Subdivision, Interpolation and Splines

by

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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and has not previously, in its entirety or in part, been submitted at any university for a degree.

Summary

In this thesis we study the underlying mathematical principles of stationary subdivision, which can be regarded as an iterative recursion scheme for the generation of smooth curves and surfaces in computer graphics. An important tool for our work is Fourier analysis, from which we state some standard results, and give the proof of one non-standard result. Next, since cardinal spline functions have strong links with subdivision, we devote a chapter to this subject, proving also that the cardinal B-splines are refinable, and that the corresponding Euler-Frobenius polynomial has a certain zero structure which has important implications in our eventual applications. The concepts of a stationary subdivision scheme and its convergence are then introduced, with as motivating example the de Rahm-Chaikin algorithm. Standard results on convergence and regularity for the case of positive masks are quoted and graphically illustrated.

Next, we introduce the concept of interpolatory stationary subdivision, in which case the limit curve contains all the original control points. We prove a certain set of sufficient conditions on the mask for convergence, at the same time also proving the existence and other salient properties of the associated refinable function. Next, we show how the analysis of a certain Bezout identity leads to the characterisation of a class of symmetric masks which satisfy the abovementioned sufficient conditions. Finally, we show that specific special cases of the Bezout identity yield convergent interpolatory symmetric subdivision schemes which are identical to choosing the corresponding mask coefficients equal to certain point evaluations of, respectively, a fundamental Lagrange interpolation polynomial and a fundamental cardinal spline interpolant. The latter procedure, which is known as the Deslauriers-Dubuc subdivision scheme in the case of a polynomial interpolant, has received attention in recent work, and our approach provides a convergence result for such schemes in a more general framework.

Throughout the thesis, numerical illustrations of our results are provided by means of graphs.

Opsomming

In hierdie tesis ondersoek ons die onderliggende wiskundige beginsels van stasionêre onderverdeling, wat beskou kan word as 'n iteratiewe rekursiewe skema vir die generering van gladde krommes en oppervlakke in rekenaargrafika. 'n Belangrike stuk gereedskap vir ons werk is Fourieranalise, waaruit ons sekere standaardresultate formuleer, en die bewys gee van een nie-standaard resultaat. Daarna, aangesien kardinale latfunksies sterk bande het met onderverdeling, wy ons 'n hoofstuk aan hierdie onderwerp, waarin ons ook bewys dat die kardinale B-latfunksies verfynbaar is, en dat die ooreenkomstige Euler-Frobenius polinoom 'n sekere nulpuntstruktuur het wat belangrike implikasies het in ons uiteindelike toepassings. Die konsepte van 'n stasionêre onderverdelingskema en die konvergensie daarvan word dan bekendgestel, met as motiverende voorbeeld die de Rahm-Chaikin algoritme. Standaardresultate oor konvergensie en regulariteit vir die geval van positiewe maskers word aangehaal en grafies geïllustreer.

Vervolgens stel ons die konsep van interpolerende stasionêre onderverdeling bekend, in welke geval die limietkromme al die oorspronklike kontrolepunte bevat. Ons bewys 'n sekere versameling van voldoende voorwaardes op die masker vir konvergensie, en bewys terselfdertyd die bestaan en ander toepaslike eienskappe van die ge-assosieerde verfynbare funksie. Daarna wys ons hoedat die analise van 'n sekere Bezout identiteit lei tot die karakterisering van 'n klas simmetriese maskers wat die bovermelde voldoende voorwaardes bevredig. Laastens wys ons dat spesifieke spesiale gevalle van die Bezout identiteit konvergente interpolerende simmetriese onderverdelingskemas lewer wat identies is daaraan om die ooreenkomstige maskerkoëffisiente gelyk aan sekere puntevaluasies van, onderskeidelik, 'n fundamentele Lagrange interpolasiepolinoom en 'n kardinale latfunksie-interpolant te kies. Laasgenoemde prosedure, wat bekend staan as die Deslauriers-Dubuc onderverdelingskema in die geval van 'n polinoominterpolant, het aandag ontvang in onlangse werk, en ons benadering verskaf 'n konvergensieresultaat vir sulke skemas in 'n meer algemene raamwerk.

Deurgaans in die tesis word numeriese illustrasies van ons resultate met behulp van grafieke verskaf.

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Contents

Preface

Subdivision is a powerful paradigm for the generation of curves and surfaces in computer graphics. Since many real-world objects are inherently smooth, and since much of computer graphics involves modelling the real world, curves and surfaces which are smooth often need to be generated in computer graphics applications. For example, computer-aided design (CAD), high-quality character fonts, data plots, and artists' sketches all contain smooth curves and surfaces. Similarly, the path of a camera or object in an animation sequence is almost always smooth, and a path through intensity or colour space must often be smooth.

Subdivision methods constitute a large class of recursive schemes for computing smooth curves and surfaces for computer graphics. These methods are easy to implement, computationally efficient, and useful in a variety of applications because of their connection with multiresolutional analysis. For an extensive treatment of computer graphics, and in particular including also subdivision methods, we refer to [18, Chapter 11].

Our emphasis in this thesis is a study of the basic mathematical principles underlying stationary subdivision algorithms which are used for generating smooth plane curves. Pioneer work in this area was the method first proposed in 1956 by de Rahm [8], and later, in 1974, popularised by Chaikin [2] because of its rapid convergence. This de Rahm-Chaikin algorithm also provides us with an instructive introductory example of a stationary subdivision algorithm, and is explained and analysed in Chapter 3.1 below. As is clear from this example, the basic idea of stationary subdivision is to start off with an initial set of given data points in the plane, called the initial control points. A particular subdivision algorithm then consists of computing a "denser" data set, called the new control points, by means of a rule which calculates each new control point as a linear combination of the initial control points. If the rule is well-chosen, as is the case with the de Rahm-Chaikin algorithm, the ever more dense data sets (or control point sets) obtained by repeatedly using the same algorithm, approach some smooth curve in the limit. Thus stationary subdivision can be regarded as an iterative refinement process which converges to a smooth curve. For a more general setting, including in particular multidimensional subdivision schemes for the generation of smooth surfaces, we refer to [24], [30] and other sources cited there.

Despite the simplicity of stationary subdivision schemes themselves, the analysis of the associated limiting curves can be a formidable task. Whereas the de Rahm-Chaikin algorithm was later shown by Riesenfeld [27] (see also our Chapter 3.2) to converge to a quadratic cardinal spline function, which can be expressed explicitly in terms of smoothly joined quadratic polynomial pieces, alternative (equally simple) stationary subdivision algorithms often produce limit curves which are not expressible in analytical form, and which can even exhibit fractal features.

An important mathematical tool in our work is Fourier analysis, some fundamental results of which we provide in Chapter 1. Note in particular that, whereas some standard Fourier analysis results are merely stated in Chapter 1, we provide a proof, in Theorem 1.1, of the fact that the Fourier transform is a one-to-one map of a certain restricted function space onto itself, a result which is employed in our later analysis. The analogous standard Fourier analysis result [3, Theorem 2.17] states that the above result holds with our restricted function space replaced by the larger space of square-(Lesbegue) integrable functions on the real line. It is therefore necessary to carefully verify that our result of Theorem 1.1 does indeed hold.

Cardinal spline functions are strongly linked to subdivision, for example in the above mentioned sense of providing the limit curve for the de Rahm-Chaikin algorithm. Hence we proceed in Chapter 2 to define and analyse the concept of cardinal splines. In particular, we introduce the cardinal B-spline basis, and we show that these cardinal B-splines are refinable, which ties up with subdivision in the sense that the limit curve of a convergent subdivision scheme can always be expressed in terms of a refinable function. Moreover, we prove in Theorem 2.5 that the Euler-Frobenius polynomial, which has as its coefficients the integer values of the cardinal B-spline, is a symmetric polynomial with only negative zeros, a property which is crucial for out later applications in Chapter 6.

In Chapter 3 we use the motivating de Rahm-Chaikin example to introduce the general concepts of a stationary subdivision scheme (SSS) and its convergence. Standard convergence and regularity results on schemes with positive masks are quoted from [25], and numerical illustrations are provided.

It is sometimes useful, when applying subdivision techniques, if the limit curve actually contains the initial control points. This situation can be achieved if, at each step of the subdivision algorithm, the new "denser" data set contains all of the previous data set. Such a SSS is called *interpolatory* , and we proceed in Chapter 4 to introduce and analyse such schemes. Indeed, the theoretical core of this thesis is provided by Theorem 4.2, which gives a set of sufficient conditions for the convergence of interpolatory stationary subdivision schemes, as well as by the subsequent Theorem 4.3 in which we characterise, by means of the analysis of a Bezout identity, a class of interpolatory subdivision schemes which satisfy the sufficient conditions of Theorem 4.2. The (unavoidably long) proof of Theorem 4.2 given here is a considerably more detailed version of the one given in [24], whereas Theorem 4.3 and its proof, as developed also in the current research project [16], is partially based on fundamental results proved in [23]. The analysis provided in Chapter 4 culminates in Algorithm 4.4, which shows that, starting with any symmetric polynomial B of even degree and with only negative zeros including minus one, we can explicitly construct a convergent symmetric interpolatory SSS.

In the final Chapters 5 and 6, we proceed to construct specific interpolatory stationary subdivision schemes which are obtained from special choices of the polynomial B in Algorithm 4.4. First, in Chapter 5, we consider the most simple choice where B has all its zeros at minus one. The resulting subdivision scheme is then shown to be equivalent to the Deslauriers-Dubuc SSS, as introduced in [17] by means of a procedure where the mask coefficients are chosen as certain point evaluations of a fundamental Lagrange interpolation polynomial. Similarly, in Chapter 6, we first explicitly construct a compactly supported fundamental cardinal spline interpolant, and proceed to show that the SSS obtained by choosing the mask coefficients equal to the values at the half integers of this spline interpolant is equivalent to the SSS obtained by choosing the polynomial B in Algorithm 4.4 in such a way that it has the Euler-Frobenius polynomial as one of its factors.

The broader framework of our approach, as based on Theorems 4.2 and 4.3, has a significant advantage with respect to the issue of convergence in the following sense. In previous work, it was shown by a direct ad hoc method in [24] that the Deslauriers-Dubuc SSS is convergent, whereas in [22], where the based-on-spline-interpolation SSS of our Chapter 6 was introduced, no convergence proof was provided. Our approach of identifying the respective stationary subdivision schemes of Chapters 5 and 6 with a specific choice of the polynomial B in Algorithm 4.4 allows us, from Theorem 4.2 and Theorem 4.3, to deduce immediately that in both cases the SSS is indeed convergent.

In both Chapters 5 and 6 we explicitly calculate several special cases, and numerically illustrate the theory by means of graphs.

Chapter 1

Preliminaries

1.1 Notation

Throughout this thesis the following notation is used:

$\delta_{j,k}$: the Kronecker delta, as defined by

$$\delta_{j,k} := \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \quad k, j \in \mathbb{Z};$$

$\lfloor x \rfloor$: is the largest integer $\leq x$;

$L^p(\mathbb{R})$: for $p \in [1, \infty)$, the Banach space of measurable (complex-valued) functions on $\mathbb{R}$ which are such that the Lebesgue integral $\int_{-\infty}^{\infty} |f(t)|^p dt$ is finite;

$||\cdot||_p$: the corresponding norm, as given by

$$||f||_p := \left(\int_{-\infty}^{\infty} |f(t)|^p dt \right)^{1/p}, \quad f \in L^p(\mathbb{R}); \tag{1.1}$$

$\langle \cdot, \cdot \rangle$: the inner product

$$\langle f, g \rangle := \int_{-\infty}^{\infty} f(t)\overline{g(t)} dt, \quad f, g \in L^2(\mathbb{R}), \tag{1.2}$$

for the Hilbert space $L^2(\mathbb{R})$;

$\{\alpha_k\}$: the bi-infinite sequence $\{..., \alpha_{-1}, \alpha_0, \alpha_1, ...\}$;

$\sum_k \alpha_k$: if $\{\alpha_k\} \subset \mathbb{C}$, then $\sum_k \alpha_k := \sum_{k=-\infty}^{\infty} \alpha_k$, denotes the resulting bi-infinite series;

ℓ^p: for $p \in [1, \infty)$, ℓ^p is the Banach space of bi-infinite sequences $\{\alpha_k\} \subset \mathbb{C}$, which are such that the series $\sum_k |\alpha_k|^p$ converges;

$||\{\alpha_k\}||_{\ell^p}$: the corresponding norm, as given by,

$$||\cdot||_{\ell^p} := \left(\sum_k |\alpha_k|^p\right)^{1/p}, \quad \{\alpha_k\} \in \ell^p; \tag{1.3}$$

ℓ^∞: the Banach space of bi-infinite sequences $\{\alpha_k\} \subset \mathbb{C}$, which are such that the norm

$$||\{\alpha_k\}||_{\ell^\infty} = \sup\{|\alpha_k| : k \in \mathbb{Z}\} < \infty;$$

Π_k: the linear space of polynomials of degree $\leq k$;

$C(\mathbb{R})$: the space of complex-valued functions which are continuous on $\mathbb{R}$;

$C_u(\mathbb{R})$: the space of complex-valued functions which are uniformly continuous on $\mathbb{R}$;

$C^k(\mathbb{R})$: for $k \in \mathbb{N}$, the space of complex-valued functions which have k continuous derivatives on $\mathbb{R}$;

$C^{-1}(\mathbb{R})$: the space of complex-valued functions which are piece-wise continuous on $\mathbb{R}$;

$f * h$: the *convolution* of two real-valued functions, $f, h \in L^1(\mathbb{R})$, as given by

$$(f * h)(t) = \int_{-\infty}^{\infty} f(x)h(t-x)dx = \int_{-\infty}^{\infty} h(x)f(t-x)dx, \quad t \in \mathbb{R}. \tag{1.4}$$

A function $f : \mathbb{R} \to \mathbb{C}$ has *finite support* (or *compact support*) if there exists a bounded interval $[\alpha, \beta] \subset \mathbb{R}$ such that

$$f(t) = 0, \quad t \notin [\alpha, \beta].$$

If $[\alpha, \beta]$ is the smallest closed interval for which the above condition holds, then $[\alpha, \beta]$ is called the *support* of f, and we write $\text{supp}\, f = [\alpha, \beta]$. Observe in particular that, if $f \in C(\mathbb{R})$, and f has finite support, then $f \in L^p(\mathbb{R})$, $p \in [1, \infty)$.

A sequence $\{c_k\} \subset \mathbb{C}$ has *finite support* if there exist integers M and N, with $M \leq N$, such that

$$c_k = 0, \quad k \notin \{M, M+1, ..., N\}.$$

Observe that if $\{c_k\}$ is a finitely supported sequence, then $\{c_k\} \in \ell^p, \quad 1 \leq p \leq \infty$.

1.2 Results from Fourier Analysis.

The following results from Fourier Analysis will be needed in our work. Our presentation is a simplified version of the analysis given in [3, Chapter 2], and will suffice for our more restricted requirements than those in [3].

The *Fourier transform* $\mathcal{F}$ is defined by

$$(\mathcal{F}f)(\omega) := \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt, \quad \omega \in \mathbb{R}, \quad f \in L^1(\mathbb{R}). \tag{1.5}$$

We also use the notation

$$\widehat{f} := \mathcal{F}f, \quad f \in L^1(\mathbb{R}). \tag{1.6}$$

Now define the linear space $\mathcal{A}$ by

$$\mathcal{A} := \left\{ f \in L^1(\mathbb{R}) : \widehat{f} \in L^1(\mathbb{R}) \right\} \cap C_u(\mathbb{R}). \tag{1.7}$$

We claim that

$$\mathcal{A} \subset L^2(\mathbb{R}), \tag{1.8}$$

for suppose $f \in \mathcal{A}$, so that $f \in L^1(\mathbb{R}) \cap C_u(\mathbb{R})$, and thus $f(t) \to 0$, $|t| \to \infty$ [29, Exercise 8(a), p.345]. Moreover, since $|f(t)|^2 \leq |f(t)|$ for all $t \in \{t \in \mathbb{R} : |f(t)| \leq 1\}$, we have that $f \in L^2(\mathbb{R})$, whence $\mathcal{A} \subset L^2(\mathbb{R})$.

According to [3, Theorem 2.5], a function $f \in \mathcal{A}$ can be recovered from its Fourier transform $\widehat{f}$ by means of the inversion formula

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \widehat{f}(\omega) d\omega, \quad t \in \mathbb{R}, \quad f \in \mathcal{A}. \tag{1.9}$$

Moreover, from [3, Theorem 2.13], we see that the *Parseval identity*

$$\langle f,g\rangle = \frac{1}{2\pi}\left\langle \widehat{f},\widehat{g}\right\rangle, \quad f,g\in\mathcal{A}, \tag{1.10}$$

holds, having noted also from (1.8) that $\mathcal{A}\subset L^2(\mathbb{R})$.

Observe in particular from (1.10) that

$$||f||_2 = \frac{1}{\sqrt{2\pi}}\,||\widehat{f}||_2, \quad f\in\mathcal{A}. \tag{1.11}$$

The following result can now be proved.

Theorem 1.1 *The Fourier transform $\mathcal{F}$ is a one-one map of $\mathcal{A}$ onto itself, where*

$$f(t) = \left(\mathcal{F}^{-1}\widehat{f}\right)(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\omega}\widehat{f}(\omega)d\omega, \quad t\in\mathbb{R}, \quad f\in\mathcal{A}, \tag{1.12}$$

with $\mathcal{F}^{-1}$ denoting the inverse Fourier transform.

Proof. First, we show that $\mathcal{F}$ maps $\mathcal{A}$ into itself. Since, as proved in [3, Theorem 2.2(ii)], $f\in\mathcal{A}$ implies $\widehat{f}\in C_u(\mathbb{R})$, we have to show that

$$\mathcal{F}\widehat{f}\in L^1(\mathbb{R}), \quad f\in\mathcal{A}. \tag{1.13}$$

Suppose therefore that $f\in\mathcal{A}$, and use (1.5) and (1.9) to deduce that

$$\left(\mathcal{F}\widehat{f}\right)(t) = \int_{-\infty}^{\infty} e^{-it\omega}\widehat{f}(\omega)d\omega = 2\pi f(-t), \quad t\in\mathbb{R},$$

whence

$$\int_{-\infty}^{\infty}\left|\left(\mathcal{F}\widehat{f}\right)(t)\right|dt = 2\pi\int_{-\infty}^{\infty}|f(-t)|\,dt = 2\pi\int_{-\infty}^{\infty}|f(t)|dt,$$

and it follows that (1.13) holds, since $f\in\mathcal{A}\subset L^1(\mathbb{R})$. Hence $\mathcal{F}$ maps $\mathcal{A}$ into itself.

The result of our theorem will therefore follow from (1.9) if we can show that, for a given $g\in\mathcal{A}$, there exists a unique $f\in\mathcal{A}$ such that $g=\widehat{f}$. To this end, we define the function g_- by $g_-(t) := g(-t)$, $t\in\mathbb{R}$. Then, since $g\in\mathcal{A}$, we see immediately that $g_-\in L^1(\mathbb{R})\cap C_u(\mathbb{R})$. Moreover, using (1.5), we get, for $\omega\in\mathbb{R}$,

$$\begin{aligned}
\widehat{g}_-(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t}g_-(t)dt && (1.14)\\
&= \int_{-\infty}^{\infty} e^{-i\omega t}g(-t)dt && (1.15)\\
&= \int_{-\infty}^{\infty} e^{i\omega t}g(t)dt && (1.16)\\
&= \widehat{g}(-\omega), && (1.17)
\end{aligned}$$

and thus

$$\int_{-\infty}^{\infty} |\widehat{g}_{-}(t)|dt = \int_{-\infty}^{\infty} |\widehat{g}(-t)|dt = \int_{-\infty}^{\infty} |\widehat{g}(t)|dt < \infty, \tag{1.18}$$

since $g \in \mathcal{A} \subset L^1(\mathbb{R})$. Hence $g_- \in \mathcal{A}$.

Now define $f := \frac{1}{2\pi}\widehat{g}_-$. Then, since $\mathcal{F}$ maps $\mathcal{A}$ into itself, we have $f \in \mathcal{A}$. Moreover, using (1.5) and (1.9), we get

$$\widehat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\omega}\widehat{g}_-(\omega)d\omega = g_-(-t) = g(t), \quad t \in \mathbb{R},$$

and thus $g = \widehat{f}$. It remains to prove the uniqueness of f. Suppose therefore that $h \in \mathcal{A}$ is such that $\widehat{h} = g$. Then, with the function v defined by $v := f - h$, we have $v \in \mathcal{A}$ and $\widehat{v} = 0$. But then also $||\widehat{v}||_2 = 0$, so that (1.11) implies $||v||_2 = 0$ and thus $v = 0$, i.e. $f = h$. ■

The following classical result in the theory of Fourier series was proved in [1].

Theorem 1.2 *Suppose f is a measurable complex-valued function on $\mathbb{R}$ such that the Lebesgue integral*

$$\int_0^{2\pi} |f(t)|^2 dt < \infty$$

and such that f is 2π-periodic on $\mathbb{R}$:

$$f(t + 2\pi) = f(t), \quad t \in \mathbb{R}.$$

Then f has the Fourier series representation

$$f(t) = \sum_k c_k e^{ikt}, \quad t \in \mathbb{R}, \tag{1.19}$$

with the sequence $\{c_k\}$ denoting the Fourier coefficients

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}dt, \quad k \in \mathbb{Z}, \tag{1.20}$$

and where (1.19) should be interpreted as

$$\int_0^{2\pi} \left[f(t) - \sum_{k=-M}^{N} c_k e^{ikt}\right]^2 dt \to 0, \quad M, N \to \infty. \tag{1.21}$$

The next result has to do with a periodisation of a given function f. For the proof, we refer to [3, Lemma 2.4, Theorem 2.25 and Corollary 2.26].

Theorem 1.3 *Suppose $f \in \mathcal{A}$ is such that*

$$f(t), \widehat{f}(t) = \mathcal{O}\left(\frac{1}{1+|t|^\alpha}\right), \quad t \in \mathbb{R}, \tag{1.22}$$

for some $\alpha > 1$. Then the series $\sum_k f(t+2\pi k)$ converges for all $t \in \mathbb{R}$ to a 2π-periodic function $g \in C(\mathbb{R})$:

$$g(t) := \sum_k f(t+2\pi k), \quad t \in \mathbb{R}. \tag{1.23}$$

Moreover

$$\sum_k f(t+2\pi k) = \frac{1}{2\pi}\sum_k \widehat{f}(k)e^{ikt}, \quad t \in \mathbb{R}, \tag{1.24}$$

or, equivalently,

$$\sum_k f(t+k) = \frac{1}{2\pi}\sum_k \widehat{f}(2\pi k)e^{i2\pi kt}, \quad t \in \mathbb{R}, \tag{1.25}$$

with the right-hand sides of (1.24) and (1.25) denoting the Fourier series of the respective functions of t in the left-hand sides.

The identities (1.24) and (1.25) are known as the *Poisson summation formula.*

For the Fourier transform, as defined by (1.5), and the convolution operation $*$ defined by (1.4), we have, for $f, h \in L^1(\mathbb{R})$, with f and h both real-valued, and $\omega \in \mathbb{R}$, that

$$\begin{aligned}
\widehat{(f*h)}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t}(f*h)(t)dt \\
&= \int_{-\infty}^{\infty} e^{-i\omega t}\left[\int_{-\infty}^{\infty} f(x)h(t-x)dx\right]dt \\
&= \int_{-\infty}^{\infty} f(x)\left[\int_{-\infty}^{\infty} e^{-i\omega t}h(t-x)dt\right]dx \\
&= \int_{-\infty}^{\infty} f(x)\left[\int_{-\infty}^{\infty} e^{-i\omega(x+t)}h(t)dt\right]dx \\
&= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\int_{-\infty}^{\infty} e^{-i\omega t}h(t)dt \\
&= \widehat{f}(\omega)\,\widehat{h}(\omega),
\end{aligned}$$

and thus

$$\widehat{f * h} = \widehat{f}\,\widehat{h}. \tag{1.26}$$

Also, as is easily proved from (1.4), for real-valued functions $f, g, h \in L^1(\mathbb{R})$, the convolution operation $*$ is associative:

$$f * (g * h) = (f * g) * h. \tag{1.27}$$

Chapter 2

Cardinal Spline Functions

A recurring theme in this thesis will be that of cardinal spline functions, which we proceed to define as follows. For $m \in \mathbb{N}$, $j \in \mathbb{Z}$, we define the linear space $S_m(\mathbb{Z}/2^j)$ as the space of functions $f : \mathbb{R} \to \mathbb{R}$ which are such that

$$\left.\begin{array}{c} f|_{[\frac{k}{2^j},\frac{k+1}{2^j}]} = p_k \in \Pi_{m-1}, \quad k \in \mathbb{Z}, \\ \text{with } f \in C^{m-2}(\mathbb{R}). \end{array}\right\} \tag{2.1}$$

The space $S_m(\mathbb{Z})$ is then called the *cardinal spline* space, for which we now proceed to construct a basis.

2.1 The Cardinal B-splines

We define the sequence $\{N_m : m \in \mathbb{N}\}$ of real-valued functions on $\mathbb{R}$ recursively by

$$N_1(t) = \begin{cases} 1, & t \in [0,1), \\ 0, & t \notin [0,1), \end{cases} \tag{2.2}$$

and

$$N_m = N_{m-1} * N_1, \quad m = 2, 3, ..., \tag{2.3}$$

with the convolution operator $*$ defined by (1.4). Observe from (2.2), (2.3) and (1.4) that then

$$N_m(t) = \int_0^1 N_{m-1}(t-x)dx, \quad t \in \mathbb{R}, \quad m = 2, 3, \dots . \tag{2.4}$$

The following properties can be proved.

Theorem 2.1 *The sequence $\{N_m : m \in \mathbb{N}\}$, as defined recursively by (2.2) and (2.3), satisfies the following properties:*

(i) $N_m(t) > 0, \quad t \in (0, m), \quad m \in \mathbb{N};$ (2.5)

(ii) $\operatorname{supp} N_m = [0, m], \quad m \in \mathbb{N};$ (2.6)

(iii) $\sum_k N_m(t-k) = 1, \quad t \in \mathbb{R}, \quad m \in \mathbb{N};$ (2.7)

(iv) $N_m(\cdot - k) \in S_m(\mathbb{Z}), \quad k \in \mathbb{Z}, \quad m \in \mathbb{N};$ (2.8)

(v) $N'_m(t) = N_{m-1}(t) - N_{m-1}(t-1), \quad t \in \mathbb{R}, \quad m = 2, 3, ...;$ (2.9)

(vi) $N_m\left(\frac{m}{2} - t\right) = N_m\left(\frac{m}{2} + t\right), \quad t \in \mathbb{R}, \quad m = 2, 3, ...;$ (2.10)

(vii) $N_m(t) = \frac{t}{m-1} N_{m-1}(t) + \frac{m-t}{m-1} N_{m-1}(t-1), \quad t \in \mathbb{R}, \quad m = 2, 3, ...;$ (2.11)

(viii) $\widehat{N}_m(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega}\right)^m = e^{-im\omega/2}\left(\frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}\right)^m, \quad \omega \in \mathbb{R}, \quad m \in \mathbb{N};$ (2.12)

(ix) $\widehat{N}_m(2\pi k) = \delta_{k,0}, \quad k \in \mathbb{Z}, \quad m \in \mathbb{N};$ (2.13)

(x) $\int_{-\infty}^{\infty} N_m(t)dt = 1, \quad m \in \mathbb{N}.$ (2.14)

Remarks:

(a) The value $\omega = 0$ in (2.12) should be interpreted as the limit for $\omega \to 0$, i.e.

$$\widehat{N}_m(0) = 1. \tag{2.15}$$

(b) Observe that the symmetry condition (2.10) has the equivalent formulation

$$N_m(m - t) = N_m(t), \quad t \in \mathbb{R}, \quad m = 2, 3, \ldots. \tag{2.16}$$

Proof: For the proofs of property (i) through to (vii), we refer the reader to [3, Theorem 4.3], and we proceed here to prove (viii), (ix) and (x).

To prove (viii), we first use the definition (1.5) of a Fourier transform and (2.2), to deduce that

$$\widehat{N}_1(\omega) = \int_0^1 e^{-i\omega t}dt = \frac{1 - e^{-i\omega}}{i\omega} = e^{-i\omega/2}\frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}, \quad \omega \in \mathbb{R}. \tag{2.17}$$

Now observe from (2.3) that

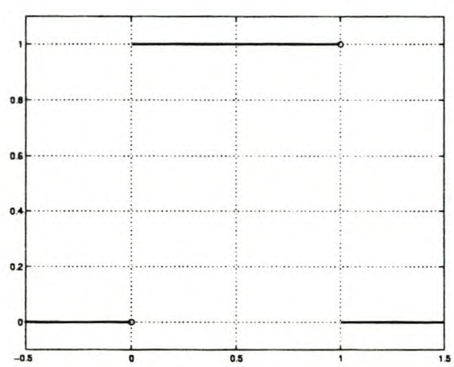
$$N_m = N_1 * N_1 * \cdots * N_1 \qquad (m \text{ times}), \tag{2.18}$$

keeping in mind also the associativity (1.27) of the convolution operation. Thus, from (1.26), we have

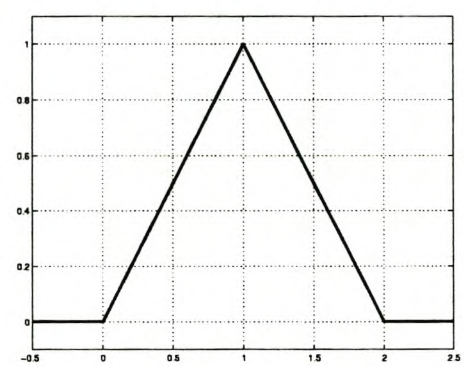
$$\widehat{N}_m = \left(\widehat{N}_1\right)^m . \tag{2.19}$$

The results (viii) and (ix) then immediately follow from (2.17) and (2.19). Also, (x) is a consequence of (2.15) and (1.5). ■

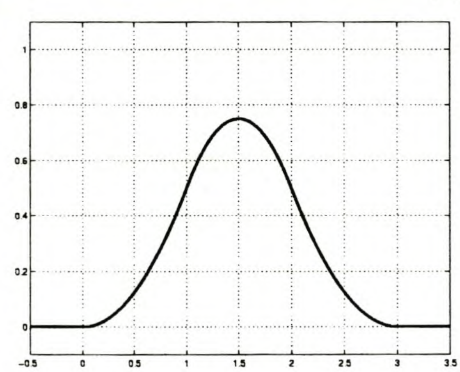
The function N_m of Theorem 2.1 will be called the *cardinal B-spline* of *order* m. Some examples are drawn in Fig. 2.1.



(a) m=1



(b) m=2



(c) m=3

Figure 2.1: *Cardinal B-splines* N_m.

2.2 The Refinement Equation

A function $\phi : \mathbb{R} \to \mathbb{R}$ will be called a *refinable function* if there exists a finitely supported sequence $\{a_j\} \subset \mathbb{R}$ such that

$$\phi(t) = \sum_j a_j \phi(2t - j), \quad t \in \mathbb{R}. \tag{2.20}$$

The equation (2.20) is then called the *refinement equation* and the sequence $\{a_j\}$ is called the corresponding *refinement mask*.

In the following theorem we prove that the cardinal B-splines are refinable.

Theorem 2.2 *The sequence $\{N_m : m \in \mathbb{N}\}$, as defined recursively by (2.2) and (2.3), satisfies the refinement equation:*

$$N_m(t) = \sum_{k=0}^{m} \frac{1}{2^{m-1}} \binom{m}{k} N_m(2t - k), \ \ t \in \mathbb{R}, \quad m \in \mathbb{N}. \tag{2.21}$$

Proof. Since (2.2) gives

$$N_1(t) = N_1(2t) + N_1(2t - 1), \quad t \in \mathbb{R}, \tag{2.22}$$

we see that (2.21) holds for $m = 1$.

Suppose next that $m \geq 2$. Noting from (2.17) that, for $\omega \in \mathbb{R}$,

$$\widehat{N}_1(\omega) = \left[\frac{1 + e^{-i\omega/2}}{2}\right] \left[\frac{1 - e^{-i\omega/2}}{i\omega/2}\right] = \left[\frac{1 + e^{-i\omega/2}}{2}\right] \widehat{N}_1(\omega/2),$$

it follows from (2.19) that

$$\widehat{N}_m(\omega) = \left[\frac{1 + e^{-i\omega/2}}{2}\right]^m \widehat{N}_m(\omega/2), \quad \omega \in \mathbb{R}. \tag{2.23}$$

From (2.8), (2.1), (2.6) and (2.12) we see that $N_m \in C_u(\mathbb{R}) \cap L^1(\mathbb{R})$, and $\widehat{N}_m \in L^1(\mathbb{R})$, since $m \geq 2$; hence $N_m \in \mathcal{A}$, as defined by (1.7). Hence we can appeal to Theorem 1.1 to deduce, using also (2.23), that, for $t \in \mathbb{R}$,

$$N_m(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\omega} \widehat{N}_m(\omega) d\omega$$

$$
\begin{aligned}
&= \frac{1}{2^m}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\omega}\left(1+e^{i\omega/2}\right)^m \widehat{N}_m\left(\frac{\omega}{2}\right)d\omega \\
&= \frac{1}{2^{m-1}}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it2\omega}\left(1+e^{-i\omega}\right)^m \widehat{N}_m(\omega)d\omega \\
&= \frac{1}{2^{m-1}}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it2\omega}\left[\sum_{k=0}^{m}\left(\begin{array}{c} m \\ k \end{array}\right)\left(e^{-i\omega}\right)^k\right] \widehat{N}_m(\omega)d\omega \\
&= \frac{1}{2^{m-1}}\sum_{k=0}^{m}\left(\begin{array}{c} m \\ k \end{array}\right)\left[\frac{1}{2\pi}\int_{\infty}^{\infty} e^{i(2t-k)\omega}\widehat{N}_m(\omega)d\omega\right] \\
&= \frac{1}{2^{m-1}}\sum_{k=0}^{m}\left(\begin{array}{c} m \\ k \end{array}\right) N_m(2t-k),
\end{aligned}
$$

which is the desired refinement equation (2.21). ■

Observe, that the function $\phi = N_m$ therefore satisfies the refinement equation (2.20) with refinement mask $\{a_k\} = \{a_k^{(m)}\}$ given by

$$
a_k^{(m)} = \begin{cases} \dfrac{1}{2^{m-1}}\left(\begin{array}{c} m \\ k \end{array}\right), & k = 0,1,...,m, \\ 0, & k \notin \{0,1,...,m\}. \end{cases} \tag{2.24}
$$

Setting $m = 3$ in (2.21), we get, for the case of quadratic cardinal B-spline,

$$
N_3(t) = \frac{1}{4}N_3(2t) + \frac{3}{4}N_3(2t-1) + \frac{3}{4}N_3(2t-2) + \frac{1}{4}N_3(2t-3), \quad t \in \mathbb{R}, \tag{2.25}
$$

as graphically illustrated in Figure 2.2.

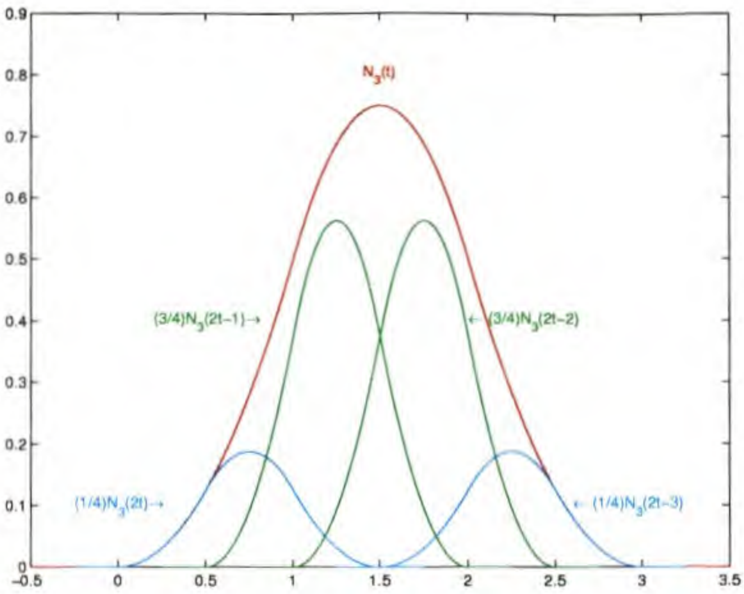


Figure 2.2: *Illustration of the refinement equation for N_3.*

2.3 A Basis for $S_m(\mathbb{Z})$

We proceed to prove that the sequence $\{N_m(\cdot - k) : k \in \mathbb{Z}\}$ is a basis of $S_m(\mathbb{Z})$ in the sense of the following of theorem, our proof of which follows the guideline of the proof given in [25, Theorem 2.11].

Theorem 2.3 *Let $m \in \mathbb{N}$. Then, for each $f \in S_m(\mathbb{Z})$, there exists a unique sequence $\{c_k\} \subset \mathbb{R}$ such that*

$$f(t) = \sum_k c_k N_m(t-k), \quad t \in \mathbb{R}, \tag{2.26}$$

where the sequence $\{N_m : m \in \mathbb{N}\}$ is defined recursively by (2.2) and (2.3).

Proof. First note from (2.2) that

$$N_1(k) = \delta_{k,0}, \quad k \in \mathbb{Z}, \tag{2.27}$$

and thus, from (2.11) and (2.27),

$$N_2(k+1) = \delta_{k,0}, \quad k \in \mathbb{Z}. \tag{2.28}$$

Hence the representation (2.26) holds with $c_k = f(k), \quad k \in \mathbb{Z}$, if $m = 1$ and with $c_k = f(k+1), \quad k \in \mathbb{Z}$, if $m = 2$, with uniqueness of the sequence $\{c_k\}$ easily established in each case by using (2.27) and (2.28).

We now prove that the property (2.26) holds for general $m \in \mathbb{N}$ by means of an inductive proof. Suppose therefore that (2.26) holds for an integer $m \geq 2$, and suppose $f \in S_{m+1}(\mathbb{Z})$. Then (2.1) and (2.8) imply that $f' \in S_m(\mathbb{Z})$, so that, according to the inductive hypothesis, there exists a unique sequence $\{\gamma_k\} \subset \mathbb{R}$ such that

$$f'(t) = \sum_k \gamma_k N_m(t-k), \quad t \in \mathbb{R}. \tag{2.29}$$

If we now define the sequence $\{\beta_k\} \subset \mathbb{R}$ by

$$\beta_k = \begin{cases} \sum_{j=1}^{k} \gamma_j, & k \geq 1, \\ 0, & k = 0, \\ -\sum_{j=k+1}^{0} \gamma_j, & k \leq -1, \end{cases} \tag{2.30}$$

we see that

$$\beta_k - \beta_{k-1} = \gamma_k, \quad k \in \mathbb{Z}, \tag{2.31}$$

which, substituted into (2.29), gives, for $t \in \mathbb{R}$,

$$\begin{aligned} f'(t) &= \sum_k (\beta_k - \beta_{k-1}) N_m(t-k) \\ &= \sum_k \beta_k N_m(t-k) - \sum_k \beta_{k-1} N_m(t-k) \\ &= \sum_k \beta_k N_m(t-k) - \sum_k \beta_k N_m(t-k-1) \\ &= \sum_k \beta_k \left[N_m(t-k) - N_m(t-k-1) \right] \\ &= \sum_k \beta_k N'_{m+1}(t-k), \end{aligned} \tag{2.32}$$

by virtue of (2.9). Integration of (2.32) then yields

$$f(t) - f(0) = \int_0^t \sum_k \beta_k N'_{m+1}(\tau - k) d\tau$$

$$\begin{aligned} &= \sum_k \beta_k \int_0^t N'_{m+1}(\tau - k)d\tau \\ &= \sum_k \beta_k \left[N_{m+1}(t-k) - N_{m+1}(-k)\right], \end{aligned}$$

and thus, using also (2.7), we find

$$f(t) = \sum_k \left[f(0) - \sum_j \beta_j N_{m+1}(-j) + \beta_k\right] N_{m+1}(t-k), \quad t \in \mathbb{R}.$$

Hence if we define

$$c_k := f(0) - \sum_j \beta_j N_{m+1}(-j) + \beta_k, \quad k \in \mathbb{Z}, \tag{2.33}$$

then (2.26) holds with m replaced with $m+1$.

To complete our inductive proof, it remains to prove the uniqueness of the sequence $\{c_k\}$. Suppose therefore $\{d_k\}$ is a sequence for which (2.26) holds with $\{c_k\}$ replaced by $\{d_k\}$, and with m replaced by $m+1$. Differentiation then yields, together with (2.9), and for $t \in \mathbb{R}$,

$$\begin{aligned} f'(t) &= \sum_k d_k N'_{m+1}(t-k) \\ &= \sum_k d_k \left[N_m(t-k) - N_m(t-k-1)\right] \\ &= \sum_k d_k N_m(t-k) - \sum_k d_k N_m(t-k-1) \\ &= \sum_k d_k N_m(t-k) - \sum_k d_{k-1} N_m(t-k) \\ &= \sum_k (d_k - d_{k-1}) N_m(t-k). \end{aligned}$$

Since $\{\gamma_k\}$, from the inductive hypothesis, is the unique sequence in $\mathbb{R}$ for which (2.29) holds, the sequence $\{d_k\}$ must therefore satisfy

$$d_k - d_{k-1} = \gamma_k, \quad k \in \mathbb{Z}. \tag{2.34}$$

But then (2.31) and (2.34) imply

$$d_k - d_{k-1} = \beta_k - \beta_{k-1}, \quad k \in \mathbb{Z}. \tag{2.35}$$

Since (2.30) gives $\beta_0 = 0$, we find from (2.35) that $\{d_k\}$ must satisfy

$$d_k = d_0 + \beta_k, \quad k \in \mathbb{Z}. \tag{2.36}$$

Inserting (2.36) into (2.26) yields, together with (2.7), and for $t \in \mathbb{R}$,

$$\begin{aligned} f(t) &= \sum_k (d_0 + \beta_k) N_{m+1}(t-k) \\ &= d_0 + \sum_k \beta_k N_{m+1}(t-k). \end{aligned} \tag{2.37}$$

Setting $t = 0$ in (2.37) then gives

$$d_0 = f(0) - \sum_j \beta_j N_{m+1}(-j). \tag{2.38}$$

Now observe from (2.36), (2.38) and (2.33) that $d_k = c_k, \quad k \in \mathbb{Z}$. ■

Since the definition (2.1) implies that, for $j \in \mathbb{Z}$, $f \in S_m(\mathbb{Z}/2^j)$ if and only if $f\left(\frac{1}{2^j}\cdot\right) \in S_m(\mathbb{Z})$, we have the following immediate consequence of Theorem 2.3.

Corollary 2.4 *For $j \in \mathbb{Z}$, suppose $f \in S_m(\mathbb{Z}/2^j)$. Then there exists a unique sequence $\{c_k\} \subset \mathbb{R}$ such that*

$$f(t) = \sum_k c_k N_m\left(2^j t - k\right), \quad t \in \mathbb{R},$$

with, in particular,

$$N_m(2^j \cdot -k) \in S_m(\mathbb{Z}/2^j), \quad k \in \mathbb{Z}, \quad j \in \mathbb{Z}. \tag{2.39}$$

Observe that (2.39) is an extension of the result (2.8).

2.4 The Euler-Frobenius Polynomial

We call a polynomial p of degree n *a symmetric polynomial* if

$$z^n p\left(\frac{1}{z}\right) = p(z), \quad z \in \mathbb{C},$$

or, equivalently, in the notation

$$p(z) = \sum_{j=0}^{n} p_j z^j, \quad z \in \mathbb{C},$$

if

$$p_{n-j} = p_j, \quad j = 0, 1, ..., n.$$

Similarly, a Laurent polynomial A is called *symmetric* if

$$A\left(\frac{1}{z}\right) = A(z), \quad z \in \mathbb{C} \setminus \{0\},$$

or, equivalently, in the notation

$$A(z) = \sum_j a_j z^j, \quad z \in \mathbb{C} \setminus \{0\},$$

if

$$a_j = a_{-j}, \quad j \in \mathbb{Z}.$$

We define the *Euler-Frobenius polynomial* $\widetilde{N}_m$ by

$$\widetilde{N}_m(z) = \sum_{j=0}^{m-2} N_m(j+1)z^j, \quad z \in \mathbb{C}, \quad m = 2, 3, \ldots, \tag{2.40}$$

where N_m denotes the cardinal B-spline as defined by (2.2) and (2.3). The following properties of $\widetilde{N}_m$, which we will need later in this thesis, can be proved. Our result, and its proof, is an extension of the analysis in [3, Section 6.4], in which only cardinal spline functions of even order were considered. The proof below is joint work with Birgit Rohwer, and may also be found in her thesis [28].

Theorem 2.5 *For an integer $m \geq 2$, the Euler-Frobenius polynomial $\widetilde{N}_m$ is a symmetric polynomial of degree $m-2$. Moreover, we have*

$$\widetilde{N}_2(z) = 1, \quad z \in \mathbb{C}, \tag{2.41}$$

$$\widetilde{N}_3(z) = \frac{1}{2}(z+1), \quad z \in \mathbb{C}, \tag{2.42}$$

whereas, for $m \geq 4$, there exists a sequence $\{\lambda_{m,j} : j = 1, 2, ..., \lfloor \frac{m-2}{2} \rfloor\} \subset \mathbb{R}$, such that

$$-1 < \lambda_{m,j} < 0, \quad j = 1, 2, ..., \left\lfloor \frac{m-2}{2} \right\rfloor, \tag{2.43}$$

and

$$\widetilde{N}_m(z) = \frac{1}{(m-1)!} \begin{cases} P_m(z), & \text{if } m \text{ is even}, \\ (z+1)P_m(z), & \text{if } m \text{ is odd}, \end{cases} \qquad z \in \mathbb{C}, \tag{2.44}$$

where the polynomial P_m is given by the formula

$$P_m(z) = \prod_{j=1}^{\lfloor \frac{m-2}{2} \rfloor} (z - \lambda_{m,j}) \left(z - \frac{1}{\lambda_{m,j}} \right), \quad z \in \mathbb{C}. \tag{2.45}$$

Proof. The fact that $\widetilde{N}_m$ has degree $m-2$ is clear from (2.40) and (2.5). Moreover, using (2.16), we find for $z \in \mathbb{C}$ that

$$\begin{aligned} z^{m-2}\widetilde{N}_m\left(\frac{1}{z}\right) &= z^{m-2}\sum_{j=0}^{m-2} N_m(j+1)z^{-j} \\ &= \sum_{j=0}^{m-2} N_m(m-(j+1))z^j \\ &= \sum_{j=0}^{m-2} N_m(j+1)z^j = \widetilde{N}_m(z), \end{aligned}$$

whence, $\widetilde{N}_m$ is a symmetric polynomial.

Next, we observe that the recursion formula (2.11) yields

$$N_{m+1}(k+1) = \frac{k+1}{m}N_m(k+1) + \frac{m-k}{m}N_m(k), \quad k \in \mathbb{Z}, \quad m = 1, 2, ..., \tag{2.46}$$

which, together with (2.2), (2.63) and (2.40), give the formulas (2.41) and (2.42).

Next, suppose $m \geq 4$, and define the sequence $\{e_n : n = 0, 1, ...\}$ by

$$e_n(\omega) := e^{-i\frac{n\omega}{2}}\widetilde{N}_{n+2}(e^{i\omega}), \quad \omega \in \mathbb{R}, \quad n = 0, 1, 2, ..., \tag{2.47}$$

which, together with (2.41) and (2.42), give

$$e_0(\omega) = 1, \quad \omega \in \mathbb{R}, \tag{2.48}$$

and

$$e_1(\omega) = \cos\left(\frac{\omega}{2}\right), \quad \omega \in \mathbb{R}. \tag{2.49}$$

We claim that the recursion formula

$$e_{n+1}(\omega) = \left(\cos\frac{\omega}{2}\right) e_n(\omega) - \frac{2}{n+2}\left(\sin\frac{\omega}{2}\right) e_n'(\omega), \quad \omega \in \mathbb{R}, \quad n = 0, 1, ..., \tag{2.50}$$

holds. Observe from (2.48) and (2.49) that (2.50) holds for $n = 0$. In general, for $n \in \{0, 1, ...\}$, we find from (2.47) and (2.40) that, for $\omega \in \mathbb{R}$,

$$\begin{aligned}
&\left(\cos\frac{\omega}{2}\right) e_n(\omega) - \frac{2}{n+2}\left(\sin\frac{\omega}{2}\right) e_n'(\omega) \\
= \;& \frac{1}{2}\left(e^{i\frac{\omega}{2}} + e^{-i\frac{\omega}{2}}\right)\sum_{k=0}^{n} N_{n+2}(k+1) e^{i\left(k-\frac{n}{2}\right)\omega} \\
&\quad - \frac{2}{2i(n+2)}\left(e^{i\frac{\omega}{2}} - e^{-i\frac{\omega}{2}}\right)\sum_{k=0}^{n} N_{n+2}(k+1) i\left(k - \frac{n}{2}\right) e^{i\left(k-\frac{n}{2}\right)\omega} \\
= \;& \frac{1}{2}\left(e^{i\frac{\omega}{2}(1-n)} + e^{-i\frac{\omega}{2}(1+n)}\right)\sum_{k=0}^{n} N_{n+2}(k+1) e^{ik\omega} \\
&\quad - \frac{1}{2}\left(e^{i\frac{\omega}{2}(1-n)} - e^{-i\frac{\omega}{2}(1+n)}\right)\sum_{k=0}^{n} \frac{2k-n}{n+2} N_{n+2}(k+1) e^{ik\omega} \\
= \;& e^{i\frac{\omega}{2}(1-n)}\sum_{k=0}^{n}\frac{n+1-k}{n+2} N_{n+2}(k+1) e^{ik\omega} + e^{-i\frac{\omega}{2}(1+n)}\sum_{k=0}^{n}\frac{k+1}{n+2} N_{n+2}(k+1) e^{ik\omega} \\
= \;& e^{i\frac{\omega}{2}(1-n)}\sum_{k=1}^{n+1}\frac{n+2-k}{n+2} N_{n+2}(k) e^{i(k+1)\omega} + e^{-i\frac{\omega}{2}(1+n)}\sum_{k=0}^{n}\frac{k+1}{n+2} N_{n+2}(k+1) e^{ik\omega} \\
= \;& e^{-i\frac{\omega}{2}(n+1)}\sum_{k=0}^{n+1}\left[\frac{k+1}{n+2} N_{n+2}(k+1) + \frac{n+2-k}{n+2} N_{n+2}(k)\right] e^{ik\omega} \\
= \;& e^{-i\frac{\omega}{2}(n+1)}\sum_{k=0}^{n+1} N_{n+3}(k+1) e^{ik\omega} \\
= \;& e_{n+1}(\omega),
\end{aligned}$$

by virtue, also of (2.46) and (2.6). Hence the recursion formula (2.50) is satisfied.

Our next step is to show that there exists a sequence $\{\beta_{n,j} : j = 0, 1, ..., n\} \subset \mathbb{R}$ such that

$$e_n(\omega) = \sum_{j=0}^{n} \beta_{n,j}\left(\cos\frac{\omega}{2}\right)^j, \quad \omega \in \mathbb{R}, \ n = 0, 1, ..., \tag{2.51}$$

and where

$$\beta_{n,n} = \frac{2^n}{(n+1)!}, \quad n = 0, 1, \dots . \tag{2.52}$$

Observe from (2.48) and (2.49) that (2.51), (2.52) hold for $n = 0$ and $n = 1$. The proof that (2.51), (2.52) hold for all $n \in \{0, 1, ...\}$ is by means of induction, the inductive step from n to $n + 1$ being given, from (2.50), by

$$\begin{aligned} e_{n+1}(\omega) &= \left(\cos\frac{\omega}{2}\right) e_n(\omega) - \frac{2}{n+2}\left(\sin\frac{\omega}{2}\right) e_n'(\omega) \\ &= \left(\cos\frac{\omega}{2}\right) \sum_{j=0}^{n} \beta_{n,j} \left(\cos\frac{\omega}{2}\right)^j + \frac{1}{n+2}\left(\sin\frac{\omega}{2}\right) \sum_{j=0}^{n} j\beta_{n,j} \left(\cos\frac{\omega}{2}\right)^{j-1} \sin\frac{\omega}{2} \\ &= \sum_{j=1}^{n+1} \beta_{n,j-1} \left(\cos\frac{\omega}{2}\right)^j + \frac{1}{n+2}\left(1 - \cos^2\frac{\omega}{2}\right) \sum_{j=0}^{n-1} (j+1)\beta_{n,j+1} \left(\cos\frac{\omega}{2}\right)^j \end{aligned}$$

from which we see that there exists a sequence $\{\beta_{n+1,j} : j = 0, 1, ..., n + 1\} \subset \mathbb{R}$ such that

$$e_{n+1}(\omega) = \sum_{j=0}^{n+1} \beta_{n+1,j} \left(\cos\frac{\omega}{2}\right)^j, \quad \omega \in \mathbb{R},$$

with

$$\beta_{n+1,n+1} = \frac{2}{n+2}\beta_{n,n}. \tag{2.53}$$

Hence (2.51), (2.52) hold for all $n \in \{0, 1, ...\}$, where the formula (2.52) is easily proved by means of (2.53) and the fact that $\beta_{0,0} = 1$, $\beta_{1,1} = 1$, by virtue of (2.48) and (2.49).

Now define the polynomial sequence $\{U_n : n = 0, 1, ...\}$ by means of

$$U_n(z) = \sum_{j=0}^{n} \beta_{n,j} z^j, \quad z \in \mathbb{C}, \ \ n = 0, 1, ..., \tag{2.54}$$

whence, from (2.51), we have

$$e_n(\omega) = U_n\left(\cos\frac{\omega}{2}\right), \quad \omega \in \mathbb{R}, \ \ n = 0, 1, \ldots . \tag{2.55}$$

Observe from (2.52) and (2.54) that

$$U_0(z) = 1, \quad z \in \mathbb{C}, \tag{2.56}$$

$$U_1(z) = z, \quad z \in \mathbb{C}. \tag{2.57}$$

Now observe from (2.55) that, for $\omega \in \mathbb{R}$,

$$e_n'(\omega) = U_n'\left(\cos\frac{\omega}{2}\right)\left(-\frac{1}{2}\sin\frac{\omega}{2}\right),$$

which, together with (2.50) and (2.55), imply that for $n \in \{0,1,...\}$ and $\omega \in \mathbb{R}$,

$$U_{n+1}\left(\cos\frac{\omega}{2}\right) = \left(\cos\frac{\omega}{2}\right) U_n\left(\cos\frac{\omega}{2}\right) + \frac{1-\left(\cos\frac{\omega}{2}\right)^2}{n+2} U_n'\left(\cos\frac{\omega}{2}\right),$$

and thus, since U_{n+1} and U_n are polynomials, and since if a polynomial vanishes identically on $[-1,1]$ then it vanishes identically on $\mathbb{C}$, we see that the sequence $\{U_n : n = 0,1,...\}$ satisfies the recursion formula

$$U_{n+1}(z) = z\,U_n(z) + \left(\frac{1-z^2}{n+2}\right) U_n'(z), \quad z \in \mathbb{C}, \ n = 0,1,..., \tag{2.58}$$

with U_0 given by (2.56). Observe also from (2.54), (2.52) that U_n is a polynomial of degree n, and with real coefficients, for $n = 0,1,...$. Now define the sequence $\{u_n : n = 0,1,...\}$ by

$$u_n(z) = \frac{U_n(iz)}{i^n}, \quad z \in \mathbb{C}, \ n = 0,1,..., \tag{2.59}$$

so that, using (2.59), (2.56) and (2.57), we have

$$u_0(z) = 1, \quad z \in \mathbb{C}, \tag{2.60}$$

and

$$u_1(z) = z, \quad z \in \mathbb{C}. \tag{2.61}$$

Since (2.59) implies

$$u_n'(z) = \frac{U_n'(iz)}{i^{n-1}}, \quad z \in \mathbb{C}, \quad n = 0,1,..., \tag{2.62}$$

it follows from replacing z by iz in (2.58), and using (2.62), that the sequence $\{u_n : n = 0,1,...\}$ satisfies the recursion formula

$$u_{n+1}(z) = zu_n(z) - \frac{1+z^2}{n+2}\,u_n'(z), \quad z \in \mathbb{C}, \quad n = 0,1,...,. \tag{2.63}$$

and with u_0 given by (2.60).

It is clear from (2.63) and (2.60) that, for each $n \in \{0,1,...\}$, u_n is a polynomial with real coefficients, that is there exists a sequence $\{\gamma_{n,k} : k = 0,1,...,n\} \subset \mathbb{R}$ such that

$$u_n(z) = \sum_{j=0}^{n} \gamma_{n,j} z^k, \quad z \in \mathbb{C}, \quad n = 0,1,..., \tag{2.64}$$

with, from (2.59), (2.54) and (2.52)

$$\gamma_{n,n} = \beta_{n,n} = \frac{2^n}{(n+1)!}, \quad n = 0, 1, ..., \tag{2.65}$$

so that the polynomial u_n has degree n for $n = 0, 1, \ldots$.

We claim that

$$\left.\begin{array}{r} u_{2k} \text{ is an even function} \\ u_{2k+1} \text{ is an odd function} \end{array}\right\} k = 0, 1, \ldots . \tag{2.66}$$

Noting from (2.60), (2.61), that (2.66) holds for $k = 0$, we proceed to prove (2.66) inductively, with the inductive step for k to $k+1$ proceeding as follows. From (2.63) we have, for $z \in \mathbb{C}$,

$$\begin{aligned} u_{2k+2}(-z) &= -z\, u_{2k+1}(-z) - \frac{1+z^2}{n+2}\, u'_{2k+1}(-z) \\ &= z\, u_{2k+1}(z) - \frac{1+z^2}{n+2}\, u'_{2k+1}(z) \end{aligned} \tag{2.67}$$

since $u_{2k+1}(-z) = -u_{2k+1}(z), \quad z \in \mathbb{C}$, and thus $u'_{2k+1}(-z) = u'_{2k+1}(z), \quad z \in \mathbb{C}$, from the inductive hypothesis. Hence (2.67) and (2.63) show that $u_{2k+2}(-z) = u_{2k+2}(z), \quad z \in \mathbb{C}$, i.e. u_{2k+2} is an even function. Next, we use (2.63) again to obtain, for $z \in \mathbb{C}$,

$$\begin{aligned} u_{2k+3}(-z) &= -z\, u_{2k+2}(-z) - \frac{1+z^2}{n+2} u'_{2k+2}(-z) \\ &= -\left[z\, u_{2k+2}(z) - \frac{1+z^2}{n+2} u'_{2k+2}(z)\right], \end{aligned} \tag{2.68}$$

since u_{2k+2} is an even function, i.e. $u_{2k+2}(-z) = u_{2k+2}(z),\ z \in \mathbb{C}$, and thus $u'_{2k+2}(-z) = -u'_{2k+2}(z),\ z \in \mathbb{C}$. It follows from (2.68) and (2.63) that $u_{2k+3}(z) = -u_{2k+3}(-z),\ z \in \mathbb{C}$, i.e. u_{2k+3} is an odd function, so that we have advanced the inductive hypothesis from k to $k+1$, and it follows that (2.66) is true.

We next prove the statement:

$$u_n \text{ has } n \text{ real simple zeros for } n = 1, 2, \ldots . \tag{2.69}$$

Using (2.63) and (2.61), we find that

$$u_2(z) = \frac{1}{3}(2z^2 - 1), \quad z \in \mathbb{C}, \tag{2.70}$$

and thus, from (2.61) and (2.70), the statement (2.69) holds for $n = 1$ and $n = 2$. We proceed to show that (2.69) holds for all $n \in \mathbb{N}$ by means of induction.

Suppose therefore that, for a given $k \in \mathbb{N}$, the polynomial u_{2k} has $2k$ real simple zeros. Observe also that $u_{2k}(0) \neq 0$, for if $z = 0$ is a zero of u_{2k}, then (2.66) implies that $z = 0$ is a zero of order ≥ 2 of u_{2k}, contradicting the inductive hypothesis that all the zeros of u_{2k} are simple. Hence, using also (2.66), we deduce that there exists a sequence $\{\xi_j : j = 1, 2, ..., k\} \subset \mathbb{R}$, with

$$0 < \xi_1 < \xi_2 < \cdots < \xi_k, \tag{2.71}$$

such that

$$u_{2k}(\pm\xi_j) = 0, \quad j = 1, 2, ..., k. \tag{2.72}$$

Moreover, since the zeros of u_{2k} are all simple, and since $\lim_{t\to\infty} u_{2k}(t) = \infty$ from (2.64) and (2.65), we also have

$$(-1)^{k+j} u'_{2k}(\xi_j) > 0, \quad j = 1, 2, ..., k. \tag{2.73}$$

Now use (2.63) and (2.72) to obtain

$$u_{2k+1}(\xi_j) = -\frac{1+\xi_j^2}{2k+2} u'_{2k}(\xi_j), \quad j = 1, 2, ..., k,$$

which, together with (2.73), yield

$$(-1)^{k+j+1} u_{2k+1}(\xi_j) > 0, \quad j = 1, 2, ..., k. \tag{2.74}$$

It follows from the intermediate value theorem, together with (2.74), that there exists, if $k \geq 2$, a sequence $\{\eta_j : j = 1, 2, ..., k-1\} \subset \mathbb{R}$ such that the interlacing property

$$0 < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_{k-1} < \eta_{k-1} < \xi_k, \tag{2.75}$$

holds, and

$$u_{2k+1}(\eta_j) = 0, \quad j = 1, 2, ..., k-1. \tag{2.76}$$

Observing from (2.74) that $u_{2k+1}(\xi_k) < 0$, and using the fact that from (2.64) and (2.65), $\lim_{t\to\infty} u_{2k+1}(t) = \infty$, it similarly follows that there exists a real number $\eta_k \in (\xi_k, \infty)$, such that $u_{2k+1}(\eta_k) = 0$. Hence, exploiting also the fact that u_{2k+1} is an odd function, and thus $u_{2k+1}(0) = 0$, it follows, with the choice $\eta_0 = 0$ and recalling also (2.75) and (2.76), that we have established the existence of a strictly increasing sequence $\{-\eta_k, -\eta_{k-1}, ..., -\eta_1, \eta_0, \eta_1, ..., \eta_k\} \subset \mathbb{R}$, such that

$$u_{2k+1}(-\eta_j) = 0, \quad u_{2k+1}(0) = 0, \quad u_{2k+1}(\eta_j) = 0, \quad j = 1, 2, ..., k. \tag{2.77}$$

Since u_{2k+1} is a polynomial of degree $2k+1$, it follows that u_{2k+1} has $2k+1$ real simple zeros, with

$$(-1)^{k+j} u'_{2k+1}(\eta_j) > 0, \quad j = 0, 1, ..., k, \tag{2.78}$$

having recalled once again the fact that $\lim\limits_{t\to\infty} u_{2k+1}(t) = \infty$.

Next, we use (2.63) and (2.76) to deduce that

$$u_{2k+2}(\eta_j) = -\frac{1+\eta_j^2}{2k+3} u'_{2k+1}(\eta_j), \quad j = 0, 1, ..., k,$$

which, together with (2.78), yield

$$(-1)^{k+j+1} u_{2k+2}(\eta_j) > 0, \quad j = 0, 1, ..., k. \tag{2.79}$$

As before, the intermediate value theorem, together with (2.79), imply the existence of a sequence $\{\zeta_j : j = 0, 1, 2, ..., k-1\} \subset \mathbb{R}$ with

$$0 = \eta_0 < \zeta_0 < \eta_1 < \zeta_1 < \cdots < \zeta_{k-1} < \eta_k, \tag{2.80}$$

such that

$$u_{2k+2}(\zeta_j) = 0, \quad j = 0, 1, ..., k-1. \tag{2.81}$$

Moreover, (2.79) gives $u_{2k+2}(\eta_k) < 0$, and thus, since (2.64), (2.65) imply $\lim\limits_{t\to\infty} u_{2k+2}(t) = \infty$, there exists a real number $\zeta_k \in (\eta_k, \infty)$, such that $u_{2k+2}(\zeta_k) = 0$. Since u_{2k+2} is an even function, it follows, by recalling also (2.80), (2.81), that there exists a sequence $\{\zeta_j : j = 0, 1, ..., k\} \subset \mathbb{R}$, with

$$0 < \zeta_0 < \zeta_1 < \cdots < \zeta_k, \tag{2.82}$$

such that

$$u_{2k+2}(\pm\zeta_j) = 0, \quad j = 1, 2, ..., k. \tag{2.83}$$

Hence u_{2k+2} has $2k+2$ real simple zeros, thereby concluding the inductive proof of the statement (2.69).

Combining the results (2.64), (2.65), (2.66) and (2.69), we deduce that there exists, for each integer $n \geq 2$, a sequence $\{\alpha_{n,j} : j = 1, ..., \lfloor \frac{n}{2} \rfloor\}$, with

$$0 < \alpha_{n,1} < \alpha_{n,2} < \cdots < \alpha_{n,\lfloor \frac{n}{2} \rfloor}, \tag{2.84}$$

such that

$$u_n(z) = \frac{2^n}{(n+1)!} \begin{cases} Q_n(z), & z \in \mathbb{C}, \quad \text{if } n \text{ is even}, \\ z\, Q_n(z), & z \in \mathbb{C}, \quad \text{if } n \text{ is odd}, \end{cases} \tag{2.85}$$

where the polynomial Q_n is defined by

$$Q_n(z) = \prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left(z^2 - \alpha_{n,j}^2\right), \quad z \in \mathbb{C}. \tag{2.86}$$

Now observe that (2.59) implies

$$U_n(z) = i^n u_n(-iz), \quad z \in \mathbb{C}, \quad n = 0, 1, \ldots. \tag{2.87}$$

Substituting (2.85) and (2.86) into (2.87) then gives, for $n \geq 2$,

$$U_n(z) = \frac{2^n}{(n+1)!} \begin{cases} R_n(z), & z \in \mathbb{C}, \quad \text{if } n \text{ is even}, \\ z\, R_n(z), & z \in \mathbb{C}, \quad \text{if } n \text{ is odd}, \end{cases} \tag{2.88}$$

where the polynomial R_n is defined by

$$R_n(z) = \prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} \left(z^2 + \alpha_{n,j}^2\right), \quad z \in \mathbb{C}. \tag{2.89}$$

Now recall (2.55) to deduce from (2.88) that, for $n \geq 2$,

$$e_n(\omega) = \frac{2^n}{(n+1)!} \begin{cases} R_n\left(\cos \frac{\omega}{2}\right), & \omega \in \mathbb{R}, \quad \text{if } n \text{ is even}, \\ \left(\cos \frac{\omega}{2}\right) R_n\left(\cos \frac{\omega}{2}\right), & \omega \in \mathbb{R}, \quad \text{if } n \text{ is odd}, \end{cases} \tag{2.90}$$

and with the polynomial R_n given as in (2.89). Setting $z = \cos\frac{\omega}{2}$ in (2.89), we see that, for $\omega \in \mathbb{R}$,

$$\begin{aligned} z^2 + \alpha_{n,j}^2 &= \left(\cos\frac{\omega}{2}\right)^2 + \alpha_{n,j}^2 \\ &= \left[\frac{e^{i\omega/2} + e^{-i\omega/2}}{2}\right]^2 + \alpha_{n,j}^2 \\ &= \frac{1}{4}\left[e^{i\omega} + (2 + 4\alpha_{n,j}^2) + e^{-i\omega}\right] \\ &= \frac{1}{4e^{i\omega}}\left[\left(e^{i\omega}\right)^2 + 2(1 + 2\alpha_{n,j}^2)e^{i\omega} + 1\right] \\ &= \frac{e^{-i\omega}}{4}\left(e^{i\omega} - \mu_{n,j}\right)\left(e^{i\omega} - \frac{1}{\mu_{n,j}}\right), \end{aligned} \tag{2.91}$$

where

$$\mu_{n,j} = -1 + 2\alpha_{n,j}\left(\sqrt{1 + \alpha_{n,j}^2} - \alpha_{n,j}\right), \tag{2.92}$$

with clearly

$$-1 < \mu_{n,j} = -1 + \frac{2}{\sqrt{1 + \frac{1}{\alpha_{n,j}^2}} + 1} < 0, \tag{2.93}$$

having noted also (2.84).

Finally, we combine (2.47), (2.90), (2.89), (2.91) and (2.92) to deduce that, for $m \geq 4$, m even, and $\omega \in \mathbb{R}$,

$$\begin{aligned} \widetilde{N}_m(e^{i\omega}) &= e^{i\frac{m-2}{2}\omega}\frac{2^{m-2}}{(m-1)!}\prod_{j=1}^{\frac{m-2}{2}}\frac{1}{4}e^{-i\omega}\left(e^{i\omega} - \mu_{m-2,j}\right)\left(e^{i\omega} - \frac{1}{\mu_{m-2,j}}\right) \\ &= \frac{1}{(m-1)!}\prod_{j=1}^{\frac{m-2}{2}}\left(e^{i\omega} - \mu_{m-2,j}\right)\left(e^{i\omega} - \frac{1}{\mu_{m-2,j}}\right), \end{aligned} \tag{2.94}$$

whereas, for $m \geq 5$, m odd, and $\omega \in \mathbb{R}$,

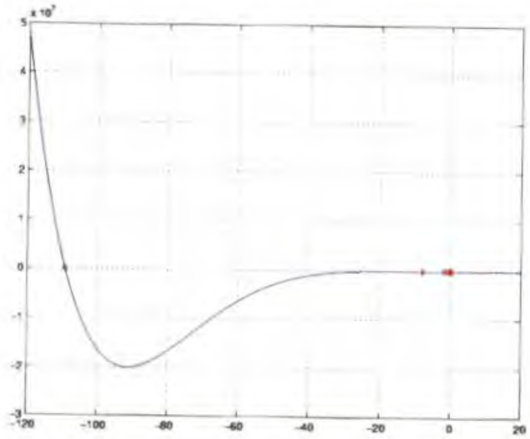
$$\begin{aligned} \widetilde{N}_m(e^{i\omega}) &= e^{i\frac{m-2}{2}\omega}\frac{2^{m-2}}{(m-1)!}\frac{e^{i\omega/2} + e^{-i\omega/2}}{2}\prod_{j=1}^{\frac{m-3}{2}}\frac{1}{4}e^{-i\omega}\left(e^{i\omega} - \mu_{m-2,j}\right)\left(e^{i\omega} - \frac{1}{\mu_{m-2,j}}\right) \\ &= \frac{1}{(m-1)!}(e^{i\omega} + 1)\prod_{j=1}^{\frac{m-3}{2}}\left(e^{i\omega} - \mu_{m-2,j}\right)\left(e^{i\omega} - \frac{1}{\mu_{m-2,j}}\right). \end{aligned} \tag{2.95}$$

If we now define, for $m \geq 4$, the sequence $\{\lambda_{m,j} : j = 1, 2, ..., \lfloor\frac{m-2}{2}\rfloor\} \subset \mathbb{R}$ by

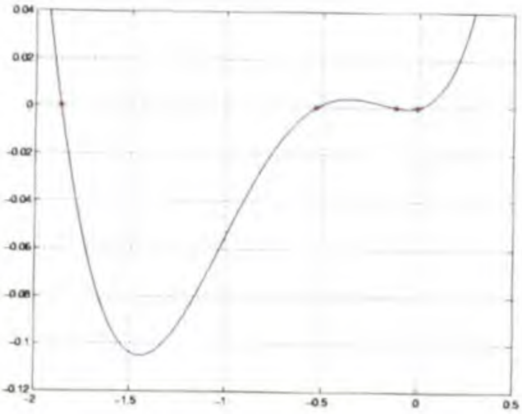
$$\lambda_{m,j} := \mu_{m-2,j}, \quad j = 1, 2, ..., \left\lfloor\frac{m-2}{2}\right\rfloor, \tag{2.96}$$

we see from (2.96), (2.93), (2.94) and (2.95) that the result (2.43), (2.44), (2.45) holds on the unit circle $|z| = 1$. Since $\widetilde{N}_m$ and P_m are polynomials, it follows that (2.44) and (2.45) hold for all $z \in \mathbb{C}$. ■

A numerical illustration of the position of $\widetilde{N}_8$ has been graphed in Fig. 2.3. Observe that N_8 has 6 real simple zeros, of which three lie in the interval $(-1, 0)$, whereas the remaining three lie in the interval $(-\infty, -1)$, as implied by Theorem 2.5.



(a) zeros in the interval $[-120, 20]$



(b) zeros in the interval $[-2, \frac{1}{2}]$

Figure 2.3: *The zeros of the Euler-Frobenius polynomial $\widetilde{N}_8$.*

2.5 Riesz Stability of N_m

First, we define the concept of Riesz stability.

Definition 2.6 Suppose $\phi \in C^{-1}(\mathbb{R})$, and suppose ϕ is compactly supported. We say that ϕ is *Riesz stable* if there exist two real numbers α and β, called the *Riesz bounds,* and with $0 < \alpha \leq \beta < \infty$, such that

$$\alpha||\{c_k\}||_{\ell^2}^2 \leq \left\| \sum_k c_k \phi(\cdot - k) \right\|_2^2 \leq \beta||\{c_k\}||_{\ell^2}^2, \quad \{c_k\} \in \ell^2. \tag{2.97}$$

Now observe from (2.2) that, for $\{c_k\} \in \ell^2$,

$$\begin{aligned} \int_{-\infty}^{\infty} \left| \sum_k c_k N_1(t-k) \right|^2 dt &= \sum_l \int_l^{l+1} \left| \sum_k c_k N_1(t-k) \right|^2 dt \\ &= \sum_l |c_l|^2 \int_l^{l+1} dt \\ &= \sum_l |c_l|^2, \end{aligned}$$

whence, recalling the definition (1.1) and (1.3), the first order cardinal B-spline $N_1 \in C^{-1}(\mathbb{R})$ is Riesz stable with Riesz bounds $\alpha = \beta = 1$.

To investigate the Riesz stability of the cardinal B-spline N_m, with $m \geq 2$, we proceed as follows.

According to a result proved in [3, Theorem 3.24], the *Riesz stability condition* (2.97) has the following equivalent Fourier transform formulation.

Theorem 2.7 *Suppose $\phi \in \mathcal{A}$, and suppose ϕ is a compactly supported function such that*

$$\phi(t), \widehat{\phi}(t) = \mathcal{O}\left(\frac{1}{1+|t|^{\varepsilon}}\right), \quad t \in \mathbb{R}, \tag{2.98}$$

for some $\varepsilon > 1$. Then ϕ is Riesz stable, in the sense of (2.97), with Riesz bounds α and β, if and only if the Fourier transform $\widehat{\phi}$ of ϕ satisfies

$$\alpha \leq \sum_k \left| \widehat{\phi}(\omega + 2\pi k) \right|^2 \leq \beta, \qquad \omega \in \mathbb{R}. \tag{2.99}$$

We show next how the result of Theorem 2.5 can be used to prove that the cardinal B-spline N_m is Riesz stable.

Theorem 2.8 *For $m \in \mathbb{N}$, there exist real numbers α_m and β_m, with $0 < \alpha_m \leq \beta_m$, such that*

$$\alpha_m||\{c_k\}||^2_{\ell^2} \leq \left\| \sum_k c_k N_m(\cdot - k) \right\|_2^2 \leq \beta_m||\{c_k\}||^2_{\ell^2}, \quad \{c_k\} \in \ell^2. \tag{2.100}$$

Proof. As noted above for the case $m = 1$, (2.100) holds with equality, and with Riesz bounds $\alpha_1 = \beta_1 = 1$. Suppose therefore $m \geq 2$. But then from (2.8), (2.1), (2.6) and (2.12) we have that $N_m \in \mathcal{A} \subset L^2(\mathbb{R})$ and that N_m and $\widehat{N}_m$ satisfy condition (2.98), hence it follows from Theorem 2.7 that (2.100) is equivalent to the condition

$$\alpha_m \leq \sum_k \left| \widehat{N}_m(\omega + 2\pi k) \right|^2 \leq \beta_m, \quad \omega \in \mathbb{R}. \tag{2.101}$$

Now define the sequence $\{\gamma_l^{(m)}\} \subset \mathbb{R}$ by

$$\gamma_l^{(m)} := \langle N_m, N_m(\cdot + l) \rangle = \int_{-\infty}^{\infty} N_m(x) N_m(x + l) dx, \quad l \in \mathbb{Z}, \tag{2.102}$$

and define the sequence $\{u_l\}$ of compactly supported functions in $C(\mathbb{R})$ by

$$u_l(t) := N_m(t + l), \quad t \in \mathbb{R}, \quad l \in \mathbb{Z}, \tag{2.103}$$

Whence, for $\omega \in \mathbb{R}$,

$$\begin{aligned} \widehat{u}_l(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} N_m(t + l) dt \\ &= \int_{-\infty}^{\infty} e^{-i\omega(t-l)} N_m(t) dt \\ &= e^{i\omega l} \int_{-\infty}^{\infty} e^{-i\omega t} N_m(t) dt, \end{aligned}$$

and thus, from (1.5),

$$\widehat{u}_l(\omega) = e^{i\omega l} \widehat{N}_m(\omega), \quad \omega \in \mathbb{R}. \tag{2.104}$$

Then, for $l \in \mathbb{Z}$, we have from (2.8), (2.1), (2.6) and (2.12) that $N_m, u_l \in \mathcal{A}$ and hence, from (2.102), (1.10), (1.2), (2.104) and (1.5), we obtain

$$\begin{aligned}
\gamma_l^{(m)} &= \langle N_m, u_l \rangle \\
&= \frac{1}{2\pi} \left\langle \widehat{N}_m, \widehat{u}_l \right\rangle \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \widehat{N}_m(\omega) \right|^2 e^{-i\omega l} d\omega \\
&= \frac{1}{2\pi} \sum_k \int_{2\pi k}^{2\pi(k+1)} \left| \widehat{N}_m(\omega) \right|^2 e^{-i\omega l} d\omega \\
&= \frac{1}{2\pi} \sum_k \int_0^{2\pi} \left| \widehat{N}_m(\omega + 2\pi k) \right|^2 e^{-i\omega l} d\omega \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_k \left| \widehat{N}_m(\omega + 2\pi k) \right|^2 \right] e^{-i\omega l} d\omega. \qquad (2.105)
\end{aligned}$$

Observe, from (2.8), (2.1), (2.6) and (2.12), that N_m and $\widehat{N}_m$ satisfy the condition (2.98), and hence, from Theorem 1.3, the function G, defined by

$$G(\omega) := \sum_k \left| \widehat{N}_m(\omega + 2\pi k) \right|^2, \quad \omega \in \mathbb{R}, \qquad (2.106)$$

is a real-valued continuous function on $\mathbb{R}$ which is 2π-periodic:

$$G(\omega + 2\pi) = G(\omega), \quad \omega \in \mathbb{R}, \qquad (2.107)$$

with

$$G(\omega) = \sum_k \left[\frac{1}{2\pi} \int_0^{2\pi} G(t) e^{-ikt} dt \right] e^{ik\omega}, \quad \omega \in \mathbb{R}. \qquad (2.108)$$

Thus, from (2.108), (2.106) and (2.105), we have

$$\sum_k \left| \widehat{N}_m(\omega + 2\pi k) \right|^2 = \sum_k \gamma_k^{(m)} e^{ik\omega}, \quad \omega \in \mathbb{R}. \qquad (2.109)$$

But from (2.102) and (2.16), we have, for $k \in \mathbb{Z}$, that

$$\begin{aligned}
\gamma_k^{(m)} &= \int_{-\infty}^{\infty} N_m(x) N_m(x+k) dx \\
&= \int_{-\infty}^{\infty} N_m(m-x) N_m(x+k) dx \\
&= \int_{-\infty}^{\infty} N_m(x) N_m(m+k-x) dx \\
&= (N_m * N_m)(m+k), \qquad (2.110)
\end{aligned}$$

having also used (1.4). But, from (2.18), and using (1.27), we have

$$\begin{aligned} N_{2m} &= \underbrace{N_1 * N_1 * \cdots * N_1}_{2m \text{ times}} \\ &= \underbrace{(N_1 * \cdots * N_1)}_{m \text{ times}} * \underbrace{(N_1 * \cdots * N_1)}_{m \text{ times}} \\ &= N_m * N_m. \end{aligned} \tag{2.111}$$

Now, from (2.110), (2.111) and (2.109), we have that for $\omega \in \mathbb{R}$,

$$\begin{aligned} \sum_k \left|\widehat{N}_m(\omega + 2\pi k)\right|^2 &= \sum_k N_{2m}(m+k)e^{ik\omega} \\ &= \sum_k N_{2m}(k+1)e^{i(k+1-m)\omega} \\ &= e^{i(1-m)\omega} \sum_k N_{2m}(k+1)e^{ik\omega} \\ &= e^{i(1-m)\omega} \sum_{k=0}^{2m-2} N_{2m}(k+1)e^{ik\omega} \\ &= e^{i(1-m)\omega} \widetilde{N}_m(e^{i\omega}), \end{aligned} \tag{2.112}$$

from (2.6), and where $\widetilde{N}_{2m}$ is the Euler-Frobenius polynomial as defined in (2.40).

Now combine (2.112), (2.44) and (2.45) to obtain, for $\omega \in \mathbb{R}$,

$$\begin{aligned} \sum_k \left|\widehat{N}_m(\omega + 2\pi k)\right|^2 &= \frac{e^{i\omega(1-m)}}{(m-1)!} \prod_{j=1}^{m-1} \left(e^{i\omega} - \lambda_{2m,j}\right)\left(e^{i\omega} - \frac{1}{\lambda_{2m,j}}\right) \\ &= \left|\frac{e^{i\omega(1-m)}}{(m-1)!} \prod_{j=1}^{m-1} \left(e^{i\omega} - \lambda_{2m,j}\right)\left(e^{i\omega} - \frac{1}{\lambda_{2m,j}}\right)\right| \\ &= \frac{1}{(m-1)!} \prod_{j=1}^{m-1} \left|e^{i\omega} - \lambda_{2m,j}\right| \left|e^{i\omega} - \frac{1}{\lambda_{2m,j}}\right| \\ &= \frac{1}{(m-1)!} \prod_{j=1}^{m-1} \left|1 - \lambda_{2m,j}e^{-i\omega}\right| \left|e^{i\omega} - \frac{1}{\lambda_{2m,j}}\right| \\ &= \frac{1}{(m-1)!} \prod_{j=1}^{m-1} \left|\left(1 - \lambda_{2m,j}e^{-i\omega}\right)\left(e^{i\omega} - \frac{1}{\lambda_{2m,j}}\right)\right| \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\left|e^{i\omega}+e^{-i\omega}-\left(\lambda_{2m,j}+\frac{1}{\lambda_{2m,j}}\right)\right| \\
&= \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\left|2\cos\omega-\left(\lambda_{2m,j}+\frac{1}{\lambda_{2m,j}}\right)\right| \\
&= \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\left|\frac{\lambda_{2m,j}^2-2\lambda_{2m,j}\cos\omega+1}{\lambda_{2m,j}}\right| \\
&= \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\frac{\left|\lambda_{2m,j}^2-2\lambda_{2m,j}\cos\omega+1\right|}{\left|\lambda_{2m,j}\right|}. \qquad (2.113)
\end{aligned}
$$

But, since (2.43) gives

$$
-1<\lambda_{2m,j}<0, \quad j=1,2,...,m-1, \qquad (2.114)
$$

we see that, for $\omega \in \mathbb{R}$,

$$
\lambda_{2m,j}^2-2\lambda_{2m,j}\cos\omega+1 \geq \lambda_{2m,j}^2+2\lambda_{2m,j}+1=(\lambda_{2m,j}+1)^2,
$$

which, together with (2.113), yields, for $\omega \in \mathbb{R}$,

$$
\sum_k \left|\widehat{N}_m(\omega+2\pi k)\right|^2 \geq \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\frac{(\lambda_{2m,j}+1)^2}{\left|\lambda_{2m,j}\right|}, \qquad (2.115)
$$

and thus

$$
\sum_k \left|\widehat{N}_m(\omega+2\pi k)\right|^2 \geq \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\frac{(\lambda_{2m,j}+1)^2}{-\lambda_{2m,j}}, \quad \omega\in\mathbb{R}, \qquad (2.116)
$$

by virtue of (2.114) and (2.115). Now observe from (2.112) and (2.40) that, for $\omega \in \mathbb{R}$,

$$
\begin{aligned}
\sum_k \left|\widehat{N}_m(\omega+2\pi k)\right|^2 &= \left|e^{i\omega(1-m)}\sum_{k=0}^{m-2}N_{2m}(k+1)e^{i\omega k}\right| \\
&= \left|\sum_{k=0}^{m-2}N_{2m}(k+1)e^{i\omega k}\right| \\
&\leq \sum_{k=0}^{m-2}N_{2m}(k+1) \\
&\leq \sum_k N_{2m}(k+1)=1, \qquad (2.117)
\end{aligned}
$$

from (2.5) and (2.7). Hence, combining (2.116) and (2.117), we see that the condition (2.101) is satisfied, with Riesz bounds

$$\alpha_m = \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\frac{(\lambda_{2m,j}+1)^2}{-\lambda_{2m,j}}, \tag{2.118}$$

and

$$\beta_m = 1, \tag{2.119}$$

and therefore the Riesz stability condition also holds for $m \geq 2$ with α_m and β_m as given by (2.118) and (2.119), having recalled also the fact that $\lambda_{m,j} < 0$ from (2.114). ■

Chapter 3

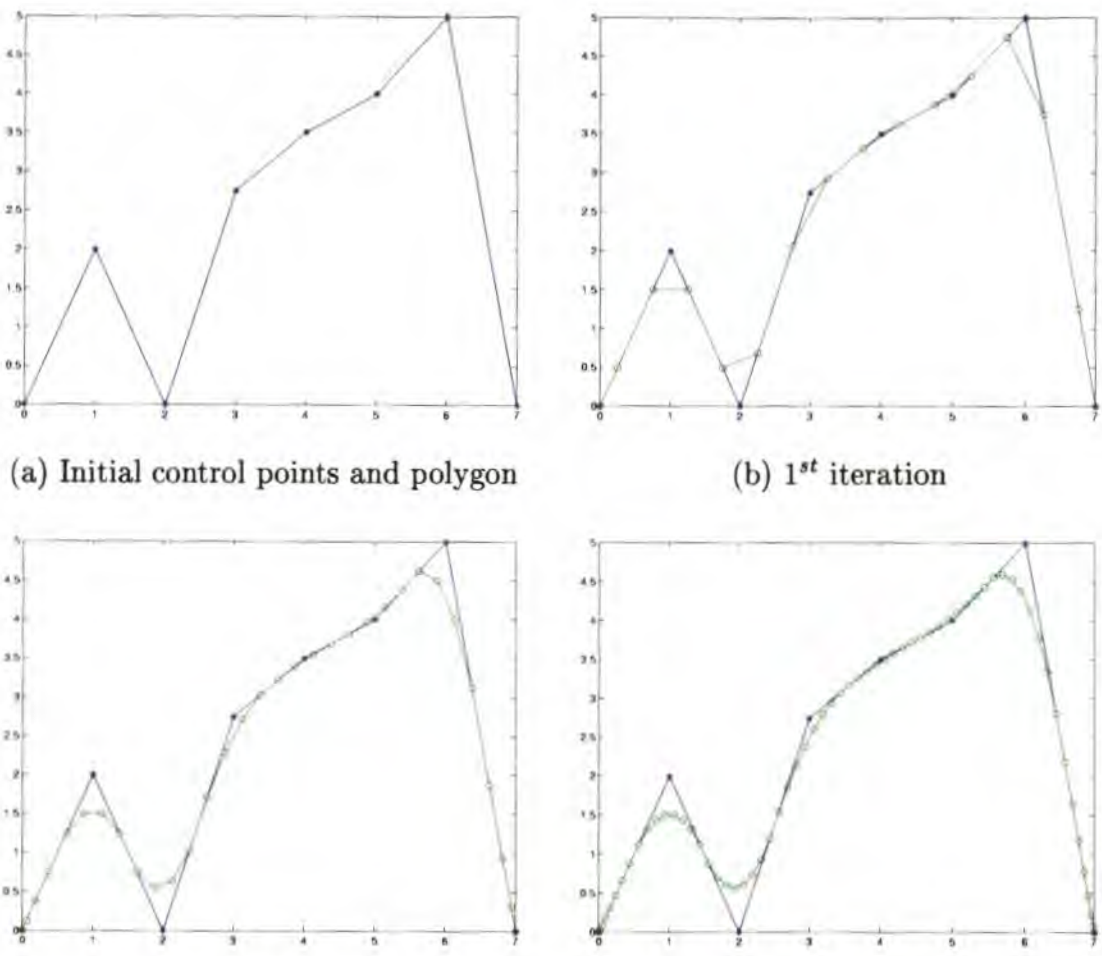
Stationary Subdivision

In this chapter we introduce the concepts of a stationary subdivision scheme (SSS) and its corresponding refinable function. In Section 3.1, we define the de Rahm-Chaikin "corner-cutting" (or "corner-smoothing") algorithm. We also provide a geometric proof for the convergence of this algorithm. In Section 3.2 we consider the more general Lane-Riesenfeld subdivision algorithm, which we introduce as an algorithm for the efficient computation of cardinal spline functions. A convergence result for the Lane-Riesenfeld algorithm then rigorously also yields the convergence for the de Rahm-Chaikin algorithm. Building on these specific concepts, we next formulate, in Section 3.3, a general stationary subdivision operator. Then, in Section 3.4, we quote some results pertaining to the convergence of the subdivision scheme, and the smoothness of the corresponding refinable function, for a certain class of subdivision schemes.

3.1 The de Rahm-Chaikin Algorithm

Consider the following simple iterative procedure: begin with an ordered sequence $\{c_j : j \in \mathbb{Z}\} \in \mathbb{R}^2$, which we shall call the *control points*. Then join these control points with straight-line segments, thereby forming the *initial control polygon* with respect to $\{c_j\}$, as illustrated by the example drawn in Figure 3.1(a). On each segment of the control polygon we then choose two new control points, each a quarter of the length from the ends of the segment.

These new control points yield a new control polygon, as illustrated in the example drawn in Figure 3.1(b), on a finer lattice.



(a) Initial control points and polygon

(b) 1^{st} iteration

(c) 2^{nd} iteration

(d) 3^{rd} iteration

Figure 3.1: *Illustration of de Rahm-Chaikin algorithm: initial ($*$) and new ($\circ$) control points*

Repeated application of the above procedure, as illustrated in Figure 3.1, is called the *de Rahm-Chaikin algorithm* (see [8], [25, Section 2.1]), which, as shown in Section 3.2 below, can be interpreted as a special case of the Lane-Riesenfeld algorithm (see [21], [25, Section 2.3]), for the computation of cardinal spline functions of any given order. A rigorous proof of the convergence of the de Rahm-Chaikin algorithm will therefore follow from the main convergence theorem of Section 3.2. We present the following geometric convergence argument, which extends the geometric argument in [25, pp.55-58], and which is due to Riesenfeld [27].

We shall show geometrically that, if the initial control points $\{c_k\}$ satisfy the condition $\sup_k ||c_{k+1} - c_k|| < \infty$, where $||\cdot||$ denotes the Euclidean norm in $\mathbb{R}^2$, then, in the limit, the de Rahm-Chaikin algorithm produces a sequence that converges to points on a C^1-smooth curve.

With the notation $\{c_k^{(0)}\} := \{c_k\} \subset \mathbb{R}^2$, we see that the de Rahm-Chaikin algorithm generates the sequence $\{c_k^{(j)}\} \subset \mathbb{R}^2, j = 0, 1, \ldots$, where

$$\left.\begin{aligned} c_{2k-1}^{(j+1)} &= \tfrac{3}{4}c_k^{(j)} + \tfrac{1}{4}c_{k+1}^{(j)}, \\ c_{2k}^{(j+1)} &= \tfrac{1}{4}c_k^{(j)} + \tfrac{3}{4}c_{k+1}^{(j)}, \end{aligned}\right\} \quad k \in \mathbb{Z}, \quad j = 0, 1, \ldots. \tag{3.1}$$

Now define the sequence of midpoints $\{d_k^{(j)}\} \subset \mathbb{R}^2, j = 0, 1, \ldots$, by

$$d_k^{(j)} = \frac{1}{2}\left[c_k^{(j)} + c_{k+1}^{(j)}\right], \quad k \in \mathbb{Z}, \ \ j = 0, 1, \ldots, \tag{3.2}$$

and denote by $\{Q_k^{(j)}\}, j = 0, 1, \ldots$, the sequence of Bernstein-Bézier curves given by

$$Q_k^{(j)}(t) = d_k^{(j)}(1-t)^2 + 2c_{k+1}^{(j)}t(1-t) + d_{k+1}^{(j)}t^2, \ \ t \in [0,1], \ \ k \in \mathbb{Z}, \ \ j = 0, 1, \ldots. \tag{3.3}$$

Observe from (3.3) that then

$$\left.\begin{aligned} &Q_k^{(j)}(0) = d_k^{(j)}, && Q_k^{(j)}(1) = d_{k+1}^{(j)}, \\ &\left(\tfrac{d}{dt}Q_k^{(j)}\right)(0) = 2\left[c_{k+1}^{(j)} - d_k^{(j)}\right], && \left(\tfrac{d}{dt}Q_k^{(j)}\right)(1) = 2\left[d_{k+1}^{(j)} - c_{k+1}^{(j)}\right], \end{aligned}\right\} \quad k \in \mathbb{Z}, \quad j = 0, 1, \ldots.$$

Hence, if we define the sequence of curves $Q^{(j)}$, $j = 0, 1, \ldots$, by

$$Q^{(j)} := \bigcup_{k \in \mathbb{Z}} Q_k^{(j)}, \quad j = 0, 1, \ldots, \tag{3.4}$$

we see from (3.4) and (3.2) that, for a given $j \in \mathbb{Z}$, the curve $Q^{(j)}$ is a C^1-smooth curve, with

$$d_k^{(j)} \in Q_k^{(j)} \subset Q^{(j)}, \quad k \in \mathbb{Z}, \ \ j = 0, 1, \ldots, \tag{3.5}$$

and, if we write, for a given $j \in \mathbb{Z}$, $C^{(j)}$ for the control polygon formed by the sequence $\{c_k^{(j)}\}$, then, for each $j \in \mathbb{Z}$, the curve $Q^{(j)}$ is tangential to $C^{(j)}$ at the midpoint sequence $\{d_k^{(j)}\}$.

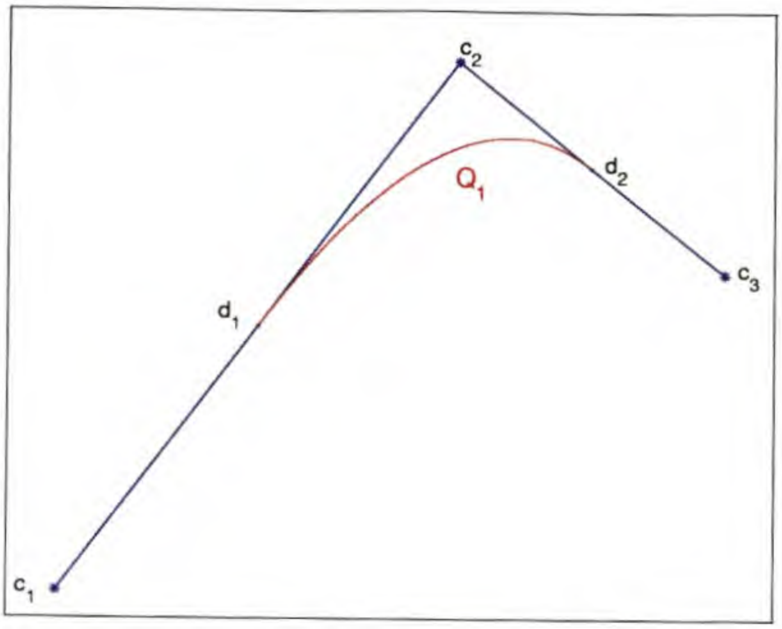


Figure 3.2: *The control points* $\{c_1, c_2, c_3\}$, *the corresponding midpoints* $\{d_1, d_2\}$, *and the Bernstein-Bézier curve* Q_1.

We shall use the notation

$$\{d_k\} := \{d_k^{(0)}\}, \quad \{Q_k\} := \{Q_k^{(0)}\}, \text{ and } Q := Q^{(0)}.$$

An example is drawn in Figure 3.2. We claim that

$$Q^{(j+1)} \subset Q^{(j)}, \quad j = 0, 1, \ldots. \tag{3.6}$$

From (3.4), we see that (3.6) will follow if we can show that

$$Q_k^{(j+1)} \subset Q^{(j)}, \quad k \in \mathbb{Z}, \ j = 0, 1, \ldots. \tag{3.7}$$

To prove (3.7), we first observe from (3.3), (3.1) and (3.2) that, for $j \in \{0, 1, \ldots\}$, $k \in \mathbb{Z}$ and $t \in \left[0, \frac{1}{2}\right]$,

$$\begin{aligned}
Q_{2k-1}^{(j+1)}(2t) &= \frac{1}{2}\left[c_{2k-1}^{(j+1)} + c_{2k}^{(j+1)}\right](1-2t)^2 + 2c_{2k}^{(j+1)}(2t)(1-2t) + \frac{1}{2}\left[c_{2k}^{(j+1)} + c_{2k+1}^{(j+1)}\right](2t)^2 \\
&= \frac{1}{2}\left[c_k^{(j)} + c_{k+1}^{(j)}\right](1-2t)^2 + 2\left[\frac{1}{4}c_k^{(j)} + \frac{3}{4}c_{k+1}^{(j)}\right](2t)(1-2t) \\
&\quad + \frac{1}{2}\left[\frac{1}{4}c_k^{(j)} + \frac{3}{2}c_{k+1}^{(j)} + \frac{1}{4}c_{k+2}^{(j)}\right]4t^2
\end{aligned}$$

$$
\begin{aligned}
&= d_k^{(j)}(1-4t+4t^2) + \left[d_k^{(j)} + c_{k+1}^{(j)}\right](2t-4t^2) + \left[d_k^{(j)} + 2c_{k+1}^{(j)} + d_{k+1}^{(j)}\right]t^2 \\
&= d_k^{(j)}(1-t)^2 + 2c_{k+1}^{(j)}t(1-t) + d_{k+1}^{(j)}t^2,
\end{aligned}
$$

which, together with (3.3), yields

$$
Q_{2k-1}^{(j+1)}(2t) = Q_k^{(j)}(t), \quad t \in \left[0, \frac{1}{2}\right], \quad k \in \mathbb{Z}, \;\; j = 0, 1, \dots . \tag{3.8}
$$

Next, we use (3.3), (3.1) and (3.2) to deduce $j \in \{0, 1, \dots\}$ and $k \in \mathbb{Z}$, and $t \in \left[\frac{1}{2}, 1\right]$,

$$
\begin{aligned}
Q_{2k}^{(j+1)}(2t-1) &= \frac{1}{2}\left[c_{2k}^{(j+1)} + c_{2k+1}^{(j+1)}\right](1-(2t-1))^2 + 2c_{2k+1}^{(j+1)}(2t-1)(1-(2t-1)) \\
&\quad + \frac{1}{2}\left[c_{2k+1}^{(j+1)} + c_{2k+2}^{(j+1)}\right](1-2t)^2 \\
&= \left[\frac{1}{2}c_k^{(j)} + 3c_{k+1}^{(j)} + \frac{1}{2}c_{k+2}^{(j)}\right](1-t)^2 + \left[3c_{k+1}^{(j)} + c_{k+2}^{(j)}\right](2t-1)(1-t) \\
&\quad + \frac{1}{2}\left[c_{k+1}^{(j)} + c_{k+2}^{(j)}\right](1-2t)^2 \\
&= \left[d_k^{(j)} + 2c_{k+1}^{(j)} + d_{k+1}^{(j)}\right](1-2t+t^2) + 2\left[d_{k+1}^{(j)} + c_{k+1}^{(j)}\right](-1+3t-2t^2) \\
&\quad + d_{k+1}^{(j)}(1-4t+4t^2) \\
&= d_k^{(j)}(1-t)^2 + 2c_{k+1}^{(j)}t(1-t) + d_{k+1}^{(j)}t^2,
\end{aligned}
$$

which together with (3.3). yield

$$
Q_{2k}^{(j+1)}(2t-1) = Q_k^{(j)}(t), \quad t \in \left[\frac{1}{2}, 1\right], \quad k \in \mathbb{Z}, \;\; j = 0, 1, \dots . \tag{3.9}
$$

Combining (3.8), (3.9) and (3.4) then shows that (3.7) holds.

Hence (3.6) is true, and thus

$$
Q^{(j)} \subset Q, \quad j = 0, 1, \dots . \tag{3.10}
$$

It follows from (3.5) and (3.10) that

$$
d_k^{(j)} \in Q, \quad k \in \mathbb{Z}, \quad j = 0, 1, \dots, \tag{3.11}
$$

that is, all the midpoints $d_k^{(j)}$, $k \in \mathbb{Z}$, $j = 0, 1, \dots$, lie on the C^1-smooth curve Q which meets the initial control polygon $C := C^{(0)}$ tangentially at the midpoint sequence $\{d_k\}$.

We claim that

$$\sup_k ||c_{k+1}^{(j)} - c_k^{(j)}|| \leq \left(\frac{1}{2}\right)^j M, \quad j = 0, 1, ..., \tag{3.12}$$

where

$$M := \sup_k ||c_{k+1} - c_k||. \tag{3.13}$$

Noting from (3.13) that (3.12) holds for $j = 0$, we proceed to prove (3.12) inductively as follows. Assuming that (3.12) holds for a given $j \in \{0, 1, ...\}$, we use (3.1) to obtain, for $k \in \mathbb{Z}$,

$$||c_{2k}^{(j+1)} - c_{2k-1}^{(j+1)}|| = \frac{1}{2}||c_{k+1}^{(j)} - c_k^{(j)}|| \leq \left(\frac{1}{2}\right)^{j+1} M,$$

and thus

$$\sup_k ||c_{2k}^{(j+1)} - c_{2k-1}^{(j+1)}|| \leq \left(\frac{1}{2}\right)^{j+1} M. \tag{3.14}$$

Similarly, from (3.1), we get for $k \in \mathbb{Z}$, that

$$\begin{aligned} ||c_{2k+1}^{(j+1)} - c_{2k}^{(j+1)}|| &= \frac{1}{4}||c_{k+2}^{(j)} - c_k^{(j)}|| \\ &= \frac{1}{4}\left(||c_{k+2}^{(j)} - c_{k+1}^{(j)}|| + ||c_{k+1}^{(j)} - c_k^{(j)}||\right) \\ &= \frac{1}{4}\left(\left(\frac{1}{2}\right)^j M + \left(\frac{1}{2}\right)^j M\right) \\ &= \left(\frac{1}{2}\right)^{j+1} M, \end{aligned}$$

and thus

$$\sup_k ||c_{2k+1}^{(j+1)} - c_{2k}^{(j+1)}|| \leq \left(\frac{1}{2}\right)^{j+1} M. \tag{3.15}$$

Combining (3.14) and (3.15) then concluded the inductive proof of (3.12).

Now observe from (3.2) that, for $k \in \mathbb{Z}$, $j \in \{0, 1, ...\}$,

$$||c_k^{(j)} - d_k^{(j)}|| = \frac{1}{2}||c_{k+1}^{(j)} - c_k^{(j)}||,$$

which, together with (3.12), gives the estimate

$$||c_k^{(j)} - d_k^{(j)}|| \leq \left(\frac{1}{2}\right)^{j+1} M, \quad k \in \mathbb{Z}, \quad j = 0, 1, \ldots. \tag{3.16}$$

Combining (3.16) and (3.11) then shows that, for a given $k \in \mathbb{Z}$, $c_k^{(j)} \to Q$ for $j \to \infty$, thereby providing a geometric proof for the fact that the de Rahm-Chaikin algorithm converges to the C^1-smooth limit curve Q. An example is drawn in Figure 3.3.

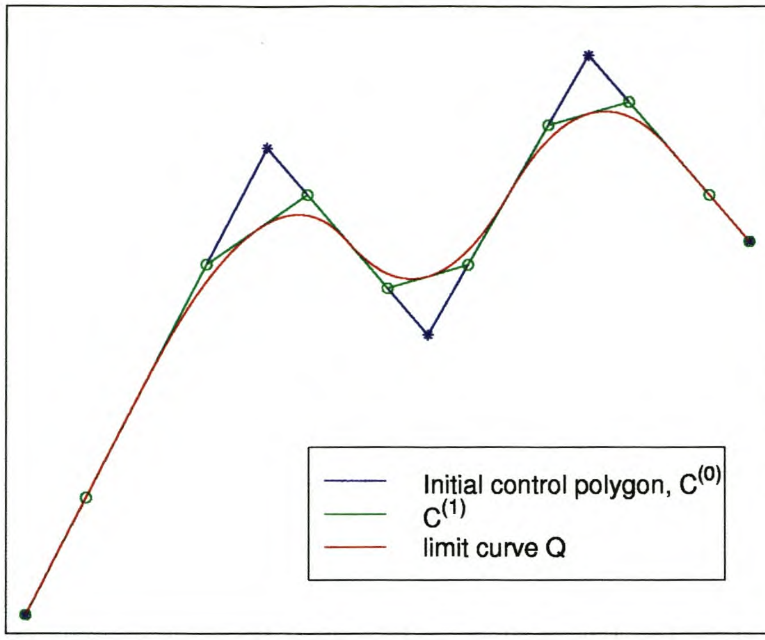


Figure 3.3: *An illustration on the convergence of the de Rahm-Chaikin algorithm.*

3.2 Lane-Riesenfeld Subdivision

In this section we introduce an iterative algorithm for the computation of a cardinal spline.

Suppose therefore, for an integer $m \geq 2$, we consider a cardinal spline function $f \in S_m(\mathbb{Z})$. According to Theorem 2.3 there exists a unique sequence $\{c_j\} \subset \mathbb{R}$ such that

$$f(t) = \sum_j c_j N_m(t-j), \quad t \in \mathbb{R}. \tag{3.17}$$

Our purpose here is to construct (recursively) a sequence $c^{(r)} = \{c_j^{(r)}\}$, $r = 0, 1, ...$, with $c^{(0)} = \{c_j\}$, of sequences in $\mathbb{R}$ such that $c^{(r)}$ uniformly approaches points on the curve f in the sense that, for each fixed $j \in \mathbb{Z}$, we have $\left|f\left(\frac{j}{2^r}\right) - c_{j-1}^{(r)}\right| \to 0$, $r \to \infty$, keeping in mind also the fact that the sequence $\{\frac{j}{2^r} : j \in \mathbb{Z},\ r = 0, 1, ...\}$ is dense in $\mathbb{R}$.

With the usual convention that $\binom{m}{k} = 0$ if $k < 0$ or $k > m$, we next observe from (2.21), (see also (2.24)) that, with the sequence $\{a_k^{(m)}\}$ defined by

$$a_k^{(m)} := \frac{1}{2^{m-1}} \begin{pmatrix} m \\ k \end{pmatrix}, \quad k \in \mathbb{Z}, \quad m \in \mathbb{N}, \tag{3.18}$$

we have

$$N_m(t) = \sum_k a_k^{(m)} N_m(2t-k), \quad t \in \mathbb{R} \quad m \in \mathbb{N}. \tag{3.19}$$

But then (3.17) and (3.19) give, for $t \in \mathbb{R}$,

$$\begin{aligned} f(t) &= \sum_j c_j \sum_k a_k^{(m)} N_m(2t-2j-k) \\ &= \sum_j c_j \sum_k a_{k-2j}^{(m)} N_m(2t-k) \\ &= \sum_k \left[\sum_j a_{k-2j}^{(m)} c_j \right] N_m(2t-k). \end{aligned} \tag{3.20}$$

We now define the operator S_m mapping the space of real sequences into itself by $S_m c = \{(S_m c)_k\}$, with

$$(S_m c)_k = \sum_j a_{k-2j}^{(m)} c_j, \quad k \in \mathbb{Z}, \quad c = \{c_j\} \subset \mathbb{R}. \tag{3.21}$$

Then (3.20) and (3.21) imply

$$f(t) = \sum_j (S_m c)_j N_m(2t-j), \quad t \in \mathbb{R}. \tag{3.22}$$

Now use (3.19) again to deduce from (3.22) that, for $t \in \mathbb{R}$ and $c = \{c_j\} \subset \mathbb{R}$,

$$\begin{aligned} f(t) &= \sum_j (S_m c)_j \sum_k a_k^{(m)} N_m(4t-2j-k) \\ &= \sum_j (S_m c)_j \sum_k a_{k-2j}^{(m)} N_m(4t-k) \\ &= \sum_k \left[\sum_j a_{k-2j}^{(m)} (S_m c)_j \right] N_m(4t-k) \\ &= \sum_k \left(S_m^2 c\right)_k N_m(4t-k). \end{aligned} \tag{3.23}$$

Repeating the procedure, which led to (3.22, and (3.23) therefore yields the following result.

Proposition 3.1 *The cardinal spline function f given by (3.17) satisfies the identity*

$$f(t) = \sum_j c_j^{(r)} N_m(2^r t - j), \quad t \in \mathbb{R}, \quad r = 0, 1, ..., \tag{3.24}$$

where $c^{(0)} = c = \{c_j\}$ *and*

$$c_j^{(r)} = \left(S_m c^{(r-1)}\right)_j = (S_m^r c)_j\,, \quad j \in \mathbb{Z}, \quad r \in \mathbb{N}, \tag{3.25}$$

and with the operator S_m *defined by (3.21) and (3.18).*

Computing from (3.18), we find the values

$$\left.\begin{aligned} a_0^{(1)} &= 1, \\ a_1^{(1)} &= 1, \\ a_k^{(1)} &= 0, \quad k \notin \{0,1\}; \end{aligned}\right\} \tag{3.26}$$

$$\left.\begin{aligned} a_0^{(2)} &= 1/2, \\ a_1^{(2)} &= 1, \\ a_2^{(2)} &= 1/2, \\ a_k^{(2)} &= 0, \quad k \notin \{0,1,2\}; \end{aligned}\right\} \tag{3.27}$$

$$\left.\begin{aligned} a_0^{(3)} &= 1/4, \\ a_1^{(3)} &= 3/4, \\ a_2^{(3)} &= 3/4, \\ a_3^{(3)} &= 1/4, \\ a_k^{(3)} &= 0, \quad k \notin \{0,1,2,3\}, \end{aligned}\right\} \tag{3.28}$$

and

$$\left.\begin{aligned} a_0^{(4)} &= 1/8, \\ a_1^{(4)} &= 1/2, \\ a_2^{(4)} &= 3/4, \\ a_3^{(4)} &= 1/2, \\ a_4^{(4)} &= 1/8, \\ a_k^{(4)} &= 0, \quad k \notin \{0,1,2,3,4\}. \end{aligned}\right\} \tag{3.29}$$

In general, the recursive algorithm (3.25), (3.21), (3.18) is known as the *Lane-Riesenfeld subdivision algorithm*, see also [25, pp. 61-67] and [21].

Also, note from (3.21), (3.28) and (3.25) that, for the choice $m = 3$ in (3.21), we obtain precisely the de Rahm-Chaikin algorithm (3.1). Hence the de Rahm-Chaikin algorithm can be interpreted as the case $m = 3$ of the Lane-Riesenfeld subdivision algorithm, but where the B-spline coefficient sequence $\{c_j\}$ in (3.17) is replaced by the control points $\{c_j\} \subset \mathbb{R}^2$.

For a given sequence $c = \{c_j\} \subset \mathbb{R}$, we define the backward difference operator Δ which maps the space of real sequences into itself by

$$\left.\begin{aligned} &\Delta c = \{(\Delta c)_j\} \subset \mathbb{R}, \\ \text{with} \quad & \\ &(\Delta c)_j := c_j - c_{j-1}, \quad j \in \mathbb{Z}. \end{aligned}\right\} \tag{3.30}$$

Observe in particular that, if $c \in \ell^\infty$, then $\Delta c \in \ell^\infty$.

The following fundamental inequality then holds.

Theorem 3.2 *Suppose, for a given integer $m \geq 2$, that $f \in S_m(\mathbb{Z})$, and suppose that, in the B-spline representation (3.17) of f, we have $c = \{c_j\} \in \ell^\infty$. Then the Lane-Riesenfeld sequence $\{c^{(r)} : r = 0, 1, ...\}$ defined by $c^{(0)} = c$ and (3.25), (3.21), (3.18), satisfies the inequality*

$$\left| f\left(\frac{j}{2^r}\right) - c_{j-1}^{(r)} \right| \leq \frac{m-2}{2^r} ||\Delta c||_{\ell^\infty}, \quad j \in \mathbb{Z}, \quad r = 0, 1, \ldots . \tag{3.31}$$

Proof. Using (3.24) and (2.7), we get, for $j \in \mathbb{Z}$ and $r \in \{0, 1, ...\}$,

$$\begin{aligned} \left| f\left(\frac{j}{2^r}\right) - c_{j-1}^{(r)} \right| &= \left| \sum_k c_k^{(r)} N_m(j-k) - c_{j-1}^{(r)} \right| \\ &= \left| \sum_k c_{j-k}^{(r)} N_m(k) - c_{j-1}^{(r)} \right| \\ &= \left| \sum_k \left(c_{j-k}^{(r)} - c_{j-1}^{(r)} \right) N_m(k) \right| \\ &= \left| \sum_{k=1}^{m-1} \left(c_{j-k}^{(r)} - c_{j-1}^{(r)} \right) N_m(k) \right| \\ &\leq \sum_{k=1}^{m-1} \left| c_{j-k}^{(r)} - c_{j-1}^{(r)} \right| N_m(k), \end{aligned} \tag{3.32}$$

after having recalled also (2.5) and (2.6).

We proceed to prove that $\Delta c^{(r)} \in \ell^\infty$, with

$$\left|\left|\Delta c^{(r)}\right|\right|_{\ell^\infty} \leq \frac{1}{2^r} \left|\left|\Delta c\right|\right|_{\ell^\infty}, \quad r = 0, 1, \ldots . \tag{3.33}$$

Noting that (3.33) holds trivially for $r = 0$, we next observe from (3.25) and (3.21) that

$$c_j^{(1)} - c_{j-1}^{(1)} = \sum_k a_{j-2k}^{(m)} c_k - \sum_k a_{j-1-2k}^{(m)} c_k, \quad j \in \mathbb{Z}. \tag{3.34}$$

But, from (3.18), we have,

$$a_k^{(m)} = \frac{1}{2^{m-1}} \left[\begin{pmatrix} m-1 \\ k \end{pmatrix} + \begin{pmatrix} m-1 \\ k-1 \end{pmatrix} \right], \quad k \in \mathbb{Z} \quad m \geq 2,$$

and thus

$$a_k^{(m)} = \frac{1}{2} \left[a_k^{(m-1)} + a_{k-1}^{(m-1)} \right], \quad k \in \mathbb{Z}, \quad m \geq 2. \tag{3.35}$$

It follows from (3.34) and (3.35) that, for $j \in \mathbb{Z}$,

$$\begin{aligned} c_j^{(1)} - c_{j-1}^{(1)} &= \sum_k \frac{1}{2} \left[a_{j-2k}^{(m-1)} + a_{j-1-2k}^{(m-1)} \right] c_k - \sum_k \frac{1}{2} \left[a_{j-1-2k}^{(m-1)} + a_{j-2-2k}^{(m-1)} \right] c_k \\ &= \frac{1}{2} \left[\sum_k a_{j-2k}^{(m-1)} c_k - \sum_k a_{j-2-2k}^{(m-1)} c_k \right]. \end{aligned} \tag{3.36}$$

We claim that

$$\left| \sum_k a_{j-2k}^{(m-1)} c_k - \sum_k a_{j-2-2k}^{(m-1)} c_k \right| \leq \left|\left|\Delta c\right|\right|_{\ell^\infty}, \quad j \in \mathbb{Z}. \tag{3.37}$$

Suppose first $m = 2$ in (3.37). Then, for $j = 2l$, $l \in \mathbb{Z}$, we have

$$\begin{aligned} \sum_k a_{2l-2k}^{(1)} c_k - \sum_k a_{2l-2-2k}^{(1)} c_k &= \sum_k a_{2k}^{(1)} c_{l-k} - \sum_k a_{2k}^{(1)} c_{l-1-k} \\ &= c_l - c_{l-1}, \end{aligned} \tag{3.38}$$

by virtue of (3.26), whereas, for $j = 2l + 1$, $l \in \mathbb{Z}$, we have

$$\begin{aligned} \sum_k a_{2l+1-2k}^{(1)} c_k - \sum_k a_{2l-1-2k}^{(1)} c_k &= \sum_k a_{2k+1}^{(1)} c_{l-k} - \sum_k a_{2k+1}^{(1)} c_{l-k-1} \\ &= c_l - c_{l-1}, \end{aligned} \tag{3.39}$$

again from (3.26). We deduce from (3.38) and (3.39) that (3.37) holds if $m = 2$. Suppose next that

$$\left|\sum_k a_{j-2k}^{(r)} c_k - \sum_k a_{j-2-2k}^{(r)} c_k\right| \leq ||\Delta c||_{\ell^\infty}, \quad j \in \mathbb{Z}. \tag{3.40}$$

for an integer $r \geq 1$. But then, using also (3.35) with m replaced by $r+1$, we have

$$\begin{aligned}
&\sum_k a_{j-2k}^{(r+1)} c_k - \sum_k a_{j-2-2k}^{(r+1)} c_k \\
= \;& \sum_k \frac{1}{2}\left(a_{j-2k}^{(r)} + a_{j-1-2k}^{(r)}\right) c_k - \sum_k \frac{1}{2}\left(a_{j-2-2k}^{(r)} + a_{j-3-2k}^{(r)}\right) c_k \\
= \;& \frac{1}{2}\left[\sum_k a_{j-2k}^{(r)} c_k - \sum_k a_{j-2-2k}^{(r)} c_k\right] + \frac{1}{2}\left[\sum_k a_{j-1-2k}^{(r)} c_k - \sum_k a_{j-3-2k}^{(r)} c_k\right] \\
\leq \;& \frac{1}{2}||\Delta c||_{\ell^\infty} + \frac{1}{2}||\Delta c||_{\ell^\infty} = ||\Delta c||_{\ell^\infty},
\end{aligned}$$

after having used the inductive hypothesis (3.40). Hence (3.40) also holds with r replaced by $r+1$, and it follows that (3.37) is true.

Combining (3.36) and (3.37) then yields

$$\left|c_j^{(1)} - c_{j-1}^{(1)}\right| \leq \frac{1}{2}||\Delta c||_{\ell^\infty}, \quad j \in \mathbb{Z},$$

and thus $\Delta c^{(1)} \in \ell^\infty$, with

$$||\Delta c^{(1)}|| \leq \frac{1}{2}||\Delta c||_{\ell^\infty},$$

that is (3.33) also holds for $r = 1$.

Similarly, it follows that $\Delta c^{(2)} \in \ell^\infty$, with

$$||\Delta c^{(2)}||_{\ell^\infty} \leq \frac{1}{2}||\Delta c^{(1)}||_{\ell^\infty},$$

and in general, $\Delta c^{(r)} \in \ell^\infty$, with

$$||\Delta c^{(r)}||_{\ell^\infty} \leq \frac{1}{2}||\Delta c^{(r-1)}||_{\ell^\infty}, \quad r = 1, 2, ... \tag{3.41}$$

The inequality (3.33) then follows from (3.41).

Now observe in (3.32) that, for $j \in \mathbb{Z}$, $r \in \{0, 1, ...,\}$ and $k \in \{0, 1, ..., m-1\}$, we have

$$\begin{aligned} \left|c_{j-k}^{(r)} - c_{j-1}^{(r)}\right| &= \left|\sum_{l=2}^{k}\left(c_{j-l}^{(r)} - c_{j-l+1}^{(r)}\right)\right| \\ &\leq \sum_{l=2}^{k}\left|c_{j-l}^{(r)} - c_{j-l+1}^{(r)}\right| \\ &\leq (k-1)\left|\left|\Delta c^{(r)}\right|\right|_{\ell^\infty} \\ &\leq (m-2)\left|\left|\Delta c^{(r)}\right|\right|_{\ell^\infty}, \end{aligned} \tag{3.42}$$

since $\Delta c^{(r)} \in \ell^\infty$, and $k \leq m-1$. Then, by combining (3.32), (3.42) and (3.33), and recalling from (2.6), (2.7) that

$$\sum_{k=1}^{m-1} N_m(k) = 1,$$

the desired inequality (3.31) follows. ■

Hence, recalling also the fact that the sequence $\left\{\frac{j}{2^r} : j \in \mathbb{Z}, r = 0, 1, ...\right\}$ is dense in $\mathbb{R}$, we see that the Lane-Riesenfeld sequence $\{c^{(r)} : r = 0, 1, ...\}$ converges to the cardinal spline curve f in the sense of (3.31) if $c \in \ell^\infty$.

Observe in particular that the choice

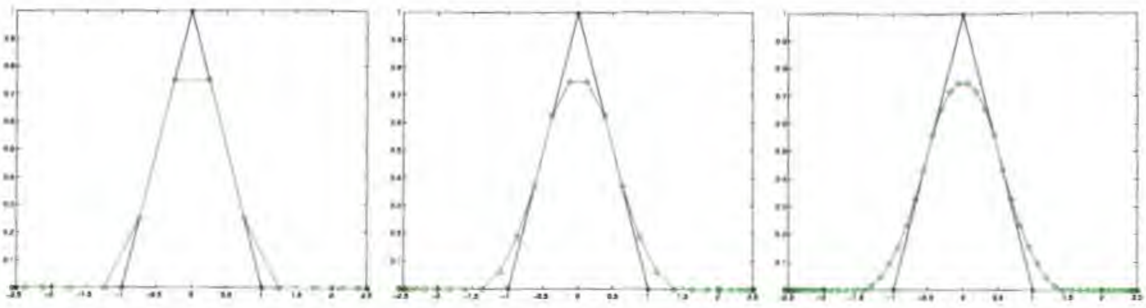
$$c_j = \delta_{j,0}, \quad j \in \mathbb{Z}, \tag{3.43}$$

in (3.17) yields $f = N_m$, and since $c = \{c_j\} \in \ell^\infty$ for this case, with $||c||_{\ell^\infty} = 1$ and $||\Delta c||_{\ell^\infty} = 1$, we get from (3.31) that

$$\left|N_m\left(\frac{j}{2^r}\right) - c_{j-1}^{(r)}\right| \leq \frac{m-2}{2^r}, \quad j \in \mathbb{Z}, \quad r = 0, 1, ..., \tag{3.44}$$

that is, the Lane-Riesenfeld subdivision algorithm converges exponentially fast to the B-spline curve N_m.

We have therefore constructed a most efficient algorithm for the computation of cardinal B-spline. For example, for the computation of the (centralised) quadratic B-spline N_3, we used (3.43), (3.25), (3.21), (3.28) to produce the graphs in Figure 3.4 (cf. Figure 2.1(c)).



(a) $\{c_j^{(1)}\}$ (b) $\{c_j^{(2)}\}$ (c) $\{c_j^{(3)}\}$

Figure 3.4: *Initial sequence* $\{c_j\} = \{\delta_{j,0}\}$ *(*) and new control points* ($\circ$)

If we choose $c = c^{(0)} = \{c_j\} \subset \mathbb{R}^2$, the Lane-Riesenfeld subdivision scheme (3.25), (3.21), (3.18) can be used as a "corner-cutting" algorithm, with as a special case for $m = 3$ the de Rahm-Chaikin algorithm, with respect to the control polygon $\{c_j\} \subset \mathbb{R}^2$. Indeed, if we write $c_j = (c_{1,j}, c_{2,j}) \in \mathbb{R}^2$, $j \in \mathbb{Z}$, and assume that $\{c_{1,j}\}, \{c_{2,j}\} \in \ell^\infty$, we see from Theorem 3.2 that the limit curve $\mathbf{f} = (f_1, f_2)$ is given by

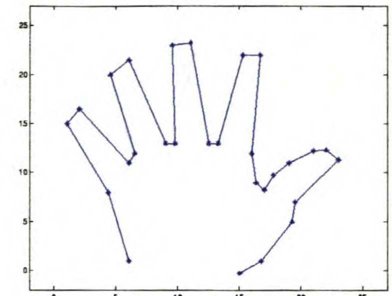
$$f_n(t) = \sum_j c_{n,j} N_m(t-j), \quad t \in \mathbb{R}, \quad n = 1, 2.$$

Since $S_m(\mathbb{Z}) \subset C^{m-2}(\mathbb{R})$, the smoothness of the limit curve f can be increased by increasing the value of m. Illustrative examples are shown in Figures 3.5 and 3.6.

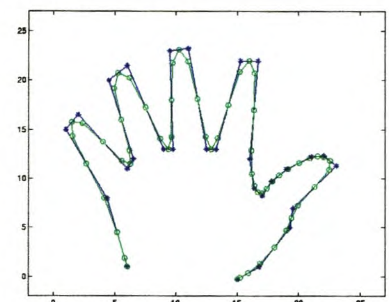
For $m = 4$, we use the subdivision algorithm (3.25), (3.21), (3.29), whereas for $m = 5$ we use (3.25), (3.21) and the sequence $\{a_k^{(5)}\}$ given, according to (3.18), by

$$\left.\begin{aligned} a_0^{(5)} &= 1/16, \\ a_1^{(5)} &= 5/16, \\ a_2^{(5)} &= 5/8, \\ a_3^{(5)} &= 5/8, \\ a_4^{(5)} &= 5/16, \\ a_5^{(5)} &= 1/16, \\ a_k^{(5)} &= 0, \quad k \notin \{0,1,2,3,4,5\}. \end{aligned}\right\} \tag{3.45}$$

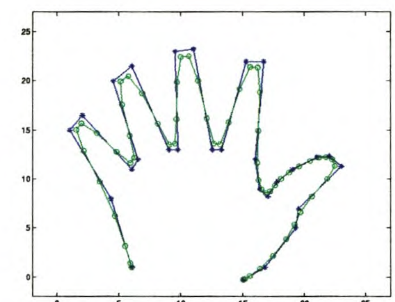
As a numerical illustration, we apply the above Lane-Riesenfeld subdivision algorithm for $m = 4$ and for $m = 5$ to the same set of initial control points, as given in Figure 3.5.



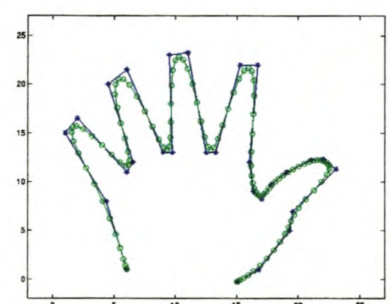
(a) Initial control polygon



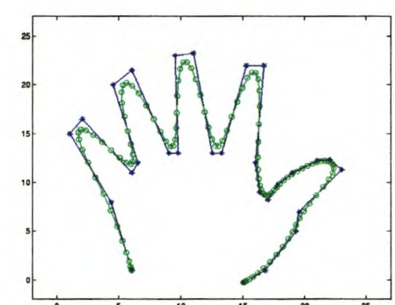
(b) m=4: 1^{st} iteration



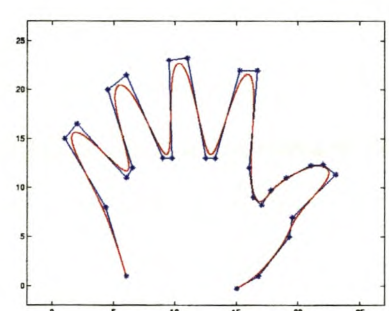
(c) m=5: 1^{st} iteration



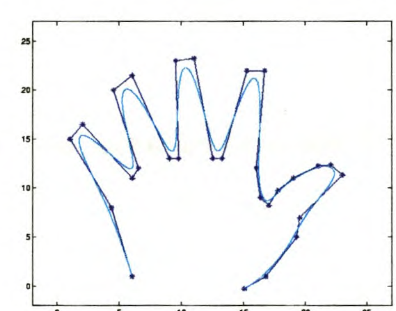
(d) m=4: 2^{nd} iteration



(e) m=5: 2^{nd} iteration

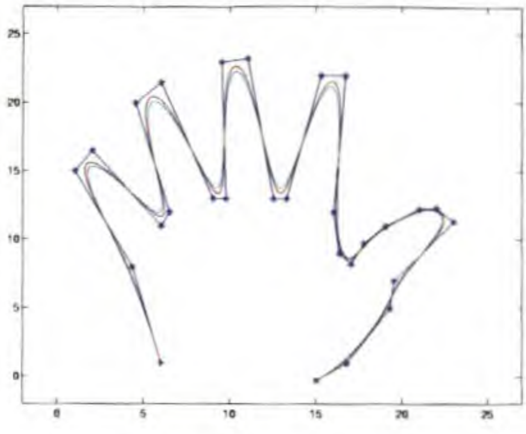


(f) m=4: limit curve

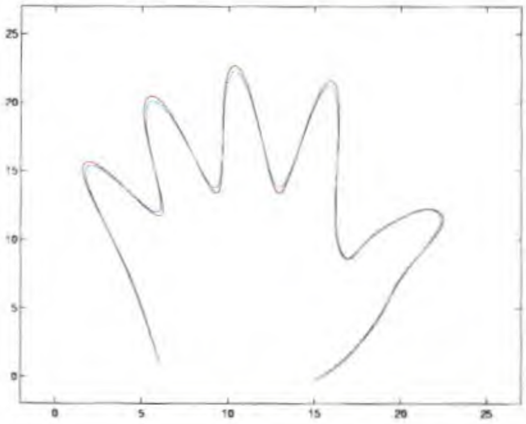


(g) m=5: limit curve

Figure 3.5: *Comparison of the Lane-Riesenfeld algorithm, with* $m = 4$ *and* $m = 5$.



(a) $\{c_j\}$ and limit curves



(b) Limit curves

Figure 3.6: *Comparison of two limit curves: $m = 4$ (red) and $m = 5$ (blue)*

Observe that the limit curves of the Lane-Riesenfeld algorithm, with both $m = 4$ and $m = 5$, appear to be smooth. In fact, from Theorem 3.2, we have that the limit curve in Figure 3.5(f) is in $C^2(\mathbb{R})$, while the limit curve in Figure 3.5(g) is in $C^3(\mathbb{R})$. Note also, that although the Lane-Riesenfeld algorithm, with both $m = 4$ and $m = 5$, has a "corner-cutting" action, when $m = 4$ the limit curve seems to turn closer to the corners of the control polygon, than when $m = 5$, as illustrated in Figure 3.6.

3.3 A General Formulation

We proceed to introduce a class of subdivision algorithms, which include Lane-Riesenfeld subdivision as a special case. To this end, we suppose $a = \{a_k\} \subset \mathbb{R}$ is a given finitely supported sequence, and define the operator S_a which maps the space of real sequences into itself by $S_a c = \{(S_a c)_k\}$, where

$$(S_a c)_k = \sum_j a_{k-2j} c_j, \quad k \in \mathbb{Z}, \quad c = \{c_j\} \subset \mathbb{R}. \tag{3.46}$$

We then call S_a a *stationary subdivision scheme* (SSS) corresponding to the given *mask* $\{a_k\}$. Observe that the Lane-Riesenfeld subdivision scheme (3.25), (3.21), (3.18) is based on precisely such a sequence $\{a_k\}$ and corresponding operator S_a. Moreover, we shall use the Laurent polynomial A defined by

$$A(z) := \sum_k a_k z^k, \quad z \in \mathbb{C} \setminus \{0\}, \tag{3.47}$$

as the *symbol* associated with the SSS defined by $a = \{a_k\}$ and S_a.

Observe in particular for the Lane-Riesenfeld SSS that, from (3.18) and (3.47), the corresponding symbol $A = \widetilde{A}_m$ is given, for $z \in \mathbb{C}$, by

$$\widetilde{A}_m(z) = \sum_k a_k^{(m)} z^k = \frac{1}{2^{m-1}} \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} z^k, \tag{3.48}$$

and thus

$$\widetilde{A}_m(z) = 2\left(\frac{1+z}{2}\right)^m, \quad z \in \mathbb{C}, \tag{3.49}$$

that is, $\widetilde{A}_m$ is a polynomial of degree m. Hence, from (3.49), we see that

$$\widetilde{A}_m(1) = 2, \qquad \widetilde{A}_m(-1) = 0, \tag{3.50}$$

which, together with (3.48), yield, respectively,

$$\sum_j a_j^{(m)} = 2, \tag{3.51}$$

and

$$\sum_k a_{2k}^{(m)} = \sum_k a_{2k+1}^{(m)}, \tag{3.52}$$

and thus

$$\sum_k a_{2k}^{(m)} = \sum_k a_{2k+1}^{(m)} = 1. \tag{3.53}$$

Based on Theorem 3.2, we next introduce the concept of the convergence for a SSS.

Definition 3.3 For a given finitely supported sequence $a = \{a_k\}$ let S_a denote the SSS defined by (3.46). We shall say that S_a is a *uniformly convergent* SSS if, for each given sequence $c = \{c_k\} \in \ell^\infty$, there exists a function $f \in C(\mathbb{R})$ (which is non-trivial for at least one $c \in \ell^\infty$) such that, for each $\varepsilon > 0$, there exists an integer $m \in \mathbb{N}$ for which it holds that

$$\left| f\left(\frac{j}{2^r}\right) - (S_a^r c)_j \right| < \varepsilon, \quad j \in \mathbb{Z}, \quad r \geq m. \tag{3.54}$$

We then call f the *limit function* of the SSS *defined by* S_a.

3.4 Existence, Uniqueness and Regularity Results for Positive Masks

According to Theorem 3.2, the Lane-Riesenfeld SSS is uniformly convergent in the sense of Definition 3.3, with a limiting function f which is a B-spline series. The following more general existence and convergence theorem, as proved in [25, Theorem 2.5 and Proposition 2.1], holds.

Theorem 3.4 *Suppose $a = \{a_k\}$ is a finitely supported mask sequence such that, for a given integer $n \geq 2$, we have*

$$(i) \quad a_k > 0, \quad k = 0, 1, ..., n, \tag{3.55}$$

$$(ii) \quad a_k = 0, \quad k \notin \{0, 1, ..., n\}, \tag{3.56}$$

$$(iii) \quad \sum_k a_{2k} = \sum_k a_{2k+1} = 1. \tag{3.57}$$

Then there exists a unique refinable function $\phi \in C(\mathbb{R})$ such that

$$(a)\quad \phi(t) > 0, \quad t \in (0,n); \tag{3.58}$$

$$(b)\quad \operatorname{supp}\phi = [0,n]; \tag{3.59}$$

$$(c)\quad \phi(t) = \sum_k a_k \phi(2t-k), \quad t \in \mathbb{R}; \tag{3.60}$$

$$(d)\quad \sum_k \phi(t-k) = 1, \quad t \in \mathbb{R}. \tag{3.61}$$

Moreover, the SSS defined by (3.46) then converges uniformly in the sense of Definition 3.3, with the limit function f given, for $c = \{c_k\} \in \ell^\infty$, by

$$f(t) = \sum_k c_k \phi(t-k), \quad t \in \mathbb{R}. \tag{3.62}$$

Hence if we choose the mask sequence $a = a^{(m)} = \{a_k^{(m)}\}$ as in (3.18), then the conditions (3.55), (3.56) and (3.57) are satisfied with $n = m$, by virtue of (3.18) and (3.53), and we see that the above result of Theorem 3.4 is consistent with (2.5), (2.6), (2.21), (2.7) as well as (3.17), (3.31), showing that, for this special case, $\phi = N_m$.

Observe, from the result of Theorem 3.4, that an efficient numerical algorithm for the computation of the refinable function ϕ of Theorem 3.4 is given by the following SSS: If we define the sequence $\{c_k^{(j)}\}$, $\quad j = 0, 1, ...,$ by

$$\begin{aligned} c_k^{(0)} &= \delta_{k,0}, \quad k \in \mathbb{Z}, && (3.63)\\ c_k^{(j+1)} &= \sum_l a_{k-2l} c_l^{(j)}, \quad k \in \mathbb{Z}, \quad j = 0,1,..., && (3.64) \end{aligned}$$

then

$$\left|\phi\left(\frac{j}{2^r}\right) - c_j^{(r)}\right| \to 0, \quad r \to \infty, \tag{3.65}$$

uniformly in j, keeping in mind that the sequence $\left\{\frac{j}{2^r} : j \in \mathbb{Z}, r = 0,1,...\right\}$ is dense in $\mathbb{R}$.

An alternative iterative algorithm for the numerical computation is the so called *cascade algorithm* as given by

$$\phi_1(t) = \begin{cases} t, & 0 \le t \le 1, \\ 2-t, & 1 < t \le 2, \\ 0, & t \notin [0,2], \end{cases} \tag{3.66}$$

and

$$\phi_{j+1}(t) = \sum_k a_k \phi_j(2t-k), \quad t \in \mathbb{R}, \quad j = 1, 2, \ldots . \tag{3.67}$$

Then, according to [25, Theorem 2.5, second proof on pp. 78-82], we have the following theorem.

Theorem 3.5 *Suppose $a = \{a_k\}$ is such that the conditions of Theorem 3.4 are satisfied. Then the sequence $\{\phi_j : j = 1, 2, \ldots\} \subset C(\mathbb{R})$, as generated by the cascade algorithm (3.66), (3.67), converges uniformly to the refinable function ϕ.*

The following result on the support interval of a refinable function ϕ also holds.

Theorem 3.6 *Suppose $\{a_k\}$ is a finitely supported sequence satisfying (3.56) for an integer $n \geq 2$, and suppose the cascade algorithm (3.66), (3.67) converges uniformly to a refinable function $\phi \in C(\mathbb{R})$ satisfying (3.60). Then*

$$\operatorname{supp} \phi = [0, n]. \tag{3.68}$$

Proof. For the function ϕ_1 given by (3.66), $\operatorname{supp} \phi_1 = [0, 2]$. We claim that

$$\operatorname{supp} \phi_j = \left[0, \frac{2 + (2^{j-1} - 1)n}{2^{j-1}}\right], \quad j \in \mathbb{N}. \tag{3.69}$$

and will prove it inductively. Hence we must show that if (3.69) holds for a given $j \in \{1, 2, \ldots\}$, then

$$\operatorname{supp} \phi_{j+1} = \left[0, \frac{2 + (2^j - 1)n}{2^j}\right]. \tag{3.70}$$

Since (3.56), (3.60) and (3.67) imply

$$\phi_{j+1}(t) = \sum_{k=0}^{n} a_k \phi_j(t-k), \quad t \in \mathbb{R}, \tag{3.71}$$

we see that ϕ_{j+1} is finitely supported. Suppose $\operatorname{supp} \phi_{j+1} = [\alpha, \beta]$. Then (3.71) and (3.69) give $\alpha = 0$ and

$$2\beta - n = \frac{2 + (2^{j-1} - 1)n}{2^{j-1}},$$

so that

$$\beta = \frac{2 + (2^{j-1} - 1)n + 2^{j-1}n}{2^j} = \frac{2 + (2^j - 1)n}{2^j},$$

and it follows that (3.70) holds. Hence (3.69) is true for $j \in \mathbb{N}$. Observe that, since $n \geq 2$, we have

$$\frac{2 + (2^{j-1} - 1)n}{2^{j-1}} = n + \frac{2-n}{2^{j-1}} \leq n,$$

and thus from (3.69) we see that

$$\operatorname{supp} \phi_j \subset [0, n], \quad j = 1, 2, \ldots. \tag{3.72}$$

Moreover, from (3.69),

$$\lim_{j \to \infty} \operatorname{supp} \phi_j = \lim_{j \to \infty} \frac{2 + (2^{j-1} - 1)n}{2^{j-1}} = \lim_{j \to \infty} \left[n + \frac{2-n}{2^{j-1}} \right] = n. \tag{3.73}$$

But

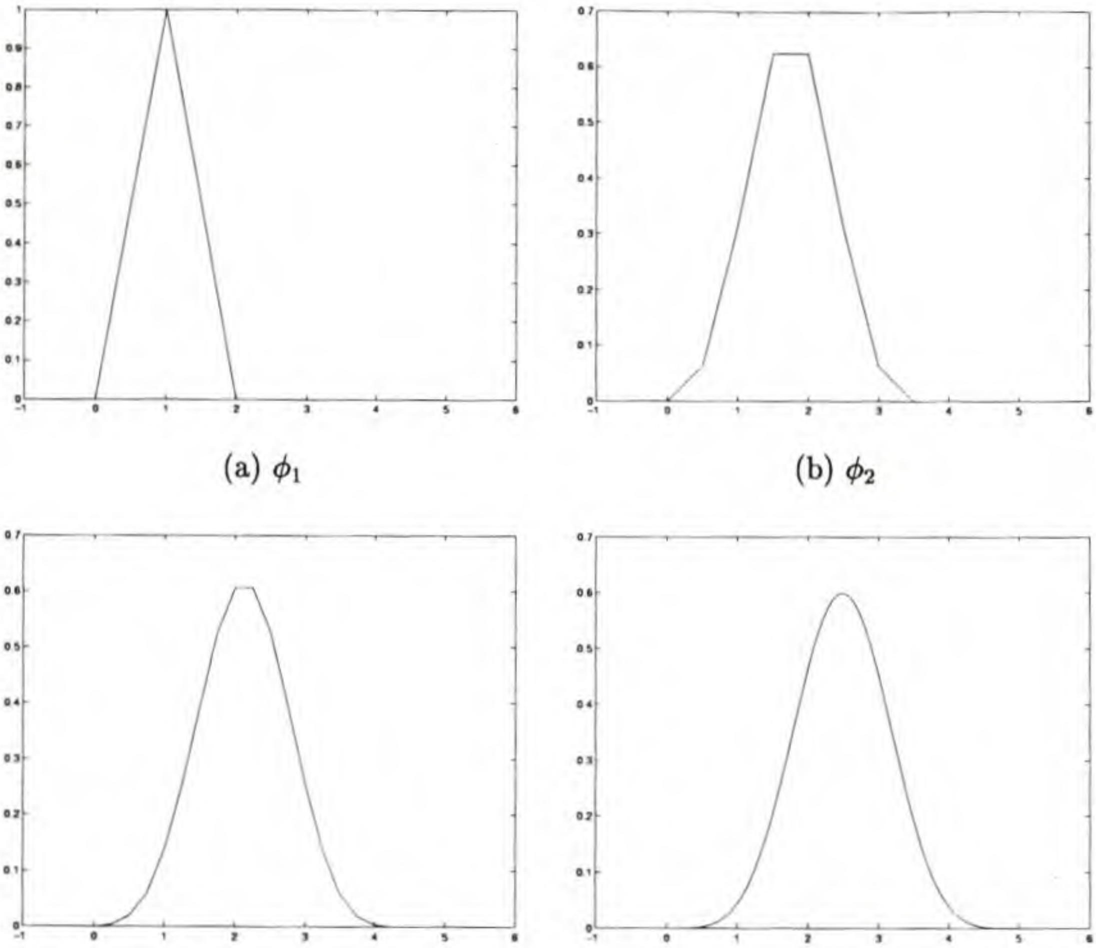
$$\operatorname{supp} \phi = \operatorname{supp} \lim_{j \to \infty} \phi_j = \lim_{j \to \infty} \operatorname{supp} \phi_j,$$

and the desired result follows from (3.73). ∎

The cascade algorithm provides us with an alternative method to the Lane-Riesenfeld SSS for the computation of cardinal B-splines (see Figure 3.4). According to Theorem 3.4 and 3.5, and the properties (2.5), (2.6), (2.7), (2.21) in Theorem 2.1 and 2.2 of the cardinal B-spline N_m, if we choose $a_k = a_k^{(m)} = \frac{1}{2^{m-1}} \binom{m}{k}$, $k \in \mathbb{Z}$, in the cascade algorithm (3.66), (3.67), then the sequence ϕ_j converges uniformly to N_m for $j \to \infty$. The case $m = 5$, converging to the cardinal B-spline N_5, is shown graphically in Figure 3.7.

Next, we investigate the regularity, (or smoothness class) of the refinable function ϕ in Theorem 3.4. First observe that the condition (3.57) in Theorem 3.4 means that the polynomial A defined by (3.47) has a zero at $z = -1$. Moreover, the polynomial $\widetilde{A}_m$ for the Lane-Riesenfeld SSS, as given by (3.49), with associated refinable function $\phi = N_m \in S_m(\mathbb{Z}) \subset C^{m-2}(\mathbb{R})$, shows that the order of zero at $z = -1$ of $\widetilde{A}_m$ is equal to m.

A *Hurwitz polynomial* is defined as a polynomial for which all the zeros lie in the open left half plane of $\mathbb{C}$. Observe that $\widetilde{A}_m$ is a Hurwitz polynomial, since the only zero of $\widetilde{A}_m$ is an $m-$fold zero at $z = -1$.



(a) ϕ_1 (b) ϕ_2

(c) ϕ_3 (d) ϕ_8

Figure 3.7: *The cascade algorithm for the computation of the cardinal B-spline N_5*

The next result, as proved in [25, Theorem 2.7], shows that, for a positive mask, the smoothness of a refinable function is indeed directly linked to the order of the zero at $z = -1$ of the polynomial A, provided that A is a Hurwitz polynomial.

Theorem 3.7 *If, in Theorem 3.4, the sequence $\{a_k\}$ is such that the polynomial A defined, according to (3.47), by*

$$A(z) = \sum_{k=0}^{n} a_k z^k, \quad z \in \mathbb{C}, \tag{3.74}$$

is a Hurwitz polynomial, then the refinable function in Theorem 3.4 is such that $\phi \in C^r(\mathbb{R})$, where $r = \min\{n-2, k-1\}$, if and only if A has a zero of order k at $z = -1$.

Observe that the result of Theorem 3.7 is consistent with Proposition 3.1 and the fact that $N_m \in C^{m-2}(\mathbb{R})$, since, in this case, $r = \min\{m-2, m-1\} = m-2$. As a further illustration of Theorem 3.7, we consider the following two examples:

(a) Define the mask $\{a_k\}$ by

$$\left.\begin{aligned} a_0 &= 1/3,\\ a_1 &= 2/3,\\ a_2 &= 2/3,\\ a_3 &= 1/3, \end{aligned}\right\} \quad a_k = 0, \quad k \notin \{0,1,2,3\}, \tag{3.75}$$

so that, from (3.74), we get

$$A(z) = \frac{1}{3}(1+z)(1+z+z^2), \quad z \in \mathbb{C}. \tag{3.76}$$

Hence the condition of Theorem 3.4 and 3.7 are satisfied, with $n = 3$ and $k = 1$. Hence, both the SSS (3.63), (3.64) and the cascade algorithm (3.66), (3.67) converge uniformly to the refinable function ϕ of Theorem 3.4 with, according to Theorem 3.7, $\phi \in C(\mathbb{R})$, since here $r = \min\{1, 0\} = 0$. Moreover, Theorem 3.6 implies that $\operatorname{supp}\phi = [0,3]$. All the above conclusions are illustrated graphically in Figure 3.8, in which specifically the SSS (3.63), (3.64) was used.

(b) Next we define the mask $\{a_k\}$ by

$$\left.\begin{aligned} a_0 &= 1/6,\\ a_1 &= 1/2,\\ a_2 &= 2/3,\\ a_3 &= 1/2,\\ a_4 &= 1/6, \end{aligned}\right\} \quad a_k = 0, \quad k \notin \{0,1,2,3,4\}, \tag{3.77}$$

so that, from (3.74),

$$A(z) = \frac{1}{6}(1+z)^2(1+z+z^2), \quad z \in \mathbb{C}. \tag{3.78}$$

Hence the conditions of Theorem 3.4 and 3.7 are satisfied with $n = 4$ and $k = 2$. Hence, appealing to Theorems 3.4, 3.5, 3.6 and 3.7, both the SSS (3.63), (3.64) and the cascade algorithm (3.66), (3.67) converge uniformly to a refinable function ϕ. Also, supp $\phi = [0,4]$ and $\phi \in C^1(\mathbb{R})$, since in this case $r = \min\{2,1\}$, as illustrated graphically in Figure 3.8, in which specifically the SSS (3.63), (3.64) was used for the numerical computation.

Observe that in Figure 3.8, a numerical illustration of Theorem 3.7 is given where the refinable function ϕ corresponding to the mask (3.75) is such that $\phi \in C(\mathbb{R})\backslash C^1(\mathbb{R})$ while the refinable function ϕ corresponding to the mask (3.77) is such that $\phi \in C^1(\mathbb{R})$.

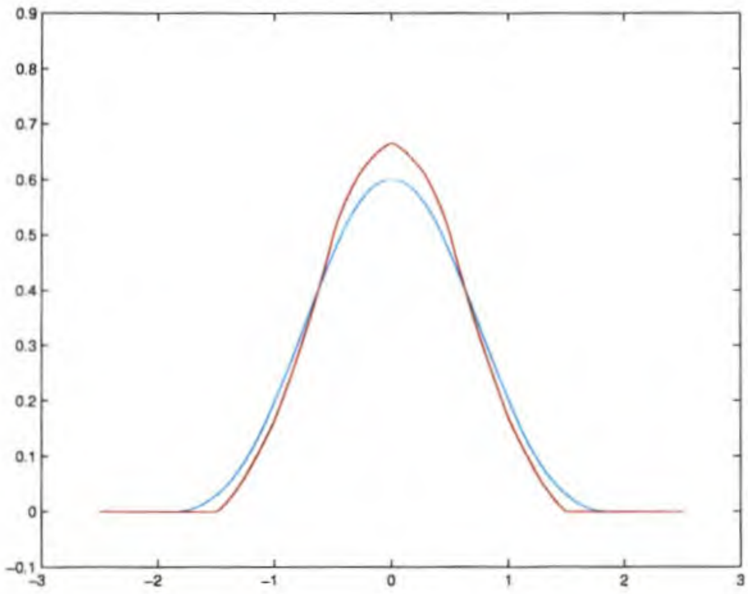


Figure 3.8: *Comparison of ϕ for masks (3.75) (red) and (3.77) (blue).*

Chapter 4

Interpolatory Subdivision

4.1 The Interpolatory Condition

In Section 3.3, we defined the concept of a stationary subdivision scheme (SSS) by means of the subdivision operator S_a, which, for a given finitely supported sequence $\{a_k\}$, maps the space of real sequences into itself, with, for a given real sequence $c = \{c_j\}$, the sequence $S_a c = \{(S_a c)_k\}$ defined, according to (3.46), by

$$(S_a c)_k = \sum_j a_{k-2j} c_j, \quad k \in \mathbb{Z}, \quad c = \{c_j\} \subset \mathbb{R}. \tag{4.1}$$

We consider here the subclass of stationary subdivision schemes which are *interpolatory* in the sense that, in (4.1), we have

$$(S_a c)_{2k} = c_k, \quad k \in \mathbb{Z}, \quad c = \{c_k\} \subset \mathbb{R}. \tag{4.2}$$

Hence, if we define the sequence $c^{(r)}$, $r = 0, 1, ...$, of real sequences by

$$\left.\begin{aligned} c^{(0)} &= c, \\ c^{(r)} &= S_a^r c, \quad r = 1, 2, ..., \end{aligned}\right\} \tag{4.3}$$

it follows from (4.2) and (4.3) that, if S_a denotes an interpolatory SSS, then

$$c_{2k}^{(r+1)} = c_k^{(r)}, \quad k \in \mathbb{Z}, \quad r = 0, 1, ..., \tag{4.4}$$

that is, at each step of the subdivision algorithm we retain all the previous control points, and add one extra control point between each pair of these retained control points.

A necessary and sufficient condition on the mask sequence $\{a_k\}$ for a SSS to be interpolatory is given in the following theorem.

Proposition 4.1 *The SSS defined by (4.1) is interpolatory in the sense of (4.2) if and only if the mask* $\{a_k\}$ *satisfies*

$$a_{2k} = \delta_{k,0}, \qquad k \in \mathbb{Z}, \tag{4.5}$$

or, equivalently, if the corresponding symbol A, as defined by (3.47), satisfies the identity

$$A(z) + A(-z) = 2, \qquad z \in \mathbb{C} \setminus \{0\}. \tag{4.6}$$

Proof. Suppose the sequence $\{a_k\}$ satisfies (4.5). Then, from (4.1), we have for $k \in \mathbb{Z}$ that

$$(S_a c)_{2k} = \sum_j a_{2k-2j} c_j = \sum_j a_{2j} c_{k-j} = \sum_j \delta_{j,0} c_{k-j} = c_k,$$

thereby yielding (4.2).

If (4.2) holds for all real sequences $c = \{c_j\}$, we can choose $c_j = \delta_{j,0}$, $j \in \mathbb{Z}$, in (4.2) to deduce from (4.1) that, for $k \in \mathbb{Z}$,

$$\delta_{k,0} = \sum_j a_{2k-2j} \delta_{j,0} = a_{2k},$$

so that (4.5) holds.

It remains to prove the equivalence of (4.5) and (4.6). First observe that (4.5) holds, then (3.47) gives, for $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned} A(z) + A(-z) &= \sum_k a_k z^k + \sum_k a_k (-z)^k \\ &= \left[\sum_k a_{2k} z^{2k} + \sum_k a_{2k+1} z^{2k+1}\right] + \left[\sum_k a_{2k} z^{2k} - \sum_k a_{2k+1} z^{2k+1}\right], \end{aligned}$$

and thus

$$A(z) + A(-z) = 2 \sum_k a_{2k} z^{2k}, \qquad , z \in \mathbb{C} \setminus \{0\}. \tag{4.7}$$

Now suppose that (4.5) holds. Then (4.7) gives

$$A(z) + A(-z) = 2\sum_k \delta_{k,0} z^{2k} = 2, \quad z \in \mathbb{C} \setminus \{0\},$$

so that (4.6) holds. If (4.6) holds, then (4.7) implies

$$2 = A(z) + A(-z) = 2\sum_k a_{2k} z^{2k}, \quad z \in \mathbb{C} \setminus \{0\},$$

and thus

$$\sum_k a_{2k} z^{2k} = 1, \quad z \in \mathbb{C} \setminus \{0\},$$

thereby yielding (4.5). ■

Observe from (3.26), (3.27) that $a_k = a_k^{(1)}$, $j \in \mathbb{Z}$, as in (3.26), and $a_k = a_{k+1}^{(2)}$, $k \in \mathbb{Z}$, as in (3.27), are examples of an interpolatory SSS. These two examples, however, both produce limit curves which, according to Theorem 3.2, are in $S_1(\mathbb{Z}) \subset C^{-1}(\mathbb{R}) \setminus C(\mathbb{R})$, and $S_2(\mathbb{Z}) \subset C(\mathbb{R}) \setminus C^1(\mathbb{R})$, respectively, and are therefore not smooth. In the following sections we shall develop a method for the construction of convergent interpolatory stationary subdivision schemes for which the limit curves are indeed smooth.

4.2 An Existence and Convergence Theorem

Following [24, Theorem 4.1 and Corollary 4.1], we next state and prove (in considerably more detail than was done in [24]) a set of sufficient conditions on a mask for the convergence of an interpolatory SSS. The theorem can also be used as an existence theorem for a solution ϕ of the refinement equation (2.20).

The degree of a Laurent polynomial

$$A(z) = \sum_k a_k z^k, \ \ z \in \mathbb{C} \setminus \{0\},$$

is defined as the smallest non-negative integer n for which $a_k = 0$, $|k| \geq n + 1$.

Theorem 4.2 *For a non-negative integer N, let $a = \{a_k\}$ denote a finitely supported sequence in $\mathbb{R}$ such that the Laurent polynomial A defined by (3.47) has degree $2N+1$, and satisfies the following properties:*

$$(i)\quad A(z) + A(-z) = 2, \qquad z \in \mathbb{C} \setminus \{0\}, \tag{4.8}$$

$$(ii)\quad A(-1) = 0, \tag{4.9}$$

$$(iii)\quad A(e^{ix}) > 0,\ -\pi < x < \pi. \tag{4.10}$$

Then there exists a refinable function $\phi \in C(\mathbb{R})$ such that

$$(a)\quad \phi(t) = 0, \quad t \notin (-2N-1, 2N+1); \tag{4.11}$$

$$(b)\quad \phi(t) = \sum_k a_k \phi(2t-k), \qquad t \in \mathbb{R}; \tag{4.12}$$

$$(c)\quad \phi(k) = \delta_{k,0}, \qquad k \in \mathbb{Z}; \tag{4.13}$$

$$(d)\quad \sum_k \phi(t-k) = 1, \qquad t \in \mathbb{R}; \tag{4.14}$$

$$(e)\quad \int_{-\infty}^{\infty} \phi(t)dt = 1; \tag{4.15}$$

$$(f)\quad \phi, \hat{\phi} \in \mathcal{A}; \tag{4.16}$$

$$(g)\quad \hat{\phi}(2\pi k) = \delta_{k,0}, \qquad k \in \mathbb{Z}; \tag{4.17}$$

$$(h)\quad \hat{\phi}(x) \geq 0, \qquad x \in \mathbb{R}; \tag{4.18}$$

(*i*) ϕ *is Riesz stable, in the sense that there exist real numbers α and β, with $0 < \alpha \leq \beta$, such that*

$$\alpha||\{c_k\}||_{\ell^2}^2 \leq ||\sum_k c_k \phi(\cdot - k)||_2^2 \leq \beta||\{c_k\}||_{\ell^2}^2, \qquad \{c_k\} \in \ell^2. \tag{4.19}$$

Moreover, the SSS defined by (4.1) is interpolatory in the sense of (4.2), and converges uniformly, with

$$(S_a^r c)_k = \sum_j c_j \phi\left(\frac{k}{2^r} - j\right), \quad j \in \mathbb{Z}, \quad r \in \mathbb{N}, \quad c = \{c_j\} \in \ell^\infty. \tag{4.20}$$

Proof. Suppose the order of the zero at $z = -1$ of the Laurent polynomial A is equal to n; hence there exists a Laurent polynomial B such that

$$A(z) = 2\left(\frac{1+z}{2}\right)^n B(z), \quad z \in \mathbb{C} \setminus \{0\}, \tag{4.21}$$

with

$$B(1) = 1, \tag{4.22}$$

and

$$B(e^{ix}) \neq 0, \quad -\pi \leq x \leq \pi, \tag{4.23}$$

from (4.10), and the fact that (4.8), (4.9) imply

$$A(1) = 2. \tag{4.24}$$

Our first step is to prove the existence of a function $\phi \in C(\mathbb{R})$ which satisfies the refinement equation (4.12). To this end, we define the sequence $\{P_r : r \in \mathbb{N}\}$ by

$$P_r(x) = \frac{1}{2^r} \prod_{j=1}^{r} A\left(e^{-i\frac{x}{2^j}}\right), \quad x \in \mathbb{R}, \quad r \in \mathbb{N}, \tag{4.25}$$

and proceed to investigate the convergence of the infinite product

$$\lim_{r\to\infty} P_r(x) = \prod_{j=1}^{\infty} \left[\frac{1}{2} A\left(e^{-i\frac{x}{2^j}}\right)\right], \quad x \in \mathbb{R}. \tag{4.26}$$

Suppose first that $x \in \{2^l(2k+1)\pi : l \in \mathbb{N}, k \in \mathbb{Z}\}$. Then it is clear from (4.9), (4.25) that

$$\lim_{r\to\infty} P_r(x) = 0. \tag{4.27}$$

Since

$$\{x \in \mathbb{R} : x = 2k\pi, \ k \in \mathbb{Z} \setminus \{0\}\} = \left\{x \in \mathbb{R} : x = 2^l(2k+1)\pi, \ l \in \mathbb{N}, \ k \in \mathbb{Z}\right\}, \tag{4.28}$$

as can easily be shown, we deduce from (4.27) and (4.28) that

$$\lim_{r\to\infty} P_r(2\pi k) = 0, \quad k \in \mathbb{Z} \setminus \{0\}. \tag{4.29}$$

Also, note from (4.24) and (4.25) that

$$P_r(0) = 1, \quad r \in \mathbb{N}, \tag{4.30}$$

which, together with (4.29), implies that

$$\lim_{r\to\infty} P_r(2\pi k) = \delta_{k,0}, \quad k \in \mathbb{Z}. \tag{4.31}$$

Next, suppose $x \in \mathcal{B}$, where

$$\mathcal{B} = \{x \in \mathbb{R} : x \neq 2k\pi, k \in \mathbb{Z} \setminus \{0\}\}. \tag{4.32}$$

Then from (4.10), (4.25) and (4.28), we have that $A\left(e^{-i\frac{x}{2^j}}\right) > 0$ for $j \in \mathbb{N}$, so that we can write

$$P_r(x) = \exp\left\{\ln \prod_{j=1}^{r} \frac{A\left(e^{-i\frac{x}{2^j}}\right)}{2}\right\}, \quad x \in \mathcal{B}, \ r \in \mathbb{N},$$

and thus

$$P_r(x) = \exp\left\{\sum_{j=1}^{r} \ln\left[\frac{A\left(e^{-i\frac{x}{2^j}}\right)}{2}\right]\right\}, \quad x \in \mathcal{B}, \ r \in \mathbb{N}. \tag{4.33}$$

If we can now show that the absolute series

$$\sum_{j=1}^{\infty} f_j(x), \tag{4.34}$$

where

$$f_j(x) = \left|\ln \frac{A\left(e^{-i\frac{x}{2^j}}\right)}{2}\right|, \quad x \in \mathcal{B}, \quad j \in \mathbb{Z}, \tag{4.35}$$

converges, it would follow from (4.33) that the infinite product (4.26) converges. From (4.24) and (4.35) we have, for $j \in \mathbb{N}$ and $x \in \mathcal{B}$,

$$f_j(x) = \left|\int_{A(1)}^{A(e^{-ix/2^j})} \frac{1}{t}\, dt\right| \leq \frac{1}{\min\{2, A(e^{-i\frac{x}{2^j}})\}} \left|A(e^{-i\frac{x}{2^j}}) - A(1)\right|. \tag{4.36}$$

Since (4.24) gives the value $A(1) = 2$, and since the Laurent polynomial A is continuous at $z = 1$, there exists a $\delta > 0$ such that

$$|z - 1| < \delta \ \Rightarrow \ |A(z) - 2| < 1. \tag{4.37}$$

Also, from the fact that

$$|e^{iy} - 1| = 2\left|\sin\left(\frac{y}{2}\right)\right| \leq |y|, \quad y \in \mathbb{R}, \tag{4.38}$$

we get

$$|e^{-i\frac{x}{2^j}} - 1| \leq \frac{|x|}{2^j}, \quad x \in \mathbb{R}, \quad j \in \mathbb{N}. \tag{4.39}$$

Hence if we choose the positive integer J_x to satisfy

$$J_x \geq \max\left\{1, \frac{\ln|x| - \ln\delta}{\ln 2}\right\}, \tag{4.40}$$

then, we see from (4.36), (4.37) and (4.39) that

$$f_j(x) \leq \left|A(e^{-i\frac{x}{2^j}}) - A(1)\right|, \quad x \in \mathcal{B}, \quad j > J_x. \tag{4.41}$$

Since the degree of the Laurent polynomial A is $2N+1$, we have

$$A(z) = \sum_{k=-2N-1}^{2N+1} a_k z^k, \quad z \in C \setminus \{0\}, \tag{4.42}$$

and thus, for a fixed $j \in \mathbb{N}$,

$$\begin{aligned} \left|A(e^{-i\frac{x}{2^j}}) - A(1)\right| &= \left|\sum_{k=-2N-1}^{2N+1} a_k \left(e^{-i\frac{kx}{2^j}} - 1\right)\right| \\ &\leq \sum_{k=-2N-1}^{2N+1} |a_k| \left|e^{-i\frac{kx}{2^j}} - 1\right| \\ &\leq \left[\sum_{k=-2N-1}^{2N+1} |ka_k|\right] \frac{|x|}{2^j}, \end{aligned} \tag{4.43}$$

by virtue of (4.39). But then (4.41) and (4.43) yield

$$f_j(x) \leq M\frac{|x|}{2^j}, \quad x \in \mathcal{B}, \quad j > J_x, \tag{4.44}$$

with

$$M := \sum_{k=-2N-1}^{2N+1} |ka_k|. \tag{4.45}$$

Since $\sum_{k=1}^{\infty} \frac{1}{2^k}$ is a convergent series (with $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$), we deduce from (4.44), (4.35), (4.33), (4.32) and (4.31), that for each fixed $x \in \mathbb{R}$, the infinite product (4.26) does indeed converge.

Moreover, according to (4.40), (4.44) and (4.29), the convergence is uniform on compact sets in $\mathbb{R}$; hence it follows that the limit function g defined by

$$g(x) := \lim_{r\to\infty} P_r(x), \quad x \in \mathbb{R}, \tag{4.46}$$

is in $C(\mathbb{R})$, with, according to (4.46) and (4.31),

$$g(2\pi k) = \delta_{k,0}, \qquad k \in \mathbb{Z}. \tag{4.47}$$

We claim that $g \in L^1(\mathbb{R})$. To prove this fact, we observe from (4.46) and (4.25) that, for $x \in \mathbb{R}$,

$$g(x) = \frac{1}{2}A(e^{-i\frac{x}{2}}) \lim_{k\to\infty} \prod_{j=1}^{k-1} \left[\frac{1}{2}A(e^{-i\frac{x}{2.2^j}})\right],$$

and thus

$$g(x) = \frac{1}{2}A(e^{-i\frac{x}{2}})g\left(\frac{x}{2}\right), \quad x \in \mathbb{R}. \tag{4.48}$$

It follows from (4.48) and (4.25) that

$$g(x) = P_r(x)g\left(\frac{x}{2^r}\right), \quad x \in \mathbb{R}, \quad r \in \mathbb{N}. \tag{4.49}$$

Note also from (4.46), (4.25), (4.9), (4.10) and (4.24) that

$$g(x) \geq 0, \quad x \in \mathbb{R}. \tag{4.50}$$

Now use (4.49) and (4.50) to obtain, for $r \in \mathbb{N}$,

$$\begin{aligned}
\int_{-2^r\pi}^{2^r\pi} |g(x)|dx &= \int_{-2^r\pi}^{2^r\pi} g(x)dx \\
&= \int_{-2^r\pi}^{2^r\pi} P_r(x)g\left(\frac{x}{2^r}\right)dx \\
&= \int_{-\pi}^{\pi} G_r(x)g(x)dx,
\end{aligned} \tag{4.51}$$

where

$$G_r(x) = \prod_{j=1}^{r} A(e^{-ix2^{j-1}}), \quad x \in \mathbb{R}, \quad r \in \mathbb{N}, \tag{4.52}$$

and clearly also

$$G_r(x) \geq 0, \quad x \in \mathbb{R}, \ \ r \in \mathbb{N}. \tag{4.53}$$

Since $g \in C(\mathbb{R})$, there exists a positive number R such that

$$|g(x)| \leq R, \quad |x| \leq \pi, \tag{4.54}$$

so that (4.51), (4.53), (4.54) imply

$$\int_{-2^r\pi}^{2^r\pi} |g(x)|dx \leq R \int_{-\pi}^{\pi} G_r(x)dx, \quad r \in \mathbb{N}. \tag{4.55}$$

Next, we define the sequence of Laurent polynomials

$$A_r(z) = \prod_{j=1}^{r} A(z^{2^{j-1}}), \quad z \in \mathbb{C} \setminus \{0\}, \tag{4.56}$$

and we define the sequence $\{\alpha_{r,j} : j \in \mathbb{Z}, \ r \in \mathbb{N}\}$ by

$$A_r(z) = \sum_j \alpha_{r,j} z^j, \quad z \in \mathbb{C} \setminus \{0\}, \ r \in \mathbb{N}. \tag{4.57}$$

We claim that then

$$\alpha_{r,2^r j} = \delta_{j,0}, \quad j \in \mathbb{Z}, \ r \in \mathbb{N}. \tag{4.58}$$

We prove (4.58) by induction. For $r = 1$, we see from (4.56) that $A_1(z) = A(z)$, $z \in \mathbb{C} \setminus \{0\}$, and thus, using (4.57) and (3.47)

$$\alpha_{1,j} = a_j, \quad j \in \mathbb{Z}. \tag{4.59}$$

But, according to (4.8) and Proposition 4.1, we have

$$a_{2j} = \delta_{j,0}, \quad j \in \mathbb{Z}. \tag{4.60}$$

It follows from (4.59) and (4.60) that (4.58) holds for $r = 1$. The inductive step from r to $r + 1$ follows by noting from (4.56) that, for $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned} \sum_k \alpha_{r+1,k} z^k = A_{r+1}(z) &= A_r(z) A(z^{2^r}) = \sum_k \alpha_{r,k} z^k \sum_j a_j z^{j2^r} \\ &= \sum_j a_j \left[\sum_k \alpha_{r,k} z^{k+j2^r} \right] = \sum_j a_j \left[\sum_k \alpha_{r,k-j2^r} z^k \right] \\ &= \sum_k \left[\sum_j a_j \alpha_{r,k-j2^r} \right] z^k, \end{aligned}$$

and thus, for $k \in \mathbb{Z}$,

$$\alpha_{r+1,2^{r+1}k} = \sum_j a_j \alpha_{r,2^{r+1}k-j2^r} = \sum_j a_j \alpha_{r,2^r(2k-j)} = \sum_j a_j \delta_{2k-j,0} = a_{2k},$$

from the inductive hypothesis, before appealing once again to (4.60).

Now observe from (4.52), (4.56) and (4.57) that

$$G_r(x) = A_r(e^{-ix}) = \sum_j \alpha_{r,j}(e^{-ijx}), \quad x \in \mathbb{R}, \tag{4.61}$$

and thus

$$\int_{-\pi}^{\pi} G_r(x)dx \;=\; \sum_j \alpha_{r,j} \int_{-\pi}^{\pi} (e^{-ijx})dx = 2\pi\alpha_{r,0}. \tag{4.62}$$

It follows from (4.62) and (4.58) that

$$\int_{-\pi}^{\pi} G_r(x)dx = 2\pi, \qquad r \in \mathbb{N}. \tag{4.63}$$

Now insert (4.63) into (4.55) to deduce that

$$\int_{-2^r\pi}^{2^r\pi} |g(x)|dx \leq 2\pi R, \qquad r \in \mathbb{N},$$

and thus $g \in L^1(\mathbb{R})$, with

$$\int_{-\infty}^{\infty} |g(x)|dx \leq 2\pi R. \tag{4.64}$$

Now define the function $\phi : \mathbb{R} \to \mathbb{R}$ by the formula

$$\phi(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} g(x)dx, \quad t \in \mathbb{R}, \tag{4.65}$$

for which it follows (cf. [3, proof of Theorem 2.2(ii)]), by virtue of the fact that $g \in L^1(\mathbb{R})$, that $\phi \in C(\mathbb{R})$. Moreover, using (4.65), (4.48) and (3.47), we have, for $t \in \mathbb{R}$,

$$\begin{aligned}
\phi(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \left[\frac{1}{2} \sum_k a_k e^{-ikx/2} \right] g\left(\frac{x}{2}\right) dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{2itx} \left[\sum_k a_k e^{-ikx} \right] g(x)dx \\
&= \sum_k a_k \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(2t-k)x} g(x)dx \\
&= \sum_k a_k \phi(2t-k),
\end{aligned}$$

thereby proving the existence of a refinable function $\phi \in C(\mathbb{R})$ for which (4.12) holds.

Next, we show that the interpolation property (4.13) holds. To this end, we first show that there exists a real number $\tau > 0$ such that

$$g(x) \geq \tau, \quad |x| \leq \pi. \tag{4.66}$$

Clearly, from (4.46), (4.50), (4.26) and (4.21) it holds for $t \in \mathbb{R}$ that

$$g(x) = |g(x)| = \left|\prod_{j=1}^{\infty}\left[\frac{1+e^{-i\frac{x}{2^j}}}{2}\right]^n\right| \prod_{j=1}^{\infty}\left|B(e^{-i\frac{x}{2^j}})\right|. \tag{4.67}$$

But, for any positive integer M and $x \neq 0$,

$$\begin{aligned}\prod_{j=1}^{M}\left[\frac{1+e^{-i\frac{x}{2^j}}}{2}\right] &= \prod_{j=1}^{M}\frac{1-e^{i\frac{x}{2^{j-1}}}}{2(1-e^{-i\frac{x}{2^j}})} \\ &= \frac{1-e^{-ix}}{2^M(1-e^{-i\frac{x}{2^M}})}.\end{aligned} \tag{4.68}$$

Also, using L'Hôpital's rule, we have, for $x \in \mathbb{R}$, that

$$\begin{aligned}\lim_{M\to\infty} 2^M(1-e^{-i\frac{x}{2^M}}) &= \lim_{M\to\infty}\frac{1-e^{-ix2^{-M}}}{2^{-M}} \\ &= \lim_{M\to\infty}\frac{-e^{-ix2^{-M}}(-ix)2^{-M}(-\ln 2)}{2^{-M}(-\ln 2)} \\ &= \lim_{M\to\infty} ixe^{-ix2^{-M}} = ix,\end{aligned}$$

which, together with (4.68), gives, for $x \in \mathbb{R}\setminus\{0\}$,

$$\prod_{j=1}^{\infty}\left[\frac{1+e^{-i\frac{x}{2^j}}}{2}\right] = \frac{1-e^{-ix}}{ix} = e^{i\frac{x}{2}}\left[\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right],$$

and thus

$$\prod_{j=1}^{\infty}\left[\frac{1+e^{-i\frac{x}{2^j}}}{2}\right]^n = \left[\prod_{j=1}^{\infty}\frac{1+e^{-i\frac{x}{2^j}}}{2}\right]^n = \begin{cases} e^{-in\frac{x}{2}}\left[\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}\right]^n, & x \in \mathbb{R}\setminus\{0\}, \\ 1, & x = 0.\end{cases} \tag{4.69}$$

Since B is a Laurent polynomial for which (4.23) holds on the unit circle $|z| = 1$, there exists a positive number γ such that

$$|B(e^{ix})| \geq \gamma, \quad -\pi \leq x \leq \pi, \tag{4.70}$$

Also, using (4.22), it be can proved as in the argument which yielded (4.43) that there exists a real number $c > 0$ such that

$$|1 - B(e^{-\mathbf{i}\frac{x}{2^j}})| \leq \frac{c|x|}{2^j}, \quad x \in \mathbb{R}, \quad j \in \mathbb{N},$$

and thus

$$|1 - B(e^{-\mathbf{i}\frac{x}{2^j}})| \leq \frac{c\pi}{2^j}, \quad |x| \leq \pi, \quad j \in \mathbb{N}. \tag{4.71}$$

If the positive integer j_0 is chosen to satisfy

$$2^{j_0} \geq \pi c e, \tag{4.72}$$

then, using (4.71),

$$|1 - B(e^{-\mathbf{i}\frac{x}{2^j}})| \leq e^{-1}, \quad |x| \leq \pi, \quad j \geq j_0. \tag{4.73}$$

From (4.70),

$$\begin{aligned}\left|\prod_{j=1}^{\infty} B(e^{-\mathbf{i}\frac{x}{2^j}})\right| &= \prod_{j=1}^{j_0-1} |B(e^{-\mathbf{i}\frac{x}{2^j}})| \prod_{j=j_0}^{\infty} |B(e^{-\mathbf{i}\frac{x}{2^j}})| \\ &\geq \gamma^{j_0-1} \prod_{j=j_0}^{\infty} \left(1 - |1 - B(e^{-\mathbf{i}\frac{x}{2^j}})|\right). \end{aligned} \tag{4.74}$$

Now, employ the inequality

$$1 - t \geq e^{-et}, \quad 0 \leq t \leq e^{-1},$$

in (4.74) to obtain, for $|x| \leq \pi$, having recalled also (4.73),

$$\begin{aligned}\prod_{j=j_0}^{\infty} \left(1 - |1 - B(e^{-\mathbf{i}\frac{x}{2^j}})|\right) &\geq \prod_{j=j_0}^{\infty} \exp\left(-e|1 - B(e^{-\mathbf{i}\frac{x}{2^j}})|\right) \\ &= \exp\left(-e\sum_{j=j_0}^{\infty} |1 - B(e^{-\mathbf{i}\frac{x}{2^j}})|\right) \\ &\geq \exp\left(-ec\pi \sum_{j=j_0}^{\infty} \frac{1}{2^j}\right), \end{aligned} \tag{4.75}$$

by virtue of (4.71). But

$$\sum_{j=j_0}^{\infty} \frac{1}{2^j} = \frac{1}{2^{j_0}} \sum_{j=0}^{\infty} \frac{1}{2^j} = 2^{1-j_0},$$

which, with (4.75), give

$$\prod_{j=j_0}^{\infty} \left(1 - |1 - B(e^{-i\frac{x}{2^j}})|\right) \geq e^{-2^{1-j_0}\pi ce} \geq e^{-2}, \tag{4.76}$$

by virtue of (4.72). Now combine (4.67), (4.69), (4.74), (4.76), and the inequality

$$\frac{|\sin\left(\frac{x}{2}\right)|}{|\frac{x}{2}|} \geq \frac{2}{\pi}, \quad |x| \leq \pi,$$

to deduce that (4.66) does indeed hold, with

$$\tau = \left(\frac{2}{\pi}\right)^n \gamma^{j_0-1} e^{-2}. \tag{4.77}$$

Now recall the function G_r as given by (4.52), for which, according to (4.61), we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} G_r(x) e^{i2^r jx} dx = \frac{1}{2\pi} \sum_k \alpha_{r,k} \int_{-\pi}^{\pi} e^{ix(2^r j-k)} dx = \alpha_{r,2^r j}, \quad j \in \mathbb{Z}, \quad r \in \mathbb{N}. \tag{4.78}$$

Combining (4.58) and (4.78) then gives

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} G_r(x) e^{i2^r jx} dx = \delta_{j,0}, \quad j \in \mathbb{Z}, \quad r \in \mathbb{N}. \tag{4.79}$$

But, according to (4.52), and (4.25), we have for $j \in \mathbb{Z}$, $r \in \mathbb{N}$, that

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} G_r(x) e^{i2^r jx} dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{l=1}^{r} A(e^{-ix2^{l-1}}) e^{i2^r jx} dx \\
&= \frac{1}{2\pi} \int_{-2^r\pi}^{2^r\pi} \frac{1}{2^r} \prod_{l=1}^{r} A(e^{-i\frac{x}{2^{r-l+1}}}) e^{ixj} dx \\
&= \frac{1}{2\pi} \int_{-2^r\pi}^{2^r\pi} \frac{1}{2^r} \prod_{l=1}^{r} A(e^{-i\frac{x}{2^l}}) e^{ixj} dx \\
&= \frac{1}{2\pi} \int_{-2^r\pi}^{2^r\pi} P_r(x) e^{ixj} dx.
\end{aligned} \tag{4.80}$$

Introducing the notation

$$\chi_{[a,b]}(x) = \begin{cases} 1, & x \in [a,b], \\ 0, & x \notin [a,b], \end{cases}$$

it follows from (4.80) and (4.79) that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} P_r(x)\chi_{[-2^r\pi,2^r\pi]}(x)e^{ixj}dx = \delta_{j,0}, \quad j \in \mathbb{Z}, \quad r \in \mathbb{N}. \tag{4.81}$$

Now use (4.49), (4.50) to deduce that, for $|x| \leq 2^r\pi$ and $j \in \mathbb{Z}$, $r \in \mathbb{N}$, we have

$$\left|P_r(x)\chi_{[-2^r\pi,2^r\pi]}(x)e^{ijx}\right| = |P_r(x)| = \frac{|g(x)|}{\left|g(\frac{x}{2^r})\right|} = \frac{g(x)}{g(\frac{x}{2^r})} \leq \tau^{-1}g(x), \tag{4.82}$$

by virtue of (4.66). Also, it is clear from (4.46) that

$$\lim_{r\to\infty}\left[P_r(x)\chi_{[-2^r\pi,2^r\pi]}(x)e^{ijx}\right] = g(x)e^{ijx}, \quad x \in \mathbb{R}, \ j \in \mathbb{Z}. \tag{4.83}$$

Thus, since $g \in L^1(\mathbb{R})$, (4.82) and (4.83) imply that the Lebesgue bounded convergence theorem [19, Theorem 2.9.1], can be applied to the integral in (4.81) to deduce that, for $j \in \mathbb{Z}$,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ijx}g(x)dx &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{r\to\infty}\left[P_r(x)\chi_{[-2^r\pi,2^r\pi]}(x)e^{ijx}\right]dx \\ &= \lim_{r\to\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}P_r(x)\chi_{[-2^r\pi,2^r\pi]}(x)e^{ijx}dx\right] = \delta_{j,0}.\end{aligned} \tag{4.84}$$

The desired result (4.13) is then an immediate consequence of (4.84) and (4.65).

To prove the compact support property (4.11), we first prove that

$$\phi\left(\frac{j}{2^k}\right) = 0, \ \ j \in \mathbb{Z}, \ \ |j| > 2^k(2N+1), \ \ k = 0, 1, 2\dots. \tag{4.85}$$

From (4.13), we get that (4.85) holds for $k = 0$, since N is a non-negative integer. For the inductive step from k to $k+1$, suppose that (4.85) holds, and let $|j| > 2^{k+1}(2N+1)$. Then, using (4.12), and the fact the $a_j = 0$, $|j| > 2N+2$, we get

$$\phi\left(\frac{j}{2^{k+1}}\right) = \sum_l a_l\phi\left(\frac{j}{2^k} - l\right) = \sum_{|l|\leq 2N+1} a_l\phi\left(\frac{j-2^kl}{2^k}\right) = 0,$$

since

$$|j - 2^kl| \geq |j| - 2^k|l| \geq 2^{k+1}(2N+1) - 2^k(2N+1) = 2^k(2N+1),$$

thereby proving (4.85) for all $k \in \{0, 1, ...\}$. By the continuity of ϕ, together with the fact that the sequence $\left\{\frac{j}{2^k} : \ j \in \mathbb{Z}, \ k \in \mathbb{N}\right\}$ is dense in $\mathbb{R}$, it follows from (4.85) that (4.11) holds.

Next, to establish the properties (4.15), (4.16), (4.17), and (4.18), we exploit the fact that ϕ is in $C(\mathbb{R})$, with the compact support property (4.11), and thus $\phi \in L^1(\mathbb{R})$, to deduce that the Fourier transform $\widehat{\phi}$ of ϕ is well-defined. But the (4.12) implies, for $x \in \mathbb{R}$, that

$$\begin{aligned}
\widehat{\phi}(x) &= \int_{-\infty}^{\infty} e^{-ixt} \left[\sum_k a_k \phi(2t-k)\right] dt \\
&= \sum_k a_k \int_{-\infty}^{\infty} e^{-ixt} \phi(2t-k) dt \\
&= \frac{1}{2} \sum_k a_k \int_{-\infty}^{\infty} e^{-ix\left(\frac{t+k}{2}\right)} \phi(t) dt \\
&= \frac{1}{2} \sum_k a_k \left(e^{-i\frac{x}{2}}\right)^k \int_{-\infty}^{\infty} e^{-i\frac{x}{2}t} \phi(t) dt,
\end{aligned}$$

and thus, from (3.47),

$$\widehat{\phi}(x) = \frac{1}{2} A\left(e^{-i\frac{x}{2}}\right) \widehat{\phi}\left(\frac{x}{2}\right), \quad x \in \mathbb{R}. \tag{4.86}$$

It follows from (4.86) and (4.25) that

$$\widehat{\phi}(x) = P_r(x) \widehat{\phi}\left(\frac{x}{2^r}\right), \quad x \in \mathbb{R}, \quad r \in \mathbb{N}. \tag{4.87}$$

But then, with the function g defined by (4.46), we have, using also (4.49), for $x \in \mathbb{R}$ and $r \in \mathbb{N}$,

$$g\left(\frac{x}{2^r}\right) \widehat{\phi}(x) = P_r(x) g\left(\frac{x}{2^r}\right) \widehat{\phi}\left(\frac{x}{2^r}\right),$$

and thus

$$g\left(\frac{x}{2^r}\right) \widehat{\phi}(x) = g(x) \widehat{\phi}\left(\frac{x}{2^r}\right), \quad x \in \mathbb{R}, \quad r \in \mathbb{N}. \tag{4.88}$$

Recalling from (4.66) the fact that there exists a real number $\tau > 0$ for which (4.66) is satisfied, we deduce from (4.88) that, for the function $h : [-\pi, \pi] \to \mathbb{R}$ defined by

$$h(x) := \frac{\widehat{\phi}(x)}{g(x)}, \quad x \in [-\pi, \pi], \tag{4.89}$$

we have

$$h(x) = h\left(\frac{x}{2^r}\right), \quad x \in [-\pi, \pi], \quad r \in \mathbb{N}. \tag{4.90}$$

Moreover, since $\phi \in C(\mathbb{R})$ and (4.11) together imply $\phi \in L^1(\mathbb{R})$, we have, from [3, Theorem 2.2(ii)], that $\widehat{\phi} \in C(\mathbb{R})$. Also, since $g \in C(\mathbb{R})$, as has previously been noted, and thus, noting also (4.66), we have $h \in C[-\pi,\pi]$. Hence, for $x \in [-\pi,\pi]$, we have, from (4.90),

$$h(0) = h\left(\lim_{r\to\infty} \frac{x}{2^r}\right) = \lim_{r\to\infty} h\left(\frac{x}{2^r}\right) = h(x),$$

whence, using also (4.89),

$$\widehat{\phi}(x) = \frac{\widehat{\phi}(0)}{g(0)} g(x), \quad x \in [-\pi,\pi]. \tag{4.91}$$

Now use (4.86), (4.91) and (4.48) to deduce that, for $x \in [-\pi,\pi]$,

$$\begin{aligned}
\widehat{\phi}(2x) &= \frac{1}{2}A(e^{-ix})\widehat{\phi}(x) \\
&= \frac{\widehat{\phi}(0)}{g(0)}\left[\frac{1}{2}A(e^{-ix})g(x)\right] \\
&= \frac{\widehat{\phi}(0)}{g(0)} g(2x),
\end{aligned}$$

whence

$$\widehat{\phi}(x) = \frac{\widehat{\phi}(0)}{g(0)} g(x), \quad x \in [-2\pi, 2\pi]. \tag{4.92}$$

Successively repeating the procedure which led from (4.91) to (4.92), then eventually yields the result

$$\widehat{\phi}(x) = \frac{\widehat{\phi}(0)}{g(0)} g(x), \quad x \in \mathbb{R}. \tag{4.93}$$

Since $g \in L^1(\mathbb{R})$, we see from (4.93) that $\widehat{\phi} \in L^1(\mathbb{R})$. Moreover, $\phi \in C(\mathbb{R})$ and (4.11) imply $\phi \in C_u(\mathbb{R}) \cap L^1(\mathbb{R})$. Hence we have established the fact that $\phi \in \mathcal{A}$. Moreover, from (1.5) and (4.65), we have

$$\widehat{g}(x) = \int_{-\infty}^{\infty} e^{-ixt} g(t)dt = 2\pi\phi(-x), \quad x \in \mathbb{R},$$

and thus $\widehat{g} \in L^1(\mathbb{R})$ by virtue of the fact that $\phi \in L^1(\mathbb{R})$. Also, (4.93) and the fact that $\widehat{\phi} \in C_u(\mathbb{R})$ give $g \in C_u(\mathbb{R})$. Hence we also have $g \in \mathcal{A}$. It then follows from (4.65) and Theorem 1.1 that

$$\widehat{\phi} = g. \tag{4.94}$$

The property (4.18) therefore follows from (4.94) and (4.47), whereas (4.15) is an immediate consequences of (4.18) and (1.5), with $\omega = 0$.

Also, (4.16) is immediately clear from (4.94), and the fact that $\phi, g \in \mathcal{A}$.

Next, to prove (4.14), we define the function F by means of

$$F(t) := \sum_k \phi(t-k), \quad t \in \mathbb{R}, \tag{4.95}$$

for which, from (4.11) and the fact that $\phi \in C(\mathbb{R})$, we have $F \in C(\mathbb{R})$. Now use (4.12) to deduce that, for $t \in \mathbb{R}$,

$$\begin{aligned} F(t) &= \sum_k \sum_l a_l \phi(2t-2k-l) \\ &= \sum_k \sum_l a_{l-2k} \phi(2t-l) \\ &= \sum_l \left[\sum_k a_{l-2k} \right] \phi(2t-l). \end{aligned} \tag{4.96}$$

But (4.24), (4.9) and (3.47) imply

$$\sum_k a_{2k} = \sum_k a_{2k+1} = 1,$$

whence, for $l \in \mathbb{Z}$,

$$\sum_k a_{2l-2k} = \sum_k a_{2k} = 1,$$

and

$$\sum_k a_{2l+1-2k} = \sum_k a_{2k+1} = 1,$$

and thus

$$\sum_k a_{l-2k} = 1, \quad l \in \mathbb{Z}. \tag{4.97}$$

It follows from (4.95), (4.96) and (4.97) that

$$F(t) = F(2t), \quad t \in \mathbb{R},$$

and thus

$$F(t) = F\left(\frac{x}{2^r}\right), \quad t \in \mathbb{R}, \quad r \in \mathbb{N}. \tag{4.98}$$

Exploiting the fact that $F \in C(\mathbb{R})$, it follows from (4.98) that, for any fixed $t \in \mathbb{R}$,

$$F(t) = \lim_{r\to\infty} F\left(\frac{x}{2^r}\right) = F\left(\lim_{r\to\infty} \frac{x}{2^r}\right) = F(0). \tag{4.99}$$

But, from (4.95) and (4.13), we have

$$F(0) = \sum_k \phi(-k) = \sum_k \phi(k) = \sum_k \delta_{k,0} = 1, \tag{4.100}$$

and the desired result (4.14) follows from (4.95), (4.99) and (4.100).

To prove the Riesz stability property (4.19) of ϕ, we first show that the sequence $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is algebraically linearly independent in the sense that, if $\{c_k\} \subset \mathbb{R}$ is such that

$$\sum_k c_k \phi(t-k) = 0, \quad t \in \mathbb{R}, \tag{4.101}$$

then $c_k = 0$, $k \in \mathbb{Z}$. But, from (4.13), we see that (4.101) implies, for $j \in \mathbb{Z}$,

$$0 = \sum_k c_k \phi(j-k) = \sum_k c_k \delta_{j,k} = c_j,$$

and it follows that the sequence$\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is indeed algebraically linearly independent. According to a standard result as proved in [20, Theorem 5.1], we then deduce that ϕ is indeed Riesz stable in the sense of (4.19).

It remains to prove the convergence of the stationary subdivision scheme. We see from (4.12) and (4.1) that, for $j \in \mathbb{Z}$, $r \in \mathbb{N}$,

$$\begin{aligned}
\sum_l c_l \phi\left(\frac{j}{2^r} - l\right) &= \sum_l c_l \sum_k a_k \phi\left(\frac{j}{2^{r-1}} - 2l - k\right) \\
&= \sum_l c_l \sum_k a_{k-2l} \phi\left(\frac{j}{2^{r-1}} - k\right) \\
&= \sum_k \left(\sum_l a_{k-2l} c_l\right) \phi\left(\frac{j}{2^{r-1}} - k\right) \\
&= \sum_k (S_a c)_k \, \phi\left(\frac{j}{2^{r-1}} - k\right) \\
&= \cdots \\
&= \sum_k (S_a^r c)_k \, \phi(j-k) = (S_a^r c)_j,
\end{aligned}$$

after having recalled also (4.13). ■

A useful consequence of Theorem 4.2, is that it shows that, for a given finitely supported sequence $\{a_k\}$ satisfying (3.47), (4.8), (4.9), (4.10), there exists a refinable function ϕ with the properties (4.11) to (4.19). Hence ϕ is an interpolatory scaling function from which wavelets in $L^2(\mathbb{R})$ can be constructed by means of a multiresolutional approach (see e.g. [3], [4], [6], [22], [23], [24], [25]). We shall not further pursue this aspect here.

Theorem 4.2 therefore shows that, if a SSS is interpolatory in the sense of (4.2), i.e. according to Proposition 4.1, if the corresponding symbol A satisfies (4.8), then the additional constraints (4.9) and (4.10) on A ensures convergence of the SSS. In our next section, we proceed to identify a class of such interpolatory stationary subdivision schemes.

4.3 Construction based on a Bezout Identity

In this section, we identify a class of symmetric Laurent polynomials A which satisfy the sufficient conditions for convergence of Theorem 4.2. We follow, in the proof of the theorem below, the guidelines of a technique introduced in [16], in which also the results and methods of [23, Lemmas 2.3 and 2.4] were employed.

Theorem 4.3 *For a non-negative integer N, suppose B is a symmetric polynomial of degree $2N+2$, with $B(-1)=0$, and suppose B has only negative zeros. Then there exists a unique polynomial C of degree $2N$ such that the Bezout identity*

$$B(z)C(z) - B(-z)C(-z) = z^{2N+1}, \qquad z \in \mathbb{C}, \tag{4.102}$$

is satisfied. Also, C is a symmetric polynomial of degree $2N$, with

$$C(z) \neq 0, \qquad |z| = 1, \tag{4.103}$$

and, writing $C(z) = \sum_{j=0}^{2N} c_j z^j$, $z \in \mathbb{C}$, we have

$$(-1)^{N+j} c_j > 0, \qquad j = 0, 1, .., 2N. \tag{4.104}$$

Moreover, the Laurent polynomial A defined by

$$A(z) = 2z^{-2N-1} B(z)C(z), \qquad z \in \mathbb{C} \setminus \{0\}, \tag{4.105}$$

is symmetric, has degree $2N+1$, and satisfies the conditions (4.8), (4.9) and (4.10) of Theorem 4.2.

Proof. Since the polynomial B is of degree $2N+2$, the polynomial B_- defined by means of

$$B_-(z) := B(-z), \quad z \in \mathbb{C}, \tag{4.106}$$

has degree $2N+2$. Moreover, since B has only negative zeros, we have $\gcd\{B, B_-\} = 1$. Hence, according to the Euclidean algorithm, there exist polynomials u and v such that

$$B(z)u(z) + B(-z)v(z) = 1, \quad z \in \mathbb{C}, \tag{4.107}$$

and thus also

$$B(-z)u(-z) + B(z)v(-z) = 1, \quad z \in \mathbb{C}. \tag{4.108}$$

Hence, if we define the polynomial p by

$$p(z) = \frac{1}{2}\left[u(z) + v(-z)\right], \quad z \in \mathbb{C}, \tag{4.109}$$

it follows from (4.107), (4.108) and (4.109) that

$$B(z)p(z) + B(-z)p(-z) = 1, \quad z \in \mathbb{C}. \tag{4.110}$$

It follows from (4.110) that

$$B(z)z^{2N+1}p(z) + B(-z)z^{2N+1}p(-z) = z^{2N+1}, \quad z \in \mathbb{C}. \tag{4.111}$$

Also, according to the Euclidean algorithm, there exist unique polynomials q and r, with $r \in \Pi_{2N+1}$, such that

$$z^{2N+1}p(-z) = q(z)B(z) + r(z), \quad z \in \mathbb{C}. \tag{4.112}$$

If we define the polynomial Q by

$$Q(z) = z^{2N+1}p(z) + q(z)B(-z), \quad z \in \mathbb{C}, \tag{4.113}$$

we find from (4.112), (4.111) and (4.113) that

$$B(z)Q(z) + B(-z)r(z) = z^{2N+1}, \quad z \in \mathbb{C}. \tag{4.114}$$

It is clear from (4.114), and the fact that $r \in \Pi_{2N+1}$, that $Q \in \Pi_{2N+1}$. In fact, we claim that

$$Q(z) = -r(-z), \quad z \in \mathbb{C}. \tag{4.115}$$

To prove (4.115), we note from (4.112) that

$$-z^{2N+1}p(z) = q(-z)B(-z) + r(-z), \quad z \in \mathbb{C},$$

which, together with (4.113), implies that, for $z \in \mathbb{C}$,

$$\begin{aligned} Q(z) &= -q(-z)B(-z) - r(-z) + q(z)B(-z) \\ &= B(-z)\left[q(z) - q(-z)\right] - r(-z). \end{aligned} \tag{4.116}$$

Now, if we define the polynomial w by

$$w(z) = q(z) - q(-z), \quad z \in \mathbb{C},$$

and suppose w is not the zero polynomial, then, since $Q \in \Pi_{2N+1}$, and since $r \in \Pi_{2N+1}$, we see from (4.116) that

$$2N + 1 \geq \deg Q = \deg B + \deg w = 2N + 2 + \deg w \geq 2N + 2,$$

which is a contradiction. Hence w is the zero polynomial, and it follows from (4.116) that (4.115) holds.

Substituting (4.115) into (4.114) then gives

$$B(z)\left(-r(-z)\right) + B(-z)r(z) = z^{2N+1}, \quad z \in \mathbb{C}.$$

Thus the polynomial C defined by

$$C(z) = -r(-z), \quad z \in \mathbb{C}, \tag{4.117}$$

satisfies (4.102), and $C \in \Pi_{2N+1}$. Exploiting the fact that B is a polynomial of even degree $2N + 2$, we see from (4.102) that it is not possible for the polynomial C to have degree $2N + 1$, i.e. $C \in \Pi_{2N}$.

To prove the uniqueness of C, suppose that there exists a $C^* \in \Pi_{2N}$, with $C^* \neq C$, which satisfies (4.102) with C replaced by C^*. Then, if we define the polynomial $\widetilde{C}$ by $\widetilde{C} = C - C^*$, we have $\widetilde{C} \neq 0$, and

$$B(z)\widetilde{C}(z) = B(-z)\widetilde{C}(-z), \quad z \in \mathbb{C}, \tag{4.118}$$

with $\widetilde{C} \in \Pi_{2N}$. But , since $\gcd\{B, B_-\} = 1$, the polynomials B and B_- have no common factors, so that $B(z)$ must be a factor of $\widetilde{C}(-z)$, which is not possible, since $\widetilde{C} \in \Pi_{2N}$, while the degree of B is $2N+2$. Hence the polynomial C given by (4.117) is the unique solution of (4.102).

Next, to prove (4.104), we write

$$C(z) = \sum_{j=0}^{2N} c_j z^j, \quad z \in \mathbb{C}, \tag{4.119}$$

$$B(z) = \sum_{j=0}^{2N+2} b_j z^j, \quad z \in \mathbb{C}, \tag{4.120}$$

and setting

$$b_j = 0, \;\; j \notin \{0, 1, ..., 2N+2\},$$

it follows that the identity (4.102) is equivalent to the linear system of equations given by

$$\sum_{j=0}^{2N} b_{2i+1-j} c_j = \delta_{i,N+1} \quad i = 0, 1, ..., 2N. \tag{4.121}$$

Define the matrix F by

$$F = (F_{ij}), \qquad F_{ij} = b_{2i+1-j}, \quad i, j = 0, 1, ..., 2N, \tag{4.122}$$

and introduce the notation

$$F\begin{pmatrix} r_1, & r_2, & \cdots & r_p \\ k_1, & k_2, & \cdots & k_p \end{pmatrix}, \quad 1 \leq p \leq 2N+1, \tag{4.123}$$

for the determinant of the submatrix of F formed by rows $r_1,\ r_2, ..., r_p$ and columns $k_1,\ k_2, ..., k_p$. An application of Cramer's rule to (4.121) yields

$$c_j = \frac{(-1)^{N+j}}{\det(F)} F\begin{pmatrix} r_1, & r_2, & \cdots & r_{2N} \\ k_{j,1}, & k_{j,2}, & \cdots & k_{j,2N} \end{pmatrix}, \quad j = 0, 1, ..., 2N, \tag{4.124}$$

where

$$r_n = \begin{cases} n, & n = 1, 2, ..., N-1, \\ n+1, & n = N, N+1, ..., 2N, \end{cases} \tag{4.125}$$

and

$$k_{j,n} = \begin{cases} n, & n = 1, 2, ..., j, \\ n+1, & n = j+1, j+2, ..., 2N, \end{cases} \qquad j = 0, 1, ..., 2N. \tag{4.126}$$

The polynomial B has only negative zeros, and therefore B is a Hurwitz polynomial. Since the matrix F is constructed, as in (4.122), from the Hurwitz polynomial B, and since (4.125) and (4.126) give the inequalities $0 \leq 2r_n - k_{j,n} \leq 2N + 2$, $n = 1, 2, ..., 2N$, $j = 0, 1, ..., 2N$, the result [25, Lemma 2.4] can be employed to deduce that $\det(F) > 0$ and

$$F \begin{pmatrix} r_1, & r_2, & \cdots & r_{2N} \\ k_{j,1}, & k_{j,2}, & \cdots & k_{j,2N} \end{pmatrix} > 0, \qquad j = 0, 1, ..., 2N, \tag{4.127}$$

and (4.104) then follows from (4.124). Observe also that (4.104), together with the fact that $C \in \Pi_{2N}$, implies $\deg C = 2N$.

To prove that C is a symmetrical polynomial, we replace, for $z \in \mathbb{C} \setminus \{0\}$, the variable z in (4.102) by $\dfrac{1}{z}$, and use the fact that B is a symmetrical polynomial, i.e. $z^{2N+2}B\left(\dfrac{1}{z}\right) = B(z)$, $z \in \mathbb{C}$, to deduce that (4.102) holds with C replaced by $C_0 \in \Pi_{2N}$, as defined by $C_0(z) = z^{2N}C\left(\dfrac{1}{z}\right)$, $z \in \mathbb{C}$. From the uniqueness of a solution in Π_{2N} of (4.102), it follows that $C = C_0$, i.e. C is a symmetric polynomial.

Next, to prove (4.103), we first define the Laurent polynomials G and H by

$$G(z) := \frac{B(z)}{z^{N+1}}, \qquad H(z) := \frac{C(z)}{z^N}, \qquad z \in \mathbb{C} \setminus \{0\}. \tag{4.128}$$

Observe, from the fact that B and C are both symmetric polynomials, that both G and H are symmetric Laurent polynomials. Using the fact that, for each $k \in \{0, 1, ...\}$, there exists a sequence $\{\beta_{k,j} : j = 0, 1, ..., k\} \subset \mathbb{R}$, with $\beta_{k,k} \neq 0$, such that

$$\cos(kx) = \sum_{j=0}^{k} \beta_{k,j} (\cos x)^j, \qquad x \in \mathbb{R}, \tag{4.129}$$

and exploiting the symmetry of the polynomial B, according to which, in (4.120),

$$b_{2N+2-j} = b_j, \quad j = 0, 1, ..., N+1, \tag{4.130}$$

we see from (4.128) and (4.129) that, for $\omega \in \mathbb{R}$, we have

$$\begin{aligned}
G(e^{-i\omega}) &= e^{i\omega(N+1)} \sum_{k=0}^{2N+2} b_k e^{-i\omega k} \\
&= \sum_{k=0}^{2N+2} b_k e^{i\omega(N+1-k)} \\
&= b_{N+1} + \sum_{k=0}^{N} b_k e^{i\omega(N+1-k)} + \sum_{k=N+2}^{2N+2} b_k e^{i\omega(N+1-k)} \\
&= b_{N+1} + \sum_{k=0}^{N} b_k e^{i\omega(N+1-k)} + \sum_{k=0}^{N} b_{2N+2-k} e^{-i\omega(N+1-k)} \\
&= b_{N+1} + \sum_{k=0}^{N} b_k e^{i\omega(N+1-k)} + \sum_{k=0}^{N} b_k e^{-i\omega(N+1-k)} \\
&= b_{N+1} + 2\sum_{k=0}^{N+1} b_k \cos\left[(N+1-k)\omega\right] \\
&= b_{N+1} + 2\sum_{k=1}^{N+1} b_{N+1-k} \cos(k\omega) \\
&= b_{N+1} + 2\sum_{k=1}^{N+1} b_{N+1-k} \sum_{l=0}^{k} \beta_{k,l} (\cos\omega)^l \\
&= b_{N+1} + 2\sum_{k=1}^{N+1} b_{N+1-k} \sum_{l=0}^{k} \beta_{k,l} \left(1 - 2\sin^2\frac{\omega}{2}\right)^l \\
&= b_{N+1} + 2\sum_{k=1}^{N+1} b_{N+1+k} \sum_{l=0}^{k} \beta_{k,l} \left(1 - 2\sin^2\frac{\omega}{2}\right)^l,
\end{aligned}$$

after using (4.130) again. Hence, since $\deg B = 2N+2$ gives $b_{2N+2} \neq 0$, and since $\beta_{N+1,N+1} \neq 0$, it follows that there exists a polynomial p of degree $N+1$ such that

$$G(e^{-i\omega}) = p\left(\sin^2\left(\frac{\omega}{2}\right)\right), \quad \omega \in \mathbb{R}. \tag{4.131}$$

Since

$$\sin^2\left(\frac{\omega+\pi}{2}\right) = \cos^2\left(\frac{\omega}{2}\right) = 1 - \sin^2\left(\frac{\omega}{2}\right), \quad \omega \in \mathbb{R},$$

and

$$G(-e^{-i\omega}) = G(e^{-i(\omega+\pi)}), \quad \omega \in \mathbb{R},$$

it follows from (4.131) that

$$G(-e^{-i\omega}) = p\left(1 - \sin^2\left(\frac{\omega}{2}\right)\right), \quad \omega \in \mathbb{R}. \tag{4.132}$$

Similarly, we can show from (4.128) that there exists a polynomial q of degree N such that

$$H(e^{-i\omega}) = q\left(\sin^2\left(\frac{\omega}{2}\right)\right), \quad \omega \in \mathbb{R}, \tag{4.133}$$

and

$$H(-e^{-i\omega}) = q\left(1 - \sin^2\left(\frac{\omega}{2}\right)\right)), \quad \omega \in \mathbb{R}. \tag{4.134}$$

Hence, from (4.102), (4.131), (4.132), (4.133), (4.134) it follows that the polynomials p and q satisfy the identity

$$p(x)q(x) + p(1-x)q(1-x) = 1, \quad x \in [0,1],$$

and therefore

$$p(z)q(z) + p(1-z)q(1-z) = 1, \quad z \in \mathbb{C}. \tag{4.135}$$

Next, observe that

$$\sin^2\left(\frac{\omega}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos\omega = \frac{1}{2} - \frac{1}{4}\left(z + \frac{1}{z}\right) = -\frac{(z-1)^2}{4z}, \quad z = e^{-i\omega}, \quad \omega \in \mathbb{R}, \tag{4.136}$$

and thus, from (4.131),

$$G(z) = p\left(-\frac{(z-1)^2}{4z}\right), \quad |z| = 1.$$

Hence

$$z^{N+1}G(z) = z^{N+1}p\left(-\frac{(z-1)^2}{4z}\right), \quad |z| = 1,$$

so that (4.128) yields

$$B(z) = z^{N+1}p\left(-\frac{(z-1)^2}{4z}\right), \quad |z| = 1, \tag{4.137}$$

and thus also, since both the left and the right hand sides of (4.137) are polynomials in z, we have

$$B(z) = z^{N+1} p\left(-\frac{(z-1)^2}{4z}\right), \quad z \in \mathbb{C}. \tag{4.138}$$

If we now consider the coordinate transformation

$$x = \frac{1}{2} - \frac{1}{4}\left(z + \frac{1}{z}\right) = -\frac{(z-1)^2}{4z}, \quad z \in \mathbb{C} \setminus \{0\}, \tag{4.139}$$

a routine calculus procedure shows that $x \geq 1$ if and only if $z < 0$, (see also Figure 4.1). We

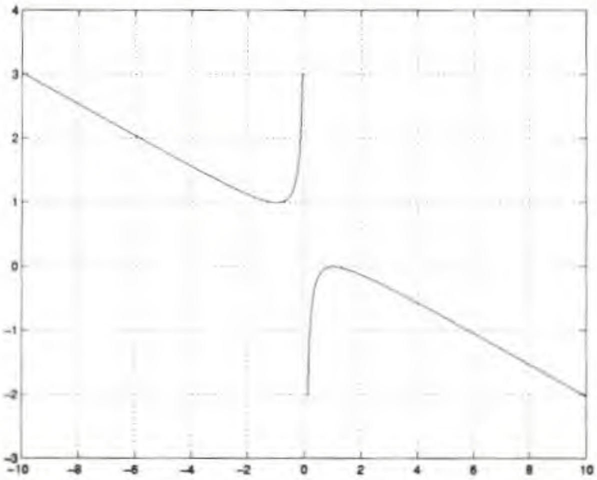


Figure 4.1: *The transformation* $x = -\frac{(z-1)^2}{4z}$, $z \in \mathbb{R}$

claim that the zeros of p all lie in the interval $[1, \infty)$. For suppose $x_0 \in \mathbb{C} \setminus [1, \infty)$ is such that $p(x_0) = 0$, and choose $z_0 \in \mathbb{C} \setminus \{0\}$ to satisfy

$$z_0^2 + (4x_0 - 2)z_0 + 1 = 0,$$

which is equivalent to the equation

$$x_0 = -\frac{(z_0 - 1)^2}{4z_0}.$$

Thus, since $x_0 \notin [1, \infty)$, we also have $z_0 \notin (-\infty, 0)$. Also, (4.138) then gives $B(z_0) = z_0^{N+1} p(x_0) = 0$, which is a contradiction, since B has only negative zeros. Hence, the zeros of p all lie in the interval $[1, \infty)$.

Since $\deg p = N+1$, it follows that, counting multiplicities, p has $N+1$ zeros in $[1,\infty)$. Hence the polynomial $w = pq$ has at least $N+1$ zeros in $[1,\infty)$. Thus, from Rolle's theorem, the polynomial w' has at least N zeros in $[1,\infty)$. Also, from (4.135), $w'(x) = w'(1-x), \quad x \in \mathbb{R}$, so that w' also has at least N zeros in $(-\infty,0]$. Hence, w' has at least $2N$ zeros in $(-\infty,0] \cup [1,\infty)$. Since $\deg w = \deg pq = 2N+1$, we have $w' \in \Pi_{2N}$. Thus w' has at most $2N$ zeros, which must all lie in $(-\infty,0] \cup [1,\infty)$. It follows that

$$w'(x) \neq 0, \quad x \in (0,1). \tag{4.140}$$

Now observe from (4.131), (4.133) and (4.128) that

$$w(0) = p(0)q(0) = G(1)H(1) = B(1)C(1), \tag{4.141}$$

and

$$w(1) \;=\; p(1)q(1) = G(-1)H(-1) = -B(-1)C(-1). \tag{4.142}$$

But $B(-1) = 0$, so that (4.102) yields

$$B(1)C(1) = 1, \tag{4.143}$$

and thus (4.141) and (4.142) give $w(0) = 1$, $w(1) = 0$. Suppose there exists a point $x_0 \in (0,1)$ such that $w(x_0) = 0$. Then there must exist a point $\xi_0 \in [x_0, 1)$ such that $w'(\xi_0) = 0$, which contradicts (4.140). Hence

$$w(x) = p(x)q(x) \neq 0, \quad x \in (0,1). \tag{4.144}$$

But, since the zeros of p all lie in $[1,\infty)$, we have $p(x) \neq 0$, $x \in (0,1)$, so that (4.144) gives

$$q(x) \neq 0, \quad x \in (0,1). \tag{4.145}$$

It follows from (4.128), (4.133) and (4.145) that

$$C(e^{i\omega}) \neq 0, \quad -\pi < \omega < \pi. \tag{4.146}$$

Now note that (4.143) implies

$$C(1) \neq 1, \tag{4.147}$$

whereas (4.119) and (4.104) give

$$(-1)^N C(-1) = \sum_{j=0}^{2N} (-1)^{N+j} c_j > 0,$$

and thus

$$C(-1) \neq 0. \tag{4.148}$$

The desired property (4.103) the follows by combining (4.146), (4.147) and (4.148).

Now consider the Laurent polynomial A defined by (4.105), and write

$$A(z) = \sum_k a_k z^k, \quad z \in \setminus\{0\}.$$

Then $a_{2N+1} = 2b_{2N+2}c_{2N} \neq 0$ and $a_k = 0$ if $k \geq 2N+2$, since $\deg B = 2N+2$ and $\deg C = 2N$. Also, since B has only negative zeros, we have from (4.120), $0 \neq B(0) = b_0$, whereas $c_0 \neq 0$ from (4.104), hence $a_{-2N-1} = \frac{1}{2}b_0c_0 \neq 0$ and clearly $a_k = 0$, $k \leq -2N-2$. Hence the Laurent polynomial A has degree $2N+1$. Moreover, for $z \in \mathbb{C} \setminus \{0\}$, and using the fact that B and C are symmetric polynomials, we have

$$\begin{aligned} A\left(\frac{1}{z}\right) &= 2z^{2N+1} B\left(\frac{1}{z}\right) C\left(\frac{1}{z}\right) \\ &= 2z^{2N+1} \frac{B(z)}{z^{2N+2}} \frac{C(z)}{z^{2N}} \\ &= 2z^{-2N-1} B(z) C(z) = A(z), \end{aligned}$$

so that A is a symmetric Laurent polynomial.

We see from (4.105) and (4.102) that the Laurent polynomial A satisfies (4.8). Also, since $B(-1) = 0$, we have $A(-1) = 0$.

Now observe that, for $z \in \mathbb{C} \setminus \{0\}$, and using the fact that $a_j = a_{-j}$, $j \in \mathbb{Z}$,

$$\begin{aligned} A(z) &= \sum_{j=-2N-1}^{2N+1} a_j z^j \\ &= a_0 + \sum_{j=1}^{2N+1} a_j z^j + \sum_{j=-2N-1}^{-1} a_j z^j \\ &= a_0 + \sum_{j=1}^{2N+1} a_j \left(z^j + z^{-j}\right). \end{aligned}$$

Thus, for $\omega \in \mathbb{R}$,

$$A(e^{i\omega}) = a_0 + 2\sum_{j=1}^{2N+1} a_j \cos(j\omega).$$

Hence, $A(e^{i\omega}) \in \mathbb{R}$, $\omega \in \mathbb{R}$.

Moreover, the fact that B only has negative zeros implies $B(e^{i\omega}) \neq 0$, $-\pi < \omega < \pi$, which, together with (4.103) and (4.105), show that $A(e^{i\omega}) \neq 0$, $-\pi < \omega < \pi$. Also, according to (4.143) and (4.105), we have $A(1) = 2$, and it follows that (4.10) holds. ■

The results of Theorem 4.2 and 4.3 show that a convergent, symmetric, interpolatory SSS can be generated by choosing an arbitrary even degree, symmetric polynomial B, with $B(-1) = 0$, and with only negative zeros, and then solving the Bezout identity (4.102) for the polynomial C, before constructing the Laurent polynomial A by means of (4.105). Moreover, the constructive nature of our proof of Theorem 4.3 suggests the following algorithm for the construction of a symmetrical convergent interpolatory SSS.

Algorithm 4.4

1. Choose a non-negative integer N.

2. Choose a symmetric polynomial B of degree $2N+2$, such that $B(-1) = 0$ and B has only negative zeros.

3. Find polynomials u and v which are such that

$$B(z)u(z) + B(-z)v(z) = 1, \qquad z \in \mathbb{C}.$$

4. Define the polynomial p by

$$p(z) = \frac{1}{2}[u(z) + v(-z)], \qquad z \in \mathbb{C}.$$

5. Apply the Euclidean algorithm to find the (unique) polynomial $r \in \Pi_{2N+1}$ which is such that

$$z^{2N+1}p(-z) = q(z)B(z) + r(z), \quad z \in \mathbb{C},$$

with q denoting a (unique) polynomial.

6. Define the polynomial C by

$$C(z) = -r(-z), \qquad z \in \mathbb{C}.$$

7. Define the Laurent polynomial A by

$$A(z) = 2z^{2N-1}B(z)C(z), \quad z \in \mathbb{C} \setminus \{0\}.$$

8. Define the sequence $\{a_k\}$ by

$$\sum_k a_k z^k = A(z), \qquad z \in \mathbb{C} \setminus \{0\}.$$

9. Define the SSS by

$$(S_a c)_k = \sum_j a_{k-2j} c_j, \quad \{c_j\} \in \ell^\infty.$$

In the rest of this thesis we consider specific choices for the polynomial B.

Chapter 5

Deslauriers-Dubuc Subdivision

5.1 Constructing the Mask from a Bezout Identity

As was done in [16], we first consider the case where the polynomial B in Theorem 4.3 is chosen as

$$B(z) = (1+z)^{2N+2}, \qquad z \in \mathbb{C}. \tag{5.1}$$

Since then

$$z^{2N+2} B\left(\frac{1}{z}\right) = B(z), \qquad z \in \mathbb{C},$$

it follows that B is a symmetric polynomial of degree $2N+2$. Also, $B(-1) = 0$, and $z = -1$ is the only zero of B. Thus the polynomial B satisfies the conditions of Theorem 4.3, so that if we define C as the unique polynomial of degree $2N$ which satisfies the Bezout identity

$$(1+z)^{2N+2}C(z) - (1-z)^{2N+2}C(-z) = z^{2N+1}, \qquad z \in \mathbb{C}, \tag{5.2}$$

then, according to Theorem 4.3 and Theorem 4.2, the Laurent polynomial A defined by

$$A(z) = 2z^{-2N-1}(1+z)^{2N+2}C(z), \qquad z \in \mathbb{C} \setminus \{0\}, \tag{5.3}$$

yields a convergent symmetric interpolatory SSS.

To derive, following [16], an explicit formulation of the Laurent polynomial A, we introduce, for a given non-negative integer N, the Lagrange fundamental polynomials $\{L_{N,j} : j =$

$-N, ..., N+1\}$ of degree $2N+1$ with respect to the interpolation points $\{-N, ..., N+1\}$, that is

$$L_{N,j}(x) = \prod_{j \neq l = -N}^{N+1} \frac{x-l}{j-l}, \quad x \in \mathbb{R}, \quad j = -N, ..., N+1, \tag{5.4}$$

so that

$$L_{N,j}(k) = \delta_{j,k}, \qquad j, k = -N, ..., N+1. \tag{5.5}$$

Also, recall that then

$$\sum_{j=-N}^{N+1} p(j) L_{N,j}(x) = p(x), \quad x \in \mathbb{R}, \quad p \in \Pi_{2N+1}. \tag{5.6}$$

Proposition 5.1 *Suppose the Laurent polynomial A is given by (5.3), as obtained in Theorem 4.3 with the polynomial B chosen as in (5.1). Then, writing*

$$A(z) = \sum_j a_j z^j, \quad z \in \mathbb{C} \setminus \{0\}, \tag{5.7}$$

we have

$$a_{2j+\varepsilon} = L_{N,-j}\left(\frac{\varepsilon}{2}\right), \quad j = -N-1, ..., N, \quad \varepsilon \in \{0, 1\}, \tag{5.8}$$

and

$$a_j = 0, \qquad |j| \geq 2N+2, \tag{5.9}$$

with the Lagrange fundamental polynomial sequence $\{L_{N,j} : j = -N, ..., N+1\}$ defined as in (5.4).

Proof. Suppose the Laurent polynomial A is defined by (5.7), (5.8), (5.9). We claim that then

$$A^{(k)}(-1) = 0, \qquad k = 0, 1, ..., 2N+1. \tag{5.10}$$

To prove (5.10), we first observe from (5.7) and (5.9) that

$$A(-1) = \sum_j a_{2j} - \sum_j a_{2j+1}. \tag{5.11}$$

But (5.8) implies

$$a_{2j} = L_{N,-j}(0) = \delta_{j,0}, \quad j = -N-1, ..., N,$$

from (5.5), which, together with (5.9), give

$$a_{2j} = \delta_{j,0}, \quad j \in \mathbb{Z}, \tag{5.12}$$

and thus

$$\sum_j a_{2j} = 1. \tag{5.13}$$

Next, we use (5.8) and (5.9) to get

$$\sum_j a_{2j+1} = \sum_{j=-N-1}^{N} L_{N,-j}\left(\frac{1}{2}\right) = \sum_{j=-N}^{N+1} L_{N,j}\left(\frac{1}{2}\right) = 1, \tag{5.14}$$

as obtained by choosing the polynomial p in (5.6) as $p(x) = 1$, $x \in \mathbb{R}$. It follows from (5.11), (5.13) and (5.14) that

$$A(-1) = 0, \tag{5.15}$$

so that (5.10) holds for $k = 0$. Next, we use (5.7) to obtain, for a fixed $k \in \{1, ..., 2N+1\}$, that

$$A^{(k)}(z) = \sum_j q_k(j) a_j z^{j-k}, \quad z \in \mathbb{C} \setminus \{0\}, \tag{5.16}$$

where the polynomial q_k is given by

$$q_k(x) = \prod_{l=0}^{k-1}(x - l), \quad x \in \mathbb{R}. \tag{5.17}$$

Hence, from (5.16),

$$\begin{aligned} A^{(k)}(-1) &= (-1)^k \sum_j q_k(2j) a_{2j} + (-1)^{k+1} \sum_j q_k(2j+1) a_{2j+1} \\ &= (-1)^{k+1} \sum_j q_k(2j+1) a_{2j+1}, \end{aligned} \tag{5.18}$$

from (5.12) and the fact that $q_k(0) = 0$ by virtue of the definition (5.17).

Now use (5.18), (5.8) and (5.9) to find

$$\begin{aligned} A^{(k)}(-1) &= (-1)^{k+1} \sum_{j=-N-1}^{N} q_k(2j+1) L_{N,-j}\left(\frac{1}{2}\right) \\ &= (-1)^{k+1} \sum_{j=-N}^{N+1} q_k(-2j+1) L_{N,j}\left(\frac{1}{2}\right). \end{aligned} \tag{5.19}$$

Let the polynomial $\tilde{q}_k$ be defined by

$$\tilde{q}_k(x) = q_k(-2x+1), \quad x \in \mathbb{R}. \tag{5.20}$$

Since (5.17) gives $q_k \in \Pi_k \subset \Pi_{2N+1}$, we have $\tilde{q}_k \in \Pi_k \subset \Pi_{2N+1}$, and thus, from (5.6) and (5.20),

$$\begin{aligned} A^{(k)}(-1) &= (-1)^{k+1} \sum_{j=-N}^{N+1} \tilde{q}_k(j) L_{N,j}\left(\frac{1}{2}\right) \\ &= (-1)^{k+1} \tilde{q}_k\left(\frac{1}{2}\right) \\ &= (-1)^{k+1} q_k(0) = 0, \end{aligned}$$

from (5.17), thereby proving (5.10) for $k \in \{1, 2, ..., 2N+1\}$.

Next, note from (5.8) and (5.4) that

$$\begin{aligned} a_{2N+1} &= L_{N,-N}\left(\frac{1}{2}\right) = \prod_{l=-N+1}^{N+1} \frac{\frac{1}{2}-l}{j-l} \neq 0, \\ a_{-2N-1} &= L_{N,N+1}\left(\frac{1}{2}\right) = \prod_{l=-N}^{N} \frac{\frac{1}{2}-l}{j-l} \neq 0, \end{aligned}$$

and it follows from (5.9) that the Laurent polynomial A defined by (5.7) has degree $2N+1$. Hence there is a polynomial H of degree $4N+2$ such that

$$A(z) = \frac{1}{z^{2N+1}} H(z), \quad z \in \mathbb{C} \setminus \{0\}, \tag{5.21}$$

that is

$$H(z) = z^{2N+1} A(z), \quad z \in \mathbb{C}. \tag{5.22}$$

An inductive method based on (5.10) then yields

$$H^{(k)}(-1) = 0, \quad k = 0, 1, ..., 2N+1.$$

Hence there exists a polynomial C of degree $2N$ such that

$$H(z) = (1+z)^{2N+2} C(z), \quad z \in \mathbb{C}, \tag{5.23}$$

and it follows from (5.21) and (5.23) that

$$A(z) = z^{-2N-1}(1+z)^{2N+2}C(z), \qquad z \in \mathbb{C} \setminus \{0\}. \tag{5.24}$$

Moreover, since (5.12) holds, it follows from Proposition 4.1 that

$$A(z) + A(-z) = 2, \qquad z \in \mathbb{C} \setminus \{0\}, \tag{5.25}$$

and thus, from (5.24) and (5.25), that C satisfies the identity

$$(1+z)^{2N+2}C(z) - (1-z)^{2N+2}C(-z) = z^{2N+1}, \qquad z \in \mathbb{C}. \tag{5.26}$$

The uniqueness of C as a solution in Π_{2N} of (5.26), as ensured by Theorem 4.3, then shows that the Laurent polynomial A defined in (5.3) is identical to the one defined by (5.7), (5.8), (5.9). ■

In the literature (see e.g. [24], [17]) the stationary subdivision operator S_a defined by

$$(S_a c)_k = \sum_j a_{k-2j} c_j, \quad k \in \mathbb{Z}, \ c = \{c_j\} \subset \mathbb{R}, \tag{5.27}$$

with mask sequence $\{a_k\}$ defined by (5.8), (5.9) is known as the *Deslauriers-Dubuc* SSS. According to Proposition 5.1, we have shown here that Deslauriers-Dubuc subdivision has the alternative formulation (5.27), (5.7), (5.3), with C denoting the (unique) polynomial of degree $2N$ which satisfies the Bezout identity (5.2).

The fact that the Deslauriers-Dubuc subdivision algorithm is a convergent SSS was proved in a direct manner in [17] and [24, Lemma 3.1 and Theorem 4.1] . Our Theorem 4.3 and Proposition 5.1 enable us to characterize Deslauriers-Dubuc subdivision as a special case of a more general context, so that Theorem 4.2 yields convergence. Hence, we have the following theorem.

Theorem 5.2 *The Deslauriers-Dubuc SSS (5.27), (5.8), (5.9) is a symmetric convergent interpolatory SSS.*

5.2 An Explicit Formulation

Next, we calculate, as in [16], an explicit formulation for the Deslauriers-Dubuc mask coefficient sequence $\{a_j\}$.

Since, for $j \in \{-N, ..., N+1\}$, we have

$$\prod_{j \neq l=-N}^{N+1} \left(\frac{1}{2} - l\right) = \frac{1}{2^{2N+1}} \frac{1}{1-2j} \prod_{l=-N-1}^{N} (2l+1) = \frac{(-1)^N}{2^{4N+1}} \frac{1}{2j-1} \left[\frac{(2N+1)!}{N!}\right]^2,$$

and

$$\prod_{j \neq l=-N}^{N+1} (j-l) = (-1)^{N+1+j}(N+j)!(N+1-j)!\,,$$

we have from (5.4) that

$$L_{N,j}\left(\frac{1}{2}\right) = \frac{N+1}{2^{4N+1}} \begin{pmatrix} 2N+1 \\ N \end{pmatrix} \frac{(-1)^{j+1}}{2j-1} \begin{pmatrix} 2N+1 \\ N+j \end{pmatrix}, \quad j = -N, ..., N+1. \tag{5.28}$$

Hence, from (5.8), (5.9), (5.12), (5.28), we have the following explicit formulation.

Theorem 5.3 *The Deslauriers-Dubuc mask coefficient sequence $\{a_j\}$, as defined by (5.8), (5.9), is given explicitly by*

$$\left.\begin{array}{ll} a_{2j+1} = \dfrac{N+1}{2^{4N+1}} \begin{pmatrix} 2N+1 \\ N \end{pmatrix} \dfrac{(-1)^j}{2j+1} \begin{pmatrix} 2N+1 \\ N+j+1 \end{pmatrix}, & j = -N-1, ..., N, \\ \\ a_{2j} = \delta_{j,0}, & j = -N, ..., N, \\ \\ a_j = 0, & |j| \geq 2N+2. \end{array}\right\} \tag{5.29}$$

5.3 Examples

We can now calculate explicitly by means of (5.29) to find that the Deslauriers-Dubuc Laurent polynomial A, for $z \in \mathbb{C} \setminus \{0\}$, is given by

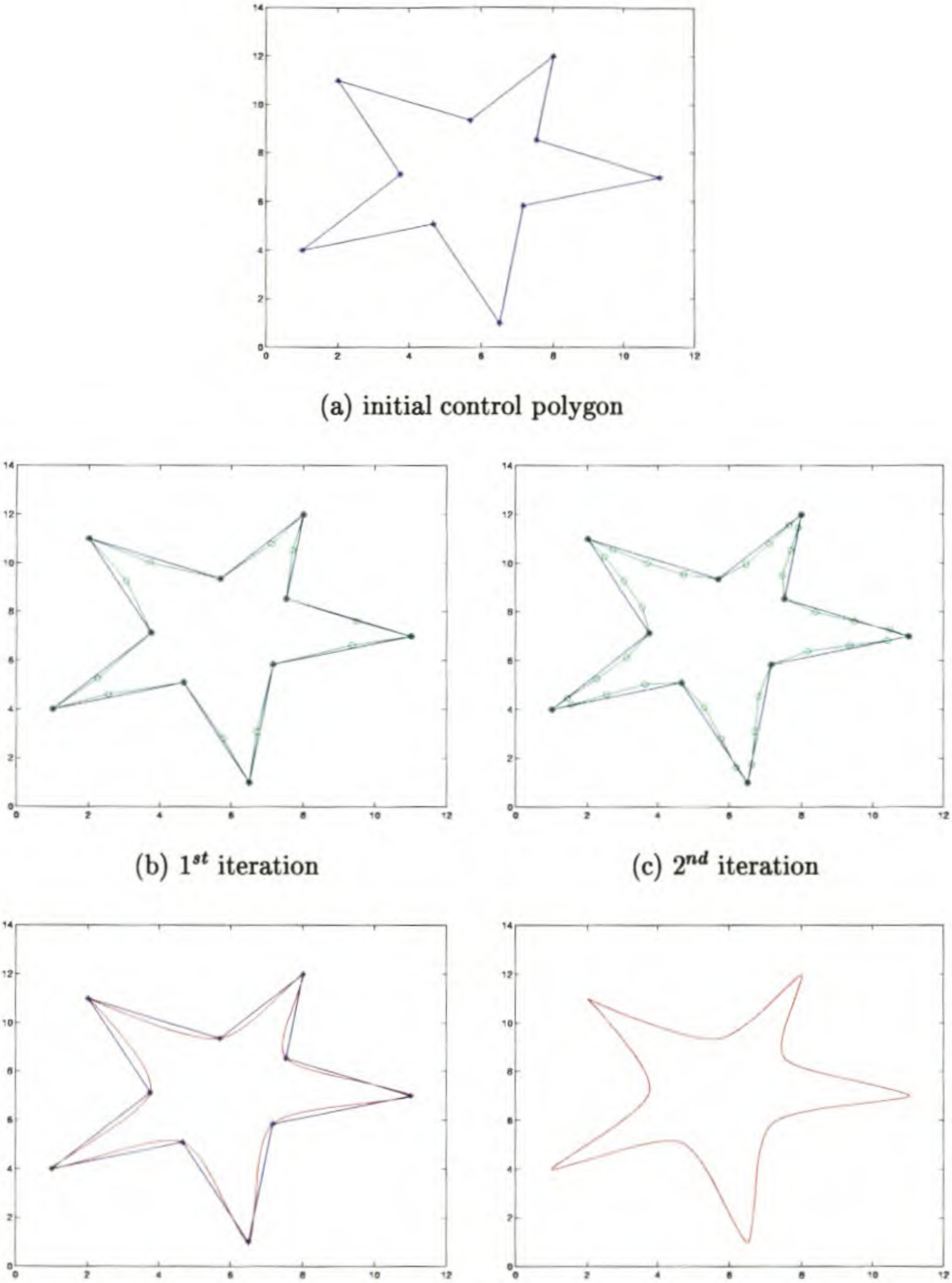
$$A(z) \;=\; \frac{1}{2}\left(\frac{1}{z} + 2 + z\right), \quad \text{if } N = 0, \tag{5.30}$$

$$A(z) = \frac{1}{16}\left(-\frac{1}{z^3} + \frac{9}{z} + 16 + 9z - z^3\right), \quad \text{if } N = 1, \tag{5.31}$$

$$A(z) = \frac{1}{256}\left(\frac{3}{z^5} - \frac{25}{z^3} + \frac{150}{z} + 256 + 150z - 25z^3 + 3z^5\right), \quad \text{if } N = 2. \tag{5.32}$$

Note that (5.30) is identical to the symbol of the mask (3.27). At each iteration this SSS keeps all the previous control points and adds the midpoints of each segment of the control polygon as the new control points , hence the limit curve is identical to the original control polygon. The Deslauriers-Dubuc stationary subdivision schemes defined by (5.31) and (5.32) yield more interesting interpolatory limit curves, as illustrated by the examples shown in Figure 5.1 and 5.2.

Note in particular the apparent smoothness of the limiting curves in Figures 5.1(c) and 5.2(c). Note also, by comparing the limit curve in Figure 5.2(c) with the limit curves in Figure 3.6, the differences between a typical "corner-cutting" SSS and a typical interpolatory "corner-smoothing" SSS can be observed.



(a) initial control polygon

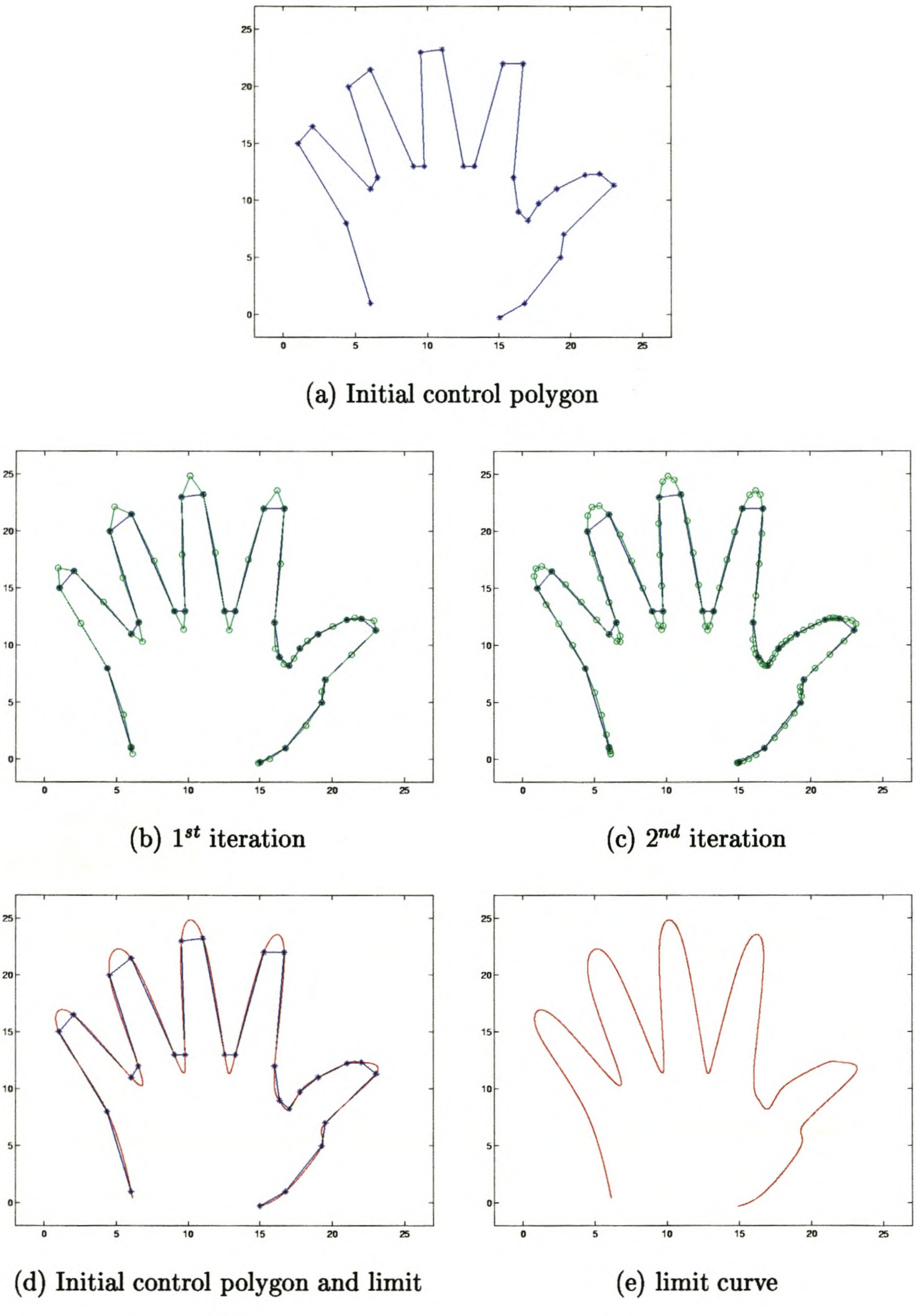
(b) 1^{st} iteration

(c) 2^{nd} iteration

(d) Initial control polygon and limit curve

(e) limit curve

Figure 5.1: *Deslauriers-Dubuc subdivision with $N = 1$: initial control polygon ($*$) and new control points ($\circ$)*



(a) Initial control polygon

(b) 1^{st} iteration

(c) 2^{nd} iteration

(d) Initial control polygon and limit curve

(e) limit curve

Figure 5.2: *Deslauriers-Dubuc subdivision with $N = 2$: initial control polygon ($*$) and new control points ($\circ$)*

Chapter 6

Spline Interpolation and Subdivision

In this chapter we shall show that a mask sequence chosen as the values at the half integers $\mathbb{Z}/2$ of a certain fundamental spline interpolant yields a symmetric convergent interpolatory SSS.

6.1 A Fundamental Cardinal Spline Interpolant

First, we show how the Bezout identity existence result of Theorem 4.3 can be used to construct a compactly supported fundamental spline interpolant, such that its associated spline approximation operator has the favourable approximation property of exactness on polynomials of optimal degree. A fundamental existence and uniqueness theorem for this particular spline interpolant was proved in [7], whereas an explicit construction technique by means of a blending formula was developed in [5]. Also, compactly supported spline interpolants of the same general kind, and which are identical to the one considered here for orders $m \leq 3$, were developed and analyzed in a sequence of 5 papers [9, 10, 11, 13, 14] and shown to have applications in numerical quadrature in [12] and [15]. We follow here the approach of [16], which is novel in the sense that the linear system to be solved for the B-spline coefficient sequence is formulated and analyzed as a Bezout identity. Our result is as follows.

Theorem 6.1 *For a given integer $m \geq 1$, there exists a spline function $Q_m \in S_m(\mathbb{Z}/2)$ such that*

$$(a) \quad Q_m(t) = 0, \quad t \notin (-m+1, m-1); \tag{6.1}$$

$$(b) \quad Q_m(k) = \delta_{k,0}, \quad k \in \mathbb{Z}; \tag{6.2}$$

$$(c) \quad \sum_k p(k) Q_m(t-k) = p(t), \quad t \in \mathbb{R}, \quad p \in \Pi_{m-1}; \tag{6.3}$$

$$(d) \quad \int_{-\infty}^{\infty} Q_m(t)dt = 1. \tag{6.4}$$

Proof. If $m = 1$, we see from (2.2) that the choice $Q_1 = N_1$ satisfies properties (a), (b), (c) and (d).

Suppose next $m \geq 2$, and recall from (2.40) that the Euler-Frobenius polynomial $\widetilde{N}_m$ is defined by

$$\widetilde{N}_m(z) = \sum_{k=0}^{m-2} N_m(k+1)z^k, \quad z \in \mathbb{C}. \tag{6.5}$$

Now define the polynomial B_m by

$$B_m(z) = \left(\frac{1+z}{2}\right)^m \widetilde{N}_m(z), \quad z \in \mathbb{C}, \tag{6.6}$$

so that B_m is a polynomial of degree $2m-2$. Also, for $z \in \mathbb{C}$, we have

$$\begin{aligned} z^{2m-2} B_m\left(\frac{1}{z}\right) &= \left(\frac{1+z}{2}\right)^m z^{m-2} \widetilde{N}_m\left(\frac{1}{z}\right) \\ &= \left(\frac{1+z}{2}\right)^m \widetilde{N}_m(z) = B_m(z), \end{aligned}$$

since, according to Theorem 2.5, $\widetilde{N}_m$ is a symmetric polynomial of degree $m-2$. Hence B_m is a symmetric polynomial of degree $2m-2$. Also, according to Theorem 2.5 the polynomial $\widetilde{N}_m$ has only negative zeros, and thus, from (6.6), B_m has only negative zeros. Moreover, (6.6) shows that $B_m(-1) = 0$. It follows that we can appeal to Theorem 4.3 with $N = m-2$ to deduce that there exists a (unique) polynomial C_m of degree $2m-4$ such that the Bezout identity

$$B_m(z)C_m(z) - B_m(-z)C_m(-z) = z^{2m-3}, \quad z \in \mathbb{C}, \tag{6.7}$$

is satisfied.

Now define the sequence $\{q_{m,j}\} \subset \mathbb{R}$ by means of the equation

$$\sum_j q_{m,j} z^j = 2z^{-2m+2} \left(\frac{1+z}{2}\right)^m C_m(z), \quad z \in \mathbb{C} \setminus \{0\}, \tag{6.8}$$

whence, since the polynomial C_m has degree $2m-4$, we have

$$q_{m,j} = 0, \quad j \notin \{-2m+2, ..., m-2\}. \tag{6.9}$$

We claim that the spline function $Q_m \in S_m(\mathbb{Z}/2)$ defined by

$$Q_m(t) = \sum_{j=-2m+2}^{m-2} q_{m,j} N_m(2t-j), \quad t \in \mathbb{R}, \tag{6.10}$$

then satisfies the desired properties (a), (b), (c), (d) of the theorem.

First, observe that (6.1) is a direct consequence of (6.10) and (2.6). Next, to prove the interpolation property (6.2), we define the Laurent polynomial D_m by

$$D_m(z) = 2\, z^{-2m+3} B_m(z) C_m(z), \quad z \in \mathbb{C} \setminus \{0\}, \tag{6.11}$$

for which, according to (6.7), we have

$$D_m(z) + D_m(-z) = 2, \quad z \in \mathbb{C} \setminus \{0\}. \tag{6.12}$$

Hence, if we write

$$D_m(z) = \sum_k d_{m,k} z^k, \quad z \in \mathbb{C} \setminus \{0\}, \tag{6.13}$$

we see from (6.12), (6.13) that

$$d_{m,2k} = \delta_{k,0}, \quad k \in \mathbb{Z}. \tag{6.14}$$

But, using (6.11), (6.6), (6.8) and (6.5) we see that, for $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned} D_m(z) &= \left[\sum_j q_{m,j} z^j\right] \left[z\, \widetilde{N}_m(z)\right] \\ &= \left[\sum_j q_{m,j} z^j\right] \left[\sum_k N_m(k) z^k\right] \\ &= \sum_k \left[\sum_j q_{m,j} N_m(k-j)\right] z^k. \end{aligned} \tag{6.15}$$

It follows from (6.10), (6.9), (6.13), (6.14) and (6.15) that, for $k \in \mathbb{Z}$,

$$Q_m(k) = \sum_j q_{m,j} N_m(2k - j) = d_{m,2k} = \delta_{k,0}, \tag{6.16}$$

so that (6.2) holds.

Next, to prove the polynomial reproduction property (6.3), we first use the definition (1.5) of the Fourier transform to deduce from (6.10) that, for $\omega \in \mathbb{R}$,

$$\begin{aligned}
\widehat{Q}_m(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} \sum_{j=-2m+2}^{m-2} q_{m,j} N_m(2t-j)dt \\
&= \frac{1}{2} \sum_{j=-2m+2}^{m-2} q_{m,j} \int_{-\infty}^{\infty} e^{-i\omega\left(\frac{t+j}{2}\right)} N_m(t)dt \\
&= \frac{1}{2} \sum_{j=-2m+2}^{m-2} q_{m,j} \left(e^{-i\omega/2}\right)^j \widehat{N}_m\left(\frac{\omega}{2}\right),
\end{aligned}$$

and thus, from (6.9),

$$\widehat{Q}_m(\omega) = \frac{1}{2}\left[\sum_j q_{m,j}\left(e^{-i\omega/2}\right)^j\right] \widehat{N}_m\left(\frac{\omega}{2}\right), \quad \omega \in \mathbb{R}. \tag{6.17}$$

Hence, from (6.8) and (6.17), we have

$$\widehat{Q}_m(\omega) = e^{i(m-1)\omega}\left(\frac{1+e^{-i\omega/2}}{2}\right)^m C_m(e^{-i\omega/2})\widehat{N}_m\left(\frac{\omega}{2}\right), \quad \omega \in \mathbb{R},$$

and thus, using also (2.23), we get

$$\widehat{Q}_m(\omega) = e^{i(m-1)\omega} C_m(e^{-i\omega/2})\widehat{N}_m(\omega), \quad \omega \in \mathbb{R}. \tag{6.18}$$

Inserting the formula (2.12) into (6.18) then gives

$$\widehat{Q}_m(\omega) = e^{i(m-1)\omega} C_m(e^{-i\omega/2})\left(\frac{1-e^{-i\omega}}{i\omega}\right)^m, \quad \omega \in \mathbb{R}, \tag{6.19}$$

with

$$\widehat{Q}_m(0) = C_m(1). \tag{6.20}$$

But, since $B_m(-1) = 0$ from (6.6), we see from (6.7) that

$$B_m(1)C_m(1) = 1. \tag{6.21}$$

Moreover, (6.6) yields

$$B_m(1) = \widetilde{N}_m(1) = \sum_k N_m(k+1) = 1, \tag{6.22}$$

from (6.5) and (2.7). It follows from (6.22) and (6.21) that

$$C_m(1) = 1, \tag{6.23}$$

which, together with (6.20), yields the value

$$\widehat{Q}_m(0) = 1. \tag{6.24}$$

Using (6.19), we see furthermore that

$$\widehat{Q}_m^{(l)}(2\pi k) = 0, \quad k \in \mathbb{Z} \setminus \{0\}, \quad l = 0, 1, ..., m-1. \tag{6.25}$$

For a fixed $t \in \mathbb{R}$, and for a given $k \in \{0, 1, ..., m-1\}$, we now define the function u by means of the formula

$$u(x) = x^k Q_m(t-x), \quad x \in \mathbb{R}. \tag{6.26}$$

Observe, from (6.1) and (6.10) that u is a compactly supported function, with $u \in S_m(\mathbb{Z}/2) \subset C(\mathbb{R})$. Hence also $u \in L^1(\mathbb{R})$, so that we may apply the Fourier transform (1.5) to u to obtain, for $\omega \in \mathbb{R}$,

$$\begin{aligned}
\widehat{u}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega x} x^k Q_m(t-x)dx \\
&= \int_{-\infty}^{\infty} e^{-i\omega(t-x)} (t-x)^k Q_m(x)dx \\
&= e^{-i\omega t} \int_{-\infty}^{\infty} e^{i\omega x} \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} t^{k-j} (-1)^j x^j Q_m(x)dx \\
&= e^{-i\omega t} \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (-1)^j t^{k-j} \int_{-\infty}^{\infty} e^{i\omega x} x^j Q_m(x)dx \\
&= e^{-i\omega t} \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (-i)^j t^{k-j} \int_{-\infty}^{\infty} e^{i\omega x} (-ix)^j Q_m(x)dx.
\end{aligned} \tag{6.27}$$

From the definition (1.5) of the Fourier transform, we derive the formula

$$\widehat{Q}^{(l)}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} (-ix)^l Q_m(x)dx, \quad \omega \in \mathbb{R}, \quad l = 0, 1, ..., \tag{6.28}$$

having noted also (6.1). But then (6.27) and (6.28) give

$$\widehat{u}(\omega) = e^{-i\omega t} \sum_{l=0}^{k} \binom{k}{l} (-i)^l t^{k-l} \widehat{Q}_m^{(l)}(-\omega), \quad \omega \in \mathbb{R}. \tag{6.29}$$

Combining (6.25) and (6.29) then shows that

$$\widehat{u}(2\pi j) = \begin{cases} e^{-i2\pi jt} \sum_{l=0}^{k} \binom{k}{l} (-i)^l t^{k-l} \widehat{Q}_m^{(l)}(0), & \text{if } j = 0, \\ 0, & \text{if } j \in \mathbb{Z} \setminus \{0\}. \end{cases} \tag{6.30}$$

It is clear from (6.29) and (6.19), since $m \geq 2$ and C_m is a polynomial, and since $u \in C(\mathbb{R})$ with u compactly supported, that the condition

$$u(t),\ \widehat{u}(t) = \mathcal{O}\left(\frac{1}{1+|t|^{\alpha}}\right), \quad t \in \mathbb{R},$$

where $\alpha = m \geq 2$ is satisfied. It therefore follows from Theorem 1.3 that the Poisson summation formula (1.25) may be applied to u to get

$$\sum_k u(k) = \sum_k \widehat{u}(2\pi k),$$

and thus, from (6.26) and (6.30),

$$\sum_l l^k Q_m(t-l) = \sum_{l=0}^{k} \binom{k}{l} (-i)^l \widehat{Q}_m^{(l)}(0) t^{k-l}, \quad t \in \mathbb{R}, \quad k = 0, 1, ..., m-1. \tag{6.31}$$

We see from (6.31) that the function v defined by

$$v(t) = \sum_j j^k Q_m(t-k), \quad t \in \mathbb{R}, \tag{6.32}$$

is in $\Pi_k \subset \Pi_{m-1}$. Moreover, from (6.2) and (6.32), we have

$$v(l) = \sum_j j^k Q_m(l-j) = l^k, \quad l \in \mathbb{Z}. \tag{6.33}$$

Hence v is a polynomial in Π_{m-1} which interpolates the polynomial p defined by $p(t) = t^k$, $t \in \mathbb{R}$, at all the integers. Thus $v = p$, whence, from (6.32),

$$\sum_j j^k Q_m(t-j) = t^k, \quad t \in \mathbb{R}, \quad k = 0, 1, ..., m-1, \tag{6.34}$$

which is equivalent to the condition (6.3).

Finally, observe that (6.4) is a consequence of (6.24) and the definition of a Fourier transform (1.5), with $\omega = 0$. ■

Our theorem above also provides an explicit construction procedure for the fundamental spline interpolant Q_m as follows:

Algorithm 6.2 (The construction of Q_m)

1. Compute the sequence $\{N_m(j) : j = 1, ..., m-1\}$ recursively by means of (2.11);
2. Define the polynomial $\widetilde{N}_m$ by means of (6.5);
3. Execute steps 3 - 6 of Algorithm 4.4, with the polynomial $B = B_m$, where
$$B_m(z) = \left(\frac{1+z}{2}\right)^m \widetilde{N}_m(z);$$
4. Define the sequence $\{q_{m,j} : j = -2m+2, ..., m-2\}$ by means of (6.8);
5. Define $Q_m \in S_m(\mathbb{Z}/2)$ by means of (6.10).

Examples

m=2 For $m = 2$, we have from (6.5) that

$$\widetilde{N}_2(z) = 1, \quad z \in \mathbb{C},$$

then using Algorithm 6.2, we get that

$$C_2(z) = 1, \quad z \in \mathbb{C}.$$

Hence, using (6.8) we get that the sequence $\{q_{2,k} : k = -2, -1, 0\}$, is given by $\{1/2, 1, 1/2\}$, and therefore the fundamental spline interpolant Q_2 (see Figure 6.1) is

$$Q_2(t) = \frac{1}{2}N_2(2t+2) + N_2(2t+1) + \frac{1}{2}N_2(2t), \quad t \in \mathbb{R}. \tag{6.35}$$

Observe from (6.35) and (2.2), (2.11) that, as is also clear by comparing Figures 6.1 and 2.1(b), we have $Q_2 = N_2(\cdot + 1)$.

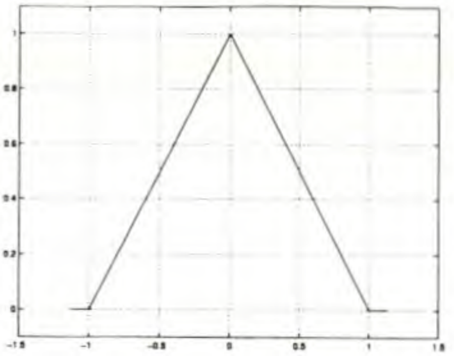


Figure 6.1: *The fundamental spline interpolant Q_2*

m=3 Setting $m = 3$ in (6.5) yields

$$\widetilde{N}_3(z) = \frac{1}{2} + \frac{1}{2}z, \quad z \in \mathbb{C},$$

so that using Algorithm 6.2, we obtain

$$C_3(z) = -\frac{1}{2}(1 - 4z + z^2), \quad z \in \mathbb{C}.$$

Then the sequence $\{q_{3,k} : k = -4, -3, -2, -1, 0, 1\}$, as defined by (6.8), is given by

$$\frac{1}{8}\{-1,\ 1,\ 8,\ 8,\ 1,\ -1\}.$$

The function Q_3 is then computed from (6.10), as shown in Figure 6.2, in which we also observe that $\operatorname{supp} Q_3 = [-2, 2]$ as implied by (6.1).

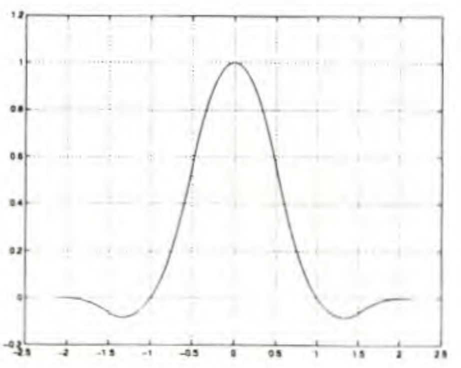


Figure 6.2: *The fundamental spline interpolant Q_3*

m=4 For $m = 4$, we have that

$$\widetilde{N}_4(z) = \frac{1}{6} + \frac{2}{3}z, + \frac{1}{6}z^2 \quad z \in \mathbb{C},$$

and hence, from Algorithm 6.2, we obtain

$$C_4(z) = \frac{1}{6}(1 - 8z + 20z^2 - 8z^3 + z^4), \quad z \in \mathbb{C},$$

Then, from (6.8), we have that the sequence $\{q_{4,k} : k = -6, ..., 2\}$ is given by

$$\frac{1}{48}\{1, -4, -6, 28, 58, 28, -6, -4, 1\},$$

and Q_4 is then computed from (6.10), and illustrated graphically in Figure 6.3. Note that, as implied by (6.1), supp $Q_4 = [-3, 3]$.

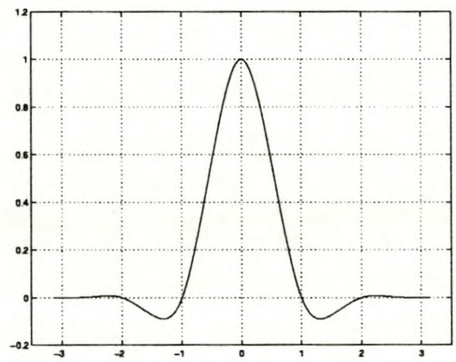


Figure 6.3: *The fundamental spline interpolant Q_4*

m=5 For $m = 5$, we get that

$$\widetilde{N}_5(z) = \frac{1}{24} + \frac{11}{24}z + \frac{11}{24}z^2 + \frac{1}{24}z^3, \quad z \in \mathbb{C},$$

and hence, from Algorithm 6.2, we get

$$C_5(z) = \frac{1}{855}(-38 + 608z - 2655z^2 + 5320z^3 - 2655z^4 + 608z^5 - 38z^6), \quad z \in \mathbb{C}.$$

Then, from (6.8), we have that the sequence $\{q_{5,k} : -8, ..., 3\}$ is given by

$$\frac{1}{1440}\{-4, 44, -15, -315, 175, 1555, 1555, 175, -315, -15, 44, -4\},$$

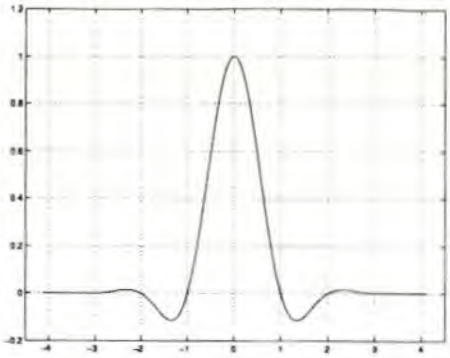


Figure 6.4: *The fundamental spline interpolant* Q_5

and Q_5 is then defined by (6.10) and drawn in Figure 6.4.

Note: The graphs in Figures 6.1 to 6.4 were generated by using specifically the Lane-Riesenfeld algorithm (3.25), (3.21), (3.18), with the choice

$$c_j = \begin{cases} q_{m,j}, & j = -2m+2, ..., m-2, \\ 0, & j \notin \{-2m+2, ...m-2\}, \end{cases}$$

and with the sequence $\{c_j^{(r)}\}$, $r = 1, 2, ...$ defined by

$$c_j^{(r)} = \left(S_m c^{(r-1)}\right)_j, \quad j \in \mathbb{Z}, \quad r = 1, 2, ..., \tag{6.36}$$

for, respectively, $m = 2, 3, 4, 5$. Then from (3.44), we know that this method converges exponentially fast to a continuous function f, such that

$$f(t) = \sum_j c_j N_m(t - j), \qquad t \in \mathbb{R},$$

which then yields the desired curve if t is replaced by $2t$.

6.2 The Corresponding SSS

A link between spline interpolation and interpolatory stationary subdivision is now obtained by first noting the following result.

Proposition 6.3 *For a given integer $m \geq 2$, the sequence $\{a_{m,k}\}$ defined by*

$$a_{m,k} = Q_m\left(\frac{k}{2}\right), \quad k \in \mathbb{Z}, \tag{6.37}$$

with Q_m denoting the fundamental spline interpolant of Theorem 6.1, is identical to the sequence $\{a_k\} = \{a_{m,k}\}$ obtained from (3.47) and Theorem 4.3, with $N = m - 2$, and with the polynomial B_m chosen as

$$B_m(z) = \left(\frac{1+z}{2}\right)^m \tilde{N}_m(z), \quad z \in \mathbb{C}, \tag{6.38}$$

where $\tilde{N}_m$ denotes the Euler-Frobenius polynomial defined by (6.5).

Proof. Combining (3.47), (6.10), (6.9), (6.5) and (6.6), we find, with $\{a_{m,k}\}$ chosen as in (6.37), that the resulting Laurent-polynomial A_m satisfies, for $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}
A_m(z) &= \sum_k Q_m\left(\frac{k}{2}\right) z^k \\
&= \sum_k \sum_j q_{m,j} N_m(k-j) z^k \\
&= \sum_j q_{m,j} \sum_k N_m(k+1) z^{k+1+j} \\
&= z\left[\sum_j q_{m,j} z^j\right]\left[\sum_k N_m(k+1) z^k\right] \\
&= z\tilde{N}_m(z) 2z^{-2m+2}\left(\frac{1+z}{2}\right)^m C_m(z) \\
&= 2z^{-2m+3} B_m(z) C_m(z),
\end{aligned}$$

which agrees precisely with the situation of Theorem 4.3 with $N = m-2$, having recalled also, from the proof of Theorem 6.1, that the polynomial B_m is, from Theorem 2.5, a symmetrical polynomial of degree $2m - 2$ with $B(-1) = 0$ and with only negative zeros, and that the polynomials B_m and C_m satisfy the Bezout identity (6.7). ■

The following result therefore holds.

Theorem 6.4 *The SSS defined by (3.46), where, for a given integer $m \geq 2$, the mask sequence $\{a_{m,k}\}$ is defined by*

$$a_{m,k} = Q_m\left(\frac{k}{2}\right), \quad k \in \mathbb{Z}, \tag{6.39}$$

with Q_m denoting the fundamental spline interpolant of Theorem 6.1, satisfies the conditions of Theorem 4.2.

The specific choice (6.39) for the mask sequence $\{a_k\} = \{a_{m,k}\}$ was recently also considered in [22, p. 263]. The resulting refinable function ϕ satisfying (4.12) was then employed in [22] for the construction of orthonormal wavelets, including the case of Daubechies wavelets, as was first constructed in [6]. However, whereas in [22] the existence of the refinable function ϕ was assumed (see e.g. [22, Theorem 3.4]), our Theorems 6.4, 4.2 and 4.1 ensures the existence of a refinable function ϕ with the properties (a) to (i) of Theorem 4.2.

We have the following algorithm for computing the masks as given by (6.39).

Algorithm 6.5

1. Execute steps 1 to 3 of Algorithm 6.2;

2. Define the Laurent polynomial A_m by

$$A_m(z) = 2z^{-2m+3}B_m(z)C_m(z), \quad z \in \mathbb{C} \setminus \{0\}, \quad m = 2, 3, \ldots; \tag{6.40}$$

3. Define the sequence $\{a_k\}$ by

$$\sum_k a_k z^k = A_m(z);$$

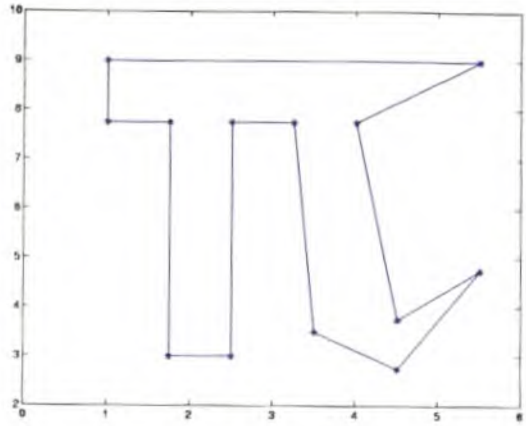
4. Define the SSS by

$$(S_a c)_k = \sum_l a_{k-2l} c_l, \quad \{c_l\} \in \ell^\infty.$$

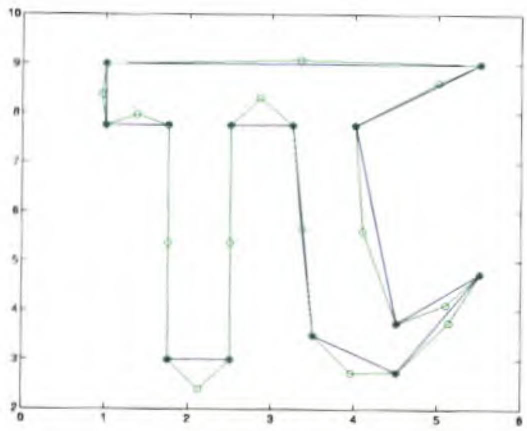
Using Algorithm 6.5, we obtained the following symbols of the SSS given in Theorem 6.4, for $m = 2, 3, 4$ and for $z \in \mathbb{C} \setminus \{0\}$:

$$
\begin{aligned}
A_2(z) &= \frac{1}{2}(z + 2 + z)\,, && (6.41)\\
A_3(z) &= \frac{1}{16}\left(-z^{-3} + 9z + 16 + 9z - z^3\right) && (6.42)\\
A_4(z) &= \frac{1}{288}(z^{-5} - 21z^{-3} + 164z^{-1} + 288 + 164z - 21z^3 + z^5), && (6.43)\\
A_5(z) &= \frac{1}{34560}(-4z^{-7} + 425z^{-5} - 3411z^{-3} + 20270z^{-1} + 34560 \\
&\qquad +20270z - 3411z^3 + 425z^5 - 4z^7). && (6.44)
\end{aligned}
$$

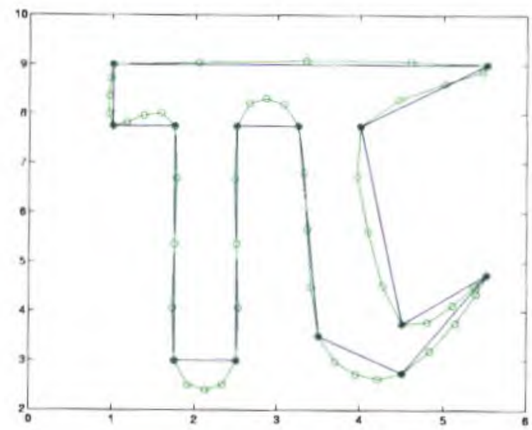
Note, from (6.41), that A_2 is identical to the symbol given by (5.30). We previously noted that this mask yields an uninteresting limit curve and so will not be illustrated graphically. Note, also, from (6.42), that A_3 is identical to the symbol given by (5.31), since both (6.42) and (5.31) derive from the same Bezout identity, hence the SSS based on (6.43) is equivalent to Deslauriers-Dubuc subdivision with $N = 1$, an application of which is shown in Figure 6.5 (cf. also Figure 5.1). However, (6.43) and (6.44) have no corresponding Deslauriers-Dubuc symbols. In Figures 6.6 and 6.7, we show, graphically, how we have applied the SSS based on the symbols (6.43) and (6.44), respectively, to different initial control polygons. Observe that the SSS based on (6.42), (6.43) and (6.44) are all interpolatory and "corner-smoothing" and seem to have smooth limit curves.



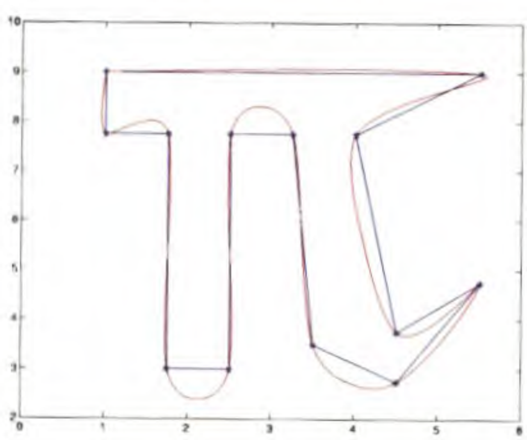
(a) Initial control polygon



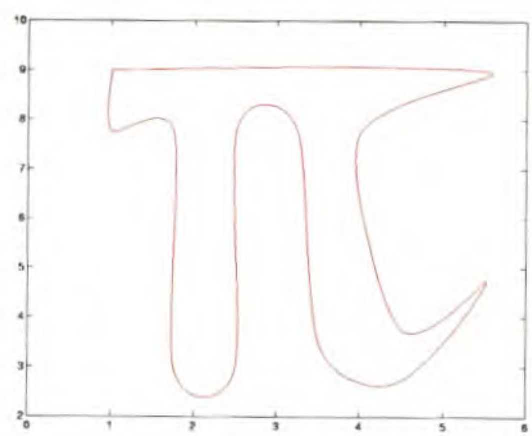
(b) 1^{st} iteration



(c) 2^{nd} iteration

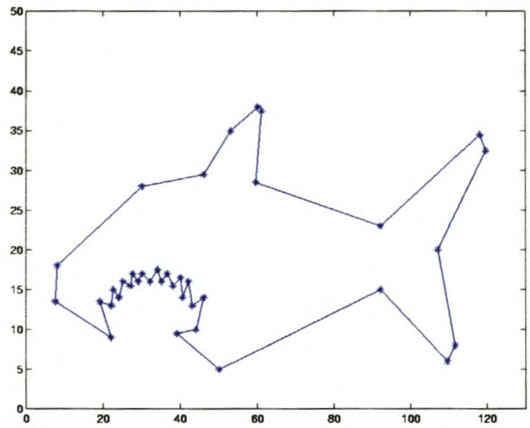


(d) Initial control polygon and limit curve

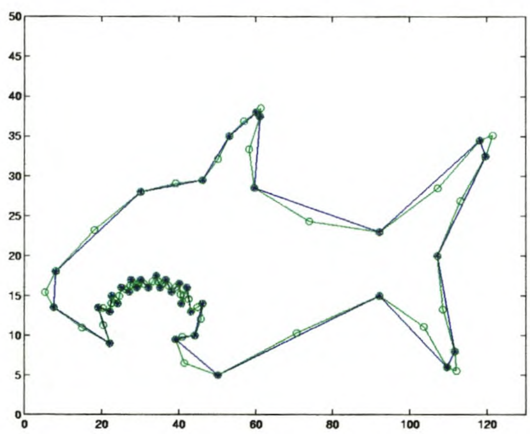


(e) Limit curve

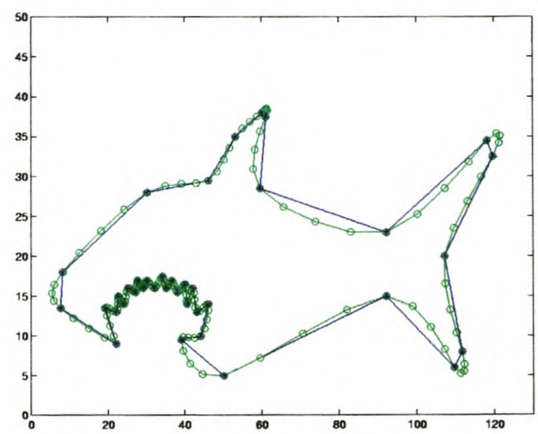
Figure 6.5: *The SSS with symbol (6.42): initial ($*$) and new control points ($\circ$).*



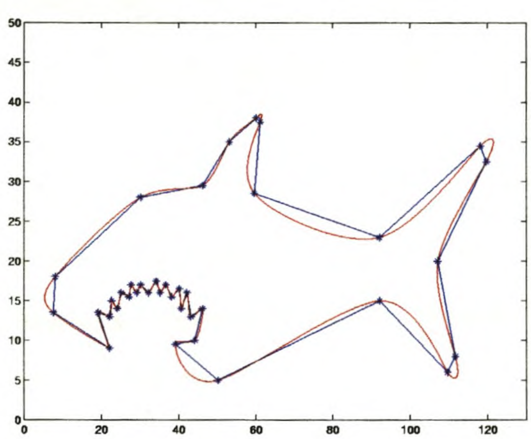
(a) Initial control polygon



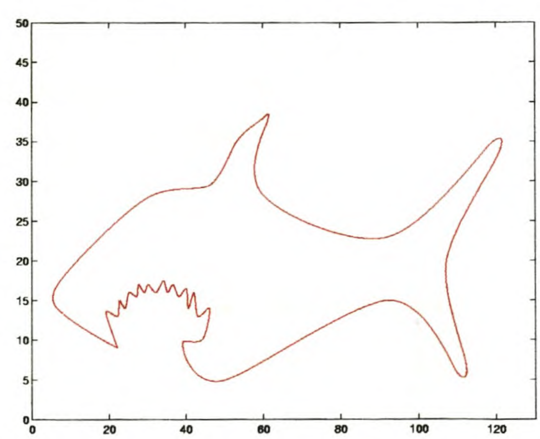
(b) 1^{st} iteration



(c) 2^{nd} iteration

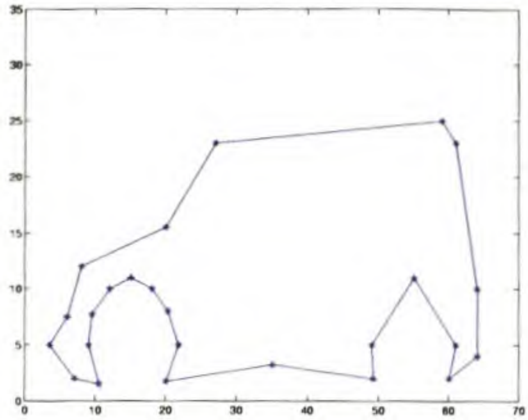


(d) Initial control polygon and limit curve

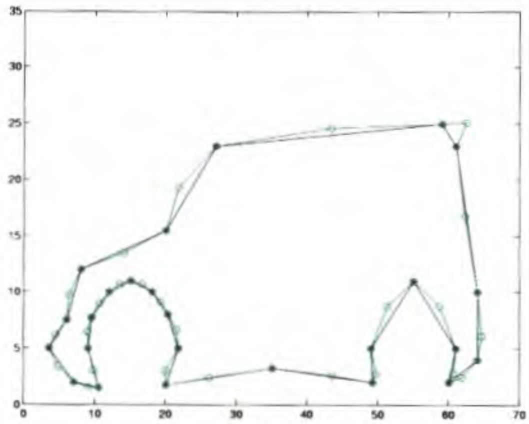


(e) Limit curve

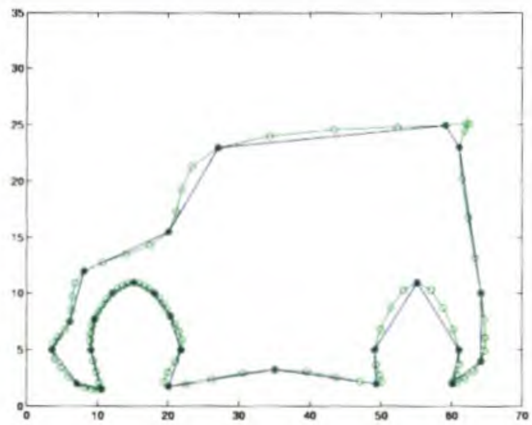
Figure 6.6: *The SSS with symbol (6.43): initial ($*$) and new control points ($\circ$).*



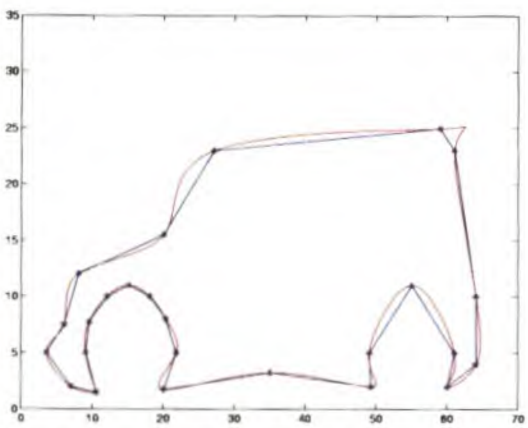
(a) Initial control polygon



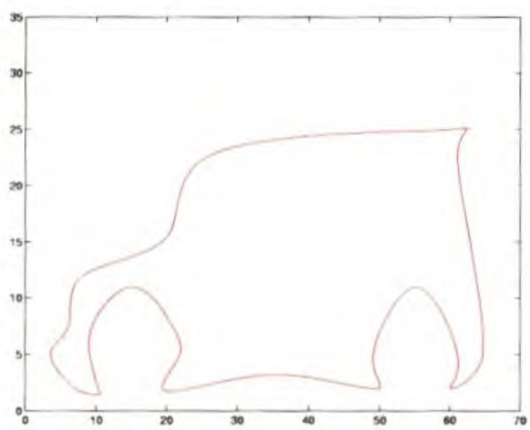
(b) 1^{st} iteration



(c) 2^{nd} iteration



(d) Initial control polygon and limit curve



(e) Limit curve

Figure 6.7: *The SSS with symbol (6.44): initial ($*$) and new control points ($\circ$).*

Chapter 7

Conclusions

The main thrust of this thesis is to provide a general framework for the convergence of an interpolatory SSS, in particular including also the cases where the mask coefficients are chosen as certain point evaluations of a fundamental Lagrange interpolation polynomial or of a fundamental cardinal spline interpolant. The latter type of construction procedure has been receiving attention in recent work, and our combination of the results of Theorems 4.2 and 4.3 is an attempt to unify and clarify the corresponding convergence theory.

It should be pointed out that all of our results on subdivision are based on an initial control point sequence $\{c_j^{(0)} : j \in \mathbb{Z}\}$, i.e. we assume the availability of infinitely many initial data points. In real-world applications where only a finite number of given data is available, it is therefore necessary to either infinitely extend the data in some appropriate manner in both directions, or to modify the subdivision algorithm so as to be applicable to a finite amount of data. The observed phenomenon in Figure 5.1(e) of the star-shaped limit curve with one non-smooth corner is due to the fact that such modifications for endpoint behaviour were not done in this thesis. The non-smooth corner in Figure 5.1(e) in fact corresponds to both the first and last indexed data points. It is intended to pursue further research in this area with the view to removing such deficiencies near endpoints.

Further important issues which are not addressed here, and which we plan to explore in future work, include, in the context of Theorem 4.2, the polynomial reproduction property (analogous to (6.3)), as well as the regularity, or continuity class, of the refinable function

ϕ. In fact, the availability of a refinable Riesz-stable function ϕ means that, using standard methods like the ones employed in [4] and [6], the multiresolutional analysis generated by ϕ can be used to construct wavelets of different kinds. Work in this regard has recently been done in [22] where, using the refinable function ϕ corresponding to our Chapter 6, a class of wavelets including as a special case the orthonormal Daubechies wavelet was obtained. Our own preliminary investigations along similar lines are promising.

Finally, it should again be emphasised that our work here is restricted to the univariate case in which a SSS is used to generate a smooth plane curve. An extremely important practical application area is the use of a multidimensional SSS for the smoothing of the corners of a given polyhedral surface. In fact, these methods are sometimes referred to as "wood carver" algorithms because the repeated smoothing operations are analogous to sculpting wood. The extension of the results of this thesis to the multidimensional setting seems a further sensible course of action towards relevant and exciting new research.

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