ASPECTS OF QUANTUM FIELD THEORIES

By

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DECLARATION

I, the undersigned, hereby declare that the work contained in this study project is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

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Date

ABSTRACT

In this study project we give a general introduction to quantum field theories. In the first chapter we revise the operator formalism of quantum mechanics as well as the second quantization scheme that is used to describe many-particle systems. In the second chapter we develop the idea of path integrals within a quantum mechanics framework. We then apply path integral formalism developed in chapter two to introduce quantum field theories. We describe a field theory for scalar fields in chapter three and then a field theory for fermion fields in chapter four as well as the renormalization techniques in chapter five. In chapter six we show how scattering amplitudes are related to Green's functions which are derived from the path integral formalism. In the last chapter we give a brief introduction to gauge field theories.

OPSOMMING

In hierdie werkstuk, gee ons 'n algemene inleiding tot kwantum velde-teorieë. In die eerste hoofstuk hersien ons die operator formalisme van kwantum meganika asook die tweede kwantisasie skema wat gebruik word om veel-deeltjie sisteme te beskryf. Ons ontwikkel in die tweede hoofstuk die idee van padintegrale binne die raamwerk van kwantum meganika. Ons gebruik dan hierdie padintegraal formalisme, wat in die tweede hoofstuk ontwikkel is, om kwantum velde-teorieë te verduidelik. Ons beskryf 'n velde teorie vir skalaar velde in hoofstuk drie en dan 'n velde teorie vir fermion velde in hoofstuk vier asook die renormalisasie tegnieke in hoofstuk vyf. In hoofstuk ses toon ons die die verband tussen verstrooiings-amplitudes en Green's funksies, wat uit die padintegraal formalisme herlei word. In die laaste hoofstuk, gee ons 'n kort inleiding tot ykveldteorieë.

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Notation

In this work we take $c = \hbar = 1$.

The following notation is used for commutation relations.

$$\text{Commutators}: [a,b]_- \equiv ab - ba. \text{ Anti-commutators}: [a,b]_+ \equiv ab + ba.$$

For relativistic variables we use the standard notation. The metric is

$$g \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The four-vector inner product is denoted as

$$px \equiv p^\mu x_\mu = g^{\mu\nu} p_\nu x_\mu = p_0 x_0 - \vec{p} \cdot \vec{x},$$

where $\vec{p} \cdot \vec{x}$ is the three vector inner product.

The gamma matrices, γ^μ are defined as

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\ \gamma^i &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \end{aligned}$$

where σ^i; $i = 1, 2, 3$ are the Pauli spin matrices.

We also use the Feynman notation that $\not{p} = \gamma^\mu p_\mu$.

We will use a bar notation ($\bar{x}$) to denote that a variable is in Euclidian space, otherwise the variable is in Minkowski space.

CHAPTER 1
Operator formalism and Second Quantization

In non-relativistic quantum mechanics, the method of first quantization describes the motion of single particles and is suitable for massive particles with kinetic energy less than the particle's rest mass, $E_k \ll m_o$. First quantization also can be used for particles where the kinetic energy is higher than the rest mass, but care must be taken with the interpretation of the results within this quantization framework, or else difficulties like the Klein paradox could occur. To describe many-particle systems within the first quantization framework, one must construct a N-body wave function within the configuration space. In principle, the N-body wave-function contains all possible information about the system, but a direct solution of the Schrödinger equation using this wave-function is impractical.

We wish to introduce a new quantization scheme, namely second quantization, where many-particle systems are described more naturally. The advantages of second quantization is that the second quantized operators incorporate the specific statistics of the particles, where in first quantization, the statistics need to be inserted manually when constructing the N-body wave-function. Another advantage is that calculations are much simpler since one concentrates on the few matrix elements of interest, thus avoiding the need for dealing with the many-particle wave-function and the coordinates of all the remaining spectator particles.

In the second quantization framework, we interpret single particle wave functions (from first quantization) as classical fields[1]. They are then transformed into quantized operator fields. The requirement that the energy of the free states be positive restricts states of half spin to anti-symmetric states and states of integer spin to symmetric states.

In the following section, the second quantization scheme is applied to the Schrödinger equation.

1.1 Schrödinger's Theory

We regard the Schrödinger wave function, $\psi(x)$, as a classical field. The Lagrange density for a free particle is given by

$$\mathcal{L} = \frac{1}{2}\psi^*(x) i \frac{\overleftrightarrow{\partial}}{\partial t}\psi(x) - \frac{1}{2m}\vec{\nabla}\psi^*(x)\cdot\vec{\nabla}\psi(x),$$

[1] A field is the generalization of the notion of a coordinate of a particle to a continuum of particles, with the possibility of one or more particles occurring at each point of space.

where $\overleftrightarrow{\frac{\partial}{\partial t}} = \overrightarrow{\frac{\partial}{\partial t}} - \overleftarrow{\frac{\partial}{\partial t}}$.

If $\psi^*(x)$ and $\psi(x)$ are independent fields, then the Euler -Lagrange relations produce the free particle Schrödinger equation,

$$i\frac{\partial\psi(x)}{\partial t} = -\frac{\nabla^2\psi(x)}{2m}.$$

The momentum conjugates of ψ and ψ^* are $\pi(x)^*$ and $\pi(x)$ respectively, where

$$\pi^*(x) = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial t}\right)} = \frac{i}{2}\psi^*(x), \quad \text{and} \quad \pi(x) = \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi^*}{\partial t}\right)} = -\frac{i}{2}\psi(x).$$

The Hamiltonian density can now be calculated by

$$\mathcal{H} = \pi^*\frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial t}\pi - \mathcal{L} = \frac{1}{2m}\vec{\nabla}\psi^*(x)\cdot\vec{\nabla}\psi(x).$$

We calculate the total Hamiltonian by integrating over the total volume, which yields (after an integration by parts)

$$H = \int d^3r\, \psi^*(x)\left(-\frac{\nabla^2}{2m}\right)\psi(x), \tag{1.1}$$

where we have assumed that the integral over the total derivative is zero because of boundary conditions.

The solutions of the Scrödinger equation
$H\psi_n(x) = E_n\psi_n(x)$ are then

$$\psi_n^+(x) = \frac{1}{\sqrt{L^3}}e^{i\vec{p}_n\cdot\vec{r} - iE_n^0 t} = \frac{1}{\sqrt{L^3}}e^{-ip\cdot r}.$$

So far all of the above has been classical field theory, but we can now go over to the second quantization formalism by quantizing the fields. This is done by expanding the classical field in terms of the eigensolutions of the classical field equations, $\psi(x) = \sum_n a_n\psi_n^+(x)$. We interpret the expansion coefficients as annihilation and creation operators, where a_n is the annihilation operator which destroys a particle of momentum p_n; and the complex conjugate, $a_n^\dagger$, is the creation operator which creates a particle of momentum p_n. For a single particle, the creation and annihilation operators satisfy $[a_n, a_{n'}^\dagger]_\pm = \delta_{nn'}$. For identical many-particle systems, the specific commutator relation that holds depends on the the spin of the particle (See later).

The expansion of the classical field can be substituted into equation (1.1) and using the fact that $\int d^3r\ \psi_n^{+*}(x)\psi_{n'}^{+}(x) = \delta_{nn'}$, the Hamiltonian in second quantization form is obtained:

$$H = \sum_n E_n a_n^\dagger a_n. \tag{1.2}$$

1.1.1 Commutation and Anti-commutation Relations

The quantization of a classical field leads to the introduction of creation and annihilation operators on Fock states, which describe many particles. These particles are identical, which means that the Fock states of a quantum field must be symmetric or anti-symmetric. This implies that the creation and annihilation operators must satisfy commutation relations for symmetric states (integer spin) or anti-commutation relations for anti-symmetric states (half integer spin).

The Schrödinger equation can describe particles with both symmetric and anti-symmetric states, thus we can be quantize by using either commutation relations or anti-commutation relations for operators. The Klein-Gordon equation and the Dirac equation however, describe particles of only symmetric and anti-symmetric states respectively. We can use the same procedure that was used to quantize the Schrödinger theory to quantize the Klein-Gordon and Dirac theories if suitable care is taken as to which commutation or anti-commutation relations are used.

1.2 Klein-Gordon Theory

The Lagrange density for the complex Klein-Gordon theory, which describes spin 0 particles (scalar theory), is given by

$$\mathcal{L} = \frac{\partial \phi^*}{\partial x^\mu}\frac{\partial \phi}{\partial x_\mu} - m^2\phi^*\phi.$$

The equations of motion are $\Box\phi(x) + m^2 = 0$ and the conjugate momenta are

$$\pi^*(x) = \frac{\partial \phi^*}{\partial t} \quad \text{and} \quad \pi(x) = \frac{\partial \phi}{\partial t}.$$

$$\mathcal{H} = \pi^*\pi + \nabla_i\phi^* \cdot \nabla_i\phi + m^2\phi^*\phi.$$

Once again the Hamiltonian can be obtained by integrating over the spatial coordinates to give (after partial integration)

$$H = \int d^3r \left(\frac{\partial \phi^*}{\partial t} \frac{\partial \phi}{\partial t} - \phi^* \left(\nabla^2 \phi - m^2 \phi \right) \right)$$

or

$$H = -\int d^3r \left(\phi^* \frac{\overleftrightarrow{\partial}}{\partial t} \left(\frac{\partial \phi}{\partial t} \right) \right),$$

where the Klein-Gordon equations of motions were used to go from the first line to the second.

We can expand the Klein-Gordon fields in terms of the positive and negative energy solutions[2] of the Klein-Gordon equation and quantize by imposing commutator/anti-commutator relations on the coefficients. This transforms the coefficients into creation and annihilation operators. Thus

$$\begin{aligned} \phi(x) &= \sum_n (a_n \phi_n^+(x) + c_n^\dagger \phi_{-n}^-(x)) \\ \phi^\dagger(x) &= \sum_n (a_n^\dagger \phi_n^{+*}(x) + c_n \phi_{-n}^{-*}(x)) \end{aligned}$$

The coefficients are interpreted as follows

- a_n destroys a particle with momentum p_n and positive charge.
- c_n destroys an antiparticle with momentum p_n and negative charge.
- $a_n^\dagger$ creates a particle with momentum p_n and positive charge.
- $c_n^\dagger$ creates an antiparticle with momentum p_n and negative charge.

Using the orthogonal relations (A.2) between ϕ^+ and ϕ^-, we can rewrite the Hamiltonian

$$H = -\int d^3r \, \phi^\dagger(x) \frac{\overleftrightarrow{\partial}}{\partial t} \frac{\partial \phi}{\partial t}$$

as

$$\begin{aligned} H &= -\int d^3r \sum_n (a_n^\dagger \phi_n^{+*}(x) + c_n \phi_{-n}^{-*}(x)) \\ &\quad \times (-i) \frac{\overleftrightarrow{\partial}}{\partial t} \sum_{n'} E_{n'} (a_{n'} \phi_{n'}^+(x) + c_{n'}^\dagger \phi_{-n'}^-(x)) \\ &= \sum_n E_n (a_n^\dagger a_n + c_n c_n^\dagger). \end{aligned}$$

[2]See section (A.1) for the solution of the free particle Klein-Gordon equation.

The last term above is not in the form of a number operator, so we rewrite $c_n c_n^\dagger = \pm c_n^\dagger c_n + 1$, where the plus is for the operators satisfying commutation relations and the minus is for anti-commutation relations. Thus we obtain

$$H = \sum_n E_n(a_n^\dagger a_n \pm c_n^\dagger c_n) + \langle 0|H|0\rangle$$

We can remove the zero energy term by defining $:H:\equiv H - \langle 0|H|0\rangle$ as the normal ordered product. This gives

$$:H: = \sum_n E_n(a_n^\dagger a_n \pm c_n^\dagger c_n)$$

We would like this Hamiltonian and thus the total energy to be positive definite.[3] This is only satisfied if the creation and annihilation operators satisfy commutation relations. These relations are

$$[a_n, a_{n'}^\dagger]_- = [c_n, c_{n'}^\dagger]_- = \delta_{nn'}$$

with all other commutators zero. Since the Klein-Gordon equation describes particles of 0-spin, they satisfy Bose-Einstein statistics.

1.3 Dirac Theory

The Lagrange density for the Dirac theory, which describes spin-$\frac{1}{2}$ particles, is given by

$$\mathcal{L} = \bar{\varphi}(\frac{i}{2}\gamma^\mu \frac{\overleftrightarrow{\partial}}{\partial x^\mu} - m)\varphi,$$

where $\bar{\varphi} = \varphi^\dagger\gamma^0$.

The equations of motion are $(-i\gamma^\mu \frac{\partial}{\partial x^\mu} - m)\varphi(x) = 0$ and the conjugate momenta are

$$\bar{\pi}(x) = \frac{i}{2}\bar{\varphi}\gamma^0 \quad \text{and} \quad \pi(x) = -\frac{i}{2}\gamma^0\varphi.$$

The Hamiltonian density is

$$\begin{aligned}\mathcal{H} &= \bar{\varphi}(-\frac{i}{2}\gamma^i \frac{\overleftrightarrow{\partial}}{\partial x^i} + m)\varphi(x) \\ &= \varphi^\dagger(-\frac{i}{2}\alpha^i \frac{\overleftrightarrow{\partial}}{\partial x^i} + \beta m)\varphi(x).\end{aligned}$$

[3]We would like the energy to be positive definite as we interpret negative energy particles as being *anti-particles* with a *positive energy*. If we had kept the interpretation of there being only particles – with positive and negative energies, then runaway transitions would occur from the positive to the negative states, and the ground state would then have an infinite negative energy.

We integrate by parts over the spatial coordinates to obtain the Hamiltonian

$$H = \int d^3r\ \varphi^\dagger \left(-i\alpha\cdot\vec{\nabla} + \beta m\right)\varphi.$$

To second quantize, expand φ in terms of the positive and negative energy solutions[4]

$$\varphi(x) = \sum_{n,s}(b_{n,s}\varphi^+_{n,s}(x) + d^\dagger_{n,s}\varphi^-_{-n,-s}(x))$$

and regard the coefficients as creation and annihilation operators. The coefficients are interpreted as follows

- $b_{n,s}$ destroys a particle with spin projection s and momentum p_n.
- $d_{n,s}$ destroys an antiparticle with spin projection s and momentum p_n.
- $b^\dagger_{n,s}$ creates a particle with spin projection s and momentum p_n.
- $d^\dagger_{n,s}$ creates an antiparticle with spin projection s and momentum p_n.

Using this expansion and the canonical equal time anti-commutation relations for classical Dirac fields, $[\varphi_\alpha(r,t),\varphi^\dagger_\beta(r',t)]_+ = \delta^3(r-r')\delta_{\alpha\beta}$, the Hamiltonian reduces to

$$\begin{aligned}
H &= -\int d^3r \sum_{n,s}(b^\dagger_{ns}\phi^{+\dagger}_{ns}(x) + d_{ns}\phi^{-\dagger}_{-n-s}(x)) \\
&\qquad \times \sum_{n',s'} E_{n'}(b_{n's'}\phi^+_{n's'}(x) - d^\dagger_{n's'}\phi^-_{-n'-s'}(x)) \\
&= \sum_{n,s} E_n(b^\dagger_{ns}b_{ns} - d_{ns}d^\dagger_{ns}).
\end{aligned}$$

Once again, we rewrite the last term in terms of the number operator and use the definition of the normal product to obtain

$$:H: = \sum_{n,s} E_n(b^\dagger_{ns}b_{ns} \mp d^\dagger_{ns}d_{ns}),$$

where the minus sign is for the operators satisfying commutation relations and the plus for those satisfying anti-commutation relations. The requirement that the energy must be positive definite is satisfied only if the operators satisfy anti-commutation relations. These relations are

$$[b_{ns}, b^\dagger_{n's'}]_+ = [d_{ns}, d^\dagger_{n's'}]_+ = \delta_{nn'}\delta_{ss'},$$

[4]See (A.2) for the solution of the free particle Dirac equation.

with all other anti-commutators zero. Since the Dirac equation describes spin-$\frac{1}{2}$ particles they satisfy Fermi-Dirac statistics.

1.4 Symmetries

If a symmetry is observed in nature, then the action S must be invariant under the symmetry transformation. This imposes a constraint on the form that the Lagrangian can take. With Noether's Theorem we can identify the conserved current associated with the invariance of the action under general infinitesimal transformations. This is Noether's Theorem:

Noether's Theorem : For every continuous transformation of the field functions and coordinates which leaves the action unchanged, there is a definite combination of the field functions and their derivatives which is conserved.

Infinitesimal transformations of the coordinates and fields can be written as

$$\begin{aligned} x'^{\mu} &= x^{\mu} + \lambda_i^{\mu}(x)\epsilon^i, \\ \phi'_{\alpha}(x') &= \phi_{\alpha}(x) + \Omega_{\alpha i}\epsilon^i, \end{aligned}$$

where λ and Ω are known functions of x and the ϵ^i are known parameters which describe the transformation.

1.4.1 Transformations of states and operators

In quantum mechanics a symmetry is a group of transformations which preserve matrix elements. We can therefore represent each transformation in a symmetry group by a unitary matrix, which operates on the quantum mechanical states, transforming them under the symmetry. Thus the transformation of states is given by

$$U(\theta)|n\rangle = e^{-iQ\theta}|n\rangle = |n'\rangle,$$

where θ is a continuous real parameter and Q is the generator of the transformations. The transformation of operators is obtained by using the invariance of the matrix elements

$$\langle m|\mathcal{O}|n\rangle = \langle m'|\mathcal{O}'|n'\rangle = \langle m|U^{\dagger}(\theta)\mathcal{O}'U(\theta)|n\rangle.$$

We see that the transformation of operators is

$$\mathcal{O}' = U(\theta)\mathcal{O}U^{\dagger}(\theta) = e^{-iQ\theta}\mathcal{O}e^{iQ\theta}.$$

If $\mathcal{O}$ is independent of θ, then

$$-i\frac{\partial \mathcal{O}'(\theta)}{\partial \theta} = [Q, \mathcal{O}'(\theta)]_-.$$

Thus the infinitesimal change in any operator $\mathcal{O}$ under a symmetry transformation is given by the commutator of the generators of the symmetry group with that operator. If the symmetry is *also* a symmetry of the Lagrangian, then the transformations $U(\theta)$ will commute with the Hamiltonian and thus the generator of transformations, Q, will be a constant of the motion. In this case, Q will be the conserved charge that is associated with the symmetry.

1.5 Interactions

Now that we have shown how states and operators transform, we can introduce the two main view points of quantum mechanics, namely the Heisenberg and Scrödinger pictures. We then introduce the interaction picture, which is a combination of the Heisenberg and Scrödinger pictures.

1.5.1 Heisenberg Picture

In this picture, the time development is carried by the field operators ϕ_α according to

$$\begin{aligned} \dot{\phi}_\alpha &= i[H, \phi_\alpha(x)]_- \\ \dot{\pi}_\alpha &= i[H, \pi_\alpha(x)]_- \end{aligned}$$

If H is time independent, then

$$\begin{aligned} \phi_\alpha(x,t) &= e^{iHt}\phi_\alpha(x,0)e^{-iHt}, \\ \pi_\alpha(x,t) &= e^{iHt}\pi_\alpha(x,0)e^{-iHt}. \end{aligned}$$

The state vectors, $|a\rangle$ are time independent.

1.5.2 Scrödinger Picture

We can now define the operators in the Scrödinger picture by means of a unitary transformation

$$\begin{aligned} \phi^{\mathrm{S}}_\alpha(x) &= e^{-iHt}\phi_\alpha(x,t)e^{iHt} = \phi_\alpha(x,0), \\ \pi^{\mathrm{S}}_\alpha(x) &= e^{-iHt}\pi_\alpha(x,t)e^{iHt} = \pi_\alpha(x,0). \end{aligned}$$

Here ϕ_α^{S} and π_α^{S} are the time independent field operators in the Schrödinger picture, and are equivalent to the field operators in the Heisenberg picture at time $t = 0$.

The state vectors $|a,t\rangle^{\mathrm{S}} = e^{-iHt}|a\rangle$ are dependent on time according to the Schrödinger equation

$$i\frac{\partial}{\partial t}|a,t\rangle^{\mathrm{S}} = H^{\mathrm{S}}|a,t\rangle^{\mathrm{S}},$$

where $H^{\mathrm{S}} = H(t=0) = H$.

Note that the Schrödinger and Heisenberg pictures coincide at time $t = 0$.

1.5.3 Interaction Picture

The interaction picture is an intermediate step between the Schrödinger and Heisenberg pictures. We can separate an interactive Hamiltonian into a part that is easily soluble, usually the free Hamiltonian, and a part containing interactions that is more difficult to solve; $H = H_0 + H_{\mathrm{I}}$.

Define the interaction field operators $\phi_\alpha^{\mathrm{in}}$ and conjugate momenta π_α^{in} as

$$\begin{aligned} \phi_\alpha^{\mathrm{in}}(x,t) &= e^{iH_0^{\mathrm{S}}t}\phi_\alpha^{\mathrm{S}}(x)e^{-iH_0^{\mathrm{S}}t} \\ \pi_\alpha^{\mathrm{in}}(x,t) &= e^{iH_0^{\mathrm{S}}t}\pi_\alpha^{\mathrm{S}}(x)e^{-iH_0^{\mathrm{S}}t}, \end{aligned}$$

where H_0^{S} is the free field Hamiltonian in the Schrödinger picture.

Define the state vectors $|a,t\rangle^{\mathrm{in}}$ by

$$|a,t\rangle^{\mathrm{in}} = e^{iH_0^{\mathrm{S}}t}|a,t\rangle^{\mathrm{S}}.$$

The equation of motion for $|a,t\rangle^{\mathrm{in}}$ is obtained by differentiating and using the Schrödinger equation to give

$$\frac{\partial|a,t\rangle^{\mathrm{in}}}{\partial t} = iH_0^{\mathrm{S}}e^{iH_0^{\mathrm{S}}t}|a,t\rangle^{\mathrm{S}} - ie^{iH_0^{\mathrm{S}}t}H^{\mathrm{S}}|a,t\rangle^{\mathrm{S}}.$$

By noting that $H_0^{\mathrm{in}}(t) = e^{iH_0^{\mathrm{S}}t}H_0^{\mathrm{S}}e^{-iH_0^{\mathrm{S}}t} = H_0^{\mathrm{S}}$, $H^{\mathrm{in}}(t) = e^{iH_0^{\mathrm{S}}t}H^{\mathrm{S}}e^{-iH_0^{\mathrm{S}}t}$ and $H^{\mathrm{in}} = H_0^{\mathrm{in}} + H_{\mathrm{I}}^{\mathrm{in}}$, the equations of motion are rewritten as

$$i\frac{\partial|a,t\rangle^{\mathrm{in}}}{\partial t} = H_{\mathrm{I}}^{\mathrm{in}}(t)|a,t\rangle^{\mathrm{in}} \tag{1.3}$$

The time dependence of the states in the interaction picture are thus determined by the interaction Hamiltonian, $H_{\mathrm{I}}^{\mathrm{in}}(t)$, and the time dependence of the operators are determined by the free Hamiltonian, since

$$\dot{\phi}_\alpha^{\text{in}}(x,t) = i[H_0^{\text{S}}, \phi_\alpha^{\text{in}}(x,t)] = i[H_0^{\text{in}}, \phi_\alpha^{\text{in}}(x,t)].$$

1.6 Time evolution operator

The time evolution operator, U, is an operator that transforms a state at time t_0 into a corresponding state at time t_1. Thus

$$|a,t_1\rangle^{\text{in}} = U(t_1,t_0)|a,t_0\rangle^{\text{in}}. \tag{1.4}$$

The properties of U are given by

$$\begin{aligned} &U(t_0,t_0) = 1 \\ &U(t_1,t)U(t,t_0) = U(t_1,t_0) \\ &U^\dagger(t,t_0)U(t,t_0) = 1 \\ &i\frac{\partial U(t,t_0)}{\partial t} = H_{\text{I}}^{\text{in}} U(t,t_0) \end{aligned} \tag{1.5}$$

The last equation of (1.5) is obtained from (1.3) and (1.4). We can convert this into an integral equation using the boundary condition $U(t_0,t_0) = 1$. Thus

$$\begin{aligned} U(t_1,t_0)|_{t_0}^{t} &= \int_{t_0}^{t} dt_1(-iH_{\text{I}}^{\text{in}}U(t_1,t_0)) \\ U(t,t_0) &= 1 - i\int_{t_0}^{t} dt_1(H_{\text{I}}^{\text{in}}U(t_1,t_0)). \end{aligned} \tag{1.6}$$

1.6.1 Dyson time series

We can now use perturbation theory to solve (1.6). If H_{I}[5] is small, then we can apply the standard iterative solution to obtain the Dyson time series :

$$\begin{aligned} U(t,t_0) &= 1 - i\int_{t_0}^{t} dt_1 H_{\text{I}}(t_1) + (-i)^2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_{\text{I}}(t_1)H_{\text{I}}(t_2) \\ &\quad + \ldots + (-i)^n \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n H_{\text{I}}(t_1)\cdots H_{\text{I}}(t_n) + \ldots. \end{aligned} \tag{1.7}$$

To simplify (1.7), note that the nth order term can be written as

$$\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n H_{\text{I}}(t_1)\cdots H_{\text{I}}(t_n) = \frac{1}{n!}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n T[H_{\text{I}}(t_1)\cdots H_{\text{I}}(t_n)],$$

[5]From now on we will work only in the interaction picture and thus shall suppress the $^{\text{in}}$ superscripts.

where $T[A(t_1)B(t_2)] = A(t_1)B(t_2)\theta(t_1 - t_2) + B(t_2)A(t_1)\theta(t_2 - t_1)$ is the time ordering operator.

Thus (1.7) gives

$$\begin{aligned} U(t, t_0) &= 1 + \textstyle\sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n T[H_I(t_1) \cdots H_I(t_n)] \\ &\equiv T e^{-i\int_{t_0}^{t} d^4x \mathcal{H}(x)}, \end{aligned} \tag{1.8}$$

where $H_I(t) = \int d^3x \mathcal{H}(x)$.

1.7 Interacting Field Theories

We can now introduce an interacting field theory, e.g. the ϕ^3 theory. This theory includes 3 kinds of scalar particles, two charged and one neutral. The Lagrangian is given by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{\text{int}}$, where

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x_\mu} - \frac{1}{2}\mu^2\phi^2, \\ \mathcal{L}_i &= \frac{1}{2}\frac{\partial\Phi_i^\dagger}{\partial x^\mu}\frac{\partial\Phi_i}{\partial x_\mu} - \frac{1}{2}m_i^2\Phi_i^\dagger\Phi_i, \qquad i = 1, 2 \\ \mathcal{L}_{\text{int}} &= -\lambda_1\Phi_1^\dagger(x)\Phi_1(x)\phi(x) - \lambda_2\Phi_2^\dagger(x)\Phi_2(x)\phi(x) - \frac{\lambda}{3!}\phi^3(x). \end{aligned}$$

Thus we can calculate the Hamiltonian in the second quantization representation to obtain (after normal ordering)

$$H = \sum_k E_k(a_{ik}^\dagger a_{ik} + c_{ik}^\dagger c_{ik}) + \sum_k E_k a_k^\dagger a_k + \int d^3x \mathcal{H}_{\text{int}},$$

where

$$\mathcal{H}_{\text{int}} = \lambda_1\Phi_1^\dagger(x)\Phi_1(x)\phi(x) + \lambda_2\Phi_2^\dagger(x)\Phi_2(x)\phi(x) + \frac{\lambda}{3!}\phi^3(x).$$

The time translation operator, (1.8), would contain the above interaction Hamiltonian density for the ϕ^3 theory.

1.8 Green's functions

A wide range of observables may be expressed as the expectation value of products of an operator at different space time points. This leads to the introduction of the n-point Green's function, which is the ground state (vacuum) expectation value of a time-ordered product of

n field operators. The Green's function is thus given by

$$G^n(x_1, \ldots, x_n) = \langle 0|T(\varphi(x_1)\ldots\varphi(x_n))|0\rangle. \tag{1.9}$$

As we will show later, the n-point Green's function can be directly related to the scattering amplitude involving n incoming and outgoing particles. The scattering amplitude can be measured in experiment, thus the Green's function is important as it relates theory to experimental results.

CHAPTER 2
Path Integrals in Quantum Mechanics

2.1 Quantum mechanics

If a physical system is in an initial state $|\zeta, t_1\rangle$ at a time t_1, then the probability that the system will be in a different state $|\zeta', t_2\rangle$ at a later time t_2, is given by $P = |\langle\zeta', t_2|\zeta, t_1\rangle|^2$. We need to be able to calculate the transformation function, or probability function, $\langle\zeta', t_2|\zeta, t_1\rangle$.

The time dependent wave function is the transformation function $\psi(q,t) = \langle q, t|\psi\rangle$. Associated with the coordinate operator $q(t)$ is the momentum operator $p(t)$. The transformation function between the eigenstates of the coordinate operator and the momentum operator, $\langle q|p\rangle$ is a fundamental quantity and is given by

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi}}e^{iqp} \quad \text{and} \quad \langle p|q\rangle = \langle q|p\rangle^* = \frac{1}{\sqrt{2\pi}}e^{-iqp},$$

with

$$\int dq|q\rangle\langle q| = 1 = \int dp|p\rangle\langle p|.$$

Thus we can expand $\langle\zeta', t_2|\zeta, t_1\rangle$ by

$$\begin{aligned}\langle\zeta', t_2|\zeta, t_1\rangle &= \int dq' \int dq'' \langle\zeta', t_2|q'', t_2\rangle\langle q'', t_2|q', t_1\rangle\langle q', t_1|\zeta, t_1\rangle \\ &= \int dq' \int dq'' \langle\zeta'|q''\rangle\langle q'', t_2|q', t_1\rangle\langle q'|\zeta\rangle.\end{aligned}$$

Since we can calculate $\langle\zeta'|q''\rangle$ and $\langle q'|\zeta\rangle$, a knowledge of $\langle q'', t_2|q', t_1\rangle$ enables us to calculate $\langle\zeta', t_2|\zeta, t_1\rangle$.

2.2 Functional integral representation

We wish to represent the basic transformation function, $\langle q'', t_2|q', t_1\rangle$, in an integral form. To do this, we divide the time interval into N smaller intervals, $T = (t_2 - t_1) = (N+1)\Delta t$, and then insert N resolutions of the identity, one for each time slice, into the transformation function. This yields

$$\begin{aligned}\langle q'', t_2|q', t_1\rangle &= \int dq_1 \ldots dq_N \langle q_{N+1}, t_1 + (N+1)\Delta t|q_N, t_1 + N\Delta t\rangle \\ &\quad \times\langle q_N, t_1 + N\Delta t|\cdots|q_1, t_1 + \Delta t\rangle\langle q_1, t_1 + \Delta t|q_0, t_1\rangle, \qquad (2.1)\end{aligned}$$

where $q_{N+1} = q''$ and $q_0 = q'$.

This decomposition involves integrating over all possible functions defined in the interval t_1, t_2, because the variables $q_1 \dots q_N$ are completely arbitrary. We now need to calculate the infinitesimal transformation functions $\langle q_k, t + \Delta t | q_{k-1}, t\rangle$. This may be achieved by an approximation that becomes exact as $\Delta t \to 0$ or $N \to \infty$. In this limit, we form a functional integral, where we integrate over an infinite number of variables.

To calculate $\langle q_k, t+\Delta t | q_{k-1}, t\rangle$, introduce a complete set of intermediate eigenstates of the momentum operator

$$\langle q_k, t + \Delta t | q_{k-1}, t\rangle = \int dp_k \langle q_k, t + \Delta t | p_k, t\rangle \langle p_k, t | q_{k-1}, t\rangle.$$

Assuming the form of the Hamiltonian to be $H = \frac{p^2}{2m} + V(q)$, we have

$$\begin{aligned}
& \langle q_k, t + \Delta t | p_k, t\rangle \\
= \; & \langle q_k | e^{-iH\Delta t} | p_k\rangle = \langle q_k | (1 - iH\Delta t - \frac{H^2}{2!}\Delta t^2 + \dots) | p_k\rangle \\
= \; & \langle q_k | [1 - i(\frac{p^2}{2m} + V(q))\Delta t - \frac{1}{2!}(\frac{p^2}{2m} + V(q))^2 \Delta t^2 + \dots] | p_k\rangle \\
= \; & \langle q_k | [1 - i :(\frac{p^2}{2m} + V(q)): \Delta t - \frac{1}{2!} :(\frac{p^2}{2m} + V(q)):^2 \Delta t^2 \\
& + \frac{1}{2!}[V(q), \frac{p^2}{2m}]_- \Delta t^2 + \dots] | p_k\rangle \\
& \text{(the ordering, : :, is } p \text{ to the right and } q \text{ to the left)} \\
= \; & [1 - i(\frac{p_k^2}{2m} + V(q_k))\Delta t - \frac{1}{2!}(\frac{p_k^2}{2m} + V(q_k))^2 \Delta t^2 \\
& + \frac{1}{2!}(\frac{\partial V(q_k)}{\partial q_k}\frac{p_k}{m} + \frac{1}{2m}\frac{\partial^2 V(q_k)}{\partial q_k^2}))\Delta t^2 + \dots] \langle q_k | p_k\rangle \\
= \; & [e^{-i(\frac{p_k^2}{2m} + V(q_k))\Delta t} + \frac{1}{2!}(\frac{\partial V(q_k)}{\partial q_k}\frac{p_k}{m} + \frac{1}{2m}\frac{\partial^2 V(q_k)}{\partial q_k^2}))\Delta t^2] \langle q_k | p_k\rangle \\
= \; & [e^{-i(\frac{p_k^2}{2m} + V(q_k))\Delta t + O(\Delta t^2)}] \frac{1}{\sqrt{2\pi}} e^{ip_k q_k},
\end{aligned}$$

where we work to lowest order in Δt. The term of order Δt^2 occurs due to corrections that take place in the ordering of the Hamiltonian. Because the the integral $\int dp$ occurs before the limit $\Delta t \to 0$ is taken, we cannot ignore the $O(\Delta t^2)$ term. If this term contains a function of p_k such that $O(f(p_k)\Delta t^2) \leq O(\Delta t)$, then that part of the second order term must not be neglected. The functions of p_k that must be taken into account are of order p_k^2, since $O(p_k^2 \Delta t^2) = O(\frac{2m}{\Delta t}\Delta t^2) = O(\Delta t)$.

If we assume that the $O(\Delta t^2)$ is independent of p_k^2 terms, then we obtain

$$\langle q_k, t+\Delta t | p_k, t\rangle \approx \frac{1}{\sqrt{2\pi}} e^{+i[p_k q_k - H(p_k,q_k)\Delta t]}.$$

Thus

$$\begin{aligned}\langle q_k, t+\Delta t | q_{k-1}, t\rangle &= \int dp_k \langle q_k, t+\Delta t|p_k, t\rangle\langle p_k, t|q_{k-1}, t\rangle \\ &\approx \int \frac{dp_k}{2\pi} e^{+i\left(p_k(q_k-q_{k-1}) - H(p_k,q_k)\Delta t\right)}.\end{aligned}$$

The basic transformation function (2.1) becomes

$$\langle q'', t_2|q', t_1\rangle = \int \frac{dp_{N+1}}{2\pi} \int \prod_{k=1}^{N} \frac{dq_k dp_k}{2\pi} e^{i\sum_{k=1}^{N+1}\left(p_k(q_k-q_{k-1}) - H(p_k,q_k)\Delta t\right)}. \tag{2.2}$$

We can write (2.2) in a more compact form by noting that in the limit $N \to \infty$,

$$\begin{aligned}\sum_{k=1}^{N+1} p_k(q_k - q_{k-1}) &= \sum_{k=1}^{N+1} \frac{p(t_k)(q(t_k) - q(t_k-\Delta t))}{\Delta t}\Delta t \\ &= \int_{t_1}^{t_2} dt\, p(t)\dot{q}(t),\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{N+1} H(q_k, p_k, t_1 + k\Delta t)\Delta t &= \sum_{k=1}^{N+1} H(q(t_k), p(t_k), t_k)\Delta t \\ &= \int_{t_1}^{t_2} dt\, H(q(t), p(t), t).\end{aligned}$$

If we define the measure as

$$\frac{dp_{N+1}}{2\pi} \prod_{k=1}^{N} \frac{dq_k dp_k}{2\pi} \equiv [dp][dq],$$

then (2.2) becomes

$$\begin{aligned}\langle q'', t_2|q', t_1\rangle &= \int [dp][dq] e^{i\int_{t1}^{t2} dt[p(t)\dot{q}(t) - H(p(t),q(t),t)]} \\ &= \int [dp][dq] e^{i\int_{t1}^{t2} dt L} \\ &= \int [dp][dq] e^{iS},\end{aligned} \tag{2.3}$$

where S is the action.

If there are no velocity dependent forces, i.e $H = \frac{p^2}{2m} + V(q,t)$, then the momenta can be integrated out in (2.3) to give

$$\langle q'', t_2 | q', t_1 \rangle = \int [dq] e^{i \int_{t1}^{t2} dt [\frac{1}{2} m \dot{q}(t)^2 - V(q(t))]}$$

where the regular measure is now

$$[dq] = \sqrt{\frac{m}{2\pi i \Delta t}} \prod_{k=1}^{N} \sqrt{\frac{m}{2\pi i \Delta t}} dq_k.$$

This new form of the functional integral involves $L = \frac{1}{2} m \dot{q}(t)^2 - V(q(t))$, which is the classical Lagrangian in the second form, thus

$$\langle q'', t_2 | q', t_1 \rangle = \int [dq] e^{i \int_{t1}^{t2} dt L} = \int [dq] e^{iS} \tag{2.4}$$

The N variables $q_1, \ldots, q_N$ specify an unique path. The functional integral is an integral over all these variables, weighted by e^{iS}. Thus the functional integral can be considered as a sum over all possible paths.

2.3 External Forces

In general we wish to be able to calculate the matrix elements $\langle\ q'', t'' |\ F[\hat{q}]\ | q', t'\ \rangle$, where $F[\hat{q}]$ is a general function of the operator $\hat{q}$. We thus need to calculate $\langle q'', t'' | \hat{q}_k | q', t' \rangle$, where $t'' > t_k > t'$. This can be done by inserting M resolutions of the identity between $\hat{q}_k$ and $|q', t'\rangle$, and $N - M$ resolutions of the identity between $\langle q'', t''|$ and $\hat{q}_k$ so that the following is obtained,

$$\begin{aligned}
\langle q'', t'' | \hat{q}_k | q', t' \rangle \quad = \quad & \int dq_N \ldots dq_{M+1} \int dq_M \ldots dq_1 \langle q'', t'' | q_N, t_k + (N-M)\Delta t \rangle \\
& \times \langle q_N, t_k + (N-M)\Delta t | \cdots | q_{M+1}, t_k + \Delta t \rangle \\
& \times \langle q_{M+1}, t_k + \Delta t | \hat{q}_k | q_M, t' + M\Delta t \rangle \\
& \times \langle q_M, t' + M\Delta t | \cdots | q_1, t' + \Delta t \rangle \langle q_1, t' + \Delta t | q', t' \rangle,
\end{aligned}$$

where $t'' = t' + (N+1)\Delta t$ and $t_k = t' + M\Delta t$.

We can thus calculate the eigenvalue of $\hat{q}_k$ to obtain

$$\langle q'', t'' | \hat{q}_k | q', t' \rangle \quad = \quad \int dq_N \ldots dq_1 \ q_k \langle q'', t'' | q_N, t' + N\Delta t \rangle$$

$$
\begin{aligned}
&\times\langle q_N, t' + N\Delta t|\cdots|q_{k+1}, t' + (k+1)\Delta t\rangle \\
&\times\langle q_{k+1}, t' + (k+1)\Delta t|q_k, t' + k\Delta t\rangle \\
&\times\langle q_k, t' + k\Delta t|\cdots|q_1, t' + \Delta t\rangle\langle q_1, t' + \Delta t|q', t'\rangle \\
=\ & \int [dq]\ q(t) e^{i\int_{t'}^{t''} dt L_0}, \qquad (2.5)
\end{aligned}
$$

where we have used (2.4) for the last equality.

If we add an external source term to (2.4), then the Lagrangian will be altered to $L = L_0 + qf$, and the transition function with external sources will be given by

$$\langle q'', t''|q', t'\rangle^f \equiv \int [dq] e^{i\int_{t'}^{t''} dt[L_0 + f(t)q(t)]}. \qquad (2.6)$$

Taking the functional derivative[6] of (2.6) with respect to $f(t)$ and then setting the source to zero gives (2.5). Thus we can write the matrix element of an operator in terms of a functional integral,

$$
\begin{aligned}
\langle q'', t''|\hat{q}(t)|q', t'\rangle &= \frac{1}{i}\frac{\delta}{\delta f(t)}\langle q'', t_2|q', t_1\rangle^f|_{f(t)=0} \\
&= \int [dq]\ q(t) e^{i\int_{t'}^{t''} dt L_0}.
\end{aligned}
$$

We can also write the matrix element of the product of n time ordered operators in terms of a functional integral by taking n functional derivatives of (2.6) with respect to $f(t_i)$ and then setting the sources to zero. Thus

$$
\begin{aligned}
\langle q'', t''|T[\hat{q}(t_1)\ldots\hat{q}(t_n)]|q', t'\rangle &= \frac{1}{i}\frac{\delta}{\delta f(t_1)}\cdots\frac{1}{i}\frac{\delta}{\delta f(t_n)}\langle q'', t_2|q', t_1\rangle^f|_{f(t)=0} \\
&= \int [dq]\ q(t_1)\ldots q(t_n) e^{i\int_{t'}^{t''} dt L_0}.
\end{aligned}
$$

Any time ordered functional of operators $T(F[\hat{q}])$ can now be expressed in a functional integral form by

$$\langle q'', t''|T\ F[\hat{q}]|q', t'\rangle = \int [dq]\ F[q] e^{i\int_{t'}^{t''} dt L_0}. \qquad (2.7)$$

2.4 Classical Limit

Reverting to ordinary units, the phase that appears in the path integral becomes

$$e^{\frac{i}{\hbar}\int_{t'}^{t''} dt L_0} = e^{\frac{i}{\hbar}S}.$$

[6] See appendix (B.1)

In the classical limit $\hbar \to 0$. In this case, the phase will oscillate wildly and the dominant contribution to the path integral will occur when $\delta S \simeq 0$. In this limit, called the stationary phase limit, we are provided with a valid zero order approximation to the path integral.

The condition $\delta S \simeq 0$ yields the Euler Lagrange equations of motion. The solution of these equations define the classical trajectory, $q_{cl}(t)$ between q' and q''. We can now expand $q(t)$ around the classical trajectory, $q(t) = q_{cl}(t) + \tilde{q}(t)$, so that the transformation function becomes

$$
\begin{aligned}
& \langle q'', t''|q', t'\rangle \\
= & \int [dq] e^{\frac{i}{\hbar} S[q_{cl}(t)+\tilde{q}(t)]} \\
= & \int [d\tilde{q}] e^{\frac{i}{\hbar} S[q_{cl}(t)+\tilde{q}(t)]} \\
= & \int [d\tilde{q}] e^{\frac{i}{\hbar}\left(S[q_{cl}(t)] + \frac{\delta S}{\delta q(t)}|_{q_{cl}} \tilde{q}(t) + \frac{\delta^2 S}{\delta q(t_1)\delta q(t_2)}|_{q_{cl}} \tilde{q}(t_1)\tilde{q}(t_2) + \ldots\right)}
\end{aligned}
$$

where an integral over t is implied in the last two terms of the exponent.

But $\frac{\delta S}{\delta q(t)}|_{q_{cl}} = 0$, thus

$$
\begin{aligned}
& \langle q'', t''|q', t'\rangle \\
= & \int [d\tilde{q}] e^{\frac{i}{\hbar}\left(S[q_{cl}(t)] + \frac{\delta^2 S}{\delta q(t_1)\delta q(t_2)}|_{q_{cl}} \tilde{q}(t_1)\tilde{q}(t_2) + \ldots\right)}.
\end{aligned}
$$

We introduce sources so that we can calculate normalized expectation values,

$$
\begin{aligned}
& \langle q(t)\rangle^f \\
= & \frac{\langle q'', t''|q(t)|q', t'\rangle^f}{\langle q'', t''|q', t'\rangle^f} \\
= & \frac{\frac{\hbar}{i}\frac{\delta}{\delta f(t)} \int [d\tilde{q}] e^{\frac{i}{\hbar}\left(S[q_{cl}] + \frac{\delta^2 S}{\delta q(t_1)\delta q(t_2)}|_{q_{cl}} \tilde{q}(t_1)\tilde{q}(t_2) + \ldots + \int_{t'}^{t''} dt (q_{cl}+\tilde{q}) f\right)}}{\int [d\tilde{q}] e^{\frac{i}{\hbar}\left(S[q_{cl}] + \frac{\delta^2 S}{\delta q(t_1)\delta q(t_2)}|_{q_{cl}} \tilde{q}(t_1)\tilde{q}(t_2) + \ldots + \int_{t'}^{t''} dt (q_{cl}+\tilde{q}) f\right)}} \\
= & \frac{\hbar}{i}\frac{\delta}{\delta f(t)} \ln\left[e^{\frac{i}{\hbar}\left(S[q_{cl}] + \int_{t'}^{t''} dt\, q_{cl} f\right)} \int [d\tilde{q}] e^{\frac{i}{\hbar}\left(\frac{\delta^2 S}{\delta q(t_1)\delta q(t_2)}|_{q_{cl}} \tilde{q}(t_1)\tilde{q}(t_2) + \ldots + \int_{t'}^{t''} dt \tilde{q} f\right)}\right] \\
= & \frac{\delta}{\delta f(t)}\left(S[q_{cl}] + \int_{t'}^{t''} dt\, q_{cl} f\right) \\
& + \frac{\hbar}{i}\frac{\delta}{\delta f(t)} \ln\left[\int [d\tilde{q}] e^{\frac{i}{\hbar}\left(\frac{\delta^2 S}{\delta q(t_1)\delta q(t_2)}|_{q_{cl}} \tilde{q}(t_1)\tilde{q}(t_2) + \ldots + \int_{t'}^{t''} dt \tilde{q} f\right)}\right] \\
\approx & \frac{\delta S[q_{cl}]}{\delta f(t)} + q_{cl}
\end{aligned}
$$

$$= \quad q_{cl},$$

since $S[q_{cl}$ is independent of the sources $f(t)$. We thus see that in the classical, stationary phase limit, the expectation value of an operator, $\langle q(t)\rangle^f$, is just the operator's classical value. This result holds even if the sources are now set to zero.

2.5 Ground state to ground state amplitudes.

In section 1.8 we saw that the Green's functions are defined in terms of the ground state expectation value of time ordered operators. With this in mind, we would like to derive a path integral formalism for the ground state to ground state transition amplitudes in the presence of an external source. We will take the external source, $f(t)$, to be zero for times less than t_- and more than t_+.

Using the completeness relations of the eigenstates $|q_+, t_+\rangle$ and $|q_-, t_-\rangle$ we see that the transition amplitude $\langle q'', t''|q', t'\rangle$ in the presence of a source becomes

$$\langle q'', t''|q', t'\rangle^f = \int dq_+ \int dq_- \langle q'', t''|q_+, t_+\rangle\langle q_+, t_+|q_-, t_-\rangle^f \langle q_-, t_-|q', t'\rangle \tag{2.8}$$

provided that $t'' > t_+ > t_- > t'$.

We note that $\langle q'', t''|q_+, t_+\rangle$ can be expanded in terms of a complete set of energy eigenstates at time t_+ to give

$$\begin{aligned}\langle q'', t''|q_+, t_+\rangle &= \langle q''|e^{iHt''}|q_+, t_+\rangle \\ &= \sum_n e^{iE_n t''}\langle q''|n, t_+\rangle\langle n, t_+|q_+, t_+\rangle\end{aligned}$$

and similarly $\langle q_-, t_-|q', t'\rangle$ can be expanded at time t_- to give

$$\begin{aligned}\langle q_-, t_-|q', t'\rangle &= \langle q_-, t_-|e^{-iHt'}|q'\rangle \\ &= \sum_n e^{-iE_n t'}\langle q_-, t_-|n, t_-\rangle\langle n, t_-|q', t'\rangle\end{aligned}$$

If we analytically continue to imaginary time, that is $t \to i\bar{t}$, then

$$\langle q'', t''|q_+, t_+\rangle \to \langle q'', \bar{t}''|q_+, \bar{t}_+\rangle = \sum_n e^{-E_n \bar{t}''}\langle q'', \bar{t}''|n, \bar{t}_+\rangle\langle n, \bar{t}_+|q_+, \bar{t}_+\rangle.$$

If we take the limit $\bar{t}'' \to \infty$ then $e^{-E_0 \bar{t}''}$ becomes the dominant term[7] and we can ignore the

[7]The eigenvalue spectrum of H must have a lower bound otherwise the system will be unstable and will

rest of the expansion terms. Thus

$$\lim_{\bar{t}''\to\infty}\langle q'',\bar{t}''|q_+,\bar{t}_+\rangle \simeq e^{-E_0\bar{t}''}\langle q'',\bar{t}''|0,\bar{t}_+\rangle\langle 0,\bar{t}_+|q_+,\bar{t}_+\rangle.$$

Similarly, if we take the limit $\bar{t}' \to -\infty$ then we obtain

$$\lim_{\bar{t}'\to-\infty}\langle q_-,\bar{t}_-|q',\bar{t}'\rangle \simeq e^{E_0\bar{t}'}\langle q_-,\bar{t}_-|0,\bar{t}_-\rangle\langle 0,\bar{t}_-|q',\bar{t}'\rangle.$$

If we take the transition function (2.8), normalize by the sourceless transition function and take the limits $t'' \to i\infty, t' \to -i\infty$; then

$$\begin{aligned}
&\lim_{\substack{t''\to i\infty\\ t'\to -i\infty}} \frac{\langle q'',t''|q',t'\rangle^f}{\langle q'',t''|q',t'\rangle^0}\\
=&\lim_{\substack{t''\to i\infty\\ t'\to -i\infty}} \Big[\frac{\int dq_+\int dq_-\langle q'',t''|q_+,t_+\rangle^0\langle q_+,t_+|q_-,t_-\rangle^f}{\int dq_+\int dq_-\langle q'',t''|q_+,t_+\rangle^0\langle q_+,t_+|q_-,t_-\rangle^0}\\
&\qquad\times\frac{\langle q_-,t_-|q',t'\rangle^0}{\langle q_-,t_-|q',t'\rangle^0}\Big]\\
=&\lim_{\substack{t''\to i\infty\\ t'\to -i\infty}} \Big[\frac{\int dq_+\int dq_- e^{iE_0t''}\langle q''|0,t_+\rangle^0\langle 0,t_+|q_+,t_+\rangle^0\langle q_+,t_+|q_-,t_-\rangle^f}{\int dq_+\int dq_- e^{iE_0t''}\langle q''|0,t_+\rangle^0\langle 0,t_+|q_+,t_+\rangle^0\langle q_+,t_+|q_-,t_-\rangle^0}\\
&\qquad\times\frac{\langle q_-,t_-|0,t_-\rangle^0\langle 0,t_-|q',t'\rangle^0 e^{-iE_0t'}}{\langle q_-,t_-|0,t_-\rangle^0\langle 0,t_-|q',t'\rangle^0 e^{-iE_0t'}}\Big]\\
=&\lim_{\substack{t''\to i\infty\\ t'\to -i\infty}} \frac{e^{iE_0t''}\langle q''|0,t_+\rangle^0\langle 0,t_+|0,t_-\rangle^f\langle 0,t_-|q',t'\rangle^0 e^{-iE_0t'}}{e^{iE_0t''}\langle q''|0,t_+\rangle^0\langle 0,t_+|0,t_-\rangle^0\langle 0,t_-|q',t'\rangle^0 e^{-iE_0t'}}\\
=&\frac{\langle 0,t_+|0,t_-\rangle^f}{\langle 0,t_+|0,t_-\rangle^0}.
\end{aligned}$$

We can now take the limit of $t_+ \to i\infty$ and $t_- \to -i\infty$ so that the external source is defined at all times. We then obtain the normalized ground state to ground state transition amplitude

$$\frac{\langle 0,i\infty|0,-i\infty\rangle^f}{\langle 0,i\infty|0,-i\infty\rangle^0} \equiv \frac{W[f]}{W[0]} \equiv Z[f], \tag{2.9}$$

where $Z[f]$ is the normalized generating functional ($W[f]$ is the unnormalized generating functional) with an external source f.

continuously be decaying into lower energy levels. Thus $E_0 < E_1 < E_2 < \cdots$

From (2.6) we see that

$$
\begin{aligned}
&\lim_{\substack{t_+ \to i\infty \\ t_- \to -i\infty}} \lim_{\substack{t'' \to i\infty \\ t' \to -i\infty}} \frac{\langle q'', t'' | q', t' \rangle^f}{\langle q'', t'' | q', t' \rangle^0} \\
&= \lim_{\substack{t_+ \to i\infty \\ t_- \to -i\infty}} \lim_{\substack{t'' \to i\infty \\ t' \to -i\infty}} \frac{\int [dq] e^{i \int_{t'}^{t''} dt [L + f(t) q(t)]}}{\int [dq] e^{i \int_{t'}^{t''} dt [L]}} \\
&= \frac{\int [dq] e^{i \int_{-i\infty}^{i\infty} dt [L + f(t) q(t)]}}{\int [dq] e^{i \int_{-i\infty}^{i\infty} dt [L]}}.
\end{aligned}
$$

Thus the normalized generating functional in the path integral formalism is given by

$$
Z[f] = \frac{\int [dq] e^{i \int dt [L + f(t) q(t)]}}{\int [dq] e^{i \int dt [L]}}, \tag{2.10}
$$

where the time integral in the exponent is over the imaginary region $(-i\infty, i\infty)$.

2.6 Green's Functions as path integrals.

We would like to write the Green's function in terms of a path integral. From (2.7) we see that

$$
\langle q'', t'' | T \ F[\hat{q}] | q', t' \rangle = \int [dq] \ F[q] e^{i \int_{t'}^{t''} dt L}.
$$

We project out the ground states by using the imaginary time damping technique of the previous section. If we take the limits $t'' \to i\infty, t' \to -i\infty$ then we obtain

$$
\langle 0, \infty | T \ F[\hat{q}] | 0, -\infty \rangle = \int [dq] \ F[q] e^{i \int_{-i\infty}^{i\infty} dt L}.
$$

The normalized Green's function is given by (1.9)

$$
G(q(t_1) \cdots q(t_n)) \equiv \frac{\langle 0, \infty | T[q(t_1) \cdots q(t_n)] | 0, -\infty \rangle}{\langle 0, \infty | 0, -\infty \rangle},
$$

where the field operators in quantum mechanics are the coordinate operators.

Thus

$$
G(q(t_1) \cdots q(t_n)) \quad = \quad \frac{\langle 0, \infty | T[q(t_1) \cdots q(t_n)] | 0, -\infty \rangle}{\langle 0, \infty | 0, -\infty \rangle}
$$

$$
\begin{aligned}
&= \frac{\int [dq] q(t_1) \cdots q(t_n) e^{i \int dt L}}{\int [dq] e^{i \int dt [L]}} \\
&= \frac{1}{i} \frac{\delta}{\delta f(t_1)} \cdots \frac{1}{i} \frac{\delta}{\delta f(t_n)} \frac{\int [dq] e^{i \int dt [L + f q]} \Big|_{f(t)=0}}{\int [dq] e^{i \int dt [L]}} \\
&= \left(\frac{1}{i}\right)^n \frac{\delta}{\delta f(t_1)} \cdots \frac{\delta}{\delta f(t_n)} Z[f] \Big|_{f(t)=0} \qquad (2.11)
\end{aligned}
$$

2.7 Path integrals in field theory

We have seen in this chapter that we can calculate all the results of quantum mechanics (which can be obtained with a knowledge of the transformation function) by means of functional integrals. We wish to generalize this method to apply to many-particle systems.

In analogy with the definition of a generating functional in quantum mechanics, we can define a generating functional for a field theory. The unnormalized generating functional is defined as

$$
W[0] \equiv \int [d\phi] e^{i \int dt L} = \int [d\phi] e^{i \int d^4 x \mathcal{L}},
$$

where $\mathcal{L}$ is the Lagrange density and the time component of the integral is integrated over $(-i\infty, i\infty)$.

We can introduce sources for the fields, ρ to obtain the generating functional in the presence of a external source

$$
W[\rho] = \int [d\phi] e^{i \int d^4 x [\mathcal{L} + \rho \phi]}. \qquad (2.12)
$$

CHAPTER 3
Quantum Field Theory of Scalar Fields

3.1 Free field propagators

From section 1.7 we know that the Lagrange density for neutral scalar fields is given by

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2. \tag{3.1}$$

Inserting 3.1 into (2.12) we have

$$\begin{aligned} W[\rho] &= \int [d\phi] e^{i\int d^4x \frac{1}{2}[(\partial_\mu\phi)^2 - m^2\phi^2 + \rho\phi]} \\ &= \int [d\phi] e^{i\int d^4x [\frac{1}{2}\phi(-\partial_\mu\partial^\mu - m^2)\phi + \rho\phi]}, \end{aligned} \tag{3.2}$$

where we have integrated by parts in the second step. As shown previously the range of the time component of the integral in the exponent is from $-i\infty$ to $i\infty$.

We now continue from Minkowski space to Euclidean space by using a Wick rotation, where we transform $x^0 = -i\bar{x}^4$ and $x^i = \bar{x}^i$, with $i = 1, 2, 3$. Thus (3.2) becomes

$$W_E[\rho] = \int [d\phi] e^{-\int d^4\bar{x}[\frac{1}{2}\phi(-\bar{\partial}^\mu\bar{\partial}^\mu + m^2)\phi/partial - \rho\phi]}. \tag{3.3}$$

$W_E[\rho]$ is a damped integral because $-\bar{\partial}^\mu\bar{\partial}^\mu + m^2$ is positive definite. We can thus calculate the integral directly since it is Gaussian.[8] Writing (3.3) in matrix form[9] $M(\bar{x}, \bar{x}') = (-\bar{\partial}^\mu\bar{\partial}^\mu + m^2)\delta(\bar{x} - \bar{x}')$, and then completing the square we obtain

$$\begin{aligned} W_E[\rho] &= \int [d\phi] e^{\frac{1}{2}[-(\phi - \rho M^{-1})M(\phi - M^{-1}\rho) + \rho M^{-1}\rho]} \\ &\propto (\det M)^{-\frac{1}{2}}\, e^{\frac{1}{2}\rho M^{-1}\rho}. \end{aligned}$$

If we define the normalized generating functional as

$$Z[\rho] \equiv W[\rho]/W[0], \tag{3.4}$$

then we have, for a free field,

$$Z_0[\rho] = e^{\frac{1}{2}\rho M^{-1}\rho}. \tag{3.5}$$

[8] See appendix (B.4).

[9] An integral $\int d^4\bar{x}$ is implied with multiplication of matrices.

We can easily determine M^{-1} to be

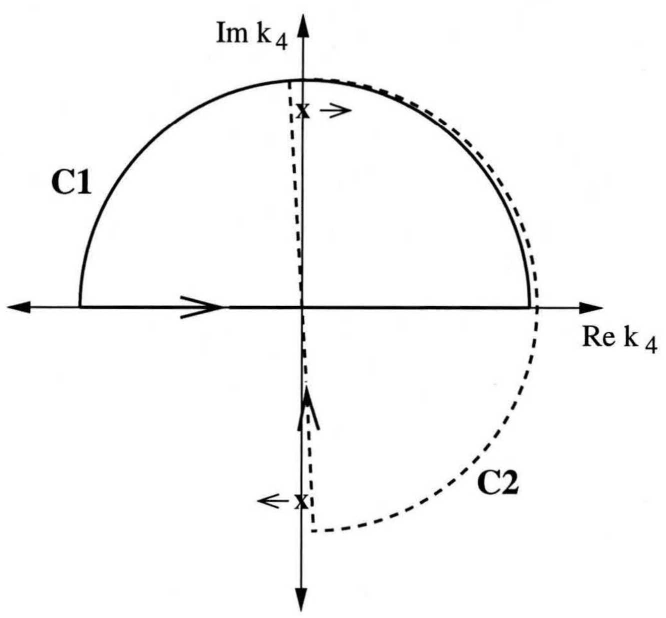
$$\begin{aligned} M^{-1}(\bar{x},\bar{x}') &= \int \frac{d^4\bar{k}}{(2\pi)^4} \frac{e^{i\bar{k}(\bar{x}'-\bar{x})}}{(\bar{k}^2+m^2)} \\ &\equiv \Delta_E(\bar{x}'-\bar{x}). \end{aligned} \tag{3.6}$$

It can be seen that there are no poles in (3.6), thus M^{-1} is well defined. We define $\Delta_E(\bar{x}'-\bar{x})$ as the Euclidean space propagator.

Since we have calculated the Gaussian integral to give (3.5) we can continue back from Euclidean space to Minkowski space.

We first concentrate on (3.6). Expanding in terms of the time and spatial components we obtain

$$M^{-1}(\bar{x},\bar{x}') = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}(\vec{x}'-\vec{x})} \int_{-\infty}^{\infty} \frac{d\bar{k}_4}{2\pi} \frac{e^{i\bar{k}_4(\bar{x}_4'-\bar{x}_4)}}{(\bar{k}_4^2+\vec{k}^2+m^2)}. \tag{3.7}$$



The last integral of (3.7) can be written with complex analysis (for $\bar{x}_4' - \bar{x}_4 > 0$) as

$$\begin{aligned} I(\bar{x}_4',\bar{x}_4) &= \int_{C1} \frac{d\bar{k}_4}{2\pi} \frac{e^{i\bar{k}_4(\bar{x}_4'-\bar{x}_4)}}{(\bar{k}_4^2+\vec{k}^2+m^2)} \\ &= -\lim_{\epsilon\to 0+} \int_{C2} \frac{d\bar{k}_4}{2\pi} \frac{e^{i\bar{k}_4(\bar{x}_4'-\bar{x}_4)}}{(\bar{k}_4^2+\vec{k}^2+m^2-i\epsilon)}, \end{aligned}$$

where the contours $C1$ and $C2$ are defined as in the figure.

The $-i\epsilon$ term appears in the second integral because the poles which occur at $\bar{k}_4 = i\sqrt{\vec{k}^2+m^2}$ and $\bar{k}_4 = -i\sqrt{\vec{k}^2+m^2}$ are shifted to the right and left respectively when the contour is changed from C1 to C2. We can go over to Minkowski coordinates to obtain

$$\begin{aligned} I(ix_0', ix_0) &= -\lim_{\epsilon\to 0+}\int_{C2}\frac{d\bar{k}_4}{2\pi}\frac{e^{-\bar{k}_4(x_0'-x_0)}}{(\bar{k}_4^2+\vec{k}^2+m^2-i\epsilon)} \\ &= -\lim_{\epsilon\to 0+}\int_{-i\infty}^{i\infty}\frac{d\bar{k}_4}{2\pi}\frac{e^{-\bar{k}_4(x_0'-x_0)}}{(\bar{k}_4^2+\vec{k}^2+m^2-i\epsilon)} \end{aligned}$$

The first integral is a damped integral if $Re(\bar{k}_4) > 0$, thus the arc part of contour C2 will have a zero contribution to the integral.[10]

The momenta can now be mapped to Minkowski space to give

$$\begin{aligned} I(ix_0', ix_0) &= -\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty}\frac{idk_0}{2\pi}\frac{e^{-ik_0(x_0'-x_0)}}{(-k_0^2+\vec{k}^2+m^2-i\epsilon)} \\ &= \lim_{\epsilon\to 0+} i\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\frac{e^{-ik_0(x_0'-x_0)}}{(k_0^2-\vec{k}^2-m^2+i\epsilon)} \end{aligned}$$

Inserting this last result into (3.7) we obtain the Minkowski propagator

$$\begin{aligned} \Delta(x'-x) &\equiv M^{-1}((ix_0', \vec{x}'); (ix_0, \vec{x})) \\ &= \lim_{\epsilon\to 0+} i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik^\mu(x_\mu'-x_\mu)}}{(k^2-m^2+i\epsilon)} \end{aligned} \tag{3.8}$$

We can write out the matrices in (3.5) and then continue back to Minkowski space

$$\begin{aligned} Z_0[\rho] &= e^{\frac{1}{2}\int d^4\bar{x}d^4\bar{x}'\rho(\bar{x})M^{-1}(\bar{x},\bar{x}')\rho(\bar{x}')} \\ &\to e^{-\frac{1}{2}\int d^4xd^4x'\rho(x)M^{-1}((ix_0,\vec{x});(ix_0',\vec{x}'))\rho(x')} \\ &= e^{-\frac{1}{2}\rho\Delta(x-x')\rho}, \end{aligned} \tag{3.9}$$

where $\Delta(x-x')$ is given by (3.8).

3.2 Self interacting ϕ^4 theory

We can introduce interactions by adding a term $\propto \phi^4$ to (3.1). Thus the interaction Lagrange density becomes

$$\mathcal{L} = \mathcal{L}_0 - \frac{\lambda}{4!}\phi^4, \quad \lambda > 0. \tag{3.10}$$

[10]Note that if $\bar{x}_4' - \bar{x}_4 < 0$, then the same result is obtained if the contour C1 is rotated in an anti-clockwise direction and the poles are shifted in the opposite direction as previously.

The generating functional is thus

$$\begin{aligned} Z[\rho] = \frac{W'[\rho]}{W'[0]} &= \frac{\int[d\phi]e^{i\int d^4x[\mathcal{L}_0-\frac{\lambda}{4!}\phi^4+\rho\phi]}}{\int[d\phi]e^{i\int d^4x[\mathcal{L}_0-\frac{\lambda}{4!}\phi^4]}} \\ &= \frac{e^{-i\frac{\lambda}{4!}\int d^4x\frac{1}{i^4}\frac{\delta^4}{\delta\rho^4}}\int[d\phi]e^{i\int d^4x[\mathcal{L}_0+\rho\phi]}/\int[d\phi]e^{i\int d^4x\mathcal{L}_0}}{e^{-i\frac{\lambda}{4!}\int d^4x\frac{1}{i^4}\frac{\delta^4}{\delta\rho^4}}\int[d\phi]e^{i\int d^4x[\mathcal{L}_0+\rho\phi]}\big|_{\rho=0}/\int[d\phi]e^{i\int d^4x\mathcal{L}_0}} \\ &= \frac{e^{-i\frac{\lambda}{4!}\int d^4x\frac{1}{i^4}\frac{\delta^4}{\delta\rho^4}}Z_0[\rho]}{e^{-i\frac{\lambda}{4!}\int d^4x\frac{1}{i^4}\frac{\delta^4}{\delta\rho^4}}Z_0[\rho]\big|_{\rho=0}} \qquad (3.11) \\ &\equiv \frac{W[\rho]}{W[0]} \end{aligned}$$

Thus to calculate the generating functional for an interacting theory, we just need to be able to differentiate Gaussian forms, $Z_0 = e^{-\frac{1}{2}\rho\Delta\rho}$. Unfortunately, the functional derivatives are contained in the exponent, so (3.11) cannot be calculated exactly. We can however make an expansion of the exponent in terms of λ, and thus calculate (3.11) by means of perturbation theory. After a perturbation expansion we have

$$W[\rho] = \Big(1 - i\frac{\lambda}{4!}\int d^4x\frac{\delta^4}{\delta\rho(x)^4} - \frac{\lambda^2}{2!4!4!}\int d^4x\frac{\delta^4}{\delta\rho(x)^4}\int d^4y\frac{\delta^4}{\delta\rho(y)^4} - \ldots\Big)Z_0[\rho].$$

To order λ^0, $Z[\rho] = Z_0[\rho]$ is just the free field generating functional. We can calculate $Z[\rho]$ to any order in λ. We shall calculate it to first order in λ as an example.

3.2.0.1 Example : $Z[\rho]$ to first order in λ

To first order in λ, we need to calculate

$$W[\rho] = \Big(1 - \frac{i\lambda}{4!}\int d^4x\frac{\delta^4}{\delta\rho(x)^4}\Big)Z_0[\rho]$$

We proceed[11] by explicitly calculating the functional derivatives,

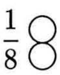
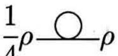
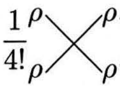
$$\begin{aligned} \frac{1}{i}\frac{\delta}{\delta\rho_x}e^{-\frac{1}{2}\rho_y\Delta_{yy'}\rho_{y'}} &= i\Delta_{xy}\rho_y e^{-\frac{1}{2}\rho_y\Delta_{yy'}\rho_{y'}} \\ \Big(\frac{1}{i}\frac{\delta}{\delta\rho_x}\Big)^2 e^{-\frac{1}{2}\rho_y\Delta_{yy'}\rho_{y'}} &= (\Delta_{xx} - (\Delta_{xy}\rho_y)^2)e^{-\frac{1}{2}\rho_y\Delta_{yy'}\rho_{y'}} \\ \Big(\frac{1}{i}\frac{\delta}{\delta\rho_x}\Big)^3 e^{-\frac{1}{2}\rho_y\Delta_{yy'}\rho_{y'}} &= (3i\Delta_{xx}\Delta_{xy}\rho_y - i(\Delta_{xy}\rho_y)^3)e^{-\frac{1}{2}\rho_y\Delta_{yy'}\rho_{y'}} \\ \Big(\frac{1}{i}\frac{\delta}{\delta\rho_x}\Big)^4 e^{-\frac{1}{2}\rho_y\Delta_{yy'}\rho_{y'}} &= (3\Delta_{xx}^2 - 6\Delta_{xx}(\Delta_{xy}\rho_y)^2 + (\Delta_{xy}\rho_y)^4)e^{-\frac{1}{2}\rho_y\Delta_{yy'}\rho_{y'}} \end{aligned}$$

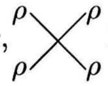
[11] We use a short notion by showing the parameters as indexes – $\Delta(x-y) \equiv \Delta_{xy}$

Thus

$$W[\rho] = \Big(1 - \int d^4x \Big[\frac{3i\lambda}{4!}\Delta_{xx}^2 - \frac{6i\lambda}{4!}\Delta_{xx}(\Delta_{xy}\rho_y)^2 + \frac{i\lambda}{4!}(\Delta_{xy}\rho_y)^4\Big]\Big) Z_0[\rho] \tag{3.12}$$

This can be represented diagrammatically as

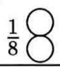
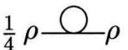
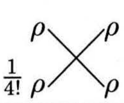
$W[\rho] = \Big(1 + \frac{1}{8}$  $- \frac{1}{4}$  $+ \frac{1}{4!}$  $\Big) Z_0[\rho]. \qquad (3.13)$

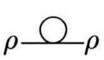
where — $\equiv \Delta(x-y)$ is the propagator,  is a vertex term where the following rules apply :

- the vertex is associated with a $-i\lambda$ term,
- an integral over the vertex is implied.

The vertex and the propagators are the basic building blocks of Feynman diagrams. The second diagram in (3.13) is obtained by taking the vertex and then joining two of the external legs to form the loop.

$Z[\rho] = \frac{W[\rho]}{W[0]}$

$= \Big(1 + \frac{1}{8}$  $- \frac{1}{4}$  $+ \frac{1}{4!}$  $\Big) Z_0[\rho] \Big/ \Big(1 + \frac{1}{8}$  $\Big)$

$= \Big(1 - \frac{1}{4}$  $+ \frac{1}{4!}$  $\Big) Z_0[\rho]$

$$= \Big(1 + \frac{i\lambda}{4}\int d^4x \Delta_{xx}(\Delta_{xy}\rho_y)^2 - \frac{i\lambda}{4!}\int d^4x (\Delta_{xy}\rho_y)^4\Big) Z_0[\rho] \tag{3.14}$$

where the denominator in the second equality was expanded using the binomial theorem, and only terms of order λ were kept.

It can be seen that the vacuum diagrams (diagrams with no external legs) have cancelled out in (3.14). This occurs to all orders in λ in perturbation theory for *normalized* generating functionals.

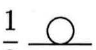
3.3 Green's functions

The n-point Green's function, in analogy to (2.11), is given

$$G^n(x_1,\ldots,x_n) = \frac{1}{i^n}\frac{\delta^n Z[\rho]}{\delta\rho(x_1)\cdots\delta\rho(x_n)}\bigg|_{\rho=0}, \tag{3.15}$$

where $Z[\rho]$ is given by (3.11).

We can calculate the two point Green's function to first order in λ by using (3.14) in (3.15). The result is

$$G^2(x_1,x_2) = \text{---} + \frac{1}{2}$$  .

We can now introduce Feynman rules. The Feynman rules are a method whereby the n-point Green's function can be obtained directly to a certain order in λ, without having to go through the differentiation process as in Example 3.2.0.1.

3.3.1 Feynman rules for ϕ^4 theory

The Feynman rules for the ϕ^4 theory can be formulated as follows

- $\overline{x_1 \quad x_2} \longrightarrow \Delta(x_1 - x_2)$
-  $\longrightarrow -i\lambda$
- Integrate over every vertex by $\int d^4x$.
- To order n, the topological weight of each diagram is obtained by counting the number of topological ways of drawing non-equivalent diagrams and then normalizing this number by dividing by $(4!)^n n!$

Other Feynman diagrams can be obtained by joining the legs of the vertex to form diagrams with loops, etc.

3.3.2 Connected and disconnected Green's functions

The Green's function as defined in (3.15) gives both connected and disconnected Green's functions. A disconnected Green's function is one that is made up of connected Green's functions of lower order by multiplying them together, for example

$$G^n(x_1,\ldots,x_n) = G^{n-1}(x_1,\ldots,x_{n-1}) \times G^1(x_n).$$

We can obtain a generating functional, $X[\rho]$, for the connected Green's functions by equating

$$\begin{aligned} Z[\rho] &= e^{iX[\rho]}, & Z[0] &= 1 \\ \Rightarrow \quad X[\rho] &= -i \ln Z[\rho]. & X[0] &= 0 \end{aligned} \tag{3.16}$$

The connected Green's functions is given by an equation similar to (3.15),

$$G_c^n(x_1, \ldots, x_n) = \frac{1}{i^{n-1}} \frac{\delta^n X[\rho]}{\delta\rho(x_1) \cdots \delta\rho(x_n)}\bigg|_{\rho=0}. \tag{3.17}$$

To calculate $X[\rho]$, we still need to calculate $Z[\rho]$ with perturbation theory and then we expand $-\ln Z[\rho]$ to the desired order in λ.

3.3.2.1 Example : Connected Green's functions

For the free field we have

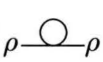
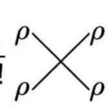
$$\begin{aligned} Z_0[\rho] &= e^{-\frac{1}{2}\rho\Delta\rho} \\ \Rightarrow \quad X[\rho] &= \frac{i}{2}\rho\Delta\rho \end{aligned} \tag{3.18}$$

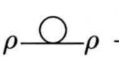
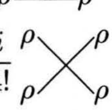
The free field connected Green's functions are

$$\begin{aligned} G_c^1 &= \frac{\delta X[\rho]}{\delta\rho(x_1)}\bigg|_{\rho=0} = 0 \\ G_c^2 &= \frac{1}{i}\frac{\delta^2 X[\rho]}{\delta\rho(x_1)\delta\rho(x_2)}\bigg|_{\rho=0} = \Delta(x_1 - x_2) \\ G_c^n(x_1, \ldots, x_n) &= 0, \qquad \forall\, n > 2. \end{aligned}$$

Thus it can be seen that all the free field Green's functions of order $n > 2$ must be disconnected.

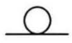
For the ϕ^4 theory $Z[\rho]$ to order λ is given by (3.14). Thus

$X[\rho] \;=\; -i \ln Z_0[\rho] - i \ln\Big(1 - \frac{1}{4}$  $+ \frac{1}{4!}$  $\Big)$

$\approx\; \frac{i}{2}\rho - \rho + \frac{i}{4}$  $- \frac{i}{4!}$  .

The connected Green's functions for the ϕ^4 theory are

$$
\begin{aligned}
G_c^1(x_1) &= \frac{\delta X[\rho]}{\delta\rho(x_1)}\bigg|_{\rho=0} = 0. \\
G_c^2(x_1,x_2) &= \frac{1}{i}\frac{\delta^2 X[\rho]}{\delta\rho(x_1)\delta\rho(x_2)}\bigg|_{\rho=0}
\end{aligned}
$$

 $+\frac{1}{2}$  . (3.19)

$$
\begin{aligned}
G_c^3(x_1,x_2,x_3) &= 0. \\
G_c^4(x_1,x_2,x_3,x_4) &= \frac{1}{i}\frac{\delta^4 X[\rho]}{\delta\rho(x_1)\delta\rho(x_2)\delta(x_3)\delta(x_4)}\bigg|_{\rho=0}
\end{aligned}
$$

$=$  . (3.20)

$$
G_c^n(x_1,\ldots,x_n) = 0, \qquad \forall\, n > 4. \tag{3.21}
$$

To order λ, all the Green's functions of order $n > 4$ are disconnected. The n-point Green's functions calculated to higher orders in λ, are in general connected.

3.4 One particle irreducible Green's functions.

We now introduce the quantity $\varphi_c(x)$, referred to as the classical field, and is defined as

$$
\begin{aligned}
\varphi_c &\equiv \frac{\delta X[\rho]}{\delta\rho(x)} = \frac{-i}{Z[\rho]}\frac{\delta Z[\rho]}{\delta\rho(x)} \\
&= \frac{\langle 0|\varphi|0\rangle^\rho}{\langle 0|0\rangle^\rho} = \langle\varphi\rangle^\rho.
\end{aligned}
$$

Thus φ is just the vacuum expectation value of the field operator in the presence of a source ρ. If $\rho = 0$, then

$$
\varphi_c = \langle\varphi\rangle \equiv \varphi_0,
$$

where φ_0 is the classical field.

We define an effective action corresponding to the classical field by means of a Legendre transformation,

$$
\begin{aligned}
\Gamma[\varphi_c] &= X[\varphi] - \int d^4x\rho(x)\varphi_c(x) \\
\text{where}\quad \frac{\delta\Gamma}{\delta\rho} &= 0 \quad\text{and}\quad \frac{\delta\Gamma}{\delta\varphi_c} = -\rho.
\end{aligned}
$$

In the case of a free field φ_c can be obtained from (3.18) to give $\varphi_c = i\Delta\rho$. If we insert this into the Klein-Gordon equation, we get

$$(\partial^\mu \partial_\mu + m^2)\varphi_c = K i \Delta \rho = \rho,$$

where we set $K \equiv \partial^\mu \ \partial_\mu + m^2$ and use matrix notation. This result is obtained because φ_c satisfies the Klein-Gordon equation in the presence of a source term. We write the effective action as

$$\begin{aligned}
\Gamma[\varphi_c] &= X[\rho] - \rho\varphi_c \\
&= \frac{i}{2}\rho\Delta\rho - \rho\varphi_c \\
&= \frac{i}{2}(K\varphi_c)\Delta(K\varphi_c) - (K\varphi_c)\varphi_c \\
&= \frac{1}{2}(K\varphi_c)\varphi_c - (K\varphi_c)\varphi_c \\
&= -\frac{1}{2}\varphi_c K \varphi_c \\
&= \frac{1}{2}\int d^4x(\partial^\mu\varphi_c\partial_\mu\varphi_c - m^2\varphi_c^2),
\end{aligned}$$

which is the action of the classical free field theory, justifying the reason why Γ is referred to as the effective action.

In the case of an interacting field we are unable to calculate φ_c and Γ exactly and must use perturbation theory to do so - there will be quantum corrections to the free field effective action.

In general we can make a functional expansion

$$\Gamma[\varphi_c] = \sum_{n=1}^{\infty} \frac{1}{n!}\Gamma^n(1,\ldots,n)\varphi_c(1)\cdots\varphi_c(n).$$

The coefficients in the expansion

$$\Gamma^n(1,\ldots,n) = i^n \frac{\delta^n \Gamma[\varphi_c]}{\delta\varphi_c(1)\cdots\delta\varphi_c(n)}\Big|_{\varphi_c=0} \tag{3.22}$$

are known as the one particle irreducible (OPI) Green's functions.(In terms of Feynman rules, these are diagrams such that cutting any line does not lead to a disconnected diagram.)

For a free field theory

$$\Gamma^2 = i^2 \frac{\delta^2(-\frac{1}{2}\varphi_c K\varphi_c)}{\delta\varphi_c(1)\delta\varphi_c(2)} = K = -i\Delta^{-1}.$$

We can relate the OPI Green's functions to the connected Green's functions by noting that

$$\frac{\delta\varphi_c(x_1)}{\delta\rho(x_2)} = \frac{\delta^2 X}{\delta\rho(x_2)\delta\rho(x_1)} \quad \text{and} \quad \frac{\delta\rho(x_1)}{\delta\varphi_c(x_2)} = -\frac{\delta^2\Gamma}{\delta\varphi_c(x_1)\delta\varphi_c(x_2)},$$

but

$$\frac{\delta\varphi_c(x_1)}{\delta\rho(x_2)} = \Big(\frac{\delta\rho(x_1)}{\delta\varphi_c(x_2)}\Big)^{-1}$$

thus

$$\frac{\delta^2 X}{\delta\rho(x_2)\delta\rho(x_1)} = \Big(-\frac{\delta^2\Gamma}{\delta\varphi_c(x_1)\delta\varphi_c(x_2)}\Big)^{-1}.$$

The second order connected Green's function can be related to the second order OPI Green's function by

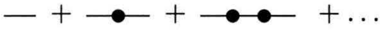
$$G_c^2(1,2) = \frac{1}{i}\frac{\delta^2 X}{\delta\rho(x_2)\delta\rho(x_1)}\Big|_{\rho=0} = i\Big(\frac{\delta^2\Gamma}{\delta\varphi_c(x_1)\delta\varphi_c(x_2)}\Big|_{\varphi_c}\Big)^{-1} = -i(\Gamma^2(1,2))^{-1} \tag{3.23}$$

For the free field $\Gamma^2(1,2) = -i(G_c^2(1,2))^{-1} = -i\Delta^{-1}$ which is the result that was obtained above.

If we consider interactions, the 2 point OPI Green's function is modified by $\Gamma^2 = -i\Delta^{-1} + \Sigma(\lambda)$. $\Sigma(\lambda)$ is the proper self energy which contains all the quantum corrections to the free field theory.

The second order connected Green's function can be written in terms of Σ,

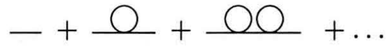
$$\begin{aligned} G_c^2(1,2) &= -i(\Gamma^2(1,2))^{-1} = -i(-i\Delta^{-1} + \Sigma(\lambda))^{-1} \\ &\approx \Delta + \Delta(-i\Sigma)\Delta + \Delta(-i\Sigma)\Delta(-i\Sigma)\Delta + \dots \end{aligned}$$

$=$ 

$=$, (3.24)

where $-i\Sigma = \bullet$ is the sum of all the OPI self energy loops. The last thick propagator denotes that we have summed up all the self energy loops to obtain a dressed propagator, where the mass parameter has been modified.

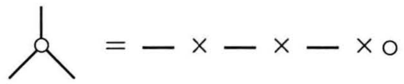
For $\Sigma(\lambda)$ to first order in λ, $\Sigma = \bigcirc$ and

$G_c^2 =$ 

The higher order connected Green's functions can also be related to the OPI Green's functions, for instance

$$G_c^3(1,2,3) = G_c^2(1,i)G_c^2(2,j)G_c^2(3,k)\Gamma^3(i,j,k),$$

where integration over seperate indices is implied. This can be represented diagramatically as



where $\circ \equiv \Gamma^3$. All the self interactions on the legs are summed up and dressed legs are obtained. The irreducible graphs are also summed up and this leads to the vertex corrections, which is the dressed 3 point OPI Green's function.

In general, any n-point Green's function will be proportional to the n-point OPI Green's function and Green's functions of lower order.

3.5 Feynman Rules in Momentum space

In section (3.3.1) we obtained the Feynman rules in coordinate space. If we Fourier transform these rules we will obtain the momentum space Feynman rules.

The propagator in momentum space is

$$\begin{aligned}\Delta(p) &= \int d^4x \Delta(x) e^{ipx} \\ &= i \int d^4x \frac{d^4k}{(2\pi)^4} \frac{e^{i(k+p)x}}{k^2 - m^2 + i\epsilon} \\ &= \frac{i}{p^2 - m^2 + i\epsilon}.\end{aligned} \tag{3.25}$$

The vertex in momentum space is obtained by taking the Fourier transform of $I(x) = -i\lambda \int d^4y \Delta(x_1 - y)\Delta(x_2 - y)\Delta(x_3 - y)\Delta(x_4 - y)$. The Fourier transform is

$$\begin{aligned}I(p) &= -i\lambda \int d^4y d^4x_1 d^4x_2 d^4x_3 d^4x_4 \Delta(x_1 - y)\Delta(x_2 - y)\Delta(x_3 - y)\Delta(x_4 - y) \\ &\quad \times e^{ip_1x_1} e^{ip_2x_2} e^{ip_3x_3} e^{ip_4x_4} \\ &= -i\lambda \int d^4y d^4x_1' d^4x_2' d^4x_3' d^4x_4' \; e^{ip_1x_1'}\Delta(x_1') \; e^{ip_2x_2'}\Delta(x_2') \; e^{ip_3x_3'}\Delta(x_3') \\ &\quad \times e^{ip_4x_4'}\Delta(x_4') \; e^{i(p_1+p_2+p_3+p_4)y} \\ &= -i\lambda \int d^4y e^{i(p_1+p_2+p_3+p_4)y} \Delta(p_1)\Delta(p_2)\Delta(p_3)\Delta(p_4) \\ &= -i\lambda(2\pi)^4 \delta(p_1 + p_2 + p_3 + p_4)\Delta(p_1)\Delta(p_2)\Delta(p_3)\Delta(p_4).\end{aligned}$$

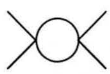
Thus the vertex is associated with the term $-i\lambda(2\pi)^4\delta(p_1 + p_2 + p_3 + p_4)$. The Dirac-delta implies a conservation of momentum for the propagators entering and exiting the vertex.

The n-point Green's function in momentum space is given by

$$G^n(p_1,\ldots,p_n) \equiv \int dx_1 \cdots dx_n G^n(x_1,\ldots,x_n) e^{ip_1x_1} \cdots e^{ip_nx_n}.$$

The two point momentum Green's function is thus

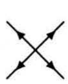
$$\begin{aligned} G^2(p_1,p_2) &= \int dx_1 dx_2 e^{ip_1x_1} e^{ip_2x_2} \Delta(x_1 - x_2) \\ &= i \int dx_2 \frac{e^{i(p_2x_2 - p_1x_2)}}{p_1^2 - m^2 + i\epsilon} \\ &= (2\pi)^4 \delta(p_1 - p_2)\Delta(p). \end{aligned}$$

If a loop occurs, i.e.  then only the external propagators will be Fourier transformed in the n-point Green's function. The internal propagators will still be in coordinate space, but they can be written in terms of momentum using (3.8). If we use the Dirac deltas that occur at the vertices, then the Green's function obtained (for the 1 loop, 2 vertex diagram) is

$$\begin{aligned} G^4(x_1,x_2,x_3,x_4) &= \tfrac{(-i\lambda)^2}{2!}(2\pi)^4\delta(p_1+p_2+p_3+p_4)\Delta(p_1)\Delta(p_2)\Delta(p_3)\Delta(p_4) \\ &\quad \times \int \tfrac{d^4q_1}{(2\pi)^4}\Delta(q_1)\Delta(q_1+p_2+p_3) \end{aligned}$$

Thus, if loops occur we need to make sure that momentum conservation holds at each vertex when choosing the internal momenta. We then have to integrate with $\int\frac{d^4q_i}{(2\pi)^4}$ for each free momentum in the loop.

We can now summarize the Feynman rules in momentum space :

- p_1 $\longrightarrow \Delta(p) \equiv \frac{i}{p^2-m^2+i\epsilon}$.
-  $\longrightarrow -i\lambda$.
- Choose the internal momenta so that momentum conservation holds at each vertex. Integrate over all the remaining internal momenta with

$$\int \frac{d^4p}{(2\pi)^4}.$$

- A factor of $(2\pi)^4\delta(p_1,\ldots,p_n)$ is added for the momentum conservation of the external legs.

- To order n, the topological weight of each diagram is obtained by counting the number of topological ways of drawing non-equivalent diagrams and then normalizing this number by $(4!)^n n!$

CHAPTER 4
Quantum Field Theory of Fermion Fields

4.1 Fermion propagator

The action for the Dirac equation derived from the Lagrangian in section (1.3) is

$$S = \int d^4x[\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi],$$

where the elements of the four vector ψ are Grassmann variables.[12]

The unnormalized generating functional for fermions is

$$\begin{aligned} W[\bar{\sigma}, \sigma] &= \int \prod_{\alpha=1}^{4} [d\bar{\psi}_\alpha][d\psi_\alpha] e^{i\int d^4x[\bar{\psi}_\alpha (i\gamma^\mu \partial_\mu - m)^\alpha_\beta \psi^\beta + \bar{\sigma}_\alpha \psi^\alpha + \bar{\psi}_\alpha \sigma^\alpha]} \\ &= \int [d\bar{\psi}][d\psi] e^{i\int d^4x[\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \bar{\sigma}\psi + \bar{\psi}\sigma]}, \end{aligned} \tag{4.1}$$

where σ is the source term for $\bar{\psi}$ and $\bar{\sigma}$ is the source term for ψ. The indices α and β are the indices for the spinor components, which will be implied from now on.

We can carry through exactly the same analytic continuation as in section (3.1) to obtain the fermion free field generating functional

$$Z_0[\bar{\sigma}, \sigma] = e^{-\bar{\sigma} S \sigma}, \tag{4.2}$$

where S is the fermion propagator given by

$$\begin{aligned} S(x - x') &= i\int \frac{d^4k}{(2\pi)^4} \frac{1}{\gamma^\mu k_\mu - m + i\epsilon} e^{ik(x-x')} \\ &= i\int \frac{d^4k}{(2\pi)^4} \frac{1}{\not{k} - m + i\epsilon} e^{ik(x-x')} \end{aligned} \tag{4.3}$$

The propagator in momentum space is obtained from a Fourier transformation and is

$$S(p) \equiv \frac{i}{\not{p} - m + i\epsilon}. \tag{4.4}$$

The $(n + m)$ point Green's function are defined as

$$G^{n+m} = \left(\frac{1}{i}\right)^{n+m} \frac{\overrightarrow{\delta^n}}{\delta\bar{\sigma}_1 \ldots \delta\bar{\sigma}_n} Z[\bar{\sigma}, \sigma] \frac{\overleftarrow{\delta^m}}{\delta\sigma_1 \ldots \delta\sigma_m}\Big|_{\bar{\sigma},\sigma=0}. \tag{4.5}$$

[12] See appendix (B.2) for an introduction to Grassmann algebra.

We can carry through all our previous results, but we must make sure of the direction of the derivatives as we are now using Grassmann variables.

4.2 Yukawa theory

We can introduce a scalar-fermion interaction of the form $g\varphi\bar{\psi}\psi$. This is known as the Yukawa interaction.

The Yukawa Lagrange density becomes

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}(\partial^\mu\varphi\partial_\mu\varphi - m^2\varphi^2) + g\varphi\bar{\psi}\psi + \bar{\sigma}\psi + \bar{\psi}\sigma + \rho\varphi, \tag{4.6}$$

where we have introduced the sources $\bar{\sigma}, \sigma, \rho$.

The generating functional is then given by

$$\begin{aligned} W[\rho,\bar{\sigma},\sigma] &= \int [d\varphi][d\bar{\psi}][d\psi] e^{i\int d^4x\mathcal{L}} \\ &= e^{ig\int d^4x \frac{1}{i}\frac{\delta}{\delta\rho}\frac{1}{i}\frac{\overleftarrow{\delta}}{\delta\sigma}\frac{1}{i}\frac{\overrightarrow{\delta}}{\delta\bar{\sigma}}} Z_0 \end{aligned} \tag{4.7}$$

where $Z_0 = e^{-\frac{1}{2}\rho\Delta\rho - \bar{\sigma}S\sigma}$ is the free field generating functional.

The propagators for this theory can easily be obtained by calculating the two point Green's function to give the scalar field propagator,(3.25) and the fermion field propagator, (4.4). We represent them graphically as :

$-\underset{p}{\rightarrow}- \longrightarrow \Delta(p) \equiv \frac{i}{p^2-m^2+i\epsilon}$, the scalar field propagator;

$\underset{p}{\longrightarrow} \longrightarrow S(p) \equiv \frac{i}{\not{p}-m+i\epsilon}$, the fermion field propagator.

The term associated with the vertex is obtained from the three point Green's function

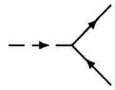
$$G^3 = \frac{1}{i^3}\frac{\delta}{\delta\rho}\frac{\overrightarrow{\delta}}{\delta\bar{\sigma}} Z[\bar{\sigma},\sigma]\frac{\overleftarrow{\delta}}{\delta\sigma}\Big|_{\bar{\sigma},\sigma,\rho=0}.$$

Expanding (4.7) in terms of g,

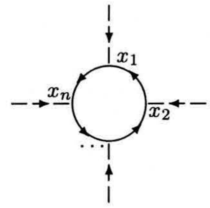
$$\begin{aligned} Z &= (1 + ig\frac{1}{i}\frac{\delta}{\delta\rho}\frac{1}{i}\frac{\overleftarrow{\delta}}{\delta\sigma}\frac{1}{i}\frac{\overrightarrow{\delta}}{\delta\bar{\sigma}} + \left(ig\frac{1}{i}\frac{\delta}{\delta\rho}\frac{1}{i}\frac{\overleftarrow{\delta}}{\delta\sigma}\frac{1}{i}\frac{\overrightarrow{\delta}}{\delta\bar{\sigma}}\right)^2 + \ldots)Z_0 \\ &= [1 - g(-\Delta\rho)(\bar{\sigma}S)(-S\sigma) + O(g^2)]Z_0, \end{aligned}$$

we see that

$$G^3 = \frac{1}{i^3}(-g\Delta(p_1)S(p_2)S(p_3)) = -ig\Delta(p_1)S(p_2)S(p_3).$$

This can be represented graphically by  where the rule applies that a factor of $-ig$ is associated with the vertex.

There is an extra complication if fermion loops occur due to the anti-commutative nature of Grassmann variables. If we consider the following n vertex loop



then we can read off the following term

$$(-ig)^n S(x_1, x_2) S(x_2, x_3) \cdots S(x_{n-1}, x_n) S(x_n, x_1) \Delta(x_1, y_1) \cdots \Delta(x_n, y_n), \tag{4.8}$$

where x_1 labels vertex points and y_1 labels external legs. This term should give us the lowest order contribution to the connected n point Green's function.

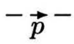
If we manipulate (4.8), we obtain

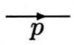
$$\begin{aligned}
& (-ig)^n S(x_1, x_2) S(x_2, x_3) \cdots S(x_{n-1}, x_n) S(x_n, x_1) \Delta(x_1, y_1) \cdots \Delta(x_n, y_n) \\
= \; & (-ig)^n \Big(\frac{1}{i^2} \frac{\overrightarrow{\delta}}{\delta \bar{\sigma}(x_1)} \frac{\overleftarrow{\delta}}{\delta \sigma(x_2)} \cdots \frac{1}{i^2} \frac{\overrightarrow{\delta}}{\delta \bar{\sigma}(x_n)} \frac{\overleftarrow{\delta}}{\delta \sigma(x_1)} \Big) \\
& \times \Big(\frac{1}{i^2} \frac{\delta}{\delta \rho(x_1)} \frac{\delta}{\delta \rho(y_1)} \cdots \frac{1}{i^2} \frac{\delta}{\delta \rho(x_n)} \frac{\delta}{\delta \rho(y_n)} \Big) Z_0 \Big|_{\sigma=\bar{\sigma}=\rho=0} \\
= \; & \frac{1}{i^n} \frac{\delta}{\delta \rho(y_1)} \cdots \frac{\delta}{\delta \rho(y_n)} \Big[(-1)^{n-1} (-ig)^n \\
& \times \frac{1}{i} \frac{\overleftarrow{\delta}}{\delta \sigma(x_1)} \frac{1}{i} \frac{\overrightarrow{\delta}}{\delta \bar{\sigma}(x_1)} \frac{1}{i} \frac{\delta}{\delta \rho(x_1)} \cdots \frac{1}{i} \frac{\overleftarrow{\delta}}{\delta \sigma(x_n)} \frac{1}{i} \frac{\overrightarrow{\delta}}{\delta \bar{\sigma}(x_n)} \frac{1}{i} \frac{\delta}{\delta \rho(x_n)} \Big] Z_0 \Big|_{\sigma=\bar{\sigma}=\rho=0} \\
= \; & -\frac{1}{i^n} \frac{\delta}{\delta \rho(y_1)} \cdots \frac{\delta}{\delta \rho(y_n)} \underbrace{\Big[(ig)^n \prod_{j=1}^{n} \frac{1}{i} \frac{\overleftarrow{\delta}}{\delta \bar{\sigma}(x_j)} \frac{1}{i} \frac{\overrightarrow{\delta}}{\delta \bar{\sigma}(x_j)} \frac{1}{i} \frac{\delta}{\delta \rho(x_j)} \Big] Z_0}_{\text{nth order expansion term of } Z} \Big|_{\sigma=\bar{\sigma}=\rho=0} \\
= \; & -G_c^n(y_1, \ldots, y_n) + \ldots,
\end{aligned}$$

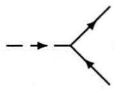
where we have ignored the higher order contributions. Instead of resulting in the n point Green's function, the loop term has resulted in the negative n point Green's function. We must thus associate a -1 with every fermion loop to correct this sign difference.

We can now summarize the Feynman rules for the Yukawa theory.

4.2.1 Feynman rules for Yukawa theory

-  $\longrightarrow \Delta(p) \equiv \frac{i}{p^2-m^2+i\epsilon}$. Scalar field propagator.

-  $\longrightarrow S(p) \equiv \frac{i}{\not{p}-m+i\epsilon}$. Fermion field propagator.

-  $\longrightarrow -ig$.

- Multiply by (-1) for each closed fermion loop.

- Choose the internal momenta so that momentum conservation holds at each vertex. Integrate over all free internal momenta with

$$\int \frac{d^4p}{(2\pi)^4}.$$

- A factor of $(2\pi)^4\delta(p_1+\ldots+p_n)$ is added for the momentum conservation of the external legs.

- To order n, the topological weight of each diagram is obtained by counting the number of topological ways of drawing non-equivalent diagrams and then normalizing this number by $n!$

CHAPTER 5
Renormalization of a $\lambda\varphi^4$ theory

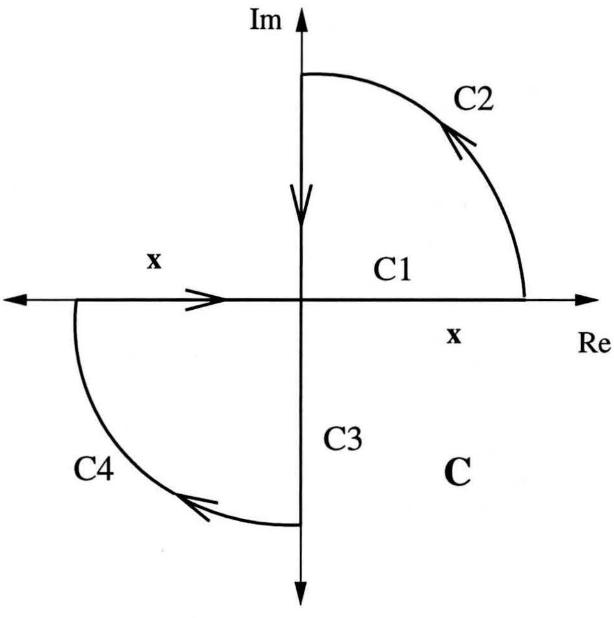
The momentum space two point Green's function for the ϕ^4 theory is given by (3.19),

$$\begin{aligned} G_c^2(p) &= \underset{p}{\longrightarrow} + \underset{p\quad p}{\overset{q}{\longrightarrow\!\!\bigcirc\!\!\longrightarrow}} \\ &= \Delta(p) + [\Delta(p)]^2 \underbrace{\left[\frac{-i\lambda}{2}\int \frac{d^4q}{(2\pi)^4}\Delta(q)\right]}_{\equiv -i\delta m^2} + \ldots \end{aligned} \tag{5.1}$$

Consider the term

$$\begin{aligned} -i\delta m^2 &= \frac{-i\lambda}{2}\int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2 - m^2 + i\epsilon} \\ &= \frac{\lambda}{2}\int \frac{d^3q}{(2\pi)^3}\int \frac{dq_0}{(2\pi)}\frac{1}{q_0^2 - (\vec{q}^2 + m^2) + i\epsilon} \end{aligned}$$

To integrate q_0, consider the contour C.



According to Jordan's Lemma the integral

$$\int_C \frac{dq_0}{(2\pi)}\frac{1}{q_0^2 - (\vec{q}^2 + m^2) + i\epsilon}$$

over this contour is zero. This integral can split into integrals over $C1, C2, C3$ and $C4$. If we set $q_0 = Re^{i\theta}$, where $\theta \in [0, \frac{\pi}{2}]$ for $C2$ and $\theta \in [\pi, \frac{3\pi}{2}]$ for $C4$, then it can be seen that the integrals over contours $C2$ and $C4$ become zero as $R \to \infty$. Thus

$$\begin{aligned}
& \int_{-\infty}^{\infty} \frac{dq_0}{(2\pi)} \frac{1}{q_0^2 - (\vec{q}^2 + m^2) + i\epsilon} \\
= & -\int_{-i\infty}^{i\infty} \frac{dq_0}{(2\pi)} \frac{1}{q_0^2 - (\vec{q}^2 + m^2) + i\epsilon} \\
= & -i \int_{-\infty}^{\infty} \frac{d\bar{q}_0}{(2\pi)} \frac{1}{\bar{q}^2 + m^2},
\end{aligned}$$

where we have transformed from Minkowski to Euclidean space in the last equality. The limit $\epsilon \to 0$ can be taken as there are no poles on the imaginary axis after this transformation.

We thus obtain

$$\delta m^2 = \frac{\lambda}{2} \int_{-\infty}^{\infty} \frac{d^4\bar{q}}{(2\pi)^4} \frac{1}{\bar{q}^2 + m^2} > 0. \tag{5.2}$$

This is just the integral of a function $f(\bar{q}^2)$, so we can use (C.3) to integrate out the angular degrees of freedom,

$$\begin{aligned}
\delta m^2 &= \frac{\lambda}{2} \frac{2\pi^2}{\Gamma(2)} \int_0^{\infty} \frac{d\bar{q}}{(2\pi)^4} \frac{\bar{q}^3}{\bar{q}^2 + m^2} \\
&= \frac{\lambda}{2^4\pi^2} \int_0^{\infty} d\bar{q} \frac{\bar{q}^3}{\bar{q}^2(1 + \frac{m^2}{\bar{q}^2})} \to \infty \quad \text{as } \bar{q}^2.
\end{aligned}$$

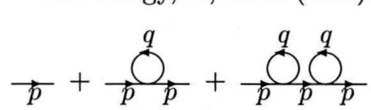
There is thus a divergence in the loop term of (5.1) at high momenta (small length scales). This divergence, known as an ultra violet divergence, occurs because it is implicitly assumed that the theory works for all length scales, including those of the order of a Planck length and less.

A technique for dealing with these divergences in a systematic manner is known as renormalization. Using this technique, the divergences are isolated and interpreted as renormalizations of the parameters of the theory (the mass m and the coupling constant λ). Since this technique cannot be applied to all field theories, those theories where the technique can be applied are known as *renormalizable* field theories.

5.1 Mass renormalization

In section (3.4), we saw that we could write the two point connected Green's function in terms of the proper self energy, Σ, as in (3.24). If we set $\Sigma = \delta m^2$, we can write

$$G_c^2 =$$



$$+ \dots$$

$$= \Delta(p) + \Delta(p)(-i\delta m^2)\Delta(p) + \Delta(p)(-i\delta m^2)\Delta(p)(-i\delta m^2)\Delta(p) + \dots$$

$$
\begin{aligned}
&= \frac{\Delta(p)}{1 + i\Delta(p)\delta m^2} \\
&= \frac{i}{p^2 - (m^2 + \delta m^2) + i\epsilon} \\
&\equiv \frac{i}{p^2 - (m_R^2) + i\epsilon},
\end{aligned}
$$

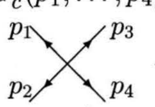
where we have renormalized by absorbing the divergent term δm^2 into the bare mass parameter m^2 to form a renormalized mass parameter m_R^2. We would like m_R^2 to be finite, because the experimentally measured value is. Since δm^2 is divergent, m^2 must cancel out this divergence in some way so that we obtain a finite value.

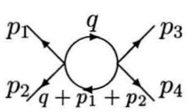
Just as there is renormalization of the mass in the propagator, there is also renormalization of other parameters.

5.2 Renormalization of the coupling constant.

If we consider the four point connected Green's function

$$G_c^4(p_1, \ldots, p_4)$$

$=$ 

$+$ 

$+ \ldots$

$$
\begin{aligned}
&= \Big[-i\lambda \prod_{i=1}^{4} \Delta(p_i) + \\
&\quad \prod_{i=1}^{4} \Delta(p_i) \underbrace{\Big[\frac{(-i\lambda)^2}{2} \int \frac{d^4q}{(2\pi)^4} \Delta(p_1 + p_2 + q)\Delta(q)\Big]}_{\equiv -i\delta\lambda} + \ldots \Big] 2\pi\delta(p_1 + \cdots + p_4) \\
&= -i \prod_{i=1}^{4} \Delta(p_i)(\lambda + \delta\lambda) 2\pi\delta(p_1 + \cdots + p_4) \\
&= -i \prod_{i=1}^{4} \Delta(p_i)\lambda_R 2\pi\delta(p_1 + \cdots + p_4).
\end{aligned}
$$

The correction to the vertex term from the $O(\lambda^2)$ term is given by

$$\delta\lambda = \frac{i\lambda^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{[q^2 - m^2 + i\epsilon][(q + p)^2 - m^2 + i\epsilon]},$$

where $p = p_1 + p_2$. Using the Feynman parameterization[13], the integral becomes

$$\delta\lambda = \frac{i\lambda^2}{2} \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{1}{[(q^2 - m^2 + i\epsilon)x + ((q + p)^2 - m^2 + i\epsilon)(1 - x)]^2}$$

[13]The Feynman parameterization is $\frac{1}{ab} \equiv \int_0^1 dx \frac{1}{(ax + b(1 - x))^2}$.

$$= \frac{i\lambda^2}{2}\int_0^1 dx \int \frac{d^4q'}{(2\pi)^4}\frac{1}{\left[q'^2+p^2x(1-x)-m^2+i\epsilon\right]^2},$$

where we have set $q = q' - p(1-x)$. We use Jordan's lemma as before to obtain

$$\delta\lambda = -\frac{\lambda^2}{2}\int_0^1 dx \int \frac{d^4\bar{q}'}{(2\pi)^4}\frac{1}{\left[\bar{q}'^2-p^2x(1-x)+m^2\right]^2}. \tag{5.3}$$

This integral is once again a function of q'^2 only, so we can integrate out the angular degrees of freedom to obtain

$$\delta\lambda = \frac{-\lambda^2}{2^4\pi^2}\int_0^1 dx \int_0^\infty d\bar{q}' \frac{\bar{q}'^3}{[\bar{q}'^2-p^2x(1-x)+m^2]^2} \to \infty \text{ as } \ln\bar{q}'.$$

We would like the renormalized parameter $\lambda_R = \lambda + \delta\lambda$ to be finite, because the experimentally measured coupling constant is finite. Since $\delta\lambda$ diverges, this means that λ must also diverge in some way to cancel out the divergence.

5.3 Renormalization schemes.

We saw in the previous two sections that renormalization is a reparameterization of the theory in terms of physically measurable parameters. By using this technique, the divergences of the correction terms cancel with those of the bare parameters.

The theory parametrized by the bare parameters is given by

$$\mathcal{L}_B = \frac{1}{2}((\partial_\mu\phi_B)^2 - m_B^2\phi_B^2) - \frac{\lambda_B}{4!}\phi_B^4.$$

If we assume a multiplicative renormalization, then the bare parameters are proportional to the renormalized parameters

$$\begin{aligned} \phi_B &= Z_3^{1/2}\phi_R \\ m_B^2 &= \frac{Z_0}{Z_3}m_R^2 \\ \lambda_B &= \frac{Z_1}{Z_3^2}\lambda_R \end{aligned}$$

This parameterization is chosen so that form of the Lagrangian that is obtained when substituting these variables is a simple one, thus[14]

$$\begin{aligned}
\mathcal{L}_B &= \underbrace{\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4}_{\text{renormalized Lagrange density}} \\
&\quad \underbrace{+\frac{1}{2}(Z_3-1)(\partial_\mu\phi)^2 - \frac{1}{2}m_2(Z_0-1)\phi^2 - \frac{\lambda}{4!}(Z_1-1)\phi^4}_{\text{counter terms}} \\
&= \mathcal{L} + \mathcal{L}_C.
\end{aligned}$$

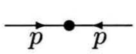
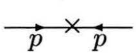
The counter terms are necessary for the divergences to cancel. We must fix Z_0, Z_1, Z_3 by means of a renormalization scheme. The two most generally used are the minimal substitution scheme and the physical renormalization scheme.

The renormalized generating functional can be written as

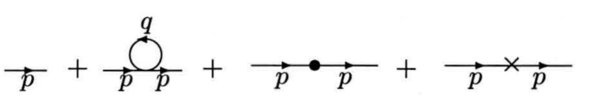
$$\begin{aligned}
Z[\rho] &= \int [d\psi] e^{i\int d^4x[\mathcal{L}+\mathcal{L}_C+\mathcal{L}_{int}+\rho\phi]} \\
&= e^{\int d^4x \mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta\rho}\right)+\mathcal{L}_C\left(\frac{1}{i}\frac{\delta}{\delta\rho}\right)} Z_0[\rho]
\end{aligned}$$

where we have extracted $\mathcal{L}_C$ as an interaction term.

The additional Feynman rules that result from the counter terms are

-  $\longrightarrow i(Z_3-1)p^2$, the wave function counter term.
-  $\longrightarrow -i(Z_0-1)m^2$, the mass counter term.
-  $\longrightarrow i(Z_1-1)\lambda$, the vertex counter term.

The renormalized two point Green's function is thus

$G_c^2(p) =$ 

$$\begin{aligned}
&= \Delta(p) + \Delta^2(p)(-i\delta m^2) \\
&\quad +\Delta^2(p)[i(Z_3-1)p^2] + \Delta^2(p)[-i(Z_0-1)m^2]. \qquad (5.4)
\end{aligned}$$

If we had only calculated the two point Green's function to zeroth order in λ, then there would be no divergences to cancel, thus there would be no counter terms. This implies that (Z_0-1) and (Z_3-1) are of $O(\lambda)$.

[14] From now on we drop the subscript on the renormalized parameters.

5.3.1 Regularization

Before we can fix Z_3 and Z_0, we must first regularize the divergent integrals. We will use dimensional regularization[15]. If we change the dimensions from 4 to 2ω, then the coupling constant no longer stays dimensionless. This can be seen from the action $S = \int d^4x\mathcal{L} \sim \int d^4x\lambda\phi^4$, which is a dimensionless quantity. Here $d^4x\lambda\phi^4$ has dimensions $[m]^{-4}1[m]^4$, where $[\ ]$ denotes the dimension. After changing dimensions from 4 to 2ω, the interaction term in the action becomes $d^{2w}x\lambda\phi^4$, with dimensions $[m]^{-2\omega}[m]^{2\omega-4}[m]^4$. The coupling constant gains a dimension of $[m]^{2\omega-4}$ to keep the action dimensionless. We would like the coupling constant to be dimensionless in 2ω dimensions.We introduce an arbitrary mass μ so that $\lambda_{2\omega} = \mu^{4-2\omega}\lambda$ is dimensionless.

If we dimensionally regularize δm^2 then (5.2) becomes

$$\delta m^2 = \frac{\mu^{4-2\omega}\lambda}{2}\int \frac{d^{2\omega}\bar{q}}{(2\pi)^{2\omega}}\frac{1}{\bar{q}^2+m^2}.$$

From (C.4) we obtain the result

$$\delta m^2 = \frac{\mu^{4-2\omega}\lambda}{2}\frac{m^{2\omega-2}\Gamma(1-\omega)}{(4\pi)^{\omega}},$$

which is well defined if ω is not an integer. We would like to see how δm^2 diverges when $\omega \to 2$, so we set $\omega = 2-\epsilon$ with $\epsilon \to 0$. Thus

$$\delta m^2 = \frac{\lambda m^2}{32\pi^2}\Big(\frac{\mu}{m}\Big)^{2\epsilon}(4\pi)^{\epsilon}\Gamma(\epsilon-1).$$

Using the expansion $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon)$, where γ is Euler's constant, and the expansion $a^{\epsilon} = 1 + \epsilon \ln a$, we see that δm^2 expanded in terms of ϵ is

$$\delta m^2 = -\frac{\lambda m^2}{32\pi^2}\Big(\frac{1}{\epsilon} + 1 + \ln(\frac{\mu^2}{m^2}4\pi) - \gamma\Big) + O(\epsilon).$$

It can now be explicitly seen that δm^2 will diverge as $\epsilon \to 0$ because of the $\frac{1}{\epsilon}$ term.

We now have for (5.4),

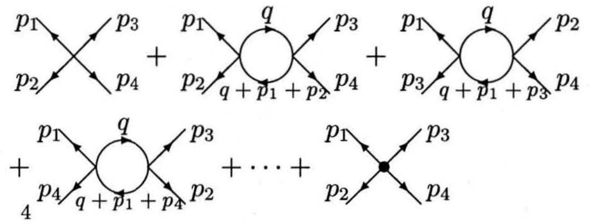
$$\begin{aligned} G_c^2(p) &= \Delta(p) + \Delta^2(p)\Big[i\frac{\lambda m^2}{32\pi^2}\Big(\frac{1}{\epsilon} + 1 + \ln(\frac{\mu^2}{m^2}4\pi) - \gamma\Big) \\ &\quad + i(Z_3-1)p^2 - i(Z_0-1)m^2\Big] \end{aligned} \tag{5.5}$$

[15] See appendix C for derivation.

We are able to use different renormalizing schemes to determine Z_3 and Z_0 in such a way that the $\frac{1}{\epsilon}$ term cancels out and no divergences occur.

We now consider the vertex renormalization. The four point renormalized Green's function is given by

$$G^4(p_1,p_2,p_3,p_4) =$$



$$= \prod_{i=1}^{4} \Delta(p_i) \underbrace{\big(-i\lambda - i\delta\lambda - i\lambda(Z_1 - 1)\big)}_{\Gamma^4(p_1,p_2,p_3,p_4)}, \tag{5.6}$$

where the second, third and fourth diagrams are not equivalent and are known as the s, t and u channel contributions respectively.

If we dimensionally regularize $\delta\lambda$ given by (5.3), then we have

$$\delta\lambda = -\frac{\lambda^2\mu^{4-2\omega}}{2} \sum_{i=1}^{3} \int_0^1 dx \int \frac{d^{2\omega}\bar{q}'}{(2\pi)^{2\omega}} \frac{1}{[\bar{q}'^2 - k_i^2 x(1+x) + m^2]^2}$$

where we have introduced the arbitrary mass so that $\delta\lambda$ is dimensionless in the arbitrary dimension, and the momenta for each channel $k_1^2 = s^2 = (p_1+p_2)^2$, $k_2^2 = t^2 = (p_1+p_3)^2$ and $k_3^2 = u^2 = (p_1+p_4)^2$.

From (C.5) we obtain,

$$\delta\lambda = -\frac{\lambda^2\mu^{4-2\omega}}{2(4\pi)^{\omega}} \Gamma(2-\omega) \sum_{i=1}^{3} \int_0^1 dx [m^2 - k_i^2 x(1+x)]^{\omega-2}$$

If set $\omega = 2 - \epsilon$ and expand in terms of ϵ, we have

$$\delta\lambda = -\frac{\lambda^2}{32\pi^2}\Big(\frac{3}{\epsilon} + \sum_{i=1}^{3} \int_0^1 dx [\ln\big(\frac{4\pi\mu^2}{m^2 - k_i^2 x(1+x)}\big) - \gamma]\Big).$$

We have now for (5.6)

$$\begin{aligned} & G_c^4(p_1,p_2,p_3,p_4) \\ = \; & \prod_{i=1}^{4} \Delta(p_i)\Big[-i\lambda + 3i\frac{\lambda^2}{32\pi^2}\frac{1}{\epsilon} + \\ & i\frac{\lambda^2}{32\pi^2} \sum_{i=1}^{3} \int_0^1 dx [\ln(\frac{4\pi\mu^2}{m^2 - k_i^2 x(1+x)}) - \gamma] - i\lambda(Z_1 - 1)\Big]. \end{aligned} \tag{5.7}$$

We can use renormalization schemes to fix Z_1 so that the $\frac{1}{\epsilon}$ divergence in (5.7) cancels out.

5.3.2 Minimal subtraction

In the minimal subtraction, we choose Z_0, Z_1, Z_3 so that the divergences cancel. It can be seen from (5.5) that the divergent term $\frac{1}{\epsilon}$ is independent of p and only dependent on m^2, so we choose $Z_3 = 1$ and $Z_0 = 1 + \frac{\lambda}{32\pi^2\epsilon}$. In this case we have

$$G_c^2 = \Delta(p) + \Delta^2(p)i\frac{\lambda m^2}{32\pi^2}(1 + \ln(\frac{4\pi\mu^2}{m^2}) - \gamma).$$

If we work to all orders in λ, then for every loop term there will be a correction term, thus $i\frac{\lambda m^2}{32\pi^2}(1 + \ln(\frac{4\pi\mu^2}{m^2}) - \gamma)$ is just the self energy correction $-i\Sigma$. Thus we have

$$G_c^2 = \frac{i}{p^2 - m^2(1 - \frac{\lambda}{32\pi^2}\ln(\frac{4\pi\mu^2}{m^2}) - \gamma) + i\epsilon}.$$

The renormalized mass term is finite and is given by

$$m_R^2 = m^2(1 - \frac{\lambda}{32\pi^2}\ln(\frac{4\pi\mu^2}{m^2})) = f(m, \lambda, \mu).$$

To cancel the divergence out in (5.7), we need to choose $Z_1 = 1 + \frac{3\lambda}{32\pi^2\epsilon}$, and we have

$$G_c^4 = \prod_{i=1}^{4}\Delta(p_i)\Big[-i\lambda + i\frac{\lambda^2}{32\pi^2}\sum_{i=1}^{3}\int_0^1 dx\Big(\ln\big(\frac{4\pi\mu^2}{m^2 - k_i^2 x(1+x)}\big) - \gamma\Big)\Big].$$

5.3.3 Physical renormalization scheme.

In this renormalization scheme we choose Z_0, Z_1, Z_3 so that the correct values for the experimentally measured (physical) parameters are reproduced by theory. The conditions that must be placed on the theory to reproduce these parameters are

- $\Gamma^2(p^2 = m_f^2) = 0$ to obtain the physical mass, m_f.
- $\frac{\partial\Gamma^2}{\partial p^2}\Big|_{p^2=m_f^2} = -1$ ensures the wavefunction is normalized.
- $\Gamma^4(p_1, p_2, p_3, p_4)\Big|_{p^2=m_f^2} = \lambda_f$ to obtain the physical coupling constant.

The two point OPI is given by $\Gamma^2 = -i\Delta^{-1} + \Sigma$ so that

$$\Gamma^2(p^2) = -(p^2 - m^2) - \frac{\lambda_f m^2}{32\pi^2}(\frac{1}{\epsilon} + 1 + \ln(\frac{\mu^2}{m^2}4\pi) - \gamma) - (Z_3 - 1)p^2 - (Z_0 - 1)m^2.$$

Using the first two conditions we find that $Z_3 = 1$ and

$$Z_0 = 1 + \frac{\lambda_f m_f^2}{32\pi^2\epsilon} + \frac{\lambda_f m_f^2}{32\pi^2}(1 + \ln(\frac{4\pi\mu^2}{m_f^2}) - \gamma).$$

Under these conditions the renormalized mass is just the physical mass and the two point Green's function is

$$G_c^2 = \frac{i}{p^2 - m_f^2 + i\epsilon}$$

From (5.6) and (5.7) we can see that

$$\Gamma^4 = \Big[-i\lambda_f + \frac{3i\lambda_f^2}{32\pi^2}\frac{1}{\epsilon} + \frac{i\lambda_f^2}{32\pi^2}\sum_{i=1}^{3}\int_0^1 dx[\ln(\frac{4\pi\mu^2}{m^2 - k_i^2 x(1+x)}) - \gamma] - i\lambda_f(Z_1 - 1)\Big].$$

Using the third condition, we see that

$$Z_1 = 1 + \Big[\frac{3\lambda_f^2}{32\pi^2\epsilon} + \frac{\lambda_f^2}{32\pi^2}\sum_{i=1}^{3}\int_0^1 dx(\ln(\frac{4\pi\mu^2}{m^2 - k_i^2 x(1+x)}) - \gamma)\Big|_{k_i^2 = m_f^2}\Big],$$

where λ_f is the physical coupling constant. We thus have

$$G_c^4 = \prod_{i=1}^{4} \Delta(p_i)(-i\lambda_f).$$

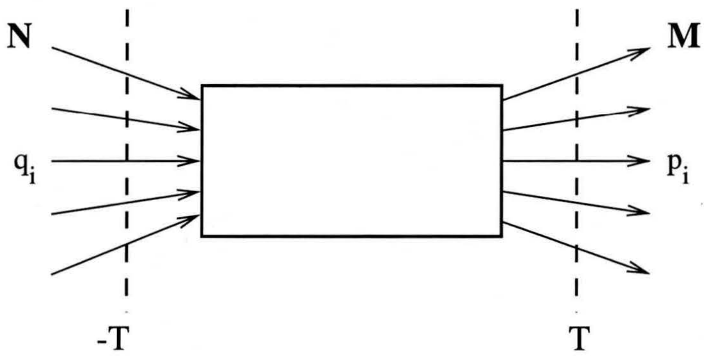
CHAPTER 6
Scattering Amplitudes

In the previous chapter we saw how we could renormalize the parameters of the theory, so that we could write the Green's functions in terms of physically measurable quantities. We would now like a way of relating these Green's functions with the S-matrix, as any quantity that is measured in an experiment is done so in terms of the S-matrix. We can relate the Green's functions with the S-matrix by using the reduction formula.

We first concentrate on scalar fields and then we later consider fermion fields.

6.1 Scalar fields.

Consider a general scattering process where there are N incoming scaler particles with momenta q_i and M scattered particles with momenta p_i.



It is reasonable to assume that for a time $t < -T$ there are no interactions between the incoming particles, and at a time $t > T$ there are no interactions between the scattered particles.[16] Thus for $t < -T$ and $t > T$ the particles are free and are solutions of the Klein-Gordon equation[17], with mass equal to the physical mass.

We can label the incoming state with $|\alpha'\rangle_{in}$ and the scattered state with $|\beta'\rangle_{out}$, where

$$\begin{aligned} |\alpha'\rangle_{in} &= \lim_{t\to-\infty} |\alpha', t\rangle_{in}, \qquad \alpha' = q_1, \ldots, q_N \\ &= N_{in} a^\dagger_{in}(\vec{q}_1) \cdots a^\dagger_{in}(\vec{q}_N)|0\rangle \\ &= N_{in}|q_1 \cdots q_N\rangle. \end{aligned}$$

[16]Self interactions are included at all times so that the propagators for the particles are dressed and the mass of the particles are the physical mass.

[17]See appendix A.1 for the solution of the free Klein-Gordon equation.

$$
\begin{aligned}
|\beta'\rangle_{out} &= \lim_{t\to\infty} |\beta', t\rangle_{out}, \qquad \beta' = p_1, \dots, p_M \\
&= N_{out} a^\dagger_{out}(\vec{p}_1) \cdots a^\dagger_{out}(\vec{p}_M)|0\rangle \\
&= N_{out}|p_1 \cdots p_M\rangle.
\end{aligned}
$$

We know that the creation and annihilation operators satisfy the commutation relation $[a(\vec{k}), a^\dagger(\vec{k}')]_- = 2(2\pi)^3 k_0 \delta^3(\vec{k} - \vec{k}')$.

The normalization factors can be calculated from the conditions that ${}_{in}\langle\alpha'|\alpha'\rangle_{in} = 1$ and ${}_{out}\langle\beta'|\beta'\rangle_{out} = 1$. These conditions give

$$
N_{in} = \frac{1}{(2V)^{N/2}} \prod_{i=1}^{N} \frac{1}{\sqrt{(q_0)_i}} \quad \text{and} \quad N_{out} = \frac{1}{(2V)^{M/2}} \prod_{i=1}^{M} \frac{1}{\sqrt{(p_0)_i}}.
$$

The free field operators that are solutions of the Klein Gordon equation are

$$
\varphi_{in}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k_0} [a_{in}(\vec{k}) e^{ikx} + a^\dagger_{in}(\vec{k}) e^{-ikx}], \tag{6.1}
$$

$$
\varphi_{out}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k_0} [a_{out}(\vec{k}) e^{ikx} + a^\dagger_{out}(\vec{k}) e^{-ikx}]. \tag{6.2}
$$

We introduce a field $\varphi(x)$ that is not a solution of the Klein Gordon equation at a finite time, because there are interactions in this region. In the limit $t \to \infty, -\infty$ we see that φ must reduce to $\varphi_{out}, \varphi_{in}$ as there are no interactions in these limits. This requirement leads to the weak asymptotic boundary conditions :

$$
\lim_{x_0\to-\infty} {}_{out}\langle\beta'|\varphi(x)|\alpha'\rangle_{in} = {}_{out}\langle\beta'|\varphi_{in}(x)|\alpha'\rangle_{in}. \tag{6.3}
$$

$$
\lim_{x_0\to\infty} {}_{out}\langle\beta'|\varphi(x)|\alpha'\rangle_{in} = {}_{out}\langle\beta'|\varphi_{out}(x)|\alpha'\rangle_{in}. \tag{6.4}
$$

6.1.1 Scattering Matrix

The S-matrix is the transition amplitude from a state $|\alpha', t_1\rangle$ at time t_1 to a state $|\beta', t_2\rangle$ at time t_2. If we take the the initial state to be at time $t_1 \to -\infty$ and the final state at time $t_2 \to \infty$, then we can define the S-matrix elements as

$$
\begin{aligned}
S_{\beta'\alpha'} &= \lim_{t\to\infty} \langle\beta', t|\alpha', -t\rangle \\
&= {}_{out}\langle\beta'|\alpha'\rangle_{in} \\
&= N_{in} N_{out} \; {}_{out}\langle\vec{p}_1 \cdots \vec{p}_M|\vec{q}_1 \cdots \vec{q}_N\rangle_{in}.
\end{aligned}
$$

We need to express this transition amplitude in terms of the ground state expectation values of field operators so that we can relate it to the Green's function as defined in the first chapter.

We first concentrate on one of the incoming particles with momentum q. If we write $|\alpha'\rangle_{in} = |\alpha q\rangle_{in}$ where $\alpha = q_1 \cdots q_{N-1} \neq q$, then (dropping the normalization constants)

$$\begin{aligned} S_{\beta'\alpha'} &= {}_{out}\langle\beta'|\alpha, q\rangle_{in} = {}_{out}\langle\beta'|a^\dagger_{in}(\vec{q})|\alpha\rangle_{in} \\ &= {}_{out}\langle\beta'|a^\dagger_{out}(\vec{q})|\alpha\rangle_{in} + {}_{out}\langle\beta'|[a^\dagger_{in}(\vec{q}) - a^\dagger_{out}(\vec{q})]|\alpha\rangle_{in} \\ &= {}_{out}\langle\beta' - q|\alpha\rangle_{in} + {}_{out}\langle\beta'|[a^\dagger_{in}(\vec{q}) - a^\dagger_{out}(\vec{q})]|\alpha\rangle_{in}. \end{aligned} \tag{6.5}$$

The first term in the last equality is the transition amplitude for when the particle with momentum q undergoes no interactions. The second term is the transition amplitude for when the particle does take part in an interaction.

We can obtain an expression for $a^\dagger_{in}$ from a linear combination of the Fourier transforms of φ given by (6.1) and $\frac{\partial\varphi}{\partial t}$ so that

$$a^\dagger_{in}(\vec{q}) = \int d^3x[q_0\varphi_{in}(x) + i\frac{\partial\varphi_{in}(x)}{\partial t}]e^{iqx} = -i\int d^3x[e^{iqk}\overleftrightarrow{\partial}_0\varphi_{in}(x)]. \tag{6.6}$$

Similarly, using (6.2) we have

$$a^\dagger_{out} = -i\int d^3x[e^{iqk}\overleftrightarrow{\partial}_0\varphi_{out}(x)].$$

Inserting this into (6.5) we have

$$\begin{aligned} S_{\beta'\alpha'} &= {}_{out}\langle\beta' - q|\alpha\rangle_{in} + {}_{out}\langle\beta'|\int d^3x e^{iqk}\overleftrightarrow{\partial}_0(\varphi_{in}(x) - \varphi_{out}(x))|\alpha\rangle_{in} \\ &= {}_{out}\langle\beta' - q|\alpha\rangle_{in} + i(\lim_{x_0\to\infty} - \lim_{x_0\to-\infty})\int d^3x e^{iqk}\overleftrightarrow{\partial}_0 {}_{out}\langle\beta'|\varphi(x)|\alpha\rangle_{in} \\ &= {}_{out}\langle\beta' - q|\alpha\rangle_{in} + i\int_{-\infty}^{\infty} dx_0 \int d^3x\partial_0[e^{iqk}\overleftrightarrow{\partial}_0 {}_{out}\langle\beta'|\varphi(x)|\alpha\rangle_{in}] \\ &= {}_{out}\langle\beta' - q|\alpha\rangle_{in} + i\int d^4x[(\partial_0^2 e^{iqk}) - e^{iqx}\partial_0^2]_{out}\langle\beta'|\varphi(x)|\alpha\rangle_{in} \\ &= {}_{out}\langle\beta' - q|\alpha\rangle_{in} - i\int d^4x e^{iqk}[(\partial_0^2 - \nabla^2 + m^2]_{out}\langle\beta'|\varphi(x)|\alpha\rangle_{in}, \\ &\qquad \text{(after integrating by parts and using the asymptotic b.c.)} \\ &= {}_{out}\langle\beta' - q|\alpha\rangle_{in} - i\int d^4x e^{iqk}K_x\, {}_{out}\langle\beta'|\varphi(x)|\alpha\rangle_{in}, \end{aligned} \tag{6.7}$$

where $K_x = \Box_x + m^2$ is the Klein Gordon operator.

We have now written the incoming particle with momentum q in terms of the field $\varphi(x)$.

We can repeat the process for another incoming particle in the transition amplitudes in (6.7) and thus write all the incoming particles in terms of field operators.

We would now like to write the scattered particles in terms of field operators. Consider a state $|\beta'\rangle_{out} = |\beta p\rangle_{out}$ where $\beta = p_1 \cdots p_{M-1} \neq p_{,}$, then

$$\begin{aligned} & {}_{out}\langle\beta'|\varphi(x)|\alpha\rangle_{in} \\ = \ & {}_{out}\langle\beta p|\varphi(x)|\alpha\rangle_{in} \ = \ {}_{out}\langle\beta|a_{out}(p)\varphi(x)|\alpha\rangle_{in} \\ = \ & {}_{out}\langle\beta|[a_{out}(p)\varphi(x) - \varphi(x)a_{in}(p)]|\alpha\rangle_{in} \ + \ {}_{out}\langle\beta|\varphi(x)|\alpha - p\rangle_{in}. \end{aligned} \tag{6.8}$$

We can obtain $a_{in}(p)$ and $a_{out}(p)$ in terms of the field operator by taking the complex conjugate of (6.6). If we insert this into (6.8) we have

$$\begin{aligned} & {}_{out}\langle\beta'|\varphi(x)|\alpha\rangle_{in} \\ = \ & {}_{out}\langle\beta|\varphi(x)|\alpha - p\rangle_{in} + i\int d^3y e^{-ipy} \overleftrightarrow{\partial_0} {}_{out}\langle\beta|\varphi_{out}(y)\varphi(x) - \varphi(x)\varphi_{in}(y)|\alpha\rangle_{in} \\ = \ & {}_{out}\langle\beta|\varphi(x)|\alpha - p\rangle_{in} + (\lim_{y_0\to\infty} - \lim_{y_0\to-\infty})i\int d^3y e^{-ipy} \overleftrightarrow{\partial_0} {}_{out}\langle\beta|T[\varphi(x)\varphi(y)]|\alpha\rangle_{in} \\ = \ & {}_{out}\langle\beta|\varphi(x)|\alpha - p\rangle_{in} - i\int d^4y e^{-ipy} K_y \, {}_{out}\langle\beta|T[\varphi(x)\varphi(y)]|\alpha\rangle_{in} \end{aligned} \tag{6.9}$$

The first term of (6.9) is where the scattered particle with momentum p did not take take part in any interaction and the second term is where it did take part in an interaction. The process can be repeated for other scattered particles in the transition amplitudes in (6.9) until all the scattered particles are written in terms of the field operators.

If we consider a process where all the incoming momenta are different from all the scattered momenta, $q_i \neq p_j \ \forall i, j$ then all particles take part in the interaction and all the direct terms reduce to zero. This simplifies the S-matrix to

$$\begin{aligned} S_{\beta'\alpha'} &= {}_{out}\langle\beta'|\alpha'\rangle_{in} \\ &= (-i)^{N+M}\int \prod_{i=1}^{M} d^4y_i \prod_{j=1}^{n} d^4x_j e^{-ip_iy_i} e^{iq_jx_j} K_{y_i} Kx_j \\ &\qquad \times \underbrace{\langle 0|T[\varphi(y_1)\cdots\varphi(y_M)\varphi(x_1)\cdots\varphi(x_N)]|0\rangle}_{G^{N+M}(y_1\cdots y_M, x_1\cdots x_N)} \\ &= (i)^{N+M}\int \prod_{i=1}^{M} d^4y_i \prod_{j=1}^{n} d^4x_j (p_i^2 - m_i^2)(q_j^2 - m_j^2) e^{-ip_iy_i} e^{iq_jx_j} \\ &\qquad \times G^{N+M}(y_1\cdots y_M, x_1\cdots x_N) \end{aligned}$$

(where we have integrated by parts)

$$
\begin{aligned}
&= (i)^{N+M}(2\pi)^4\delta(p_1+\cdots+p_M-q_1-\cdots-q_N)\\
&\quad\times G^{N+M}(-p_1\cdots-p_M,q_1\cdots q_N)\prod_{i=1}^{M}(p_i^2-m_i^2)\prod_{j=1}^{n}(q_j^2-m_j^2),
\end{aligned}
$$

where

$$
(2\pi)^4\delta(p_1+\cdots+p_n)G^n(p_1\cdots p_n)=\int\prod_{i=1}^{n}d^4x_ie^{ip_ix_i}G^n(x_1\cdots x_n)
$$

is the Fourier transform of $G^n(x_1\cdots x_n)$.

Thus

$$
\begin{aligned}
S_{\beta\alpha} &= (2\pi)^4\delta(p_1+\cdots+p_M-q_1-\cdots-q_N)M_{\beta\alpha}, \qquad (6.10)\\
M_{\beta\alpha} &= (i)^{N+M}\prod_{i=1}^{M}(p_i^2-m_i^2)\prod_{j=1}^{n}(q_j^2-m_j^2)G^{N+M}(-p_1\cdots-p_M,q_1\cdots q_N).
\end{aligned}
$$

We note that $M_{\beta\alpha}$ can be obtained from the Feynman rules for the Green's function by removing the external legs and then placing the external momenta on the mass shell. All connected, disconnected and one particle reduced diagrams contribute to the S-matrix, except for diagrams containing single propagators, since

$$
\begin{aligned}
M_{\beta\alpha} &= (p^2-m^2)(p^2-m^2)\Delta(p)|_{p^2=m^2}\\
&= (p^2-m^2)(p^2-m^2)\frac{1}{(p^2-m^2)}|_{p^2=m^2}=0.
\end{aligned}
$$

If we now take the normalization into account we have

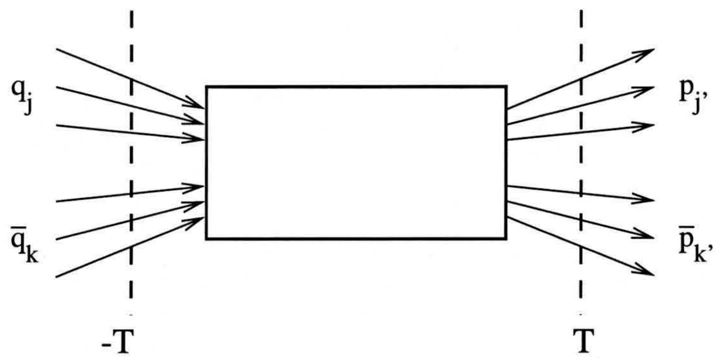
$$
\begin{aligned}
S_{\beta\alpha} &= \frac{1}{V^{(N+M)/2}}\prod_{i=1}^{M}\frac{1}{\sqrt{2p_{0i}}}\prod_{i=1}^{M}\frac{1}{\sqrt{2q_{0i}}}\\
&\quad\times(2\pi)^4\delta(p_1+\cdots+p_M-q_1-\cdots-q_N)M_{\beta\alpha}. \qquad (6.11)
\end{aligned}
$$

We have managed to express the S-matrix, which can be measured experimentally, in terms of the Green's function.

6.2 Fermions.

We consider a scattering process, this time of fermions.

We assume that interactions only take place if the time is within the range $-T<t<T$.



Outside this range, the particles are free and are solutions of the Dirac equation[18] with physical mass. Once again we can introduce the incoming state $|\alpha'\rangle_{in}$ and the scattered state with $|\beta'\rangle_{out}$, where

$$
\begin{aligned}
|\alpha'\rangle_{in} &= N_{in} d^{\dagger}_{in}(\bar{q}_k, \bar{s}_k) \cdots d^{\dagger}_{in}(\bar{q}_1, \bar{s}_1) b^{\dagger}_{in}(q_j, s_j) \cdots b^{\dagger}_{in}(q_1, s_1)|0\rangle \\
&= N_{in}|(\bar{q}_k, \bar{s}_k) \cdots (\bar{q}_1, \bar{s}_1), (q_j, s_j) \cdots (q_1, s_1)\rangle. \\
|\beta'\rangle_{out} &= N_{out} d^{\dagger}_{out}(\bar{p}_{k'}, \bar{s}_{k'}) \cdots d^{\dagger}_{out}(\bar{p}_{1'}, \bar{s}_{1'}) b^{\dagger}_{out}(p_{j'}, s_{j'}) \cdots b^{\dagger}_{out}(p_{1'}, s_{1'})|0\rangle \\
&= N_{out}|(\bar{p}_{k'}, \bar{s}_{k'}) \cdots (\bar{p}_{1'}, \bar{s}_{1},)(p_{j'}, s_{j'}) \cdots (p_{1'}, s_{1'})\rangle,
\end{aligned}
$$

with

- $\bar{q}_k$ denoting an incoming anti–particle with momentum q_k,
- q_j denoting an incoming particle with momentum q_j,
- $\bar{p}_{k'}$ denoting a scattered anti–particle with momentum $p_{k'}$,
- $p_{j'}$ denoting a scattered particle with momentum $p_{j'}$.

The normalization is the same as for the scalar scattering.

As before, we can introduce field operators

$$
\begin{aligned}
\psi_{in}(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \sum_s [b_{in}(p,s)\psi^+(p,s) + d^{\dagger}_{in}(p,s)\psi^-(p,s)] \\
\bar{\psi}_{in}(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \sum_s [b^{\dagger}_{in}(p,s)\bar{\psi}^+(p,s) + d_{in}(p,s)\bar{\psi}^-(p,s)] \\
\psi_{out}(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \sum_s [b_{out}(p,s)\psi^+(p,s) + d^{\dagger}_{out}(p,s)\psi^-(p,s)]
\end{aligned}
$$

[18]See appendix A.2 for the solution of the free Dirac equation.

$$\bar{\psi}_{out}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p_0} \sum_s [b^\dagger_{out}(p,s)\bar{\psi}^+(p,s) + d_{out}(p,s)\bar{\psi}^-(p,s)]$$

where we have the weak asymptotic boundary conditions :

$$\begin{aligned}
\lim_{x_0\to-\infty} {}_{out}\langle\beta'|\psi(x)|\alpha'\rangle_{in} &= {}_{out}\langle\beta'|\psi_{in}(x)|\alpha'\rangle_{in}.\\
\lim_{x_0\to-\infty} {}_{out}\langle\beta'|\bar{\psi}(x)|\alpha'\rangle_{in} &= {}_{out}\langle\beta'|\bar{\psi}_{in}(x)|\alpha'\rangle_{in}.\\
\lim_{x_0\to\infty} {}_{out}\langle\beta'|\psi(x)|\alpha'\rangle_{in} &= {}_{out}\langle\beta'|\psi_{out}(x)|\alpha'\rangle_{in}.\\
\lim_{x_0\to\infty} {}_{out}\langle\beta'|\bar{\psi}(x)|\alpha'\rangle_{in} &= {}_{out}\langle\beta'|\bar{\psi}_{out}(x)|\alpha'\rangle_{in}.
\end{aligned}$$

6.2.1 S-matrix elements

The S-matrix is now given by (ignoring normalization for now)

$$\begin{aligned}
S_{\beta'\alpha'} &= {}_{out}\langle\beta'|\alpha'\rangle_{in}\\
&= \langle(\bar{q}_k,\bar{s}_k)\cdots(\bar{q}_1,\bar{s}_1),(q_j,s_j)\cdots(q_1,s_1)|\\
&\quad \times(\bar{p}_{k'},\bar{s}_{k'})\cdots(\bar{p}_{1'},\bar{s}_{1},)(p_{j'},s_{j'})\cdots(p_{1'},s_{1'})\rangle
\end{aligned} \tag{6.12}$$

We are now going to express the matrix elements in (6.12) as the ground state expectation value of field operators.

6.2.1.1 Removing an incoming particle.

We first express the incoming particles as field operators. To do this we write $|\alpha'\rangle_{in} = |(p,s)\alpha\rangle_{in}$ where α represents all incoming particles and anti–particles except for an incoming particle with momentum p. Thus

$$\begin{aligned}
S_{\beta'\alpha'} &= {}_{out}\langle\beta'|b^\dagger_{out}(p,s)|\alpha\rangle_{in} +_{out} \langle\beta'|b^\dagger_{in}(p,s) - b^\dagger_{out}(p,s)|\alpha\rangle_{in}\\
&= {}_{out}\langle\beta' - (p,s)|\alpha\rangle_{in} +_{out} \langle\beta'|b^\dagger_{in}(p,s) - b^\dagger_{out}(p,s)|\alpha\rangle_{in}
\end{aligned} \tag{6.13}$$

We can express $b^\dagger_{in}$ and $b^\dagger_{out}$ in terms of field operators,

$$\begin{aligned}
b^\dagger_{in}(p,s) &= \int d^3x\psi^\dagger_{in}(x)\psi^+(x)\\
b^\dagger_{out}(p,s) &= \int d^3x\psi^\dagger_{out}(x)\psi^+(x)
\end{aligned}$$

Using these equations in (6.13) we have

$$\begin{aligned} S_{\beta'\alpha'} &= {}_{out}\langle\beta' - (p,s)|\alpha\rangle_{in} \\ &\quad + \int d^3x\ {}_{out}\langle\beta'|\psi^\dagger_{in}(x)\psi^+(x) - \psi^\dagger_{out}(x)\psi^+(x)|\alpha\rangle_{in} \\ &= {}_{out}\langle\beta' - (p,s)|\alpha\rangle_{in} - \int d^4x\ \partial_0\ {}_{out}\langle\beta'|\psi^\dagger(x)\psi^+(x)|\alpha\rangle_{in}, \end{aligned} \tag{6.14}$$

where we used the asymptotic boundary conditions as before. If we write $\psi^\dagger = \bar{\psi}\gamma^0$, then we have, after differentiating and integrating by parts (as in the scalar case),

$$S_{\beta'\alpha'} = {}_{out}\langle\beta' - (p,s)|\alpha\rangle_{in} - i\int d^4x\ {}_{out}\langle\beta'|\bar{\psi}(x)|\alpha\rangle_{in}\ (-i\overleftarrow{\not{\partial}_x} - m)\psi^+(x). \tag{6.15}$$

The first term of (6.15) is a direct term where the particle does not take part in the interaction. In the second term, the particle takes part in the interaction.

6.2.1.2 Removing an incoming anti–particle.

We can now repeat the above steps to express an incoming anti–particle with momentum $\bar{p}$ in terms of the field operators. If we write $|\alpha'\rangle_{in} = |(\bar{p},\bar{s})\alpha\rangle_{in}$. The S-matrix elements are

$$\begin{aligned} S_{\beta'\alpha'} &= {}_{out}\langle\beta'|d^\dagger_{out}(\bar{p},\bar{s})|\alpha\rangle_{in} + {}_{out}\langle\beta'|d^\dagger_{in}(\bar{p},\bar{s}) - d^\dagger_{out}(\bar{p},\bar{s})|\alpha\rangle_{in} \\ &= {}_{out}\langle\beta' - (\bar{p},\bar{s})|\alpha\rangle_{in} + {}_{out}\langle\beta'|d^\dagger_{in}(\bar{p},\bar{s}) - d^\dagger_{out}(\bar{p},\bar{s})|\alpha\rangle_{in}. \end{aligned} \tag{6.16}$$

Using

$$\begin{aligned} d^\dagger_{in}(\bar{p},\bar{s}) &= \int d^3x\psi^{\dagger -}(\bar{p},\bar{s})(x)\psi_{in}(x) \\ d^\dagger_{out}(\bar{p},\bar{s}) &= \int d^3x\psi^{\dagger -}(\bar{p},\bar{s})(x)\psi_{out}(x) \end{aligned}$$

in (6.16) and repeating the same analysis as before, we find

$$S_{\beta'\alpha'} = {}_{out}\langle\beta' - (\bar{p},\bar{s})|\alpha\rangle_{in} + i\int d^4x\ \bar{\psi}^-(\bar{p},\bar{s})(x)(i\overrightarrow{\not{\partial}_x} - m)\ {}_{out}\langle\beta'|\psi(x)|\alpha\rangle_{in}. \tag{6.17}$$

6.2.1.3 Removing a scattered particle.

If we express a scattered particle with momentum p in terms of field operators, we have $|\beta'\rangle_{out} = |(p,s)\beta\rangle_{out}$ and

$$S_{\beta'\alpha'} = {}_{out}\langle\beta|\alpha' - (p,s)\rangle_{in} + {}_{out}\langle\beta|b_{out}(p,s) - b_{in}(p,s)|\alpha'\rangle_{in}. \tag{6.18}$$

Using

$$\begin{aligned} b_{out}(p,s) &= \int d^3x \psi^{\dagger +}(p,s)(x)\psi_{out}(x) \\ b_{in}(p,s) &= \int d^3x \psi^{\dagger +}(p,s)(x)\psi_{in}(x) \end{aligned}$$

in (6.18) we obtain

$$S_{\beta'\alpha'} = {}_{out}\langle\beta|\alpha' - (p,s)\rangle_{in} - i\int d^4x \ \bar{\psi}^{+}(p,s)(x)(i \overrightarrow{\not\partial_x} - m) \ {}_{out}\langle\beta|\psi(x)|\alpha'\rangle_{in}. \qquad (6.19)$$

6.2.1.4 Removing a scattered anti–particle.

Finally, we can express a scattered anti–particle with momentum $\bar{p}$ as a field operator. We have $|\beta'\rangle_{out} = |(\bar{p},\bar{s})\beta\rangle_{out}$ and

$$S_{\beta'\alpha'} = {}_{out}\langle\beta|\alpha' - (\bar{p},\bar{s})\rangle_{in} + {}_{out}\langle\beta|d_{out}(\bar{p},\bar{s}) - d_{in}(\bar{p},\bar{s})|\alpha'\rangle_{in}. \qquad (6.20)$$

Using

$$\begin{aligned} d_{out}(\bar{p},\bar{s}) &= \int d^3x \psi^{\dagger}_{out}(x)\psi^{-}(\bar{p},\bar{s})(x) \\ d_{in}(\bar{p},\bar{s}) &= \int d^3x \psi^{\dagger}_{in}(x)\psi^{-}(\bar{p},\bar{s})(x) \end{aligned}$$

in (6.20) we obtain

$$S_{\beta'\alpha'} = {}_{out}\langle\beta|\alpha' - (\bar{p},\bar{s})\rangle_{in} + i\int d^4x \ {}_{out}\langle\beta|\bar{\psi}(x)|\alpha'\rangle_{in} \ (-i \overleftarrow{\not\partial_x} - m)\bar{\psi}^{-}(\bar{p},\bar{s})(x). \qquad (6.21)$$

We can now iterate the process to express all the particles in terms of field operators and thus end up with the ground state expectation value of these operators. We must be careful when going through this iterative process as the field operators are fermionic. As an example we consider a case where $n+m$ field operators (n particles and m anti–particles removed) are present and we wish to express an incoming particle as another field operator. Thus we have

$${}_{out}\langle\beta|T(\psi(x_1)\dots\psi(x_n)\bar{\psi}(y_1)\dots\bar{\psi}(y_m))|(p,s)\alpha\rangle_{in}$$

where $T(\psi(x)\psi(y)) = \psi(x)\psi(y)\theta(x_0-y_0) - \psi(y)\psi(x)\theta(y_0-x_0)$ is the time ordered operator for fermion fields. When we extract the particle we obtain

$${}_{out}\langle\beta|T(\psi(x_1)\dots\psi(x_n)\bar{\psi}(y_1)\dots\bar{\psi}(y_m))|(p,s)\alpha\rangle_{in}$$

$$
\begin{aligned}
&= {}_{out}\langle\beta|T(\ldots)b^{\dagger}_{in}(p,s)|\alpha\rangle_{in} \\
&= (-1)^{m+n}\ {}_{out}\langle\beta-(p,s)|T(\ldots)|\alpha\rangle_{in} \\
&\quad - {}_{out}\langle\beta|(-1)^{m+n}b^{\dagger}_{out}(p,s)T(\ldots)|\alpha\rangle_{in} + {}_{out}\langle\beta|T(\ldots)b^{\dagger}_{in}(p,s)|\alpha\rangle_{in} \\
&= (-1)^{m+n}\ {}_{out}\langle\beta-(p,s)|T(\ldots)|\alpha\rangle_{in} \\
&\quad -\int d^3y\ {}_{out}\langle\beta|(-1)^{m+n}\psi^{\dagger}_{out}(y)T(\ldots)-T(\ldots)\psi^{\dagger}_{in}(y)|\alpha\rangle_{in}\ \psi^{+}(p,s)(y) \\
&= (-1)^{m+n}\ {}_{out}\langle\beta-(p,s)|T(\ldots)|\alpha\rangle_{in} \\
&\quad -(\lim_{y_0\to\infty}-\lim_{y_0\to\infty})\int d^3y\ {}_{out}\langle\beta|T(\ldots\psi^{\dagger}(y)|\alpha\rangle_{in}\ \psi^{+}(p,s)(y) \\
&= (-1)^{m+n}\ {}_{out}\langle\beta-(p,s)|T(\ldots)|\alpha\rangle_{in} \\
&\quad -\int d^4y\ \partial_{y_0}\ {}_{out}\langle\beta|T(\ldots\psi^{\dagger}(y)\psi^{+}(p,s)(y)|\alpha\rangle_{in} \\
&= (-1)^{m+n}\ {}_{out}\langle\beta-(p,s)|T(\ldots)|\alpha\rangle_{in} \\
&\quad -i\int d^4y\ {}_{out}\langle\beta|T(\ldots\bar{\psi}(y)|\alpha\rangle_{in}\ (-i\overleftarrow{\not\partial}_y-m)\psi^{+}(p,s)(y)
\end{aligned}
\tag{6.22}
$$

We thus have a generalized result of (6.15). We can obtain similar results for incoming anti–particles, scattered particles and scattered anti–particles. They are

$$
\begin{aligned}
&{}_{out}\langle\beta|T(\psi(x_1)\ldots\psi(x_n)\bar{\psi}(y_1)\ldots\bar{\psi}(y_m))|(\bar{p},\bar{s})\alpha\rangle_{in} \\
&= (-1)^{m+n}\ {}_{out}\langle\beta-(\bar{p},\bar{s})|T(\ldots)|\alpha\rangle_{in} \\
&\quad +i\int d^4y\ \bar{\psi}^{-}(\bar{p},\bar{s})(y)(-1)^{m+n}(i\overrightarrow{\not\partial}_y-m)\ {}_{out}\langle\beta|T(\psi(y)\ldots)|\alpha\rangle_{in}
\end{aligned}
\tag{6.23}
$$

$$
\begin{aligned}
&{}_{out}\langle(p,s)\beta|T(\psi(x_1)\ldots\psi(x_n)\bar{\psi}(y_1)\ldots\bar{\psi}(y_m))|\alpha\rangle_{in} \\
&= (-1)^{m+n}\ {}_{out}\langle\beta|T(\ldots)|\alpha-(p,s)\rangle_{in} \\
&\quad -i\int d^4y\ \bar{\psi}^{+}(p,s)(y)(i\overrightarrow{\not\partial}_y-m)_{out}\langle\beta|T(\psi(y)\ldots)|\alpha\rangle_{in}
\end{aligned}
\tag{6.24}
$$

$$
\begin{aligned}
&{}_{out}\langle(\bar{p},\bar{s})\beta|T(\psi(x_1)\ldots\psi(x_n)\bar{\psi}(y_1)\ldots\bar{\psi}(y_m))|\alpha\rangle_{in} \\
&= (-1)^{m+n}\ {}_{out}\langle\beta|T(\ldots)|\alpha-(\bar{p},\bar{s})\rangle_{in} \\
&\quad +i\int d^4y\ (-1)^{m+n}{}_{out}\langle\beta|T(\ldots\bar{\psi}(y))|\alpha\rangle_{in}(-i\overleftarrow{\not\partial}_y-m)\psi^{-}(\bar{p},\bar{s})(y)
\end{aligned}
\tag{6.25}
$$

In this way we can express all the incoming and scattered particles and anti–particles as field operators.

If we assume that all the incoming momenta are different from the scattered momenta, then we have

$$\begin{aligned} S_{\beta\alpha} &= (-1)^{m+n} \int \prod_{a=1}^{k'} dy'_a \int \prod_{b=1}^{j'} dy_b \int \prod_{c=1}^{k} dx'_c \int \prod_{d=1}^{j} dx_d \\ & \times \Big[\bar{\psi}^{+}(p_b, s_b)(y_b)(i \overrightarrow{\not\partial}_{y_b} - m) \bar{\psi}^{-}(\bar{q}_c, \bar{s}_c)(x'_c)(i \overrightarrow{\not\partial}_{x'_c} - m) \\ & \times \langle 0 | T(\psi(y_1) \ldots \psi(y_{j'}) \psi(x'_1) \ldots \psi(x'_k) \bar{\psi}(y'_1) \ldots \bar{\psi}(y'_{k'}) \bar{\psi}(x_1) \ldots \bar{\psi}(x_j)) | 0 \rangle \\ & \times (-i \overleftarrow{\not\partial}_{y'_a} - m) \psi^{-}(\bar{p}_a, \bar{s}_a)(y'_a)(-i \overleftarrow{\not\partial}_{x_d} - m) \psi^{+}(q_d, s_d)(x_d) \Big] \end{aligned}$$

We note that the free field solutions are

$$\begin{aligned} \psi^{+}(p,s)(x) &= u(p,s)e^{-ipx} \\ \psi^{-}(p,s)(x) &= v(p,s)e^{ipx} \\ \bar{\psi}^{+}(p,s)(x) &= \bar{u}(p,s)e^{ipx} \\ \bar{\psi}^{-}(p,s)(x) &= \bar{v}(p,s)e^{-ipx} \end{aligned}$$

where u, v are Dirac spinors.If rewrite in terms of the Dirac spinors, do a partial integration so that the Dirac equations operate on the plane waves and then form the Fourier transform of the Green's function, we can write the fermion S-matrix as

$$\begin{aligned} S_{\beta\alpha} &= (-1)^{n+m} \prod_{a=1}^{k} (\bar{\not q}_a + m) v(\bar{q}_a \bar{s}_a) \prod_{b=1}^{j} \bar{u}(q_b, s_b)(\not q_b - m) \\ & \times \prod_{c=1}^{k'} \bar{v}(\bar{p}_c \bar{s}_c)(\not p_c + m) \prod_{d=1}^{j'} (\bar{\not p}_d - m) u(p_d, s_d) \\ & \times (2\pi)^4 \delta(q_1 + \ldots + q_j + \bar{q}_1 + \ldots + \bar{q}_k - \bar{p}_1 - \ldots - \bar{p}_{j'} - p_1 - \ldots - p_{k'}) \\ & \times G(q_1, \ldots, q_j, \bar{q}_1, \ldots, \bar{q}_k, -\bar{p}_1, \ldots, -\bar{p}_{j'}, -p_1, \ldots, -p_{k'}), \qquad (6.26) \end{aligned}$$

where we have suppressed the spinor indexes.

This S-matrix element for fermions can be obtained from the momentum space Green's function by removing all the external legs and then placing all external momenta on the mass shell. A spinor u is associated with incoming particles and a spinor $\bar{v}$ with incoming anti–particles. A spinor v is associated with scattered anti–particles and a spinor $\bar{u}$ with scattered particles.

We are thus able to relate experimental results with theoretical predictions using the reduction formulas, equation (6.11) for scalrs and equation (6.26) for fermions.

CHAPTER 7
Quantum Field Theory of Gauge Fields

Consider N fermion fields, $\psi_{i\alpha}(x)$, where $i = 1 \cdots N$ denote colour indices and $\alpha = 1, 2, 3, 4$ denote spinor indices. Instead of working with N four vector fields, we can work with one $4N$ vector field $\Psi(x)$, where

$$\Psi(x) \equiv \left.\begin{bmatrix} \psi_1(x) \\ \vdots \\ \psi_N(x) \end{bmatrix}\ \right\} 4N \text{ column vector.}$$

We would like to form a unitary transformation on Ψ that mixes the colour indices, $\psi_1 \ldots \psi_N$, but does not mix the spinor indexes. To do this we need a transformation of the form $(U \otimes I_4)\Psi$, where U is an element of a N dimensional representation of a matrix group with generators T^a.

To allow for the most general unitary transformation of $\Psi(x)$, take T^a to be the generators of $SU(N)$ in the fundamental representation. Then $[T_a, T_b]_- = if^c_{ab}T_c$, $T_a^\dagger = T_a$ and we normalize so that $Tr(T_aT_b) = \frac{1}{2}\delta_{ab}$. An element of the group, G is given by $U = e^{-igT_a\Lambda^a}$.

7.1 Gauge invariance.

If we introduce a global gauge transformation[19], $\Psi' \to e^{-ig\Lambda^aT_a}\Psi$, then the Lagrange density $\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$ stays invariant under this transformation. This can easily be seen since

$$\begin{aligned} \bar{\Psi}' &= \Psi'^\dagger\gamma^0 \to \Psi^\dagger e^{ig\Lambda^aT_a}\gamma^0 \\ &= \bar{\Psi}e^{ig\Lambda^aT_a} \quad \text{because } T_a^\dagger = T_a \text{ and } [T^\mu, \gamma_\mu] = 0. \\ \text{Thus}\ \ \mathcal{L}' &= \bar{\Psi}'(i\gamma^\mu\partial_\mu - m)\Psi' \\ &\to \bar{\Psi}e^{ig\Lambda^aT_a}(i\gamma^\mu\partial_\mu - m)e^{-ig\Lambda^aT_a}\Psi \\ &= \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi = \mathcal{L}. \end{aligned}$$

This invariance of the Lagrange density leads to the conservation of charge from Noether's theorem.

[19]These transformations leave the coordinates unchanged and transform the spinors by a phase transformation.

We would now like to generalize this transformation so that it is coordinate dependent. Consider the local gauge transformation, $\Psi'(x) \to U(x)\Psi(x)$, then $\mathcal{L}$ is no longer invariant under this transformation. This is because the derivatives in $\mathcal{L}$ now see the transformation operator, thus

$$\begin{aligned} \mathcal{L}' &= \bar{\Psi}'(i\gamma^\mu \partial_\mu - m)\Psi' \\ &\to \bar{\Psi}(i\gamma^\mu U^\dagger \partial_\mu U - m)\Psi \\ &= \bar{\Psi}(i\gamma^\mu \partial_\mu + i\gamma^\mu U^\dagger(\partial_\mu(U)) - m)\Psi \end{aligned}$$

We would like to enforce local gauge invariance in $\mathcal{L}$, so we must alter the theory. To do this, we introduce a covariant derivative defined as $D_\mu \equiv \partial_\mu + igT_a A^a_\mu$, where A^a_μ is a gauge field. We now use the covariant derivative in the Lagrangian, thus $\mathcal{L}' \equiv \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi$. Thus we have

$$\begin{aligned} \mathcal{L}' &= \bar{\Psi}'(i\gamma^\mu D'_\mu - m)\Psi' \\ &\to \bar{\Psi}(i\gamma^\mu U^\dagger D'_\mu U - m)\Psi. \end{aligned}$$

We see that to enforce local gauge invariance in $\mathcal{L}$, we need to ensure that the covariant derivative transforms as $D'_\mu \to U D_\mu U^\dagger$. This condition thus enables us to determine how the gauge fields must transform. We consider[20] $D'_\mu = \partial_\mu + igT \cdot A'_\mu$. We know that

$$\begin{aligned} D'_\mu &\to U D_\mu U^\dagger \\ &= U(\partial_\mu + igT \cdot A_\mu)U^\dagger \\ &= \partial_\mu + U(\partial_\mu U^\dagger) + igUT \cdot A_\mu U^\dagger. \end{aligned}$$

We thus see that $T \cdot A'_\mu \to UT \cdot A_\mu U^\dagger - \frac{i}{g} U(\partial_\mu U^\dagger)$.

We can obtain the infinitesimal form of the gauge transformation, $U(x) = e^{-ig\Lambda \cdot T} \simeq (1 - ig\Lambda \cdot T)$, where Λ is small. Thus

$$\begin{aligned} T \cdot A_\mu &\to (1 - ig\Lambda \cdot T)T{\cdot}A_\mu(1 + ig\Lambda \cdot T) + (\partial_\mu \Lambda \cdot T)(1 + ig\Lambda \cdot T) + O(\Lambda^2) \\ &= T \cdot A_\mu + ig[T \cdot A_\mu, \Lambda \cdot T] + (\partial_\mu \Lambda) \cdot T \\ &= T \cdot A_\mu + (\partial_\mu \Lambda) \cdot T + igA^a_\mu \Lambda^b [T_a, T_b]_-. \end{aligned}$$

[20] We use $T \cdot A_\mu$ as short notation for $T_a A^a_\mu$.

In component form this reads :

$$A_\mu^c \to A_\mu^c + \partial_\mu \Lambda^c - igf_{ab}^c A_\mu^a \Lambda^b.$$

If we have an Abelian theory, then $f_{ab}^c = 0$ and the transformation reduces to $A_\mu^c \to A_\mu^c + \partial_\mu \Lambda^c$, which is the gauge invariant condition of Maxwell's equations. Thus an Abelian theory is just Maxwell theory.

7.2 Gauge Field Lagrange density.

We have introduced gauge fields into the fermion Lagrange density by means of the covariant derivative. We must now investigate the dynamics of these gauge fields which is determined by a gauge field Lagrange density. This Lagrange density must also be locally gauge invariant. We introduce an anti–symmetric tensor of rank 2 which satisfies the locally gauge invariant condition,

$$\begin{aligned} F_{\mu\nu} &= F_{\mu\nu}^a T_a = -\frac{i}{g}[D_\mu, D_\nu] \\ &= -\frac{i}{g}[\partial_\mu + igA_\mu, \partial_\nu + igA_\nu] \\ &= -\frac{i}{g}[ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2[A_\mu, A_\nu]] \\ &= (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c)T_c + igA_\mu^a A_\nu^b f_{ab}^c T_c, \end{aligned}$$

where we see that $F_{\mu\nu}$ describes the dynamics of the the gauge fields.

$F_{\mu\nu}$ transforms as

$$\begin{aligned} F'_{\mu\nu} &= -\frac{i}{g}[D'_\mu, D'_\nu] \\ &\to -\frac{i}{g}[UD_\mu U\dagger, UD_\nu U\dagger] \\ &= -\frac{i}{g}U[D_\mu, D_\nu]U\dagger \end{aligned}$$
$$= UF_{\mu\nu}U\dagger.$$

If we wish to to introduce a gauge invariant Lagrange density for the gauge fields, then the Lagrange density must contain a combination of $F_{\mu\nu}$. The simplest Lorentz and gauge invariant term is $\mathrm{T}r(F_{\mu\nu}F^{\mu\nu})$, since

$$\mathrm{T}r(F'_{\mu\nu}F^{\mu\nu\prime}) \to \mathrm{T}r(UF_{\mu\nu}U\dagger UF^{\mu\nu}U\dagger)$$

$$\begin{aligned} &= \mathrm{Tr}(U\dagger U F_{\mu\nu}F^{\mu\nu}) \\ &= \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}). \end{aligned}$$

Thus the required fermion Lagrange density for a non-Abelian gauge theory is

$$\begin{aligned} \mathcal{L} &= \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{2}Tr(F_{\mu\nu}F^{\mu\nu}) \\ &= \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a. \end{aligned} \tag{7.1}$$

The factor of $\frac{1}{4}$ appears in the last line because $Tr(T_aT_b) = \frac{1}{2}\delta_{ab}$.

7.3 Quantization of gauge theories.

We consider a case where there are no fermions in (7.1). In this case we obtain to the Yang Mills Lagrange density, $\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}Tr(F_{\mu\nu}F^{\mu\nu})$. We introduce sources for the gauge fields and write the unnormalized generating functional for the gauge fields as

$$\begin{aligned} W[J] &= \int [dA_\mu] e^{i\int d^4x(-\frac{1}{2}Tr(F_{\mu\nu}F^{\mu\nu})+2TrJ^\mu A\mu)} \\ &= \int [dA^a_\mu] e^{i\int d^4x(-\frac{1}{4}(F^a_{\mu\nu}F^{\mu\nu}_a)+J^\mu_a A^a\mu)}. \end{aligned}$$

The non quadratic terms contained in $F_{\mu\nu}F^{\mu\nu}$ can be extracted from the action by means of differential operators with respect to the sources, like the interaction term of previous chapters. We are then left with a Gaussian integral,

$$\begin{aligned} W[J] &\propto \int [dA^a_\mu] e^{i\int d^4x[-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)+J^\mu_a A^a_\mu]} \\ &= \int [dA^a_\mu] e^{\frac{i}{2}\int d^4x[A^a_\nu\partial_\mu\partial^\mu A^\nu_a + A^a_\nu\partial_\mu\partial^\nu A^\mu_a + J^\mu_a A^a_\mu]} \\ &= \int [dA^a_\mu] e^{\frac{i}{2}\int d^4x[A^\alpha_a(\partial_\mu\partial^\mu g_{\alpha\beta} - g_{\alpha\gamma}\partial_\beta\partial^\gamma)A^\beta_a + J^\mu_a A^a_\mu]} \\ &= \int [dA^a_\mu] e^{\frac{i}{2}\int d^4x[A^\alpha_a(\Box g_{\alpha\beta} - \partial_\alpha\partial_\beta)A^\beta_a + J^\mu_a A^a_\mu]} \\ &= \int [dA^a_\mu] e^{\frac{i}{2}\int d^4x\int d^4x'[A^\alpha_a(x)K_{\alpha\beta}A^\beta_a(x') + J^\mu_a A^a_\mu]}, \end{aligned}$$

where $K_{\alpha\beta} = (\Box g_{\alpha\beta} - \partial_\alpha\partial_\beta)\delta(x - x')$.

Unfortunately this Gaussian integral does not exist as the matrix K does not have an inverse since it is a projection operator. The origin of this problem is that we are integrating over unphysical fields. The Lagrange density is by construction gauge invariant, so it does not depend on some of the gauge fields components. Integration over all the gauge fields thus leads to a singularity. We solve this problem by first integrating out the unphysical degrees of freedom. This is done by gauge fixing with the help of the Faddeev-Popov procedure.

7.4 The Faddeev-Popov procedure

We shall illustrate the Faddeev-Popov (FP) procedure using functions.

7.4.0.5 Example with functions.

Suppose we have an integral $\int_{R^2} d^2x e^{f(x^2)}$, where $f(x)$ is a function of distance only, and not the angle θ.

We can transform to polar coordinates and then integrate out the unphysical degree of freedom, θ. Thus

$$\int_{R^2} d^2x e^{f(x^2)} = \int rdr \int d\theta e^{f(r^2)} = 2\pi \int_0^\infty rdre^{f(r^2)}.$$

We know the transformation to polar coordinates and can thus integrate out the θ degree of freedom. However, we usually have a manifold $M \subset R^n$ and we do not know how to transform from R^n to M. We need to have a procedure that will allow us to integrate out the unphysical degrees of freedom without explicitly knowing the exact transformation necessary to do so.

Choose $g(x)$ so that $g(x(\theta)) = 0$ has only one solution, say θ_0, where $x(\theta)$ is obtained by rotating x through an angle θ. Thus

$$\int_0^{2\pi} d\theta \delta(g(\theta)) = \frac{1}{|\frac{dg}{d\theta}|}\Big|_{\theta=\theta_0}.$$

This implies that

$$\Big|\frac{dg}{d\theta}\Big|_{\theta=\theta_0} \int_0^{2\pi} d\theta \delta(g(\theta)) = 1 \tag{7.2}$$

If we have the integral $\int_{R^2} d^2x e^{f(x^2)}$, then we can insert (7.2) to obtain

$$\begin{aligned}
\int_{R^2} d^2x e^{f(x^2)} &= \int_{R^2} d^2x \Big|\frac{dg}{d\theta}\Big|_{\theta=\theta_0} \int_0^{2\pi} d\theta \delta(g(x(\theta))) e^{f(x^2)} \\
&= \int_{R'^2} d^2x' \Big|\frac{dg}{d\theta}\Big|_{\theta=\theta_0}(x') \int_0^{2\pi} d\theta \delta(g(x')) e^{f(x'^2)} \\
&= \int_0^{2\pi} d\theta \int_{R'^2} d^2x' \Big|\frac{dg}{d\theta}\Big|_{\theta=\theta_0}(x') \delta(g(x')) e^{f(x'^2)},
\end{aligned}$$

where we have changed from the x coordinates to the x' coordinates by setting $x' = x(\theta)$.

Since $\delta(g(x'))$ is independent of θ, we can integrate out the θ degree of freedom to give

$$\int_{R^2} d^2x e^{f(x^2)} \propto 2\pi \int_{R'^2} \underbrace{d^2x' \left|\frac{dg}{d\theta}\right|_{\theta=\theta_0} (x')\delta(g(x'))}_{measure} e^{f(x'^2)}.$$

We have managed to integrate out the θ degree of freedom without knowing what transformation is needed. However, we had to introduce a more complex measure.

7.4.1 The Gribov problem.

In the derivative of the F.P. procedure, it was assumed that $g(x(\theta)) = 0$ has only one solution. There is however, *no* continuous and single valued function of x with this property.

7.4.1.1 Proof:

We assume there is a continuous and single valued function, $h(\theta) = f(x(\theta))$, then $h(0) = h(2\pi)$. If we plot any function $h(\theta)$ with the above conditions, it is easy to see that the function must cut the θ axis at least twice. This implies that there is no continuous and single valued function with only one zero point.

This fundamental error in the F.P. procedure is bypassed by using perturbation theory to expand around one of the zero points. If the perturbation is small enough (i.e. the convergence radius is small) then the other zero points can be ignored in the procedure. Thus the application of the F.P. procedure means that we must work in the perturbative regime.

7.5 FP procedure in gauge field theory

We can now implement the Faddeev-Popov procedure in the Yang Mills theory.

We return to the Yang Mills Lagrange density given by $\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a$ and the generating functional $W[0] = \int [dA^a_\mu] e^{i\int d^4x(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a)}$, where we integrate over all gauge fields A^a_μ, and thus all equivalent gauge configurations. As in the example above, we want to transform to "polar coordinates" involving integration over the gauge group and non-equivalent gauge configurations $\tilde{A}_\mu$, so that the measure becomes $[d\mu[U]][d\tilde{A}^a_\mu]$, where $[d\mu[U]]$ is a left invariant measure. We know $[d\mu[U]]$, but do not know $d\tilde{A}^a_\mu$.

Assume that a gauge condition[21] $G^a[{}^UA] = 0$ exists with only one solution, $U = 1$. Then

[21] We use the notation ${}^UA$ to denote the gauge fields that have been transformed by U, thus ${}^UA \equiv U^\dagger AU$.

we can write in analogy with (7.2), the identity

$$\Delta_{\mathrm{FP}}[A] \int [d\mu[U]] \delta(G[\,^{U}A]) \propto 1, \tag{7.3}$$

where $\Delta_{\mathrm{FP}}[A] \equiv \left|\frac{\delta G}{\delta U}\right|_{U=1}$ is called the Faddeev-Popov determinant and $\delta(G) = \prod_a \delta(G^a)$. The Faddeev-Popov determinant is gauge invariant.

7.5.0.2 Proof:

For a fixed gauge transformation U_0, the gauge fields transform as $A \to \,^{U_0}A$. Using (7.3) we have

$$\Delta_{\mathrm{FP}}[^{U_0}A] \int [d\mu(U)] \delta(G[\,^{U}(^{U_0}A)]) \propto 1.$$

But

$$\begin{aligned}
^{U}(^{U_0}A) &= \,^{U}(U_0 A U_0^{-1} + \frac{i}{g}(\partial_\mu U_0)U_0^{-1}) \\
&= U(U_0 A U_0^{-1} + \frac{i}{g}(\partial_\mu U_0)U_0^{-1})U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1} \\
&= (UU_0)A(UU_0)^{-1} + \frac{i}{g}(\partial_\mu(UU_0))(UU_0)^{-1} \\
&= \,^{UU_0}A.
\end{aligned}$$

Thus

$$\begin{aligned}
& \Delta_{\mathrm{FP}}[^{U_0}A] \int [d\mu(U)] \delta(G[\,^{UU_0}A)]) \\
= \; & \Delta_{\mathrm{FP}}[^{U_0}A] \int [d\mu(U'U_0^{-1})] \delta(G[\,^{U'}A)]) \\
= \; & \Delta_{\mathrm{FP}}[^{U_0}A] \int [d\mu(U')].\delta(G[\,^{U'}A)]) \propto 1
\end{aligned}$$

because a left invariant measure has the property $d\mu(g_0 \cdot g) = d\mu(g)$.

$$\Rightarrow \Delta_{\mathrm{FP}}[\,^{U_0}A] = \Delta_{\mathrm{FP}}[A].$$

The last equality shows that Δ_{FP} is gauge invariant.

We can insert (7.3) into the sourceless generating functional to obtain

$$\begin{aligned} W[0] &\propto \int [dA^a_\mu]\Delta_{\mathrm{FP}}[A] \int [d\mu[U]\delta(G(^U A))e^{iS[A]} \\ &\propto \Big[\int [d\mu[U]]\Big[\int [dA'^a_\mu]\Delta_{\mathrm{FP}}[A']\delta(G(A'))e^{iS[A']}\Big] \\ &\propto \int [dA^a_\mu]\Delta_{\mathrm{FP}}[A]\delta(G(A))e^{iS[A]}. \end{aligned}$$

This is obtained because of the gauge invariance of Δ_{FP} and S, and then having integrated out the unphysical degrees of freedom U, where the result is now part of the normalization. We can now introduce source terms, as all the remaining degrees of freedom are physical. Thus we have a generating functional for the gauge fields,

$$W[J] \propto \int [dA^a_\mu]\Delta_{\mathrm{FP}}[A]\delta(G(A))e^{i(S[A]+J^a_\mu A^\mu_a)}. \tag{7.4}$$

We wish to rewrite (7.4) so that it is just an integral of an exponent. Firstly, consider the $\delta(G(A))$ term. Since (7.4) does not depend on G, we can shift $G^a[A]$ with a constant to $G^a(A) = \tilde{G}^a(A) - \lambda^a$, and then integrate with respect to λ^a. Thus

$$\begin{aligned} &\int [d\lambda] e^{\frac{-i}{2\xi}\int d^4x \lambda^a\lambda^a} W[J] \\ \propto &\int [d\lambda][dA^a_\mu]\Delta_{\mathrm{FP}}[A] e^{\frac{-i}{2\xi}\int d^4x\lambda^a\lambda^a}\delta(\tilde{G}^a(A)-\lambda^a)e^{i(S[A]+JA)} \\ \propto &\int [dA^a_\mu]\Delta_{\mathrm{FP}}[A]e^{\frac{-i}{2\xi}\int d^4x \tilde{G}^a(A)\tilde{G}^a(A)}e^{i(S[A]+JA)} \end{aligned}$$

The integral in the first line contains a normalization factor which we can drop. We obtain

$$W[J] \propto \int [dA^a_\mu]\Delta_{\mathrm{FP}}[A]e^{i(S[A]+JA)}$$

where the action, $S[A] = \int d^4x(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{1}{2\xi}\tilde{G}^a(A)\tilde{G}^a(A))$, now includes a gauge fixing term.

We now rewrite the FP determinant in the form of an exponential. We know that $\det M = \int d\bar{\eta}_i d\eta_j e^{\bar{\eta}_i M_{ij}\eta_j}$, where $\bar{\eta}_i, \eta_j$ are Grassmann numbers. We can thus write the FP determinant as

$$\begin{aligned} \Delta_{\mathrm{FP}}[A] &= \Big|\frac{\delta G^a[^{U(\lambda)}A]}{\delta\lambda^b}\Big|_{\lambda=0} \\ &= \det[M^{ab}(x,x')] \end{aligned}$$

$$= \int [d\bar{\eta}_a][d\eta_b] e^{-i\int d^4x d^4x' \bar{\eta}_a(x) M^{ab} \eta_b(x')}.$$

To write the FP determinant as an exponential, we needed to introduce Grassmann fields known as the Faddeev-Popov ghost terms. The generating functional now becomes

$$W[J] \propto \int [dA^a_\mu][d\bar{\eta}^a][d\eta^b] e^{i(S[A]+JA+\bar{\rho}\eta+\bar{\eta}\rho)}, \tag{7.5}$$

where the action has again been altered to include the ghost terms and sources for the ghost terms have been included. Thus

$$\begin{aligned} S &= S_{\mathrm{YM}} + S_{\mathrm{gf}} + S_{\mathrm{gh}} \\ &= \int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \frac{-1}{2\xi} G^a G^a - \bar{\eta}^a M^{ab} \eta^b\right) \end{aligned}$$

7.5.1 Gauge fixing.

We have managed to find a generating functional for gauge fields. Before we can continue, we must choose a suitable gauge. The most commonly used gauge is the Lorentz gauge, $G^a[A_\mu] = \partial^\mu A_\mu$. If we use this gauge, then the gauge fixing part of the action becomes $S_{\mathrm{gf}} = \frac{-1}{2\xi}(\partial^\mu A^a_\mu)(\partial^\mu A^a_\mu)$. We calculate the ghost action, S_{gh}.

$$\begin{aligned} G^a[{}^U A_\mu] &= \partial^\mu [A^a_\mu + \partial_\mu \lambda^a + g f^{abc} \lambda^b A^c_\mu] \\ \Rightarrow M^{pq} &= \frac{\delta G^p[{}^{U(\lambda)}A]}{\delta \lambda^q}\Big|_{\lambda=0} = (\partial_\mu \partial^\mu \delta^{pq} + g f^{pqc} \partial^\mu A^c_\mu) \\ \Rightarrow S_{\mathrm{gh}} &= -\int d^4x \bar{\eta}^a M^{ab} \eta^b \\ &= -\int d^4x \bar{\eta}^a (\partial_\mu \partial^\mu \delta^{ab} + g f^{abc} (\partial^\mu A^c_\mu)) \eta^b \\ &= \int d^4x [(\partial^\mu \bar{\eta}^a)(\partial_\mu \eta^a) - g f^{abc} \bar{\eta}^a \eta^b \partial^\mu A^c_\mu], \end{aligned}$$

after integrating by parts to obtain the last step.

7.6 Feynman rules for the Lorentz gauge.

Now that we have chosen a gauge, we can determine the propagators and the vertex terms for the theory. From the previous section we can read off the Lagrange densities associated with a gauge theory in the Lorentz gauge,

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \tag{7.6}$$

$$\mathcal{L}_{\text{gf}} = \frac{-1}{2\xi}(\partial^\mu A^a_\mu)^2 \tag{7.7}$$
$$\mathcal{L}_{\text{gh}} = (\partial^\mu \bar{\eta}^a)(\partial_\mu \eta^a) - g f^{abc} \bar{\eta}^a \eta^b \partial^\mu A^c_\mu \tag{7.8}$$

7.6.1 Propagators.

7.6.1.1 Gauge propagator.

To calculate the propagator for the gauge fields, A , we need terms that are quadratic in A. Thus

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{gh}} \\
&\approx -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a) - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + J^\mu A^a_\mu \\
&= \frac{1}{2}(A^a_\nu \partial_\mu \partial^\mu g^{\nu\alpha} A^a_\alpha - A^a_\nu \partial^\mu \partial^\nu A^a_\mu + \frac{1}{\xi} A^a_\mu \partial^\mu \partial^\nu A^a_\nu) + J^\mu A^a_\mu \\
&= \frac{1}{2} A^a_\nu M^{\nu\mu}_{ab} A^b_\mu + J^\mu A^a_\mu,
\end{aligned}$$

where $M^{\nu\mu}_{ab} = (\Box g^{\nu\mu} - (1 - \frac{1}{\xi})\partial^\mu \partial^\nu)$. We can complete the Gauss integral to give the free field generator for the gauge fields $W_0[J] \propto e^{-\frac{1}{2}JDJ}$, where the propagator is $D \equiv iM^{-1}$.

We must determine M^{-1} for

$$M^{\mu\nu} = (\Box g^{\nu\mu} - (1 - \frac{1}{\xi})\partial^\mu \partial^\nu)\delta(x - x') = -\int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')}[p^2 g^{\mu\nu} - (1 - \frac{1}{\xi})p^\mu p^\nu].$$

If we assume that the inverse has the form $Ag^{\mu\nu} + Bp^\mu p^\nu$, then we can solve for A and B from the definition of the inverse $M^{\mu\nu} M^{-1}_{\nu\lambda} \equiv \delta^\mu_\lambda$. The solution is $A = \frac{1}{p^2}$ and $B = \frac{(\xi-1)}{p^4}$. Thus the inverse is

$$(M^{-1})^{\mu\nu}(x, x') = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')}[\frac{g^{\mu\nu}}{p^2} - \frac{(1-\xi)}{p^4} p^\mu p^\nu].$$

If we go through the analytic continuation as before (the ϵ prescription) we obtain the propagator (in coordinate and momentum space),

$$\begin{aligned}
D^{\mu\nu}(x) &= i\int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')}[-g^{\mu\nu} + \frac{(1-\xi)}{p^2} p^\mu p^\nu]\frac{1}{p^2 + i\epsilon} \\
D^{\mu\nu}(p) &= \frac{i}{p^2 + i\epsilon}[-g^{\mu\nu} + \frac{(1-\xi)}{p^2} p^\mu p^\nu] \qquad (7.9)
\end{aligned}$$

Note, that if there is no gauge fixing then $\xi \to \infty$ and the propagator does not exist.

If we choose $\xi = 1$, then we obtain the Feynman gauge propagator

$$D^{\mu\nu}(p) = \frac{-ig^{\mu\nu}}{p^2 + i\epsilon}.$$

7.6.1.2 Ghost propagator.

To calculate the ghost propagator, we need to complete the Gaussian integral,

$$\begin{aligned} W_0[\bar{\rho}, \rho] &\propto \int [d\bar{\eta}][d\eta] e^{-i\bar{\eta}M\eta + \bar{\rho}\eta + \bar{\eta}\rho} \\ &\propto e^{\bar{\rho}\Delta_{\text{gh}}\rho}, \end{aligned}$$

where $\Delta_{\text{gh}} \equiv iM^{-1}$ is the ghost propagator. We have

$$M = \partial^\mu \partial_\mu \delta^{ab} + g f^{abc} \partial^\mu A^c_\mu = -\int \frac{d^4p}{(2\pi)^4} p^2 e^{ip(x-x')} \delta^{ab}.$$

and the inverse

$$M^{-1} = \frac{\delta^{ab}}{p^2 + i\epsilon}.$$

Thus the propagator is given by

$$\Delta_{\text{gh}} \equiv \frac{i\delta^{ab}}{p^2 + i\epsilon}. \tag{7.10}$$

The free field generating functional is thus given by

$$W_0[J, \bar{\rho}, \rho] \propto e^{-\frac{1}{2}JDJ - \bar{\rho}\Delta_{\text{gh}}\rho}. \tag{7.11}$$

At this point we can add fermions to the theory by adding the fermion generating functional to obtain

$$W_0[\bar{\sigma}, \sigma, J, \bar{\rho}, \rho] \propto e^{-\frac{1}{2}JDJ - \bar{\sigma}S\sigma - \bar{\rho}\Delta_{\text{gh}}\rho}. \tag{7.12}$$

We can summarize the Feynman rules for the propagators.

- $\alpha, i \quad \beta, j \longrightarrow \quad S_{\alpha\beta} \equiv \frac{i\delta^{ij}}{\not{p} - m + i\epsilon}$, the fermion propagator.
- $\mu, a \quad \nu, b \longrightarrow \quad D^{\mu\nu} \equiv \frac{i}{p^2 + i\epsilon}[-g^{\mu\nu} + \frac{(1-\xi)}{p^2} p^\mu p^\nu]$, the gauge propagator.
- $a \quad b \longrightarrow \quad \Delta_{\text{gh}} \equiv \frac{i\delta^{ab}}{p^2 + i\epsilon}$, the ghost propagator.

7.6.2 Vertex terms.

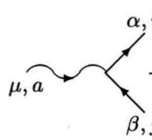
7.6.2.1 Fermion field - Gauge field vertex

The interaction term between fermion fields and gauge fields which is extracted from the action is

$$-igA_\mu^a\bar{\psi}\gamma^\mu T^a\psi = -ig\frac{1}{i^3}\frac{\delta}{\delta J}\frac{\overleftarrow{\delta}}{\delta\sigma}\gamma^\mu T^a\frac{\overrightarrow{\delta}}{\delta\bar{\sigma}}.$$

The contribution to the vertex term can be read off from the three point Greens function to be $(-1)^3\frac{1}{i^3}(-ig)\frac{1}{i^3}\gamma^\mu T^a = -ig\gamma^\mu T^a$. The factor of $(-1)^3$ comes from the above derivatives working on (7.12), and the $\frac{1}{i^3}$ term comes from the derivatives of the sources when calculating the Greens function.

Thus the vertex term is



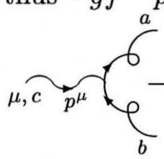
$$\longrightarrow \quad -ig(\gamma^\mu)_{\alpha\beta}(T^a)_{ij}.$$

7.6.2.2 Ghost field - Gauge field vertex

The interaction term between the ghosts and the gauge fields is

$$-igf^{abc}\bar{\eta}^a\eta^b\partial^\mu A_\mu^c = -igf^{abc}\bar{\eta}^a\eta^b(-ip^\mu)A_\mu^c$$

The vertex term is thus $-gf^{abc}p^\mu$, where p^μ is the momentum of the gauge field.



$$\longrightarrow \quad -gf^{abc}p^\mu.$$

7.6.2.3 3 Point Yang Mills vertex

The Yang Mills Lagrange density (7.6) has two non-quadrature terms that form interactions. The first term leads to a three point vertex and is

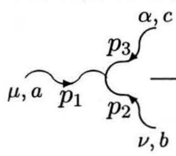
$$\frac{ig}{2}(\partial_\mu A_n u^a - \partial_n u A_m u^a)A^{\mu b}A^{\nu c}f^{abc} = igf^{abc}g_{\nu\alpha}A^{\mu a}A^{\nu b}(\partial_\mu A^{\alpha c})$$

This term is not symmetrical with respect to the indices $(\mu, a); (\nu, b); (\alpha, c)$; and since the Greens function is we must symmetrize over all permutations of these indices before we can read off the vertex term.

$$\begin{aligned} & igf^{abc}g_{\nu\alpha}A^{\mu a}A^{\nu b}(\partial_\mu A^{\alpha c}) \\ = \; & \frac{ig}{3!}\Big[f^{abc}g_{\nu\alpha}A^{\mu a}A^{\nu b}(\partial_\mu A^{\alpha c}) + f^{bac}g_{\mu\alpha}A^{\nu b}A^{\mu a}(\partial_\nu A^{\alpha c}) + \end{aligned}$$

$$
\begin{aligned}
& f^{cba} g_{\nu\mu} A^{\alpha c} A^{\nu b}(\partial_\alpha A^{\mu a}) + f^{acb} g_{\alpha\nu} A^{\mu a} A^{\alpha c}(\partial_\mu A^{\nu b}) + \\
& f^{bca} g_{\alpha\mu} A^{\nu b} A^{\alpha c}(\partial_\nu A^{\mu a}) + f^{cab} g_{\mu\nu} A^{\alpha c} A^{\mu a}(\partial_\alpha A^{\nu b})\Big] \\
= \; & \frac{ig}{3!} f^{abc}\Big[g_{\nu\alpha}\Big(A^{\mu a} A^{\nu b}(\partial_\mu A^{\alpha c}) - A^{\mu a} A^{\alpha c}(\partial_\mu A^{\nu b})\Big) + \\
& g_{\mu\alpha}\Big(A^{\nu b} A^{\alpha c}(\partial_\nu A^{\mu a}) - A^{\nu b} A^{\mu a}(\partial_\nu A^{\alpha c})\Big) + \\
& g_{\mu\nu}\Big(A^{\alpha c} A^{\mu a}(\partial_\alpha A^{\nu b}) - A^{\alpha c} A^{\nu b}(\partial_\alpha A^{\mu a})\Big)\Big]
\end{aligned}
$$

We can now read off the contribution to the vertex, which is



$$\longrightarrow \quad g f^{abc}[g_{\nu\alpha}(p_3 - p_2)_\mu + g_{\mu\alpha}(p_1 - p_3)_\nu + g_{\mu\nu}(p_2 - p_1)_\alpha].$$

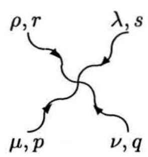
7.6.2.4 4 Point Yang Mills vertex

The second non quadrature term in the Yang Mills Lagrange density leads to a four point vertex and is given by

$$\frac{-ig^2}{4} f^{pqa} f^{rsa} g_{\mu\lambda} g_{\nu\rho} A^{\lambda p} A^{\rho q} A^{\mu r} A^{\nu s}.$$

This term must also be symmetrized over all permutations of (λ, p); (ρ, q); (μ, r) and (ν, s).

The contribution to the four point vertex is then given by



$$
\begin{aligned}
\longrightarrow \quad \frac{-ig^2}{4}\Big[& f^{pqa} f^{rsa}(g_{\mu\lambda} g_{\nu\rho} - g_{\nu\lambda} g_{\mu\rho}) \\
& f^{pra} f^{qsa}(g_{\rho\lambda} g_{\nu\mu} - g_{\nu\lambda} g_{\rho\mu}) \\
& f^{psa} f^{rqa}(g_{\mu\lambda} g_{\rho\nu} - g_{\rho\lambda} g_{\mu\nu})\Big].
\end{aligned}
$$

7.7 Conclusion

Now that we have the Feynman rules for the gauge fields, we can apply all the previously obtained results to renormalize the theory. Care must be taken when regularizing not to break the gauge symmetry. After renomalization, the S-matrix elements can be calculated.

Field theories like Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD) can be introduced at this point. The results obtained from these theories, using perturbation expansions are usually very good. The calculation of the Lamb shift and the anomalous magnetic moment of the electron with the aid of QED are two examples of the excellent results that are obtained.

Further detailed discussions of these theories and others, which is beyond the scope of this work, may be found in advanced field theory texts.

APPENDIX A
Free particle Equations

A.1 The free particle Klein-Gordon equation.

The Klein-Gordon equation for a free particle is given by

$$(\Box + m^2)\phi(x) = 0, \tag{A.1}$$

where $\phi(x)$ is a generalized field. Solutions of this equation can be obtained by separation of variables. Using periodic boundary conditions, the solutions are

$$\phi_n^{\pm}(x) = Ne^{i(\vec{p}_n\cdot\vec{x}\mp E_n t)}$$

where the energy $E_n = \sqrt{m^2 + p_n^2}$ is always positive. Here ϕ^+ is the positive energy solution and ϕ^- is the negative energy solution.

The norm for both of these solutions is

$$\int d^3r \phi_n^{(\pm)*} \frac{\overleftrightarrow{\partial}}{\partial t} \phi_m^{(\pm)} = \pm 2E_n V N^2 \delta_{nm}.$$

If we wish the norm to be unitary then we need to choose the normalization as $N = (2E_nV)^{-\frac{1}{2}}$.

The orthogonal relations that hold are

$$\begin{aligned}
\int d^3r\ \phi_n^{+*}(x) \frac{\overleftrightarrow{\partial}}{\partial t} \phi_{n'}^{+}(x) &= \delta_{nn'}, \\
\int d^3r\ \phi_n^{-*}(x) \frac{\overleftrightarrow{\partial}}{\partial t} \phi_{n'}^{-}(x) &= -\delta_{nn'}, \\
\int d^3r\ \phi_n^{+*}(x) \frac{\overleftrightarrow{\partial}}{\partial t} \phi_{n'}^{-}(x) &= 0,
\end{aligned} \tag{A.2}$$

The normalized orthogonal solutions are then

$$\phi_n^{\pm}(x) = \frac{1}{\sqrt{2E_nV}} e^{i(\vec{p}_n\cdot\vec{x}\mp E_n t)}$$

where the positive energy solutions have a norm $+1$ and the negative energy solutions have a norm -1.

The most general solution of the Klein-Gordon equation is give by a linear combination of the positive and negative energy solutions,

$$\begin{aligned} \phi(x) &= \sum_n (a_n\phi_n^+(x) + c_n^\dagger\phi_{-n}^-(x)) \\ \phi^\dagger(x) &= \sum_n (a_n^\dagger\phi_n^{+*}(x) + c_n\phi_{-n}^{-*}(x)), \end{aligned} \tag{A.3}$$

where a_n, c_n are complex expansion coefficients.

A.2 The Dirac equation for a free particle.

The Dirac equation for the free particle is given by

$$(i\gamma^\mu p_\mu - m)\psi = 0. \tag{A.4}$$

We write $\psi(x) = Nwe^{-px}$ with w a four column vector and substitute this into (A.4) to obtain

$$p_0\gamma^0 w = -(p_i\gamma^i + m)w.$$

If we write $w = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ with ϕ, χ as two column vectors, we have

$$p_0\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} mI & \vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -mI \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix}. \tag{A.5}$$

Multiplying by $\begin{pmatrix} mI & \vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -mI \end{pmatrix}$ and noting that $(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p}) = p^2$ we see that

$$\begin{aligned} p_0^2\begin{pmatrix} \phi \\ \chi \end{pmatrix} &= p_0\begin{pmatrix} mI & \vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -mI \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} \\ &= \begin{pmatrix} mI & \vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -mI \end{pmatrix}\begin{pmatrix} mI & \vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -mI \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} \\ &= \begin{pmatrix} m^2+\vec{p}^2 & 0 \\ 0 & m^2+\vec{p}^2 \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix}. \end{aligned}$$

This implies that the eigenvalue solutions are

$$p_0 = \pm\sqrt{m^2+\vec{p}^2} \equiv \pm E.$$

We now solve the eigenstate equation to obtain ϕ and χ.

A.2.1 Positive energy solutions. (Particles)

We take $\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \phi \\ \alpha\vec{\sigma}\cdot\vec{p}\phi \end{pmatrix}$.

Substituting this into (A.5) and solving for $E(p_0)$ and α we obtain $\alpha = \frac{1}{E+m}$. We thus obtain two independent positive energy eigensolutions

$$w_+(\vec{p}, s) = \begin{pmatrix} \phi^s \\ \frac{1}{E+m}\vec{p}\cdot\vec{\sigma}\phi^s \end{pmatrix}$$

where ϕ^s is a two component spinor with $\phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The complete positive energy solutions for Dirac equation are

$$\begin{aligned} \psi^+_{p,s}(x) &= w_+(\vec{p}, s)e^{-ipx} \\ &= \begin{pmatrix} \phi^s \\ \frac{1}{E+m}\vec{p}\cdot\vec{\sigma}\phi^s \end{pmatrix} e^{-i(Ex_0 - \vec{p}\cdot\vec{x})} \end{aligned}$$

with $E = p_0 = \sqrt{\vec{p}^2 + m^2}$.

Choosing $\phi^\dagger\phi = 1$ and normalizing the states to unity determines the normalization constant N,

$$\begin{aligned} \int d^3r\psi^{+\dagger}\psi^+ &= N^2\phi^\dagger\Big[1 + \frac{p^2}{(E+m)^2}\Big]\phi V \\ &= N^2V\frac{2E}{E+m} \\ &\Rightarrow N = \frac{\sqrt{E+m}}{\sqrt{2EV}}. \end{aligned}$$

It is customary to write the positive energy solution in terms of the positive energy Dirac spinor $u(\vec{p}, s)$ which is defined as $u(\vec{p}, s) = \sqrt{E+m}\begin{pmatrix} 1 \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} \end{pmatrix}\phi^s$.

The positive energy solution is then given by

$$\psi^+_{p,s} = \frac{1}{\sqrt{2EV}}u(\vec{p}, s)e^{-ipx}.$$

A.2.2 Negative energy solutions. (Anti-particles)

We take

$$\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \alpha\vec{\sigma}\cdot\vec{p}\chi \\ \chi \end{pmatrix},$$

and substitute this into (A.5) to obtain $\alpha = -\frac{1}{E+m}$. The two independent negative energy eigenstates are

$$w_-(\vec{p},s) = \begin{pmatrix} -\frac{1}{E+m}\vec{p}\cdot\vec{\sigma}\chi^s \\ \chi^s \end{pmatrix}$$

where χ^s is a two component spinor with $\chi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\chi^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The complete negative energy solution (with the same normalization as before) is

$$\begin{aligned} \psi^-_{p,s}(x) &= w_-(\vec{p},s)e^{-ipx} \\ &= \frac{\sqrt{E+m}}{\sqrt{2EV}}\begin{pmatrix} -\frac{\vec{p}\cdot\vec{\sigma}}{E+m} \\ 1 \end{pmatrix}\chi^s e^{-i(-Ex_0-\vec{p}\cdot\vec{x})} \end{aligned}$$

If we change the sign of $\vec{p}$ to $-\vec{p}$, then the plane wave associated with the negative energy solution is e^{ipx}, where we now have $p_0 = E > 0$. We can interpret the negative energy solution as the complex conjugate of the positive energy solution with $p_0 > 0$. Thus

$$\begin{aligned} \psi^-_{-p,s}(x) &= \frac{\sqrt{E+m}}{\sqrt{2EV}}\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m} \\ 1 \end{pmatrix}\chi^s e^{ipx} \\ &= \frac{1}{\sqrt{2EV}}v(\vec{p},s)e^{ipx} \end{aligned}$$

where $v(\vec{p},s) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m} \\ 1 \end{pmatrix}\chi^s$ is the negative energy Dirac spinor.

A.2.3 Orthogonality.

The orthogonal relations that hold between the Dirac spinors are

$$u^\dagger(\vec{p},s)u(\vec{p},s') = 2E\delta_{ss'}$$

$$v^\dagger(\vec{p},s)v(\vec{p},s') = 2E\delta_{ss'}$$

$$v^\dagger(-\vec{p},s)u(\vec{p},s') = 0 = u^\dagger(\vec{p},s)v(-\vec{p},s')$$

Therefore at time $t = t'$ the following orthogonal relations hold for the solutions:

$$\int d^3r\ \psi^{+\dagger}_{p,s}(x)\psi^{+}_{p',s'}(x) = 2(2\pi)^3\sqrt{\vec{p}^2+m^2}\delta(\vec{p}-\vec{p}\,')\delta_{ss'},$$
$$\int d^3r\ \psi^{-\dagger}_{p,s}(x)\psi^{-}_{p',s'}(x) = 2(2\pi)^3\sqrt{\vec{p}^2+m^2}\delta(\vec{p}-\vec{p}\,')\delta_{ss'},$$
$$\int d^3r\ \psi^{+\dagger}_{p,s}(x)\psi^{-}_{p',s'}(x) = 0 \tag{A.6}$$

A.2.4 General Solution

The general solution to the Dirac equation is given by a linear combination of the positive and negative energy solutions,

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\sum_s \frac{1}{2p_0}[b_{p,s}\psi^{+}_{p,s}(x) + d^{*}_{p,s}\psi^{-}_{p,s}], \tag{A.7}$$
$$\bar{\psi}(x) = \psi^{\dagger}(x)\gamma^0 \tag{A.8}$$
$$= \int \frac{d^3p}{(2\pi)^3}\sum_s \frac{1}{2p_0}[b^{*}_{p,s}\bar{\psi}^{+}_{p,s}(x) + d_{p,s}\bar{\psi}^{-}_{p,s}]. \tag{A.9}$$

APPENDIX B
Mathematical Formalisms

B.1 Functional Derivatives

The derivative of the functional[22] $F[f(x)]$ with respect to the function $f(y)$ is defined (in analogy with ordinary differentiation) as

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\epsilon \to 0} \frac{F[f(x) + \epsilon\delta(x-y)] - F[f(x)]}{\epsilon}.$$

Consider the functional $F[f] = \int dx f(x)$. Then

$$\begin{aligned} \frac{\delta F[f]}{\delta f(y)} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big[\int (f(x) + \epsilon\delta(x-y))dx - \int f(x)dx\Big] \\ &= \int \delta(x-y)dx = 1. \end{aligned}$$

As another example, consider $F[f] = \int G(x,y)f(y)\, dy$. Then

$$\begin{aligned} \frac{\delta F[f]}{\delta f(z)} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big[\int G(x,y)(f(y) + \epsilon\delta(y-z))\, dy - \int G(x,y)f(y)\, dy\Big] \\ &= \int G(x,y)\delta(y-z)\, dy = G(x,z). \end{aligned}$$

In the general case, the functional derivative of a functional of the form

$$F[f] = \int dt_1 K_1(t_1)f(t_1) + \frac{1}{2!}\int dt_1 dt_2 K_2(t_1,t_2)f(t_1)f(t_2) + \ldots,$$

with $K_n(t_1 \ldots t_n)$ completely symmetrical in $t_1, \ldots, t_n$; is given by

$$\frac{\delta F[f]}{\delta f(t)} = K_1(t) + \int dt_1 K_2(t_1,t)f(t_1) + \ldots$$

Higher order derivatives may be obtained by taking the functional derivative of lower order functional derivatives.

B.2 Grassmann Algebras.

Consider a set of n real anti-commuting elements $\{\chi_i\}_{i=1}^n$ with $[\chi_i, \chi_j]_+ = 0$. This commutation relation implies a nilpotency condition, $\chi_i^2 = 0$.

[22] A functional $F[f]$ is a rule for going from a function to a number. Thus *functional* : *function* $\to$ *number*.

A Grassmann algebra is formed by the vector space

$$\mathcal{G} = \text{span}\{1, \chi_i, \chi_i\chi_j, \ldots, \chi_1\cdots\chi_n\},$$

with the elements χ_i being the generators of the algebra. No higher orders of these bases occur due to the nilpotency requirement on the generators. An element of this algebra $\mathcal{G}$ is called a Grassmann number.The most general form of a Grassmann number is

$$\begin{aligned} g &= a^0 + a^i\chi_i + a^{ij}\chi_i\chi_j + \ldots + a^{1\ldots n}\chi_1\cdots\chi_n \\ &= \sum_k \sum_{\alpha_1\cdots\alpha_k} a^{\alpha_1\cdots\alpha_k}\chi_{\alpha_1}\cdots\chi_{\alpha_k}. \end{aligned}$$

We can introduce complex Grassmann algebras, by forming a set of complex generators $\{\eta_i\}_{i=1}^n$ from two sets of real generators $\{\chi_i, \zeta_i\}_{i=1}^n$ with

$$\begin{aligned} \eta_i &= \chi_i + i\zeta_i \\ \eta_i^* &= \chi_i - i\zeta_i. \end{aligned}$$

Since $\{\eta_i\}_{i=1}^n$ satisfies the anti-commutator relations, we can form a complex Grassmann algebra

$$\mathcal{G} = \text{span}\{1, \eta_i, \eta_i\eta_j, \ldots, \eta_1\cdots\eta_n\},$$

with a general complex Grassmann number given by

$$\begin{aligned} g &= a^0 + a^i\eta_i + a^{ij}\eta_i\eta_j + \ldots + a^{1\ldots n}\eta_1\cdots\eta_n \\ &= \sum_k \sum_{\alpha_1\cdots\alpha_k} a^{\alpha_1\cdots\alpha_k}\eta_{\alpha_1}\cdots\eta_{\alpha_k}. \end{aligned}$$

The complex conjugate of the product of two complex Grassmann variables is defined as

$$(\eta_i\eta_j)^* \equiv \eta_j^*\eta_i^* = -\eta_i^*\eta_j^*.$$

B.2.1 Grading of Grassmann numbers.

We introduce even and odd Grassmann numbers. An even Grassmann number is given by

$$g = a^0 + a^{ij}\eta_i\eta_j + a^{ijkl}\eta_i\eta_j\eta_k\eta_l + \ldots$$

and an odd Grassmann number is given by

$$g = a^i \eta_i + a^{ijk} \eta_i \eta_j \eta_k + \dots$$

Consider

$$\begin{aligned}
& (\eta_{i_1} \dots \eta_{i_p})(\eta_{j_1} \dots \eta_{j_q}) \\
= \; & (-1)^p \eta_{j_1} (\eta_{i_1} \dots \eta_{i_p})(\eta_{j_2} \dots \eta_{j_q}) \\
= \; & (-1)^{pq} (\eta_{j_1} \dots \eta_{j_q})(\eta_{i_1} \dots \eta_{i_p}) \\
\Rightarrow \; & (\eta_{i_1} \dots \eta_{i_p})(\eta_{j_1} \dots \eta_{j_q}) - (-1)^{pq} (\eta_{j_1} \dots \eta_{j_q})(\eta_{i_1} \dots \eta_{i_p}) = 0
\end{aligned}$$

Thus even Grassmann numbers commute with any other Grassmann number and odd Grassmann numbers anti-commute only with other odd Grassmann numbers. We thus introduce the parity of a Grassmann number A,

$$\begin{aligned}
\epsilon_A &= 0 \text{ if } A \text{ is even} \\
&= 1 \text{ if } A \text{ is odd}
\end{aligned}$$

and see that Grassmann numbers satisfy graded commutation relations

$$AB - (-1)^{\epsilon_A \epsilon_B} BA = 0.$$

Note that $A^2 = 0$ only if A is odd. This is a generalization of the nilpotency requirement given above.

B.3 Calculus of a complex Grassmann algebra.

B.3.1 Differentiation.

With normal complex numbers, we differentiate

$$\begin{aligned}
& \frac{d}{dz_i}(z_1 \dots z_i \dots z_n) \\
= \; & \frac{d}{dz_i} z_i (z_1 \dots z_{i-1} z_{i+1} \dots z_n) = \frac{d}{dz_i}(z_1 \dots z_{i-1} z_{i+1} \dots z_n) z_i \\
= \; & (z_1 \dots z_{i-1} z_{i+1} \dots z_n).
\end{aligned}$$

We follow the same method when we differentiate Grassmann variables, but we must be careful of the ordering. Since the Grassmann variables anti-commute (they are odd), we obtain two different results if we ordered the variable being differentiated on the left or the right. This leads to the concept of a left and a right derivative.

Left Derivative

$$\begin{aligned}(\eta_1 \dots \eta_i \dots \eta_n)\frac{\overleftarrow{\partial}}{\partial \eta_i} &= (-1)^{n-i-1}(\eta_1 \dots \eta_{i-1}\eta_{i+1} \dots \eta_n)\eta_i \frac{\overleftarrow{\partial}}{\partial \eta_i} \\ &= (-1)^{n-i-1}(\eta_1 \dots \eta_{i-1}\eta_{i+1} \dots \eta_n).\end{aligned}$$

Right Derivative

$$\begin{aligned}\frac{\overrightarrow{\partial}}{\partial \eta_i}(\eta_1 \dots \eta_i \dots \eta_n) &= \frac{\overrightarrow{\partial}}{\partial \eta_i}(-1)^{i-1}\eta_i(\eta_1 \dots \eta_{i-1}\eta_{i+1} \dots \eta_n) \\ &= (-1)^{i-1}(\eta_1 \dots \eta_{i-1}\eta_{i+1} \dots \eta_n).\end{aligned}$$

The left and right derivatives differ by a sign factor.

We can introduce the product rule (for right derivatives)

$$\frac{\overrightarrow{\partial}}{\partial \eta_i}(AB) = (\frac{\overrightarrow{\partial}}{\partial \eta_i}A)B + (-1)^{\epsilon_A}A(\frac{\overrightarrow{\partial}}{\partial \eta_i})$$

where A, B are complex Grassmann variables.

B.3.2 Integration.

We introduce integration on a Grassmann algebra by defining the Berezin integral

$$\begin{aligned}\int d\eta_i &\equiv 0 \\ \int d\eta_i \eta_j &= \delta_{ij}.\end{aligned}$$

We can also introduce multiple integral such as

$$\int d\eta_n \dots d\eta_1 g(\eta) = n! a^{1\dots n}.$$

Integration and differentiation give the same results, thus integration can be expressed in terms of differentiation

$$\int d\eta_1 \dots d\eta_n \; g(\eta) \equiv \frac{\partial^n g}{\partial \eta_1 \dots \partial \eta_n}.$$

B.3.3 Change of variables.

In the Grassmann algebra, we can replace the generators $\eta_1, \ldots, \eta_n$ with a new set of generators $\bar{\eta}_1, \ldots, \bar{\eta}_n$ where $\bar{\eta}_i = S_{ij}\eta_j$.

Thus $\int d\eta_1 \ldots d\eta_n F(\eta) = \int d\bar{\eta}_1 \ldots d\bar{\eta}_n \Delta F(\eta(\bar{\eta}))$ where Δ is a factor similar to the Jacobian that appears for transformation of normal variables. This can be written in differential form as

$$\frac{\vec{\partial}}{\partial \eta_1} \cdots \frac{\vec{\partial}}{\partial \eta_n} F(\eta) = \frac{\vec{\partial}}{\partial \bar{\eta}_1} \cdots \frac{\vec{\partial}}{\partial \bar{\eta}_n} [\Delta F(\eta(\bar{\eta}))].$$

Using the chain rule $\frac{\vec{\partial} F(\bar{\eta})}{\partial \bar{\eta}_i} = \sum_k \frac{\vec{\partial} \eta_k}{\partial \bar{\eta}_i} \frac{\vec{\partial} F(\eta(\bar{\eta}))}{\partial \eta_k}$ we see that

$$\begin{aligned}
& \frac{\vec{\partial}}{\partial \bar{\eta}_1} \cdots \frac{\vec{\partial}}{\partial \bar{\eta}_n} F(\eta(\bar{\eta})) \\
= & \sum_{k_1 \ldots k_n} \frac{\vec{\partial}\, \eta_{k_1}}{\partial \bar{\eta}_1} \cdots \frac{\vec{\partial}\, \eta_{k_n}}{\partial \bar{\eta}_n} \frac{\vec{\partial}}{\partial \eta_{k_1}} \cdots \frac{\vec{\partial}}{\partial \eta_{k_n}} F(\eta) \\
= & \sum_{k_1 \ldots k_n} S^{-1}_{k_1,1} \cdots S^{-1}_{k_n,n} \frac{\vec{\partial}}{\partial \eta_{k_1}} \cdots \frac{\vec{\partial}}{\partial \eta_{k_n}} F(\eta) \\
= & \sum_{\pi} \epsilon_\pi S^{-1}_{\pi(1),1} \cdots S^{-1}_{\pi(n),n} \frac{\vec{\partial}}{\partial \eta_1} \cdots \frac{\vec{\partial}}{\partial \eta_n} F(\eta) \\
& \text{(where we sum over all permutations of } k_1 \ldots k_n) \\
= & (\det S)^{-1} \frac{\vec{\partial}}{\partial \eta_1} \cdots \frac{\vec{\partial}}{\partial \eta_n} F(\eta)
\end{aligned}$$

$$\Rightarrow \quad \frac{\vec{\partial}}{\partial \eta_1} \cdots \frac{\vec{\partial}}{\partial \eta_n} F(\eta) = \det S \frac{\vec{\partial}}{\partial \bar{\eta}_1} \cdots \frac{\vec{\partial}}{\partial \bar{\eta}_n} F(\eta(\bar{\eta})).$$

We see that Δ is the determinant of the transformation matrix S.

The change of variables in integral form is thus

$$\int d\eta_1 \ldots d\eta_n F(\eta) = \det S \int d\bar{\eta}_1 \ldots \bar{\eta}_n F(\eta(\bar{\eta})).$$

B.3.4 Translation invariance.

If we have a Grassmann function $F(\eta)$ which we can decompose as $F(\eta) = f_1 + \eta f_2$, where f_1, f_2 are independent of η, then

$$\int d\eta F(\eta) = \int d\eta (f_1 + \eta f_2) = f_2$$

If we translate η by η_0 then

$$\int d\eta F(\eta+\eta_0) = \int d\eta(f_1 + (\eta+\eta_0)f_2) = f_2$$
$$\Rightarrow \int d\eta F(\eta) = \int d\eta F(\eta+\eta_0).$$

The integral over Grassmann variables is thus invariant under translations.

B.3.5 Partial Integration.

We can now derive partial integration using the chain rule. Thus

$$\begin{aligned}\int d\eta \frac{\vec{\partial}}{\partial\eta}(f)g &= \int d\eta[\frac{\vec{\partial}}{\partial\eta}(fg) - (-1)^{\epsilon_f} f\frac{\vec{\partial}\, g}{\partial\eta}] \\ &= -(-1)^{\epsilon_f}\int d\eta f\frac{\vec{\partial}\, g}{\partial\eta},\end{aligned}$$

because $\frac{\vec{\partial}(fg)}{\partial\eta}$ is independent of η, as η can only occur once in fg due to the nilpotency requirement on Grassmann numbers.

Thus, we see that partial integration is given by

$$\int d\eta \frac{\vec{\partial}}{\partial\eta}(f)g = -(-1)^{\epsilon_f}\int d\eta f\frac{\vec{\partial}\, g}{\partial\eta}.$$

B.4 Gaussian integrals.

We wish to calculate many variable Gaussian integrals of the form

$$I = \int dx_1\dots dx_n e^{-\frac{1}{2}x^T M x}$$

where M is a $n\times n$ matrix.

We diagonalize M by setting $M = R^T D R$, where R is an orthogonal $n\times n$ matrix.Thus we have

$$I = \int dx_1\dots dx_n e^{-\frac{1}{2}x^T R^T M R x}.$$

We change variables to $x' = Rx$ so that

$$I = \int dx'_1\dots dx'_n J e^{-\frac{1}{2}x'^T D x'}.$$

with the Jacobian given by $J = |\frac{\partial(x_1\dots x_n)}{\partial(x'_1\dots x'_n)}| = \det R^T$.

Since R is orthogonal, $R^T R = 1$ and $\det R^T = \det R$ thus $\det R^T R = \det R^T \det R = (\det R^T)^2 = 1$. This implies that $\det R^T = \pm 1$, and as the value of the volume integral must be positive, we choose $\det R^T = 1$.

This gives

$$\begin{aligned} I &= \int dx'_1 \dots dx'_n e^{-\frac{1}{2}x'^T D x'} \\ &= \prod_{i=1}^{n} \int dx'_i e^{-\frac{1}{2}d_i x_i^{2'}}. \end{aligned}$$

with d_i the diagonal elements of D. The many variable Gaussian integral reduces to the products of single variable Gaussian integrals with the solution

$$\int dy e^{-\frac{1}{2}ay^2} = \sqrt{\frac{2\pi}{a}}.$$

We now have

$$I = \prod_{i=1}^{n} \sqrt{\frac{2\pi}{d_i}} = \frac{(2\pi)^{n/2}}{\sqrt{\prod_i d_i}}$$

but $\prod_i d_i = \det D = \det(RMR^T) = \det M$, thus the final solution of the many body Gaussian integral is

$$I = \frac{(2\pi)^{n/2}}{(\det M)^{\frac{1}{2}}} \propto \frac{1}{(\det M)^{\frac{1}{2}}}.$$

If we have complex variables then (by the same arguments as above) the Gaussian integral is given by

$$\int dx_1 \dots dx_n dx_1^* \dots dx_n^* e^{-\frac{1}{2}x^\dagger M x} \propto \frac{1}{(\det M)}.$$

B.5 Grassmann Gaussian integrals

We now have a Gaussian integral where we the variables are Grassmann variables. Consider

$$\int \prod_k d\eta_k d\eta_k^* e^{\eta_i^* M_{ij} \eta_j}.$$

This integral exists, since the exponent is finite as it cuts off after the quadratic expansion of the exponent.

We first consider the integral where M is diagonal

$$\int \prod_k d\eta_k d\eta_k^* e^{\sum_i \eta_i^* M_{ii} \eta_i} = \int \prod_k d\eta_k d\eta_k^* e^{\alpha_k \eta_k^* \eta_k}.$$

The $\eta_k^*\eta_k$ term is even, so we can commute it past the $d\eta_1 d\eta_1^* d\eta_2 d\eta_2^* \ldots$ term. We can reorder the integral so that

$$\prod_k \int d\eta_k d\eta_k^* e^{\alpha_k \eta_k^* \eta_k} = \prod_k \int d\eta_k d\eta_k^* (1 + \alpha_k \eta_k^* \eta_k) = \prod_k \alpha_k$$

If we wish to calculate a non-diagonal Gaussian integral then we must first diagonalize the Gaussian term. We set $D = S^{-1}MS$, thus

$$\int \prod_k d\eta_k d\eta_k^* e^{\eta^\dagger S D S^{-1} \eta}$$

We can change coordinates so that $\bar{\eta}_j = S_{ij}^{-1}\eta_j$ and $\bar{\eta}_j^\dagger = \eta_i^\dagger S_{ij}$. This leads to

$$\begin{aligned}
\int \prod_k d\eta_k d\eta_k^* e^{\eta^\dagger M \eta} &= \int \prod_k d\bar{\eta}_k d\bar{\eta}_k^* \det S \det S^{-1} e^{\bar{\eta}^\dagger S D S^{-1} \bar{\eta}} \\
&= \det S \det S^{-1} \prod_k d_{kk} \\
&= \det S \det S^{-1} \det D = \det M.
\end{aligned}$$

The Grassmann Gaussian integral is thus

$$\int \prod_k d\eta_k d\eta_k^* e^{\eta^\dagger M \eta} = \det M.$$

APPENDIX C
Regularization

C.1 Integrals

In field theory we usually have to contend with integrals that are of the form

$$\int d^n x f(|x|^{\frac{1}{2}}) = \int d\Omega \int_0^\infty \mu(r) f(r) dr \qquad r = |x|^{\frac{1}{2}}, \tag{C.1}$$

where $|x|^2 = \sum x_i^2$, $\mu(r)$ is a new measure and Ω is the angular variables of the integrand.

The measure $\mu(r)$ can be calculated by means of scaling. If x scales by a positive λ such that $x_i = \lambda x_i'$, then $d^n x_i = \lambda^n d^n x_i'$. Since $r = |x|^{\frac{1}{2}}$ then $r = \lambda r'$ and $dr = \lambda dr'$. The measure then becomes

$$\begin{aligned} d\Omega \mu(r) dr &= \lambda^n \mu(r') dr' d\Omega \\ &= \lambda^n \mu(r') \frac{dr}{\lambda} d\Omega \\ \mu(r) &= \lambda^{n-1} \mu(r') \\ \Rightarrow \mu(\lambda r') &= \lambda^{n-1} \mu(r'). \end{aligned}$$

The only solution to this last equality is $\mu(r') = r'^{n-1}$.

Thus (C.1) becomes

$$\int d^n x f(|x|^{\frac{1}{2}}) = \int d\Omega \int_0^\infty r^{n-1} f(r) dr. \tag{C.2}$$

To calculate $\int d\Omega$, choose $f(|x|^{\frac{1}{2}}) = e^{-\frac{1}{2}|x|^2}$. Then

$$\int d^n x e^{-\frac{1}{2}|x|^2} = \prod_{i=1}^{n} \int dx_i e^{-\frac{1}{2}x_i^2} = (2\pi)^{\frac{n}{2}}.$$

But

$$\begin{aligned} \int d^n x e^{-\frac{1}{2}|x|^2} &= \int d\Omega \int_0^\infty r^{n-1} e^{-\frac{1}{2}r^2} dr \\ &= \int d\Omega \int dx (2x)^{\frac{n}{2}-1} e^{-x} \\ &= \frac{2^{\frac{n}{2}}}{2} \Gamma(\frac{n}{2}) \int d\Omega. \end{aligned}$$

Thus the integral of the angular dependents is

$$\int d\Omega = \frac{2(\pi)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})},$$

and the integral we wish to calculate, (C.2), becomes

$$\int d^n x f(|x|^{\frac{1}{2}}) = \frac{2(\pi)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^\infty r^{n-1} f(r) dr. \tag{C.3}$$

C.2 Regularization

We cannot manipulate integrals that diverge, so we make these integrals finite by using a regularization technique. The regularized integrals become infinite under certain limit conditions. The aim is to see how the integral behaves when this limit is taken. There are mainly two kinds of regularization; the cutoff method and dimensional regularization.

C.2.1 Cut off method.

If we have an integral that diverges in the ultraviolet limit, then we can introduce a cut off value Λ in the integral so that, for example,

$$\begin{aligned} \int_0^\infty dq \frac{q^3}{(q^2+m^2)} &= \lim_{\Lambda\to\infty} \int_0^\Lambda dq \frac{q^3}{(q^2+m^2)} \\ &\to \Lambda^2 + \Lambda + 1 + O(\frac{1}{\Lambda}). \end{aligned}$$

We need to introduce terms in the theory that cancel out $\Lambda^2 + \Lambda$ and then we can take the implied limit $\Lambda \to \infty$ without there being any divergences. This method does not respect the symmetries of the theory.

C.2.2 Dimensional Regularization.

This is the standard regularization technique. To calculate Feynman integrals that are divergent in the UV limit, we usually have to calculate the following

$$\int \frac{d^n k}{(2\pi)^n} f(k^2) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^\infty \frac{dk}{(2\pi)^n} k^{n-1} f(k^2).$$

This integral can be well defined in arbitrary dimensions, say $n = 2\omega \in$ C. We can then calculate the integral in 2ω dimensions and then take the limit where $\omega \to 2$ to see how the integral diverges.

Define the integral in two dimensions as

$$\begin{aligned}
&\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} f(k^2) \\
&= \frac{2\pi^{\omega}}{\Gamma(\omega)} \int_0^{\infty} \frac{dk}{(2\pi)^{2\omega}} k^{2\omega-1} f(k^2) \\
&= \frac{2}{(4\pi)^{\omega}\Gamma(\omega)} \int_0^{\infty} dk\ k^{2\omega-1} f(k^2).
\end{aligned}$$

In general, we can manipulate $f(k^2)$ into a form $f(k^2) = \frac{1}{(k^2+b)^{\alpha}}$; $\alpha, b > 0$. Thus

$$\begin{aligned}
&\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k^2+b)^{\alpha}} \\
&= \frac{2}{(4\pi)^{\omega}\Gamma(\omega)} \int_0^{\infty} dk\ \frac{k^{2\omega-1}}{(k^2+b)^{\alpha}} \\
&= \frac{2}{(4\pi)^{\omega}\Gamma(\omega)b^{\alpha}} \int_0^{\infty} dk\ \frac{k^{2\omega-1}}{(\frac{k^2}{b}+1)^{\alpha}}.
\end{aligned}$$

This can be written in the form of the beta function.

The beta function is given by

$$\begin{aligned}
\beta(x,y) &= \int_0^1 dt\ t^{x-1}(1-t)^{y-1} \\
&= 2\int_0^{\infty} ds\ \frac{s^{2x-1}}{(s^2+1)^{x+y}}\ ; \qquad t = \frac{s^2}{(1+s^2)} \\
&= 2\int_0^{\infty} ds\ \frac{s^{2\omega-1}}{(s^2+1)^{\alpha}}\ ; \qquad x = \omega, y = \alpha-\omega.
\end{aligned}$$

but $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{\Gamma(\omega)\Gamma(\alpha-\omega)}{\Gamma(\alpha)}$.

$$\begin{aligned}
&\Rightarrow \frac{2}{(4\pi)^{\omega}\Gamma(\omega)b^{\alpha}} \int_0^{\infty} dk\ \frac{k^{2\omega-1}}{(\frac{k^2}{b}+1)^{\alpha}} \\
&= \frac{2}{(4\pi)^{\omega}\Gamma(\omega)} b^{\omega-\alpha} \int_0^{\infty} dt \frac{t^{2\omega-1}}{(t^2+1)^{\alpha}} \\
&= \frac{b^{\omega-\alpha}}{(4\pi)^{\omega}\Gamma(\omega)} \beta(\omega, \alpha-\omega) \\
&= \frac{b^{\omega-\alpha}}{(4\pi)^{\omega}\Gamma(\omega)} \frac{\Gamma(\omega)\Gamma(\alpha-\omega)}{\Gamma(\alpha)}
\end{aligned}$$

Thus

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k^2+b)^{\alpha}} = \frac{b^{\omega-\alpha}}{(4\pi)^{\omega}} \frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)} \tag{C.4}$$

will have poles and diverge when $\alpha - \omega \to -n$, a negative integer, otherwise the integral is convergent.

We can use (C.4) to calculate further integrals such as

$$\begin{aligned}
&\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k^2 + 2k \cdot p + b)^\alpha} \\
&= \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{[(k+p)^2 + b - p^2]^\alpha} \\
&= \int \frac{d^{2\omega}k'}{(2\pi)^{2\omega}} \frac{1}{(k'^2 + b - p^2)^\alpha} \\
&= \frac{(b-p^2)^{\omega-\alpha}}{(4\pi)^\omega} \frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)}.
\end{aligned} \tag{C.5}$$

We can also differentiate the above equation with respect to p to obtain additional identities

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu}{(k^2 + 2k \cdot p + b)^\alpha} = \frac{(b-p^2)^{\omega-\alpha}}{(4\pi)^\omega} \frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)} p_\mu \,,$$

$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu k_\nu}{(k^2 + 2k \cdot p + b)^\alpha} = \frac{(b-p^2)^{\omega-\alpha}}{(4\pi)^\omega} \frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)} p_\mu p_\nu \,.$$

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