# Spectrum preserving linear mappings between Banach algebras

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## Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

# Abstract

Let $A$ and $B$ be unital complex Banach algebras with identities 1 and $1'$ respectively. A linear map $T : A \to B$ is *invertibility preserving* if $Tx$ is invertible in $B$ for every invertible $x \in A$. We say that $T$ is *unital* if $T1 = 1'$. If $Tx^2 = (Tx)^2$ for all $x \in A$, we call $T$ a *Jordan homomorphism*. We examine an unsolved problem posed by I. Kaplansky:

> *Let $A$ and $B$ be unital complex Banach algebras and $T : A \to B$ a unital invertibility preserving linear map. What conditions on $A, B$ and $T$ imply that $T$ is a Jordan homomorphism?*

Partial motivation for this problem are the Gleason-Kahane-Żelazko Theorem (1968) and a result of Marcus and Purves (1959), these also being special instances of the problem. We will also look at other special cases answering Kaplansky's problem, the most important being the result stating that if $A$ is a von Neumann algebra, $B$ a semi-simple Banach algebra and $T : A \to B$ a unital bijective invertibility preserving linear map, then $T$ is a Jordan homomorphism (B. Aupetit, 2000).

For a unital complex Banach algebra $A$, we denote the spectrum of $x \in A$ by Sp $(x, A)$. Let $\sigma(x, A)$ denote the union of Sp $(x, A)$ and the bounded components of $\mathbb{C} \setminus$ Sp $(x, A)$. We denote the spectral radius of $x \in A$ by $\rho(x, A)$.

A unital linear map $T$ between unital complex Banach algebras $A$ and $B$ is invertibility preserving if and only if Sp $(Tx, B) \subset$ Sp $(x, A)$ for all $x \in A$. This leads one to consider the problems that arise when, in turn, we replace the invertibility preservation property of $T$ in Kaplansky's problem with Sp $(Tx, B) =$ Sp $(x, A)$ for all $x \in A$, $\sigma(Tx, B) = \sigma(x, A)$ for all $x \in A$, and $\rho(Tx, B) = \rho(x, A)$ for all $x \in A$. We will also investigate some special cases that are solutions to these problems. The most important of these special cases says that if $A$ is a semi-simple Banach algebra, $B$ a primitive Banach algebra with minimal ideals and $T : A \to B$ a surjective linear map satisfying $\sigma(Tx, B) = \sigma(x, A)$ for all $x \in A$, then $T$ is a Jordan homomorphism (B. Aupetit and H. du T. Mouton, 1994).

# Opsomming

Gestel $A$ en $B$ is unitale komplekse Banach algebras met identiteite 1 en $1'$ onderskeidelik. 'n Lineêre afbeelding $T : A \to B$ is *omkeerbaar-behoudend* as $Tx$ omkeerbaar in $B$ is vir elke omkeerbare element $x \in A$. Ons sê dat $T$ *unitaal* is as $T1 = 1'$. As $Tx^2 = (Tx)^2$ vir alle $x \in A$, dan noem ons $T$ 'n *Jordan homomorfisme*. Ons ondersoek 'n onopgeloste probleem wat deur I. Kaplansky voorgestel is:

*Gestel $A$ en $B$ is unitale komplekse Banach algebras en $T : A \to B$ is 'n unitale omkeerbaar-behoudende lineêre afbeelding. Watter voorwaardes op $A, B$ en $T$ impliseer dat $T$ 'n Jordan homomorfisme is?*

Gedeeltelike motivering vir hierdie probleem is die Gleason-Kahane-Żelazko Stelling (1968) en 'n resultaat van Marcus en Purves (1959), wat terselfdertyd ook spesiale gevalle van die probleem is. Ons sal ook na ander spesiale gevalle kyk wat antwoorde lewer op Kaplansky se probleem. Die belangrikste van hierdie resultate sê dat as $A$ 'n von Neumann algebra is, $B$ 'n semi-eenvoudige Banach algebra is en $T : A \to B$ 'n unitale omkeerbaar-behoudende bijektiewe lineêre afbeelding is, dan is $T$ 'n Jordan homomorfisme (B. Aupetit, 2000).

Vir 'n unitale komplekse Banach algebra $A$, dui ons die spektrum van $x \in A$ aan met Sp $(x, A)$. Laat $\sigma(x, A)$ die vereniging van Sp $(x, A)$ en die begrensde komponente van $\mathbb{C} \setminus$ Sp $(x, A)$ wees. Ons dui die spektraalradius van $x \in A$ aan met $\rho(x, A)$.

'n Unitale lineêre afbeelding $T$ tussen unitale komplekse Banach algebras $A$ en $B$ is omkeerbaar-behoudend as en slegs as Sp $(Tx, B) \subset$ Sp $(x, A)$ vir alle $x \in A$. Dit lei ons om die probleme te beskou wat ontstaan as ons die omkeerbaar-behoudende eienskap van $T$ in Kaplansky se probleem vervang met Sp $(Tx, B) =$ Sp $(x, A)$ vir alle $x \in A$, $\sigma(Tx, B) = \sigma(x, A)$ vir alle $x \in A$ en $\rho(Tx, B) = \rho(x, A)$ vir alle $x \in A$, onderskeidelik. Ons sal ook 'n paar spesiale gevalle van hierdie probleme ondersoek. Die belangrikste van hierdie spesiale gevalle sê dat as $A$ 'n semi-eenvoudige Banach algebra is, $B$ 'n primitiewe Banach algebra met minimale ideale is, en $T : A \to B$ 'n surjektiewe lineêre afbeelding is sodanig dat $\sigma(Tx, B) = \sigma(x, A)$ vir alle $x \in A$, dan is $T$ 'n Jordan homomorfisme (B. Aupetit en H. du T. Mouton, 1994).

# Acknowledgements

# Contents

# Introduction

Let $R$ and $R'$ be unital rings with identities 1 and $1'$ respectively. A map $\phi : R \to R'$ is called *unital* if $\phi(1) = 1'$ and is called *invertibility preserving* if $\phi(x)$ is invertible in $R'$ for every invertible $x \in R$. For example, every homomorphism and every anti-homomorphism of rings are invertibility preserving linear maps. By an anti-homomorphism $\phi$, we mean that $\phi$ is additive and $\phi(xy) = \phi(y)\phi(x)$ for all $x, y \in R$. An additive map $\phi : R \to R'$ is called a *Jordan homomorphism* if $\phi(x^2) = \phi(x)^2$ for every $x \in R$. A bijective Jordan homomorphism is called a *Jordan isomorphism*. Clearly, homomorphisms and anti-homomorphisms of rings are Jordan homomorphisms. However, not every Jordan homomorphism is necessarily a homomorphism or an anti-homomorphism of rings.

Let $R$ be a unital ring and $R'$ a unital ring such that $2x \neq 0$ for all nonzero $x \in R'$. Then every surjective Jordan homomorphism $\phi : R \to R'$ is a unital invertibility preserving map. I. Kaplansky raised the question as to when the converse of this result is true. More precisely, what conditions on $R, R'$ and $\phi$ imply that every unital invertibility preserving additive map $\phi : R \to R'$ is a Jordan homomorphism? This problem was also motivated by a result of Marcus and Purves (1959):

*Every unital linear invertibility preserving map on $M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices, is a Jordan automorphism* (see Corollary 3.1.11).

Further motivation came from the famous Gleason-Kahane-Żelazko Theorem (1968):

*A linear functional on a unital complex Banach algebra is multiplicative if and only if it is unital and invertibility preserving* (see Theorem 3.2.7).

This thesis is about Kaplansky's problem. Most of the work on this problem was done by mathematicians working in functional analysis. Therefore, we will consider linear maps on Banach algebras rather than additive maps on rings. Moreover, we restrict our attention to the case where $A$ and $B$ are unital complex Banach algebras and $T : A \to B$ is a linear invertibility preserving map. Also, we shall refer to a unital complex Banach algebra as a Banach algebra throughout.

For a Banach algebra $A$, we denote the spectrum of $x \in A$ by $\mathrm{Sp}\,(x)$. Whenever there might be a risk of confusion, we denote the spectrum of $x \in A$ by $\mathrm{Sp}\,(x, A)$. The *full spectrum* of $x \in A$, denoted by $\sigma(x)$, is the union of $\mathrm{Sp}\,(x)$ and the bounded components of $\mathbb{C} \setminus \mathrm{Sp}\,(x)$. We let the spectral radius of $x \in A$ be denoted by $\rho(x)$. The notations $\sigma(x, A)$ and $\rho(x, A)$ are defined in a similar way to $\mathrm{Sp}\,(x, A)$. In this thesis, if $A$ and

ii

$B$ are two sets, we mean by $A \subset B$ that $A$ is contained in $B$. Also, when we write $A \subsetneq B$, we mean that $A$ is properly contained in $B$. We let $\mathcal{L}(X)$ denote the Banach algebra of bounded linear operators on a Banach space $X$.

This thesis consists of five chapters. Chapter 1 is a reminder of certain concepts and results from spectral theory and ring theory that are relevant to the subsequent chapters. These results are well known and therefore we omit their proofs, except in those cases where satisfactory references could not be found.

In Chapter 2, we introduce a few concepts that we will need to begin our study in Chapter 3, the most important being those of a Jordan homomorphism and of an invertibility preserving linear map. We also give a few elementary properties of these concepts.

In Chapter 3, we begin our study of Kaplansky's problem. We start this chapter by giving a proof of the 1959-result of Marcus and Purves. This proof is a straight forward application of results due to B. Aupetit, given in [5].

Next, the Gleason-Kahane-Żelazko Theorem is proved. This is done by proving a stronger result (see Theorem 3.2.3), namely:
*Let $A$ be a Banach algebra and $f$ a linear functional on $A$ having no exponentials as zeroes. Then $f$ is multiplicative if and only if $f$ is bounded and unital.*
The proof of this result follows the lines of [22], Theorem 2. We also show how the Gleason-Kahane-Żelazko Theorem is extended to general linear operators (see Theorem 3.2.9).

In 1996, A. R. Sourour proved the following result:
*Let $X$ and $Y$ be Banach spaces and $T : \mathcal{L}(X) \to \mathcal{L}(Y)$ a unital bijective invertibility preserving linear mapping. Then $T$ is a Jordan isomorphism* (see Theorem 3.3.1).

In the third section of Chapter 3, we see how the above results are used to obtain the following reasonable conjecture arising from Kaplansky's problem:
*Let $A$ and $B$ be semi-simple Banach algebras. If $T : A \to B$ is a unital bijective invertibility preserving linear mapping, then $T$ is a Jordan isomorphism.*

A unital linear mapping $T$ between Banach algebras $A$ and $B$ is invertibility preserving if and only if $\mathrm{Sp}\,(Ta, B) \subset \mathrm{Sp}\,(a, A)$ for every $a \in A$. Thus we can reformulate the conjecture as follows: Let $A$ and $B$ be semi-simple Banach algebras. If $T : A \to B$ is a unital bijective linear mapping such that $\mathrm{Sp}\,(Ta, B) \subset \mathrm{Sp}\,(a, A)$ for every $a \in A$, then $T$ is a Jordan isomorphism. So we can ask a question that is somewhat easier than Kaplansky's original question: When must a spectrum preserving linear mapping between Banach algebras be a Jordan homomorphism? By a *spectrum preserving map*,

iii

we mean a map $T : A \to B$ satisfying $\mathrm{Sp}\,(Ta, B) = \mathrm{Sp}\,(a, A)$ for every $a \in A$ (where $A$ and $B$ are Banach algebras).

In the fourth section of Chapter 3, we explain how to obtain the following reasonable conjecture arising from this question:

*Let $A$ and $B$ be semi-simple Banach algebras. If $T : A \to B$ is a surjective spectrum preserving linear mapping, then $T$ is a Jordan isomorphism.*

We then give some results supporting this conjecture, namely the Marcus-Moyls Theorem (Theorem 3.4.7) and the Jafarian-Sourour Theorem (Theorem 3.4.10), for the cases $A = M_n(\mathbb{C}) = B$ and $A = \mathcal{L}(X)$, $B = \mathcal{L}(Y)$, where $X$ and $Y$ are Banach spaces, respectively.

We conclude this chapter by looking at spectral radius preserving linear mappings, i.e. linear mappings $T$ that satisfy $\rho(Ta, B) = \rho(a, A)$ for every $a \in A$. We investigate a reasonable conjecture that gives conditions as to when a spectral radius preserving linear map is a Jordan homomorphism.

In Chapter 4, we study the following problem: If $A$ and $B$ are semi-simple Banach algebras and if $T : A \to B$ is a surjective linear mapping with the property that $\sigma(Ta, B) = \sigma(a, B)$ for every $a \in A$ (such a $T$ is called a *full spectrum preserving* linear map), is it true that $T$ is a Jordan homomorphism? This chapter is based on a paper of B. Aupetit and H. du. T. Mouton, namely [4].

We begin by defining the notion of a rank one element of a semi-simple Banach algebra (see Definition 4.1.1). We then see how rank one elements are characterized spectrally (see Theorem 4.2.3). This, along with other results, brings us to one of the main results of this chapter (see Corollary 4.2.19), namely, a result due to B. Aupetit and H. du T. Mouton (1994):

*Let $A$ and $B$ be semi-simple Banach algebras. If $B$ is a primitive Banach algebra with minimal ideals and $T : A \to B$ is a surjective full spectrum preserving linear map, then $T$ is a Jordan homomorphism.*

An important consequence of this result is the result by Jafarian and Sourour. This means that we have an algebraic proof of the Jafarian-Sourour Theorem. It would be of interest to know if Sourour's theorem (Theorem 3.3.1) can be proved using algebraic techniques only (instead of operator theoretic techniques), as this could yield insight into solving Kaplansky's problem.

In Chapter 5, the following result by B. Aupetit (2000) is proved:

*Let $A$ be a von Neumann algebra, $B$ a semi-simple Banach algebra and $T : A \to B$ a unital bijective invertibility preserving linear mapping. Then $T$ is a Jordan isomorphism* (see Corollary 5.2.19).

In 1999, M. Brešar and P. Šemrl have obtained a new proof of the Marcus-Purves Theorem. Although their proof is still partly matrix theoretic, it is simpler than the proof that Marcus and Purves originally gave in 1959. Part of the strategy of Brešar and Šemrl's proof is to find a spectral characteri-

zation of idempotent square complex matrices. They used this to show that any unital invertibility preserving linear map $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ (as in the hypothesis of the Marcus-Purves result) is idempotent preserving. It then follows that $T$ is a Jordan automorphism.

We start Chapter 5 by discussing Brešar and Šemrl's proof of the Marcus-Purves Theorem. It will also be seen in this chapter that this proof led Brešar and Šemrl to give the following strategy for proving the main result of this chapter: If one can find a suitable spectral characterization of idempotents in a Banach algebra which would imply that a unital bijective invertibility preserving linear map $T$ preserves idempotents, then it will follow that $T$ is a Jordan isomorphism.

In 2000, B. Aupetit implemented this strategy, culminating in Theorem 5.2.7, Corollary 5.2.17 and Corollary 5.2.20. Theorem 5.2.7 is Aupetit's spectral characterization of idempotents in a semi-simple Banach algebra. The ingredients of the proof of this result are, among other things, the holomorphic functional calculus, upper-semicontinuity of the spectrum and a characterization of the radical of a Banach algebra which is based on results on subharmonicity.

In this thesis we do not give any matrix or operator theoretic proofs. The definitions, theorems and other results are numbered succesively in each chapter. When we refer to Theorem 3.2.1, we mean Result 1 of Section 2 of Chapter 3. The symbol $\nabla$ indicates the end of a proof.

# Chapter 1

# Preliminaries

The aim of this chapter is to remind the reader of some general Banach algebra theory that will be used in the rest of the text. No proofs will therefore be given, except in those cases where satisfactory references could not be found. Familiarity with the basics of real analysis, complex analysis, functional analysis and ring theory is assumed.

## 1.1 Banach algebras

**Definition 1.1.1** *A complex algebra is a vector space $A$ over $\mathbb{C}$ together with a mapping $(x, y) \mapsto xy$ of $A \times A$ into $A$ that satisfies the following axioms (for all $x, y, z \in A, \alpha \in \mathbb{C}$):*

*(i) $x(yz) = (xy)z$,*

*(ii) $x(y + z) = xy + xz, (x + y)z = xz + yz$,*

*(iii) $(\alpha x)y = \alpha(xy) = x(\alpha y)$.*

*If, in addition, $A$ is a Banach space with norm $\|.\|$ satisfying $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$, and $A$ has identity element $1$ with $\|1\| = 1$, we say that $A$ is a unital complex Banach algebra.*

*A subset $B$ of a complex algebra $A$ is called a subalgebra of $A$ if $B$ is closed under addition, multiplication and scalar multiplication.*

We shall only be working with unital complex Banach algebras in this text. Therefore we shall refer to a unital complex Banach algebra as a Banach algebra throughout.

An example of a Banach algebra is $\mathcal{L}(X)$, the space of all bounded linear operators on a Banach space $X$.

1

**Definition 1.1.2** *Let A be a Banach algebra. An element p of A having the property $p^2 = p$ is called an idempotent of A.*

We call an idempotent $p$ in a Banach algebra $A$ *non-trivial* if $p \neq 0$ and $p \neq 1$. If $p$ and $q$ are idempotents in $A$ such that $pq = qp = 0$, then $p$ and $q$ are known as *orthogonal idempotents*. Let $n \in \mathbb{N}$ and suppose that $\{p_i : i = 1, \ldots, n\}$ is a set of idempotents of $A$. We say that the $p_i$ are *mutually orthogonal idempotents* if $p_i p_j = 0$ for all $i \neq j$. A nonzero idempotent $p$ in $A$ is called a *minimal idempotent* if $pAp$ is a division algebra.

## 1.2 Ideals of a Banach algebra

**Definition 1.2.1** *Let I be a non-empty subset of a Banach algebra A. Then I is called a left ideal of A if the following conditions hold:*

*(i) If $a, b \in I$, then $a + b \in I$.*

*(ii) If $a \in I$ and $\alpha \in \mathbb{C}$, then $\alpha a \in I$.*

*(iii) If $a \in I$ and $r \in A$, then $ra \in I$.*

*Similarly, a non-empty subset I of A is called a right ideal of A if the following conditions hold:*

*(i) If $a, b \in I$, then $a + b \in I$.*

*(ii) If $a \in I$ and $\alpha \in \mathbb{C}$, then $\alpha a \in I$.*

*(iii) If $a \in I$ and $r \in A$, then $ar \in I$.*

*If I is a left ideal and a right ideal of A, then we call I a two-sided ideal of A.*

Every two-sided ideal of a Banach algebra $A$ is a subalgebra of $A$.

**Definition 1.2.2** *Let A be a Banach algebra. A proper left (right, two-sided) ideal M of A is a maximal left (right, two-sided) ideal of A if A has no left (right, two-sided) ideal I such that $M \subsetneq I \subsetneq A$.*
*Similarly, a nonzero left (right, two-sided) ideal M is a minimal left (right, two-sided) ideal of A if there exists no left (right, two-sided) ideal I of A such that $\{0\} \subsetneq I \subsetneq M$.*

2

**Definition 1.2.3** *Let A be a Banach algebra. If A has minimal left (right) ideals, we define the left (right) socle of A to be the sum of all minimal left (right) ideals of A. If A has no minimal left (right) ideals, the left (right) socle of A is zero. If the left socle of A coincides with the right socle of A, it is called the socle of A and is denoted by Soc(A). We say that the socle exists.*

It is clear that Soc $(A)$ is a two-sided ideal of $A$.

**Theorem 1.2.4** *([2], Theorem 3.1.3) If A is a Banach algebra, then the following sets are identical:*

(i) *the intersection of all maximal left ideals of A,*

(ii) *the intersection of all maximal right ideals of A.*

**Definition 1.2.5** *The radical of A , denoted by Rad (A), is defined to be the two identical sets given in Theorem 1.2.4. We say that A is semi-simple if Rad $(A) = \{0\}$.*

Clearly, the radical of a Banach algebra $A$ is a two-sided ideal of $A$. We have the following useful characterization of the socle of a semi-simple Banach algebra.

**Theorem 1.2.6 (J. C. Alexander)** *([1], Theorem 7.2) Let A be a semi-simple Banach algebra and $u \in A$. Then uAu is finite-dimensional if and only if $u \in Soc (A)$.*

## 1.3 Homomorphisms and isomorphisms between Banach algebras

Important mappings between Banach algebras are those preserving addition, multiplication and scalar multiplication. These are called *homomorphisms*.

**Definition 1.3.1** *Let A and B be Banach algebras. A map $\phi : A \to B$ is called an (algebra) homomorphism if the following conditions hold for all $a, b \in A$ and $\alpha \in \mathbb{C}$:*

(i) $\phi(a + b) = \phi(a) + \phi(b)$,

(ii) $\phi(\alpha a) = \alpha\phi(a)$,

(iii) $\phi(ab) = \phi(a)\phi(b)$,

3

*(iv)* $\phi(1) = 1,$

and $\phi : A \to B$ *is called an (algebra) anti-homomorphism if the following conditions hold for all* $a, b \in A$ *and* $\alpha \in \mathbb{C}$:

*(i)* $\phi(a + b) = \phi(a) + \phi(b),$

*(ii)* $\phi(\alpha a) = \alpha \phi(a),$

*(iii)* $\phi(ab) = \phi(b)\phi(a),$

*(iv)* $\phi(1) = 1.$

**Definition 1.3.2** *Let $A$ be a Banach algebra. We say that a linear functional $f$ is multiplicative if $f$ is nonzero and $f(xy) = f(x)f(y)$ for all $x, y \in A$.*

If $A$ and $B$ are Banach algebras with identity elements 1 and $1'$ respectively, then the linear mapping $T : A \to B$ is called *unital* if $T1 = 1'$.

By definition, a homomorphism between Banach algebras is unital. The following result says that a multiplicative linear functional on a Banach algebra is unital.

**Lemma 1.3.3** *Every multiplicative linear functional $f$ on a Banach algebra $A$ is unital.*

**Proof.** Since $f$ is multiplicative, $f$ is nonzero. Therefore there exists $y \in A$ such that $f(y) \neq 0$. Due to the multiplicativity of $f$, it follows that $f(y) = f(y.1) = f(y)f(1)$. Hence $f(1) = \frac{f(y)}{f(y)} = 1$. $\nabla$

**Definition 1.3.4** *Let $A$ and $B$ be Banach algebras. A map $\phi : A \to B$ is called an (algebra) isomorphism if $\phi$ is a bijective homomorphism. The map $\phi : A \to B$ is called an (algebra) anti-isomorphism if $\phi$ is a bijective anti-homomorphism.*

Basically, isomorphisms between Banach algebras $A$ and $B$ ensure that $A$ and $B$ have the same algebraic properties. For a Banach algebra $A$, we call an isomorphism $\phi : A \to A$ an *automorphism*.

**Definition 1.3.5** *Let $T$ be a bounded linear operator from $X$ into $Y$, where $X$ and $Y$ are normed spaces. The adjoint operator $T^\times$ that maps $Y'$ into $X'$ is defined by $(T^\times g)(x) = g(Tx)$ for all $x \in X$ and $g \in Y'$, where $X'$ and $Y'$ are the dual spaces of $X$ and $Y$ respectively.*

4

**Proposition 1.3.6** *Let $X$ and $Y$ be Banach spaces. If $T : \mathcal{L}(X) \to \mathcal{L}(Y)$ is a linear mapping defined as $Ta = bab^{-1}$ for some invertible linear operator $b : X \to Y$, then $T$ is an isomorphism. If $Ta = ca^{\times}c^{-1}$ for some invertible linear operator $c : X \to Y$, then $T$ is an anti-isomorphism.*

**Proof.** Suppose that $Ta = bab^{-1}$ for every $a \in \mathcal{L}(X)$. Clearly, $T$ is a bijection. Let $x, y \in A$. Then $T(xy) = bxyb^{-1} = (bxb^{-1})(byb^{-1}) = TxTy$, implying that $T$ is a homomorphism. Thus $T$ is an isomorphism. The case $Ta = ca^{\times}c^{-1}$ is dealt with in a similar manner. $\nabla$

**Lemma 1.3.7** *If $X$ is a Banach space, then $C = \{t^{\times} : t \in \mathcal{L}(X)\}$ is anti-isomorphic to $\mathcal{L}(X)$ under the mapping $t \to t^{\times}$.*

**Proof.** Since $(t + s)^{\times} = t^{\times} + s^{\times}$, $(ts)^{\times} = s^{\times}t^{\times}$ and $(\lambda t)^{\times} = \lambda t^{\times}$ for all $t, s \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$, the mapping $t \mapsto t^{\times}$ is an anti-homomorphism. Clearly, the mapping is surjective. Since $\|t\| = \|t^{\times}\|$ for every $t \in \mathcal{L}(X)$, the mapping is injective. Hence the result follows. $\nabla$

An important class of homomorphisms are known as *representations* of a Banach algebra $A$.

**Definition 1.3.8** *Let $A$ be a Banach algebra and $X$ a Banach space. A mapping $\pi : A \to \mathcal{L}(X)$ is called a continuous representation of $A$ on $X$ if $\pi$ is a homomorphism. We say that a subspace $Y$ of $X$ is invariant under $\pi$ if $\pi(x)Y \subset Y$ for every $x \in A$. We call $\pi$ an irreducible representation of $A$ on $X$ if $\{0\}$ and $X$ are the only invariant subspaces under $\pi$.*

**Lemma 1.3.9** *([2], Exercise 13, p. 90) Let $A$ be a Banach algebra and let $\pi$ be an irreducible representation of $A$ on a Banach space $X$ of dimension $n$. Then $\pi(A)$ is isomorphic to $M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the Banach algebra of $n \times n$ complex matrices.*

A Banach algebra $A$ admits a *separating family* $\mathcal{S}$ of irreducible representations if there exists $\pi \in \mathcal{S}$ such that $\pi(x) \neq 0$ for all nonzero $x$ in $A$. The following result is related to the Noether-Skolem Theorem.

**Theorem 1.3.10** *([23], Theorem 36.2) If $V_n$ is an $n$-dimensional vector space, then every automorphism $T : \mathcal{L}(V_n) \to \mathcal{L}(V_n)$ is of the form $Ta = u^{-1}au$ for some invertible element $u$ of $\mathcal{L}(V_n)$. Similarly, every anti-automorphism $T : \mathcal{L}(V_n) \to \mathcal{L}(V_n)$ is of the form $Ta = u^{-1}a^t u$ for some invertible element $u$ of $\mathcal{L}(V_n)$.*

5

**Corollary 1.3.11** *Let $V_n$ be an n-dimensional vector space. If $T : \mathcal{L}(V_n) \to \mathcal{L}(V_n)$ is a surjective linear map that is either a homomorphism or an anti-homomorphism, then $T$ is either of the form $Ta = u^{-1}au$ or $Ta = u^{-1}a^t u$ for some invertible $u$ of $\mathcal{L}(V_n)$.*

**Proof.** Since $V_n$ is finite dimensional and $T$ is surjective, it follows from a standard result of linear algebra that $T$ is injective. Hence $T$ is bijective. Therefore, due to the fact that $T$ is a homomorphism or an anti-homomorphism, it follows that $T$ is an automorphism or an anti-automorphism. The result now follows from Theorem 1.3.10. $\nabla$

**Definition 1.3.12** *Let $\mathcal{B}$ be an algebra of linear operators on a vector space $X$. We say that $\mathcal{B}$ is k-fold transitive on $X$ if for arbitrary linearly independent vectors $x_1, \ldots, x_k$ and arbitrary vectors $y_1, \ldots, y_k$ in $X$, there exists $T \in \mathcal{B}$ such that $Tx_i = y_i$ $(i = 1, \ldots, k)$. If $\mathcal{B}$ is k-fold transitive for every $k$, then we say that $\mathcal{B}$ is strictly dense in $X$.*

*We say that a representation $\pi$ of a Banach algebra $A$ on a Banach space $X$ is strictly dense if the range of $\pi$ is a strictly dense algebra in $X$.*

**Theorem 1.3.13** *([31], Theorem 2.5.19) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be strictly dense subalgebras of $\mathcal{L}(X_1)$ and $\mathcal{L}(X_2)$ respectively, where $X_1$ and $X_2$ are Banach spaces, and let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be an isomorphism. If $\mathcal{A}_1$ and $\mathcal{A}_2$ have minimal ideals, then there exists a bounded invertible linear operator $u : X_1 \to X_2$ such that $\phi(x) = uxu^{-1}$ for every $x \in \mathcal{A}_1$.*

## 1.4 Primitive, prime and semi-prime Banach algebras

The aim of this section is to recall some of the well-known results about primitive, prime and semi-prime Banach algebras that will be used in Chapter 4.

Let $L_1$ and $L_2$ be two left (right) ideals of an algebra $A$. Recall that the product $L_1 L_2$ of $L_1$ and $L_2$ is defined as the set of all finite sums of elements of the form $ab$ with $a \in L_1$ and $b \in L_2$. In fact, $L_1 L_2$ is generated by the set of all products of the form $ab$, where $a \in L_1$ and $b \in L_2$.

**Definition 1.4.1** *A two-sided ideal $I$ of a Banach algebra $A$ is primitive if it is the kernel of an irreducible representation of $A$.*

The radical of a Banach algebra $A$ can be characterized in terms of primitive ideals of $A$, as the following result confirms.

6

**Theorem 1.4.2** *([8], Proposition 14(i), p. 124) The radical of a Banach algebra A is the intersection of all primitive ideals of A.*

**Definition 1.4.3** *Let I be a two-sided ideal of a Banach algebra A. Suppose that I has the property that if $A_1$ and $A_2$ are two-sided ideals of A such that if $A_1 A_2 \subset I$, then $A_1 \subset I$ or $A_2 \subset I$. Then we call I a prime ideal of A.*

**Theorem 1.4.4** *([31], Theorem 2.2.9) Let A be a Banach algebra. Then, if I is a primitive ideal of A and $B_1$ and $B_2$ are left ideals of A such that $B_1 B_2 \subset I$, then $B_1 \subset I$ or $B_2 \subset I$.*

An important characterization of prime ideals is the following:

**Theorem 1.4.5** *([27], Theorem 4.3) Let A be a Banach algebra and I a two-sided ideal of A. The following statements are equivalent.*

*(i) I is a prime ideal.*

*(ii) If $a, b \in A$ and $aAb \subset I$, then $a \in I$ or $b \in I$.*

*(iii) If U and V are left ideals in A such that $UV \subset I$, then $U \subset I$ or $V \subset I$.*

Two important classes of Banach algebras are *primitive* and *prime* Banach algebras.

**Definition 1.4.6** *A Banach algebra A is called*

*(i) primitive if $\{0\}$ is a primitive ideal of A,*

*(ii) prime if $\{0\}$ is a prime ideal of A.*

The following result follows easily from Theorem 1.4.2.

**Corollary 1.4.7** *Every primitive Banach algebra is semi-simple.*

The next result follows from Theorem 1.4.5(ii).

**Theorem 1.4.8** *([19], p. 47) A Banach algebra A is prime if and only if $aB = \{0\}$ implies $a = 0$ for any nonzero left ideal B of A.*

The following result follows from Theorem 1.4.5(iii).

**Lemma 1.4.9** *([27], p. 71) Let A be a Banach algebra. The following conditions are equivalent:*

*(i) A is prime,*

7

*(ii) for any two left ideals $L_1$ and $L_2$ of $A$, the condition $L_1 L_2 \subset \{0\}$ implies $L_1 = \{0\}$ or $L_2 = \{0\}$.*

The following result states that the class of prime Banach algebras includes the primitive ones.

**Theorem 1.4.10** *Every primitive Banach algebra is prime.*

**Proof.** If $A$ is a primitive Banach algebra, then $\{0\}$ is a primitive ideal of $A$. By Theorem 1.4.4, if $L_1 L_2 \subset \{0\}$ for left ideals $L_1$ and $L_2$ of $A$, then $L_1 = \{0\}$ or $L_2 = \{0\}$. It follows from Lemma 1.4.9 that $A$ is prime. $\nabla$

**Definition 1.4.11** *An algebra $A$ is semi-prime if $\{0\}$ is the only two-sided ideal $J$ of $A$ with $J^2 = \{0\}$.*

**Lemma 1.4.12** *([8], Lemma 4, p. 155) Let $A$ be a semi-prime Banach algebra and let $L$ be a left ideal of $A$. If $L^2 = \{0\}$, then $L = \{0\}$.*

The following result is a characterization of semi-prime Banach algebras.

**Theorem 1.4.13** *([29], p. 657) For a Banach algebra $A$, the following statements are equivalent:*

*(i) $A$ is semi-prime.*

*(ii) If $u \neq 0$, then there exists $x \in A$ such that $uxu \neq 0$.*

**Proof.** Suppose that $A$ is semi-prime. Assume that $u \neq 0$. Then $Au$ is a nonzero left ideal of $A$. By Lemma 1.4.12, $(Au)^2 \neq \{0\}$. Therefore there exist $w, x \in A$ such that $wuxu \neq 0$. Thus $uxu \neq 0$.

Conversely, suppose that (ii) holds. Let $J$ be a two-sided ideal of $A$ having the property that $J^2 = \{0\}$, and let $a \in J$. Since $xa \in J$ for every $x \in A$, it follows that $axa = 0$ for every $x \in A$. By (ii), $a = 0$. Hence $J = \{0\}$. Therefore $A$ is semi-prime. $\nabla$

**Lemma 1.4.14** *([8], Proposition 5, p. 155) Every semi-simple Banach algebra is semi-prime.*

Corollary 1.4.7 and Lemma 1.4.14 together imply the next result.

**Corollary 1.4.15** *Every primitive Banach algebra is semi-prime.*

Before we continue, we should take notice of an important characterization of minimal ideals.

8

**Proposition 1.4.16** *([8], Proposition 6, p. 155) Let $A$ be a semi-prime Banach algebra. Then $L$ is a minimal left ideal of $A$ if and only if $L = Ap$ for some minimal idempotent $p$ of $A$.*

**Lemma 1.4.17** *([8], Proposition 10, p. 156) If $A$ is a semi-prime Banach algebra, then the socle of $A$ exists.*

**Theorem 1.4.18** *Every primitive Banach algebra $A$ with minimal ideals has the property that $a\,Soc\,(A) = \{0\}$ implies $a = 0$.*

**Proof.** By Corollary 1.4.15 and Lemma 1.4.17, Soc $(A)$ exists. Since $A$ has minimal ideals, Soc $(A) \neq \{0\}$. It follows from Theorem 1.4.10 that $A$ is prime. Recalling that Soc $(A)$ is a two-sided ideal of $A$, we obtain from Theorem 1.4.8 that if $a\,$Soc $(A) = \{0\}$, then $a = 0$. $\nabla$

## 1.5   Useful results from spectral theory

In this section, we give results that will be of importance throughout the text.

**Definition 1.5.1** *Let $a \in A$, where $A$ is a Banach algebra. We define the spectrum of $a$, denoted by $Sp\,(a)$, as*

$$Sp\,(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible in } A\}.$$

*The spectral radius of $a$, denoted by $\rho(a)$, is the number*

$$\rho(a) = \sup\{|\lambda| : \lambda \in Sp\,(a)\}.$$

*The full spectrum of $a$, denoted by $\sigma(a)$, is defined to be the union of $Sp\,(a)$ with the bounded components of $\mathbb{C} \setminus Sp\,(a)$.*

Another term for a bounded component of $\mathbb{C} \setminus$ Sp $(a)$ is a *hole* of Sp $(a)$. Whenever there might be a risk of confusion, we indicate the spectrum of an element $a$ in a Banach algebra $A$ as Sp $(a, A)$ instead of Sp $(a)$. If $f$ is a linear functional on a Banach algebra $A$, then Sp $(f(x)) = \{f(x)\}$ for every $x \in A$. Note that the definition of the spectrum of $x \in A$ also makes sense if $A$ is a unital complex algebra, i.e. $A$ does not have to be a Banach algebra.

**Lemma 1.5.2** *([2], p. 36) Let $A$ be a Banach algebra. Then $Sp\,(xy) \cup \{0\} = Sp\,(yx) \cup \{0\}$ for every $x, y \in A$.*

Lemma 1.5.2 implies the following result.

**Corollary 1.5.3** *Let $A$ be a Banach algebra. Then $\rho(xy) = \rho(yx)$ for every $x, y \in A$.*

**Lemma 1.5.4** *([2], p. 36) Let $A$ be a Banach algebra. Then $\rho(x) \leq \|x\|$ for all $x \in A$.*

If $M$ is a commutative subset of $A$ with the property that $M$ is not properly contained in any commutative subset of $A$, we call $M$ a *maximal commutative subset* of $A$.

**Theorem 1.5.5** *([8], Proposition 3, Theorem 4, p. 75) Each commutative subset of a Banach algebra $A$ is contained in a maximal commutative subalgebra of $A$. Each maximal commutative subalgebra $M$ of $A$ is closed and contains the identity element of $A$. Furthermore, $Sp\,(x, M) = Sp\,(x, A)$ for all $x \in M$.*

**Lemma 1.5.6** *([2], Exercise 3.9) Let $A$ be a Banach algebra and let $x_1, \ldots, x_n$ in $A$ be such that $x_i x_j = 0$ for all $i \neq j$. Then $Sp\,(x_1 + \cdots + x_n) \setminus \{0\} = \left( Sp\,(x_1) \cup \cdots \cup Sp\,(x_n) \right) \setminus \{0\}$.*

**Lemma 1.5.7** *If $f$ is a multiplicative linear functional on a Banach algebra $A$, then $f(x) \in Sp\,(x)$ for every $x \in A$.*

**Proof.** Suppose that $f$ is a multiplicative linear functional on $A$. Assume that $f(x) \notin \mathrm{Sp}\,(x)$ for some $x \in A$. Then there exists $y \in A$ such that $\left( x - f(x)1 \right) y = y \left( x - f(x)1 \right) = 1$. Therefore $xy - f(x)y = 1 = yx - yf(x)$ and so $xy = 1 + f(x)y$. This implies that $f(xy) = f\left( 1 + f(x)y \right)$, i.e. $f(x)f(y) = f(1) + f\left( f(x)y \right) = 1 + f(x)f(y)$. This is a contradiction. Hence $f(x) \in \mathrm{Sp}\,(x)$ for every $x \in A$. $\nabla$

**Proposition 1.5.8** *Let $f$ be a linear functional on a Banach algebra $A$. If $f(x) \in \sigma(x)$ for every $x \in A$, then $f$ is bounded and unital.*

**Proof.** By hypothesis, $|f(x)| \leq \rho(x) \leq \|x\|$ for every $x \in A$. Hence $f$ is bounded. Furthermore, $f$ is unital since $f(1) \in \sigma(1) = \{1\}$. $\nabla$

**Corollary 1.5.9** *Let $f$ be a linear functional on a Banach algebra $A$. If $f(x) \in Sp\,(x)$ for every $x \in A$, then $f$ is bounded and unital.*

It follows from Lemma 1.5.7 and Corollary 1.5.9 that

10

**Corollary 1.5.10** *If $f$ is a multiplicative linear functional on a Banach algebra A, then $f$ is bounded and unital.*

**Definition 1.5.11** *Let A be a Banach algebra. An element $a \in A$ is called quasi-nilpotent if $Sp\,(a) = \{0\}$. The set of quasi-nilpotent elements of A is denoted by QN(A).*

**Theorem 1.5.12** *([2], Theorem 3.1.4) If $X$ is a Banach space, then $\mathcal{L}(X)$ is semi-simple.*

A *unital standard operator algebra* on a Banach space $X$ is a closed subalgebra of $\mathcal{L}(X)$ containing the space of finite rank operators $\mathcal{F}(X)$ and the unit element $I$ of $\mathcal{L}(X)$.

**Theorem 1.5.13** *Every unital standard operator algebra $\mathcal{B}$ on a Banach space $X$ is semi-simple.*

**Proof.** Let $x$ be a nonzero fixed element in $X$ and let $I_x = \{T : T \in \mathcal{B}, Tx = 0\}$. Then $I_x$ is a left ideal of $\mathcal{B}$:

Let $T \in I_x$ and $S \in \mathcal{B}$. Since $T \in \mathcal{B}$, we have $ST \in \mathcal{B}$. Also, $(ST)x = S(Tx) = S(0) = 0$. Thus $ST \in I_x$.

We will now show that $I_x$ is a maximal left ideal of $\mathcal{B}$. Suppose that there exists a left ideal $\mathcal{T}$ of $\mathcal{B}$ with $I_x \subsetneqq \mathcal{T}$. Then $\mathcal{T}x = \{Tx : T \in \mathcal{T}\}$ is a vector subspace of $X$ different from 0 which is invariant under all $S \in \mathcal{B}$. It follows that $\mathcal{T}x = X$:

Assume that $\mathcal{T}x \neq X$. Then there exists $\eta_2 \notin \mathcal{T}x$. Since $\mathcal{T}x \neq 0$, there exists $\eta_1 \in \mathcal{T}x$ with $\eta_1 \neq 0$. By the Hahn-Banach Theorem, there exists a finite rank operator $S \in \mathcal{L}(X)$ such that $S\eta_1 = \eta_2$. Since $S \in \mathcal{F}(X) \subset \mathcal{B}$, we have $S(\mathcal{T}x) \subset \mathcal{T}x$. This implies that $\eta_2 \in \mathcal{T}x$. This is a contradiction. Therefore $\mathcal{T}x = X$.

Since $x \in X$, there exists $U \in \mathcal{T}$ such that $Ux = x$. For an arbitrary $T \in \mathcal{B}$, we have that $TU - T \in I_x$:

Observe that $TU \in \mathcal{B}$ since $U \in \mathcal{T} \subset \mathcal{B}$. So $TU - T \in \mathcal{B}$. Since $Ux = x$, we see that $(TU - T)x = 0$, implying that $TU - T \in I_x$.

Therefore $T \in \mathcal{T} + I_x \subset \mathcal{T}$ and so $\mathcal{B} = \mathcal{T}$. Hence $I_x$ is a maximal left ideal of $\mathcal{B}$.

It follows that Rad $(\mathcal{B}) \subset \cap_{x \in X} I_x = \{0\}$, i.e. $\mathcal{B}$ is semi-simple. $\nabla$

The proof of Theorem 1.5.13 is similar to the proof of Theorem 1.5.12 given in [2]. Theorem 1.5.14 and Theorem 1.5.15 are useful characterizations of the radical of a Banach algebra $A$.

11

**Theorem 1.5.14** *([2], p. 36) The radical of a Banach algebra $A$ is the set $\{x \in A : xA \subset QN(A)\}$. Alternatively, the radical of $A$ is also $\{x \in A : Ax \subset QN(A)\}$.*

**Theorem 1.5.15 (J. Zemánek)** *([2], Theorem 5.3.1) Let $A$ be a Banach algebra. Then the following properties are equivalent.*

*(i) $a \in Rad\ (A)$.*

*(ii) $\rho(a + x) = 0$ for all $x \in QN(A)$.*

*(iii) There exists $C \geq 0$ such that $\rho(x) \leq C\|x - a\|$ for all $x$ in a neighbourhood of $a \in A$.*

Theorem 1.5.15 contains two characterizations of the radical of a Banach algebra. We will use it in Chapters 3, 4 and 5. The main ingredients of the proof are subharmonic techniques.

**Theorem 1.5.16** *([2], Theorem 4.1.2) Let $A$ be a commutative Banach algebra. Then*

$$Sp\ (x) = \{f(x) : f \text{ is a multiplicative linear functional on } A\}$$

*for every $x \in A$.*

Two characterizations of the radical of a commutative Banach algebra are the following:

**Corollary 1.5.17** *([2], Remark 1, p. 71) If $A$ is a commutative Banach algebra, then*

*(i) $Rad\ (A) = QN(A)$,*

*(ii) $Rad\ (A) = \cap\{Ker(f) : f \text{ is a multiplicative linear functional on } A\}$.*

**Theorem 1.5.18** *([2], Theorem 5.3.2) Let $A$ be a semi-simple Banach algebra and let $a \in A$. There exists $\alpha \in \mathbb{C}$ such that $a = \alpha 1$ if and only if $Sp\ (a + q)$ consists of one element for all quasi-nilpotent elements $q$ of $A$.*

**Theorem 1.5.19** *([2], Theorem 3.2.1) Suppose that $A$ is a Banach algebra and $x \in A$ with $\|x\| < 1$. Then $1 - x$ is invertible and $(1 - x)^{-1} = \sum_{k=0}^{\infty} x^k$.*

The group of invertible elements of a Banach algebra is open. This is the next result.

**Theorem 1.5.20** *([2], Theorem 3.2.3) Let $A$ be a Banach algebra and suppose that $a \in A$ is invertible. If $\|x - a\| < \frac{1}{\|a^{-1}\|}$, then $x$ is invertible in $A$.*

The following result is known as the resolvent equation for Banach algebras. We will use it in Chapter 5.

**Theorem 1.5.21** *([24], p. 379) Let $A$ be a Banach algebra and let $x \in A$. If $\lambda, \mu \notin Sp(x)$, then*

$$(\mu 1 - x)^{-1} - (\lambda 1 - x)^{-1} = (\lambda - \mu)(\lambda 1 - x)^{-1}(\mu 1 - x)^{-1}.$$

There are some useful properties of the spectrum as well as the spectral radius of an element of a Banach algebra, namely

**Theorem 1.5.22 (I. M. Gelfand)** *([2], Theorem 3.2.8) Let $A$ be a Banach algebra and $x \in A$. Then*

*(i) the mapping $\lambda \mapsto (x - \lambda 1)^{-1}$ is analytic on $\mathbb{C} \setminus Sp(x)$,*

*(ii) $Sp(x)$ is compact and nonempty,*

*(iii) $\rho(x) = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}}$.*

**Lemma 1.5.23** *([3], Lemma 2.5) Let $A$ be a semi-simple Banach algebra and let $p \in A$ be an idempotent element. Then*

*(i) $pAp$ is a closed subalgebra of $A$ with identity $p$,*

*(ii) $pAp$ is semi-simple,*

*(iii) $Sp(pxp, pAp) \subset Sp(pxp, A)$ for every $x \in A$.*

**Proof.** (i) We first show that $pAp$ is a subalgebra of $A$ with identity $p$. Let $a, b \in A$. Then $(pap)(pbp) = papbp \in pAp$. We also observe that $pap + pbp = p(a + b)p \in pAp$ and $\lambda(pap) = p(\lambda a)p \in pAp$. This proves that $pAp$ is a subalgebra of $A$. Let $a \in A$. Then $(pap)p = pap$ and $p(pap) = pap$. Hence $p$ is the identity of $pAp$.

Next, we show that $pAp$ is closed in $A$. Let $a \in \overline{pAp}$. Then there exists a sequence $(px_np)$ in $pAp$ that converges to $a$. Therefore $p(px_np)p$ converges to $pap$, i.e. $px_np$ converges to $pap$. By uniqueness of limits, $a = pap \in pAp$. Thus $pAp$ is closed in $A$.

(ii) It follows from Theorem 1.5.22(iii) that $\rho(x, A) = \rho(x, pAp)$ for every $x \in pAp$, so that we can write unambiguously $\rho(x)$. Let $a \in \mathrm{Rad}(pAp)$. Then $a = pbp$ for some $b \in A$. This implies that $pap = pbp = a$.

13

By Theorem 1.5.14, $\rho(apxp) = 0$ for every $x \in A$. Therefore, by Corollary 1.5.3, $0 = \rho(apxp) = \rho(papx) = \rho(ax)$ for all $x \in A$, i.e. $a \in \text{Rad}(A)$. Since $A$ is semi-simple, $a = 0$, implying that $pAp$ is semi-simple.

(iii) Suppose that $\lambda \notin \text{Sp}(pxp, A)$. Then there exists $y \in A$ such that $(pxp - \lambda 1)y = y(pxp - \lambda 1) = 1$. Therefore $(pxp - \lambda p)pyp = pyp(pxp - \lambda p) = p$. Hence, it follows that $pxp - \lambda p$ is invertible in $pAp$, i.e. $\lambda \notin \text{Sp}(pxp, pAp)$. This implies that $\text{Sp}(pxp, pAp) \subset \text{Sp}(pxp, A)$. $\nabla$

**Theorem 1.5.24 (I. M. Gelfand and S. Mazur)** *([2], Corollary 3.2.9) If $A$ is a Banach algebra such that every nonzero element of $A$ is invertible, then $A$ is isomorphic to $\mathbb{C}$.*

**Theorem 1.5.25** *([2], Corollary 3.2.10) Let $A$ be a Banach algebra $x, y \in A$. If $xy = yx$, then $\rho(x + y) \leq \rho(x) + \rho(y)$ and $\rho(xy) \leq \rho(x)\rho(y)$.*

We let $H(\Omega)$ denote the set of all analytic functions on an open set $\Omega$ of $\mathbb{C}$.

**Theorem 1.5.26 (Holomorphic Functional Calculus)** *([2], Theorem 3.3.3) Let $A$ be a Banach algebra and $x \in A$. Let $\Omega$ be an open set containing $\text{Sp}(x)$ and let $\Gamma$ be an arbitrary smooth contour included in $\Omega$ and surrounding $\text{Sp}(x)$. Then the mapping $f \mapsto f(x) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda 1 - x)^{-1} \, d\lambda$ from $H(\Omega)$ into $A$ has the following properties:*

*(i) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$,*

*(ii) $(f_1 f_2)(x) = f_1(x)f_2(x) = f_2(x)f_1(x)$,*

*(iii) $1(x) = 1$ and $I(x) = x$, where $I(\lambda) = \lambda$,*

*(iv) $\text{Sp}(f(x)) = f(\text{Sp}(x))$.*

Theorem 1.5.26(iv) is known as the *spectral mapping theorem*. The next result can easily be obtained from the spectral mapping theorem.

**Lemma 1.5.27** *([2], p. 40) Let $A$ be a Banach algebra and $p$ a non-trivial idempotent of $A$. Then $\text{Sp}(p) = \{0, 1\}$.*

Recall that a subset $K$ of $\mathbb{C}$ is *disconnected* if there exist nonempty disjoint subsets $S$ and $T$ of $\mathbb{C}$ such that $K \subset S \cup T$, $K \cap S \neq \emptyset$ and $K \cap T \neq \emptyset$.

**Theorem 1.5.28** *([2], Theorem 3.3.4) Let $A$ be a Banach algebra and suppose that $x \in A$ has a disconnected spectrum. Let $U_0$ and $U_1$ be disjoint open sets such that $\text{Sp}(x) \subset U_0 \cup U_1$, $\text{Sp}(x) \cap U_0 \neq \emptyset$ and $\text{Sp}(x) \cap U_1 \neq \emptyset$. Then there exists a non-trivial projection $p$ commuting with $x$, such that $\text{Sp}(px) = (\text{Sp}(x) \cap U_1) \cup \{0\}$ and $\text{Sp}(x - px) = (\text{Sp}(x) \cap U_0) \cup \{0\}$.*

14

This theorem is of special significance in the case where $\alpha$ is an isolated point of Sp $(x)$. Let $\Gamma$ be a circle with center $\alpha$, separating $\alpha$ from the rest of Sp $(x)$. Then the idempotent

$$p = \frac{1}{2\pi i} \int_\Gamma (\lambda 1 - x)^{-1} d\lambda$$

is known as the *spectral idempotent* associated with $x$ and $\alpha$.

**Theorem 1.5.29** *([2], Corollary 3.3.5) Let $A$ be a Banach algebra and $x \in A$. If $\alpha \notin Sp\ (x)$, then*

$$\text{dist}(\alpha, Sp\ (x)) = \frac{1}{\rho\big((\alpha 1 - x)^{-1}\big)}.$$

If $K$ is a compact subset of the complex plane and $r > 0$, then we define the set $K + r$ to be the set $\{\lambda : \text{dist}(\lambda, K) \leq r\}$. We denote the *Hausdorff distance* $\Delta(K_1, K_2)$ between compact subsets $K_1$ and $K_2$ of the complex plane by

$$\Delta(K_1, K_2) = \max \Big( \sup_{z \in K_1} \text{dist}(z, K_2), \sup_{z \in K_2} \text{dist}(z, K_1) \Big).$$

**Theorem 1.5.30** *([2], Theorem 3.4.2) Let $A$ be a Banach algebra. Then the spectrum function $x \mapsto Sp\ (x)$ is upper-semicontinuous on $A$, that is for every open set $U$ containing $Sp\ (x)$, there exists $\delta > 0$ such that if $\|x - y\| < \delta$, then $Sp\ (y) \subset U$.*

For a Banach algebra $A$, we say that the mapping $x \mapsto \text{Sp}\ (x)$ is *continuous* at $a \in A$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x - a\| < \delta$ implies $\Delta\Big(\text{Sp}\ (x), \text{Sp}\ (a)\Big) < \epsilon$. If $x \mapsto \text{Sp}\ (x)$ is continuous at every $a \in A$, we say that it is continuous.

If the only connected sets of Sp $(x)$ are the one-point sets, we say that Sp $(x)$ is *totally disconnected*.

**Theorem 1.5.31 (J. D. Newburgh)** *([2], Corollary 3.4.5) Let $A$ be a Banach algebra and $a \in A$. If the spectrum of $a$ is totally disconnected, then $x \mapsto Sp\ (x)$ is continuous at $a$.*

We will only apply Theorem 1.5.31 to cases where Sp $(a)$ is finite or countable.

**Theorem 1.5.32** *([2], Theorem 3.4.17, Theorem 3.4.18) Let $f$ be an analytic function from a domain $D$ of $\mathbb{C}$ into a Banach algebra $A$. Suppose that $Sp\ (f(\lambda)) = \{0, \alpha(\lambda)\}$ for all $\lambda \in D$ or $Sp\ (f(\lambda)) = \{\alpha(\lambda)\}$ for all $\lambda \in D$, where $\alpha$ is a mapping from $D$ into $\mathbb{C}$. Then $\alpha$ is analytic on $D$.*

15

The *capacity* of a subset of the complex plane is a difficult concept. For our purposes, it suffices only to know that the capacity of a subset $A$ of the complex plane is in some sense a measure of the size of $A$. Both closed disks and closed line segments have nonzero capacities. A discrete subset of the complex plane has zero capacity. If $A$ and $B$ are subsets of the complex plane and $A \subset B$, then the capacity of $A$ is less than or equal to the capacity of $B$.

**Theorem 1.5.33** *([2], Theorem 3.4.25) Let $f$ be an analytic function from a domain $D$ of $\mathbb{C}$ into a Banach algebra $A$. Then either the set $\lambda \in D$ such that $Sp\left(f(\lambda)\right)$ is finite is a Borel set having zero capacity, or there exist an integer $n \geq 1$ and a closed discrete subset $E$ of $D$ such that the number of elements of $Sp\left(f(\lambda)\right)$ is $n$ for every $\lambda \in D \setminus E$ and the number of elements of $Sp\left(f(\lambda)\right)$ is less then $n$ for every $\lambda \in E$.*

Let $f$ be an analytic function on a domain $D$ of $\mathbb{C}$. Then, by Theorem 1.5.33, if the set of $\lambda \in D$ such that $Sp\left(f(\lambda)\right)$ is finite is not a Borel set having zero capacity, then $Sp\left(f(\lambda)\right)$ is finite for every $\lambda \in D$.

**Corollary 1.5.34** *Let $A$ be a Banach algebra. If $D$ is a domain in $\mathbb{C}$, $f : D \to A$ an analytic function and the number of elements of $Sp\left(f(\lambda)\right)$ is at most $n$ for every $\lambda$ in a subset of $D$ with nonzero capacity, then $Sp\left(f(\lambda)\right)$ has at most $n$ elements for every $\lambda \in D$.*

**Proof.** Let $f : D \to A$ be an analytic function and suppose that the number of elements of $Sp\left(f(\lambda)\right)$ is at most $n$ for every $\lambda$ in a subset $E$ of $D$ having nonzero capacity. It follows from Theorem 1.5.33 that there exists $m \geq 1$ and a closed discrete subset $F$ of $D$ such that $Sp\left(f(\lambda)\right)$ has $m$ elements for every $\lambda \in D \setminus F$ and $Sp\left(f(\lambda)\right)$ has fewer than $m$ elements for every $\lambda \in F$ only. Assume that $m > n$. Then $Sp\left(f(\lambda)\right)$ has fewer than $m$ elements for every $\lambda$ in the subset $E$ of $D$ having nonzero capacity. But $Sp\left(f(\lambda)\right)$ has fewer than $m$ elements for every $\lambda \in F$ only and $F$ has zero capacity. This is a contradiction. Hence $m \leq n$. So $Sp\left(f(\lambda)\right)$ has at most $n$ elements for every $\lambda \in D$. $\nabla$

**Theorem 1.5.35 (Jacobson density theorem)** *([2], Theorem 4.2.5) Let $\pi$ be a continuous irreducible representation of a Banach algebra $A$ on a Banach space $X$. If $x_1, \ldots, x_n$ are linearly independent elements in $X$ and*

$y_1, \ldots, y_n$ are arbitrary elements in $X$, then there exists $a \in A$ such that $\pi(a)x_i = y_i$ for $i = 1, \ldots, n$.

**Theorem 1.5.36** *([2], Theorem 5.5.2) Let $A$ and $B$ be Banach algebras, with $B$ semi-simple. Suppose that $T : A \to B$ is a surjective linear mapping such that $\rho(Tx, B) \le \rho(x, A)$. Then $T$ is continuous.*

**Definition 1.5.37** *A two-sided ideal $I$ of a Banach algebra $A$ is called an inessential ideal of $A$ if $Sp\ (x)$ is finite or a sequence converging to zero for every $x \in I$.*

**Corollary 1.5.38** *If $A$ is a semi-simple Banach algebra, then Soc $(A)$ is an inessential ideal.*

**Proof.** Let $x \in $ Soc $(A)$. It follows from Theorem 1.2.6 that $xAx$ is finite-dimensional. Hence $x$ is algebraic and so, by the spectral mapping theorem, Sp $(x)$ is finite. Therefore Soc $(A)$ is an inessential ideal. $\nabla$

**Definition 1.5.39** *Let $A$ be a Banach algebra and $x \in A$. We say that $\lambda \in \mathbb{C}$ is a Riesz point of $Sp\ (x)$ relative to a two-sided ideal $I$ if $\lambda$ is an isolated point of $Sp\ (x)$ and if the spectral idempotent associated with $x$ and $\lambda$ is in $I$.*

We denote the set of Riesz points of Sp $(x)$ relative to a two-sided ideal $I$ by $\mathcal{R}_I(x)$. For a two-sided ideal $I$ of $A$, let $D_I(x)$ denote the set

$$\mathcal{D}_I(x) = \text{Sp } (x) \setminus \mathcal{R}_I(x).$$

We write $D(x)$ instead of $D_I(x)$ if there is no risk of confusion as to which two-sided ideal $I$ we are refering to. If $K$ is a subset of the complex plane, then $\sigma(K)$ denotes the union of $K$ with the bounded components of $\mathbb{C} \setminus K$.

**Theorem 1.5.40** *([2], Theorem 5.7.4(ii)) Let $I$ be an inessential ideal of a Banach algebra $A$. If $x \in A$ and $y \in I$, then the unbounded components of $\mathbb{C} \setminus D(x)$ and $\mathbb{C} \setminus D(x + y)$ coincide, i.e. $\sigma\big(D(x)\big) = \sigma\big(D(x + y)\big)$.*

A subset $U$ of a real vector space $X$ is *absorbing* if there exists $a \in U$ such that for every $x \in X$, there exists $r > 0$ such that $a + \lambda x \in U$ for $-r \le \lambda \le r$. For example, every open set is absorbing.

**Theorem 1.5.41** *([2], Theorem 5.4.2) Let $A$ be a Banach algebra. If $A$ contains an absorbing set $U$ such that $Sp\ (x)$ is finite for every $x \in U$, then $A/Rad(A)$ is finite dimensional.*

17

It follows from the proof of [2], Theorem 5.4.2 that if $A$ contains an absorbing set $U$ such that the number of elements of Sp $(x)$ is at most $n$ for every $x \in U$ and some fixed $n \in \mathbb{N}$, then the dimension of $A/\mathrm{Rad}(A)$ is at most $n^4$.

**Corollary 1.5.42** *([2], Exercise 4, p. 114) Let $A$ be a semi-simple Banach algebra. If $U$ is a nonempty open subset of $A$ such that Sp $(x)$ consists of one point for every $x \in U$, then $A$ is isomorphic to $\mathbb{C}$.*

# 1.6   C*-algebras and von Neumann algebras

C*-algebras and von Neumann algebras are special kinds of representations on a Hilbert space. They form an important study in their own right. The main result of this section is Theorem 1.6.15 which is needed in Chapter 5. We first give concepts and results that we need in order to prove Theorem 1.6.15.

**Definition 1.6.1** *A mapping $x \mapsto x^*$ from a Banach algebra $A$ into itself is called an involution on $A$ if it satisfies the following properties for all $x, y \in A$ and $\lambda \in \mathbb{C}$:*

*(i) $(x + y)^* = x^* + y^*$,*

*(ii) $(\lambda x)^* = \bar{\lambda} x^*$,*

*(iii) $(xy)^* = y^* x^*$,*

*(iv) $(x^*)^* = x$.*

Another name for a Banach algebra with involution is a *-algebra.

**Definition 1.6.2** *A Banach algebra $A$ with involution, having the property that $\|xx^*\| = \|x\|^2$ for all $x \in A$, is called a C*-algebra.*

*Let $H$ be a Hilbert space and let $M \subset \mathcal{L}(H)$. Define $M'$ to be the set $M' = \{t \in \mathcal{L}(H) : ts = st \text{ for all } s \in M\}$. A von Neumann algebra $A$ in $H$ is a *-subalgebra of $\mathcal{L}(H)$ such that $A = A''$.*

Every von Neumann algebra is a C*-algebra because $\|TT^*\| = \|T\|^2$ for all $T \in \mathcal{L}(H)$. A von Neumann algebra is also known as a *W*-algebra*.

We call an element $x$ of a C*-algebra *self-adjoint* if $x = x^*$, and *normal* if $xx^* = x^*x$. A *-isomorphism* between C*-algebras $A$ and $B$ is an isomorphism $\phi$ between $A$ and $B$ such that $\phi(x)^* = \phi(x^*)$ for every $x \in A$.

18

**Definition 1.6.3** *If $H$ is a Hilbert space with inner product $\langle . \rangle$, we define the weak-operator topology on $\mathcal{L}(H)$ as the topology defined by the seminorm $T \mapsto |\langle Tx, y \rangle|$ with $x, y \in H$.*

An equivalent characterization of a von Neumann algebra is the following result.

**Theorem 1.6.4** *([14], A3 and A4, p. 374-375) A Banach algebra $A$ is a von Neumann algebra on a Hilbert space $H$ if and only if it is a subalgebra of $\mathcal{L}(H)$ that is closed in the weak-operator topology and contains the identity element of $\mathcal{L}(H)$.*

**Proposition 1.6.5** *([14], p.4) Let $A$ be $C^*$-algebra. Every $t \in A$ can be uniquely expressed in the form $t = t_1 + it_2$, where $t_1$ and $t_2$ are self-adjoint elements of $A$.*

**Corollary 1.6.6** *Let $A$ be a von Neumann algebra. Every $t \in A$ can be uniquely expressed in the form $t = t_1 + it_2$, where $t_1$ and $t_2$ are self-adjoint elements of $A$.*

**Proposition 1.6.7** *([15], p.2) If $H$ is a Hilbert space, then $\mathcal{L}(H)$ is a von Neumann algebra.*

**Theorem 1.6.8** *([13], Corollary I.9.13) Every $C^*$-algebra is semi-simple.*

It follows that every von Neumann algebra is semi-simple.

**Definition 1.6.9** *A compact Hausdorff space $X$ is called Stonean if the closure of each open set of $X$ is open.*

**Definition 1.6.10** *A subset $B$ of a $C^*$-algebra $A$ is normal if $B \cup B^*$ is a commutative subset of $A$, where $B^* = \{x^* : x \in B\}$.*

Let $K$ be a compact subset of a topological space. We denote by $C(K)$ the set of all complex valued continuous functions on $K$.

**Lemma 1.6.11** *([8], Proposition 7, p. 190) Each normal subset of a $C^*$-algebra $A$ is contained in a maximal normal subset of $A$. Every maximal normal subset is a closed commutative $^*$-subalgebra of $A$ containing the identity element of $A$.*

Theorem 1.6.12, Lemma 1.6.13 and Lemma 1.6.14 given below are important results that are needed in order to prove Theorem 1.6.15.

19

**Theorem 1.6.12** *([32], Theorem 1.2.1) Let $A$ be a commutative $C^*$-algebra. Then $A$ is $^*$-isomorphic to $C(X)$, where $X$ is the compact Hausdorff space of all maximal ideals of $A$ (called the spectrum space of $A$).*

**Lemma 1.6.13** *([32], Proposition 1.3.1) Let $X$ be a Stonean space. Then every element in $C(X)$ can be uniformly approximated by finite linear combinations of mutually orthogonal idempotents of $C(X)$.*

**Lemma 1.6.14** *([32], Lemma 1.7.5) If $M$ is any maximal commutative $C^*$-subalgebra of a von Neumann algebra, then its spectrum space is Stonean.*

A result that will be needed in Chapter 5 is

**Theorem 1.6.15** *Let $A$ be a von Neumann algebra. If $x \in A$ is self-adjoint, then $x$ is the limit of a sequence of finite linear combinations of mutually orthogonal idempotents of $A$.*

**Proof.** Let $x \in A$ be self-adjoint. Since $x$ is normal, it follows from Lemma 1.6.11 that $x$ is contained in a maximal normal subset $B$ of $A$. Furthermore, $B$ is a closed commutative $^*$-subalgebra of $A$ containing the identity element of $A$. Hence $B$ is a commutative $C^*$-algebra. It follows from Theorem 1.6.12 that $B$ is $^*$-isomorphic to $C(X)$, where $X$ is the spectrum space of $B$. Since $A$ is a von Neumann algebra, it follows from Lemma 1.6.14 that $X$ is Stonean.

Let $\phi : C(X) \to B$ denote the isomorphism between $C(X)$ and $B$. Then $x = \phi(z)$ for some $z \in C(X)$. It follows from Lemma 1.6.13 that $z = \lim_{m \to \infty} x_m$, where $x_m = \sum_{i=1}^{n_m} \lambda_{i,m} p_{i,m}$ with $\lambda_{i,m} \in \mathbb{C}$ for all $i, m$ and the $p_{i,m}$ are mutually orthogonal idempotents in $C(X)$. Then $x = \phi(\lim_{m \to \infty} x_m)$. By Theorem 1.5.36, $\phi$ is continuous and so $x = \lim_{m \to \infty} \phi(x_m)$. It follows that $x = \lim_{m \to \infty} \sum_{i=1}^{n_m} \lambda_{i,m} \phi(p_{i,m})$. Since $\phi$ is an isomorphism, the $\phi(p_{i,m})$ are mutually orthogonal idempotents in $B$. This completes the proof. $\nabla$

We cannot extend Theorem 1.6.15 to $C^*$-algebras. For instance, consider a $C^*$-algebra $A$ having no non-trivial idempotents. Suppose that Theorem 1.6.15 holds for $A$. Then, if $x \in A$ is self-adjoint, it follows from Theorem 1.6.15 that $x$ is the limit of a sequence of scalar multiples of the identity element of $A$. This implies that every self-adjoint element of $A$ is a scalar multiple of the identity element of $A$.

Now $C([0,1])$ is a $C^*$-algebra with involution $f^*(\lambda) = \overline{f(\lambda)}$ for every $f \in C([0,1])$, having no non-trivial idempotents. The function $f$ defined by $f(\lambda) = \lambda^2$ for every $\lambda \in [0,1]$ is self-adjoint. But $f$ is certainly not a scalar multiple of the identity element of $C([0,1])$. This is a contradiction. So Theorem 1.6.15 does not hold in general for $C^*$-algebras.

20

If $a$ is a normal $n \times n$ matrix, then it is well known that $a = \sum_{i=1}^{k} \lambda_i p_i$, where $\lambda_1, \ldots, \lambda_k$ denote the distinct eigenvalues of $a$ and $p_1, \ldots, p_k$ denote self-adjoint mutually orthogonal idempotents. This is extended in the next result.

**Lemma 1.6.16** *([2], Corollary 6.2.8) Let $A$ be a $C^*$-algebra and let $x$ be a normal element of $A$ having a finite spectrum $\{\lambda_1, \ldots, \lambda_k\}$. Then there exist self-adjoint mutually orthogonal idempotents $p_1, \ldots, p_k$ in the commutative closed subalgebra generated by $1, x, x^*$ such that $p_1 + \cdots + p_k = 1$ and $x = \sum_{i=1}^{k} \lambda_i p_i$.*

The following example will be of importance in Chapter 5.

**Example 1.6.17** *There exists a $C^*$-algebra $A$ not isomorphic to a von Neumann algebra and having the property that every self-adjoint element of $A$ is the limit of a sequence of linear combinations of mutually orthogonal idempotents in $A$.*

**Proof.** Let $H$ be a finite dimensional Hilbert space and $p$ a non-trivial self-adjoint idempotent of $\mathcal{L}(H)$. It follows from Lemma 1.5.23 that $p\mathcal{L}(H)p$ is a closed subalgebra of $\mathcal{L}(H)$ with identity element $p$.

Furthermore, if $t \in p\mathcal{L}(H)p$, then $t = psp$ for some $s \in \mathcal{L}(H)$. Since $p$ is self-adjoint, $(psp)^* = ps^*p \in p\mathcal{L}(H)p$. This implies that $p\mathcal{L}(H)p$ is closed under involution. It follows from [8], Theorem 1, p. 18, that we can assume that $\|p\| = 1$. Therefore $p\mathcal{L}(H)p$ is a $C^*$-algebra.

Also, since $p$ is the identity element of $p\mathcal{L}(H)p$, it is clear that the identity element of $\mathcal{L}(H)$ is not in $p\mathcal{L}(H)p$. It follows from Theorem 1.6.4 that $p\mathcal{L}(H)p$ is not a von Neumann algebra. By definition, it is also not isomorphic to a von Neumann algebra.

If $t \in p\mathcal{L}(H)p$ is self-adjoint, then $t$ has a finite spectrum because $H$ is finite dimensional and hence, by Lemma 1.6.16, $t$ is the limit of a sequence of linear combinations of mutually orthogonal idempotents in $A$. $\nabla$

## 1.7 An extension of Liouville's theorem

The following result from complex analysis is a version of Liouville's theorem that we will need in Chapter 3.

**Theorem 1.7.1 (Extended Liouville Theorem)** *([6], Theorem 16.10) If $f$ is an entire function and at least one of the four inequalities*

$$-A|z|^n \leq \operatorname{Re} f(z),$$

21

$$\begin{aligned}
\mathrm{Re}\ f(z) &\leq A|z|^n, \\
-A|z|^n &\leq \mathrm{Im}\ f(z), \\
\mathrm{Im}\ f(z) &\leq A|z|^n
\end{aligned}$$

*holds for sufficiently large $z$, then $f$ is a polynomial of degree less than or equal to $n$.*

The following result is needed in Chapter 4.

**Corollary 1.7.2** *If $h : \mathbb{C} \to \mathbb{C}$ is an entire function such that $\lim_{\lambda \to \infty} \frac{h(\lambda)}{\lambda} = k$, then $h(\lambda) = h(0) + \lambda k$ for every $\lambda \in \mathbb{C}$.*

**Proof.** Let $\epsilon > 0$. Since $\lim_{\lambda \to \infty} \frac{h(\lambda)}{\lambda} = k$, there exists $R > 0$ such that if $|\lambda| > R$, then $|\frac{h(\lambda)}{\lambda} - k| < \epsilon$. Therefore, if $|\lambda| > R$, then $|\frac{h(\lambda)}{\lambda}| < \epsilon + |k|$. This implies that if $|\lambda| > R$, then $\mathrm{Re}\ h(\lambda) \leq |h(\lambda)| < (\epsilon + |k|)|\lambda|$. Therefore it follows from Theorem 1.7.1 that there exist $\alpha, \beta \in \mathbb{C}$ such that $h(\lambda) = \alpha + \lambda\beta$. Clearly, $\alpha = h(0)$. Since $\lim_{\lambda \to \infty} \frac{h(\lambda)}{\lambda} = k$, it follows that $\beta = k$. This proves the result. $\nabla$

22

# Chapter 2

# Jordan homomorphisms and invertibility preserving maps

The aim of this chapter is to give a few elementary properties of Jordan homomorphisms and of invertibility preserving, spectrum preserving, full spectrum preserving and spectral radius preserving linear mappings.

## 2.1 Jordan homomorphisms

**Definition 2.1.1** *Let $A$ and $B$ be Banach algebras and $T : A \to B$ a linear mapping. We say that $T$ is a Jordan homomorphism if $Tx^2 = (Tx)^2$ for all $x \in A$. We call $T$ a Jordan isomorphism if $T$ is a bijective Jordan homomorphism.*

If a nonzero linear functional $f$ on a Banach algebra $A$ is a Jordan homomorphism, we call $f$ a *Jordan functional* on $A$. Examples of Jordan homomorphisms are algebra homomorphisms.

**Corollary 2.1.2** *Let $A$ and $B$ be Banach algebras. If $T : A \to B$ is a Jordan isomorphism on a Banach algebra, then $T^{-1}$ is a Jordan isomorphism.*

The following result is an equivalent characterization of a Jordan homomorphism.

**Lemma 2.1.3** *([34], p. 13) Let $A$ and $B$ be Banach algebras and suppose that $T : A \to B$ is a linear operator. Then $T$ is a Jordan homomorphism if and only if $T(xy + yx) = TxTy + TyTx$ for all $x, y \in A$.*

**Proof.** Assume that $T(xy + yx) = TxTy + TyTx$ for all $x, y \in A$. Then $T(x^2 + x^2) = (Tx)^2 + (Tx)^2$ for all $x \in A$, i.e. $Tx^2 + Tx^2 = (Tx)^2 + (Tx)^2$

23

for all $x \in A$. Hence $2Tx^2 = 2(Tx)^2$ for all $x \in A$. So $Tx^2 = (Tx)^2$ for all $x \in A$, i.e. $T$ is a Jordan homomorphism.

Now suppose $T$ is a Jordan homomorphism, i.e. $Tx^2 = (Tx)^2$ for all $x \in A$. Then $T(x+y)^2 = \left(T(x+y)\right)^2$ for all $x, y \in A$, i.e.

$$T\Big((x+y)(x+y)\Big) = T(x+y)T(x+y)$$

for all $x, y \in A$, i.e.

$$T(x^2 + xy + yx + y^2) = (Tx + Ty)(Tx + Ty)$$

for all $x, y \in A$, i.e.

$$Tx^2 + T(xy + yx) + Ty^2 = (Tx)^2 + TxTy + TyTx + (Ty)^2$$

for all $x, y \in A$. It follows that $T(xy+yx) = TxTy+TyTx$, since $Tx^2 = (Tx)^2$ and $Ty^2 = (Ty)^2$ for all $x, y \in A$. $\nabla$

**Lemma 2.1.4** *([11], p. 186) Let $A$ and $B$ be Banach algebras and $T : A \to B$ a Jordan homomorphism. Then $T(aba) = TaTbTa$ for all $a, b \in A$.*

**Proof.** It follows from Lemma 2.1.3 that

$$
\begin{aligned}
T\Big(a(ab + ba) + (ab + ba)a\Big) &= TaT(ab + ba) + T(ab + ba)Ta \\
&= Ta(TaTb + TbTa) + (TaTb + TbTa)Ta \\
&= (Ta)^2Tb + TaTbTa + TaTbTa + Tb(Ta)^2 \\
&= (Ta^2)Tb + TaTbTa + TaTbTa + Tb(Ta^2) \\
&= T(a^2b + ba^2) + TaTbTa + TaTbTa
\end{aligned}
$$

since $T$ is a Jordan homomorphism.

Clearly, $T\Big(a(ab + ba) + (ab + ba)a\Big) = T(a^2b + ba^2) + T(aba) + T(aba)$. Therefore

$$T(a^2b + ba^2) + T(aba) + T(aba) = T(a^2b + ba^2) + TaTbTa + TaTbTa.$$

Hence $2T(aba) = 2TaTbTa$, i.e. $T(aba) = TaTbTa$ for all $a, b \in A$. $\nabla$

A Banach algebra $A$ is *simple* if the only two-sided ideals of $A$ are $A$ and $\{0\}$. The following two results are due to I. N. Herstein.

**Theorem 2.1.5** *([17], Theorem I) A Jordan automorphism of a simple Banach algebra is an automorphism or an anti-automorphism.*

**Theorem 2.1.6** *([19], Theorem 3.1) Let $A$ and $B$ be Banach algebras. If $B$ is prime and $T : A \to B$ is a Jordan homomorphism, then $T$ is either a homomorphism or an anti-homomorphism.*

24

## 2.2 Invertibility preserving, spectrum preserving and full spectrum preserving linear mappings

**Definition 2.2.1** *Let $A$ and $B$ be Banach algebras. We say that a linear map $T : A \to B$ is invertibility preserving if $Tx$ is invertible in $B$ for every invertible element $x \in A$, and $T$ is spectrum preserving if $Sp\,(Tx, B) = Sp\,(x, A)$ for every $x \in A$. We say that $T$ is full spectrum preserving if $\sigma(Tx, B) = \sigma(x, A)$ for all $x \in A$. We call $T$ spectral radius preserving if $\rho(Tx, B) = \rho(x, A)$ for all $x \in A$.*

A linear functional $f$ on a Banach algebra $A$ is invertibility preserving if and only if $f(x) \neq 0$ for all invertible $x \in A$. Note that $f$ is spectral radius preserving if and only if $|f(x)| = \rho(x)$ for all $x \in A$.

**Corollary 2.2.2** *Let $A$ be a Banach algebra and $T : A \to B$ a linear operator. If $T$ is spectrum preserving, then $T$ is full spectrum preserving and hence spectral radius preserving.*

**Lemma 2.2.3** *([11], p. 187) Let $A$ and $B$ be Banach algebras. A unital linear map $T : A \to B$ preserves invertibility if and only if $Sp\,(Ta, B) \subset Sp\,(a, A)$ for every $a \in A$.*

**Proof.** Suppose that $T$ is invertibility preserving and let $a \in A$ and $\lambda \in Sp\,(Ta, B)$. Then $Ta - \lambda T1$ is not invertible in $B$ (since $T$ is unital), i.e. $T(a - \lambda 1)$ is not invertible in $B$ (since $T$ is linear). Hence $a - \lambda 1$ is not invertible in $A$ since $T$ is invertibility preserving. So $\lambda \in Sp\,(a, A)$. Therefore $Sp\,(Ta, B) \subset Sp\,(a, A)$. This holds for every $a \in A$.

Conversely, suppose that $Sp\,(Ta, B) \subset Sp\,(a, A)$ for all $a \in A$. So $Ta - \lambda T1$ not invertible in $B$ implies that $a - \lambda 1$ is not invertible in $A$, i.e. $T(a - \lambda 1)$ not invertible in $B$ implies that $a - \lambda 1$ is not invertible in $A$. This is true for all $a \in A$. Hence $T((a + \lambda 1) - \lambda 1)$ not invertible in $B$ implies that $(a + \lambda 1) - \lambda 1$ not invertible in $A$, i.e. $Ta$ not invertible in $B$ implies that $a$ is not invertible in $A$. So $T$ is invertibility preserving. $\nabla$

Lemma 2.2.3 gives rise to Corollaries 2.2.4 and 2.2.5 given below.

**Corollary 2.2.4** *If $f$ is a unital linear functional on a Banach algebra $A$, then $f$ is invertibility preserving if and only if $f(x) \in Sp\,(x)$ for all $x \in A$.*

**Corollary 2.2.5** *Let $A$ and $B$ be Banach algebras. Then a unital linear spectrum preserving map $T : A \to B$ is invertibility preserving.*

25

The following result is elementary. We give a proof for completeness.

**Lemma 2.2.6** *Let A and B be Banach algebras. If $\phi : A \to B$ is a homomorphism or an anti-homomorphism, then $\phi$ is invertibility preserving.*

**Proof.** Let $\phi$ be a homomorphism on $A$ and $a$ an invertible element of $A$. Then there exists $b \in A$ such that $ab = ba = 1$. Since $\phi$ is a homomorphism, it follows that

$$1 = \phi(1) = \phi(ab) = \phi(a)\phi(b) \text{ and}$$

$$1 = \phi(1) = \phi(ba) = \phi(b)\phi(a).$$

Hence $\phi(a)$ is invertible in $B$, i.e. $\phi$ is invertibility preserving.

Similarly, if $\phi$ is an anti-homomorphism, then

$$1 = \phi(1) = \phi(ab) = \phi(b)\phi(a) \text{ and}$$

$$1 = \phi(1) = \phi(ba) = \phi(a)\phi(b).$$

This implies $\phi(a)$ is invertible in $B$. Hence $\phi$ is invertibility preserving. $\nabla$

**Corollary 2.2.7** *If $f$ is a multiplicative linear functional on a Banach algebra $A$, then $f$ is invertibility preserving.*

Our next result is rather trivial. We give a proof for completeness.

**Lemma 2.2.8** *Let A and B be Banach algebras. If $\phi : A \to B$ is an isomorphism or an anti-isomorphism, then $\phi$ is spectrum preserving.*

**Proof.** Let $\phi$ be an isomorphism. Then $\phi$ is a homomorphism and thus, by Lemma 2.2.6, is invertibility preserving. It follows from Lemma 2.2.3 that $\operatorname{Sp}\left(\phi(a), B\right) \subset \operatorname{Sp}(a, A)$ for all $a \in A$. Since $\phi$ is bijective, $\phi^{-1}$ exists and is also a homomorphism. This implies, by Lemma 2.2.6, that $\phi^{-1}$ is invertibility preserving. So $\operatorname{Sp}\left(\phi^{-1}(b), A\right) \subset \operatorname{Sp}(b, B)$ for all $b \in B$. Hence $\operatorname{Sp}\left(\phi^{-1}(\phi(a)), A\right) \subset \operatorname{Sp}\left(\phi(a), B\right)$ for all $a \in A$, i.e. $\operatorname{Sp}(a, A) \subset \operatorname{Sp}\left(\phi(a), B\right)$. It follows that $\operatorname{Sp}\left(\phi(a), B\right) = \operatorname{Sp}(a, A)$. Therefore $\phi$ is spectrum preserving.

Now suppose that $\phi$ is an anti-isomorphism. Then, by Lemma 2.2.6, $\phi$ is invertibility preserving. The same argument as above now applies. Thus $\phi$ is spectrum preserving. $\nabla$

**Corollary 2.2.9** *If $f$ is a bijective multiplicative linear functional on a Banach algebra $A$, then $f$ is spectrum preserving.*

26

**Corollary 2.2.10** *Let $A$ and $B$ be Banach algebras. If $\phi : A \to B$ is an isomorphism or an anti-isomorphism, then $\phi$ is full spectrum preserving and hence spectral radius preserving.*

**Theorem 2.2.11** *([34], Proposition 1.3) Let $A$ and $B$ be Banach algebras. Then every Jordan homomorphism $T : A \to B$ having 1 in its range is a unital invertibility preserving map.*

**Proof.** It follows from Lemma 2.1.3 that $T(a1 + 1a) = TaT1 + T1Ta$, i.e. $Ta + Ta = TaT1 + T1Ta$ for every $a \in A$.

By hypothesis, there exists $u \in A$ such that $1 = Tu$. Therefore

$$
\begin{aligned}
2.1 &= Tu + Tu \\
&= TuT1 + T1Tu \\
&= T1 + T1 \\
&= 2T1.
\end{aligned}
$$

Hence $T1 = 1$, i.e. $T$ is unital.

Let $a \in A$ be invertible in $A$. Then, by Lemma 2.1.4, it follows that

$$Ta = TaT(a^{-1})Ta. \tag{2.2.12}$$

Also, by Lemma 2.1.3,

$$
\begin{aligned}
2.1 &= 2T(1) \\
&= T(2.1) \\
&= T(aa^{-1} + a^{-1}a) \\
&= TaT(a^{-1}) + T(a^{-1})Ta.
\end{aligned}
$$

Let $p_1 = TaT(a^{-1})$ and $p_2 = T(a^{-1})Ta$. Then $2.1 = p_1 + p_2$. It follows from (2.2.12) that $TaT(a^{-1}) = TaT(a^{-1})TaT(a^{-1})$ and so $p_1$ is an idempotent. Similarly, $p_2$ is an idempotent. Therefore, since $p_2 = 2.1 - p_1$, it follows that $4.1 - 4p_1 + p_1 = 2.1 - p_1$. This implies that $2(p_1 - 1) = 0$. Therefore $p_1 = 1$ and so $p_2 = 1$, i.e. $TaT(a^{-1}) = T(a^{-1})Ta = 1$. It follows that $Ta$ is invertible. Hence $T$ is invertibility preserving. $\nabla$

**Corollary 2.2.13** *Let $A$ and $B$ be Banach algebras. Then every surjective Jordan homomorphism $T : A \to B$ is a unital invertibility preserving map.*

In fact, we have the following

**Theorem 2.2.14** *Let $A$ and $B$ be Banach algebras. If $T : A \to B$ is a Jordan isomorphism, then $T$ is unital and spectrum preserving.*

27

**Proof.** It follows from Corollary 2.2.13 that $T$ is a unital and invertibility preserving map and hence, by Lemma 2.2.3, that Sp $(Ta, B) \subset$ Sp $(a, A)$ for all $a \in A$.

Let $\lambda \in$ Sp $(a, A)$. Then $a - \lambda 1$ is not invertible in $A$. Since $T$ is a Jordan isomorphism, it follows from Corollory 2.1.2 that $T^{-1}$ is a Jordan isomorphism. So, since $T^{-1}(Ta - \lambda T1) = T^{-1}T(a - \lambda 1) = a - \lambda 1, Ta - \lambda T1$ is not invertible by Corollary 2.2.13, as $a - \lambda 1$ is not invertible. Therefore $\lambda \in$ Sp $(Ta, B)$. Hence Sp $(a, A) \subset$ Sp $(Ta, B)$. It follows that Sp $(a, A) =$ Sp $(Ta, B)$ for every $a \in A$, i.e. $T$ is spectrum preserving. $\nabla$

# Chapter 3

# Kaplansky's problem and some conjectures

Corollary 2.2.13 states that every surjective Jordan homomorphism between Banach algebras is a unital invertibility preserving linear map.

An interesting question is whether or not the converse of Corollary 2.2.13 is true. Stated slightly differently, if $T$ is a unital invertibility preserving linear map, is $T$ a Jordan homomorphism? This motivates a problem posed by I. Kaplansky:

> What conditions on $A, B$ and $T$ imply that every unital invertibility
> preserving linear map $T : A \to B$ is a Jordan homomorphism?

This problem is in general still unsolved. Kaplansky's problem was also motivated by a result of M. Marcus and R. Purves (Theorem 3.1.9) and the Gleason-Kahane-Żelazko Theorem (Theorem 3.2.7).

We discuss the Marcus-Purves Theorem as well as the Gleason-Kahane-Żelazko Theorem in the first two sections. In the subsequent sections we give some conjectures arising from Kaplansky's problem.

## 3.1  The Marcus-Purves Theorem

Let $M_n(\mathbb{C})$ be the Banach algebra of all $n \times n$ complex matrices and $a^t$ the transpose of the complex matrix $a$. In this section, we prove the Marcus-Purves Theorem (1959) which states that every unital invertibility preserving linear mapping from $M_n(\mathbb{C})$ into itself is either of the form $Ta = bab^{-1}$ or $ba^tb^{-1}$ for some invertible $n \times n$ complex matrix $b$. We do not give the original proof of the Marcus-Purves result (given in [26]) because this proof is entirely matrix theoretic. Instead, we examine a proof given by B. Aupetit

29

(1979). This proof is more algebraic and less matrix theoretic. We use Proposition 3.1.1 and Proposition 3.1.7 to prove Theorem 3.1.8. It will be seen that the Marcus-Purves Theorem follows from Theorem 3.1.8. The proofs of Proposition 3.1.1 and Proposition 3.1.7 are part of Aupetit's original proof of Theorem 3.1.8, as given in [5].

**Proposition 3.1.1** *If $A$ is a Banach algebra and $T : A \to M_n(\mathbb{C})$ a surjective linear mapping that is unital and invertibility preserving, then there exist $\alpha, \beta \in \mathbb{C}$ such that*

$$\det\left(T(e^{\lambda x}e^{\mu y})e^{-\lambda Tx}e^{-\mu Ty}\right) = e^{\alpha\lambda + \beta\mu}$$

*for all $\lambda, \mu \in \mathbb{C}$.*

**Proof.** Since $T$ is linear, unital and invertibility preserving, it follows that $\mathrm{Sp}\left(Tx, M_n(\mathbb{C})\right) \subset \mathrm{Sp}\left(x, A\right)$, so that $\rho\left(Tx, M_n(\mathbb{C})\right) \leq \rho(x, A)$, for all $x \in A$. Since $M_n(\mathbb{C})$ is semi-simple, it follows from Theorem 1.5.36 that $T$ is continuous. Therefore, for any $x, y \in A$, the function $(\lambda, \mu) \mapsto T(e^{\lambda x}e^{\mu y})$ is analytic. Also, $(\lambda, \mu) \mapsto e^{-\lambda Tx}e^{-\mu Ty}$ is analytic, and so is the determinant. It follows that the function

$$(\lambda, \mu) \mapsto \phi(\lambda, \mu) = \det\left(T(e^{\lambda x}e^{\mu y})e^{-\lambda Tx}e^{-\mu Ty}\right)$$

is analytic in $\lambda, \mu$.

Since $e^{\lambda x}e^{\mu y}$ is invertible for all $\lambda, \mu \in \mathbb{C}$ and $T$ is invertibility preserving, it follows that $T(e^{\lambda x}e^{\mu y})$ is invertible for all $\lambda, \mu \in \mathbb{C}$. So, for all $\lambda, \mu \in \mathbb{C}$, $T(e^{\lambda x}e^{\mu y})e^{-\lambda Tx}e^{-\mu Ty}$ is invertible because $e^{-\lambda Tx}e^{-\mu Ty}$ is invertible for all $\lambda, \mu \in \mathbb{C}$. Therefore $\phi(\lambda, \mu) = \det\left(T(e^{\lambda x}e^{\mu y})e^{-\lambda Tx}e^{-\mu Ty}\right) \neq 0$ for all $\lambda, \mu \in \mathbb{C}$. Therefore the function $(\lambda, \mu) \to \phi(\lambda, \mu)$ has no zeros.

Since $\phi$ is analytic and has no zeros, it follows that there exists an entire function $\psi(\lambda, \mu)$ such that

$$e^{\psi(\lambda, \mu)} = \phi(\lambda, \mu). \tag{3.1.2}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of $T(e^{\lambda x}e^{\mu y})e^{-\lambda Tx}e^{-\mu Ty}$, counting multiplicity. Then

$$
\begin{aligned}
|\phi(\lambda, \mu)| &= \left|\det\left(T(e^{\lambda x}e^{\mu y})e^{-\lambda Tx}e^{-\mu Ty}\right)\right| \\
&= |\lambda_1 \ldots \lambda_n| \\
&= |\lambda_1| \ldots |\lambda_n|
\end{aligned}
$$

30

$$\leq \left( \rho \Big( T(e^{\lambda x} e^{\mu y}) e^{-\lambda Tx} e^{-\mu Ty} \Big) \right)^n$$

$$\leq \| T(e^{\lambda x} e^{\mu y}) e^{-\lambda Tx} e^{-\mu Ty} \|^n$$

$$\leq \| T \|^n e^{\Big( |\lambda| n(\|x\| + \|Tx\|) + |\mu| n(\|y\| + \|Ty\|) \Big)},$$

since $T$ is continuous. Thus, by (3.1.2),

$$|e^{\psi(\lambda,\mu)}| \leq \| T \|^n e^{\Big( |\lambda| n(\|x\| + \|Tx\|) + |\mu| n(\|y\| + \|Ty\|) \Big)},$$

i.e.

$$e^{\mathrm{Re}\ \psi(\lambda,\mu)} \leq \| T \|^n e^{\Big( |\lambda| n(\|x\| + \|Tx\|) + |\mu| n(\|y\| + \|Ty\|) \Big)},$$

so that

$$\mathrm{Re}\ \psi(\lambda,\mu) \leq \ln \| T \|^n + |\lambda| n(\|x\| + \|Tx\|) + |\mu| n(\|y\| + \|Ty\|),$$

since the logarithmic function is increasing. Let $L = n(\|x\| + \|Tx\|)$, $M = n(\|y\| + \|Ty\|)$ and $N = \ln \| T \|^n$. Then

$$\mathrm{Re}\ \psi(\lambda,\mu) \leq L|\lambda| + M|\mu| + N. \tag{3.1.3}$$

Let $g(\lambda,\mu) = \psi(\lambda,\mu) - N - M|\mu|$. Then $\mathrm{Re}\ g(\lambda,\mu) = \mathrm{Re}\ \psi(\lambda,\mu) - N - M|\mu|$. So, by (3.1.3), $\mathrm{Re}\ g(\lambda,\mu) \leq L|\lambda|$. For a fixed $\mu$, it follows from Theorem 1.7.1 that $g(\lambda,\mu)$ is a polynomial in $\lambda$ of degree 0 or 1. Hence $\psi(\lambda,\mu)$ is a polynomial in $\lambda$ of degree 0 or 1. So

$$\psi(\lambda,\mu) = f_1(\mu)\lambda + f_2(\mu), \tag{3.1.4}$$

where $f_1$ and $f_2$ are entire functions. Let $\lambda$ be a positive real number. Then, by (3.1.3) and (3.1.4),

$$\Big( \mathrm{Re}\ f_1(\mu) \Big)\lambda + \mathrm{Re}\ f_2(\mu) = \mathrm{Re}\ \psi(\lambda,\mu) \leq L\lambda + M|\mu| + N.$$

Hence

$$\Big( \mathrm{Re}\ f_1(\mu) \Big)\lambda - L\lambda \leq N - \mathrm{Re}\ f_2(\mu) + M|\mu|,$$

so that

$$\Big( \mathrm{Re}\ f_1(\mu) - L \Big)\lambda \leq N + M|\mu| - \mathrm{Re}\ f_2(\mu)$$

and hence

$$\mathrm{Re}\ f_1(\mu) - L \leq \frac{N + M|\mu| - \mathrm{Re}\ f_2(\mu)}{\lambda}.$$

31

Therefore it follows that $\lim_{\lambda\to\infty}\left(\text{Re } f_1(\mu) - L\right) \leq 0$ and so Re $f_1(\mu) \leq L$. Thus, by Liouville's theorem, the mapping $\mu \to e^{f_1(\mu)}$ is constant, implying that $f_1$ is constant, say $f_1(\mu) = \alpha$ for all $\mu \in \mathbb{C}$. It follows from (3.1.4) that

$$\psi(\lambda, \mu) = \alpha\lambda + f_2(\mu). \qquad (3.1.5)$$

A similar reasoning, with $\lambda$ fixed, shows that $\psi(\lambda, \mu) = g_1(\lambda)\mu + g_2(\lambda)$, where $g_1$ and $g_2$ are entire and $g_1(\lambda) = \beta$ (say). Therefore

$$\psi(\lambda, \mu) = \beta\mu + g_2(\lambda).$$

It follows from (3.1.5) that $\alpha\lambda - g_2(\lambda) = \beta\mu - f_2(\mu)$ for all $\lambda, \mu \in \mathbb{C}$. Taking $\mu = 0$, we have that $\alpha\lambda - g_2(\lambda) = -f_2(0)$ for all $\lambda \in \mathbb{C}$. This implies that $\alpha\lambda - g_2(\lambda)$ is constant. Therefore $\alpha\lambda - g_2(\lambda) = \beta\mu - f_2(\mu)$ is constant, say

$$\alpha\lambda - g_2(\lambda) = \beta\mu - f_2(\mu) = k. \qquad (3.1.6)$$

It follows from (3.1.5) and (3.1.6) that

$$\begin{aligned}\psi(\lambda, \mu) &= \alpha\lambda + f_2(\mu) \\ &= \alpha\lambda + \beta\mu - k.\end{aligned}$$

Thus

$$\begin{aligned}e^{\psi(\lambda,\mu)} &= e^{\alpha\lambda + \beta\mu}e^{-k} \\ &= \gamma e^{\alpha\lambda + \beta\mu}, \text{ where } \gamma = e^{-k}.\end{aligned}$$

Therefore $\gamma e^{\alpha\lambda + \beta\mu} = e^{\psi(\lambda,\mu)} = \phi(\lambda, \mu) = \det\left(T(e^{\lambda x}e^{\mu y})e^{-\lambda Tx}e^{-\mu Ty}\right)$. By taking $\lambda = 0$ and $\mu = 0$, it follows that $\gamma = 1$, since $T$ is unital. $\nabla$

**Proposition 3.1.7** *Let $A$ is a Banach algebra and $T : A \to M_n(\mathbb{C})$ a surjective linear mapping. If there exist $\alpha, \beta \in \mathbb{C}$ such that*

$$\det\left(T(e^{\lambda x}e^{\mu y})e^{-\lambda Tx}e^{-\mu Ty}\right) = e^{\alpha\lambda + \beta\mu}$$

*for all $\lambda, \mu \in \mathbb{C}$, then $T$ is a Jordan homomorphism.*

The proof of this result involves matrix theory. We thus omit the proof as it falls outside our scope. Proposition 3.1.1 and Proposition 3.1.7 now imply our next result.

**Theorem 3.1.8 (B. Aupetit)** *([5], Theorem 1) If $A$ is a Banach algebra and $T : A \to M_n(\mathbb{C})$ a surjective linear mapping that is unital and invertibility preserving, then $T$ is a homomorphism or an anti-homomorphism.*

32

**Proof.** By Proposition 3.1.1, there exist $\alpha, \beta \in \mathbb{C}$ such that, for all $\lambda, \mu \in \mathbb{C}$,

$$\det \left( T(e^{\lambda x} e^{\mu y}) e^{-\lambda T x} e^{-\mu T y} \right) = e^{\alpha\lambda + \beta\mu}.$$

Therefore, by Proposition 3.1.7, $T$ is a Jordan homomorphism. Since $M_n(\mathbb{C})$ is a prime Banach algebra, the result follows from Theorem 2.1.6. $\nabla$

We formally state the Marcus-Purves Theorem, which follows from Theorem 3.1.8.

**Theorem 3.1.9 (M. Marcus and R. Purves)** *([26], Theorem 2.1) If $T$ : $M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital linear mapping preserving invertible elements, then $T$ is either of the form $Ta = bab^{-1}$ or $Ta = ba^t b^{-1}$ for some invertible complex matrix $b$.*

This result follows from Theorem 3.1.8 and Corollary 1.3.11 since the bijectivity of $T$ is implied by [26], Lemma 2.3. Therefore Theorem 3.1.8 by Aupetit extends the Marcus-Purves Theorem. Marcus and Purves originally proved their result without assuming that $T$ is unital. They have thus proved a result that is slightly stronger than Theorem 3.1.9. The next corollary follows from Proposition 1.3.6.

**Corollary 3.1.10** *If $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital linear mapping preserving invertibility, then $T$ is an automorphism or an anti-automorphism.*

**Corollary 3.1.11** *If $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital linear mapping preserving invertibility, then $T$ is a Jordan automorphism.*

Corollary 3.1.11 is an answer to Kaplansky's question if $A = B = M_n(\mathbb{C})$. This is why Theorem 3.1.9 inspired Kaplansky's problem. It follows from Theorem 2.1.5 that Corollary 3.1.11 implies Corollary 3.1.10. Also, it follows from Corollary 1.3.11 that Corollary 3.1.10 implies Theorem 3.1.9. Therefore Corollary 3.1.11 is an equivalent formulation of the Marcus-Purves result. We give another proof of the Marcus-Purves result in Chapter 5.

Another solution to Kaplansky's problem is the result below. It is in fact the first part of the proof of Theorem 3.1.8. We restate it for clarity.

**Corollary 3.1.12** *If $A$ is a Banach algebra and $T : A \to M_n(\mathbb{C})$ is a surjective linear mapping that is unital and invertibility preserving, then $T$ is a Jordan homomorphism.*

Corollary 3.1.12 extends Corollary 3.1.11 since the bijectivity of $T$ in Corollary 3.1.11 is implied by [26], Lemma 2.3.

33

## 3.2 The Gleason-Kahane-Żelazko Theorem

In 1967–1968 J.P. Kahane and W. Żelazko proved that a linear functional $f$ on a commutative Banach algebra $A$ is multiplicative if and only if $f(x) \in$ Sp $(x)$ for every $x \in A$ (see [22]). Later, still in 1968, Żelazko proved that the commutativity of $A$ can be dropped (see [35]). A.M. Sinclair gave a short and elegant proof of Lemma 3.2.1 below. Lemma 3.2.1 is an important part of the proof of the above mentioned result of Żelazko.

**Lemma 3.2.1** *([8], Proposition 16.6) Every Jordan functional $f$ on a Banach algebra $A$ is multiplicative.*

**Proof.** If $f$ is a Jordan functional on $A$, then $f\left((x + y)^2\right) = \left(f(x + y)\right)^2$ for all $x, y \in A$. Expanding both sides, we get

$$f(x^2 + xy + yx + y^2) = \left(f(x) + f(y)\right)^2,$$

so that

$$f(x^2) + f(xy + yx) + f(y^2) = \left(f(x)\right)^2 + 2f(x)f(y) + \left(f(y)\right)^2.$$

Therefore, since $f$ is a Jordan functional,

$$f(xy + yx) = 2f(x)f(y) \tag{3.2.2}$$

for all $x, y \in A$.

We will now show that $f$ is unital. Let $x \in A$. Then, by (3.2.2), $2f(x) = f(x) + f(x) = f(x.1 + 1.x) = 2f(x)f(1)$. In particular, if $y \in A$ is not in the kernel of $f$, then $2f(y) = 2f(y)f(1)$. It follows that $f(1) = 1$, i.e. $f$ is unital.

Assume that $f$ is not multiplicative. We are going to show that there exist $a, b \in A$ such that $f(a) = 0$ and $f(ab) = 1$. Since $f$ is not multiplicative, there exist $x, y \in A$ such that $f(xy) \neq f(x)f(y)$, i.e. $f(xy) - f(x)f(y) \neq 0$. Let $a = x - f(x)1$. Then, since $f$ is unital,

$$
\begin{aligned}
f(a) &= f\left(x - f(x)1\right) \\
&= f(x) - f\left(f(x)1\right) \\
&= f(x) - f(x)f(1) \\
&= f(x) - f(x) \\
&= 0.
\end{aligned}
$$

34

Also,

$$
\begin{aligned}
f(xy) - f(x)f(y) &= f(xy) - f\big(f(x)y\big) \\
&= f\big(xy - f(x)y\big) \\
&= f\big((x - f(x)1)y\big) \\
&= f(ay).
\end{aligned}
$$

Since $f(xy) - f(x)f(y) \neq 0$, we have $f(ay) \neq 0$, say $f(ay) = k \neq 0$. Therefore $\frac{1}{k}f(ay) = 1$ and so $f(a.\frac{y}{k}) = 1$. Let $b = \frac{y}{k}$. Hence we have shown that there exist $a, b \in A$ such that $f(a) = 0$ and $f(ab) = 1$.

Since, by (3.2.2), $f(ab) + f(ba) = f(ab + ba) = 2f(a)f(b) = 2.0.f(b) = 0$ and $f(ab) = 1$, it follows that $f(ba) = -1$. Let $c = bab$. Then, using (3.2.2) and the fact that $f$ is a Jordan functional,

$$
\begin{aligned}
0 &= 2f(a)f(c) \\
&= f(ac + ca) \\
&= f(ac) + f(ca) \\
&= f\big((ab)^2\big) + f\big((ba)^2\big) \\
&= f(ab)^2 + f(ba)^2 \\
&= 1^2 + (-1)^2 \\
&= 2.
\end{aligned}
$$

This is a contradiction. Therefore $f$ is multiplicative. $\nabla$

The proof of Lemma 3.2.1 was due to A. M. Sinclair. Our next result is slightly stronger than Theorem 3.2.6. The proof is similar to that of [22], Theorem 2.

**Theorem 3.2.3** *Let $A$ be a Banach algebra and $f$ a linear functional on $A$ having no exponentials as zeros. Then $f$ is multiplicative if and only if $f$ is bounded and unital.*

**Proof.** If $f$ is a multiplicative linear functional on $A$, then it follows from Corollary 1.5.10 that $f$ is bounded and unital.

Suppose that $f$ is bounded and unital. Let $x \in A$ be arbitrary and $\phi(\lambda) = f(e^{\lambda x})$. Since the function $\lambda \mapsto e^{\lambda x}$ is entire and $f$ is bounded and linear, we have that $\phi$ is entire. Since $f$ has no zeros that are exponentials, $\phi(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$. It follows that $\phi(\lambda) = e^{\psi(\lambda)}$ for some entire function $\psi$. Also,

$$
|\phi(\lambda)| = |f(e^{\lambda x})| \leq \|f\| \|e^{\lambda x}\| \leq \|f\| e^{|\lambda| \|x\|}
$$

35

and

$$|\phi(\lambda)| = |e^{\psi(\lambda)}| = e^{\mathrm{Re}\ \psi(\lambda)}.$$

So

$$e^{\mathrm{Re}\ \psi(\lambda)} \leq \|f\|e^{|\lambda|\|x\|} = e^{\ln\|f\|+|\lambda|\|x\|}.$$

Since the exponential function is increasing, it follows that $\mathrm{Re}\ \psi(\lambda) \leq \ln\|f\| + |\lambda|\|x\|$.

Let $g(\lambda) = \psi(\lambda) - \ln\|f\|$. Then $\mathrm{Re}\ g(\lambda) = \mathrm{Re}\ \psi(\lambda) - \ln\|f\|$, so that $\mathrm{Re}\ g(\lambda) \leq |\lambda|\|x\|$. Furthermore, $g$ is entire since $\psi$ is entire. By Theorem 1.7.1, $g$ is a polynomial of degree 0 or 1. This implies that $\psi$ is a polynomial of degree 0 or 1, say $\psi(\lambda) = \alpha\lambda + \beta$ for some $\alpha, \beta \in \mathbb{C}$.

Since $\phi(0) = f(e^{0.x}) = f(1) = 1$, it follows that $1 = \phi(0) = e^{\psi(0)}$. Therefore $\psi(0) = 2\pi mi$ for some $m \in \mathbb{Z}$, i.e. $\beta = 2\pi mi$, so that $\psi(\lambda) = \alpha\lambda + 2\pi mi$. It follows that

$$\phi(\lambda) = e^{\psi(\lambda)} = e^{\alpha\lambda} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}\lambda^n.$$

Since $f$ is bounded and linear, it follows from the definition of $\phi$ that

$$\phi(\lambda) = \sum_{n=0}^{\infty} \frac{f(x^n)}{n!}\lambda^n.$$

Hence, by comparing coefficients, we have $f(x^n) = \alpha^n$. Thus, in particular, $f(x) = \alpha$ and so $\alpha^n = f(x)^n$, so that $f(x^n) = f(x)^n$ for all $n \in \mathbb{N}$. Hence $f(x^2) = f(x)^2$, implying that $f$ is a Jordan functional on $A$. It follows from Lemma 3.2.1 that $f$ is multiplicative. $\nabla$

**Corollary 3.2.4** *Let $A$ be a Banach algebra and $f$ a linear functional on $A$ having no exponentials as zeros. Then $f$ is multiplicative if and only if $f(x) \in \sigma(x)$ for all $x \in A$.*

**Proof.** Suppose that $f$ is multiplicative. Then, by Lemma 1.5.7, $f(x) \in \mathrm{Sp}\ (x)$ for all $x \in A$. Therefore $f(x) \in \sigma(x)$ for all $x \in A$.

Conversely, assume that $f(x) \in \sigma(x)$ for all $x \in A$. Then, by Proposition 1.5.8, $f$ is bounded and unital. It follows from Theorem 3.2.3 that $f$ is multiplicative. $\nabla$

**Corollary 3.2.5** *Let $A$ be a Banach algebra and $f$ a linear functional on $A$ having no exponentials as zeros. Then $f$ is multiplicative if and only if $f$ is unital and $|f(x)| \leq \rho(x)$ for all $x \in A$.*

36

**Proof.** Suppose that $f$ is multiplicative. Then, by Corollary 3.2.4, $f(x) \in \sigma(x)$ for all $x \in A$. Therefore $f$ is unital and $|f(x)| \le \rho(x)$ for all $x \in A$.

Conversely, assume that $f$ is unital and that $|f(x)| \le \rho(x)$ for all $x \in A$. Then $f$ is bounded since $\rho(x) \le \|x\|$ for every $x \in A$. Hence, by Theorem 3.2.3, $f$ is multiplicative. $\nabla$

A consequence of Theorem 3.2.3 is the Gleason-Kahane-Żelazko Theorem given below.

**Theorem 3.2.6 (Gleason-Kahane-Żelazko)** *([22], Theorem 2; [35], Theorem 2 and [16]) Let $A$ be a Banach algebra. Then a linear functional $f$ on $A$ is multiplicative if and only if $f(x) \in Sp\ (x)$ for every $x \in A$.*

**Proof.** Suppose that $f$ is multiplicative. By Lemma 1.5.7, $f(x) \in \text{Sp}\ (x)$ for all $x \in A$.

Conversely, assume that $f(x) \in \text{Sp}\ (x)$ for all $x \in A$. Then it follows from Corollary 1.5.9 that $f$ is bounded and unital. Furthermore, $f(e^x) \in \text{Sp}\ (e^x)$ for all $x \in A$, implying that $f$ has no exponentials as zeros. By Theorem 3.2.3, the result follows. $\nabla$

It follows from Lemma 2.2.3 that an equivalent formulation of Theorem 3.2.6 is

**Theorem 3.2.7 (Gleason-Kahane-Żelazko)** *Let $A$ be a Banach algebra and $f$ a linear functional on $A$. Then $f$ is multiplicative if and only if $f$ is unital and invertibility preserving.*

Theorem 3.1.8 by Aupetit is also an extension of the Gleason-Kahane-Żelazko Theorem (Theorem 3.2.7) for surjective linear functionals.

From Theorem 3.2.7, we see that an answer to Kaplansky's question in the case $B = \mathbb{C}$ is

**Corollary 3.2.8** *A linear functional $f$ on a Banach algebra $A$ that is unital and invertibility preserving is a Jordan functional on $A$.*

In fact, since the kernel of an invertibility preserving linear functional on a Banach algebra $A$ contains no exponentials of $A$, it follows that the proof of Corollary 3.2.8 is already contained in the proof of Theorem 3.2.3 given above.

The Gleason-Kahane-Żelazko Theorem, namely Theorem 3.2.6, can be extended to general linear operators. This is the following result.

**Theorem 3.2.9** *([22], Theorem 4) Let $A$ and $B$ be Banach algebras and suppose that $B$ is commutative and semi-simple. If $T : A \to B$ is a linear mapping such that $Sp\,(Tx, B) \subset Sp\,(x, A)$ for every $x \in A$, then $T$ is multiplicative.*

**Proof.** Let $f$ be a multiplicative linear functional on $B$ and let $F(x) = f(Tx)$ for all $x \in A$. Then $F$ is a linear functional on $A$. We also have, by Theorem 3.2.6, that $F(x) = f(Tx) \in Sp\,(Tx, B) \subset Sp\,(x, A)$ for all $x \in A$. Hence, by Theorem 3.2.6, $F$ is a multiplicative linear functional on $A$. So $F(xy) = F(x)F(y)$ for all $x, y \in A$. Hence $f(T(xy)) = f(Tx)f(Ty) = f(TxTy)$ for all $x, y \in A$. It follows that $f(T(xy) - TxTy) = 0$ for all $x, y \in A$ and for all multiplicative linear functionals $f$ on B, i.e. $T(xy) - TxTy \in \text{Ker}(f)$ for all multiplicative linear functionals $f$ on $B$. Therefore, by Corollary 1.5.17(ii), $T(xy) - TxTy \in \text{Rad } B = \{0\}$. It follows that $T$ is multiplicative. $\nabla$

It follows from Lemma 2.2.3 that an equivalent formulation of Theorem 3.2.9 is

**Corollary 3.2.10** *Let $A$ and $B$ be Banach algebras and suppose that $B$ is commutative and semi-simple. If $T : A \to B$ is linear, unital and invertibility preserving, then $T$ is multiplicative.*

In fact, we have the following solution to Kaplansky's problem.

**Corollary 3.2.11** *Let $A$ and $B$ be Banach algebras and suppose that $B$ is commutative and semi-simple. If $T : A \to B$ is linear, unital and invertibility preserving, then $T$ is a Jordan homomorphism.*

We conclude this section with another result due to Aupetit, namely Theorem 3.2.12. In fact, it is another solution to Kaplansky's problem.

**Theorem 3.2.12** *([5], Theorem 2) Let $A$ and $B$ be Banach algebras. Suppose that $B$ has a separating family of finite dimensional irreducible representations. If $T : A \to B$ is a surjective linear mapping that is unital and invertibility preserving, then $T$ is a Jordan homomorphism.*

**Proof.** Let $\pi$ be an irreducible representation of $B$ belonging to the separating family $\mathcal{S}$ having finite dimensional irreducible representations. By Lemma 1.3.9, $\pi(B)$ is isomorphic to $M_n(\mathbb{C})$ for a certain $n \in \mathbb{N}$. Since $T$ is surjective, $(\pi \circ T)(A) = \pi\Big(T(A)\Big) = \pi(B) \cong M_n(\mathbb{C})$, i.e. $\pi \circ T$ maps onto $M_n(\mathbb{C})$. Furthermore, $\pi \circ T$ is unital because $(\pi \circ T)(1) = \pi(T1) = \pi(1) = 1$.

Let $a$ be an invertible element of $A$. Then $Ta$ is an invertible element in $B$. Since $\pi$ is a homomorphism, it follows from Lemma 2.2.6 that $\pi(Ta)$ is

38

invertible. This implies that $\pi \circ T$ is invertibility preserving. By Theorem 3.1.8, $\pi \circ T$ is a homomorphism or an anti-homomorphism. Hence, for any $x \in A$,

$$
\begin{aligned}
\pi\Big(Tx^2 - (Tx)^2\Big) &= \pi(Tx^2) - \pi\Big((Tx)^2\Big) \\
&= (\pi \circ T)x^2 - \pi(Tx.Tx) \\
&= (\pi \circ T)x^2 - \pi(Tx)\pi(Tx) \\
&= (\pi \circ T)x^2 - \Big((\pi \circ T)x\Big)^2 \\
&= (\pi \circ T)x^2 - (\pi \circ T)x^2 \\
&= 0.
\end{aligned}
$$

Therefore $Tx^2 - (Tx)^2$ is in the kernel of every $\pi$ in $\mathcal{S}$. In particular, $Tx^2 - (Tx)^2$ is in the kernel of $\pi_1$, where $\pi_1$ has the property that $\pi_1(x) \neq 0$ for every nonzero $x \in B$. Hence $Tx^2 - (Tx)^2 = 0$ for every $x \in A$, i.e. $Tx^2 = (Tx)^2$ for every $x \in A$. So $T$ is a Jordan homomorphism. $\nabla$

Since $\mathcal{L}(\mathbb{C})$ is isomorphic to $\mathbb{C}$, we have that $\mathbb{C}$ has a separating family of finite dimensional irreducible representations, namely $\mathcal{S} = \{\pi\}$, where $\pi(x) = x$ for every $x \in \mathbb{C}$. Therefore the Gleason-Kahane-Żelazko Theorem for surjective linear functionals follows from Theorem 3.2.12.

## 3.3 A reasonable conjecture arising from Kaplansky's problem

The results in Section 3.2, together with the following result, will help us to formulate Kaplansky's question into a reasonable conjecture.

**Theorem 3.3.1 (A. R. Sourour)** *([34], Theorem 1.1) Let $X$ and $Y$ be Banach spaces and let $T$ be a unital bijective linear map from $\mathcal{L}(X)$ onto $\mathcal{L}(Y)$. The following conditions are equivalent:*

*(i) $T$ preserves invertibility.*

*(ii) $T$ is a Jordan isomorphism.*

*(iii) $T$ is either an isomorphism or an anti-isomorphism.*

*(iv) Either $Y$ is isomorphic to $X$ and $Ta = b^{-1}ab$ for every $a \in \mathcal{L}(X)$, where $b$ is an isomorphism from $Y$ to $X$; or*
    *$Y$ is isomorphic to $X'$, where $X'$ denotes the dual space of $X$, and*

39

$Ta = c^{-1}a^{\times}c$ *for every* $a \in \mathcal{L}(X)$, *where* $c$ *is an isomorphism from* $Y$ *to* $X'$.

We omit the proof since it is operator theoretic.

**Corollary 3.3.2** *Let* $X$ *and* $Y$ *be Banach spaces. A unital bijective linear map* $T : \mathcal{L}(X) \to \mathcal{L}(Y)$ *preserves invertibility if and only if it is an isomorphism or an anti-isomorphism.*

This yields the following solution to Kaplansky's problem if $A = \mathcal{L}(X)$ and $B = \mathcal{L}(Y)$, where $X$ and $Y$ are Banach spaces.

**Corollary 3.3.3** *Let* $X$ *and* $Y$ *be Banach spaces. If* $T : \mathcal{L}(X) \to \mathcal{L}(Y)$ *is a unital bijective linear mapping preserving invertibility, then it is a Jordan isomorphism.*

Note that $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ are semi-simple. This, along with Corollary 3.3.3, suggests the following reasonable conjecture:

**Conjecture C1** Let $A$ and $B$ be semi-simple Banach algebras and $T : A \to B$ a unital bijective linear map preserving invertibility. Then $T$ is a Jordan isomorphism.

The following example illustrates that at least the surjectivity of $T$ is essential.

**Example 3.3.4** *([34], Example 2) Let* $H$ *be a Hilbert space. There exists a unital linear invertibility preserving map* $T : \mathcal{L}(H) \to \mathcal{L}(H \oplus H)$ *such that* $T$ *is not a Jordan homomorphism and not surjective.*

**Proof.** Let $H$ be a Hilbert space and $e$ the unit element of $\mathcal{L}(H)$. There exists a nonzero linear functional $f$ on $\mathcal{L}(H)$ such that $f(e) = 0$:

Since $e \neq 0$, the set $\{e\}$ is linearly independent. Therefore there exists a basis $B$ of $H$ containing $e$. Let $b \in B$. Clearly, $\{e, b\}$ is a linearly independent set. Let $Z = \operatorname{span}\{e, b\}$ and let $x \in Z$ with $x = \alpha_1 e + \alpha_2 b$, where $\alpha_1, \alpha_2 \in \mathbb{C}$.

Define a function $f : Z \to \mathbb{C}$ by $f(x) = \alpha_2$. Note that $f$ is well defined since $\{e, b\}$ is a basis for $Z$. Clearly, $f$ is linear. Also, $f(e) = 0$ and $f(b) = 1$. Thus $f$ is not identical to zero. By the Hahn-Banach Theorem, $f$ can be extended to a (nonzero) linear functional on $H$.

Define $T : \mathcal{L}(H) \to \mathcal{L}(H \oplus H)$ by

$$Ta = \begin{pmatrix} a & f(a)e \\ 0 & a \end{pmatrix}.$$

40

It follows that $T$ is unital, linear and invertibility preserving:

$$\begin{aligned}
T(\alpha a + \beta b) &= \begin{pmatrix} \alpha a + \beta b & f(\alpha a + \beta b)e \\ 0 & \alpha a + \beta b \end{pmatrix} \\
&= \alpha \begin{pmatrix} a & f(a)e \\ 0 & a \end{pmatrix} + \beta \begin{pmatrix} b & f(b)e \\ 0 & b \end{pmatrix} \\
&= \alpha Ta + \beta Tb.
\end{aligned}$$

Hence $T$ is linear.

We now show that $T$ is invertibility preserving: Suppose that $a$ is invertible in $\mathcal{L}(H)$. Then there exists $b \in \mathcal{L}(H)$ such that $ab = ba = e$. It follows that

$$\begin{pmatrix} b & -f(a)b^2 \\ 0 & b \end{pmatrix}$$

is the inverse of

$$Ta = \begin{pmatrix} a & f(a)e \\ 0 & a \end{pmatrix}.$$

This implies that $Ta$ is invertible and hence $T$ is invertibility preserving.

Also, $T$ is unital since

$$Te = \begin{pmatrix} e & f(e)e \\ 0 & e \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix},$$

as $f(e) = 0$.

However, $T$ is not a Jordan homomorphism:

$$Ta^2 = \begin{pmatrix} a^2 & f(a^2)e \\ 0 & a^2 \end{pmatrix}.$$

But,

$$(Ta)^2 = \begin{pmatrix} a^2 & 2f(a)a \\ 0 & a^2 \end{pmatrix}.$$

Observe that $f(a^2)e \neq 2f(a)a$ for at least one $a \in \mathcal{L}(H)$: Suppose that $f(a^2)e = 2f(a)a$ for all $a \in \mathcal{L}(H)$. Since $f$ is nonzero, there exists $x \in \mathcal{L}(H)$ such that $f(x) \neq 0$. In particular, $f(x^2)e = 2f(x)x$. So $x = \frac{f(x^2)}{2f(x)}e = \lambda e$, with $\lambda = \frac{f(x^2)}{2f(x)}$. Now $f(x) = f(\lambda e) = \lambda f(e) = 0$. This is a contradiction. So $f(a^2)e \neq 2f(a)a$ for at least one $a \in \mathcal{L}(H)$.

It follows that there exists $a \in A$ such that $(Ta)^2 \neq Ta^2$, i.e. $T$ is not a Jordan homomorphism. It is also clear that $T$ is not surjective. $\nabla$

41

Therefore, in general, $T$ might not be a Jordan homomorphism if the bijectivity, or at least the surjectivity, of $T$ is omitted (note that $\mathcal{L}(H)$ and $\mathcal{L}(H \oplus H)$ are semi-simple).

Note that Corollary 3.1.11 is a special case of Conjecture C1, since it follows from [26], Lemma 2.3 that $T$ is bijective. Also, Corollary 3.2.8, in the case that $A$ is semi-simple and $f$ is bijective, is a special case of Conjecture C1. Another instance of Conjecture C1 is Corollary 3.3.3. Furthermore, if we assume that $A$ is semi-simple and $B$ is commutative, it is clear that Corollary 3.2.11 is an instance of Conjecture C1. Corollary 3.1.11 follows directly from Corollary 3.3.3 since the bijectivity of $T$ is implied by [26], Lemma 2.3. If $\mathcal{L}(Y)$ in Corollary 3.3.3 is replaced by the complex field $\mathbb{C}$, then, since $\mathbb{C}$ is isomorphic to $\mathcal{L}(\mathbb{C})$, Corollary 3.3.3 follows directly from Corollary 3.2.8.

Conjecture C1 is still unsolved even for the class of C*-algebras. Some progress in this direction has been made in [12] and [30]. Recently, B. Aupetit proved Conjecture C1 for the class of von Neumann algebras. We will discuss this in detail in Chapter 5.

## 3.4 Spectrum preserving and full spectrum preserving linear mappings

Lemma 2.2.3 shows that Conjecture C1 can be reformulated as

**Conjecture C1** Let $A$ and $B$ be semi-simple Banach algebras and $T : A \to B$ a unital bijective linear map such that, for all $a \in A$, Sp $(Ta, B) \subset$ Sp $(a, A)$. Then $T$ is a Jordan isomorphism.

Conjecture C1 therefore motivates the following question(*) that is somewhat easier than Kaplansky's original question:

When must a spectrum preserving unital linear map between Banach algebras be a Jordan homomorphism? Consider the following

**Example 3.4.1** *([4], p. 92) There exists a Banach algebra $A$, which is not semi-simple, having the property that there exists a surjective linear spectrum preserving map $T : A \to A$ which is not a Jordan homomorphism.*

**Proof.** Let $A$ be the subalgebra of $M_4(\mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

with $a, b, c \in M_2(\mathbb{C})$. Then $A$ is not semi-simple because every matrix of the form

$$\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix},$$

with $w \in M_2(\mathbb{C})$, is in Rad $(A)$. The reason is as follows: Let $a$ be such a $4 \times 4$ matrix and let $b \in A$ be arbitrary. Then $ab$ has the property that all its diagonal entries are zero, i.e. 0 is the only eigenvalue of $ab$. Therefore $ab$ has a spectral radius of zero. So, by Theorem 1.5.14, $a \in$ Rad $(A)$.

Define a linear mapping $T$ from $A$ onto $A$ by

$$T \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c^t \end{pmatrix},$$

where $c^t$ is the transpose of $c$. Clearly, $T$ is linear, bijective and unital. Furthermore, $T$ is invertibility preserving: Let

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

be an invertible complex matrix. Then

$$\begin{pmatrix} a & b \\ 0 & c^t \end{pmatrix}$$

is invertible. Thus $T$ is invertibility preserving. Similarly, $T^{-1}$ is invertibility preserving. It follows from Lemma 2.2.3 that $T$ is spectrum preserving.

However, $T$ is not a Jordan homomorphism. The reason is that

$$T \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^2 - \left( T \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right)^2$$

is in the radical of $A$, but not in general zero. $\nabla$

This implies that in order to obtain from (*) a reasonable conjecture, we have to require that $A$ and $B$ be semi-simple.

**Theorem 3.4.2** *([4], Corollary 3.5) Let $A$ and $B$ be Banach algebras. If $A$ is semi-simple and $T : A \to B$ is linear and full spectrum preserving, then $T$ is injective. If, in addition, $T$ is surjective, then $T$ is unital.*

**Proof.** Let $a \in \mathrm{Ker}(T)$, i.e. $Ta = 0$. Then Sp $(a, A) =$ Sp $(Ta, B) =$ Sp $(0, B) = \{0\}$. Let $q \in \mathrm{QN}(A)$. It follows that $\sigma(a + q, A) = \sigma(T(a + q), B) = \sigma(Ta + Tq, B) = \sigma(Tq, B) = \sigma(q, A) = \{0\}$. Therefore $\sigma(a+q, A) =$

43

$\{0\}$ for all $q \in \text{QN}(A)$. It follows from Theorem 1.5.15 that $a \in \text{Rad }(A) = \{0\}$. Therefore $a = 0$. Hence $T$ is injective.

Now suppose that, in addition, $T$ is surjective. Then there exists $b \in A$ such that $b = T^{-1}(1)$. Due to the injectivity of $T$, it follows that $b = T^{-1}(1)$ is unique. Hence $\sigma(b, A) = \sigma(Tb, B) = \sigma(1, B) = \{1\}$. Let $q \in \text{QN}(A)$. It follows that $\sigma(b + q, A) = \sigma(T(b + q), B) = \sigma(Tb + Tq, B) = \sigma(1 + Tq, B) = 1 + \sigma(Tq, B) = 1 + \sigma(q, A) = \{1\}$, so that $\text{Sp }(b + q, A) = \{1\}$. Then, by Theorem 1.5.18, $b = \alpha 1$ for some $\alpha \in \mathbb{C}$. Hence $\{1\} = \text{Sp }(b, A) = \{\alpha\}$. So $\alpha = 1$ and hence $T1 = 1$. $\nabla$

**Corollary 3.4.3** *Let $A$ and $B$ be Banach algebras. If $A$ is semi-simple and $T : A \to B$ is linear and spectrum preserving, then $T$ is injective. If, in addition, $T$ is surjective, then $T$ is unital.*

So, in the light of Conjecture C1, Corollary 3.4.3, Example 3.3.4 and Example 3.4.1, (*) becomes

**Conjecture C2** If $A$ and $B$ are semi-simple Banach algebras and $T : A \to B$ is a surjective spectrum preserving linear map, then $T$ is a Jordan isomorphism.

We thus see that C1 is stronger than C2, i.e. if C1 is true, then C2 is true. Affirmative answers to this conjecture are contained in Theorems 3.4.7 (Marcus-Moyls Theorem) and 3.4.10 (Jafarian-Sourour Theorem). The next three results, due to B. Aupetit, are needed to prove the Marcus-Moyls Theorem.

**Lemma 3.4.4** *([5], Lemma 3) Let $A$ and $B$ be Banach algebras. If $T : A \to B$ is a linear mapping such that $\text{Sp }(Tx, B) \subset \text{Sp }(x, A)$ for all $x \in A$, then $Sx = (T1)^{-1}Tx$ is a linear mapping from $A$ into $B$ such that $S1 = 1$ and $\text{Sp }(Sx, B) \subset \text{Sp }(x, A)$ for all $x \in A$.*

**Proof.** Since $\text{Sp }(T1, B) \subset \text{Sp }(1, A) = \{1\}$, it follows that $T1$ is invertible. Therefore $S$ is well defined. Furthermore, $S$ is unital since $S1 = (T1)^{-1}T1 = 1$.

If $0 \in \text{Sp }(Sx, B)$, then $0 \in \text{Sp }\left((T1)^{-1}Tx, B\right)$. Thus $(T1)^{-1}Tx$ is not invertible. So $Tx$ is not invertible. This implies that $0 \in \text{Sp }(Tx, B) \subset \text{Sp }(x, A)$.

Let $0 \neq \lambda \in \text{Sp }(Sx, B)$, i.e. $\lambda \in \text{Sp }\left((T1)^{-1}Tx, B\right)$. Then $(T1)^{-1}Tx - \lambda 1$ is not invertible in $B$. Hence $(T1)^{-1}T(-\frac{x}{\lambda}) - (-1)1$ is not invertible in $B$. Thus $-1 \in \text{Sp }\left((T1)^{-1}Ty, B\right)$, where $y = -\frac{x}{\lambda}$. It follows from the spectral

mapping theorem that $0 \in \mathrm{Sp}\ (1 + (T1)^{-1}Ty, B)$, i.e. $1 + (T1)^{-1}Ty$ is not invertible in $B$. Therefore $(T1)\left(1 + (T1)^{-1}Ty\right) = T(1 + y)$ is not invertible in $B$ since $T1$ is invertible in $B$, i.e. $0 \in \mathrm{Sp}\ \left(T(1 + y)\right)$. Due to the fact that $\mathrm{Sp}\ \left(T(1 + y), B\right) \subset \mathrm{Sp}\ (1 + y, A)$, we then have that $1 + y$ is not invertible in $A$, i.e. $1 - \frac{x}{\lambda}$ is not invertible in $A$. So $\lambda 1 - x$ is not invertible in $A$. Hence $\lambda \in \mathrm{Sp}\ (x, A)$. We have thus shown that $\mathrm{Sp}\ (Sx, B) \subset \mathrm{Sp}\ (x, A)$. $\nabla$

Lemma 3.4.4 says that we may assume without loss of generality that $T$ is unital.

**Corollary 3.4.5** *([5], Corollary 1) If $A$ is a Banach algebra and $T : A \to M_n(\mathbb{C})$ a surjective linear mapping such that $\mathrm{Sp}\ (Tx, M_n(\mathbb{C})) \subset \mathrm{Sp}\ (x, A)$ for all $x \in A$, then $Tx = (T1)Sx$ for all $x \in A$, where $S$ is a homomorphism or an anti-homomorphism.*

**Proof.** Define a linear mapping $S : A \to M_n(\mathbb{C})$ by $Sx = (T1)^{-1}Tx$. Then, by Lemma 3.4.4, $S$ is unital and invertibility preserving. Since $T$ is surjective, $S$ is surjective. By Theorem 3.1.8, $S$ is a homomorphism or an anti-homomorphism. Therefore, since $Sx = (T1)^{-1}Tx$, we have $Tx = (T1)Sx$ and the proof is complete. $\nabla$

**Corollary 3.4.6** *([5], Corollary 2) If $T$ is a linear mapping from $M_n(\mathbb{C})$ onto itself that preserves the determinant, then $Tx = (T1)Sx$, where $Sx$ is either of the form $uxu^{-1}$ or $ux^tu^{-1}$ for a particular $u \in M_n(\mathbb{C})$ that is invertible.*

**Proof.** Note that $(T1)^{-1}$ exists since $T$ is determinant preserving. We observe that the following implications are true.

$$
\begin{aligned}
\lambda \in \mathrm{Sp}\ (x, M_n(\mathbb{C})) \quad &\Longleftrightarrow \quad \det(x - \lambda 1) = 0 \\
&\Longleftrightarrow \quad \det\left(T(x - \lambda 1)\right) = 0 \\
&\Longleftrightarrow \quad \det(Tx - \lambda T1) = 0 \\
&\Longleftrightarrow \quad \det\left((T1)\left((T1)^{-1}Tx - \lambda 1\right)\right) = 0 \\
&\Longleftrightarrow \quad \det(T1) \det\left((T1)^{-1}Tx - \lambda 1\right) = 0 \\
&\Longleftrightarrow \quad \det\left((T1)^{-1}Tx - \lambda 1\right) = 0 \ (\text{since } \det(T1) = 1) \\
&\Longleftrightarrow \quad \det(Sx - \lambda 1) = 0 \\
&\Longleftrightarrow \quad \lambda \in \mathrm{Sp}\ \left(Sx, M_n(\mathbb{C})\right).
\end{aligned}
$$

It follows that $S$ is spectrum preserving.

It is clear that $S$ is unital because $S1 = (T1)(T1)^{-1} = 1$. Since $T$ is surjective, $S$ is surjective. Since $S$ is unital, $Sx = (S1)Sx$ for every $x \in M_n(\mathbb{C})$. It follows from Corollary 3.4.5 that $S$ is a homomorphism or an anti-homomorphism. The result now follows from Corollary 1.3.11. $\nabla$

**Theorem 3.4.7 (M. Marcus and B.N. Moyls)** *([25], Theorem 3) If $T$ : $M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear mapping preserving eigenvalues and their multiplicities, then $T$ is either of the form $Ta = bab^{-1}$ or $Ta = ba^t b^{-1}$ for some invertible complex matrix $b$.*

Corollary 3.4.6 implies the Marcus-Moyls Theorem, since the surjectivity of $T$ is implied by [26], Lemma 2.3, and the unitality of $T$ is implied by [25], Lemma 10. The Marcus-Moyls Theorem is a special case of the Marcus-Purves Theorem. Therefore it follows from an earlier remark that the Marcus-Moyls Theorem is also a consequence of Theorem 3.1.8 by Aupetit.

**Corollary 3.4.8** *If $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear mapping preserving eigenvalues and their multiplicities, then $T$ is either an isomorphism or an anti-isomorphism.*

The following result is a special case of Conjecture C2 since the surjectivity of $T$ is implied by [26], Lemma 2.3.

**Corollary 3.4.9** *If $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear mapping preserving eigenvalues and their multiplicities, then $T$ is a Jordan isomorphism.*

**Theorem 3.4.10 (A.A. Jafarian and A.R. Sourour)** *([20], Theorem 2) Let $X$ and $Y$ be Banach spaces. If $T : \mathcal{L}(X) \to \mathcal{L}(Y)$ is a spectrum preserving surjective linear mapping, then either*

*(i) there exists a bounded invertible linear operator $b : X \to Y$ such that $Ta = bab^{-1}$ for every $a \in \mathcal{L}(X)$, or*

*(ii) there exists a bounded invertible linear operator $c : X' \to Y$ such that $Ta = ca^{\times}c^{-1}$ for every $a \in \mathcal{L}(X)$.*

The original proof of this result is operator theoretic. We thus omit this proof since it falls outside the scope of this text. However, we shall prove Theorem 3.4.10 in Chapter 4. Theorem 3.4.10 is a generalization of Theorem 3.4.7 under the additional assumption that $T$ is surjective.

Keeping Lemma 2.2.8 and Proposition 1.3.6 in mind, we have the following corollary to Theorem 3.4.10.

46

**Corollary 3.4.11** *([20], Corollary 3) Let $X$ and $Y$ be Banach spaces. If $T : \mathcal{L}(X) \to \mathcal{L}(Y)$ is a surjective linear map, then $T$ is spectrum preserving if and only if it is an isomorphism or an anti-isomorphism.*

**Corollary 3.4.12** *Let $X$ and $Y$ be Banach spaces. If $T : \mathcal{L}(X) \to \mathcal{L}(Y)$ is a surjective spectrum preserving linear map, then $T$ is an isomorphism or an anti-isomorphism.*

The following corollary is a special case of Conjecture C2.

**Corollary 3.4.13** *Let $X$ and $Y$ be Banach spaces. If $T : \mathcal{L}(X) \to \mathcal{L}(Y)$ is a surjective spectrum preserving linear map, then $T$ is a Jordan isomorphism.*

It would be of interest to know if Conjecture C2 is true for linear functionals. The following result is a characterization of bijective multiplicative linear functionals.

**Corollary 3.4.14** *Let $A$ be a Banach algebra and $f$ a bijective linear functional on $A$. Then $f$ is multiplicative if and only if $f$ is spectrum preserving.*

**Proof.** Suppose that $f$ is multiplicative. Then it follows from Corollary 2.2.9 that $f$ is spectrum preserving.

Conversely, suppose that $f$ is spectrum preserving. Then $\{f(x)\} = \text{Sp}\,(x)$ for all $x \in A$, so that $f(x) \in \text{Sp}\,(x)$ for all $x \in A$. Hence, by Theorem 3.2.6, $f$ is multiplicative. $\nabla$

**Corollary 3.4.15** *Let $A$ be a Banach algebra and $f$ a bijective spectrum preserving linear functional on $A$. Then $f$ is multiplicative.*

Therefore, Conjecture C2 is true for linear functionals. This is our next result.

**Corollary 3.4.16** *Let $A$ be a Banach algebra and $f$ a bijective spectrum preserving linear functional on $A$. Then $f$ is a Jordan isomorphism.*

A more general problem can be studied by replacing the spectrum preservation property of $T$ in C2 with full spectrum preservation. In the light of Theorem 3.4.2 and Conjecture C2, the following seems to be a reasonable conjecture:

**Conjecture C3** If $A$ and $B$ are semi-simple Banach algebras and if $T : A \to B$ is a surjective linear mapping with the property that $T$ is full spectrum preserving, then $T$ is a Jordan isomorphism.

Observe that C3 is stronger than C2, i.e. if C3 is true, then so is C2. We will investigate some special cases of Conjecture C3 in Chapter 4.

## 3.5 Spectral radius preserving linear maps

We can study an even more general problem than Conjecture C3. Under what conditions is a unital linear spectral radius preserving map between Banach algebras a Jordan homomorphism?

This problem is also in general unsolved. In order to formulate a reasonable conjecture from this, we need

**Lemma 3.5.1** *Let $A$ and $B$ be Banach algebras. If $A$ is semi-simple and $T : A \to B$ is a spectral radius preserving linear mapping, then $T$ is injective.*

**Proof.** Let $a \in \mathrm{Ker}(T)$, i.e. $Ta = 0$, and let $q \in \mathrm{QN}(A)$. Then $\rho(a+q, A) = \rho(T(a+q), B) = \rho(Ta + Tq, B) = \rho(Tq, B) = \rho(q, A) = 0$. So $\rho(a+q) = 0$ for all $q \in \mathrm{QN}(A)$. Hence, by Theorem 1.5.15, $a \in \mathrm{Rad}\,(A) = \{0\}$ since $A$ is semi-simple. $\nabla$

Lemma 3.5.1, along with Conjecture C3, enables us to formulate a reasonable conjecture, namely

**Conjecture C4** Let $A$ and $B$ be semi-simple Banach algebras. If $T : A \to B$ is linear, surjective, unital and spectral radius preserving, then $T$ is a Jordan isomorphism.

Observe that C4 is stronger than C3 and C2. Progress has been made in establishing the truth of C4, namely Corollary 3.5.4, Corollary 3.5.6, Corollary 3.5.8 and Corollary 3.5.12. We first consider the conjecture for linear functionals.

**Corollary 3.5.2** *Let $A$ be a Banach algebra and $f$ a linear functional on $A$ that is spectral radius preserving. If $f$ is unital, then $f$ is multiplicative.*

**Proof.** Let $a$ be an invertible element of $A$. Then $a \notin QN(A)$ and so $\rho(a) \neq 0$. Consequently, $|f(a)| = \rho(a) \neq 0$, i.e. $f(a) \neq 0$. Hence $f$ is invertibility preserving, so that, by Theorem 3.2.7, $f$ is multiplicative. $\nabla$

A converse of Corollary 3.5.2 can be obtained if $f$ is bijective. This is our next result.

**Corollary 3.5.3** *Let $A$ be a Banach algebra and $f$ a unital bijective linear functional on $A$. Then $f$ is multiplicative if and only if $f$ is spectral radius preserving.*

48

**Proof.** Suppose that $f$ is multiplicative. Then, by Corollary 3.4.14, $f$ is spectrum preserving and hence spectral radius preserving.

Conversely, suppose that $f$ is unital and spectral radius preserving. Then, by Corollary 3.5.2, $f$ is multiplicative. $\nabla$

Corollary 3.5.3 gives us the following special case of Conjecture C4.

**Corollary 3.5.4** *Let $A$ be a Banach algebra and $f$ a unital bijective spectral radius preserving linear functional on $A$. Then $f$ is a Jordan functional on $A$.*

This concludes our observations about linear functionals. We now consider general linear operators.

**Corollary 3.5.5** *Let $A$ and $B$ be Banach algebras, with $B$ isomorphic to $\mathbb{C}$. Suppose that $T : A \to B$ is a linear mapping that is unital and spectral radius preserving. Then $T$ is multiplicative.*

**Proof.** Let $f$ be a bijective multiplicative linear functional on $B$ and let $F(x) = f(Tx)$ for all $x \in A$. Then $F$ is a linear functional on $A$ and, by Lemma 1.3.3 and Corollary 3.5.3, $f$ is spectral radius preserving.

Since $T$ is spectral radius preserving, it follows that $|F(x)| = |f(Tx)| = \rho(Tx) = \rho(x)$. Hence $F$ is spectral radius preserving. Furthermore, $F(1) = f(T1) = f(1) = 1$, implying that $F$ is unital. It follows from Corollary 3.5.2 that $F$ is multiplicative, i.e. $F(xy) = F(x)F(y)$ for all $x, y \in A$, i.e. $f\Big(T(xy)\Big) = f(Tx)f(Ty)$ for all $x, y \in A$. Hence $f\Big(T(xy)\Big) = f(TxTy)$ for all $x, y \in A$. So $f(Txy - TxTy) = 0$ for all $x, y \in A$. Since $f$ is injective, it follows that $T(xy) - TxTy = 0$ for all $x, y \in A$. Therefore $T(xy) = TxTy$ for all $x, y \in A$, i.e. $T$ is multiplicative. $\nabla$

The proof of Corollary 3.5.5 follows the lines of [22], Theorem 4. The following result is a special case of Conjecture C4.

**Corollary 3.5.6** *Let $A$ and $B$ be Banach algebras, with $B$ isomorphic to $\mathbb{C}$. Suppose that $T : A \to B$ is a linear mapping that is unital and spectral radius preserving. Then $T$ is a Jordan homomorphism.*

**Theorem 3.5.7 (M. Nagasawa)** *([2], Theorem 4.1.17) Let $A$ and $B$ be commutative and semi-simple Banach algebras. If $T : A \to B$ is a unital surjective spectral radius preserving linear mapping, then $T$ is an isomorphism from $A$ onto $B$.*

49

The next result is a special case of Conjecture C4, namely for commutative Banach algebras.

**Corollary 3.5.8** *Let $A$ and $B$ be commutative and semi-simple Banach algebras. If $T : A \to B$ is a unital surjective spectral radius preserving linear mapping, then $T$ is a Jordan isomorphism.*

Theorem 3.5.9 below has been proved by Brešar and Šemrl in [9].

**Theorem 3.5.9** *([9], Theorem 1) Let $X$ be a Banach space. Suppose that a surjective linear map $T : \mathcal{L}(X) \to \mathcal{L}(X)$ is spectral radius preserving. Then there exists a complex number $\alpha$ such that $|\alpha| = 1$ and either*

*(i) there exists a bounded invertible linear operator $b : X \to X$ such that $Ta = \alpha bab^{-1}$ for every $a \in \mathcal{L}(X)$, or*

*(ii) there exists a bounded invertible linear operator $c : X' \to X$ such that $Ta = \alpha ca^{\times}c^{-1}$ for every $a \in \mathcal{L}(X)$.*

**Corollary 3.5.10** *If $\alpha = 1$ in Theorem 3.5.9, then $T$ is a homomorphism or an anti-homomorphism.*

**Corollary 3.5.11** *Let $X$ be a Banach space. Suppose that a map $T : \mathcal{L}(X) \to \mathcal{L}(X)$ is unital, linear and surjective. Then $T$ is spectral radius preserving if and only if $T$ is an automorphism or an anti-automorphism.*

**Proof.** Suppose that $T$ is spectral radius preserving. Since $T$ is unital, it follows that $\alpha = 1$ in Theorem 3.5.9. Hence, by Corollary 3.5.10, T is a homomorphism or an anti-homomorphism. It follows from Lemma 3.5.1 that $T$ is injective. Therefore $T$ is an automorphism or an anti-automorphism.

Conversely, suppose that $T$ is an automorphism or an anti-automorphism. Then, by Corollary 2.2.10, $T$ is spectral radius preserving. $\nabla$

The result below is a special case of Conjecture C4.

**Corollary 3.5.12** *Let $X$ be a Banach space. Suppose that a map $T : \mathcal{L}(X) \to \mathcal{L}(X)$ is unital, linear, surjective and spectral radius preserving. Then $T$ is a Jordan isomorphism.*

Note that this result is also similar to the Jafarian-Sourour result.

# Chapter 4

# The solution of a conjecture for primitive Banach algebras with minimal ideals

The following conjecture was introduced in Chapter 3: Let $A$ and $B$ be semi-simple Banach algebras and $T : A \to B$ a surjective full spectrum preserving linear mapping. Then $T$ is a Jordan isomorphism (Conjecture C3). In this chapter we see how this conjecture is proved for the case where $B$ is a primitive Banach algebra with minimal ideals (Corollary 4.2.19). This result is used to obtain a stronger result, namely, if $B$ is a primitive Banach algebra with minimal ideals, then $T$ is not only a Jordan homomorphism, but either a homomorphism or an anti-homomorphism (Corollary 4.2.20). The main results of this chapter are Corollary 4.2.19 and Corollary 4.2.20.

All of the results in this chapter, unless stated otherwise, are due to B. Aupetit and H. du T. Mouton.

## 4.1 Rank one elements of Banach algebras

In this section we prove a result that we need in order to prove Corollary 4.2.19 and Corollary 4.2.20 (the main results of this chapter). This result is Corollary 4.1.14.

**Definition 4.1.1** *Let $A$ be a Banach algebra. If $a \in A$ and $Sp\ (xa)$ contains at most one nonzero point for every $x \in A$, we say that $a$ is a rank one element of $A$.*

We denote the set of rank one elements of $A$ by $\mathcal{F}_1(A)$. Since $Sp\ (xy) \cup \{0\} = Sp\ (yx) \cup \{0\}$, it follows that $\mathcal{F}_1(A)$ is closed under multiplication.

51

**Definition 4.1.2** *An element $u$ of a Banach algebra $A$ is called one-dimensional if there exists a linear functional $f_u$ on $A$ such that $uxu = f_u(x)u$ for every $x \in A$.*

Lemma 4.1.3 and Lemma 4.1.4 are about properties of one-dimensional elements and are due to J. Puhl.

**Lemma 4.1.3** *([29], Lemma 2.7) Let $u$ be a one-dimensional element of a Banach algebra $A$. Then, for every $x, y \in A$, $xuy$ is one-dimensional.*

**Proof.** Since $u$ is one-dimensional, there exists a linear functional $f_u$ on $A$ such that $utu = f_u(t)u$ for every $t \in A$. Therefore $xuytxuy = xf_u(ytx)uy = f_u(ytx)xuy$. Let $g_{xuy}(t) = f_u(ytx)$ for every $t \in A$. Then $g_{xuy}$ is a linear functional on $A$. So $xuytxuy = g_{xuy}(t)xuy$, implying that $xuy$ is a one-dimensional element of $A$. $\nabla$

**Lemma 4.1.4** *([29], Lemma 2.8) If $u$ is a one-dimensional element of a Banach algebra $A$, then $Sp\,(u)$ contains at most one nonzero point.*

**Proof.** Since $u$ is a one-dimensional element of $A$, there exists a linear functional $f_u$ on $A$ such that $uxu = f_u(x)u$ for every $x \in A$. Hence $u^2 = f_u(1)u$. Let $g(z) = z^2 - f_u(1)z$. Then $g(u) = 0$. It follows from the spectral mapping theorem that

$$\{0\} = Sp\,\Big(g(u)\Big) = \{g(\lambda) : \lambda \in Sp\,(u)\}.$$

Let $\lambda \in Sp\,(u)$. Then $g(\lambda) = 0$, i.e. $\lambda^2 - f_u(1)\lambda = \lambda(\lambda - f_u(1)) = 0$. Therefore $\lambda = 0$ or $\lambda = f_u(1)$. This says that $Sp\,(u) \subset \{0, f_u(1)\}$. $\nabla$

This elegant proof makes use of the holomorphic functional calculus. Puhl proved Lemma 4.1.4 by using a more direct method, instead of using the holomorphic functional calculus.

**Theorem 4.1.5** *([8], Proposition 3, p. 157) If $p$ is a minimal idempotent in a Banach algebra $A$, then $p$ is a one-dimensional element of $A$.*

**Proof.** By definition, $pAp$ is a division algebra. Furthermore, $pAp$ is a Banach algebra having $p$ as its identity element. It follows from Theorem 1.5.24, that $pAp$ is isomorphic to $\mathbb{C}$. Hence $pAp = \mathbb{C}p$. Therefore there exists a linear functional $f$ on $A$ such that $pap = f(a)p$ for every $a \in A$, showing that $p$ is a one-dimensional element of $A$. $\nabla$

Just as minimal ideals can be characterized in terms of minimal idempotents (Proposition 1.4.16), they can also be characterized in terms of one-dimensional elements (due to Puhl), as our next result shows.

52

**Proposition 4.1.6** *([29], Remark 2.5) Let $A$ be a semi-prime Banach algebra. Then $L$ is a minimal left ideal of $A$ if and only if $L = Au$ for some nonzero one-dimensional element $u$ of $A$.*

**Proof.** Let $L$ be a minimal left ideal of $A$. It follows from Proposition 1.4.16 that $L = Ap$ for some minimal idempotent $p$ of $A$. By Theorem 4.1.5, $p$ is a one-dimensional element of $A$. Hence $L = Au$ for some nonzero one-dimensional element $u$ of $A$.

Conversely, suppose that $L = Au$ for some nonzero one-dimensional element $u$ of $A$. Clearly, $L$ is a left ideal of $A$. We show that $L$ is a minimal left ideal of $A$. Let $I$ be a nonzero left ideal of $A$ such that $I \subset L$. Each nonzero $z \in I$ is of the form $z = z_0 u$, with $z_0 \in A$. By Theorem 1.4.13, there exists $y_0 \in A$ such that $zy_0z \neq 0$. Since $u$ is one-dimensional, it follows that $f_u(y_0z_0)z_0u = z_0uy_0z_0u = zy_0z \neq 0$. Therefore, if $x \in A$, then $\frac{1}{f_u(y_0z_0)}xuy_0z_0u = \frac{1}{f_u(y_0z_0)}f_u(y_0z_0)xu = xu$. Since $\frac{1}{f_u(y_0z_0)}xuy_0z_0u = \frac{1}{f_u(y_0z_0)}xuy_0z \in I$, it follows that $xu \in I$. Since $x \in A$ is arbitrary, $xu \in I$ for every $x \in A$. Hence $L \subset I$ and so $I = L$. Therefore $L$ is a minimal left ideal of $A$. $\nabla$

An important characterization of the socle of a semi-prime Banach algebra in terms of one-dimensional elements is the following result by Puhl.

**Lemma 4.1.7** *([29], p. 659) Let $A$ be a semi-prime Banach algebra. Then the set of all finite sums of one-dimensional elements of $A$ coincides with $Soc\,(A)$.*

**Proof.** Let $x$ be a finite sum of one-dimensional elements of $A$. Since $Soc\,(A)$ is the sum of all minimal left ideals of $A$, it follows from Proposition 4.1.6 that $x \in Soc\,(A)$.

Conversely, suppose that $x \in Soc\,(A)$. By definition, $x$ is contained in the sum of all minimal left ideals of $A$. By Lemma 4.1.3 and Proposition 4.1.6, $x$ is a finite sum of one-dimensional elements of $A$. $\nabla$

Lemma 4.1.8, Lemma 4.1.9 and Lemma 4.1.10 below are due to H. du T. Mouton and H. Raubenheimer. We use these results to prove Proposition 4.1.11.

**Lemma 4.1.8** *([28], Proposition 2.4) Let $A$ be a Banach algebra, $a \in A$ and suppose that $\alpha$ is a nonzero isolated point of $Sp\,(a)$. If $p$ is the spectral idempotent associated with $a$ and $\alpha$, then there exists $c \in A$ such that $p = ac = ca$.*

**Proof.** Let $\Gamma$ be a circle centered at $\alpha$, separating $\alpha$ from zero and the rest of the spectrum of $a$. For $\lambda \in \Gamma$, we have

$$(\lambda 1 - a)^{-1} = \frac{1}{\lambda}1 + \frac{1}{\lambda}a(\lambda 1 - a)^{-1}.$$

Hence

$$p = \frac{1}{2\pi i}\int_\Gamma (\lambda 1 - a)^{-1}\, d\lambda = \Big(\frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda}\, d\lambda\Big)1 + \frac{a}{2\pi i}\int_\Gamma \frac{1}{\lambda}(\lambda 1 - a)^{-1}\, d\lambda.$$

By Cauchy's theorem, $\int_\Gamma \frac{1}{\lambda}\, d\lambda = 0$. Hence

$$p = \frac{a}{2\pi i}\int_\Gamma \frac{1}{\lambda}(\lambda 1 - a)^{-1}\, d\lambda = a\Big(\frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda}(\lambda 1 - a)^{-1}\, d\lambda\Big).$$

Let $c = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda}(\lambda 1 - a)^{-1}\, d\lambda$. Then $p = ac = ca$. $\nabla$

**Lemma 4.1.9** *([28], Lemma 2.8) Let $A$ be a Banach algebra and let $B$ be a subalgebra of $A$ such that $Sp\,(b, A)$ consists of zero and possibly one other point for every $b \in B$. Then there are no nonzero orthogonal idempotents in $B$.*

**Proof.** Suppose that $B$ does have nonzero orthogonal idempotents, namely $p$ and $q$. If $p + 2q - 1$ is invertible in $A$, then $(p + 2q - 1)p = 0$ implies that $p = 0$. This is a contradiction and so $1 \in Sp\,(p + 2q, A)$. Similarly, $2 \in Sp\,(p + 2q, A)$. Hence $\{1, 2\} \subset Sp\,(p + 2q, A)$.

Since $p + 2q \in B$, it follows by hypothesis that $Sp\,(p + 2q, A)$ has at most one nonzero point. This is a contradiction. Therefore, $B$ has no nonzero orthogonal idempotents. $\nabla$

**Lemma 4.1.10** *([28], Lemma 2.9) Let $A$ be a semi-simple Banach algebra and let $0 \neq a \in A$ be such that $Sp\,(xa)$ consists of zero and possibly one other point for every $x \in A$. Then $A$ has a minimal idempotent $p \in Aa$.*

**Proof.** There exists an $xa \in Aa$ with a nonzero isolated point in $Sp\,(xa, A)$. (If not, then every element of $Aa$ is a quasi-nilpotent element of $A$, implying that $a \in Rad\,(A) = \{0\}$.) Let $p$ be the spectral idempotent of $xa$ associated with this point. It follows from Lemma 4.1.8 that there exists $c \in A$ such that $p = cxa$. Hence $p \in Aa$.

It remains to show that $p$ is a minimal idempotent of $A$. Let $B = pAp$. Since $p \in Aa$, it follows that $B \subset Aa$. Therefore, since $a \in \mathcal{F}_1(A)$, it is clear that $Sp\,(pyp, B)$ consists of at most two points for every $pyp \in B$.

Suppose that $Sp\,(pyp, B)$ contains two points. It follows from Theorem 1.5.28 that there exists a non-trivial idempotent $pzp$ in $B$. But $pzp$ and

54

$p - pzp$ are orthogonal idempotents in $Aa$. Lemma 4.1.9 implies that this is a contradiction. Therefore, Sp $(pyp, B)$ consists of one point only. This, together with Corollary 1.5.42 and the fact that $pAp$ is semi-simple, implies that $pAp = \mathbb{C}p$. Thus $pAp$ is a division algebra. Therefore $p$ is a minimal idempotent. $\nabla$

We are now ready to prove Proposition 4.1.11, which says that in a semi-simple Banach algebra, rank one elements are the same as one-dimensional elements. No satisfactory reference for this result could be found. The proof is based on the proof of [28], Theorem 2.2.

**Proposition 4.1.11** *Every one-dimensional element of a Banach algebra $A$ is rank one. Conversely, if $A$ is semi-simple, then every rank one element of $A$ is a one-dimensional element of $A$.*

**Proof.** Let $u$ be a one-dimensional element of $A$. It follows from Lemma 4.1.3 that $xu$ is one-dimensional. According to Lemma 4.1.4, Sp $(xu)$ contains at most one nonzero point for every $x \in A$. Hence $u \in \mathcal{F}_1(A)$.

Conversely, let $a \in \mathcal{F}_1(A)$, where $A$ is a semi-simple Banach algebra. The result clearly holds if $a = 0$. Therefore we consider the case $a \neq 0$.

Suppose that $a$ is invertible. Then $Aa = A$ and, since $a \in \mathcal{F}_1(A)$, Sp $(x)$ has at most one nonzero point for every $x \in A$. Therefore, for every invertible $b \in A$, Sp $(b)$ consists of one point only. Since $A$ is semi-simple and the set of invertible elements of $A$ is open, it follows from Corollary 1.5.42 that $A = Aa = \mathbb{C}1$. Therefore there exists a linear functional $f_a$ on $A$ such that $axa = f_a(x)a$, implying that $a$ is a one-dimensional element of $A$.

Suppose that $a$ is not invertible. Then $Aa \neq A$ or $aA \neq A$. We consider only the case $Aa \neq A$. The other case is dealt with in a similar manner. Suppose that at least one element of $Aa$ is invertible, say $ba$ is invertible for some $b \in A$. This implies that there exists $c \in A$ such that $cba = 1$, implying that $xcba = x$ for every $x \in A$. Hence $Aa = A$. This is a contradiction. So every element of $Aa$ is not invertible. Thus $0 \in$ Sp $(xa)$ for every $x \in A$. Also, since $a \in \mathcal{F}_1(A)$, Sp $(xa)$ has at most one nonzero point for every $x \in A$. By Lemma 4.1.10, there exists a minimal idempotent $p \in Aa$.

Suppose that $ap \neq a$. Clearly, $A(ap - a) \subset Aa$. By Lemma 4.1.10, there exists a minimal idempotent $q \in A(ap - a)$. Hence there exists $x \in A$ such that $q = x(ap - a)$. Thus $qp = x(ap - a)p = 0$. Let $w = q - pq$. Then $qw = q(q - pq) = q^2 - qpq = q \neq 0$ since $qp = 0$. Hence $w \neq 0$. Furthermore, $w^2 = w$ and $pw = pq - ppq = 0$. Similarly, $wp = 0$. This implies that $Aa$ has orthogonal idempotents $p$ and $w$. Since Sp $(xa)$ consists of 0 and at most one other point, it follows from Lemma 4.1.9 that we have obtained a contradiction. So $a = ap$.

55

By Lemma 4.1.5, $p$ is one-dimensional. It follows from Lemma 4.1.3 that $a = ap$ is one-dimensional. $\nabla$

The following result shows that our definition of rank one elements of a Banach algebra does make sense.

**Proposition 4.1.12** *([29], Proposition 2.6) Every rank one bounded linear operator on a Banach space $X$ is a rank one element of $\mathcal{L}(X)$.*

**Proof.** Let $T$ be a rank one bounded linear operator on $X$. Then there exists a linear functional $g$ on $X$ and $b \in \mathcal{L}(X)$ such that $Tx = g(x)b$ for every $x \in X$. Let $S \in \mathcal{L}(X)$ be arbitrary. Then

$$
\begin{aligned}
(TST)x &= (TS)Tx \\
&= (TS)\Big(g(x)b\Big) \\
&= g(x)\Big((TS)b\Big) \\
&= g(x)\Big(g(Sb)b\Big) \\
&= g(Sb)g(x)b \\
&= g(Sb)Tx
\end{aligned}
$$

for every $x \in X$. Let $f(S) = g(Sb)$ for every $S \in \mathcal{L}(X)$. Clearly, $f$ is a linear functional on $X$. It follows that there exists a linear functional $f$ on $\mathcal{L}(X)$ such that $(TST)x = f(S)Tx$ for every $x \in X$, implying that $TST = f(S)T$. So $T$ is a one-dimensional element of $\mathcal{L}(X)$. Hence it follows from Proposition 4.1.11 that $T \in \mathcal{F}_1(\mathcal{L}(X))$. $\nabla$

The next result says that the socle of a semi-simple Banach algebra $A$ contains all finite rank elements of $A$. This result is used in the proof of Proposition 4.1.15, Theorem 4.2.3 and Theorem 4.2.10.

**Corollary 4.1.13** *([4], p. 93) If $A$ is a semi-simple Banach algebra, then $\mathcal{F}_1(A) \subset Soc\ (A)$.*

**Proof.** Let $a \in \mathcal{F}_1(A)$. It follows from Proposition 4.1.11 that $a$ is one-dimensional. By Lemma 1.4.14 and Lemma 4.1.7, $a \in Soc\ (A)$. $\nabla$

The following result is a useful characterization of the socle of a semi-simple Banach algebra in terms of rank one elements. It follows from Proposition 4.1.11 that, if $A$ is semi-simple, then Corollary 4.1.14 is the same as Lemma 4.1.7. Corollary 4.1.14 is used to prove Theorem 4.2.10, containing the bulk of the proofs of Corollaries 4.2.19 and 4.2.20.

56

**Corollary 4.1.14** *([4], p. 93) Let A be a semi-simple Banach algebra. Then Soc (A) is equal to the set of all finite sums of rank one elements of A.*

**Proof.** By Lemma 1.4.14 and Lemma 1.4.17, $A$ is semi-prime and Soc $(A)$ exists. The result follows from Lemma 4.1.7 and Proposition 4.1.11. $\nabla$

An important example of a primitive Banach algebra with minimal ideals is given in our next result. We use Proposition 4.1.15 and Theorem 4.1.16 to prove an extension of the Jafarian-Sourour result, namely Corollary 4.2.21.

**Proposition 4.1.15** *([4], p. 92) If X is a Banach space, then $\mathcal{L}(X)$ is a primitive Banach algebra with minimal ideals.*

**Proof.** Consider the map $\pi : \mathcal{L}(X) \to \mathcal{L}(X)$ defined by $\pi(x) = x$. Then $\pi$ is a continuous irreducible representation on $\mathcal{L}(X)$ and $\mathrm{Ker}(\pi) = \{0\}$. Therefore, by definition, $\{0\}$ is a primitive ideal, implying that $\mathcal{L}(X)$ is a primitive Banach algebra.

Since $\mathcal{L}(X)$ has nonzero rank one operators, it follows from Corollary 4.1.13 and Proposition 4.1.12 that Soc $(\mathcal{L}(X)) \neq \{0\}$. This implies that $\mathcal{L}(X)$ has minimal ideals. $\nabla$

The following result will be useful for the proof of Theorem 4.2.21. In fact, it is part of the proof of [4], Corollary 3.5.

**Theorem 4.1.16** *Let X and Y be Banach spaces and let $T : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a linear mapping.*

(i) *If T is an isomorphism, then there exists a bounded invertible linear operator $b : X \to Y$ such that $Ta = bab^{-1}$ for every $a \in \mathcal{L}(X)$.*

(ii) *If T is an anti-isomorphism, then there exists a bounded invertible linear operator $c : X' \to Y$ such that $Ta = ca^{\times}c^{-1}$ for every $a \in \mathcal{L}(X)$.*

**Proof.** (i) Suppose that $T$ is an isomorphism. It follows from Proposition 4.1.15, Theorem 1.3.13 and Theorem 1.5.35 that there exists a bounded invertible linear operator $b : X \to Y$ such that $Ta = bab^{-1}$ for all $a \in \mathcal{L}(X)$.

(ii) Suppose that $T$ is an anti-isomorphism. It follows from Theorem 1.5.35 that the set $C = \{t^{\times} : t \in \mathcal{L}(X)\}$ is a strictly dense subalgebra of $\mathcal{L}(X')$ on $X'$. Furthermore, Lemma 1.3.7 yields that $C$ is anti-isomorphic to $\mathcal{L}(X)$. Let $\phi : C \to \mathcal{L}(X)$ denote this anti-isomorphism, i.e. $\phi(t^{\times}) = t$ for all $t \in \mathcal{L}(X)$. Clearly, $T \circ \phi : C \to \mathcal{L}(Y)$ is an isomorphism. It follows from Theorem 1.3.13 and Theorem 1.5.35 that there exists a bounded invertible linear operator $c : X' \to Y$ such that $(T \circ \phi)x = cxc^{-1}$ for all $x \in C$. Let $a \in \mathcal{L}(X)$. Since $\phi$ is surjective, $a = \phi(a^{\times})$. Hence $Ta = (T \circ \phi)(a^{\times}) = ca^{\times}c^{-1}$. Since $a \in \mathcal{L}(X)$ is arbitrary, this completes the proof. $\nabla$

57

## 4.2 Aupetit and Mouton's solution

Aupetit and Mouton's strategy for proving Corollary 4.2.19 and Corollary 4.2.20 is as follows:

Let $T$ be a surjective full spectrum preserving linear mapping between semi-simple Banach algebras $A$ and $B$. First, a spectral characterization of rank one elements of a semi-simple Banach algebra is obtained (Theorem 4.2.3). This characterization of rank one elements is used to show that $T$ maps the set of rank one elements of $A$ onto the set of rank one elements of $B$ (Theorem 4.2.9). This fact is used to prove that $(Ta^2 - (Ta)^2)x = 0$ for every $x \in \text{Soc}(B)$ and every $a \in A$ (Theorem 4.2.10). Using this result, the main results of this chapter can be proved without too much effort.

We use Propositions 4.2.1 and 4.2.2 to prove Theorem 4.2.3, which is a characterization of rank one elements of a semi-simple Banach algebra. The proofs of Propositions 4.2.1 and 4.2.2 are contained in the proof of [4], Theorem 2.2(1).

**Proposition 4.2.1** *Let $A$ be a Banach algebra. If $z = \mu 1 - (y - \lambda 1)^{-1}$ with $\mu = -\frac{1}{\lambda}$ and $|\lambda| > 2\rho(y)$, then $\rho(z) \leq |\mu|$.*

**Proof.** Since $z = \mu 1 - (y - \lambda 1)^{-1}$, it follows from the spectral mapping theorem that

$$
\begin{aligned}
\text{Sp}(z) &= \left\{ \mu - \frac{1}{\gamma - \lambda} : \gamma \in \text{Sp}(y) \right\} \\
&= \left\{ -\frac{\gamma}{\lambda(\gamma - \lambda)} : \gamma \in \text{Sp}(y) \right\},
\end{aligned}
$$

so that $\rho(z) = \sup \left\{ \frac{|\gamma|}{|\lambda||\gamma - \lambda|} : \gamma \in \text{Sp}(y) \right\}$. Since $|\lambda| > 2\rho(y)$, it follows that $|\gamma| < |\lambda|$ for every $\gamma \in \text{Sp}(y)$. Therefore $|\gamma - \lambda| \geq |\lambda| - |\gamma| > \rho(y)$. Thus $\frac{|\gamma|}{|\lambda||\gamma - \lambda|} < \frac{1}{|\lambda|} = |\mu|$ for all $\gamma \in \text{Sp}(y)$, so that $\rho(z) \leq |\mu|$. $\nabla$

**Proposition 4.2.2** *Let $A$ be a Banach algebra. Suppose that, if $F$ is a two-element subset of $\mathbb{C} \setminus \{0\}$, then $\bigcap_{t \in F} \sigma(x + ta) \subset \sigma(x)$ for every $x \in A$. Then if $y \in A$ and $|\lambda| > 2\rho(y)$, then $Sp((\lambda 1 - y)a)$ contains at most one nonzero point.*

**Proof.** We will first prove that if $x \in A$ and $\mu \notin \sigma(x)$, then $\text{Sp}\left((\mu 1 - x)^{-1}a\right)$ contains at most one nonzero point. Let $F$ be a two-element subset of $\mathbb{C} \setminus \{0\}$. Then $\mu 1 - x$ is invertible and $\mu \notin \sigma(x + t_1 a)$ for some $t_1 \in F$, so that $\mu 1 - (x + t_1 a)$ is invertible. It follows from the relation

$$
\mu 1 - (x + ta) = (\mu 1 - x)\left(1 - t(\mu 1 - x)^{-1}a\right)
$$

58

that $1 - t_1(\mu 1 - x)^{-1}a$ is invertible, so that $\frac{1}{t_1} \notin \mathrm{Sp}\left((\mu 1 - x)^{-1}a\right)$. Without loss of generality, this implies that every two-element subset of $\mathbb{C} \setminus \{0\}$ contains a point of $\mathbb{C} \setminus \mathrm{Sp}\left((\mu 1 - x)^{-1}a\right)$. So $\mathrm{Sp}\left((\mu 1 - x)^{-1}a\right)$ contains at most one nonzero point. This holds for every $x \in A$.

Let $y \in A$ be arbitrary and let $\lambda \in \mathbb{C}$ with $|\lambda| > 2\rho(y)$. Define $z \in A$ to be $z = \mu 1 - (y - \lambda 1)^{-1}$, where $\mu = -\frac{1}{\lambda}$. Then $y - \lambda 1 = (\mu 1 - z)^{-1}$. It follows from Proposition 4.2.1 that $\rho(z) \leq |\mu|$. Hence $\mu \notin \sigma(z)$. Therefore, by the first part of the proof, $\mathrm{Sp}\left((\lambda 1 - y)a\right)$ contains at most one nonzero point. $\nabla$

The following result by B. Aupetit and H. du T. Mouton is an important spectral characterization of rank one elements of a semi-simple Banach algebra. The original proof in [4] used analytic multifunction techniques, namely [2], Theorem 7.1.7. We give a proof that is a slight adjustment of Aupetit and Mouton's proof in [4]. It is a slight adjustment in the sense that our proof is the same as Aupetit and Mouton's proof, except that where Aupetit and Mouton have used analytic multifunction techniques, we use subharmonic techniques.

**Theorem 4.2.3** *([4], Theorem 2.2(1)) Let $A$ be a semi-simple Banach algebra and $a \in A$. Then $a \in \mathcal{F}_1(A)$ if and only if, for every two-element subset $F \subset \mathbb{C} \setminus \{0\}$, we have $\cap_{t \in F}\sigma(x + ta) \subset \sigma(x)$ for every $x \in A$.*

**Proof.** Let $a \in \mathcal{F}_1(A)$, $x \in A$ and $F$ a two-element subset of $\mathbb{C} \setminus \{0\}$. If $\mu \notin \sigma(x)$, then $\{\frac{1}{t} : t \in F\} \nsubseteq \mathrm{Sp}\left((\mu 1 - x)^{-1}a\right)$ because $\mathrm{Sp}\left((\mu 1 - x)^{-1}a\right)$ contains at most one nonzero point. Hence there exists $t_0 \in F$ such that $\frac{1}{t_0} \notin \mathrm{Sp}\left((\mu 1 - x)^{-1}a\right)$, i.e. $\frac{1}{t_0}1 - (\mu 1 - x)^{-1}a$ is invertible. But then, since $\mu \notin \sigma(x)$,

$$\mu 1 - (x + ta) = (\mu 1 - x)\left(1 - t(\mu 1 - x)^{-1}a\right)$$

implies that $\mu 1 - (x + t_0 a)$ is invertible and consequently $\mu \notin \mathrm{Sp}\,(x + t_0 a)$.

Suppose that $\mu \in \sigma(x + t_0 a)$. Then $\mu$ belongs to a hole of $\mathrm{Sp}\,(x + t_0 a)$. Since, by Corollary 4.1.13 and Corollary 1.5.38, $t_0 a \in \mathrm{Soc}\,(A)$ and $\mathrm{Soc}\,(A)$ is an inessential ideal, it follows from Theorem 1.5.40 that $\sigma\left(D(x + t_0 a)\right) = \sigma\left(D(x)\right)$. It follows that $\sigma(x + t_0 a)$ and $\sigma(x)$ differ at most by isolated points of $\mathrm{Sp}\,(x + t_0 a)$ and $\mathrm{Sp}\,(x)$. Since $\mu$ is in a hole of $\mathrm{Sp}\,(x + t_0 a)$ and $\mu \notin \sigma(x)$, we see that $\sigma(x + t_0 a)$ and $\sigma(x)$ differ by a point that is not an isolated point of $\mathrm{Sp}\,(x + t_0 a)$ or $\mathrm{Sp}\,(x)$. This is a contradiction. Thus $\mu \notin \sigma(x + t_0 a)$.

This implies that $\mu \notin \cap_{t \in F}\sigma(x + ta)$. Therefore $\cap_{t \in F}\sigma(x + ta) \subset \sigma(x)$ for every $x \in A$ and every two-element subset $F$ of $\mathbb{C} \setminus \{0\}$.

Conversely, suppose that, for every two-element subset $F \subset \mathbb{C} \setminus \{0\}$, we have that $\cap_{t \in F} \sigma(x + ta) \subset \sigma(x)$ for every $x \in A$. Let $y \in A$ be arbitrary and $|\lambda| > 2\rho(y)$. It follows from Proposition 4.2.2 that Sp $((\lambda 1 - y)a)$ contains at most one nonzero point.

(i) If $a$ is invertible, then Sp $((\lambda 1 - y)a)$ consists of a single nonzero point because $|\lambda| > 2\rho(y)$. Since the set $\{\lambda \in \mathbb{C} : |\lambda| > 2\rho(y)\}$ has nonzero capacity and $\lambda \mapsto (\lambda 1 - y)a$ is an analytic function, it follows from Corollary 1.5.34 that Sp $((\lambda 1 - y)a)$ consists of one element for every $\lambda \in \mathbb{C}$. In particular, Sp $((1 - y)a)$ consists of one element. This implies that Sp $(ya)$ consists of one point. Since $y \in A$ is arbitrary, it follows that $a \in \mathcal{F}_1(A)$.

(ii) Suppose that $a$ is not invertible. Then $Aa \neq A$ or $aA \neq A$. Consider the case $Aa \neq A$, the other case being dealt with in a similar manner. As a result, no element of $Aa$ is invertible. In particular, $0 \in$ Sp $\left((\lambda 1 - y)a\right)$. Therefore Sp $\left((\lambda 1 - y)a\right)$ has at most two elements. Since the set $\{\lambda \in \mathbb{C} : |\lambda| > 2\rho(y)\}$ has nonzero capacity and $\lambda \mapsto (\lambda 1 - y)a$ is an analytic function, it follows from Corollary 1.5.34 that Sp $((\lambda 1 - y)a)$ consists of at most two elements for every $\lambda \in \mathbb{C}$. Taking $\lambda = 0$, it follows that Sp $(ya)$ consists of at most two elements. Recall that $y \in A$ is arbitrary. Also, $0 \in$ Sp $(ya)$ for every $y \in A$ since no element of $Aa$ is invertible. This implies that Sp $(ya)$ has at most one nonzero point for every $y \in A$. Hence $a \in \mathcal{F}_1(A)$. $\nabla$

The next result characterizes all semi-simple Banach algebras that are isomorphic to $\mathbb{C}$.

**Theorem 4.2.4** *([4], p. 95) A semi-simple Banach algebra $A$ contains invertible elements of rank one if and only if $A$ is isomorphic to $\mathbb{C}$.*

**Proof.** Let $a$ be an invertible rank one element of $A$. Then for each $x \in A$ and $\mu \notin \sigma(x)$, it follows that Sp $\left((\mu 1 - x)^{-1}a\right)$ consists of a single nonzero point. For an arbitrary $y \in A$ and $\lambda \in \mathbb{C}$ with $|\lambda| > 2\rho(y)$ and $\mu = -\frac{1}{\lambda}$, let $x = \mu 1 - (y - \lambda 1)^{-1}$. Then, by Proposition 4.2.1, $\rho(x) \leq |\mu|$ and so $\mu \notin \sigma(x)$. Therefore Sp $\left((\lambda 1 - y)a\right)$ consists of a single element because Sp $\left((\lambda 1 - y)a\right) =$ Sp $\left(-(\mu 1 - x)^{-1}a\right)$. Since the set $\{\lambda \in \mathbb{C} : |\lambda| > 2\rho(y)\}$ has nonzero capacity, it follows from Corollary 1.5.34 that Sp $\left((\lambda 1 - y)a\right)$ consists of a single element for every $\lambda \in \mathbb{C}$. In particular, Sp $\left((1 - y)a\right)$ consists of a single element. This says that every element of $Aa$ has a spectrum consisting of one element. By Corollary 1.5.42, $A$ is isomorphic to $\mathbb{C}$ since $Aa$ is open.

60

Conversely, suppose that $A$ is isomorphic to $\mathbb{C}$. Denote this isomorphism by $\phi : \mathbb{C} \to A$. Then $\phi$ is spectrum preserving and so every element of $A$ has a spectrum consisting of one element. Therefore $A$ contains invertible elements of rank one. $\nabla$

If $A$ is not isomorphic to $\mathbb{C}$ and $a \in \mathcal{F}_1(A)$, then it follows from Theorem 4.2.4 that Sp $(a)$ consists of 0 and possibly one other point. This enables us to define a function $t : \mathcal{F}_1(A) \to \mathbb{C}$ by Sp $(a) = \{0, t(a)\}$ if $A$ is not isomorphic to $\mathbb{C}$ and Sp $(a) = \{t(a)\}$ if $A$ is isomorphic to $\mathbb{C}$. This function is used to prove Theorem 4.2.10. Some properties of $t$ are given in Lemma 4.2.5, Lemma 4.2.6 and Lemma 4.2.7. By standard arguments, the following result follows from Theorem 1.5.31.

**Lemma 4.2.5** *Let $A$ be a semi-simple Banach algebra. The mapping $t : \mathcal{F}_1(A) \to \mathbb{C}$, defined by Sp $(a) = \{0, t(a)\}$ if $A$ is not isomorphic to $\mathbb{C}$ and Sp $(a) = \{t(a)\}$ if $A$ is isomorphic to $\mathbb{C}$, is continuous.*

Lemma 4.2.6 and Lemma 4.2.7 say that $t$, as defined above, is in some sense linear.

**Lemma 4.2.6** *Let $A$ be a semi-simple Banach algebra and $a \in \mathcal{F}_1(A)$. Then $t(\lambda a) = \lambda t(a)$ for every $\lambda \in \mathbb{C}$.*

**Proof.** Suppose that $A$ is not isomorphic to $\mathbb{C}$. Then, by definition, Sp $(a) = \{0, t(a)\}$ and Sp $(\lambda a) = \{0, t(\lambda a)\}$. Furthermore, Sp $(\lambda a) = \lambda$Sp $(a) = \{0, \lambda t(a)\}$. Therefore $t(\lambda a) = \lambda t(a)$. The case where $A$ is isomorphic to $\mathbb{C}$ is dealt with in a similar manner. $\nabla$

**Lemma 4.2.7** *([4], Lemma 2.3) Let $A$ be a semi-simple Banach algebra. Suppose that $a, b \in \mathcal{F}_1(A)$ such that $a + \lambda b \in \mathcal{F}_1(A)$ for all $\lambda \in \mathbb{C}$. Then $t(a + b) = t(a) + t(b)$.*

**Proof.** Let $h(\lambda) = t(a + \lambda b)$ and $f(\lambda) = a + \lambda b$ for every $\lambda \in \mathbb{C}$. Then Sp $\big(f(\lambda)\big) =$ Sp $(a + \lambda b) = \{0, h(\lambda)\}$ or $\{h(\lambda)\}$. It is clear that $f$ is entire. By Theorem 1.5.32, $h$ is entire. Hence the mapping $\lambda \mapsto \frac{t(a+\lambda b)}{\lambda}$ is analytic and thus continuous for every $\lambda \neq 0$. It follows from Lemma 4.2.5 and Lemma 4.2.6 that

$$
\begin{aligned}
\lim_{\lambda \to \infty} \frac{h(\lambda)}{\lambda} &= \lim_{\lambda \to \infty} t\Big(\frac{a}{\lambda} + b\Big) \\
&= t\Big( \lim_{\lambda \to \infty} \Big(\frac{a}{\lambda} + b\Big)\Big) \\
&= t(b).
\end{aligned}
$$

61

By Corollary 1.7.2, $h(\lambda) = h(0) + \lambda t(b)$, i.e. $t(a + \lambda b) = t(a) + \lambda t(b)$. Taking $\lambda = 1$, the result follows. $\nabla$

The following result, which is needed to prove Theorem 4.2.10, simply follows from the fact that $\mathcal{F}_1(A)$ is closed under multiplication.

**Lemma 4.2.8** *([4], p. 96) Let $A$ be a Banach algebra. If $a = cx$ and $b = dx$ with $x \in \mathcal{F}_1(A)$, then $a, b \in \mathcal{F}_1(A)$ and $a + \lambda b \in \mathcal{F}_1(A)$ for every $\lambda \in \mathbb{C}$.*

Any surjective full spectrum preserving linear mapping between semi-simple Banach algebras preserves rank one elements. This is the next result.

**Theorem 4.2.9** *([4], Theorem 3.1) Let $A$ and $B$ be semi-simple Banach algebras and $T : A \to B$ a surjective full spectrum preserving linear mapping. Then $T(\mathcal{F}_1(A)) = \mathcal{F}_1(B)$.*

**Proof.** We first show that $T(\mathcal{F}_1(A)) \subset \mathcal{F}_1(B)$. Let $a \in \mathcal{F}_1(A)$. Then for every $x \in A$ and every two-element subset $F \subset \mathbb{C} \setminus \{0\}$, we have from Theorem 4.2.3 that $\cap_{t \in F} \sigma(x + ta) \subset \sigma(x)$. Since $T$ is full spectrum preserving, it follows that $\cap_{t \in F} \sigma(Tx + tTa) \subset \sigma(Tx)$ for every $x \in A$. Therefore, since $T$ is surjective, we have $\cap_{t \in F} \sigma(y + tTa) \subset \sigma(y)$ for every $y \in B$. According to Theorem 4.2.3, this implies that $Ta$ is of rank one, i.e. $Ta \in \mathcal{F}_1(B)$.

It follows from Theorem 3.4.2 that $T$ is injective. Thus $T$ is bijective and so $T^{-1}$ exists. Furthermore, $T^{-1}$ is surjective and full spectrum preserving. Therefore, by the first part of the proof, $T^{-1}(\mathcal{F}_1(B)) \subset \mathcal{F}_1(A)$. So $T\left(T^{-1}(\mathcal{F}_1(B))\right) \subset T(\mathcal{F}_1(A))$. Hence $T(\mathcal{F}_1(A)) = \mathcal{F}_1(B)$. $\nabla$

The bulk of the proofs of the main results, namely Corollaries 4.2.19 and 4.2.20, is contained in the next result.

**Theorem 4.2.10** *([4], Theorem 3.2) If $A$ and $B$ are semi-simple Banach algebras and $T : A \to B$ a surjective full spectrum preserving linear map, then $(Ta^2 - (Ta)^2)x = 0$ for every $x \in \mathrm{Soc}\,(B)$ and every $a \in A$.*

**Proof.** Let $b \in \mathcal{F}_1(A)$ and $c \in A$ with $0 \notin \sigma(c)$. Since $T$ is full spectrum preserving, it follows that $0 \notin \sigma(Tc)$.

We first show that

$$0 \in \sigma(c + b) \text{ if and only if } c + b \text{ is not invertible.}$$

If $c + b$ is not invertible, then $0 \in \mathrm{Sp}\,(c + b) \subset \sigma(c + b)$. Conversely, suppose that $0 \in \sigma(c + b)$. We need to show that $0 \in \mathrm{Sp}\,(c + b)$. Assume that $0 \notin \mathrm{Sp}\,(c + b)$. Then $0$ lies in a hole of $\mathrm{Sp}\,(c + b)$. Recalling that $\mathcal{F}_1(A) \subset$

Soc $(A)$ and that Soc $(A)$ is an inessential ideal, it follows from Theorem 1.5.40 that $\sigma\big(D(c)\big) = \sigma\big(D(c+b)\big)$. Therefore $\sigma(c)$ and $\sigma(c+b)$ differ by at most isolated points of Sp $(c)$ and Sp $(c+b)$. Since 0 is in a hole of Sp $(c+b)$, it is not an isolated point of Sp $(c+b)$. Since $0 \notin \sigma(c)$, it follows that $0 \notin$ Sp $(c)$ and so 0 is also not an isolated point of Sp $(c)$. Therefore $\sigma(c)$ and $\sigma(c+b)$ differ by at least one point that is not an isolated point of Sp $(c)$ and Sp $(c+b)$. This is a contradiction. Thus $0 \in$ Sp $(c+b)$, implying that $c+b$ is not invertible.

Therefore, since $0 \notin \sigma(c)$, the following statements are equivalent for $b \in \mathcal{F}_1(A)$ and $c \in A$:

(i) $0 \in \sigma(c+b)$,

(ii) $c+b$ is not invertible,

(iii) $1 + c^{-1}b$ is not invertible,

(iv) $-1 \in$ Sp $(c^{-1}b)$,

(v) $t(c^{-1}b) = -1$

because $c^{-1}b \in \mathcal{F}_1(A)$. By Theorem 4.2.9, $Tb \in \mathcal{F}_1(B)$. Since $T$ is full spectrum preserving, $0 \in \sigma(c+b)$ if and only if $0 \in \sigma(Tc+Tb)$. Hence, by the equivalence of (i) and (v), $t(c^{-1}b) = -1$ if and only if $t((Tc)^{-1}Tb) = -1$. Therefore, by Lemma 4.2.6, the following statements are equivalent:

(i) $t(c^{-1}b) = \alpha \neq 0$,

(ii) $t(-\frac{1}{\alpha}c^{-1}b) = -1$,

(iii) $t\big((-\alpha Tc)^{-1}Tb)\big) = -1$,

(iv) $t\big((Tc)^{-1}Tb\big) = \alpha$.

This shows that

$$t(c^{-1}b) = t\big((Tc)^{-1}Tb\big) \tag{4.2.11}$$

for every $b \in \mathcal{F}_1(A)$ and $c \in A$ satisfying $0 \notin \sigma(c)$.

Let $a \in A$ and $b \in \mathcal{F}_1(A)$. For all $\lambda \in \mathbb{C}$ with $|\lambda| > \rho(a)$, we have that $0 \notin \sigma(\lambda 1 - a)$. Therefore, by (4.2.11),

$$t\big(((\lambda 1 - a)^{-1}b)\big) = t\big((\lambda 1 - Ta)^{-1}Tb\big). \tag{4.2.12}$$

63

Recall that $(\lambda 1 - a)^{-1} = \frac{1}{\lambda} \sum_{k=0}^{\infty} (\frac{a}{\lambda})^k$ for all $\lambda \in \mathbb{C}$ with $|\lambda| > \rho(a) = \rho(Ta)$. Therefore, by expanding both sides of (4.2.12), it follows that

$$t\Big(\frac{b}{\lambda} + \frac{ab}{\lambda^2} + \frac{a^2 b}{\lambda^3} + \cdots\Big) = t\Big(\frac{Tb}{\lambda} + \frac{TaTb}{\lambda^2} + \frac{(Ta)^2 Tb}{\lambda^3} + \cdots\Big). \qquad (4.2.13)$$

By Lemma 4.2.8, Lemma 4.2.7 is applicable to finite sums in (4.2.13), and since $t$ is continuous (Lemma 4.2.5), it follows from Lemma 4.2.6 that

$$\sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} t(a^n b) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} t((Ta)^n Tb) \qquad (4.2.14)$$

for $|\lambda| > \rho(a)$. Hence, by comparing coefficients, we have that

$$t(a^n b) = t((Ta)^n Tb)$$

for every $n \in \mathbb{N}$. Since $T$ is full spectrum preserving, $t(a^n b) = t(T(a^n b))$ for every $n \in \mathbb{N}$ and so, $t(T(a^n b)) = t((Ta)^n Tb)$ for every $n \in \mathbb{N}$. Therefore

$$t(T(ab)) = t(TaTb) \qquad (4.2.15)$$

and

$$t\Big(T(a^2 b)\Big) = t\Big((Ta)^2 Tb\Big). \qquad (4.2.16)$$

Replacing $a$ by $a^2$, it follows from (4.2.15) that $t\Big(T(a^2 b)\Big) = t(Ta^2 Tb)$, so that, by (4.2.16), we get $t(Ta^2 Tb) = t\Big((Ta)^2 Tb\Big)$. Therefore, by Lemma 4.2.7,

$$0 = t(Ta^2 Tb) - t\Big((Ta)^2 Tb\Big) = t\Big(Ta^2 Tb - (Ta)^2 Tb\Big)$$

because $Ta^2 Tb + \lambda(Ta)^2 Tb = \Big(Ta^2 + \lambda(Ta)^2\Big) Tb \in \mathcal{F}_1(B)$ for every $\lambda \in \mathbb{C}$. Hence

$$t\Big((Ta^2 - (Ta)^2)Tb\Big) = 0 \qquad (4.2.17)$$

for every $b \in \mathcal{F}_1(A)$.

   Let $u = Ta^2 - (Ta)^2$. Then, by Theorem 4.2.9 and (4.2.17), $t(ud) = 0$ for every $d \in \mathcal{F}_1(B)$. Suppose that $ud \neq 0$ for some $d \in \mathcal{F}_1(B)$. Since $B$ is semi-simple, Rad $(B) = \{0\}$ and so, by Theorem 1.5.14, there exists $0 \neq x \in B$ such that Sp $(udx) \neq \{0\}$, implying that $t(udx) \neq 0$. But $dx \in \mathcal{F}_1(B)$ and so, by (4.2.17), $t(udx) = 0$. This is a contradiction. Thus $ud = 0$ for every $d \in \mathcal{F}_1(B)$. It follows from Corollary 4.1.14 that $ux = 0$, i.e. $\Big(Ta^2 - (Ta)^2\Big)x = 0$, for every $x \in$ Soc $(B)$ and every $a \in A$. $\nabla$

64

**Corollary 4.2.18** *([4], Corollary 3.3) Let A and B be semi-simple Banach algebras and $T : A \to B$ a surjective full spectrum preserving linear map. If B has the property that $bSoc\ (B) = \{0\}$ implies $b = 0$, then T is a Jordan homomorphism.*

**Proof.** It follows from Theorem 4.2.10 that $\left(Ta^2 - (Ta)^2\right) Soc\ (B) = \{0\}$ for every $a \in A$. Therefore, by hypothesis, $Ta^2 - (Ta)^2 = 0$ and so $Ta^2 = (Ta)^2$, for every $a \in A$. Hence $T$ is a Jordan homomorphism. $\nabla$

We have the following solution to Conjecture C3, which is the first main result of this chapter. In fact, it is part of the proof of [4], Corollary 3.4.

**Corollary 4.2.19** *Let A and B be semi-simple Banach algebras. If B is a primitive Banach algebra with minimal ideals and $T : A \to B$ is a surjective full spectrum preserving linear map, then T is a Jordan homomorphism.*

**Proof.** Since $B$ is a primitive Banach algebra with minimal ideals, it follows from Theorem 1.4.18 that $aSoc\ (B) = \{0\}$ implies $a = 0$. Therefore, by Corollary 4.2.18, we have that $T$ is a Jordan homomorphism. $\nabla$

Recall that a primitive Banach algebra is semi-simple. We are now ready to prove the second main result of this chapter.

**Corollary 4.2.20 (B. Aupetit and H. du T. Mouton)** *([4], Corollary 3.4) Let A and B be semi-simple Banach algebras and $T : A \to B$ a surjective full spectrum preserving linear mapping. If, in addition, B is a primitive Banach algebra with minimal ideals, then T is a homomorphism or an anti-homomorphism.*

**Proof.** By Corollary 4.2.19, $T$ is a Jordan homomorphism. It follows from Theorem 1.4.10 that $B$ is prime. We get from Theorem 2.1.6 that $T$ is a homomorphism or an anti-homomorphism. $\nabla$

Theorem 3.4.10 can be extended to full spectrum linear mappings. This is the next result.

**Corollary 4.2.21** *([4], Corollary 3.5) Suppose that X and Y are Banach spaces. Let $T : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a full spectrum preserving surjective linear mapping. Then either*

(i) *there exists a bounded invertible linear operator $b : X \to Y$ such that $Ta = bab^{-1}$ for every $a \in \mathcal{L}(X)$, or*

*(ii) there exists a bounded invertible linear operator $c : X' \to Y$ such that $Ta = ca^{\times}c^{-1}$ for every $a \in \mathcal{L}(X)$.*

**Proof.** By Theorem 1.5.12, $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ are semi-simple. Since, by Proposition 4.1.15, $\mathcal{L}(Y)$ is a primitive Banach algebra with minimal ideals, it follows from Corollary 4.2.20 that $T$ is a homomorphism or an anti-homomorphism. It follows from Theorem 3.4.2, $T$ is injective and hence bijective. Hence $T$ is an isomorphism or an anti-isomorphism. The result follows from Theorem 4.1.16. $\nabla$

Clearly, the Jafarian-Sourour result follows from Corollary 4.2.21. The proof of Corollary 4.2.21 is entirely algebraic since it does not use any operator theoretic techniques. Therefore we now have an algebraic proof of the Jafarian-Sourour result.

# Chapter 5

# The solution of Kaplansky's problem for von Neumann algebras

In Chapter 3 we have introduced the following conjecture: Let $A$ and $B$ be semi-simple Banach algebras. If $T : A \to B$ is a unital bijective invertibility preserving linear mapping, then $T$ is a Jordan isomorphism (Conjecture C1). It was also remarked that this conjecture is still unsolved for the case where $A$ is a C*-algebra. It follows from Corollary 3.2.11 that if $A$ is a C*-algebra, $B$ a commutative semi-simple Banach algebra and $T : A \to B$ a surjective spectrum preserving linear mapping, then $T$ is a Jordan isomorphism.

Recently, B. Aupetit showed that the conjecture is true for von Neumann algebras. The aim of this chapter is to discuss Aupetit's solution.

## 5.1 The Marcus-Purves Theorem revisited

In [11], M. Brešar and P. Šemrl have obtained a proof of the Marcus-Purves Theorem that is simpler than the original proof that Marcus and Purves gave in 1959 (see Theorem 5.1.5 below). Their proof relies on a spectral characterization of idempotent elements of $M_n(\mathbb{C})$. We will see later on that Brešar and Šemrl's proof holds the key to a stategy in proving Conjecture C1 for von Neumann algebras. It is this strategy that Aupetit used to solve the problem. We begin with a spectral characterization of idempotent square matrices.

**Theorem 5.1.1** *([11], Theorem, p.189) A square complex matrix a is an idempotent if and only if $Sp\,(a) \subset \{0, 1\}$ and for every matrix t there exists*

67

*a positive number $K_t$ such that*

$$\rho(\alpha a + t) \leq |\alpha| + K_t$$

*and*

$$\rho\Big(\alpha(e - a) + t\Big) \leq |\alpha| + K_t$$

*for every $\alpha \in \mathbb{C}$, where $e$ denotes the identity square complex matrix.*

We omit the proof since it involves matrix theory. The bulk of the proof of Theorem 5.1.5 is contained in the proofs of Propositions 5.1.2 and 5.1.3. In fact, Propostions 5.1.2 and 5.1.3 are contained in the proof of [11], Corollary on page 190. These results are stronger in the sense that they hold for more general Banach algebras instead of just matrix algebras.

**Proposition 5.1.2** *Let $A$ and $B$ be Banach algebras and suppose that $T : A \to B$ is an idempotent preserving linear map. If $x \in A$ and $n \in \mathbb{N}$ are such that $x = \sum_{i=1}^{n} \lambda_i p_i$, where $p_i$ $(i = 1, \ldots, n)$ are idempotents such that $p_i p_j = p_j p_i = 0$ for all $i \neq j$ $(i, j = 1, \ldots, n)$, then $Tx^2 = (Tx)^2$.*

**Proof.** It is clear that $p_i + p_j$ is an idempotent if $i \neq j$. Since $T$ preserves idempotents, it follows that $\Big(T(p_i + p_j)\Big)^2 = T(p_i + p_j)$ for all $i \neq j$, i.e. $(Tp_i + Tp_j)^2 = Tp_i + Tp_j$ for all $i \neq j$. But

$$
\begin{aligned}
(Tp_i + Tp_j)^2 &= (Tp_i)^2 + Tp_j Tp_i + Tp_i Tp_j + (Tp_j)^2 \\
&= Tp_i + Tp_j Tp_i + Tp_i Tp_j + Tp_j,
\end{aligned}
$$

since $p_i$ and $p_j$ are idempotents and, therefore, so are $Tp_i$ and $Tp_j$. It follows that

$$Tp_i Tp_j + Tp_j Tp_i = 0$$

for all $i \neq j$, so that

$$
\begin{aligned}
(Tx)^2 &= \left(T\left(\sum_{i=1}^{n} \lambda_i p_i\right)\right)^2 \\
&= \left(\sum_{i=1}^{n} \lambda_i Tp_i\right)^2 \\
&= \lambda_1^2 (Tp_1)^2 + \cdots + \lambda_n^2 (Tp_n)^2 \\
&\quad + \text{ a linear combination of } (Tp_i Tp_j + Tp_j Tp_i) \ (i \neq j) \\
&= \lambda_1^2 (Tp_1)^2 + \cdots + \lambda_n^2 (Tp_n)^2.
\end{aligned}
$$

68

Since $p_1, \ldots, p_n$ are idempotents, $T$ is idempotent preserving, and $p_i$ $(i = 1, \ldots, n)$ are mutually orthogonal, it follows that

$$
\begin{aligned}
Tx^2 &= T\left(\sum_{i=1}^{n} \lambda_i p_i\right)^2 \\
&= T(\lambda_1^2 p_1^2 + \cdots + \lambda_n^2 p_n^2 + \text{ a linear combination of } p_i p_j \ (i \neq j)) \\
&= T(\lambda_1^2 p_1^2 + \cdots + \lambda_n^2 p_n^2) \\
&= \lambda_1^2 T p_1^2 + \cdots + \lambda_n^2 T p_n^2 \\
&= \lambda_1^2 T p_1 + \cdots + \lambda_n^2 T p_n \\
&= \lambda_1^2 (T p_1)^2 + \cdots + \lambda_n^2 (T p_n)^2.
\end{aligned}
$$

Therefore $Tx^2 = (Tx)^2$. $\nabla$

**Proposition 5.1.3** *Let $A$ be a $C^*$-algebra, $B$ a Banach algebra and $T : A \to B$ a linear map. If $Ts^2 = (Ts)^2$ for all self-adjoint elements $s$ of $A$, then $T$ is a Jordan homomorphism.*

**Proof.** Let $q$ and $s$ be self-adjoint elements of $A$. Then $q + s$ is self-adjoint and therefore $T(q+s)^2 = \left(T(q+s)\right)^2$. This implies that

$$
Tq^2 + T(qs + sq) + Ts^2 = (Tq)^2 + TqTs + TsTq + (Ts)^2.
$$

Since $Tq^2 = (Tq)^2$ and $Ts^2 = (Ts)^2$, it follows that

$$
T(qs + sq) = TqTs + TsTq. \tag{5.1.4}
$$

Let $u$ be an arbitrary element of $A$. Then, by Proposition 1.6.5, $u = q + is$, where $q$ and $s$ are self-adjoint elements of $A$. The assumption, together with (5.1.4), implies that

$$
\begin{aligned}
Tu^2 &= T(q + is)^2 \\
&= T(q^2 + iqs + isq - s^2) \\
&= T(q^2 + i(qs + sq) - s^2) \\
&= Tq^2 + iT(qs + sq) - Ts^2 \\
&= (Tq)^2 + iTqTs + iTsTq - (Ts)^2.
\end{aligned}
$$

Also,

$$
\begin{aligned}
(Tu)^2 &= \left(T(q + is)\right)^2 \\
&= (Tq + iTs)^2 \\
&= (Tq)^2 + iTqTs + iTsTq - (Ts)^2.
\end{aligned}
$$

69

This implies that $(Tu)^2 = Tu^2$. This is true for all $u \in A$, so $T$ is a Jordan homomorphism. $\nabla$

We now give Brešar and Šemrl's proof of the Marcus-Purves Theorem.

**Theorem 5.1.5 (M. Marcus and R. Purves)** *([26]; [11], Corollary, p.190) Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a unital linear mapping preserving invertibility. Then $T$ is a Jordan automorphism.*

**Proof.** By [26], Lemmas 2.3 and 2.4, $T$ is bijective and Sp $(Ta) = $ Sp $(a)$ for every $a \in M_n(\mathbb{C})$. In particular, $\rho(Ta) = \rho(a)$ for every $a \in M_n(\mathbb{C})$.

We first show that $T$ is idempotent preserving. Let $a$ be an idempotent element of $M_n(\mathbb{C})$. By Theorem 5.1.1, Sp $(a) \subset \{0, 1\}$ and for every $b \in M_n(\mathbb{C})$, there exists a positive number $K_b$ such that $\rho(\alpha a + b) \le |\alpha| + K_b$ and $\rho\Big(\alpha(e - a) + b\Big) \le |\alpha| + K_b$ for all $\alpha \in \mathbb{C}$.

It follows that Sp $(Ta) = $ Sp $(a) \subset \{0, 1\}$. Furthermore, if $b \in M_n(\mathbb{C})$, then since $T$ is bijective, there exists a unique $v \in M_n(\mathbb{C})$ such that $Tv = b$ and so $\rho(\alpha Ta + b) = \rho\Big(T(\alpha a + v)\Big) = \rho(\alpha a + v) \le |\alpha| + K_v$ for all $\alpha \in \mathbb{C}$ and some positive number $K_v$, depending on $b$. This is true for any $b \in M_n(\mathbb{C})$. Similarly, since $T$ is unital,

$$
\begin{aligned}
\rho\Big(\alpha(e - Ta) + b\Big) &= \rho\Big(T(\alpha(e - a) + v)\Big) \\
&= \rho\Big(\alpha(e - a) + v\Big) \\
&\le |\alpha| + K_v,
\end{aligned}
$$

for all $\alpha \in \mathbb{C}$. Hence, by Theorem 5.1.1, $Ta$ is an idempotent element of $M_n(\mathbb{C})$.

Let $h \in M_n(\mathbb{C})$ be a Hermitian matrix. Then $h = \sum_{i=1}^{n} \lambda_i p_i$, where $\lambda_i \in \mathbb{R}$ for all $i \in \mathbb{N}$ and $p_i$ $(i = 1, \ldots, n)$ are idempotent elements of $M_n(\mathbb{C})$ such that $p_i p_j = p_j p_i = 0$ if $i \ne j$. It follows from Proposition 5.1.2 that $(Th)^2 = Th^2$.

Recall that $M_n(\mathbb{C})$ represents $\mathcal{L}(\mathbb{C}^n)$, the Banach algebra of bounded linear operators on $\mathbb{C}^n$. Since $\mathbb{C}^n$ is a Hilbert space, it follows from Proposition 1.6.7 that $\mathcal{L}(\mathbb{C}^n)$ is a von Neumann algebra. Hence, by Proposition 5.1.3, the map $T$ is a Jordan homomorphism. It was stated above that $T$ is bijective. Therefore $T$ is a Jordan automorphism. $\nabla$

Propositions 5.1.2 and 5.1.3 enable us to prove Theorem 5.1.6, creating the foundation of Brešar and Šemrl's strategy. In fact, the proof is similar to that of Theorem 5.1.5, except that that we do not have to prove that $T$

preserves idempotents. Observe that the result is stronger than Theorem 5.1.5.

**Theorem 5.1.6** *Let $A$ be a $C^*$-algebra with the property that every self-adjoint element of $A$ is the limit of a sequence of linear combinations of mutually orthogonal idempotents of $A$. If $B$ is a Banach algebra, then every linear continuous idempotent preserving map $T : A \to B$ is a Jordan homomorphism.*

**Proof.** Let $s$ be a self-adjoint element of $A$. By hypothesis, there exists a sequence $(s_n)$ converging to $s$, where for every $n \in \mathbb{N}$, $s_n$ is a finite linear combination of idempotents $p_i$ satisfying $p_i p_j = p_j p_i = 0$ for all $i \neq j$. By Proposition 5.1.2, $T s_n^2 = (T s_n)^2$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} T s_n^2 = \lim_{n\to\infty} (T s_n)^2$.

Since $T$ is continuous, it follows that $\lim_{n\to\infty} T s_n^2 = T(\lim_{n\to\infty} s_n^2) = T s^2$. Similarly, $\lim_{n\to\infty} (T s_n)^2 = \left( T(\lim_{n\to\infty} s_n) \right)^2 = (T s)^2$. Therefore $T s^2 = (T s)^2$. This is true for every self-adjoint element $s$ of $A$. It follows from Proposition 5.1.3 that $T$ is a Jordan homomorphism. $\nabla$

The following result follows easily from Theorem 5.1.6 and Theorem 1.6.15.

**Corollary 5.1.7** *([11], p. 190) If $A$ is a von Neumann algebra and $B$ a Banach algebra, then every linear continuous idempotent preserving map $T : A \to B$ is a Jordan homomorphism.*

By Theorem 1.5.36, every bijective unital invertibility preserving linear map from a von Neumann algebra onto a semi-simple Banach algebra is continuous.

This led Brešar and Šemrl to suggest the following strategy for proving Conjecture C1 for the case where $A$ is a von Neumann algebra: If one can find a suitable spectral characterization of idempotents in a Banach algebra which would imply that a unital bijective invertibility preserving linear map $T$ preserves idempotents, then it will follow from Corollary 5.1.7 that $T$ is a Jordan isomorphism. Brešar and Šemrl succesfully used their strategy to prove the main result for unital standard operator algebras. This has culminated in Theorem 5.1.8 and Corollary 5.1.9.

**Theorem 5.1.8** *([10], Theorem 1) Let $A$ be a Banach algebra and let $\mathcal{B}$ be a unital standard operator algebra on a Banach space $X$. Assume that $T : A \to \mathcal{B}$ is a unital surjective linear mapping preserving invertibility. Then $T$ preserves idempotents.*

71

We omit the proof since it is partly operator theoretic.

**Corollary 5.1.9** *([10], Corollary 1) Let $A$ be a von Neumann algebra and $\mathcal{B}$ a unital standard operator algebra on a Banach space $X$. Assume that $T : A \to \mathcal{B}$ is a unital surjective linear mapping preserving invertibility. Then $T$ is a Jordan homomorphism.*

**Proof.** By Theorem 1.5.13, $\mathcal{B}$ is semi-simple. Therefore it follows from Theorem 1.5.36 that $T$ is continuous. It follows from Theorem 5.1.8 that $T$ preserves idempotents. The result now follows from Corollary 5.1.7. $\nabla$

Note that this proof is based on the strategy that Brešar and Šemrl have given. Brešar and Šemrl have also used Theorem 5.1.8 to give a shorter proof of the next result, which is the first step of the proof of Sourour's theorem.

**Theorem 5.1.10** *([34], Lemma 3.2; [10], Corollary 2) Let $X$ and $Y$ be Banach spaces and let $T : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a unital bijective invertibility preserving linear mapping. Then $T$ maps every operator of rank one onto an operator of rank one.*

We omit the proof since it is operator theoretic. The second step of Sourour's theorem is

**Theorem 5.1.11** *([34], Theorem 3.4) Let $X$ and $Y$ be Banach spaces. If a linear mapping $T : \mathcal{L}(X) \to \mathcal{L}(Y)$ maps every operator of rank one onto an operator of rank one, then $T$ is a Jordan homomorphism.*

Again, we omit the proof since it is operator theoretic. Sourour's theorem (Corollary 3.3.3) follows immediately from Theorems 5.1.10 and 5.1.11.

## 5.2 Aupetit's solution

Aupetit proved Conjecture C1 for the case where $A$ is a von Neumann algebra using the strategy of Brešar and Šemrl stated in Section 5.1. This has culminated in Theorem 5.2.7, Corollary 5.2.17 and Corollary 5.2.20. Before discussing Aupetit's solution, we need Lemma 5.2.1.

**Lemma 5.2.1** *([3], Lemma 2.2) Let $A$ be a Banach algebra and $e \in A$ an idempotent element. For every $x \in A$, we have the following:*

*(i) If $e = 0$, then*

$$Sp\,(x) \subset \overline{D}(0, \|x\|) = \overline{D}\Big(0, (\|e\| + \|1 - e\|)\|x - e\|\Big).$$

72

*(ii) If $e = 1$, then*

$$Sp\ (x) \subset \overline{D}(1, \|x - 1\|) = \overline{D}\Big(1, (\|e\| + \|1 - e\|)\|x - e\|\Big).$$

*(iii) If $e$ is a non-trivial idempotent, then*

$$Sp\ (x) \subset \overline{D}\Big(0, (\|e\| + \|1 - e\|)\|x - e\|\Big) \cup \overline{D}\Big(1, (\|e\| + \|1 - e\|)\|x - e\|\Big).$$

**Proof.** (i) If $e = 0$, then it is clear that

$$\mathrm{Sp}\ (x) \subset \overline{D}(0, \|x\|) = \overline{D}\Big(0, (\|e\| + \|1 - e\|)\|x - e\|\Big).$$

(ii) If $e = 1$, then it is clear that

$$\mathrm{Sp}\ (x) \subset \overline{D}(1, \|x - 1\|) = \overline{D}\Big(1, (\|e\| + \|1 - e\|)\|x - e\|\Big).$$

(iii) Suppose that $e$ is a non-trivial idempotent. In this case, $\mathrm{Sp}\ (e) = \{0, 1\}$. Let $\lambda \neq 0, 1$. Then we have that $(\lambda 1 - e)^{-1}$ exists and we now show that

$$(\lambda 1 - e)^{-1} = \frac{1}{\lambda - 1}e + \frac{1}{\lambda}(1 - e). \qquad (5.2.2)$$

First, we observe that

$$
\begin{aligned}
\Big(\frac{1}{\lambda - 1}e + \frac{1}{\lambda}(1 - e)\Big)(\lambda 1 - e) &= \frac{1}{\lambda - 1}e(\lambda 1 - e) + \frac{1}{\lambda}(1 - e)(\lambda 1 - e) \\
&= \frac{1}{\lambda - 1}(\lambda e - e) + \frac{1}{\lambda}(\lambda 1 - e - \lambda e + e) \\
&= \frac{1}{\lambda - 1}(\lambda - 1)e + \frac{1}{\lambda}\lambda(1 - e) \\
&= e + (1 - e) \\
&= 1.
\end{aligned}
$$

Similarly, $(\lambda 1 - e)\Big(\frac{1}{\lambda - 1}e + \frac{1}{\lambda}(1 - e)\Big) = 1$. Therefore (5.2.2) follows.

Let $p(\alpha) = \frac{1}{\lambda - 1}\alpha + \frac{1}{\lambda}(1 - \alpha)$ for every $\alpha \in \mathbb{C}$. Then $p(x) = \frac{1}{\lambda - 1}x + \frac{1}{\lambda}(1 - x)$ for every $x \in A$. It follows from the spectral mapping theorem that

$$
\begin{aligned}
\mathrm{Sp}\ \Big((\lambda 1 - e)^{-1}\Big) &= \mathrm{Sp}\ \Big(\frac{1}{\lambda - 1}e + \frac{1}{\lambda}(1 - e)\Big) \\
&= \mathrm{Sp}\ \Big(p(e)\Big) \\
&= p\Big(\mathrm{Sp}\ (e)\Big) \\
&= \Big\{\frac{1}{\lambda}, \frac{1}{\lambda - 1}\Big\}.
\end{aligned}
$$

73

Therefore $\rho\left((\lambda 1 - e)^{-1}\right) = \max\left(\frac{1}{|\lambda|}, \frac{1}{|\lambda-1|}\right)$. This, together with (5.2.2), implies that

$$
\begin{aligned}
\|(\lambda 1 - e)^{-1}\| &\leq \frac{1}{|\lambda - 1|}\|e\| + \frac{1}{|\lambda|}\|1 - e\| \\
&\leq \rho\left((\lambda 1 - e)^{-1}\right)\|e\| + \rho\left((\lambda 1 - e)^{-1}\right)\|1 - e\| \\
&= (\|e\| + \|1 - e\|)\rho\left((\lambda 1 - e)^{-1}\right),
\end{aligned}
$$

so that

$$
\frac{1}{\rho\left((\lambda 1 - e)^{-1}\right)} \leq \frac{\|e\| + \|1 - e\|}{\|(\lambda 1 - e)^{-1}\|}, \tag{5.2.3}
$$

for $\lambda \neq 0, 1$.

Suppose that the result is false. Then there exists $x \in A$ and $\alpha \in \mathrm{Sp}\,(x)$ such that

$$
\alpha \notin \overline{D}\left(0, (\|e\| + \|1 - e\|)\|x - e\|\right) \cup \overline{D}\left(1, (\|e\| + \|1 - e\|)\|x - e\|\right),
$$

i.e. $|\alpha| > (\|e\| + \|1 - e\|)\|x - e\|$ and $|\alpha - 1| > (\|e\| + \|1 - e\|)\|x - e\|$. Since $\mathrm{Sp}\,(e) = \{0, 1\}$, we have that

$$
\mathrm{dist}(\alpha, \mathrm{Sp}\,(e)) > (\|e\| + \|1 - e\|)\|x - e\|. \tag{5.2.4}
$$

In particular, $\alpha \notin \mathrm{Sp}\,(e)$ (since otherwise, $\mathrm{dist}(\alpha, \mathrm{Sp}\,(e)) = 0$), so that $\alpha 1 - e$ is invertible. It follows from Theorem 1.5.29, and from (5.2.3) and (5.2.4) that

$$
(\|e\| + \|1 - e\|)\|x - e\| < \frac{\|e\| + \|1 - e\|}{\|(\alpha 1 - e)^{-1}\|},
$$

i.e. $\|x - e\| < \frac{1}{\|(\alpha 1 - e)^{-1}\|}$. So $\|(\alpha 1 - e)^{-1}(x - e)\| \leq \|x - e\|\|(\alpha 1 - e)^{-1}\| < 1$. Thus, by Theorem 1.5.19, $\left(1 - (\alpha 1 - e)^{-1}(x - e)\right)$ is invertible. Since $\alpha 1 - x = (\alpha 1 - e)\left(1 - (\alpha 1 - e)^{-1}(x - e)\right)$, we get that $\alpha 1 - x$ is invertible. Hence $\alpha \notin \mathrm{Sp}\,(x)$. This is a contradiction. Thus the result follows. $\nabla$

The following lemma is part of the proof of [3], Theorem 1.1 (see Theorem 5.2.7).

**Lemma 5.2.5** *Let $A$ be a Banach algebra and let $\Gamma_0, \Gamma_1$ be circles of centres 0 and 1 respectively with radii less than $\frac{1}{2}$ each, which bound the two disjoint*

74

*open disks $\Delta_0$ and $\Delta_1$. We define, with $a \in A$ and $Sp\,(a) = \{0, 1\}$,*

$$p_0 = \frac{1}{2\pi i} \int_{\Gamma_0} (\lambda 1 - a)^{-1} d\lambda,$$

$$p_1 = \frac{1}{2\pi i} \int_{\Gamma_1} (\lambda 1 - a)^{-1} d\lambda,$$

$$a_0 = \frac{1}{2\pi i} \int_{\Gamma_0} \lambda(\lambda 1 - a)^{-1} d\lambda,$$

$$a_1 = \frac{1}{2\pi i} \int_{\Gamma_1} \lambda(\lambda 1 - a)^{-1} d\lambda.$$

*Clearly, $p_0$ and $p_1$ are the spectral idempotents associated with $a$ and $0$, and with $a$ and $1$, respectively. Then*

*(i) $a = a_0 + a_1, p_0 + p_1 = 1$,*

*(ii) $a_0 = p_0 a = a p_0, a_1 = p_1 a = a p_1, p_0 p_1 = p_1 p_0 = 0$ and $a_0 a_1 = a_1 a_0 = 0$,*

*(iii) $\{1, a, p_0, p_1\}$ is a commutative set,*

*(iv) $Sp\,(a_0) = \{0\}$ and $Sp\,(a_1) = \{0, 1\}$,*

*(v) there exists a quasi-nilpotent element $q$ of $A$ commuting with $p_1$ having the property that $a_1 = p_1 + q$.*

**Proof.** (i) It follows from Theorem 1.5.26 that

$$
\begin{aligned}
a_0 + a_1 &= \frac{1}{2\pi i} \int_{\Gamma_0} \lambda(\lambda 1 - a)^{-1} \, d\lambda + \frac{1}{2\pi i} \int_{\Gamma_1} \lambda(\lambda 1 - a)^{-1} \, d\lambda \\
&= \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} \lambda(\lambda 1 - a)^{-1} \, d\lambda \\
&= a.
\end{aligned}
$$

Similarly, $p_0 + p_1 = 1$.

(ii) We first show that $a_0 = p_0 a$. Observe that

$$p_0 a = p_0(a_0 + a_1) = p_0 a_0 + p_0 a_1.$$

Let $\mu, \lambda \notin Sp\,(a)$ and $\mu \neq \lambda$ throughout. Then, by Lemma 1.5.21,

$$
\begin{aligned}
p_0 a_1 &= \left( \frac{1}{2\pi i} \int_{\Gamma_0} (\lambda 1 - a)^{-1} \, d\lambda \right) \left( \frac{1}{2\pi i} \int_{\Gamma_1} \mu(\mu 1 - a)^{-1} \, d\mu \right) \\
&= -\frac{1}{4\pi^2} \int_{\Gamma_0} \int_{\Gamma_1} \mu(\lambda 1 - a)^{-1}(\mu 1 - a)^{-1} \, d\mu \, d\lambda
\end{aligned}
$$

75

$$
= -\frac{1}{4\pi^2} \int_{\Gamma_0} \int_{\Gamma_1} \mu \frac{(\lambda 1 - a)^{-1} - (\mu 1 - a)^{-1}}{\mu - \lambda} \, d\mu \, d\lambda
$$

$$
= -\frac{1}{4\pi^2} \left( \int_{\Gamma_0} \int_{\Gamma_1} \frac{\mu(\lambda 1 - a)^{-1}}{\mu - \lambda} \, d\mu \, d\lambda - \int_{\Gamma_0} \int_{\Gamma_1} \frac{\mu(\mu 1 - a)^{-1}}{\mu - \lambda} \, d\mu \, d\lambda \right)
$$

$$
= -\frac{1}{4\pi^2} \left( \int_{\Gamma_0} \left( \int_{\Gamma_1} \frac{\mu}{\mu - \lambda} \, d\mu \right) (\lambda 1 - a)^{-1} \, d\lambda \right.
$$

$$
\left. - \int_{\Gamma_1} \mu \left( \int_{\Gamma_0} \frac{1}{\mu - \lambda} \, d\lambda \right) (\mu 1 - a)^{-1} \, d\mu \right)
$$

$$
= 0
$$

because, by Cauchy's theorem, $\int_{\Gamma_1} \frac{\mu}{\mu - \lambda} \, d\mu = 0$ and $\int_{\Gamma_0} \frac{1}{\mu - \lambda} \, d\lambda = 0$. Similarly, $a_1 p_0 = 0$. Therefore it remains to show that $p_0 a_0 = a_0$. By Theorem 1.5.22(i), the mapping $\mu \mapsto (\mu 1 - a)^{-1}$ is analytic on $\mathbb{C} \setminus \mathrm{Sp}\,(a) = \mathbb{C} \setminus \{0, 1\}$. Hence the mapping $\mu \mapsto \mu(\mu 1 - a)^{-1}$ is analytic on $\mathbb{C} \setminus \{0, 1\}$. It follows from Cauchy's theorem that we can find a circular contour $\Gamma$, with radius less than that of $\Gamma_0$, such that $\int_{\Gamma_0} \mu(\mu 1 - a)^{-1} \, d\mu = \int_{\Gamma} \mu(\mu 1 - a)^{-1} \, d\mu$. Thus

$$
p_0 a_0 = \left( \frac{1}{2\pi i} \int_{\Gamma_0} (\lambda 1 - a)^{-1} \, d\lambda \right) \left( \frac{1}{2\pi i} \int_{\Gamma_0} \mu(\mu 1 - a)^{-1} \, d\mu \right)
$$

$$
= \left( \frac{1}{2\pi i} \int_{\Gamma_0} (\lambda 1 - a)^{-1} \, d\lambda \right) \left( \frac{1}{2\pi i} \int_{\Gamma} \mu(\mu 1 - a)^{-1} \, d\mu \right)
$$

$$
= -\frac{1}{4\pi^2} \int_{\Gamma_0} \int_{\Gamma} \mu(\lambda 1 - a)^{-1}(\mu 1 - a)^{-1} \, d\mu \, d\lambda
$$

$$
= -\frac{1}{4\pi^2} \int_{\Gamma_0} \int_{\Gamma} \mu \frac{(\lambda 1 - a)^{-1} - (\mu 1 - a)^{-1}}{\mu - \lambda} \, d\mu \, d\lambda
$$

$$
= -\frac{1}{4\pi^2} \left( \int_{\Gamma_0} \int_{\Gamma} \frac{\mu(\lambda 1 - a)^{-1}}{\mu - \lambda} \, d\mu \, d\lambda - \int_{\Gamma_0} \int_{\Gamma} \frac{\mu(\mu 1 - a)^{-1}}{\mu - \lambda} \, d\mu \, d\lambda \right)
$$

$$
= -\frac{1}{4\pi^2} \left( \int_{\Gamma_0} \left( \int_{\Gamma} \frac{\mu}{\mu - \lambda} \, d\mu \right) (\lambda 1 - a)^{-1} \, d\lambda \right.
$$

$$
\left. - \int_{\Gamma} \left( \int_{\Gamma_0} \frac{1}{\mu - \lambda} \, d\lambda \right) \mu(\mu 1 - a)^{-1} \, d\mu \right)
$$

$$
= a_0,
$$

since by Cauchy's theorem, $\int_{\Gamma} \frac{\mu}{\mu - \lambda} \, d\mu = 0$ and $\int_{\Gamma_0} \frac{1}{\mu - \lambda} \, d\lambda = -\int_{\Gamma_0} \frac{1}{\lambda - \mu} \, d\lambda = -2\pi i$. Therefore $a_0 = p_0 a$. Similarly, $a_0 = a p_0$ and $a_1 = p_1 a = a p_1$.

Using a similar argument as in showing that $p_0 a_1 = 0$, it follows that $a_0 a_1 = a_1 a_0 = 0$ and $p_1 p_0 = p_0 p_1 = 0$.

(iii) It follows from (ii) that the set $\{1, a, p_0, p_1\}$ is commutative.

(iv) Let $M$ be a maximal commutative subalgebra of $A$ containing the set $\{1, a, p_0, p_1\}$. Then $a_0, a_1 \in M$. Since $\mathrm{Sp}\,(x, A) = \mathrm{Sp}\,(x, M)$ for every

$x \in M$, we may write unambiguously Sp $(x)$ for every $x \in M$, and it follows from Theorem 1.5.16 that

$$\text{Sp }(x) = \{\chi(x) : \chi \text{ is a multiplicative linear functional on M}\}. \qquad (5.2.6)$$

Therefore, since Sp $(a) = \{0, 1\}$, it follows for an arbitrary multiplicative linear functional $\chi$ on $M$ that $\chi(a) = 0$ or 1. By Cauchy's theorem,

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{\lambda}{\lambda - 0} \, d\lambda = 0.$$

Let $g(\lambda) = \lambda$. Then it follows from Cauchy's integral formula that

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\Gamma_1} \frac{\lambda}{\lambda - 1} \, d\lambda &= \frac{1}{2\pi i} \int_{\Gamma_1} \frac{g(\lambda)}{\lambda - 1} \, d\lambda \\
&= g(1) \\
&= 1.
\end{aligned}$$

Let $\chi$ be an arbitrary multiplicative linear functional on $M$. By Corollary 1.5.10, $\chi$ is continuous. Therefore it follows that

$$\chi(a_1) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\lambda}{\lambda - \chi(a)} \, d\lambda = \left\{ \begin{array}{ll} 0 & \text{if } \chi(a) = 0 \\ 1 & \text{if } \chi(a) = 1. \end{array} \right.$$

This implies that Sp $(a_1) = \{0, 1\}$. Similarly,

$$\chi(a_0) = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{\lambda}{\lambda - \chi(a)} \, d\lambda = \left\{ \begin{array}{ll} 0 & \text{if } \chi(a) = 0 \\ 0 & \text{if } \chi(a) = 1. \end{array} \right.$$

This implies that Sp $(a_0) = \{0\}$.

(v) Let $\chi$ be a multiplicative linear functional on $M$, where $M$ is a maximal commutative subalgebra of $A$ containing the set $\{1, a, p_0, p_1\}$. If $\chi(a) = 0$, then it follows from Cauchy's theorem that $\chi(p_1) = 0 = \chi(a_1)$. Considering the case $\chi(a) = 1$, it follows once again from Cauchy's theorem that

$$\begin{aligned}
\chi(a_1) - \chi(p_1) &= \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\lambda}{\lambda - 1} \, d\lambda - \frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{\lambda - 1} \, d\lambda \\
&= \frac{1}{2\pi i} \int_{\Gamma_1} 1 \, d\lambda \\
&= 0.
\end{aligned}$$

So $\chi(a_1) = \chi(p_1)$. Hence, by (5.2.6), $a_1 = p_1 + q$, with $q \in M$ and $\rho(q) = 0$. This completes the proof. $\nabla$

77

We are now ready to prove Aupetit's spectral characterization of idempotent elements of semi-simple Banach algebras. Recall that this is the first step of Brešar and Šemrl's strategy for proving the main result.

**Theorem 5.2.7 (B. Aupetit)** *([3], Theorem 1.1) Let A be a semi-simple Banach algebra. The following properties are equivalent:*

*(i) a is an idempotent element of A,*

*(ii) $Sp\,(a) \subset \{0,1\}$ and there exist $r, C > 0$ such that $Sp\,(x) \subset Sp\,(a) + C\|x - a\|$ for $\|x - a\| < r$.*

**Proof.** (i) $\Rightarrow$ (ii): Suppose that $a$ is an idempotent in $A$. Then $Sp\,(a) \subset \{0,1\}$. Let $C = \|a\| + \|1 - a\|$.

If $Sp\,(a) = \{0\}$, then $a = 0$ because $a$ is an idempotent. Thus, by Lemma 5.2.1(i), $Sp\,(x) \subset \overline{D}\big(0, (\|a\| + \|1 - a\|)\|x - a\|\big)$. Therefore $Sp\,(x) \subset Sp\,(a) + C\|x - a\|$ for all $x \in A$. The case $a = 1$ is dealt with in a similar manner using Lemma 5.2.1(ii).

Finally, we consider the case $Sp\,(a) = \{0,1\}$. Let $\lambda \in Sp\,(x)$ for any $x \in A$. By Lemma 5.2.1(iii), $|\lambda| \leq (\|a\| + \|1 - a\|)\|x - a\|$, or $|\lambda - 1| \leq (\|a\| + \|1 - a\|)\|x - a\|$. Since $Sp\,(a) = \{0,1\}$, $\text{dist}(\lambda, Sp\,(a)) = \inf(|\lambda|, |\lambda - 1|) \leq C\|x - a\|$, and hence $Sp\,(x) \subset Sp\,(a) + C\|x - a\|$ for all $x \in A$.

(ii) $\Rightarrow$ (i): Suppose that (ii) holds. Consider the case $Sp\,(a) = \{0\}$. Then there exist $r, C > 0$ such that $Sp\,(x) \subset \{0\} + C\|x - a\|$ if $\|x - a\| < r$. Hence $\rho(x) \leq C\|x - a\|$ if $\|x - a\| < r$. By Theorem 1.5.15, we have that $a \in \text{Rad}\,(A) = \{0\}$, i.e. $a = 0$, implying that $a$ is an idempotent element in $A$.

Suppose that $Sp\,(a) = \{1\}$. It follows from the spectral mapping theorem that $Sp\,(a - 1) = \{0\}$. Replacing $a$ by $a - 1$ in the previous argument, it follows that $a - 1 \in \text{Rad}\,(A) = \{0\}$, i.e. $a = 1$. Once again, this implies that $a$ is an idempotent element in $A$.

We now consider the case where $Sp\,(a) = \{0,1\}$. Let $\Gamma_0$ and $\Gamma_1$ be circles with centers 0 and 1 respectively and radii less than $\frac{1}{2}$ each, which bound the disjoint open disks $\Delta_0$ and $\Delta_1$ respectively. We set

$$p_0 = \frac{1}{2\pi i} \int_{\Gamma_0} (\lambda 1 - a)^{-1} d\lambda,$$

$$p_1 = \frac{1}{2\pi i} \int_{\Gamma_1} (\lambda 1 - a)^{-1} d\lambda,$$

$$a_0 = \frac{1}{2\pi i} \int_{\Gamma_0} \lambda(\lambda 1 - a)^{-1} d\lambda,$$

$$a_1 = \frac{1}{2\pi i} \int_{\Gamma_1} \lambda(\lambda 1 - a)^{-1} d\lambda.$$

78

Let $y$ be an arbitrary element in $A$, and let $u = p_0(a+y)p_0$ and $v = a_1$. It follows from Lemma 5.2.5(ii) that $a_0 = p_0 a p_0$ and $p_0 a_1 = 0 = a_1 p_0$. Therefore we have for $x = a_0 + a_1 + p_0 y p_0 = p_0(a+y)p_0 + a_1$ that $x = u + v$ and $uv = 0 = vu$. Therefore, by Lemma 1.5.6, Sp $(u+v)\setminus\{0\} = ($Sp $(u) \cup$ Sp $(v))\setminus\{0\}$ and so $\{0\}\cup$ Sp $(u+v) = $ Sp $(u)\cup$ Sp $(v) = \{0,1\}\cup$ Sp $(u)$ since, by Lemma 5.2.5 (iv), Sp $a_1 = \{0,1\}$. Hence

$$\{0\}\cup \text{Sp }(x) = \{0,1\}\cup \text{Sp }\Big(p_0(a+y)p_0\Big). \qquad (5.2.8)$$

By Lemma 5.2.5 (iv), $a_0$ is quasi-nilpotent. It therefore follows from Theorem 1.5.30 that there exists an $r_1 > 0$ such that if $\|p_0 y p_0\| = \|p_0(a+y)p_0 - a_0\| < r_1$, then Sp $\Big(p_0(a+y)p_0\Big) \subset \Delta_0$. In this case, we then have from (5.2.8) that Sp $(x)$ consists of 1 and a piece contained in $\Delta_0$. In fact, if $\|p_0 y p_0\| < r_1$, then

$$\{0\}\cup \text{Sp }(x) = \{0,1\}\cup\Delta_0', \text{where } \Delta_0' = \text{Sp }\Big(p_0(a+y)p_0\Big)\subset \Delta_0. \quad (5.2.9)$$

We now prove that there exist $r, C > 0$ such that, if $\|p_0 y p_0\| < r$, then

$$\rho\Big(p_0(a+y)p_0\Big) \le C\|p_0 y p_0\|. \qquad (5.2.10)$$

Since $x = a_0 + a_1 + p_0 y p_0$, it follows from Lemma 5.2.5(i) that $x = a + p_0 y p_0$. So $x - a = p_0 y p_0$. By assumption (ii), there exist $r_2, C > 0$ such that

$$\text{Sp }(x) \subset \{0,1\} + C\|p_0 y p_0\| \qquad (5.2.11)$$

if $\|p_0 y p_0\| < r_2$. Let $r = \min(r_1, r_2)$ and let $\|p_0 y p_0\| < r$. Also, let $\lambda \in$ Sp $\Big(p_0(a+y)p_0\Big)$ and $\lambda \ne 0$. Then, by (5.2.9), $\lambda \in$ Sp $(x)$ and so, by (5.2.11), dist$(\lambda, \{0,1\}) \le C\|p_0 y p_0\|$, i.e. $\inf(|\lambda|, |\lambda - 1|) \le C\|p_0 y p_0\|$. Since $\lambda \in$ Sp $\Big(p_0(a+y)p_0\Big)$, it follows from (5.2.9) that $\lambda \in \Delta_0$ and so $\inf\{|\lambda|, |\lambda - 1|\} = |\lambda|$. Hence $|\lambda| \le C\|p_0 y p_0\|$. So if $\|p_0 y p_0\| < r$, then Sp $\Big(p_0(a+y)p_0\Big) \subset \overline{D}(0, C\|p_0 y p_0\|)$. This yields (5.2.10).

By Lemma 1.5.23(i) and (ii), $A_0 = p_0 A p_0$ is a closed semi-simple subalgebra of $A$ with identity $p_0$. It follows from (5.2.10) that $\rho\Big(p_0(a+y)p_0\Big) \le C\|p_0(a+y)p_0 - p_0 a p_0\|$, for $y$ such that $\|p_0(a+y)p_0 - p_0 a p_0\| = \|p_0 y p_0\| < r$. Any element of $p_0 A p_0$ can be written in the form $p_0(a+y)p_0$ for some $y \in A$. By Theorem 1.5.15, $a_0 = p_0 a p_0 \in$ Rad $A_0$. Therefore $a_0 = 0$. Hence $a = a_1$.

We will now prove that $a = p_1$. It follows from Lemma 5.2.5(v) that there exists a quasi-nilpotent element $q$ of $A$ such that $a_1 = p_1 + q$. It follows

79

from Lemma 5.2.5(ii) that $p_1 a p_1 = a_1$. Let $A_1 = p_1 A p_1$. Then, by Lemma 1.5.23(i) and (ii), $A_1$ is a closed semi-simple subalgebra of $A$. Since $p_1$ is the identity element of $A_1$ and $q$ is quasi-nilpotent, it follows from the spectral mapping theorem that

$$\text{Sp } (p_1 a p_1, A_1) = \{1\}. \tag{5.2.12}$$

Since $p_1 a p_1 = a_1 = a$, the assumption (ii) implies that if $\|p_1(x - a)p_1\| < r_2$, then $\text{Sp } (p_1 x p_1, A) \subset \{0, 1\} + C\|p_1(x - a)p_1\|$. It follows from Lemma 1.5.23(iii) that $\text{Sp } (p_1 x p_1, A_1) \subset \text{Sp } (p_1 x p_1, A)$. So, if $\|p_1(x - a)p_1\| < r_2$, then

$$\text{Sp } (p_1 x p_1, A_1) \subset \{0, 1\} + C\|p_1(x - a)p_1\|.$$

Hence, if $\|p_1 x p_1 - a\| < r_2$ and $\lambda \in \text{Sp } (p_1 x p_1, A_1)$, then $\text{dist}(\lambda, \{0, 1\}) \leq C\|p_1(x - a)p_1\|$. It follows from Theorem 1.5.30 and (5.2.12) that there exists $r_3 > 0$ such that if $\|p_1 x p_1 - a\| < r_3$, then $\text{Sp } (p_1 x p_1, A_1) \subset \Delta_1$. Let $s = \min(r_2, r_3)$. Then, if $\|p_1(x - a)p_1\| < s$ and $\lambda \in \text{Sp } (p_1 x p_1, A_1)$, then $\text{dist}(\lambda, \{0, 1\}) = |\lambda - 1|$. Therefore, if $\|p_1(x - a)p_1\| < s$, we have

$$\text{Sp } (p_1 x p_1, A_1) \subset \{1\} + C\|p_1(x - a)p_1\|. \tag{5.2.13}$$

We now show that $p_1(a - p_1)p_1 = 0$. Let $\lambda \in \text{Sp } (p_1(x - p_1)p_1, A_1)$. By the spectral mapping theorem, $\lambda + 1 \in \text{Sp } (p_1 x p_1, A_1)$. It follows from (5.2.13) that $|\lambda| = |(\lambda + 1) - 1| \leq C\|p_1(x - a)p_1\|$ if $\|p_1(x - a)p_1\| < s$. In other words,

$$\rho(p_1(x - p_1)p_1) \leq C\|p_1(x - p_1)p_1 - p_1(a - p_1)p_1\|$$

if $\|p_1(x - p_1)p_1 - p_1(a - p_1)p_1\| = \|p_1(x - a)p_1\| < s$. Any element of $p_1 A p_1$ can be written in the form $p_1(x - p_1)p_1$ for some $x \in A$. Thus, by Theorem 1.5.15, it follows that $p_1(a - p_1)p_1 \in \text{Rad } (A_1) = \{0\}$. Hence $p_1(a - p_1)p_1 = 0$.

This implies that $a - p_1 = 0$. We conclude that $a = p_1$, and so $a$ is an idempotent element of $A$. This completes the proof. $\nabla$

Theorem 5.2.7 enabled Aupetit to execute the second step of Brešar and Šemrl's strategy for proving the main result. He achieved this in the next result. The proof follows the lines of [3], Theorem 1.2. (Aupetit only proved Corollary 5.2.17 in his paper [3]. He did remark that his proof can be extended to yield the given proof of Theorem 5.2.14.)

**Theorem 5.2.14** *Let $A$ and $B$ be semi-simple Banach algebras and let $T : A \to B$ be a unital bijective invertibility preserving linear map. Then $T$ preserves idempotents.*

**Proof.** Let $e$ be an idempotent element of $A$. We prove that $Te$ is an idempotent element of $B$. If $e = 0$, then $Te = 0$, i.e. $Te$ is an idempotent

80

element in $B$. Now consider the case $e = 1$. By hypothesis, $Te = 1$ and hence is an idempotent element of $B$.

We now consider the case where $e$ is a non-trivial idempotent of $A$. Then $\mathrm{Sp}\,(e, A) = \{0, 1\}$ and so $\mathrm{Sp}\,(Te, B) \subset \{0, 1\}$. Since $e$ is an idempotent, it follows from Theorem 5.2.7 that there exist $r, C > 0$ such that

$$\mathrm{Sp}\,(x, A) \subset \{0, 1\} + C\|x - e\|$$

for $\|x - e\| < r$. By Theorem 1.5.36, $T$ is continuous. Therefore, by the bounded inverse theorem, $T^{-1}$ is continuous and so there exists an $\alpha > 0$ such that

$$\|x - e\| \leq \frac{1}{\alpha}\|Tx - Te\|. \tag{5.2.15}$$

This implies that $C\|x - e\| \leq \frac{C}{\alpha}\|Tx - Te\|$. Therefore

$$\mathrm{Sp}\,(x, A) \subset \{0, 1\} + \frac{C}{\alpha}\|Tx - Te\|$$

for $\|x - e\| < r$.

Let $z \in B$. Then $z = Tw$ for some $w \in A$ because $T$ is surjective. Since $T$ is unital and invertibility preserving, we have that

$$\mathrm{Sp}\,(z, B) = \mathrm{Sp}\,(Tw, B) \subset \mathrm{Sp}\,(w, A) \subset \{0, 1\} + \frac{C}{\alpha}\|z - Te\|$$

for $\|w - e\| < r$. Hence it follows from (5.2.15) that if $\|z - Te\| < \alpha r$, then

$$\mathrm{Sp}\,(z, B) \subset \{0, 1\} + \frac{C}{\alpha}\|z - Te\|. \tag{5.2.16}$$

Consider the case $\mathrm{Sp}\,(Te, B) = \{0, 1\}$. Then the result follows from (5.2.16) and Theorem 5.2.7.

It remains to consider the case where $\mathrm{Sp}\,(Te, B) = \{\lambda_0\}$ with $\lambda_0 = 0$ or $1$. Let $B(\lambda_0, \frac{1}{2})$ be an open ball with center $\lambda_0$ and radius $\frac{1}{2}$. By Theorem 1.5.30, there exists $s > 0$ such that if $\|z - Te\| < s$, then $\mathrm{Sp}\,(z, B) \subset B(\lambda_0, \frac{1}{2})$. Let $s_1 = \min(s, \alpha r)$ and let $\lambda \in \mathrm{Sp}\,(z, B)$ with $\|z - Te\| < s_1$. Then, by (5.2.16), $|\lambda - \lambda_0| = \mathrm{dist}(\lambda, \{0, 1\}) \leq \frac{C}{\alpha}\|z - Te\|$. Thus, if $\|z - Te\| < s_1$ and $\lambda \in \mathrm{Sp}\,(z, B)$, then $\mathrm{dist}(\lambda, \mathrm{Sp}\,(Te, B)) \leq \frac{C}{\alpha}\|z - Te\|$. In other words, if $\|z - Te\| < s_1$, then

$$\mathrm{Sp}\,(z, B) \subset \mathrm{Sp}\,(Te, B) + \frac{C}{\alpha}\|z - Te\|.$$

It follows from Theorem 5.2.7 that $Te$ is an idempotent. $\nabla$

Clearly, Theorem 5.2.14 extends Theorem 5.1.8 if $T$ (in Theorem 5.1.8) is also injective. The next result follows easily from Theorem 5.2.14, Corollary 3.4.3 and Lemma 2.2.3.

81

**Corollary 5.2.17** *([3], Theorem 1.2) Let A and B be semi-simple Banach algebras and let $T : A \to B$ be a surjective spectrum preserving linear map. Then T preserves idempotents.*

The following result is a special case of Conjecture C1.

**Theorem 5.2.18** *Let A be a $C^*$-algebra such that every self-adjoint element of A can be expressed as a limit of a sequence of linear combinations of mutually orthogonal idempotents in A. If B is a semi-simple Banach algebra and $T : A \to B$ a unital bijective invertibility preserving linear mapping, then T is a Jordan isomorphism.*

**Proof.** It follows from Theorem 1.5.36 that $T$ is continuous. By Theorem 5.2.14, $T$ preserves idempotents. The result follows from Theorem 5.1.6. $\nabla$

It follows from Example 1.6.17 that Theorem 5.2.18 is stronger than the following corollary.

**Corollary 5.2.19** *Let A be a von Neumann algebra and B a semi-simple Banach algebra. If $T : A \to B$ is a unital bijective invertibility preserving linear mapping, then T is a Jordan isomorphism.*

**Proof.** Since $A$ is a von Neumann algebra, the result follows from Theorem 1.6.15 and Theorem 5.2.18. $\nabla$

It is clear that Corollary 5.2.19 extends Corollary 5.1.9 if $T$ is bijective. Recall the following conjecture from Chapter 3: Let $A$ and $B$ be semi-simple Banach algebras. If $T$ is a surjective spectrum preserving linear map, then $T$ is a Jordan isomorphism (Conjecture C2). Clearly, the following result by Aupetit is an instance of this conjecture.

**Corollary 5.2.20** *([3], Theorem 1.3) Let A be a von Neumann algebra and B a semi-simple Banach algebra. If $T$ is a spectrum preserving linear mapping from A onto B, then T is a Jordan isomorphism.*

**Proof.** It follows from Corollary 3.4.3 that $T$ is injective and unital. We now obtain the result from Lemma 2.2.3 and Corollary 5.2.19. $\nabla$

Corollary 5.2.19 makes it possible to give an algebraic proof of a special case of Sourour's theorem, namely

**Corollary 5.2.21** *Let H be a Hilbert space, Y a Banach space and $\phi : \mathcal{L}(H) \to \mathcal{L}(Y)$ a unital bijective linear mapping preserving invertibility. Then $\phi$ is a Jordan isomorphism.*

In fact, using Corollary 5.2.20, we now also have an algebraic proof of a special case of the Jafarian-Sourour result.

**Corollary 5.2.22** *Let $H$ be a Hilbert space and $Y$ a Banach space. If $T :$ $\mathcal{L}(H) \to \mathcal{L}(Y)$ is a spectrum preserving surjective linear mapping, then $T$ is a Jordan isomorphism.*

Recall that in Chapter 4, it was proved algebraically that if $T$ is a full spectrum preserving linear mapping from $\mathcal{L}(X)$ into $\mathcal{L}(Y)$, where $X$ and $Y$ are Banach spaces, then $T$ is a Jordan isomorphism. This clearly extends Corollary 5.2.22, as well as the Jafarian-Sourour result.

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84

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# Index