IMPROVING MATHEMATICS TEACHING AND LEARNING THROUGH GENERATING AND SOLVING ALGEBRA PROBLEMS

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DECLARATION

I the undersigned hereby declare that the work contained in this thesis is my own original work and has not previously in its entirety or in part been submitted at any university for a degree.

Signature: Date:

ABSTRACT

In many countries, due to a growing criticism of the inadequacy of mathematics curricula, reforms have been undertaken across the world for meeting new social and technological needs and many researchers have begun to pay attention to the way mathematics is learned and taught. In the same vein, this study aims to investigate innovative and appropriate teaching strategies to introduce in the Rwandan educational system in order to foster students' mathematical thinking and problem solving skills. For this, a classroom-based research experiment was undertaken, focusing on meticulous observation, description and critical analysis of mathematics teaching and learning situations.

In the preparation of the research experiment, three mathematics teachers were helped to acquire proficiency in doing mathematics and to refine their teaching strategies, as well as to enable them to create a mathematics classroom culture that fosters students' understanding of mathematics through the problem solving process.

Three classes of 121 students of the second year, their ages ranging from 14 years to 16 years, chosen from three different secondary schools in Rwanda, participated in this research experiment. Students were taught an experimental programme based on solving contextualised algebra problems in line with the constructivist approach towards mathematics teaching and learning. Twenty-four mathematics lessons were observed in the three classes and students' learning activities were systematically recorded, focusing on teacher-students and student-student interaction.

The participating teachers experienced many difficulties in implementing new teaching strategies based on a problem solving approach but were impressed and encouraged by their students' abilities to generate different and unexpected ways of solving problem situations. However, the construction of mathematical models of non-routine problems constituted the most difficult task for many students because it required a high level of abstraction, characterising algebraic reasoning. Despite evident cognitive obstacles, a substantial improvement in students' systematic reasoning with respect to the different steps in the problem solving process, namely formulating a mathematical model, solving a model, verifying the solution and interpreting the answer, was progressively observed during the experiment. Many students had to overcome a language problem, which inhibited their understanding

and interpretation of mathematical problem situations and deeply affected their active participation in classroom discussions.

In this study, small group work and group discussions gave rise to excellent and successful teaching and learning situations which were appreciated and continuously improved up by the teachers. They provided students with opportunities for learning to argue about their mathematical thinking and to communicate mathematically. This kind of classroom organisation created an ideal learning environment for students but an uncomfortable teaching situation for teachers. It required much effort from the teachers to transform the mathematics classroom into a forum of discussion in setting up stimulating and challenging tasks for students, in working efficiently with different groups and in moderating the whole class discussion.

It was unrealistic to expect spectacular changes in teaching practices established over years to take place during a period of a month. This type of change requires sufficient time and support. However, teachers did develop a new and practical vision of mathematics teaching strategies focusing on students' full engagement in exploring and grappling with problematic situations in order to solve problems. Teachers made remarkable efforts in internalising and adopting their new role of mediators of students' mathematics learning and in being more flexible in their teaching styles. They learned to communicate with their students, to accept students' explanations and suggestions, to encourage their logical disagreement and to consider their errors and misconceptions constructively.

Students' results in the pre-test and the post-test showed their low performance in building mathematical models especially when they had to use symbols but revealed a significant progress in the students' ways of thinking which was observed through the variety and originality of their strategies, their systematic work and their perseverance in solving algebra problems. Students also developed positive attitudes to do mathematics; this was exhibited by their pride and satisfaction to accomplish non-routine tasks by themselves.

Teachers' comments indicated that they work under pressure to cover an overloaded mathematics curriculum and have poor support from educational authorities. For them, mathematics is socially considered as a difficult subject. For many students, mathematics is a gatekeeper to access higher levels of education; to fail in mathematics unfortunately implies to fail at school and in life. Students' negative attitudes towards mathematics were mainly due to their repeated failures in mathematics, but also to some mathematics teachers who intimidate and discourage their students.

Both educational authorities and teachers should make efforts to rethink an appropriate mathematics curriculum and alternative teaching strategies in order to efficiently prepare students to meet new societal and technological requirements.

OPSOMMING

As gevolg van toenemende kritiek oor die kwaliteit van wiskundekurrikula, is bewegings vir hervorming wêreldwyd geïnisieer om nuwe sosiale en tegnologiese behoeftes aan te spreek en baie navorsing is gedoen oor die wyse waarop wiskunde geleer en onderrig word. In lyn hiermee, is die doel van hierdie studie om innoverende en geskikte onderrigstrategieë te ondersoek om in die Rwandese onderwysstelsel in te voer om leerders se wiskundige denke en probleemoplossingsvaardighede te ontwikkel. Om dit te bereik, is 'n klaskamergebaseerde navorsingseksperiment uitgevoer, met die klem op fyn waarneming, beskrywing en kritiese ontleding van wiskunde leer- en onderrigsituasies.

As voorbereiding tot die navorsingseksperiment is drie wiskunde-onderwysers gehelp om vaardighede te verwerf in die doen van wiskunde en om hul onderrigstrategieë te verfyn, asook om hulle in staat te stel om 'n wiskunde-klaskamerkultuur te vestig wat leerders se begryping van wiskunde deur die probleemoplossingsproses ontwikkel.

Drie klasse van 121 leerders in die tweede jaar, tussen 14 en 16 jaar oud, is uit drie verskillende hoërskole in Rwanda gekies om aan die navorsing deel te neem. Die leerders is deur middel van 'n eksperimentele program onderrig wat gebaseer is op die oplossing van gekontekstualiseerde algebraprobleme in ooreenstemming met 'n konstruktivistiese benadering tot wiskunde-leer en -onderrig. Vier-en-twintig wiskundelesse is in die drie klaskamers waargeneem en leerders se leeraktiwiteite is stelselmatig opgeskryf, met die klem op onderwyser-leerder en leerder-leerder interaksie.

Die betrokke onderwysers het baie probleme ondervind om nuwe onderrigstrategieë gebaseer op 'n probleemoplossingsbenadering te implementeer, maar was baie beïndruk en begeesterd deur hul leerders se vermoë om verskillende en onverwagte planne te beraam om probleme op te los. Die opstelling van wiskundige modelle vir nie-roetine probleme was vir baie leerders die moeilikste taak omdat dit 'n hoë vlak van abstraksie wat kenmerkend is van algebraïese denke verteenwoordig. Ten spyte van kognitiewe struikelblokke was daar nogtans 'n merkbare verbetering in leerders

se logiese redeneringsprosesse soos geopenbaar in die toepassing van die verskillende stappe van die probleemoplossingsproses, naamlik die formulering van 'n wiskundige model, die oplossing van die model, verifiëring van die oplossing en interpretasie van die antwoord. Baie studente is gekniehalter deur 'n taalprobleem wat hul begrip en interpretasie van wiskundige probleemsituasies en hul vrymoedigheid om aan klaskamergesprekke deel te neem, aan bande gelê het.

In hierdie studie het kleingroepwerk en groepbesprekings suksesvolle onderrig- en leersituasies geskep wat deur die onderwysers raakgesien en verder uitgebou is. Dit het geleenthede geskep vir die leerders om oor hul wiskundige denke te argumenteer en om wiskundig te kommunikeer. Hierdie soort klaskamerorganisasie het 'n ideale leeromgewing vir leerders geskep maar 'n ongemaklike onderrigomgewing vir onderwysers. Dit het baie van onderwysers geverg om die wiskundeklaskamer in 'n gespreksforum te omskep deur stimulerende en uitdagende probleme aan leerders te stel, deur met verskillende groepe te werk en deur die algemene klaskamerbesprekings te fasiliteer.

Dit was onrealisties om binne die bestek van 'n maand grootskaalse veranderinge in onderwyspraktyke wat oor 'n tydperk van jare posgevat het, te verwag. Hierdie soort verandering benodig genoeg tyd en ondersteuning. Onderwysers het nogtans 'n nuwe en praktiese visie ontwikkel van wiskunde-onderrigstrategieë wat fokus op leerders se betrokkenheid by die ondersoek en oplossing van probleme wat vir hulle uitdagend en nie-roetine was. Onderwysers het daadwerklike pogings aangewend om hul nuwe rolle as mediators te internaliseer en te aanvaar, en om meer soepel onderrigstyle te ontwikkel. Hulle het geleer om met hul leerders te kommunikeer, om leerders se verduidelikings en voorstelle te aanvaar, om logiese argumentering aan te moedig en om foute en wankonsepte konstruktief te benader.

Leerders se resultate in die voor- en na-toetse dui op swak vermoë om wiskundige modelle te bou veral wanneer hulle simbole moes gebruik, maar wys beduidende vordering in leerders se denke, wat gemanifesteer het in die verskeidenheid en oorspronklikheid van hul strategieë, hul sistematiese werk en hul voortgesette pogings om algebraprobleme op te los. Leerders het ook positiewe instellings teenoor die doen van wiskunde ontwikkel; dit is getoon deur hul trots en tevredenheid wanneer hulle self nie-roetine take opgelos het.

Onderwysers se kommentaar openbaar dat hulle onder druk werk om 'n oorlaaide wiskundekurrikulum af te handel en dat hulle min ondersteuning van onderwyshoofde kry. Hulle sê ook dat wiskunde deur die breë gemeenskap as 'n moeilike vak beskou word. Vir baie leerders is wiskunde 'n hekwagter wat toegang tot verdere onderwys en opleiding beheer; om in wiskunde te faal beteken om op skool te faal en om in die lewe te faal. Leerders se negatiewe instellings teenoor wiskunde was hoofsaaklik as gevolg van hul herhaalde mislukkings in skoolwiskunde maar ook as gevolg van sommige wiskunde-onderwysers wat hul leerders intimideer en ontmoedig.

Beide onderwyshoofde en onderwysers behoort pogings aan te wend om te besin oor 'n geskikte wiskundekurrikulum en alternatiewe onderrigstrategieë om leerders meer doeltreffend voor te berei om aan nuwe sosiale en tegnologiese eise te voldoen.

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CHAPTER ONE: STATEMENT OF THE PROBLEM

1.1 INTRODUCTION

At the beginning of the twentieth century, the famous mathematician, Poincaré, asked this pertinent question: "How does it happen that so many refuse to understand mathematics?", quoted by Libeskind (1980: 59). A century later, the situation has not improved, it is even worse. For many students and teachers, mathematics is identified with calculating complicated arithmetical or algebraic expressions, solving sophisticated equations, memorising and applying formulas without understanding rather than solving interesting and challenging problems. As a result, when students are exposed to some mathematical activity, they have little or no belief in their capacity to do mathematics and they frequently adopt negative attitudes which lead them to dislike mathematics or to reject it completely.

Many centuries ago, Brahmagupta, said: "If you want to shine in the company of learned persons, pose mathematical problems and solve them" (in Burkhardt, Groves, Schoenfeld & Stacey; 1984: 290). In the light of this advice, early opportunities should be given to students to solve a wide variety of problems, which are meaningful to them and which stimulate their thinking and reasoning.

Consequently, the most important issue in mathematics teaching and learning becomes how teachers could best change their ways of teaching with the ultimate aim of promoting a positive image of mathematics in their students and enhancing their willingness to be actively engaged in solving problems.

1.2 MOTIVATION FOR THE STUDY

For many reasons, the current Rwandan educational system is often criticised by parents, employers and higher educationists for a low level of students' performance, knowledge and skills at the end of their secondary education. Their major complaint is that students' knowledge and skills, especially in Mathematics, do not meet the multiple demands of our moving and changing society.

It should be recognised that there was a significant shortage of mathematics teachers in Rwanda since 1994, but the use of traditional teaching methods in mathematics classrooms should be one of the most important reasons for the present critical situation. Thus, it seems urgent to introduce innovative and appropriate teaching strategies that enable teachers to change their classroom practices and to prepare their students to benefit from modern technological facilities for attaining new societal purposes. In accordance with Brahmagupta's advice: "If you want the present generation of children to face the future with confidence, give them a thorough training in mathematical problem solving" (in Burkhardt et al., 1984: 290), the attempt to develop the students' problem solving abilities through solving algebra problems constitutes an ambitious challenge for this study.

1.3 AIM OF THE STUDY

The main aim of this study is to investigate how to foster students' mathematical ways of thinking when they are exposed to various and interesting problems situated in their cultural context and how to develop their problem solving skills. For this purpose, it is intended to observe, to describe and to critically analyse teaching and learning situations in order to stimulate teachers' self-confidence in their own understanding of mathematical concepts and their abilities to refine their teaching strategies.

It is therefore also necessary to investigate appropriate ways to create a safe learning environment that supports mathematical thinking, in which teachers and students negotiate and collaborate to improve students' learning while solving algebra problems.

1.4 KEY RESEARCH QUESTIONS

In many countries, it is agreed that problem solving activities should constitute the focus of mathematics teaching at all levels. Consequently, the following questions should arise in our current educational system:

- Is it possible to design an innovative mathematics teaching and an active learning environment, which will promote students' learning processes through a problem solving-based approach in Algebra?

- What are mathematics teachers' beliefs about the mathematical content, their teaching strategies, the students' learning processes?
- Do they really understand what they teach?
- Are they able to identify suitable tasks, which challenge students to think mathematically and empower them to transfer their existing knowledge and skills to problem solving situations?
- How do they develop a creative and cooperative atmosphere in the classroom?
- What obstacles do they find in trying to put a problem-solving approach into practice when teaching algebra?
- What steps are more difficult in their teaching process and how do they deal with students' learning difficulties?
- Do they understand the needs for a change in their traditional way of teaching? Are they ready to participate actively in this experiment?

▪ How can we identify specific ways in which their teaching strategies can improve and what is the impact of this change on students' mathematical knowledge and skills?

This study attempts to find answers to these questions.

1.5 RESEARCH METHODOLOGY

1.5.1 Planning research activities

Before undertaking the research, preliminary activities were planned: the selection of participants, the preparatory meeting with the teachers' team, the organisation of a workshop and the administration of a questionnaire on teachers' attitudes and beliefs about mathematics teaching and learning.

Three mathematics teachers of the second year classes in the Secondary School were chosen from three different schools, according to three categories of in-service mathematics teachers: qualified mathematics teachers (teachers who have at least an Honour's Degree), unqualified teachers with a good background in Mathematics and others (graduates in other subjects).

A meeting with the teachers' team was set up to negotiate their willingness to participate and to collaborate in the research by creating good relationships and mutual trust. It is also a good opportunity to listen to their strategies in teaching algebra problems, the problems they encounter, and their needs for improving their own understanding of mathematical concepts and pedagogical knowledge.

The organisation of a workshop seemed indispensable to give to the teachers' team a wide briefing on a new platform of good and solid teaching and learning focused on students' involvement in constructing and modifying their own knowledge through well-selected and planned tasks which are mathematically problematic. Teachers were progressively initiated to adopt new practices based on good planning of learning activities and the establishment of the appropriate mathematical classroom culture.

Finally, a questionnaire on teachers' attitudes and beliefs about the mathematical content itself, their own teaching strategies and their students' learning process was administered to 23 mathematics teachers from different schools.

1.5.2 Research experiment

Three classes totalling 121 students were selected to participate in the experiment. They were taught an experimental programme (see Chapter Four) on solving problems by using algebra. Eight lessons were given in each class.

During the experiment, we observed teachers at work, described and recorded students' activities with a special interest in the teacher-students and student-student interaction.

A short discussion with teachers after each lesson was necessary to improve on inadequate mathematical content knowledge or pedagogical practices and to encourage them to persevere in their efforts to innovate their teaching strategies.

A weekly meeting with the teachers' team allowed us to take note of realised progress, to discuss ways to cope with pedagogical barriers and to plan for the next lessons.

1.5.3 Measuring instruments

To assess the impact of the experiment on teaching practices and on the learning process, a pre-test, continuous assessment and a post-test were planned.

The pre-test was a valuable tool for assessing students' strategies when they are working out problems before the intervention and getting useful information on their abilities and difficulties to use in planning the teaching.

In addition, ongoing assessment during the intervention phase constituted the basis for an objective analysis of the students' progress. Teachers can improve their teaching strategies by identifying students' successful problem solving methods and encouraging their eventual positive attitudes such as efficient and effective organisation of data, systematic working and perseverance with mathematical activities. They can also detect their students' weaknesses in mathematical knowledge, and their common errors in solving problems, thus providing a good source of information for adapting their teaching to the students' levels.

The aim of the post-test was to evaluate the effects of teachers' classroom intervention on students' mathematical knowledge and skills. This should be achieved by comparing the students' results in the pre-test and the post-test and in carefully analysing any changes in their problem solving skills.

Significant changes in teaching practices were expected from the teachers' team. This outcome should be identified by the improvement of students' problem solving strategies, showing that they slowly but surely try to become better thinkers and competent problem-solvers. The ultimate purpose remains to enable students, when they are faced with a non-routine problem, to reflect on it, to develop their own ideas and strategies, to invent their own imaginative solutions, to persevere until they get a meaningful answer, to explain and to justify the result of their thinking.

1.6 OVERVIEW OF THE CHAPTER

This chapter explained briefly the main concern of this study, which is to investigate how to promote mathematics teaching and learning in the Rwandan educational system and to create a rich and pleasant learning environment through a problem solving-based approach. It described how different activities were planned, namely the selection and preparation of the participating teachers, the organisation of a pre-test and a post-test and the classroom intervention as an important teaching activity through which a progressive change in teachers' practices should make an substantial impact on the improvement of students' thinking and problem solving skills.

CHAPTER TWO: PROBLEM SOLVING AND MATHEMATICAL MODELLING PROCESSES

2.1 INTRODUCTION

Much has been written about problem solving and many publications on this subject referred to Polya as the pioneer in this field. Without contradicting different ideas developed under the name of problem solving, in this chapter problem solving will be viewed as a process through which students develop their ability to solve various and complex mathematics problems. Three major parts will be considered: the process involved in solving a problem situation, the mathematical modelling process and the search for a means to solve algebra problems.

2.2 THE PROBLEM SOLVING PROCESS

2.2.1 Problem solving as the essence of the mathematical activity

In the mathematics education literature, there is a great diversity of approaches to the problem solving concept. For instance, Krulik and Rudnick (1984: 124) defined problem solving as "a process by which the individual approaches, considers, analyzes and eventually achieves an answer to their confrontation". For Reitman (1965), a problem solver is a person who perceives and accepts a goal without immediate means of reaching it. In conformity with this idea, Burkhardt and Schoenfeld (1984: 3) stipulated:

> *To us, the essence of problem solving is that the solver faces an unfamiliar task, one for which they could not reasonably be expected to know immediately a path to a solution. Indeed, the finding of a solution path is the essence of problem solving.*

Most of the time, it has been observed that, when challenged with a well-selected task, students strive to get a reasonable solution. Their mind is ultimately focused on a mathematical activity, which requires identifying the nature of the task, making its mental representation, formulating some conjectures, looking for efficient strategies, monitoring and evaluating the result of their reasoning. During this activity, they systematically organise their thinking for a better understanding and a refined reasoning towards a suitable solution. All these aspects of mathematical reasoning and thinking are typical and fundamental in a problem solving activity. For this reason, in mathematics education, problem solving is considered as the heart or the cornerstone of mathematics teaching and learning. In many countries, it is even considered as a significant and integrant part of the mathematics curriculum. For instance, in its "Agenda for action", in the USA the National Council of Teachers of Mathematics (NCTM) stated this recommendation: "Problem solving must be the focus of school mathematics in the 1980s" (NCTM, 1980).

2.2.2 The problem solving process

In Polya's writings, his main concern was to teach students to think, to develop their thinking abilities when they are solving problems. He expected students to engage in thinking about the various heuristics available to them for solving problems in following these steps: understanding the problem, making a plan, carrying out the plan and looking back. By heuristics, one can understand some kinds of procedures or some techniques available to students, variable according to the particular structure of the problem situation and helpful while they are generating a solution of a given problem; they do not guarantee a solution but provide a more probable method for discovering the solution to a problem. They are synonymous with strategies such as making a table, drawing a picture and so on.

These steps suggest that the problem solving activity should be viewed as a dynamic and cyclical process. In fact, in a mathematics classroom, when a problem is presented, students should be invited to read it carefully with understanding, to identify the nature of the problem, all the relevant information and the question underlying the problem. They attempt to make a plan in order to understand the problem better by collecting, organising and representing data but also by choosing an efficient strategy. Thus, they can try to carry out their plan by using the selected strategy and solving the problem by means of their computational skills or logical reasoning. If they are unable to do so, a new plan could be made or they can go back to develop a new understanding. Finally, they should check the effectiveness of their solution by verifying if all the relevant information in the problem has been used or if their solution really makes sense, otherwise they should review the whole process. This last step provides the opportunities for students to learn from the problem and to extend their successful strategies to other problems.

In the search of a solution, in accordance with a constructivist view, students organise their thinking on the basis of their previous mathematical knowledge, construct their own new mathematical knowledge and apply it to come closer to the solution. Dewey (1933) and Schoenfeld (1985) emphasised the metacognitive aspect of problem solving by stating that while students were making progress in their attempt to solve a problem, they should constantly think about their own cognition, they should control and regularly adjust their cognitive processes. The teacher's responsibility should be to create learning situations and contexts which promote "reflective inquiry" and allow students to construct appropriate knowledge during the problem solving process (Hiebert, Carpenter, Fennena, Fuson, Human, Murray, Olivier & Wearne; 1996).

2.2.3 Learning through problem solving

In mathematics teaching and learning, problem solving can serve as a vehicle for learning new mathematical ideas and skills (Schroeder & Lester, 1989; Murray, Olivier & Human, 1998). In fact, faced with a rich and challenging problem, students struggle, grapple with it and try to arrive at a solution.

While they are deeply involved in searching a solution Murray, Olivier and Human (1998: 169) have noticed that:

> *Learning occurs when students grapple with problems for which they have no routine methods. Problems therefore come before the teaching of the solution method. The teacher should not interfere with the students while they are trying to solve the problem, but students are encouraged to compare their methods with each other, discuss the problem, etc.*

Describing the process of learning through solving problems, Davis (in Murray et al., 1998: 171-172) stressed this idea as follows:

> *Instead of starting with 'mathematical' ideas, and then 'applying' them, we would start with problems or tasks, and as a result of working on these problems the children would be left with a residue of 'mathematics' – we would argue that mathematics is what you have left over after you have worked on problems. We reject the notion of 'applying' mathematics, because of the suggestion that you start with mathematics and then look around for ways to use it.*

This argument highlighted the importance of the "residue" in mathematics learning (Hiebert et al., 1996).

It should also be recognised that through problem solving, students have the opportunity to reinforce their understanding and to make sense of mathematics. They become confident in their abilities to tackle difficult problems, to face unfamiliar situations, they acquire good habits of perseverance in doing mathematics and experience a feeling of a task well-accomplished.

2.2.4 The mathematical classroom culture

A mathematical classroom culture is normally constructed through the interaction of the teacher and his students during a mathematical activity where the teacher "negotiates" with them social norms for a safe learning environment. Together, they create a mathematics community in which students generate mathematics ideas, learn to communicate their findings mathematically, justify their points of view, listen to and try to understand others' mathematical arguments and establish connections among different mathematics concepts.

The teacher is responsible to create such supportive and challenging learning environments by setting worthwhile and intriguing tasks connected to the students' everyday life or arising from purely mathematical contexts. Murray et al. (1998: 175) suggest that: "The classroom culture and the quality of students' interactions when solving problems have a greater influence on students' mathematical constructions than the facilitatory skills of the teacher during discussions".

For McNeal and Simon (2000: 500):

> *Negotiation of norms and practices is ongoing and often implicit in everyday classroom mathematical activity. The community develops expectations and modes of operating, often without the conscious awareness of the participants. Meanings are established as a result of community members' interpretations of classroom interaction. Brousseau (1983) calls this largely implicit process the establishment of the " contrat didactique".*

They suggest that the mathematical classroom culture should be a mutual adaptation between the members of the mathematics community. The teacher can play an important role in developing his students' problem solving dispositions by establishing and stimulating such an environment conducive to better mathematics understanding and learning.

2.3 THE MATHEMATICAL MODELLING PROCESS

2.3.1 Mathematical modelling as an activity

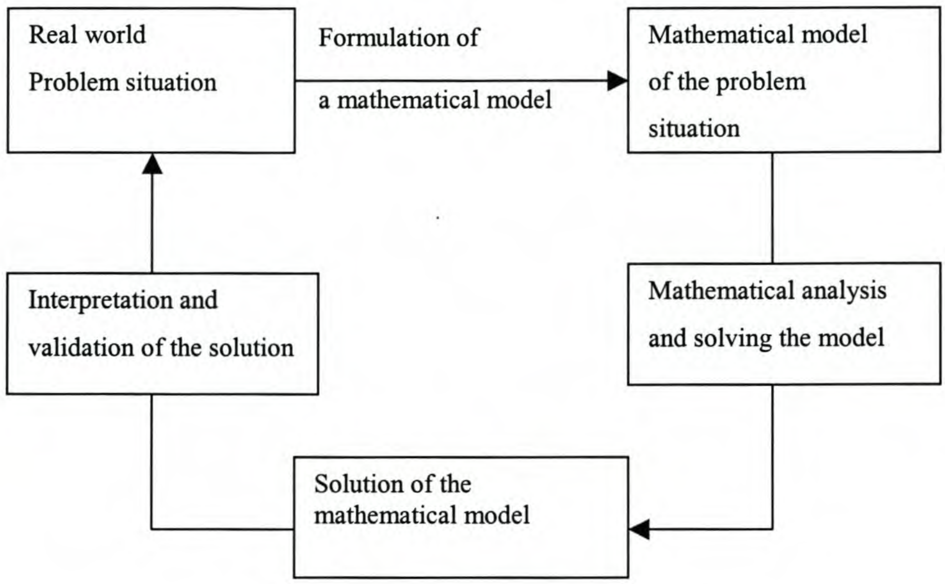
Mathematics is one of the approaches used in solving real world problems. In practice, a problem situation is interpreted, represented and formulated in terms of mathematical symbols. This mathematical formulation of the real world problem, commonly called mathematical modelling, greatly helps to visualise and to understand the given problem situation. Treilibs (in Williams, 1980: 32) briefly described mathematical modelling as follows:

> *By mathematical modelling, we shall mean the mathematizing of reality in order to solve real world problems by applying appropriate mathematical techniques and then the translating of the mathematical results into real world solutions.*

In a mathematics classroom, the use of mathematical modelling allows students to organise messy information in order to clarify the problem situation and to solve the model with facility. It should be noted that mathematical modelling constitutes an indispensable instrument in all areas where mathematics is applied.

2.3.2 The mathematical modelling process

Mathematical modelling is considered to be a process involving many steps. The most important and very difficult one is the translation of the problem from real-life into a mathematical model. It should be stressed that the effectiveness of the whole process depends on the appropriateness of the mathematical model. Therefore, mathematical techniques such as graphs, diagrams and formulas can be used to build the mathematical model. Writing an equation is one of the ways of modelling a problem. The solution of this mathematical model is generally obtained by applying appropriate mathematical rules and procedures. The next step consists of the interpretation of a solution in real world terms and the validation of the solution against the real situation.



The mathematical modelling process

2.3.3 Mathematical modelling as a teaching strategy

As has been discussed above, by the mathematical modelling process a problem situation is translated into a model using symbols or drawings; this model is solved by means of mathematical techniques; the solution is checked and translated back into real-life terms. This translation requires a good level of abstraction from students and constitutes the main difficulty they encounter in the mathematical modelling process. In order to enable students to model problem situations efficiently, Cobb and Bauersfeld (1995), and Mouton (1995) proposed to create an "inquiry mathematics classroom" in which the teacher directs the learning process so that the students can make sense of mathematics. Classrooms are organised as communities of inquiry in which significant and challenging problems are set and "students can construct authentic mathematical knowledge and can reach a stage of reflection that leads to decisive action" (Mouton, 1995: 37).

As mathematical modelling deals with the solving of real world problems, it is advisable to give students opportunities to represent real world situations mathematically and to construct meaning for mathematical concepts before they try to solve practical and contextualised problems.

Modelling as a teaching strategy can provide the context within which students are exposed to many mathematically problematic situations and they are encouraged to reflect, to interact with one another and to communicate their thinking during group discussions. Group work and collaboration constitute important aspects of modelling as a teaching strategy. In this way, students have the opportunity to share their ideas, to communicate mathematically and to develop skills needed for better understanding and independent learning.

In the modelling classroom, (Mouton, 1995: 46) suggested that:

> *Both the teacher and the pupils must change their expectations of the traditional contract in the mathematics classroom. The teacher must acknowledge that knowledge in the mathematics classroom is not transmitted; rather it is through activity and negotiation that the child gains knowledge and develops a feel for the well-organized system of relationships on which mathematics is based.*

The teacher's role becomes that of guide, mediator, "negotiator of meaning" while students engage actively in group work without constant teacher's help.

Cobb, Wood, Yackel, Nicholls, Wheatly, Trigatti and Perlwitz (1991: 7) stressed this idea in these terms:

> *The teacher's role in initiating and guiding mathematical negotiations is a highly complex activity that includes highlighting conflicts between alternative interpretations or solutions, helping students develop productive small-group collaborative relationships, facilitating mathematical dialogue between students.*

It is important to note that the teacher has the challenging task of setting learning experiences that engage students productively and support their reflection and the communication of their perceptions or interpretations.

2.3.4 The need for mathematical modelling in solving algebra problems

Mathematical modelling is an important tool used to interpret complex phenomena and to solve problems using algebra. It serves as a link between two different worlds: the real world illustrated by the problem situation and the abstract world of mathematics symbols. Depending on the type of model required, the passage from one world to another requires a high level of abstraction and makes mathematical modelling a difficult but challenging activity for students (Murthy, Page & Rodin: 1990).

In fact, when students are working on real world problems in the abstract, they need appropriate skills to build good models and to obtain a suitable and realistic answer.

This mental process enables them to understand, to cope with real world situations and to develop an openness towards new situations. Unfortunately, it seems to be out of reach for many students due to the way they have been taught. Through mathematical modelling, students can see how the mathematics they learn can be applied in real life and how it is useful and enjoyable.

Furthermore, skill in mathematical modelling is strongly linked to problem solving. It enhances and consolidates students' problem solving skills. This assertion has been clearly explained by Kapur (in Burkhardt et al., 1984: 293) as follows:

> *Problem solving and mathematical modelling are two sides of the same coin, usually problem solving refers to the solution of problems within mathematics and mathematical modelling is the attempt to use mathematics to solve those problems of daily life, society, science and technology in whose solution mathematics can play a significant role. Both require creativity, originality, courage to think boldly, recognition of patterns, willingness to try out new solutions, a capacity to learn from experience, intellectual curiosity and an eagerness to face intellectual challenges.*

In order to develop these positive attitudes and aptitudes towards mathematics the teachers have the difficult task to train their students by regularly exposing them to a variety of interesting applications of mathematics in other disciplines such as Physics, Chemistry, Biology, Geography, Economics, Banking, etc.

2.4 SOLVING ALGEBRA PROBLEMS

2.4.1 Non-routine problems

Many mathematics textbooks contain many problems which require only a simple application of formulas already dealt with in the mathematics classroom. According to Burkhardt (in Burkhardt et al., 1984: 124), in this situation, students are asked to perform rote learning in doing *exercises* instead of *solving problems*, because: "A problem is a situation, quantitative or otherwise, that confronts an individual and for which no solution or path to a solution appears evident."

It is important to note that in choosing each mathematics problem, a teacher indirectly takes the responsibility for developing problem solving skills in his students' minds. Thus, it is suitable that selected problems should be related to real-life, interesting and mathematically enriching for students, relevant to their experience, arousing their curiosity and inviting their involvement. Such problems should not be too easy nor too difficult for students. Ideally, such problems should be accessible to all students in order to stimulate each student's self-confidence in his capacity to solve problems and to make progress at his own pace. Logically, if the process of problem solving requires a high level of mathematical thinking and the use of various and not obvious strategies in tackling and solving the problem, the technical aspect must be reasonable and in line with the students' levels of expertise. It should also be emphasised that well-selected problems might leave behind a "residue" of mathematics useful to solve new problems in a new context (Hiebert et al., 1996).

2.4.2 Problem solving strategies

In general, problem solving as a process cannot be formally taught in a mathematics classroom, but it is possible, even commendable, to introduce this activity while students are solving non-routine problems. However, students should be familiarised with quite simple but helpful specific heuristics or strategies for good problem solvers. These strategies are essentially based on Polya's (1957) model, which consists of understanding the problem, making and carrying out a plan and looking back.

We provide below a summary of Polya's (1957) four phases of problem solving and heuristics:

1. **UNDERSTANDING THE PROBLEM**

   - **First**. You have to *understand* the problem.

   - What is the unknown? What are the data? What is the condition?

   - Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?

   - Draw a figure. Introduce suitable notation.

   - Separate the various parts of the condition. Can you write them down?

2. **DEVISING A PLAN**

   - **Second**. Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

   - *Have you seen it before?* Have you seen the same problem in the slightly different form?

- *Do you know a related problem? Do you know a theorem that could be useful?*

- *Look at the unknown!* And try to think of a familiar problem having the same or a similar unknown.

- *Here is a problem related to yours and solved before. Could you use it?*

- Could you use its results? Could you use its method?

-Could you restate the problem? Could you restate it still differently?

Go back to the definitions.

- If you cannot solve the proposed problem try to first solve some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how it varies? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or data, or both if necessary, so that the new unknown and the new data are nearer to each other?

- Did you use all data? Did you use the whole condition?

- Have you taken into account all the essential notions involved in the problem?

3. CARRYING OUT THE PLAN

- **Third.** *Carry out* your plan.

Carrying out your plan of solution, *check each step*. Can you see clearly that the step is correct? Can you *prove* that it is correct?

4. LOOKING BACK

- **Fourth.** *Examine* the solution obtained.

Can you *check the result?* Can you *check the argument?*

Can you derive the solution differently? Can you see it at a glance?

Can you use the result, or the method, for some other problem?

Paradoxically, many problems which are difficult when they are solved using formal methods become simple when problem solving strategies are adopted. Some common problem solving heuristics or strategies are: guessing and checking or guessing and improving, drawing a picture, making an organised list or a table, looking for a pattern, using logical reasoning, working backwards, solving a simpler problem and so on.

In many situations, especially when problems get more difficult, these heuristics are not usually used on their own but in combination with other strategies.

Once again, it should be stressed that problem solving strategies constitute a precious tool to develop mathematical thinking and reasoning in students' minds in such a way that students can realise that any problem may be solved in many ways. The teacher has the responsibility to pose problems that allow students to use multiple paths to answer a problem.

2.4.3 Cognitive obstacles

In Mathematics, particularly in Algebra, it has been observed that students have many difficulties to understand the basic notions of algebra such as negative numbers, variable, function, generality, the equal sign and so on. They make persistent errors in the construction of meanings of algebraic expressions and in the meaning of the letters contained in these expressions. In the historical development of Mathematics, it is well known that it has taken many centuries for the best mathematicians "to move from the idea of a letter symbol as an unknown in an equation to a letter symbol as an arbitrary number in an identity and a parameter in a formula" (Malati Algebra Rationale, 2000: 3).

Herscovics (1989) distinguished three kinds of obstacles faced by students in learning algebra: epistemological, psycho-genetic and didactical obstacles.

The psycho-genetic obstacles are inherent to the cognitive development of the students, their maturity and previous experience. They often occur if students do not possess the necessary pre-requisite cognitive structures in Mathematics and are associated with the students' process of accommodation of new concepts.

The didactical obstacles are the result or the consequence of the mathematics teaching that students have previously received or the mathematics curriculum itself. In the study of algebra, some notions are often introduced in such a way that students cannot relate their existing knowledge to the new material. As a result, there is a significant gap between the new content and their prior knowledge, which can persist if there is no remediation.

The epistemological obstacles are mainly related to the nature of mathematical knowledge. They are due to a kind of cognitive structure, which takes place in the students' mind and is strengthened during the conception of a notion. During the learning process, it recurs and even becomes a permanent error if it is not rooted out in time.

In an attempt to give a sound explanation of these students' cognitive obstacles, Sfard (1991, 1995) carried out research on the historical development of Algebra. She developed a theory in which she analysed the role of mathematical concepts in mathematical thinking. In her study, she realised that abstract notions such as number, variable, function and so on, can be conceived in two different but complementary ways: structurally (conceptually) as objects and operationally (procedurally) as processes. For Kieran (in Grouws, 1992: 392), "procedural refers to arithmetic operations carried out on numbers to yield numbers. Structural refers to a different set of operations that are carried out, not on numbers but on algebraic expressions". All mathematical concepts are endowed with a "process-object" duality and the students' difficulties with basic algebraic notions should be rooted in this dual meaning of concepts. Sfard (1991, 1994, 1995) elaborated a theory on how students should learn algebra in describing that the development of understanding in algebra in the individual follows the same stages that can be observed in the historical development of Algebra.

According to Eves (1990) and Burton (1991), the development of algebraic notation is characterised by three stages. In the first stage, rhetorical algebra (practised from the earliest times till the sixteen century), no symbols were used; all equations were posed and solved completely in prose form. The second stage, syncopated algebra (Diophantus, in 250 A.D.), was characterised by the use of some abbreviations for the frequently recurring quantities and operations. The third stage was symbolic algebra (Viète, 1540-1603), which is the algebra we use today. It was with the development of symbolic algebra that the symbols became objects in their own right, rather than simply shorthand for describing computational procedures (Malati Algebra Rationale, 2000)

To Sfard, the historical development of algebra from rhetorical to symbolic notation should be reproduced in the individual to achieve an understanding of algebra. In fact, the development of mathematical understanding of concepts, from computational operations to abstract objects, takes place through three stages, namely interiorisation, condensation and reification. During the first stage, called interiorisation, some process is performed on familiar mathematical objects. In the second stage, called condensation, the process is refined and made more manageable. It lasts as long as a new entity is conceived only procedurally or operationally. In the third stage, reification, a giant ontological leap is taken. "Reification is an act of turning computational operations into permanent object-like entities" (Sfard as quoted by Malati Algebra Rationale, 2000: 4). In this stage, the individual must move from the operational or computational orientation to a structural orientation.

In the same line Kieran (1992: 392-393) stated that:

> *Reification involves the sudden ability to see something familiar in a new light. Reification seems to be a leap: a process solidifies into an object, into a static structure. The new entity is detached from the process that produced it.*

It seems that cognitive obstacles should be connected to the difficulty in reification (Sfard, 1991, 1994; Kieran, 1989, 1992; Herscovics, 1989). For instance, it is frequently observed that students could solve complex problems by applying a series of procedures but could not translate the series into an algebraic expression. Furthermore, they often perform the processes to solve a word problem, but have difficulty when they are asked to set up an equation to describe the problem situation.

Sfard suggested that this theory of reification could be applied to develop a curricular approach to algebra. She proposed that for many students the operational conception might be the first step in the acquisition of new mathematical notions because the structural approach is more abstract than the operational and students could hardly arrive at a structural conception without previous operational understanding.

In the light of this theory, it seems necessary to rethink the way algebra is taught by taking into consideration the students' progress in understanding algebraic concepts, and eventually on the basis of research results, in modifying the way algebra is presented in many textbooks.

2.4.4 Factors affecting problem solving difficulties

Students fail to solve problems for many reasons: some believe that they are unable to solve a problem without the help of their teacher. The others, more motivated, start working on the problem without analysing it deeply to come to a better understanding. These students tend to make their decisions after reading only the first few words of a given problem; they use superficial aspects of this problem instead of integrating it into a coherent structure (Silver, 1979). Most of the time, they hardly translate a problem situation from natural language to symbols in trying to build the mathematical model and if their attempt is unsuccessful they are discouraged instead of trying another way to tackle the problem.

Nathan and Koedinger (2000) in their analysis of students' problem solving strategies pointed out two important factors affecting problem solving difficulties: the position of the unknown quantity in the problem and the linguistic presentation of the problem. Riley, Greeno and Heller (1983) also found that problem difficulty is strongly affected by the role of the unknown quantity within the problem statement. Nathan and Koedinger (in Nathan & Koedinger, 2000) formulated six problem types categorised according to the placement of the unknown quantity or their presentation formats: result-unknown and start-unknown problems or verbal stories problems with context, word equation problems with no context and symbolic equations.

In result-unknown problems, the unknown is the result of mathematical operations described in the problem. They are considered to be arithmetic because they can be solved through the direct application of arithmetic operations.

Problem 1: *When Ted got home from his waiter job, he took the 1500F he earned that day and subtracted the 450F he received in tips. Then he divided the remaining money by the 6 hours he worked and found his hourly wage. How much per hour did Ted make?*

Problem 2: *Starting with 1500, if I subtract 450 and then divide by 6, I get a number. What is it?*

Problem 3: *Solve the equation:* $(1500 - 450) / 6 = x$

In start-unknown problems, the unknown value refers to a quantity needed to specify a relationship. They are considered to be algebraic because they can be solved through the application of standard algebraic procedures.

Problem 4: *When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the 450F he made in tips and found he earned 1500F. How much per hour did Ted make?*

Problem 5: *I think of a number, multiply it by 6 and add 450. I get 1500. What is my number?*

Problem 6: *Solve the equation:* $6x + 450 = 1500$

Story problems are presented in a verbal format with contextual information about the problem situation (problem 1 and 4). Word problems, presented also in a verbal format, describe the relationship among pure quantities with no story context (problem 2 and 5). Symbolic equation problems are typically described as number sentences (problem 3 and 6).

Researchers have found that start-unknown problems are harder for students than result-unknown problems; symbolic equation problems are harder than both word equation problems and story problems and verbal algebra problems are equal in difficulty to symbolic arithmetic problems (Nathan & Koedinger, 2000). It is noticeable that students are able to solve arithmetic problems more easily because their arithmetic skills precede algebraic skills and require only computational skills rather than algebraic reasoning. Problems with contexts are more meaningful for them than symbolic equations and are relatively easy to solve. In fact, these kinds of problems activate the students' everyday life knowledge and consequently, facilitate their problem solving performance. Different contexts for presenting a problem improve students' abilities to use qualitatively different problem solving strategies.

The students' ability to comprehend the words and the terms used in a verbal story problem greatly affect how the problem is solved. Indeed, some words appear to make the semantic aspect of a problem difficult to understand for many students and the same problem should become easy to solve if it was clearly formulated for them. A good teaching strategy will consist in asking students to rephrase or to retell the problem in their own words or to draw a picture to illustrate the problem.

2.4.5 Ways to overcome problem solving difficulties

It is recognised that, in mathematics lessons, many students experience difficulties in solving non-routine problems and do not like to undertake such an activity because they are time consuming and students have to grapple with them before getting suitable solutions. Indeed, good problems must really stimulate students to think mathematically in reflecting on possible paths towards an answer, looking for patterns, making conjectures and building mathematical models helpful to solve problems. More specifically, problem solving should be considered in relation to fostering students' mathematical thinking.

A way to achieve this purpose is to expose students to a broad variety of non-routine problems and to give them opportunities to solve these problems by using different problem solving strategies. Sometimes students fail to solve problems because they are not aware of the existence of these strategies and do not have sufficient background of efficient methods to use when they are faced with a new problem.

Teachers should select interesting and contextualised problems related to students' everyday life and allowing them to build different ways of thinking. Simple problems could be used merely to illustrate methods essential for solving complex problems, which require reflection and students' involvement. Each solved problem should be for them a source of a new experience in mathematical reasoning.

Frequently, teachers are tempted to overcome students' difficulties in solving problems by giving explanations too soon and by asking leading questions in order to facilitate their students' understanding. They forget that the full engagement of their students in searching for a solution is highly valuable in the problem solving and learning process.

Teachers must try to create a safe learning environment, an "inquiry classroom culture" in which discussion is encouraged. They can encourage the participation of all students by valuing each student's ideas, by accepting students' disagreement, by making sure that all students are included in the discussion. Group discussions between teacher and students or students themselves constitute an important factor in generating a constructive and creative atmosphere for successful mathematics teaching and learning. The quality of students' mathematical reasoning as well as their ability to communicate mathematically are considerably enhanced by discussion.

2.4 OVERVIEW OF THE CHAPTER

In this chapter, problem solving was presented as a dynamic and cyclical process based on Polya's model through which students constructed mathematical models of real world problems. The teacher negotiated social norms with students in order to create a mathematical community in which learning with understanding could take place.

For successful mathematics teaching and learning, the teacher should select interesting, challenging and contextualised problems, which require students' involvement and allow them to build different ways of thinking using various problem solving strategies. He should also give students frequent opportunities to communicate mathematically during group discussions.

In facilitating classroom discussions in small groups or with the whole class, the teacher could detect students' cognitive obstacles and misconceptions inhibiting the mathematics learning process and could help the students to identify and to clear them up.

CHAPTER THREE: THE PRE-TEST

3.1 BRIEF DESCRIPTION OF THE ACTIVITY

Before the classroom intervention, a pre-test comprising of five problems was administered to 121 students in collaboration with the teachers' team (see Appendix A, p. 75). The problems were related to contexts familiar to the students and represented the following mathematical content domains: algebraic operations (Anita's operation, "stella", noted $*$), linear relationships (Number of cows), geometric context (Perimeter of a window, Radius of a semi-circle) and Venn diagram representations (Meat-lovers and vegetarians).

As a prerequisite, the students had to have received formal instruction in solving linear equations. They were allowed 90 minutes to complete the test. No particular indications were given to them except some recommendations to read the problems carefully before solving them and to explain all the steps of their reasoning clearly.

3.2 MAIN OBJECTIVES OF THE PRE-TEST

This pre-test was conceived as a diagnostic test indispensable for obtaining useful information on the students' abilities and strategies when they are faced with contextualised problems. In addition, a thorough analysis of their work should allow teachers to detect their students' weaknesses in mathematical knowledge and to identify the kinds of errors commonly made in order to take them into consideration during the classroom intervention. It also allows teachers to examine how the phenomenon of transfer of students' prior knowledge takes place when they are exposed to a new mathematical situation.

3.3 PRELIMINARY DATA COLLECTION AND ANALYSIS

A preparatory meeting with the three teachers was organised to discuss with them how to proceed to obtain useful data from the students' work. First of all, it was agreed that no marks should be given to students; teachers only had to study their students' work in outlining different strategies used in solving problems, to observe which steps proved to be relatively complicated for many students, to list and to categorise their common errors. The researcher undertook the same activity for the three classes and everybody reported his data during a workshop session. These data are reported in Appendix B (p. 84) and the detailed analysis is described in Chapter Six of this study.

In examining the teachers' comments on different strategies used by their students, it seemed evident that the context of each problem had a great effect on an algebraic or arithmetical approach adopted by students in solving problems. For them, it was easy to use an arithmetical route when the problem was formulated in a geometric context or when it required simple logical reasoning. Faced with a story problem, students referred to their algebraic knowledge to solve it. In this particular case, the translation of a given problem situation in a symbolic language was a hard task for many students. Furthermore, it was noticeable that students were mainly engaged in trying to build and to solve mathematical models quickly and only that a few of them could interpret their answer.

The common and repeated students' errors encountered denoted a lack of mastery of the equivalent equation method and computations involving negative numbers, for example: $x + 24 = 3x \Leftrightarrow x - 3x = 24$ or decimal numbers, for example: $3x + 3.14x = 9.14 \Leftrightarrow \frac{1}{3}3x + x\frac{157}{50}\frac{50}{157} = 9.14\frac{1}{3}\frac{50}{157} \Leftrightarrow x + x = \frac{457}{471}$. This student was unable to solve the equation including a decimal number as a coefficient of an unknown, substituted the decimal number by a fraction ($3.14 = \frac{157}{50}$) but failed to use the equivalent equation method efficiently. It should be noted that equivalent equations have the same truth set and the equivalent equation method consists of finding progressively simpler equivalent equations from a given equation. Practically, an equivalent equation is obtained if we add the same number to both sides of a given equation or if we subtract the same number from both sides. An equivalent equation will be also obtained if we multiply or divide both sides of a given equation by the same non-zero number.

Many syntactic errors, related to mathematical rules and procedures applied in operating with algebraic expressions or in solving equations, were noted. For instance, a student has solved this equation as follows: if $4 * x = x$ then $4 * = x - x = 0$, where the operation, noted ($*$), is defined by: $x * y = xy + (x - y)$. Some students were unable to understand the new operation and to use it correctly in solving a simple equation, they deliberately substituted it with familiar operations, namely addition, subtraction or multiplication which facilitated the solution of their equation. It seemed that this operation ($*$) does not make sense to students and they really had many difficulties to cope with abstract notions. Many semantic errors were due to wrong interpretations of words such as more, and, or and both, used in a mathematical context.

Some mistakes such as $4 * x = 4x + (4 - x) = 4x + 3x = x$, were frequently made by students. An incorrect concept of the perimeter of a real object was also observable in solving problems with a geometric context, for example: $4x + \frac{3.14x}{2} = 9.14$ (the inner side is included). Teachers should therefore be aware of these kinds of errors in planning their teaching activities.

3.4 IMPLICATIONS FOR MATHEMATICS TEACHING

This brief analysis of data suggests that during the intervention phase, teachers should prepare their students on the problem solving process so that they can become aware of each step of the process. In addition, for developing their students' mathematical thinking, teachers should give them opportunities to think for themselves when they are tackling a problem, trying to understand it and make personal sense of it, looking for one or more plausible strategies for getting a good solution. Teachers should recognise that these strategies adopted by their students are more important than the solutions and should accept and encourage students' alternative solutions. They should realise that students' errors or misconceptions are part of the problem solving process; they constitute guidelines for new problems and are indispensable for adjusting their teaching strategies.

A particular emphasis should be put on the mastery of the equivalent equation method, the source of many syntactic errors and misconceptions. Teachers should help their students to identify and to correct their own errors or misconceptions in order to eradicate them. Different tasks should be posed to different students taking into account their special weaknesses or errors. Classroom discussions are good opportunities to remove students' misconceptions through their interaction. Any learning situation should be deeply analysed and adequately used for the students' benefit.

CHAPTER FOUR: CLASSROOM INTERVENTION

4.1 A THEORETICAL BACKGROUND ON MATHEMATICS TEACHING AND LEARNING

In much of the mathematics education literature, mathematics teaching and mathematics learning are considered to be two sides of the same coin. They are so closely interrelated that it seems inadequate to consider them separately. In fact, one cannot talk about mathematics teaching without addressing how to develop students' understanding of mathematics, their ability to solve problems and their dispositions towards mathematics. Orton and Frobisher (1996: 11) argued, "We teach in order that others may learn". It is obvious that in mathematics classrooms, the choice of a specific teaching method will depend on the kind of learning outcomes to promote.

In cognitive and mathematics education, the theoretical foundation in the learning process is indubitably Piaget's work. In fact, in searching to understand how children think and how they gradually acquire knowledge, Piaget has developed a learning theory in which he proposed the learning process as a direct interaction of the child with the environment without necessarily the intervention of a mediator (Cilliers, 1999). He suggested that children learn slowly and their thought processes are very different from those of adults, so time should be given to let new concepts be formed gradually in their minds with respect to their successive stages of cognitive development. In addition, he argued that children must have real and relevant practical experiences if they are to build up and to internalise a concept until they are able to reason abstractly (Cornelius, 1982). It should be noticed that in practice, this kind of learning might not always be efficient because it depends on the quality of subsequent reflection by the child.

In order to promote effective learning, Vygotsky and Feuerstein emphasised the importance of the social context in the learning process and the role that should be played by the mediator (Cilliers, 1999).

Feuerstein proposed mediated learning in which an adult or a more competent peer should take place between the child and the environment so that the conditions of interaction might be modified or changed. In this situation, the mediator assists the child, selects and interprets relevant objects and processes for him. As a result, the child becomes more receptive to direct exposure of stimuli and can benefit from it independently of the mediator (Lomofsky, 1994).

Vygotsky in his theory of mediated activity argued that a child' s potential for learning was realised when he was in interaction with more capable others (teacher, parents, peers and so on). He stressed that this assistance or guidance occurs in the "zone of proximal development" (ZPD), which is the difference between what an individual can do alone, and what can be achieved with help from more capable others. For him, learning is optimised where the social interaction is encouraged especially through language and communication, for instance when children are playing and interacting with others at school or outside school (Cilliers, 1999).

Developing students' capacity to think through learning experiences remains the primary goal in Cognitive and Mathematics Education. In fact, in every learning situation, teachers want their students to acquire knowledge but also to be able to analyse facts and phenomena from different points of view, to generate and organise ideas, to communicate and defend their own opinions, to make comparisons, to draw inferences and to evaluate arguments (Chance, 1986). This requires critical and creative thinking skills, which usually affect all forms of communication and are practised in every interaction with the environment.

Three main theoretical approaches to mathematics teaching and learning will now be discussed: the behaviourist, the constructivist and the social constructivist views.

4.1.1 A behaviourist view of teaching and learning

Traditionally, mathematics teaching has relied on direct transmission by the teacher of fundamental knowledge and skills to be acquired and applied by the students. Much of what is taught consists of routines and it seems obvious that students must memorise and practise such routines in order to reinforce and to reproduce, if it is required, what has been transmitted by their teacher. Students' progress through the curriculum is entirely under the control of the teacher. As a result, the teacher's role becomes that of lecturer and explainer while students absorb a logical and well-structured body of mathematical knowledge from their teacher. During mathematics lessons, the teacher gives more rules, more formulas, more practice for enabling students to perform the computational procedures needed to find the correct answer. This learning approach focuses on the product of the learning process and is mainly centred on students' efforts to gain and to accumulate knowledge and on teacher's effort to transmit this knowledge.

According to Illich (1971: 28), the biggest problem with traditional teaching is:

> *School is an institution built on the axiom that learning is the result of teaching. And institutional wisdom continues to accept this axiom, despite overwhelming evidence to the contrary.*

Even if traditional methods may sometimes be appropriate and effective under some circumstances, they do not necessarily lead to students' understanding, creativity, or ability to solve problems. These traditional teaching methods are basically intended to improve memory and recall but not to promote critical and creative thinking and are based on a behaviourist learning theory, which is associated with methods of training and conditioning (Orton & Frobisher, 1996).

In many countries, the aim of mathematics teaching has been reconceived and directed to the development of students' ways of thinking and problem solving skills. New teaching methods, which foster such abilities, and which are based on a different philosophy of learning have been adopted. Emphasis in mathematics teaching and learning has been placed on understanding rather than on repeating remembered routines and demonstrated basic skills. For instance, in their recommendations, the National Council of Teachers of Mathematics (NCTM, 1989) and the Mathematical Sciences Educational Board (MSEB, 1988) stressed that whatever students learn, they should learn with understanding.

Skemp (1976) suggested two kinds of understanding mathematics, namely instrumental and relational understanding. He stipulated that when students are learning many of the procedures of mathematics, such as extracting the square root of a number, they think that they understand but all they know are "rules without reasons". Performing a mathematical skill without understanding the principle is to have learned instrumentally, but "knowing what to do and why" is to have learned relationally. He argued in favour of relational understanding as a goal of teaching, in the words "what I have always meant by understanding ... knowing what to do and why...", because it leads to mathematics understanding which is "... more adaptable to new tasks" and "... is easier to remember". Perhaps many students never master much mathematics because they have been taught it as if it consists of procedures difficult to understand (Orton & Frobisher, 1996). This idea was also well expressed by

Hiebert et al. (1997) in these terms: "If we want students to know what mathematics is, as a subject, they must understand it. Knowing mathematics, really knowing it, means understanding it". Hiebert and Carpenter (in Hiebert et al., 1997) added that: "We understand something if we see how it is related or connected to other things we know".

4.1.2 A constructivist view

In the constructivist theory of learning it is stated that knowledge cannot be transferred "ready-made" from teacher to students (Olivier, 1989; Orton & Frobisher, 1996) but that understanding has to be constructed by the student's own efforts. "We understand something best and most thoroughly when we worked it out or at least checked through it in a meaningful way ourselves" (Orton and Frobisher, 1996: 18). However, Orton and Frobisher (1996) added that constructivism does not imply that students can make progress only on their own, or that the teacher has no contribution to make. In fact, students should be actively involved in their own learning and the teacher has a primary role to play in designing problematic tasks which will provoke students' thinking and mental activity, which are likely to lead to the construction of meaning and in creating interesting and stimulating situations so that learning with understanding might be fostered for each student. For many teachers, it is a demanding task to use a constructivist approach in their mathematics classrooms because in constructing their own knowledge, especially through problem solving, students can use different mathematical strategies unknown to the teacher or may ask questions that the teacher cannot answer, so they have a strong resistance to changing their teaching strategies. This implies that teachers need to construct their own mathematical and pedagogical knowledge in order to have a good background in mathematics, to involve students in mathematical activity with confidence and to be able to give appropriate answers to students' questions.

4.1.3 A social constructivist view

In the social constructivist approach, the importance of social interaction and communication in each learning situation is well emphasised (Cobb, Yackel & Wood, 1992; Confrey, 1985). In this learning theory, it is mainly required to create effective learning environments where students are engaged in a rich mathematical dialogue with their peers as well as with their teacher. Students need to be actively involved in questioning, conjecturing, explaining, justifying, and listening to others' arguments. Teachers should organise such classroom environment to encourage mathematical discussion, to stimulate the participation of all students by valuing each student's ideas and not immediately putting a stop to students' disagreements. The quality of students' mathematical thinking and reasoning as well as their ability to communicate mathematically are considerably enhanced by group discussions.

As a consequence, teaching mathematics through patterns, especially in algebra, seems to be one of the most attractive, exciting and effective ways to students, as it facilitates students' understanding, both procedurally and structurally. In fact, students learn best when they are intellectually challenged to formulate mathematical conjectures leading to generalisations and a construction of conceptual structures. The teacher's role should be to provide stimulating problems in which students can recognise patterns useful to motivate and to help them to develop an understanding of the mathematical concepts that underlie many algebra problems.

4.2 EFFECTIVE MATHEMATICS TEACHING

It is certainly recognised that each mathematics teacher plays an important role in his classroom to improve the quality of mathematics teaching and learning. But his/her students' involvement in mathematical activities remains of paramount importance in the learning and teaching process (*From now on, he will mean he/she and him will mean him/her)*. Knowing exactly what to teach and how to teach it is certainly indispensable for him, but it is also vital to understand how his students learn. Brown and Atkins (1988: 1) suggested that: "Often an indirect but powerful way of improving your teaching is to improve the ways in which students learn". It is obviously a hard task for a teacher to imagine exactly what takes place in his students' heads while they are learning new mathematical concepts or solving unfamiliar problems. But trying to know students' mental processes during these activities could facilitate good planning.

Mostly, it is agreed that an individual constructs his own meaning by connecting new information and concepts to what he already knows. To be efficient, it is imperative for teachers to understand what their students already know. For this, it is advisable during a mathematics lesson, to ask the kind of questions that reveal students' prior knowledge in order to design learning experiences that relate to and build on their knowledge.

In addition, effective teaching involves observing students, and listening carefully to their ideas and explanations. As a result, teachers should motivate their students to engage in mathematical thinking and reasoning and provide learning opportunities that challenge students at all levels of understanding and encourage them to communicate mathematically. This requires a continuing effort to learn and improve about mathematics and pedagogy, benefiting from interactions with students and engaging in continual professional development.

Mathematics teachers should carefully select the most important concepts and skills to emphasise and should plan how the learning environment will be structured in order to concentrate on the quality of students' understanding rather than on the quantity of information required by the curriculum. As each student is expected to understand mathematics, it is the teachers' responsibility to support students' efforts in choosing suitable and motivating tasks, which can arouse their curiosity and draw them into mathematics.

Mathematics teachers should listen carefully to how students go about solving problems. They should try to know their students' strengths and weaknesses and should develop a teaching strategy based on this knowledge. Teachers who listen to students and who plan instruction based on what they learn from listening or from students' errors, can transform their students' mathematics learning. Moreover, in listening to students' mathematical explanations, teachers can find that students know a great deal of mathematics at an informal level and by building on it they can help students to construct more sophisticated concepts.

4.3 WORKSHOP SESSION

As was planned in the research experiment, a preliminary meeting with the participating teachers was organised and this was followed by a two-day workshop session. The ultimate goal was to prepare teachers to become effective and competent collaborators during the classroom intervention. With the teachers' team, the researcher's attitude should be that of facilitator and collaborator ready to help them to acquire proficiency in doing mathematics but also to enable them to create a mathematics classroom culture that fosters students' understanding of mathematics and their mathematical thinking skills through the problem solving process.

The workshop session was mainly focused on empowering teachers in giving them opportunities to increase in mathematical knowledge and to gain new insights in teaching strategies in order to become more powerful, more self-confident, effective and dynamic in their teaching practices. According to Carl (1995: 7):

> *Empowerment is that process of development and growth through which a person goes which enables him/her to take independent decisions and to act autonomously with a view to making a contribution towards the development of his environment. This process is coupled with the development of applicable skills, attitudes and knowledge within a positive and democratic climate. These persons are therefore regarded as professionals in their own right as they are able to make a contribution to change through their particular power.*

During this session, it was envisaged to help teachers to gradually change the way they teach in order to meet their students' needs in mathematical knowledge, to improve their problem solving skills and to develop a positive attitude towards mathematics.

Practical work on problem resources and discussions on methodological issues in mathematics teaching based on students' active learning were also planned. The following items were discussed: general objectives in teaching, specific objectives in mathematics teaching, important methodological principles in mathematics teaching, the didactical triangle with emphasis on teacher-centred methods, student-centred methods and subject-centred methods, and basic principles in the preparation and the construction of a lesson. Each teacher received a syllabus in which all these elements figured.

It is important to note that, during the discussion, particular emphasis was put on the problem solving and the mathematical modelling processes. In fact, it was essential for teachers to know how to develop in their students the ability to solve problems not only through a mastery of computational skills but through an understanding of the problem in using alternative strategies for finding an appropriate solution. As the method to use in a problem solving situation is not obvious, teachers should encourage their students to try independently different approaches for tackling the problem. This requires from students to acquire progressively a suitable level of critical and creative thinking.

For a successful problem solving process, it is necessary to make a mathematical model of a given problem situation: messy information is obtained from a real-life problem, described mathematically by means of a drawing or an equation, analysed meaningfully for additional information by solving the equation and finally the mathematical solution is interpreted in terms of the given problem to be validated.

Another important activity consisted in working with them on the content of the problems proposed for the experiment with emphasis on Polya's four phases for solving problems. In this way, teachers should overcome the probable lack of their mathematical knowledge and should be progressively confident in their teaching. The choice of the different problems takes into account the students' cultural environment and their everyday life activities such as shopping, travelling, playing games, guessing, and visualising geometrical situations. It is important to use challenging problems which are not too easy or too difficult for keeping students' interest and enthusiasm, and requiring a systematic investigation and more students' input.

It should also be possible to stimulate teachers to formulate problems themselves to address students' difficulties; their creativity and involvement in generating non-routine problems and facilitating their solutions constitute a good indication of successful mathematics teaching and learning.

Once again, teachers should understand that they have a great responsibility to encourage their students to build themselves different ways of thinking, to negotiate with them new social norms for a safe learning environment where they can work individually, in small groups or as a whole classroom according to the nature of their task, discuss their alternative strategies and justify their points of view, listen to each other's ideas and establish meaningful connections among different mathematical concepts.

It should be noted that during our initial discussions on various aspects of mathematics and mathematics teaching, teachers seemed interested in some innovative ideas on mathematics teaching, namely the problem solving approach and the mathematical modelling process but were unsure of their ability to put them into practice. This point was noted for later follow-up activities. When working on the content of the problem resources, the teachers experienced the problem solving process as learners and had difficulties in building some mathematical models because they wanted to arrive at a problem solution quickly without taking into account all the relevant information of the problem and without following all the steps required for the whole process. However, they performed quite well in solving equations. Important modifications on the formulation of some problems were made after the teachers' experiences.

For the sake of a successful intervention, it was agreed to prepare and to plan together the content of appropriate problems for each week. In order to give students enough time to explore various problem solving strategies independently, mathematics lessons ran over two consecutive periods, giving sufficient time for teachers to evaluate students' strengths or weaknesses and their eventual progress or lack of understanding in mathematics.

4.4 TEACHERS AND STUDENTS AT WORK

4.4.1 The new role of the teacher

In accordance with our planning, the experiment was conducted in three second year classes (C_1, C_2 and C_3) selected from three different secondary schools: an urban religious school, known for its strict discipline and selective recruitment (for C_1), a private and gender-mixed school with no rigorous selection of students (for C_2) and a typical country secondary school with 95% female students selected at national level (for C_3).

Contact with school administrators was made to gain their support as this was felt to be necessary for working successfully with the selected teachers. The intervention was scheduled to last a month and weekly activities were limited to only two consecutive lessons in each class.

During the first teaching week, each teacher was observed teaching the problems: "Teddy's dress" (Problem 2), "Consecutive even numbers" (Problem 6) and "I think of a number" (Problem 7) (see Appendix A, p. 75). The teachers preferred to start with easy problems in order to familiarise students with the mathematical modelling process. During the mathematics lessons, it should be observed that all teachers adopted practically the same approach in setting up problems, clarifying them by asking students to read them carefully and to underline key words, allowing students to ask questions and giving appropriate answers to their questions. Then, sufficient time was provided to students for their investigations. Meanwhile, teachers moved around in the class, going from group to group, trying to discuss with students for a while and taking note of some students' strategies or mistakes. The key problem that they had to face was how to organise the whole class discussion successfully. In fact, during this important period, when students were given opportunities to communicate their results, different teaching approaches were clearly observed.

In class C_1, the teacher invited some students to present and explain their different solutions. As the problems seemed relatively easy for them, these students wrote on the board different mathematical models and solved them correctly. The teacher only had to intervene in asking them to check their solutions and to interpret the answers. The other students had to agree on or to react to some explanations of solutions which were not easy to follow before making a collective decision on their "best" solution, easy to understand and obtained in a shorter way. In order to motivate good solvers, students' models were labelled with their names.

For instance, in solving Problem 6: "The sum of two consecutive even numbers is 150. What are those numbers?" five models were proposed:

Eric's model: $x + x + 2 = 150$

Victor's model: $x + y = 150$ and $y = x + 2$

Clement's model: $x + 2 = 150 - x$

Didier's model: $2x - 2 = 150$

Emmanuel's solution: The difference between the two numbers is equal to 2.

As 150 - 2 = 148. Thus, 148 is the double of the smallest number. The smallest number is 74 and the greatest is 76.

It is important to note that Victor formed and solved a system of two linear equations by substitution without prior instruction. Each solver seemed to take pride in his solution and felt confident in his mathematical ability.

At the end of the lesson, the teacher was also satisfied with his students' performance attained without much intervention except when some mistakes were made in manipulating numbers with negative signs. The teacher's instruction was eased by his students' mathematical expertise but also by his willingness to implement some pedagogical practices introduced during the workshop session.

In class C_2, the teaching approach was different during the classroom debate. One student was invited to solve Problem 7: "I think of a number, add 27 and divide the sum by 7. My result is the same when I subtract 9 from the original number. What is my number?"

After writing his mathematical model correctly: $\frac{(x+27)}{7} = x - 9$, he was unable to explain the following step. Instead of helping him try out by means of judicious questions, the teacher immediately suggested inserting brackets and applying the equivalent equation method. As the student did not understand what to do, the teacher developed, for the whole class, well formulated explanations on rules and procedures used to obtain equivalent equations. Then, the student carried out the following operations prompted by the teacher:

$$\frac{(x+27)}{7} = x - 9$$

$$\Leftrightarrow x + 27 = 7\,(x - 9)$$

$$\Leftrightarrow x + 27 = 7x - 63$$

$$\Leftrightarrow x - 7x = -63 - 27$$

$$\Leftrightarrow -6x = -90$$

$$\Leftrightarrow x = 15$$

At the end, the student wrote without verification: my number is 15. Even if this student got the right answer, it was clear that the teacher did not make an effort to identify that he had a cognitive difficulty, perhaps that he did not understand the equivalent equation method. The other students also listened to the teacher's explanations but without enthusiasm except for a few "strong" students who seemed to understand the procedure. This teacher was really worried about his class. Though he was competent in mathematical knowledge, he used to teach mathematics in its rigour and maintained his classroom in an atmosphere of order and discipline so rigid that there was no good communication with his students. After the lesson, he confessed that in general his students were weak in mathematics so that it was difficult to go quickly with them and this was why he was obliged to give so many explanations.

In class C_3, the teacher wanted his students to work in a systematic way in strictly following these steps in their reasoning: choice of the unknown, mathematical modelling, solution of the equation and verification. Invited to explain the solution of Problem 7 (see p. 36) in front of the class, a student immediately wrote the following equation:

$x + 27 / 7 - 9 = x$. Thinking that his mathematical model was correct, he continued as follows: $x - x = - 27.\ 7 + 9$. Then, $x = - 180$

The teacher tried to correct him but no progress was made because he could not identify his student's cognitive problem. This student, like many other students in this class, had a language problem, which is why he constructed his model with respect to the succession of the numerical data combined with the corresponding operations. He solved the model using, in a rote fashion, the equivalent equation method reduced to "change a side change a sign". The teacher asked him to re-read the problem for better understanding; this was a good suggestion because the problem solving process cannot take place if a student does not understand the problem.

Intuitively, the teacher asked him to translate the first phrase in symbols and to formulate the second phrase in his own words. The student said: "My result is equal to the difference between the original number and 9".

"What is the original number?" asked the teacher.

"x", said the student.

Then, he wrote the correct model:

$$\frac{(x+27)}{7} = x - 9$$

This student translated the problem, word by word, into an equation without understanding its semantic structure.

To solve it, the teacher involved the whole class and the student had to write what the other students said until he arrived at the solution. During the whole lesson, the teacher wanted his students to work quickly and systematically in order to cover sufficient material but he seemed worried because it took much time to solve this problem. Despite his attempt to involve his students in the process of problem solving, he did not realise that they needed much time to think deeply about the problem situation, to try out many strategies, sometimes to make mistakes in building their models. He thought that what was easy for him should also be easy for his students.

In our weekly meeting, teachers gave their comments on the previous week's lessons. For them, the problem solving approach inspires a good teaching strategy because it involves students to think about the problem and to generate their own solutions, they believed it to be appropriate for "strong" students but not for all. They added that students couldn't continue to work on the problem if they were unable to build a model; the teacher is obliged to give more hints to the "weak" students. They said that it was time consuming and difficult to solve many problems during the given time. Though the teacher of class C_1 apparently had no problem to experiment with this teaching approach, two major problems arose for the other teachers, namely the active participation of the whole class in the problem solving process and the language problem. To overcome these difficulties, it was agreed to continue to reinforce the mathematical modelling process but also to initiate small group work in the three classes in order to stimulate students' collaboration and to develop their ways of thinking and their communication skills.

Additionally, when carefully examining with the teachers the students' work collected after the lessons, the following errors were revealed in Problem 7: the passage from the equation $\frac{(x+27)}{7} = x - 9$ to its equivalent equation $x + 27 = 7\,(x - 9)$ and the use of the equal sign. For instance, a student wrote: if $\frac{(x+27)}{7} = x - 9$ then $\frac{27x}{7} = \mathrm{x} - 9$. Chalouh and Herscovics (in Coxford, 1988: 34) characterised this kind of error as a cognitive obstacle, which was created in students' minds when symbols were juxtaposed in arithmetic and algebra, known as "concatenation". In fact, in arithmetic the concatenation of two numbers denotes addition while in algebra, it denotes multiplication. In arithmetic, 50 + 4 = 54 or 5 + ¼ = 5 ¼. In algebra, 5 x b = 5b but"5 + b" is a "closed" algebraic expression. In this case, the problem of reluctance to accept "lack of closure" is added to the student's difficulty. As stipulated by Orton and Frobisher (1996), many students regarded algebraic expressions such as $x + 27$ as "not closed" and replaced two numbers connected by operations by the "result" of the operation. The equal sign is also viewed as an instruction "to do something" rather than an indication of an equivalence between two algebraic expressions (Herscovics

& Kieran, 1980). As an illustration, in solving the same equation, another student wrote: $\frac{(x+27)}{7} = x - 9 = 7x + 27 - 9x$. Teachers should be aware of such misconceptions and should help their students to identify and correct their errors.

4.4.2 Working in small groups

From a constructivist perspective, group discussions are greatly valued as a very good way to enhance students' learning through social interaction and communication. For this, it is important to organise small groups during a mathematics lesson in order to give students opportunity to work together, to share ideas and to develop their thinking skills. The teacher has the responsibility to set up several small groups not too big, in order to allow every student to be engaged in the discussion and to promote meaningful exchange between students. He should also choose interesting and challenging tasks for stimulating students' motivation and enthusiasm in searching appropriate solutions of various problem situations. It was observed that in small groups students have the opportunity to interact, to construct and to negotiate meaning of mathematical concepts. Orton et al. (1996: 60) stated that:

> *If we believe that children are not passive receivers of knowledge, but are continually involved in constructing meaning, and that meaning may need to be negotiated, this demands a forum within which such negotiation and construction can take place.*

Small groups increase participation and develop the students' responsibility for their own learning, they are less intimidating for students so that communication skills are indirectly enhanced because they are often invited to talk in debating interesting issues with the others. During group discussions, students develop their language skills, promote interpersonal relationships and develop more positive attitudes towards mathematics.

The teacher as a facilitator should establish mathematics classrooms with social structures and norms in order to promote communication and collaboration among students. He should encourage them to be more involved in working together and more productive in good results. He should also choose the best moment for gathering ideas from different groups in order to reach a suitable consensus in the discussion with the whole class.

In conformity with the propositions adopted during the meeting, during the next mathematics lessons teachers attempted a new classroom organisation in grouping students by pairs. Some norms were clearly defined namely students had to work together in discussing efficient ways to tackle a problem without much noise, in sharing their points of views, listening carefully to other's ideas and challenging them for improvement, in checking their answer after solving the problem. As usual, a worksheet was given to each group for reporting the results of their investigations.

In class C_1, it was noticed that students were not really accustomed to work in groups because when the teacher finished giving important hints to clarify the first problem, many students automatically took pieces of paper to work individually, forgetting their teacher's suggestions. The teacher had to intervene by asking them to collaborate effectively and to report the group's work on the given worksheets. He began to understand the importance of social norms in implementing the problem solving process in a mathematical classroom. He had planned to solve "Philip's shopping problem" (Problem 4) and "Mathematics competition problem" (Problem 9). Problem 4 seemed so complex that its model was not easy to formulate.

"Philip went to a store, spent a half of his money and 1000F more. He went to a second store, spent a half of his remaining money and 500F more. As he had no money left, how much money had he in his pocket just before his shopping?"

Many groups decided to decompose it into simpler problems.

The method used by one group:

First store: $\frac{x}{2} + 1000$, remaining money: $x - (\frac{x}{2} + 1000) = \frac{x}{2} - 1000$

Second store: $\frac{1}{2}(\frac{x}{2} - 1000) + 500 = \frac{x}{4} - 500 + 500 = \frac{x}{4}$

Mathematical model: $\frac{x}{2} + 1000 + \frac{x}{4} = x$

$$\Leftrightarrow \frac{x}{2} + \frac{x}{4} - x = -1000$$

$$\Leftrightarrow \frac{3}{4}x - x = -1000$$

$$\Leftrightarrow - \frac{x}{4} = - 1000$$

$$\Leftrightarrow x = 4000$$

After a quick verification, the reporter wrote: Philip had 4000 F

Another group proposed this complicated but correct model:

$$\frac{x}{2} + 1000 + \frac{1}{2}\ (x - (\frac{x}{2} + 1000)) + 500 = x$$

To solve this equation, students faced computational problems consisting in manipulating successive operations with brackets and fractions. In giving this problem, the teacher wanted to reinforce the mathematical modelling process and to diversify strategies used by students in solving mathematics problems. It was also an opportunity for him to correct their frequent errors in computations. It was really interesting to hear students spontaneously giving simple and clear explanations to their partners who were unable to understand how the model was built or how to manipulate combined algebraic symbols (brackets, negative sign,...).

Problem 9 appeared to be very interesting for students judged by their warm discussions and the various strategies used to solve it.

"In a mathematics competition, 5 points were scored for each correct answer and 2 points were deducted from the score for each incorrect answer. How many correct answers did Jimmy get if he worked out 10 problems and scored 36 points?"

In fact, four categories were presented:

Model 1: $5x - 2\ (10 - x) = 36$

Model 2: $5x - 0y = 50$ and $5x - 7y = 36$

Model 3: $x + y = 10$ and $5x - 2y = 36$

Model 4: $x + y = 10$ and $5x - 7y = 36$

Model 5: The student used the arithmetical method of "false assumption" learned earlier in primary school and got a correct answer.

It is important to notice that Model 2 was independently constructed by two groups simultaneously using arithmetical (false assumption) and algebraic (two linear equations) reasoning. When interviewing one member of the two groups, he clearly explained the model and its solution as follows:

Jimmy got 5 points for each correct answer. If all the answers were correct then he would get 50 points. That means $5x - 0y = 50$

As he had x correct answers and y wrong answers, then he got 36 points:

That means $5x - 7y = 36$, because he lost 7 points by each wrong answer.

$5x - 0y = 50$

$-5x + 7y = -36$

$7y = 14$ and $y = 2$. Then, $x = 8$

He got 8 correct answers and 2 wrong answers.

Without this explanation, the model would have been considered wrong and immediately rejected by the class. But these students' reasoning was based on their own logic without prior instruction in solving a system of two linear equations. The teacher elicited the students' thinking but concluded, after a discussion with the whole class, that Model 1 was the best.

In class C_2, the teacher adopted the classroom organisation of grouping students by pairs. He proposed to solve Problem 6 because during the previous lesson, they did not have sufficient time to solve it. While the students worked on the problem, the teacher moved about in the class to assist "weak" groups and to encourage the others. When he judged that time was up to work together he asked students to stop working in groups and to present their results with explanations.

This class identified four different models:

Model 1: $x + (x + 2) = 150$

Model 2: $x + y = 150$ and $x - y = 2$

Model 3: $x + x = 150$

Method 4: guess and test

The solution of Model 3 was not clear. In fact, invited to explain it, the reporter said:

"$x + x = 150$, the first x represents the first number and the second x represents the second number. Then, the difference between $2x$ is equal to 2".

"What $2x$ means for you?" asked the teacher. He answered: "$2x$ means the two numbers". He continued his reasoning as follows: $150 - 2 = 148$. Then $2x = 148$ and $x = 74$. The other $x = 76$.

This student had a serious language problem but he also believed that all variables should be represented by x. It was the reason why x represented two different quantities.

Model 2 was easily solved by substitution and students seemed to understand the procedure.

In this class two different categories of students could be identified: those who were very "strong" in mathematics and had no language problem and those "weak" in mathematics and in language. This constituted a big difficulty for the teacher who often had to manage his students functioning at different levels.

At the end of the lesson, the teacher seemed satisfied with some perceptible progress in his students' work due to the new organisation of his class. He realised that it was helpful for the whole class to work together in small groups because the "weak" students learned from their exchange with the others and improved their language skills if they were asked to explain their solutions in their own words. For him, this was a good way to remediate his students' cognitive difficulties.

In class C_3, the teacher was aware of his students' difficulties, and proposed as a test two equations including fractions, which were quickly solved by the students, before he presented Problem 4. Preliminary indications on the meaning of some words were given before he invited students to tackle the problem in pairs. Some groups seemed to understand the problem and started to discuss suitable strategies to use while the others tried individually to write some seemingly senseless calculations. The teacher then interrupted their activities and reasoned aloud with the whole class by asking some students to mediate the problem for the others. As a result, all groups started to work more actively on the problem by exchanging ideas and finally reported the results of their discussions.

When the group discussion took place, it should be noted that many groups decided to construct first simpler models and then to reconstitute the complete equation to solve. Except for their difficulties to manipulate the combined symbols included in the equation, the solution was easily obtained and this strategy was well integrated by students. Despite the low level of mathematical knowledge of this class, the teacher experienced success with a new organisation of his class and it was noticeable that many students enjoyed it by their interventions and questions for better understanding.

During the weekly meeting, the three teachers admitted that working in small groups was a successful learning situation in the problem solving process. Even if students made many errors in their attempt to search independently for a solution of a given problem, they were interested in doing mathematics and opportunities were given to them to correct their deficiencies in language progressively. Teachers realised that their students were not used to collaborate in mathematics; it is why "strong" students resisted for a while to collaborate with the others in the C_1 and C_2 classes. As usual, before planning other activities, specific students' errors were analysed, namely the use of successful arithmetical strategies in algebra and the combination of many symbols in the same equation that caused confusion for some students. For instance, a student built this model in solving Problem 4: $\frac{2}{2} - (\frac{x}{2} + 1000) = x$. For him, x represents Philip's money before his shopping: $x = \frac{2}{2}$, meaning two parts where $\frac{x}{2}$, meaning one part, the money spent in the first store. This shows again how arithmetical and algebraic reasoning interfere in the students' minds.

In the following lessons, it was decided to diversify the types of problems to solve with emphasis on those which might induce new strategies in students. They opted to experiment with a problem involving patterns. In addition, for promoting more interaction between students, it was decided to group them in fours. Three problems were selected for mathematics classroom activities: "Refund for returned bottle racks" (Problem 3), "Growth for a seedling" (Problem 19) and "Square garden" (Problem 20) (see Appendix A p. 75). Before they started to work on these problems, students received a briefing on the new class organisation consisting of the formation of groups of four and collaboration among them to getting better results.

Groups were voluntarily composed in the three classes. But it should be observed that in C_1, "weak" students moved spontaneously to work with "strong" students; in C_2, a special group of "strong" students who were friends was formed while the others were arbitrarily organised and in C_3, four students put two desks close together and started to work. According to the teachers' comments, students had never solved problems with patterns, so they preferred to pose Problem 3 first, because it introduced the search for a functional rule, which created the need for a model to solve. The functional rule was $C = 2000x$ and the corresponding model became $36000 = 2000x$. Step-by-step, all groups got a right answer.

Problem 19 was presented in tabular form. Students enthusiastically looked for patterns, formulated a functional relationship and solved the model. Each group seemed satisfied in quickly showing a right answer to their teachers, a proof that they were still dependent on their teachers. On the basis of this model constructed by one group: $\frac{14}{21} = \frac{20}{x}$, the teacher of class C_3 anticipated by drawing students' attention on the notion of "proportion" in mathematics.

Problem 20 was a little difficult but interesting in shape patterns. For the sake of involving more of his students in this activity, the teacher of class C_2 dispersed the "strong" students' group by asking them to reinforce the other groups. This initiative produced immediate good results because all groups appeared strengthened and began to work effectively and efficiently. Group discussions were animated and everybody wanted to give his idea, which was accepted or rejected by the others. Good explanations were given to those who did not understand how to look for patterns, to generalise and to solve the model at hand. Some exclamations such as: "Aaah! I understand now", "We are the first", "Look! Look teacher!" were heard from different groups.

In order to reduce the number of groups and to interact easily with his students, the teacher in class C_3 suggested that they work in groups of eight. No positive results appeared from this modification of the group size because it took students time to reorganise themselves, there was no cohesion and no consistent activity of the groups, only the team leader helped by a few students did all the work, and the others were passive listeners. A real decrease in students' interaction was observed. However, one group obtained good results in building this model: $(n + 2)^2 - n^2 = 200$ but was unable to solve it. The teacher was obliged to develop the further explanations necessary for its solution.

In class C_1, the results were so satisfying that the teacher was very excited when he showed his students' findings to the researcher. In this class, two models were built: $4n + 4 = 200$ and $4(n - 1) + 8 = 200$. He had to propose extra problems for the fast groups. Students were so interested in doing such mathematical activity that they preferred to continue their discussions instead of taking their break. The pattern approach seemed to be one of the most effective ways that arouses students' interest and curiosity but also empowers them procedurally and structurally.

4.5 THE INFLUENCE OF THE INTERVENTION AND CHANGE IN TEACHERS' PRACTICE

It is widely recognised that for many teachers introducing the problem solving based approach in their mathematics classrooms is a demanding task. To Burkhardt (in Burkhardt et al., 1984: 18), "teaching for problem solving is mathematically, pedagogically and personally" more difficult for teachers. This is mainly due to their lack of self-confidence, self-knowledge and experience in problem solving skills but also due to the difficulties they encounter in creating a learning situation based not on the mathematical content itself but on the student mathematical activity through problem solving.

In this study, with the classroom intervention, this objective was to observe and to describe how teachers shifted slowly from their traditional and secure ways of teaching to a new and uncomfortable teaching situation centred on the problem solving process. Logically, it was difficult to expect from teachers spectacular change of teaching styles that have been established in mathematics classrooms over years, as changes require plenty of time and sufficient support.

However, a careful analysis of teachers' comments on their teaching strategies before and after the intervention revealed some progress.

Describing the effect of the experiment in his classroom, the teacher of class C_1 said:

> *Before the intervention, I began my class with direct exposition, stimulated students' understanding by well-prepared questions and good examples, and ended my lesson by many applications taken from official textbooks. Some students, especially the brightest, participated during the lesson by answering my questions or by asking interesting questions related to the content while the others listened carefully but passively to my explanations. They used to solve problems in applying directly the methods they have been taught. They had no habit of working together in groups.*

Then, he added:

> *With this teaching approach students worked differently. I introduced my lesson in giving them useful indications for solving a problem. For this, I asked them to understand first the problem situation, to build a model, to solve*

> *it and to validate their answers. They seemed very interested in their task, they freely discussed in groups, and seemed pushed to "discover" something using different ways not necessarily my way of solution. The "weak" students learned much from others in small groups. I implemented this teaching strategy successfully in Geometry. But it was time consuming and required much effort to organise the classroom, to work with different students' groups and to facilitate group discussions. It does not really fit to our current school curriculum.*

In her turn, the teacher of class C_2 declared:

> *My teaching style has been always based on the students' activity. I began with a mini-exposition, gave examples and ended with many exercises. I tried to do my best to be sure that my students should understand mathematics; sometimes I was obliged to "force" them to understand it as I do.*

She continued her comments in these terms:

> *Now, I realised that students were able to construct their knowledge with less help. I learned to let them take the responsibility for their own learning while working in small groups and I was impressed by the various strategies they generated. I think that this approach contributed greatly to increase their thinking and reasoning skills. But it took plenty of time and we have to cover a broad mathematics curriculum. Problems were contextualised and even easy for students who did not have a language problem. For the others, I was obliged to give more explanations for being sure that they have understood the problem before they tackle it. Despite the language problem, I noticed that they enjoyed solving mathematics problems as we progressed.*

The teacher of class C_3 commented:

> *I usually practised the standard teaching method of solving problems. I selected a problem in our mathematics textbooks, and asked students to read it. Sufficient explanations were given according to the complexity of the problem and I solved it on the board by involving the whole class. Students were invited to solve a similar problem individually, which was immediately corrected, and many exercises were given as homework.*

She pursued in indicating that:

> *During the intervention, students had to solve problems posed in a different manner without prior instruction. For the first time, it was difficult for them to find the appropriate strategies to solve them. But when I encouraged them to try and to persevere, some of them got good solutions using different ways of thinking. When I invited them to work in groups, they collaborated efficiently and the "strong" students helped the others to understand and to participate actively in the discussions. I often had to intervene in order to provoke their thinking and to facilitate their interaction by means of warm and productive mathematical discussions. Sometimes, I guided some groups who required my assistance when they seemed "blocked" in their reasoning. Students felt challenged to get good solutions to problems but many of them faced the problem of language, which affected their active participation during the whole class discussion. I appreciated how students worked together in small groups; it helped them to be more involved in mathematics lessons. Next year,*

I will see how to improve the formation of groups, but it will depend on the size of the class.

In the light of these different comments, it is clear that some progress was made by the teachers. In fact, there was a qualitative improvement in their teaching styles, changing from transmitting mathematical knowledge to facilitating students' interaction in constructing their own knowledge; from providing too much explanation and forcing students' understanding to giving only some hints. They have realised the necessity to give students opportunities to work independently on clear and challenging problems. Teachers have understood that their primary role is to create a rich and contextualised environment, which promotes students' critical and creative thinking, and to mediate students' learning through dialogue. Teachers have learned that the problem solving process could take place when students worked together, exchanged ideas, tried a variety of ways to find the best solution for a given problem. They have learned to be more flexible in their teaching styles in trying to communicate with their students, in encouraging logical disagreement, in accepting students' explanations and suggestions, and considering their mistakes as important learning opportunities. They have appreciated the positive effect of collaborative work on their students' learning and have experimented with it in other areas.

On the other hand, during the intervention, students slowly became familiar with various strategies to solve problems; they were motivated to learn in new ways and showed a will to learn by themselves. They seemed more involved in doing mathematics, gained confidence in their mathematical abilities and developed positive attitudes to learning mathematics, which could be observed by their perseverance and their satisfaction to achieve something by themselves. Students learned to socialise, to share their learning strategies, to listen to others' opinions, to make efforts to learn from each other. They improved their communication skills in making remarkable efforts to explain their mathematical reasoning, learned new ways to work with others and taught each other. They enjoyed working in a relaxed learning environment.

However, the language problem was crucial for better learning in the three classes and teachers had to face it during the whole teaching process.

Additionally, teachers wanted at any cost to cover the prescribed curriculum and showed that the problem solving teaching approach was time consuming and did not allow them to progress in their usual fashion. Students remained dependent to their teachers' solutions and asked them for more assistance. Teachers still had great concern for maintaining control on their students' thinking and reasoning. It should be observed that a kind of "disequilibrium" between their previous teaching experience and the present teaching approach was created in the teachers' minds, predicting substantial dispositions to change.

4.6 OVERVIEW OF THE CHAPTER

This chapter reported on the teachers' classroom intervention. It discussed different views of mathematics teaching and learning with emphasis on a constructivist approach. It described how teachers slowly shifted from their traditional ways of teaching to a new but demanding teaching style centred on the problem solving approach. Progressively, they changed their role of transmitters of mathematical knowledge and skills to the new role of mediators of learning and facilitators of students' interaction. They successfully experienced collaborative work and appreciated its positive effect on their students' active learning.

Despite a crucial language problem, students had opportunities to learn in new and meaningful ways, to share their learning strategies, to listen to others' points of view and to work in a relaxed learning environment; important factors that have greatly enhanced their thinking skills when they worked on algebra problems even if they still remained dependent on their teachers while doing and learning mathematics.

CHAPTER FIVE: THE POST-TEST

5.1 BRIEF DESCRIPTION OF THE ACTIVITY

The post-test took place one week after the classroom intervention. It was administered to 116 students; the other 5 were absent for various reasons. The two tests were practically similar: the same content domains, namely linear relationships (Number of cows, Age of Susan's mother), geometric context (Perimeter of a window, Radius of a semi-circle, Area of a door), Venn diagram representation (Meat-lovers, Participants in a conference) and algebraic operations (Anita's operation, Student's operation) but with different semantic structures for the problems (see Appendix A, p.75). The test took an hour. Students were asked to read the problems carefully for understanding and to write all details related to their ways of thinking and reasoning while solving the problems. All the students' worksheets were collected for a thorough analysis.

5.2 MAIN OBJECTIVES OF THE POST-TEST

This post-test was organised in order to evaluate the effects of the teachers' intervention on the students' mathematical learning. The intention was to compare the students' results in the pre-test and the post-test, to describe and to analyse qualitatively the progress they made in their problem-solving skills.

In this analysis, it is indispensable to examine closely how students generate variables from a given problem situation, choose the most realistic and appropriate relationships between variables, identify specific questions needed to solve the problem efficiently, build a mathematical model underlying the structural aspect of the problem, solve the model, verify their solution and finally validate the answer.

5.3 DATA COLLECTION

In order to study the impact of the experiment on the improvement of students' problem solving strategies, the students' work in the post-test was meticulously examined as an indication of the level of their mathematics understanding. The data were categorised into four steps, namely formulating a mathematical model (A), solving the model (B), verifying the solution (C) and interpreting the answer (D). As the first step was the most important in the problem solving process but also the most difficult for students, it was subdivided into intermediate steps corresponding to possible students' subskills in the formulation of a mathematical model. For this, it is important to note how students generate variables, generate relationships, connect relationships and build a model using numbers, symbols or drawing pictures. The data, expressed in percentages (%), were presented in tabular form for analysis and interpretation. They are recorded in detail in Appendix B (p. 84) with pertinent observations on various students' attempts to construct mathematical models. Students' common errors were identified and reported as helpful indicators of their level of understanding. This is extremely useful in helping teachers to address these errors.

CHAPTER SIX: ANALYSIS AND INTERPRETATION OF RESULTS

6.1 ANALYSIS AND INTERPRETATION OF RESULTS

This study was designed to investigate how to improve students' thinking skills through mathematics teaching based on a problem solving approach. As noted earlier, all data related to the pre-test and the post-test are presented in Appendix B (p. 84). The present analysis of results was carried out only on the basis of the number of correct responses in the pre-test and the post-test.

In the pre-test, the first problem "Travelling in the same bus" was proposed in order to evaluate students' attitudes when they faced any mathematical problem situation. The students' responses revealed that, in class C_1, many students (91 %) tried to think about the problem before they solved it. In the other classes, the results were quite satisfying for a large proportion of students (53 % and 54 % respectively in classes C_2 and C_3) while the remaining students mechanically worked on this simple problem using a complicated way and failed to get a correct answer. They immediately attacked the problem without scrutinising all the relevant information and without trying to make sense of it.

As regards to the problems on algebraic operations (Anita's operation and Student's operation), they were designed in order to test the students' level of abstraction and creativity. The results indicated that many students (respectively 87 %, 34 % and 50 % in the three classes) were able to manipulate "procedurally" with numbers any operation ($*$), different from the four well-known basic operations in mathematics, for example: $4 * 5 = 4 \times 5 + (4 - 5) = 21$. However, this mathematical ability strongly decreased when they had to solve an equation involving such an operation, for example: $4 * x = x$; where the operation ($*$) is defined by $x * y = xy + (x - y)$. In many cases, it was simply substituted by the additive, subtractive or multiplicative operations as follows: $4 * x = x \Leftrightarrow 4x = x$, $4 * x = x \Leftrightarrow 4 + x = x$ or $4 * x = x \Leftrightarrow 4 - x = x$. In the post-test, only one student was able to propose his own algebraic operation, to form and to solve his equation correctly. He defined the operation ($*$) as follows: $a * b = \frac{a}{b} - 1$. Therefore, the equation to solve became $2 * b = 1$, which was equivalent to the equation $\frac{2}{b}$ - 1 = 1. He continued in writing that 2 - b = b, equivalent to 2 = 2b, but without specifying that b was not equal to zero because division by zero is not allowed. Therefore, b = 1. These results showed that students were not used to handle structural algebraic expressions and their experience in abstract concepts was limited.

The results of the two tests for the other problems were analysed according to four important steps in the problem solving process: formulation of a mathematical model (A), solving a model (B), verification of the solution (C) and interpretation of the answer (D). In order to understand more deeply and to clarify the students' ways of thinking the following subskills were taken into account in the mathematical modelling: generating variables (A_1), generating relationships (A_2), connecting relationships (A_3) and finally building a model (A_4). As it was already noticed, the construction of a model of a problem situation constituted the most difficult task for many students because it required a high level of abstraction characterising the algebraic reasoning. According to the nature of the problem, three ways were used by the students, namely arithmetical reasoning (A_{41}), algebraic reasoning (A_{42}) and pictorial representation (A_{43}). The number of students, given in percentages, who performed successfully the two tests is reported in Table 6.1 (p. 56) according to the four mentioned steps A, B, C and D. Similar items in mathematical content domains (linear relationships, geometric context and Venn diagram representation) were grouped for the pre-test and the post-test.

The analysis of the students' results for the two problems "Number of cows" and "Age of Susan's mother" indicated that students had a low performance in the post-test in all three classes when required to build models (36%, 27% and 8%) but there is a substantial improvement in their systematic reasoning with respect to the different steps in the problem solving process. In fact, in the pre-test the problem seemed easy to solve and required little mathematical background, and the linguistic presentation was so clear that it became easy for many students (87% in class C_1) to determine which operation to apply in order to get a reasonable model. "If I had 24 cows more, I would have three times my herd".

This was simply translated by the equation: $x + 24 = 3x$. In the post-test, they had to understand and to interpret correctly the semantic structure of the problem before they could formulate an appropriate model.

"I am 24 years older than Susan and 5 years from now, she will be a half of the age I am now". This problem allowed them to generate a variety of models:

$$(x - 24) + 5 = \frac{x}{2}$$

$$x + 24 = 2(x + 5)$$

$$\frac{x}{2} + 24 - 5 = x$$

This is a proof that a few of them benefited from the problem solving process. The others failed to construct correct models but improved in solving linear equations (57%, 37% and 39%), and in verifying and interpreting their answers (79%, 67% and 62%). It is noticeable that although teachers had stressed the importance of self-control in mathematical thinking during the intervention, the two steps of verification and interpretation of solutions were often confused by students. Some of them thought that they had finished a problem when they had written a simpler equation like $x = 38$ and neglected to check or to validate their answers.

For the problems taken in a geometric context "Perimeter of a window", "Radius of a semi-circle" and "Area of a door", the students tended, successfully in class C_1 (79%), poorly for the others (34% and 12%), to refer to their mathematical knowledge in geometry in order to represent the problem situation. The solution was explicitly deduced by the application of geometric formulas. At this level, they showed that they did not master the concept of the perimeter of a specific object which is different from classical geometric figures (square, rectangle, circle and so on). A lack of a-2 dimension plane visualisation was also evident. In fact, for them, the perimeter of the window was equal to the sum of the perimeter of both the rectangle and the semi-circle, they forgot to subtract the inner side and this explained why they performed poorly in the pre-test. In the post-test, the results revealed a little improvement in formulating models symbolically (7%, 3% and 4%) and pictorially (32%, 73% and 54%), but their performance in solving these models was low (11%, 13% and 12%) because they faced many difficulties when manipulating fractions and decimal numbers in the equation: $4x + x + \frac{3.14x}{2} = 6.57$. Once again, the interpretation of the results (71%, 46% and 42%) was considered without previously checking if the answer was plausible. This was essentially due to complicated calculations they had to carry out in solving wrong models.

Looking at the problems related to Venn diagram representation "Meat-lovers" and "Participants in a conference", the results obtained in the pre-test and the post-test indicated that students used three ways to build models but had a tendency to use numbers (45%, 22% and 19% in the pre-test; 25%, 7% and 3% in the post-test) instead of symbols (12%, 6% and 0% in the pre-test; 11%, 7% and 10% in the post-test) or diagrams (9%, 0% and 0% in the pre-test; 32%, 0% and 27% in the post-test), as was expected. The solutions were easily deduced from models or by arithmetical reasoning in the post-test (68%, 53% and 60%) and a remarkable improvement in checking and validating the solutions (93%, 83% and 87%) was noticeable.

This meant that mathematics teaching using the problem solving approach has had a significant effect on the promotion of students' thinking and metacognitive abilities. But it should be recognised that some students still showed lower proficiency in identifying the mathematical structure hidden in a problem situation for building correct models, an indispensable step in the problem solving process. They often tended to use non-algebraic strategies, generally inefficient in solving complex problems.

STUDENTS' PERFORMANCE STEP-BY-STEP AND ITEM-BY-ITEM IN PERCENTAGE

CATEGORIES / PROBLEMS		A						B	C	D
		A_1	A_2	A_3	A_4					
					A_{41}	A_{42}	A_{43}			
PRE-TEST Number of cows	C_1	93	93	93	-	87	-	75	48	81
	C_2	69	66	63	-	37	-	31	-	37
	C_3	54	54	54	-	27	-	42	11	34
POST-TEST Age of Susan's mother	C_1	89	89	82	-	36	-	57	21	79
	C_2	87	80	77	-	27	-	37	33	67
	C_3	100	100	88	-	8	-	39	12	62
PRE-TEST Perimeter of a window	C_1	60	60	48	24	-	79	24	-	66
	C_2	34	34	7	-	-	34	-	-	9
	C_3	15	15	12	-	-	12	-	-	27
Radius of a semi-circle	C_1	27	27	27	6	-	79	12	-	6
	C_2	6	6	6	-	-	34	6	-	-
	C_3	4	4	4	-	-	12	23	-	31
POST-TEST Area of a door	C_1	82	82	82	-	7	32	11	4	71
	C_2	30	30	30	-	3	73	13	-	46
	C_3	88	84	81	-	4	54	12	-	42
PRE-TEST Meat-lovers and vegetarians	C_1	12	12	12	45	12	9	63	-	58
	C_2	37	37	37	22	6	-	28	-	25
	C_3	19	19	19	19	-	-	30	4	38
POST-TEST Participants in a conference	C_1	11	11	11	25	11	32	68	11	93
	C_2	50	50	50	7	7	-	53	17	83
	C_3	27	27	27	3	10	27	60	8	87

Table 6.1

Observations: *1. In Table 6.1, all values are expressed in percentages (%).*

2. Meaning of abbreviations:

A: Formulation of a mathematical model

A_1 : *Generating variables*

A_2 : *Generating relationships*

A_3 : *Connecting relationships*

A_4 : *Building a mathematical model*

A_{41} : *Using numbers*

A_{42} : *Using symbols*

A_{43} : *Using pictures*

B: Solving a model

C: Verifying the solution

D: Interpreting the answer

6.2 FROM ARITHMETIC TO ALGEBRAIC REASONING

During the administration of the two tests no specific indication was given to students on how to solve problems but the analysis of their worksheets showed that they adopted a variety of methods. For instance, "Meat-lovers" and "Participants in a conference" were categorised as "Venn diagram representation" problems, and their solutions require the representation by means of diagrams and the deduction of algebraic equations, for example: $34 + 31 = 40 + x$ for the first problem, and $(59 - x) + x + (45 - x) = 84$ for the second. Surprisingly, in the pre-test, a large proportion of students (45%, 22% and 19%) referred to arithmetic reasoning to solve the "Meat-lovers" problem. During the intervention, this way of reasoning was particularly exhibited when students solved the "Mathematics competition" problem. They deflected from algebraic reasoning previously used in various problem situations to their successful method of "false assumption". In solving this problem, some students even adopted a combination of two ways of reasoning leading to the following system of linear equations: $5x - 0y = 50$ and $5x - 7y = 36$. In listening to their explanations, it was possible to understand that they thought correctly in arithmetic but used algebraic symbolism incorrectly. In the post-test, some students (25%, 7% and 3%) persisted in using arithmetic reasoning in solving the "Participants in a conference" problem.

Stacey and MacGregor (2000: 150-151) explained that this phenomenon was due to "the cognitive discontinuities involved in the shift from using methods of arithmetic to algebraic reasoning". In fact, the move from arithmetic into algebraic reasoning implied three major difficulties in the students' mind. The first one was a change from calculating with numbers to operating with unknowns, described as a "cognitive gap" (Herscovics & Linchevski, 1994) or a "didactic cut" between arithmetic and algebra (Filloy & Rojano, 1989), a difficulty observed for instance when students were attempting to use arithmetic reasoning in solving equations with variables on both sides.

The second difficulty came from the distinction made between two ways of interpreting an algebraic expression such as $x - 4$ as a procedure or operation, "subtracting 4 from x", or as an object in its own, "the result obtained in performing the subtraction", or a concept in a mathematical structure (Kieran, 1992; Sfard, 1991).

It should be noted that for many researchers, procedural thinking was associated with arithmetic operations and structural thinking with algebra.

The third difficulty was due to the ways in which problems were solved using arithmetic (involving calculations with numbers), and algebra (requiring manipulations or operations on symbols). For Stacey and MacGregor (2000), students have difficulties in reconciling algebraic and arithmetic approaches to solve problems. These students believed that problems could be solved by direct calculation and the "compulsion to calculate" prevented them from attempting an algebraic approach in favour of logical arithmetic reasoning.

It is important to note that students had a tendency to solve easy problems by arithmetic reasoning because this kind of reasoning was greatly influenced by their perception of the structure of the problem situation. Unfortunately, it became inefficient when the problem complexity increased. This reasoning should help to illustrate strategies essential to tackle problematic situations and to facilitate students' acquisition of formal algebraic reasoning indispensable to their advanced mathematics learning.

Teachers should help their students to build their reasoning on arithmetical strategies in order to be gradually familiarised with procedures and the basic concepts of algebra.

Students should have opportunities to work on problems that cannot be solved without algebra so that they can feel that they are doing algebra in symbolising contextualised problem situations, in manipulating symbolic expressions and in performing with fluency the equivalent equation method, if it is required, during the problem solving process.

6.3 TEACHERS' BELIEFS ABOUT MATHEMATICS TEACHING AND LEARNING

In this study, it was expected that substantial change in mathematics teaching and learning would require improvement in teachers' perceptions of the nature of mathematics, the learning of mathematics and their mathematics teaching strategies (see pp. 46-48). In fact, teachers' beliefs greatly influence the learning environment, which in turn could motivate or inhibit students' involvement during the learning process. In order to consider the impact of these important factors in mathematics teaching and learning, a questionnaire was administered to 23 mathematics teachers: the three participating teachers and 20 other in-service teachers with a rich experience in mathematics teaching. The teachers' points of view were useful to have a global vision on how mathematics as a subject is perceived, how it is taught and learned in school mathematics in order to make sound recommendations for curriculum improvement.

The data were collected and organised using tabular form consisting of a list of statements to which the teachers were asked to answer whether they agreed on a five point scale: No, I strongly disagree (1); no, I disagree (2); I cannot say anything (3); yes, I agree (4) and Yes, I strongly agree (5). Conventionally, the "Yes" responses (Yes and yes) and similarly, the "No" responses (No and no) were added together in the analysis but specific data for "yes" and "no" responses were given in brackets as a reference. Additionally, teachers were asked to write a few sentences describing their opinions about their experience in teaching mathematics, the mathematics instruction they have received and the ways in which they thought they would prepare their students to meet new societal purposes. Their beliefs were grouped under three headings: the nature of mathematics, the teaching strategies and the mathematics learning (Table 6.2, Table 6.3 and Table 6.4, pp. 62-63).

The data indicated that nearly all the teachers (96%) declared that mathematics does not involve memorising and following rules, neither does it consist in doing computational exercises (78%) nor does it have little to do with the real world (74%), points of view totally different from common societal beliefs, and sources of students' negative attitudes towards mathematics. However, the teachers fully (82%) agreed that students dislike mathematics because they are unable to obtain good results and this poor performance in mathematics often creates students' anxiety (87%); some of them (65%) recognised that students have experienced negative situations with their mathematics teacher or have a fear of mathematics. A reasonable proportion of them (65%) advocated that this belief was due to poor support from teachers but disagreed (56%) that mathematics is particularly difficult or irrelevant for many students. They strongly (92%) affirmed that students are motivated to continue to learn mathematics if they have a good teacher, if they have good results in mathematics or if they envisage an interesting career (87%) at the end of their studies.

According to their perceptions of their teaching strategies, many teachers (78%) emphasised the constructivist view of learning as the basis of mathematics teaching while many (66%) also declared that mathematics teaching must be focused on the teacher's competence; fewer of them (56%) emphasised mastery of rules and procedures. They were totally (100%) aware that students' mistakes are valuable opportunities for them to adjust their teaching strategies and that classroom activities should be focused on students' specific needs in mathematical knowledge. For a large proportion (74%), it is important to pose suitable tasks, to encourage and to facilitate classroom discussions, to regularly assess students' understanding and to sort them into ability groups (69%) on the basis of examination and test results. However, they still maintained that the teacher's role remains that of explainer (60%), transmitter of knowledge (57%) and most (83%) considered getting correct answers to be the most important goal in mathematics teaching.

As regards the learning of mathematics, they unanimously (100%) agreed that the construction of students' knowledge occurred through teacher-students and student-student interaction and refuted (65%) the idea of learning based on imitating teachers' instruction. Nearly all teachers (96%) were convinced that students learn more if they deal with everyday life problems or if they work in groups with others (83%). Although a large majority of teachers (70%) recognised that mathematics problems can be solved in several ways, a few of them (48%) did not accept that recalling or using the correct rule necessarily means to know mathematics.

These various points of view indicated that teachers were willing to change their teaching practices and seemed committed to make their students learn in a more meaningful way.

A.Teachers' beliefs about mathematics subject

Beliefs	**Yes**	**yes**	**No**	**no**	**-**
Mathematics learned at school has little or nothing to do with the real world	13	(9)	74	(22)	13
Mathematics is nothing more than doing computational exercises	13	(4)	78	(35)	9
Mathematics involves memorising and following rules	-	-	96	(65)	4
Students dislike mathematics because of:					
- negative situation with teachers	65	(43)	26	(22)	9
- poor support from teachers	57	(35)	39	(26)	4
- lack of ability to obtain good results	82	(43)	13	(9)	4
- difficulty and non relevance with the real world	31	(22)	56	(17)	13
- fear of mathematics	65	(26)	26	(13)	9
Students continue to learn mathematics because of:					
- good teacher	92	(48)	9	(9)	-
- good results in mathematics	87	(48)	8	(4)	4
- interesting career	87	(30)	9	(9)	4
- enjoying mathematics	52	(22)	17	(17)	30
Mathematics seems boring and useless	43	(39)	47	(17)	9
Poor performance in mathematics creates students' anxiety	87	(57)	9	-	4

Table 6. 2

All values in Table 6. 2, Table 6. 3 and Table 6. 4 are expressed in percentages (%).

B. Teachers' beliefs about mathematics teaching strategies

Beliefs	Yes	yes	No	no	-
Mathematics must be focused on:					
- students' performance and mastery of mathematics rules and procedures	44	(35)	56	(43)	-
- students' constructions of mathematical knowledge	78	(35)	21	(17)	
- teacher's competence	66	(35)	34	(17)	
Ways to teach mathematics:					
- to explain clearly important concepts	60	(17)	30	(26)	9
- to orientate students to correct answers	83	(48)	13	(9)	4
A good teacher poses suitable problems and facilitates discussions	74	(43)	22	(9)	4
Mathematics concepts must be transmitted by the teacher to students	57	(35)	43	(26)	-
Students' mistakes are precious situations to learn from	100	(35)	-	-	-
Giving different activities according to different students' needs	96	(22)	-	-	4
Sorting students into ability groups by exam and test results	69	(26)	13	(4)	17
Necessity to create opportunities for assessing students' understanding	74	(39)	8	(4)	7

Table 6. 3

C.Teachers' beliefs about mathematics learning

Beliefs	Yes	yes	No	no	-
Students construct their knowledge through interaction teacher-students and students-students	100	(48)	-	-	-
Students learn more by working in groups with others	83	(31)	13	(9)	4
Context and collaboration are important for effective learning	61	(35)	13	(13)	26
Learning is imitating all the teacher's instructions	30	(17)	65	(52)	4
Mathematics problems can be solved in several ways	70	(22)	26	(26)	4
Students are motivated if they deal with everyday life problems	96	(39)	-	-	4
Knowing mathematics means to recall and to use correct rule	35	(13)	48	(31)	17

Table 6. 4

The teachers' comments and opinions were summarised and classified into three themes: their experience in teaching mathematics, the instruction they have received in mathematics and some suggestions helpful to prepare their students adequately to live in a modern society.

Referring to their experience in teaching mathematics, most of them deplored the fact that the current mathematics curriculum is so overloaded and inappropriate to their students' level that they had to work under pressure to cover the prescribed content and do not have sufficient time to prepare and to plan their lessons effectively, and to help their students to do mathematics with understanding. As a result, students learn mathematical rules and procedures that enable them to pass their examinations and after that they immediately forget what they have learned. When they face a complex problem they do not make efforts to grapple with it; they immediately give up if they have no help from their teacher. Their repeated failures often generate negative attitudes towards mathematics.

Some teachers attempted to adapt their mathematics teaching to their students' level and finally, they realised after some years that their own mathematical knowledge was practically reduced to what they teach. Frequent change in the mathematics curriculum obliged some of them to teach what they have not learned and mathematics teaching became a very hard task for them.

One teacher said:

> *I taught what I have not learned so that I have many problems to communicate it.*

For the same problem, another teacher declared:

> *Sometimes, I was obliged to teach what I have not learned. So, I worked hard in trying to learn it by myself. For being confident in my teaching, I solved all problems before I presented them to my students. My preparations took me much time.*

To overcome these pedagogical difficulties, they requested refresher mathematics courses and much support in didactical material. As practical advice conducive to good mathematics teaching, experienced teachers proposed to become closer to their students, to correct their mistakes, to encourage them to work together, to allow and to facilitate mathematics discussions between students and to demystify the mathematics as a subject in their students' minds.

Many teachers realised that they had a good preparation in mathematics especially at secondary school level, which has allowed them to continue their studies at university level. However, those who experienced mathematics instruction based on memorising definitions, theorems and formulas with abusive use of symbols, long and complicated demonstrations, and bad assessment methods affirmed that their success was due to their own efforts:

> *Every time I read mathematics books, I used to do many exercises with other students and this helped me to acquire a good habit of working in mathematics.*

They deplored that they had no opportunity to develop their creative thinking in mathematics lessons; they had to follow their teachers' explanations and reasoning:

> *We have learned to calculate, to master formulas not to reason logically and to think creatively. Our teachers never insisted on the utility of mathematics.*

They greatly criticised that mathematics as a subject constituted a gatekeeper for students towards access to higher levels of education. They condemned mathematics teachers who intimidated and discouraged their students in telling them that mathematics was difficult. Many students gave up learning mathematics and disliked it during their studies because they had no support from their teachers.

In order to prepare their students to meet new societal and technological requirements with confidence, they suggested reviewing the current mathematics curriculum and to make it more realistic, not extensive but adapted to new and specific students' needs. Regular meetings and workshops should be helpful for many teachers for improving their mathematical knowledge and pedagogical qualities. As a result, they should be able to stimulate young students to like mathematics, to be interested in doing mathematics and to persevere in their tertiary education for being good scientists.

Teachers should adopt new teaching methods which facilitate students' learning with understanding, allow them to "discover" mathematical concepts by themselves and to be aware of their applications in various technological areas. Scientific information should be disseminated by national media and a solid link of mathematics with other scientific subjects should be established. Students and parents should be sensitised on the importance of mathematics in everyday life in order to stimulate more young people to pursue scientific and technological studies.

6.4 IMPLICATIONS FOR MATHEMATICS TEACHING AND LEARNING

A deeper analysis of the data, showed that students encountered many difficulties when they tried to deal with the abstract concepts of algebra during the problem solving process. Many of them referred to their previous arithmetic strategies to solve algebra problems but they often failed to construct adequate mathematical models and were unable to pursue the whole mathematical reasoning process. As learning mathematics requires students to understand concepts, it appears very difficult for many students to move easily from a procedural to a structural understanding. The hard task for a mathematics teacher becomes mainly to bridge the gap between the students' procedural thinking based on arithmetic strategies and their structural thinking characterising algebraic methods.

To overcome these difficulties, it is advisable to help students to build their mathematical thinking on their prior arithmetic reasoning and to familiarise them gradually with the basic concepts of algebra through solving non-routine algebra problems. Teachers should assist their students carefully in their intellectual effort towards an algebraic symbolism.

Students should be presented with stimulating and challenging problems which "provoke" their thinking and engage them in the problem solving activity. Selected problems might closely resemble real situations in their richness and their complexity so that students would gain new experience in solving them. It is what they learn in struggling with a problem that is important in the problem solving process and not necessarily getting a right answer. Teachers should create a classroom atmosphere which promotes persistent effort and favourable attitudes towards mathematics learning. In such an environment, students should be encouraged to explore problems spontaneously, using several ways of thinking.

In order to improve their students' thinking skills, teachers need to integrate opportunities for increasing critical and creative thinking into lessons especially in promoting collaborative learning. In fact, when students work together, they develop their critical thinking through discussion, explanation and justification of their points of view, consideration and evaluation of others' ideas. The classroom should be a forum for discussion where students are allowed to argue their mathematical thinking and the role of the teacher will not be to transmit well-elaborated mathematical knowledge but to serve as a moderator of classroom discussions and a mediator of learning.

Students' mistakes should be detected and rapidly cleared up from students' minds otherwise they inhibit the learning process. It is recognised that they constitute important sources of information for teachers in helping them to discover what their students may not understand and where they need assistance. Teachers can modify their teaching strategies and organise the learning material according to their students' needs and progress.

Teachers should identify students' anxiety towards mathematics as early as possible and eliminate its effect on mathematics learning to avoid setting ambiguous and unrealistic goals for students.

CHAPTER SEVEN: CONCLUSIONS AND SUGGESTIONS

7.1 CONCLUSIONS

In undertaking this study, the main concern was to investigate the possibility to introduce in the Rwandan educational system innovative mathematics teaching strategies based on the problem solving approach in order to promote students' learning processes in mathematics.

Significant changes were expected in teachers' practices with positive impact on students' mathematical knowledge and skills.

According to the teachers' comments before and after the classroom intervention, it seems obvious that the participating teachers developed a new vision of mathematics teaching focusing on their students' active involvement in exploring and solving various mathematical problem situations. They made remarkable efforts in adopting new teaching strategies in mathematics, in gradually changing their authoritative role of instructor to the delicate and demanding role of facilitator of students' mathematical constructions and classroom discussions and in organising their mathematics classrooms so that group work and collaborative learning could take place.

Many factors contributed to the impact of the experiment on teachers' practices. In fact, teachers learned and practised some innovative ideas in teaching mathematics and were continuously encouraged to be more confident in their mathematical knowledge. In turn, they showed a growing interest in improving their pedagogical qualities and were fully engaged to be cooperative and devoted to the experiment.

During the classroom intervention, teachers experienced how a safe and pleasant learning environment has positive effect on students' mathematical activity and productivity and realised with excitement that students' mathematical potentialities were often under-utilised.

Through the results in the pre-test and the post-test, it seems difficult to draw conclusions about the improvement of students' problem solving skills during the experiment. One month was not enough for students to internalise new learning experiences and to become familiar with the problem solving process. However, some encouraging indicators of significant progress in their ways of thinking were revealed by the variety and the originality of their strategies, their systematic work and perseverance in solving algebra problems. For many students, the language problem was crucial, it inhibited their mathematics learning process especially when they had to communicate their ideas. Teachers should pay attention of this learning difficulty. As the teacher is the key determinant for improving mathematics teaching and learning in his classroom, his ability to choose problematic and contextualised tasks, to create and to manage meaningful learning opportunities so that all students participate actively in problem solving discussions, is primordial. One can conclude that a problem solving-based approach might be effectively integrated into classroom practice if sufficient follow-up supervision and support was provided by the educational authorities.

From this study, two main questions should be raised:

- Referring to the pressure on teachers to cover an overloaded curriculum, can we hope for lasting changes from the participating teachers in mathematics teaching strategies based on this problem solving approach?
- Is it possible now to try out the problem solving approach on a large scale in second year classes and to envisage, in the near future, its implementation at a national level?

Once again, it is for educational authorities to give the right answer to these important questions. Nevertheless, in accordance with existing research results, any initiative undertaken in order to promote gradual changes in mathematics teaching practices through practical activities, such as regular workshops and pedagogical meetings with in-service teachers, should be welcomed and encouraged by the authority in waiting to rethink deeply on a more appropriate mathematics curriculum improvement.

7.2 SUGGESTIONS

It is universally recognised that the prosperity of a nation depends greatly on the quality of its educational system. Without pretending to advocate the teachers' claims indicated in their comments, it seems urgent to review the current mathematics curriculum in Rwanda in order to meet new societal needs in this technological age. With the increasing use of various technological aids, people no longer need the computational skills required for every educated person many years ago. Modern society is so technologically oriented that people need to estimate, to communicate mathematically, to have flexible reasoning abilities and to see more than one way to solve real-life problems. Nowadays, people work together and take time in thinking and solving problems, so it is of great importance to build better mathematics programmes that can foster students' problem solving skills. A particular emphasis should be put on students' thinking and understanding rather than rote learning.

The mathematics curriculum should not be too extensive but sound, more realistic and accessible to all students. It should allow teachers to easily practise new teaching methods that foster students' independent learning. Teachers should incorporate non-routine problems taken from various sources and not necessarily in official mathematics textbooks in their mathematics lessons. They should be encouraged to build themselves a resource of problems of interest to their students but related to well-defined curriculum design.

Refresher mathematics courses should be organised in order to empower teachers and to enable them to accomplish their new and demanding task efficiently. Regular meetings between mathematics teachers in the same school or different schools should be helpful to share their teaching experience and to work together as a team.

Students should have opportunities to invent and to create strategies, to sharpen their thinking and to release their intellectual potentialities when they are faced with problematic situations. Student teachers should be progressively familiarised with innovative teaching methods in such a way that, later on, they should be able to implement them with facility. Parents should be concerned in the school programme and be effectively involved in helping teachers to prepare their children to do and to like mathematics. With the educational authority support and the teachers' goodwill to try innovative teaching strategies, there are good reasons to hope to have, in the near future, effective teachers and successful problem solvers in our mathematics classrooms.

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APPENDIX A:

PRE-TEST

PROBLEM RESOURCES

POST-TEST

TEACHER QUESTIONNAIRE

PRE-TEST

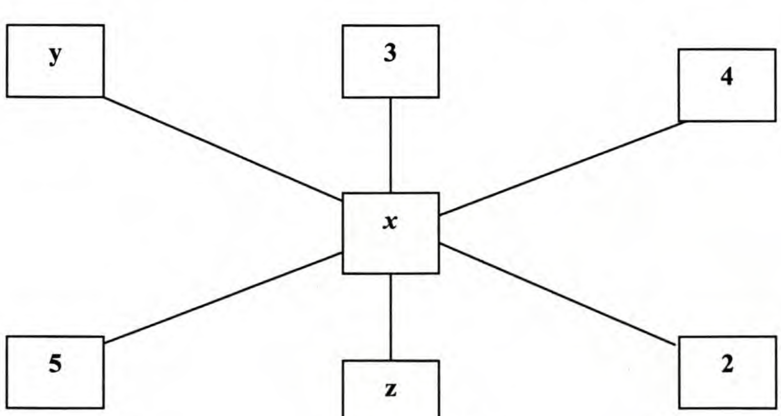
1. A journey in a bus from Butare to Kigali usually takes 2 hours. How much time will it take 3 friends if they are travelling in the same bus from Kigali to Butare?
2. In an algebra lesson, Anita suggested a mathematical operation "*stella*" defined by: $x * y = xy + (x - y)$.

   a. Find $4 * 5$.

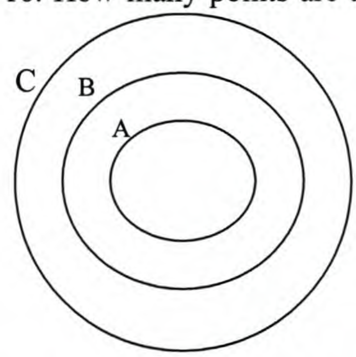
   b. If $4 * x = 4$, then solve this equation.
3. Tony asked to a cowherd how many cows he had. He said: "If I had 24 cows more, I would have three times my herd". Can you help Tony to find how many cows the cowherd had?
4. A window has the shape of a square with a semi-circle attached at the top:
   a. Assume that the side of the square is 1.2 m; find the perimeter of the window.
   b. Assume that the perimeter of another window of the same shape is 9.14 m; find the radius of the semi-circle.
5. In Shema's classroom, there are 40 students. All the students are meat-lovers or vegetarians. If 34 of them are meat-lovers and 31 are vegetarians, how many students are both meat-lovers and vegetarians?

PROBLEM RESOURCES

1. In a bookshop, a notebook and a blue pen cost 120 F. The same notebook and two blue pens cost 150 F. What is the price of a notebook?
2. Teddy was counting her pocket money. "If I had 1500 F more, I would have five times my money and would buy a dress", she said. What is the price of a dress?
3. In a store, a deposit of 2000 F is charged on a bottle rack taken and is refunded when it is returned.
   a. What would be the refund for returning 3 bottle racks?
   b. Describe how the storeowner would determine the amount of refund for any number of returned bottle racks.
   c. Write an equation for the amount of refund according to the number of bottle racks returned.
   d. Solve your equation to find out how many bottle racks have not yet been returned if the storeowner has a deposit of 36000 F.
4. Philip went to a store, spent half of his money and 1000 F more. He went to a second store, spent half of his remaining money and 500 F more. As he had no money left, how much money had he in his pocket just before his shopping?
5. When I open my mathematics book, I see two page numbers. If the sum of the two page numbers is 293, what are the two page numbers?
6. The sum of two consecutive even numbers is 150. What are those numbers?
7. I think of a number, add 27 and divide the sum by 7. My result is the same when I subtract 9 from the original number. What is my number?
8. I think of a number, subtract 8, multiply the difference by 2, divide the result by 4, and add 4 and one half of the original number. Given that the result is equal to the original number, what is my secret number?
9. In a mathematics competition, 5 points were scored for each correct answer and 2 points were deducted from the score for each incorrect answer. How many correct answers did Jimmy get if he worked out 10 problems and scored 36 points?
10. The counting numbers are arranged from 1 to 7 in the squares (below) so that the sum of the numbers along each line is 10. Find the numbers x, y and z.



11. In a dart game, suppose that a, b and c represent the number of points assigned to the target regions A, B and C respectively. The sum of b and c is 11, the sum of a and b is 19 and the sum of a and c is 16. How many points are assigned to the central region A?



12. A bus can transport 60 passengers. It starts out empty from the bus station, picks up some passengers at Nyabugogo, 11 other passengers at Gitarama and the number of all passengers becomes twice at Ruhango. If there are 4 empty seats in the bus, how many passengers has it taken at Nyabugogo?

13. A third of Peter's age two years ago is equal to a fifth of his age in four year's time. What is Peter's age now?

14. Charles walks to his work at 5 km/h and walks to his home at 4 km/h. If the total time taken is 54 minutes, find the distance from his home to his work.

15. A rectangular classroom is twice as long as it is wide. If the perimeter is 43.2 m, how many square tiles of 30 cm on each side are needed to tile the whole classroom?

16. A goat is tied up by a rope to one of the 4 corners of a square barn that measures 10 m on each side. The rope is as long as the side of the square. On how many square metres of land is the goat able to graze?

17. T wo water pipes are used to fill a swimming pool. The first pipe alone takes 6 hours to fill the pool; the second pipe alone takes 4 hours to fill the pool. If the two pipes are opened at the same time, how long will it take all together to fill the pool?

18. At the end of a school year, 30 students planned to visit three touristic regions of the country: Birunga Park, Nyungwe Forest and Akagera Park. Of this group, 16 visited Birunga Park, 14 visited Nyungwe Forest, 11 visited Akagera Park, 5 visited Birunga Park and Akagera Park, 8 visited only Nyungwe Forest, 5 visited only Akagera Park and 7 visited only Birunga Park. How many students visited all three regions?

19. During Botany lessons, students regularly measured the growth of a seedling as shown in the table:

Days	0	2	4	6	8	10	12	14
Height (mm)	0	3	6	9	12	15	18	21

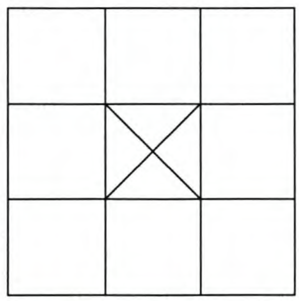
a. What is the height of the seedling:- after 20 days?

- after 30 days?
- after x days?

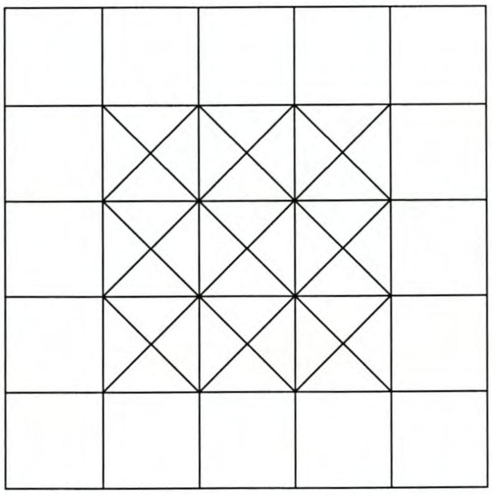
b. In how many days, the seedling will be: - 33 mm high?

- h mm high?

20. The city council of NGOMA decided to make BUTARE CITY attractive by making square gardens surrounded by square tiles as shown below:



1-dimension garden



3-dimension garden

a. How many tiles are needed to make a 4-dimension garden?

b. Complete the table:

Dimension	1	2	3	4	6	12
Number of tiles	8	12	16	20	...	...

c. If 200 tiles are bought, what is the largest garden can they make with these tiles?

d. Describe the formula used to find tiles needed for an n-dimension garden.

e. Find the dimension of the garden made with 400 tiles.

21. Helen built square shapes using matches. The following table shows the number of matches used to build different shapes.

Number of matches	4	7	10	13	16
Shape number	1	2	3	4	5

a. How many matches did she use to build 6 square shapes?

b. How many square shapes did she build with 31 matches?

c. Which of the following equations can be used to find the number of square shapes built with 31 matches? Explain.

$3n + 1 = 31$

$8n - 4 = 31$

$2n + 1 = 31$

$4 + 3(n-1) = 31$

d. How many square shapes did she build with 100 matches?

22. Two students Olivier and Eric are sitting on the two extremities of a rigid metal bar, which is 2 m long such that they balance the bar. If their masses are directly proportional to their respective ages, 14 and 16 years, where is the fulcrum of the bar located?

POST- TEST

1. A doctor wanted to know the age of Susan's mother. She told him: "I am 24 years older than Susan and 5 years from now, she will be a half of the age I am now". What is her age?
2. In a conference room, there are 84 participants. The registration form shows the following information: - 59 speak English

   - 45 speak French

   How many participants speak only English?

3. A door has the shape of a rectangle with a semi-circle attached at the top. Assume that the length is twice the width and the perimeter of the door is 6.57 m; find the area of the door.
4. Given two numbers a and b, propose your own mathematical operation between the two numbers. By means of your operation, formulate an equation and show that it works.

TEACHER QUESTIONNAIRE

Important note:

We are trying together to find how to improve mathematics teaching so we need your collaboration.

Answer the following questions as honestly as possible. Any comments on the questionnaire will be much appreciated.

Please note that the correct answer is the one that reflects your true opinion.

A. Please indicate your answer on the response sheet by writing a number (1, 2, 3, 4 or 5) according to the degree to which you agree or disagree with the following statements:

(1) No, I strongly disagree

(2) no, I disagree

(3) I cannot say anything

(4) yes, I agree

(5) Yes, I strongly agree

1. For many teachers, mathematics teaching must mainly be focused on:
   - students' performance and mastery of mathematics rules and procedures
   - students' personal construction of mathematics knowledge
   - teacher's competence
2. The best way to teach mathematics is to explain clearly important concepts and to have students practice them.
3. Mathematics is best learnt when the teacher orientates his students to correct answer.
4. A good teacher is someone who poses problems and facilitates discussion in his classroom.
5. Mathematics concepts must be transmitted by teachers to students.
6. Students' knowledge is constructed through the interaction teacher – students and students-students.
7. Students learn more by working with other students.
8. The context and collaboration are important for effective learning.
9. Any mathematics task can be solved in several possible ways.
10. For a teacher, students' mistakes are precious situations to learn from.
11. It is necessary to create many opportunities for assessing students' understanding.
12. Examination and test results enable teachers to sort the students into ability groups.
13. Students should be given different activities according to their different needs.

14. Knowing mathematics means to recall and use the correct rule when it is asked by the teacher.
15. The mathematics learnt at school has little or nothing to do with the real world.
16. Learning is imitating all the teacher's instructions.
17. Mathematics is nothing more than doing computational exercises.
18. Students are motivated if they are dealing with problems taken from their everyday life.
19. Mathematics involves memorising and following rules.
20. Mathematics anxiety is related to poor performance in mathematics.
21. For many students, mathematics seems boring and useless.
22. Many students dislike mathematics because:
    - they have experienced a negative situation with a teacher.
    - they are unable to obtain good results.
    - it is so difficult and has no relevance to the real world for them.
    - they have poor support from teachers.
    - they have a fear of mathematics.
23. Some students continue to study mathematics because:
    - they have had a good teacher.
    - they obtain good results in mathematics.
    - they aim at interesting careers that require mathematics.
    - they enjoy mathematics and it is challenging for them.

B. Write a few sentences about your own experience as a mathematics teacher.

C. Make some remarks or comments on the teaching you have received in mathematics.

D. Suggest some guidance required for a good mathematics teacher who wants to prepare his students to meet the new societal and technological requirements.

APPENDIX B:

DATA COLLECTION

PRE-TEST

POST-TEST

RESULTS OF THE PRE - TEST

Categories / Problems		Correct answer with explanation	Correct answer without explanation	Wrong answer with explanation (1)	Wrong answer without explanation	Total	Observations
P_1 Travelling in the same bus	C_1	82	9	6	3	100	(1): 2h + 2h = 4h (two ways) 2h x 3= 6h (each 2h)
	C_2	22	31	47	-	100	(1): 2h x 2 = 4h 2h x 3 = 6h 2h $+ x$ = 2h $3(2 + x) = 6 + 3x$
	C_3	46	8	42	4	100	(1): 2h x 3 = 6h 2h x 2 = 4h 4h x 2 = 8h $3x + x$ = 2h

OBSERVATIONS: All the results in the pre-test and the post-test are expressed in percentages

Categories / Problems		A Formulating a mathematical model											B Solving a model			C Verifying the solution	D Interpreting the answer	Observations
					Building a model													C: Correctly I: Incorrectly W: Wrongly GV: Generating variables GR: Generating relationships CR: Connecting relationships
					Using numbers			Using symbols			Drawing							
		GV	GR	CR	C (1)	I (2)	W (3)	C (4)	I (5)	W (6)	C	W	C	I	W (7)			
P_2 Anita's operation (∗) $(x*y)=x.y+(x-y)$	C_1	-	-	-	87	9	3	18	15	67	-	-	21	-	79	6	15	(1): $4*5=4.5+(4-5)=19$ (2): $4*5=4.5+(4-5)=20-1=21$ $4*5=4.5+(4-5)=9-1=8$ (3): $4*5=9$ (4): $4*x=3x+4=x$ (5): $4*x=3x+4$ (meaning of an equation) (6): $4*x=4x=x$ (∗ = x) $4*x=4+x$ (∗ = +) .$4*x=4-x$ (∗ = -) $x*4=x.4+(x-4)=x$ (2 meanings of x - commutativity of ∗?) (7): CSCS operation sense: If $4*x=x$ then $*x= x/4$ If $4*x=x$ then $*x-x=-4 \Rightarrow * = -4$ If $4*x=x$ then $4*=x-x=0 \Rightarrow x=0$
	C_2	-	-	-	34	28	38	-	10	90	-	-	-	-	10	-	-	(2): $4*5=20+(4-5)=-21$ $4*5=4.5+(4-5)=20+(4-5) = 80-100=-20$ (5): $4*x=x.4+(x-4)=5x-4$ (6):$4*x=4x+(4-x)=4x+3x=x$ $4*x=4x=x$; $4*x=4+x=x$ $4*x=x.x+4.x=x^2+4=4x^2$ (7): If $4*x=x$ then $4x=x$ $x=4$ or $x=3$ If $4*x=x$ then $*x=-4$ or $x-x=4*$
	C_3	-	-	-	50	34	16	-	4	96	-	-			-	-	-	(2): $4*5=-21$; $4*5=-19$; $4*5=-20$; $4*5=21$ (3): $4*5=4.5=20$ $4*5=4-5=-1$ (6): If $4*x=4$ then $4.x=*x$ If $4*x=x$ then $*x-x=-4$ or $*x=2$ S_{313} $4*x=x=x.4+(x-4)=4x+(x-4)=5x-4$ $5x=-4$ thus $x=-4/5$

Categories / Problems		A Formulating a mathematical model											B Solving a model			C Verifying the model	D Interpreting the answer	Observations
		GV	GR	CR	Building a model													
					Using numbers			Using symbols			Drawing							
					C	I	W (1)	C (2)	I	W (3)	C	W	C	I (4)	W			
P_3 Number of cows of a herd	C_1	93	93	93	-	-	7	87	-	7	-	-	75	18	-	48	81	(1): 24.3=72 ; 24:3=8 (2): x+24=3x (3): 24+x=24.3 (x: more) 3(x+24)=x+24 (4): x+24=3x⇒x-3x=24 -2x=24⇒x=12
	C_2	69	66	63	-	-	18	37	3	23	-	-	31	9	25	-	37	(1): 24.3=72 (2): x+24=3x (3): 24+x=x x x x x x+24=x+3x x=24.3 24+3x=3x x.3=24 24+3x=x 24-3x=24+3x
	C_3	54	54	54	-	-	46	27	-	27	-	-	42	11	-	11	34	(1): 24.3=72 (2): x+24=3x (3): 3x=24 (more) (4): -2x=-24 ⇒-x=-24/-2⇒x=12

Categories / Problems		A Formulating a mathematical model											B Solving a model			C Verifying the solution	D Interpreting the answer	Observations
					Building a model													
		GV	GR	CR	Using numbers			Using symbols			Drawing							
					C	I (1)	W (2)	C	I	W	C	W	C	I	W			
P_{4a} Perimeter of a window	C_1	60	60	48	24	45	30	-	-	7	79	-	24	45	30	-	66	(1): concept of perimeter 4th side (2): computational errors π =13.4 using wrong formulas P=3c+3.14c
	C_2	34	34	7	-	9	91	-	-	-	34	16	-	9	91	-	9	(1): concept of perimeter 4th side (2): using wrong formulas
	C_3	15	15	12	-	23	77	-	-	-	12	-	-	23	77	-	27	(1): concept of perimeter 4th side (2): using wrong formulas

Categories / Problems		A Formulating a mathematical model											B Solving a model			C Verifying the solution	D Interpreting the answer	Observations
					Building a model													
					Using numbers			Using symbols			Drawing							
		GV	GR	CR	C	I	W	C (1)	I (2)	W (3)	C	W	C	I	W (3)			
P_{4b} Radius of a semi - circle	C_1	27	27	27	-	-	73	6	21	-	79	-	12	-	15	-	6	(1): $3x+(.3.14x):2=9.14$ $2x3.14x):2+2x3x=9.14$ (2): $3x+3.14x=9.14$ $4c+3.14c=9{,}14$ (3): decimal as coefficient of variable
	C_2	6	6	6	-	-	94	-	-	6	34	16	6	-	-	-	-	(2): using wrong formulas $r=\frac{p}{6.28}$ $r=\frac{3.14p}{2}$
	C_3	4	4	4	-	-	96	-	-	65	12	-	23	-	42	-	31	(2): using wrong formulas $r=\frac{p}{6.28}$ $r=p:8$ (square only) $x+x+x+1/2x=9.14$

Categories / Problems		A Formulating a mathematical model											B Solving a model			C Verifying the solution	D Interpreting the answer	Observations
		GV	GR	CR	Building a model													
					Using numbers			Using symbols			Drawing							
					C	I	W (2)	C (1)	I	W (2)	C	W	C	I	W			
P_5 Meat – lovers and vegetarians	C_1	12	12	12	45	-	27	12	-	-	9	6	63	-	36	-	58	(1): $x+6=31$ $34+31-x=40$ $34+31=40+x$ $x=34+31-40$ (2): Wrong interpretation of words: both-and- or
	C_2	37	37	37	22	-	41	6	-	31	-	-	28	-	72	-	25	(1): $34+31=40+x$ (2): $x\,(31+34)=40$ $31+34+x=40+34+31+x$ Wrong interpretation of words
	C_3	19	19	19	19	-	62	-	-	19	-	-	30	-	70	4	38	(2): $x+34-31=40$ $40x=34+31$ $2x=40$ $x+34+31=40$ Wrong interpretation of words

RESULTS OF THE POST-TEST

Categories / Problems		A Formulating of a mathematical model												B Solving a model			C Verifying the solution	D Interpreting the answer	Observations
		GV	GR	CR	Building a model														
					Using numbers			Using symbols			Drawing								
					C	I	W	C (1)	I	W (2)	C	W		C	I	W (3)			
P_1 Age of Susan's mother	C_1	89	89	82	-	-	11	36	46	-	-	-		57	21	4	21	79	(1): x-24+5=x/2 (x+24)/2=x+5 x+24=2(x+5) x-24=x/2-5 (2): x-24+5=(x+5)/2 x-24+5= (x+24)/2 x- (24+5)=x- (24-5) (x+24)/5=x/2 (more) (x+24)+5=2(x+5) (now) (24+x)=-5+x/2 x+24=x/2+5 x+24=5x/2 x+5=(x+24+5)/2 (now)

C_2	87	80	77	-	-			13	27	27	23	-	-	37	30	17	33	67	(1): x-24+5=x/2 (x+24):2=x+5 x+24=2(x+5) x/2+24-5=x (2): 24+x=5x x+24+5=x/2 24+x-5=x/2 2(x-24)=x+24+5 2(x+24)=5+x (3): 5+x=12+x/2 5+x=24+2x Sign (CSCS)
C_3	100	100	88	-	-			-	8	79	23	-	-	39	27	34	12	62	(1): x+5=(x+24)/2 x-24+5=x/2 (2): x+24=5x/2 x+24=(x+24)/2+5 x+24=x/2+5 x+24+5=x/2 x+24-5=x/2 x+24=5x/2 x+24=(x+5)/2 (3): x-24+5=x/2 ⇒24+5=x/2+x

Categories / Problems		A Formulating a mathematical model											B Solving a model			C Verifying the solution	D Interpreting the answer	Observations
		GV	GR	CR	Building a model													
					Using numbers			Using symbols			Drawing							
					C	I	W	C (1)	I	W (2)	C	W	C	I	W			
P2 Participants in a conference	C_1	11	11	11	25	21	7	11	-	-	32	4	68	21	11	11	93	(1): $(59-x)+(45-x)+x=84$ $84=(59+45)-x$ (2): Wrong interpretation of words
	C_2	50	50	50	7	26	17	7	30	13	-	-	53	-	47	17	83	(1): $(59-x)+45=84$ $x+45=84$ (2): Wrong interpretation of words $x+84=59+45$ $x=(45+59)-84$ $(59-x)+(45-x)=84$
	C_3	27	27	27	3	10	3	10	10	7	27	17	60	27	13	8	87	(1): $x+39+25=84$ $(59-x)+x+(45-x)=84$ (2): Wrong interpretation of words $x+59+45=84$ $84-x=59$

Categories / Problems		A Formulating a mathematical model											B Solving a model			C Verifying the solution	D Interpreting the answer	Observations
		GV	GR	CR	Building a model													
					Using numbers			Using symbols			Drawing							
					C	I	W	C (1)	I (2)	W	C	W	C	I	W			
P_3 Area of a door	C_1	82	82	82	-	-	18	7	68	25	32	4	11	46	43	4	71	(1): $4x+x+(3.14x)/2=6.57$ $(2l+l)\,2+(3.14l)/2=6.57$ (2): $(x+2x)2=6.57$ $6l=6.57$ $5l=6.57$ concept of perimeter errors in formulas visualisation in a 2- dimension plane
	C_2	30	30	30	-	10	60	3	20	7	73	7	13	17	30	-	46	(1): $5x+1.57x=6.57$ (2): $p=l.6$ $p=lxlxlxlxl=5l$ $2(2x+x)=6,57$ $2x+x=6,57$ concept of perimeter
	C_3	88	84	81	-	7	27	4	35	31	54	4	12	15	38	-	42	(1): $2x+4x+1,57x-x=6.57$ (2): $(x/2+x)\,2=6.57$ $x+2x=3.285$ $(2x+x)\,2=6.57$ $x+5z=6.57$ $x+4x=6.57$ $x+2x+2+3.14x=6.57$ $(L=x+2)$ $6.57-6x=(6.28x)/2$ concept of perimeter