and Tankov [40]. Other references on Lévy copulas see the textbook by Cont and Tankov [22], Nielsen and Lindner [5], Chen [17]. Positive Lévy copulas are discussed in [71], [72]. Application of

Lévy copulas to finance include the calculation of option prices [74], [75], calculation of the VaR [10] and application to insurance [34].

6.3.1 Definition and Basic Properties

Lévy copulas are tools used to model complex dependence structure of multivariate jump processes. As previously stated, the key idea for modeling the dependence among the jumps of Lévy processes is that, the Lévy measure plays the same role as the probability measure and tail integral plays the role of distribution function for random variables. Hence, to model the dependence, we must construct copulas for Lévy measure. The principle difference from the random variable case is that Lévy measure are not necessarily finite: they may have a non-integrable singularity at zero [73].

Define

$$J(x) = \begin{cases} [0, x), & \text{if } x \geq 0 \\ (x, 0], & \text{if } x < 0. \end{cases}$$

Every finite measure μ on \mathbb{R}^d is associated with a distribution function F such that

$$F(x_1, \dots, x_d) = \prod_{i=1}^d \text{sgn}(x_i) \mu \left(\prod_{i=1}^d J(x_i) \right). \quad (6.20)$$

However, for the jumps of Lévy processes, the Lévy measure ν may be infinite. So, instead of using $J(x)$ we use $I(x)$ defined by

$$I(x) = \begin{cases} [x, \infty), & x \geq 0 \\ (-\infty, x], & x < 0, \end{cases}$$

for any $x \in \mathbb{R}^d$. The tail integral is then defined by

Definition 6.3.1. *Let X be an \mathbb{R}^d -valued Lévy process with Lévy measure ν . The tail integral of ν is the function $U : (\mathbb{R}^d \setminus \{0\}) \rightarrow \mathbb{R}$ defined by*

$$U(x_1, \dots, x_d) = \prod_{i=1}^d \text{sgn}(x_i) \nu \left(\prod_{i=1}^d I(x_i) \right). \quad (6.21)$$

Since the Lévy measures are not necessarily finite (they are not in general integrable in a neighborhood of zero), tail integrals are defined on $\mathbb{R}^d \setminus \{0\}$. The tail integral is defined

such that $(-1)^d U$ is d -increasing and left-continuous in each orthant. Moreover, due to the discontinuity at zero, tail integral does not determine the Lévy measure uniquely unless we know that the Lévy measure does not charge the coordinate axes (the tail integral does not reflect the mass on the coordinate axes). For example, if the components of a Lévy process are independent, the tail integral is zero.

For modeling dependence, the margins play an important role. The marginal tail integrals of the Lévy measure are defined similarly to the margins of a distribution function.

Definition 6.3.2. *Let X be an \mathbb{R}^d -valued Lévy process and let $I \subset \{1, \dots, d\}$ non-empty. The I -marginal tail integral U^I of X is the tail integral of the process $X^I = (X_i)_{i \in I}$. We denote the one-dimensional tail integral by $U_i = U^{\{i\}}$.*

Figure (6.1) depicts the one-dimensional tail integral of the CGMY Lévy measure with parameters $C = 6$, $G = 8$, $M = 10$ and $Y = 0.3625$ (on the left) and the tail integral of the variance gamma Lévy measure with parameters $\theta = 0.003$, $\alpha = 0.3$ and $\sigma = 0.023$ (on the right). As you can see the univariate tail integral is a decreasing function on both sides of zero.

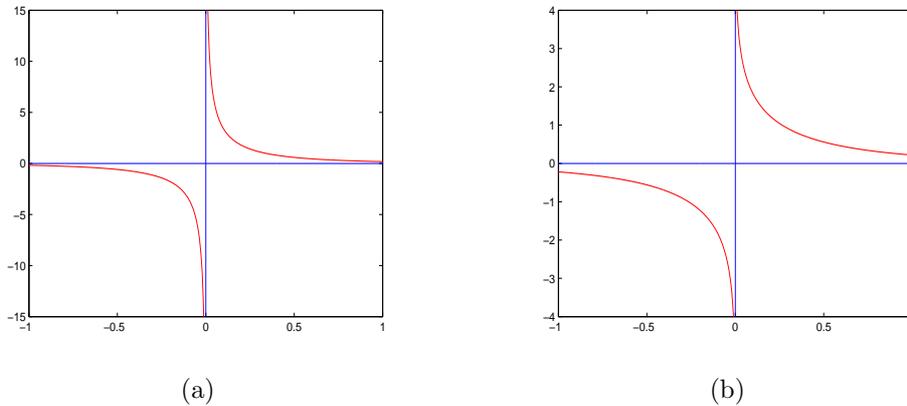


FIG. 6.1. On the left: The tail integral of the CGMY Lévy measure. On the right: The tail integral of the variance gamma Lévy measure.

It is known in statistics that any probability measure can be characterized by its distribution function. In a similar way, it can be shown (cf. [40] Lemma 3.5) that the Lévy measure is uniquely determined by the set $\{U^I : I \in \{1, \dots, d\}\}$ of its marginal tail integral and vice versa.

Tankov defines Lévy copulas on the analogy of distributional copula with the main difference being the domain of definition. This is because the Lévy measures are not necessarily finite: they may have a non-integrable singularity at zero.

Definition 6.3.3. *A function $F : \bar{\mathbb{R}}^d \rightarrow \mathbb{R}$ is called Lévy copula if*

1. $F(u_1, \dots, u_d) \neq \infty$ for $(u_1, \dots, u_d) \neq (\infty, \dots, \infty)$,
2. $F(u_1, \dots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \dots, d\}$,
3. F is d -increasing,
4. $F^{\{i\}}(u) = u$ for any $i \in \{1, \dots, d\}$, $u \in \mathbb{R}$.

The notion of groundedness guarantees that F defines a measure on \mathbb{R}^d ; indeed a Lévy copula is a d -dimensional measure with Lebesgue margins (cf. subsection 6.3.4, [73]).

An important result in the concept of Lévy copula is the version of Sklar's theorem for Lévy processes. This theorem parallels the Sklar's theorem in the context of tail integral instead of probability distributions. It states that, any multidimensional Lévy process can be written in terms of a Lévy copula taking the marginal tail integrals as arguments, and conversely, any Lévy copula taking marginal tail integrals as arguments generates a multidimensional Lévy process.

Theorem 6.3.1. *(Sklar's theorem for Lévy processes).*

Let ν be a Lévy measure on $\mathbb{R}^d \setminus \{0\}$. Then there exists a Lévy copula F such that the tail integrals of ν satisfy:

$$U^I((x_i)_{i \in I}) = F^I((U_i(x_i))_{i \in I}), \quad (6.22)$$

for non-empty $I \subset \{1, \dots, d\}$ and any $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$. The Lévy copula F is unique on $\prod_{i=1}^d \overline{\text{Ran}U_i}$.

Conversely, if F is a d -dimensional Lévy copula and ν_1, \dots, ν_d are Lévy measures on $\mathbb{R} \setminus \{0\}$ with tail integrals U_i , $i = 1, \dots, d$ then there exists a unique Lévy measure on $\mathbb{R}^d \setminus \{0\}$ with dimensional marginal tail integrals U_1, \dots, U_d and whose marginal tail integrals satisfy equation (6.22) for any non-empty $I \subset \{1, \dots, d\}$ and any $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^d$.

The first part assigns Lévy copulas the capability to represent all types of dependence between the jumps of a Lévy process, and the second part makes the construction of multidimensional Lévy models possible by specifying separately jump dependence structure and one-dimensional laws for the components.

The results of theorem 6.3.1 serve as necessary and sufficient conditions for building multivariate models with jumps, since one only needs to take d one-dimensional Lévy processes (note that the components need not to be of the same nature) and one Lévy copula, possibly from a parametric Lévy copula to construct a multivariate model. This is the approach followed in chapter 4 to construct two-dimensional exponential model with variance gamma and CGMY margins.

Proof. For simplicity, we consider the case $d = 2$ and the Lévy measures of the components are infinite and have no atoms. In this case, the marginal tail integrals are continuous and so the Lévy copula is unique. The proof in general case can be found in [73] or [40].

Denote the Lévy measure of X_t^1 and X_t^2 by ν_1 and ν_2 respectively. For the purpose of this proof we need the following definition of the generalized inverse tail integral of the one-dimensional tail integral $U_i(x)$ $i = 1, 2$.

Definition 6.3.4. *For the one-dimensional tail integral, the generalized inverse $U_i^{-1}(x)$ of $U_i(x)$ is define by*

$$U_i^{-1}(u) = \begin{cases} \inf\{x > 0 : u \geq U_i(x)\}, & u \geq 0 \\ \inf\{x < 0 : u \geq U_i(x)\} \wedge 0, & u < 0. \end{cases}$$

We first prove the existence of the Lévy copula F . Consider the function $F : D = \text{Ran}(U_1) \times \text{Ran}(U_2) \rightarrow (-\infty, \infty)$ defined by

$$F(x_1, x_2) = U(U_1^{(-1)}(x_1), U_2^{(-1)}(x_2)).$$

Suppose that $x_1 = 0$ and z is such that $U_1(z) = 0$. Then

$$\begin{aligned} F(0, x_2) &= \lim_{a \rightarrow 0^-} (F(0, x_2) - F(a, x_2)) \\ &= \lim_{a \rightarrow 0^-} \left(U(U_1^{(-1)}(0), U_2^{(-1)}(x_2)) - U(U_1^{(-1)}(a), U_2^{(-1)}(x_2)) \right) \\ &= U(z, U_2^{(-1)}(x_2)) - U(-z, U_2^{(-1)}(x_2)). \end{aligned}$$

Since $U(x_1, x_2) \geq 0$ and $U(x_1, x_2) \leq U(x_1, 0) = U_1(x_1)$, it follows that

$$\begin{aligned} 0 &\leq U(z, U_2^{(-1)}(x_2)) - U(-z, U_2^{(-1)}(x_2)) \\ &\leq U(z, 0) - U(-z, 0) \\ &= U_1(z) - U_1(-z) = 0. \end{aligned}$$

Therefore, F is grounded. The margins of F are obtain as

$$\begin{aligned} F^{\{1\}} &= F(x_1, \infty) - \lim_{a \rightarrow -\infty} F(x_1, a) \\ &= U(U_1^{(-1)}(x_1), U_2^{(-1)}(\infty)) - \lim_{a \rightarrow -\infty} U(U_1^{(-1)}(x_1), U_2^{(-1)}(a)) \\ &= U(U_1^{(-1)}(x_1), 0^+) - U(U_1^{(-1)}(x_1), 0^-). \end{aligned}$$

Now let $a, b \in D$ with $a_k \leq b_k$, $k = 1, 2$ and denote

$$B = (a_1, b_1] \times (a_2, b_2].$$

Then definition (6.2.1) entails that

$$F(x_1, x_2) = V_F(B) \geq 0,$$

which means that F is 2-increasing. To check the uniqueness, suppose that there exists another copula F_2 satisfying (6.22). Then for every $x_i \in \mathbb{R}^2$, $i = 1, 2$

$$F_1(x_1, x_2) = F_2(x_1, x_2). \quad (6.23)$$

From the continuity of $U_i(x)$, it follows that for all t_1, t_2 in \mathbb{R}^2 , there exists x_1, x_2 in \mathbb{R}^2 such that

$$U_1(x_1) = t_1, U_2(x_2) = t_2.$$

This means that for all t_1, t_2 ,

$$F_1(t_1, t_2) = F_2(t_1, t_2). \quad (6.24)$$

Therefore the Lévy copula F is unique.

For the converse statement we need to show that ν is a Lévy measure and that its marginal tail integral $U^{\{i\}}$, $i = 1, 2$ satisfy $U^{\{i\}}(x_i) = F^{\{i\}}(U_i(x_i))$. Hence,

$$\begin{aligned} U^{\{1\}}(x_1) &= \text{sgn} x_1 \nu(I(x_1) \times \mathbb{R}) \\ &= \text{sgn} x_1 (\nu(I(x_1) \times (0, \infty)) + \nu(I(x_1) \times (-\infty, 0))) \\ &= U(x_1, 0^+) - U(x_1, 0^-) \\ &= F^{\{1\}}(U_1(x_1)). \end{aligned}$$

Similarly $U^{\{2\}}(x_2) = F^{\{2\}}(U_2(x_2))$, which proves that the marginal tail integrals of ν equal U_1 and U_2 .

It remains to prove that ν is a Lévy measure on \mathbb{R}^2 . Since the marginals ν_i of ν are Lévy measure on \mathbb{R} , we have $\int (x_i^2 \wedge 1) \nu_i(dx_i) < \infty$ for all $i = 1, 2$. This implies that

$$\begin{aligned} \int (|x|^2 \wedge 1) \nu(dx) &\leq \int \sum_{i=1}^2 (x_i^2 \wedge 1) \nu(dx) \\ &= \sum_{i=1}^2 \int (x_i^2 \wedge 1) \nu_i(dx_i) < \infty, \end{aligned}$$

and hence ν is a Lévy measure on \mathbb{R}^2 . The uniqueness follows from the fact that it is uniquely determined by its marginal tail integrals. \square

When the tail integrals and the Lévy copula are sufficiently smooth, the Lévy density of the Lévy measure ν can be obtained by differentiating the Lévy copula times the one-dimensional Lévy densities.

Definition 6.3.5. *Let F be a d -dimensional Lévy copula continuous on $[-\infty, \infty]^d$, and let U_1, \dots, U_d be one-dimensional tail integrals with Lévy densities ν_1, \dots, ν_d respectively. Then the Lévy density of a Lévy measure ν with marginal Lévy densities ν_1, \dots, ν_d is defined as*

$$\nu(x_1, \dots, x_d) = \frac{\partial^d F(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d} \Big|_{u_1=U_1(x_1), \dots, u_d=U_d(x_d)} \times \nu_1(x_1) \times \dots \times \nu_d(x_d). \quad (6.25)$$

Unlike the tail integral $U(x_1, \dots, x_d)$ which is only singular at the origin, the Lévy density can be singular in any point. However, it must be integrable everywhere except at the origin.

We close this subsection by stating a theorem that shows the relationship between the Lévy copula F of a Lévy process X and the distributional copula C_t of its distribution at a given time t . The proof can be found in [40].

Theorem 6.3.2. *Let X be an \mathbb{R}^d -valued Lévy process with marginal tail integrals U_1, \dots, U_d , and denote by F its Lévy copula in the sense of theorem (6.3.1). Denote by $C_t^{(\alpha_1, \dots, \alpha_d)} : [0, 1]^d \rightarrow [0, 1]$ a distributional copula of $(-\alpha_1 X_t^1, \dots, -\alpha_d X_t^d)$ (or, equivalently, a survival copula of $(\alpha_1 X_t^1, \dots, \alpha_d X_t^d)$) for $t \in (0, \infty)$, $\alpha \in \{-1, 1\}^d$. Then*

$$F(u_1, \dots, u_d) = \lim_{t \rightarrow 0} \frac{1}{t} C_t^{(\text{sgn } u_1, \dots, \text{sgn } u_d)}(t|u_1|, \dots, t|u_d|) \cdot \prod_{i=1}^d \text{sgn } u_i, \quad (6.26)$$

for any $(u_1, \dots, u_d) \in \prod_{i=1}^d \overline{\text{Ran}}(U_i)$.

6.3.2 Examples of Lévy Copulas

Lévy copulas allow to characterize the possible dependence structure between components of multidimensional Lévy process. In this section, we derive Lévy copulas corresponding to extreme cases—namely complete independence and complete dependence. We start with describing the independence of components of a d -dimensional Lévy process via a Lévy copula. We first restate lemma 6.1.1 in terms of tail integrals.

Lemma 6.3.1. *The components X^1, \dots, X^d of an \mathbb{R}^d -valued Lévy process X are independent if and only if the tail integrals of the Lévy measure satisfy $U^I((x_i)_{i \in I}) = 0$ for all $I \subset \{1, \dots, d\}$ with $\text{card } I \geq 2$ and all $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$.*

Proof. \Rightarrow Let ν be defined by equation (6.1) where ν_i is the Lévy measure of X^i for $i = 1, \dots, d$. Then all marginal tail integral of ν coincide with those of the Lévy measure of X . Therefore, ν is the Lévy measure of X which entails by proposition 6.1.1 that X^1, \dots, X^d are independent.

\Leftarrow Let $I \subset \{1, \dots, d\}$ with $\text{card } I \geq 2$ and $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$. Then the components of the Lévy process $(X^i)_{i \in I}$ are independent as well. By proposition 6.1.1, we conclude that $U^I((x_i)_{i \in I}) = 0$. \square

The following result gives criterion under which the components of an \mathbb{R}^d -valued Lévy process X are independent when the dependence structure is modeled by Lévy copula.

Theorem 6.3.3. *The components X_1, \dots, X_d of an \mathbb{R}^d -valued Lévy process X are independent if it has a Lévy copula of the form*

$$F_{\perp}(x_1, \dots, x_d) = \sum_{i=1}^d x_i \prod_{j \neq i} 1_{\infty}(x_j).$$

Proof. \Rightarrow By definition 6.3.3, $F(u_1, \dots, u_d) = 0$ for at least one $i \in \{1, \dots, d\}$. Therefore, by formula (6.22) $U((x_i)_{i \in I}) = 0$ for all $(x_i)_{i \in I} \in \mathbb{R} \setminus \{0\}$. From lemma 6.3.1, we conclude that the components of (X_t) are independent.

\Leftarrow By lemma 6.3.1, the tail integrals of X satisfy $U^I((x_i)_{i \in I}) = 0$ for all $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$ and $I \subset \{1, \dots, d\}$ with $\text{card } I \geq 2$. Since also $F_{\perp}^I((u_i)_{i \in I}) = 0$, we conclude that F_{\perp} is a Lévy copula for X . \square

Recall that the jumps of a d -dimensional Lévy process are completely dependent if the trajectory of the components can be constructed from the trajectory of the one component. The following theorem describes the complete jump dependence of an \mathbb{R}^d -valued Lévy process X in terms of Lévy copula F .

Theorem 6.3.4. *The components X_1, \dots, X_d of an \mathbb{R}^d -valued Lévy process X are completely positive dependent if and only if it has a Lévy copula of the form*

$$F_{\parallel}(x_1, \dots, x_d) = \min(|x_1|, \dots, |x_d|) 1_K(x_1, \dots, x_d) \prod_{i=1}^d \text{sgn}(x_i).$$

Conversely, if F_{\parallel} is a Lévy copula of X , then the Lévy measure of X is supported by an ordered subset of K . If in addition, the tail integrals U_i of X^i are continuous and satisfy $\lim_{x \rightarrow 0} U_i(x_i) = 0$, $i = 1, \dots, d$, then the jumps of X are completely dependent.

Proof. The proof is given in appendix A.5. □

To characterize complete negative dependence, we need the following definition.

Definition 6.3.6. Define $S^- = \{x \in \mathbb{R}^2 : \text{sgn}(x_1) \neq \text{sgn}(x_2)\}$. Then the jumps of $\{X_t\}$ are completely negative dependent if there exists a decreasing set $D \subset S^-$ such that $\Delta X_t \subset D$, $t \geq 0$.

We define the complete negative dependence of an \mathbb{R}^2 -valued Lévy process via Lévy copula.

Theorem 6.3.5. The components X_1, X_d of an \mathbb{R}^2 -valued Lévy process X are completely negative dependent if and only if it has a Lévy copula of the form

$$F(u_1, u_2) = \min(|u_1|, |u_2|)1_{S^-}(u_1, u_2).$$

Figure (6.2) and (6.3) depict the complete negative dependence, the independence, and the complete positive dependence Lévy copula for the bivariate variance gamma Lévy measure with parameters $\theta_1 = \theta_2 = 0.003, \alpha_1 = \alpha_2 = 0.3, \sigma_1 = 0.025$ and $\sigma_2 = 0.023$, and the bivariate CGMY Lévy measure with parameters $C_1 = 5.5, C_2 = 6, G_1 = 7, G_2 = 8, M_1 = M_2 = 10$ and $Y_1 = Y_2 = 0.3625$. The complete positive dependence Lévy copula has a kink; it does not possess a continuous density. The complete negative dependence is made of pyramidal bricks, while the independence Lévy copula is zero everywhere.

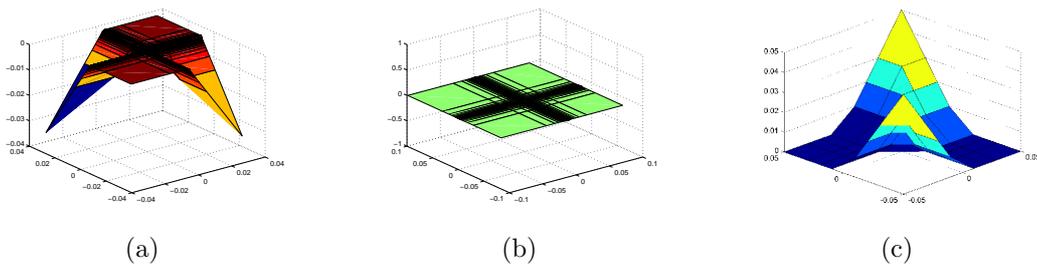


FIG. 6.2. The surface of the complete negative dependence Lévy copula (left), the independence Lévy copula (center), and the complete positive dependence Lévy copula (right) for the variance gamma Lévy measure

The dependence structure of stable Lévy processes can be parametrized in a simple way by Lévy copula. It turns out that, Lévy copula of stable Lévy processes are homogeneous function of order one, as shown in the following theorem

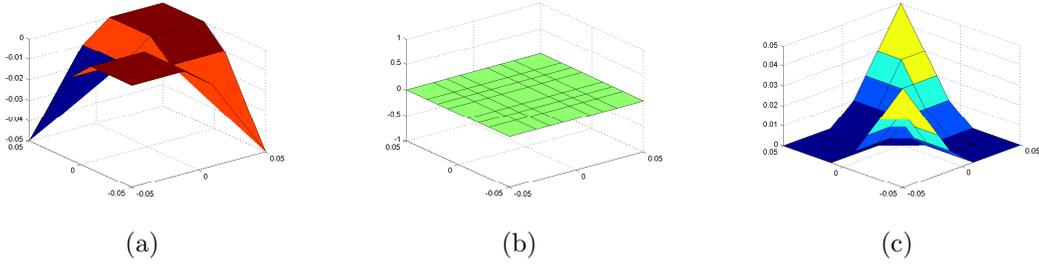


FIG. 6.3. The surface of the complete negative dependence Lévy copula (left), the independence Lévy copula (center), and the complete positive dependence Lévy copula (right) for the CGMY Lévy measure

Theorem 6.3.6. *Let $X = (X_1, \dots, X_d)$ be an \mathbb{R}^d -valued Lévy process and let $\alpha \in (0, 2)$. X is α -stable Lévy process if and only if its components X_1, \dots, X_d are α -stable and it has a Lévy copula F that is homogeneous function of order one:*

$$\forall r > 0, \quad \forall u_1, \dots, u_d, \quad F(ru_1, \dots, ru_d) = rF(u_1, \dots, u_d). \quad (6.27)$$

6.3.3 The Family of Archimedean Lévy Copulas

Lévy copulas can be used to construct multivariate models using formula (6.22). However, this method is not very useful when there is not enough information about the dependence structure between components. In this case, one has to rely on parametric Lévy copulas. All types of dependence structure can be parametrized through the Lévy copula's parameters by adjusting them to the desired level. The important feature of this method is that the number of parameters does not depend on the dimension of the Lévy process.

A popular class of such parametric Lévy copulas is the Archimedean Lévy copulas. The importance of this class of parametric Lévy copulas is that it contains a large number of Lévy copulas while enjoying a certain number of interesting properties. The following result allows to construct Archimedean Lévy copula in the analogy of Archimedean copulas (see [54]).

Definition 6.3.7. *Let $\phi : [-1, 1] \rightarrow [-\infty, \infty]$ be strictly continuous function with $\phi(1) = \infty$, $\phi(0) = 0$, and $\phi(-1) = -\infty$, having derivative of orders up to d on $(-1, 0)$ and $(0, 1)$, and satisfying*

$$\frac{\partial^d \phi(e^x)}{\partial x^d} \geq 0, \quad \frac{\partial^d \phi(e^x)}{\partial x^d} \leq 0, \quad x \in (-\infty, 0).$$

Let

$$\tilde{\phi}(u) := 2^{d-2}(\phi(u) - \phi(-u)),$$

for $u \in [-1, 1]$. Then

$$F(u_1, \dots, u_d) := \phi\left(\prod_{i=1}^d \tilde{\phi}^{-1}(u_i)\right)$$

defines a Lévy copula.

The function ϕ is called the *generator* of the Archimedean Lévy copula F . An example of the Archimedean Lévy copula is the Clayton Lévy copula [40].

Example 6.3.2. Let ϕ be the generator given by

$$\phi(x) = \eta(-\log|x|)^{-1/\theta} 1_{\{x>0\}} - (1-\eta)(-\log|x|)^{-1/\theta} 1_{\{x<0\}},$$

with $\theta > 0$ and $\eta \in (0, 1)$. Then

$$\begin{aligned} \tilde{\phi}(x) &= 2^{d-2}(-\log|x|)^{-1/\theta} \operatorname{sgn}x, \quad \text{and} \\ \tilde{\phi}^{-1}(u) &= \exp(-|2^{2-d}u|^{-\theta}) \operatorname{sgn}u. \end{aligned}$$

ϕ produces a general Clayton Lévy copula given by

$$F(u_1, \dots, u_d) = 2^{2-d} \left(\sum_{i=1}^d |u_i|^{-\theta} \right) (\eta 1_{\{u_1 \dots u_d \geq 0\}} - (1-\eta) 1_{\{u_1 \dots u_d < 0\}}). \quad (6.28)$$

F defines a two-parametric Lévy copula. We refer to (6.28) as the *bidirectional Clayton Lévy copula*. It is named Clayton Lévy copula because it resembles the distributional Clayton copula in terms of its construction. The Clayton Lévy copula (6.28) covers the whole range of dependence through two parameters only.

In [73], it is shown that F is a Lévy copula for any $\theta > 0$ and $\eta \in [0, 1]$. The parameter η determines the dependence of the sign of jumps, while the parameter θ is responsible for the dependence of absolute values of the jumps in different components. When $\eta = 1$, the two components jump in the same direction, and when $\eta = 0$, positive jumps in one component are accompanied by negative jumps in the other and vice versa. The two components are independent when $\eta = 1$ and $\theta \rightarrow 0$. They are completely dependent when $\eta = 1$ and $\theta \rightarrow \infty$.

To clarify the above argument (to show the role of the parameters) we consider $d = 2$ and plot the contour plot of the CGMY Lévy density for different values of θ and η . In this case (6.28) becomes

$$F(u, v) = (|u|^{-\theta} + |v|^{-\theta})^{-\theta} (\eta 1_{uv \geq 0} - (1 - \eta) 1_{uv < 0}). \quad (6.29)$$

We compute the Lévy density of the bivariate CGMY Lévy measure as in definition (6.3.5). Differentiating (6.29) with respect to u and v , we obtain

$$\nu(x_1, x_2) = (\theta + 1)(u^{-\theta-1}v^{-\theta-1})(u^{-\theta} + v^{-\theta})^{-\frac{1+\theta}{\theta}} \Big|_{u=U_1(x_1), v=U_2(x_2)} \nu_1(x_1)\nu_2(x_2), \quad (6.30)$$

where $U_1(x_1)$ and $U_2(x_2)$ are the marginal tail integrals of the Lévy measure ν .

Figures (6.4) and (6.5) show the contour plot of the Lévy density of the bivariate CGMY Lévy measure with dependence structure given by the Clayton Lévy copula (6.29). In figure (6.4), we fixed θ and then change η . This parameter determines the sign of the jumps. In (6.4(a)) $\eta = 0$; positive jumps in the first component correspond to negative jumps in the second component and vis-versa. In (6.4(b)), the two components jump in the same direction. Figure (6.4(c)) correspond to $\eta = 0.5$. In this case (for the values of $\eta \in (0, 1)$), positive jumps of one components correspond to both positive and negative jumps of the other.

In figure (6.5), we fixed η and then changed θ . This parameter is responsible for the dependence structure between components. In the left panels $\eta = 0$ and in the right panels $\eta = 1$. In figure (6.5(a)) and (6.5(b)), the two components are independent. As θ increases, figures (6.5(c)), (6.5(d)), (6.5(e)), and (6.5(f)), the dependence structure gets stronger and stronger.

6.3.4 Probabilistic Interpretation of Lévy Copulas

Although Lévy copulas are not distribution functions, their derivatives have an interesting probability interpretation [73]. Let F be a Lévy copula on $(-\infty, \infty]^d$ satisfying

$$\lim_{(x_i)_{i \in I} \rightarrow \infty} F(u_1, \dots, u_d) = F(u_1, \dots, u_d) \Big|_{(x_i)_{i \in I} = \infty} \quad (6.31)$$

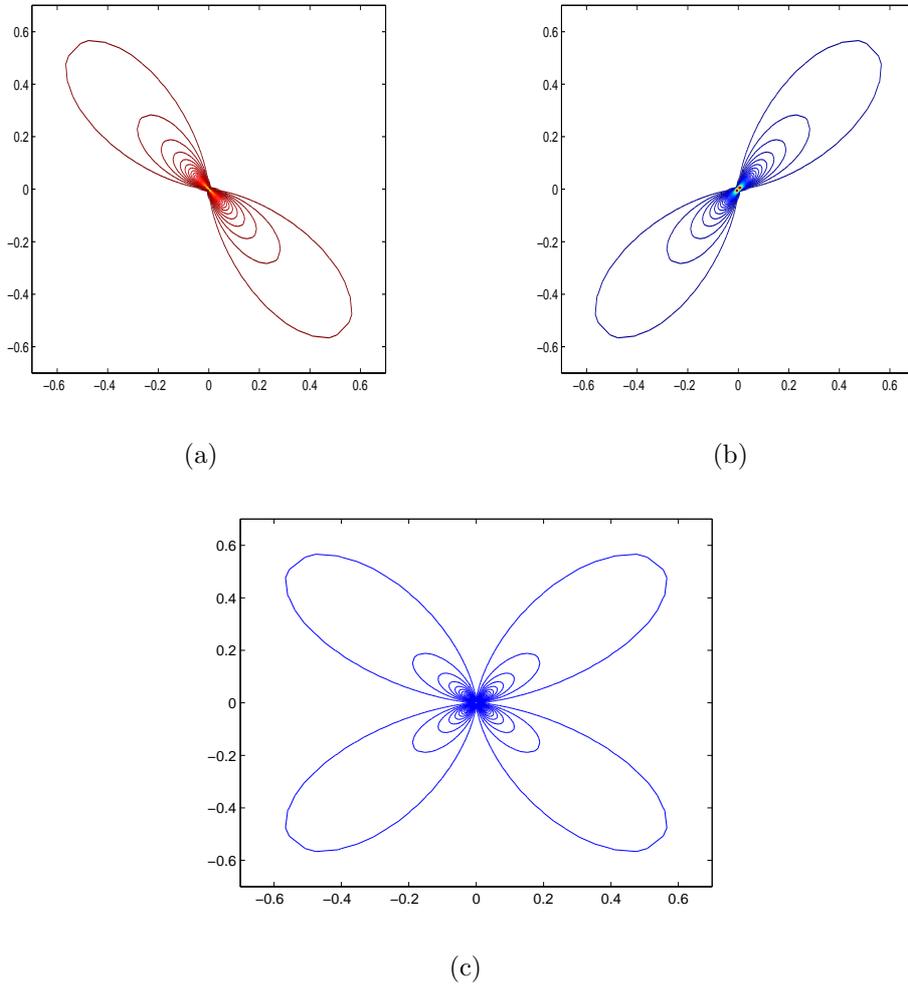


FIG. 6.4. Lévy density of the bivariate CGMY process with dependence structure given by the Clayton Lévy copula of equation (6.29).

for all $I \subset \{1, \dots, d\}$. Since F is d -increasing and continuous, there exists a positive measure μ on \mathbb{R}^d with Lebesgue margins such that for all $a, b \in \mathbb{R}^d$ with $a \leq b$,

$$V_F((a, b]) = \mu((a, b]). \quad (6.32)$$

Define $f : (u_1, \dots, u_n) \mapsto (U_1^{-1}(u_1), \dots, U_d^{-1}(u_d))$ and let μ be defined by (6.32). For each $A \in \mathcal{B}(\mathbb{R}^d)$, the relation between the Lévy measure ν and the measure μ is

$$\begin{aligned} \nu(A) &= \mu(\{u \in \mathbb{R}^d : f(u) \in A\}) \\ &= \mu(f^{-1}(A)). \end{aligned}$$

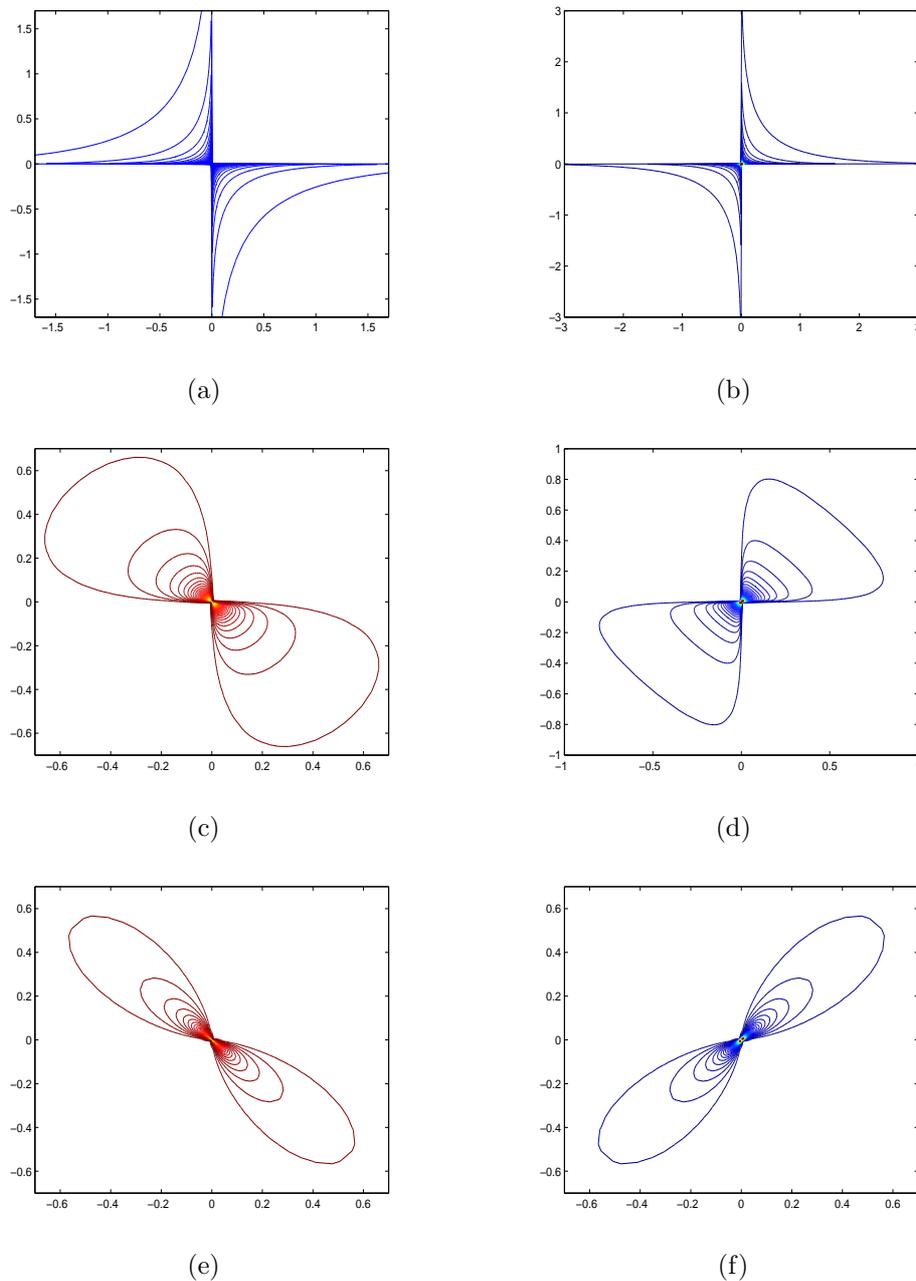


FIG. 6.5. Lévy density of the bivariate CGMY process with dependence structure given by the Clayton Lévy copula of equation (6.29). In figure 6.5(a) and 6.5(b) $\theta = 0.5$, in 6.5(c) and 6.5(d) $\theta = 0.8$, and 6.5(e) and 6.5(f) $\theta = 3$.

ν is then the image measure of μ by f (cf. lemma 5.4 in [73]). By [73] (see section 5.2 and references therein), there exists a family indexed by $\xi \in \mathbb{R}$, of positive random measures

$K(\xi, dx_2, \dots, dx_d)$ on \mathbb{R}^{d-1} , such that $\xi \mapsto K(\xi, dx_2, \dots, dx_d)$ is a Borel measure and

$$\mu(dx_1, dx_2, \dots, dx_d) = \lambda(dx_1) \otimes K(x_1, dx_2, \dots, dx_d).$$

$K(\xi, x_2, \dots, x_d)$ is called the family of conditional probability distribution associated to the Lévy copula F . Denoting

$$F_\xi(x_2, \dots, x_d) = K(\xi, (-\infty, x_2], \dots, (-\infty, x_d]),$$

there exists a set $N \in \mathbb{R}$ of zero Lebesgue measure such that for every fixed $\xi \in \mathbb{R} \setminus N$, F_ξ is a probability distribution function, satisfying

$$F_\xi(x_2, \dots, x_d) = \text{sgn}(\xi) \frac{\partial}{\partial \xi} V_F((\xi \wedge 0, \xi \vee 0] \times (-\infty, x_2] \times \dots \times (-\infty, x_d]), \quad (6.33)$$

in every point (x_2, \dots, x_d) where F_ξ is continuous. F_ξ determines the distribution of jump sizes of $d-1$ components at a given time t conditionally on the jump size of the one of the components.

The function F_ξ is very useful in the simulation of multidimensional Lévy process (of the corresponding Poisson random measure) when the dependence structure is given by the Lévy copulas. The simulation is done by first simulating the jumps in the first component and the jumps in the other components are simulated conditioning on the jumps in the first one (see section 5.2, [73]).

6.4 Simulation of Lévy Processes via Lévy Copula

For multidimensional Lévy processes there is no explicit formula for simulating the increments except for Brownian motion. To simulate multidimensional Lévy processes, one has to use the approximation method like compound Poisson process or series representation. One can also use Poisson and Gaussian approximation combined with series representations (see [21]).

Series representation also serves as a method for simulating multivariate Lévy processes when the dependence between components is given by a Lévy copula. The use of series representation in conjunction with Lévy copula originates from [73] (see also [74], [17]). In this setting, the simulation is based on the conditional distribution associated to the

Lévy copula. The idea is to simulate the jumps of the first component, then the jumps of the other components are simulated conditioning on the jumps of the first component. However, sometimes the simulation is too expensive because this method requires to have the analytical form of the inverse of the tail integral or some fast method to compute it numerically.

Chen [17] also argues that the Tankov's method for simulating multidimensional Lévy processes when the dependence structure is given by a Lévy copula has bias: the loss of jump mass when the dependence level is low and the numerical complexity in high dimension since the Tankov's algorithm is based on conditional probability that needs to be computed recursively. To overcome this problem, he therefore suggested a new method named SRLMD (series representation for Lévy processes with pre-specified marginals and pre-specified dependence) which is also based on series representation and avoids the conditional probability argument.

The idea behind the simulation of Lévy processes with dependence structure given by a Lévy copula is to first simulate the Poisson random measure M on $[0, 1]$ with intensity measure $dt \times \mu(dx)$ and then construct the Lévy process using the Lévy-Itô decomposition. Since the Lévy copula defines a positive measure μ on \mathbb{R}^d with Lebesgue marginals (see section 6.3.4), the conditional distribution of μ defines then the conditional distribution associated to the Lévy copula F .

For Lévy processes of finite variation, the simulation is simple compared to Lévy processes of infinite variation as shown in the following theorem.

Theorem 6.4.1. *(Simulation of multidimensional Lévy processes, finite variation).*

Let ν be a Lévy measure on \mathbb{R}^d with marginal tail integrals U_i , $i = 1, \dots, d$ and Lévy copula $F(x_1, \dots, x_d)$. Let $\{V_i\}$ be a sequence of independent random variables, uniformly distributed on $[0, 1]$. Introduce d random sequences $\Gamma_i^1, \dots, \Gamma_i^d$, independent from $\{V_i\}$ as follows

- . Γ_i^1 is a sequence of jump times of a Poisson process with intensity 1.
- . Conditionally on Γ_i^1 , the random vector $\Gamma_i^2, \dots, \Gamma_i^d$ is independent from Γ_j with $j \neq i$ and has distribution function (6.33).

Then

$$(Z_t)_{0 \leq t \leq 1}, \text{ where } Z_t^k = \sum_{i=1}^{\infty} U_k^{(-1)}(\Gamma_i^k) 1_{[0, t]}(V_i), \quad k = 1, \dots, d, \quad (6.34)$$

is a Lévy process on the time interval $[0, t]$ with characteristic function

$$e^{i\langle u, Z_t \rangle} = \exp\left(t \int_{\mathbb{R}^d} (e^{i\langle u, z \rangle} - 1) \nu(dx)\right). \quad (6.35)$$

Proof. By (6.33), $K(x_1, \cdot)$ is a probability distribution for almost all x_1 . So, Γ_i^k are well defined. Let

$$Z_{\tau, t}^k = \sum_{-\tau \leq \Gamma_i^1 \leq \tau} U_k^{(-1)}(\Gamma_i^k) 1_{V_i \leq t}, \quad k = 1, \dots, d.$$

By [73] (and reference therein),

$$Z_{\tau, t}^k = \int_{[0, t] \times [-\tau, \tau] \times \mathbb{R}^{d-1}} U_k^{(-1)}(x_k) M(ds \times dx_1 \dots dx_d),$$

where M is a Poisson random measure on $[0, 1] \times \mathbb{R}^d$ with intensity measure $dt \times \mu(dx_1, \dots, dx_d)$, and μ is defined by (6.32). By Lemma 5.4 in [73] and reference therein,

$$Z_{\tau, t}^k = \int_{[0, t] \times \mathbb{R}^d} x_k N_\tau(ds \times dx_1 \dots dx_d), \quad (6.36)$$

for some Poisson random measure N_τ on $[0, 1] \times \mathbb{R}^d$ with intensity measure $ds \times \mu(dx_1, \dots, dx_d)$, where

$$\nu_\tau = 1_{(-\infty, U_1^{(-1)}(-\tau)] \cup [U_1^{(-1)}(-\tau), \infty)}(x_1) \nu(dx_1 \dots dx_d). \quad (6.37)$$

The Lévy-Itô decomposition (2.2.1) implies that $Z_{\tau, t}$ is a Lévy process on the time interval $[0, 1]$ with characteristic function

$$\mathbb{E} [e^{i\langle u, Z_{\tau, t} \rangle}] = \exp\left(\int_{\mathbb{R}^d} (e^{i\langle u, z \rangle} - 1) \nu_\tau(dz)\right).$$

Now consider a bounded continuous function such that $h(x) \equiv x$ on a neighborhood of 0. Since $\lim_{\tau \rightarrow \infty} U_1^{(-1)}(\tau) = 0$ and $\lim_{\tau \rightarrow \infty} U_1^{(-1)}(-\tau) = 0$, by dominated convergence,

$$\int_{\mathbb{R}^d} h^2(x) \nu_\tau(dx) \xrightarrow{\tau \rightarrow \infty} \int_{\mathbb{R}^d} h^2(x) \nu(dx),$$

and

$$\int_{\mathbb{R}^d} h(x) \nu_\tau(dx) \xrightarrow{\tau \rightarrow \infty} \int_{\mathbb{R}^d} h(x) \nu(dx).$$

Moreover, for every $f \in C_b(\mathbb{R}^d)$ such that $f(x) \equiv 0$ on a neighborhood of 0,

$$\int_{\mathbb{R}^d} f(x) \nu_\tau(dx) = \int_{\mathbb{R}^d} f(x) \nu(dx)$$

starting from sufficiently large τ . Corollary VII.3.6 in [35] allows us to conclude that $(Z_{\tau, t})_{0 \leq t \leq 1}$ converges in law to a Lévy process with characteristic function given by (6.35). \square

When the Lévy process is of infinite variation, we have to introduce a centering term into the series because it is no longer a sum of the jumps. The following theorem describe the simulation procedure.

Theorem 6.4.2. (*Simulation of multidimensional Lévy process, infinite variation case.*) Let ν be a Lévy measure on \mathbb{R}^d with marginal tail integrals U_i , $i = 1, \dots, d$ and Lévy copula $F(x_1, \dots, x_d)$. Let $\{V_i\}$ and $\Gamma_i^1, \dots, \Gamma_i^d$ be as in theorem 6.4.1. Let

$$A_k(\tau) = \int_{|x| \leq 1} x_k dF(\tau \wedge U_1(x_1), \dots, U_d(x_d)), \quad k = 1, \dots, d. \quad (6.38)$$

Then the process

$$(Z_{\tau,t})_{0 \leq t \leq 1}, \quad \text{where } Z_{\tau,s}^k = \sum_{\Gamma_i^1 \leq \tau} U_k^{(-1)}(\Gamma_i^k) 1_{V_i \leq s} - s A_k(\tau), \quad (6.39)$$

converges in law as $\tau \rightarrow \infty$ to a Lévy process on the time interval $[0, 1]$ with characteristic function

$$\mathbb{E}[e^{i\langle u, Z_t \rangle}] = \exp\left(t \int_{\mathbb{R}^d} (e^{i\langle u, z \rangle} - 1 - \langle u, z \rangle) 1_{|z| \leq 1} \nu(dx)\right). \quad (6.40)$$

Proof. The proof of this theorem is similar to that of theorem 6.4.1 but we have now to add a centering term into the series. In this case $Z_{\tau,t}^k$ is represented by

$$\begin{aligned} Z_{\tau,t}^k &= \int_{[0,t] \times \{x \in \mathbb{R}^d : |x| \leq 1\}} x_k \{N_\tau(ds \times dx_1 \dots dx_d) - ds \nu_\tau(dx_1 \dots dx_d)\} \\ &\quad + \int_{[0,t] \times \{x \in \mathbb{R}^d : |x| > 1\}} x_k N_\tau(ds \times dx_1 \dots dx_d), \end{aligned}$$

where N_τ is a Poisson random measure on $[0, t] \times \mathbb{R}^d$ with intensity measure $ds \nu_\tau$, and ν_τ is defined by (6.37). This entails that $(Z_{\tau,t})$ is a Lévy process (compound Poisson) with characteristic function

$$\mathbb{E}[e^{i\langle u, Z_{\tau,t} \rangle}] = \exp\left(t \int_{\mathbb{R}^d} (e^{i\langle u, z \rangle} - 1 - \langle u, z \rangle) 1_{|z| \leq 1} \nu_\tau(dx)\right).$$

Again corollary VII.3.6 in [35] allows us to conclude that $(Z_{\tau,t})_{0 \leq t \leq 1}$ converges in distribution to a Lévy process with characteristic function given by (6.40). \square

6.4.1 Simulation of Variance Gamma and CGMY Processes

To illustrate how the technique discussed above works, we simulated the trajectories of a two-dimensional variance gamma process and a two-dimensional CGMY process with

dependence structure given by the Clayton Lévy copula of example 6.3.2. The conditional distribution F_ξ and its inverse were calculated in [74] and are given by

$$F_\xi(x_2) = \{(1 - \eta) + (1 + |\frac{\xi}{x_2}|^\theta)^{-1-\frac{1}{\theta}}(\eta - 1_{x_2 < 0})\}1_{\xi \geq 0} \quad (6.41)$$

$$+ \{\eta + (1 + |\frac{\xi}{x_2}|^\theta)^{-1-\frac{1}{\theta}}(1_{x_2 \geq 0} - \eta)\}1_{\xi < 0},$$

and

$$F_\xi^{-1}(u) = B(\xi, u) | \xi | \{C(\xi, u)^{-\frac{\theta}{\theta+1}} - 1\}^{-\frac{1}{\theta}},$$

with

$$B(\xi, u) = \text{sgn}(u - 1 + \eta)1_{\xi \geq 0} + \text{sgn}(u - \eta)1_{\xi < 0},$$

and

$$C(\xi, u) = \left\{ \frac{u - 1 + \eta}{\eta} 1_{u \geq 1 - \eta} + \frac{1 - \eta - u}{1 - \eta} 1_{u < 1 - \eta} \right\} 1_{\xi \geq 0}$$

$$+ \left\{ \frac{u - \eta}{1 - \eta} 1_{u \geq \eta} + \frac{\eta - u}{\eta} 1_{u < \eta} \right\} 1_{\xi < 0}.$$

In both cases the number of jumps for each trajectory was limited to 1000 and the inverse tail integrals of the variance gamma Lévy measure and CGMY Lévy measure were computed numerically. The variance gamma is of finite variate, while the CGMY process is of finite variation for $Y < 1$, and is of infinite variation if $Y \in (1, 2)$. We fixed Y to 0.5, and thus we used theorem 6.4.1 in our simulation. The simulated trajectories of the two-dimensional variance gamma process with no drift and two-dimensional CGMY process are depicted in figures (6.6) and (6.7) respectively.

In the left graphs, the copula's parameters are $\theta = 0.5$ and $\eta = 0.25$, which correspond to weak dependence between the components. As you can see from the figures, the two processes jump in opposite directions. In the right graphs, $\theta = 5$ and $\eta = 0.75$. The dependence structure between the two components is strong both in terms of sign and absolute values. Thus the two processes jump mostly in the same direction. In figure (6.8), the parameters were estimated from INTC and IBM. Similarly to the case of one dimensional the sample paths look like the stock prices.

Table (6.1) represent the amount of time (in seconds) taken for simulating the two-dimensional variance gamma and two-dimensional CGMY processes on the Intel Celeron

computer, in the case of weak dependence as well as strong dependence. However, this computational time can be reduced if a faster method is used to invert the tail integrals of the variance gamma and CGMY Lévy measures. Note that the tail integrals of the variance gamma and CGMY Lévy measure must be computed numerical.

Dependence pattern	VG process	CGMY process
Weak dependence	0.34318	9.9706
Strong dependence	0.34306	9.6508

TABLE. 6.1. Time (in seconds) taken to simulate the two-dimensional variance gamma process and the two-dimensional CGMY process.

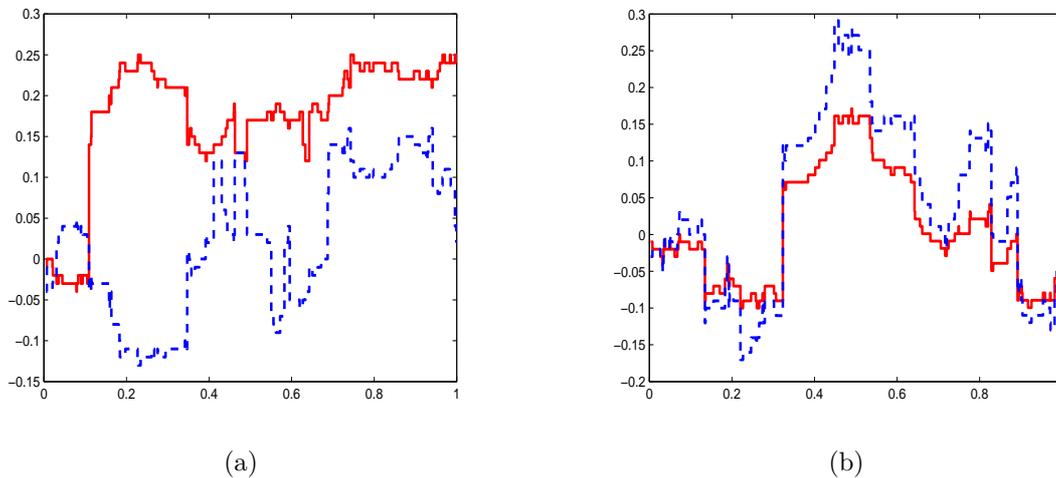


FIG. 6.6. Trajectories of two-dimensional variance gamma process with dependence structure given by the Clayton Lévy copula of example 6.3.2. In both graphs, the variance gamma are driftless and have parameters $C_1 = 9.5567$, $C_2 = 9.713$, $G_1 = 15.1567$, $G_2 = 12.713$, $M_1 = 10.5193$, and $M_2 = 11.5193$. The left panel corresponds to weak dependence with the Lévy copula's parameters $\theta = 0.5$ and $\eta = 0.25$. In the right panel $\theta = 5$ and $\eta = 0.75$ which correspond to a strong dependence.

To summarize, we discussed the dependence structure between the components of a d -dimensional Lévy process. Since the continuous part of a Lévy process is completely characterized by the covariance matrix, we have concentrated on the pure jump part that must be studied using the Lévy measure. The dependence structure of the pure jump part of a Lévy process is completely characterized by the Lévy copula function. Together with marginals Lévy measures they completely describe multivariate Lévy measure on $\mathbb{R}^d \setminus \{0\}$.

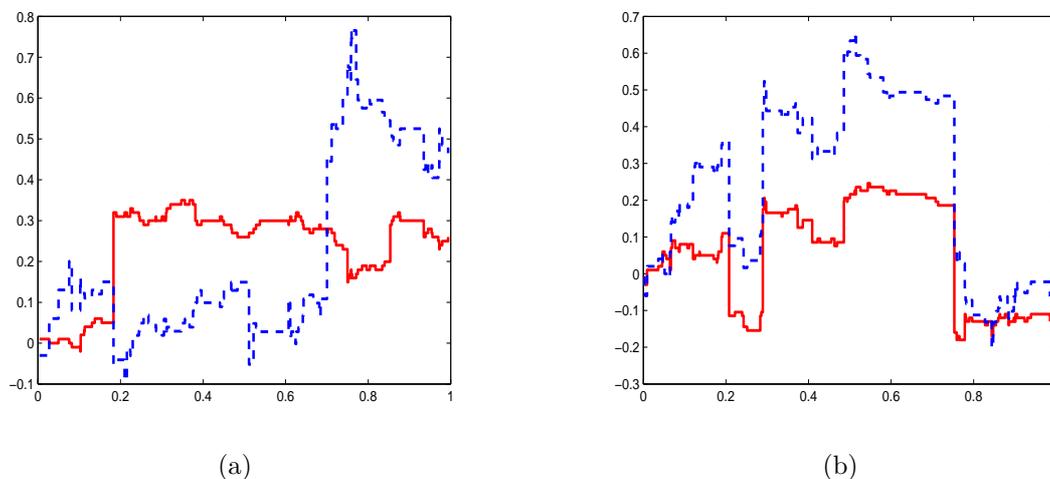


FIG. 6.7. Trajectories of two-dimensional CGMY process with dependence structure given by the Clayton Lévy copula of example 6.3.2. In both graphs, the CGMY parameters are $C_1 = 10.56$, $C_2 = 11.25$, $G_1 = 15.75$, $G_2 = 15.25$, $M_1 = 14.75$, $M_2 = 13.895$ and $Y_1 = Y_2 = 0.5$. The left panel corresponds to weak dependence with the Lévy copula's parameters $\theta = 0.5$ and $\eta = 0.25$. In the right panel $\theta = 5$ and $\eta = 0.75$ which correspond to a strong dependence.

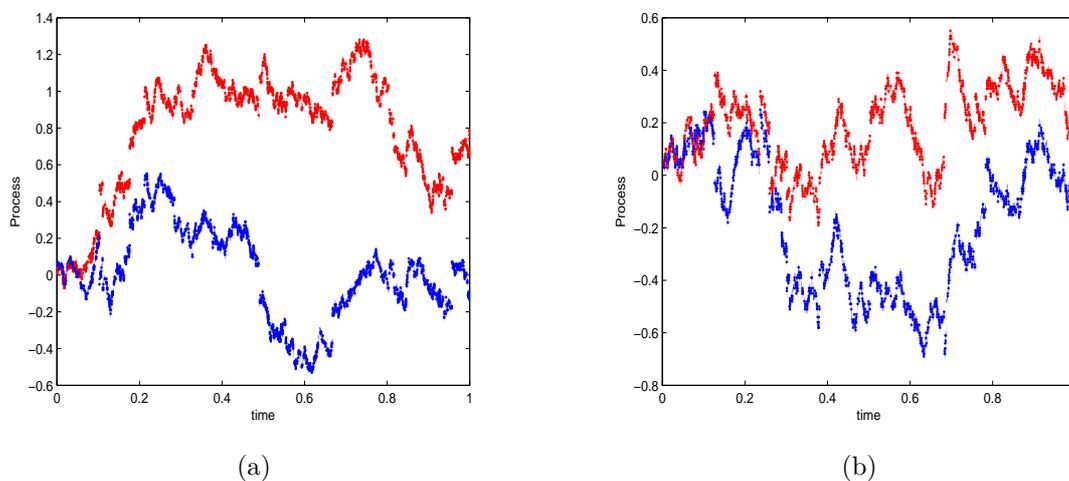


FIG. 6.8. On the left: The trajectories of the bivariate variance gamma with the parameters estimated by MLE on INTC and IBM. On the right: The trajectories of the bivariate CGMY process.

We stated the Sklar's theorem for Lévy processes and we discussed method for constructing parametric Lévy copulas which turns out to be useful in finance.

We also discussed simulation of d -dimensional Lévy processes when the dependence structure is given by the Lévy copulas. The simulation is based on series representations where one has to simulate the jumps of the first components and the jumps of the other are simulated conditioning on the jumps of the first one. We showed that when the Lévy process's parameters are calibrated from the market data, the sample paths of the Lévy process look like the sample paths of the stock prices.

Chapter 7

Multivariate Model and Option Pricing

One of the open problems in mathematical finance is the extension of the risk-neutral valuation technique to the multivariate case, that is, the case of options written on more than one underlying asset. Multivariate options can be in the form of calls (or puts) that gives the right to buy (or to sell) the best or the worst performer of a number of underlying assets, an option on the difference between the prices of underlying assets, or an option on the maximum or minimum of the underlying assets.

The computation of the prices and hedges for multivariate options is difficult because closed form formulas are not available in most exponential Lévy models. The key point in evaluating multivariate options is the determination of dependence between the underlying assets. For example, when pricing basket options, one needs to estimate the dependence structure from the historical time series of asset returns and the risk-neutral marginals to price the option. Therefore, one needs to be able to separate the dependence structure from the margins.

The separation of the dependence structure from the margins is very important in mathematical finance. It allows to price multivariate products consistently with the information stemming from the marginal ones. It is also a big advantage in the calibration procedure, because it reduces considerably the computational complexity: It can now be done in two steps instead of one.

In this chapter, we construct multivariate models with jumps and we use these models to price multi-asset options. The construction of multivariate models with jumps is not easy. However, some effort has been made to construct multivariate models, and different parametric copulas have been used to price multivariate options. Xia [78] used a linear combination of independent variance gamma processes to model and price multi-asset options. Luciano and Schoutens [45] constructed a multivariate variance gamma model by time-changing a multivariate Brownian motion by a univariate gamma process. They obtained closed form formula for the marginal distributions and the joint distribution and used the distributional copula function to price multivariate options in equity and credit risk. Chen [17] used a VG copula to characterize the dependence structure between DELL, IBM, INTC and MSFT and to price basket options on these underlying assets.

We suggest a different method which is based on the concept of Lévy copulas. We follow Tankov [73] and we model the dependence structure between the underlying assets by a Lévy copula. This yields models with comparably few parameters, which, particularly in the light of sparse data at hand, may be a viable alternative to other more complex models. Our model has the advantage that the dependence structure between the underlying assets can be separated from the univariate marginal assets S^i , $i = 1, \dots, d$.

The objective of this chapter is to discuss the effect of the dependence structure to the pricing of multivariate models constructed using Lévy copula and to present a case study of the pricing of multi-asset options. We consider the same set of data (the six stocks considered in chapter 4) and show the existence of dependence between the stock prices by estimating the Lévy copula's parameters. To stress the importance of dependence in option pricing, we consider different set of dependence and then compare with the option prices in the Black-Scholes framework.

We will only consider the European call options. The simplest of such options is the European call option on the weighted average. Given the weights ω_1 and ω_2 with $\omega_1 + \omega_2 = 1$, a strike price K and maturity time T , the payoff of European call options on two names is defined by

$$C(t, T, K) = \left(\sum_{i=1}^2 \omega_i S^i(T) - K \right)^+. \quad (7.1)$$

Other examples of multivariate options that we would like to price are the rainbow op-

tions. Rainbow options are multivariate contingent claims whose payoff is the maximum or minimum between underlying assets. For example, the payoff of the rainbow call option on the maximum between two assets is given by

$$C(S^1, S^2, t; K, T) = \max(\max(S^1(T), S^2(T)) - K, 0), \quad (7.2)$$

and that of call option on the minimum between two assets is given by

$$C(S^1, S^2, t; K, T) = \max(\min(S^1(T), S^2(T)) - K, 0). \quad (7.3)$$

The payoff of put option on the maximum between two assets is given by

$$C(S^1, S^2, t; K, T) = \max(K - \max(S^1(T), S^2(T)), 0), \quad (7.4)$$

and the payoff of put option on the minimum between two assets is given by

$$C(S^1, S^2, t; K, T) = \max(K - \min(S^1(T), S^2(T)), 0). \quad (7.5)$$

The models are to be variance gamma and CGMY models. Note that in the Black-Scholes framework, analytical pricing solutions for all these options are available (see for example [69], [18], [36]).

7.1 Construction of Multivariate Lévy Model

In the one-dimensional problem, models based on Lévy processes have proved capable to model the skewness and kurtosis observed from the time series data of financial market [16], [49], [66]. As for the case of multivariate option, the dependence between underlying assets described by a Gaussian structure is not realistic, mainly because the joint normal distribution does not exhibit tail dependence. Therefore, our aim is to build more realistic models, incorporating jumps, and non-Gaussian dependence structure.

Here we opt to work with the exponential-Lévy model consisting of a d -dimensional Lévy process X_t with triplet (A, ν, γ) , a constant risk-free rate r , and a vector of positive, constant initial prices $S_0 = S_0^i$, $i = 1, \dots, d$. We model the price process as the exponential of the d -dimensional Lévy process X_t . The univariate marginals are given by

$$S^i(t) = S_0^i \exp(rt + X_t^i), \quad t \geq 0. \quad (7.6)$$

According to Cont and Tankov (cf. proposition 9.9 [22]), our model is arbitrage-free because the Lévy processes X_t^i are neither almost surely increasing nor almost surely decreasing. Since this is an exponential-Lévy model which is arbitrage-free, there exists an equivalent martingale measure \mathbb{Q} under which the discounted stock price processes are martingales [22]. However, the model belongs to the class of incomplete market models; the equivalent martingale measure is not unique. Among the possible equivalent martingale measure, we select the mean-correcting one (the change of measure is done in analogy to the Black-Scholes setting), in which the (historical) mean parameter is changed into a new parameter in order to make the model risk-neutral [66]. More precisely, the risk-neutral dynamics for the asset prices are given by

$$S^i(t) = S_0^i \exp((r + \omega_i)t + X_t^i), \quad (7.7)$$

where r is the continuously compounded interest rate, and ω_i a parameter used to ensure the martingale property of the discounted stock price process $e^{-rt}S_t^i$.

The general question that arises is, how can we model the dependence between the underlying assets such that all the dependence structure is captured? In other words, how can we model the dependence structure between the components of the Lévy process $X_t = (X_t^1, \dots, X_t^d)$? To answer this question, we turn to the modeling of the dependence structure between the components of the Lévy process $X_t = (X_t^1, \dots, X_t^d)$.

In chapter 6, we learnt that the dependence structure between components of a multivariate pure jump Lévy process can be reduced to the Lévy measure. The Lévy measure controls the jumps behavior of a Lévy process and can be interpreted economically as follows: The Lévy measure determines the frequency and size of moves/jumps (downwards and upwards) of the stock prices. Since we are interested in large moves/jumps, it is convenient to work with tail integral of the Lévy measure and to model dependence between jumps by a Lévy copula (see chapter 6). Here, we only have to substitute the Lévy process in the exponential Lévy model by a d -dimensional Lévy process with dependence structure given by a Lévy copula to obtain a d -dimensional exponential Lévy model.

We will focus our attention on the multivariate variance gamma model and multivariate CGMY model, that is, in (7.6) the Lévy process X_t will be either the variance gamma process or the CGMY process. Because a bivariate model is particularly useful to illustrate how dependence modeling via Lévy copula works, we will focus on two-dimensional case.

7.1.1 Bivariate Variance Gamma Model

The variance gamma process is obtained by time-changing a Brownian motion with drift by a gamma process (cf. subsection, 3.3.2), [48], [49]. A multidimensional variance gamma process is obtained by time-changing a d -dimensional Brownian motion by a common gamma process. Therefore, multivariate variance gamma model is obtained by exponentiating a multidimensional variance gamma process. More precisely, we suppose that under the risk-neutral probability, the prices S_t^1 and S_t^2 of two risky assets is given by

$$S_t^i = \exp((r + \omega_i)t + X_t^i), \quad (7.8)$$

where r is the constant continuously compound interest rate and X_t^i , $i = 1, 2$ are variance gamma processes and $\omega_i = \alpha^{-1} \log(1 - \frac{1}{2}\sigma_i^2\alpha - \theta_i\alpha)$, where θ is the drift of the Brownian motion, σ its volatility, and α the variance of the gamma process. The parameter θ controls over the skewness and α the kurtosis.

For this type of model, a jump in the time change leads to a jump in the processes and hence all moves/jumps (small and the big ones) occur simultaneously. However the jump-sizes are caused by the individual Brownian motion. We now turn to the modeling of the dependence structure between X_t^1 and X_t^2 .

For financial markets, any sensible model must allow dependence of positive jumps and negative jumps, that is, modeling dependence in every quadrant of \mathbb{R}^2 . The model should allow the dependence of positive jumps of the first component with the positive jumps of the second, dependence of the negative jumps of the first component and positive jumps of the second and so on. In this case, the Lévy measure is supported on $\mathbb{R}^2 \setminus \{0\}$ and a parametric copula can be constructed to capture the dependence pattern. We consider the bidirectional Clayton Lévy copula (for the sake of the reader, we recall the bidirectional Clayton Lévy copula in equation (7.9)) to model the dependence structure between the underlying assets.

$$F(u, v) = (|u|^{-\theta} + |v|^{-\theta})^{-\theta} (\eta 1_{uv \geq 0} - (1 - \eta) 1_{uv < 0}). \quad (7.9)$$

To our knowledge the Clayton Lévy copula (7.9) is the only example of bidirectional parametric Lévy copula that is discussed in the literature. Secondly, it has a simple parametrization—only two parameters. The Clayton Lévy copula covers the whole range of dependence

with independence if $\eta = 1$ and $\theta \rightarrow 0$ and complete dependence if $\eta = 1$ and $\theta \rightarrow \infty$. By varying θ and η , the dependence of returns changes smoothly between the two extremes.

Figure (7.1) depicts the scatter plot of log-returns in a two-dimensional variance gamma model with different patterns. In each graph, we simulated a sample of 1000 realizations of the couple (X_t^1, X_t^2) using the procedure described in section 6.4. We then simulated the two stock price processes by

$$S^i = S^i \exp((r + \omega_i)t + X_t^i), \quad i = 1, 2. \quad (7.10)$$

The log-returns are then obtained by

$$r_t^i = \log \left(\frac{S_{t+1}^i}{S_t^i} \right).$$

In both graphs the variance gamma processes are driftless and the marginals' parameters are $\theta_1 = \theta_2 = -0.039$, $\alpha_1 = \alpha_2 = 0.106$, $\sigma_1 = 0.25$, and $\sigma_2 = 0.3$. In the parametrization (3.35) (see also [49]), they correspond to $C_1 = C_2 = 9.4340$, $G_1 = 14.9189$, $G_2 = 18.0101$, $M_1 = 14.0522$ and $M_2 = 16.7621$. For $\theta < 0$, the risk-neutral distribution is negatively skewed which is a consequence of risk aversion in facing the risk of price jump. Moreover, the risk-neutral density should present a significant excess kurtosis.

In the left panel, $\theta = 0.5$ and $\eta = 0.25$ which corresponds to weak tail dependence. The returns have the same sign but their absolute values are weakly correlated. In the right panel $\theta = 5$ and $\eta = 0.75$, and there is a strong tail dependence. Although the signs of returns may be different, the probability that the returns will be large in absolute value simultaneously in both components is very high.

7.1.2 Bivariate CGMY Model

For a single name, the CGMY process successfully explains the physical returns from financial time series data and also nicely captures the risk-neutral measure from the option surface as we have seen in chapter 4. The goal of this subsection is to extend the univariate CGMY model to the bivariate case.

We model the bivariate CGMY model by

$$S_t^i = S_0^i \exp((r + \omega_i)t + X_t^i), \quad (7.11)$$

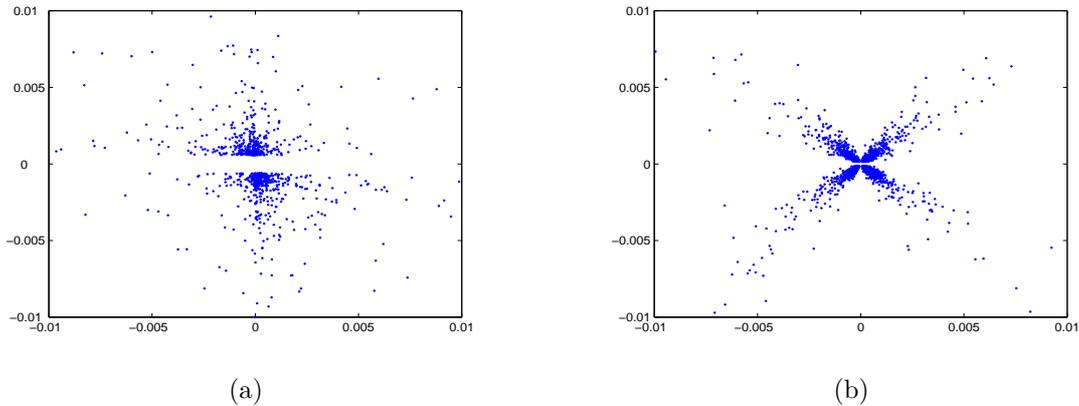


FIG. 7.1. The scatter plot of the log-returns of two-dimensional variance gamma model with different patterns. In the left panel $\theta = 0.5$ and $\eta = 0.25$ which corresponds to weak tail dependence. In the right panel $\theta = 5$ and $\eta = 0.75$ which correspond to a strong dependence.

where r is the constant continuously compound interest rate, X_t^i are CGMY processes, and $\omega_i = -C_i \Gamma(Y_i) ((M_i - 1)^{Y_i} - M_i^{Y_i} + (G_i + 1)^{Y_i} - G_i^{Y_i})$.

We now turn to the construction of the bivariate CGMY process. We have seen in chapter 6 that parametric copulas allow to construct multidimensional Lévy processes and all dependence structure can be parametrized through the copula's parameters. Moreover, the number of parameters does not depend on the dimension of the process. Thus, to construct bivariate CGMY process, we compute the marginal tail integral of the CGMY Lévy measure and then glue/couple them using a parametric Lévy copula.

Given 2 one-dimensional tail integrals $U_1(x_1), U_2(x_2)$ with Lévy measures $\nu_1(x_1), \nu_2(x_2)$ respectively, and the Lévy copula F , we define the 2-dimensional tail integral by (cf. subsection 6.3.1 equation (6.22))

$$U(x_1, x_2) = F(U_1(x_1), U_2(x_2)) \quad (7.12)$$

For the CGMY Lévy measure supported on $\mathbb{R} \setminus \{0\}$, the marginal tail integrals are divided into two tail integrals corresponding to the positive and negative real axis, which results in a function decreasing on both sides of zero. Hence, the marginal tail integrals are given by

$$U_i^+(x_i) = \nu_i([x_i, \infty)) \text{ for } x > 0, \quad (7.13)$$

and

$$U_i^-(x_i) = -\nu_i((-\infty, x_i)) \text{ for } x < 0. \quad (7.14)$$

The two dimensional tail integral of the Lévy measure $\nu(x_1, x_2)$ can then be defined for $x_i \neq 0$ as

$$U(x_1, x_2) = F(U_1(x_1), U_2(x_2)), \quad (7.15)$$

where [77]

$$U_i(x) = U_i^+(x)1_{x>0} + U_i^-(x)1_{x<0},$$

Lévy copula and tail integrals together lead to a consistent method for modeling the dependence structure of multivariate model. In this setting some complex models which were difficult to parametrize can be dealt with in a very simple manner.

Madan and Yor [79] showed that the CGMY process can be written as a Brownian motion time changed by one sided $\frac{Y}{2}$ -stable process. One can then take a d -dimensional Brownian motion and time-change it by a univariate subordinator to construct a multivariate CGMY model. In this case the jump in the time-change leads to a jump in the processes and all jumps (small and the big ones) occur simultaneously. With the Lévy copulas, one can then model their dependence structure.

Figure (7.2) depicts the scatter plot of log-returns in a two-dimensional CGMY model with different patterns. In each graph, we used the same procedure as in the bivariate variance gamma model to generate the bivariate returns data. In both graphs the marginals' parameters are $C_1 = 2.5567$, $C_2 = 2.5263$, $G_1 = 4.713$, $G_2 = 5.358$, $M_1 = 4.5193$, $M_2 = 5.2591$, $Y_1 = 0.5915$, and $Y_2 = 0.6105$. The parameters were chosen such that the risk-neutral distribution presents heavy tails. Moreover, if $G < M$, the left tail is heavier than the right tail, which is consistent with the risk-neutral distribution implied from the option data [16]. The parameter Y was chosen such that the Lévy process be completely monotonic(this property translate that large jumps occur at a small rate than small jumps), infinite activity, and has a finite variation a feature which is attractive from the point of view that it allows the separation of the up and down tick modeling of the market [47].

In the left panel $\theta = 0.5$ and $\eta = 0.25$ which corresponds to weak tail dependence. The returns have the same sign but their absolute values are weakly correlated. In the right panel $\theta = 5$ and $\eta = 0.75$ and there a strong tail dependence. Although the signs of

returns may be different, the probability that the returns will be large in absolute value simultaneously in both components is very high.

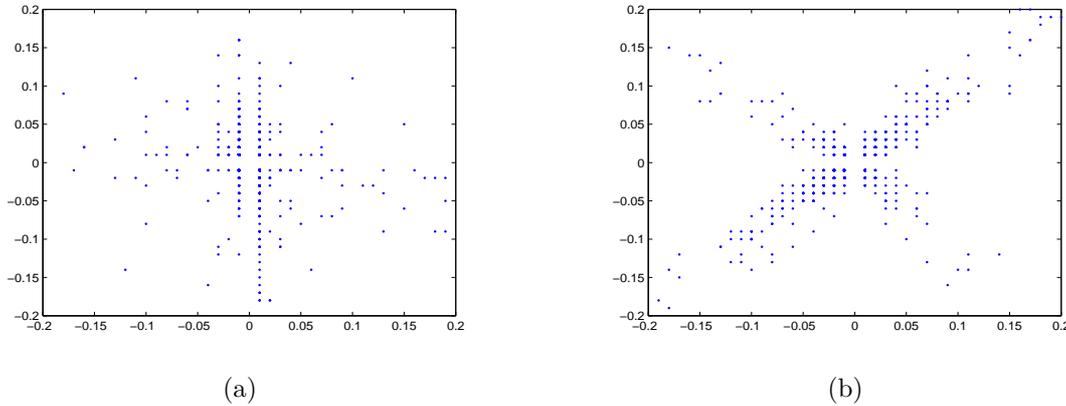


FIG. 7.2. The scatter plot of log-returns of two-dimensional CGMY model with different pattern. In the left panel $\theta = 0.5$ and $\eta = 0.25$. Thus, there is a weak tail dependence. In the right panel $\theta = 5$ and $\eta = 0.75$ which correspond to a strong tail dependence.

7.2 Statistical Inferences

Statistical properties are very important because they allow us to study the performance of the model on a dataset. For multivariate models, the estimation of parameters is a subtle issue. However, the separation of dependence structure from the margins simplifies the estimation procedure since it can be done in two steps: estimate the marginals parameter, then the copula parameters.

Kallsen and Tankov proved (see theorem 6.3.2 or theorem 6.1 in [40]) that for all points where the Lévy copula is unique, the Lévy copula can be recovered from the ordinary copula at small fixed time t . We fixed a small time $t > 0$ and extract the Lévy copula from the distributional copula. We first model the marginals with variance gamma or CGMY processes. The marginal dynamics of the stock price process is given by

$$S_t^{(i)} = S_0 \exp \left((m + \omega^i)t + X^{(i)}(t) \right), \quad (7.16)$$

where $X(t)$ is the variance gamma or CGMY process,

$$\omega = -\frac{1}{t} \ln(\phi(-i))$$

and m is the mean rate of return on stock under the statistical probability measure and ϕ is the characteristic function of the variance gamma or CGMY processes.

To estimate the model parameters, we first estimate the marginal parameters using MLE. The density function of the variance gamma and CGMY process is obtained by inverting the characteristic function using FFT. The second step is to transform the log-return data into uniformly distributed random variables. We use the estimated parameters and theorem 4.3.1 to get the CDF $u_i^j = F(r_i^j)$, where r_i^j is the i -th daily log-return for name j . The fact that the marginal distribution are uniform is a sign that the influences of the original marginal distributions have been removed from the data. The only remaining feature is the way the numbers u_i, v_i are paired, and we claim that the dependence between the log-returns is captured by the way these coupling are done.

Let F_X and F_Y be the marginal distribution function of the process X_t and Y_t at time t and let C_t be an ordinary copula. The bivariate distribution function $F_t(x, y)$ of the two-dimensional process at time t is given as

$$F_t(x, y) = C_t(F_X(x), F_Y(y)) \quad (7.17)$$

The density function of the bivariate process is given by

$$\frac{\partial^2 F_t(x, y)}{\partial x \partial y} = \frac{\partial^2 C_t(u, v)}{\partial u \partial v} \Big|_{u=F_X(x), v=F_Y(y)} \frac{\partial F_X(x)}{\partial x} \frac{\partial F_Y(y)}{\partial y} \quad (7.18)$$

In equation (7.18), we need the density function of the copula C_t and the one-dimensional density function of the process considered. Notice that equation (7.18) can easily be extended to d -dimensional case.

7.2.1 Empirical Study

In this subsection, we analyze the performance of the bivariate variance gamma model and the bivariate CGMY model on a dataset of six names with ticker symbols INTC, IBM, AMZN, DELL, FDX, and ABC. The dataset was discussed in section 4.2. In the copula estimation procedure, we only need to estimate the copula's parameters as the marginal ones have already been estimated in section 4.2 and the results were reported in tables (4.1) and (4.2). We then compute the marginals CDF using the corresponding marginals

parameters. Therefore, we fit the Clayton ordinary copula

$$C(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}, \quad \theta > 0. \quad (7.19)$$

In our estimation procedure we faced the problem of initial guess. We tried different values and we found that for initial values of θ greater than 1 the estimated copula's parameters for all pairs is greater than 6 for the case of variance gamma and 10 for the case of CGMY distribution. For initial values of θ less than 0.1, the estimated copula's parameters are the same for all pairs in both case. We fixed the initial value to 0.5. We report the estimated parameters on the pairs for variance gamma and CGMY distribution with dependence structure given by the Clayton copula in tables (7.1) and (7.2) respectively.

Tickers	INTC	IBM	AMZN	DELL	FDX	ABC
INTC	4.071	3.127	1.975	1.745	1.921	3.028
IBM		4.071	2.942	1.958	1.693	2.4612
AMZN			4.071	2.438	2.660	2.905
DELL				4.071	2.317	3.009
FDX					4.071	2.578
ABC						4.071

TABLE. 7.1. Estimated parameter on pairs with dependence structure given by the Clayton copula of equation (7.19).

Tickers	INTC	IBM	AMZN	DELL	FDX	ABC
INTC	5.1250	4.6705	2.1705	3.7150	3.2150	2.3642
IBM		5.1250	3.6705	2.2390	3.2629	3.4422
AMZN			5.1250	3.6523	3.6318	2.9464
DELL				5.1250	2.7519	2.8172
FDX					5.1250	3.4750
ABC						5.1250

TABLE. 7.2. Estimated parameter on pairs with dependence structure given by the Clayton copula of equation (7.19).

In figures (7.3) and (7.4), we plotted the scatter plot of the daily log-returns on pairs INTC-IBM and AMZN-DELL for the two-dimensional variance gamma model and two-dimensional CGMY model with dependence structure given by the Clayton Lévy copula of equation (7.19). We fixed η to 0.5 (for $\eta \in (0, 1)$ positive jumps of one component

correspond to both positive and negative jumps of the other) and the values of θ correspond to those of table (7.1) and (7.2) respectively. These graphs clearly show that there is a high probability that the returns will be large in absolute value simultaneously in both components.

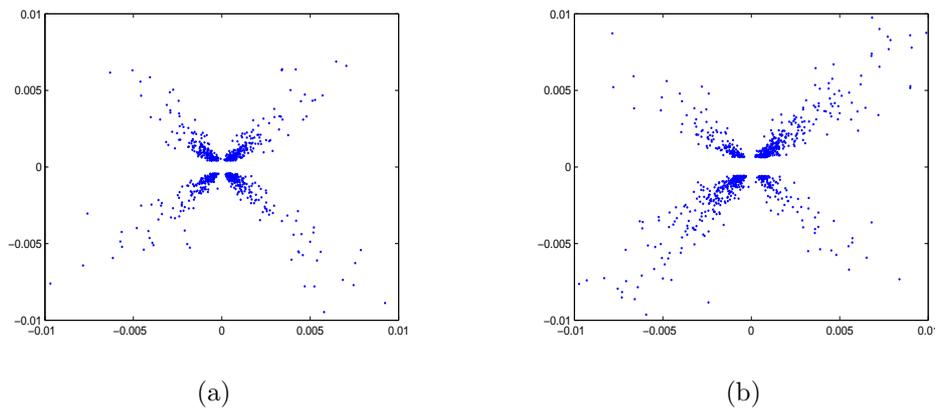


FIG. 7.3. On the left: Scatter plot of the daily log-returns of INTC and IBM for the period January 3rd 2007 to December 3th 2008. On the right: Scatter plot of the daily log-returns of AMZN and DELL for the same period.

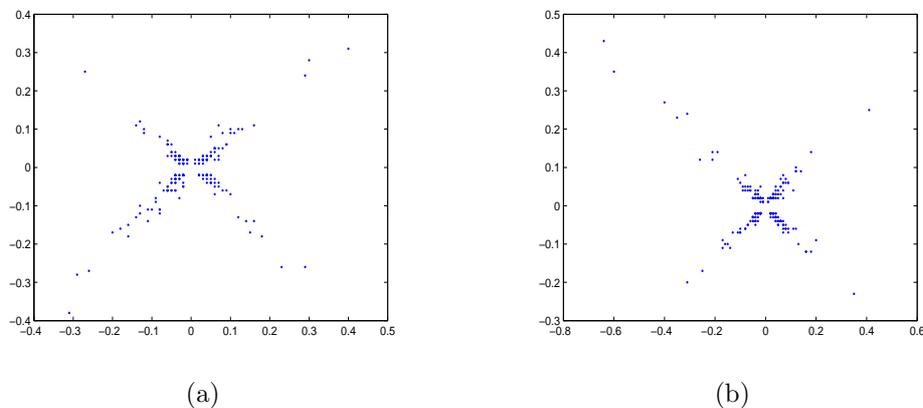


FIG. 7.4. On the left graph: Scatter plot of the daily log-returns of INTC and IBM, for the period of January 3rd 2007 to December 30th 2008. On the right : Scatter plot of the daily log-returns of AMZN and DELL for the same period

7.3 Pricing Multi-Asset Options

Pricing financial contracts is based on absence of arbitrage. For financial market models with assets driven by Lévy processes are in general incomplete. This means that not every contingent claim can be hedged completely, and hence one is forced to think about hedging strategies which cover the risk. In a complete market, the hedging problem of a final payoff is solved by investing in a portfolio and pursuing a self-financing trading strategy which produces the final payoff.

In many cases, single names are liquid assets and the risk can be minimized. As for the case of multivariate contingent claims, one faces the problem of hedging a large variety of different risks connected to derivative products which are often exotic and written on underlying assets that might not be actively traded on liquid markets. As a result, the hedging activity may rely on transaction on the over-the-counter (OTC) market, where counter-party risk component can be relevant. Accounting for counter-party risk in a derivative transaction imposes two condition on the value of the derivative contract: The contract ends in the money, and the counter-party survives until the contract is exercised [20]. As the copulas technique enables us to separate the dependence modeling from the marginal modeling, it is not difficult to foresee that they can serve as great help in the evaluation and hedging strategy of these products.

We now turn to the problem of pricing bivariate variance gamma and bivariate CGMY models constructed in section 7.1. In order to price the options, we need the historical dependence and the marginal risk-neutral measure. We perform a measure change on the marginals and we model the stock price process by

$$S^i(t) = S^i(0) \exp(rt + X_t^i + \omega_i t),$$

The bivariate options are priced by risk-neutral marginals as variance gamma or CGMY processes and the dependence is modeled by the Clayton Lévy copula of equation (7.9).

To price options, we use Monte Carlo simulation method. However, as plain Monte Carlo method brings an error in option price, we apply a variance reduction scheme [32], [55], which uses the technique of control variate to reduce the error.

Denote $V_T = e^{-rT} C(S^1, S^2, t; K, T)$ the discounted payoff of the bivariate option. To

obtain a more accurate option price, it is important to choose control variates which are highly correlated with V_T and with a convenient computable expectation value. We use the discounted European option on individual stock $V_T^i = e^{-rT}(S_T^i - K)^+$ as the control variate. The Monte Carlo estimate of the bivariate option price is then given by

$$\tilde{V}_0 = \bar{V}_T + a_1(\mathbb{E}[V_T^1] - \bar{V}_T^1) + a_2(\mathbb{E}[V_T^2] - \bar{V}_T^2), \quad (7.20)$$

where a bar over the random variable denotes the sample mean of that random variable; that is,

$$\bar{V}_T = \frac{1}{N} \sum_{i=1}^N V_T^i,$$

with V_T^i independent and have the same law as V_T . We follow the Carr and Madan [15] FFT method to compute the individual stock price and use Nelder-Mead simplex (direct search) method to minimize the difference between the model price and the market price. In this procedure, we only consider options of one maturity time T . The coefficient a_1 and a_2 are chosen so as to minimize the variance of (7.20). In the most cases, the optimal value of a_i is the quotient of the covariance of V_T and V_T^i and variance of V_T^i which is estimated by

$$a_i = \frac{\sum_{i=1}^N (V_T - \bar{V}_T)(V_T^i - \bar{V}_T^i)}{\sum_{i=1}^N (V_T^i - \bar{V}_T^i)^2}.$$

Using these estimated values introduces bias in the estimator of V_T but for sufficiently large samples this bias is small compared to the Monte Carlo error [32].

We now proceed to compare the prices of call option written on two assets with two sets of dependence (weak dependence and strong dependence) between underlying assets. We again consider our six stocks whose historical data were discussed in section 4.2. We only present the results for the pairs INTC-IBM and AMZN-DELL, because similar results for other pairs were obtained and so are omitted. We consider three types of options: Rainbow options on the minimum, rainbow option on the maximum, and option on the weighted average between two underlying assets. In all cases we consider the same sets of strike prices and the maturity time is $T = 1$. The models are to be the bivariate variance gamma model and the bivariate CGMY model. We then compare the option prices obtained under the variance gamma and CGMY models with the option prices in the Black-Scholes setting for the three types of options considered.

Figures (7.5)-(7.8) depict the prices of three options considered for the bivariate variance gamma and bivariate CGMY models with the corresponding option prices in the Black-Scholes framework. Figures (7.5) and (7.6) correspond to the variance gamma model, and figures (7.7) and (7.8) correspond to the option prices under the CGMY model. In all figures, the top left graph corresponds to option prices on the minimum, top right graph corresponds to option prices on the maximum, and bottom graph corresponds to option prices on the weighted average between two assets, where the weights ω_1 and ω_2 were chosen to be 0.5. In all graphs, the plus line represents strong tail dependence, the solid line represents weak tail dependence and the zeros line represents the Black-Scholes option prices.

In all cases, the prices were evaluated by simulating the sample path 1000 times and the initial prices are $S^1 = S^2 = 1$ and $r = 0.09$. The results of copula parameter estimation indicated that the pairs are strongly dependent with $\theta = 3.127$ for the pair INTC-IBM and $\theta = 2.438$ for the pair AMZN-DELL in the case of variance gamma model, and $\theta = 4.6705$ for the pair INTC-IBM and $\theta = 3.6523$ for the pair AMZN-DELL in the case of CGMY model (cf. table (7.1) and (7.2)). The correlation between AMZN-DELL is $\rho = 0.6258$ and that of INTC-IBM is $\rho = 0.6863$ which shows that the association between the stocks is quite strong. The univariate risk-neutral parameters for the variance gamma and CGMY models are given in table (4.4) and (4.5) respectively. The parameter η was fixed to 0.75.

Our objective is to compare the option prices in the case of strong dependence, weak dependence, and under the Black-Scholes framework. Recall that in the Black-Scholes setting, analytic pricing formula are available (see for example [36]). Since our calibration results indicated that the two stocks are strongly dependent, we fixed $\theta = 0.5$ and $\eta = 0.25$ in the case of weak dependence. The risk-neutral parameters under the Black-Scholes model were estimated by fitting the option data to the Black-Scholes model. This resulted into $\sigma_1 = 0.0295$ in the case of INTC, $\sigma_2 = 0.0193$ in the case of IBM, $\sigma_3 = 0.0290$ in the case of AMZN, and $\sigma_4 = 0.0126$ in the case of DELL. The correlation coefficient between the payoff and the control variates (see chapter 4 in [32] for the detail) is $\rho = 0.88041$ when the underlying assets are INTC and IBM, and $\rho = 0.8503$ when the underlying assets are AMZN and DELL for the variance gamma model. In the case of CGMY model the correlation is given by $\rho = 0.84902$ when the underlying assets are INTC and IBM, and $\rho = 0.8435$ when the underlying assets are AMZN and DELL.

The difference between prices computed with or without tail dependence is clearly seen in all cases with the option prices in the case of strong dependence being higher than the option prices in the weak dependence. This feature explains the importance of considering dependence structure between underlying assets in option pricing. Neglecting tail dependence leads to an error in option pricing. Even though, the option prices obtained without tail dependence is lower than the option prices obtained with tail dependence in both variance gamma and CGMY models, they are nevertheless higher than the option prices obtained in the Black-Scholes framework. This tells us that, apart from the problem of modeling the dependence structure between the underlying assets, the Gaussian models do not price the options accurately. Hence, alternative models are needed since the Gaussian models give an approximation of the options prices.

Figure (7.9) depicts the absolute percentage error between option prices in the case of strong dependence, weak dependence, and in the Black-Scholes framework for the bivariate variance gamma model (Note that similar results were obtained for the case of CGMY model). The percentage error was computed as follows: For example, the percentage error between the Black-Scholes (BS) option prices and the option prices in the case of strong dependence (SD) is given by

$$Error = \frac{BS - SD}{BS} \times 100.$$

From the figure, we see that neglecting the dependence between assets when there is a strong dependence leads to a high error in option pricing. For example, in the left graph $\theta = 2.438$ and the highest error is about 12%, while in the right graph $\theta = 3.127$ and the highest error is 16%. An interesting question that one can ask himself is the reason why deep in the money the error is high and as we increase the strike prices the error reduces to zeros. Because of the time constraint, we leave this question to our future research.

We also want to stress that, although in the Lévy copulas settings, pricing multivariate models is computationally expensive, Lévy copulas allow us to model the possible dependence structure between underlying assets and the option prices obtained are more accurate than the option prices in the case of weak dependence as well as in the Black-Scholes framework. Therefore, we recommend practitioners to use Lévy copulas to model dependence structure between underlying assets whenever big jumps are observed in the historical data.

Table (7.3) shows the amount of time taken to compute the bivariate option prices in all cases considered. However, we do believe that this amount of time can be reduced if a more faster method is used to invert the tail integrals of the variance gamma and CGMY Lévy measures in the simulation of the couple (X_t^1, X_t^2) .

Options	VG+BS	CGMY+BS
Weighted Av.	1.402509	12.056210
Max	1.408011	10.923508
Min	1.400869	11.302056

TABLE. 7.3. The amount of time taken to price the bivariate options for the variance gamma model, CGMY model, and Black-Scholes model. The time is given in seconds.

In conclusion, we have constructed multivariate models with jumps and we used the models to price INTC, IBM, AMZN, DELL, FDX, and ABC pairwise. The calibration results indicated the existence of strong dependence between these underlying assets. The option prices computed with tail dependence is higher than the option prices computed without tail dependence, and that of the Black-Scholes models. Neglecting tail dependence leads to an error in option pricing. This error gets worse when the dependence structure between assets is very high. This explains the importance of modeling correctly the dependence in option pricing. We recommend practitioners to use Lévy copula models whenever big jumps are observed in the historical stock prices.

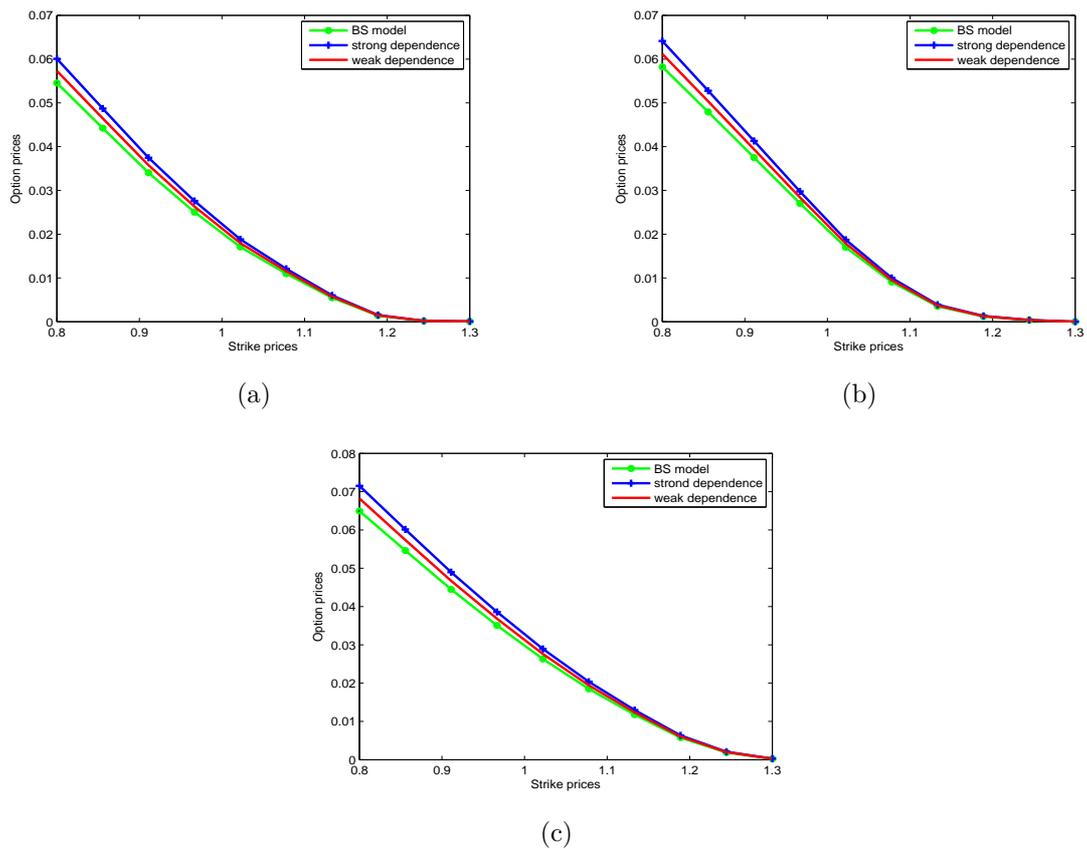


FIG. 7.5. Bivariate VG option prices with AMZN and DELL margins: The top left graph depicts option prices on the minimum, the top right graph represents the option prices on the maximum, and the bottom graph represents the option prices on the weighted average. The plus line represents strong tail dependence, the solid line represents weak dependence, and the zeros line represents the option prices in the Black-Scholes framework.

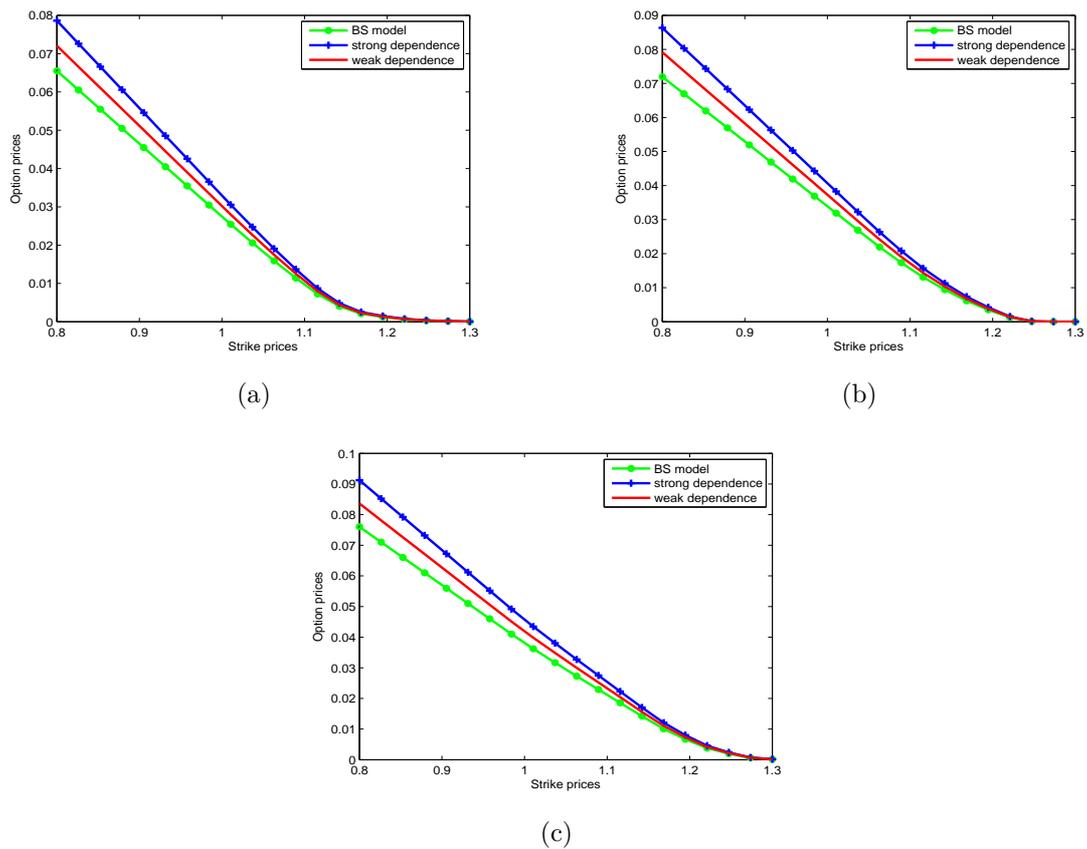


FIG. 7.6. Bivariate VG option prices with INTC and IBM margins: The top left graph depicts option prices on the minimum, the top right graph represents the option prices on the maximum, and the bottom graph represents the option prices on the weighted average. The plus line represents strong tail dependence, the solid line represents weak dependence, and the zeros line represents the option prices in the Black-Scholes framework.

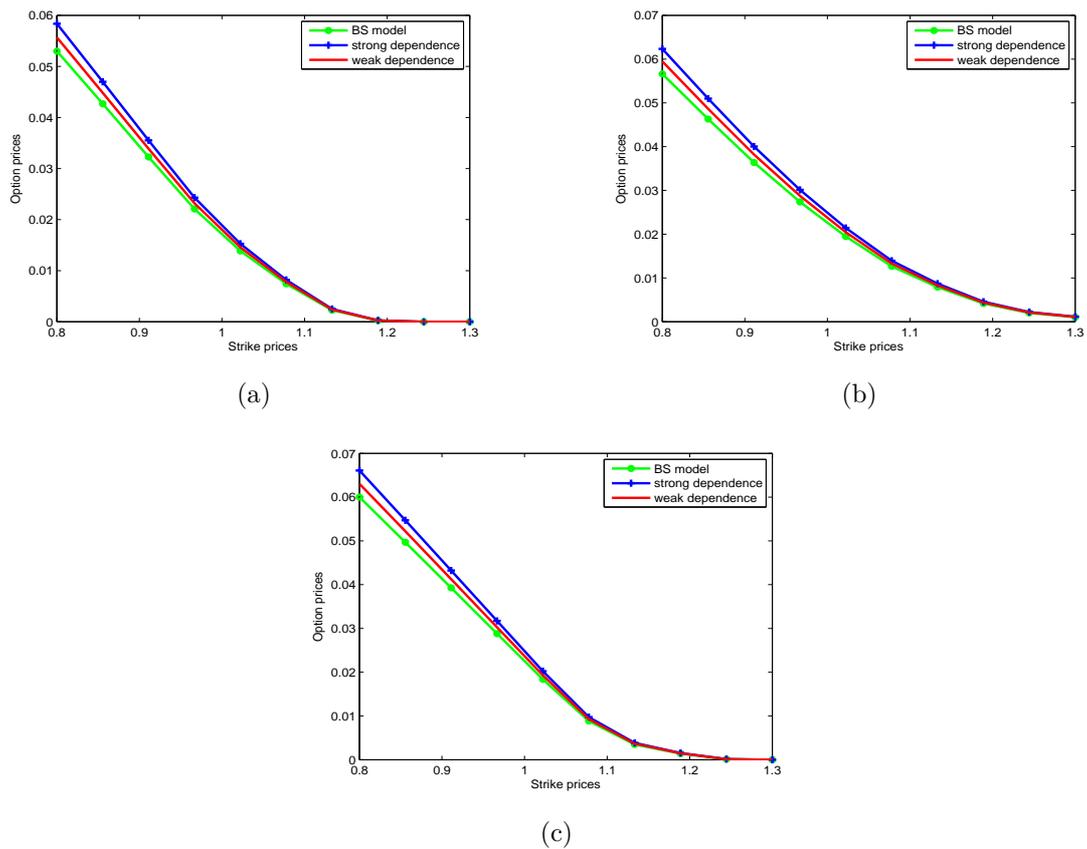


FIG. 7.7. Bivariate CGMY option prices with AMZN and DELL margins: The top left graph depicts option prices on the minimum, the top right graph represents the option prices on the maximum, and the bottom graph represents the option prices on the weighted average. The plus line represents strong tail dependence, the solid line represents weak dependence, and the zeros line represents the option prices in the Black-Scholes framework.

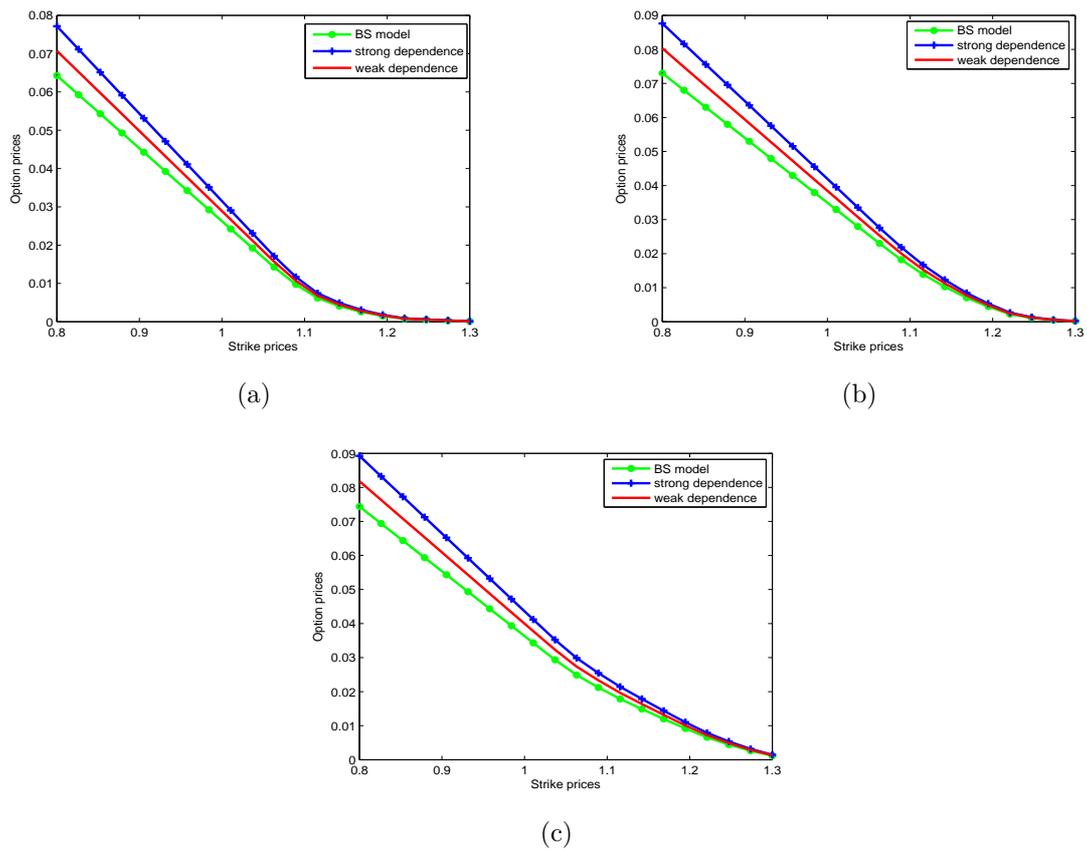


FIG. 7.8. Bivariate CGMY option prices with INTC and IBM margins: The top left graph depicts option prices on the minimum, the top right graph represents the option prices on the maximum, and the bottom graph represents the option prices on the weighted average. The plus line represents strong tail dependence, the solid line represents weak dependence, and the zero line represents the option prices in the Black-Scholes framework.

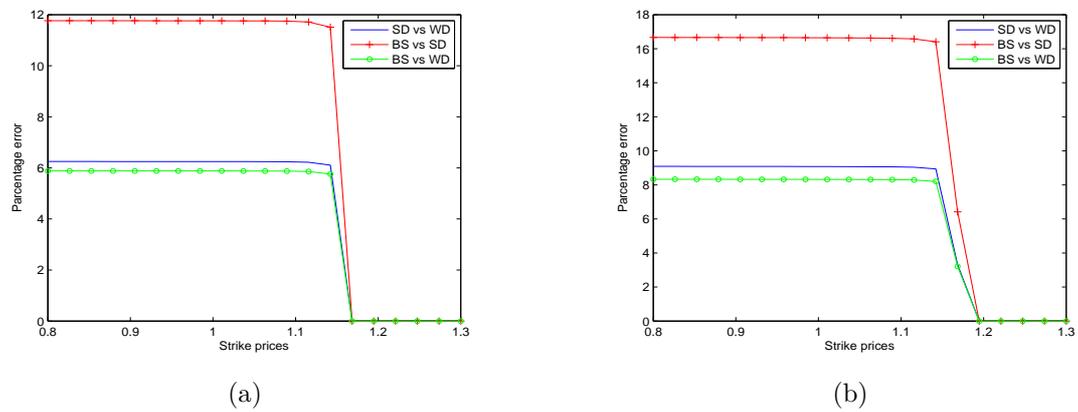


FIG. 7.9. The percentage errors between strong dependence and weak dependence under the variance gamma model, and the option prices under Black-Scholes model. The left graph correspond to the pairs INTC-IBM, and the right graph correspond to the pairs AMZN-DELL.

Chapter 8

Conclusions

This dissertation examined the application of pure jump Lévy processes of infinite activity, in particular variance gamma and CGMY processes to derivative pricing. Infinite activity processes are suitable for financial modeling, as they are quite tractable, and the fact that the arrival rate of jumps in each finite time interval is infinite results in a process rich enough to describe the asset price behavior without the need for a diffusion component.

The numerical results on six stocks show that the variance gamma distribution and the CGMY distribution capture the skewness and the tail behavior of the distribution of the asset returns better than the normal distribution. The simulation of the asset prices under the variance gamma and CGMY processes using the estimated parameters shows that these processes reproduce the dynamics of the asset prices, a feature which is not observed under the Brownian motion in the Black-Scholes model.

Option pricing under the variance gamma and the CGMY models for European call option is tractable using the fast Fourier transform method. Our results show that the calibration accuracy is much better for the variance gamma and CGMY models than that of the Black-Scholes model and the volatility smile is quite well reproduced.

In our study we have just focused on the pricing of European options and we have not discussed how to hedge them. Nevertheless, pricing and hedging are tightly related. This is a point of interest since exponential Lévy models are incomplete and hence one can not replicate the options. Our future work is to study the hedging approximations where one

tries to minimize the residual hedging error that are associated to the jump risk. It could also be interesting to consider models including jumps and stochastic volatility. Another issue that is not discussed here is to check the accuracy of the calibration procedure to the prices of stock index options of several maturities at the same time. It is not obvious that considering several maturity times we will obtain approximately the same set of parameters.

This dissertation also extends the application of Lévy processes to the pricing of multi-asset options. For pricing multivariate options dependence plays a crucial role. We modeled the dependence structure between underlying assets by a Lévy copula. Lévy copulas completely characterize the possible dependence structure of a Lévy process in the sense that, for every multivariate Lévy process there exists a Lévy copula that describes the dependence structure between its components and, for every Lévy copula and every d one-dimensional Lévy processes there exists a d -dimensional Lévy process with dependence structure given by this Lévy copula and with margins given by these d one-dimensional Lévy processes.

Multivariate models are then constructed by taking d one-dimensional Lévy processes and one Lévy copula possibly from the parametric family. This is the approach we followed to construct two-dimensional exponential Lévy model with variance gamma and CGMY margins. The dependence structure between underlying asset was given by the bidirectional Clayton Lévy copula. The simulation methods discussed in chapter 5 of this dissertation allow to compute the option prices using Monte Carlo methods. In order to reduce the error in Monte Carlo estimator, we employed the method of control variates.

Although Lévy copula models are computationally expensive and few references are available, we tried to study the applicability of these models to the actual data. We applied the models to the pairs of six stocks. The empirical results showed the existence of strong dependence between the pairs. As for the option pricing, our results showed that, choosing different set of dependence parameters results in different set of option prices with option prices in weak dependence pattern being lower than in strong dependence for all types options considered. This is a reasonable feature and coincide with the financial market behavior in which the market prices are dependent in the bad moments than in the good ones. The Black-Scholes model does not capture this feature leading to lower option prices. It is therefore necessary to have convenient methods to characterize the dependence structure between jumps in order to price the options accurately.

The future work in this direction is to investigate the method of hedging approximation in multivariate models. Even in a univariate model, perfect hedging is impossible since exponential Lévy models are incomplete. The incompleteness of the model is much involved in a multivariate model and so one needs to hedge the risk associated to different jumps occurring in different underlying assets.

Another point of interest is to investigate the effect of dependence structure to option pricing using another Lévy copula. It could also be interesting to study the consistency of Lévy copula models to option pricing.

Even though Lévy processes are difficult to deal with, there is a certain amount of benefit to be gained from their use. Lévy processes provide better models of financial market data. Their dependence structure allows one to model the possible dependence structure between underlying assets more flexibly than does reliance on the linear correlation, and to price multi-asset options consistently. Therefore, we strongly recommend practitioners to use Lévy copula models whenever big jumps are observed in the historical data.

Appendix A

A.1 The Variance Gamma Density

The density function of the variance gamma process can be obtained by first conditioning on the realization of the gamma process as a normal density function and then integrating out the density of the gamma process (3.14).

$$\begin{aligned}
 f_X(g) &= \int_0^\infty f_{X(t)|G_t(t)=g}(x) f_{G(t)} dg \\
 &= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 g}} \exp\left(-\frac{(x-\theta g)^2}{2\sigma^2 g}\right) \frac{g^{\frac{t}{\alpha}-1} e^{-\frac{g}{\alpha}}}{\alpha^{\frac{t}{\alpha}} \Gamma(\frac{t}{\alpha})} dg \\
 &= \frac{1}{\sqrt{2\pi\sigma^2} \alpha^{\frac{t}{\alpha}} \Gamma(\frac{t}{\alpha})} \int_0^\infty g^{\frac{t}{\alpha}-1-\frac{1}{2}} \exp\left(-\frac{x^2}{2\sigma^2 g} + \frac{\theta x}{\sigma^2} - \frac{\theta^2 g}{\sigma^2} - \frac{g}{\alpha}\right) dg \\
 &= \frac{-\frac{\theta x}{\sigma^2}}{\sqrt{2\pi\sigma^2} \alpha^{\frac{t}{\alpha}} \Gamma(\frac{t}{\alpha})} \int_0^\infty g^{(\frac{t}{\alpha}-\frac{1}{2})-1} \exp\left(-\frac{1}{2} \left(\frac{x^2}{\sigma^2 g} + \left(\frac{\theta^2}{\sigma^2} + \frac{2}{g}\right)g\right)\right) dg
 \end{aligned}$$

Now let

$$p = \frac{\sqrt{2\sigma^2/\alpha + \theta^2}}{x} g. \quad (\text{A.1})$$

Then the following relations hold

$$\begin{aligned}
 dp &= \frac{\sqrt{2\sigma^2/\alpha + \theta^2}}{x} dg \\
 g &= \frac{x p}{\sqrt{2\sigma^2/\alpha + \theta^2}} \\
 \frac{x^2}{\sigma^2 g} &= \frac{\sqrt{2\sigma^2/\alpha + \theta^2}}{p \sigma^2}
 \end{aligned} \quad (\text{A.2})$$

Plugging the above relations into the exponential term, we get

$$\begin{aligned} \exp \left[-\frac{1}{2} \left\{ \frac{x^2}{\sigma^2 g} + g \left(\frac{\theta^2}{\sigma^2} + \frac{2}{\alpha} \right) \right\} \right] &= \exp \left[-\frac{1}{2} \left\{ \frac{\sqrt{2\sigma^2/\alpha + \theta^2} x}{p\sigma^2} + \frac{xp}{\sqrt{2\sigma^2/\alpha + \theta^2}} \left(\frac{\theta^2}{\sigma^2} + \frac{2}{\alpha} \right) \right\} \right] \\ &= \exp \left[-\frac{x}{2} \sqrt{\frac{2\sigma^2}{\alpha} + \theta^2} \left(\frac{1}{p\sigma^2} + \frac{p}{\frac{2\sigma^2 + \theta^2\alpha}{\sigma^2\alpha}} \right) \right] \\ &= \exp \left[-\frac{x\sqrt{\frac{2\sigma^2}{\alpha} + \theta^2}}{2\sigma^2} \left(\frac{1}{p} + p \right) \right] \end{aligned}$$

Differentiating relations (A.2) we obtain the $g^{\frac{t}{\alpha} - \frac{3}{2}} dg$ term given by

$$\begin{aligned} g^{(\frac{t}{\alpha} - \frac{1}{2}) - 1} dg &= \left(\frac{xp}{\sqrt{2\sigma^2/\alpha + \theta^2}} \right)^{\frac{t}{\alpha} - \frac{3}{2}} \frac{xdp}{\sqrt{2\sigma^2/\alpha + \theta^2}} \\ &= \left(\frac{1}{2\sigma^2/\alpha + \theta^2} \right)^{\frac{t}{2\alpha} - \frac{3}{4} + \frac{1}{2}} x^{\frac{t}{\alpha} - \frac{1}{2}} p^{\frac{t}{\alpha} - \frac{3}{2}} dp \\ &= \left(\frac{x^2}{2\sigma^2/\alpha + \theta^2} \right)^{\frac{t}{2\alpha} - \frac{1}{4}} p^{\frac{t}{\alpha} - \frac{3}{2}} dp \end{aligned} \quad (\text{A.3})$$

Putting everything together the probability density f_X becomes

$$f_X(x) = \frac{2 \exp(\frac{x\theta}{\sigma^2})}{\alpha^{t/\alpha} \Gamma(t/\alpha) \sigma \sqrt{2\pi}} \left(\frac{x^2}{2\sigma^2/\alpha + \theta^2} \right)^{\frac{t}{2\alpha} - \frac{1}{4}} \frac{1}{2} \int_0^\infty \exp \left[-\frac{1}{2} \frac{x\sqrt{\frac{2\sigma^2}{\alpha} + \theta^2}}{\sigma^2} \left(\frac{1}{p} + p \right) \right] p^{(\frac{t}{\alpha} - \frac{1}{2}) - 1} dp \quad (\text{A.4})$$

To complete the integration, we introduce the modified Bessel function of the third kind K_φ , which has the following representation

$$K_\varphi(\zeta) = \frac{1}{2} \int_0^\infty t^{\varphi-1} \exp \left[-\frac{\zeta}{2} \left(\frac{1}{t} + t \right) \right] dt \quad (\text{A.5})$$

Letting

$$\zeta = \frac{\sqrt{x^2(2\sigma^2/\alpha + \theta^2)}}{\sigma^2}, \quad \varphi = \frac{t}{\alpha} - \frac{1}{2}, \quad (\text{A.6})$$

and $p = t$ then, the probability density of the variance gamma process becomes

$$f_X(x) = \sqrt{\frac{2}{\pi}} \frac{\exp(\frac{x\theta}{\sigma^2})}{\alpha^{t/\alpha} \Gamma(t/\alpha) \sigma} \left(\frac{x^2}{2\sigma^2/\alpha + \theta^2} \right)^{\frac{\varphi}{2}} K_\varphi(\zeta) \quad (\text{A.7})$$

A.2 The Characteristic Function of the CGMY Process

From the Lévy-Khintchine theorem, we have

$$\phi(u, t) = \exp \left(t \int_{-\infty}^{\infty} (e^{iux} - 1) \kappa_{CGMY}(x) dx \right). \quad (\text{A.8})$$

Writing (A.8) as the sum of two integrals of the form

$$\int_0^{\infty} (e^{iux} - 1) C \frac{\exp(-\beta x)}{x^{1+Y}} dx, \quad (\text{A.9})$$

where β equals G and M , respectively, with iu replaced by $-iu$ for $\beta = G$, we get

$$\begin{aligned} & \int_0^{\infty} \frac{C}{x^{1+Y}} (\exp[-(\beta - iu)x] - \exp(-\beta x)) dx \\ &= C \int_0^{\infty} (\beta - iu)^Y w^{-Y-1} \exp(-w) dw \\ & \quad - C \int_0^{\infty} \beta^Y w^{-Y-1} \exp(-w) dw, \quad \text{where } w = \beta x \\ &= C \Gamma(-Y) ((\beta - iu)^Y - \beta^Y) \end{aligned}$$

Substituting β by M and G and evaluating the case $\beta = G$ at $-iu$, we get

$$\phi_{CGMY}(u, t) = \exp [t C \Gamma(-Y) ((M - iu)^Y - M^Y + (G + iu)^Y - G^Y)]. \quad (\text{A.10})$$

A.3 Proof of Theorem (4.3.1)

Theorem A.3.1. *Let X be a random variable with characteristic function $\phi(x)$ and let $e^{-\alpha x}$ be the dampening factor. X has density function $f(x)$ and distribution function $F(x)$. Then, the c.d.f. function $F(x)$ is given by:*

$$F(x) = \frac{e^{\alpha x}}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \frac{\phi(u + i\alpha)}{\alpha - iu} du$$

Proof. The characteristic function of X is given by

$$\phi(u) = \mathbb{E}[e^{iuX}] = \int_{-\infty}^{\infty} e^{iux} f(x) dx.$$

For the distribution function $F(x)$, the Fourier transform of $e^{-\alpha x}F(x)$ is given by

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{iux} e^{-\alpha x} F(x) &= \int_{-\infty}^{\infty} e^{(iu-\alpha)x} \left(\int_{-\infty}^x f(y) dy \right) dx \\
&= \int_{-\infty}^{\infty} \left(\int_y^{\infty} e^{-(\alpha-iu)x} f(y) dx \right) dy \\
&= \int_{-\infty}^{\infty} f(y) \frac{e^{-(\alpha-iu)y}}{\alpha-iu} dy \\
&= \frac{1}{\alpha-iu} \int_{-\infty}^{\infty} f(y) e^{iy(u+i\alpha)} dy \\
&= \frac{\phi(u+i\alpha)}{\alpha-iu}
\end{aligned}$$

Taking the inverse Fourier transform, the distribution function $F(x)$ is then given by

$$F(x) = \frac{e^{\alpha x}}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \frac{\phi(u+i\alpha)}{\alpha-iu} du$$

which complete the proof. □

A.4 Calculation of Correlation of Returns in Equation (6.14)

The covariance of X_t^1 and X_t^2 is given by

$$\begin{aligned}
\rho(X_t^1, X_t^2) &= \text{cov}(X_t^1, X_t^2) \\
&= \mathbb{E}[(B^1(Z_t) + \mu_1 Z_t)(B^2(Z_t) + \mu_2 Z_t) | Z_t] \\
&= \mathbb{E}[(B^1(Z_t)B^2(Z_t) + \mu_1 Z_t B^2(Z_t) + \mu_2 Z_t B^1(Z_t) + \mu_1 \mu_2 Z_t^2) | Z_t] \\
&= \sigma_1 \sigma_2 \rho \mathbb{E}[Z_t] + \mu_1 \mu_2 \text{Var}[Z_t]
\end{aligned}$$

The variance of X_t^1 is equal to

$$\begin{aligned}
\text{var}[X_t^1] &= \mathbb{E}[(B^1(Z_t) + \mu_1 Z_t)^2 | Z_t] \\
&= \mathbb{E}[(B^1(Z_t))^2 + 2\mu_1 Z_t B^1(Z_t) + \mu_1^2 Z_t^2 | Z_t] \\
&= \sigma_1^2 \mathbb{E}[Z_t] + \mu_1^2 \text{var}[Z_t]
\end{aligned}$$

Similarly,

$$\text{var}[X_t^2] = \sigma_2^2 \mathbb{E}[Z_t] + \mu_2^2 \text{var}[Z_t]$$

Putting everything together, we obtain

$$\rho(X_t^1, X_t^2) = \frac{\sigma_1 \sigma_2 \rho \mathbb{E}[Z_t] + \mu_1 \mu_2 \text{Var}[Z_t]}{(\sigma_1^2 \mathbb{E}[Z_t] + \mu_1^2 \text{var}[Z_t])^{1/2} (\sigma_2^2 \mathbb{E}[Z_t] + \mu_2^2 \text{var}[Z_t])^{1/2}}$$

A.5 Proof of Theorem (6.3.4)

Proof. Let's first prove that F_{\parallel} is a Lévy copula. Properties (1) and (2) of definition (6.3.3) are obvious. We now prove property (3). A Lévy copula defines a positive measure say μ on \mathbb{R}^d with Lebesgue margins. Define the positive measure as

$$\mu(B) = \lambda(\{x \in \mathbb{R} : (x, \dots, x) \in B\}), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where λ denotes the Lebesgue measure on \mathbb{R} . For $a, b \in \mathbb{R}^d$, with $a \leq b$, we have

$$V_{F_{\parallel}}((a, b]) = \mu((a, b]),$$

and therefore F_{\parallel} is d -increasing.

\Rightarrow The proof is based on the fact that an ordered set can be represented as a disjoint union of some increasing set and countable number of segments that are parallel to some coordinate axis. To prove that there are countable number of segments with upper bound \bar{x} and lower bound \underline{x} , define a segment parallel to the k^{th} coordinate axis as

$$S(x, k) = \{x \in \mathbb{R}^d : x_k = x\} \cap S. \quad (\text{A.11})$$

Since for segments $S_i = S(x_i, k)$, with length $\geq \epsilon$ (the length of $S(x, k) = \sum_{i=1}^d (\bar{x}_i - \underline{x}_i) \neq 0$ for $\bar{x}_i \neq \underline{x}_i$), are subset of S , then they are countable number of segments greater or equal to ϵ . Therefore, they are countable number of segments of non-zero length which we denote by S_n , $n \in \mathbb{N}$.

Now we let $S^* = S \setminus \bigcup_{i=1}^{\infty} S_n$. S^* is ordered because it is a subset of S . Let $x, y \in S^*$. If $x_k = y_k$ for some k , then either x and y are the same or they are in the same segment of type (A.11). Therefore, either $x_k < y_k$ for every k or $x_k > y_k$ for every k which entails that S^* is increasing and hence we obtain the desired representation for S :

$$S = S^* \cup \bigcup_{i=1}^n S_n$$

Now let $x \in (0, \infty)^d$. Clearly $U(x) \leq U_k(x_k)$ for some k . On the other hand, since S is an ordered set, we have

$$\{y \in \mathbb{R}^d : x_k \leq y_k\} \cap S = \{y \in \mathbb{R}^d : x \leq y\} \cap S$$

for some k . Indeed, suppose that this is not the case. Then there exist points $z^1, \dots, z^d \in S$ such that for every k , $z_k^k \geq x_k$ and there exists $j(k)$ with $z_{j(k)}^k < x_{j(k)}$. Choosing greatest elements of z^1, \dots, z^d (this is possible because they all belong to an ordered set) and call it z^k . Then $z_{j(k)}^k < x_{j(k)}$. However by construction of z^1, \dots, z^d we also have $z_{j(k)}^{j(k)} \geq x_{j(k)}$, which is a contradiction that z^k is the greatest element. Therefore,

$$U(x) = \min(U_1(x_1), \dots, U_d(x_d)).$$

Similarly, it can be shown that for every $x \in (-\infty, 0)^d$,

$$U(x) = (-)^d \min(|U_1(x_1)|, \dots, |U_d(x_d)|).$$

Since $U(x) = 0$ for any $x \notin K$, we show that

$$U(x) = F_{\parallel}(U_1(x_1), \dots, U_d(x_d))$$

for any $x \in (\mathbb{R} \setminus \{0\})^d$. Since the marginal Lévy measure of X are also supported by non-decreasing sets and the margins of F_{\parallel} have the same form as F_{\parallel} , we have

$$U^I((x_i)_{i \in I}) = F_{\parallel}^I(U^I((x_i)_{i \in I})) \quad (\text{A.12})$$

for any $I \in \{1, \dots, d\}$ and any $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$.

\Leftarrow : If the Lévy copula of (X_t) is given by F_{\parallel} , its tail integral is of the form (A.12). Suppose that $S := \text{sup } \nu$ is not an ordered set. Then, there exist two points $u, v \in S$ such that $u_p > v_p$ and $u_q < v_q$ for some p and q . Moreover we can have either $u_i \geq 0$ and $v_i \geq 0$ for all i or $u_i \leq 0$ and $v_i \leq 0$ for all i . Suppose that $u_i \leq 0$ and $v_i \leq 0$, the other case being analogous. Let $x = \frac{u+v}{2}$. Since $u, v \in S$, we have $\nu(\{z \in \mathbb{R}^d : z_p < x_p, z_q \geq x_q\}) > 0$ and $\nu(\{z \in \mathbb{R}^d : z_p \geq x_p, z_q < x_q\}) > 0$. However

$$\begin{aligned} \nu(\{z \in \mathbb{R}^d : z_p < x_p, z_q \geq x_q\}) > 0 &= U_q(x_q) - U_{\{p,q\}}(x_p, x_q) \\ &= U_q(x_q) - \min(U_p(x_p), U_q(x_q)) \end{aligned}$$

and

$$\nu(\{z \in \mathbb{R}^d : z_p \geq x_p, z_q < x_q\}) > 0 = U_p(x_p) - \min(U_p(x_p), U_q(x_q)),$$

which is a contradiction because these two expressions cannot be simultaneously positive.

It remains to show that for every n , $\nu(S_n) = 0$. Assume that the tail integrals U_i of (X_t^i) are continuous and satisfy $\lim_{x \rightarrow 0} U_i(x) = \infty$, $i = 1, \dots, d$. Suppose that $\xi(n) \neq 0$, then

$$\nu(S_n) = \lim_{\epsilon \downarrow 0} (U_{k(n)}(\xi(n) - \epsilon) - U_{k(n)}(\xi(n))) = 0$$

because $U_{k(n)}$ is continuous. Suppose now that $\xi(n) = 0$. Since S_n does not reduce to a single point, we must have either $x_p > 0$ or $x_p < 0$ for some $x \in S_n$ and some p . Suppose that $x_p > 0$, the other case being analogous. Since S is ordered, we have

$$\nu(\{x \in \mathbb{R}^d : x_{k(n)} \geq \epsilon\} \cap S) \leq \nu(\{\xi \in \mathbb{R}^d : \xi_p \geq x_p\} \cap S) < \infty$$

uniformly in $\epsilon > 0$. This implies $\lim_{\epsilon \downarrow 0} U_{k(n)} < \infty$ in contradiction to $\lim_{x \rightarrow 0} U_i(x) = \infty$. Hence, $\xi(n) > 0$ for any n . Therefore, $\nu(\mathbb{R}^d \setminus S^*) = 0$ and the proof is complete. \square

A.6 Proof of Theorem (5.3.1)

Define a stochastic process

$$Y(s) = \sum_{\{i: \Gamma_i \leq s\}} H(\Gamma_i, V_i) - A(s), \quad s \geq 0. \quad (\text{A.13})$$

Y is has càdlàg path and can be written as

$$Y(s) = \int_{[0, s] \times S} H(r, v) [M(dr, dv) - drF(dv)], \quad (\text{A.14})$$

where $M = \sum_{i=1}^{\infty} \delta_{\Gamma_i, V_i}$ is a marked Poisson process with mean measure $Leb \times F$ (here F is the common distribution of V_i). It follows that Y is a process with independent increments. Furthermore, $Y(s)$ is a centered compound Poisson random variable with Lévy measure ν_s given by

$$\nu_s(B) = \int_0^s \sigma(r; B) dr \nearrow \nu(B), \quad \text{a.s. } s \nearrow \infty. \quad (\text{A.15})$$

Since ν is a Lévy measure on \mathbb{R}^d , then

$$\lim_{s \rightarrow \infty} Y(s) = Y(\infty) \text{ exists a.s.} \quad (\text{A.16})$$

Indeed $\lim_{s \rightarrow \infty} \mathcal{L}(Y(s))$ exists. By the independence of increments of Y ,

$$Y(s) = \sum_{i=1}^k (Y(s_i) - Y(s_{i-1}))$$

converges a.s. as $k \rightarrow \infty$, for each increasing sequence $s_k \nearrow \infty$ with $s_0 = 0$. Moreover,

$$\mathbb{E}[\exp(iuY(\infty))] = \phi(y)e^{-iu a}. \quad (\text{A.17})$$

The converse also implies that $\lim_{s \rightarrow \infty} \mathcal{L}(Y(s))$ exists. Hence the Lévy measures ν_s converge vaguely to some Lévy measure on each continuity set bounded away from the origin. By (A.15), that Lévy measure must coincide with the measure ν .

Now if ν is a Lévy measure and $\lim_{s \rightarrow \infty} A(s)$ exists, then by (A.16) we have

$$\sum_{i=1}^n H(\Gamma_i, V_i) = Y(\Gamma_n) + A(\Gamma_n) \rightarrow Y(\infty) + a \quad (\text{A.18})$$

a.s. as $n \rightarrow \infty$. Conversely, if $\sum_{i=1}^{\infty} H(\Gamma_i, V_i)$ converges a.s., then

$$Y(s) + A(s) = \sum_{\{i: \Gamma_i \leq s\}} H(\Gamma_i, V_i)$$

converges a.s. to the same limit as $s \rightarrow \infty$. Since the Lévy measure of $Y(s) + A(s)$ is ν_s , we get that ν is also a Lévy measure by the same argument as above. Hence by (A.16), $A(s) = (Y(s) + A(s)) - Y(s)$ converges as $n \rightarrow \infty$ which concludes (i).

(ii) Since

$$\sum_{i=1}^n H(\Gamma_i, V_i) - A(\Gamma_n) = Y(\Gamma_n) \rightarrow Y(\infty) \quad (\text{A.19})$$

a.s. as $n \rightarrow \infty$, it is enough to show that $A(\Gamma_n) - A(n) \rightarrow 0$ a.s. as $n \rightarrow \infty$. We have

$$|A(\Gamma_n) - A(n)| \leq \int_{\Gamma_n \wedge n}^{\Gamma_n \vee n} \int_{\mathbb{R}^d} (|x| \wedge 1) \sigma(r; dx) dr. \quad (\text{A.20})$$

Put

$$g(r) = \int_{\mathbb{R}^d} (|x| \wedge 1) \sigma(r; dx) dr = \mathbb{E}[|H(r, V_1)| \wedge 1].$$

By (5.5), g is increasing and square integrable. Indeed, by Jensen's inequality,

$$\begin{aligned} \int_0^\infty [g(r)]^2 dr &\leq \int_0^\infty \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \sigma(r; dx) dr \\ &= \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty. \end{aligned} \quad (\text{A.21})$$

From (A.20),

$$\begin{aligned} |A(\Gamma_n) - A(n)| &\leq |\Gamma_n - n| g(\Gamma_n \wedge n) \\ &= g(n/2) |\Gamma_n - n| [g(\Gamma_n \wedge n) / g(n/2)]. \end{aligned}$$

Since $n^{-1}\Gamma_n \rightarrow 1$ a.s. and g is nonincreasing,

$$\lim_{n \rightarrow \infty} \sup g(\Gamma_n \wedge n)/g(n/2) \leq 1.$$

By (A.21) and the Hájek-Rényi-Chow inequality, we get for every $\epsilon > 0$,

$$\mathbb{P}[\sup_{n \geq k} g(n/2)|\Gamma_n - n| \geq \epsilon] \leq \epsilon^{-2}(kg^2(n/2)) + \sum_{n > k} g^2(n/2) \rightarrow 0$$

as $k \rightarrow \infty$. Hence $\lim_{n \rightarrow \infty} g(n/2)|\Gamma_n - n| = 0$ a.s., which completes (ii) and thus the proof of theorem (5.3.1).

Bibliography

- [1] Anderson T. W., and Darling D. A., *Asymptotic theory of certain "goodness of fit criteria"*, **23** (1952), Annals of Mathematical Statistics, p. 193-212.
- [2] Asmussen S., and Rosiński J., *Approximation of small jumps of Lévy processes with a view towards simulation*, **38** (2000), J. Appl. Probab., p. 482-493.
- [3] Bachelier L., *Théorie de la spéculation*, **17** (1900), Annales de l'Ecole Normale Supérieure, p. 21-86.
- [4] Barndorff-Nielsen O. E., *Normal inverse Gaussian distribution and the modeling of stock returns*, **300** (1995), Research Report.
- [5] Barndorff-Nielsen O. E. and Lindner A., *Lévy copula: Dynamics and transforms of epsilon type*, **34** (2004), Scandinavian Journal of Statistics, p. 298-316.
- [6] Barndorff-Nielsen O. E. (b), *Exponentially decreasing distributions for logarithm of particle size*, **353** (1977), Preceedings of the Royal Society London, p. 151-157.
- [7] Barndorff-Nielsen O.E., Mikosch T., and Resnick S. I., *Lévy processes: Theory and applications*, Birkhäuser Boston, 2001.
- [8] Bedendo M., Campolongo F., Joossens E., and Saita F., *Pricing multiasset equity options: How relevant is the dependence function ?*, **34** (2009), Journal of Banking and Finance p. 788-801.
- [9] Black F., and Scholes M., *The pricing of options and corporate liabilities*, **81** (1973), J. Polit. Econ., p. 637-654.

-
- [10] Böcker K., and Klüppelberg C., *Modeling and measuring multivariate operational risk with Lévy copulas*, **3** (2008), Journal of Operational Risk, p. 3-27.
- [11] Boyarchenko S. I., and Levendorskiĭ S. Z. (a), *Generalizations of the Black-Scholes equation for truncated Lévy processes*, (1999), Working Paper.
- [12] Boyarchenko S. I., and Levendorskiĭ S. Z. (b) , *Option pricing for truncated Lévy processes*, **3** (2000), International Journal for Theory and Applications in Finance, p. 549-552.
- [13] Breeden D. T., and Litzenberger R. H., *Prices of state-contingent claims implicit in option prices*, **51** (1978), Journal of Business, p. 621-651.
- [14] Carmona R., and Durrleman V., *Generalizing the Black-Scholes formula to multivariate contingent claims*, **9** (2005), Journal of Computational Finance p. 43-68.
- [15] Carr P., and Madan D., *Option valuation using the Fast Fourier Transform*, **75** (1998), J. Comput. Finance, p. 61-73.
- [16] Carr P., Geman H., Madan D. B., and Yor M., *The fine structure of asset returns: An empirical investigation*, **75** (2000), Journal of Business, p. 305-332.
- [17] Chen Q., *Dependence structure in Lévy processes and its application in finance*, Phd, University of Maryland, 2008.
- [18] Chen R. R., Chung S. L., and Yang T., *Option pricing in a multi-asset, complete market economy*, **37** (2002), Journal of Financial and Quantitative Analysis, p. 649-666.
- [19] Cherubini U., and Luciano E., *Multivariate option pricing with copulas*, 2001, Working Paper Series.
- [20] Cherubini U., Luciano E., and Vecchiato W., *Copula methods in finance*, John Wiley & Sons Ltd, 2004.
- [21] Cohen S., and Rosiński J., *Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered stable processes*, **13** (2007), Bernoulli p. 195-210.

-
- [22] Cont R., and Tankov P., *Financial modelling with jumps processes*, Chapman and Hall/CRC, 2004.
- [23] Cont R., Tankov P., and Voltchkova E., *Hedging with options in models with jumps*, 2007, Stochastic Analysis and Application p. 197-217.
- [24] Delbaen F., and Schachermayer W., *A general version of the fundamental theorem of asset pricing*, **300** (1994), Mathematische Annalen, p. 520-563.
- [25] Eberlein K., Keller U., and Prause K., *New insights into smile, mispricing and value at risk: The hyperbolic model*, **71** (1998), Journal of Business, p. 371-406.
- [26] Embrechts P., Lindskog F., and McNeil A., *Modeling dependence with copulas and applications to risk management*, 2001, Available from <http://www.risklab.ch/papers/DependenceWithCopulas.pdf>.
- [27] Embrechts P., McNeil A., and Straumann D. (a), *Correlation and dependence in risk management: Properties and pitfalls*, (2002), In Risk Management: Value at Risk and Beyond, p. 176-223.
- [28] Embrechts P., McNeil A., and Straumann D. (b), *Correlation: Pitfalls and alternatives*, **12** (1999), Risk, p. 69-71.
- [29] Geman H., *Pure jump Lévy processes for asset price modeling*, **75** (2002), Journal of Banking and Finance.
- [30] Geman H., Madan D., and Yor M., *Asset prices are Brownian motion: Only in business time*, (2001), Quantitative Analysis in Financial Markets, p. 103-146.
- [31] Gentle J. E., *Random number generation and Monte Carlo methods*, Springer-Verlag New York, 1998.
- [32] Glasserman P., *Monte Carlo methods in financial engineering*, Springer, New York, 2003.
- [33] Gradshteyn I. S., and Ryzhik I. M., *Table of integrals, series and products*, Academic Press, New York, 1995.

-
- [34] Hunting M., *Lévy processes and Lévy copulas with application in insurance*, MSc, University of Belgen, Norway, 2007.
- [35] Jacod J., and Shiryaev A. N., *Limit theorems for stochastic processes*, Springer Berlin, 2nd ed., 2003.
- [36] James P., *Option theory*, John Wiley & Sons Ltd, 2003.
- [37] Ji M., and Zapatero F., *Empirical performance of Lévy option pricing models*, Available from http://marchallinside.usc.edu/zapatero/Papers/new_sum4.pdf.
- [38] Joe H., *Multivariate models and dependence concepts*, Chapman and Hall, 1997.
- [39] Johnson H., *Options on the maximum or the minimum of several assets*, **22** (1987), Journal of Financial and Quantitative Analysis.
- [40] Kallsen J., and Tankov P., *Characterization of dependence of multidimensional Lévy processes using Lévy copulas*, **97** (2006), Journal of Multivariate Analysis, p. 1551-1572.
- [41] Koponen I., *Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process*, **52** (1995), Physical Review, p. 1197-1199.
- [42] Lindskog F., and McNeil A., *Common Poisson shock models: Applications to insurance and credit risk modelling*, 2001, Astin Bulletin, p. 209-238.
- [43] Liuren W., *Modeling financial security returns using Lévy processes*, **15** (2008), Journal of Financial Engineering.
- [44] Luciano E., *Copulas and dependence models in credit risk: Diffusions versus jumps*, (2007), Working Papers Applied Mathematics Series.
- [45] Luciano E., and Schoutens W., *A multivariate jump-driven financial asset model*, **6** (2005), Quantitative Finance, p. 385-402.
- [46] Madan D., and Milne F., *The variance gamma (VG) model for share market returns*, **1** (1991), Mathematical Finance, p. 39-55.
- [47] Madan D. B., *Purely discontinuous asset price processes*, (1999), Handbooks in Mathematical Finance: Option Pricing, Interest Rates and Risk Management.

-
- [48] Madan D. B., and Seneta E., *The variance gamma model for share market returns*, **63** (1990), Journal of Business, p. 511-524.
- [49] Madan D. B., Carr P., and Chang E., *The variance gamma process and option pricing*, **2** (1998), European Financial Review, p. 125-105.
- [50] Malevergne Y., and Sornette D., *Extreme financial risk: From dependence to risk management*, Springer-Verlag Berlin Heidelberg, 2006.
- [51] Merton R., *Option pricing when underlying stock returns are discontinuous*, **3** (1976), Journal of Financial Economics, p. 125-144.
- [52] Michele L. B., Svetlozar T. R., Young S. K., and Frank J. F. , *Tempered infinitely divisible distributions and processes*, (2008), Technical Report.
- [53] Moosbrucker T., *Pricing CDOs with correlated variance gamma distributions*, (2006), Availalbe from http://www.fmpm.ch/docs/9th/papers_2006_web/9112b.pdf.
- [54] Nelsen R. B., *An introduction to copulas*, Springer Science+Business Media, Inc, 2006.
- [55] Pellizzari P., *Efficient Monte Carlo pricing of European options using mean value control variates*, **24** (2001), J. Decision in Economics and Finance, p. 107-126.
- [56] Poirrot J., and Tankov P., *Monto Carlo option pricing for tempered stable (CGMY) processes*, **13** (2006), Asia Pacific Financial Markets, p. 327-344.
- [57] Protter P. E., *Stochastic integral and differential equations*, Wiley, 2000.
- [58] Rosenberg J. V. (a), *Pricing multivariate contingent claims using estimated risk-neutral density functions*, **17** (1998), Journal of International Money and Finance, p. 229-247.
- [59] Rosenberg J. V. (b), *Semiparametric pricing of multivaiate contingent claims*, 1999, Working Paper Series.
- [60] Rosenberg J. V. (c), *Nonparametric pricing of multivariate contingent claims*, **10** (2001), Journal of Derivatives, p. 9-26.

-
- [61] Rosiński J (a)., *Series representations of Lévy processes from the perspectives of point processes*, in *Lévy processes-theory and applications*, (2001), O. Berndorff-Nielsen, T. Mikosch, and S. Resnick, eds, Birkhauser, Boston, p. 401-415.
- [62] Rosiński J., and Sinclair J. , *Generalized tempered stable processes*, (2009), Available from www.math.utk.edu/~rosinski/Manuscripts/GTS-2009.pdf.
- [63] Rosiński J (b)., *Simulation of Lévy processes*, (2007), To appear in *Encyclopedia of Statistics in Quality and Reliability: Computationally Intensive Methods and Simulation*.
- [64] Rosiński J (c)., *Tempering stable processes*, **117** (2007), *Stochastic Processes and their Applications*, p. 677-707.
- [65] Sato K. I., *Lévy processes and infinitely divisible distributions*, Cambridge University Press, 1999.
- [66] Schoutens W., *Lévy processes in finance: Pricing financial derivatives*, Wiley, 2003.
- [67] Samuelson, P. A., *Rational theory of warrant pricing*, **6** (1965), *Indust. Manag. Rev.*, p. 13-31.
- [68] Shiryaev A. N., *Essentials of stochastic finance*, (1999), World Scientific.
- [69] Stulz R. M., *Option on the minimum or the maximum of two risky assets: Analysis and applications*, **10** (1982), *Journal of Financial Economics*, p. 161-185.
- [70] Svetlozar T. R., Rosella G., and Michele L. B. , *Tempered stable models in finance: Theory and application*, (2008), Technical Report.
- [71] Tankov P. (a), *Dependence structure of spectrally positive multidimensional Lévy processes*, (2003), Unpublished manuscript (cf. www.cmap.polytechnique.fr/~tankov).
- [72] Tankov P. (b), *Dependence structure of multivariate Lévy processes with applications in risk management*, 2003, Unpublished Manuscript. Available at www.cmap.polytechnique.fr/~tankov.
- [73] Tankov P. (d), *Lévy processes in finance: Inverse problems and dependence modeling*, PhD, Ecole Polytechnique, 2004.

-
- [74] Tankov P. (e), *Simulation and option pricing in Lévy copula models*, 2006, Preprint, available from <http://www.institut.math.jussieu.fr/~tankov/>.
- [75] Tappe K., *Ordinary and Lévy copulas in finance: Models, methods and tools for risk management and option pricing*, PhD, Mathematik und Naturwissenschaften der Bergischen Universität Wuppertal a Genehmigte, 2008.
- [76] Vollmer T., *Bivariate Lévy processes and approximations*, 2007, Available from <http://www.time.vollmeer-net.de>.
- [77] Wannewetsch J., *Lévy processes in finance: The change of measure and non-linear dependence*, Phd, Rheinischen Friedrich-Wilhelms-Universität Bonn, 2005.
- [78] Xia Q., *Extending the Lévy processes to multiasset products pricing*, PhD, University of Maryland, 2006.
- [79] Yor M., and Madan D. B., *CGMY and Meixner subordinators are absolutely continuous with respect to one sided stable subordinators*, 2005, Publication du Laboratoire de Probabilités et Modèles Aléatoires.
- [80] Young S. K., Svetlozar T. R., Michele L. B., and Frank J. F. , *A new tempered stable distribution and its application to finance*, (2008), Risk Assessment: Decisions in Banking and Finance, p. 51-84.
- [81] Zhang B., *A new Lévy based short-rate model for the fixed income market and its estimation with particle filter*, PhD, University of Maryland, 2006.