RELIABILITY MODELLING OF PERFORMANCE FUNCTIONS CONTAINING CORRELATED BASIC VARIABLES, WITH APPLICATION TO CONSTRUCTION PROJECT RISK MANAGEMENT

By

Gregory Mark Ker-Fox
B.Eng. (Civil), M.Eng.

Dissertation presented to
the Department of Civil Engineering,
University of Stellenbosch in fulfilment
of the requirements for the degree of
Doctor of Philosophy.

Supervisor
Prof. JV Retief

Stellenbosch November 2002

Declaration

I the undersigned hereby declare that the work contained in this dissertation is my own original work and has not previously in its entirety or in part been submitted for a degree.

………………………..   Date: ………………………..

G.M. Ker-Fox

Synopsis

Correlation mechanisms describing systematic variations and common sensitivities are critical contributors to uncertainty in quantitative functions modelling project performance in terms of probabilistic or basic variables. Current reliability methods transform dependent vectors to an equivalent set of independent standard normal variates. A simple method is developed for dealing with correlation in the original variable space.

An algebraic description of the direction cosine (or alpha) for performance functions under conditions of dependence is formally derived and numerically validated. The resultant General First Order Second Moment (GFOSM) method for correlated basic variables is shown to be equivalent to the orthogonal transformation method. Geometric and physical interpretations of the general direction cosine are developed, with alpha found to be equivalent to the correlation between a basic variable and performance function. Corresponding inequalities and normalizing conditions are also developed for alpha.

Expressions for a number of applications utilising the general dependent form for the direction cosine are derived and demonstrated. The current definition of the direction cosine as an importance factor is validated for dependent conditions, and conditions established under which this descriptor is no longer adequate. Expressions are derived to measure the significance of a variable in terms of stochastic importance and function sensitivity, to establish reliability index sensitivity to the omission of non-critical items, quantifying variable elasticity and an elasticity index. The general FOSM method for correlated basic variables is applied to system analysis to generate modal correlation coefficients between failure modes.

The general direction cosine is stable for multivariate linear functions and functions of limited curvature across a range of reliabilities and correlation levels. This characteristic further simplifies the process by providing for deterministic reliability modelling of performance functions containing dependent variables, avoiding the solution of the more complex joint density function.

The extension of the current theory and the treatment of performance functions in the original vector space develop invaluable insight into the correlation mechanisms driving risk and reliability. This will assist project managers to better understand areas that can affect project performance, to focus management attention, develop mitigation strategies and to allocate resources for the optimal management of project risk.

Sinopsis

Korrelasie meganismes wat sistematiese afwykings en gemeenskaplike sensitiwiteite veroorsaak, is kritieke bydraers tot onsekerheid in kwantitatiewe funksies wat projek prestasie modelleer in terme van probabilistiese of basiese veranderlikes. Huidige betroubaarheidsmetodes transformeer afhanklike vektore tot 'n ekwivalente stel van standaard normaal onafhanklike veranderlikes. 'n Eenvoudige metode is ontwikkel om die effekte van korrelasie in die oorspronklike vektorspasie te hanteer.

'n Algebraise beskrywing van die rigtingscosines (genoem alfa) vir prestasiefunksies onder omstandighede van afhanklikheid is formeel afgelei en numeries gevalideer. Dit is bewys dat die resulterende Algemene Eerste Orde Tweede Moment metode vir gekorreleerde basiese veranderlikes ekwivalent is aan die tradisionele Ortogonale Transformasie metode. Geometriese en fisiese interpretasies vir die algemene rigtingscosinus is ontwikkel, met bewys dat alfa ekwivalent is aan die korrelasie tussen 'n basiese veranderlike en die prestasiefunksie. Ooreenstemmende ongelykhede en normaliserings-kondisies is ook vir alfa ontwikkel.

Uitdrukkings vir 'n aantal toepassings wat gebruik maak van die algemene afhanklike vorm van die rigtingscosinus is afgelei en gedemonstreer. Die huidige definisie van die rigtingscosinus as 'n belangrikheidsfaktor is gevalideer vir kondisies van afhanklikheid en omstandighede is uitgewys wanneer dit onvoldoende is. Uitdrukkings is afgelei om stochastiese belangrikheid te meet asook funksie sensitiwiteit, die sensitiwiteit van die betroubaarheidsindeks tot die weglating van nie kritiese veranderlikes, sowel as die kwantifisering van elastisiteit en die elastisiteitsindeks. Die Algemene Eerste Orde Tweede Moment metode vir gekorreleerde veranderlikes is toegepas op sisteem analise om die korrelasie tussen falingsmodes te genereer.

Die algemene rigtingscosinus is stabiel vir liniêre funksies en funksies met 'n beperkte kromming oor 'n reeks betroubaarheidswaardes en korrelasie vlakke. Hierdie kenmerk vereenvoudig die metode verder deur voorsiening te maak vir deterministiese betroubaarheidsmodellering van prestasie funksies met afhanklike veranderlikes, deur die oplossing van die meer komplekse gesamentlike-digtheidsfunksies te vermy.

Die uitbreiding van die huidige teorie en die hantering van prestasie funksies in die oorspronklike vektor spasie ontwikkel waardevolle insig in die korrelasie meganismes wat risiko en betroubaarheid oorheers. Hierdie insig sal projekbestuurders in staat stel om kritieke gebiede wat projek prestasie kan affekteer beter te verstaan, om hulle aandag daarop te fokus, om teenmaatreël-strategieë te ontwikkel en hulpbronne toe te ken vir die optimale bestuur van projek risiko.

Dedication

Dedicated to my wife, Liezl
and parents, Rod and Anita

Acknowledgements

1. God, for giving me the opportunity and potential to complete this work.
2. My wife Liezl, for the late nights typing, sacrifices and unfailing encouragement throughout my studies. Without her support this dissertation would certainly not have been completed.
3. My parents, Rod and Anita, for their support and teaching me the value of education.
4. My brother, Brad and extended family and friends, for their enthusiasm and encouragement.
5. My supervisor, Prof. Retief, it was a great privilege to have studied under him for six years, while completing a Masters degree and PhD. I am very grateful for his patience, guidance and wisdom and hope that I will do him proud during my career.
6. My mentor, Prof. Hugo, for awarding me an internship, introduction to the construction industry and induction into the field of technology management.
7. To the Institute for Transport Technology, the National Department of Transport and the Harry Crossley Bursary fund for the financial support during my studies.
8. My staff at TRAC South Africa; Janine, Christa, Wayne, Frik and Jan, for giving me the time I needed to focus on my studies.
9. The risk management research team; Toit, André, Bianca and Katriene, for their help and support.
10. The Department of Civil Engineering Chairman, Prof. Dunaiski and his staff for the opportunity to study at Stellenbosch University and in particular Delysia and Rodney for their support.

Table of Contents

List of Figures

List of Tables

1. INTRODUCTION

Engineers and project managers are under increasing pressure to improve project performance. The large number of projects that fail to meet budget and schedule targets is an indication that there is a need for radical change. The Latham Report (LATHAM, 1994) produced by the United Kingdom's Construction Industry called for a 30% real cost reduction by the year 2000. The Egan Report produced by the Construction Task Force of the UK Department of the Environment (EGAN, 1998) called for greater customer focus, an annual cost reduction of 10%, a 20% reduction in project defects and an increase of 20% in the predictability of project cost and duration. Although not explicitly described in the reports, many of the recommendations are being addressed in terms of the processes and tools for effective risk management.

The formal management of project risk has received increasing attention in recent years. Identifying events that potentially could give rise to risk in projects, modelling the likely consequence of this uncertainty and developing formal mitigation strategies is recognised as a credible means to improve project performance.

1.1 King II Report on Corporate Governance – Risk Management

The King Committee on Corporate Governance launched the King Report on Corporate Governance for South Africa - 2002 (King II Report) in March 2002. The draft report and code applies to all companies listed on the Johannesburg Stock Exchange. King II included four new chapters addressing issues related to compliance for risk management.

The report repeatedly emphasised that the board of directors is accountable for risk management; *"Directors have an obligation to demonstrate that they have dealt comprehensively with the issues of risk management and internal control... The total process of risk management, which includes a related system of internal control, is the responsibility of the board. Management is accountable to the board for designing, implementing and monitoring the process of risk management and integrating it into the day to day activities of the company...and providing assurance that it has done so."*

In terms of the value to investors of a comprehensive and transparent approach to managing risk, the report said; *"Any vulnerability in the achievement of the companies objectives, whether caused by internal or external risk factors, should be detected in good time, reported by the systems of control in place and met with appropriate interventions. Not only will this* *improve its risk profile, thereby enhancing the companies attraction as a worthwhile investment but it will also enhance the positive influence of risk on the business."*

The report had as a key recommendation to the assimilation of risk in the control environment; *"Companies should develop a system of risk management and internal control that builds more robust business operations. The systems should demonstrate the company's key risks are being managed in a way that enhances shareowners' and relevant stakeholders' interests. The system should incorporate mechanisms to deliver the following:*

- *A demonstrable system of dynamic risk identification;*
- *A demonstrable system of risk mitigation activities;*
- *A system of documented risk communications;*
- *A system of documenting the costs of non-compliance and losses;*
- *A documented system of internal control and risk management;*
- *A register of key risks that could affect share owner and relevant stakeholder interests.*
- *The board must identify key risk areas and key performance indicators of the company and monitor these factors as part of a regular review of the process and procedures to ensure the effectiveness of its internal systems of control, so that its decision-making capability and the accuracy of its reporting and financial results are maintained at a high level at all times."*

The board is required to formally disclose in its annual report that the; *"The board is responsible for disclosure in relation to risk management and should as a minimum disclose:*

- *That it is accountable for the process of risk management and the system of internal control,...*
- *That there is ongoing process for identifying, evaluating and managing the significant risks faced by the company,...*
- *That there is an adequate system of internal control in place to mitigate the significant risks faced by the company,...*
- *That there is a documented and tested process in place that will allow the company to continue its critical business processes in the event of a disastrous incident impacting its activities."*

For companies listed on the Johannesburg Stock Exchange, it is now required in terms of corporate governance to deliberately and comprehensively address the area of risk management to protect share owner and stakeholder interests. Furthermore the board of directors are made directly accountable for risk management within the company and must demonstrate that these processes are in place and communicate key risk issues to shareholders.

1.2 Motivation

Functions describing project performance in the construction environment are vulnerable to systematic variations. This is due typically to the large number of variables which demonstrate common sensitivities to risk events such as site conditions, weather conditions, material quality and availability, foreign exchange, inflation, productivity, accuracy of the cost, schedule and quantity estimates, political conditions, relationships between participating parties, contract conditions, estimating conditions and levels of risk aversion of estimators and planners and a host of other factors. If these events affect variables independently, the consequence may well be containable. However due to interrelationships and shared sensitivities, the impact of the aforementioned risk events can cascade into conditions that threaten the successful execution of the project.

The statistical measure for these interrelationships is the correlation coefficient. Due to the relatively complex nature of probabilistic modelling functions with large numbers of dependent variables, this area of risk and reliability has as yet not received extensive attention. This is particularly true for project risk analysis, which often demonstrates correlation mechanisms as strong drivers in risk exposure.

A direct method to model project risk in performance functions containing correlated basic variable has been proposed (KER-FOX, 1998). This initial development created an opportunity to explore the proposed formulation and its potential utility.

1.3 Objective

The objective of the study is to theoretically derive and numerically validate the general dependent form for the direction cosine (KER-FOX, 1998). This is followed by an extensive literature survey, to identity similar developments in First Order Second Moment reliability modelling which address correlated variables in the original vector space. The formulation of the general dependent form for the direction cosine is then to be reconciled with the current theory. Having derived, validated and reconciled the expression, attention is focused on the fundamental understanding of the direction cosine. The potential simplifications of the solution of the limit state functions as well as applications to system analysis are to be explored. Any further utility of the methodology, resulting from different formulations of the direction cosine describing quantitative risk behaviour, as well as opportunities to reduce the solution of performance functions to a deterministic process, are also to be investigated.

1.4 Scope

Functions describing project performance such as cost estimates, schedules, return on investment and cash flows are well suited to the First Order Second Moment (FOSM) reliability method. This is an approximate method utilising the mean, standard deviation and correlation coefficients to model performance reliability. These functions are particularly vulnerable to the effects of correlation, and therefore respond well to simple rational methods for dealing with the effects of dependence. The applications utilised in this study to demonstrate the different methodologies are all typical construction project functions representing a bill of quantities type cost estimate and activity schedule. However it should be noted that the FOSM reliability method is a generic risk modelling tool, initially developed for structural reliability based design, which would be well suited to many fields requiring risk analysis and management.

1.5 Study Themes

Three distinct themes are developed in the study. These are generally grouped in different chapters (not necessarily sequential) as will be described in the following section. Throughout the dissertation, the original formulations have been given a border. The themes are

- Mathematical derivation and validation of a general first order second moment methodology for addressing correlated basic variables in the original variable space. This method avoids the transformation of correlated basic variables into an equivalent independent space. Chapters 4 and 6 fall under this theme.
- Developing theoretical insight into the mechanics of reliability modelling, derived from an algebraic description of the direction cosine for correlated basic variables. Chapters 7 and 8 focus on developing theoretical insight, drawn from the general correlated form of the direction cosine.
- Exploring the utility of the GFOSM method for dependent variables, with regards to risk descriptors, sensitivities and modal correlation for systems analysis. Further simplifications lead to a deterministic reliability based design method. Chapters 5, and 9 to 12 explore the utility of an analytical form of alpha with correlated variables.

Chapter 13 consolidates the utility theme by demonstrating applications to a hypothetical construction project.

1.6 Chapter Layout

1.6.1 Chapter 2: Limit State Reliability Modelling

Chapter 2 describes the limit state reliability method background and theory, with attention to the performance and limit state function formulation, first order second moment analysis, the reliability index, direction cosine, failure point co-ordinates and the generic algorithm for independent variables. The application of the methodology to project cost estimating (KER-FOX, 1998), is referenced. Current extensions to the basic theory addressing non-normal distributions, as well as the Orthogonal and Rosenblatt methods for transforming dependent variables into an independent variable space are described.

1.6.2 Chapter 3: Correlation Effects

As was referred to in the motivation, the study focuses on a method for modelling correlation. Chapter 3 develops the concept of regression, covariance and correlation as techniques for accounting for interrelationships and dependencies. Additional to the basic theory, the mathematical relationship between correlation and regression is derived. A parametric study is conducted of a simple performance function to highlight the importance of including the effects of correlation on reliability modelling. Finally the potential impact of correlation on project risk functions is discussed.

1.6.3 Chapter 4: Analytical Solution of the Direction Cosine under Conditions of Dependence

The direction cosine described in chapter 2 is a critical descriptor for the FOSM methodology, since it carries information describing each variables performance at the failure point. Addressing the effects of correlation is often complex, requiring transformation into an independent variable space. Methods have been developed for estimating equivalent direction cosines and direct extraction of the failure point. However an algebraic form for the direction cosine for dependent variables has eluded researchers. A general dependent form for the direction cosine is formally derived in chapter 4 and reconciled with the current theory. The expression is consistent with the First Order Second Moment Reliability Method approximations, and can be applied to linear and non-linear multivariate functions containing normally and non-normally distributed correlated basic variables.

1.6.4 Chapter 5: General First Order Second Moment (GFOSM) Method for Correlated Basic Variables

The current FOSM algorithm must be applied to independent variables. Under conditions of dependence, variables must be transformed into an equivalent independent variable space. The current algorithm is extended to include the general dependent form for the direction cosine. This facilitates the application of the algorithm in the original variable space, significantly simplifying the probabilistic solution of the limit state function. A demonstration application is given at the end of the chapter.

1.6.5 Chapter 6: Numerical Validation of the Proposed General Dependent form of the Direction Cosine

The theoretical derivation of the general direction cosine is validated numerically by comparing results for a number of limit functions, to those obtained through the orthogonal transformation procedure. The validity of the functions of different curvatures, increasing number of variables, increasing levels of correlation and different distribution types across a range of reliabilities is explored. As a control, selected functions were solved with a commercial software package utilising the Rosenblatt transformation procedure. The proposed general form for alpha produced identical results to the orthogonal transformation procedure for all cases.

1.6.6 Chapter 7: Interpretation of the Direction Cosine

Chapter 7 is an important chapter in terms of the contribution of this study, since it explores the fundamental understanding of alpha afforded by the development and validation of a general direction cosine. The term is divided into dependent and independent components that provide for the description of correlation mechanisms. Geometric and physical interpretations of alpha are developed which give rise to risk indicators discussed in chapter 8. An important mathematical contribution is the development of an inequality for the direction cosine, as well as a normalising condition.

1.6.7 Chapter 8: Stability of the Direction Cosine

An explicit description of the direction cosine under conditions of dependence, affords an opportunity to observe the stability of this quantity across a range of reliabilities. A number of conditions are identified under which the direction cosine remains stable. Under such conditions, the values for alpha at a known level of reliability can be used to estimate the direction cosines corresponding to other failure point co-ordinates. This will prove invaluable in simplifying the solution of the limit state function, discussed in chapter 12.

1.6.8 Chapter 9: Risk Indicators

In addition to avoiding the transformation of dependent variables into an independent variable space, the general dependent form of the direction cosine facilitates the validation of existing risk indicators, identifying conditions under which they are not sufficient and developing additional utility. Risk indicators such as the importance factor, stochastic importance, elasticity, elasticity index, contingency allocation and limit function sensitivity are developed for correlated basic variables.

1.6.9 Chapter 10: Omission Sensitivity

Omission sensitivity factors are applied to limit functions to test the sensitivity of the reliability index to the exclusion of non-critical variables. This is a useful tool for simplifying complex performance functions, by focussing only on the significant variables. However the current method requires the transformation of dependent variables and the solution of the limit function before omission sensitivities could be established.

The omission sensitivity factor is developed as a function of the general direction cosine, thus avoiding transformation into an independent variable space. Furthermore the stability of the direction cosine investigated in chapter 7 produces an approximate value for alpha as a function of the mean values. This allows the omission sensitivity to be established and simplifications affected prior to the solution of the limit function. Reconciliation with the current theory as well as a demonstration application are given at the end of the chapter.

1.6.10 Chapter 11: System Reliability

Project systems consist of a number of performance functions. System analysis considers the effect of modal correlation between different performance functions on the reliability of the system as a whole. A method is derived for measuring the modal correlation as a function of the general direction cosines. New insight into system reliability is developed regarding the drivers of correlation mechanisms. An illustrative example is given at the end of the chapter.

1.6.11 Chapter 12: Further Simplifications for Deterministic Reliability Based Design

The stability of the direction cosine presented in chapter 7 provides for the direct solution of the limit function, without full probabilistic analysis. This is a significant development since it reduces the complexity of the FOSM reliability method by an order of magnitude, placing this powerful tool well within in the reach of project risk managers. Advanced analysis of non-linear functions containing large numbers of correlated variables can easily be performed on a standard spreadsheet.

1.6.12 Chapter 13: Application to Construction Project Risk Management

The theory developed in the study is consolidated in chapter 13 by way of an application. The system to be analysed consists of a construction project consisting of 5 activities. Three performance functions are modelled representing a cost estimate and two potential critical schedule paths. The results of a spreadsheet analysis are discussed to illustrate the utility of the tools presented. An omission sensitivity is conducted which facilitates the simplification of the complex performance functions. The generic algorithm is then applied. Results from the simplified limit functions are compared with results obtained from the solution of the full variable set. Indicators for omission sensitivity, importance factors, elasticity, the elasticity index and sensitivity are presented graphically to indicate critical variables and correlation mechanisms. The modal correlation matrix between the limit functions is generated and utilised in establishing reliability bounds for the system. Partial safety factors are calculated utilising the proposed deterministic reliability based design method. The results are found to compare well with the solution of the full variable set.

1.6.13 Chapter 14: Conclusions

Chapter 14 provides conclusions from theory derived in the study as well as presenting some of the core contributions to the solution of performance functions containing correlated basic variables. The proposed general direction cosine is found to withstand a theoretical and numerical validation process. This provides opportunities for better understanding this important reliability parameter. A direct result of the formulation is the development of a general first order second moment method for correlated basic variables. Further simplifications are developed which provide for deterministic reliability based design. A number of existing risk indicators, described by the direction cosine are validated for correlated basic variables. Extensions to avoid transformation into independent variables spaces are developed as well as a number of new risk descriptors. Finally the general direction cosine is used to develop an algebraic description of modal correlation between functions forming part of a system. The treatment of correlated basic variables in the original variable space significantly improves the efficiency of the first order reliability method as well as developing additional insight into the behaviour of variables under systematic risk.

1.6.14 Chapter 15 and 16: References and Appendix

A list of reference material is given in chapter 15. Appendix I describes a sample of the orthogonal transformation and GFOSM methods utilised in the numerical validation of the general direction cosine. Appendix II is a printout of the spreadsheet used to generate the results discussed in the application (Chapter 12).

2. LIMIT STATE RELIABILITY MODELLING

Engineers have the responsibility of assuring a system's performance. The decisions associated with planning, design, construction, maintenance and operation of infrastructure are usually made under conditions of uncertainty, due to inherent variability, incomplete information or imperfect models. Furthermore, projects must be executed under the constraints of time and economy, which places additional demands on the decision making process. In light of this uncertainty, a rational efficient description of project risk and performance is desirable.

2.1 Performance Reliability

ANG AND TANG [4] define reliability as the probabilistic assurance of performance. The analysis of reliability considers a system's supply capacity (X_1) in terms of the demand requirement (X_2), with X_1 and X_2 both modelled stochastically. This relationship can be described as a simple performance function $g(X)$, where a letter in upper case indicates a random variable, such that

$$g(X) = X_1 - X_2 \tag{2.1}$$

The function is said to be in a safe state if the supply exceeds the demand $P_s[g(X)>0]$. Conversely the function is in the domain of failure if the demand exceeds the supply $P_F[g(X)<0]$. The so called limit state surface separates the two regions and is found by setting the performance function equal to zero, $g(X) = 0$. Of course there are an infinite number of points on the failure surface that satisfy the limit state function. The objective of reliability analysis is to establish the most likely point of failure from which the corresponding limiting performance reliability is estimated.

2.2 First Order Second Moment analysis

The applications of advanced tools for risk modelling are often not feasible due to insufficient data for the stochastic description of variables. The complex nature of engineering projects also limits the availability of quantified sources of uncertainty and degrees of variability, necessary for developing comprehensive risk models. An approximate method, applied at an appropriate level, overcomes the interrelated restrictions between modelling and data.

First Order Second Moment (FOSM) reliability analysis is inherently the variational analysis of a performance calculation, as expressed in units of standard deviations of the variables. This methodology was introduced and popularised in the development of structural design code specifications (FREUDENTAL, GARRELTS AND SHINAZUKA, 1966; CORNELL, 1968; HASOFER AND LIND, 1974; PALOHEIMO AND HANNUS, 1974; RACKWITZ AND FIESSLER, 1978; DITLEVSEN, 1979).

Key properties of FOSM analysis that make it particularly attractive to project risk analysis, are the following:

- Project risk and reliability is evaluated in terms of an explicit objective, expressed in terms of a performance function such as cost and schedule paths.
- A First Order Second Moment approximation significantly simplifies statistical modelling and proves to be computationally efficient.
- The most probable conditions at the limit of acceptable performance are determined.
- The relative contribution of the different variables as sources of uncertainty can be determined quantitatively.
- The FOSM reliability method can be applied to a multivariate function in an *n* dimensional hyperspace.
- Although the method is derived as a linearised solution of the performance function in terms of normally distributed variables, approximations are readily available for extension to non-linear functions, non-normal distributions and correlated variables.
- Limit state theory can be extended to system analysis. A rational basis for evaluating a number of potential failure modes is provided with due consideration given to conditional variation or modal correlation.

2.2.1 Second Moment Formulation

It is difficult to define the joint density functions necessary for statistically establishing reliability due to the fore mentioned lack of sufficient data. The available data may only allow for the realistic determination of the first two moments and the covariance between pairs of variables. By implication, the data is not sufficient to infer skewness or flatness for the probability density functions. For this reason, the calculation of reliability should be restricted to the first two moments, mean and variance, or *Second-Moment* formulation (CORNELL, 1969). The normal distribution is particularly useful since this distribution is completely described by the first two moments. The measure of interdependence between variables or covariance can also be measured from the data and is consistent with the second moment approximation.

2.2.2 *First Order Approximation*

The function given by $g(X) = X_1 - X_2$, in reality may consist of a number of variables. The generalised form of this multivariate performance function can be given as

$$g(X) = g(X_1, X_2, X_3 ... X_n) \tag{2.2}$$

where $(X_1, X_2, X_3 ... X_n)$ is a vector of the variables under consideration and the function $g(X)$ models the performance of these variables.

To simplify the analytical development of the Taylor series, the original normal variates, X_i, have been transformed to their standard normal variates (HASOFER and LIND, 1974), X'_i, where

$$X_i^{'} = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \tag{2.3}$$

The general function, $g(X^{'})$, can be linearised by expanding it in a Taylor series (KREYSZIG, 1988) around the standard normal design point, $x^{'*}$, (FREUDENTAL et al, 1966). This linear approximation is therefore a tangent to the non-linear performance function at the design point. The general form of the Taylor series given below includes the first and second order terms, where *HOT* represents the higher order terms and *(*)* indicates that the function is evaluated at the design point.

$$g(X^{'}) = g(x_1^{'*}, x_2^{'*}, x_3^{'*} ... x_n^{'*}) + \sum_{i=1}^{n} (X_i^{'} - x_i^{'*}) \left(\frac{\partial g}{\partial X_i^{'}} \right)_*$$

$$+ \sum_{j=1}^{n} \sum_{i=1}^{n} (X_i^{'} - x_i^{'*})(X_j^{'} - x_j^{'*}) \Big/ \left(\frac{\partial^2 g}{\partial X_i^{'} \partial g X_j^{'}} \right) + ...HOT \tag{2.4}$$

Truncating the Taylor series at the first order terms develops *First-Order* approximations for the first and second moments. Furthermore, given that on the failure surface $g(x^{'*}) = 0$ and that the mean value for standard normal variates is zero, the first order approximation of the mean value for the general function is given as (ANG AND TANG, 1984)

$$\therefore \mu_g \approx -\sum_{i=1}^{n} x_i^{'*} \left(\frac{\partial g}{\partial X'_i} \right)_* \tag{2.5}$$

Similarly, given that the basic variables are independent, a first order approximation of the standard deviation for the general function in terms of the standard normal variates is (ANG AND TANG, 1984),

$$Var[g(X)] = E\left[\left(g(x'^*) + \sum_{i=1}^{n}\left(\frac{\partial g}{\partial X'_i}\right)_*(X'_i - x'^*_i) - g(x'^*)\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X'_i}\right)_*^2(X'_i - x'^*_i)^2\right)\right]$$

$$= \sum_{i=1}^{n}\left(\frac{\partial g}{\partial X'_i}\right)_*^2 Var(X'^*_i) \tag{2.6}$$

For a standard normal function $Var(X_i) = 1$, therefore the function variance is

$$Var[g(X)] = \sum_{i=1}^{n}\left(\frac{\partial g}{\partial X'_i}\right)_*^2$$

The first order approximation for the standard deviation of a general function with independent basic variables is therefore given by,

$$\sigma_g \approx \left[\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X'_i}\right)_*^2\right]^{\frac{1}{2}} \tag{2.7}$$

2.2.3 Reliability index

The probability of success or reliability is measured as the area under the probability density function or integrand between infinity and x_i. The values corresponding to the standard normal density function are widely documented. The FOSM interpretation is the margin of safety of the performance function in the standard normal space,

$$P_S(g(X) > 0) = \Phi\left(-\frac{\mu_g}{\sigma_g}\right) \tag{2.8}$$

HASSOFER AND LIND (1974) initially proposed the term $\left(\mu_g / \sigma_g\right)$ as the reliability index (denoted, β) for linear functions. Utilising the general form of the Taylor series expansion for the limit function, DITLEVSEN (1984) proposed a general First Order approximation of the Reliability Index, taken at the design point (x'^*)

$$\beta = \frac{\mu_g}{\sigma_g} = \frac{-\sum_{i=1}^{n} x'^{*} \left(\frac{\partial g}{\partial X'_i} \right)_*}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial X'_i} \right)_*^2}} \tag{2.9}$$

2.2.4 Failure Point Co-ordinates

SHINOZUKA (1983), used the method of Lagrange Multipliers to verify the above expression for the reliability index. By this method SHINOZUKA was also able to prove that the design point in the standard normal space, x'^{*}, on the failure surface, $g(X') = 0$, represented the minimum distance from the origin of the axes to the limit state surface. These co-ordinates, when inserted into the joint density function $f_{X'}(x')$ for the random vector, X', yield the highest probability of failure. This proved that the design point was also the most probable failure point or point of maximum likelihood.

The gradient vector is a vector with components given by the partial derivatives with respect to each variable (KREYZIG, 1988)

$$\nabla g = grad \quad g = \left[\left(\frac{\partial g}{\partial X'_1} \right)_*, \left(\frac{\partial g}{\partial X'_2} \right)_*, \ldots, \left(\frac{\partial g}{\partial X'_n} \right)_* \right] \tag{2.10}$$

Each component indicates the rate of change of the vector described by the underlying function, in the direction of the axes under observation. The gradient vector is also by definition normal to the tangent plane at the point of expansion.

As indicated in Figure 2.1, the limit function is always equal to zero and the performance function value increases towards the origin of the axes, corresponding to the safe state. The sign convention for the gradient vector is therefore positive towards the origin.

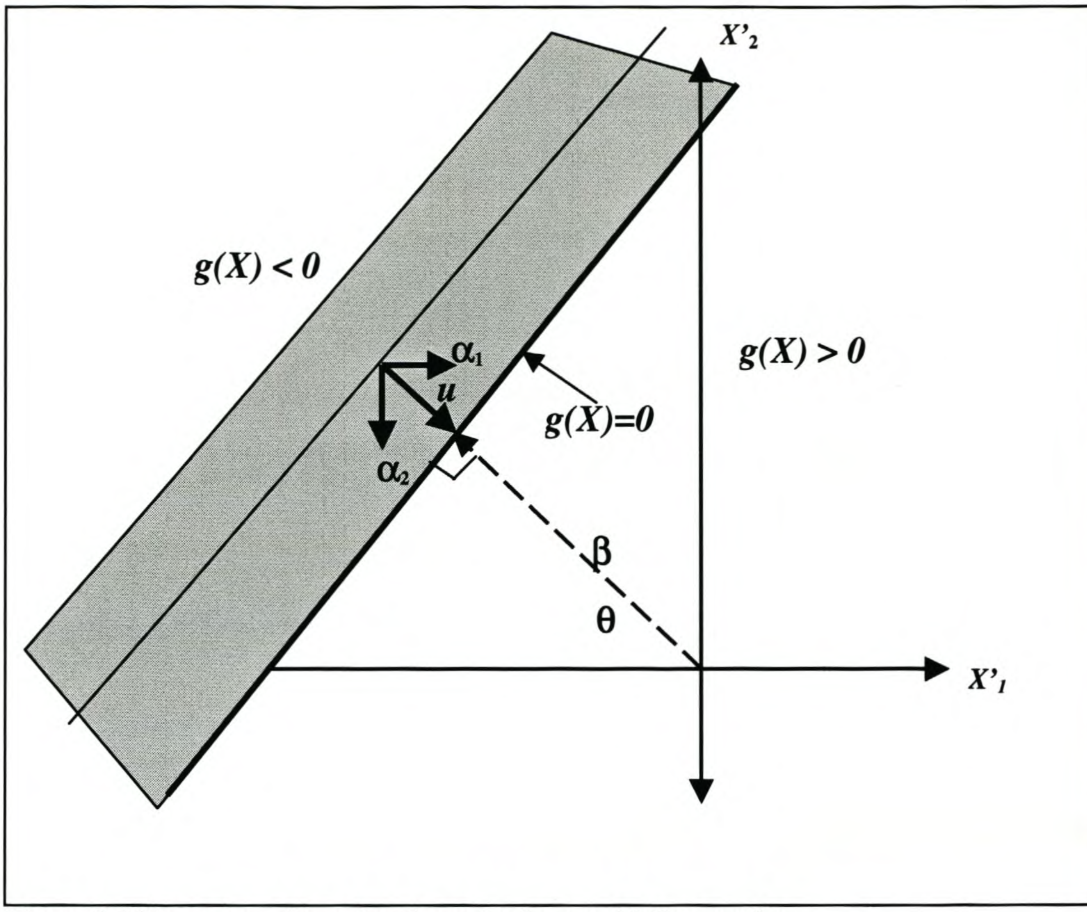


Figure 2.1 Linear two-dimensional standard normal failure surface

By Pythagoras' theorem, the gradient vector has length,

$$|\nabla g| = \sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i^{'}}\right)_*^2} \tag{2.11}$$

The gradient vector can be normalized to represent a unit vector (denoted u) with the same direction, by diving Equation (2.10) by (2.11).

$$u = \frac{1}{|\nabla g|}\nabla g \tag{2.12}$$

As can be seen in Figure 2.1, the components of the unit vector are equivalent to the cosine of the angle between the unit vector and the axes under observation, and referred to as the direction cosine or alpha (denoted α_i),

$$\alpha_i = \cos(\theta) = \frac{\left(\frac{\partial g}{\partial X_i^{'}}\right)_*}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i^{'}}\right)_*^2}} \tag{2.13}$$

As components of the unit normal vector (and by virtue of the normalising process) the direction cosines are subject to the condition

$$\sum_{i=1}^{n}\alpha_i^2 = \sum_{i=1}^{n}\left(\frac{\left(\frac{\partial g}{\partial X_i^{'}}\right)_*}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i^{'}}\right)_*^2}}\right)^2 = \frac{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i^{'}}\right)_*^2}{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i^{'}}\right)_*^2} = 1 \tag{2.14}$$

From Figure 2.1, β is the position vector from the origin of the standard normal axes to the design point on the limit function. As such it forms the radius of a circle that touches the linear approximation of the limit function at the design point. From basic geometry, the reliability index is therefore normal to this tangent hyperplane at the design point and increases away from the origin. Consequently the reliability index lies in an equal but opposite direction to the gradient vector and corresponding unit vector.

From Figure 2.1 the expression for a co-ordinate of the failure point, $x_i^{'*}$, corresponding to the standard normal variate, $X_i^{'}$, is the component of the reliability index vector in the direction of the $X_i^{'}$ axes. This is simply the negative value (due to the direction of the unit vector) of the product of the reliability index and relevant direction cosine

$$x_i^{'*} = -\alpha_i\beta \tag{2.15}$$

The resultant standard normal co-ordinate can be transformed into the original variable space by substituting Equation (2.3) into Equation (2.15)

$$-\alpha_i\beta = \frac{x_i - \mu_{X_i}}{\sigma_{X_i}}$$

$$x_i^* = \mu_{X_i} - \alpha_i\beta\sigma_{X_i} \tag{2.16}$$

This indicates that the axes of the standard normal variates have been moved by μ_{X_i} in the positive direction, with the scale of the axes transformed from units of standard deviations to units corresponding to X_i.

By the chain rule for partial differentiation, it is shown that the direction cosine is invariant through transformation into the standard normal vector space, provided the original variables are independent

$$\left(\frac{\partial g}{\partial X_i^{'}}\right)_* = \left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial X_i}{\partial X_i^{'}}\right)_* = \left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{X_i} \tag{2.17}$$

Therefore the direction cosine written in terms of the original independent variables, is

$$\alpha_i = \frac{\left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{X_i}}{\sqrt{\sum_i^n \left(\frac{\partial g}{\partial X_i}\right)_*^2 \sigma_{X_i}^2}} \tag{2.18}$$

This facilitates the description of the failure point co-ordinate in Equation (2.16) in terms of the original variables and the unknown reliability index.

2.2.5 Algorithm for independent basic variables

The failure point co-ordinates can be calculated directly for linear independent functions, since the partial derivatives are not functions of variables. RACKWITZ FIESSLER (1978) developed an iterative algorithm to solve the FOSM reliability method for non-linear performance functions containing independent basic variables, described below

i) Define the limit state function, $g(X) = 0$.

ii) Assume initial values for the components of the failure point $(x_1^*; x_2^*; x_3^* \ldots x_n^*)$,

$$x_i^* = \mu_{X_I}$$

iii) Evaluate $\left(\frac{\partial g}{\partial X'_i}\right)_*$ at the failure point x'^*. The standard normal partial derivatives can be transformed into the original variable space by the chain rule,

$$\left(\frac{\partial g}{\partial X_i^{'}}\right)_* = \left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial X_i}{\partial X_i^{'}}\right)_* = \left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{X_i}$$

iv) From the results for the partial derivative, calculate the direction cosine α_i,

$$\alpha_i = \frac{\left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{X_i}}{\sqrt{\sum_i^n \left(\frac{\partial g}{\partial X_i}\right)_*^2 \sigma_{X_i}^2}}$$

v) Substitute α_i into Equation (2.16) to find an expression for the component of the failure point x_i^* in terms of the still unknown Reliability Index, β.

$$x_i^* = \mu_{x_i} - \alpha_i \beta \sigma_{x_i}$$

vi) Substitute the set of equalities for the components of the failure point $(x_1^*; x_2^*; x_3^* ... x_n^*)$ into the original limit state function $g(x^*) = 0$.

vii) Solve for the Reliability Index, β.

viii) Utilising the resultant values for the failure point co-ordinates, repeat steps (iii) to (vii) until convergence is achieved for β.

The performance function can also be designed to some desired level of reliability, by setting the beta value and solving for a quantity in the performance function. The same procedure as above is followed until convergence is achieved for the solution.

2.3 The Limit State Cost Function (LSCF)

The FOSM reliability method can be applied to project cost estimating. The bill of quantities type cost function measures the performance of the available budget against the estimated cost. Setting the budget equal to the cost develops the Limit State Cost Function (LSCF) (KER-FOX, 1998).

As an illustration, the method was applied to the tender adjudication. Admeasure type 'bill of quantities' contracts transfer risk, due to the variability of the quantity estimates, to the client. A bill of quantities multiplied by each item's respective tendered unit rate can be considered to be a one-dimensional financial model of the project. The LSCF is the difference between the available budget and estimated cost

$$g(A) = m - \sum_{i=1}^{n} A_i r_i = 0 \qquad (2.19)$$

where $g(A)$ represents the LSCF as a function of the variable quantity estimates (A_i) and the deterministic values for the budget and tendered unit rates are represented by m and r_i respectively.

The magnitude of the available contingency fund is the difference between the budget and tender value for the project cost and can be allocated to items on the bill of quantities by applying First Order Second Moment reliability analysis (KER-FOX, 1998). The co-ordinates of the failure point, corresponding to each of the variable quantities, indicate the most likely combination of values that will result in the project cost and budget being equal. The itemised contingency is found by multiplying the tendered unit rate by the difference between the co-ordinate of the failure point and the corresponding estimated quantity.

Profiles for each tender describing the tendered item's cost and the likely demand for the available contingency, describe probabilistically the implications of awarding the contract to a particular bidder. This methodology develops insight into the tendering strategies employed, thereby supporting the tender adjudication process.

2.4 Equivalent Normal Distributions

If a variable, X_i, is found to have a non-normal distribution, the equivalent normal mean $(\mu_{X_i}^{NE})$ and standard deviation $(\sigma_{X_i}^{NE})$ can be found which correspond to the variable's actual Probability Density Function $[f_{X_i}(x_i^*)]$ and Cumulative Distribution Function $[F_{X_i}(x_i^*)]$ at the failure point x_i^* on the failure surface, $g(X)=0$ (DITLEVSEN, 1981). The procedure described for normal distributions can then be applied utilising these equivalent normal parameters, $\mu_{X_i}^{NE}$ and $\sigma_{X_i}^{NE}$.

Equating the non-normal Cumulative Distribution Function (CDF), $F_{X_i}(x_i^*)$, and the CDF for the Standard Normal Function $\Phi(X')$, taken at the failure point, x^*, the following generalised expression for the equivalent normal mean value can be found

$$\Phi(x_i'^*) = \Phi\left(\frac{x_i^* - \mu_{X_i}^{NE}}{\sigma_{X_i}^{NE}}\right) = F_{X_i}(x_i^*) \tag{2.20}$$

By taking the standard normal inverse (Φ^{-1}) of the non-normal CDF and rearranging the above equation, the equivalent normal mean value is given by

$$\Phi^{-1}\left[\Phi\left(\frac{x_i^* - \mu_{X_i}^{NE}}{\sigma_{X_i}^{NE}}\right)\right] = \left(\frac{x_i^* - \mu_{X_i}^{NE}}{\sigma_{X_i}^{NE}}\right) = \Phi^{-1}\left[F_{X_i}\left(x_i^*\right)\right] \tag{2.21a}$$

$$\therefore \mu_{X_i}^{NE} = x_i^* - \sigma_{X_i}^{NE}\Phi^{-1}[F_{X_i}(x_i^*)] \tag{2.21b}$$

Equating the non-normal Probability Density Function (PDF), $f_{X_i}(x_i^*)$ and the PDF for the Standard Normal Density Function $\left(\frac{1}{\sigma_{X_i}}\right)\phi(x_i^{*})$, taken at the failure point, x_i^*, a generalised expression for the equivalent normal standard deviation value can be found

$$\frac{1}{\sigma_{X_i}^{NE}}\phi\left(\frac{x_i^* - \mu_{X_i}}{\sigma_{X_i}^{NE}}\right) = f_{X_i}(x_i^*) \tag{2.22}$$

From Equation (2.22) the standard normal failure point co-ordinate (represented by the term contained within the large brackets), is equivalent to the standard normal inverse of the non-normal cumulative function. Substituting Equation (2.21a) into Equation (2.22), an expression for the equivalent normal standard deviation is given by,

$$\sigma_{X_i}^{NE} = \frac{\phi\{\Phi^{-1}[F_{X_i}(x_i^*)]\}}{f_{X_i}(x_i^*)} \tag{2.23}$$

As can be seen from Equation (2.23) and (2.21b) the equivalent normal mean value is a function of the equivalent normal standard deviation. The standard deviation must therefore be solved first and then the mean value. If the probability density or cumulative density functions for the non-normal functions are sensitive to the failure point co-ordinates, then new equivalent normal parameters will be calculated for each iteration.

2.5 Correlated Variates

Correlation may exist between pairs of variables in the performance function. In order to apply the Limit State Design procedure described above, these dependent variates must be transformed to a set of uncorrelated variates. This is consistent with the First Order Second Moment approach to Reliability Based Design.

2.5.1 Orthogonal Transformation method

The orthogonal transformation procedure develops the matrix T, which will transforms a correlated set of standard normal variables *X'* into an equivalent independent set through,

$$Y = TX' \tag{2.24}$$

Since the variable set Y is independent, this requires that the covariance matrix for Y be diagonal (i.e. all entries not on the diagonal, $i \neq j$, are zero). The procedure for achieving an uncorrelated set of transformed variates is therefore developed from the covariance matrix $[C]$ of the original variables $X_1, X_2, X_3,...X_n$. This matrix is given by

$$[C] = \begin{bmatrix} \sigma_{X_i}^2 & Cov(X_1, X_2) & .. & Cov(X_1, X_n) \\ Cov(X_2, X_1) & \sigma_{X_2} & .. & Cov(X_2, X_n) \\ : & : & : & : \\ Cov(X_n, X_1) & Cov(X_n, X_2) & .. & \sigma_{X_n}^2 \end{bmatrix} \tag{2.25}$$

where $Cov(X_i, X_j)$ represents the covariance between variables X_i and X_j. The covariance for the corresponding pair of reduced standard normal variates, X'_i and X'_j, is given by

$$\begin{aligned} Cov(X'_i, X'_j) &= \mathrm{E}\left[(X'_i - \mu_{X'_i})(X'_j - \mu_{X'_j})\right] \\ &= \frac{\mathrm{E}\left[(X_i - \mu_{X_i})(X_j - \mu_{X_j})\right]}{\sigma_{X_i}\sigma_{X_j}} \\ &= \frac{Cov(X_i; X_j)}{\sigma_{X_i}\sigma_{X_j}} = \rho_{ij} \end{aligned} \tag{2.26}$$

This indicates that the covariance matrix for the reduced standard normal variates $[C']$ is given by the matrix of correlation coefficients (ρ_{ij}) for the original variates. This covariance matrix for reduced variates $[C']$ is given by

$$[C'] = \begin{bmatrix} 1 & \rho_{12} & .. & \rho_{1n} \\ \rho_{21} & 1 & & \rho_{2n} \\ : & : & & : \\ \rho_{n1} & \rho_{n2} & .. & 1 \end{bmatrix} \tag{2.27}$$

The transformation matrix [T] is orthogonal due to the fact that the reduced covariance matrix $[C']$ is real $(-1 \leq \rho_{ij} \leq 1)$ and symmetric $(\rho_{ij} = \rho_{ji})$ (ANG AND TANG, 1984). The orthogonal transformation matrix is the set of eigenvectors derived from the eigenvalues (λ) of the reduced covariance matrix $[C']$ such that

$$\mathrm{T}^{\mathrm{T}}[C']\mathrm{T} = \lambda \tag{2.28}$$

By Cramer's Theorem, the homogeneous system of linear equations has non-trivial solutions if and only if the determinant of the coefficients is zero (KREYSZIG, 1988). In matrix notation this is written as

$$D(\lambda) = \det(C' - \lambda \mathrm{I}) = 0 \tag{2.29}$$

Where $D(\lambda)$ is the characteristic determinant and $(C' - \lambda \mathrm{I})$ the characteristic equation with $[C']$ the covariance matrix, $[\lambda]$ the eigenvalues and [I] a unit matrix. The n eigenvalues (λ) are calculated by solving for n roots of the corresponding characteristic determinant $D(\lambda)$. For each eigenvalue, λ_i, an eigenvector can be determined corresponding to the system

$$(C' - \lambda \mathrm{I})x' = 0 \tag{2.30}$$

This is done by substituting each eigenvalue (λ_i) into the system $(C' - \lambda \mathrm{I})X' = 0$. The eigenvectors are found by applying Gauss elimination (KREYSZIG, 1988). It is convenient to select a single variable, say X'_1, and express the other variables, X'_i, in terms of X'_1. The normalised eigenvectors form the columns of the orthogonal transformation matrix, T.

Equation (2.24) represents the orthogonal transformation of reduced variates, X'_i, to a set of uncorrelated transformed variates, Y. By definition, the inverse of an orthogonal matrix $[\mathrm{T}^{-1}]$ is equal to the transpose of the matrix $[\mathrm{T}^{\mathrm{T}}]$.

$$\mathrm{T}^{-1} = \mathrm{T}^{\mathrm{T}} \tag{2.31}$$

This provides for the expression of the reduced variates, $x_i^{'}$, in terms of the uncorrelated transformed variates, Y, such that

$$X' = T^T Y \tag{2.32}$$

The original variates, X , can be derived by substituting Equation (2.3) into Equation (2.32).

$$X = [\sigma_X]X' + \mu_X = [\sigma_X]T^T Y + \mu_X \tag{2.33}$$

Where the term $[\sigma_X]$ is the diagonal matrix of the standard deviations, σ_{X_i} and the term, μ_X , is the vector space of mean values, μ_{X_i} , for the original variates, X_i. The limit state function $g(X) = 0$ can now be written in terms of the uncorrelated transformed variates, Y_i, by substituting Equation (4.80) into the limit state function, such that

$$g(X) = g([\sigma_x]\mathrm{T}^\mathrm{T}\mathrm{Y} + [\mu_x]) = 0 \tag{2.34}$$

Due to the fact that the variable Y is independent, the covariance matrix of Y will be a diagonal matrix with non-zero entries equal to $\sigma^2_{y_i}$. The covariance between two variables, Y_i and Y_j, is the expected value for their joint second moments about the respective means, μ_{Y_i} and μ_{Y_j}. As a function of the standard normal variable, X', the resultant mean values for Y are zero. Since $\mu_{\mathrm{Y}_i} = \mu_{\mathrm{Y}_j} = 0$, the covariance matrix, $[C_\mathrm{Y}]$, can be written in matrix notation as

$$[C_Y] = \mathrm{E}(\mathrm{Y}\mathrm{Y}^\mathrm{T}) = \mathrm{E}(\mathrm{T}^\mathrm{T} X' X'^\mathrm{T}\, \mathrm{T}) = \mathrm{T}^\mathrm{T}\mathrm{E}(X' X'^\mathrm{T})\mathrm{T} \tag{2.35}$$

The middle term can be recognised as the covariance matrix of the reduced variates $[C']$,

$$\mathrm{E}(X' X'^\mathrm{T}) = [C']$$

$$\therefore [C_\mathrm{Y}] = \mathrm{T}^\mathrm{T}[C']\mathrm{T} = [\lambda] \tag{2.36}$$

The covariance matrix for Y, $[C_\mathrm{Y}]$, is therefore the diagonal matrix of eigenvalues. Consequently, the eigenvalues are the respective variances for the variates $\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \ldots \mathrm{Y}_n$. As a result the standard deviation for the uncorrelated transformed variable, Y_i, is given by

$$\sigma_{y_i} = \sqrt{\lambda_i} \tag{2.37}$$

The direction cosine is expressed in terms of standard normal variates, Y', such that

$$\alpha_{\mathrm{Y}_i} = \frac{\dfrac{\partial g}{\partial \mathrm{Y}'_i}}{\sqrt{\sum_i^n \left(\dfrac{\partial g}{\partial \mathrm{Y}'_j}\right)^2}} \tag{2.38}$$

As with the variable, X_i, by the Chain Rule the partial derivative $\frac{\partial g}{\partial \mathrm{Y}'_i}$ can be written in terms of the uncorrelated transformed variable, Y_i

$$\frac{\partial g}{\partial \mathrm{Y}'_i} = \frac{\partial g}{\partial \mathrm{Y}_i}\sigma_{\mathrm{Y}_i} \tag{2.39}$$

The direction cosine, α_{Y_i}, therefore becomes

$$\alpha_{\mathrm{Y}_i} = \frac{\frac{\partial g}{\partial \mathrm{Y}_i}\sigma_{\mathrm{Y}_i}}{\sqrt{\sum_j^n \left(\frac{\partial g}{\partial \mathrm{Y}_i}\sigma_{\mathrm{Y}_i}\right)^2}} \tag{2.40}$$

The same procedure can now be followed as was described for uncorrelated variates with the limit state function written in terms of the uncorrelated transformed variates, $\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3, \ldots \mathrm{Y}_n$, such that

$$g(\mathrm{Y}) = 0 \tag{2.41}$$

Once the Reliability Index, β, and the components of the failure point for the uncorrelated transformed variates, y_i^*, have been found the components of the failure points for the original variates x_i^* can be found by substituting y_i^* into the expression

$$X^* = [\sigma_x]\mathrm{T}^{\mathrm{T}}\mathrm{Y} + \mu_x \tag{2.42}$$

2.5.2 *Rosenblatt Transformation*

The orthogonal transformation procedure is only exact for normally distributed variables, since it essentially controls the covariance of the transformation matrix. ROSENBLATT (1952) proposed an alternative transformation procedure, which is exact for non-normal variables as well. HOHENBICHLER AND RACKWITZ (1981) introduced the Rosenblatt transformation to the FOSM reliability method. Essentially the method involves a sequential transformation of the conditional cumulative density function. In this way an equivalent cumulative probability can be established,

$$y_1 = P(X_1 \le x_1) = F_1(x_1)$$

$$y_2 = P(X_2 \le x_2 | X_1 = x_1) = F_2(x_2 | x_1)$$

$$\vdots$$

$$y_n = P(X_n \le x_n | X_1 = x_1,..., X_{n-1} = x_{n-1}) = F_2(x_2 | x_1,..., x_{n-1}) \tag{2.43}$$

The conditional joint density functions can be calculated from the multivariate density functions given by

$$f_i(x_i | x_1, x_2, x_3 ... x_{n-1}) = \frac{f_{X_i}(x_1, x_2, x_3 ... x_i)}{f_{X_{i-1}}(x_1, x_2, x_3, ... x_{i-1})} \tag{2.44}$$

The cumulative density function is then found by integrating f_i over the domain of X_i

$$F_{X_i}(x_i | x_1, x_2, x_3 ... x_{n-1}) = \int_{-\infty}^{x_i} f_i(x_i | x_1, x_2, x_3 ... x_{n-1}) \partial x_i$$

Having calculated all the conditional cumulative density functions in this way, the results may be inverted to obtain x_i

$$x_1 = F^{-1}(y_1)$$

$$x_2 = F^{-1}(y_2 | x_1)$$

$$\vdots$$

$$\vdots$$

$$x_n = F^{-1}(y_n | x_1, x_2, x_3 ... x_{n-1}) \tag{2.45}$$

This method can also be used to transform from a dependent non-normal to an independent standard normal variable set, by utilising y_i as an intermediate variable,

$$F_1(x_1^{'}) = y_1 = F_1(x_1)$$

$$F_2(x_2^{'} | x_1^{'}) = y_2 = F_1(x_2 | x_1)$$

$$\vdots$$

$$F_n(x_n^{'} | x_1^{'},..., x_{n-1}^{'}) = y_n = F_n(x_n | x_1,..., x_{n-1}) \tag{2.46}$$

3. CORRELATION EFFECTS

First Order Second Moment reliability based structural design utilises well-established limit state functions describing performance in terms of material properties, design detail and loading conditions. Cost estimates and schedules are quantitative models utilised as an integral part of project planning and management, with cash flow and return on investment functions supporting project feasibility studies and performance appraisal. However these are deterministic functions expressed in terms of a best estimate for each variable, with no explicit description of variability. As a result no insight regarding project risk and reliability is extracted from the available data. Discrepancies between forecasted and actual values are dealt with as a frustration or an unavoidable part of the project management process.

A rational probabilistic method applied to the given functions provides a basis for capturing data and transforming it into useful information. In addition to variabilities, interrelationships between variables can also be measured and modelled. This serves as an invaluable decision support tool, developing vital insight into the drivers of project risk and reliability. Some of the basic theory applied to data analysis is discussed below, with particular attention to correlation effects.

3.1 Regression Analysis

When considering two data sets for variables *X* and *Y*, the average or central value for each of the data sets is given by

$$\bar{x}_i = \frac{1}{n}\sum^{n} x_i \tag{3.1}$$

$$\bar{y}_i = \frac{1}{n}\sum^{n} y_i \tag{3.2}$$

In addition to describing the mean or central value for the data points, it would be useful to describe the extent to which the data points vary from the central value. One possibility is the cumulative difference between observed and average values. However, the sum of the difference between the average and actual data values is always equal to zero,

$$\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n} x_i - n\bar{x}_i = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i = 0 \tag{3.3}$$

Simply subtracting the observed data points from the average is therefore not a good measure of dispersion. Intuitively the contributions of deviations from the mean, whether above or below the mean should contribute positively to the measure of dispersion.

The squared deviation from the mean is known as the residual sum of squares and provides a rational measure of dispersion about the central value. Dividing this quantity by *(n - 1)* gives an unbiased estimate of the dispersion known as the variance. A more convenient measure of dispersion in the same dimension as the original variable, is the square root of the variance referred to as the standard deviation. The population standard deviation, denoted σ_X, is estimated by finding the standard deviation for a sample set of data points, denoted s_x. The sample standard deviation and Variance relate to the average and data values as follows,

$$\begin{aligned}
Var(X) = s_x^2 &= \frac{1}{(n-1)}\sum(x_i - \bar{x})^2 \\
&= \frac{1}{(n-1)}\left[\sum x_i^2 - 2\sum x_i\bar{x} + n\bar{x}^2\right] \\
&= \frac{1}{(n-1)}\left[\sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2\right] \\
&= \frac{1}{(n-1)}\left[\sum x_i^2 - n\bar{x}^2\right] \qquad (3.4)
\end{aligned}$$

Similarly, Variance for the variable Y is given by,

$$Var(Y) = s_Y^2 = \frac{1}{(n-1)}\left[\sum y_i^2 - n\bar{y}^2\right]$$

Assuming a linear relationship between the variables *X* and *Y*, such that

$$Y = a_0 + b_y X \qquad (3.5)$$

Where a_0 is a constant value, and indicates the value at which the graph crosses the *Y* axis (i.e. $Y = a_0$ for $X = 0$). The co-efficient of the *X* variable (b_y) indicates the gradient of the linear curve (i.e. for each unit that *X* moves in the horizontal direction, *Y* moves b_y units in the vertical direction).

The objective of performing a linear regression is to fit a straight line through the data set, such that the distance between the data points and graph co-ordinate has been minimised. An expression for the regression coefficient is developed by minimising the residual sum of squares and solving for a_0 and b_y. We will consider b_y as the regression of *Y* on *X* for the Equation (3.5). The regression of *X* on *Y* is denoted b_x for the function

$$X = a_1 + b_x Y \tag{3.6}$$

By the method of least squares, the residual sum of the errors squared (denoted Δ^2) is minimised and solved for the constants a_0 and b_X.

$$\Delta^2 = \sum_{i=1}^{n} (y_i - Y_i) \tag{3.7}$$

In the proof given below $\sum$ indicates $\sum_{i=1}^{n}$ where y_i is the observed data point corresponding to x_i, and Y_i is the regression curve co-ordinate corresponding to x_i. Substituting Equation (3.5) into (3.7)

$$\Delta^2 = \sum [y_i - (a_0 + b_y x_i)]^2 \tag{3.8}$$

The residual sum of squares is minimised by setting the partial derivatives with respect to a_0 and b_y equal to 0,

$$\frac{\partial \Delta^2}{\partial a_0} = \sum 2[y_i - a_0 - b_y x_i](-1)$$

$$= 2\sum y_i + 2na_0 + 2b_y \sum x_i = 0 \tag{3.9}$$

$$\therefore a_0 = \frac{2\sum y_i}{2n} - \frac{2b_y \sum x_i}{2n}$$

$$= \bar{y}_i - b_y \bar{x}_i \tag{3.10}$$

$$\frac{\partial \Delta^2}{\partial b_y} = \sum 2[y_i - a_0 - b_y x_i](-x_i)$$

$$= -2\sum y_i x_i + 2a_0 \sum x_i + 2b_y \sum x_i^2 = 0 \tag{3.11}$$

Substituting Equation (3.10) into Equation (3.11)

$$-2\sum y_i x_i + \sum x_i\left[\bar{y} - b_y \bar{x}\right] + 2b_y \sum x_i^2 = 0$$

$$-\sum y_i x_i + \sum x_i\left[\bar{y} - b_y \bar{x}\right] + b_y \sum x_i^2 = 0$$

$$-b_y \bar{x} 2 \sum x_i + b_y \sum x_i^2 = \bar{y} \sum x_i + \sum y_i x_i$$

$$\therefore b_y = \frac{-\bar{y}\sum x_i + \sum y_i x_i}{-\bar{x}\sum x_i + \sum x^2}$$

$$= \frac{\sum y_i x_i - n\bar{y}\bar{x}}{\sum x_i^2 + n\bar{x}^2} \qquad (3.12)$$

We also show that

$$\sum (x_i - \bar{x})(y_i - \bar{y}) = \sum x_i y_i - \sum x_i \bar{y} - \sum y_i \bar{x} + n\bar{x}\bar{y}$$

$$= \sum x_i y_i - 2n\bar{x}\bar{y} + n\bar{x}\bar{y} = \sum x_i y_i - n\bar{x}\bar{y} \qquad (3.13)$$

and

$$\sum (x_i - \bar{x})^2 = \sum x_i^2 - 2\bar{x}\sum x_i + n\bar{x}^2$$

$$= \sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 = \sum x_i^2 - n\bar{x}^2 \qquad (3.14)$$

The results of the expressions given above are recognised respectively as the numerator and denominator of Equation (3.12). Substituting the left hand side of (3.13) and the left hand side of (3.14) into Equation (3.12 a) convenient formulation for the regression co-efficient is (EDWARDS, 1976),

$$b_y = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} \qquad (3.15)$$

3.2 Covariance and the correlation coefficient

The joint second moment about the means is given as the

$$Cov(XY) = E[(X - \mu_x)(Y - \mu_y)] \tag{3.16}$$

A non-biased point estimate for covariance taken from a sample data set is given by

$$Cov(XY) = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{(n-1)} \tag{3.17}$$

By Schwarz's inequality, the joint density function squared is less than or equal to the product of the square of the individual density functions.

$$[Cov(XY)]^2 \leq \sigma_X^2 \sigma_Y^2$$

A convenient and widely used expression for the measure of the linear interrelationship between two variables X and Y is given as the correlation co-efficient, calculated as the covariance divided by the product of the standard deviations,

$$\rho_{XY} = \frac{Cov(XY)}{\sigma_X \sigma_Y}$$

The inequality given above, develops maximum and minimum values for correlation,

$$-1 \leq \rho_{XY} \leq 1$$

The non-biased point estimate for correlation is given by

$$\begin{aligned} \rho_{XY} &= \frac{Cov(X,Y)}{s_X s_Y} = \frac{1}{(n-1)} \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{s_X s_Y} \\ &= \frac{1}{(n-1)} \left[\frac{\sum x_i y_i - \sum x_i \bar{y} - \sum \bar{x} y_i + n\overline{xy}}{s_X s_Y} \right] \\ &= \frac{1}{(n-1)} \left[\frac{\sum x_i y_i - n\overline{xy}}{s_X s_Y} \right] \end{aligned} \tag{3.18}$$

3.3 Relationship between correlation and regression

It was shown that the regression co-efficient is the gradient of the best-fit linear relationship through a sample data set relating the performance of a dependent variable against a control variable. Correlation is the extent to which a linear relationship exists between two variables.

The regression co-efficient formulation in Equation (3.15) can be written as a function of the covariance and standard deviation,

$$\begin{aligned}\frac{Cov(XY)}{s_x^2} &= \left[\frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{(n-1)}\right]\left[\frac{(n-1)}{\sum(x_i-\bar{x})^2}\right] \\ &= \left[\frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sum(x_i-\bar{x})^2}\right] \\ &= b_y \end{aligned} \tag{3.19}$$

Since the correlation co-efficient is also a function of the covariance and standard deviations, a relationship can be developed between the correlation and regression co-efficient (EDWARDS, 1976),

$$b_y = \frac{Cov(XY)}{s_x^2} = \rho_{XY}\frac{s_X}{s_Y} \tag{3.20a}$$

For the case where variables X and Y have the same standard deviations, the relationship in 3.20 reduces to

$$b_y = b_x = \rho_{XY} \tag{3.20b}$$

3.4 Parametric study of correlation

In order to conduct a parametric study of correlation we consider a simple linear function $g(x)$ composed of n identical normally distributed variables, X_i with parameters $N(\mu_x,\sigma_x)$

$$g(X) = \sum_{i=1}^{n} X_i \tag{3.21}$$

3.4.1 Independent basic variables

In the case of independent basic variables, the expected value for the function $g(X)$ is

$$\mu_g = \sum_{i=1}^{n} \mu_X = n\mu_X \tag{3.22}$$

And the standard deviation, given that the variables X_i are independent

$$\sigma_g = \sqrt{\sum \sigma_x^2} = \left(\sqrt{n}\right)\sigma_X \tag{3.23}$$

The variability of the total function can be described as the non-dimensional co-efficient of variation (c.o.v.)

$$\Omega_g = \frac{\sigma_g}{\mu_g} = \left(\frac{1}{\sqrt{n}}\right)\left(\frac{\sigma_X}{\mu_X}\right) \tag{3.24}$$

Similarly the c.o.v. for the variable X_i is

$$\Omega_i = \frac{\sigma_X}{\mu_X} \tag{3.25}$$

A comparison can be drawn between the variability of the independent variable X_i and function $g(X)$ by dividing Ω_g by Ω_i.

$$V = \frac{\Omega_g}{\Omega_i} = \left(\frac{1}{\sqrt{n}}\right)\left(\frac{\sigma_X}{\mu_X}\right)\left(\frac{\mu_X}{\sigma_X}\right) = \frac{1}{\sqrt{n}} \tag{3.26}$$

An increase in the number of variables therefore decreases the function variability, due to the compensating differences or central limit theorem. Since the variability of each variable is independent, the positive and negative errors tend to have a dampening effect on variance. The result is that the variability of the total function is proportionately less than the variability of each individual variable. Of course, the consequence of the total function variability is more severe than the individual variable variability.

3.4.2 Dependent basic variables

We now consider the case of dependence, given that the basic variables are all correlated with $\rho_{ij} = \rho$. Continuing with the function given in Equation (3.21), the mean is

$$\mu_g = \sum_{i=1}^{n} \mu_{X_i} = n\mu_X \tag{3.27}$$

and variance

$$\sigma_g^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\sigma_X\sigma_X = \sum_{i=1}^{n}\sum_{j=i}^{n} \sigma_X\sigma_X + \sum_{i=1}^{n}\sum_{j\neq i}^{n} \rho_{ij}\sigma_X\sigma_X$$

$$= n\sigma_X^2 + n(n-1)\rho\sigma_X^2 \tag{3.28}$$

The c.o.v. for $g(X)$ is now,

$$\Omega_g = \frac{\sqrt{n\sigma_X^2 + n(n-1)\rho\sigma_X^2}}{n\mu_X} \tag{3.29}$$

The ratio of the function variability for dependent variables, is

$$V = \frac{\Omega_g}{\Omega_i} = \left[\frac{\sqrt{n\sigma_X^2 + n(n-1)\rho\sigma_X^2}}{n\mu_X}\right]\left[\frac{\mu_X}{\sqrt{\sigma_X^2}}\right]$$

$$= \sqrt{\frac{n + n(n-1)\rho}{n^2}} = \sqrt{\left(\frac{1}{n} + \frac{(n-1)}{n}\rho\right)} \tag{3.30}$$

The term $\frac{1}{n}$ represents the independent component of variability and the term containing the correlation coefficient ρ, the dependent component. The dependent component dominates this index from

$$\frac{(n-1)}{n}\rho > \frac{1}{n}$$

$$\rho > \frac{1}{(n-1)} \tag{3.31}$$

Therefore, functions having as few as 10 dependent variables will demonstrate a dominate contribution to total variance for correlation greater than *0.11*. As a rule of thumb in the past, correlation coefficients of less than 0.3 have been discarded as negligible.

For a large number of dependent variables,

$\lim\limits_{n \to \infty} \frac{1}{n} \approx 0$ and $\lim\limits_{n \to \infty} \frac{n-1}{n} \approx 1$, therefore Equation (3.30) becomes

$$V \approx \sqrt{\rho} \tag{3.32}$$

Therefore the ratio of individual to total variability for a large number of items is totally dominated by correlation, even for levels of correlation that may have been deemed negligible. It should be noted that negative correlation would have a dampening effect on the contribution of dependence to total variation.

3.5 The importance of correlation for project risk functions

Structural reliability based design applications utilise the orthogonal or Rosenblatt transformation methods for addressing dependent variables. Although the effect of correlation on the result is considered, correlation mechanisms are not specifically represented as part of the solution of the limit state function. In addition to this, low levels of correlation are often discarded, since their impact on variability is negligible. While the value of a numeric description of the direction cosine with dependent variables has been cited as a valuable contribution to solving functions of dependent variables, exact analytical or easy numerical solutions for alpha are seldom available (HOHENBICHLER and RACKWITZ, 1986). Furthermore, one can conclude that since correlation is not a dominant factor for performance functions, the transformation methods have been adequate for the purpose of structural reliability based design.

As is clear from the parametric study, the effects of correlation dominate total variability for functions containing a large number of dependent variables. Project risk functions, such as those for cost estimating and scheduling, typically are composed of large numbers of variables. Furthermore, factors influencing project performance such as inflation, foreign exchange, labour productivity, plant and operator efficiency, weather and site conditions, accuracy of the quantity and cost estimates, material shrinkage, communication between the project participants, political conditions and a host of other factors impact to a greater or lesser extent on many of the project activities. This will result in the representative variables contained in the different limit functions demonstrating systematic variations from the expected values. The correlation coefficient, applied in statistical modelling, measures the extent of these interrelationships.

The factors influencing common sensitivities and conditional variance are so many and varied, that few if any project activities remain independent. It is therefore imperative to rationally address the effects of correlation when modelling project risk and reliability.

4. ANALYTICAL SOLUTION OF THE DIRECTION COSINE UNDER CONDITIONS OF DEPENDENCE

The reliability index or Beta(β) is the most important result obtained from the First Order Second Moment reliability method. Beta is an alternative measure of performance reliability and relates to the probability of failure through the inverse standard normal function $\beta = 1 - \phi^{-1}(P_F)$. A second important parameter utilized in the FOSM methodology is the direction cosine or alpha (α_t). Alpha enables the evaluation of the most likely component conditions at the limit of acceptable performance. Valuable parametric insight can be developed from the direction cosine into the importance of independent basic variables, since alpha can be described analytically under these conditions. However the evaluation of alpha becomes difficult when the basic variables are correlated. This has necessitated the transformation of the basic variables into a representative set of independent variables. However apart from the computational challenge, transformation limits insight at a parametric. PALOHEIMO AND HANNUS (1974) and HOHENBICHLER AND RACKWITZ (1986) conducted extensive studies into the direction cosine and approximations under conditions of dependence.

4.1 Equivalent Direction cosine for dependent functions

HOHENBICHLER AND RACKWITZ (1986) showed that an equivalent alpha value could be derived, for dependent basic variables, from the solution of the failure point derived from the transformation into an independent standard normal variable space, $X^* = T(Y^*, \tau_o)$.

$$\alpha_E = \frac{1}{\beta}\left[Y^* \frac{\partial}{\partial \tau_j} T^{-1}(Y^*, \tau_o) \right] = -\frac{x'^*}{\beta} \tag{4.1}$$

The expression contained within the square brackets in Equation (4.1) is a concise formulation for the transformation of the independent solution into the correlated standard normal hyperspace. The resultant coordinate of the failure point in the standard normal form is divided by the reliability index to find the required equivalent direction cosine. In short the

4.2 Mathematical derivation of a direct method for extracting the failure point co-ordinates

SEIFI, PONAMBALAM AND VLACH (1999) derived a method for extracting the co-ordinates of the failure point, without having to first transform the basic variables into an independent standard normal space. By definition the reliability index β is the minimum distance from the origin of a standard normal hyperspace to the failure surface. In matrix notation this can be written

$$\min \beta = \left(X^{T} X^{'}\right)^{1/2} \tag{4.2}$$

Consider an original variable set $X \sim \mathrm{N}(\mu_X, C_X)$ with mean μ_X and covariance matrix C_X. There exists an orthogonal matrix T which transforms X into an independent vector Y

$$Y = TX \text{, where by definition for orthogonal } TT^T = T^T T = \mathrm{I} \text{, } T^T = T^{-1} \tag{4.3}$$

from the definition of the original variable set it follows that $Y \sim \mathrm{N}(\mu_Y, C_Y)$, where $(\mu_Y = T\mu_X)$ and $\left(C_Y = TC_X T^T\right)$. C_Y is a diagonal matrix containing the variances for Y. By the chain rule the relationship between the gradient vectors G_X and G_Y can be derived,

$$\left(\frac{\partial g}{\partial X_j}\right) = \sum_{i=1}^{n} \frac{\partial g}{\partial Y_i} \frac{\partial Y_i}{\partial X_i} \tag{4.4}$$

$$\text{leading to the equality } G_X = T^T G_Y \quad \text{or} \quad G_Y = TG_X \tag{4.5}$$

Y is transformed into the standard normal space Y', through the expression

$$Y^{'} = C_Y^{-1/2}(Y - \mu_Y) \tag{4.6}$$

Where $C_Y^{-1/2}$ is a diagonal matrix containing the inverses of the standard deviations of Y'. By the chain rule, the standard normal gradient vector therefore relates to the gradient vector for the independent variates $G_{Y'} = C_Y^{1/2} G_Y$. The standard deviation for Y' is

$$\sigma_{Y'} = \left(G_{Y'}^{\mathrm{T}} G_{Y'}\right)^{1/2} \tag{4.7}$$

By definition, Beta is the distance from the origin of the independent standard normal vector space to the design point on the limit state surface. From this relationship an expression for Beta can be derived in the original dependent variable space.

$$\begin{aligned}
\beta &= \left(Y'^{T}Y'\right)^{1/2} \\
&= \left[(Y-\mu_Y)^{T} C_Y^{-1} (Y-\mu_Y)\right]^{1/2} \\
&= \left[(X-\mu_X)^{T} T^{T} \left(TC_X T^{T}\right)^{-1} T (X-\mu_X)\right]^{1/2} \\
&= \left[(X-\mu_X)^{T} \left(T^{T}T\right) C_X^{-1} \left(T^{T}T\right)(X-\mu_X)\right]^{1/2} \\
&= \left[(X-\mu_X)^{T} C_X^{-1} (X-\mu_X)\right]^{1/2}
\end{aligned} \tag{4.8}$$

The above derivation proves that beta is invariant under the transformation into the independent standard normal space. The direction cosine for the independent standard normal variates, given in matrix notation is

$$\alpha = \frac{G_{y'}}{\left(G_{Y'}^{T} G_{Y'}\right)^{1/2}} \tag{4.9}$$

The above equation is substituted into the well-known expression for the standard normal failure point $Y^{*} = -\alpha\beta$. The expression for directly extracting the failure point co-ordinates for dependent basic variables is derived below.

$$Y' = -\alpha\beta = -\beta\left[\frac{G_{Y'}}{\left(G_{Y'}^{T} G_{Y'}\right)^{1/2}}\right] \tag{4.10}$$

$$C_Y^{-1/2}\left(Y^{*}-\mu_Y\right) = \frac{-C_Y^{-1/2} G_Y}{\left[G_Y^{T} C_Y G_Y\right]^{1/2}}\beta$$

$$Y^{*}-\mu_Y = \frac{-C_Y G_Y}{\left[G_Y^{T} C_Y G_Y\right]^{1/2}}\beta$$

$$T\left(X^{*}-\mu_X\right) = \frac{-\left(TC_X T^{T}\right) \mathrm{A} G_X}{\left[G_X^{T} T^{T} \left(TC_X T^{T}\right) \mathrm{A} G_X\right]^{1/2}}\beta$$

$$T\left(X^{*}-\mu_X\right) = \frac{-TC_X G_X}{\left[G_X^{T} C_X G_X\right]^{1/2}}\beta$$

$$X^{*}-\mu_X = \frac{-C_X G_X}{\left[G_X^{T} C_X G_X\right]^{1/2}}\beta \tag{4.11}$$

It has been shown that Beta can be described in terms of the original variables. The design point is therefore also invariant under transformation into an independent standard normal variable space. The failure point co-ordinate X^*, derived from the original dependent variable space is give by (SEIFI, PONAMBALAM AND VLACH, 1999),

$$X^* = \mu_X - \frac{C_X G_X}{\left[G_X^{\mathrm{T}} C_X G_X\right]^{1/2}} \beta \qquad (4.12)$$

4.3 Development of an analytical form of the direction cosine for correlated basic variables

As was shown in Equation (2.15), the failure point in the standard normal space is given as a function of the reliability index and direction cosine $x_i^{'*} = -\alpha_i \beta$. The direction cosine as the object of this function is

$$\alpha_i = -\frac{x_i^{'*}}{\beta} \qquad (4.13)$$

The beta and alpha parameters are defined in the independent standard normal variables space by Equations (2.9) and (2.13) respectively. Equation (4.13) can be written in terms of the standard normal failure point co-ordinates by substituting Equation (2.9) and (2.13),

$$\frac{\left(\dfrac{\partial g}{\partial X_i^{'}}\right)_*}{\sqrt{\displaystyle\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i^{'}}\right)_*^2}} = \frac{x_i^{'*}\sqrt{\displaystyle\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i^{'}}\right)_*^2}}{E\left(X^*\right)} \qquad (4.14)$$

Equation (4.14) is transformed from the standard normal into the original variable space, by applying Equation (2.17)

$$\frac{\left(\dfrac{\partial g}{\partial X_i}\right)_* \sigma_{X_i}}{\sqrt{\displaystyle\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*^2 \sigma_{X_i}^2}} = \frac{-\left(X_i^* - \mu_{X_i}\right)\sqrt{\displaystyle\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*^2 \sigma_{X_i}^2}}{\sigma_{X_i} E(X)} \qquad (4.15)$$

Multiplying Equation (4.15) through by $\dfrac{\left(\dfrac{\partial g}{\partial X_i}\right)_* \sigma_{X_i}}{\sqrt{\sum_{i=1}^{n}\left(\dfrac{\partial g}{\partial X_i}\right)_*^2 \sigma_{X_i}^2}}$

$$\frac{\left(\dfrac{\partial g}{\partial X_i}\right)_*^2 \sigma_{X_i}^2}{\sum_{i=1}^{n}\left(\dfrac{\partial g}{\partial X_i}\right)_*^2 \sigma_{X_i}^2} = \frac{-\left(X_i^* - \mu\right)}{E\left(X^*\right)}\left(\frac{\partial g}{\partial X_i}\right)_* \tag{4.17}$$

Introducing the factor R_i^2 as the proportional contribution of a basic variable to total variance or stochastic importance. For independent basic variables, this factor is

$$R_i^2 = \frac{\left(\dfrac{\partial g}{\partial X_i}\right)_*^2 \sigma_{X_i}^2}{\sum_{i=1}^{n}\left(\dfrac{\partial g}{\partial X_i}\right)_*^2 \sigma_{X_i}^2} \tag{4.18}$$

We also note that for X_i independent, this right hand side of Equation (4.18) is equivalent to the direction cosine squared, and subject the normalising condition in Equation (2.14)

$$\alpha_i^2 = \frac{\left(\dfrac{\partial g}{\partial X_i}\right)_*^2 \sigma_{X_i}^2}{\sum_{i=1}^{n}\left(\dfrac{\partial g}{\partial X_i}\right)_*^2 \sigma_{X_i}^2} \tag{4.19}$$

We now develop the dependent form for the factor R_i^2. Consider a performance function Z, containing several random variables

$$Z = g(X_1, X_2, X_3 .. X_n) = 0 \tag{4.20}$$

Expanding $g(X)$ in a Taylor series evaluated at the failure point x^*,

$$Z = g(X) = g\left(x_1^*, x_2^*, x_3^* ... x_n^*\right) + \sum_{i=1}^{n}\left(X_i - x_i^*\right)\left(\frac{\partial g}{\partial X_i}\right)_*$$

$$+\sum_{j-1}^{n}\sum_{i=1}^{n}\left(X_i - x_i^*\right)\left(X_j - x_j^*\right)\bigg/\left(\frac{\partial^2 g}{\partial X_i \partial g X_j}\right)+\ldots \tag{4.21}$$

Truncating the above expression at the first order terms, and given that at the limit state surface $g(X^*) = 0$, an appropriate expression for expected value is

$$E(Z) = \sum\left(\frac{\partial g}{\partial X_i}\right)_* \left(X_i - x^*\right) \tag{4.22}$$

The first order general form for the variance of a function containing correlated basic variables is derived as follows,

$$VAR\left(X^*\right) = E\left[(X - X^*)^2\right]$$

$$= E\left[\left(g\left(x^*\right) + \sum_{i=1}^{n}\left(X_i - x_i^*\right)\left(\frac{\partial g}{\partial X_i}\right)_* - g\left(x^*\right)\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n}\left(X_i - x_i^*\right)\left(\frac{\partial g}{\partial X_i}\right)_*\right)^2\right]$$

$$= E\Bigg[\left(X_1 - x_1^*\right)^2\left(\frac{\partial g}{\partial X_1}\right)_*^2 + \left(X_2 - x_2^*\right)^2\left(\frac{\partial g}{\partial X_2}\right)_*^2 + \left(X_3 - x_3^*\right)^2\left(\frac{\partial g}{\partial X_3}\right)_*^2 +$$

$$\ldots + .\left(X_n - x_n^*\right)^2\left(\frac{\partial g}{\partial X_n}\right)^2$$

$$+\left(X_1 - x_1^*\right)\left(X_2 - x_2^*\right)\left(\frac{\partial g}{\partial X_1}\right)_*\left(\frac{\partial g}{\partial X_2}\right)_*$$

$$+\left(X_1 - x_1^*\right)\left(X_3 - x_3^*\right)\left(\frac{\partial g}{\partial X_1}\right)_*\left(\frac{\partial g}{\partial X_3}\right)_* \ldots$$

$$+\left(X_n - x_n^*\right)\left(X_{n-1} - x_{n-1}^*\right)\left(\frac{\partial g}{\partial X_n}\right)_*\left(\frac{\partial g}{\partial X_{n-1}}\right)_*\Bigg]$$

$$= \sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*^2 Var(X_i)_* + \sum_{i\neq j}^{n}\sum_{j\neq i}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* Cov\left(X_i, X_j\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j} \tag{4.23}$$

By inspection each variable X_i contributes $\sum_{j}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_*\rho_{ij}\sigma_{X_i}\sigma_{X_j}$ to the total variance in Equation (4.23). The left hand side of Equation (4.17) is recognised as the proportional contribution of variable X_i to the total variance of a function of independent basic variables, defined earlier as R_i^2. The general format for the variance in Equation (4.23), enables the extension of (4.17) for a function containing correlated basic variables. The general dependant form for R_i^2 is therefore the contribution of variable X_i to the total variance divided by the total variance,

$$R_i^2 = \frac{\left(\frac{\partial g}{\partial X_i}\right)_*\sigma_{X_i}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)_*\rho_{ij}\sigma_{X_j}}{\sum_{i}^{n}\sum_{j}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_*\rho_{ij}\sigma_{X_i}\sigma_{X_j}} \tag{4.24}$$

Substituting Equation (4.24) develops the dependent form of Equation (4.17),

$$\frac{\left(\frac{\partial g}{\partial X_i}\right)_*\sigma_{X_i}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)_*\rho_{ij}\sigma_{X_j}}{\sum_{i}^{n}\sum_{j}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_*\rho_{ij}\sigma_{X_i}\sigma_{X_j}} = \frac{-\left(X_i^* - \mu_{X_i}\right)}{E\left(X^*\right)}\left(\frac{\partial g}{\partial X_i}\right)_* \tag{4.25}$$

Dividing both sides of the Equation (4.25) by $\dfrac{\left(\frac{\partial g}{\partial X_i}\right)_*\sigma_{X_i}}{\sqrt{\sum_{i}^{n}\sum_{j}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_*\rho_{ij}\sigma_{X_i}\sigma_{X_j}}}$,

$$\frac{\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)_*\rho_{ij}\sigma_{X_j}}{\sqrt{\sum_{i}^{n}\sum_{j}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_*\rho_{ij}\sigma_{X_i}\sigma_{X_j}}} = \frac{-\left(X_i^* - \mu_{X_i}\right)}{\sigma_{X_i}}\frac{\sqrt{\sum_{i}^{n}\sum_{j}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_*\rho_{ij}\sigma_{X_i}\sigma_{X_j}}}{E\left(X^*\right)} \tag{4.26}$$

The general dependent form for the reliability index is given by the expected value divided by the function standard deviation, which in the case of correlated basic variables is given by Equation (4.13)

$$\beta = \frac{E(X^*)}{\sqrt{\sum_{i}^{n}\sum_{j}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}} \tag{4.27}$$

Substituting Equation (4.27) and by the standard normal transformation given by Equation (2.3), Equation (4.26) now becomes

$$\frac{\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_j}}{\sqrt{\sum_{i}^{n}\sum_{j}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}} = \frac{-x_i^{'*}}{\beta} \tag{4.28}$$

The right hand side of Equation (4.28) is recognised as the right hand side of Equation (4.13). The left hand side of Equation (4.13) is the independent form of α_i in the original variable space. Consequently the left hand side of Equation (4.28) gives the analytical first order form of the direction cosine for a performance function with correlated basic variables. By the chain rule given in Equation (2.17), it can be seen that the general dependent form of the direction cosine is invariant under transformation into the standard normal space (KER-FOX, 1998),

$$\boxed{\alpha_i = \frac{\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_j}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}} = \frac{\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j^{'}}\right)_* \rho_{ij}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i^{'}}\right)_*\left(\frac{\partial g}{\partial X_j^{'}}\right)_* \rho_{ij}}}} \tag{4.29}$$

4.4 Reconciliation of the dependent form for the direction cosine

The general form of the direction cosine given in Equation (4.29) can be reconciled with the original expression for the direction cosine given in Equation (2.18), in that for independent variables $\rho_{ii} = 1$ and $\rho_{ij} = 0$ for $i \neq j$, Equation (4.29) then reduces to

$$\alpha_i = \frac{\left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{X_i}}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*^2 \sigma_{X_i}^2}} \tag{4.30}$$

Transforming Equation (4.29) from the standard normal into the original variables space given in Equation (2.3), and making the failure point co-ordinate the object of the equation,

$$x_i^* = \mu_{X_i} - \frac{\sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{X_j} \sigma_{X_j} \sigma_{X_i}}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_i} \right)_* \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_i} \sigma_{X_j}}} \beta \tag{4.31}$$

This equation can be represented in matrix notion where G_X is the gradient vector and C_X the covariance matrix,

$$X^* = \mu_X - \frac{C_X G_X}{\left[G_X^{\mathrm{T}} C_X G_X \right]^{1/2}} \beta \tag{4.32}$$

This expression is recognised as being the same as the method for extracting the failure point coordinates for functions containing correlated basic variables derived in section 4.2 by SEIFI, PONAMBALAM AND VLACH (1999).

4.5 Reflection on the proposed contribution to the current theory

The current theory defines the direction cosine in the independent variable space, and is suitable for linear and non-linear performance functions with normal and non-normally distributed variables. HOHENBICHLER AND RACKWITZ (1986) described the analytical description of the direction cosine as only being possible for fortunate cases such as those for which the performance functions contained uncorrelated variables.

A general form of the direction cosine is derived which facilitates the treatment of dependent variables in the original variable space. Since the linear approximation of non-linear functions and the development of normal equivalent parameters to represent non-normal distributions is not affected by correlation, the general direction cosine is suitable for application to the full spectrum of FOSM reliability modelling.

5. GENERAL FIRST ORDER SECOND MOMENT (GFOSM) METHOD FOR CORRELATED BASIC VARIABLES

5.1 Extending the iterative algorithm for correlated variables

The expression for the direction cosine for performance functions containing correlated basic variables enables the extension of the algorithm (RACKWITZ AND FISSLER, 1976) described in Section 2.2.5. This is a significant development since the current methods of transforming correlated basic variable to equivalent independent standard normal variables, apart from being computationally demanding, restricts the parametric insight which can be developed into the effects of correlation mechanisms on the reliability index.

The extended algorithm for solution of the General First Order Second Moment (GFOSM) reliability method is given below.

(i) For those variables with non-normal distributions, equivalent normal mean and standard deviations must be found for application of the GFOSM Reliability method.

$$\mu_{X_i}^{NE} = x_i^* + \sigma_{X_i}^{NE}\Phi^{-1}[F_{X_i}(x_i^*)] \tag{5.1}$$

$$\sigma_{X_i}^{NE} = \frac{\phi\{\Phi^{-1}[F_{X_i}(x_i^*)]\}}{f_{X_i}(x_i^*)} \tag{5.2}$$

(ii) For the first iteration, the failure point is assumed to be equal to the normal equivalent mean value for the particular variable

$$x_i^* = \mu_{X_i}{}^{NE} \tag{5.3}$$

(iii) The partial derivates evaluated at the failure point (denoted *) in the standard normal space are

$$\left(\frac{\partial g}{\partial X_i^{'}}\right)_* = \left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{X_i} \tag{5.4}$$

(iv) A substitute factor K_i is introduced, which improves computational efficiency for the general algorithm

$$K_i = \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j^{'}} \right)_* \rho_{ij} \tag{5.5}$$

For convenience, in matrix notation this can be calculated as

$$K = \rho G_X \tag{5.6}$$

Where ρ is the correlation matrix and G_X the gradient vector for $\left(\frac{\partial g}{\partial X_i^{'}} \right)_*$

(v) The variance is calculated as a function of K factors found in step (iv),

$$Var[g(X)] = \sum K_i \left(\frac{\partial g}{\partial X_i^{'}} \right)_* \tag{5.7}$$

(vi) The direction cosine is also calculated as a function of K_i

$$\alpha_i = \frac{\sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{i_j} \sigma_{X_j}}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_i} \right)_* \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{i_j} \sigma_{Xi} \sigma_{X_j}}} = \frac{K_i}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i^{'}} \right)_* K_i}} \tag{5.8}$$

(vii) The failure point co-ordinates are found by substituting Equation (5.9) into the limit state function $g(x^*) = 0$ and solving for the unknown β.

$$x_i^* = \mu_i - \sigma_{X_i} \alpha_i \beta \tag{5.9}$$

Setting the assumed failure point in *(ii)*, equal to the failure point calculated in *(vii)*, repeat steps *(iii)* – *(vii)* until convergence is achieved for β.

5.2 Reflection on the proposed contribution to the current theory

The iterative algorithm proposed above for correlated basic variables is similar to the one described in section 2.2.5. However the method for dealing with correlation is different. Orthogonal and Rosenblatt transformations are used extensively in the application of the original FOSM method to transform dependent variables to an equivalent independent variable set. RACKWITZ FISSLER's (1976) iterative algorithm is then applied to solve for the failure point in the transformed independent hyperspace. The inverse of the transformation is applied to express the solution into the original variable space.

The general method for correlated basic variables considerably simplifies the treatment of performance functions containing dependent variables by avoiding transformation to an equivalent independent space. It is computationally more efficient than the orthogonal transformation method, significantly reducing the number and complexity of the calculations required, while maintaining a consistent result. This will be explored in more detail in the following chapter.

5.3 Application

The generic algorithm described above together with some of descriptive factors described in the previous chapter will be demonstrated by way of a simple example.

Consider a low-level non-linear function $g(X)$,

$$g(X) = 18000 - X_1(X_2 + X_3 + X_4) = 0$$

This simple function would be typical of the cost make-up of an item to be tendered where X_1 represents the estimated quantity, and X_2, X_3 and X_4 the unit rates associated with labour, plant and material costs. The reliability of the cost estimates is measured against the intended tender price of R18 000.

The variables are all normally distributed with mean, standard deviation and correlation matrix given below.

Table 5.1 (a) and (b) Distribution Parameters and Correlation Matrix

	μ_i	σ_i
X_1	350	100
X_2	5	2
X_3	3	2
X_4	25	15

	X_1	X_2	X_3	X_4
X_1	1	0.60	0.30	0.60
X_2	0.60	1	0.30	0.00
X_3	0.30	0.30	1	0.40
X_4	0.60	0.00	0.40	1

The partial derivatives are given below:

$$\frac{\partial g}{\partial X_1^{'}} = -\left(x_2^{'*} + x_3^{'*} + x_4^{'*}\right)\sigma_{x_1}$$

$$\frac{\partial g}{\partial X_2^{'}} = -\left(x_1^{'*}\right)\sigma_{x_2}$$

$$\frac{\partial g}{\partial X_3^{'}} = -\left(x_1^{'*}\right)\sigma_{x_3}$$

$$\frac{\partial g}{\partial X_4^{'}} = -\left(x_1^{'*}\right)\sigma_{x_4}$$

For the first iteration the failure point is assumed to be equal to the mean value

$x_1^{'*} = \mu_{x_1} = 350$; $\quad x_2^{'*} = 5$; $\quad x_3^{'*} = 3$; $\quad x_4^{'*} = 25$

The partial derivatives are calculated as functions of the assumed failure point co-ordinates

$$\frac{\partial g}{\partial X_1^{'}} = -\left(x_2^{'*} + x_3^{'*} + x_4^{'*}\right)\sigma_{x_1} = (5 + 3 + 25)100 = -3\,300$$

$$\frac{\partial g}{\partial X_2^{'}} = -(350 * 2) = -700$$

$$\frac{\partial g}{\partial X_3^{'}} = -(350 * 2) = -700$$

$$\frac{\partial g}{\partial X_4^{'}} = -(350 * 15) = -5250$$

The K_i factors introduced in section are calculated as follows

$$K_1 = \sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j^{'}}\right)\rho_{1j} = (-3300)(1.0) + (-700)(0.6) + (-700)(0.3) + (-5250)(0.6) = -7080$$

$K_2 = -2890$; $\quad K_3 = -4000$; $\quad K_4 = -7510$

The terms developed below are convenient for calculating the variance when modelling in a spreadsheet environment, as will be demonstrated

$$K_1\left(\frac{\partial g}{\partial X_1^{'}}\right) = (-7080)(-3300) = 23364000$$

$$K_2\left(\frac{\partial g}{\partial X_2^{'}}\right) = 2023000 \qquad K_3\left(\frac{\partial g}{\partial X_3^{'}}\right) = 2800000 \qquad K_4\left(\frac{\partial g}{\partial X_4^{'}}\right) = 39427500$$

The variance calculated as a function of the terms above is,

$$\text{Var(X)} = \sum_{i=1}^{n} K_i\left(\frac{\partial g}{\partial X_i}\right) = 23364000 + 2023000 + 2800000 + 39427500 = 6.76 \text{ x } 10^7$$

The direction cosine is now calculated as a function of K_i and *Var(X)*.

$$\alpha_1 = \frac{K_1}{\sqrt{Var}} = \frac{-7080}{\sqrt{6.76x10^7}} = -0.86$$

$\alpha_2 = -0{,}35$; $\qquad \alpha_3 = -0{,}49$; $\qquad \alpha_4 = -0{,}91$

The new failure point co-ordinates are modelled as a function of the unknown β,

$$x_1^{'*} = \mu_{x_1} - \alpha_1\sigma_{x_1}\beta = 350 - (-0{,}86)(100)(\beta) = 350 + 86\beta$$

$$x_2^{'*} = 5 - (-0{,}35)(2)(\beta) = 5 + 0.70\beta$$

$$x_3^{'*} = 3 - (-0{,}49)(2)(\beta) = 3 + 0.98\beta$$

$$x_4^{'*} = 25 - (-0{,}91)(15)(\beta) = 25 + 13.6\beta$$

These co-ordinates are inserted into the limit state function and solved for the unknown β

$$g(\beta) = 18000 - x_1^{'*}\left(x_2^{'*} + x_3^{'*} + x_4^{'*}\right)$$

$$= 18000 - (350 + 86\beta)[(5 + 0.70\beta) + (3 + 0.98\beta) + (25 + 13.6\beta)] = 0$$

$\beta = 0.7045$ satisfies the above condition.

The new failure point co-ordinates, as functions of the solution beta, can now be calculated,

$$x_1^{'*} = 350 + 86(0.7045) = 410.66$$

$$x_2^{'*} = 5 + 0.70(0.7045) = 5.50$$

$$x_3^{'*} = 3 + 0.98(0.7045) = 3.69$$

$$x_4^{'*} = 25 + 13.6(0.7045) = 34.65$$

For the second iteration the assumed failure point co-ordinates are set equal to the new failure point co-ordinates, and the procedure is repeated. The results for β are compared to the results for the first iteration. If the betas for iteration 1 and iteration 2 are not the same, the procedure is repeated until convergence is achieved. The spreadsheet solutions for iteration 1, 2 and 3 are given below. As can be seen, the beta values converge within 3 iterations for this application.

Table 5.2 General FOSM Algorithm for Correlated Basic Variables

Iteration	Variable	Assumed x*	μ	σ	(dg/dX'i)	K(i)	K(i)(dg/dX'i)	α_i	New x*
1	A1	18000							
	X1	350.00	350.00	100.00	-3300.00	-7080.00	23364000.00	-0.86	410.66
	X2	5.00	5.00	2.00	-700.00	-2890.00	2023000.00	-0.35	5.50
	X3	3.00	3.00	2.00	-700.00	-4000.00	2800000.00	-0.49	3.69
	X4	25.00	25.00	15.00	-5250.00	-7510.00	39427500.00	-0.91	34.65

Var = 6.76E+07

β = 0.7045

g(X) = 0.00

Iteration	Variable	Assumed x*	μ	σ	(dg/dX'i)	K(i)	K(i)(dg/dX'i)	α_i	New x*
2	A1	18000							
	X1	410.66	350.00	100.00	-4383.20	-8818.32	38652463.60	-0.87	411.54
	X2	5.50	5.00	2.00	-821.32	-3697.63	3036930.74	-0.37	5.52
	X3	3.69	3.00	2.00	-821.32	-4846.63	3980617.08	-0.48	3.68
	X4	34.65	25.00	15.00	-6159.88	-9118.33	56167805.06	-0.90	34.55

Var = 1.02E+08

β = 0.7043

g(X) = 0.000

Iteration	Variable	Assumed x*	μ	σ	(dg/dX'i)	K(i)	K(i)(dg/dX'i)	α_i	New x*
3	A1	18000							
	X1	411.54	350.00	100.00	-4373.79	-8818.45	38570078.32	-0.87	411.51
	X2	5.52	5.00	2.00	-823.08	-3694.29	3040707.32	-0.37	5.52
	X3	3.68	3.00	2.00	-823.08	-4851.40	3993108.83	-0.48	3.68
	X4	34.55	25.00	15.00	-6173.13	-9126.64	56339929.75	-0.90	34.55

Var = 1.02E+08

β = 0.7043

g(X) = 0.000

6. NUMERICAL VALIDATION OF THE PROPOSED GENERAL DEPENDENT FORM OF THE DIRECTION COSINE

6.1 Overview of validation procedure

The proposed general form for the direction cosine was validated by comparing the results with the results from the orthogonal transformation of dependent basic variables into an independent standard normal variable space.

The results of the two methods were compared to establish any systematic variations. The same results would be considered a validation of the general procedure (for the given performance functions conditions) utilising the proposed general form of the direction cosine in the original dependent vector space. A selection of the cases was also solved with a commercial software package (COMREL, 1997) utilising the ROSEBLATT (1952) transformation method, to ensure that no logic errors had unknowingly been programmed into the orthogonal procedure.

6.2 Performance function

The performance function used to validate the proposed method was a typical Bill of Quantities type model, in which the project cost is simply the sum of the product of the estimated quantities (A_i) and unit rates (R_i). The performance of the cost is measured against the budget a_0

$$g(A,R) = a_0 - \sum_{i=1}^{n} A_i X_i = 0 \tag{6.1}$$

6.3 Validation conditions

The validation procedure was conducted for 36 cases, over a range of reliabilities, which described likely statistical complications for the given performance functions. The different categories covered by the 36 cases are described below. The full set of conditions used in the validation procedure are given in Table 6.1

Table 6.1 Performance function conditions tested in the validation procedure

Input file	Unit Rate (X)			Quantity(A)			Conditions			
Case No.	μ (X)	σ (X)	Distr.(X)	Parameter 1	Parameter 2	Distr.(A)	ρ (XiXj)	No. Items	Function	Distr. Category
1	10	3	Normal	1000	0	Constant	0.05	3	Linear	Normal
2	10	3	Normal	1000	0	Constant	0.2	3	Linear	Normal
3	10	3	Normal	1000	0	Constant	0.6	3	Linear	Normal
4	10	3	Normal	1000	200	Normal	0.05	3	Non-Linear	Normal
5	10	3	Normal	1000	200	Normal	0.2	3	Non-Linear	Normal
6	10	3	Normal	1000	200	Normal	0.6	3	Non-Linear	Normal
7	10	3	Normal	6.89	0.04	Lognormal	0.05	3	Non-Linear	Non-Normal
8	10	3	Normal	6.89	0.04	Lognormal	0.2	3	Non-Linear	Non-Normal
9	10	3	Normal	6.89	0.04	Lognormal	0.6	3	Non-Linear	Non-Normal
10	10	3	Normal	900	1100	Triangular	0.05	3	Non-Linear	Non-Normal
11	10	3	Normal	900	1100	Triangular	0.2	3	Non-Linear	Non-Normal
12	10	3	Normal	900	1100	Triangular	0.6	3	Non-Linear	Non-Normal
13	10	3	Normal	1000	0	Constant	0.05	10	Linear	Normal
14	10	3	Normal	1000	0	Constant	0.2	10	Linear	Normal
15	10	3	Normal	1000	0	Constant	0.6	10	Linear	Normal
16	10	3	Normal	1000	200	Normal	0.05	10	Non-Linear	Normal
17	10	3	Normal	1000	200	Normal	0.2	10	Non-Linear	Normal
18	10	3	Normal	1000	200	Normal	0.6	10	Non-Linear	Normal
19	10	3	Normal	6.89	0.04	Lognormal	0.05	10	Non-Linear	Non-Normal
20	10	3	Normal	6.89	0.04	Lognormal	0.2	10	Non-Linear	Non-Normal
21	10	3	Normal	6.89	0.04	Lognormal	0.6	10	Non-Linear	Non-Normal
22	10	3	Normal	900	1100	Triangular	0.05	10	Non-Linear	Non-Normal
23	10	3	Normal	900	1100	Triangular	0.2	10	Non-Linear	Non-Normal
24	10	3	Normal	900	1100	Triangular	0.6	10	Non-Linear	Non-Normal
25	10	3	Normal	1000	0	Constant	0.05	50	Linear	Normal
26	10	3	Normal	1000	0	Constant	0.2	50	Linear	Normal
27	10	3	Normal	1000	0	Constant	0.6	50	Linear	Normal
28	10	3	Normal	1000	200	Normal	0.05	50	Non-Linear	Normal
29	10	3	Normal	1000	200	Normal	0.2	50	Non-Linear	Normal
30	10	3	Normal	1000	200	Normal	0.6	50	Non-Linear	Normal
31	10	3	Normal	6.89	0.04	Lognormal	0.05	50	Non-Linear	Non-Normal
32	10	3	Normal	6.89	0.04	Lognormal	0.2	50	Non-Linear	Non-Normal
33	10	3	Normal	6.89	0.04	Lognormal	0.6	50	Non-Linear	Non-Normal
34	10	3	Normal	900	1100	Triangular	0.05	50	Non-Linear	Non-Normal
35	10	3	Normal	900	1100	Triangular	0.2	50	Non-Linear	Non-Normal
36	10	3	Normal	900	1100	Triangular	0.6	50	Non-Linear	Non-Normal

6.3.1 Linear and Non-linear functions

A function is said to be linear if the first derivative is a constant. In other words any partial derivative of a linear function is not a function of variables.

The unit rates in Equation (6.1) are modelled throughout as normally distributed. The performance function given in Equation (6.1) is in a linear form if the quantity estimates are treated as constants. Conversely the limit function is non-linear if the variability of the quantity estimates is modelled with some form of probability density function.

It is important to note that the non-linear version of the performance function is considered to be of limited curvature, since the function reduces to a constant in the second derivative. The validation is therefore not particularly extensive regarding linearity. However, this type of function should be sufficient for most reliability functions for which the first order second moment methodology is suitable.

6.3.2 Number of variables

It was shown in chapter three that a large number of variables in the performance function is an important driver for increasing the impact of correlation, even those levels considered to be insignificant. Each item in the performance function consists of two potential variables; the quantity estimate and unit rate. The function was tested for 3, 10 and 50 cost items. It is important to note that for the non-linear case these varying numbers of cost items represented 6, 20 and 100 variables respectively. The validation procedure therefore extensively tests the credibility of the general form for alpha under increasing number of variables.

6.3.3 Levels of correlation

Correlation levels below 0,3 are often considered to be insignificant and discarded. However as was demonstrated in section 3.4, large numbers of variables magnify the cumulative effect of correlation on total variance, even those levels considered to be negligible. For this reason the performance functions were tested for correlation levels of 0.05; 0.2 and 0.6. Correlation was only modelled between the normally distributed unit rate variables (i.e. usually half the number of basic variables).

6.3.4 Distribution types

The FOSM reliability method was developed specifically for normally distributed variables. However, approximates are available for finding equivalent normal parameters for representing non-normal distributions. Consistent results under non-normal conditions would indicate that the general method did not weaken the normal equivalent approximation used with the orthogonal transformations.

As was noted earlier, the unit rates were treated throughout as normally distributed. Four different distribution types were utilised to represent the quantity estimates namely constant, normal, lognormal and triangular.

The normal equivalent parameters used to represent these distributions are derived below. Parameters 1 and 2 are given in Table 6.1.

Constant – Quantity estimates represented by constant values are not variable and serve as the co-efficient of the unit rates.

$\mu_i^{NE} = \textit{Parameter 1}$ and $\sigma_i^{NE} = 0$

Normal Distribution– No approximates are necessary for normal approximations with the first two moments equal to the given parameters

μ_i^{NE} = *Parameter 1*

σ_i^{NE} = *Parameter 2*

Lognormal Distribution – lognormal distributions are common in cost functions since cost distributions demonstrate a tendency to bias (i.e. actual costs tend to be higher rather than lower than the estimated costs). This phenomenon is represented by the positively skewed lognormal function. This distribution can be well approximated with a normal tail, since lognormal populations can be fitted to the normal distribution simply by taking the population's logarithmic values. Utilising the definition for equivalent normal distributions given in section 2.4, the first two moments of the lognormal distribution are given by

λ_i = Parameter 1

ζ_i^2 = Parameter 2

The CDF and PDF for the lognormal distribution are respectively (ANG AND TANG, 1986)

$$F_{x_i}(x_i) = \Phi\left(\frac{\ln(x_i) - \lambda_i}{\zeta_i}\right)$$

$$f_{x_i}(x_i) = \frac{1}{x_i \zeta_i}\phi\left(\frac{\ln(x_i) - \lambda_i}{\zeta_i}\right)$$

From Equation (2.23) and (2.21a), the equivalent normal moments are given by

$$\begin{aligned}
\sigma_{x_i}^{NE} &= \frac{\phi\{\Phi^{-1}[F_{x_i}(x_i^*)]\}}{f_{x_i}(x_i^*)} \\
&= \frac{x_i^*\zeta_i}{\phi\left(\frac{\ln(x_i^*) - \lambda_i}{\zeta_i}\right)}\phi\left\{\Phi^{-1}\left[\Phi\left(\frac{\ln(x_i^*) - \lambda_i}{\zeta_{i_i}}\right)\right]\right\} \\
&= \frac{x_i^*\zeta_i}{\phi\left(\frac{\ln(x_i^*) - \lambda_i}{\zeta_i}\right)}\phi\left(\frac{\ln(x_i^*) - \lambda_i}{\zeta_i}\right) \\
&= x_i^*\zeta_i
\end{aligned}$$

$$\mu_i^{NE} = x_i^* - \sigma_{x_i}^{NE}\Phi^{-1}\left[F_{x_i}\left(x_i^*\right)\right]$$

$$= x_i^* - \sigma_{x_i}^{NE}\Phi^{-1}\left[\Phi\left(\frac{\ln(x_i^*) - \lambda_i}{\zeta_i}\right)\right]$$

$$= x_i^* - x_i^*\zeta_i\left(\frac{\ln(x_i^*) - \lambda_i}{\zeta_i}\right)$$

$$= x_i^*\left(1 - \ln(x_i^*) - \lambda_i\right)$$

Triangular Distribution – The triangular distributions are usually represented by a minimum, most likely and maximum value (a , u and b respectively), with the first representative normal calculated from the simple formulation (ANG and TANG, 1975) given below

$$\mu_i^{NE} = \frac{1}{3}(a + b + u)$$

$$\sigma_i^{NE} = \frac{1}{18}\left(a^2 + b^2 + u^2 - ab - au - bu\right)$$

Only two parameters are given, representing a negatively skewed distribution

a = Parameter 1

u = b = Parameter 2

6.3.5 Different levels of reliability

Different values for the reliability index were set, between 0 and 3 in increments of 0,5 (where reliability indices of 0 and 3 correspond to a 50 % and 99.999 % probabilities of success respectively). The limit state function was solved for the unknown budget required to produce the desired levels of reliability. In this way each of the conditions could be investigated at low, operational and high levels of reliability.

6.4 Comparison of results

Sample solutions for the orthogonal and general methods are given in Appendix I. The results obtained for the 36 conditions given in table 6.1 are discussed below.

6.4.1 Total performance

As was noted earlier for each case, a solution for the unknown budget corresponding to set beta values between 0 and 3 (increased in increments of 0.5) is found. The results for the orthogonal transformation and general FOSM methods were recordered and represented graphically.

Cases 1 to 12 (those containing three items) were also solved using the Rosenblatt Transformation method programmed into the commercial software package ComRel. This independent validation was conducted to ensure that no logic errors had unknowingly been modelled into the other two methodologies used.

The results for the three methods, corresponding to Case 1 are tabulated below

Table 6.2 Comparison of methodologies for Case 1

		Required Budget		
Beta	**Prob.(Success)**	**Orthogonal Trans.**	**General Method**	**Rosenblatt Trans.**
0.00	0.500	R 30,000	R 30,000	R 30,000
0.50	0.691	R 32,725	R 32,725	R 32,725
1.00	0.841	R 35,450	R 35,450	R 35,450
1.50	0.933	R 38,175	R 38,175	R 38,175
2.00	0.977	R 40,900	R 40,900	R 40,900
2.50	0.994	R 43,624	R 43,624	R 43,624
3.00	0.999	R 46,349	R 46,349	R 46,349

These results are represented graphically on the following page..

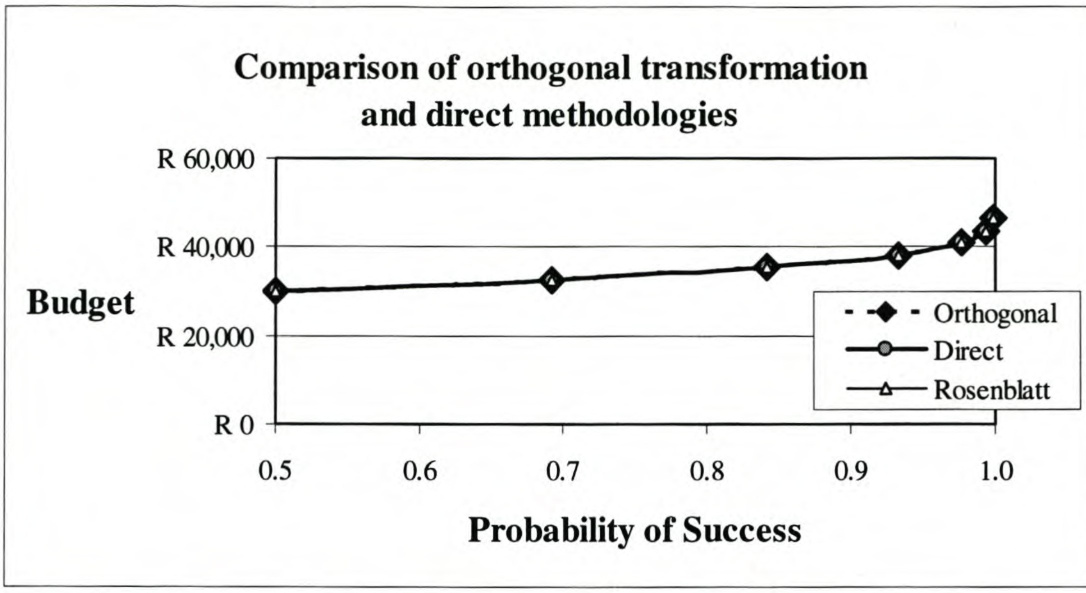


Figure 6.1 Comparison of performance reliabilities for different reliability methods

It is clear from the table and figure given above that the general FOSM method produced identical results to the Orthogonal and Rosenblatt transformation methods for Case 1.

It was found that the proposed general FOSM methodology produced identical budget requirements to the results obtained with the orthogonal transformation method for all 36 cases across the given range of reliability index values.

The Rosenblatt transformation method, which is considered to be exact, produced identical results for the cases involving the quantities modelled by constant and normal distributions. However slight discrepancies were noted for the lognormal and triangular distributions. This error has been well documented and is consistent with approximating non-normal distributions with normal equivalent distribution tails. What is significant is that both the orthogonal transformation and general method produced the same error. This adds to the argument that the discrepancy is caused in the normal approximation, and not the general form of the direction cosine.

An example of this discrepancy is tabulated below on the following page. The example represents Case 12, with the quantity estimate taking a triangular distribution, 3 items and correlation coefficients between the unit rates of 0,6 (this was the most severe error).

Table 6.3 Comparison of required budget for the three different methodologies

Required Budget				
Beta	Prob.(Success)	Orthogonal Trans.	General Method	Rosenblatt Trans.
0.00	0.500	R 29,000	R 29,000	R 28,750
0.50	0.691	R 32,753	R 32,753	R 32,490
1.00	0.841	R 36,520	R 36,520	R 36,250
1.50	0.933	R 40,303	R 40,303	R 40,030
2.00	0.977	R 44,102	R 44,102	R 43,850
2.50	0.994	R 47,921	R 47,921	R 47,695
3.00	0.999	R 51,761	R 51,761	R 51,585

6.4.2 Failure point co-ordinates

In addition to comparing the required budget totals, the individual failure point co-ordinates were also compared. The discrepancy between the co-ordinates in the original variable space was measured as the sums of the proportional difference squared (Note: In the non-linear cases there are two variables (A_i and X_i) for each of the *n* items). The equation for the error is given below

$$\text{Error }(\beta)=\sum_{i=1}^{2n}\left(\frac{x_{i\ Orthogonal}^{*}-x_{i\ Direct}^{*}}{x_{i\ Direct}^{*}}\right)^{2} \tag{6.2}$$

For case 1, $\beta=3$ the Error is given

$$\text{Error }(\beta=3)=\left(\frac{15.45-15.45}{15.45}\right)^{2}*3=0.0000$$

The general FOSM method and orthogonal transformation method produces the same failure point co-ordinates corresponding to the different levels of correlation for all 36 cases.

6.4.3 Equivalent Direction Cosine

An equivalent direction cosine (α_{i_E}) was found which would have been necessary to generate the failure point co-ordinate calculated using the orthogonal transformation method (HOHENBICHLER AND RACKWITZ, 1986). This was compared with the value of the direction cosine produced with the proposed expression for the direction cosine. The consistent results obtained indicate that the proposed general form of the direction cosine is an accurate algebraic representation of this quantity. As an example, for case 1,

$$\alpha_{1_E}=-x_1^{\prime *}\big/\beta=-\left(\frac{x_{1\ (orthogonal)}^{*}-\mu_{x_i}}{\sigma_{x_i}}\right)\frac{1}{\beta}=-\left(\frac{15{,}45-10}{3}\right)\frac{1}{3}=-0{,}61$$

Similarly $\alpha_{2_E} = \alpha_{3_E} = -0{,}61$

These results are consistent with those obtained using the general FOSM method. Since all the unit rates produce the same direction cosine, as do the quantity estimates, only 1 sample was recorded corresponding to each beta values. The results for case 1 are tabulated below.

Table 6.4 Comparison of the equivalent and general forms of the direction cosine

Beta	Prob. (Success)	Equivalent (Orthogonal)		General FOSM	
		$\alpha_E(X)$	$\alpha_E(A)$	$\alpha(X)$	$\alpha(A)$
0.00	0.500	NA	NA	-0.61	0.00
0.50	0.691	-0.61	NA	-0.61	0.00
1.00	0.841	-0.61	NA	-0.61	0.00
1.50	0.933	-0.61	NA	-0.61	0.00
2.00	0.977	-0.61	NA	-0.61	0.00
2.50	0.994	-0.61	NA	-0.61	0.00
3.00	0.999	-0.61	NA	-0.61	0.00

As can be seen all the equivalent direction cosines were the same as the general form values. The same was true for all 36 cases. An interesting observation is that the alpha values remain relatively stable over a range of reliabilities. This trend was demonstrated in many of the 36 cases.

6.5 Reflection on the proposed contribution to the current theory

The general form for a direction cosine has therefore been mathematically derived and reconciled with the existing independent form as well as a recent method (SEIFI, PONAMBALAM AND VLACH, 1999) derived for directly extracting the failure point coordinates. A numerical exercise testing 36 performance functions describing likely statistical complications compared results obtained through the orthogonal transformation into an independent variable space to the general FOSM method. The two methods were found to produce identical solutions for the reliability, failure point coordinate and direction cosine. The proposed general form for the direction cosine is therefore validated.

7. INTERPRETATION OF THE DIRECTION COSINE

7.1 Components of the direction cosine

The difficulty in describing the direction cosine for correlated variables has restricted its physical interpretation for dependent functions. A validated general dependent form of the direction cosine affords an opportunity to better understand the mathematical behaviour of this important parameter utilised in FOSM reliability modelling.

From Equation (4.29), one can observe that the general form of the direction cosine is composed of an independent and a dependent component, denoted below as α_{ii} and $\widehat{\alpha}_i$ respectively. An arbitrary component is denoted α_{ij}.

$$\alpha_i = \frac{\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_j}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}}$$

$$= \frac{\left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{X_j}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}} + \frac{\sum_{j\neq i}^{n}\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_j}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}}$$

$$\boxed{\alpha_i = \alpha_{ii} + \widehat{\alpha}_i} \tag{7.1}$$

7.1.1 Independent component

The independent component describes that portion of the direction cosine, and consequently the portion of the failure point coordinate in the standard normal space, contributed by the variable under observation. This independent component is represented by

$$\boxed{\alpha_{ii} = \frac{\left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{X_i}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}}} \tag{7.2}$$

7.1.2 Dependent component

The components of the direction cosine that through correlation also affect the magnitude of α_i can be considered as the dependent contribution from other basic variables. This dependent component, denoted $\hat{\alpha}_i$, can also be derived as a function of the independent components of the basic variables X_j and the correlations between variables X_j and X_i.

$$\hat{\alpha}_i = \frac{\sum_{j \neq i}^{n} \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_j}}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_i} \right)_* \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_i} \sigma_{X_j}}}$$

$$\boxed{\hat{\alpha}_i = \sum_{j \neq i}^{n} \alpha_{jj} \rho_{ij} = \alpha_i - \alpha_{ii}} \tag{7.3}$$

7.2 Correlation Mechanisms

The general form of the direction cosine α_i can also be written in terms of the independent contributions to the direction cosine of basic variables X_j and the correlations between variables X_j and X_i.

$$\boxed{\alpha_i = \frac{\sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_j}}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_i} \right)_* \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_i} \sigma_{X_j}}} = \sum_{j=1}^{n} \alpha_{jj} \rho_{ij} = \sum_{j=1}^{n} \alpha_{ij}} \tag{7.4}$$

A correlation component for the direction cosine α_i (denoted α_{ij}) contributed by a variable X_j is given by

$$\boxed{\alpha_{ij} = \alpha_{jj} \rho_{ij}} \tag{7.5}$$

This formulation disaggregates the direction cosine to expose correlation mechanisms and develops the elasticity in discussed under Section 9.4.

7.3 Normalizing condition

Under independent conditions, the direction cosine, as a component of the unit vector, is subject to the condition

$$\sum_{i=1}^{n} \alpha_i^2 = 1 \tag{7.6}$$

This condition is however only true for independent basic variables. This is easily verified by substituting the general expression for the direction cosine

$$\sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} \left[\frac{\sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_j}}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_i} \right)_* \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_i} \sigma_{X_j}}} \right]^2$$

$$= \frac{\sum_{i=1}^{n} \left[\sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_j} \right]^2}{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_i} \right)_* \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_i} \sigma_{X_j}} \tag{7.7}$$

If the basic variables of the above equation are independent then, $\rho_{ij} = 0$ for $i \neq j$ and $\rho_{ii} = 1$. Equation (7.7) then reduces to,

$$\sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} \left[\frac{\left(\frac{\partial g}{\partial X_i} \right)_* \sigma_{X_j}}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i} \right)_*^2 \sigma_{X_i}^2}} \right]^2 = \frac{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i} \right)_*^2 \sigma_{X_i}^2}{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i} \right)_*^2 \sigma_{X_i}^2} = 1 \tag{7.8}$$

However Equation (7.7) does not readily reduce to untiy if some or all of the basic variables are correlated, i.e.

$$\sum \alpha_i^2 \neq 1; \ \rho_{ij} \neq 0 \text{ for } j \neq i \tag{7.9}$$

Equation (4.24) gives the general dependent form for the proportional contribution to Variance, denoted R_i^2,

$$R_i^2 = \frac{\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}} \tag{7.10}$$

This is also the general dependent form for stochastic importance. The expression can be reformulated to produce two familiar expressions.

$$R_i^2 = \left[\frac{\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_j}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}}\right]\left[\frac{\left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{X_i}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}}\right] \tag{7.11}$$

The above formulation is recognised as the product of alpha and the independent component of alpha given in Equations (4.29) and (7.2) respectively. Substituting these expressions, a convenient formulation for the stochastic importance is

$$\boxed{R_i^2 = \alpha_i\alpha_{ii}} \tag{7.12}$$

We also note that this relationship satisfies the normalising condition

$$\begin{aligned}
\sum_{i=1}^{n}\alpha_i\alpha_{ii} &= \sum_{i=1}^{n}\left[\frac{\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_j}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_j}\right)\rho_{ij}\sigma_{X_i}\sigma_{X_j}}}\right]\left[\frac{\left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{X_i}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_j}\right)\rho_{ij}\sigma_{X_i}\sigma_{X_j}}}\right] \\
&= \sum_{i=1}^{n}\left[\frac{\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}\right] \\
&= \left[\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}\right] = 1
\end{aligned} \tag{7.13}$$

This can easily be reconciled with the original condition, in that for independent basic variables the direction cosine is composed only of the independent component $(\alpha_i = \alpha_{ii})$, therefore

$$\boxed{\sum_{i=1}^{n} \alpha_i \alpha_{ii} = \sum_{i=1}^{n} \alpha_i^2 = 1} \tag{7.14}$$

7.4 Geometric interpretation of the direction cosine

The geometric representation of the direction cosine becomes difficult when the basic variables are correlated, since alpha is multidimensional. Furthermore, it has been shown that the reliability index is the minimum distance from the origin of the independent standard normal vector space to the failure point. However since the sum of the direction cosines squared does not equal one and $x_i^{'*} = -\alpha_i \beta$, beta is no longer the distance from the origin to the failure point in the standard normal vector space.

We introduce a position vector **F** with components represented by the function, $f_i(x^{'*})$ such that

$$f_i(x^{'*}) = \sqrt{x_i^{'*}(-\alpha_{ii}\beta)} \tag{7.15}$$

Then the length of this vector is given by

$$\boxed{\sum_{i=1}^{n} \left[f_i(x^{'*})\right]^2 = \sum_{i=1}^{n} \left[\sqrt{x_i^{'*}(-\alpha_{ii}\beta)}\right]^2}$$

$$= \sum_{i=1}^{n} x_i^{'*}(-\alpha_{ii}\beta) \qquad \textit{substituting } x_i^{'*} = -\alpha_i \beta$$

$$= \sum_{i=1}^{n} (-\alpha_i \beta)(-\alpha_{ii}\beta) \qquad \textit{extracting } \beta$$

$$= \beta^2 \sum_{i=1}^{n} \alpha_i \alpha_{ii} \qquad \sum_{i=1}^{n} \alpha_i \alpha_{ii} = 1 \textit{ Equation (7.14)} \tag{7.16}$$

$$\boxed{\therefore \beta = \sqrt{\sum_{i=1}^{n} \left[f_i(x^{'*})\right]^2}} \tag{7.17}$$

The reliability index for general dependent and independent performance functions is therefore the length of the vector **F**. This expression can be reconciled with the known geometric definition for the reliability index for independent basic variables,

$$f_i(x^{'*}) = \sqrt{x_i^{'*}(-\alpha_{ii}\beta)} \qquad \textit{for } X_i \textit{ independent, } \alpha_i = \alpha_{ii}$$

$$\therefore f_i(x^{'*}) = \sqrt{x_i^{'*}(-\alpha_i\beta)} \qquad \textit{by definition, } x_i^{'*} = -\alpha_i\beta$$

$$\therefore f_i(x^{'*}) = \sqrt{\left(x_i^{'*}\right)^2} = x_i^{'*} \qquad \textit{substituting into Equation (7.16)}$$

$$\therefore \beta = \sqrt{\sum_{i=1}^{n}\left(f_i(x^{'*})\right)^2} = \sqrt{\sum_{i=1}^{n}\left(x_i^{'*}\right)^2} \qquad \textit{for } X_i \textit{ independent} \tag{7.18}$$

HOHENBICHLER and RACKWITZ (1986) formulated the performance function as a function of the reliability index and alpha values

$$g(X) = \beta + \sum_{i=1}^{n} x_i^{'*}\alpha_i = 0 \tag{7.19}$$

Utilising Equation (7.17) as a point of departure, a general form of the above equation can be developed for functions containing correlated basic variables.

$$\beta^2 = \sum_{i=1}^{n}\left[f_i(x^{'*})\right]^2 \qquad \textit{substituting (7.15)}$$

$$= \sum_{i=1}^{n}\left(\sqrt{x_i^{'*}(-\alpha_{ii}\beta)}\right)^2 \qquad \textit{which reduces to}$$

$$= \sum_{i=1}^{n} x_i^{'*}(-\alpha_{ii}\beta) \qquad \textit{dividing through by } \beta$$

$$\therefore \beta = -\sum_{i=1}^{n} x_i^{'*}\alpha_{ii} \qquad \textit{making } g(X) \textit{ the object of the equation} \tag{7.20a}$$

$$\boxed{\therefore g(X) = \beta + \sum_{i=1}^{n} x_i^{'*}\alpha_{ii} = 0} \tag{7.20b}$$

In the original space

$$g(X) = \beta + \sum_{i=1}^{n}\left(\frac{x_i - \mu_{X_i}}{\sigma_{X_i}}\right)\alpha_{ii} = 0 \tag{7.21}$$

This can easily be reconciled with the original expression since for independent functions, alpha is composed only of the independent component, $\alpha_i = \alpha_{ii}$.

7.5 Physical interpretation of the direction cosine

It is well documented that for a simple linear performance function with independent basic variables, the direction cosine represents the correlation between the performance function $Z = g(X) = 0$ and the independent basic variables X_i. This is derived by considering a function of the form

$$g(X) = a_0 - \sum_{j=1}^{n} a_i X_i = 0 \tag{7.22}$$

The direction cosines for the above linear equation with independent basic variables is

$$\alpha_i = \frac{a_i \sigma_{X_i}}{\sigma_z} \tag{7.23}$$

The covariance between Z and the variable X_i is

$$\begin{aligned}
Cov(Z, X_i) &= \mathrm{E}\left[\left(aX - aX^*\right)\left(X_i - X_i^*\right)\right] \\
&= \mathrm{E}\left[a_1\left(X_1 - x_1^*\right)\left(X_i - x_i^*\right) + a_2\left(X_2 - x_2^*\right)\left(X_i - x_i^*\right) + \right. \\
&\qquad \left. \ldots.. + a_n\left(X_n - x_n^*\right)\left(X_i - x_i^*\right)\right] \\
&= \mathrm{E}\left[a_i\left(X_i - x_i^*\right)^2 + \sum_{j \neq i}^{n} a_j\left(X_j - x_j^*\right)\left(X_i - x_i^*\right)\right] \\
&= a_i Var(X_i) + \sum_{j \neq i}^{n} a_j Cov(X_j, X_i)
\end{aligned}$$

Since the basic variables X_i are independent $Cov(X_j, X_i) = 0$ for $j \neq i$

$$\therefore Cov(Z, X_i) = a_i \sigma_i^2$$

The correlation coefficient is given as the Covariance divided by the product of the standard deviations,

$$\rho_{ZX_i} = \frac{Cov(Z, X_i)}{\sigma_Z \sigma_{X_i}} = \frac{a_i \sigma_{X_i}^2}{\sigma_Z \sigma_{X_i}}$$

$$= \frac{a_i \sigma_{X_i}}{\sigma_Z} = \alpha_i \qquad \textit{Equation (7.23)} \tag{7.24}$$

We can now extend this definition by considering a general performance function $Z = g(X) = 0$, with dependent basic variables. The covariance between a basic variable X_i, and Z can similarly be derived as follows:

$$Cov(\mathrm{Z}, X_i) = \mathrm{E}\left[(\mathrm{Z} - \mathrm{Z}^*)(X_i - X_i^*)\right] \tag{7.25}$$

Expanding the performance function as a Taylor series about the failure point and truncating at the first order terms, the function Z, given that at the limit state surface, $g(X^*) = 0$ is

$$\mathrm{Z} = g(X^*) + \sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i}\right)_* (X_i - X_i^*) = \sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i}\right)_* (X_i - X_i^*) \tag{7.26}$$

The general form of the variance for the limit function $Z = 0$ is

$$Var(Z) = \sigma_Z^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_j}\right) \rho_{ij} \sigma_{X_i} \sigma_{X_j} \tag{7.27}$$

Substituting 7.26 into 7.25

$$\begin{aligned}
Cov(\mathrm{Z}, X_i) &= \mathrm{E}\left[\left\{\sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j}\right)_* (X_j - X_j^*)\right\}(X_i - X_i^*)\right] \\
&= \mathrm{E}\left[\left(\frac{\partial g}{\partial X_1}\right)_* (X_1 - x_1^*)(X_i - x_i^*) + \left(\frac{\partial g}{\partial X_2}\right)_* (X_2 - x_2^*)(X_i - x_i^*) + \right. \\
&\qquad \left. \ldots.. + \left(\frac{\partial g}{\partial X_n}\right)_* (X_n - x_n^*)(X_i - x_i^*)\right] \\
&= \mathrm{E}\left[\left(\frac{\partial g}{\partial X_i}\right)_* (X_i - x_i^*)^2 + \sum_{j \neq i}^{n} \left(\frac{\partial g}{\partial X_j}\right)_* (X_j - x_j^*)(X_i - x_i^*)\right] \\
&= \left(\frac{\partial g}{\partial X_i}\right)_* Var(X_i) + \sum_{j \neq i}^{n} \left(\frac{\partial g}{\partial X_j}\right)_* Cov(X_j, X_i) \\
&= \left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{\mathrm{X}_i}^2 + \sum_{j \neq i}^{n} \left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij} \sigma_{X_i} \sigma_{X_j} \\
&= \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij} \sigma_{X_i} \sigma_{X_j}
\end{aligned} \tag{7.28}$$

The correlation co-efficient between Z and X_i is derived below

$$\rho_{ZX_i} = \frac{Cov(\mathrm{Z}, X_i)}{\sigma_Z \sigma_{X_i}}$$ *substituting Equation (7.28)*

$$= \frac{\sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_i} \sigma_{X_j}}{\sigma_Z \sigma_{X_i}} \qquad \textit{substituting Equation (7.27)}$$

$$= \frac{1}{\sigma_{X_i}} \left[\frac{\sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_j} \sigma_{X_i}}{\sqrt{\sum \sum \left(\frac{\partial g}{\partial X_i} \right)_* \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_i} \sigma_{X_j}}} \right]$$

$$= \frac{\sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_j}}{\sqrt{\sum \sum \left(\frac{\partial g}{\partial X_i} \right)_* \left(\frac{\partial g}{\partial X_j} \right)_* \rho_{ij} \sigma_{X_i} \sigma_{X_j}}}$$

This is recognised as the general form of the direction cosine in Equation (4.28)

$$\boxed{\therefore \rho_{zx_i} = \alpha_i} \tag{7.29}$$

The general form of the direction cosine is therefore equal to the correlation coefficient between the basic variable X_i and the general dependent performance function Z. Since these two parameters are equivalent, the physical interpretation of the correlation coefficient can be extended to the direction cosine. The direction cosine is therefore a measure of linear dependence between a basic variable X_i and the performance $Z = 0$.

Interestingly this demonstrates that the direction cosine, as the correlation between the performance function and a variable, is an algebraic description of the sensitivity measure often utilised in the Monte Carlo Simulation method.

7.6 An inequality for the direction cosine

According to Schwarz's inequality (HARDY, LITTELWOOD, POLYA, 1959), the integral for of the joint function $f(X)g(X)$ squared is less than or equal to the product of the integrals of the individual functions squared,

$$\left(\int f(X) g(X) \partial X \right)^2 \leq \int f(X)^2 \partial X \int g(X)^2 \partial X \tag{7.30}$$

where the functions *f(X)* and *g(X)* can be finite or infinite. This inequality is utilised to develop boundaries for the correlation coefficient (ANG and TANG, 1986) and can similarly be applied to relate the general form of the covariance between the limit function $Z=0$ and the basic variable X_i, to the respective variances for Z and X_i, where these terms are represented respectively by

$$Cov(Z,X_i) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(Z-\mu_Z)(X_i-\mu_{X_i})f_{Z,X_i}(Z,X_i)\partial X_i \partial Z \tag{7.31}$$

$$Var(Z) = \sigma_Z^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(Z-\mu_Z)^2 f_{Z,X_i}(Z,X_i)\partial X_i \partial Z \tag{7.32}$$

$$Var(X_i) = \sigma_{X_i}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X_i-\mu_{X_i})^2 f_{Z,X_i}(Z,X_i)\partial X_i \partial Z \tag{7.33}$$

Substituting Equations (7.31), (7.32) and (7.33) into (7.30),

$$\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(Z-\mu_Z)(X_i-\mu_{X_i})f_{Z,X_i}(Z,X_i)\partial X_i \partial Z\right]^2$$

$$\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(Z-\mu_Z)^2 f_{Z,X_i}(Z,X_i)\partial X_i \partial Z \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X_i-\mu_{X_i})^2 f_{Z,X_i}(Z,X_i)\partial X_i \partial Z \tag{7.34}$$

The left hand side of Equation (7.34) is recognised as $[Cov(Z,X_i)]^2$ and the right hand side as $\sigma_z^2 \cdot \sigma_{X_i}^2$. Equation (7.34) can therefore be rewritten as $[Cov(Z,X_i)]^2 \leq \sigma_Z^2\sigma_{X_i}^2$ which reduces to the familiar inequality for the correlation coefficient

$$\frac{[Cov(Z,X_i)]^2}{\sigma_Z^2\sigma_{X_i}^2} = [\rho_{ZX_i}]^2 \leq 1$$

$$\therefore -1 \leq \rho_{ZX_i} \leq 1 \tag{7.35}$$

Equation (7.29) shows that the general form of the direction cosine is equal to the correlation coefficient between the performance function Z and basic variable X_i. The boundaries for the general form of the direction cosine are therefore also equivalent and given by

$$\boxed{-1 \leq \alpha_i \leq 1} \tag{7.36}$$

7.7 Reflection on the proposed contribution to the current theory

The current theory models the direction cosine as an independent parameter. The proposed general form of the direction cosine disaggregates alpha into its constituent independent and dependent components. This is particularly useful in modelling correlation mechanisms.

As components of a unit vector, the sum of the direction cosines squared is equal to unity. A new geometric transformation maintains a normalising condition for dependent variables as well as the reliability index as the distance between the origin of the standard normal axes and the failure point. The direction cosine is further shown to represent the correlation between a performance function and variable under observation for correlated and uncorrelated variables. As such alpha is subject to the same inequality, with values falling within –1 and 1 $(-1 \le \alpha_i \le 1)$.

8. STABILITY OF THE DIRECTION COSINE

The direction cosine is a key focus for the dissertation in modelling the effects of correlation on risk and reliability. A number of observations have been made regarding the mathematical behaviour of this descriptive parameter. The remainder of the study will investigate potential applications of the dependent form for alpha.

It would be convenient, and computationally efficient to be able to use a value for the direction cosines at one level of reliability to approximate the direction cosine at other levels of reliability. This will require an understanding of the extent to which alpha changes over a range of reliabilities. Conditions under which the direction cosine remains stable creates an opportunity to significantly improve the efficiency of the FOSM reliability method. This chapter focuses on the stability of the direction cosine across a range of reliabilities.

8.1 Direction cosine sensitivity

Before observing the stability of the direction cosine for a number of given functions, an explicit description of the sensitivity of alpha will be developed. From Equation (4.29), the general form of the direction cosine is given by (where σ_g represents the standard deviation for the performance function)

$$\alpha_i = \frac{\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)\rho_{ij}\sigma_{X_j}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_*\rho_{i_j}\sigma_{Xi}\sigma_{X_j}}} = \frac{\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)\rho_{ij}\sigma_{X_j}}{\sigma_g} \tag{8.1}$$

By the quotient rule for partial differentiation

$$\frac{\partial \alpha_i}{\partial X_k} = \frac{\sum_{j=1}^{n}\left[\left(\frac{\partial^2 g}{\partial X_j \partial X_k}\right)\rho_{ij}\sigma_{X_j}\sigma_g - \left(\frac{\partial g}{\partial X_j}\right)\rho_{ij}\sigma_{X_j}\left(\frac{\partial \sigma_g}{\partial X_k}\right)\right]}{\sigma_g^2} \tag{8.2}$$

The standard deviation is given as the square rout of the variance from which, by the chain rule the partial derivate $\left(\frac{\partial \sigma_g}{\partial X_l}\right)$ can be developed.

$$\sigma_g = (Var)^{1/2}$$

$$\therefore \left(\frac{\partial \sigma_g}{\partial X_k}\right) = \frac{1}{2}(Var)^{-1/2}\left(\frac{\partial (Var)}{\partial X_k}\right) \tag{8.3}$$

The variance for the function $g(x)$ is

$$Var = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_j}\right)\rho_{ij}\sigma_{X_i}\sigma_{X_j} \tag{8.4}$$

By the multiplication rule for partial differentiation

$$\frac{\partial (Var)}{\partial X_k} = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\frac{\partial^2 g}{\partial X_i \partial X_k}\right)\left(\frac{\partial g}{\partial X_j}\right) + \left(\frac{\partial g}{\partial X_i}\right)\left(\frac{\partial^2 g}{\partial X_j \partial X_k}\right)\right]\rho_{ij}\sigma_{X_i}\sigma_{X_j} \tag{8.5}$$

Substituting Equation (8.5) into (8.3)

$$\therefore \frac{\partial \sigma_g}{\partial X_k} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\frac{\partial^2 g}{\partial X_i \partial X_k}\right)\left(\frac{\partial g}{\partial X_j}\right) + \left(\frac{\partial g}{\partial X_i}\right)\left(\frac{\partial^2 g}{\partial X_j \partial X_k}\right)\right]\rho_{ij}\sigma_{X_i}\sigma_{X_j}}{2\sigma_g} \tag{8.6}$$

Substituting the above Equation into Equation (8.2)

$$\frac{\partial \alpha_i}{\partial X_k} = \left[\frac{\sum_{j=1}^{n}\left(\frac{\partial^2 g}{\partial X_j \partial X_k}\right)\rho_{ij}\sigma_{X_j}}{\sigma_g}\right] + \left[\frac{\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)\rho_{ij}\sigma_{X_j}\left(\frac{\partial (Var)}{\partial X_k}\right)}{2\sigma_g^3}\right] \tag{8.7}$$

Substituting Equation (8.1) into (8.7)

$$\frac{\partial \alpha_i}{\partial X_k} = \left[\frac{\sum_{j=1}^{n}\left(\frac{\partial^2 g}{\partial X_j \partial X_k}\right)\rho_{ij}\sigma_{X_j}}{\sigma_g}\right] + \left[\frac{\alpha_i\left[\frac{\partial (Var)}{\partial X_k}\right]}{2\sigma_g^2}\right]$$

The limit of the second term is tends to zero,

$$\lim_{\Delta x \to 0}\left\{\alpha_i \frac{\left[\dfrac{\partial(Var)}{\partial X_k}\right]}{2\sigma_g^2}\right\} = \alpha_i \left[\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\dfrac{\partial^2 g}{\partial X_i \partial X_l}\right)\left(\dfrac{\partial g}{\partial X_j}\right)+\left(\dfrac{\partial g}{\partial X_i}\right)\left(\dfrac{\partial^2 g}{\partial X_j \partial X_l}\right)\right]\rho_{ij}\sigma_{X_i}\sigma_{X_j}}{2\sigma_g^2}\right] \approx 0$$

Therefore the sensitivity of the direction cosine to changes in the failure point co-ordinate can be approximated by,

$$\boxed{\frac{\partial \alpha_i}{\partial X_k} = \frac{\sum_{j=1}^{n}\left(\dfrac{\partial^2 g}{\partial X_j \partial X_k}\right)_* \rho_{ij}\sigma_{X_j}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\dfrac{\partial g}{\partial X_i}\right)_*\left(\dfrac{\partial g}{\partial X_j}\right)_* \rho_{i_j}\sigma_{Xi}\sigma_{X_j}}}} \tag{8.8}$$

Clearly $\dfrac{\partial \alpha_i}{\partial X_k}$ is dependent on the curvature of *g(X)*. As a result the direction cosine will be sensitive to the position of the failure point (i.e. corresponding to a level of reliability) only if *g(X)* is of a non-linear form. For linear functions and functions of limited curvature the direction cosine will remain constant across a range of reliabilities.

8.2 Numerical Investigation of the direction cosine stability

Values for the direction cosines corresponding to a range of reliabilities were generated to assist in understanding the implications of an increasing curvature of the performance function. The three functions represent a linear function, function of limited curvature and a non-linear function.

The linear case consisted of four dependent normally distributed variables, represented by
$g(X) = a_0 - (X_1 + 150X_2 + 20X_3 + 1000X_4) = 0$

This is a typical bill of quantities type cost function where the variables represent unit rates tendered against the given quantities. Each of the partial derivatives for the above performance function will be a constant. The second partial derivative $\left(\dfrac{\partial g}{\partial X_i \partial X_k}\right)$ will therefore be zero for all X_k. As a result the sensitivity of the direction cosine given in Equation (8.8), is zero. Figure 8.1 clearly illustrates that the alpha values for a linear function remain constant across a range of reliabilities.

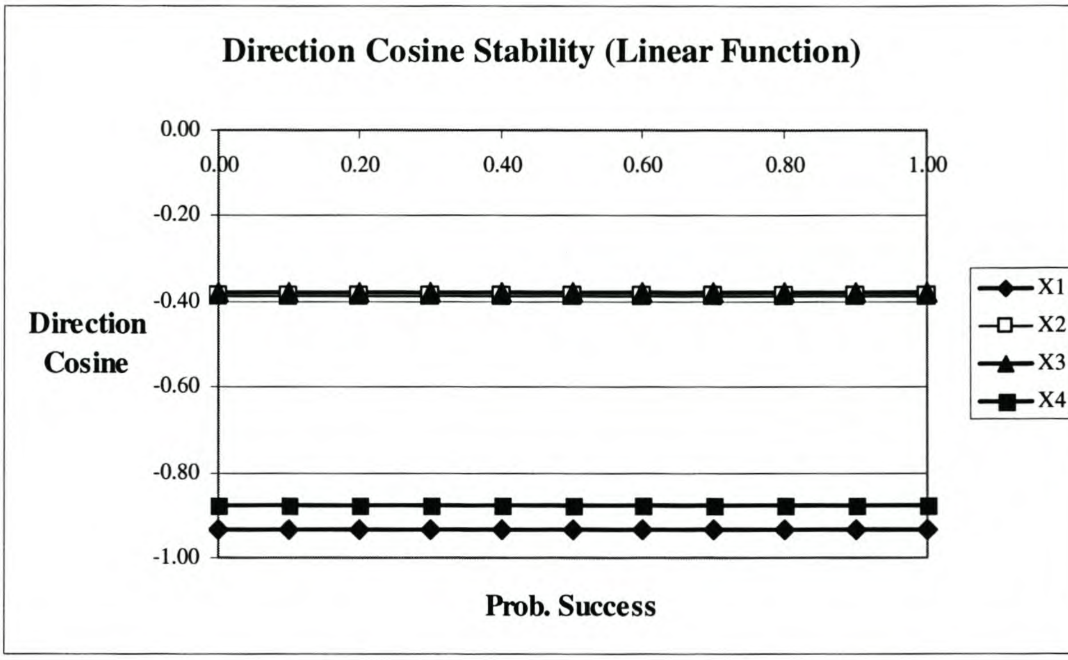


Figure 8.1 Direction Cosine Stability for Linear Functions across a range of reliabilities

A function of limited curvature (or low level of non-linearity) was demonstrated with the function

$$g(X) = a_0 - [200X_1(X_2 + X_3) + X_4] = 0$$

As with the application in chapter 5, this is a typical cost function where a 200 represents a known quantity, X_1 a shrinkage factor, X_2 and X_3 unit rates associated with the direct costs of labour and equipment and X_4 fixed costs such as overheads. The partial derivatives of the above performance produce functions of variables. The second derivatives with respect to X_k all reduce to constant values. The sensitivity of the direction cosine will consequently be low, and remain relatively constant. It should be noted the standard deviation is a function of the failure point co-ordinates, which will cause the sensitivity to change with different levels of reliability. Figure 8.2 illustrate clearly that the direction co-sine remains relatively stable over a range of reliabilities. What is particularly noteworthy is that the value at a reliability of 50 % (corresponding to a $\beta = 0$) is a very good approximation for the alpha values at higher reliabilities. This 50 % alpha value, as a function of the mean values, would have been solved deterministically.

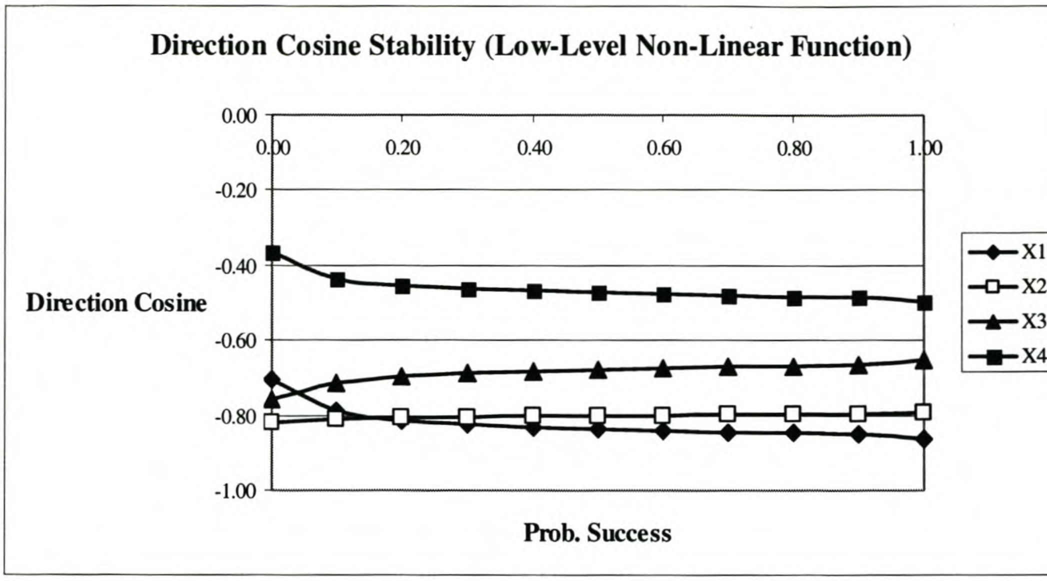


Figure 8.2 Direction Cosine Stability for Functions of Limited across a range of reliabilities

A non-linear function was represented by a performance function containing five dependent normally distributed variables

$$g(X) = a_0 - X_1 X_3 (X_4 + X_5 + X_6) = 0$$

As with the previous application this is a typical cost function where X_1 represents an variable quantity, X_3 a shrinkage factor and X_4 , X_5 and X_6 unit rates for labour, material and equipment. Alpha values corresponding to the different variables were generated across a range of reliabilities and represented graphically in Figure 8.3. The first and second derivatives of the above performance functions remain functions of variables. As a result the direction cosines for these variables are highly sensitive to changes in the failure point. As is clear from Figure 12.7 the alpha values at a reliability of 50 % provides a weak approximation for the higher levels of reliability.

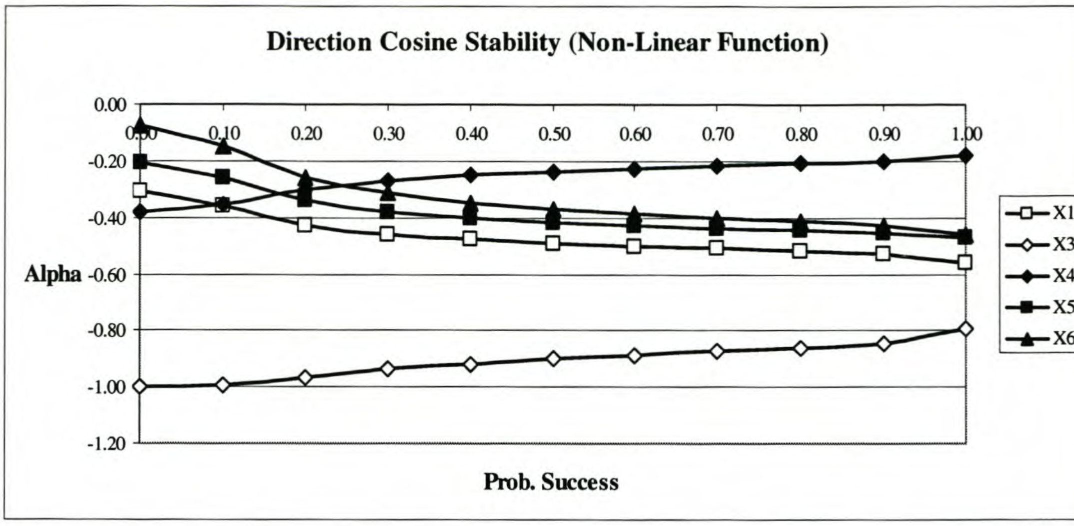


Figure 8.3 Direction Cosine Stability for Non-Linear Functions across a range of reliabilities

The conclusion that can be drawn from the three performance functions given above, is that families of functions must be tested to establish the stability of the direction cosine.

8.3 An approximate direction cosine

The general form of the direction cosine by definition is evaluated at the failure point. A stable, or relatively stable, direction cosine across a range of reliabilities indicates that a known set of direction cosines can be used to approximate that value of the direction cosine at other levels of reliability.

Furthermore, it is known that at $\beta = 0$ (50 % probability of success) the failure point falls on the origin of the standard normal variable space. This results in a set of standard normal failure point co-ordinates all having a value of zero. Transforming this value into the original variable space, the co-ordinates are equivalent to the mean values for the respective variables.

Since the direction cosine is evaluated at the failure point, alpha is a function of the mean values at $\beta = 0$. This value for the direction cosine can therefore be used to approximate alpha at higher levels of reliability, for performance functions demonstrating relatively limited curvature.

$$\alpha_{i(\beta)} \approx \alpha_{i(\beta=0)} = \frac{\sum \left(\frac{\partial g}{\partial X_j} \right)_{\mu} \rho_{ij} \sigma_{X_j}}{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_i} \right)_{\mu} \left(\frac{\partial g}{\partial X_j} \right)_{\mu} \rho_{ij} \sigma_{X_i} \sigma_{X_j}}$$

8.4 Reflection on the contribution to the current theory

The stability of the direction cosine across a range of reliabilities is investigated. A numerical observation of a number of performance functions found this descriptor to remain constant for functions of limited curvature having independent and dependent basic variables. The sensitivity of the direction cosine to the location of the failure point is derived, which supports the observation that alpha is stable for relatively linear functions. The co-ordinates of the failure point at a beta value of zero are known to be equal to the mean values. The direction cosine corresponding to $\beta = 0$ is evaluated as a direct function of the mean values. These results are then used to approximate alpha at the desired level of reliability.

9. RISK INDICATORS

The direction cosine (or alpha) is a useful quantity since it describes component performance at the failure point. A number of risk descriptors have been developed utilising the direction cosine. However most formulations are only qualified for uncorrelated variables. The analytical form of alpha for dependent functions facilitates further investigation and extensions of these risk indicators.

9.1 The direction cosine as an importance factor

HOHENBICHLER AND RACKWITZ (1986) presented the general formulation for the reliability index

$$\beta = -\sum_{i=1}^{n} x_i^{'*} \alpha_i \tag{9.1}$$

This led to the derivation of the direction cosine as a means to measure the sensitivity of the reliability index for independent basic variables.

$$\frac{\partial \beta}{\partial X_i^{'*}} = -\alpha_i \tag{9.2}$$

However it was shown in Equation (7.20a) that the formulation of the failure surface represented under general dependent conditions is a function of the independent component of α_{ii}.

$$\beta = -\sum x_i^{'*} \alpha_{ii} \tag{9.3}$$

If variables are dependent the sensitivity is developed by the chain rule as a function of partial derivatives. A new representation for the sensitivity of the reliability index with respect to the standard normal variable $X_k^{'}$ is

$$\frac{\partial \beta}{\partial X_k^{'}} = -\sum_{i=1}^{n} \left[\frac{\partial X_i^{'*}}{\partial X_k^{'*}} \alpha_{ii} + \frac{\partial \alpha_{ii}}{\partial X_k^{'}} x_i^{'*} \right] \tag{9.4}$$

The above formulation requires an investigation of partial derivatives $\left(\frac{\partial X_i^{'}}{\partial X_k^{'}} \right)$ and $\left(\frac{\partial \alpha_{ii}}{\partial X_k^{'}} \right)$.

First we shall consider the term $\left(\frac{\partial X_i^{'}}{\partial X_k^{'}}\right)$. The correlation co-efficient (ρ_{ij}) is based on the assumption of linear dependence between variables X_i and X_j,

$$X_j^{'} = bX_j^{'} + c \tag{9.5}$$

where b is the regression co-efficient and c a constant. The derivative of X_i with respect to X_k is therefore given by the regression co-efficient

$$\frac{\partial X_i^{'}}{\partial X_k^{'}} = b \tag{9.6}$$

It has also been shown (Equation 3.20a) that the relationship between the regression coefficient (of X_i on X_k) and correlation coefficient is given by

$$b_i = \rho_{ik}\frac{\sigma_{X_i^{'}}}{\sigma_{X_k^{'}}} \tag{9.7}$$

Therefore the derivative of X_i with respect to X_k, can be written as a function of the correlation co-efficient and standard derivations

$$\frac{\partial X_i^{'}}{\partial X_k^{'}} = \rho_{ik}\frac{\sigma_{X_i^{'}}}{\sigma_{X_k^{'}}} \tag{9.8}$$

Since $X_i^{'}$ and $X_k^{'}$ are standard normal variates $\sigma_{X_i^{'}} = \sigma_{X_k^{'}} = 1$, therefore the required partial derivative is

$$\frac{\partial X_i^{'}}{\partial X_k^{'}} = \rho_{ik} \tag{9.9}$$

.

The second term in Equation (9.4) $\left(\frac{\partial \alpha_{ii}}{\partial X_k} x_i^{'*}\right)$ contains the derivative of the independent component of the direction cosine with respect to the standard normal variable X_k. In Equation (7.2), the expression for the independent component (α_{ii}) is

$$\alpha_{ii} = \frac{\left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{X_i}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}} = \frac{\left(\frac{\partial g}{\partial X_i}\right)\sigma_{X_i}}{\sigma_g} \tag{9.10}$$

$$\therefore \frac{\partial \alpha_{ii}}{\partial X_k} = \frac{\left(\frac{\partial^2 g}{\partial X_i \partial X_k}\right)\sigma_{X_i}\sigma_g - \left(\frac{\partial g}{\partial X_i}\right)\sigma_{X_i}\left(\frac{\partial \sigma_g}{\partial X_k}\right)}{\sigma_g^2}$$

$$= \frac{\left(\frac{\partial^2 g}{\partial X_i \partial X_k}\right)\sigma_{X_i}}{\sigma_g} - \frac{\left(\frac{\partial g}{\partial X_i}\right)\sigma_{X_i}\left(\frac{\partial \sigma_g}{\partial X_k}\right)}{\sigma_g^2} \tag{9.11}$$

We are now required to derive the standard deviation with respect to X_k. The general first order form for the variance is given in Equation (4.23). The standard deviation is the square root of variance,

$$\sigma_g = (Var)^{\frac{1}{2}} \tag{9.12}$$

The derivative of the standard deviation is

$$\frac{\partial \sigma_g}{\partial X_j} = \frac{1}{2}(Var)^{-\frac{1}{2}}\frac{\partial (Var)}{\partial X_k} = \frac{1}{2\sigma_g^2}\frac{\partial (Var)}{\partial X_k} \tag{9.13}$$

Once again the expression requires the derivation of the total variance for the function, defined as

$$Var = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_j}\right)\rho_{ij}\sigma_{X_i}\sigma_{X_j} \tag{9.14}$$

By the multiplication rule for partial differentiation,

$$\left(\frac{\partial (Var)}{\partial X_j}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(\frac{\partial^2 g}{\partial X_i \partial X_k}\right)\left(\frac{\partial g}{\partial X_j}\right) + \left(\frac{\partial g}{\partial X_i}\right)\left(\frac{\partial^2 g}{\partial X_j \partial X_k}\right)\right]\rho_{ij}\sigma_{X_i}\sigma_{X_j} \tag{9.15}$$

Substituting Equation (9.15) into (9.13), and (9.13) into (9.11) we find that the second term is negligible,

$$\lim_{\Delta x \to 0} \left(\frac{\left(\frac{\partial g}{\partial X_i}\right) \sigma_{X_i} \left(\frac{\partial (Var)}{\partial X_k}\right)}{2\sigma_g^4} \right) \approx 0 \tag{9.16}$$

The sensitivity of the independent component of alpha is therefore given as,

$$\therefore \frac{\partial \alpha_{ii}}{\partial X_k^{'}} \approx \frac{1}{\sigma_g} \left(\frac{\partial_g^2}{\partial X_i \partial X_k} \right) \sigma_{X_i} \tag{9.17}$$

Substituting Equation (9.17) and (9.9) into (9.4), the sensitivity of the reliability index becomes

$$\boxed{\frac{\partial \beta}{\partial X_k^{'}} = -\sum_{i=1}^{n} \left[\alpha_{ii} \rho_{i_j} + \frac{\sigma_{X_i}}{\sigma_g} \left(\frac{\partial^2 g}{\partial X_i \partial X_k} \right) \right]} \tag{9.18}$$

As is the case for α_i (section 8.1), from the expression given above the stability of the independent component of alpha is dependent on the curvature of the performance function. Under conditions in which the function curvature is limited the second term is negligible, that is

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} \left[\frac{\sigma_{X_i}}{\sigma_g} \left(\frac{\partial^2 g}{\partial X_i \partial X_k} \right) \right] \approx 0 \tag{9.19}$$

The sensitivity of the reliability index can further be simplified for functions of limited curvature, from Equations (9.18), (9.19) and (8.1)

$$\boxed{\frac{\partial \beta}{\partial X_k} = -\sum_{i=1}^{n} [\alpha_{ii} \rho_{ik}] = -\alpha_k} \tag{9.20}$$

This is recognised as the importance factor developed by HOHENBICHLER AND RACKWITZ (1986).

9.2 Stochastic Importance

As was demonstrated in Equation (2.14), the direction cosines as components of the unit vector are subject to the normalising condition,

$$\sum_{i=1}^{n} \alpha_i^2 = 1 \tag{9.21}$$

In the above expression each term α_i^2 represents the stochastic importance of the basic variable X_i. Stochastic importance is the proportional contribution of the uncertainty associated with the variable X_i to the uncertainty, measured in terms of the variance of the total performance function $g(X) = 0$.

However this is only true for independent basic variables. An expression for the general dependent form of the proportional contribution to total variance R_i^2 (or stochastic importance) was developed in Equation (7.12), as a function of alpha and the independent component of alpha,

$$\boxed{R_i^2 = \frac{\sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_i}\right)_* \left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}{\sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_i}\right)_* \left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}} = \alpha_i \alpha_{ii}} \tag{9.22}$$

The proportional contribution of X_i to the total variance for $g(X)$ taken at the failure point is therefore $\alpha_i \alpha_{ii}$.

9.3 Elasticity

Each variable α_i is affected through correlation by each of the other variables. A useful importance measure would be the extent to which the variable X_i contributes to the alpha values of each of the other α_j values, and is denoted $\hat{\alpha}_i^E$

$$\widehat{\alpha}_i^E = \frac{\left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{X_j} \sum_{j \neq i}^{n} \rho_{ij}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}}$$

$$= \alpha_{ii}\sum_{j \neq i}^{n} \rho_{ij} = \alpha_{ii}\left(\sum_{j=1}^{n} \rho_{ij} - 1\right) \tag{9.23}$$

If we include the independent component of this importance measure, we obtain the convenient formulation of the importance of a variable in terms of its cumulative impact on all the importance factors. This is also the sum of the correlation components containing the independent component of the direction cosine. This can be considered to be the elasticity of variable X_i indicated by the superscript E.

$$\boxed{\alpha_i^E = \alpha_{ii}\sum_{j=1}^{n} \rho_{ij}} \tag{9.24}$$

9.4 Elasticity Index

The elasticity index is simply the ratio of the elasticity to the direction cosine.

$$\boxed{\varepsilon_i = \frac{\alpha_i^E}{\alpha_i}} \tag{9.25}$$

A value greater than one would indicate that the effect of a variable on other variables is greater than the cumulative impact on the variable under observation. The result indicates the "*influencibility*" or extent to which a particular variable is receptive to management attention. It will be difficult to reduce the importance of a variable if the direction cosine has predominantly been generated through interrelationships with other variables. Conversely if a variable has a significant effect on a number of other variables, then applying risk mitigation strategies should have a favourable effect on all the variables.

Resources should be focused on variables with a high importance factor and elasticity index greater than one.

9.5 Sensitivity

It was shown in Equation (3.20a) that the regression co-efficient of X on Y *(denoted b_X)* relates to the correlation co-efficient as follows,

$$b_X = \rho_{XY} \frac{\sigma_Y}{\sigma_X} \tag{9.26}$$

Since the direction cosine is equivalent under all conditions to the correlation between a variable X_i and a general performance function $g(X) = 0$ (Equation 2.14), and the correlation coefficient is a measure of linear interrelationship, a linear relationship between $X_i^{'}$ and $g(X^{'})$ can be assumed. This linear relationship is given by,

$$g(X^{'}) = b_g X_i^{'} + c$$

A formulation for the sensitivity as a function of the direction cosine is then,

$$\frac{\partial g}{\partial X_i^{'}} = b_g = \rho_{gX_i} \frac{\sigma_g}{\sigma_{X_i}} = \alpha_i \frac{\sigma_g}{\sigma_{X_i}} \qquad \alpha_i = \rho_{gX_i}$$

$$= \alpha_i \sigma_g \qquad \sigma_{X_i^{'}} = 1 \; \textit{for} \; X_i^{'} \; \textit{standard normal} \tag{9.27}$$

This formulation can be verified by the chain rule

$$\frac{\partial g}{\partial X_i^{'}} = \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j^{'}} \frac{\partial X_j^{'}}{\partial X_k^{'}} \right) \qquad \textit{Substituting Equation (9.9)}$$

$$= \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j^{'}} \rho_{ij} \right) \qquad \textit{Substituting Equation (4.29) and (4.23)}$$

$$= \alpha_i \sigma_g$$

From Equation (2.3) the partial derivative of the original variable with respect to the standard normal variable is given by,

$$\frac{\partial g}{\partial X_i} = \frac{\partial g}{\partial X_i^{'}} \frac{1}{\sigma_{X_i}} \tag{9.28}$$

Substituting Equation (9.27) into (9.28), an expression for the full derivative of the performance function $g(X)$ with respect to X_i is

$$\boxed{\frac{\partial g}{\partial X_i} = \sum_{j=1}^{n} \frac{\partial g}{\partial X_j} \frac{\partial X_j}{\partial X_i} = \frac{\alpha_i \sigma_g}{\sigma_{X_i}}} \tag{9.29}$$

Equation (9.29) is given as a function of the failure point coordinates and not the mean values. If the performance function were independent and evaluated at a beta value of zero (i.e. a function of the mean values), then the result from Equation (9.29) would be equivalent to the derivative of $g(X)$ with respect to the X_i.

9.6 Likely Contingency Demand

Contingency is usually an amount money added to an uncertain cost estimate to achieve an acceptable level of reliability. The General FOSM reliability method can be used to establish each term in a cost function's likely demand for the available contingency. This is achieved by subtracting the term financial value as a function of the mean values, from the corresponding term value as a function of the failure point co-ordinates. This is a relative expression of risk in the dimension of the limit function.
A contingency fund however can hide poor performance, particularly early in the project life cycle. Establishing each activity or section's probabilistic demand for the available contingency provides a tool to assist project managers to control the contingency fund. The actual draw down can be measured against the likely demand, highlighting variations that deviated significantly from the budget.

9.7 Reflection on the proposed contribution to the current theory

The direction cosine has been shown to be the sensitivity of the reliability index to the location of the failure point, and is often referred to as the importance factor. This relationship was confirmed for linear functions and function of limited curvature containing dependent variables. An additional term must be added to the direction cosine to account for the stability of the direction cosine for non-linear limit state functions.

The proposed general form of the direction cosine provides for the representation of a number of additional risk indicators as functions of alpha, these include; stochastic importance or proportional contribution to total variance, performance function sensitivity and likely contingency demand.

Elasticity was introduced as a new indicator capturing the cumulative effect of correlation mechanisms. The elasticity index expresses elasticity as a proportion of the direction cosine, and indicates the extent to which a variable will potentially respond to management.

9.8 Application

9.8.1 Importance Factors and Correlation Mechanisms

This application is a continuation of the application in chapter 5 describing a cost function. Equation (7.5) disaggregates the direction cosines into its constituent correlation components. In addition to the direction cosines, calculated under section 5.3, the independent components are also required. As an example the independent component of the direction cosine for variable X_1, is

$$\alpha_{11} = \frac{\frac{\partial g}{\partial X_1^{'}}\sigma_{x_1}}{\sqrt{Var}} = \frac{-4373.79}{\sqrt{1.02x10^8}} = -0.43$$

$\alpha_{22} = -0.08$; $\alpha_{33} = -0.08$; $\alpha_{44} = -0.08$

From which the correlation component α_{12} is given by

$$\alpha_{12} = \alpha_{11}\rho_{12} = (-0.43)(0.6) = -0.26$$

The matrix of direction cosines disaggregated into the correlation components is tabulated below.

Table 9.1 Disaggregated direction cosine and elasticity

	α_{j1}	α_{j2}	α_{j3}	α_{4j}	$\alpha_i^E = \sum_{j=1}^{n}\alpha_{ji}$
α_{1i}	-0.43	-0.26	-0.13	-0.26	-1.08
α_{2i}	-0.05	-0.08	-0.02	0.00	-0.15
α_{3i}	-0.02	-0.02	-0.08	-0.03	-0.16
α_{4i}	-0.37	0.00	-0.24	-0.61	-1.22
$\alpha_j = \sum_{i=1}^{n}\alpha_{ji}$	-0.87	-0.37	-0.48	-0.90	

The sum of each column in the matrix given above results in the direction cosine. As will be shown, the sum of each row gives the elasticities. Table 9.1 is represented graphically in Fig 9.1 below. The total length of the bar represents the magnitude of the direction cosine. The different colour blocks represent each correlation component, describing the correlation mechanisms. The cumulative effect of the correlation mechanism is the elasticity.

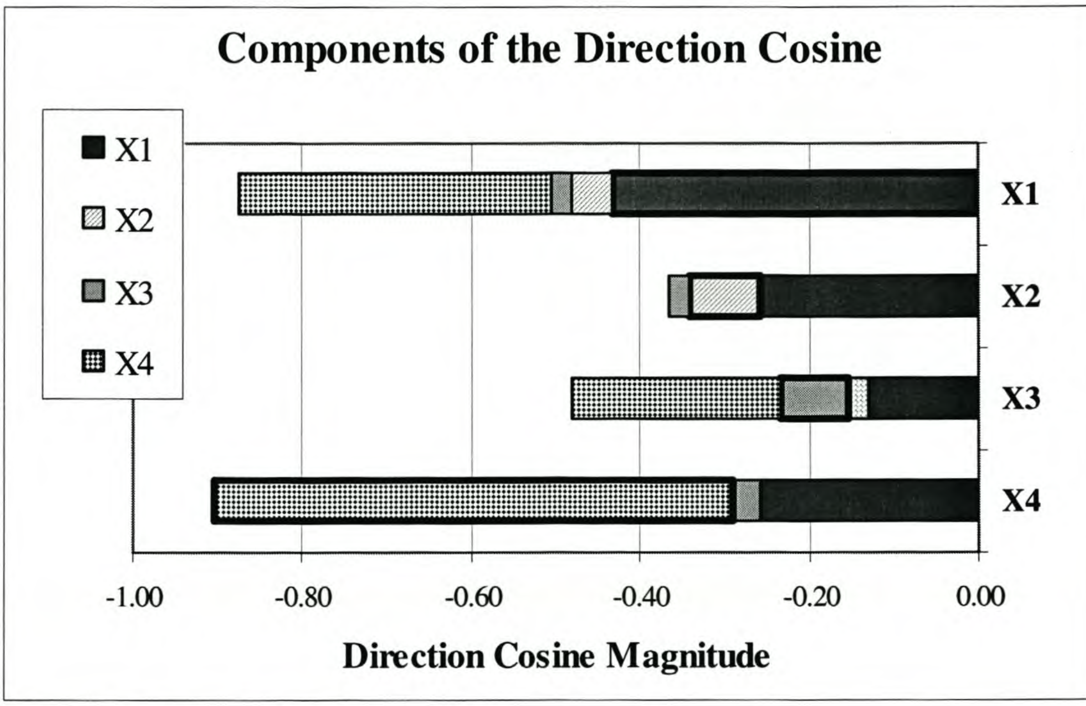


Figure 9.1 Components of the Direction Cosines for a Cost Function

From Figure 9.1 it can be seen that variables X_4 (*Material Rates*) and X_1 (*Quantity estimate*) are particularly important. Variable X_4 is made up of a dominant independent component, with a significant contribution from X_1 and has significant effects on X_1 and X_3. Variable X_1 is made up of an independent component and a dependent contribution from X_4. Variable X_1 also makes significant contributions to X_2, X_3 and X_4.

9.8.2 Elasticity and the elasticity index

The sum of the correlation components (same colour blocks in Figure 9.1) gives the cumulative effect of each variable on all the direction cosines. This alternative measure of importance was introduced in section 9.3 as the elasticity, denoted α_i^E, and represented by Equation (9.24). The elasticities for the application are calculated below

$$\alpha_1^E = \alpha_{11} \sum_{j=1}^{n} \rho_{ij} = (-0.43)(1+0.6+0.3+0.6) = -1.08$$

$$\alpha_2^E = -0.15; \qquad \alpha_3^E = -0.16; \qquad \alpha_4^E = -1.22$$

From these results it can be seen that variable X_4 has the greatest importance factor α_4 and the highest elasticity α_4^E. Variable X_1 while having a similar direction cosine has a significantly lower elasticity.

The elasticity index is taken as a proportion of the elasticity to the direction cosine.

$$\varepsilon_1 = \frac{\alpha_1^E}{\alpha_1} = \frac{-1.08}{-0.87} = 1.24$$

$$\varepsilon_2 = 0.42 \; ; \qquad \varepsilon_3 = 0.34; \qquad \varepsilon_4 = 1.35$$

From Figure 9.2 on the following page it can be seen that material *quantity estimate* *(X_4)* and *unit rate* *(X_1)* demonstrate the highest elasticity indices. X_2 and X_3 have elasticities less than one. It will therefore not be an efficient application of resources to focus on these variables to reduce risk.

From these observations, the risk manager's attention should be focussed on reducing the uncertainty or impact of the material unit rate (X_1) and quantity estimate (X_4). Reducing the uncertainty and impact of these variables will have a significant effect on the reliability of the performance function.

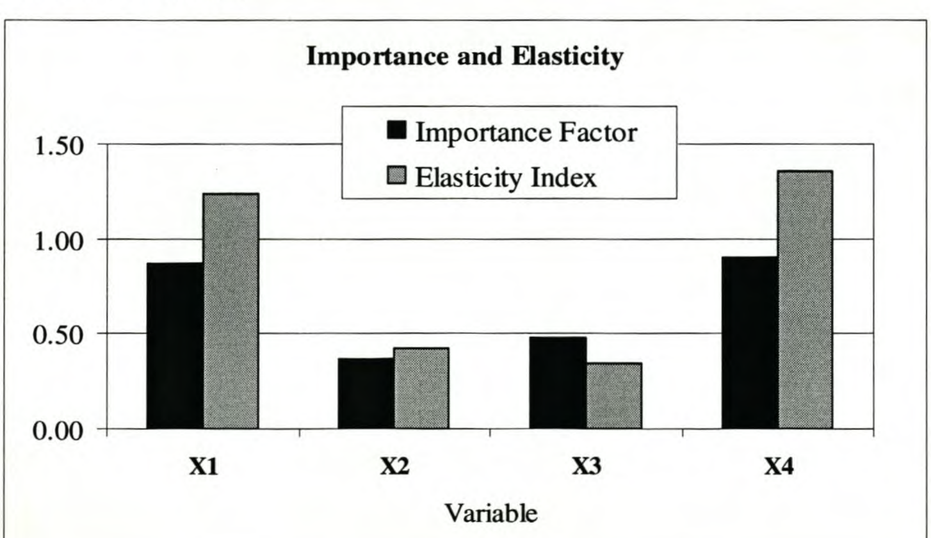


Figure 9.2 Importance Factors and Elasticities for a Cost Function

9.8.3 Stochastic Importance

Stochastic importance is calculated as the product of the direction cosine and the independent component of the direction cosine, given below for variable X_1,

$$R_1^2 = \alpha_1 \alpha_{11} = (-0.87)(-0.43) = 0.38$$

$\alpha_2 \alpha_{22} = 0.03$; $\alpha_3 \alpha_{33} = 0.04$; $\alpha_4 \alpha_{44} = 0.55$

As was demonstrated in Equation (7.13), the sum of the stochastic importance equals one,

$$\sum_{i=1}^{4} \alpha_i \alpha_{ii} = 0.38 + 0.03 + 0.04 + 0.55 = 1.00$$

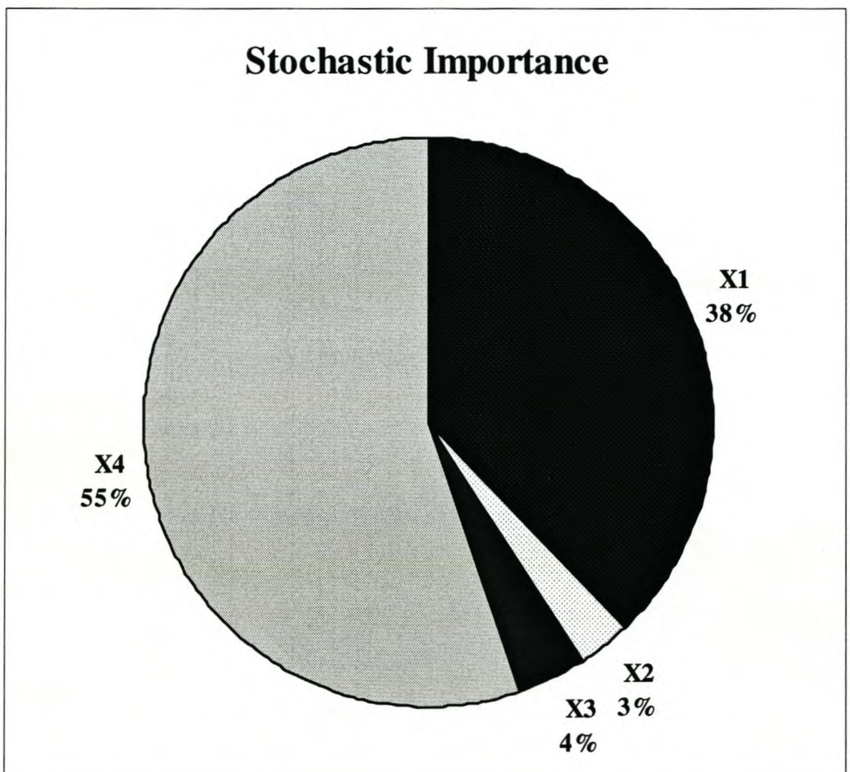


Figure 9.3 Stochastic Importance Describing proportional contribution to the variance for the total cost probability density function

9.8.4 Sensitivity

The sensitivity indicates the rate of change of the performance function in the direction of the variables under observation, taking correlation between variables into account.

$$\frac{\partial g}{\partial X_1} = \frac{\alpha_1 \sigma_g}{\sigma_{X_1}} = \frac{(-0.87)(1.02x10^8)}{100} = -88.18$$

$$\frac{\partial g}{\partial X_2} = -1847.14; \qquad \frac{\partial g}{\partial X_3} = -2425.70; \qquad \frac{\partial g}{\partial X_4} = -608.44$$

These sensitivities are represented in Figure 9.4 below. Interestingly variables X_2 and X_3 demonstrate the highest sensitivities, which is not in keeping with the other results. This will be discussed in more detail at the end of the chapter.

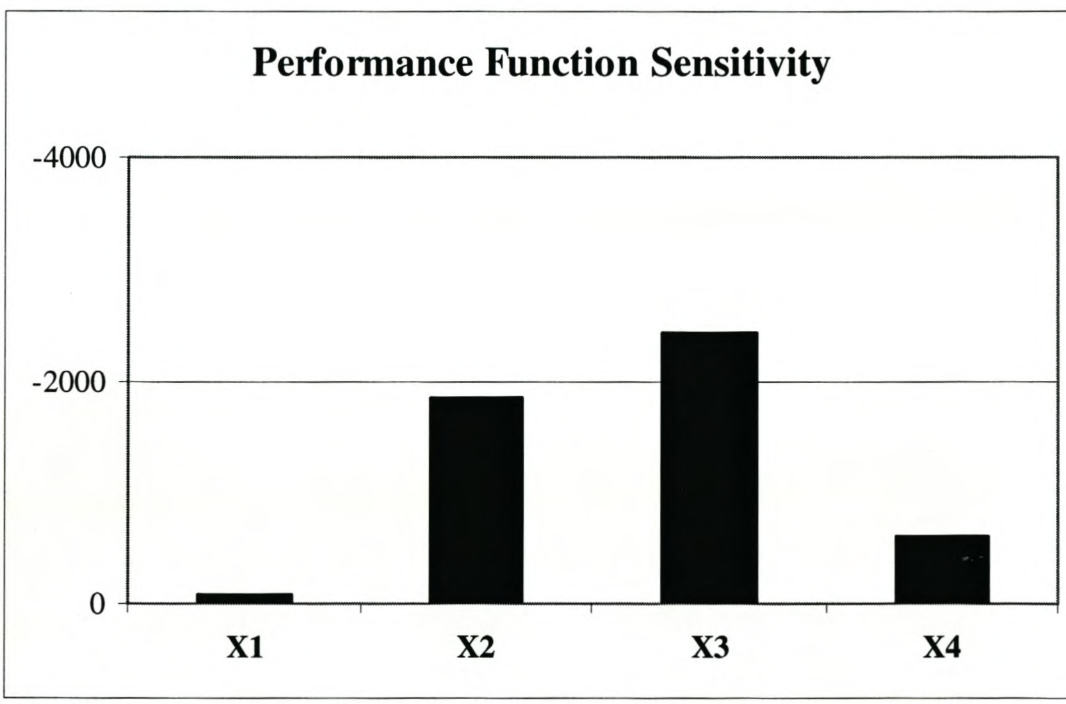


Figure 9.4 Sensitivity of the cost performance function to individual variables

9.8.5 *Likely Contingency Demand*

The available contingency is the difference between the budget and estimated cost. The likely demand by the different function terms for this contingency is a proportional representation of risk in the dimension of the performance function. Then performance function, written as a function of its terms is

$$g(X) = 18000 - X_1X_2 - X_1X_3 - X_1X_4 = 0$$

As an example the term X_1X_2 demonstrates a probabilistic demand for the available contingency given by,

$$Contingency(X_1X_2) = x_1^*x_2^* - \mu_1\mu_2 = (411.5)(5.5) - (350)(5) = 519.6$$

Similarly

$$Contingency(X_1X_3) = 463.0 \qquad Contingency(X_1X_4) = 5467.3$$

These results (given in white) together with the estimated cost (given in black) for each term in the cost function are represented graphically in Figure 9.5

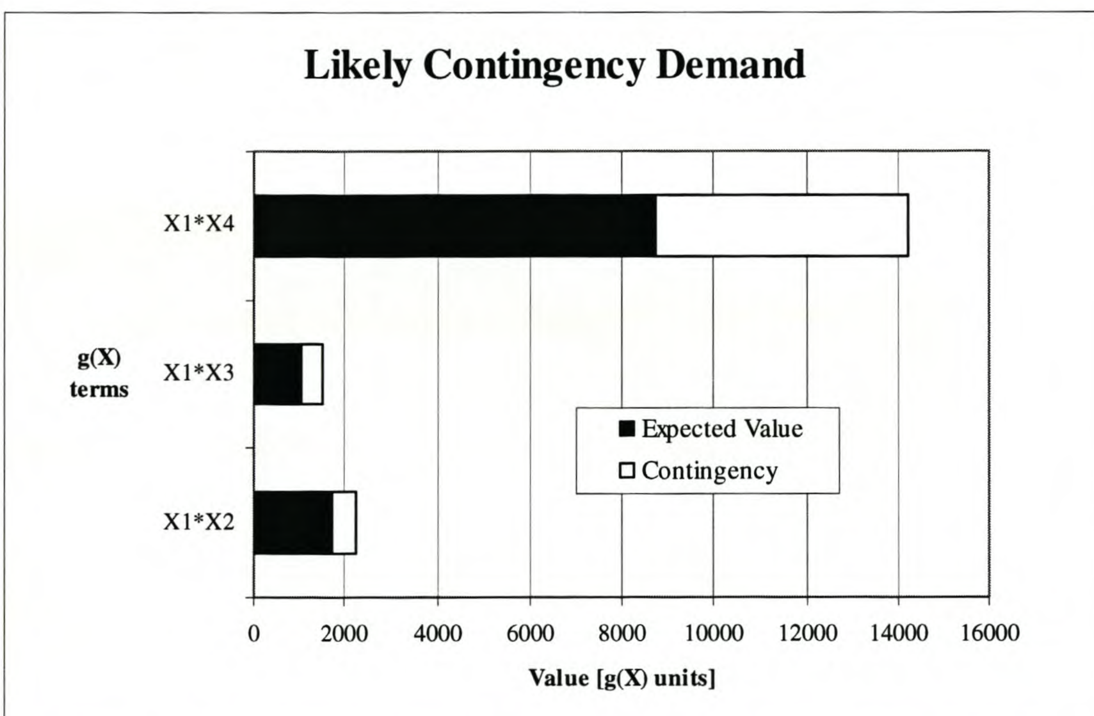


Figure 9.5 Likely contingency demand of terms in the cost performance function

It is clear that the likely contingency demand by term X_1X_4 is a factor of 10 greater than the other two terms. The risk associated with this term is therefore clearly dominant.

9.8.6 Conclusion

A summary of the risk indicator results is given in table 9.2 below.

Table 9.2 Summary table of the risk indicator results

	$(d\beta/dX')=-\alpha_i$	$(dg/dX)=\alpha_i\sigma_g/\sigma_i$	$R_i=\alpha_i\alpha_{ii}$	$\alpha_i^E=\alpha_{ii}\Sigma\rho_{ij}$	$\varepsilon_i=\alpha_i^E/\alpha_i$
X_1	0.87	-88.18	0.38	-1.08	1.24
X_2	0.37	-1847.14	0.03	-0.15	0.42
X_3	0.48	-2425.70	0.04	-0.16	0.34
X_4	0.90	-608.44	0.55	-1.22	1.35

It is clear that X_4 (representing material cost rates) is the dominant variable in terms of the importance factor, stochastic importance, elasticity and the elasticity index. Variable X_1 (representing the quantity estimate) is the second most significant variable, although with a lessor impact on stochastic importance.

The variables X_3 (plant rate) and X_2 (labour rate) result in high performance function sensitivities. This is due to the limited magnitude of the units present in the variable. A unit change in X_2 or X_3 will naturally have a greater effect on $g(X)$ than a unit change in X_1 or X_4. It is for this reason that sensitivity is not a sufficient measure of risk, when considered in isolation.

10. OMISSION SENSITIVITY

MADSEN (1988) developed an omission sensitivity factor, ψ_i, which measures the sensitivity of the reliability index to the treatment of non-critical variables as deterministic values. In this way the performance function can be simplified to focus only on the important variables. A shortcoming of the application of the ψ_i parameter is that the function must first be solved before simplification can be made. The method also treats dependent variables through transformation to an equivalent independent variable set.

The GFOSM method makes the treatment of correlated variables in the original variable space possible. The stability of the direction cosine discussed in chapter 8 further contributes to establishing omission sensitivities by producing results without having to first solve the limit function.

10.1 Omission Sensitivity for Linear Independent Functions

The omission sensitivity factor compares the reliability index calculated from the reduced number of variables $\beta(\mathrm{X}_i = \mu_i)$ to the reliability index with a full complement. An omission sensitivity close to 1 indicates that the reliability is relatively insensitive to the replacement of the variable with its mean value.

$$\psi(X_i = \mu_{X_i}) = \frac{\beta(X_i = \mu_{X_i})}{\beta} \tag{10.1}$$

Consider a linear performance function with independent basic variables,

$$g(\mathrm{X}) = a_0 - \sum_{j=1}^{n} a_j \mathrm{X}_j \tag{10.2}$$

a_o and a_j are constants and X_j variables. The expected value, standard deviation and reliability index for g(X) are given below.

$$\mathrm{E}(\mathrm{X}) = \mu_g = a_0 - \sum_{j=1}^{n} a_j \mu_{X_i} \tag{10.3}$$

$$\sigma_g^2 = \sum a_j^2 \sigma_{X_j}^2 \tag{10.4}$$

$$\beta = \frac{\mu_g}{\sigma_g} = \frac{a_0 - \sum_{j=1}^{n} a_j \sigma_{X_j}}{\sqrt{\sum_{j=1}^{n} a_j^2 \sigma_{X_j}^2}} \tag{10.5}$$

The expected value, standard deviation and reliability index for g(X), found by substituting the mean value μ_{X_i} for the variable X_i

$$E(X_i = \mu_{X_i}) = a_0 - \sum_{j=1}^{n} a_j \mu_{X_j} \tag{10.6}$$

$$\sigma_g^2(X_i = \mu_{X_i}) = \sum_{j=1}^{n} a_j^2 \sigma_{X_j}^2 - a_i^2 \sigma_{X_i}^2 \tag{10.7}$$

$$\beta(X_i = \mu_{X_i}) = \frac{a_0 - \sum_{i=1}^{n} a_i \mu_{X_i}}{\sqrt{\sum_{j=1}^{n} a_j^2 \sigma_{X_j}^2 - a_i^2 \sigma_{X_i}^2}} \tag{10.8}$$

Dividing Equation (10.8) by (10.5) leads to the omission sensitivity factor

$$\begin{aligned}
\psi(X_i = \mu_{X_i}) &= \frac{\beta(X_i = \mu_{X_i})}{\beta} \\
&= \left[\frac{a_o - \sum_{j=1}^{n} a_j \mu_{X_j}}{\sqrt{\left(\sum_{j=1}^{n} a_j^2 \sigma_{X_j}^2 - a_i^2 \sigma_{X_i}^2 \right)}} \right] \left[\frac{\sqrt{\sum_{j=1}^{n} a_j^2 \sigma_{X_j}^2}}{a_o - \sum_{j=1}^{n} a_j \mu_{X_j}} \right] \\
&= \frac{\sqrt{\sum_{j=1}^{n} a_j^2 \sigma_{X_j}^2}}{\sqrt{\sum_{j=1}^{n} a_j^2 \sigma_{X_j}^2 \left(1 - \frac{a_i^2 \sigma_{X_i}^2}{\sum_{j=1}^{n} a_j^2 \sigma_{X_j}^2} \right)}} = \frac{1}{\sqrt{\left(1 - \frac{a_i^2 \sigma_{X_i}^2}{\sum_{j=1}^{n} a_j^2 \sigma_{X_j}^2} \right)}}
\end{aligned} \tag{10.9}$$

The direction cosine for a linear function of independent standard normal variables is given by

$$\alpha_i = \frac{a_i \sigma_{X_i}}{\sqrt{\sum_{j=1}^{n} a_j^2 \sigma_{X_j}^2}} \tag{10.10}$$

Substituting Equation (10.9) into (10.10), the omission sensitivity factor for linear independent functions reduces to (MADSEN, 1988)

$$\therefore \psi(X_i = \mu_{X_i}) = \frac{1}{\sqrt{(1-\alpha_i^2)}} \tag{10.11}$$

10.2 Omission Sensitivity for Transformed Linear dependent Functions

MADSEN (1988) also proposed a number of extensions for the omission sensitivity factor for dependent and non-normal linear multivariate functions. Consider a general limit state function of the form

$$g(\mathrm{X}) = a_0 - a^{\mathrm{T}}\mathrm{X} \tag{10.12}$$

Where X is a vector of normally distributed dependent variables, with mean μ and covariance matrix C_{X}. The matrix X is transformed (by way of the Rosenblatt method) into the independent standard normal space through the expression

$$\mathrm{X}' = L(\mathrm{X} - \mu_{\mathrm{X}}) \tag{10.13}$$

Where L is a lower triangular matrix, determined by Cholesky triangulanzitation of the covariance matrix . As a result, L is related to C_{X} through

$$C_{\mathrm{X}} = (L^{-1})(L^{-1})^{\mathrm{T}} \tag{10.14}$$

The reliability index, derived from all the variables is given by

$$\beta = \frac{a_0 - a^{\mathrm{T}}\mu_{\mathrm{X}}}{\sqrt{a^{\mathrm{T}} C_{\mathrm{X}} a}} \tag{10.15}$$

Similarly the reliability index, derived by substituting variable X_i with its mean value, μ_{X_i}

$$\beta\left(\mathrm{X}_i = \mu_{\mathrm{X}_i}\right) = \frac{a_0 - a^{\mathrm{T}}\mu_{\mathrm{X}}}{\sqrt{a^{\mathrm{T}}\left(C_{\mathrm{X}} - C_{\mathrm{X}}^{j}\right)a}} \tag{10.16}$$

In the above expression, C_{X}^{j} is the $n \times n$ matrix containing the j$^{\mathrm{th}}$ row and j$^{\mathrm{th}}$ column of the covariance matrix C_{X}, with all other entries zero. The omission sensitivity factor $\psi\left(\mathrm{X}_i = \mu_{\mathrm{X}_i}\right)$ is

$$\psi\left(\mathrm{X}_i = \mu_{\mathrm{X}_i}\right) = \frac{\beta\left(\mathrm{X} = \mu_{\mathrm{X}_i}\right)}{\beta} = \frac{\sqrt{a^{\mathrm{T}}C_{\mathrm{X}}a}}{\sqrt{a^{\mathrm{T}}\left(C_{\mathrm{X}} - C_{\mathrm{x}}^{j}\right)a}} \tag{10.17}$$

The direction cosine can also be expressed in terms of the lower triangular matrix L,

$$\alpha = \frac{\left(L^{-1}\right)^{\mathrm{T}}a}{\sqrt{a^{\mathrm{T}}C_{\mathrm{X}}a}} \tag{10.18}$$

By substitution $\psi\left(\mathrm{X}_i = \mu_i\right)$ for linear dependent functions of normally distributed variables can be reduced to (MADSEN, 1988)

$$\psi\left(\mathrm{X}_i = \mu_{\mathrm{X}_i}\right) = \frac{1}{\sqrt{1 - \alpha^{\mathrm{T}}LC_{\mathrm{X}}^{j}L^{\mathrm{T}}\alpha}} \tag{10.19}$$

For independent basic variables the term $LC_{X}^{j}T^{T}$ falls away and Equation (10.19) reduces to the more familiar form

$$\psi\left(\mathrm{X}_i = \mu_{\mathrm{X}_i}\right) = \frac{1}{\sqrt{1 - \alpha_i^2}} \tag{10.20}$$

10.3 General Correlated form for the Omission Sensitivity Factor

A general dependent form of the omission sensitivity factor can be derived utilising the general form of the direction cosine (Equation 4.29). Consider a general limit state function, $g(\mathrm{X}) = 0$. This function may be of a linear or non-linear form, with dependent normal and non-normally distributed variables. It has been shown that a linear approximation of the failure domain at the design point, and the representation of non-normal distributions by an equivalent normal tail the above limit state function can be solved in the original variable space with the general form of the direction cosine.

The reliability index has been defined as $\beta = \frac{\mu_g}{\sigma_g}$ which is true for the general function g(X).

The reliability index obtained by replacing variable X_i with μ_i is

$$\beta(X_i = \mu_{X_i}) = \frac{\mu_g}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_i}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j} - \left(\frac{\partial g}{\partial X_i}\right)_*\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}} \tag{10.21}$$

Where the term $\left(\frac{\partial g}{\partial X_i}\right)_*\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}$ is the contribution made by variable X_i to the total variance. This can be rewritten in a more compact form by substituting the expression for the proportional contribution of variable X_i to the total variance σ_g^2 given in Equation (7.10) denoted R_i^2,

$$\beta(X_i = \mu_i) = \frac{E(X)}{\sqrt{(\sigma_g^2 - R_i^2\sigma_g^2)}} \tag{10.22}$$

Dividing Equation (10.22) by the general form of the reliability index $\left(\beta = \frac{\mu_g}{\sigma_g}\right)$ then develops the omission sensitivity

$$\psi(X_i = \mu_{X_i}) = \frac{\beta(X_i = \mu_{X_i})}{\beta} = \frac{\sigma_g}{\sqrt{(\sigma_g^2 - R_i^2\sigma_g^2)}} = \frac{1}{\sqrt{(1 - R_i^2)}} \tag{10.23}$$

It has been shown (Equation 7.12) that R_i^2 is the stochastic importance of variable X_i, and can be written in terms of the alpha and the independent component of alpha,

$$R_i^2 = \alpha_i\alpha_{ii} \tag{10.24}$$

The omission sensitivity factor can therefore be represented as a function of the direction cosine

$$\psi(X_i = \mu_{X_i}) = \frac{1}{\sqrt{(1 - \alpha_i\alpha_{ii})}} \tag{10.25}$$

Since R_i^2 is additive, the cumulative omission sensitivity derived from the treatment of a number of variables as deterministic values is

$$\boxed{\psi\left(X_j = \mu_{X_j}, j = 1,2,\ldots k\right) = \frac{1}{\sqrt{\left(1 - \sum_{j=1}^{k} \alpha_j \alpha_{jj}\right)}}} \tag{10.26}$$

10.4 Approximating the omission sensitivity factor

It was shown in section 8.3 that for performance functions of limited curvature, a known value for the direction cosine can be used to approximate alpha at other levels of reliability. Furthermore the failure point co-ordinates at a reliability index of zero are known to be the mean values. This failure point is therefore defined prior to the solution of the limit function. The resultant values for alpha and the independent component of alpha as a function of the mean values can therefore be used to approximate the omission sensitivity. This has important implications since the omission sensitivity can be established and simplifications effected prior to the solution of the complex performance function.

As demonstrated in Equation (8.8), the stability of the direction cosine is affected by the curvature of the performance function. The direction cosine utilised in the approximate omission sensitivity is therefore only suitable for linear functions and functions of limited curvature. However this approximate omission sensitivity will be sufficiently accurate for typical cost estimating and project management functions, since these are of limited curvature with many variables.

10.5 Reflection on the proposed contribution to the current theory

The current omission sensitivity factor requires that dependent variables be transformed into an independent variable space. This is somewhat inconvenient since it implies that the performance function with a full variable complement must first be solved for the unknown failure point co-ordinates. This is required for establishing the direction cosine values. Only once this has been solved, can the function be simplified by treating non-critical items as constant values.

A general form of the omission sensitivity is developed as a function of the analytical direction cosine for correlated basic variables. This formulation avoids the transformation of dependent variables into an independent variable space. As was demonstrated in section 8.3, a stable direction cosine provides for the approximation of alpha and the independent component of alpha as a function of the mean values. This allows the omission sensitivity to be established prior to the solution of the full performance function. As a result the general form of the omission sensitivity provides an efficient tool for simplifying complex limit state functions.

The complexity of solving limit state functions increases exponentially as the number of variables contained within the performance function increases. This is due to the fact that each variable adds a column and a row to the correlation matrix. The size of the correlation matrix is n^2 where n is the number of dependent variables. Considerable value is therefore gained by reducing the number of correlated variables in a multi-variate performance function.

10.6 Application

This application is a continuation of the application in section 5.3 and 9.8. The correct order would be for the omission sensitivity analysis to be conducted as the first step, the GFOSM method would then be solved and then the risk indicators calculated for the simplified performance function. However as part of the logical development of the dissertation, the omission sensitivity is addressed here.

The direction cosines (α_i) and independent components (α_{ii}) were calculated in the same way as chapters 5 and 9. These quantities, utilised in establishing omission sensitivities, are approximated as functions of the mean values. The performance function can therefore be solved prior to the solution of the failure point co-ordinate.

Utilising alpha and independent alpha values, the omission sensitivities corresponding to the treatment of each variable as a constant are

$$\psi(X_1 = \mu_{X_1}) = \frac{1}{\sqrt{(1-\alpha_1\alpha_{11})}} = \frac{1}{\sqrt{(1-(-0.86)(-0.40))}} = 1.24$$

$$\psi(X_2 = \mu_{X_2}) = 1.02; \qquad \psi(X_3 = \mu_{X_3}) = 1.02; \qquad \psi(X_4 = \mu_{X_4}) = 1.55$$

From the above omission sensitivities it is clear that the reliability index will be sensitive to the treatment of variables X_1 and X_4 (quantity estimates and material unit rates) as constant values. This corresponds to the high direction cosines and elasticity indices found in section 10.3. However the omission sensitivity factors for variables X_2 and X_3 are sufficiently close to 1 to indicate that these variables can be replaced by constant values.

The cumulative omission sensitivity derived by treating variables X_2 and X_3 as constant values is (from Equation 10.26)

$$\psi(X_1 = \mu_{X_1,} X_2 = \mu_{X_2,}) = \frac{1}{\sqrt{[1-((-0.35)(-0.09)+(-0.49)(-0.09))]}} = 1.04$$

The GFOSM method was solved for each of the cases given above to demonstrate what the effect of omitting variability of the respective variables would have on the reliability. A summary of these results is tabulated below.

Table 10.1 Omission sensitivities and corresponding levels of reliability

Ommitted	α_i	α_{ii}	ψ_i	β (x*=μ)	P_s
None			1.00	0.70	0.76
X_1	-0.86	-0.40	1.24	1.14	0.87
X_2	-0.35	-0.09	1.02	0.72	0.77
X_3	-0.49	-0.09	1.02	0.73	0.77
X_4	-0.91	-0.64	1.55	1.47	0.93
X_2 & X_3			1.04	0.75	0.77

Omitting the non-critical variables (X_2 and X_3) from the original performance reduces the number of variables by half. The probability of success for the simplified function was within 0.01 of the full solution. This marginal error easily justifies the considerable simplification achieved by replacing the non-critical items with constant values.

11. SYSTEM RELIABILITY

11.1 Systems Analysis

The FOSM theory described has been applicable to the evaluation of individual limit state functions. In reality reliability is not only measured in terms of single performance functions, but may be multi-dimensional with a number of failure surfaces. In structural reliability design engineers must consider structural performance in terms of shear, flexure, torsion and bending moment or combinations of these. Similarly project risk may be measured in terms of budget constraints, schedule, cash flow, return on investment, structural fit for purpose environmental impact, safety and a host of other dimensions. In order to deal with different performance functions in a rational way the FOSM reliability method was extended to include system analysis. One of the most important factors to consider in system reliability is the extent to which the occurrence of one failure event will have a systematic effect on the occurrence of other failure events. This phenomenon is referred to as modal correlation.

Consider a system composed of three limit state functions, given in Figure 11.1. System performance cannot be measured by considering each limit function in isolation. The combination of reliabilities β_1, β_2 and β_3 may well demonstrate interrelationships due to shared variables and correlation, influencing system reliability.

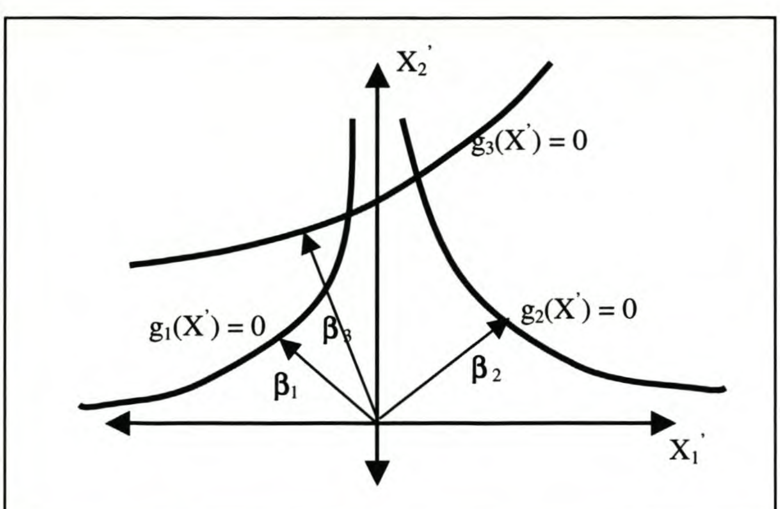


Figure 11.1 Three performance functions forming part of a system

If we consider 3 failure events, corresponding to the performance functions in Figure 11.1, these can be given by

$$E_1 = [g_1(X) < 0]; \qquad E_2 = [g_2(X) < 0]; \qquad E_3 = [g_3(X) < 0]$$

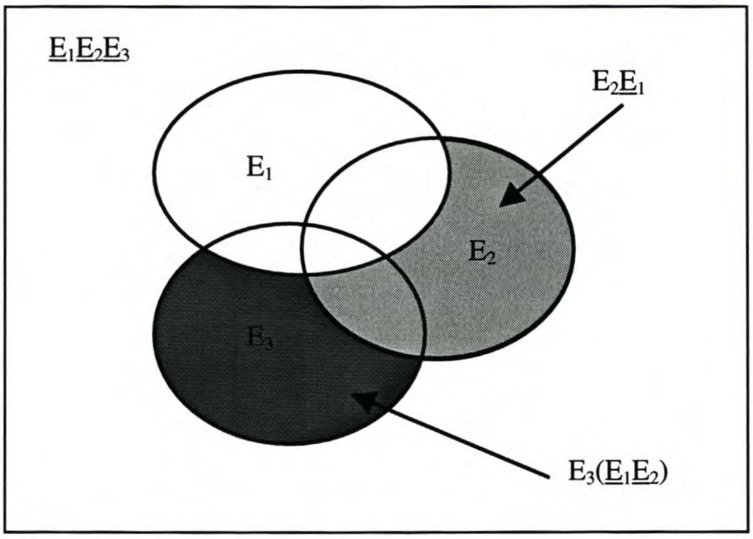


Figure 11.2 Interaction of 3 failure events

These failure events and their intersections are represented diagrammatically in Figure 11.2. The area contained within each sphere represents the failure event occurring. The area outside the elipse (enclosed by the frame) represents the safe state.

A system would be considered to be safe if none of the potential failure events occur, given by the intersection of the safe states

$$\underline{E} = \underline{E}_1 \cap \underline{E}_2 \cap \underline{E}_3 = \underline{E}_1\underline{E}_2\underline{E}_3 \tag{11.1}$$

Conversely the system fails if any one or more of the potential failure events occur, given by the union of the failure events

$$E = E_1 \cup E_2 \cup E_3 \tag{11.2}$$

System analysis requires the evaluation of the joint probabilities representing combinations of likely failure events. As a result a number of approximations have been developed to assist with establishing upper and lower bounds for the probability of system failure.

11.1.1 Uni-modal Bounds

Uni-modal Bounds take into account the effect of correlation, in that if events $\underline{E}_i$ and $\underline{E}_j$ are positively correlated, then the likelihood of $\underline{E}_i$ given that $\underline{E}_j$ has occurred, is greater than the likelihood of $\underline{E}_i$. Therefore the probability of success is bounded by the conditioned likelihood that no events fail and the most likely event to fail (CORNELL, 1967).

$$\prod_{i=1}^{k} P_{S_i} \le P_S \le \min_i P_{S_i} \tag{11.3}$$

from which the bounds for the probability of failure are developed as

$$\max_i P_{F_i} \le P_F \le 1 - \prod_{i=1}^{k} (1 - P_{F_i}) \tag{11.4}$$

The lower bound corresponds to no correlation (or minimum effect of correlation) and the upper bound to perfect correlation (or maximum effect of correlation).

Conversely if the events of $\underline{E}_i$ and $\underline{E}_j$ are negatively correlated, the likelihood of $\underline{E}_i$ occurring given that $\underline{E}_j$ has occurred is less than the likelihood of $\underline{E}_i$ occurring. The probability of success is therefore less than or equal to the likelihood that no events fail, but must be greater than zero.

$$0 \le P_S \le \prod_{i=1}^{k} P_{S_i} \tag{11.5}$$

The corresponding bounds for the probability of failure are

$$1 - \prod_{i=1}^{k} (1 - P_{F_i}) \le P_F \le 1 \tag{11.6}$$

11.1.2 Bi-modal Bounds

The uni-modal bounds described above are a relatively extreme approximation, since they consider only no correlation or perfect correlation. The method does not explicitly account for varying levels of correlation between pairs of failure modes.

KOUVIAS (1968) and HUNTER (1976) developed the concept of second order or bi-modal failure probabilities. The method takes modal correlation into account by calculating joint failure probabilities between failure mode.

$$P_{F_1} + \max\left[\sum_{i=2}^{k}\left\{P_{F_i} - \sum_{j=1}^{i=1} P(E_i E_j)\right\};0\right] \le P_F \le \sum_{i=1}^{k} P_{F_i} - \sum_{i=2}^{k} \max P(E_i E_j) \tag{11.7}$$

DITLEVSEN (1979) proposed an approximate method to overcome the difficulty of calculating the joint probabilities required in the procedure given above. Given that the events are positively correlated, the joint probability for failure events $P(E_i E_j)$ is bounded by

$$\max[P(A), P(B)] \le P(E_i E_j) \le P(A) + P(B) \tag{11.8}$$

where $P(A) = \Phi(-\beta_i)\Phi(-a) = \Phi(-\beta_i)\Phi\left(-\frac{\beta_j - \rho\beta_i}{\sqrt{1-\rho^2}}\right)$ (11.9)

and $P(B) = \Phi(-\beta_j)\Phi(-b) = \Phi(-\beta_j)\Phi\left(-\frac{\beta_i - \rho\beta_j}{\sqrt{1-\rho^2}}\right)$ (11.10)

If the failure events $\underline{E}_i$ and $\underline{E}_j$ were negatively correlated, the joint probability would be

$$0 \le P(E_i E_j) \le \min[P(A), P(B)] \tag{11.11}$$

11.2 Modal Correlation for Independent basic variables

Consider two multivariate functions $\mathrm{Y} = g(\mathrm{X})$ and $\mathrm{Z} = f(\mathrm{X})$. X is a vector space of independent basic variables. Some, but not necessarily all, variables contained in X are common to $g(\mathrm{X})$ and $f(\mathrm{X})$. The first order approximation for Y and Z, given $g(x^*) = 0$ and $f(x^*) = 0$ on the limit state surface, is

$$\mathrm{Y} = g(x^*) + \sum_{i=1}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)(\mathrm{X}_i - \mathrm{X}_i^*) = \sum_{i=1}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)(\mathrm{X}_i - \mathrm{X}_i^*) = 0 \tag{11.12}$$

$$\mathrm{Z} = f(x^*) + \sum_{j=1}^{n}\left(\frac{\partial f}{\partial \mathrm{X}_j}\right)(\mathrm{X}_j - \mathrm{X}_j^*) = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial \mathrm{X}_j}\right)(\mathrm{X}_j - \mathrm{X}_j^*) = 0 \tag{11.13}$$

The covariance between Y and Z is

$$
\begin{aligned}
Cov(\mathrm{Y},\mathrm{Z}) &= \mathrm{E}\left[(\mathrm{Y}-\mathrm{Y}^*)(\mathrm{Z}-\mathrm{Z}^*)\right] \\
&= \mathrm{E}\left[\left\{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)_*(\mathrm{X}_i-\mathrm{X}_i^*)\right\}\left\{\sum_{j=1}^{n}\left(\frac{\partial f}{\partial \mathrm{X}_j}\right)_*(\mathrm{X}_j-\mathrm{X}_j^*)\right\}\right] \\
&= \mathrm{E}\left[\sum_{i=1}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)_*\left(\frac{\partial f}{\partial \mathrm{X}_i}\right)_*(\mathrm{X}_i-\mathrm{X}_i^*)^2+\sum_{i\neq j}^{n}\sum_{j\neq i}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)_*\left(\frac{\partial f}{\partial \mathrm{X}_j}\right)_*(\mathrm{X}_i-\mathrm{X}_i^*)(\mathrm{X}_j-\mathrm{X}_j^*)\right] \\
&= \sum_{i=1}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)_*\left(\frac{\partial f}{\partial \mathrm{X}_i}\right)_* Var(\mathrm{X}_i)+\sum_{i\neq j}^{n}\sum_{j\neq i}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)_*\left(\frac{\partial f}{\partial \mathrm{X}_j}\right)_* Cov(\mathrm{X}_i,\mathrm{X}_j) \qquad (11.14)
\end{aligned}
$$

For independent basic variables $Cov(X_i, X_j) = 0$ for $i \neq j$. The covariance between Y and Z is then given by,

$$Cov(Y,Z) = \sum_{i=1}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)_*\left(\frac{\partial f}{\partial \mathrm{X}_i}\right)_* \sigma_{\mathrm{X}_i}^2 \qquad (11.15)$$

From this expression for the covariance, the correlation coefficient between performance functions Y and Z is given by

$$\rho_{YZ} = \frac{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)\left(\frac{\partial f}{\partial X_i}\right)\sigma_{X_i}^2}{\sigma_Y\sigma_Z} = \sum_{i=1}^{n}\left[\frac{\left(\frac{\partial g}{\partial X_i}\right)\sigma_{X_i}}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)^2\sigma_{X_i}^2}}\right]\left[\frac{\left(\frac{\partial f}{\partial X_i}\right)\sigma_{X_i}}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial f}{\partial X_i}\right)^2\sigma_{X_i}^2}}\right] \qquad (11.16)$$

This is recognised as the sum of the product of the direction cosines for variables common to both performance functions. The correlation between Y and Z can therefore be written in terms of the direction cosine, where $\alpha_i(Y)$ represents the direction cosine corresponding to variable X_i for the function $Y = g(X) = 0$, and $\alpha_i(Z)$ the direction cosine corresponding to variable X_i for the function $Z = f(X) = 0$,

$$\rho_{YZ} = \sum_{i=1}^{n}\alpha_i(Y)\alpha_i(Z) \qquad (11.17)$$

For dependent functions, the basic variables are first transformed into an independent variable space. The modal correlation is then calculated utilising these representative independent variables. It is important to note that in order for the transformed variables to demonstrate commonality, the full variable set containing all variables in both limit functions must be transformed. The dependent forms for the two functions are then constructed from these new independent variables. In this way the direction cosines corresponding to common variables in the independent space can be identified and utilised in establishing the modal correlation.

11.3 General formulation for Modal Correlation

It is clear that transforming variables into an independent variable space restricts the parametric insight that can be developed into the mechanisms dominating the modal correlations. This is an important shortcoming since the failure in one dimension may well increase the likelihood of failure in other dimensions. This chain reaction could result in a containable adverse event escalating into a series of events causing total failure. It would therefore not only be useful to quantify modal correlation but to understand its sources.

The general dependent form of the direction cosine given in Equation (4.29) facilitates the development of a general form for modal correlation between dependent limit functions, in the original variable space. The covariance between the two general performance functions Y and Z given in Equation (11.14), assuming dependence between the basic variables reduces to

$$
\begin{aligned}
Cov(Y,Z) &= \sum_{i=1}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)_*\left(\frac{\partial f}{\partial \mathrm{X}_i}\right)_* Var(\mathrm{X}_i)+\sum_{i\neq j}^{n}\sum_{j\neq i}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)_*\left(\frac{\partial f}{\partial \mathrm{X}_j}\right)_* Cov(\mathrm{X}_i,\mathrm{X}_j)\\
&= \sum_{i=1}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)_*\left(\frac{\partial f}{\partial \mathrm{X}_i}\right)_* \sigma_{\mathrm{X}_i}^2+\sum_{i\neq j}^{n}\sum_{j\neq i}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)_*\left(\frac{\partial g}{\partial \mathrm{X}_j}\right)_* \rho_{ij}\sigma_{\mathrm{X}_i}\sigma_{\mathrm{X}_j}\\
&= \sum_{i}^{n}\sum_{j}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)_*\left(\frac{\partial f}{\partial \mathrm{X}_i}\right)_* \rho_{ij}\sigma_{\mathrm{X}_i}\sigma_{\mathrm{X}_j}
\end{aligned}
\tag{11.18}
$$

The correlation coefficient between *Y* and *Z* can therefore be derived as

$$
\begin{aligned}
\rho_{\mathrm{YZ}} &= \frac{Cov(\mathrm{Y},\mathrm{Z})}{\sigma_{\mathrm{Y}}\sigma_{\mathrm{Z}}}\\
&= \frac{\sum_{i}^{n}\sum_{j}^{n}\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)_*\left(\frac{\partial f}{\partial \mathrm{X}_i}\right)\rho_{ij}\sigma_{\mathrm{X}_i}\sigma_{\mathrm{X}_j}}{\sigma_{\mathrm{Y}}\sigma_{\mathrm{Z}}}\\
&= \sum_{i=1}^{n}\left[\frac{\left(\frac{\partial g}{\partial \mathrm{X}_i}\right)_*\sigma_{\mathrm{X}_i}}{\sqrt{\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_*\rho_{ij}\sigma_{X_i}\sigma_{X_j}}}\right]\left[\frac{\sum_{j=1}^{n}\left(\frac{\partial f}{\partial \mathrm{X}_j}\right)_*\rho_{ij}\sigma_{\mathrm{X}_j}}{\sqrt{\left(\frac{\partial f}{\partial X_i}\right)_*\left(\frac{\partial f}{\partial X_j}\right)_*\rho_{ij}\sigma_{X_i}\sigma_{X_j}}}\right]
\end{aligned}
\tag{11.19}
$$

This is recognised as the summation of the product of the independent component of the direction cosine for Y multiplied by the corresponding direction cosine for Z. Since the correlation coefficient between X_i and X_j are shared by both functions, the formulation given above can also be reversed. This is represented by

$$\boxed{\rho_{YZ} = \sum_{i=1}^{n} \alpha_{ii}(Y)\alpha_i(Z) = \sum_{i=1}^{n} \alpha_i(Y)\alpha_{ii}(Z)} \tag{11.20}$$

The modal correlation between two performance functions containing dependent basic variables can therefore be described as functions of the direction cosines. What is particularly significant is that modal correlation is not restricted to common variables. It can be seen that all X_j variables are represented in Equation (11.19), regardless of whether they are shared by both functions. Any variable in either of the limit functions can therefore through correlation with other variables contribute to modal correlation. The only condition is that if variables are unique to one of the functions, there must be some form of correlation with variables common to both functions.

This observation introduces the concept that limit functions describing the performance of a system in a particular dimension are not unique but composed of variables belonging to a global variable set. While variables not contained in the functions will not affect the limit function solution, unique variables can through correlation impact on modal correlation and consequently on system reliability. This provides a motivation for the construction and maintenance of a generic covariance matrix for variables utilised in modelling project risk.

11.4 Reflection on the contribution to the current theory

The current theory utilises the independent form of the direction cosine to develop an expression for modal correlation. Performance functions containing correlated variables must be transformed into an independent space to enable the measurement of modal correlation. The current theory also utilises direction cosines for variables common to both functions under consideration.

The analytical form of the direction cosine for functions of correlated variables enables the development of an expression for modal correlation in the original variable space, thus avoiding transformation to an equivalent independent set of variables. It is noteworthy that all variables contained in the global variable set are utilised in constructing modal correlation. This observation indicates that a variable unique to one of the functions can through correlation also contribute to modal correlation. The only condition for a unique variable to have an impact is that it must be correlated to one or more variables common to both functions. This characteristic presents the idea of a global set of variables from which performance functions can be constructed.

11.5 Application

Two simple performance functions of limited curvature are utilised to demonstrate the proposed general form for calculating modal correlation. This can be interpreted as version of the cost functions utilised in the applications in chapter 5 and 8. These functions are given by,

$$g_1(X) = 1000000 - (200X_1(X_2 + X_3) + X_4) = 0$$

$$g_2(X) = 500000 - X_1X_3(X_4 + X_5 + X_6) = 0$$

The variables are defined as a global set of distribution parameters and a global correlation matrix. Subjective estimations of the mean and standard deviations can be made for the given project conditions. An objective data source from a number of similar projects is used in the application to measure the correlation coefficient described by Equation (3.18).

Table 11.1 Distribution parameters and Correlation Matrix

	μ_i	σ_i
X_1	350	100
X_2	5	2
X_3	3	2
X_4	25	15
X_5	4	2
X_6	200	90

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	1	0.6	0.3	0.6	0.0	0.2
X_2	0.6	1	0.3	0.0	0.5	0.3
X_3	0.3	0.3	1	0.4	0.2	0.1
X_4	0.6	0.0	0.4	1	0.0	0.7
X_5	0.0	0.5	0.2	0.0	1	0.6
X_6	0.2	0.3	0.1	0.7	0.6	1

The GFOSM solutions for performance functions $g_1(X)$ and $g_2(X)$ are tabulated below. The function $g_1(X)$returns a reliability of 86.6% (β =1.11). Similarly function $g_2(X)$ returns a reliability of 60.6% (β =0.27).

Table 11.2 GFOSM Solution for function $g_1(X)$

Iteration	Variable	Assumed x*	μ	σ	(dg/dX'i)	K(i)	K(i)(dg/dX'i)	α_i	α_{ii}	New x*
3	A1	1000000								
	X1	444.37	350	100	-225030.81	-385012.69	86639717173.72	-0.85	-0.50	444.26
	X2	6.79	5	2	-177747.65	-366090.43	65071711715.56	-0.81	-0.39	6.79
	X3	4.46	3	2	-177747.65	-298587.18	53073169193.25	-0.66	-0.39	4.46
	X4	32.58	25	15	-15.00	-206132.54	3091988.16	-0.46	0.00	32.57
	X5	4.61	4	2	0.00	-124423.35	0.00	-0.27	0.00	4.61
	X6	225.58	200	90	0.00	-116115.72	0.00	-0.26	0.00	225.58

Var = 2.05E+11
β = 1.1079
g(X) = 0.000

Table 11.3 GFOSM Solution for function $g_2(X)$

Iteration	Variable	Assumed x*	μ	σ	(dg/dX'i)	K(i)	K(i)(dg/dX'i)	a_i	a_{ii}	New x*
3	A1	500000								
	X1	370.02	350	100	-135126.08	-299879.02	40521477063.68	-0.75	-0.34	370.02
	X2	5.43	5	2	-184182.90	-321510.76	59216784704.39	-0.80	-0.46	5.43
	X3	3.17	3	2	0.00	-126731.41	0.00	-0.32	0.00	3.17
	X4	27.38	25	15	-30135.11	-237778.21	7165472351.57	-0.60	-0.08	27.38
	X5	4.27	4	2	-4018.01	-204595.86	822069123.19	-0.51	-0.01	4.27
	X6	217.22	200	90	-180810.65	-286596.12	51819632295.61	-0.72	-0.45	217.22

Var = 1.60E+11
β = 0.2667
g(X) = 0.000

In each of the iteration procedures for both functions given above, the independent component of the direction co-sine corresponding to each variable in the global set was also calculated. These values together with the product of the direction cosine and the independent component from the other function are tabulated below.

Table 11.4 Modal Correlation Coefficient compnents

	$g_1(X)$		$g_2(X)$			
	$\alpha_i(1)$	$\alpha_{ii}(1)$	$\alpha_i(2)$	$\alpha_{ii}(2)$	$\alpha_i(1)^*\alpha_{ii}(2)$	$\alpha_i(2)^*\alpha_{ii}(1)$
X_1	-0.85	-0.50	-0.75	-0.34	0.29	0.37
X_2	-0.81	-0.39	-0.80	-0.46	0.37	0.32
X_3	-0.66	-0.39	-0.32	0.00	0.00	0.12
X_4	-0.46	0.00	-0.60	-0.08	0.03	0.00
X_5	-0.27	0.00	-0.51	-0.01	0.00	0.00
X_6	-0.26	0.00	-0.72	-0.45	0.12	0.00
				$\rho_{12} = \rho_{21} =$	**0.81**	**0.81**

From Equation (11.10), the modal correlation between g_1 and g_2 is

$$\rho_{12} = \sum_{i=1}^{6} \alpha_i(g_1)\alpha_{ii}(g_2)$$

$$= (-0.85)(-0.34) + (-0.81)(-0.46) + (-0.66)(0.00) + (-0.46)(-0.08)$$

$$+ (-0.27)(-0.01) + (-0.26)(-0.45) = 0.81$$

From Table 11.4 it can be seen that the modal correlation ρ_{21}, also returns a value of 0.81.

$$\rho_{21} = \sum_{i=1}^{6} \alpha_i(g_2)\alpha_{ii}(g_1) = (-0.75)(-0.50) + (-0.80)(-0.39) + (-0.32)(-0.39) = 0.81$$

It can further be noted that the correlation ρ_{11} is, as would be expected, unity (the same is true for ρ_{22}) and given by

$$\rho_{11} = \sum_{i=1}^{6} \alpha_i(g_2)\alpha_{ii}(g_1) = (-0.85)(-0.50) + (-0.81)(-0.39) + (-0.66)(-0.39) = 1.00$$

This expression is equivalent to the normalising condition given in Equation (7.13) and the summation of stochastic importance given in Equation (9.22).

The reliability indexes for the two functions are respectively $\beta_1 = 1.11$ and $\beta_2 = 0.27$, with the modal correlation between g_1 and g_2, $\rho_{12} = 0.81$. With this known the uni-modal and bimodal bounds can be developed for the probability of system failure.

11.5.1 Uni-modal Performance Bounds

The uni-modal failure bounds for positive correlation are given in Equation (11.4).

$$\max_i P_{F_i} \le P_F \le 1 - \prod_{i=1}^{k} (1 - P_{F_i})$$

To execute this function the probabilities of failure for the two functions are required.

$$P_{F_i} = 1 - \Phi(\beta_1) = 1 - \Phi(1.11) = 0.134$$

$$P_{F_2} = 1 - \Phi(\beta_2) = 1 - \Phi(0.27) = 0.395$$

The uni-modal probability of failure inequality is therefore given by,

$$0.395 \le P_F \le 1 - (1 - 0.134)(1 - 0.395)$$

$$0.395 \le P_F \le 0.476$$

11.5.2 Bi-modal System Performance Bounds

The bimodal bounds calculated using DITLEVSENS' (1979) approximation of the joint density function are derived below as functions of the modal correlation,

$$P_{F_1} + \max\left[\sum_{i=2}^{z}\left\{P_{F_i} - \sum_{j=1}^{i=1} P\left(E_i E_j\right)\right\};0\right] \le P_F \le \sum_{i=1}^{k} P_{F_i} - \sum_{i=z}^{k} \max_{j<1} P\left(E_i E_j\right)$$

$$P(A) = \Phi(-\beta_1)\Phi\left(-\frac{\beta_2 - \rho_{12}\beta_1}{\sqrt{1-\rho_{12}^2}}\right) = \Phi(-1.11)\Phi\left(-\frac{(0.27)-(0.81)(1.11)}{\sqrt{1-(0.81)^2}}\right)$$

$$= \Phi(-1.11)\Phi(1.09) = (0.134)(0.863) = 0.116$$

$$P(B) = \Phi(-\beta_2)\Phi\left(-\frac{\beta_1 - \rho_{21}\beta_2}{\sqrt{1-\rho_{21}^2}}\right) = \Phi(-0.27)\Phi\left(-\frac{(1.11)-(0.81)(0.27)}{\sqrt{1-(0.81)^2}}\right)$$

$$= \Phi(-0.27)\Phi(-1.53) = (0.395)(0.063) = 0.025$$

From Equation (11.8) the inequality for the joint probability of failure is given by,

$$\max\left[P(A), P(B)\right] \le P(E_i E_j) \le P(A) + P(B)$$

$$0.063 \le P(E_i E_j) \le 0.111$$

Substituting these values into Equation (11.7) the performance bounds for the probability of system failure is given by,

$$0.395 + \max\{(0.134 - 0.116);0\} \le P_F \le 0.395 + 0.134 - 0.166$$

$$0.395 \le P_F \le 0.413$$

As would be expected, the bimodal bounds are narrower than the results obtained from the uni-modal method.

12. FURTHER SIMPLIFICATIONS FOR DETERMINISTIC RELIABILITY MODELLING

Partial safety factors are used extensively in the development of structural design codes (SABS 0100 – 1, 1992; SABS 0160, 1989; SABS 0162-1, 1993; SABS 0162-2, 1993). This approach is useful in that deterministic safety factors can be applied to capacity or demand variables in order to achieve a desired level of reliability without conducting reliability analysis during design.

However the partial safety factors used in the codes must be developed by first solving for the failure point corresponding to the desired level of reliability. A method is proposed which for relatively stable direction cosines develops the partial safety factor as a direct function of the reliability index. In this way deterministic reliability based design will be facilitated.

12.1 The partial safety factor

A partial safety factor is a deterministic value, which when applied to the mean value of the corresponding variable, raises a performance function to a target reliability. The failure point co-ordinate in the original variable space was defined in Equation (2.16) as,

$$x_i^* = \mu_{X_i} - \alpha_i \sigma_{X_i} \beta \tag{12.1}$$

Dividing Equation (12.1) through by μ_{X_i}, a partial safety factor is developed which relates the failure point co-ordinate to the mean value,

$$\gamma_i = \frac{x_i^*}{\mu_{X_i}} = 1 - \alpha_i \Omega_{X_i} \beta \tag{12.2}$$

Where Ω_{X_i} represents the coefficient of variation, $\Omega_{X_i} = \dfrac{\sigma_{X_i}}{\mu_{X_i}}$

12.2 Deterministic partial safety factors

The method for developing an approximate direction cosine was developed in section 8.3. The suitability of this approach requires that the performance function be of limited curvature.

Essentially it entails evaluating the direction cosine as a direct function of the mean and standard deviation values. Since these distribution parameters are known, alpha can be calculated directly without first solving the limit function. The approximate general dependent form of the direction cosine is given below as a direct function of the mean values, $x_i^* = \mu_i$.

$$\boxed{\alpha_{i(\beta)} \approx \alpha_{i(\beta=0)} = \frac{\sum\left(\frac{\partial g}{\partial X_j}\right)_\mu \rho_{ij}\sigma_{X_j}}{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_\mu\left(\frac{\partial g}{\partial X_j}\right)_\mu \rho_{ij}\sigma_{X_i}\sigma_{X_j}}} \tag{12.3}$$

Since the co-efficient of variance, $\Omega_{X_i} = \dfrac{\sigma_{X_i}}{\mu_{X_i}}$, and the direction cosine can be calculated directly as functions of the mean and covariance, the partial safety factor is a function of the reliability index.

$$\boxed{\gamma_i = f(\beta) = 1 - \alpha_{i(\beta=0)}\Omega_{X_i}\beta} \tag{12.4}$$

Therefore designing a performance function to a desired level of reliability is a simple process of calculating the partial safety factor and applying it to the mean value

$$g(X^*) = g(\beta) = 0 \tag{12.5}$$

12.3 Reflection on the contribution to the current theory

Design Code committees reduce the process of structural design to a deterministic procedure by developing partial safety factors for variables in structural performance functions, such as those describing bending moment and shear. This useful approach is limited however since it requires that predefined performance functions be evaluated for a given reliability index. The application to project risk management cannot afford this luxury, since performance functions are not generic and acceptable levels of reliability are subjectively set by individual project team.

The proposed approximate analytical form of the direction cosine for correlated variables facilitates the direct calculation of the partial safety factor as a function of the reliability index and distribution parameters. Individual variables can therefore be deterministically evaluated at desired levels of reliability, without having to solve the GFOSM method.

It must be noted however that the suitability of this method is dependent on the curvature of the performance function. Increasing levels of non-linearity will weaken the approximation. That said, most project risk management applications are concerned with operation levels of reliability, since the available information is seldom sufficient to accurately model the higher probabilities of success. The proposed deterministic method should give acceptable results for most project risk management performance functions at thesae operational levels of reliability.

12.4 Applications

The performance function utilised in the application sections of chapters 5, 9 and 10 is continued in the example. The only difference is that the function is to be designed to a level of reliability of 85% (*β = 1.0*). This implies that the function be solved for the required capacity (in this case the available budget) which would result in a reliability index of 1.0.

12.4.1 Deterministic reliability based design

Since the performance function is of limited curvature, the level of reliability or capcity of the system will not significantly affect the direction cosines. The required approximate direction cosines are therefore functions of the partial derivatives evaluated at the mean values. These values for the direction cosine were produced as part of the first iteration of the GFOSM method in section 5.3. These are

$\alpha_1 = -0.86$ $\alpha_2 = -0.35$ $\alpha_3 = -0.49$ $\alpha_4 = -0.91$

The deterministic partial safety corresponding to a reliability index of 1.0 are,

$$\gamma_1 = 1 + \alpha_i \Omega_i \beta = 1 + (-0.86)\left(100/_{350}\right)(1.0) = 1.25$$

$\gamma_2 = 1.14$ $\gamma_3 = 1.32$ $\gamma_4 = 1.55$

The failure point co-ordinates are extracted by multiplying the partial safety factor by the mean value,

$$x_1^* = \gamma_i \mu_i = (1.25)(350) = 436.10$$

$x_2^* = 5.70$ $x_3^* = 3.97$ $x_4^* = 38.70$

From these values the required capacity can be calculated,

$$Capacity = (436.10)(5.70 + 3.97 + 38.70) = 21096.67$$

12.4.2 Reliability based design by the GFOSM Method

The capacity for the function corresponding to a beta value of 1 was solved using the GFOSM method. The first and last iterations are given in Table 12.1.

Table 12.1 GFOSM Method Solution for function *g(X)*

Iteration	Variable	Assumed x*	μ	σ	(dg/dX'i)	K(i)	K(i)(dg/dX'i)	αi	New x*
1	A1	21096.67							
	X1	350.00	350.00	100.00	-3300.00	-7080.00	23364000.00	-0.86	436.10
	X2	5.00	5.00	2.00	-700.00	-2890.00	2023000.00	-0.35	5.70
	X3	3.00	3.00	2.00	-700.00	-4000.00	2800000.00	-0.49	3.97
	X4	25.00	25.00	15.00	-5250.00	-7510.00	39427500.00	-0.91	38.70
						Var =	6.76E+07		
						β =	1.0		
						g(X) =	0.0		

Iteration	Variable	Assumed x*	μ	σ	(dg/dX'i)	K(i)	K(i)(dg/dX'i)	αi	New x*
3	A1	21102.66							
	X1	437.70	350.00	100.00	-4821.23	-9548.42	46035134.56	-0.88	437.71
	X2	5.74	5.00	2.00	-875.41	-4030.77	3528553.99	-0.37	5.74
	X3	3.96	3.00	2.00	-875.41	-5210.61	4561398.60	-0.48	3.96
	X4	38.51	25.00	15.00	-6565.54	-9808.44	64397721.88	-0.90	38.51
						Var =	1.19E+08		
						β =	1.0		
						g(X) =	0.0		

The capacity required to return a reliability index of 1.0 is 21102.66, and the failure point co-ordinates

$x_1^* = 437.71$ $\quad x_2^* = 5.74$ $\quad x_3^* = 3.96$ $\quad x_4^* = 38.51$

From these results, the partial safety factors are calculated by dividing the failure point co-ordinate by the mane value.

$$\gamma_1 = {x_1^*}/{\mu_1} = {437.71}/{350} = 1.25$$

$\gamma_2 = 1.15$ $\quad \gamma_3 = 1.32$ $\quad \gamma_4 = 1.54$

12.4.3 Comparison of the Results

The approximate direction cosine, partial safety factors, capacity and failure point co-ordinates derived with the deterministic reliability based design a method proposed in the chapter and those derived from the GFOSM solution of the performance function are given below in table 12.2. As can be seen, the deterministic results provide a very good approximation of the full solution.

Table 12.2 Comparison of Deterministic and Full Solution Partial Safety Factors

	Deterministic Reliability Based Design			GFOSM		
	$\alpha_i = f(\mu)$	$\gamma_i = f(\beta)$	$x^* = \gamma_i \mu_i$	α_i	x^*	$\gamma_i = x^*/\mu$
Budget	-	-	21096.67	-	21102.66	-
X_1	-0.86	1.25	436.10	-0.88	437.71	1.25
X_2	-0.35	1.14	5.70	-0.37	5.74	1.15
X_3	-0.49	1.32	3.97	-0.48	3.96	1.32
X_4	-0.91	1.55	38.70	-0.90	38.51	1.54

13. APPLICATION TO CONSTRUCTION PROJECT RISK MANAGEMENT

An hypothetical application is presented to demonstrate the theories and methodologies developed in the dissertation. The application models a road construction project consisting of five activities; *site establishment, cut to fill, rip and compact, sub-base* and *base*. Performance functions representing a cost estimate and two schedule paths are developed. The input parameters were estimated from those utilised in a typical estimating and planning process, or simple extensions involving available data and experienced based intuitive estimation.

Three performance functions are presented as models for the cost and schedule. These will be simplified by way of the omission sensitivity factor and solved through application of the GFOSM method. Risk indicators including stochastic importance, contingency allocation, sensitivity, importance, elasticity and the correlation mechanisms are developed to demonstrate the risk behaviour within the functions. The elasticity index is established to indicate where risk management resources should be allocated. Results from the deterministic reliability based design method are compared to the full solution of the GFOSM method. Modal correlation between performance functions is calculated and used to establish system reliability. Finally the impact of a set of risk strategies on the risk profile is demonstrated.

13.1 Performance functions

The schedule paths containing the five activities are represented diagrammatically below and mathematical performance functions given on the following page.

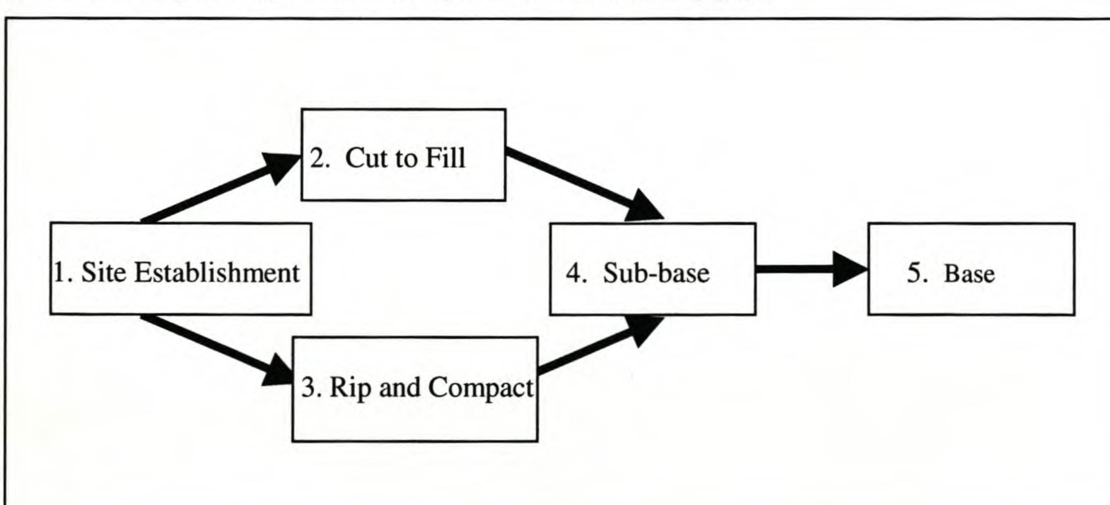


Figure 13.1 Construction Project Schedule Activities

The performance functions are utilised to represent the project in financial and duration dimensions. The function $g_1(X)$ below models the cost of the project and $g_2(X)$ and $g_3(X)$ represent the two schedule paths.

$$g_1(X) = Budget - \left\{ P\,\&\,G + \sum_{i=2}^{5} \left[Q_i (L_i + F\,\&\,E_i) / P_i + M_i (1 + S_i) \right] \right\}$$

$$g_2(X) = Duration - \{ P\,\&\,G + Q_2 / P_2 + Q_4 / P_4 + Q_5 / P_5 \}$$

$$g_2(X) = Duration - \{ P\,\&\,G + Q_3 / P_3 + Q_4 / P_4 + Q_5 / P_5 \}$$

The function variables are defined below

Budget	-	This is the amount tendered for the section of the project, including monetary contingency.
Duration	-	This is the planned duration for the section of the project, including time contingency.
$P\,\&\,G$	-	Preliminary and General (including Site Establishment).
Q_i	-	Quantity estimate [m^3]
P_i	-	Productivity [m^3/hr]
L_i	-	Labour rate [R/hr]
$F\,\&\,E_i$	-	Fuel and equipment rate [R/hr]
M_i	-	Material rate [R/m^3]
S	-	Material shrinkage due to waste, theft etc.

13.2 Distribution parameters

A construction company provided 11 historical records for similar projects from their cost estimating system. The data described the quantities and productivities corresponding to Cut to Fill, Rip and Compact, Sub-base and Base utilised in each of these historical projects. This data was used to generate the mean, standard deviation and correlation matrix for quantity and productivity. Based on their experience and the current project conditions, the quantity surveyors, cost estimators and planners refined the mean values and standard deviations, resulting in the input parameters given in table 13.1.

Table 13.1 Input Parameters

Activity	Variable	Mean	Std Dev	c.o.v.	Unit
Site Establishment	P&G(cost)	500000	25000	0.05	**R**
	P&G(time)	160	16	0.1	**hr**
Quantity	Q2	6000	600	0.1	**m^3**
	Q3	2500	1000	0.4	**m^3**
	Q4	3500	525	0.15	**m^3**
	Q5	2000	600	0.3	**m^3**
Productivity	P2	20	4	0.2	**m^3/hr**
	P3	30	4.5	0.15	**m^3/hr**
	P4	50	5	0.1	**m^3/hr**
	P5	15	4.5	0.3	**m^3/hr**
Material Shrinkage	S	10%	5%	0.5	**%**
Labour rate	L2	20	0	0	**R/hr**
	L3	25	0	0	**R/hr**
	L4	30	0	0	**R/hr**
	L5	80	0	0	**R/hr**
Fuel & Equip rate	F&E2	60	0	0	**R/hr**
	F&E3	70	0	0	**R/hr**
	F&E4	50	0	0	**R/hr**
	F&E5	80	0	0	**R/hr**
Material rate	M4	30	0	0	**R/m^3**
	M5	50	0	0	**R/m^3**

The correlation coefficients for the quantities and productivities, measured from the objective data sources are given in Tables 13.2 (a) and (b) respectively.

Table 13.2 (a) and (b) Correlation matrices for quantities and productivities

	Q2	**Q3**	**Q4**	**Q5**
Q2	1	-0.2	0.6	0.5
Q3	-0.2	1	0.7	0.2
Q4	0.6	0.7	1	0.5
Q5	0.5	0.2	0.5	1

	P2	**P3**	**P4**	**P5**
P2	1	0.4	0.3	0.5
P3	0.4	1	0.2	0.6
P4	0.3	0.2	1	0.1
P5	0.5	0.6	0.1	1

13.3 Omission Sensitivity

The omission sensitivity factors are calculated to assist in simplifying the functions. Equation (10.25) was used to establish whether ignoring the variability of the variable would have a marked effect on the reliability index. Since the omission sensitivities are calculated as functions of the mean values, it is not necessary to first solve for the failure point co-ordinates. The results of the omission sensitivities are tabulated below.

Table 13.3 Omission sensitivity factors for the cost and schedule functions

	$g_1(X)$	$g_2(X)$	$g_3(X)$
P&G(cost)	1.09	1.00	1.00
P&G(time)	1.00	1.01	1.00
Q2	1.01	1.06	1.00
Q3	1.01	1.00	1.04
Q4	1.12	1.02	1.01
Q5	1.45	1.10	1.11
P2	1.01	1.28	1.00
P3	1.00	1.00	1.02
P4	1.00	1.01	1.00
P5	1.01	1.16	1.43
S	1.02	1.00	1.00

The significant omission sensitivities ($\psi_i \geq 1.05$) in Table 13.3, have been highlighted. Since the threshold for significance is subjective, three different omission sensitivity limits were tested

- $\psi_i \leq 1.01$ - This excludes *Site Establishment* duration *[P&G (Time)]* and *Sub-base* productivity *(P4)*.
- $\psi_i \leq 1.02$ - In addition to the above two variables, *Rip and Compact* productivity and , *material shrinkage* are treated as non-critical.
- $\psi_i \leq 1.05$ - the quantity of work associated with *Rip and Compact* is added to the list of non-critical items.

The risk manager can revisit the remaining variables, particularly those with high omission sensitivities, ensuring that the parameters reasonably reflect uncertainty. It is important to note that this does not mean that the mean values represented are not important, only that the variability does not significantly affect the reliability. Considering each function independently would increase the number of non-critical variables can could be omitted.

The probabilities of success for each of the three functions were calculated for the case of a full complement of variables, and the simplified functions with three non-critical variables excluded. A table comparing the full and simplified functions is given below. The probabilities of success derived from the simplified functions were found to correspond closely to the full function reliability.

Table 13.4 Comparison of results for the full and simplified functions

	Full Function	Simplified Function		
	Omit = 1.00	Omit <= 1.01	Omit <= 1.02	Omit <= 1.05
$P(g_1(X)>0)$	0.856	0.856	0.861	0.866
$P(g_2(X)>0)$	0.752	0.756	0.756	0.756
$P(g_3(X)>0)$	0.982	0.983	0.985	0.986

From Table 13.4 it can be seen that ignoring the variability of variables with omission sensitivities less than or equal to 1.05 gives a very good approximation of the full solution of the three performance functions. The $\psi_i \leq 1.05$ omission sensitivity will be utilised as the threshold for this application.

13.4 GFOSM Method Solution

The three simplified performance functions were solved with the GFOSM algorithm described in section 5.1. The spreadsheet model for the method, together with the additional calculations utilised in the application are given in Appendix II. The failure point co-ordinates for each of the three performance functions together with the resultant reliability indices and probabilities of success are tabulated below. The capacities relate to the total cost estimate and project duration, including contingencies.

Table 13.5 Failure point co-ordinates

	Mean Values	Cost Estimate	Schedule Path 1	Schedule Path 2
Variable	μ	$g_1(X)$	$g_2(X)$	$g_3(X)$
Capacity		R850,000	750 days	750 days
β		1.11	0.69	2.20
P(g(X)>0)		0.87	0.76	0.99
P&G(cost)	500000	511518.91	500000.00	500000.00
P&G(time)	160	160.00	160.00	160.00
Q2	6000	6368.49	6185.45	6216.55
Q3	2500	2500.00	2500.00	2500.00
Q4	3500	3883.08	3638.63	3702.70
Q5	2000	2555.01	2201.83	2406.68
P2	20	19.25	17.90	15.81
P3	30	30.00	30.00	30.00
P4	50	50.00	50.00	50.00
P5	15	13.99	12.83	5.56
S	0.1	0.10	0.10	0.10

13.5 Stochastic Importance

The stochastic importance is calculated as the first risk indicator from Equation (9.22). This is similar to the omission sensitivity, since it is a function of the product of the direction cosine and the independent component of the direction cosine. However instead of indicating non-critical variables, it indicates which are dominant contributors to the total performance function uncertainty.

The pie chart given in figure 13.2 below is a graphic representation of stochastic importance indicating the proportional contribution to total variance of the variables in the cost estimate performance function. From the graph it is clear that *site establishment* and the *quantity estimates* for *sub-base* and *base* are the dominant contributors to uncertainty in the *cost function. Cut to fill quantity, Cut to fill productivity* and *Base quantity* and *Base productivity* make significant contributions to *schedule path 1*, while *Base productivity* and *Base quantity* dominate *schedule path 2* variability. *Base quantity* and *Base productivity* seem to be critical contributors to variance for all three functions, and should prove to be significant risk areas.

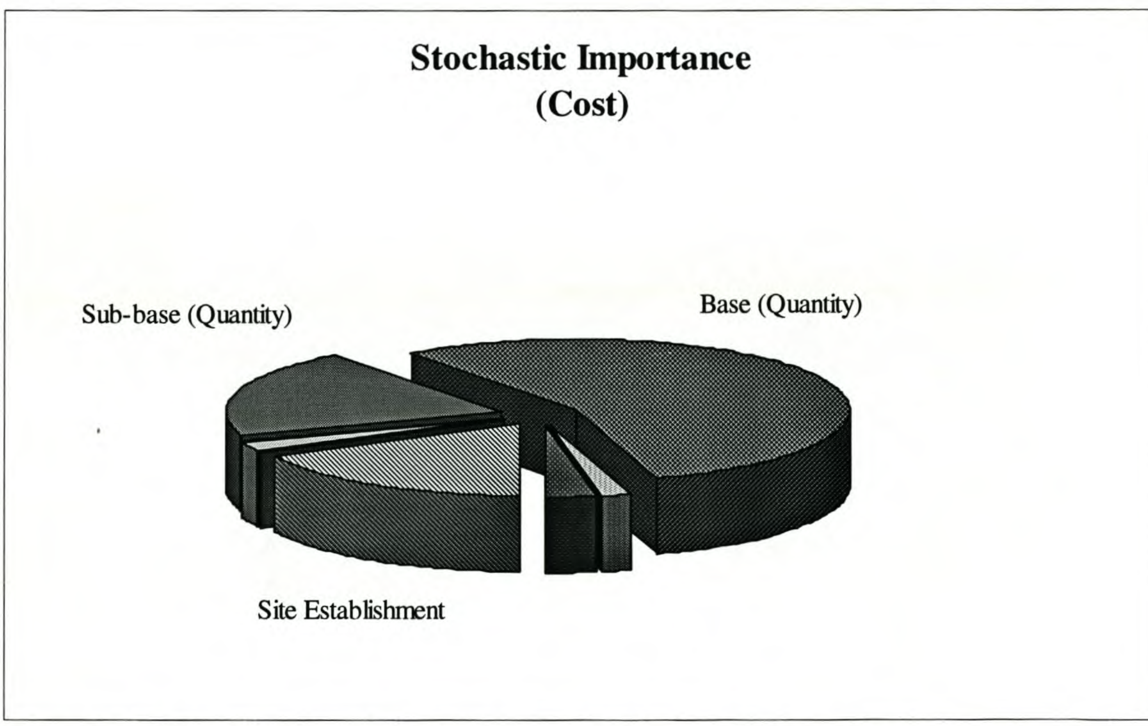


Figure 13.2 Proportional contribution to variance of cost estimate variables

13.6 Likely Contingency Demand

The difference between the allocated budget and estimated cost provides a contingency for the cost performance function. Similarly for the two schedule functions, the difference between the project duration and path length is a time contingency. This contingency is allocated to each of the activities by subtracting the expected value for each activity (as a function of means) from the limit state value (as a function of the failure point co-ordinates). This likely demand for the available contingency gives a proportional expression for risk in the dimension of the function (in this case Rands and days). In other words it represents each activity's probabilistic demand for the available contingency.

The contingency demand for schedule path 1 is given in Figure 13.3 below. The solid bar represents the best estimate and the light section the additional time required to reach the performance function reliability. From figure 13.3 it can be seen that *Cut to Fill* and *base* require the largest potions of the available time contingency. This indicates that these two activities present the greatest risk for schedule path 1. Similarly Schedule path 2 demonstrates *Rip and Compact* and *Base* as the two significant risk activities. The cost function derives its risk from *Cut to fill* and *Base*. It is interesting to note that *Base* is not the greatest contributor to the total cost, but requires the majority of the available contingency to achieve the level of reliability.

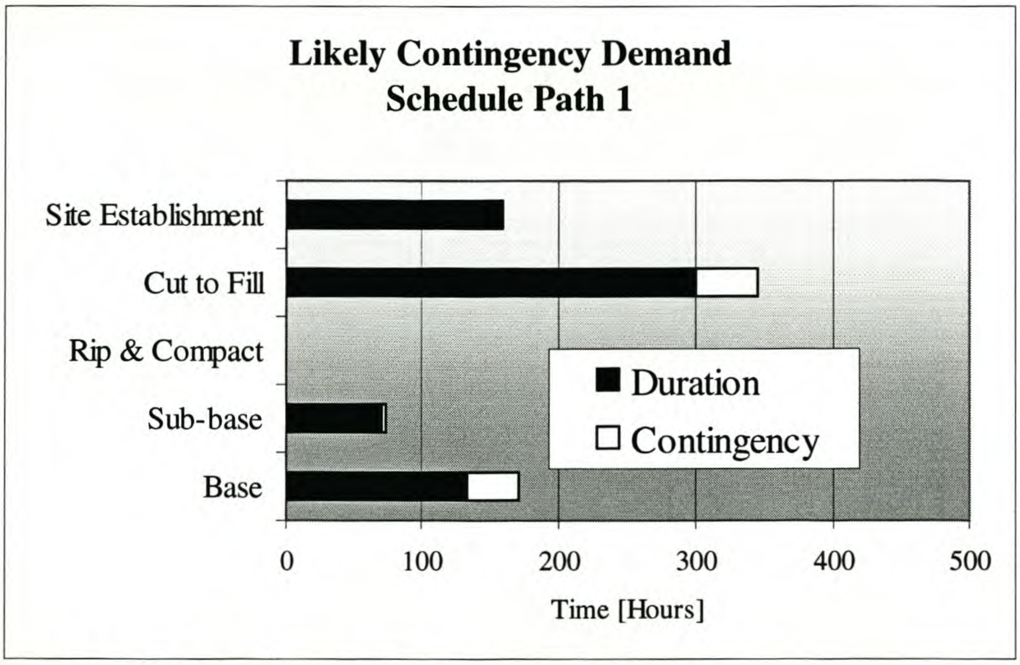


Figure 13.3 Likely Contingency Demand for Schedule Path 1

13.7 Sensitivity

The sensitivity gives the gradient or rate of change of the performance function with respect to each variable in the original variable space. The quantity is modelled from Equation (9.28) and describes the full derivative of the performance function with respect to the variable under observation, taking into account the interrelatedness between variables.

From figure 13.4 on the following page it is clear that both schedule paths have a high sensitivity to the *cut to fill productivities* (particularly Schedule path 2), with a negligible sensitivity to the other quantities. The cost function is sensitive to *cut to fill* and *base productivities*.

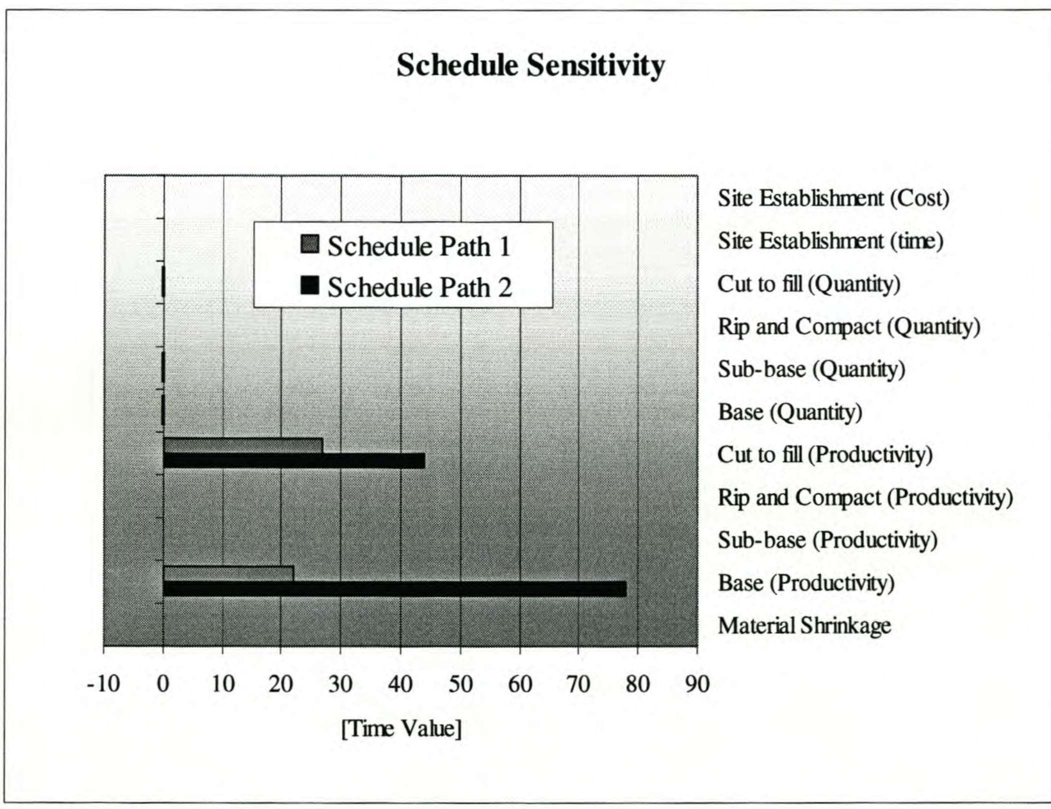


Figure 13.4 Sensitivity for the two schedule paths

13.8 Importance Factor, Elasticity and Correlation Mechanisms

The sensitivity of the reliability index to the location of the standard normal failure point co-ordinate was shown in Equation (9.20) to be equal to the negative value of the direction cosine. Furthermore disaggregating the direction into its constituent correlation components describes the correlation mechanisms in the performance function. The elasticity develops the concept of correlation mechanisms further by quantifying the cumulative effect of a variable on other variables.

Figure 13.5 on the following page gives the importance factors for the cost estimate. It is clear that *sub-base (Q4)* and *base quantities(Q5)*, *base productivity (P5)* and cost associated with *site establishment (P&G Cost)* are significant in terms of the sensitivity of the reliability index to the location of their failure point co-ordinates. Furthermore from the correlation mechanisms, the significance of *cut to fill (Q2)* and *rip and compact quantities (Q3)* is almost entirely as a result of the effects of correlation from *sub-base (Q4)* and *base quantities (Q5)*. Similarly *productivity* associated with *base (P5)* is a significant contributor to the importance of these variables. The cumulative value of the correlation components is the elasticity. From Figure 13.5, *base quantity (Q5)* and *productivity (P5)* and *sub base quantity (Q4)* have high elasticity's.

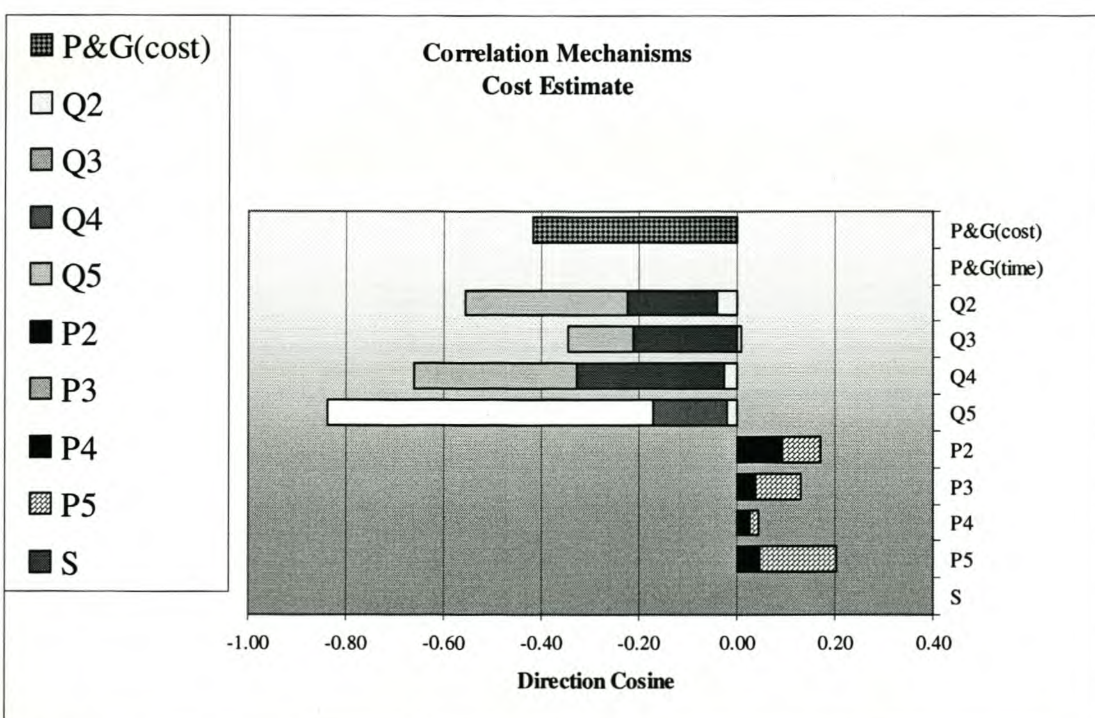


Figure 13.5 Cost Estimate Importance factors demonstrating Correlation Mechanisms

13.9 Elasticity Index

The elasticity index is the ratio of the elasticity to the direction cosine. A value greater than one would be considered a high elasticity index. From Figure 13.6 given on the following page, it can be seen that *base quantity* and *productivity* present elasticities indices significantly greater than one for all three functions. Focusing attention on reducing the uncertainty or impact of these variables would be an efficient utilization of management resources. *Sub base quantity* and *cut to fill productivity* also give good elasticity indices for the cost function and schedule path 1 respectively.

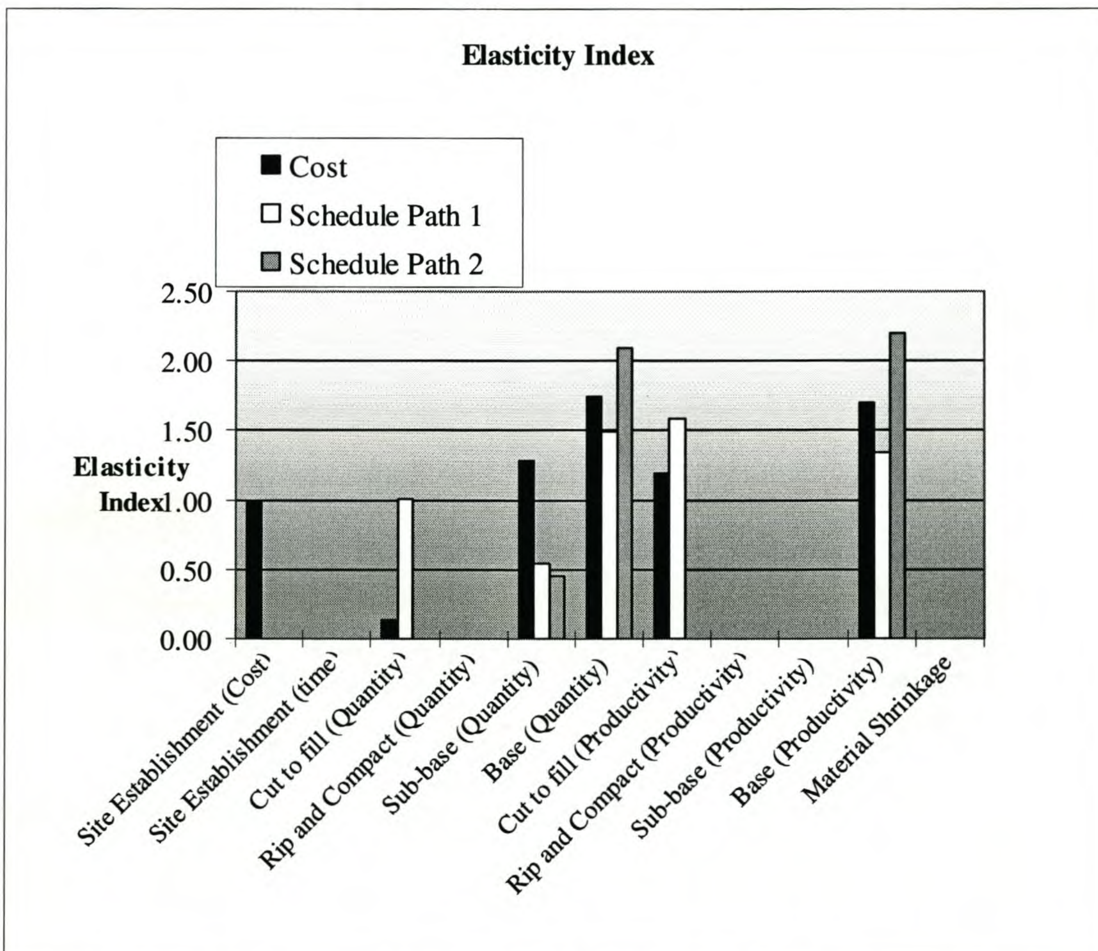


Figure 13.6 Elasticity Indices for Cost and schedule path 1 and 2 performance function

13.10 Deterministic Reliability Based Design

It is possible to generate cumulative probability curves for limit functions utilising the GFOSM method. The capacity is varied incrementally and the limit function solved for the corresponding level of reliability. The resultant reliability indices and capacities are graphed, resulting in typical cumulative distribution function given for schedule path 1 and 2 in figure 13.7 on the following page. The narrower duration range in schedule path 2 indicates that it is more certain. This function also has a lower duration than schedule path 2 over the majority of the reliability range. This path therefore has a very limited likelihood of becoming critical.

In addition to establishing function reliability, a performance function can also be designed to a desired level of reliability. This forms the basis for reliability-based design. The required beta value is set, and the limit function solved for the required capacity (budget or duration) and failure point co-ordinates.

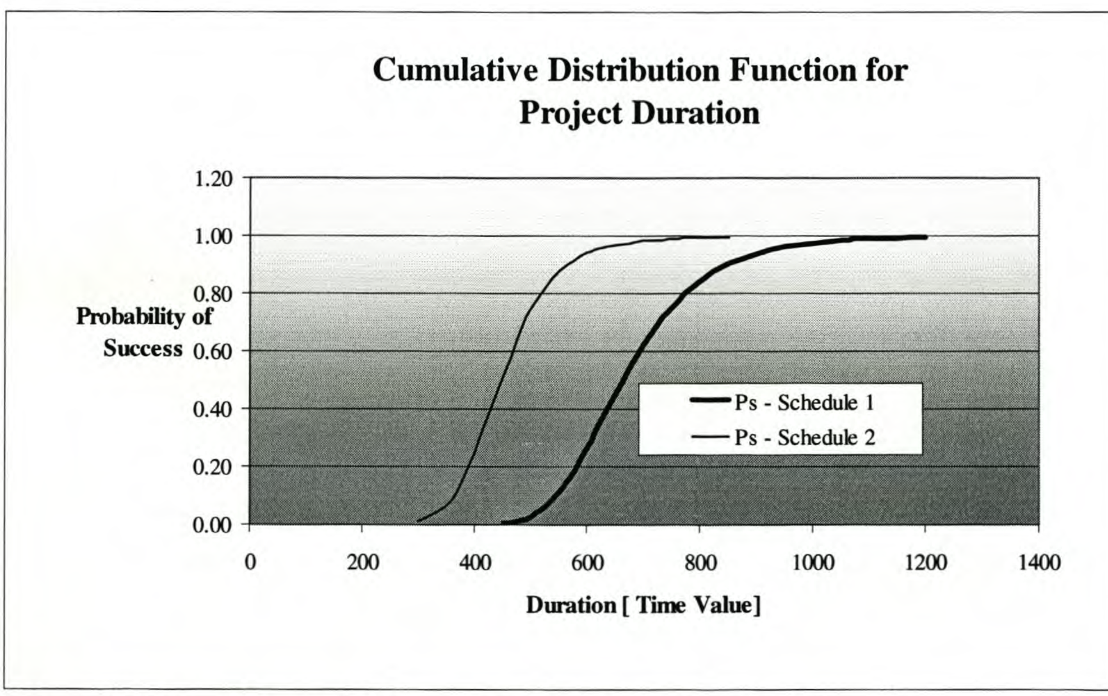


Figure 13.7 Cumulative probability curves for schedule paths 1 and 2

Partial safety factors are utilized in structural reliability based design to raise design variables contained within pre-defined performance functions to some desired level of reliability. These partial safety factors can be developed from the solution of the GFOSM method, by expressing the failure point coordinate as a proportion of the mean value, $\gamma_i = {x_i^{'*}}/{\mu_i}$.

Equation (12.2) is utilized in the deterministic approach to reliability-based design whereby the partial safety factor is solved directly as a function of the desired level of reliability, $\gamma_i = f(\beta)$. First the direction cosine is approximated as a function of the mean values. This corresponds to alpha at a beta value of 0. A comparison of the partial safety factors calculated using the GFOSM method as well as the deterministic approximation are tabulated below.

From table 13.6 it can be seen that the partial safety factors generated using the deterministic reliability based design method all correspond very well to the full solution of the GFOSM method.

Table 13.6 Comparison of deterministic and GFOSM solution partial safety factors

Variable	$g_1(X)$		$g_2(X)$		$g_3(X)$	
	$\gamma_i=f(\beta)$	$\gamma_i=x^*/\mu$	$\gamma_i=f(\beta)$	$\gamma_i=x^*/\mu$	$\gamma_i=f(\beta)$	$\gamma_i=x^*/\mu$
β	1.00	1.00	1.00	1.00	1.00	1.00
P(g(X)>0)	0.84	0.84	0.84	0.84	0.84	0.84
Capacity	R 844,851	R 846,174	796 days	796 days	536 days	543 days
P&G(cost)	1.02	1.02	1.00	1.00	1.00	1.00
P&G(time)	1.00	1.00	1.00	1.00	1.00	1.00
Q2	1.05	1.05	1.05	1.04	1.02	1.02
Q3	1.15	1.15	1.03	1.03	1.24	1.18
Q4	1.10	1.10	1.07	1.05	1.10	1.08
Q5	1.25	1.24	1.16	1.14	1.19	1.16
P2	0.97	0.97	0.86	0.85	0.94	0.92
P3	0.98	0.98	0.93	0.93	0.93	0.92
P4	1.00	1.00	0.98	1.00	0.99	1.00
P5	0.95	0.94	0.81	0.78	0.82	0.78
S	1.08	1.10	1.00	1.00	1.00	1.00

13.11 System Reliability

The previous sections in this application chapter have investigated individual function reliability and their underlying risk drivers. Analysis of the joint behaviour of the three performance functions as part of a system would provide insight into potential failure modes.

Of particular importance in establishing system reliability is the interrelatedness between the performance functions or modal correlation. Uni-modal reliability bounds are developed as a simple attempt to account for correlation, by first assuming independence and then perfect dependence between the failure modes. The resultant uni-modal bounds for the three performance functions are given below

$$0.244 \leq P_F \leq 0.354$$

Narrower failure bounds are developed by establishing the joint probabilities of failure between pairs of failure modes. This requires knowledge of the modal correlation between the performance function pairs. This modal correlation is calculated as a function of direction cosines and independent components of alpha, given in Equation (11.20). The resultant matrix of modal correlation coefficients is given in table 13.7 on below.

Table 13.7 Modal Correlation matrix for the three performance functions

	$g_1(X)$	$g_2(X)$	$g_3(X)$
$g_1(X)$	1	0.63	0.46
$g_2(X)$	0.63	1	0.82
$g_3(X)$	0.46	0.82	1

Since the direction cosines can be disaggregated into their correlation components, the modal correlation can similarly be disaggregated into their components. The make-up of the 0.82 correlation coefficient between performance functions $g_2(X)$ and $g_3(X)$ can be seen in Figure 13.8 below. From this graph it is clear that *base quantity* and *base productivity* are the main contributors to the modal correlation between these two functions. These two variables also feature strongly in the other two pairs of modal correlation coefficients.

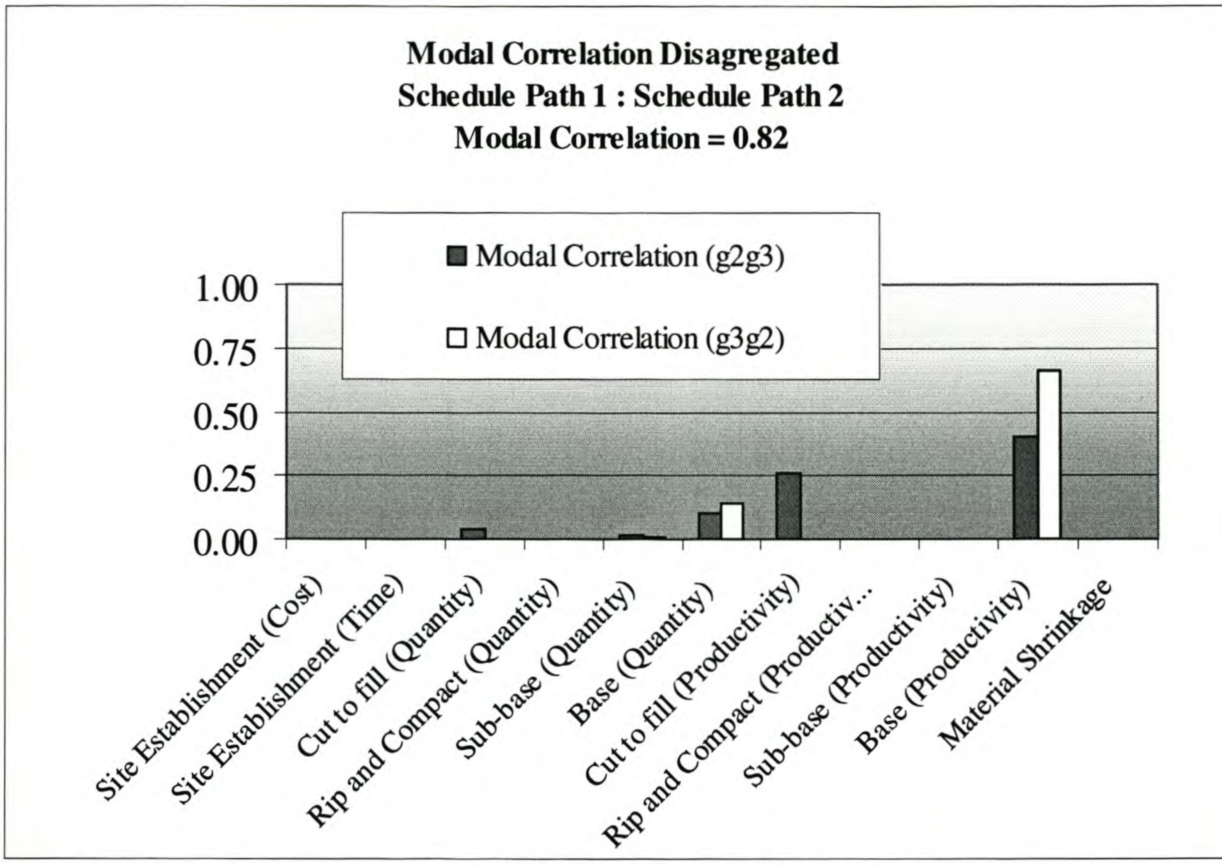


Figure 13.8 Modal Correlation between schedule path 1 and schedule path 2

The resultant bi-modal failure bounds of the given system of failure modes are

$0.263 \leq P_F \leq 0.305$

13.12 Risk Control

A number of publications have presented procedures for managing project risk (RAMP, 1999; PRAM, 1997; THOMPSON AND PERRY, 1992; STANDARDS AUSTRALIA, 1999). Essentially the project risk management steps address the identification, analysis and control of risk. The analytical tools presented in the dissertation and demonstrated in this chapter are applied as part of the second category, exploring the extent and sources of risk and uncertainty in a project. To illustrate the manner in which these results could be utilised in the management of project risk, two hypothetical risk control strategies are presented.

The analysis has demonstrated that *base quantity* and *base productivity* are critical contributors to risk in the project. As a result the project manager implements the following risk control strategies

- Additional geotechnical tests are conducted to improve the confidence in the amount and quality of the available material to be utilised in the base course. Based on the results of the additional testing, the quantity surveyor reduces the standard deviation for base quantity from a coefficient of variance (c.o.v.) of 0.3 to 0.15.
- The project manager decided to implement a strategy of task linked labour rates for the base course. In his experience, this has the effect of making the production rate more consistent. It also reduces the effects of correlation with *cut to fill* and *rip and compact*, since these activities are largely machine based. The new c.o.v. for *base productivity* is reduced from 0.3 to 0.15. The correlations between *cut to fill, rip and compact* and *base productivities* are also reduced from 0.5 and 0.6 respectively to 0.2 each.

The GFOSM method was repeated with this improved risk profile. The results are tabulated below,

Table 13.8 Comparison of improved reliability

		β	P_s
$g_1(X)$	**Original**	1.11	0.87
	Controlled	1.52	0.94
$g_2(X)$	**Original**	0.69	0.76
	Controlled	0.91	0.82
$g_3(X)$	**Original**	2.20	0.99
	Controlled	4.38	1.00

The control strategy has therefore improved the probability of success for the individual performance functions. Similarly the risk control strategy has had a significantly positive effect on system reliability, improving the original bi-modal reliability bounds from $0.263 \le P_F \le 0.305$ to $0.198 \le P_F \le 0.218$.

13.13 Application Spreadsheet

A complete spreadsheet model for the application described in Chapter 13 is given on the following page. This spreadsheet is the source for the graphs and simplified function results utilised in the chapter and indicates the sequential steps involved in the solution.

	F&E3	70	0	0	R/hr
	F&E4	50	0	0	R/hr
	F&E5	80	0	0	R/hr
Material rate	M4	30	0	0	R/m³
	M5	50	0	0	R/m³

P&G(cost)	1	0	0	0	0	0	0	0	0	0	0
P&G(time)	0	1	0	0	0	0	0	0	0	0	0
Q2	0	0	1	-0.2	0.6	0.5	0	0	0	0	0
Q3	0	0	-0.2	1	0.7	0.2	0	0	0	0	0
Q4	0	0	0.6	0.7	1	0.5	0	0	0	0	0
Q5	0	0	0.5	0.2	0.5	1	0	0	0	0	0
P2	0	0	0	0	0	0	1	0.4	0.3	0.5	0
P3	0	0	0	0	0	0	0.4	1	0.2	0.6	0
P4	0	0	0	0	0	0	0.3	0.2	1	0.1	0
P5	0	0	0	0	0	0	0.5	0.6	0.1	1	0
S	0	0	0	0	0	0	0	0	0	0	1

P2	15.0	0	175	0	0.48	19
P3	30.0	0	210	0	0.57	30
P4	50.0	0	35	0	0.10	50
P5	5.0	349.9384119	350	122457	0.95	6
S	10%	0	0	0	0.00	0

Var (X) = 135331.81

β = 2.205
g(X) = 0.00
Duration = 750

0.14	0.00
0.54	0.00
0.27	0.00
0.67	0.95
0.00	0.00

1.00	-0.48
1.00	-0.57
1.00	-0.10
1.66	-0.95
1.00	0.00

Correlation Mechanisms

Cost

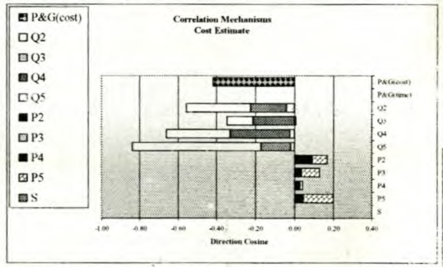
	P&G(cost)	P&G(time)	Q2	Q3	Q4	Q5	P2	P3	P4	P5	S	$\alpha_i^2=\Sigma$
P&G(cost)	-0.42	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.42
P&G(time)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Q2	0.00	0.00	-0.04	0.01	-0.02	-0.02	0.00	0.00	0.00	0.00	0.00	-0.08
Q3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Q4	0.00	0.00	-0.18	-0.21	-0.30	-0.15	0.00	0.00	0.00	0.00	0.00	-0.85
Q5	0.00	0.00	-0.33	-0.13	-0.33	-0.66	0.00	0.00	0.00	0.00	0.00	-1.46
P2	0.00	0.01	0.00	0.00	0.00	0.00	0.09	0.04	0.03	0.05	0.00	0.22
P3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.06
P4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
P5	0.00	0.00	0.00	0.00	0.00	0.00	0.08	0.09	0.02	0.16	0.00	0.34
S	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\alpha_i=\Sigma$	-0.42	0.00	-0.56	-0.34	-0.66	-0.84	0.17	0.13	0.04	0.20	0.00	

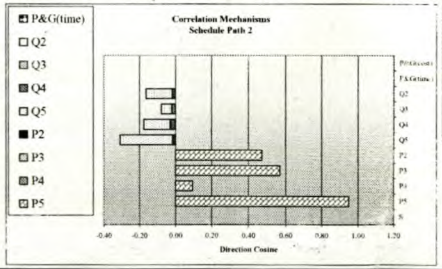
Schedule Path 1

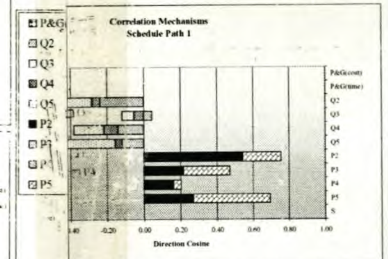
	P&G(cost)	P&G(time)	Q2	Q3	Q4	Q5	P2	P3	P4	P5	S	$\alpha_i^2=\Sigma$
P&G(cost)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
P&G(time)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Q2	0.00	0.00	-0.24	0.05	-0.14	-0.12	0.00	0.00	0.00	0.00	0.00	-0.45
Q3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Q4	0.00	0.00	-0.04	-0.05	-0.07	-0.04	0.00	0.00	0.00	0.00	0.00	-0.21
Q5	0.00	0.00	-0.16	-0.07	-0.16	-0.33	0.00	0.00	0.00	0.00	0.00	-0.73
P2	0.00	0.00	0.00	0.00	0.00	0.00	0.54	0.22	0.16	0.27	0.00	1.20
P3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
P4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
P5	0.00	0.00	0.00	0.00	0.00	0.00	0.21	0.25	0.04	0.42	0.00	0.93
S	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\alpha_i=\Sigma$	0.00	0.00	-0.45	-0.07	-0.38	-0.48	0.76	0.47	0.21	0.70	0.00	

Schedule Path 2

	P&G(cost)	P&G(time)	Q2	Q3	Q4	Q5	P2	P3	P4	P5	S	$\alpha_i^2=\Sigma$
P&G(cost)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
P&G(time)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Q2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Q3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Q4	0.00	0.00	-0.02	-0.02	-0.03	-0.01	0.00	0.00	0.00	0.00	0.00	-0.08
Q5	0.00	0.00	-0.15	-0.06	-0.15	-0.29	0.00	0.00	0.00	0.00	0.00	-0.64
P2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
P3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
P4	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
P5	0.00	0.00	0.00	0.00	0.00	0.00	0.48	0.57	0.10	0.95	0.00	2.09
S	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\alpha_i=\Sigma$	0.00	0.00	-0.16	-0.08	-0.18	-0.31	0.48	0.57	0.10	0.95	0.00	







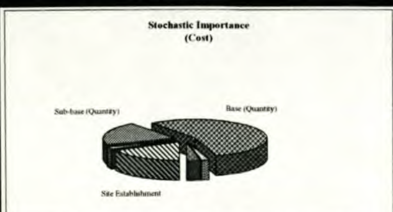
Likely Contingency Demand

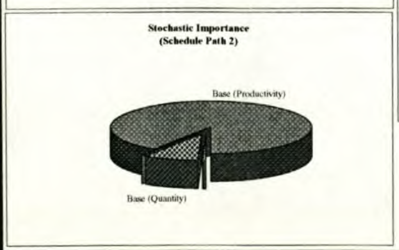
Cost	Cost	Contingency
Site Establishment	500000.00	11519
Cut to Fill	24000.00	2469
Rip & Compact	7916.67	0
Sub-base	121100.00	13255
Base	131333.33	38407
Σ	784350.00	65650.00
Capacity	850000	

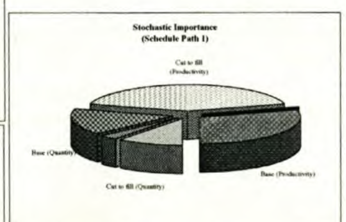
Schedule Path 1	Duration	Contingency
Site Establishment	160.00	0.00
Cut to Fill	300.00	45.54
Rip & Compact		
Sub-base	70.00	2.77
Base	133.33	38.35
Σ	663.33	86.67
Capacity	750	

Schedule Path 2	Duration	Contingency
Site Establishment	160.00	0.00
Cut to Fill		
Rip & Compact	83.33	0.00
Sub-base	70.00	4.05
Base	133.33	299.28
Σ	446.67	303.33
Capacity	750	

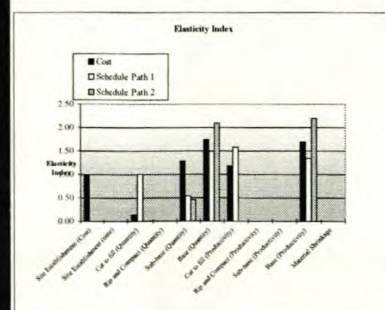
Stochastic Importance



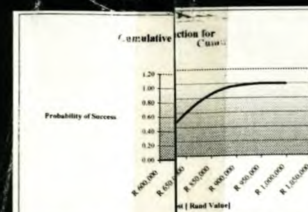


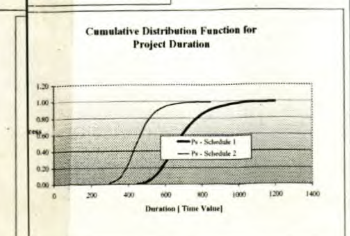


Elasticity Index



Cumulative Distributions





Modal Correlation Matrix	g1	g2	g3
g1	1.00	0.63	0.46
g2	0.63	1.00	0.82
g3	0.46	0.82	1.00

	g1	g2	g3
β	1.106	0.694	2.205
P_f	0.134	0.244	0.014

Unimodal Bounds

$0.244 < P_F < 0.354$

Bi Modal Bounds

	g1 g2	g1 g3	g2 g3
ρ_{ij}	0.63	0.46	0.82
β_i	1.106	1.106	0.694
β_j	0.694	2.205	2.205
P_{fi}	0.134	0.134	0.244
P_{fj}	0.244	0.014	0.014
a	0.504	0.028	0.002
b	0.195	0.456	0.972
P(A)	0.068	0.004	0.001
P(B)	0.047	0.006	0.013
$<=P_{fifj}$	0.068	0.006	0.013
$P_{fifj}<=$	0.115	0.010	0.014

$0.263 < P_F < 0.305$

14. CONCLUSIONS

Quantitative analysis of risk and reliability is often considered to be the most formal aspect of risk analysis. The First Order Second Moment reliability method, developed initially for structural reliability based design, provides a potentially rich toolkit for project risk managers. This approximate method significantly simplifies the modelling process by evaluating the most probable conditions at the limit of acceptable performance, while maintaining some of the more sophisticated aspects of statistical analysis such as the treatment of non-linear multi-variate functions with non-normally distributed dependent variables. The dissertation addresses the treatment of correlated basic variables in the original hyperspace.

14.1 The importance of Correlation for Project Risk Management

The effect of correlation on the limit state function is investigated. Increasing numbers of variables and levels of correlation are found to significantly impact the outcome of reliability analysis.

Performance functions utilised in structural design generally contain a limited number of variables. The well-established methods for transforming dependent variables have been sufficient for this purpose, with lower levels of correlation discarded as negligible. However project risk functions, such as those for cost estimating and scheduling, may contain many hundreds of variables. Consequently project performance is vulnerable to systematic variations. Single containable failure events can through common sensitivities and interrelationships, cascade into a series of events significantly effecting system performance. As a result it is critical when modelling project risk to comprehensively address the effects of correlation. Such a rational process develops insight into the correlation mechanisms driving risk and reliability, providing an invaluable decision support tool.

14.2 General Analytical Form of the Direction Cosine for Correlated Basic Variables

14.2.1 Proposed formulation

A unit vector is taken in the direction opposite to the reliability index. The component of the unit vector in the dimension of X_i, is referred to as the direction cosine (denoted alpha, α_i). This quantity is used to extract independent component information at the most likely point of failure on the limit state surface. A Masters thesis submitted by the candidate (Ker-Fox, 1998) proposed a general form of the direction cosine for dependent limit state functions, given below

$$\alpha_i = \frac{\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)\rho_{ij}\sigma_{X_j}}{\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_j}\right)\left(\frac{\partial g}{\partial X_i}\right)\rho_{ij}\sigma_{X_j}\sigma_{X_i}}}$$

14.2.2 Formal derivation and reconciliation of the general direction cosine

Components of the standard normal failure point co-ordinate are by definition functions of the direction cosine and reliability index, $x_i^{'*} = -\alpha_i\beta$. As a point of departure for the theoretical derivation of the general form for the direction cosine, this standard expression is reformulated to represent an independent variable's proportional contribution to total variance. From the known general form for the variance of functions containing dependent variables, a general expression for the proportional contribution to variance with correlated basic variables is developed. The resultant equality is returned to its original form, producing the proposed general dependent form of the direction cosine.

The FOSM method utilising the general form of the direction cosine reduces to the expression developed by SEIFI, PONAMBULAM AND VLACH (1999) for extracting failure point co-ordinates for limit functions containing correlated basic variables in the original variable space. This provides an independent theoretical validation for the proposed general form of the direction cosine.

14.2.3 Numerical validation of the general form of the direction cosine

The proposed formulation for alpha is validated numerically by comparing results to those obtained through the orthogonal transformation of dependent variables into an independent hyperspace. Thirty-six different cases for a typical Bill of Quantities type cost performance function are investigated. The conditions describe an increasing number of variables, increasing levels of correlation, increasing curvature of the performance function and different distribution types across a range of reliabilities. Selected cases are solved independently utilising the Rosenblatt transformation method. In addition to the required budget solution, the resultant failure point co-ordinates and equivalent direction cosines are compared.

The direct and orthogonal methods are found to produce identical results for all 36 cases. Slight discrepancies are noted for non-normal distributions when compared with results from the Rosenblatt transformation method. This is however consistent with the limitations of developing normal equivalent parameters to represent non-normal distributions.

One can conclude that the proposed general form of the direction cosine considerably simplifies the treatment of dependent performance functions, while producing results consistent with the orthogonal transformation procedure.

14.3 Mathematical Insight developed from the General Direction Cosine

14.3.1 Components of the direction cosine

The general form of the direction cosine is composed of an independent and a dependent set of components, denoted α_{ii} and $\widehat{\alpha}_i$ respectively,

$$\alpha_i = \alpha_{ii} + \widehat{\alpha}_i$$

The independent component describes the contribution to the direction cosine by the variable under observation and is defined as

$$\alpha_{ii} = \frac{\left(\frac{\partial g}{\partial X_i}\right)_* \sigma_{X_i}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial g}{\partial X_i}\right)_*\left(\frac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}}$$

This expression provides for alpha to be described in terms of correlation coefficients and the corresponding independent components of all variables contained in the performance function,

$$\alpha_i = \sum_{j=1}^{n} \alpha_{jj}\rho_{ij}$$

The dependent component describing the cumulative effect of correlation on the direction cosine, is represented as

$$\hat{\alpha}_i = \alpha_i - \alpha_{ii} = \sum_{j \neq i}^{n} \alpha_{jj}\rho_{ij}$$

In the expression given above, the correlation component of the direction cosine is given by the term

$$\alpha_{ij} = \alpha_{jj}\rho_{ij}$$

This correlation component is useful when modelling the correlation mechanisms affecting the behaviour of variables, since each direction cosine can be disaggregated into its constituent parts. Multiplying this parameter by β gives the effect of variable X_j through correlation on the standard normal failure point co-ordinate value, $x_i^{'*}$.

14.3.2 Geometric interpretation

The reliability index by definition is the minimum distance from the origin of the standard normal axes to the most likely point of failure on the limit state surface. However this formulation does not hold for correlated basic variables, which require transformation into an equivalent independent standard normal hyperspace.

It is shown that under dependent conditions, the length of a position vector with components, ($f_i(x^{'*}) = \sqrt{x_i^{'*}(-\alpha_{ii}\beta)}$ is always equal to β.

The limit state function has been geometrically formulated in terms of the reliability index and direction cosine, provided the basic variables are independent. The formulation given above facilitates a description of the performance function in the dependent variable space, as a function of the reliability index and independent component of alpha

$$g(X) = \beta + \sum_{i=1}^{n} x_i^{*} \alpha_{ii} = 0$$

This can be reconciled with the formulation in section 7.4 (HOHENBICHLER and RACKWITZ, 1986) since the direction cosine is composed only of an independent component ($\alpha_i = \alpha_{ii}$), for X_i independent.

14.3.3 Physical Interpretation

A number of authors have commented that the direction cosine measures correlation between a variable X_i and the performance function *[Z=g(X)=0]*, provided the performance function is linear and contains independent basic variables. It is shown that the general form of the direction cosine maintains this relationship for general performance functions with dependent basic variables,

$$\alpha_i = \rho_{ZX_i}$$

From this relationship, the direction cosine can be defined as a measure of linear dependence between a correlated basic variable X_i and a performance function $g(X) = 0$. This remains valid regardless of whether X_i is contained within $g(X)$ or not and provides for the development of useful risk indicators.

Interestingly the correlation between a basic variable and a performance function is often utilised in Monte Carlo modelling to indicate the potential sensitivity of the performance function to the variable. As part of the simulation this quantity is measured numerically. Given that the general direction cosine is equivalent to this correlation quantity, it is now possible to analytically represent this measure of sensitivity.

14.3.4 Bounds and conditions for the direction cosine

Schwarz's inequality, utilised in establishing minimum and maximum values for the correlation coefficient, is similarly applied to develop boundaries for the direction cosine,

$-1 \leq \alpha_i \leq 1$

Direction cosines as components of the unit vector for uncorrelated variables, conform to the condition $\sum_{i=1}^{n} \alpha_i^2 = 1$. It is shown that for X_i dependent, the sum of the product of the direction cosine and the independent component of the direction cosine is equal to unity, $\sum_{i=1}^{n} \alpha_i \alpha_{ii} = 1$. This can be reconciled with the original condition, since for X_i independent the direction cosine is composed only of the independent component, $\alpha_i = \alpha_{ii}$.

14.3.5 Stability of the direction cosine

The stability of the general form of the direction cosine over a range of reliabilities is investigated. The results indicate that alpha remains constant for linear functions and functions of limited curvature having dependent or independent basic variables. The direction cosine for beta values of zero can therefore be used to approximate the direction cosine at higher levels of reliability. Since the failure point co-ordinates at a beta value of zero are known to be equal to the mean values, the direction cosine can be calculated directly without having to first solve the limit function.

14.4 Utility of the General Direction Cosine

14.4.1 Summary of the utility of the direction cosine

A summary table of the analytical form of the direction cosine discussed above, together with the independent component of alpha and the tools formulated as functions of these two parameters are tabulated below.

Table 14.1 Summary of the utility of a general direction cosine

Utility	Mathematical representation
General Correlated Direction Cosine	$\alpha_i = \dfrac{\sum_{j=1}^{n}\left(\dfrac{\partial g}{\partial X_j}\right)\rho_{ij}\sigma_{X_j}}{\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\dfrac{\partial g}{\partial X_j}\right)\left(\dfrac{\partial g}{\partial X_i}\right)\rho_{ij}\sigma_{X_j}\sigma_{X_i}}}$
Independent component of alpha	$\alpha_{ii} = \dfrac{\left(\dfrac{\partial g}{\partial X_i}\right)_* \sigma_{X_i}}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\dfrac{\partial g}{\partial X_i}\right)_*\left(\dfrac{\partial g}{\partial X_j}\right)_* \rho_{ij}\sigma_{X_i}\sigma_{X_j}}}$
Omission Sensitivity	$\psi\left(X_j = \mu_{X_j}, j = 1,2,\ldots k\right) = \dfrac{1}{\sqrt{\left(1-\sum_{j=1}^{k}\alpha_j\alpha_{jj}\right)}}$
GFOSM	*Solves general multivariate limit state functions of linear and non-linear form, with independent and correlated normally and non-normally distributed variables in the original vector space.*
Sensitivity	$\dfrac{\partial g}{\partial X_i} = \alpha_i \dfrac{\sigma_g}{\sigma_{X_i}}$

Table 14.1 cont.

Utility	Mathematical representation
Likely Contingency Demand	*Establishes each term's probabilistic demand for the available contingency.*
Stochastic Importance	$R_i^2 = \alpha_i \alpha_{ii}$
Importance Factor	$\dfrac{\partial \beta}{\partial X'_k} = -\alpha_k$
Elasticity and Correlation Mechanisms	$\alpha_i^E = \alpha_{ii} \sum_{j=1}^{n} \rho_{ij}$
Elasticity Index	$\varepsilon_i = \dfrac{\alpha_i^E}{\alpha_i}$
Modal Correlation	$\rho_{YZ} = \sum_{i=1}^{n} \alpha_{ii}(Y)\alpha_i(Z) = \sum_{i=1}^{n} \alpha_i(Y)\alpha_{ii}(Z)$
Deterministic Reliability Based Design	$\gamma_i = f(\beta) = 1 - \Omega_i \beta \alpha_{i(\beta=0)}$

14.4.2 Omission Sensitivity

An omission sensitivity factor has been developed as a quantity to measures the sensitivity of the reliability index to the treatment of non-critical items as deterministic values (MADSEN, 1998). Values close to unity indicate a low sensitivity to the omission of variability. However the method requires the solution of the original limit function before it can be simplified and utilises transformation procedures for the treatment of correlated variables.

The dependent form of the direction cosine facilitates the development of a general form for omission sensitivity that tests the individual or cumulative effects of reducing non-critical items to deterministic values. This factor is applicable to linear and non-linear functions with correlated basic variables,

$$\psi\left(X_j = \mu_{X_j}, j = 1,2,\ldots k\right) = \frac{1}{\sqrt{\left(1 - \sum_{j=1}^{k} \alpha_j \alpha_{jj}\right)}}$$

A stable direction cosine facilitates the approximation of alpha as a function of the known mean values. This is a significant development, since sensitivities can be calculated in the original vector space and simplifications effected prior to the solution of the limit state function.

14.4.3 The General First Order Second Moment (GFOSM) Method for correlated basic variables

The current algorithm for the solution of the FOSM reliability method requires the transformation of dependent variables into an equivalent independent variable space (RACKWITZ FISSLER, 1977). The general form of the direction cosine develops a General First Order Second Moment (GFOSM) method to accommodate the treatment of correlated basic variables in the original hyperspace.

The proposed algorithm is well suited to spreadsheet type applications for linear and non-linear performance functions having any number of independent or correlated basic variables, distributed normally and non-normally. The process of solving dependent functions is considerably simplified, with invaluable parametric insight developed into the drivers of risk and reliability. The GFOSM algorithm is given below,

(i) Non-normal distributions must be represented by equivalent normal mean and standard deviations.

(ii) For the first iteration, the failure point is assumed to be equal to the mean value for the particular variable

$$x_i^* = \mu_{X_i}^{NE}$$

(iii) A substitute factor K_i is introduced, which improves computational efficiency for the general algorithm

$$\mathrm{K}_i = \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_j^{'}} \right)_* \rho_{ij}$$

(iv) The variance is calculated as a function of the K_i factors found in step (iv),

$$Var[g(X)] = \sum K_i \left(\frac{\partial g}{\partial X_i^{'}} \right)_*$$

(v) The direction cosine is also calculated as a function of K_i

$$\alpha_i = \frac{K_i}{\sqrt{Var[g(X)]}} = \frac{K_i}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i^{'}}\right)_* K_i}}$$

(vi) The failure point co-ordinates are found by substituting Equation (5.9) into the limit state function $g(x^*) = 0$ and solving for the unknown β.

$$x_i^* = \mu_i - \sigma_{X_i}\alpha_i\beta$$

Setting the assumed failure point in *(ii)*, equal to the failure point calculated in *(vii)*, repeat steps *(iii)* – *(vii)* until convergence is achieved for β.

14.4.4 Risk Indicators: Stochastic Importance

The direction cosine squared (α_i^2) represents the stochastic importance of each variable for independent functions. Furthermore as components of the unit vector, the sum of the direction cosines squared equals one. However this relationship is not consistent when basic variables are correlated. The proportional contribution to total variance, R_i^2 is a general dependent form for stochastic importance and can be written in terms of alpha and the independent component of alpha,

$$R_i^2 = \alpha_i\alpha_{ii}$$

This formulation satisfies the normalising condition $\sum_{i=1}^{n}\alpha_i\alpha_{ii} = \sum_{i=1}^{n}R_i^2 = 1$, since the sum of all the proportional components of variance must equal unity.

14.4.5 Risk Indicators: Sensitivity

It is shown that the direction cosine is equivalent under all conditions to the correlation between a variable X_i and a general performance function $g(X) = 0$. From common correlation and regression theory, an expression for the sensitivity of the performance function to changes in failure point location is given by,

$$\frac{\partial g}{\partial X_i} = \sum_{j=1}^{n} \frac{\partial g}{\partial X_j} \frac{\partial X_j}{\partial X_i} = \alpha_i \frac{\sigma_g}{\sigma_{X_i}}$$

14.4.6 Risk Indicators: Likely Contingency Demand

The General FOSM reliability method can be used to establish each term in a cost function's probabilistic demand for the available contingency, by subtracting the term value as a function of the means, from the term as a function of the failure point co-ordinates. This is a relative expression of risk in the dimension of the limit function and provides a useful tool to assist project managers to control their contingencies.

14.4.7 Risk Indicators: The Importance factor

The direction cosine has been shown to be an importance factor indicating the sensitivity of the reliability index to changes in the failure point location (HOHENBICHLER and RACKWITZ, 1986). However this definition is restricted to independent basic variables, since an equivalent direction cosine for dependent functions, $\alpha_E = -{x_i^{'*}}/{\beta}$, does not conform to the original imposed conditions {that is $\sum(\alpha_{E_i})^2 \neq 1$}.

The proposed geometric formulation for the limit state function provides for the validation of this statement for correlated basic variables, as well as conditions under which the direction cosine is not a sufficient measure of importance. A relatively complete description of the importance factor is shown to be,

$$\frac{\partial \beta}{\partial X_k^{'}} = -\left[\alpha_k + \sum_{j=1}^{n} \frac{\sigma_{X_j}}{\sigma_g}\left(\frac{\partial^2 g}{\partial X_j \partial X_k}\right)\right]$$

For linear functions and functions with limited curvature the second partial derivative is zero or negligible. Under these conditions the second term of the expression given above falls away, leaving the direction cosine as a sufficient first order approximation of the importance factor,

$$\frac{\partial \beta}{\partial X_k^{'}} = -\alpha_k$$

14.4.8 Risk Indicators: Elasticity and Correlation Mechanisms

The concept of elasticity is introduced as the cumulative contribution of a variable through correlation to the full set of direction cosines,

$$\alpha_i^E = \alpha_{ii} \sum_{j=1}^{n} \rho_{ij}$$

This formulation represents correlation mechanisms in the performance function. We recognise ($\alpha_{ij} = \alpha_{ii}\rho_{ij}$) as the correlation component contributed by variable X_i to each of the other variables. A high elasticity indicates that the variable in question has a significant impact on a number of variables.

The correlation components (α_{ij}) can be captured in an *n x n* matrix (with *i* representing the column number and *j* the row number). The sum of each row gives the corresponding elasticity index above and the sum of each column the direction cosine.

14.4.9 Risk Indicators: The Elasticity index

The elasticity index is developed as the ratio of the elasticity to the direction cosine.

$$\varepsilon_i = \frac{\alpha_i^E}{\alpha_i}$$

An index value of greater than one indicates that the cumulative effect of the variable on other variables is greater than the effect of correlation on the variable under observation. Management strategies to improve reliability can be optimised by focussing on variables with high absolute alpha values and elasticity indices greater than one.

14.4.10 System Analysis and Modal Correlation

System analysis evaluates the effect of conditional variance and modal correlation on more than one limit state function describing performance in different dimensions. Approximations have been developed for establishing boundaries for joint probabilities of failure, with dependent functions transformed into an independent vector space and the performance of common variables used to establish modal correlation.

The general form of the direction cosine facilitates the evaluation of modal correlation in the original vector space. This assists in developing insight into correlation mechanisms driving modal failure. The proposed general formulation for modal correlation is given as

$$\rho_{YZ} = \sum_{i=1}^{n} \alpha_{ii}(Y)\alpha_i(Z) = \sum_{i=1}^{n} \alpha_i(Y)\alpha_{ii}(Z)$$

Furthermore, $\rho_{YY} = \rho_{ZZ} = \sum_{i=1}^{n} \alpha_i \alpha_{ii} = 1$, which validates the imposed normalising condition and the expression for stochastic importance. For a system of potential failure modes, this results in a typical symmetric correlation matrix with all entries on the diagonal equal to unity.

An interesting observation is that a set containing the union of all variables considered in system analysis can be used to establish modal failure. This implies that any variable and not just common variables influence modal correlation, provided the unique variable and a common variable are correlated. This presents the concept of a global set of variables, with corresponding covariance matrix, from which the full complement of performance functions can be constructed.

14.4.11 Further simplification for Deterministic Reliability Based Design

Partial safety factors are utilised in reliability-based structural design codes to raise the expected values to a desired level of reliability $(x_i^* = \gamma_i \mu_{X_i})$. The partial safety factor is described as a function of the known coefficient of variance, reliability index and direction cosine.

The stability of the direction cosine and the approximate solution thereof, provides for deterministic reliability based design without probabilistic analysis. The partial safety factor is solved as a function of the reliability index, corresponding to a desired level of reliability, with the direction cosine evaluated as a direct function of the mean values,

$$\gamma_i = f(\beta) = 1 - \Omega_i \beta \alpha_i$$

The validity of this method is dependent on the stability of the direction cosine, which in turn is dependent on the form of the performance function. Functions having higher curvatures must be solved at the failure point, since approximate direction cosines may not be not be a good approximation.

14.5 Closing Remarks

The analytical form of the direction cosine for correlated basic variables in the original vector space is a significant development for reliability modelling. Complex performance functions can be simplified prior to the solution of the most likely conditions of failure. The GFOSM reliability method simplifies the solution of dependent performance functions by an order of magnitude, while still maintaining advanced functionality for project risk management. A number of innovative applications are facilitated; risk indicators such as sensitivity, contingency allocation, stochastic importance, an importance factor, elasticity and the elasticity index are made possible. The impact of correlation between variables within the same performance function is made visible by disaggregating the correlation mechanisms. The modal impact of correlation between performance functions can be observed in the context of system reliability. The stability of the direction cosine under certain conditions accommodates a deterministic approach to reliability modelling, simplifying this process even further. The result of the general dependent form of the direction cosine is to make limit state analysis considerably more accessible to risk analysts as well as increasing the utility of this powerful tool for correlated reliability modelling.

15. REFERENCES

[1] Ang, A.H.S., Tang, W.H., (1975), *Probability Concept In Engineering Planning and Design*, Volume 1-Basic Principles, John Wiley & Sons, Singapore.

[2] Ang, A.H.S., Tang, W.H., (1984), *Probability Concepts In Engineering Planning and Design*, Volume 2, Decision, Risk and Reliability, John Wiley & Sons, New York.

[3] Cornell, C.A., (1967), *Bounds on the Reliability of Structural Systems,* Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, February, pp 171 – 200.

[4] Cornell, C.A.., (1969), *A Probability-Based Structural Code,* ACI Journal, Vol. 66, December, pp 974 – 985.

[5] Ditlevsen, O, (1979), *Generalized Second Moment Reliability Index,* Journal of Structural Mechanics, No. 7, Vol. 4, March, pp 435 – 451.

[6] Ditlevsen, O, (1979), *Narrow Reliability Bounds for Structural Systems,* Journal of Structural Mechanics, No. 7, Vol. 4, pp 453 – 472.

[7] Ditlevsen, O, (1981), *Principle of Normal Tail Approximation,* Journal of the Engineering Mechanics Division, Vol. 107, No. EM6, December, pp 1191 – 1208.

[8] Ditlevsen, O., (1984), *Taylor Expansion of Series System Reliability,* Journal of Engineering Mechanics, Vol. 110, No. 2, February, pp 293 – 307.

[9] Edwards, A.L., (1976), *An Introduction to Linear Regression and Correlation,* W.H. Freeman and Company, San Francisco.

[10] Egan J., Construction Task Force, (1998), *Rethinking Construction*, UK Department of the Environment, Transport and Regions, London

[11] Freudenthal, A.M., Garrelts, J.M., Shinozuka, M., (1966), *The Analysis of Structural Safety,* Journal of Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 29, No. ST1, February, pp. 267-325.

[12] Garson, R.C., (1980), *Failure Mode Correlation in Weakest-Link Systems,* Journal of the Structural Division, Vol. 106, No. ST8, August, pp 1797 – 1810.

[13] Hardy, G.H., Littlewood, J.E., Polya, G. (1967), *Inequalities,* University Printing House, Cambridge, Great Britian.

[14] Hasofer, A.M., Lind, N.C. (1974), *Exact and Invariant Second Moment Code Format,* Journal of the Engineering Mechanics Division, Vol. 100, No. EM1, February, pp 111 – 120.

[15] Hohenbichler, M., Rackwitz, R., (1981), *Non-Normal Defendant Vectors in Structural Safety,* Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 107, No. EM6, December, pp 1227 – 1238

[16] Hunter, D, (1976), *An Upper Bound for the Probability of a Union,* Journal of Applied Probability, Vol. 13, pp 597 – 603.

[17] Institution of Civil Engineers and the Faculty and Institute of Actuaries, (1998), *Risk Analysis and Management of Projects (RAMP),* Thomas Telford, London

[18] Ker-Fox, G.M., (1998), The Limit State Cost Function (LSCF) as a Risk Management Tool for Construction Projects, University of Stellenbosch, November, pp 1 – 164.

[19] King, M.E., (2001), *King Report on Corporate Governance for South Africa (Draft),* Institute of Directors, July, pp 96 – 107.

[20] Kounias, E.G., (1968), *Bounds for the Probability of a Union, with Applications*, The Annals of Mathematical Statistics, Vol. 39, No. 6, July, pp 2154 – 2158.

[21] Kreyszig, E., (1988), *Advanced Engineering Mathematics,* Sixth Edition, Published by John Wiley & Sons, USA.

[22] Latham M., (1994), *Constructing the Team,* UK Department of the Environment, Transport and Regions, London

[23] Low, B.K., Tang, W.H., (1997), *Efficient Reliability Evaluation Using Spreadsheet,* Journal of Engineering Mechanics, Vol. 123, No. 7, July, pp 749 – 752.

[24] Madsen, H.O., (1988), *Omission Sensitivity Factors,* Structural Safety, Vol. 5, pp. 35 – 45.

[25] McCaffer, R., Baldwin, A.N., (1986), *Estimating and Tendering for Civil Engineering Works,* Collins Professional and Technical Books, Williams Collins Sons & Co. Ltd, London.

[26] Melchers, R.E., (1999) *Structural Reliability Analysis and Prediction,* Second Edition, John Wiley & Sons, England

[27] Microsoft Corporation, (1994), *Microsoft Excel, Visual Basic® User's Guide*, Version 5.0, United States of America.

[28] Paloheimo, E., Hannus, M., (1974), *Structural Design Based on Weighted Fractiles,* Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 100, No. ST7, July, pp 1367 – 1378.

[29] Rackwitz, R., Fiessler, B. (1978), *Structural Reliability under Combined Random Load Sequences,* Computers and Structures, Vol. 9, March, pp 489 – 494.

[30] Rackwitz, R., Hohenbichler, M., (1986), *Sensitivity and Importance Measures in Structural Reliability,* Civil Engineering Systems, Vol. 3, December, pp 203 – 209.

[31] RCP GmbH, (1991), Strurel A Structural Reliability Analysis Program System, COMREL & SYSREL Users Manual, Barer Strasse, Munchen

[32] Rosenblatt, M., (1952), *Remarks on a Multivariate Transformation,* The Annuals of Mathematical Statistics, Vol. 23, pp 470 – 472.

[33] SABS 0100 - 1 (1992), *The structural use of concrete. Part 1: Design,* Published by The Council of the South African Bureau of Standards, South Africa.

[34] SABS 0160 (1989), *General Procedures and loadings to be adopted in the design of buildings,* Published by The Council of the South African Bureau of Standards, South Africa.

[35] SABS 0162-1 (1993), *The structural use of steel. Part 1: Limit-states design of hot-rolled steelwork,* Published by The Council of the South African Bureau of Standards, South Africa.

[36] SABS 0162-2 (1993), *The structural use of steel. Part 2: Limit-states design of cold-formed steelwork,* Published by The Council of the South African Bureau of Standards, South Africa.

[37] Salas, S.L., Hille, E., (1990), *Calculus, One Variable,* John Wiley & Sons, Inc. Canada, 6th Edition.

[38] Seifi, A., Ponnambalam, K., Vlach, J., (1999), *Probabilistic Design of Integrated Circuits with Correlated Input Parameters,* IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, Vol. 18, No. 8, August, pp 1214 – 1219.

[39] Shinozuka, M., (1983), *Basic Analysis of Structural Safety,* Journal of Structural Engineering, Vol. 109, No. 3, March, pp. 721-740.

[40] Simon, P., Hillson, D., and Newland K., (1997), *Project Risk Analysis and Management (PRAM) Guide*, The Association of Project Management, Norwich Norfolk

[41] Sokolnikoff, I.S., Redheffer, R.M., (1958), *Mathematics of Physics and Modern Engineering*, McGraw-Hill Book Company, Inc., United States of America.

[42] Standards Australia (1999), *Risk Management*, Standards Association of Australia, Strathfield

[43] Thompson, P., and Perry, J., (1992), *Engineering Construction Risks A guide to Project Risk Analysis and Assessment,* Thomas Telford, London

16. APPENDIX I: NUMERICAL VALIDATION METHOD SAMPLES

The general direction cosine for correlated basic variables developed in chapter 4 was validated numerically in chapter 6. The validation procedure compared results obtained by applying the general First Order Second Moment algorithm for correlated basic variables (developed in chapter 5) to the results obtained by applying the orthogonal transformation of dependent variables to an equivalent independent variable space. Samples solutions for these two methods are given below.

AI.1 Orthogonal transformation method

The mathematics of the orthogonal transformation method have been described in section 2.5.1. A demonstration of the method by way of an application for Case 1 (see *Table 6.1*) is given below. This is one of the simplest examples since the performance function is linear and consists of only three variables. However it clearly demonstrates the application of the methodology.

The limit state function is given by

$$g(X) = a_0 - a_1X_1 - a_2X_2 - a_3X_3 = 0$$

Where a_0 represent the required budget, a_i the quantity estimates which are treated as constant values, and X_i the variable unit rates. The values and parameters are given below:

$$a_1 = a_2 = a_3 = 1000$$

$$\mu_{x_1} = \mu_{x_2} = \mu_{x_3} = 10$$

$$\sigma_{x_1} = \sigma_{x_2} = \sigma_{x_3} = 3$$

$$\rho_{12} = \rho_{13} = \rho_{23} = 0.05$$

The initial assumed failure point values are taken as the mean values.

$$x_1^* = x_2^* = x_3^* = 10$$

It was shown in section 2.4.1 that the matrix of correlation coefficients gives the covariance matrix in the standard normal space.

$$C' = \begin{bmatrix} 1 & 0.05 & 0.05 \\ 0.05 & 1 & 0.05 \\ 0.05 & 0.05 & 1 \end{bmatrix}$$

The eigenvalues are solved as the roots of the characteristic determinant $D(\lambda)$,

$$D(\lambda) = \det\left(C' - \lambda I\right) = 0$$

Where λ is the set of eigenvalues and I a unit matrix

$$D(\lambda) = \begin{vmatrix} (1-\lambda) & 0.05 & 0.05 \\ 0.05 & (1-\lambda) & 0.05 \\ 0.05 & 0.05 & (1-\lambda) \end{vmatrix} = 0$$

$$\therefore (1-\lambda)^3 - (1-\lambda)(0.05)^2 - (0.05)^2(1-\lambda) + (0.05)^3 + (0.05)^3 - (0.05)^2(1-\lambda) = 0$$

$$\therefore (1-\lambda)^3 - 3(1-\lambda)(0.05)^2 + 2(0.05)^2 = 0$$

Solving for the roots of the above determinant, the three eigenvalues are

$\lambda_1 = 0.95$; $\lambda_2 = 0.95$; $\lambda_3 = 1.10$

The eigenvalues are then used to generate the eigenvectors. Consider the representative system of linear equations

$$(1-\lambda)S_1 + 0.05S_2 + 0.05S_3 = 0$$

$$0.05S_1 + (1-\lambda)S_2 + 0.05S_3 = 0$$

$$0.05S_1 + 0.05S_2 + (1-\lambda)S_3 = 0$$

If we utilise the first or second eigenvalue, three identical equations are produced. This results in a non-unique set of non-trivial solutions. In addition to this complication, the determinant was found to be unstable close to zero for the performance functions containing 10 and 50 items. The commercial package Matlab deals very efficiently with this type of matrix manipulation. Matlab was therefore used to generate the eigenvalues and corresponding eigenvectors corresponding to each of the 36 cases in Table 6.1. This complication is a strong argument in favour of a general form of the direction cosine that can be used to avoid the transformation procedures.

By way of an example, which can easily be demonstrated, consider the third eigenvalue $\lambda_3 = 1.10$ Substituting this value into the system of linear equations given above,

$$(1-1.1)S_1 + 0.05S_2 + 0.05S_3 = 0$$

$$0.05S_1 + (1-1.1)S_2 + 0.05S_3 = 0$$

$$0.05S_1 + 0.05S_2 + (1-1.1)S_3 = 0$$

Which produces

$$-0.1S_1 + 0.05S_2 + 0.05S_3 = 0 \qquad \text{- (a)}$$

$$0.05S_1 - 0.1S_2 + 0.05S_3 = 0 \qquad \text{- (b)}$$

$$0.05S_1 + 0.05S_2 - 0.1S_3 = 0 \qquad \text{- (c)}$$

The objective is to express S_2 and S_3 in terms of S_1,

By Gauss elimination, $2(a)-(b)$ gives

$$-0.15S_1 + 000.15S_3 = 0; \qquad \therefore S_3 = S_1$$

Insert $S_3 = S_1$ into (c)

$$0.05S_1 + 0.05S_2 - 0.1S_1 = 0; \qquad \therefore S_2 = S_1 = S_3$$

The eigenvector corresponding to the eigenvalue, $\lambda_3 = 1.1$ is

$$T_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

It is often convenient to express the eigenvectors in their normalised form. This is done by dividing each entry by the square root of the sum of the entries squared.

$$T_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0.57735 \\ 0.57735 \\ 0.57735 \end{bmatrix}$$

The other eigenvectors (generated using Matlab) together form the columns of the transformation matrix

$$\therefore T = \begin{bmatrix} 0.707 & 0.408 & 0.577 \\ -0.707 & 0.408 & 0.577 \\ 0.000 & -0.817 & 0.577 \end{bmatrix}$$

The partial derivatives of $g(x)$ with respect to the standard normal variable $X_i^{'}$ are

$$\frac{\partial g}{\partial X_1^{'}} = \frac{\partial g}{\partial X_1^{'}}\sigma_{x_1} = a_1\sigma_{x_1} = (-1000)(3) = -3000$$

Similarly $\frac{\partial g}{\partial X_2^{'}} = \frac{\partial g}{\partial X_3^{'}} = -3000$

The gradient vector $G_{x^{'}}$ for the basic variables $X^{'}$ in the standard normal space is therefore given by

$$G_{x^{'}} = \begin{bmatrix} -3000 \\ -3000 \\ -3000 \end{bmatrix}$$

The partial derivatives of the performance function g(X) in the Y space are found by applying the chain rule to the original vector space.

$$\frac{\partial g}{\partial Y_i} = \sum_{j=1}^{n} \frac{\partial g}{\partial X_j^{'*}} \frac{\partial X_j^{'*}}{\partial Y_j}$$

The partial derivatives for the independent standard normal variables are

$$\frac{\partial g}{\partial Y_i^{'}} = \frac{\partial g}{\partial Y_i}\frac{\partial Y_i}{\partial Y_i^{'}} = \frac{\partial g}{\partial Y_i}\sigma_{Y_i}$$

In matrix notation, this relates to the transformation matrix as follows

$$G_Y = T^{\mathrm{T}} G_X$$

$$\therefore \frac{\partial g}{\partial Y_1} = 0.7071(-3000) - 0.7071(-3000) + 0(-3000) = 0$$

$$\therefore \frac{\partial g}{\partial Y_2} = 0.408(-3000) + 0.408(-3000) - 0.817(-3000) = 0$$

$$\therefore \frac{\partial g}{\partial Y_3} = 0.577(-3000) + 0.577(-3000) + 0.577(-3000) = -5449.77$$

The covariance matrix for Y is a diagonal matrix with non-zero entries as the eigenvalues. The standard deviations for Y are therefore the square root of the corresponding eigenvalues.

$$\sigma_{Y_1} = \sqrt{\lambda_1} = \sqrt{0.95} = 0.97$$

$$\sigma_{Y_2} = \sqrt{\lambda_2} = \sqrt{0.95} = 0.97$$

$$\sigma_{Y_3} = \sqrt{\lambda_3} = \sqrt{1.1} = 1.05$$

Since Y is a function of X' in the standard normal space, the mean values for Y are zero

$$\mu_{y_1} = \mu_{y_2} = \mu_{y_3} = 0$$

The partial derivatives for Y in the standard normal space are therefore

$$\frac{\partial g}{\partial Y_1'} = \frac{\partial g}{\partial Y_1}\sigma_{Y_1} = (0.00)(0.97) = 0$$

$$\frac{\partial g}{\partial Y_2'} = (0.00)(0.97) = 0$$

$$\frac{\partial g}{\partial Y_3'} = (-5196.15)(1.05) = -5449.77$$

The variance for the performance function in the independent standard normal space is given by

$$Var(Y^*) = \sum_{i=1}^{n}\left(\frac{\partial g}{\partial Y_i'}\right)^2 = (0)^2 + (0)^2 + (-5449.72)^2 = 2.97 * 10^7$$

The direction cosine for each independent standard normal variable Y_i' is calculated as

$$\alpha_{y_1} = \frac{\left(\frac{\partial g}{\partial Y_1'}\right)}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial Y_i'}\right)^2}} = \frac{0}{\sqrt{2.97*10^7}} = 0; \qquad \alpha_{y_2} = 0; \qquad \alpha_{y_3} = -1$$

The failure point co-ordinates in the transformed independent standard normal vector space are given below as functions of the unknown reliability index

$$y_1^* = \mu_{y_1} - \alpha_{y_1}\sigma_{y_1}\beta = 0 - (0)(0.97)\beta = 0; \qquad y_2^* = 0; \qquad y_3^* = 1.05\beta$$

By the definition of orthogonality, the inverse of the transformation matrix is given by the transpose $T^{-1} = T^T$. These failure point co-ordinates are then transformed into the original variable space through

$$X' = T^{-1}Y = T^T Y$$

Transforming the standard normal solution into the original variable space

$$X = X'\sigma_X + \mu_X = T^T Y \sigma_X + \mu_X$$

$$\therefore x_1 = 0 + 0 + (0.577)(1.05)\beta + 10$$

$$x_2 = 0 + 0 + (0.577)(1.05)\beta + 10$$

$$x_3 = 0 + 0 + (0.577)(1.05)\beta + 10$$

Substituting these co-ordinates into the performance function we find

$$g(X) = a_0 - (1000)(0.577)(\beta)(1.05) * 3 = 0$$

Since β is a control variable (set to 3 for the current application), we can solve for the unknown budget and failure point co-ordinates

$$a_0 = (1000)(0.577)(3)(1.05)(3) = 46349$$

The corresponding failure point co-ordinates are

$$x_1^* = x_2^* = x_3^* = (0.577)(1.05)(3) + 10 = 15.45$$

Since this is a linear function, no further iterations are necessary. The spreadsheet layout for this application is given in Table 2 on the following page.

Table AI.1 Spreadsheet application for the Orthogonal Transformation method

Orthogonal Transformation:

Iteration	Variable	Assumed x*	μ (NE)	σ (NE)	(dg/dX'i)	(dg/dYi)	σ (Y)	(dg/dY'i)	(dg/dY'i)^2	α (y*)	New y*	New x*
1	Budget		R 46,349									
	X1	10.00	10.00	3.00	-3000.00	0.00	0.97	0.00	0.00E+00	0.00	0.00	15.45
	X2	10.00	10.00	3.00	-3000.00	0.00	0.97	0.00	0.00E+00	0.00	0.00	15.45
	X3	10.00	10.00	3.00	-3000.00	-5196.15	1.05	-5449.77	2.97E+07	-1.00	3.15	15.45

Var(y*) = 2.97E+07
β = 3.00
g(y*) = 0.00

AI.2 General FOSM reliability method for correlated basic variables

The general form of the direction cosine was applied to the same performance function as described for the orthogonal transformation method, in order to solve the limit function in the original variable space.

The method will be demonstrated by way of the same application used above for the orthogonal transformation. The performance function, parametric definitions and partial derivatives are the same as those utilised in the orthogonal example. The partial derivatives in the standard normal space are similarly

It is convenient for matrix multiplication to express the correlation coefficients as a correlation matrix, given below

$$\rho = \begin{bmatrix} 1 & 0.05 & 0.05 \\ 0.05 & 1 & 0.05 \\ 0.05 & 0.05 & 1 \end{bmatrix}$$

We introduce the K_i factor (equation 5.5) corresponding to each variable X_i, such that

$$K_1 = K_2 = K_3 = \sum_{j=1}^{n} \left(\frac{\partial g}{\partial X'_j} \right) \rho_{ij}$$

$$= \left(\frac{\partial g}{\partial X_1} \right) \rho_{11} \sigma_{X_1} + \left(\frac{\partial g}{\partial X_2} \right) \rho_{12} \sigma_{X_2} + \left(\frac{\partial g}{\partial X_3} \right) \rho_{13} \sigma_{X_3}$$

$$= (-3000)1 + (-3000)0.05 + (-3000)0.05 = -3300$$

The variance is developed as a function the K_i factors,

$$Var\left(x^*\right) = \sum_{i-1}^{n} K_i \left(\frac{\partial g}{\partial X'_j} \right) = \sum_{j=1}^{n} \sum_{i=1}^{n} \left(\frac{\partial g}{\partial X'_i} \right) \left(\frac{\partial g}{\partial X'_j} \right) \rho_{ij} = 3(-33000)(-3000) = 2.97x10^7$$

It is interesting to note that the variance calculated as a function of the independent standard normal variables $\left(y^{'*}\right)$ is the same as the variance calculated in the original variable space. From equation 5.8, the direction cosine is calculated as a function of the K_i factor

$$\alpha_1 = \alpha_2 = \alpha_2 = \frac{K_1}{\sqrt{\sum_{i=1}^{n} K_i \left(\frac{\partial g}{\partial X_i^{'}} \right)}} = \frac{-33000}{\sqrt{2.97x10^7}} = -0.61$$

The co-ordinate of the failure point in the original variable space is now calculated as a function of the familiar equation 5.9

$$x_1^* = x_2^* = x_3^* = \mu_{x_1} - \alpha_1 \sigma_{x_1} \beta = 10 - (-0.61)(3)\beta = 10 + 1.83\beta$$

Substituting $\beta = 3$ into the expressions for the failure point co-ordinates

$$x_1^* = x_2^* = x_3^* = 10 + 1.83(3) = 15.45$$

The required budget is the solution of the performance function corresponding to a reliability index of 3 is

$$g(X) = a_0 - [(1000)(15.45)] * 3 = 0$$

$$\therefore a_0 = 46349$$

The results obtained for the required budget (a_0) as well as the failure point co-ordinates are the same as the results obtained for the orthogonal transformation method. The spreadsheet application for the general method is given in Table AI.3 below.

Table AI.2 Spreadsheet application of the General Method in the original variables space.

Iteration	Variable	Assumed x*	μ (NE)	σ (NE)	(dg/dX'i)	K(i)	K(i)(dg/dX'i)	α (x*)	New x*
1	Budget		R 46,349						
	X1	10.00	10.00	3.00	-3000.00	-3300.00	9.90E+06	-0.61	15.45
	X2	10.00	10.00	3.00	-3000.00	-3300.00	9.90E+06	-0.61	15.45
	X3	10.00	10.00	3.00	-3000.00	-3300.00	9.90E+06	-0.61	15.45
								Var(x*) = 2.97E+07	
								β = 3.00	
								g(x*) = 0.00	