# LEARNERS' STRATEGIES FOR SOLVING LINEAR EQUATIONS

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**THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF**

**MEd (Courses) (Curriculum Studies)**

**AT THE UNIVERSITY OF STELLENBOSCH**

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**NOVEMBER 2002**

# DECLARATION

I, THE UNDERSIGNED, HEREBY DECLARE THAT THE WORK CONTAINED IN THIS THESIS IS MY OWN ORIGINAL WORK AND HAS NOT PREVIOUSLY IN ITS ENTIRETY OR IN PART BEEN SUBMITTED AT ANY UNIVERSITY FOR A DEGREE.

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## SUMMARY

Algebra deals amongst others with the relationship between variables. It differs from Arithmetic amongst others as there is not always a numerical solution to the problem. An algebraic expression can even be the solution to the problem in Algebra. The variables found in Algebra are often represented by letters such as x, y, etc.
Equations are an integral part of Algebra. To solve an equation, the value of an unknown must be determined so that the left hand side of the equation is equal to the right hand side.
There are various ways in which the solving of equations can be taught.
The purpose of this study is to determine the existence of a cognitive gap as described by Herscovics & Linchevski (1994) in relation to solving linear equations.
When solving linear equations, an arithmetical approach is not always effective.
A new way of structural thinking is needed when solving linear equations in their different forms.

In this study, learners' intuitive, informal ways of solving linear equations were examined prior to any formal instruction and before the introduction of algebraic symbols and notation. This information could help educators to identify the difficulties learners have when moving from solving arithmetical equations to algebraic equations. The learners' errors could help educators plan effective ways of teaching strategies when solving linear equations.

The research strategy for this study was both quantitative and qualitative. Forty-two

Grade 8 learners were chosen to individually do assignments involving different types of linear equations. Their responses were recorded, coded and summarised. Thereafter the learners' responses were interpreted, evaluated and analysed. Then a representative sample of fourteen learners was chosen randomly from the same class and semi-structured interviews were conducted with them. From these interviews the learners' ways of thinking when solving linear equations, were probed.

This study concludes that a cognitive gap does exist in the context of the investigation. Moving from arithmetical thinking to algebraic thinking requires a paradigm shift. To make adequate provision for this change in thinking, careful curriculum planning is required.

# OPSOMMING

Algebra behels onder andere die verwantskap tussen veranderlikes. Algebra verskil van Rekenkunde onder andere omdat daar in Algebra nie altyd 'n numeriese oplossing vir die probleem is nie. In Algebra kan 'n algebraïese uitdrukking somtyds die oplossing van 'n probleem wees. Die veranderlikes in Algebra word dikwels deur letters soos x, y, ens. voorgestel. Vergelykings is 'n integrale deel van Algebra. Om vergelykings op te los, moet 'n onbekende se waarde bepaal word, om die linkerkant van die vergelyking gelyk te maak aan die regterkant. Daar is verskillende maniere om die oplossing van algebraïese vergelykings te onderrig.

Die doel van hierdie studie is om die bestaan van 'n sogenaamde "kognitiewe gaping" soos beskryf deur Herscovics & Linchevski (1994), met die klem op lineêre vergelykings, te ondersoek. Wanneer die oplossing van 'n linêere vergelyking bepaal word, is 'n rekenkundige benadering nie altyd effektief nie. 'n Heel nuwe, strukturele manier van denke word benodig wanneer verskillende tipes linêere vergelykings opgelos word.

In hierdie studie word leerders se intuitiewe, informele metodes ondersoek wanneer hulle lineêre vergelykings oplos, voordat hulle enige formele metodes onderrig is en voordat hulle kennis gemaak het met algebraïese simbole en notasie.
Hierdie inligting kan opvoeders help om leerders se kognitiewe probleme in verband met die verskil tussen rekenkundige en algebraïese metodes te identifiseer.Die foute wat leerders maak, kan opvoeders ook help om effektiewe onderrigmetodes te beplan, wanneer hulle lineêre vergelykings onderrig. As leerders eers die skuif van

rekenkundige metodes na algebraïese metodes gemaak het, kan hulle besef dat hul primitiewe metodes nie altyd effektief is nie.

Die navorsingstrategie wat in hierdie studie aangewend is, is kwalitatief en kwantitatief. Twee-en-veertig Graad 8 leerders is gekies om verskillende tipes lineêre vergelykings individueel op te los. Hul antwoorde is daarna geïnterpreteer, geëvalueer en geanaliseer. Daarna is veertien leerders uit hierdie groep gekies en semi-gestruktureerde onderhoude is met hulle gevoer. Vanuit die onderhoude kon 'n dieper studie van die leerders se informele metodes van oplossing gemaak word.

Die gevolgtrekking wat in hierdie studie gemaak word, is dat daar wel 'n kognitiewe gaping bestaan in die konteks van die studie. Leerders moet 'n paradigmaskuif maak wanneer hulle van rekenkundige metodes na algebraïese metodes beweeg. Hierdie klemverskuiwing vereis deeglike kurrikulumbeplanning.

## ACKNOWLEDGEMENTS

Firstly, I thank God for making this work possible. He is worthy of my sincere gratitude.

I also want to thank the following people for the role each of them played in the completion of this study:

- My wife and children for their support, interest and encouragement;
- My supervisors, Mrs JC Murray and Mr AI Olivier for their genuine supervision and support;
- My colleagues, Kitto, Clive, Regan and Jose, for the typing and original printing;
- The interviewees for their willingness to participate;
- John for the technical assistance during the interviews;
- Nicole for editing the final text.

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# CHAPTER ONE

# INTRODUCTION AND THEORETICAL FRAMEWORK

## 1.1 INTRODUCTION

Many children identify Algebra with equations (Kieran , 1981). Many children also regard Algebra (including equations) as meaningless manipulations of letters, symbols and/or indeterminate forms involving unknowns such as 3x, ½x + 4, 3 + x = 4, etc. (Herscovics & Linchevski, 1994).

This could be due to the fact that many children have a limited understanding of algebraic equations and that for them the solution processes are more important than the meaning of the actual numerical solution obtained. Ideally learners should view the unknown in an equation as represented by a letter. Then the learners must determine the numerical value of the letter from the specific data in order to make Algebra meaningful to them (Herscovics & Linchevski, 1996).

Filloy & Rojano (1989) maintained that arithmetical knowledge which has worked well up to a certain stage, would fail to be sufficient for algebraic thought in individual learners. To move from arithmetic to algebra a new kind of thinking was needed. These changes in ways of thinking were described by Filloy & Rojano (1989: 19) as "cut-points". According to them (Filloy & Rojano, 1989) there would come a stage when the arithmetical way of thinking would fail to be sufficient, e.g. when solving linear equations of the form Ax + B = Cx + D, i.e. when the unknown appeared on both sides of the equal sign. In equations like these learners would be required to understand the concept of equivalence, i.e. that the expressions on both sides of the equal sign had to have the same numerical value as solution even though they might look different. This break- down of arithmetical thought in the presence of the algebraic symbolic stage could be seen as a **cut.**

Linchevski & Herscovics (1994) referred to the demarcation between arithmetic and algebra as a **cognitive gap.** They viewed the existence of this gap as a result of **cognitive obstacles,** i.e. those obstacles (like those mentioned in 1.2) caused by earlier arithmetical thinking, exposed by the demands of algebraic thinking. Whereas Filloy & Rojano (1989) pointed out that the didactic cut existed when the unknown appeared on **both** sides of the equal sign, Linchevski & Herscovics (1994) found that learners would use the same solution methods when the unknown appeared on one side and/or on both sides of the equal sign. So the cognitive gap would appear to exist between pre-algebra and algebra.

## 1.2 AIM OF STUDY

The aim of the present study was to determine the existence of a **cognitive gap,** i.e. the inability of learners to operate on or with an unknown within the context of solving linear (first degree) equations in one unknown, prior to any formal instruction. Herscovics & Linchevski (1994) did a similar investigation and this study will simulate their methodology in as far as learners' primitive, intuitive methods (procedures) to solve equations will be examined prior to the learners having been formally taught.

This study would also like to determine more specifically the influence of the following four areas which can result in cognitive obstacles and thereby contributing to a cognitive gap.
These four areas were identified by Linchevski and Herscovics (1994) with regard to solving linear equations as follows:

- the four arithmetic operations i.e. addition, subtraction, multiplication, division and their inverses;

- the position of the unknown;
- the size of the numbers in the same type of equation;
- double or more occurrence of the unknown on the same side and on both sides of the equal sign.

## 1.3 MOTIVATION

It was decided to do the investigation prior to the formal instruction of linear equations because of the following:

- it corresponds with the problem-solving orientation in as far as it would invite learners to use their own intuitions and previously acquired knowledge to solve new problems. Teachers can use this information of learners' intuition to present Algebra in such a way that it would make sense to them and it would become viable and functional.
- the kinds of equations used would be linear equations which learners might be able to solve by making use of:
  - guess and check (trial and error);
  - substitution;
  - inverse operations or inverse flow-diagrams e.g. when solving $2x + 5 = 17$, the learners would take 5 away from 17 to get 12 and would then say 12 divided by 2 is 6 and therefore x is equal to 6.

Understanding of these methods is important if we would like learners to understand the idea of equivalent equations i.e. different equations with the same solution. Once this concept of equivalence is understood, learners would possibly be able to see the logic of solving equations with the unknown on both sides of the equal sign, e.g. $13x + 2 = 5x + 18$. In this equation the output value of $13x + 2$ and the output value of $5x + 18$ must be the same to yield the solution of both sides of the equal sign i.e. $13(2) + 2 = 5(2) + 18$. These types of equations might help to determine where the cognitive gap appears, because these types of equations could not necessarily be solved by using arithmetic methods. To get the solution of $x = 2$, learners would ideally eventually solve this equation as follows:

$$
\begin{aligned}
13x + 2 &= 5x + 18 \\
13x + 2 - 2 &= 5x + 18 - 2 \\
13x &= 5x + 16 \\
13x - 5x &= 5x - 5x + 16 \\
8x &= 16 \\
8x \div 8 &= 16 \div 8 \\
x &= 2
\end{aligned}
$$

When now verifying/validating their answers, they would use substitution to see that their answers were correct.

## 1.4 LITERATURE REVIEW

### 1.4.1 Structure vs Procedure

Many learners experience problems with the transition from Arithmetic to Algebra due to the nature of Algebra. Algebra is about transformation and equivalence, i.e. it is dynamic and changing (Nickson, 2000). In Algebra an expression can be the final outcome of a process, whereas in arithmetic, learners will be expected to get a unique numerical answer when ever a computation is done. Algebra mainly involves structure and Arithmetic mainly but not exclusively involves procedure. It must be noted that the same structure underlying algebraic manipulations also underlies the calculations in arithmetics.

The **procedural** (operational) perspective refers to Arithmetic operations performed on numbers to yield new numbers, eg. if $x = 4$ and $y = 10$ then $x + y = 14$. Another example more relevant to the present study would be to find the value of x in the equation $2x + 1 = 7$ where learners would be able to find a value for x which would make the left hand side equal to 7 by using numbers only. Thus, learners could solve this problem by using previously acquired knowledge, such as guess and check, and random substitution.

Contrary to this procedural perspective, the **structural** perspective in Algebra refers to the situation when the same set of operations is carried out on expressions to yield expressions, e.g. to solve $3x + 7 = 2x + 14$. Here one has to carry out the different operation on both sides of the equal sign to maintain the left – right equivalence of the equal sign. These operations differ from the operational perspective because the right hand is not regarded as the answer and the equal sign is not "a gives-something-sign" (see later). Furthermore, when this example is worked out according to correct algebraic structural principles, then the symmetric and transitive nature of equality is maintained as follows:

$$\begin{aligned} 3x - 2x + 7 &= 2x - 2x + 14 \\ x + 7 &= 14 \\ x + 7 - 7 &= 14 - 7 \\ x &= 7 \end{aligned}$$

In this example the solution rests on the principle of performing the same operation on both sides of the equal-sign to conserve equality. This example (like many others) might show that there is sometimes a demarcation between arithmetical (procedural) manipulations and algebraic (structural) methods of solution. As can be seen, methods such as guess and check and random substitution would not work effectively in this case.

### 1.4.2 The Equal-sign

Kieran (1981) found that both primary school and high school learners saw the equal sign as a "do-something-sign". The =-sign, which should mean equivalence between two sides, was regarded as a verb (it gives) instead of a noun (equivalence). For these learners the right hand side was the answer of the computational operations on the left hand side, e.g. $14 + 5 = 19$, i.e. the left hand side must be acted on to give the right hand side.

Kieran (1981) felt that the meaning of the equal sign had to be extended to mean equivalence, in order for learners to be able to solve equations such as the ones previously mentioned ($3x + 7 = 2x + 14$). In this equation $2x + 14$ is **not** the answer but an equivalent expression.

### 1.4.3 Understanding

One understands something when one can see the relationship between it and something that one already knows. This means that understanding is a continuing process of organising one's own knowledge. Pirie and Kieran (1994) saw understanding as a dynamic, non-linear recursive process, i.e. that understanding is built on constantly referring back to what has been learnt before. In Algebra, understanding means to be able to continually justify one's algebraic procedures. (Sfard & Linchevski, 1994). One may say that in order to make sense of algebraic processes, one must be able to see if it relates to the properties of numbers. With regard to this study, it must be noted that Kieran (1988) found that learners who had learned (and understood!) in elementary school how to solve equations by using informal methods such as substitution, covering-up and simple counting techniques were more receptive to the more formal techniques such as transforming and balancing. When learners understand the informal strategies they are in fact identifying equivalence i.e. structural techniques.

It could therefore be concluded that a learner has to understand the structure of an algebraic equation before he/she can start solving it by using algebraic methods. If he/she does not understand the algebraic structure he/she may want to revert to the procedural (input – output) methods of solution, or arithmetic methods (substitution). They may still be able to find the solution without being able to justify their answers. If this justification does not happen, the situation may arise where learners know exactly how to find the right answers, but without them knowing the mathematical structures involved, i.e. it may lead to rote learning and mere memorising of rules and algorithms. Learners should therefore always be expected to explain their thinking or to justify their thinking.

In schools, learners sometimes are expected to justify their reasoning, but usually only using previously taught theorems and/or rules. A majority of learners therefore feels that Mathematics is rule-based and about half of the learners consider that learning Mathematics is mostly memorising (Kieran, 1991).
Davis (1992) argued that one understood a new idea when the new idea fitted in with previously acquired ideas. When a learner finds a solution to an equation in some way or other, he/she must be able to recognise something familiar in the solution, e.g. $2x + 1 = 9$ is similar to $2? + 1 = 9$.

The evolution of Algebra was gradual and took place according to mainly three stages, namely:

- rhetorical: during which all equations and problems were posed and solved purely in prose form;

- syncopated: during this stage some abbreviations were used for starting and solving problems;
- symbolic: which is the Algebra that is used today. Here the symbols in Algebra do not speak for themselves, but what is represented by them is dependent on the situation in which they (the symbols) are found (MALATI , 1999).

This leads us to the concept of the "process-object" duality of Algebra (Sfard & Linchevski, 1994). For example, if one is presented with the algebraic expression 2(x + 7) + 10, it may be interpreted as:

(a) a computation process;

(b) a number as a product, i.e. a numerical value after some arithmetical operations have been carried out;

(c) a function, i.e. the relationship between variables;

(d) just a number of symbols which represent just an algebraic object in itself, i.e. it is static (Sfard, 1994).

It is this transition from the dynamic operational (input-output) to the static, structural characteristics of mathematics which causes many learners problems (MALATI, 1999). Learners would have problems with seeing the *same* mathematical concepts as processes and at another time as objects. They (the learners) would have problems with what Collis (1974) referred to as "lack of acceptance of closure" i.e. an expression like x + 2 is not something to be acted on further, but rather an algebraic answer in its own right. In this regard Sfard (1995) created a three-phase model according to which mathematical learning develops. Her three phases of conceptual developments were:

- **interiorization** - during this phase processes are performed on already familiar concepts, e.g. what should the number be in the place of the ? in 2? + 1 = 5 ?
- **condensation** – during this relatively longer phase the process and/or operation is condensed into more manageable units, as in the syncopated stage of Algebra.
- **reification** – during this phase what was previously conceived as a process now becomes an object. This stage is regarded as a higher level of mathematical thinking. In the current study, this would ideally mean solving an equation by performing the same operation on both sides of the equal sign **and** being able to justify it. If this justification does not take place, then learners would resort to memorising rules (Kieran, 1991) and the understanding of Algebra would be difficult for the majority of learners.

### 1.4.4 Equation Solving

Solving an equation is a process and it is imperative that learners must understand this process. It means using algorithms to find a value for the unknown (a number) that would make the equation true. This would have to be explained by the researcher. Kieran (1984) distinguished between two equation - solving approaches namely:

- symmetric, meaning performing the same operation on both sides of the equal sign or
- asymmetric, which means transposing i.e. changing sides and changing signs.

In the first approach equivalence is emphasised, but not in the second approach. Herscovics & Kieran (1980) maintained that when learners solved an equation, they had to transform algebraic expressions involving an unknown. It was felt by them that learners had difficulties in operating on such expressions, because learners did not treat the letter (the unknown) as representing a generalized number.

When one solves an equation, all that one does is transforming complicated algebraic expressions into simpler equivalent expressions with the aid of different applicable algorithms, until the result is obvious (Olivier, 2000).

## 1.5 RESEARCH METHOD

### 1.5.1 Context

Forty-two Grade 7 learners in the same class from a primary school in a large country town were chosen for this study. This school is a typical school in a previously disadvantaged community with learners coming from both upper-middleclass and lower middleclass townships. Eleven of them were classified as being above 80% in their mathematical achievement; fifteen as between 60% and 80%; eight between 40% and 60%; while the rest were classified as either needing help, or would not be able to pass Mathematics. These classifications were supplied by their primary school mathematics teacher. As much personal information as possible was obtained about all the learners regarding name, age, address, telephone number, family, future career plans and their attitude towards Mathematics. (Appendix A). This information would be used to try and find a representative sample for the interviews.

The researcher ascertained that these learners had not previously been exposed to working with symbols (x). They may therefore have been able to solve an equation in the form ? + 7 = 22 but not necessarily x + 7 = 22. The researcher would have to tell them they have to determine a numerical value in place of the unknown (x) to make the left hand side equal to the right hand side.

### 1.5.2 The Written Test

All 42 learners wrote a test in March 2001. On the basis of these results, 14 learners from the same Grade 8 class were selected to conduct interviews with. Each learner was given the option of using a nickname for anonymity. Everyone was given a 48-page exercise book which they would keep as part of this study.

They were encouraged to use a calculator at all times and to write down all their work. It was made clear to the learners that the assignments were not tests and that it would not be used for future gradings and/or progress purposes.

Their responses to the tests were studied during the April 2001 school holidays.

### 1.5.3 Selection of Equations for the Written Test

It was decided to restrict the equations to equations with positive whole numbers as solutions, because learners would not have learned about negative numbers yet. (Appendix B). Both small and large numbers were included to determine the influence of the sizes of numbers on the appearance of the cognitive gap.

The smaller numbers were given at the beginning of each category (indicated by the letters A, B, C, D, E and F). The objectives of each category of equations (Herscovics & Linchevski, 1994) are as follows:

A. (1 – 5) These questions were designed to investigate only the effect of the singleton i.e. where the coefficient of x was equal to one, on the learners' solution strategies. The unknown appeared on either side of the equal sign and there was only addition involved.

B. (6 – 11) In this category the coefficients of unknown were either 1 or –1. The number of operations also increased, larger numbers were included and the unknown appeared on either side of the equal sign, but only one unknown per equation.

C. (12 – 20) Now the coefficients of the unknown were no longer necessarily one, but care was taken to get whole number answers. The number of operations on both sides of the equal sign was also increased.

D. (21 – 26) This category dealt with division as an operation with the unknown as a dividend. Again, use was made of both small and larger numbers and the unknown appeared on either side of the equal sign

E. (27 – 33) In this category the unknown occurred more than once on the left hand side of the equation. Here grouping and/or cancellation in the unknown was going to be investigated, therefore more mathematical operations were included. The effect that guessing and checking would have on the learners' solution strategies was investigated. The **cognitive gap** would most likely appear in this category, because arithmetical methods would now be less effective.

F. (34 – 38) In this category the unknown appeared on **both** sides of the equal sign. In this category, arithmetic methods would be difficult to apply. Possible problems that learners might have had in category E would be highlighted here. However, since many learners would not have seen this particular type of equation before, more explanation would have to be done by the researcher. The learners would have to be reminded that the same numerical value had to be determined on either side of the equal-sign so as to make the left hand side equal to the right hand side.

Once again, it must be stated that learners were always reminded to show all their work, verify their answers, that the investigation was not a test and that we were not interested in right answers, but rather in how they got their results.

### 1.5.4 Interviews

From the initial forty two learners a subset of fourteen learners were randomly chosen with whom interviews were conducted. These learners were from the same class. This was done since the interviews were conducted during school hours.

Mainly non-examinable subjects' time was used so that learners would lose the minimum amount of instruction time. During the interviews learners were again encouraged to use a calculator and to ask questions when they were uncertain about anything. The interviews lasted about 40 minutes each.

The aims of the interviews were:

- to monitor the results of the written tests;
- to determine the learners' thought processes during the written tests;
- to probe the learners' thinking by asking them more questions and by adapting the questions in accordance with their responses;
- to determine where, how or why the cognitive gap appeared and the effect(s) it might have on the learners' solution processes.

### 1.5.5 Selection of Equations for the Interviews

As a starting point, two equations from each of the six categories of equations used in the written test were chosen (Appendix C). However, the equations were adjusted and/or added to, according to how the interview proceeded. Both the learners' verbal and written responses were recorded, transcribed and analysed. (Appendix D).

The results of the interviews will be discussed both quantitatively and generally in Chapter 3.

# CHAPTER TWO

# RESULTS OF WRITTEN TESTS

## 2.1 CODING PROCEDURES.

The results of the 38 equations used during the written test, will now be analysed. The coding system for learners' solution procedures used by Linchevski & Herscovics (1994) - slightly adapted – will mainly be used. A separate coding system will mainly be used when analysing the mistakes most commonly made by learners. All this will be done on the same table. (Appendix D).

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CODING OF SOLUTION PROCEDURES

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| | | |
|---|---|---|
| *Div.2* | = | Divides by 2, e.g. $2x = 10$<br>$x = 10/2$<br>$x = 5$ |
| *I* | = | Inverse operation, e.g. $x + 5 = 25$<br>$x + 5 - 5 = 25 - 5$<br>$x = 20$ |
| *NF* | = | Number Fact, e.g. seeing that $5 + 20 = 25$ |
| *RS* | = | Random Substitution : guess and check to get correct solution. |
| *SFT* | = | Substitutes and succeeds on First Trial. (Using substitution and obtains correct solution first time.) |

Classifying the type of solution procedure used by learners in the tests was not very accurate, since learners did not always show their thinking.

CODING SYSTEM OF MISTAKES

| | | |
|---|---|---|
| *WI* | = | Wrong inverse |
| *AE* | = | Guessing, arithmetical error or using fingers |
| *IV* | = | Ignoring the unknown |
| *NOTT* | = | Not working with or cancellation of like terms |
| *TT* | = | Seeing only two terms – one on each side of the equal sign |
| *DF* | = | Using different values for the unknown in the same equation |
| *WB* | = | Working backwards, i.e. from the "answer" |
| *NES* | = | Ignoring the negative sign |

After the coding of solution procedures and mistakes had been done, two equations from each category of equations (A, B, C, D, E and F) were chosen to compare the success rate within the specific category. A relatively easier and a relatively more difficult equation were chosen from each of the categories.

## 2.2 QUANTITATIVE DISCUSSION OF WRITTEN TESTS.

A. Equation (2), i.e. $5 + x = 25$, was solved correctly by 94.7 %, whereas (5), i.e. $100 = x + 27$, was solved correctly by 89.5 % of the learners. This might indicate that the learners knew the inverse of addition was subtraction, or that the learners simply asked themselves what must be added to 5 to give 25.

B. These equations involved subtraction as well as its inverse. Equation (6), i.e. $x - 6 = 8$, could be regarded as trivial, but still only 81.6 % of the learners got the correct answer, mainly by guessing. Equation (10), i.e.

x – 17 + 10 – 10 + 17 + 6 = 12, involved a string of simple arithmetic cancellations, but even with the help of a calculator, only 10.5 % of the learners found the correct answer.

This could indicate at least three things namely:

(a) learners did not view the inverse of subtraction as addition or

(b) they had problems with the identification or cancellation of like terms or

(c) learners could not solve the simplified equivalent equation, i.e. x + 6 + 12.

C. Equation (14) i.e. 2x – 2 = 6 could easily be solved by guessing. Yet only 44.7 % managed to solve it correctly. Equation (18), i.e. 4x + 10 = 110 –10 + 10 –10, involved the number 10 which researchers thought would be manageable at the learners' level of mathematics, but only 13.1 % solved the equation correctly. Solving the equations in this category formally involved the inverse of multiplication and the cancellation of like terms. It became obvious from their results that the learners experienced inverse and cancellation as cognitive obstacles when solving equations.

D. Equation (21), i.e. 15/x = 3, could be regarded as a trivial equation if learners could see that 15 divided by 3 is equal to 5. In this case, 76.3 % of the learners managed to get it right by using mainly number facts or guessing. Equation (26), i.e. 23 = 115/x, had the same structure as equation (21), but the equation was changed around and larger numbers were used. Here only 93.5% managed to solve the equation.

E. Equation (28), i.e. x – x + x = 7, produced surprisingly poor results, because the left hand side could be interpreted as 1 – 1 + 1 = 1. Only 34.2 % managed to solve this equation correctly. It could be that the appearance of the singleton caused the problems. Equation (31), i.e. $3x - 2x + 8x = 27 + 6$, proved to be problematic with only 5.2 % obtaining the correct answer. It should be mentioned that where it could still be possible to guess the correct answer in (28), it might be quite impossible to guess the correct answer in (31). In (31) the existence of the cognitive gap was highlighted. It was to be expected since none of these learners had prior instruction in Algebra.

F. Equation (34), i.e. $4x = x + 15$, with the unknown on both sides of the equal sign, was solved correctly by 18.4 % of the learners by random substitution or by substituting correctly the first time. In sharp contrast, no learner managed to solve equation (36), i.e. $3x + 2x + 10 = 4x - 11$, correctly. This could have been due to the fact that the unknown appeared on both sides of the equal-sign or the learners added and/or subtracted unlike terms. In this category we might observe that when the learners' intuitive, primitive methods did not work effectively, they experienced problems solving the equations.

# CHAPTER THREE

# INTERVIEWS

## 3.1 INTRODUCTION

The method used during the interviews was discussed in 1.5.4 and the results of the interviews will now be discussed.

## 3.2 QUANTITATIVE DISCUSSION OF INTERVIEWS

Unlike Herscovics and Linchevski (1994), not all the equations were solved by the learners during the interviews. Now we shall discuss the solution procedures most frequently used during the interviews by the learners, as well as the most frequent mistakes in each of the equations individually.

(1) $x + 17 = 32$

The algebraic solution of this equation involves the inverse of addition.
Some learners used their knowledge of number facts, i.e. saying $15 + 17 = 32$, because it was obvious to them. There were two learners who used the inverse in the form $17 - 32 = 15$, but when they saw the negative sign, they were quick to change the order of the operation. One learner had great problems arriving at the answer of 15. She used 19, 17, 14 on her fingers, i.e. guess and check. When she eventually arrived at the answer, she was given an example using larger numbers ($x + 107 = 282$) and then she could not solve the equation by using the inverse of addition, i.e. subtraction, even with the availability of a calculator. This indicated that the interviewer perhaps should also have given the other learners large enough numbers to see if they would go beyond simple number facts to solve the problem through mathematical reasoning, i.e. through an understanding of the structure. This inability

of learners to solve the equation algebraically was to be expected because learners had had no previous exposure to algebraic notation and/or symbols.

(2) $201 = x + 109$

This equation involved addition, but larger numbers were used and with the unknown now appearing on the right hand side of the equal sign. After the first equation, some of the learners now realised that they had to subtract 109 from 201 to get 92.

Four of the learners said that 201 was the answer or that the answer was "first" and then the "sum" followed. The same learner who had problems with the larger numbers in (1) said that she had "to turn the sum around", i.e. $x + 109 = 201$, before she could actually solve the equation. Another learner first said that $109 - 201 = -92$ and was quite surprised at the sight of the minus sign. She was then given the example of $7 = x + 2$ and then she immediately saw that $7 - 2 = 5$. This might indicate that when the numbers were larger, she could not rely on intuitive solution strategies based on her knowledge of number facts. She was therefore not using an understanding of the algebraic structure of the equation to solve it.

(3) $x - 14 = 47$

This equation – the first one involving subtraction – produced some interesting results. Half of the learners gave the answer as 33, i.e. $47 - 14 = 33$. Their answers to the question whether x would be greater or less than 47 were both greater and less. One could assume that while the learners realised that the inverse of addition was subtraction, using subtraction as the inverse of addition seemed troublesome for some learners.

Even when the equation $x - 2 = 6$ was given to one learner, he gave his

answer as 4. On inquiring further, he said that it was 8 because 8 – 2 = 6. This could show that he did not immediately see 6 + 2 as being equal to 8. When another learner was given the equation ? – 3 = 10, he gave the answer as 7.
This would seem to confirm the idea that some learners, when determining the unknown in equations, were more used to recalling addition facts than thinking about the problem, even with relatively smaller numbers.

(4) $x - 17 + 10 - 10 = 12$

This was the first of the multi-operation equations. We wanted to find out if the learners would notice that 10 – 10 = 0. This was also the equation where no learner could have subtracted grouped like terms. None of the learners rewrote the equation as x – 17 = 12 even with the interviewer's help. In this equation very much guessing with numbers took place. Two learners actually said that 10 – 10 was equal to 20, i.e. they simply ignored the negative sign between the two 10s.
Not only did some learners ignore the minus sign between the 10s, but five of them also ignored the negative sign in front of the 17. This was an example of what Linchevski & Herscovics (1994) referred to as the detachment of the negative sign, i.e. where the minus - sign preceding a number is ignored.
Even when the interviewer had told them that the solution of the equation was 29, three of them could still not prove (verify) this answer. Eight learners began with the 12 on the right hand side. These learners maintained that 12 was the "answer" and then tried to determine the value of the unknown from there.

(5) $200 = 247 - x$

In this equation the unknown (x) is the minuend. Care was taken to make the numbers relatively easy to work with. The earners were first asked to compare equation (5) with equation (3). Six learners said that the answer was first, i.e. on the left hand side. Another six learners said that the answer was 200 + 247 = 447. However, when they had to test their answer, they said 247 – 447 = -200 and were quite confused. This could be due to learners:

(a) not knowing that the inverse of subtraction was multiplication or,

(b) not understanding the underlying algebraic structure of the equation and therefore reverting to known arithmetic methods or,

(c) not having been previously exposed to algebraic notation or,

(d) becoming confused due to the unknown appearing on the right hand side of the equation.

However, when the interviewer gave one learner the simple equation of $10 = 13 - x$, she immediately saw that the value of x was 3. Then she explained that 13 –10 = 3. This again implies that the learner knew how to solve the equation arithmetically by random substitution instead of algebraic methods.

(6) $2x = 110$

In this equation, there was only one operation, i.e. multiplication, and its inverse involved, but the numbers were relatively larger. Many of the learners seemed to know that to find the answer, they had to multiply 2 with "something" to get 110. The question for them was "how to find that something". When they eventually found that "something" the learners may have realised that :

(a) if they had divided by two initially, they could have determined the solution with less difficulty.

(b) division is the inverse of multiplication.

When the interviewer referred them to earlier analogous equations, i.e. $2x = 30$ and $2x = 16$, two learners experienced the same difficulty. One of them actually said that 2 times x was the same as 2 plus x, i.e. they could distinguish between the two expressions. This could have been due to the fact that the learners had not previously been exposed to algebraic notation. When the researcher gave examples like $4x = 116$ and $8x = 200$, some of the learners seemed to determine the solution more quickly. Now, exactly the same could be true of all the other/previous equations, e.g. $x + 17 = 32$, i.e.:

1. the learners realised (because they "know what the equation says"), that something needed to be added to 17 to give 32.
2. when reflecting on how to "find that something", they may realise that:
   (a) they needed to subtract 17 from 32 and
   (b) subtraction was the inverse of addition.

(7) $3x - 12 = 6$

This was the first equation where it was expected from learners to perform inverse operations of both multiplication and subtraction. The learners therefore had to understand the meaning and structure of the equation. Two of the learners immediately divided the 12 by 3 as if there was no term on the right hand side of the equal sign. This type was an unfamiliar equation to the learners. To try and solve the equation, learners once again reverted to methods known to them, i.e. guess check and random substitution. As a result of this, one learner said that he would start with 6 / 3 and then he would divide again by 12. Another learner saw that 12 / 2 = 6, because 6 was the answer. The researcher observed that none of the learners could solve the equation , and that to try and do something, many of them started with 6, the "answer". Their methods proved to be ineffective and they soon got stuck. This implies that they could not work with the unknown in the algebraic sense and that the cognitive gap appeared.

(8) $6x + 4x = 40$

This was the first equation that involved double occurrence of the unknown on the same side of the equal sign. Simple numbers were chosen to facilitate spontaneous grouping. The interviewer once again explained that both the x's must have the same value. All of the learners seemed to understand this idea and therefore their responses were quite interesting. No one could see that $6x + 4x = 10x$, once again, due to their limited exposure to algebraic notation. This type of equation was unique to them. This was also the first time that some learners used different values for x to try and get 40. This was also the first time that some learners used systematic substitution. They would begin with 6, then 5, then 4. When they saw that 4 worked some of them then realised that they had to determine an "x" that made both sides of the equal sign 40. These results seemed to confirm our initial idea that learners could not spontaneously operate with or on the unknown and that they relied on arithmetic methods to achieve their solutions. In this equation learners could not see that $6x + 4x = 10x$, i.e. they could not identify like terms (as was identified as a cognitive obstacle in chapter 1).

(9) $3x - 2x + 8x = 45$

In this equation all the learners could see that there were three x's on the left hand side. When they were asked to "make the left hand side shorter", none of them spontaneously came up with 9x. Once again the $- 2x$ presented many problems. It must be pointed out that $3x - 2x$ was deliberately used, because it was hoped that every learner could be able to see that $3x - 2x = x$ because small numbers were used.

This example confirms the fact that learners could not group terms in the unknown spontaneously. In fact, no evidence of algebraic behaviour was witnessed and learners once again reverted to numerical methods to solve the algebraic problem. However, they eventually realised that arithmetic methods did not help (or would not help) them.

It must be noted that one learner actually said $3 - 2 + 8 = 9$ (i.e. he ignored the unknown), but he could not continue from there. There was also evidence of random substitution e.g. one learner started off with 2 and then moved on to 3, etc.

When the interviewer pointed out that the equation could be written as $9x = 45$, many learners still did not see the analogy with $2x = 110$ they had had previously. The problem of working with like terms and /or ignoring the minus – sign manifested themselves as cognitive obstacles in this equation.

(10) $14x + 10 - 3x = 55 + 11$

This equation aimed to investigate whether learners might be able to recognise like terms. Twelve learners said that they would start with the "answer" i.e. $55 + 11 = 66$. Two learners then said $14 + 10 - 3 = 21$ i.e. not recognising the unlike terms and then went on to say that $66 \div 21 = ....$

One learner simply ignored the 10 and said $14 - 3 = 11$ and $66 \div 11 = 6$. When asked about the 10, he said that he did not know.

A few just started substituting and got nowhere near the solution – some using different values for the x. This was the equation which left many of the learners embarrassed and frustrated because they simply did not know what to do, or where to start. This again could be seen as an indication of the existence of a cognitive gap.

(11) $4x = x + 21$

This was the first equation where the unknown occurred on both sides of the equal sign. Just like the results of Kieran (1984), not a single learner managed to solve this equation.

It might also have been because a singleton occurred on the right hand side.

One learner divided by 4 maybe because he saw only two terms. Another one said 21/4, i.e. once again ignoring the x on the right hand side.

The others tried random substitution with x equal to 2, 5, 10, 8, etc. When they did not get the correct answer (7), their impatience became obvious.

(12) $10x - 6x + 3x - x = 2x + 10$

Here the unknown appeared four times on the left hand side and once on the right hand side. Four of the learners simply said that they would not be able to do it.

One learner just said $10 + 6 - 3 = 13$, i.e. leaving out the x's; ignoring the negative sign in front of the 6x and ignoring the singleton. However, when she was asked to read the equation, she then saw both the negative sign and the singleton.

Another learner said $10 - 6 = 4 + 3 = 7$, i.e. ignoring the x at first and then she said that the answer was $7 - x$.

All the learners ignored the expression on the right of the equal sign.

Equations (11) and (12) confirmed the existence of the didactic cut (Filloy & Rojano, 1989) that many learners could not solve linear equations with the occurrence of the unknown on both sides of the equal sign. These equations appeared to be more difficult for learners than those before them.

### 3.3 GENERAL DISCUSSION OF INTERVIEWS

When discussing the results of this study qualitatively, it must be pointed out that it (the study) was not a teaching experiment, i.e. the information was gathered with as little as possible influence by the researcher .The initial study was conducted prior to any formal introduction to Algebra. Not all the cognitive obstacles such as:

a) the use of brackets

b) the inability of learners to select the appropriate operation for partial sums, as mentioned by Hercovics and Linchevski (1989), were investigated.

The subjects included both above average and below average learners. It was not possible to make a very sharp distinction between the mathematical abilities of the learners as they entered high school for the first time. The conditions were different from primary school. This means that not too much value could be attached to the gradings which had been sent from their primary school.

The **conditions** under which the investigation was conducted were far from optimal (unlike Hercovics and Linchevski, 1994). There were times when I had to be both interviewer and observer i.e. sometimes I had to operate the audiovisual equipment myself. The interviews were done during normal school hours which meant that it was not always quiet and/or uninterrupted.

This investigation was successful in identifying the existence of a **cognitive gap** as described by Hercovics & Linchevski (1994), i.e. the learners' inability to operate with or on the unknown. This study was restricted to linear equations with only one unknown (x). The subjects' performances indicated that they did not view the literal symbol (x) as a generalised number.

Some problem areas which were identified are the following:

The subjects depended largely on using intuitive knowledge of **number facts**, as well as guessing, random substitution and using their fingers to try and find the solutions. This could be seen as a tendency in learners to fall back to the known when faced with the unknown. Some of them even said that it was "easy to work with tens" and "five is the half of ten" and "two because two is easy", etc. As the numbers in the equation became larger they became more confused. This could indicate that they were used to procedural strategies and not to the structure of Algebra.

Even though learners had access to a calculator at all times, many still reverted to mental arithmetical operations and using their fingers. It could be due to the fact that learners had had no previous instruction in Algebra. They could therefore not cope without reverting to what was known to them, i.e. arithmetic/primitive methods.

The study also shows that when learners used estimation, they did not do it systematically. It could be that the learners had a low number sense, e.g. when learner "M" had to go from 5 to 4 in (8), she instead went from 5 to 3 and then back to 4.

When the **unknown appeared more than once** in the same equation, all the subjects experienced many problems. Even when they were told that the x's should have the same numerical value, many reverted to different values of x to get the answer. Just like in the study of Filloy and Rojano (1984), no learner could solve the equations arithmetically when the unknown appeared on both sides of the equal sign. In these equations guess and check and random substitution would not have worked effectively. The subjects seemed to know that the **inverse** of addition was subtraction, but could not recognise initially that the inverse of subtraction was addition.

The size of the numbers also played an important role.

When the numbers were small, they could easily see the solution, but when the same reasoning was required with larger numbers, they had problems finding the solution. The inverse of multiplication and division was also easily done with small numbers, but **not** with larger numbers, which could again be interpreted as a result of their poor number sense, and their lack of knowledge of the structural nature of Algebra.

When there was a box or a question mark (?) as an unknown, e.g. 2 x ? = 16 most of the subjects knew that the answer was 8 because 2 x 8 = 16. However, 2x = 16 was experienced as more difficult. This became more serious in equation (6), i.e. 2x = 110 with its relatively large number (55) as the solution. Here, in (6), some learners had answers such as 220, 10, etc.

For many of the subjects the **equal sign** meant a division between the "sum and the answer" with the answer mostly being on the right hand side. Even when the unknown was on the right hand side, it (the right hand) was still regarded as the answer. Some of them actually said that those equations where the unknown was on the right hand of the equal sign, were "turned around".

No **grouping and/or spontaneous cancellation** of like terms occurred (even in obvious cases like 10 – 10). When they were told that 10 – 10 = 0, they would still write down 10 – 10 in the equation instead of the zero. When the researcher told them to make the equation "shorter", they still did not see grouping or cancellation (with or without the x), i.e. they did not just write down zero. The longer the equation, the more confused they became even when cancellation could seem to be obvious.

What happened was that some subjects ignored the unknown (x) and then they just added/subtracted unlike terms. Even when the analogy of Rands was used, some still had many problems. Quite a few left out the unknown and then they would replace it after they had been asked to, e.g. "9 times what is equal to 45?" Another problem was the **singleton**. (This was not included intentionally). The subjects either ignored it completely or did not know that the coefficient of x was one.

The subjects had a general tendency to ignore the **negative sign**. One subject actually said that she did not like the minus-sign. Even after some had read the equation verbally correctly, they would still change the negative sign to a positive sign. When there was 10 – 10, one subject said it was 20. It was especially so in equations (9) and (10) when most of the subjects ignored the negative sign. This could have been due to the length (number of terms) of these equations.

# CHAPTER FOUR

## CONCLUSION

This study has confirmed the existence of a cognitive gap as defined by Herscovics & Linchevski (1994), even though this study was restricted to linear equations with one unknown. It shows that when learners have to deal with equations which cannot be solved effectively by primitive, intuitive, arithmetic methods, a breakdown in thinking appears.

Equations are an integral part of algebra and algebra forms a significant part of mathematics, because of its symbolic and structural nature. Since we live in a technological age, mathematics, and thus algebra, is to a lesser or greater degree necessary for many learners, irrespective of their future vocations. Since this study was conducted prior to any formal instruction, the demarcation between arithmetic and algebra has been identified in so far as learners try to use arithmetic when algebraic thinking was required. This could mean that problems that had been experienced during the earlier arithmetical phase, e.g. the learners' poor number sense, manifested themselves as cognitive obstacles in the study.

It has also been shown that the meaning of the equal sign needed more clarification. Learners have to move away from it being the separation between the statement on the left hand side and the answer on the right hand side, towards it (the equal sign) indicating equivalence of the two sides.

The teaching of the order of operations prior to this study also seemed to need more attention. Learners mostly learn addition first and then subtraction and multiplication before division. This seems to lead them to believe that addition takes precedence over subtraction and multiplication over division. This manifested itself in the equations where the inverse operation was required.

The detachment of the negative sign was also evident in those equations where the negative sign was present ( Linchevski & Herscovics, 1996).

In this study an aspect of the problem solving approach to teaching was used, i.e. learners were faced with material which they had not done before. The learners therefore had to use that which they already knew to find what was unknown to them. This approach has the advantage over the traditional methods of teaching, because the researcher (or the teachers) must respect and build on the primitive and/or intuitive methods of solution of the learners (Anthony, 1996). If this approach is not properly implemented, teachers may teach the learners rules to be used when solving equations. Learners may therefore not understand the underlying algebraic structure and could resort to memorising these rules.

Herscovics & Linchevski (1994), pointed out some of the problems that learners had to face when they were first introduced to algebra. These were:

- the pace of dealing with the new material and,
- the formal nature of the approach to instruction.

Curriculum planners will have to address these problems, because some didactical intervention will be necessary, and teachers must be made aware of the problems learners face when making the transition from arithmetical to algebraic thinking.

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**APPENDIX A**

**LEARNER'S PERSONAL INFORMATION**

Name: ..........................................................................

Nickname: ..........................................................................

Previous School: ..........................................................................

Grade: ............................

Age: ............................

Address: .....................................................

.....................................................

.....................................................

Telephone: .....................................

How many brothers? ...........

sisters? ...........

What do you intend to study after leaving school? .....................................................

Do you like Mathematics? ......................

Do you think Mathematics is important? ......................

Are you going to do Mathematics in Grade 10? ......................

## APPENDIX B: EQUATIONS FOR THE WRITTEN TEST

**THIS IS NOT A TEST.**
**Determine (calculate) the value of x in all the following equations.**
**Please write down all you calculations. Use a calculator. Test your answers.**

**A.**

1. $x + 7 = 22$
2. $5 + x = 25$
3. $x + 70 = 1100$
4. $12 = x + 3$
5. $100 = x + 27$

**B.**

6. $x - 6 = 8$
7. $17 - x = 18$
8. $240 = 200 - x$
9. $10 + 22 = 5 - x$
10. $x - 17 + 10 - 10 + 17 + 6 = 12$
11. $22 = 5 - 10 - 7 + 10 - 5 + x$

**C.**

12. $2x = 4$
13. $120 = 40x$
14. $2x - 2 = 6$
15. $2x - 2 - 6 = 8$
16. $17 + 10 + x = 50 + 2 - 1$
17. $4 + x - 2 + 11 = 101 - 3$
18. $4x + 10 = 110 - 10 + 10 - 10$
19. $10x - 65 - 5 = 30$

20. $10 + 7x - 15 = 10 - 15 + 14$

**D.**

21. $\frac{15}{x} = 3$

22. $\frac{150}{x} = 5$

23. $\frac{x}{3} = 2$

24. $\frac{x}{8} = 20$

25. $15 = \frac{x}{4}$

26. $23 = \frac{115}{x}$

**E.**

27. $x + x = 2$

28. $x - x + x = 7$

29. $6x + 4x = 20$

30. $12x + 7x + 5x + 7 = 55$

31. $3x - 2x + 8x = 27 + 6$

32. $170x - 13x = 314$

33. $100x + 9 - 90x = 69 + 10 - 69 + 30$

**F.**

34. $4x = x + 15$

35. $5x + 12 = 3x + 24$

36. $3x + 2x + 10 = 4x - 11$

37. $10x - 6x + 3x - x = 4x + 100$

38. $3x - 2x + 7x - 7 + 24 = 2x + x + 27$

## APPENDIX C: EQUATIONS USED DURING INTERVIEWS

**Solve the following equations.**

**Please show all your work and check your answers.**

**You may use a calculator.**

1. $x + 17 = 32$
2. $201 = x + 109$
3. $x - 14 = 47$
4. $x - 17 + 10 - 10 = 12$
5. $200 = 247 - x$
6. $2x = 110$
7. $3x - 12 = 6$
8. $6x + 4x = 40$
9. $3x - 2x + 8x = 45$
10. $14x + 10 - 3x = 55 + 11$
11. $4x = x + 21$
12. $10x - 6x + 3x - x = 2x + 10$

## APPENDIX D

| | Equation | Attempts | Successful | Success % | Most Frequent Procedures% | | | | | Most Frequent Mistakes % | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| No | | | | | I | NF | RS | SFT | DIV.2 | WI | AE | NOTT | DF | TT | IU | WB | NES |
| A1 | X + 7 = 22 | 38 | 37 | 97.4 | 51 | 48,7 | | | | | 100 | | | | | | |
| 2 | 5 + X = 25 | 38 | 36 | 94.7 | 44,4 | 55,5 | | | | 50 | 50 | | | | | | |
| 3 | X + 70 = 1100 | 38 | 29 | 76.3 | 55,5 | 44,5 | | | | 44,4 | 55,6 | | | | | | |
| 4 | 12 = X + 3 | 38 | 34 | 89.5 | 47,1 | 52,9 | | | | 25 | 75 | | | | | | |
| 5 | 100 = X + 27 | 38 | 34 | 89.5 | 58,8 | 41,2 | | | | | 75 | | | | | | 14,3 |
| | | | | | | | | | | | | | | | | | |
| B6 | X - 6 = 8 | 38 | 31 | 81.6 | 64,5 | 34,5 | | | | 85,7 | | | | | | | |
| 7 | 17 + X = 18 | 38 | 20 | 52.6 | 85 | 15 | | | | 55,6 | 16,7 | | | | | | 33,3 |
| 8 | 240 = 200 - X | 38 | 13 | 34.2 | 76,9 | 23,1 | | | | 100 | | | | | | | |
| 9 | 10 + 22 = 5 + X | 33 | 11 | 33.3 | 90,9 | 9,1 | | | | 33,3 | 23,8 | 28,6 | | | 14,3 | | 9,5 |
| 10 | X - 17 + 10 - 10 + 17 + 6 = 12 | 30 | 4 | 13.3 | 25 | 50 | | 25 | | 3,8 | 34,7 | 57,7 | | 3,8 | 7,6 | | |
| 11 | 22 = 5 - 10 - 7 + 10 - 5 + X | 31 | 4 | 12.9 | 25 | 50 | 25 | | | 7,4 | 29,6 | 55,6 | | | | 3,7 | |
| | | | | | | | | | | | | | | | | | |
| C12 | 2X = 4 | 35 | 24 | 68.6 | 25 | 75 | | | | 90,1 | 9,9 | | | | | | |
| 13 | 120 = 40X | 35 | 17 | 48.6 | 41,1 | 58,9 | | | | 78,9 | 10,5 | | | | | | |
| 14 | 2X - 2 = 6 | 32 | 17 | 53.1 | 5,8 | 52,9 | | 41,1 | | 60,0 | 6,7 | | | 26,7 | 6,7 | | |
| 15 | 2X - 2 - 6 = 8 | 32 | 11 | 34.4 | | 54,5 | | 45,5 | | 33,3 | | 14,3 | | 14,3 | | 4,7 | |
| 16 | 17 + 10 + X = 50 + 2 - 1 | 28 | 8 | 28.6 | | 87,5 | | 12,5 | | | 50 | 35,0 | | 5 | 5 | 5 | |
| 17 | 4 + X - 2 + 11 = 101 - 3 | 22 | 7 | 31.8 | | 57,1 | 28,6 | 14,3 | | | 46,7 | 40 | | 6,7 | | 6,7 | |
| 18 | 4X + 10 = 110 - 10 + 10 - 10 | 19 | 5 | 26.3 | 40 | 40 | 20 | | | 7,1 | 35,7 | 57,1 | | 7,1 | | | |
| 19 | 10X - 65 - 5 = 30 | 18 | 4 | 22.2 | 25 | 50 | 25 | | | 14,3 | 28,6 | 35,7 | 7,1 | 14,3 | | | |
| 20 | 10 + 7X - 15 = 10 - 15 = 14 | 19 | 3 | 15.8 | | 66,7 | | 33,3 | | 6,3 | 50 | 25 | | 6,3 | 12,5 | | |
| | | | | | | | | | | | | | | | | | |
| D21 | 15 / X = 3 | 36 | 29 | 80.6 | 27,6 | 69,0 | 3,4 | | | 51,1 | 48,9 | | | | | | |

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 22 | 150 / X= 5 | 35 | 25 | 71.4 | 40 | 60 | | | | 70 | 30 | | | | | | |
| 23 | X / 3 = 2 | 34 | 24 | 70.6 | 66,7 | 33,3 | | | | 70 | 30 | | | | | | |
| 24 | X / 8 = 20 | 34 | 12 | 35.3 | 58,3 | 41,7 | | | | 81,8 | 18,2 | | | | | | |
| 25 | 15 = X / 4 | 33 | 14 | 42.4 | 71,4 | 28,6 | | | | 73,7 | 26,3 | | | | | | |
| 26 | 23 = 115 / X | 31 | 16 | 51.6 | 56,3 | 43,8 | | | | 66,7 | 33,3 | | | | | | |
| | | | | | | | | | | | | | | | | | |
| E27 | X + X = 2 | 36 | 34 | 94.4 | | 97,1 | 2,9 | | | | 50 | | 50 | | | | |
| 28 | X - X + X = 7 | 36 | 13 | 36.1 | | 76,9 | 15,4 | 7,7 | | | 13,0 | 13,0 | 73,9 | | | | |
| 29 | 6X + 4 X = 20 | 34 | 15 | 44.1 | | 13,3 | | 86,7 | | 5,3 | 15,8 | 21,1 | 36,8 | | 5,3 | | 5,3 |
| 30 | 12X + 7X + 5X + 7 = 55 | 32 | 9 | 28.1 | | | | 100 | | | 21,7 | 26,1 | 26,1 | 8,6 | | | |
| 31 | 3X - 2X + 8X = 27 + 6 | 27 | 2 | 0.1 | | | | 50 | | 4 | 28 | 36 | 20 | 8 | 4 | | |
| 32 | 170X - 13X = 314 | 27 | 5 | 18.6 | | | | 100 | | | 31,8 | 27,3 | 13,6 | 9,1 | 13,6 | | |
| 33 | 100X + 9 - 90X = 69 + 10 - 69 + 30 | 25 | 0 | 0 | | | | | | | 36 | 32 | 20 | 8 | 4 | | |
| | | | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | | | |
| F34 | 4X = X + 15 | 31 | 7 | 22.6 | | | 14,3 | 85,7 | | | 12,5 | | 79,2 | | 8,3 | | |
| 35 | 5X + 12 = 3X + 24 | 31 | 0 | 0 | | | | | | | 19,4 | 6,5 | 45,2 | 6,5 | 9,7 | | |
| 36 | 3X + 2X + 10 = 4X - 11 | 28 | 0 | 0 | | | | | | | 28,6 | 32,1 | 25 | 3,6 | 7,1 | | |
| 37 | 10X - 6X + 3X - X = 4X + 100 | 26 | 0 | 0 | | | | | | | 46,2 | 19,2 | 26,9 | 3,8 | | | |
| 38 | 3X - 2X + 7X - 7 + 24 = 2X + X + 2 | 23 | 0 | 0 | | | | | | | 47,8 | 21,7 | 30,4 | | | | |

## APPENDIX E: TRANSCRIPTS OF INTERVIEWS

### ALLAN BLAAUW (A)

1: $x + 17 = 32$

T: How are you going to find x?.... I am going to ask questions as you write your answer.
A: I shall minus 17 from 32 and then I shall get 15. Then I add 15 to 17 again to get 32. That is why $x = 15$.
T: Okay ...

2: $201 = x + 109$

T: How does (2) differ from (1)?
A: In (1) the x was in front and in (2) it is at the back i.e. behind the equal sign.
T: Now find the value of x please.
A: (pauses)
T: In (1) you have subtracted what are you going to do in (2)? Tell me.
A: (presses on the calculator and writes) 92.
T: How did you get it?
A: $201 - 109 = 92$
T: Why did you minus again?
A: Because there is a plus.
T: Let us test. You say x is 92. Now wherever there is an x, you must write 92.
A: Must I now say 92 + 109 to get 201?
T: Yes
A: (presses 29 instead of 92 and gets the wrong answer).
T: Do it again.
A: (this time he does it right)
T: So if there is a plus then you know to minus.
A: Yes

3: $x - 14 = 47$

T: What are you going to do here?
A: (........ scratches his head ......... pauses)
T: Will x be more or less than 47?
A: Less
T: Why ........ show me.
A: Okay, $47 - 14 = 33$
T: Why do you subtract? Don't you see the minus?
A: (does the same operation i.e. $47 - 14 = 33$ again.
T: Test your answer for us.
A: $33 - 14 = 19$. No! I must add.
T: Why? What is the actual value of x?
A: (writes) $33 + 19 = 52$
T: What now? Did you get 52?

A: (seems to be confused ......... pauses) I shall take away 5.
T: Why minus 5?
A: To get 47
T: Now we have quite a few x's, but there is only one x in the sum. What is the value of the real x?
A: (silent ...)
T: Let us go back to the original problem. You said where there is a plus there you must minus. Now there is a minus. What should you do now?
A: I must plus.
T: What must you add?
A: 14 + 47 = 61 (on the calculator) So x = 61
T: Test your answer.
A: How?
T: Is 61 – 14 = 47
A: So x = 61
T: Okay.

4: x – 17 + 10 – 10 = 12

A: (changes the minus sign in front of the – 10 to a plus sign.)
T: Where are you going to begin?
A: From the front. Now I take 17 + 10 – 10 = 17
T: And then?
A: Then 17 – 5
T: Why?
A: (silent ....... fiddles on the calculator ...... writes) 12 and then I take 5 – 17 = -12
T: There is negative 12 and not 12.
A: Now I shall make it plus 12.
T: So what is the value of x?
A: (thinks ........) -5
T: What does –5 mean?
A: I want to see if I can get to 12.
T: Why do you want to get to 12?
A: 12 is the answer.
T: Now look carefully. Here is 10 – 10. How much is that?
A: 0
T: So, is it possible to write the equation as x – 17 = 12?
A: Yes
T: Can you solve the problem now? What are you going to do?
A: (quiet ........) 5
T: Why 5? Are you guessing?
A: Because 5 + 12 = 17
T: So what is the value of x?
A: 5. (then he tests and gets 17)
T: But you said the answer was 12.
A: But now I take 5 and minus 17 and then I get –12.
T: Now we are going in circles. How did you get 61 in (3)? Do you stil remember?
A: (looks at (3) for a long while ........ scratches his head and then) oh,

12 + 17 = 29.

T: Test your answer in the original equation.

A: (once again he uses 10 – 10)

T: Why do you press 10 – 10 again?

A: Because it is 0.

5: 200 = 247 – x

T: What do you say ........ Does (5) differ from (3)?

A: They are nearly the same. Now I shall take 200 and add 247 and I get 447.

T: OK. Let us use smaller numbers.
If 3 = 5 – x, what will you do?

A: I will say 3 + 5 = 8

T: Test

A: 3 = 5 – 8

T: Is that true?

A: Yes.

T: Which of 8 or 5 is greater?

A: 8

T: So, if you take 8 from 5?

A: (uses his calculator but turns it around i.e. 8 – 5)

T: You say 5 – 8 but you press 8 – 5. You must have 5 – 8.

A: (presses 8 – 5 again and looks at me in an uncertain way)

T: What now?

A: I must use my head! (scratches his head)

T: Now use 447 in the sum.

A: (once again he says 247 – 447 = -200)

T: Something is not right! Do you agree? What will we do next?

A: If I take a large number away from a small one then I shall get a minus.

T: So must the x (in the equation) be greater of less than 247?

A: Greater.

T: Why? Aren't you going to get a minus again?

A: (silent ......)

T: What if I tell you the answer is 47?

A: (tests the answer) That is right.

T: How did you get the 47?

A: I saw that 247 – 47 = 200.

6: 2x = 110

T: What does 2x mean?

A: (mutters something like "now what")

T: ....... Previously you said it was 2 times x .........

A: (silent .......)

T: Will it be greater or less than 110?

A: (............) less.

T: Why?

A: To get the answer ....... x will be 55!

T: Why?

A: The half.
T: How do you get half? Show me.
A: Okay, $110 \div 2 = 55$
T: What if I give you this one: $4x = 116$
A: (looks uncertain .......)
T: Tell me what you think.
A: I am looking for the half.
T: But there is 4x!
A: But the half is easy.
T: Okay, this is $\frac{116}{4} = 29$.

7: $3x - 12 = 6$

A: (long silence ...........)
T: Okay, write down what you are thinking.
A: I think ..........
T: Okay. How will you get the value of x?
A: (once again silent ....... scratches his head)
T: Aren't you going to begin with numbers?
A: (writes) $12 + 6 = 18$
T: Why did you add?
A: I am looking for the value of x.
T: So why didn't you ÷ , - , or x?
A: Plus helps me better and then $18 + 12 + 6 = 36$
T: Why do you have + 6 again?
A: 6 is the answer because $36 - 12 = 24$
T: Is $x = 24$?
A: Yes
T: Test your answer.
A: Okay. $3 \times 24 - 12 = 60$
T: There is 6 ! Now what?
A: (softly) If will divide by 10.
T: Okay.

8: $6x + 4x = 40$

T: Now we have two x's. They must both have the same value.
A: The same answer?
T: You say the same answer? Where will you begin?
A: ...... The two x's throw me out.
T: What if I say $x = 4$. See what you will do.
A: Where? (pointing to the two x's)
T: Both x's must have the same value.
A: (writes) $64 + 44$
T: Why?
A: I just write the 4 next to the 6 and next to the 4.
T: but 6x means 6 times x.
A: So those x's must be 40!
T: Okay. What does 6x mean again?

A: ............ times!
T: Now times for us.
A: 6 x 4 + 4 x 4 = 40
T: Now look at 2x + 5x = 14. What would you do now?
A: (mutters something like 2x is 2 times x)
T: Must these x's be the same?
A: Yes
T: How much is 2 + 5?
A: 7
T: If you could write 2x + 5x = 14 as 7x = 14, what is the answer?
A: x = 2
T: How did you get that?
A: I know that 7 x 14 = 2
T: Are you sure?
A: No it is 98.

9: 3x – 2x + 8x = 45

T: Do you see what I did in (8)? How many x's are on the left hand side?
A: 3
T: Now work out the left hand side.
A: (writes) 3x times (deletes it)
T: What did you want to write?
A: 3x x 2x + 8 x x = 45
T: No! Look again.
A: (writes) 3x + 2x + 8x = 13
T: What about the minus-sign?
A: Can I take 5 for 3 + 2?
T: But there is 3 – 2 = 1
A: Plus 8?
T: You do it.
A: It's 9.
T: 9 what?
A: 9x = 45
T: So what is x?
A: I don't know.
T: What does 9x mean?
A: I don't know.
T: Previously we said that if 9 times ........ = 45. What would come in the block?
A: ................ (long silence)
T: What is the answer of 2 x ......... = 6
A: Three!
T: How did you get 3?
A: Half of 6.
T: Didn't you say 6 ÷ 2?
A: No ... I just subtracted.
T: Okay, our answer is 5 because 45 ÷ 9 = 5.

10: 14x + 10 – 3x = 55 + 11

T: How does (9) differ from 10?
A: ........... (silent)
T: Where will you start?
A: ..................
T: Look at the right hand of the equal sign.
A: Is 55 + 11 now the answer?
T: You say! Can the answer be two numbers?
A: No, but if I plus 55 and 11 I get 66.
(writes down the 66)
T: What would you call this (the left hand side) of the equation?
A: The sum!
T: Okay..... How many x's are there in the sum?
A: Three (starts writing 14 x 2)
T: Why do you multiply by 2?
A: Because 14 x 2 = 7
T: Are you sure it is not 28?
A: (presses + 10)
T: Why did you press + 10 and not 10 x 2?
A: Because the 10 has no x.
T: Okay, why did you start with 2 and not with 3 or 4?
A: Two is easier.

11: $4x = x + 21$

T: Now there are x's on both sides. Where will you begin?
A: I can't do it !! (looks frustrated)
T: If I say that the answer is 7. Now test it.
A: Okay, 4 x 7 = 28
T: Yes, and now the other side.
A: Now 28 + 7 + 2l
T: (seeing that the time is up)
Do you think there is a way to do these sums?
A: I don't think so.

12: No attempt

**LORISTON (L)**

1: $x + 17 = 32$

T: Tell me what you are going to do.
L: I shall take the 32 and then I shall take away the 17 and the answer will be 17.
T: Now you must test your answer i.e. there where the x is, you must make it 15.
L: (writes the 15 above the x and gets 32)
Yes it is right.

2: $201 = x + 109$

T: Is there a difference between (2) and (1)?
L: No.
T: Okay. What are you going to do next?
L: I get 92
T: How?
L: I take 201 – 109 = 92
T: Why did you not take 109 – 201?
L: Because 201 is more than 109!

3: $x - 14 = 47$

T: How does (3) differ from (1)?
L: You add up.
T: What are you going to add?
L: The 14 and the 47.
T: Now use the calculator.
L: Then I get 61.
T: Now test in the original equation.
L: Okay. I'll say 61 – 14 = 47
T: So what is the value of x?
L: 61

4: $x - 17 + 10 - 10 = 12$

T: Where will you begin? What do you see?
L: I shall start with the 12.
T: Why?
L: I cannot start in front because I do not know what the answer is.
T: Okay, write down.
L: 12 – 10 = 2
T: Why do you minus?
L: Because there is + 10 (pointing at the last term)
T: Continue.
L: 2 + 10 = 12
T: Why + 10?
L: I got – 10?

T: What next? You already have 12.
L: (long silence) 10 – 10 = 0 + 17 = 17 – 12
T: Now what is the value of x?
L: 5
T: Test your answer.
L: (tests and gets –12) I say it is 12.
T: Is –12 the same as +12 ?
L: No.
T: What are you going to do next?
L: eh.....eh..... x is less than 12.
T: And .....
L: (long silence)
T: If I say to you 10 – 10 = 0, will you please do the sum from there.
L: I don't know ..... but 17 – 12 = 5.
T: Five again.
L: (looks uncertain) Yes.
T: Can't you write the sum as x – 17 + 10 = 12
L: Explain to me!
T: If 10 – 10 = 0, then you can write it as x – 17 = 12 ....... Don't you agree?
L: Yes. Now I know what the value of x is.
T: What is the value of x?
L: 5
T: But it did not work before.
L: Now it works!!
T: Let us see.
L: (uses the calculator) 5 – 17 = -12 (looks bemused)
T: What did you do in (3)?
L: I added!!
T: Can't you do the same with x – 17 = 12?
L: No, Sir, I do not know what the value of x is yet!
T: But you did not know what it was in (3) neither!!
L: Oh .....

5: 200 = 247 – x

T: Does (5) differ from (3)?
L: No ....
T: Do it.
L: I took the same i.e. 200 – 247 and then I got the answer.
T: What is the answer?
L: 47
T: Test
L: (now he writes) 247 – 200 = 47 and draws a line through the original 200 – 247 = 47.
T: Why?
L: I put the 200 behind the 247 and then I did not get a minus.

6: $2x = 110$

L: Is this times, sir?
T: You tell me ......... Previously you said it was multiplication.
L: No, sir ...... Is it 2x or 2 times x?
T: It is 2 times x. What will you do next?
L: I take the 110 and then I subtract the 2 and then I get 108.
T: Why do you subtract?
L: I want to get the answer!
T: Okay, test your answer. Remember to write your 108 in the place of x.
L: Oh ..... $x + 2 = 108$
T: Why plus? Previously you did not do it ..........
L: (silent)
T: If I give you $2x = 10$, what is x?
L: 5
T: How did you get the 5?
L: 2 times 5 is 10 <u>or</u> $10 - 2$ is ........
T: Why did you subtract?
L: (looks uncertain/surprised)

7: $3x - 12 = 6$

T: Where are you going to begin?
L: Here by the 6.
T: Why the 6?
L: Then I shall have $6 + 12$.
T: Why so you add?
L: So that I can get the answer.
T: What is the answer? What do you regard as the answer? The 6 or the x?
L: The x.
T: Okay. Continue.
L: So $18 + 3 =$
T: Why do you do that?
L: I should have taken minus.
T: Why minus?
L: There is a minus in front of the 12 therefore $18 - 3 = 15$.
T: So ..... is your answer 15. Test it.
L: (starts to work on the calculator but stops halfway)
T: What does 3x mean ........ again?
L: Can't I leave out the 12 to begin with?
T: Okay ............... why ........... write ..........
L: $6 \div 3 = 2 \div 12 = 6$
T: The answer of $2 \div 12$ is not 12.
L: No .......... (then) $6 - 12 = -6$
T: Now you have –6. Why do you subtract?
L: 12 is more than 2 and the 12 stands at the back ......
T: But you have divided!
L: No, it will not work.

T: If I say the answer is 6... See if it works.
L: Yes, it is 6!
T: What did you do?
L: Can't I put the 12 before the 6 and then say 12 – 6 = 6?
T: Why?
L: Then I won't get a negative number.
T: Once again. What does 3x mean? Isn't it 3 times x?
L: Yes.
T: So why did you first add and then you subtracted?
L: I don't know (looks sheepish)

8: 6x + 4x = 40

T: Do you see that there are two x's? Now we must find the same x in both cases. Start.
L: 40 x 4 = 160
T: Why did you do that?
L: I took the right answer i.e. 40!
T: Why didn't you also multiply the 6 by 40?
L: I will still come to the 6.
T: Okay.
L: Now I have 40 x 4 = 160 – 6 = 154
T: Why did you subtract?
L: (long silence)
T: Why did you subtract the 6? What is it that you want to get?
L: 40
T: How will you get to 40?
L: 6 times 4
T: Why 4?
L: Because there is a 4 by the 4x.
T: What about the + -sign between them?
L: (long silence)
T: If I say the answer of (8) is 4, will you believe me? Test my answer.
L: How?
T: Replace the x's with 4.
L: Both x's?
T: Yes.
L: (now he writes) 64 + 44 = 108.
T: But didn't we say that 6x is 6 times x?

9: 3x – 2x + 8x = 45

T: How many x's do you see on the left hand side?
L: There are 3.
T: Once again we must find the same everywhere. Where will you begin?
L: With 45 x 2.
T: Why do you multiply by 2?
L: I just take the two and then I multiply.
T: Do you see the x as multiplication? Remember that the x is the value which we must find like we had to find the number in the block before.

L: And now .........
T: Do you see that there are three x's on the left hand side? Can you make that side shorter?
L: (looks uncertain) 3 x 2 x 8
T: Why do you do this?
L: Sir, you have told me to multiply!
T: But there are a plus sign and a minus sign. Don't you see it?
L: Okay, Now I shall take $3 + 2 = 5 \times 8$
T: Now you only multiply the 5 by 8. Why?
L: To get 40!!
T: But there is 45 on the right hand side. Let's go to (10).

10: $14x + 10 - 3x = 55 + 11$

T: What do you do on the right hand side?
L: I must add because there is a plus i.e. $55 + 11 = 66$.
T: Do you agree with me that the sum can be written like this: $14x + 10 - 3x = 66$?
L: (nods) Now I am going to divide.
T: Show me.
L: $66 \div 3 = 22$
T: Why do you divide by 3?
L: (points at the 3 in the $- 3x$)
T: In (6) you also divided, but there were only 2 terms ..... (one on either side of the equal sign .......... didn't you?
L: .... Now $66 \div 3 = 22 + 10$
T: Why do you plus the 10?
L: To get 32 and then I shall say $32 - 14 = 18$
T: What happened to the x next to the 14?
L: (long silence)
T: Please answer me.
L: I did not understand him.
T: Why not?
L: (looks uncertain, embarrassed)

11: $4x = x + 21$

T: Read the equation for us and begin.
L: I shall say $21 \div 4$
T: Why do you divide by 4? Remember there is an x on both sides of the equal sign.
L: (long silence)
T: (now himself getting impatient) What about the x on the other side?
L: Can't I take the x as a 4?
T: Which x?
L: The one on the left hand side.
T: Why?
L: I just thought if this $x = 4$ then that x must also be 4.
T: You just thought that they must both be 4?
L: Yes.

12: $10x - 6x + 3x - x = 2x + 10$

L: I will say 10 – 6 = 4 + 3 = 7 (now looking really tired)
T: Why did you add the 3?
L: It stands there.
T: But there is also a minus x. Okay, continue.
L: Then I add it to 4 i.e. 7 + 4
T: Why do you add 4?
L: I just thought that x's value must be 4!
T: But you still have the x on the right hand side.
L: It is 12x.
T: (Thank you).

**JANINE JANSEN (JJ)**

1: x + 17 = 32

T: What will you do to find x?
JJ: I will now say 17 – 32 = -15
T: Please write it down. You must now write your answer in the place of x and then you must test it on the calculator.
JJ: (now saying) 15 + 17 = 32. Yes.
T: Are you sure?
JJ: Yes.
T: What if I say x + 3 = 8? What would x be?
JJ: Then it is 8 – 3 = 5
T: So what did you do with x + 17 = 32?
JJ: I took 32 away from 17.
T: So why did you turn it the other way here.
JJ: No, Sir! Sorry (wants to scratch out). It must be 32 – 17 = 15
T: So what is the answer?
JJ: 15

2: 201 = x + 109

T: Does (2) differ from (1)
JJ: Yes
T: How?
JJ: The answer is here first and then comes x + 109.
T: Now work it out for us.
JJ: (writes) 201 – 109 = 92
T: So x = 92? Can I assume that if there is a plus in the equation then you know that you must subtract?
JJ: Yes.

3: x – 14 = 47

JJ: (writes it down as x + 47 .....)
T: What are you going to do now?
JJ: Now I am going to plus i.e. 14 + 47 = 61
T: Why are you going to plus?
JJ: Because there is a minus.
T: Test.
JJ: Okay. 61 – 14 = 47
T: So if there is a minus what will you do?
JJ: Then I will add.

4: x – 17 + 10 – 10 = 12

T: Just look at the numbers and tell me what you see.
JJ: I see this 17 + 10 – 10 .... I see two tens and then this 12.
T: Where are you going to begin? You say that you see a plus 10 and a minus 10.

JJ: (silent) Okay, 12 + 10 – 10 + 17 = 29
T: I see that you have changed all the signs. Why did you start with 12?
JJ: Because the answer is 12.
T: Is there another method? Look again at this 10 – 10. Is there a way that you could make this left hand side shorter?
JJ: If I say 10 – 10 = 20 (presses on the calculator 10 + 10). Then I minus 3 and I get 17 and then I plus 12.
T: So what is x?
JJ: x is 12.
T: You did not previously say that x = 12. Write down what you mean.
JJ: Okay. (looking uncertain) I shall now start all over. I am going to say 10 – 10.
T: How many is 10 – 10?
JJ: 0
T: Yes.
JJ: But if I change the minus to plus then it becomes 10 + 10 = 20.
T: Okay.
JJ: Then I take 20 – 17 = 3 + 12 = 15
T: So what is x?
JJ: 15
T: Previously you said x was 29 ..... If I say to you that the 29 was correct what would you say to me?
JJ: (looks perplexed, embarrassed)
T: Let us see. You said 10 – 10 = 0 i.e. you could have written x – 17 + 10 = 12
JJ: Yes ........ it is.
T: Now you go on.
JJ: If I now make the minus 17 plus 17, then my answer will be 5.
T: Okay. Let us take out the zero. Would the sum not be x – 17 = 12? Now you can do the sum in the same way as (3) ....... do you agree?
JJ: Then you make it 12 + 17 = 29 and then x is 29!!
T: Test your answer.
JJ: I know it will be 29.

5: 200 = 247 – x

T: How does (5) differ from (3)?
JJ: It is just turned around. Here in (3) the answer is last and in (5) it is first. Then the sum will be 200 + 247 = 447.
T: Test your answer.
JJ: (tests and writes 447 + 247 = 694)
T: What have you done?
JJ: I have only said 247 + 200 = 447
T: That is not what I see on the calculator. Please try again.
JJ: Yes (then she repeats the same process i.e. 447 + 247 = 694)
T: What if I gave you this one: 10 = 13 – x. What would x be?
JJ: 3
T: How do you get 3?
JJ: I said 10 + 3 = 13
T: Could you have got the 3 on a different way? Look at your

previous work.
JJ: (long silence)
T: Is this one (10 = 13 – x) easier because the numbers are nice and small.
JJ: (Nods her head)
T: What if I say the answer of (5) is 47 and not 447!
JJ: No! Now I don't understand.
T: Okay....... 13 – 10 = 3 and therefore 247 – 200 = 4
JJ: Oh! Now I see.

6: $2x = 110$

T: What does 2x mean?
JJ: I don't know, but does mean times?
T: Look what you did here (shows her previous work). Now do (6).
JJ: I see 110 ÷ 2 = 55 and therefore 2 x 55 = 110.
T: Okay, if I gave you 8x = 200, how would you do it?
JJ: (looks uncertain) I shall say ........... 200 ÷ 8 = 25 and then she says 25 x 8 = 200.

7: $3x - 12 = 6$

JJ: (without my interference starts to press 12 ÷ 3 = 4)
T: How did you get 4?
JJ: Because 12 is a multiple of 3 and that is why I divided by 3!
T: What about the 6?
JJ: I am going to look at it now.
T: Look.
JJ: (starts with 6 + 12 = 18)
T: Why do you start with 6?
JJ: Because he is at the back.
T: Okay ........ Continue.
JJ: Then 18 ÷ 3.
T: Why do you divide by 3?
JJ: Because it appeared on the calculator!
T: But isn't it the same as (6)?
JJ: Yes, more or less.

8: $6x + 4x = 40$

T: Do you see there are two x's. We would like to find the same x in both cases. Now you show me.
JJ: I shall start with 40.
T: Why do you want to start with 40?
JJ: Because it is last and therefore it is easier, because I do not know what the value of x is but 40 ÷ 4 = 10.
T: Now tell why you divided by 4?
JJ: Because it is easy and it is also at the back.
T: But what about the 6 (here by the 6x)?
JJ: Yes, but 40 cannot be divided by correctly.
T: And now?

JJ: I don't know.
T: Do you see that there is a plus?
JJ: Yes.
T: (pauses and looks at her) How many Rand are 6 Rand + 4 Rand?
JJ: 10 Rand.
T: So how much is 6x plus 4x?
JJ: 10.
T: 10 what?
JJ: No, I don't understand!
T: When there was Rand, you said 10 Rand. Now that there are x's, what would you say?
JJ: 10x.
T: Then we have $10x = 40$ ...... Do you agree?
JJ: Yes.
T: So what is the value of x?
JJ: Oh! (sounds relieved) then $x = 4$.
T: How did you get 4?
JJ: You multiply and then you have $40 \div 4 = 10$.
T: Okay, if I said $2x + 5x = 14$, what would you do?
JJ: $2 + 5 = 7$ and $7 + 7 = 14$.
T: Yes ........ But what did you do when you had $10x = 40$?
JJ: Divided!!
T: So why are you adding here?
JJ: Must I say $14 \div 7 = 2$
T: Yes ........ but what happened to the x's?
JJ: Huh ....... Huh!

9: $3x - 2x + 8x = 45$

T: Please read what is written here.
JJ: (reads correctly)
T: Now look at (8) and show me how to do (9).
JJ: (starts to write) $3 - 2 = 1 + 8 = 9$
T: Why did you leave out the x's?
JJ: You said that if it were Rand, then it would have been R3 – R2 = R1 + R8 which gives me R9 ....... thus it actually should have been 9x. Then I shall take the 45 and divide it by 9 to get 5.
T: Good!

10: $14x + 10 - 3x = 55 + 11$

T: Okay, .... Begin.
JJ: I will say $55 + 11 = 66$ (writes it underneath each other) and then 24...
T: Where does the 24 come from?
JJ: I added the 14 and the 10 then I said $24 - 3 = 21$.
T: May I ask you something? Do you know what you are busy doing? Look again at the Rand sum.
JJ: Yes, but 24 is equal to $14 + 10$.
T: Look carefully at the 10 ....... please.
JJ: (now impatient) Oh there is no x.

T: Yes ....... and now ........ Can't you say 14 – 3 is ......
JJ: (writes 14 + 3 = 17)
T: Why did you change the minus to a plus?
JJ: Because the minus in the sum became a plus.
T: But in (9) you did not change the minus to a plus in the sum? Did you?
JJ: (looks at the sum for a long while) Now I have 11x.
T: Now rewrite the problem with the 11x and the 66 that you have got.
JJ: (starts from scratch i.e. with the 14x and the –3x)
T: No! You have already worked out what that is! What did you get when you worked it out?
JJ: 11x.
T: Okay, now write it down please.
JJ: (writes underneath another) 11x + 10 = 21x = 66
T: Why do you write the number underneath another?
JJ: Because there is not an x by the 10.
T: You now have 21x = 66. And what's next?
JJ: Now I will multiply out.
T: What are you going to multiply out? Remember that you want to find the value of x.
JJ: I shall have 66 x 21.
T: Why?
JJ: Because x is multiply and if I divide then I shall get a remainder.
T: Okay. Don't you like remainders?
JJ: No!!
T: (laughs silently) But you do have a calculator! Don't you?

11: 4x = x + 21

T: How does (11) differ from (8)?
JJ: In (11) there is no x in the answer but in (8) there is an x in the answer.
T: So you do see that we have an x on both sides of the equal sign?
JJ: Yes.
T: Remember that it must still be the same value for both x's.
JJ: All the x's?
T: Please.
JJ: Then I'll say 21 – 4.
T: Why do you begin with 21?
JJ: I am going to bring him forward.
T: Why do you subtract the 4?
JJ: Because there is a plus and I think it would be easier if I subtracted.
T: Are you referring to the plus in front of the 21?
JJ: Yes. Now 21 – 4 is 17 and 17 x 4 = 68.
T: Why do you do that?
JJ: Because there (the 4x) is an x and I want to find what x is.
T: Does the x still mean multiplication to you?
JJ: Yes.
T: How would you find k if I gave you 4k = 21?
JJ: Then I would have done the same.
T: In other words you do know that k is a letter whose value you must find.

JJ: Yes.
T: So what is the answer of (11)?
JJ: 68
T: Test.
JJ: Now I don't understand the x. I get my answer but I don't know how ........... It is $4x = x + 21$ or?
T: You said if there was a "k" then you would have done the same.
JJ: Yes, I do know how to find it but I do not know about how to get the x!!
T: You said x is 68.
JJ: Yes x is 68!
T: Just now when we had $2x = 110$, what did you do?
JJ: I multiplied.
T: So?
JJ: Therefore $4 \times 68 = 272$.
T: And what about the left hand side?
JJ: I don't know.
T: Did you not say x is 68?
JJ: Oh! $68 + 21 = 89$ (Now she is looking tired, bored etc).

12: $10x - 6x + 3x - x = 2x + 10$

JJ: (writes it down but leaves out the $-x$)
T: Can you use Rand like before to help you make the left hand side shorter?
JJ: Yes. $10 - 6 + 3 = 7x$
T: And ........... what about the $-x$?
JJ: Okay.
T: Now write it again.
JJ: $7x = 2x + 10 = 12x$
T: (realising that we are going nowhere) Do you think you will be able to do this problem.
JJ: (uncertain) No.......

**GRAHAM LOTTERING (G)**

1: $x + 17 = 32$

T: Please tell what you are or will be doing.
G: I take 32 and then I minus it with 17 and then I get 15.
T: Why do you minus?
G: To get x.
T: Okay. Test i.e. where there is x you must write 15. Please use the calculator.
G: (tests correctly i.e. $15 + 17 = 32$)

2: $201 = x + 109$

G: (starts to write with me saying anything)
T: I see that subtract again. Why?
G: Because there is a plus and I minus to get the answer. Then I plus the answer again with 109 to get 201 again – that is $201 - 109 = 92$
T: Okay. Now test your answer.
G: (tests but write $92 + 109 = 201$)
T: Why didn't you start with 201 like you had done with the 15 in (1)?
G: Because if I press 201 + 92 then I will get more than what x must be.

3: $x - 14 = 47$

G: (again starts to write immediately) $14 + 47 = 61$
T: Now I see that you have added. Why?
G: If I add I get an answer then I must minus to get the answer of 47. Then I test and I see that $x = 61$.
T: Do you accept that plus is the inverse of minus and minus is the inverse of plus?
G: Yes.

4: $x - 17 + 10 - 10 = 12$

T: What are you going to do now?
G: (Presses 17 and <u>not</u> - 17 and wants to clear when he sees that there is a minus-sign on the calculator)
T: Do not clear! Where are you going to start?
G: Now I must see what I get first before I can start with x.
T: Okay ....... start.
G: (does something on the calculator and gets 39)
T: Where does 39 come from?
G: I just took numbers.
T: So why didn't you just start with say 100?
G: 100 is too large to get 12.
T: But 39 seems like a strange number to me if you want to get 12.
G: No ...... I think I must take 29.
T: Why?
G: Because $29 - 17 + 10 - 10 = 12$ and if I ... (says something inaudible)

T: Okay. What if the sum was $x - 6 + 13 - 13 = 4$? What would you do?
G: (long silence) Then I would have done the same.
T: Okay ..... Show me.
G: (start with 10)
T: Why 10?
G: Because $10 - 6 = 4$ then I add 13 to give me $17 - 13 = 4$.
T: Right. Now I am going to give you another sum so that we can see.
$x - 117 - 1011 + 1011 = 50$
Where are you going to begin?
G: (pauses) I shall do the same. I shall want to see which number I can take to give me 50.
T: Okay.
G: (starts with 267 – 117........ (long silence)
T: Why did you start with 267?
G: I have added.
T: What have you added.
G: I had to take 267 and then subtracted 117 and then I thought I could get 50.
T: Don't you see that – 1011 + 1011 gives you 0?
G: Yes. That's why I first took 267.
T: Please explain.
G: I can see that this will give me the same number.
T: What will give you the same number?
G: Say ...... I take only 167 – 150 then I shall go beyond 0.
T: What will go beyond 0?
G: (says something to himself which does not make sense)

5: $200 = 247 - x$

G: (looks at the equation and writes $247 - 47 = 200$
T: How did you know that you had to take 47?
G: If I subtract 47 from 247 then I get 200!!
T: Wow! Let us now look at $177 = 334 - x$. Let us see if you can be just as quick.
G: (writes 177 and then a long silence)
T: How did you get 47 before? Could you just see it from the numbers?
G: Yes.
T: If I give you smaller numbers now i.e. $51 = 82 - x$, what will you do then?
G: This is the same as (5). Now I must find a number for x.
$30 + 51 = 81$ If I add one to 30 then I shall get 31.
T: So what is x?
G: $x = 31$
T: Would you agree that one could have said $82 - 51 = 31$?
G: (Looks at my work but does not answer)
T: Now do $177 = 334 - x$
G: (now correctly) $334 - 177 = 157$

6: $2x = 110$

T: What 2x mean again?
G: I want to see with what I must multiply 2 by to get 11O.
T: What will x be?
G: 55 because 2 x 55 = 110
T: Do you agree that one could have written: 110 ÷ 2 = 55?
G: Yes.

7: 3x – 12 = 6

T: Where do you begin?
G: With 12 + 6 = 18
T: Why do you say 12 + 6 and NOT 6 – 12?
G: If I plus 12 with 6 then I get 18 and then I take 18 – 12 = 6
T: Why 18 – 12?
G: Here is – 12 and that 3x is really 12 and then I did not know what to take to give 12 and then I just took 12 + 6 = 18 and now I shall write 18 where 3x stands and then I shall take 12 away from 18 to give me 6.
T: But how did you get the 18?
G: From the 12 + 6.
T: But there is also a 3x!
G: But here ..... I can take 6 be...... because 3 x 6 gives 18 and 18 – 12 = 6. (starts to count on his fingers)
T: Why do you use your fingers?
G: It is easier.
T: Okay. Let us take another one i.e. 3x – 27 = 81
G: (starts to write) 27 – 81 = .........
T: Didn't you change the minus to a plus before?
G: Yes. I am also going to plus here.
T: Oh!
G: So 21 + 81 = 108
T: What is x?
G: x is 108
T: Hmmm ....... Test.
G: (now he substitutes 3x with 108)
T: Okay.

8: 6x + 4x = 40

T: Now we have an x by the 6 and an x by the 4. They must have the same value.
G: So .... 6 times a number and 4 times a number.
T: They must be the same number.
G: (silently/ uncertain) Yes.
T: Now ....... show me. This time use your calculator.
G: (uses his calculator and gets 12)
T: How do you get 12?
G: 6 + 6
T: Don't you have to multiply the 6 by a number?

G: Yes. (then starts to write 6 x 2 = 12 and 4 x 7 = 28)
(THEN he starts to press 4 + 4 + 4 = until he gets 28)
T: Why did you keep on using the plus sign? Shouldn't it have been times?
G: Yes. (whispers something inaudible) But then I will get 12.
T: Okay. Why did you press 4 + 4 + 4 ..... etc.? Is it because you wanted to get to 40?
G: Yes.
T: I noticed that you said 6 x 2 and then 4 x 7. Why?
G: To get 40, because 12 + 28 = 40.
T: But it must be the same value of x. Must it not be?
G: (long silence/seems embarrassed) ....... Yes.

9: NOTE: Here the video camera stops and what follows is an audio cassette.
3x – 2x + 8x = 45

T: How many x's are there?
G: Three
T: Okay. Do it now!
G: (begins with 5) 2 x 5 = 10
3 x 5 = 15
8 x 5 = 40
Then I add to get 65.
T: Why did you start with 5?
G: Because it is easy!
T: Didn't you see the minus-sign?
G: Yes. But plus is easier!!

10: 14x + 10 – 3x = 55 + 11

T: Where will you begin?
G: I shall first do the answer i.e. 55 + 11 = 66.
T: Is the answer always on the right hand side?
G: Yes.
T: Okay ..... But what about the x's on the left hand side? Remember we must find the value of x.
G: (uses his calculator) 14 + 14 =
T: Why?
G: No ... actually 14 x 2.
T: Why?
G: Two is easier.
T: (smiles) Continue.
G: (then presses 14 + 14 + 10 x 2 + 3 x 2 = .....
T: Once again you add, and you put a 2 by the 10 and you failed to see the minus sign. Isn't it?
G: Yes.

11: 4x = x + 21

T: Now you have x's on both sides of the equation, but they must still have the same value. Where will you begin?
G: I will say 24 + 21 = 45
T: Why 24?
G: Because 24 + 21 = 45
T: Why 45?
G: No! ... I should have taken 23, then I plus 23 with 21 to give 44, then I know that 4 x 11 = 44 and then I shall also get 44.
T: Must the answer be 44? .... Why 44?
G: Say I am able to get the x (on the right hand side) and I add 21 then I get 44, (silence) ... then I shall guess!
T: But all the x's must be the same! What will this x (pointing at 4x) be? You said 4 x 11 and then you had a 23 ..... ? Shouldn't the x's all be the same?
G: Yes.
T: So what are we going to do?
G: (long silence)
T: Another learner found x to be 7. Would you agree?
G: (substitutes correctly) Yes.

12: $10x - 6x + 3x - x = 2x + 10$

T: Now try (12). You have seen that one can do these sums.
G: (long silence ..... looks uncertain)
T: Do you think there is a way to do (12)?
G: ....................... Yes, but I don't know how.
T: Don't you want to try?
G: (looks tired .....)
T: Thank you ......... you have been a star!

**MARALIZE MACKAY (M)**

1: x + 17 = 32

M: I shall take 17 away from 32 then I get the answer like this:
32 – 17 = 15. So x = 15.
T: Test. Remember when we test we replace x in the original equation.
M: Okay, 15 + 17 = 32

2: 201 = x + 109

T: What are you going to do?
M: The same. I shall just turn it around as follows: 201 – 109 = 92
T: Now test again.
M: Okay, 92 + 109 = 201
T: What would have happened if there was x + 109 = 201? Would the answer have been the same?
M: Yes.

3: x – 14 = 47

M: (without hesitation) I shall take the 14 and add it to the 47.
T: Write.
M: 14 + 47 = 61 – 14
T: Is the inverse of minus plus?
M: Yes.

4: x – 17 + 10 – 10 = 12

M: (first she writes 10 + 10 ..... then she thinks for a fairly long time before she writes) 17 + 10 = 27 – 10 = 17 – 5 = 12
T: So what is the value of x?
M: 5
T: Okay, ..... Now see if it is true.
M: Then 5 + 17 + 10 – 10 = 12
T: Use your calculator. What do you get?
M: -12
T: But we have +12 in the equation!
M: Then the whole sum must now change.
T: Will you now put the minus in front of the 12 or will you change the entire sum?
M: What if I put the x in front, then it is 5 – 17 = 12
T: Look on your calculator.
M: I get –12 again.
T: Is 12 and –12 the same?
M: No.
T: (underlines 10 – 10). How many is the part that I have inderlined.
M: 0
T: In other words you can replace + 10 – 10 with 0?
M: Yes.

T: Can I rewrite the sum like this now i.e. x – 17 + 0 = 12
M: Yes.
T: Now continue from there.
M: (Begins with 5 again)
T: Why do you begin with 5 again? What did you do in (3)?
M: I said 14 + 47 = 61. Then I took 61 and subtracted 14 and I got 47.
T: Now would you agree with me if I wrote (5) as x – 17 = 12?
M: So now x – 17 is 12.
T: Right ... will this x be more or less than 12?
M: Less.
T: Why less?
M: Now I understand! This sum is nearly like (3). If you take 5 minus 17 then you will get a negative because you cannot subtract a big number from a small one!
T: So what are you going to do?
M: Now I shall take 5 away from 17 to get the answer.
T: What is the answer?
M: (no immediate response) Now I shall subtract.
T: How?
M: (pauses ....)
T: (showing her (3). ) Did you guess here?
M: No! I just looked at the sum!
T: Which part is the sum?
M: x – 14 is the sum.
T: And the other side?
M: 47 is the answer.
T: Okay, let's come back to (4). What is the "sum" here?
M: x – 17 is the sum and the answer is 12.
T: So why don't you do the same in (4) as in (3)?
M: Because I will get a negative answer and that won't be the same as in (3).
T: Let us see. Do you notice that the 14 in (3) is less than the 47?
M: And in (4) the 17 is greater than 12.
T: Don't you think that there is a number from which you could deduct 17 to get 12?
M: (silence .....)
T: What about 29?
M: Yes! (looks relieved)
T: How did I get the 29?
M: (now smiling) Sir, you just knew!
T: No, I did not. I did the same as in (3). Do you see that there you said 14 + 47 = 61?
M: And here? Oh! 29 – 17 = 12

5: 200 = 247 – x

T: How does (5) differ from (2)?
M: (5) is a minus sum and (2) is a plus sum.
T: Where is x in both of the equations?
M: Both are on the right hand side.
T: Will this x be more or less than 200?

M: Less. Now 200 – 247 = -47
T: Which one of 247 or 200 is the greater?
M: 247. Oh, I worked wrong! (then she writes 247 – 200 = 47 and tests correctly)
T: Now I shall give you 113 = 702 – x. Please do it for us!
M: I would say 702 + 113 = 815 then I would test it as follows: 815 – 702 = 113.
T: Didn't you change the sum? What is the value of x?
M: 815
T: Test. Remember the story about large and small numbers!
M: I will turn the whole sum around.
T: Show me.
M: 815 – 702 = 113
T: You say you tern the sum around, but isn't that the same as changing the sum?
M: No.

6: 2x = 110

T: What are you going to do?
M: I will say 110 ÷ 2 = 55. (Then she tests as 55 x 2 = 110)
T: In other words you know that x and ÷ are inverses?
M: Yes.
T: So what if I give you 4x = 404?
M: I will say 4 x 110.
T: Why 110?
M: 110?
T: Have you guessed?
M: Yes.
T: Why don't you divide as before?
M: (then says) 404 ÷ 4 = 101 ........ Oh!

7: 3x – 12 = 6

T: What does 3x stand for? What does it mean?
M: I don't know.
T: Previously you said it was multiplication.
M: So it is 3 x 4 = 12
T: So what is x?
M: 4
T: Now replace x with 4.
M: (writes) 3 x 4 = 12 – 12 = 6
T: Are you sure? Use the calculator.
M: (once again) 3 x 4 = 12. Now I don't know! Must I now say that 12 ÷ 2 = 6?
T: Which one of the two twelve's? The –12?
M: Yes.
T: Okay, you say –12 ÷ 2 = 6. Why 6?
M: Because the answer is 6. (mutters something inaudible)
T: What about this –12?

M: Says 12 – 12 = 0
T: Is it right? So what is your answer? (silent .......) Is it 0?
M: Sir, ..... 3 x 4 = 12 but I shall now try 6 because there is not another number after the –12.
T: But in front there are things.
M: But it is not for the answer.
T: Don't you see it?
M: No.
T: What about the 6? Didn't you say just now that 6 is the answer.
M: x = 4 is the answer and 6 is the answer of the whole sum.

8: 6x + 4x = 40

T: Now we have two x's (shows here). You must now get the same x in both places.
M: So the answer of 6x must be the same as the answer of the 4x?
T: ....... Yes. Where will you begin?
M: I think with the 6. (writes 6x 6 = 36 and 4 x 6 = 24) That will give me 60.
T: And ......... now what?
M: 6 is too big.
T: And what now?
M: I am going to try 4.
T: Why not 5?
M: It looks too large.
T: Okay.
M: (Now she says) 6 x 4 = 24 and 4 x 4 = 16. If I add I will get 40.
T: So what is x?
M: x is 4.

9: 3x – 2x + 8x = 45

T: How many x's do you see here?
M: There are three.
T: Where will you begin?
M: In front.
T: Where is in front?
M: The 3x – 2x.
T: How much is 3x – 2x?
M: (silent ........)
T: If you had 3 apples and you took 2 apples away, how many would you have?
M: Five.
T: Five?

M: No. One and now I add 8 apples and I get 9.
T: Okay, now you know how to work with apples, what will you now do to this? (the left hand side)

M: It is 9.
T: 9 what?
M: 9 ........ Must I now multiply it by 8?
T: Why times 8? If you work out all the x's on the left hand side, what would you get?
M: 9
T: (getting impatient) Nine what? If I say to you that the value of x is 5 and you keep on saying nine, how do you think did I get the answer of 5?
M: Okay, $45 \div 9 = 5$
T: What is the five? Is it x?
M: Yes, $x = 5$
T: Is it (the $45 \div 9 = 5$) the same as what you did in (6) i.e. when 2x was equal to 110? In other words if $9x = 45$ what would you do?
M: Where?
T: If $9x = 45$ was the sum.
M: The same as in (6)
T: Show me.
M: I said $110 \div 2 = 55$ and here $45 \div 9 = 5$
T: Okay.

10: $14x + 10 - 3x = 55 + 11$

T: Where do we start?
M: With the answer ...... $55 + 11 = 66$
T: How many x's do you see on the left hand side?
M: Three and then $14x + 10 = 24$
T: Remember to get the same x wherever there are x's.
M: Can't I times the 24x by 2 to give me 48?
T: And then?
M: Then I take 3x times 2 to get 6 and then I get the answer.
T: Okay ..... (time is beginning to run out)

11: $4x = x + 21$
(Note: that time is now really becoming limited)

T: Do you notice that there are x's on both sides of the = -sign. Again you must find the same value for x.
M: (scratches her head........ looks very much uncomfortable and then) $21 + 8 = 29$
T: Why 8?
M: I am just guessing!
T: Do you think it will get you somewhere?
M: I don't know.

T: Okay. What will you do next?
M: Can I continue? (looks even more uncertain and writes) $21 + 7 = 28$
T: Why do you take 7?

M: To get 28 and then I shall divide 28 by 4 to give me 7.
T: Okay, test your answer.
M: 4 x 7 = 28 (stops.....)
T: What about the right hand side?
M: I have already found 7! (Now she seems frustrated)
T: But you said that you were only guessing.
M: Yes.

12: $10x - 6x + 3x - x = 2x + 10$

T: Where will you begin?
M: I do not know!
T: Aren't you going to guess again?
M: ....... I shall start with 2. (then she substitutes 2 BUT did not use the –x)
T: What about this –x?
M: I will subtract it from my answer. (softly) Can I use any number ......
T: But it must be the same number everywhere.
M: Yes. (writes) 2 x 2 + 10 = 14 (starts pressing aimlessly on the calculator)
T: Do you think there is an easier way to find x quickly?
M: No!
T: Yes, there is!!

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**SIMONE (S)**

1: $x + 17 = 32$

S: (just writes down $x = 15$)
T: Remember to tell me how you get you answers.
S: I have said $32 - 17 = 15$
T: Okay ... Now test your answer.
S: (writes) $15 + 17 = 32$
T: So what is x?
S: $x = 15$

2: $201 = x + 109$

T: How does (1) differ from (2)?
S: By (2) 201 is now the answer. So I must see what must be added to 109 to give 201.
T: What is your answer?
S: 201 !
T: Okay .... Now show me how you would do the sum.
S: I said $201 - 109 = 92$
T: So I see that when there is addition then you subtract?
S: Yes.

3: $x - 14 = 47$

T: What do you see here?
S: I see that there is a minus.
T: Okay ..... Now do the sum.
S: (writes) $47 - 14 = 33$. I must now see what I must get to get 47.
T: How are you going to get to it?
S: (long silence)
T: Tell me what you think.
S: I first want to see how I can work out the sum.
T: Which sum?
S: Number 2.
T: What do we want to get?
S: The value of x. (starts to use her calculator) I said $47 - 14 = 33$.
T: Let us test. You say x is 33?
S: Yes.
T: Let us see ......... i.e. $33 - 14$ is equal to ...
S: Nineteen .......... therefore something is wrong ....... somewhere.
T: And now what are you going to do next? In (1) and (2) there was a plus sign and there you subtracted.
S: So here I must add up ....... plus?
T: What?
S: $47 + 14$
T: Look ........
S: 61 !

T: Is x equal to 61? See if it works.
S: (uses the calculator and sees that 61 – 14 = 47)

4: x – 17 + 10 – 10 = 12

T: Now I am going to give you a chance to show me how you would work it out.
S: (writes) 17 + 10 = 27 – 10 = 12
T: Hmm .......... what?
S: Whatever I do, I must still get 12.
T: Okay. How will you do that?
S: (silent ........... looks at the calculator)
T: What about this (10 – 10)?
S: It gives me (pauses a while) 0
T: Now rewrite the sum. Remember you have just said 10 – 10 = 0.
S: What do I have to write?
T: The 0 which you have seen must be written.
S: (starts to write) 0 – 17 + 10 – 10 = 12
T: Didn't you say that 10 – 10 was going to be 0?
S: Yes. (starts to read the sum again)
T: What is the answer?
S: 0
T: Test. Remember to write the 0 in the place of the x.
S: (does the sum and gets –17)
T: And now? Is x = -17?
S: (long silence ...... looks uncertain)

5: 200 = 247 – x

T: What is the difference between (5) and (4)? Remember that we still want to find the value of x.
S: In (4) you already had the answer and in (5) I must first subtract to find the value of x.
T: Will x be more or less than 200?
S: Less.
T: Why?
S: Because if I say 247 – 230 as an example must give me 200, then a larger number above a smaller number will give me a right number.
T: Please write down what you have just said.
S: Sir, can't I work out first. (wants to use a piece of scrap paper)
T: No, please show me and remember that it is not a test.
S: (starts to write) 247 – 200 = 47. Now I first want to have a look.
T: Okay. Now test for us. What is x?
S: x = 200
T: Why 200? Because if I substitute x, then I shall get 200 = 247 – 200 and then 200 = 47. Is that true?
S: (long silence) x = 47
T: Is there another way to find the answer? (turn back to (3) ) How could you have got 47?
S: (long silence)

T: Is there another method!!
S: I don't know, Sir.
T: What would happen if I write: 247 – 200 = 47? Would you agree?
S: Yes.

6: 2x = 110

T: Read out what you see.
S: (reads correctly)
T: What does 2x mean? Is it x, ÷, - or + ?
S: I do not know, sir.
T: (turns back to previous work) Here you said it is multiplication.
S: I can see.
T: What are you going to do here in (6)?
S: 2 x 5 =
T: Are you sure?
S: (long silence and then she writes/says: 2 x 50 = 100 and 2 x 5 = 10.
If I add them then I will get 55 and 110.
T: (now shows her 2x = 16) Here your answer was 8 ..... wasn't it?
S: No, I said 16 ÷ 2.
T: Why don't you do the same in (6)? With what must you divide 110?
S: By 2.
T: Do it.
S: (does it and then tests it as 55 x 2 = 110.

7: 3x – 12 = 6

T: Tell me what you are going to try here in (7). Remember that it means
3 times x minus 12 must give 6.
S: (starts to use calculator and turns back)
T: How far are you?
S: I am still thinking.
T: Please write down what you think. (Remember it is not a test)
S: (long silence)
T: Let's go to (8).

8: 6x + 4x = 40

T: (explains that the two x's must have the same value)
S: (long silence ......... looks puzzled.)
T: What are you thinking? Aren't going to start with a number again?
S: (long silence)
T: Please tell me what is going on inside your head.
S: Can x be a plus?
T: It must be a number like all the others before.
S: (silent)
T: Why don't you try numbers like the other learners?
S: The numbers I am thinking of are all large numbers.
T: Like .........?
S: 61

T: Why 61?
S: Can't I substitute x with 6 and then I say 16?
T: Why do you want to substitute 6 with 16? Do it.
S: May I?
T: You must show me!
S: (writes 4O + 0 and says ) x = 0?
T: Now you have to test. Remember that 6x is 6 times x.
S: (writes) 6 x 0 + 4 x 0 = 0. Huh ..... now 0 + 0 will give me 40!
T: Are you saying so?
S: (looks confused, frustrated)
T: Let's go to the next one.

9: $3x - 2x + 8x = 45$

T: Please read the equation for me. Then you do it and remember all the x's must have the same value.
S: (long silence)
T: If I say x = 100 what will you say?
S: I think (starts counting on her fingers) 30 – 20 = 10 + .......
T: Where (how) did you get the 30?
S: I put 0 in the place of x.
T: But don't you have to multiply?
S: Yes. (starts to write) 3 x 10 and 2 x 5.
(Time is up)

**JUSTIN (J)**

1: $x + 17 = 32$

T: Please tell me everything that you are thinking.
J: I take the 32 and I take the 17 away and then $15 + 17 = 32$
T: Did you subtract?
J: Yes.

2: $201 = x + 109$

T: You must write down the sum and then you tell me how you would work it out. Remember it is not a test.
J: 201 minus $109 = 92$. So x is 92. Now I must just make sure.
T: How?
J: (writes $109 + 92 = 201$)
T: What did you do when you were testing your answer?
J: (pause) I added.

3: $x - 14 = 47$

T: Please write down everything that you are going to do.
J: I am going to do the same as above.
T: What do you mean with "the same"?
J: $47 - 14 = 33$
T: Test your answer.
J: (tests his answer with a calculator) x will be 33.
T: If I say that x is not 33 and I tell you the answer is 61, how would you test if 61 is right? Remember that I am not sure if 61 is right.
J: Yes, 61 is right. I should have subtracted.

4: $x - 17 + 10 - 10 = 12$

T: Do you see the minus signs?
J: Yes.
T: I am going to say nothing now. Please tell me what you are going to do.
J: Yes. (starts to write $10 - 10 = 0$
$0 + 17 = 17$)
T: Is there not a minus sign in front of the 17?
J: (does not answer but works in silence to get 29)
T: Where did you get 29 from? Test you answer please.
J: I think it is the value of x because $29 - 17 = 12 + 10 = 22 - 10 = 12$.

5: $200 = 247 - x$

T: How does (5) differ from (3)?
J: The answers. In (3) the answer was last and in (5) it is first.
T: Is there no other difference?

J: With (5) the sum is turned around.
T: Is there no other difference?
J: (long silence)
T: Okay. Do the sum for us. Let us see what you will get.
J: (begins with 447 – 247 = 200)
T: Huh ..... Will x be more or less than 200?
J: More.
T: Write it down for us.
J: (silent ............... starts to write) 200 + 247 = 447
200 = 247 – 447
T: What is the value of x?
J: 447
T: But it's then more. Now try 447 in the place of x. Please use your calculator.
J: 247 – 447 = 200
T: What do you get?
J: 200
T: Isn't there something in front of the 200?
J: Minus.
T: Yes .... what do you say now?
J: (looks unsure) The answer is -200.
T: Please write down that you saw your answer is –200.

6: 2x = 110

T: Where will you begin?
J: I am going to multiply.
T: With what are you going to multiply?
J: 205.
T: Does it work?
J: No ...... because 2 times 205 = 410
T: What are you going to do next? Must x be smaller or greater than 205?
J: Smaller.
T: What now?
J: (writes) 2 x 105 = 210. That is also not right.
T: What are you going to try now?
J: (sounds relieved) 55! Then I get 2 x 55 = 110.
T: Isn't there a quicker method? What if I write it as 2 x ? = 110? What will you do now?
J: (long silence)
T: Will you start with numbers again?
J: (softly) Yes.
T: Will you follow the same pattern?
J: No. I will say 110 ÷ 2 = 55.
T: So you agree, there is a shorter way?
J: Yes!

7: 3x – 12 = 6

T: Do you see that (7) is a mixture (combination) of (2) and (6)?

J: (looks uncertain ......... stares at the problem) I shall start with the number 8.
T: Show me.
J: I am going to test first.
T: Write down you test.
J: (writes) 3 x 8 = 24
Now I shall take 24 – 12 = 12
T: And then ........?
J: (silent)
T: Look at (6) and see if there is not another way.
J: (looks at his work)
T: Will x be more or less than 8?
J: Less
T: Let us see
J: It is going to be 6.
T: Why 6? Why did you decide on 6?
J: I saw that 3 times 6 is 18 and 18 – 12 = 6
T: Oh!

8: 6x + 4x = 40

T: Now there are 2 x's and you must find the same value for both of them.
J: If I multiply here (pointing at 6x) and I multiply there (pointing at 4x) must I get the same value for x?
T: Yes .......... Show me without writing what you will do first.
J: I shall try to find an "even" number.
T: An even number? (surprised)
J: Yes, because 40 is an even number.
T: Now write down.
J: (whispering something about an equal number, then he writes)
6 x 3 = 18
4 x 3 = 12
I shall add to get 30.
T: Is it right?
J: No.
T: What are you going to do next?
J: I will try a higher number.
T: Like what?
J: Like 4.
T: Okay .... let us see.
J: (uses his fingers) 6 x 4 = 24 and 4 x 4 = 16 (counting in units and tens to get 40)
T: Why don't you use the calculator?
J: But I will still get 40.
T: Why did you go from 3 to 4?
J: Because I first thought about 5 which normally is easy to work with but I saw it was too many.

9: 3x – 2x + 8x = 45

T: How does (9) differ from (8)?
J: Must I use the same number again? (pointing at the x's)
T: Yes. Wherever there is x you must use the same value to get 45. Where are you going to start?
J: With 4. (then he substitutes correctly to find 36)
T: Does it work?
J: No, it's too small!
T: What are you going to do next?
J: I will try a higher number like 6 (works out the left hand side and gets 54. says softly: "does not work").

10: $14x + 10 - 3x = 55 + 11$

T: Tell me where will you begin?
J: I shall begin with the answer i.e. $55 + 11 = 66$
T: In other words the right hand is your answer.
J: Yes. (writes down $14x + 10 - 3x = 66$)
T: What did you do in (9)?
J: (softly) I subtracted and I added.
T: In (10) you have 2 x's and there is also a 10. What are you going to do with the 10?
J: (writes down: $14 \times 2 = 28$) Does not work. (then whispering)
$14 \times 3 = 42 + 10 = 52$
and $52 - 3 = 49$
and $49 + 9 = 58$
T: Where does the 9 come from?
J: It is $3 \times 3 = 9$.
T: Why did you add the 9?
J: I subtracted on the other side!
T: What now?
J: It is still too few!
T: How many are you still looking for?
J Eight.
T: Where are you going to find the eight?
J: I shall try 4. (then he writes $14 \times 4 = 56 + 10 = 66$)
T: But there is also a $-3x$! What are you going to do about it?
J: I am going to try something else. First I will try $3 \times 4 = 12$ and then $66 - 12 = 54$.
T: Must your answer be 54 or 66?
J: I am now going to multiply a number which I can take over to get 66.
T: Which number?
J: Five.
T: Just remember that you must use 5 everywhere.
J: (works with 5 and gets 65)

11: $4x = x + 21$

T: How does (11) differ from (8)?
J: Now I don't have a number by x that must be multiplied. (shows to

left hand side).
T: How else?
J: The sum is shorter.
T: Are you sure? Don't you see that there is an x on both sides of the equal sign?
J: (long silence) Yes.
T: Where are you going to begin?
J: (looking tired) Writes 21 + 3 = 24.
T: Why do you first write the 21?
J: I want to find a number which can be divided by 4.
T: Okay. Where does the 3 come from?
J: Now I try to ................
T: Remember that the same number must be on both sides.
J: (uses calculator) 4 ÷ 3 = 1,33.
T: Do you think it is going to work? What will you do with the 1,333?
J: I don't know ............ Now I shall try 2. (writes 21 + 2 = 23)
T: Why do you add two? Is it the 2x?
J: Yes.
T: And now?
J: (writes) 4 ÷ 2 = 2
T: Why do you divide by 2?
J: (long silence ......... sigh) I have misunderstood the sum!

12: $10x - 6x + 3x - x = 2x + 10$

T: Show me. (notices that time is running out)
J: Starts with 10 x 2 (long silence)
T: Have you found a way to find x?
J: (shakes his head).

**MEGAN (M)**

1: $x + 17 = 32$

T: I want you to show me everything you do.
M: (writes) $32 - 17 = 15$
T: Why did you subtract?
M: If we say x plus 15 is 32 then we must add something to 17 to get 32.
T: In other words plus and minus are inverse?
M: Yes.

2: $201 = x + 109$

T: How does (2) differ from (1)?
M: Here they first give the answer and then you must do something.
T: Do you know what you must do?
M: Yes. We must subtract 109 from 201 to give 92.
T: Test your answer. You say $x = 92$. Now find out if it is right.
M: (presses 109 and then plus 92).
T: Why did you first press 109 and then 92?
M: 109 was given to me and then I found x to be 92.

3: $x - 14 = 47$

T: How does (3) differ from (1)?
M: In (1) they said $x + 17 = 32$ and now they say $x - 14$ gives you 47.
T: Will x be greater or smaller than 47?
M: It will be more.
T: Now show me how to get x.
M: (Adds 47 to 14)
T: I see that you have added. Why?
M: Because if you take a number (47) and you add 14 then you will get the answer and if you subtract the 14 again you will get 47.
T: Test your answer.
M: $47 + 14 = 61$.

4: $x - 17 + 10 - 10 = 12$

T: Read to me what you have written.
M: (reads correctly)
T: Where will you begin?
M: 10 minus 10 is zero.
T: Write the sum down and now find the value of x.
M: (writes down $12 + 17 = 29$) $x = 29$.
T: Test your answer.
M: Uses the calculator but presses the $10 - 10$ again.

5: $200 = 247 - x$

T: Is (5) different from those we had before?
M: Not really. It is just the other way round.
T: How is it the other way round?
M: We now have the answer (solution) first and then we have the sum.
T: Okay, do it for us now.
M: (writes and whispers) $247 - 200 = 47$
T: What did you say?
M: (now louder) $247 - 200 = 47$.
T: In (3) you added why do you subtract now in (5)?
M: Here 247 is greater than 200 but in (3) I added because they asked in (3) what minus 14 will give me 47. But in (5) they ask $200 = 247 - x$. In other words in (5) they have a longer notation?
T: Huh!! So will it be right if I say we must test like this? (writes) $200 + 47 = 247$. Do you agree?
M: Yes.

6: $2x = 110$

T: What does (6) say?
M: Here they actually tell you I must divide 110 by 2. It is actually 2 times x is equal to 110.
T: In other words if I multiply then it is opposite of division?
M: Yes. (writes) $110 \div 2 = 55$ and $x = 55$.
T: Should you not have said $2 \times 55 = 110$ i.e. x is 55 in $2x = 110$?
M: Yes.

7: $3x - 12 = 6$

T: Where will you begin?
M: Sir, if I really know my tables of 6 then I shall say 3 times 6 is 18 and $18 - 12 = 6$.
T: Please tell me how you got 18.
M: I am going to say $6 + 12 = 18$ and I know 3 times 6 is 18. So now I shall say $18 - 12 = 6$. So $x = 18$.
T: Are you sure?
M: No! x is 6.
T: Now look at this equation $5x - 7 = 33$. What will you do now?
M: (scratches her head) I know that 7 from 40 is 33 thus $7 + 33 = 40$. Now I know 5 times $8 = 40$.
T: In other words you first added the 7 to the 33 because here is a minus sign?
M: Yes.
T: Then you said $5x = 40$ and now? Look at (6).
M: If you take 40 and you subtract 33 then you get 7.
T: Oh......

8: 6x + 4x = 40

T: The two x's must have the same value. Do you agree?
M: Yes.
T: Now show us.
M: (says softly) If I multiply 6 with something and I multiply 4 with the same something and I add then I must get 40.
T: Now we must find that something. Do you agree?
M: Yes ....... I think.
T: Please write down or tell me what you are thinking.
M: (starts to write) 6 x 5 = 30 (says softly) You cannot multiply 4 with 5 because your answer will be too big!
T: Try it.
M: (writes 4 x 5 = 20) Too many!
T: Why too many?
M: We must get 40 and 30 + 20 = 50.
T: What now?
M: (silent)
T: Will x be more or less than 5?
M: Less than 5. Now we take 6 x 4 = 24
and 4 x 4 = 16
Yes! (looks happy) Then we add and we get 40.
So x is 4.

9: 3x – 2x + 8x = 45

T: Write down the sum.
M: (writes it down) 3x – 2x + 8x = 45
T: How many x's do you see?
M: Three. (reads the equation again) What can we multiply with 3 and then we must multiply 2 with the same and subtract and then 8 times the same and add to get 45.
T: Show me, please.
M: I think (starts to write) 3 x 5 = 15 –2 x 5 = 10 + 8 x 5 = 45.
T: Why did you start with 5?
M: (silence ...... reads the arithmetic again)
T: If I say to you the answer is NOT correct and that the value of x is 2, how will you test it?
M: (tests correctly) The answer is NOT two. Two will not work.
T: Let us try 71x – 62x + 10x = 111
What will you do to find x?
M: The numbers are BIG!! I am getting confused. (then she begins with 71 x 2)
T: Why do you start with 2?
M: (scratches her head) I do not want to start with big numbers because numbers like 5 and 3 will be totally to big!!
T: Do you think there is an easier way?

M: Yes, but I do not know at all!!

10: $14x + 10 - 3x = 55 + 11$

T: Read the equation for me.
M: (reads correctly i.e. she sees that there is no x in the second term then she immediately writes down $55 + 11 = 66$)
Now we must go and find out what the value of x is.
T: How?
M: Come we take 5. (writes down: $14 \times 5 = 70 + 10 = 80$)
T: Why don't you multiply 10 by 5 as well?
M: There is no x.
T: Why did you try 5?
M: It is half of 10 and it is easy!
T: So you just took 5 to start somewhere?
M: Yes, but I see now that my sum will be completely wrong because $80 - 3 \times 5 = 65$.
T: Will your value of x be greater or smaller than 5?
M: Greater ......... come let us try 6 ........ No ......... maybe 5½ (looks confused/ frustrated)
T: Do you think there is an easier way?
M: Yes, there must be!!

11: $4x = x + 21$

T: Is there a difference between (11) and (9)?
M: Here we have 4 times an x and if I take the same x's value and I add 21 then I must get the same answer. In (9) they say precisely how much your answer must be i.e. 45.
T: Will (11) be more difficult than (8)?
M: Not necessarily.
T: Okay. Show me.
M: Come take 5 again.
T: You like 5, hey!
M: Yes. It's easier to work with 5.
T: But is it not easier to work with 10?
M: But if we take 10 then the answer is too big. (starts to write)
$4 \times 5 = 20$
and $5 \times 21 = 26$
Thus I must take 4x to get 26, but I only have 20 therefore 26 is hopelessly too many.
T: Hopelessly?
M: Yes.
T: But 26 is not that many more than 20!
M: But we do not want 26!! We want to get an answer!
T: Okay. We now see that 5 does not work.
M: Come we try 4 (now she is talking to herself) If we say 21 .............
If we say 10 then we get 40 and then this side (right hand side) is 31 .........

T: What if we say x is Rand. Then we have R4 minus R1 is equal to R3. Would that help?
M: No. We cannot do the same here (whispering something in the line of 4 x something and we cannot have 21 minus 4 times something ..........)
T: If I get R3, can I not say $3x = 21$ and what will x be then?
M: Okay....... $3x = 21$ (long silence and then) But where does the one come from?
T: Good question ......... I will tell you later.

12: $10x - 6x + 3x - x = 2x + 10$

T: Write down and try, please.
M: (reads the equation – sounds uncertain)
T: What does he look like.
M: Must I tell the truth? Like Greek.
T: Okay, how many x's are there on the left hand side?
M: (in a soft voice) I must now first see what my answer is i.e. what the value of x is.
T: What do you think the answer would be?
M: Let us start with 5 again on the right hand side. (writes) $2 \times 5 = 10 + 10 = 20$. Now if I look at all the numbers on the left hand side then I won't get 20. I shall try ........ (writes) $50 - 30 + 15 - 5 = 30$. (looks frustrated)
T: Come we go back to (11)... Why did you only write – x in (12) but in (11), you asked where the 1 came from?
M: (softly) 1 is on the right hand side and 5 is on the left hand side.
T&M: (laugh together)
T: What are you going to try now? Greater or less than 5?
M: Very much smaller. Come we take 2 (tries 2. does not work) (tries 3. does not work) (starts to look tired.

**REDUAAN (R)**

1: $x + 17 = 32$

T: Show me how you would find the value of x.
R: (writes $32 - 17 = 15$)
T: Why did you subtract?
R: Because if I add 15 to 17 then I would get the answer.
T: Now test your answer.
R: (writes $15 + 17 = 32$)

2: $201 = x + 109$

T: You must find the value of x again. Show me.
R: (writes) $201 - 109 = 92$
$x = 92$
T: Test your answer.
R: $92 + 109 = 201$
T: Is your answer right?
R: Yes.

3: $x - 14 = 47$

T: Will the value of x be more or less than 47?
R: Less.
T: Why?
R: No, he is going to be more, because if you minus 14 from a number then (pause a fairly long while)
T: Write down what you are thinking.
R: (wants to start pressing on the calculator)
T: Before you press, please write what you are going to do.
R: $33 - 14 = 19$
T: But didn't you say that x must be more than 47? Is 19 more or less than 47?
R: Less.
T: What are you going to do now?
R: (silent for a while) and then $33 + 19 = 52$.
T: Remember we are looking for x. What did you do here in (2)?
R: I added.
T: And now?
R: (tries a number of arithmetical manipulations, but does not get 61. it seems as if he wants to stop)
T: What would you have said if I say $? - 3 = 10$? We have done something like this before. What would have come in the place of ?
R: Then $10 - 3 = ?$
T: But there stands $? - 3 = 10$
R: (long silence) Something must come here in the place of the ?
T: Yes, we are looking for an answer in the place of the ?

R: So, 3 plus what must give me 10?
T: Oh.... Your answer is 7. If we now put 7 in the place of the ? then we get 7 – 3 = 4. Is it 10? (pause) What must come in the place of the ?
R: 13
T: Right! 13 – 3 = 10. But how did you get the 13?
R: I took 10 + 3.
T: Now go back to the sum i.e. x – 14 = 47. Let us make the x a ? What now?
R: (looks anxious)
T: Another learner said the answer was 61. Do you agree?
R: In other words 47 + 14 = 61
T: What is the value of x?
R: 61
T: Test your answer.
R: (tests correctly).

4: x – 17 + 10 – 10 = 12

T: (by now I have seen that R is not very talkative) Please write down everything that you are going to do to find the value of x.
R: (writes 17 + 10 – 10 = 17)
T: Please use the calculator.
R: (presses 17 – 5 = 12)
T: What is x?
R: x = 5
T: Come we see. You say x = 5. Now test your answer.
R: (tests and gets –12)
T: Is it the same as in the sum?
R: Yes.
T: But here you have –12 and there is 12 in the sum.
R: (looks confused)

5: 200 = 247 – x

R: (writes 247 – 47 = 200)
T: How did you get 47?
R: I added the answer (47) to 47 to get 94.
T: And now? What about the 200 and the 247? Look at (4).
R: (looks at (4) but does not seem enthusiastic)
T: If I say to you the answer is 47, how would you test it?
R: (writes 200 – 47 = 153)
T: Okay. Is it right?
R: No.
T: What did you do when there was a minus sign?
R: I ............
T: If I say to you that x is 47. Now show me that I am right.
R: (again 200 – 47 = 153)
T: Let's move on.

6: $2x = 110$

T: What does 2x mean?
R: Two times something is 110.
T: How will you find that "something"?
R: (looks for a long while at the equation and says) $2 \times 12 = 24$.
T: Are you sure? Is it right?
R: No.
T: Why did you use 12?
R: I actually have 2 times 6.
T: Remember you said 2 times something must give 110. Use your calculator and see.
R: (long silence)
T: Please tell me what you think.
R: I want 210 ....... no!! (says softly) $2 \times 10 = 20$ and then $110 - 20 = 90$.
T: Why 2 times l0?
R: To subtract ........
T: Okay ..... what is x?
R: 90
T: Come let us test. (Now I start helping with the test) $2 \times 90 = 180$. But remember you said in the beginning that 2 times something must be 110.
R: (looks confused, bewildered)
T: (show him previous work) Here we had $2x = 4$ and then you said $2 \times 2 = 4$ ........ why?
R: Four divided by 2 is 2.
T: What must you do here?
R: (silence)
T: If $2 \times 3 = 6$ then $6 \div 3 = 2$ isn't it? Now look again at 2 times $x = 110$. What must you do?
R: (Still does not see that he must have $110 \div 2$).
T: Let us go on.

7: $3x - 12 = 6$

T: Begin. You do know that 3x is 3 times x.
R: Writes $3 \times 18 = 54$.
T: Test your answer.
R: $54 - 12 = 42$ and says $6 \times 12 = 72$. (scratches his head and appears confused)
T: Do you see the difference between (6) and (7)?
R: (silent)
T: What happened to the $-12$?
R: (no answer).

8: $6x + 4x = 40$

T: Here we must get the same x in both cases. In other words 6 times something plus 4 times the same something must give us 40. Now you try.

R: (writes) $6 \times 2 = 12$
$4 \times 2 = \underline{\ 8}$
$\underline{20}$

T: And now? Remember that we want to get 40. How will you do that?
R: I shall multiply by 2 i.e. 20 x 2 = 40.
T: So what is x?
R: x is 20.
T: Why 20?
R: Because 2 x 4 = 8 and 20 x 2 = 40.
T: So what is x? Is it still 20? Please explain.
R: (now writes 8 + 32 = 40)
T: Where did you get the 32?
R: (long silence)

9: $3x - 2x + 8x = 45$

T: Now I am not going to say much. You must write down everything you do.
R: (whispers softly and writes) $3 \times 2 = 6$
$6 - 2 = 4$
$6 + 4 = 10$
$8 \times 2 = 16$

T: Why do you use 2?
R: (silence) It is easier. (now he adds up everything to get 36 i.e. he does not see the negative sign in front of the 2x).
T: What is the value of x?
R: (silent, scratches with his pen) I don't know.
T: Is it 2?
R: No.
T: What are you going to try now?
R: 3 .................... (works in silence but does not seem to know what is going on).

10: $14x + 10 - 3x = 55 + 11$

T: Where will you begin this time?
R: With the answer i.e. 55 + 11 = 66.
T: Now how will you continue?
R: With 2.
T: Go on.
R: (starts to write) $14 \times 2 = 28$
$28 + 10 = 38$
$38 - 3 = 35$

T: And now?
R: 35 + 31 = 66
T: Why do you add 31?
R: I want to get the answer 66.
T: Okay.

11: $4x = x + 21$

T: Where are you going to begin?
R: With 2.
T: Show me.
R: I shall say 2 + 21 = 23.
T: And .......... go on.
R: (writes) 4 x 2 = 8 on the left hand side and then 23 + 8 = 31.
T: Why did you begin on the right hand side?
R: Because it is the answer.

12: (No more time).

**GENEVEVE WAGMAN (W)**

1: $x + 17 = 32$

T: Show me all your work.
W: (looks quite happy) 17 – 32 = 15
T: Are you sure?
W: No. It is 32 – 17 = 15
T: What did you do?
W: I subtracted.

2: $201 = x + 109$

T: Please speak as you write.
W: (long silence)
T: What are you going to do? I am going to ask many "why's" as we go on.
W: Now I am going to ........... 201 – 109 =
T: Why did you subtract?
W: (silence)
T: What do you get?
W: 92
T: Now you must test. In other words where there is x you must replace it with 92.
W: So I must add 92 to 109 and then I shall get 201.
T: So if I say to you that 1011 = x + 373, how will you find x? If you look at what you have done before.
W: Then I shall minus again.
T: Write down what you would minus.
W: (writes) 1011 – 373 = 638
T: Can I take it that you see where there is plus then you must minus?
W: Yes.

3: $x - 14 = 47$

T: How does (3) differ from (1)?
W: Now there is a minus.
T: What are you going to do now?
W: (works in silence) 47 – 14 = 33
T: What do you get?
W: 33 is x.
T: Have you tested your answer? See if 33 – 14 is equal to 47.
W: (raises her voice) But I have done so already!!
T: But here you have 33 + 14 = 47 while there is x – 14 = 47 i.e. there was a minus sign.
W: (writes) 33 – 14 = 19.
T: Is your answer 19? What if I give you smaller numbers i.e. x – 2 = 6? What will x be now?
W: 4

T: Are you sure?
W: No!!! 8!!
T: How did you get the 8?
W: Because if I take 6 from 8 then I get 2.
T: What about larger numbers .......... such as x – 901 = 475?
W: I am going to minus 901 with 475 to give 426.
T: Test your answer. Remember where there is an x you must write 426.
W: (says softly while writing) 426 – 901 = 475.
T: What do you get?
W: (silent ...........)
T: What if I say that the answer of (3) is 61. See if it is right.
W: (softly) 61 – 14 = 47 (nods affirmatively)
T: How did I get the 61?
W: (silence ...........)
T: Isn't it the same way as you got the 8 just now?
W: You have added 47 and 14.
T: Yes. That is how you obtained the 8.

4: x – 17 + 10 – 10 = 12

W: (first ignores the minus sign in front of the last 10)
T: Where would you start?
W: 17 + 10 = 27 (again no minus sign and then) 27 – 10 + 17.
T: And now. Remember we are looking for the value of x.
W: (long silence)
T: Didn't you see the minus signs? How much is 10 – 10?
W: 0
T: Now you know about the minus signs and the zero. Right? Now go on.
W: (goes on writing but looking very unsure)
3 – 17 = 14
14 + 10 = 24
24 – 10 = 14
T: Where did you get the 3?
W: I guessed!!!
T: Okay. What is the value of x?
W: I got 14.
T: Now test your answer with the calculator.
W: (tests and gets –3)
T: Can you see that 10 – 10 = 0? So now I can write x – 17 + 0 = 12. You agree?
W: mmmm
T: So x – 17 = 12. Now find x.
W: Then I am going to say 17 – 12 = 5.
T: Look what you did in (3).

5: 200 = 247 – x

T: Does (5) differ from (3)? How does it differ?
W: Yes ..... the x is now at the back (in 5) and there (in 3) it was in front.

T: Right. What are you going to do now?
W: (silent)
T: Will x be more or less than 200?
W: (pause a long while) Less.
T: Why?
W: (long silence)
T: Okay. Show me how you will find x.
W: I shall subtract 47 from 200 ...... No! I must now subtract 47 from 247 and then I get 200.
T: What if the sum was 111 = 252 – x? What would you have done?
W: I would have minused.
T: Minus for me.
W: 252 – 111 = 141.
T: Can you see the difference between this (141) and when you did 200 – 47?
W: Yes.
T: Let us do another one i.e. 41 = 92 – x.
W: (writes immediately) 92 – 41 = 51.
T: So why didn't you start with the same in (5)? Do it again now.
W: (again!) 200 – 247 = 47.
T: (now impatient) Don't you see the 252 is first and the 111 is second?
W: (No reply).

6: 2x = 110

T: What does 2x mean to you? Is it x, ÷, - or + ?
W: (silent)
T: Look what you have told me here. (show her where she said that it was multiplication) Now do the sum for us please!!
W: (still silent)
T: Let us make it easier. If I say 2x = 6, what would your answer be?
W: 4
T: Okay. So you are saying 2 x 4 = 6?
W: (looks embarrassed) No ..... 2 x 3 = 6!
T: How did you get the 3?
W: Because if I .....(pause) Because if I now multiply 2 with 3 then I get 6.
T: Right. What about 2x = 110?
W: Two times something must give me 110.
T: Yes. How will you get that something?
W: (looks / laughs sheepishly)
T: Please write down what you are thinking.
W: I don't know how.
T: What if 2x = 100? What will x be?
W: (long silence)
T: Two times how many is equal to 100?
W: (even longer silence) There will come 50. (sounds impatient)
T: How did you get the 50?
W: I just saw it.
T: In other words you just knew that 2 x 50 = 100?

W: Yes.

7: $3x - 12 = 6$

T: What does (7) read like?
W: Three times a number minus 12 must be 6 .......
T: Where are you going to begin?
W: 4 + 12 ...... No ......... 3 x 2 = 6 and then 12 minus the 6 ........ then I shall get 6.
T: Twelve minus which 6?
W: The 6 that I got here i.e. 12 – 6. Because Sir ....... 3 x 2 gives me 6. Then I minus that 6 from the 12 then I get 6.
T: What happened to this part? (i.e. the = 6)
W: I just left it out.
T: So what you are saying is that 3 x 2 = 6 and then you say minus the 6 with 12 ............ Do it on your calculator.
W: (does the calculations and gets –6)
T: It does not work .......... does it?
W: I just don't know sir (looks at me in a very questioning way).
T: If two does not work, will your answer (number) be smaller or greater than 2?
W: Greater.
T: Now try a greater number.
W: (tries 3 x 3 ........ then 3 x 0 .......
T: Let's go to (8).

8: $6x + 4x = 40$

T: As you can see there are two x's now and 6 times x plus 4 times the same x must give us 40. Now find that number for us.
W: Must the x by the 6 be the same as the x by the 4?
T: Yes.
W: Okay, then I shall try 6 x 5 = 30 plus 4 x 5 which gives me 50. Too much.
T: Will x be more or less than 5?
W: Less.
T: Show me what you will do next.
W: I shall take 3. (then she gets 30).
T: Right. Now you have got a 50 and a 30. First you tried 5 and then 3. Which number will you try next?
W: More ........ like 4.
T: Try it.
W: (Tries it and gets 40)
T: Does 4 work?
W: Yes.
T: If I say that one can write the equation as follows $6x + 4x = 10x = 40$, would you have got your answer quicker? Look at (6).
W: Then it would be 4 times.
T: How did you get 4?
W: Because I know that if a person multiplies with 10 then you just

put a zero next to the number.
T: Okay .......

9: 3x – 2x + 8x = 45

T: Did you see that we counted all the x's together in (8)? Can you do the same here?
W: (now sounding more self-confident) Okay, 3 + 2 = 5 + 8 = 13.
T: mmmm But there is a minus!
W: (pause.....)
T: (smiling) Don't you like minus signs?
W: (softly) No.
T: Use the calculator if the minuses give you problems.
W: Then it is now 3 – 2 = 1 + 8 = 9.
T: Nine what?
W: (long silence)
T: In (8) there are x's. What must be written next to the 9?
W: x ............... 9x is then equal to what?
T: You tell me!
W: 45
T: Okay, what is the value of x now?
W: x will be 4.
T: Okay. Test your answer.
W: (softly) 9 times 4 = ......... No!!
T: I will say the answer is 5.
W: Yes. Yes!!
T: How do I get the 5?
W: Don't you ............ divided by.
T: Isn't that what we have always been doing.

10: 14 x + 10 – 3x = 55 + 11

T: Show me what you will do.
W: (speaks softly) 14 – 3 = 11. So 11x = 66
T: What did you get?
W: 6
T: What happened to the 10?
W: I don't know.
T: There must be a 10 also. Do you agree? What you have without the 10 is right. But now it becomes 11x + 10 = 66. It is the same as what we had earlier. Please do it now.
W: ( ........... long silence ................. keeps staring at the sum)

11: 4x = x + 21

T: Now we have x's on both sides of the equal sign and we must get the same x on both sides. Show me how you are going to do it.
W: Okay, 4 x 12 ..........
T: Why are you multiplying?
W: No! I have just looked.

T: What did you see? Why 4 x 12? Where does the 12 com from?
W: I have just guessed.
T: Does 12 work?
W: No.
T: Show me why not?
W: (writes $4 \times 12 = 48$)
T: What about the other side?
W: Then there must also be 48. No, then x must also be 12 and $12 + 21 = 33$.
T: Okay. Once again, does 12 work?
W: No!
T: What if I say the answer is 7. See if it works.
W: (Using the calculator and writes) $4 \times 7 = 28$ and $7 + 21 = 28$. Yes, Sir!! (sounds relieved).

12: $10x - 6x + 3x - x = 2x + 10$

W: (works in silence) $10 - 6 = 4$
$4 + 3 = 7$
T: Have found x?
W: (silence)
T: What stands in front of the last x?
W: A minus.
T: Now rewrite it for us.
W: Then it becomes $7 - x$.
(Time is up).

**ROGER K. (K)**

1: $x + 17 = 32$

T: Read and tell me how you would find x.
K: I took 32 and subtracted 17 to give me 15.
T: In other words you know if there is a plus sign then you must subtract.
K: Yes. (tests correctly).

2: $201 = x + 109$

K: I say 201 – 109 = 92 and then x = 92.
T: Now you must test. Here where you see an x you must write 92.
K: (writes and presses) 201 = 92 + 109 = 201.
T: Okay.

3: $x - 14 = 47$

T: Do you think x is going to be more or less than 47?
K: More.
T: Why?
K: Because there is minus.
T: Now do it for me.
K: (writes) 47 + 14 = 61 and then tests as 61 – 14 = 47.

4: $x - 17 + 10 - 10 = 12$

T: Will you start please.
K: (begins with) 17 + 10 – 10 – 17 (i.e. ignores the x and the minus sign).
T: Remember we want to find x. Please tell me what you think.
K: (long silence) I will ...........
T: Okay. Just write down what you are going to do.
K: (starts to write) 37 – 10 = 20 + 10 – 10 = 20.
T: Now ..... what is x?
K: (long silence)
T: Is it 37?
K: No.
T: Why not?
K: (softly) There is a 12. I am now going to begin with less. (then he starts with 34 then 30 and then 29)
T: Why did you start with 34 and not 35?
K: Because the answer was 17 and the answer 17 is more than the answer 12. (then he starts with 30 and gets 13).
T: Is your answer correct?
K: No, it is not yet 12!
T: What are you going to do next?
K: Now I see that 13 is one more than 12 and then I take one number less than 30.
T: What is your answer now?

K: x is 29.

5: $200 = 247 - x$

K: Now I say that $247 - 47 = 200$. So $x = 47$.
T: How exactly did you get 47? Did you just see it?
K: Number is 247 and if you just take away 47 then you get 200.

6: $2x = 110$

T: What does 2x mean to you?
K: 2x is times.
T: How are you going to do the sum?
K: (writes down $110 \div 2 = 55$)
T: Okay. Why did you divide?
K: I must multiply 2 with something and then I just took 110 and divided it by 2.
T: You are doing well!

7: $3x - 12 = 6$

T: What do you see here? Where are you going to start?
K: 6 times 3 is 18 ......... 18 will be too many.
T: What do you mean it will be too many?
K: No, it won't be too many if I say 18 – 12 then I shall get 6 and if I see that 18 divided by 3 is 6 then x is 6.
T: Okay ..........

8: $6x + 4x = 40$

T: Remember that you must get the same value for both x's. Now if you can find it.
K: I shall try 6 times 2 which will give me 12 and then $4 \times 1 = 4$ ...... He does not work.
T: Why doesn't it work?
K: I am trying to get 40 and now I have only got 16.
T: Okay ......... next step.
K: Here is a plus (+) and I thought I had to multiply it.
T: Now try the same sum again!
K: (now writes $4 \times 4 = 16$ and adds it to 12 to get 28.)
T: Is that right?
K: No.
T: Remember you must get the same x in both cases.
K: (looks at the sum for a while and says) $6 \times 4 = 24$
and $4 \times 4 = 16$
which gives me 40. So x is 4.

9: $3x - 2x + 8x = 45$

T: Remember we want to get the same x everywhere.

K: (long silence) Writes + 2x instead of – 2x.
T: Didn't you see the minus sign?
K: (changes the signs)
T: Is there a way that you can make the left hand side shorter?
K: (long silence)
T: If you had a 3 Rand and take 2 Rand away and you add 8 Rand, how would you have simplified it?
K: Must I make it shorter?
T: Yes. How much will you get?
K: 9 Rand.
T: Is that the same as in the sum? Now we only have x's instead of Rand. Now look ................
K: So it will be 9x.
T: ...... is equal to .........?
K: x's must be the same.
T: Yes, now you have made it shorter. So there must be 9x is equal to what?
K: 45 .......... so 9x = 45.
T: What do you have now? Look at (6) again?
K: Then I have 45 ÷ 9 = 5. So x = 5.
T: Let us see .......... show me.
K: 9 x 5 = 45.

10: 14x + 10 – 3x = 55 + 11

T: Now I am not going to say much.
K: (starts to write) 14 + 10 = 24 – 3 = 21 and 21 = 55 + 11
(leaves out the x's) then 66 ÷ 21 = 3,1428.
It is not going to work!
T: Do you see that you have added the 14x to the 10.
K: (long silence ........... looks bemused)

11: 4x = x + 21

T: In (11) and (12) there are x's on both sides. Now you must be careful because the x's on both sides must still be the same.
K: Is x + 21 the answer of the sum 4x?
T: Remember the x's on both sides must be the same.
K: I am going to multiply 4 with something ........... (writes 4 x 4 = 16)
T: Why 4?
K: I am trying to find the answer of the x in x + 21.
T: Remember there is an x on the other side also!
K: (writes) 4 + 21 = 25.
T: Now what is x?
K: (silently writing) 4 x 6 = 24 and 6 + 21 = 27.
T: Why 6?
K: Because 16 is less than 24 ...........
T: Again I say that you must get the same x on both sides.
K: I will try 21 ÷ 4 ...........
T: Why?

K: Because there is a 4 on the left hand side ....... (starting to look tired).

12: 10x – 6x + 3x – x = 2x + 10

K: Must the x's be all the same?
T: Yes .......... they must be.
K: (laughs softly ....... writes) 10 x 2 – 6 x 2 = 8 + 3 x 2 = 14
T: Why 2?
K: I am only trying.
T: What about the minus after the 3x?
K: Will it not be zero and then it remains the same? (Now he rewrites the right hand side as 2 x 2 + 10 = 14.
T: Okay .........
K: So x = 2.
T: Test and see if it works.
K: (tries to test and again does not see the minus sign after the 3x) x is 2, sir.
T: Look at (8). Don't you think there is an easier way to these sums? Just look at 6x + 4x = 40.
K: Oh! 40 ÷ 10 = 4!!

**JOANNE BEUKES ( B)**

1: $x + 17 = 32$

B: So $32 - 17 = 15$ and therefore $x = 15$.
T: Why did you subtract?
B: Because there is a plus.

2: $201 = x + 109$

T: Is there a difference between (2) and (1)?
B: Now the sum is just the other way round. The answer is now first.
T: Will you follow a different method of solving now?
B: No, it is $201 - 109 = 92$ and $x = 92$.
T: Test your answer by writing 92 in place of x.
B: (turning the equation around) $109 + 92 = 201$

3: $x - 14 = 47$

T: What do you see now? Is there a difference between (2) and (3)?
B: (3) is a minus sum!
T: What are you going to do now?
B: (writes) $47 - 33 = 33$
T: Test your answer. Is it right?
B: No ........ there must be a plus in the place of the minus i.e. $47 + 14 = 61$.
T: Why did you replace the minus with a plus?
B: Because there is a minus!

4: $x - 17 + 10 - 10 = 12$

T: Where are you going to begin?
B: With 12.
T: Why 12?
B: Because the answer is 12! (then she writes $12 + 10$)
T: Why do you write $+ 10$?
B: Because there (the last term) is $-10$.
T: Okay ........... carry on.
B: Now $12 + 10 - 10 + 17 = 29$ (seems to work from the back)
So I get 29 and $x = 29$.
T: Look at the numbers again. Do you think there is a better (quicker) way?
B: Maybe there is.
T: Do you really think there is?
B: ....................... (no answer).

5: $200 = 247 - x$

T: How does (5) differ from (2)?
B: The 200 is equal to the sum.

T: Is the answer always the one without the x? In (4) you also said that 12 was the answer. Is the sum the longer part of the equation?
B: Yes. (writes 200 + 247 = 447)
T: Why did you add again?
B: Because there is (pauses a while). You might also have got 47.
T: But now you have 2 x's. What is the value of x?
B: x is 447
T: Test your answer.
B: (writes) 200 = 247 – 447 = -200.
T: You got a minus.
B: No, one must work from the back.
T: How?
B: Like 447 – 247 = 200. Thus x = 447.
T: Huh.

6: 2x = 110

T: What does the 2x stand for? Read it for me.
B: Two times x. Now I must have 110 ÷ 2 = 55.
T: Why do you divide?
B: Because 2x is 2 times x and then I turn it around to get division.
T: (not quite convinced) Say there had been 7x = 170, what would you have done?
B: Okay, then it would have been 170 ÷ 7.
T: Okay.

7: 3x – 12 = 6

T: Show me where you would begin.
B: (writes) 12 ÷ 3 = 4.
T: Why do you divide 12 by 3? Where does the 12 come from?
B: (shows the –12)
T: Okay ......... now do the sum.
B: (does some arithmetical manipulations and writes) 12 – 12 = 0. No!!
T: What now?
B: 12 ÷ 6 = 2.
T: Why 6?
B: There (on the right hand side) is a 6.
T: Now ..........
B: (uncertain) 3 x 2 – 12 = -6 (pause)
T: What do we want to find?
B: We want to get 6.
T: Why 6?
B: The answer is 6.
T: How will you find x?
B: I shall work it out.
T: Show me.
B: (writes) 24 ÷ 3 = 8

T: How do you get 24?
B: I took 12 x 2.
T: Why?
B: Three ......... what have I said? No! (writes) $3 \times 8 = 24 - 12 = 12$. Must be less than 8.
T: And now .........?
B: Now I shall try 6.
T: Why not 7?
B: 6 is an even number. (finally $3 \times 6 - 12 = 6$)

8: $6x + 4x = 40$

T: Begin.
B: I am going to start with 40 because I read from the back and then I will say $40 \div 4 = 10$ ....... then again $40 \div 6$.
T: Why $40 \div 6$? Remember that the x's must have the same value.
B: No ......... $x = 4$
T: Why?
B: Because $6 + 4 = 10$ and $40 \div 10 = 4$ (then she tests correctly)

9: $3x - 2x + 8x = 45$

T: (looks silently on)
B: In (9) there are 3 x's on the left hand side.
T: In (8) you said $6 + 4 = 10$. What happened to the x's?
B: I left out the x's because in its place there must come a number.
T: So what are you going to do in (9)?
B: I am going to try first. (writes $3 - 2 + 8 = 9$)
T: I see.
B: (writes $45 \div 9 = 5$) Thus $x = 5$. (then she tests as follows: $3 \times 5 - 2 \times 5 + 8 \times 5 = 45$) Right!!!
T: Once again, what happened to the x's?
B: I took the x's as they stand!

10: $14x + 10 - 3x = 55 + 11$

B: The answer that I have is 66.
T: Okay. Now start with the 66 and get x.
B: (writes) $14 + 10 + 3$ ... Sorry!! (and then) $14 + 10 - 3 = 21$ (and now) $66 \div 21 = 3{,}14$.
T: Didn't you see that the 10 has no x?
B: (sheepishly) I am going to leave out the 10.
T: What are you going to do??
B: I will say $66 \div 11 = 6$. Then I take $14 \times 6 + 10 - 18$ .... O gosh!!
T: What are you going to do with the 10 to get to 66?
B: But I have already taken the 10!
T: But earlier when there was a plus you subtracted ........ what now? Try again.
B: (seems confused) $14 + 3 = 17$.

T: Why do you add?
B: Because there is a minus.
T: Okay. Write it down for us.
B: 66 ÷ 17 .......... (pauses – looks at the calculator in disbelief)

T: Does it work?
B: No. (looking more embarrassed than before.)

11: $4x = x + 21$

T: Now there is an x on both sides. Show me what you will do.
B: I say 21 – 4 = 17.
T: Why do you start with the 21?
B: Because it is a number. I have taken it and I have taken 4 away.
T: Why did you minus?
B: Because there is a plus.
T: Okay. Remember there is an x on both sides and they must be the same.
B: Starts to write $4 \times 17 = 68$
$17 + 21 = 38$
$21 \times 17 = 357$
No!!!
T: Don't you see the x? (showing the x on the right hand side)
B: Yes, but I am not using it now.
T: Okay. What is the value of x?
B: x is equal to ........ (seems not to know any further)
T: Okay. What if I say the answer is 7. See if it is right.
B: (tests) $4 \times 7 = 28$
$7 + 21 = 28$
(shouts) YES!!!

12: $10x = 6x + 3x - x = 2x + 10$

B: (once again leaves out the x's and the singleton and changes the + to – thus writing 10 + 6 – 3 = 13)
T: How many x's are on the left hand side?
B: Four.
T: Can you see what you have done in (9)? What are you going to do now?
B: (substitutes all the x's with 13)
T: Okay. Thank you.

**LAURENTIA BOOYSEN (BL)**

1: $x + 17 = 32$

T: What are you going to do?
BL: 17 plus another number must give 32.
T: Which number? How will you get that number? Please use the calculator.
BL: (tries 19, 17, 14 ... on her fingers)
T: Why did you take 19?
BL: I thought he would take me to 32. But he did not ...... he brought me to 36.
T: Would you next number be smaller or ......
BL: Smaller than .......
T: Then you took 17 and then 14. What will your next number be? Greater than 14?
BL: Yes. I shall take 17 + 15 = 32.
T: So what is the value of x?
BL: 32.
T: What? Write it down.
BL: (writes the same equation) 17 + 15 = 32.
T: Do you think there is an easier way? If I gave you $x + 107 = 282$, would you use your fingers again?
BL: I think I would perhaps use a calculator.
T: Now show me!
BL: (long silence)
T: Could you not previously have subtracted i.e. to get the 15 you could have said 32 – 17 = 15.
BL: Yes.

2: $201 = x + 109$

BL: (immediately gets 92 with the help of the calculator)
T: How did you get 92?
BL: I just took 201 and then I took 109 away.
T: Why did you turn the sum around?
BL: (silence)
T: Is it easier to turn the sum around?
BL: Yes.
T: If $17 = 12 + x$, would you also turn it around?
BL: I shall minus 17 with 12.
T: Write down what you would do.
BL: 17 – 12 = 5
T: So what did you do here?
BL: I also turned it around.
T: So now you know that if there is a plus then you will minus and if there is a minus, what will you do?
BL: Plus.

3: x – 14 = 47

BL: (writes a plus instead of a minus sign)
T: But there is a minus sign.
BL: Oh! (changes the equation)
T: What are you going to do?
BL: I shall say 47 – 14.
T: Why do you minus? You said if there was a minus then you should plus.
BL: But it is easier ..... I get 33.
T: So x is = 33? Test your answer.
BL: Then 33 – 14 = 21.
T: Can it be true?
BL: No! 14 + 33 = 47.
T: But what about the minus sign?
BL: I shall change it to plus.
T: Why?
BL: Because I thought it would be easier if I added the numbers.
T: But the sum does not say so!
BL: When I got to the answer then I added him!
T: Okay. Go to (4)

4: x – 17 + 10 – 10 = 12

T: How does (4) differ from the other?
BL: There are more numbers than in (2).
T: More numbers? How do those numbers look? Will they make the sum more difficult?
BL: (silent ...... works on calculator and gets 37)
T: How did you get 37?
BL: I took 12 – 17 – 10 – 10 = -25 .....
T: Why did you use – 10 twice?
BL: Here is a 10 (first + 10) and then I minused it with 10.
T: How much is 10 – 10?
BL: It is 10!
T: Are you sure? Look on your calculator.
BL: It is 0!!
T: How are you going to write the sum shorter now?
BL: It will be 12 – 17 + 10 = -5.
T: What is the value of x?
BL: It is ........
T: Test your answer please.
BL: (having problems with the minus-sign) ...... mmmmm.
T: Does –5 work? Is your answer right or not?
BL: No ....
T: What do you want to get?
BL: I am looking for 12.
T: So how will you get 12?

BL: I am going to try again!
T: What are you going to try?
BL: Can't I take the 17 first and then I take 12 – 0 = 29.
T: Okay, let us see. You first said 10 – 10 = 10 and then you said it was 0. do you think that we can write this part (10 – 10) as 0?
BL: Yes.
T: And then x – 17 + 0 = 12?
BL: Yes.
T: Now work out the sum for us.
B: 12 – 17 + 0 = -5
T: Why –5 again? (nearly shouting) Why do you press + 0? If you add 0 to any number then the number stays the same ....... doesn't it?
BL: Yes.
T: So make the sum shorter for us.
BL: Now I am confused!! I am going to say 12 – 17 ...... because 12 is the answer.
T: Write it down.
BL: I am going to press 12 – 17 and then I shall see what my answer is. But I get –5 again!
T: so the answer is –5!!
BL: (shakes her head) No. (softly)
T: Will you allow me to write the equation as x – 17 = 12. Forget about the 0. Now look what you did in (3)!
BL: I minused ....... 47 – 14 = 33.
T: Minused?
BL: Yes.
T: Will you do the same here?
BL Yes.
T: Now do it! (decides to go on).

5: 200 = 247 – x

T: How does (5) differ from the previous equations? Look at the x.
BL: In (4) and (3) the x is right at the beginning and in (5) it is last. (meaning at the end)
T: Right. What else do you see that is different?
BL: The 200 is right in front.
T: Now do it for us!
BL: Must the <u>whole</u> answer be 200?
T: Is 200 the answer? Remember that we are trying to find x.
BL: (presses on the calculator) I got 47!
T: Is that your answer?
BL: Yes......... it is the value of x!
T: Is the value of x the answer?
BL: Yes.
T: Now test you answer.
BL: (tests) It is 447. (i.e. 200 + 247)
T: Which one of 447 or 47 is the answer?
BL: 47 ........ Sir.
T: Tell me ............. Previously in (2) you turned the problem back

to front. Why don't you do the same here?
BL: Because I minused 247 with 47.
T: Now let us look at this one 10 = 19 – x. How will you get the answer?
BL: I shall say (take) 10 – 19.
T: What do you get?
BL: I get –9 (looks embarrassed)
T: Will it work?
BL: No.
T: What did you do here (showing the 47 of (5) )? How did you get 47?
BL: I minused 247 with 200 to get 47.
T: So why do you now say 10 – 19 and <u>not</u> 19 – 10?
BL: Because 10 is the answer.
T: But didn't you say that in (5) that 200 was the answer?
BL: (rather sheepishly) Yes.

6: 2x = 110

T: Please read what you see.
BL: (reads exactly as she should)
T: What does 2x mean?
BL: (repeats)
T: What (x, ÷, + or -) is actually between the 2 and the x?
BL: Plus.
T: Sure? What did you say here (turns back to 2x = 16)? Here you said times. Do you agree that this means 2 times x?
BL: Sir, I think it is 2 times or 2 + x!
T: Is plus and times the same in Mathematics?
BL: YES.
T: Okay. (Looks at her in disbelief) Now show me how you would do the sum.
BL: I shall say 2 x 110 = 220. My answer is 220!
T: (showing her the problem) What is your answer here?
BL: It is 110!!
T: So you say x = 110.
BL: No!
T: But you replaced x with 110! What are you going to do next?
BL: I am looking for an answer!
T: How will you get that answer?
BL: I shall say 110 x 2 ....... (gets 220 again)
T: Does it work?
BL: No.
T: (goes beck to the "block"-situation)
Here you said 8 = 16 ÷ 2. so if I said 2 times x = 1014, how would you find x?
BL: Now I know that 110 ÷ 2 = 55.
T: So what is x?
BL: x = 55.
T: Now how do you now find x in 2x = 1014?
BL: I shall say 1014 ÷ = 507
T: Oh!

7: $3x - 12 = 6$

BL: I shall now have 6 ÷ 3 and then I shall divide it by 12.
T: What do you get?
BL: 0,166 ... (softly) It does not work.
T: Why did you say it does not work?
BL: No, it does work sir. I can just round it off!
T: Okay. Round it off.
BL: But ...... (long silence) I shall now start all over. (writes) 3x + 12.
T: Why do you write a plus sign?
BL: I could have written times!
T: Okay, we agreed that it meant multiplication.
BL: (starts pressing again)
T: What do you get?
BL: 0,666
T: Do you think that x should be greater of smaller than you 0,666? You know now that 3x means 3 times x.
BL: Smaller.
T: Are you sure?
BL: Now the next number ....
T: Okay.
BL: (looks unsettled) I think greater that I can divide.
T: Show me.
BL: 6 ÷ .......
T: Why do you start with 6?
BL: 6 is the answer.
T: Let's go to the next one.

8: $6x + 4x = 40$

T: Do you see that there are two x's? What does 6x stand for?
BL: It is 6 times x plus 4 times x is 40.
T: Now you must find the same x in both cases to give us 40... the same number for x.
BL: (begins with 5)
T: I see that you begin with 5. Why?
BL: Because 6 x 5 = 30 I have pressed incorrectly. It should have been 6 x 4 + 4 x 1 = 28. (then divides by 10).
T: Why did you use 10?
BL: I wanted to ........... (pauses)
T: Why did you say 6 x 4 <u>and then</u> 4 x 1?
BL: Must I use the same number?
T: Yes.
BL: Okay, then I shall say 6 x 4 + 4 x 4 = ........
T: Is what?
BL: 40.
T: Now do 5x + 2x = 14.
BL: Okay, so 5 x 4 + 2 x 4 = 28.
T: Why did you start with 4?

BL: I thought I would get half because 28 is half of 14?
T: Is 4 right?
BL: No.
T: What are you going to try next?
BL: I shall say 14 ÷ 5 x 2 + 2 x 2 ...... No it's NOT right.

9: 3x – 2x + 8x = 45

T: Read the equation for us. How many x's are there?
BL: Three.
T: Where are you going to start?
BL: With the 45?
T: And then .......?
BL: Then I shall divide by 3 to get ......... 15.
T: Why 3?
BL: Three is first on the list!
T: And then ...........?
BL: Then I shall minus it with ... 2 and then I shall plus it with 8 and then I get 21.
T: What happened to the other x's?
BL: The two x's have "melted" together.
T: Now you <u>must</u> explain!
BL: (long silence)
T: How do the x's melt together?
BL: I took the first x and minused 3 and then I multiplied to get 21.
T: Wow!! Now what is the value of x?
BL: 21.
T: Test your answer.
BL: Must I write 21 by all the x's?
T: But you did say that all the x's must be the same.
BL: (does some fingering)
T: Okay. I shall write it as 3 x 21 – 2 x 21 + 8 x 21 and ...........
BL: (looks tired and frustrated)
T: (decides to move faster)

10: 14x + 10 – 3x = 55 + 11

T: Do you think you can do this one.
BL: Okay, I shall say 55 + 11 = 66 and then 14 x 10 – 3.
T: Let's move on. (time is running out)

11: 4x = x + 21

T: Once again we must find the same value for both x's. Where will you begin?
BL: (long silence)
T: Will you be able to do it? Do you think there <u>is</u> a way to do it?
BL: No!
T: What if I say the answer is 7. Test it.
BL: (starts to press and then she stops)

T: Thank you.

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**RUZELL ESAU (E)**

1: x + 17 = 32

E: x = 15.
T: How did you get that?
E: I added.
T: Why did you add?
E: Because I saw that 15 + 17 = 32.
T: Okay. Now what would you get if I give you x + 101 = 391.
E: 200.
T: How did you get 200?
E: (silent)
T: I think it is wrong. The answer is 290.
E: Oh!
T: Do you think there was another way to get the 15 you had before?
E: (silent)
T: How would you do x + 1101 = 47207?
E: (looks uncertain/puzzled)
T: Okay. One could have found 15 as 32 – 17. So here you could have x = 47207 – 11011.

3: x – 14 = 47

T: How does (3) differ from (1)?
E: There is a minus.
T: Do it for us.
E: Now 47 – 14 = 33 <u>and</u> 47 + 14 = 61.
T: Why did you take 47 + 14? Why did you add?
E: No! (very much unsure) It does not work. I should have subtracted. Now I'll say 47 – 14 = 33.
T: Test.
E: It does not work out, sir!
T: So what is your answer?
E: No! (then she writes 47 + 14 = 61)
T: Is 61 your answer?
E: Yes.
T: Do you see what you should have done in (1) as well? Now let's go back to (2).

2: 201 = x + 109

T: What are you going to do now? You have seen that if there is a plus then you should subtract. What are you going to plus?
E: 109 plus the answer.
T: What is the answer? Is it 201?
E: (writes) 109 – 201 = -92 (then seeing the minus sign) Do I <u>have</u> to

subtract ........ sir? Because 109 – 201 does not work. The greater number must be on top.

T: (silent) ........... Use your calculator.
E: (gets –92 again!)
T: Let us see. If I gave you an equation like 7 = x = 2. What would x be?
E: (then) 7 – 2 = 5 (long pause)
T: Okay, let's go on.

4: x – 17 + 10 – 10 = 12

T: Please read to me what you are seeing.
E: (reads correctly)
T: Now you show me what you are going to do. ........ How does (4) differ from (3)?
E: It is a longer sum.
T: Remember to use your calculator and show me please. Where will you begin?
E: (somehow gets 37)
T: How did you get 37? Write your sum down.
E: (writes but did not see the negative sign in front of the 17 or the last 10)
T: You read correctly but now you say that x + 37 = 12? Is it right?
E: Oh .... (then she writes) 47 + 37 = 12
T: Okay. What is 10 – 10? Do you see the plus 10 and the minus 10?
E: It is 0.
T: Now rewrite the sum and show me where you would place the 0.
E: Must I now write down the zero?
T: Yes, you said there was a zero somewhere ....... Now bring in that zero.
E: (writes) x – 17 + 10 – 10 = 0 = 12.
T: Where did that zero come from?
E: I had 10 – 10 = 0.
T: .............. Can I help you?
E: Yes, please!
T: You said that these two tens i.e. 10 – 10 = 0. So you really had to write it as x – 17 + 0 = 12. Do you agree? Now work it out for us.
E: Okay .......... 17 – 12 = 5.
T: What did you do in (3)? There we had x – 14 = 47.
E: (says something inaudible)
T: What did you do in (3)? (wanting to pursue the issue to the full)
E: I have added 47 to 14. So ........... now I shall have 17 + 12 = 29.
T: Okay, now test your answer for us.
E: (tests her answer and gets –17)
T: Okay. Let us go on.

5: 200 = 247 – x

T: Look carefully at (5) and then you do it for us.
E: Okay. 247 – 200 = 47
T: Why did you subtract?

E: I did the same before!
T: Do you see on which side x is now?
E: Can I write it sir?
T: Yes.
E: (then she writes 247 – 200 = 47 again)
T: So what is x?
E: x = 47.

6: 2x = 110

T: What does 2x stand for?
E: (mutters something about thinking)
T: If the 2 and the x stand next to each other, what does it mean?
E: (long pause)
T: Before you said that it was times. Now get the value of x.
E: (writes) 110 x 2 = 220
T: Do you think it is right? Test.
E: No.
T: Tell me if the x must be greater or smaller than 110?
E: Smaller.
T: Which smaller number are you going to use?
E: (seems to be thinking really hard <u>and</u> starts to use her fingers)
T: Why do you use your fingers when there is a calculator.
E: (seems embarrassed) I shall now say 2 x 15 = 30.
T: Why did you use 15?
E: I just guessed! ......... No it does not work.
T: What are you going to use after the 15? A smaller or a larger number than 110?
E: Larger (then she says 2 x 120 = 240)
T: Why did you use 120? Remember you previously said that the number should be smaller than 110.
E: Okay, now we use 112.
T: But ........... right let me help you a little. If we say 2 x 2 = 36, what would you say is x?
E: (using fingers again) Two times ........ 3 is ......... 2 times 30 is ..........
T: (now really wanting to help) If I say 2 times x is 8, what is x?
E: Four, sir!!
T: How did you get 4?
E: I know 2 x 4 = 8.
T: Can't you say that 8 ÷ 2 = 4?
E: Because 2 goes into eight 4 times.
T: Now we go back to 2x = 110. Look what you have just done and tell me what you are going to do now.
E: I shall look how many times does 2 go into 110. (writes 2 ÷ 110)
T: Don't you want to write it as 110 ÷ 2?
E: Yes ... (uncertain) It is 55.
T: Now test your answer for us.
E: Then 2 x 55 = 110.
T: Is there a pattern of doing these sums? How would you have worked out 2x = 36?

E: (now more sure of herself) I will say 36 ÷ 2 = 18
T: Is 2 x 18 = 36?
E: Yes.

7: 3x – 12 = 6

T: Read the sum for us.
E: (reads correctly)
T: We still want to find x. Show us.
E: (long silence)
T: Where are you going to begin?
E: I'll say 3 times 6 = 18.
T: Why do you begin with 6 and not with 2?
E: Because 3 x 6 = 18.
T: So what is x?
E: 18.
T: Are you sure?
E: Yes! (uses the calculator and gets 3 x 18 – 12 = 42 ..... then she stares at her answer and starts using her fingers again)
T: So ...... what is the value of x?
E: x = 6
T: Why did you first use x is 18?
E: I saw that 18 – 12 = 6.

8: 6x + 4x = 40

T: Do you see that there are two x's? Now in this case we must get the same value for both of them. Let us see.
E: (begins with 6 x 10 + 4 x 10 = 100
T: Why do you begin with 10?
E: Because 4 times 10 = 40.
T: But what about the x by the 6?
E: (pauses)
T: Okay, what will you do next? A smaller or a larger number?
E: Smaller (tries 8 and gets 80) Does not work, sir!
T: Now larger or smaller?
E: Smaller. Now I see it will be 4!
T: Why 4? Why do you go from 8 to 4?
E: Because 4 is smaller than 8!
E: But 6 is also smaller than 8!
E: But 8 gave me 80. So 4 must give half of 80.
T: Oh ............ I see.

9: 3x – 2x + 8x = 45

E: (writes out 3 times x – 2 times x + 8 times x = 45)
T: How many x's do we have on the left hand side?
E: Three.
T: Okay. Now do it for us.
E: (reads the equation again)

T: What do you think your answer should be? ....... Where will you begin?
E: (long silence ............) I shall try 10.
T: Why 10?
E: (long pause ............. scratches her head .............. looks frustrated ......... starts to press on the calculator)
T: What did you try after 10?
E: 6.
T: Why 6?
E: It is smaller than 10.
T: Does it work?
E: No.
T: What are you going to do next?
E: 4.
T: Why?
E: It is even smaller.
T: Let us go to the next number.

10: $14x + 10 - 3x = 55 + 11$

T: How does (10) differ from (9)?
E: (long silence)
T: Won't you do the same as in (9)?
E: (silence)

11: $4x = x + 21$

(time is running out
T: Now we have x's on both sides of the equal sign. Do you think that you will be able to do it?
E: No, sir!!