Bounds for Ramsey numbers in multipartite graphs

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– DECLARATION –

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

Eugene Stipp ______________________

Date ______________________

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– ABSTRACT –

The notion of a classical graph theoretic Ramsey number is generalized by assuming that both the original graph whose edges are arbitrarily bicoloured and the monochromatic subgraphs to be forced are complete, balanced, multipartite graphs, instead of complete graphs as in the standard definition. Some small multipartite Ramsey numbers are found, while upper- and lower bounds are established for others. Analytic arguments as well as computer searches are used.

Keywords: Ramsey number, multipartite graph, circulant graph

– OPSOMMING –

Die klassieke grafiek-teoretiese definisie van 'n Ramsey getal word veralgemeen deur te aanvaar dat beide die oorspronklike grafiek, waarvan die lyne willekeurig met twee kleure gekleur word en die gesogte subgrafieke almal volledige, gebalanseerde, veelledige grafieke is, anders as in die standaard definisie. Klein veelledige Ramsey getalle word gevind, terwyl bo- en ondergrense vir ander daargestel word. Analitiese argumente en rekenaarsoektogte word gebruik.

Kernwoorde: Ramsey getal, veelledige grafiek, sirkulant grafiek

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Chapter 1

Introduction

1.1 The origins of Ramsey numbers

> "The origins of Ramsey theory are diffuse. Frank Ramsey was interested in decision procedures for logical systems. Issai Schur wanted to solve Fermat's last theorem over finite fields. B.L. van der Waerden solved an amusing problem – and immediately returned to his researches in algebraic geometry. The emergence of Ramsey theory as a cohesive subdiscipline of combinatorial analysis occurred only in the last decade." [58]

Issai Schur [121] proved the first theorem of what was later to be called Ramsey theory in 1916. He proved that: For every $r \in \mathcal{N}$ there exists an $n \in \mathcal{N}$ such that, given an arbitrary r-colouring of $S = \{1, 2, \ldots, n\}$; there exist $x, y, z \in S$ all the same colour, satisfying $x + y = z$. His motivation for establishing this result was the study of Fermat's Last Theorem over finite fields. In the 1920's he made the following conjecture: If the positive integers are divided into two classes, at least one of the classes contains an arithmetic progression of k terms, no matter how large the given length k is. Over lunch one day in 1926, B. L. van der Waerden told Emil Artin and Otto Schreier about this problem. Immediately after lunch they went into Artin's office in the Mathematics Department of the University of Hamburg and tried to find a proof. They solved the question of Schur's conjecture and it was later formally proved by Van der Waerden.

Ramsey proved his famous theorem in 1930 in the first 8 pages of a 20 page paper *On a problem of formal logic* [114]. Ramsey's theorem may be stated as follows: Let k, r, n be positive integers. If N is sufficiently large and if the k-sets of an N-set are coloured arbitrarily with r colours then there exists an n-set, all of whose k element subsets are the same colour. Ramsey needed this result for his researches in Mathematical Logic and he used this theorem to establish a result in a decision procedure for a certain class of statements in First Order Logic. It is ironic that it was discovered later that Ramsey's theorem was not needed for constructing the required decision procedure. This happened during the Hilbert-program, which attempted to find a general decision procedure for statements in First Order Logic. What is even more ironic is that Kurt Gödel's [63] undecidibility results (which were published the year after Ramsey died) showed that such a decision procedure could not exist. Thus Ramsey theory is named after Frank

Plumpton Ramsey because he proved a theorem he did not need, in the course of trying to do something we now know cannot be done!

The proof of Van der Waerden's theorem made a great impression on a young mathematician named Richard Rado. He may be considered the first true Ramsey theoretician, since in his PhD dissertation (under the supervision of Issai Schur) and in his subsequent work he was interested in Ramsey theory problems *per se.*

Ramsey's theorem was rediscovered in the classic 1935 paper [37] of Paul Erdös and George Szekeres. Erdös and Szekeres were young students in Budapest at the time and one of their friends in Budapest, Esther Klein, discovered that: given any 5 points in a plane, some four points form a convex quadrilateral. They soon made a general conjecture: for any δ there exists an ϵ so that given ϵ points in the plane, some δ form a convex set. Szekeres wrote in the foreword of [33]:

> "I have no clear recollection how the generalization actually came about; in the paper we attributed it to Esther, but she assures me that Paul had much more to do with it. We soon realized that a simple minded argument would not do and there was a feeling of excitement that a new type of geometric problem emerged from our circle which we were only too eager to solve. For me, [the] fact that it came from Epszi (Paul's nickname for Esther, short for epsilon) added a strong incentive to be the first with a solution and after a few weeks I was able to confront Paul with a triumphant 'E.P., open your wise mind'. What I had really found was Ramsey's Theorem, from which [the above result] easily followed. Of course, at that time none of us knew about Ramsey."

It is believed that what we now know as Ramsey theory went into a long embryonic stage from 1930 to 1973 and that it was really born at the Combinatorial Conference at Balatonfüred, Hungary during 1973. The conference proceedings [82] reveal that there were more than 24 talks devoted to what is now called Ramsey theory. Among the speakers were Richard Rado, Walter Deuber, Klaus Leeb, Ron Graham, and Paul Erdös in whose honour the conference was held. Ramsey theory found its place as a cohesive sub-discipline of *combinatorial analysis* at the Balatonfüred conference and is concerned with conditions that guarantee that a combinatorial object necessarily contains some smaller given objects. The least number of sub-objects that guarantees the existence of some smaller objects is called a *Ramsey number*. Therefore the role of Ramsey numbers is to quantify some of the general existential theorems in Ramsey theory.

The first Ramsey number was published as a result of the 1953 Putnam competition. Leo Mozer phoned Frank Harary from Edmonton asking for a graphical problem which would complete the Putnam competition which he was composing. He suggested the following problem of which the solution and commentary is given by Gleason, Greenwood and Kelly [61] in their comprehensive review and commentary on these collected problems and solutions:

> "**Problem.** Six points are in a general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are

> drawn and then painted, some segments red, some blue. Prove that some triangle has all its sides the same colour.
>
> **Solution.** Let P be any of the six points. Five of the line segments end at P, and of these at least three, say PQ, PR and PS, must have the same colour, say blue. Then, if any one of the segments QR, RS and SQ is blue we will have a blue triangle, and if not, QRS will be a red triangle. Thus in any event at least one triangle has all its sides the same colour."

The above mentioned problem is a part of the famous "party problem": What is the fewest number of people at a birthday party that will guarantee three mutual acquaintances or three mutual strangers? The answer is 6 people. Greenwood and Gleason first published this result (which is considered the first publication of a non-trivial Ramsey number) in the Canadian Journal of Mathematics in 1955 [60]. The subject has grown tremendously, in particular with regard to asymptotic bounds for various types of Ramsey numbers. The progress on evaluating the basic numbers themselves has been very unsatisfactory for a long time. Since 1990, however, considerable progress has been made in this area, mostly by using computer algorithms.

1.2 Basic graph theoretic definitions

Since the combinatorial object of study the sub-discipline of within Ramsey theory in this thesis is a graph, the object itself and terminology surrounding it is defined in this section. A **graph** G is a finite non-empty set $V(G)$ of objects called **vertices** and a (possibly empty) set $E(G)$ of 2-element subsets of $V(G)$ called **edges**. The set $V(G)$ is called the **vertex set** of G and $E(G)$ its **edge set**. See Figure 1.1(a) for a representation of a graph. The number of vertices in a graph G is called its **order**, and the number of edges is its **size**. That is, the order of G is $|V(G)|$, and its size is $|E(G)|$.

For a vertex v of G, its **neighbourhood set**, $N(v)$, is defined by $N(v) = \{u \in V(G) \mid vu \in E(G)\}$, while its **degree** $\deg(v)$ is the number of vertices adjacent to v, that is, $\deg(v) = |N(v)|$. If $e = uv$ is an edge of a graph G, then we say that e and u (and e and v) are **incident** in G. If e and f are distinct edges that are incident with a common vertex, then e and f are **adjacent edges**. The **complement** $\overline{G}$ of a graph G is that graph with $V(\overline{G}) = V(G)$, and such that uv is an edge of $\overline{G}$ if and only if uv is not an edge of G.

Two graphs G_1 and G_2 are **isomorphic** if there is a one-to-one function, say $\phi{:}V(G_1) \rightarrow V(G_2)$, such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$, in which case we write $G_1 \simeq G_2$. A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let S be a nonempty set of vertices of a graph G. Then the **subgraph induced by** S is the maximal subgraph of G with vertex set S, and is denoted $\langle S\rangle$, that is, $\langle S\rangle$ contains precisely those edges of G incident with two vertices in S. A subgraph H of a graph G is a **vertex-induced subgraph** of G, if $H = \langle S\rangle$ for some nonempty set of vertices, S, of G.

A **walk** in a graph G is an alternating sequence $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$ $(n \geq 0)$ of vertices and edges, beginning and ending with vertices, such that $e_i = v_{i-1}v_i \in E(G)$ for

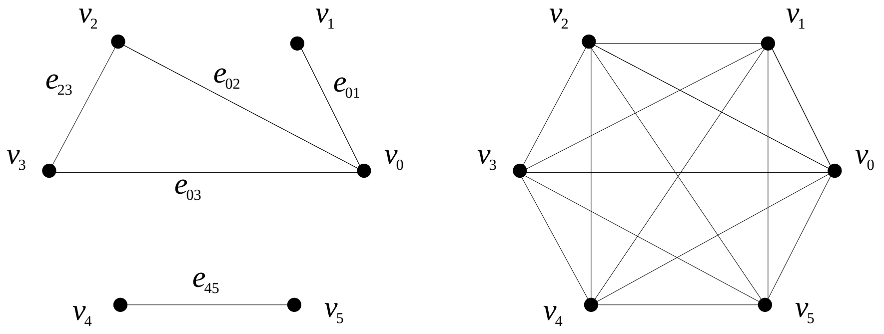
$i = 1, 2, \ldots n$. If $v_0 = v_n$ then the walk is called a **closed walk**. The walk is said to have **length** n if n (not necessarily distinct) edges are encountered. A **trail** is a walk in which no edge is repeated, while a **path** is a walk in which no vertex is repeated and the **cycle** C_n is a walk $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$ in which $n \geq 3$, $v_0 = v_n$, and the n vertices $v_1, v_2, \ldots, v_n$ are distinct. Let u and v be vertices in a graph G. We say that u is **joined to** v if G contains a $u - v$ path. The graph G itself is **connected** if u is connected to v for every pair, (u, v), of vertices of G.

A graph of order p in which every two distinct vertices are adjacent is a **complete graph** and is denoted K_p. See Figure 1.1(b) for a graphical representation of K_6. A graph G is n-partite if $V(G)$ can be partitioned into n nonempty subsets $V_1, V_2, \ldots, V_n$ such that no edge of G joins two vertices in the same set. In this case the sets $V_1, V_2, \ldots, V_n$ are called the **partite sets** of G. See Figure 1.1(c) for a graphical representation of a 2-partite graph, also called a bipartite graph. If G is an n-partite graph having partite sets $V_1, V_2, \ldots, V_n$ such that every vertex of V_i is connected to every vertex of V_j, where $1 \leq i < j \leq n$, then G is called a **complete n-partite graph**. If $|V_i| = p_i$, for $i = 1, 2, \ldots, n$, then we denote G by $K_{p_1, p_2, \ldots, p_n}$. Such a graph is also called a **complete multipartite graph**. If $p_1 = p_2 = \cdots = p_n$ then G is called a **balanced n-partite graph**. We use the notation $K_{n \times l}$ to denote a complete balanced n-partite graph with n partite sets and l vertices per partite set. See Figures 1.1 (d) and (e) for graphical representations of $K_{2\times3}$ and $K_{3\times2}$ respectively.

There are several ways in which graphs can be produced from other graphs. The **union** of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph having $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. If $G_1 \simeq G_2 \simeq G$, then we write $2G$ for $G_1 \cup G_2$. In general, if $G_1, \ldots, G_n$ are pairwise vertex-disjoint graphs that are isomorphic to G, then we write nG for $G_1 \cup \cdots \cup G_n$. Again, if G_1 and G_2 are vertex-disjoint graphs, then the **join** of G_1 and G_2, written $G_1 + G_2$, is that graph consisting of the union, $G_1 \cup G_2$, together with all edges of the type $v_1 v_2$, where $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$.

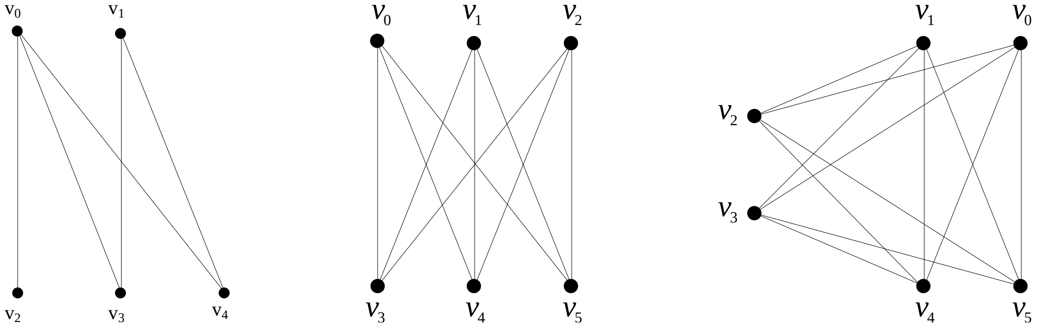
The following graphs will be referred to in the survey of literature only. A complete graph with one edge removed is denoted $K_p - e$. A **star** graph, S_i, is a bipartite graph of order i with one partite set consisting of a single vertex, i.e. $S_i \simeq K_{1,i-1}$ as seen in Figure 1.2(a). A **wheel** graph, W_i, is the result of a single vertex being connected to every vertex of a cycle C_{i-1}, of length $i - 1$ as seen in Figure 1.2(b). The **book** graph, B_i, has $i + 2$ vertices and is the result of a single vertex being connected to every vertex of a star $S_{i+1} \simeq K_{1,i}$ as seen in Figure 1.2(c). The graph mK_2 is referred to as a **stripe** graph, as seen in Figure 1.2(d). A connected graph on n vertices that does not contain a cycle is called **tree** and denoted T_n, as seen in Figure 1.2(e). Not all trees of order n are isomorphic. Note that $B_1 \simeq C_3 \simeq, B_2 \simeq K_4 - e$, $P_3 \simeq K_3 - e$, $W_4 \simeq K_4$, $C_4 \simeq K_{2,2}$ and $K_{l,l} \simeq K_{2\times l}$.

The class of circulant graphs is central to this thesis. If T vertices are ordered on the edge of an imaginary circle, and every i-th vertex is joined by an edge, then a **circulant** graph on the T vertices is obtained, which is denoted $C_T\langle i\rangle$.



(a) Graph G with $V(G) = \{v_o, v_1, \dots, v_5\}$ & $E(G) = \{e_{01}, e_{02}, e_{03}, e_{23}, e_{45}\}$

(b) K_6



(c) bipartite graph

(d) $K_{2\times3}$

(e) $K_{3\times2}$

Figure 1.1: Basic graphs

A circulant is formally defined as follows:

Definition 1.1 *Let* T, z *be natural numbers with* $z < T$ *and let* $1 \leq i_1, \dots, i_z \leq T$ *be* z *distinct integers. The* **circulant** $C_T\langle i_1, \dots, i_z\rangle$ *is the graph with vertex set*

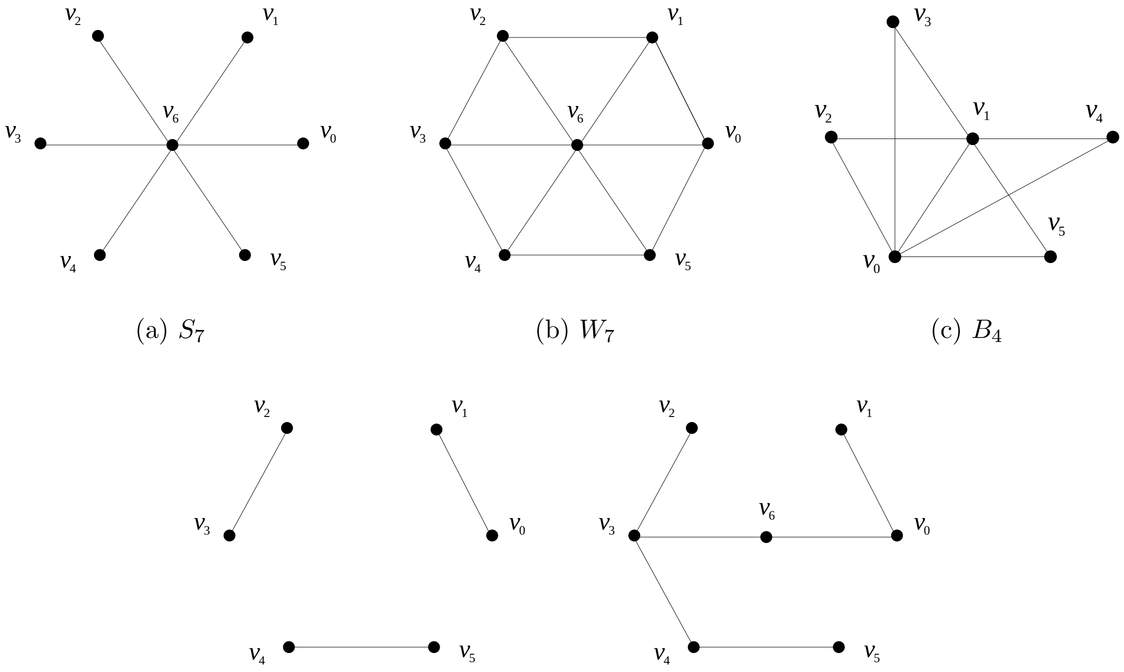
$$V(C_T\langle i_1, \dots, i_z\rangle) = \{v_0, \dots, v_{T-1}\}$$

and edge set

$$E(C_T\langle i_1, \dots, i_z\rangle) = \{v_\alpha v_{\alpha+\beta(\text{mod } T)} \mid \alpha = 0, \dots, T-1 \text{ and } \beta = i_1, \dots, i_z\}.$$

If $z = 1$*, the circulant* $C_T\langle i\rangle$ *is called an* **elementary circulant***, otherwise it is called a* **composite circulant***. If* $i = T/2$ *then* $C_T\langle i\rangle$ *is called the* **singular circulant***, otherwise it is called* **non-singular***.*

Two elementary circulants $C_{25}\langle 12\rangle$ and $C_{25}\langle 6\rangle$ are shown in Figures 1.3(a) and (b), while two composite circulants $C_8\langle 3, 4\rangle$ and $C_8\langle 1, 4\rangle$ are shown in Figure 1.3(c) and (d).



(a) S_7 (b) W_7 (c) B_4

(d) $3K_2$ (e) T_7

Figure 1.2: Special graphs used in survey of literature

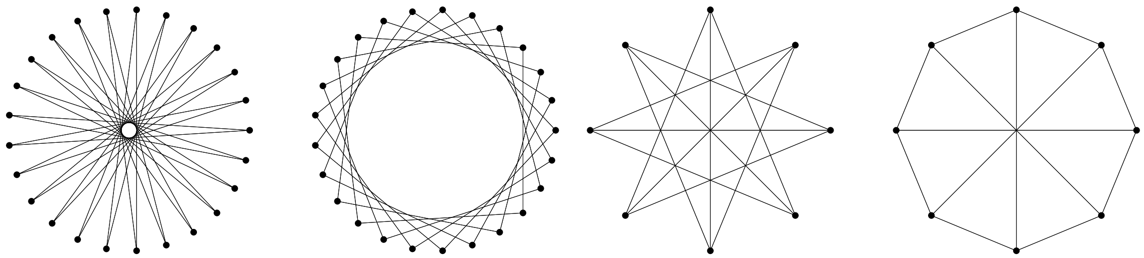
1.3 Concise survey of literature on Ramsey numbers

The survey of literature of graph theoretic Ramsey numbers will be presented in chronological order, by starting with classical 2-colour Ramsey numbers and then generalizing, progressively, to multipartite Ramsey numbers, which is the topic of this thesis.

1.3.1 Classical 2-colour Ramsey numbers

Definition 1.2 *The* **2-colour classical Ramsey number** $r(m, n)$ *is defined as the smallest natural number* p *such that if the edges of* K_p *are arbitrarily coloured using the colours red and blue, then either a red* K_m *or a blue* K_n *will be forced as a subgraph of* K_p.

Table 1.1 contains some known 2-colour Ramsey numbers, as well as upper and lower bounds for others. Known exact values appear as centered entries, upper bounds as lowered entries, and lower bounds as raised entries. Table 1.2 contains the corresponding references for the numbers listed in Table 1.2. Table 1.1 is symmetrical, since it is a well-known result that $r(m, n) = r(n, m)$. For values of m and n that are greater than 10, see [109].



(a) $C_{25}\langle 12\rangle$ (b) $C_{25}\langle 6\rangle$ (c) $C_8\langle 3,4\rangle$ (d) $C_8\langle 1,4\rangle$

Figure 1.3: Circulant graphs

The first 2-colour generalization from the above definition is as follows:

Definition 1.3 *The* **2-colour Ramsey number** $r(G_1, G_2)$*, is the least positive integer* p *such that when the edges of* K_p *are coloured arbitrarily red or blue, there necessarily exists either a red* G_1 *or a blue* G_2 *as a subgraph of* K_p*. A red* G_1 *is a* G_1 *all of whose edges are coloured red.*

In the following sections different types of graphs for G_1 and G_2 in Definition 1.3 will be considered.

1.3.2 Two colours, dropping one edge from the complete graph

Consider the case where one or both of G_1 and G_2 in Definition 1.3 are complete graphs with one edge missing, denoted by $K_p - e$. As in the previous section, Table 1.3 summarizes the known results for $r(K_m - e, K_n - e)$, while Table 1.4 contains the relevant references for the values presented in Table 1.3.

1.3.3 Other 2-colour generalizations

Many special graphs for G_1 and G_2 in Definition 1.3 have been investigated. G_i can be a path P_m, a cycle C_m, a star S_m, a wheel W_m, a book B_m, a bipartite graph $K_{m,n}$, a multipartite graph $K_{n\times l}$, complete graph K_m, or a combination of these for $i = 1, 2$. Note that the graph that is being coloured using two colours is still a complete graph K_p.

Gerènscer and Gyàrfàs [56] showed that $r(P_n, P_m) = n + \lfloor m/2 \rfloor - 1$ for all $n \geq m \geq 2$. Chvàtal and Harary [24] showed that $r(C_4, C_4) = 6$. Rosta [115] and Faudree and Schelp [54] independently showed that:

n m	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	40 43
4		18	25	35 41	49 61	55 84	69 115	80 149
5			43 49	58 87	80 143	95 216	116 316	141 442
6				102 165	109 298	122 495	153 780	167 1171
7					205 540	1031	1713	2826
8						282 1870	3583	6090
9							565 6588	12677
10								798 23581

Table 1.1: Known non-trivial values and bounds for classical two colour Ramsey numbers $r(m, n)$.

$$r(C_n, C_m) = \begin{cases} 2n-1 & \text{for } 3 \leq m \leq n,\ m \text{ odd},\ (n,m) \neq (3,3) \\ n-1+m/2 & \text{for } 4 \leq m \leq n,\ m \text{ and } n \text{ even},\ (n,m) \neq (4,4) \\ \max\{n-1+m/2, 2m-1\} & \text{for } 4 \leq m < n,\ m \text{ even and } n \text{ odd} \end{cases}$$

The Ramsey number $r(W_3, W_5) = 11$ was obtained by Clancy [27]. Bürr and Erdös [12] showed that $r(W_3, W_n) = 2n - 1$ for all $n \geq 6$. Hendry [74] proved that $r(W_4, W_5) = 17$ and that $r(W_5, W_5) = 15$ [73]. Harborth and Mengersen [67] proved the last result as well. Faudree and McKay [49] showed that $r(W_4, W_6) = 19$ and that $r(W_5, W_6) = r(W_6, W_6) = 17$.

Rousseau and Sheehan [116] proved all the following Ramsey numbers for books: $r(B_1, B_n) = 2n + 3$ for all $n > 1, r(B_3, B_3) = 14, r(B_2, B_5) = 16, r(B_3, B_5) = 17, r(B_5, B_5) = 21, r(B_4, B_4) = 18, r(B_4, B_6) = 22, r(B_6, B_6) = 26$. They also showed that $r(B_n, B_n) = 4n+2$ if $4n+1$ is prime. See [109] for results on Ramsey numbers involving combinations of paths, cycles and wheels.

1.3.4 Multi-colour Ramsey numbers

The second natural generalization of a Ramsey number involves using more than two colours for colouring the edges of the complete graph K_p.

	3	4	5	6	7	8	9	10
3	[60]	[60]	[60]	[93]	[93] [59]	[64] [102]	[93] [64]	[39] [112]
4		[60]	[92] [100]	[42] [99]	[40] [96]	[45] [96]	[111] [96]	[107] [96]
5			[41] [99]	[42] [123, 83]	[15] [123]	[107] [123]	[45] [96]	[45] [96]
6				[92] [96]	[45] [96]	[45] [96]	[45] [96]	[45] [96]
7					[97, 119] [96]	[96]	[83]	[96]
8						[9] [96]	$\star$	[83]
9							[97, 119] [120]	$\star$
10								[97, 119] [120]

Table 1.2: References for Table 1.1. $\star$ Easy to obtain by simple combinatorical arguments from other results.

Definition 1.4 *The* **multi-colour Ramsey number** $r(n_1, n_2, \ldots, n_t)$ *is the smallest natural number, p, such that if the edges of K_p are arbitrarily coloured using t different colours, then a monochromatic K_{n_i} in at least one of the colours $1 \leq i \leq t$ will be forced as a subgraph of K_p.*

The only known multi-colour Ramsey number is $r(3,3,3) = 17$, proved by Greenwood and Gleason [60]. Many bounds for multi-colour Ramsey numbers have been established; for example: $52 \leq r(3,3,3,3) \leq 64$ [21, 117], $162 \leq r(3,3,3,3,3) \leq 317$ [44, 125], $500 \leq r(3,3,3,3,3,3)$ [44], $128 \leq r(4,4,4) \leq 236$ [81], $458 \leq r(4,4,4,4)$ [124], $30 \leq r(3,3,4) \leq 31$ [93, 108], $45 \leq r(3,3,5) \leq 57$ [43, 94], $90 \leq r(3,3,9)$ [70], $108 \leq r(3,3,11)$ [71], $55 \leq r(3,4,4) \leq 79$ [94], $80 \leq r(3,4,5) \leq 161$ [45] and $87 \leq r(3,3,3,4) \leq 155$ [45].

A few more general multi-colour Ramsey numbers have been found as well. Bialostocki and Schönheim [1] showed that $r(C_4, C_4, C_4) = 11$ and Yuansheng and Rowlandson [129, 130] showed that $r(C_5, C_5, C_5) = 17$ and $r(C_6, C_6, C_6) = 12$. See Stanislaw Radziszowski's survey [109] for more results on Ramsey numbers involving more than two colours.

1.3.5 Other Ramsey numbers

The graph being coloured might even be a hypergraph, meaning that an edge joins more than two vertices. The only known hypergraph Ramsey number, where the graph that is being coloured and the monochromatic graphs that are forced are all complete graphs is the number $r(4,4;3)$ (denoting 3 vertices per edge) and was computed by McKay and Radziszowski [98]. See [109] for bounds on hypergraph Ramsey numbers.

G_1 G_2	K_3-e	K_4-e	K_5-e	K_6-e	K_7-e	K_8-e	K_9-e	$K_{10}-e$
K_3-e	3	5	7	9	11	13	15	17
K_3	5	7	11	17	21	25	31	37 38
K_4-e	5	10	13	17	28			
K_4	7	11	19	27 36	35 52			
K_5-e	7	13	22	30 39	66			
K_5	9	16	30 34	43 67	112			
K_6-e	9	17	30 39	70	135			
K_6	11	21	35 55	122	212			
K_7-e	11	28	66	135	251			
K_7	13	34	89	207				

Table 1.3: Ramsey numbers $r(G_1, G_2)$ with $G_i = K_{p_i} - e$.

G_1 G_2	$K_3 - e$	$K_4 - e$	$K_5 - e$	$K_6 - e$	$K_7 - e$	$K_8 - e$	$K_9 - e$	$K_{10} - e$
$K_3 - e$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
K_3	$\star$	[25]	[27]	[50]	[62]	[110]	[110]	[101]
$K_4 - e$	$\star$	[24]	[51]	[103]	[103]			
K_4	$\star$	[25]	[46]	[109] $\star$	† [84]			
$K_5 - e$	$\star$	[51]	[28]	[109] [109]	[84]			
K_5	$\star$	[5]	† [109]	[109] [84]	[84]			
$K_6 - e$	$\star$	[103]	[109] [109]	[84]	[84]			
K_6	$\star$	[109]	†	†	[120]			
$K_7 - e$	$\star$	[103]	†	[84]	[84]	[120]		
K_7	$\star$	†	†	[84]				

Table 1.4: References for Table 1.3, $\star$ trivial result, † easy to obtain from other results via simple combinatorial arguments.

t	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$r(K_{2,t}, K_{2,t})$	6	10	14	18	21	26	30	33	38	42	46	50	54	57	62

Table 1.5: Ramsey numbers $r(K_{2,t}, K_{2,t})$

Other generalizations from the classical definition of a Ramsey number involves introducing the concept of irredundance (see for example [3, 4, 17, 18, 29, 30, 31, 72]) or of upper domination (see for example [79, 80]). Another generalization is that of zero-sum Ramsey numbers. Progress made in the literature on zero-sum Ramsey numbers is summarized in a survey by Caro [16]. Other variations of Ramsey numbers include size Ramsey numbers, connected Ramsey numbers, chromatic Ramsey numbers, avoiding sets of graphs in some colours or Ramsey multiplicities. Information on these variations may be found in surveys [10, 22, 106].

1.3.6 Where one of the subgraphs to be forced is a cycle

Chartrand and Schuster [20] showed that $r(C_4, K_3) = 7$, while $r(C_5, K_3) = 9$ and Chvàtal and Harary showed that $r(C_4, C_4) = 6$ and $r(C_4, K_4) = 10$ in [24] and [25] respectively. Greenwood and Gleason [60] established $r(C_4, K_5) = 14$, while $r(C_4, K_6)$ was established by Exoo [43]. Jayawardene and Rousseau [86, 87, 89, 91] showed that $21 \leq r(C_4, K_7) \leq 22$, $r(C_5, K_4) = 13$, $r(C_5, K_5) = 17$, $r(C_6, K_4) = 16$, $r(C_6, K_5) = r(C_5, K_6) = 21$, $r(C_5, K_6 - e) = r(C_6, K_5 - e) = 17$ and that $r(C_4, G) \leq p + q - 1$ for any connected graph G of order p and size q. Bondy and Erdös [6] showed that $r(C_n, K_m) = (n-1)(m-1)+1$, for $n \geq m^2 - 2$ and Faudree and Schelp [54] noted that this holds for $n > 3 = m$. Sheng *et al.* [126, 127] showed that the formula also holds for $n > 4 = m$ and $n \geq 5 = m$. It was conjectured to be true for all $n > m > 3$.

Jayawardene and Rousseau [88] considered the Ramsey number $r(C_4, G)$ where G can be any order 6 graph. Chvàtal and Schwenk [26] showed that $r(C_5, W_6) = 13$ and Faudree *et al.* [53] established that $r(C_4, B_n) = 7, 9, 11, 12, 13, 16$ and 16 for $2 \leq n \leq 8$ respectively. The following results (that are very close to the field of study of this thesis) were established by Hendry [73, 74, 75]: $r(B_3, K_4) = 14$, $20 \leq r(B_3, K_5) \leq 22$, $r(C_5 + e, K_5) = r(W_5, K_5 - e) = 17$, $27 \leq r(W_5, K_5) \leq 29$ and $25 \leq r(K_5 - P_3, K_5) \leq 28$. The lower bound, $26 \leq r(K_{2,2,2}, K_{2,2,2})$, which was established during a personal communication between Radziszowski and Exoo [109], is improved in this thesis.

1.3.7 Where the subgraph to be forced is bipartite

Closer to the subject of this thesis is the case where both the graphs G_1 and G_2 in Definition 1.3 are bipartite graphs. Burr [11] showed that $r(K_{2,3}, K_{2,3}) = 10$ and Exoo and Reynolds [48] showed that $r(K_{2,3}, K_{2,4}) = 12$, while Parsons [105] showed that $r(K_{2,3}, K_{1,7}) = 13$. Harborth and Mengersen [68, 69] showed that $r(K_{2,3}, K_{3,3}) = 13$, $r(K_{3,3}, K_{3,3}) = 18$, $r(K_{2,2}, K_{2,8}) = 15$ and $r(K_{2,2}, K_{2,11}) = 18$. Lawrence [95] showed that $r(K_{2,2}, K_{1,15}) = 20$. All the values for $r(K_{2,t}, K_{2,t})$, $2 \leq t \leq 17$, in Table 1.5 as well as the general upper bound $r(K_{2,t}, K_{2,t}) \leq 4t - 2$ for all $t \geq 2$, were established by Exoo, Harborth and Mengersen [47]. They also showed that $65 \leq r(K_{2,17}, K_{2,17}) \leq 66$.

1.3.8 Where the subgraph to be forced is multipartite

Burr *et al.* [14] considered a special class of multipartite graphs for G_1 and G_2 in Definition 1.3 and concluded that:

> "An interesting case is that in which G_2 and G_2 are both complete multipartite graphs $K_{n_1,n_2,\ldots,n_k}$. There is no real hope of evaluating these numbers in general, since this includes the very difficult case in which G_1 and G_2 are complete graphs. Indeed there seems to be little hope for exact evaluations unless, in some sense, each of G_1 and G_2 is small or sparse."

The aim of this thesis is to find bounds for these numbers where both G_1 and G_2 are complete, balanced, multipartite graphs. More specifically, this thesis will consider the case where the larger graph to be coloured arbitrarily is a complete, balanced, multipartite graph.

It was established in [14] that

$$r(K_{1,m}, K_{1,n}) = \begin{cases} m+n-1 & \text{if both } m \text{ and } n \text{ are even,} \\ m+n & \text{otherwise.} \end{cases}$$

Burr *et al.* [14] also showed for $m_1 \leq m_2 \leq \cdots \leq m_k$ and sufficiently large n, that

$$r(K_{1,m_1,m_2,\ldots,m_k}, K_{1,n}) = k(r(K_{1,m_1}, K_{1,n}) - 1) + 1,$$

which may be evaluated using the above result. Chvàtal [23] showed that $r(K_{1,m}, K_{1,n}) = (m-1)(n-1)+1$ and Erdös *et al.* [35] showed that $r(K_{2,2}, K_{1,n}) \leq n + \sqrt{n}$. They proved a considerable amount of asymptotic bounds for multipartite graphs versus trees in [36], for example $r(K_{2,2}, K_{1,n-1}) > n + n^{1/2} - 5n^{3/10}$. A similar result was established by the same authors in [52].

1.3.9 Where the graph to be coloured is multipartite

The next obvious generalization of the classical definition of Ramsey numbers is that of changing the original graph to be coloured. In the previous sections of this survey of literature the original graph to be coloured was always taken to be the complete graph K_p and the Ramsey number the least natural number p that guarantees the existence of a specified subgraph. Now the original graph to be bicoloured will be a complete, balanced, multipartite graph.

Day *et al.* [32] defined a multipartite Ramsey number as follows:

n m	2	3	4	5	6
2	5^1	9^3	14^3	$\leq 19^4$	$\leq 25^4$
3		17^1	$\leq 29^4$	$\leq 41^4$	$\leq 56^4$
4			48^2	$\leq 72^4$	$\leq 101^4$
5				$\leq 115\ ^2$	$\leq 168^4$

Table 1.6: Bi-partite Ramsey numbers $b(m, n)$, [1]Due to Beineke and Schwenk [1]. [2]Due to Irving [85]. [3]Due to Hattingh and Henning [77]. [4]Due to Goddard, *et al.* [57].

Definition 1.5 *"For a graph G, a partiteness $k \geq 2$ and a number of colours c, we define the* **multipartite Ramsey number** $r_k^c(G)$ *as the minimum value m such that, given any colouring using c colours of the edges of the complete, balanced, k-partite graph with m vertices in each partite set, there must exist a monochromatic copy of G."*

They show that $r_3^2(C_4) = 3, r_4^2(C_4) = 2, r_3^3(C_4) = 7$ and $r_3^2(C_6) = 3$. Goddard *et al.* [57] used Zarankiewicz Numbers to determine bounds for bipartite Ramsey numbers. They defined the bipartite Ramsey number $b(m, n)$ as the least positive integer b such that if every edge of the bipartite graph, $K_{b,b}$ is coloured either red or blue, then the colouring necessarily contains a red $K_{m,m}$ or a blue $K_{n,n}$ as a subgraph. It was shown that $b(m, n) \leq \binom{m+n}{m} - 1$. Table 1.6 summarizes their results.

They also established the first 3-colour bipartite Ramsey number $b(2, 2, 2) = 11$. Hattingh and Henning proved a number of results concerning the situation where the original graph to be coloured is a complete bipartite graph and the monochromatic subgraphs to be forced are stars or paths [76], disjoint copies of $K_{2,2}$, stars, stripes and trees [77]. Henning and Oellermann [78] also showed that $r(nK_{2,2}) = 4n - 1$ where $nK_{2,2}$ denotes n copies of $K_{2,2}$.

1.4 Layout and structure of this thesis

1.4.1 Scope and definition

The notion of a Ramsey number is generalized in this thesis by using multipartite graphs in the definition, but with the difference that the original graph whose edges are to be bicoloured, as well as the monochromatic subgraphs attempted to be forced, are assumed to be complete, balanced, multipartite graphs. Only *symmetric*, multipartite Ramsey numbers will be considered in this thesis.

Two kinds of multipartite Ramsey numbers may be defined, depending on whether the size of a partite set is fixed, or whether the number of partite sets is fixed.

Definition 1.6 *Let n, l, k and j be natural numbers. The symmetric* **set count multipartite Ramsey number** $M_j(n, l)$ *is the least natural number k such that any bicolouring of the edges of $K_{k \times j}$ will necessarily force a monochromatic $K_{n \times l}$ as a subgraph* *of $K_{k\times j}$. The symmetric* **set size multipartite Ramsey number** $m_k(n,l)$ *is the least natural number j such that any bi-colouring of the edges of $K_{k\times j}$ will necessarily force a monochromatic $K_{n\times l}$ as a subgraph of $K_{k\times j}$.*

Note that this definition contains the symmetric 2-colour classical Ramsey numbers as special cases, since if $r(a,a)=b$, then $M_1(a,1)=b$ and $m_b(a,1)=1$. According to the above definition, the asymmetric classical Ramsey numbers are *not* special cases of the multipartite Ramsey numbers.

The only set count multipartite Ramsey numbers that have been found are the four classical Ramsey numbers $M_1(1,1)=1$, $M_1(2,1)=2$, $M_1(3,1)=6$, $M_1(4,1)=18$ of which the first two are trivial, as well as $M_1(2,2)=6$, shown by Chvátal and Harary [24] and $M_1(2,3)=18$, shown by Harborth and Mengersen [68]. The last two results were mentioned previously using the notation $r(C_4,C_4)=6$ and $r(K_{3,3},K_{3,3})=18$. The only known set size multipartite Ramsey numbers are $m_2(2,2)=5$ and $m_2(2,3)=17$, due to Beineke and Schwenk [1] and $m_3(2,2)=3$ and $m_4(2,2)=2$, due to Day *et al.* [32]. The set count multipartite Ramsey numbers $M_2(2,2), M_3(2,2), M_4(2,2)$ and $M_j(2,2)$ for all integers $j\geq 5$ are derived from the above mentioned set size multipartite Ramsey numbers in Chapter 2 of this thesis.

1.4.2 Aims for each chapter

In the second chapter of this thesis the existence of the newly defined symmetric multipartite Ramsey numbers is established and some basic properties of multipartite Ramsey numbers are proven. The class of $(2,2)$ multipartite Ramsey numbers is also completely established in Chapter 2. An algorithm for determining whether a given graph contains $K_{n\times l}$ as a subgraph is presented in Chapter 3. This algorithm is then applied in computer searches for lower bounds, using pseudo-random colourings as well as circulant colourings. The worst order complexity measure of this algorithm, as well as methods to speed up the running time of its implementation, are discussed in Chapter 3.

In Chapter 4 upper bounds for the class of $(2,l)$ multipartite Ramsey numbers as well as (weaker) general upper bounds are presented. Chapter 5 concludes the thesis with a summary of known and new multipartite Ramsey numbers, as well as relevant best known lower and upper bounds, together with a detailed description of the origin of these numbers or bounds.

Chapter 2

Existence and basic properties

> "Everything actual must first have been possible, before having actual *existence.*"
>
> *Albert Pike (1809-91)*

In this chapter the existence of the newly defined multipartite Ramsey numbers is established. Important properties of multipartite Ramsey numbers are proved and some analytical lower bounds for these numbers are presented. First the existence of set count multipartite Ramsey numbers is settled, after which the existence of set size multipartite Ramsey numbers is established, based on the existence of relevant set count multipartite Ramsey numbers. The special case where the monochromatic subgraph to be forced is $K_{2,2}$ is considered in some detail.

2.1 Boundedness of multipartite Ramsey numbers

The question of existence will be addressed for both set count and set size multipartite Ramsey numbers. General, but weak upper and lower bounds will be found for all multipartite Ramsey numbers which exist.

2.1.1 Set count multipartite Ramsey numbers

The following result proves the existence of all set count multipartite Ramsey numbers and is based on the known existence of the classical Ramsey numbers.

Proposition 2.1 *The multipartite Ramsey number $M_j(n,l)$ exists and, in fact,*

$$M_j(n,l) \leq \binom{2nl-2}{nl-1}.$$

Proof. From the existence theorem of Erdös and Szekeres [37] it follows that $r(nl,nl) \leq \binom{2nl-2}{nl-1} = t$ say. Hence, when arbitrarily colouring the edges of $K_t \equiv K_{t\times 1}$ red or blue, a red K_{nl} or a blue K_{nl} will necessarily be forced as subgraph of K_t. However, since $K_{n\times l} \subseteq K_{nl} \subseteq K_t \equiv K_{t\times 1} \subseteq K_{t\times j}$ for all $j \geq 1$, it follows that $K_{t\times j}$ necessarily contains a red $K_{n\times l}$ or a blue $K_{n\times l}$ as subgraph. □

The next two results both establish lower bounds for set count multipartite Ramsey numbers. The first is based on a condition under which a given multipartite graph could possibly contain another multipartite graph. The second result uses the concept of an expansive colouring, which is due to Day, *et al* [32].

Proposition 2.2 *$M_j(n, l) \geq \lceil l/j \rceil n$.*

Proof. It will be shown that $K_{n\times l} \nsubseteq K_{(\lceil l/j\rceil n-1)\times j}$ which concludes the proof. Consider first the case where $l \leq j$. Then $K_{(\lceil l/j\rceil n-1)\times j} \equiv K_{(n-1)\times j}$. But $K_n \nsubseteq K_{(n-1)\times j}$ and $K_n \subseteq K_{n\times l}$, therefore $K_{n\times l} \nsubseteq K_{(\lceil l/j\rceil n-1)\times j}$.

Consider now the case where $l = mj + r$ for some $m \geq 1$ and $0 \leq r < j$. We show, by attempting to construct partite sets of a subgraph $K_{n\times l}$, that $K_{n\times l} \nsubseteq K_{(\lceil l/j\rceil n-1)\times j}$. To construct a single partite set of size l from smaller partite sets of size j, at least $\lceil l/j \rceil$ partite sets of size j are needed, as seen in Figure 2.1, and there will be $j - r$ superfluous vertices for each such construction. But there must be n partite sets of size l; therefore $\lceil l/j \rceil n - 1$ partite sets of size j will not be enough to construct $K_{n\times l}$. Hence $K_{n\times l} \nsubseteq K_{(\lceil l/j\rceil n-1)\times j}$□.

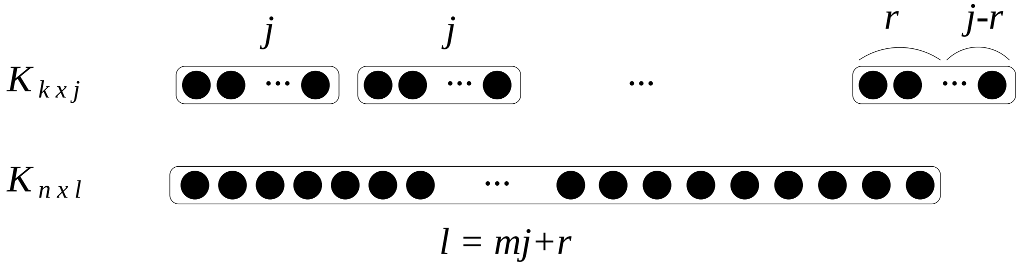


Figure 2.1: A comparison between partitions sets of $K_{n\times l}$ and $K_{k\times j}$.

In order to prove an alternative lower bound for set count multipartite Ramsey numbers, the notion of an expansive colouring is necessary. A colouring of the edges of $K_{k\times j}$ is called an **expansive colouring** if, for every pair of partite sets of $K_{k\times j}$, the edges between all vertices in these partite sets have the same colour. Therefore every expansive colouring corresponds to exactly one edge colouring of K_k. This may be seen as replacing each partite set of $K_{k\times j}$ by a single vertex. It is said that the expansive colouring of $K_{k\times j}$ is induced by the corresponding edge colouring of K_k.

Proposition 2.3 *$M_j(n, l) \geq r(n, n)$.*

Proof. Let $r(n, n) = t$. An edge colouring of $K_{(t-1)\times m}$ that does not contain a monochromatic $K_{n\times l}$ will be presented. There exists a colouring of the edges of K_{t-1} that does not contain a monochromatic K_n as subgraph. The expansive colouring of $K_{(t-1)\times m}$ induced by this colouring of K_{t-1} can therefore not contain a monochromatic n-partite graph with any number of vertices within its partite sets. □

The results of Propositions 2.1, 2.2 and 2.3 may now be summarized in the following Theorem.

Theorem 2.1 $\max\{R(n,n), \lceil l/j \rceil n\} \leq M_j(n,l) \leq \binom{2nl-2}{nl-1}$.

The following proposition establishes a link between multipartite Ramsey numbers and the case where the larger graph to be bicoloured is complete, while still seeking to force monochromatic multipartite subgraphs.

Proposition 2.4
1. If $c < M_1(n,l)$ *then* $\lfloor c/j \rfloor < M_j(n,l)$.
2. If $M_j(n,l) \leq t$ *then* $M_1(n,l) \leq tj$.

Proof. 1. If $c < M_1(n,l)$ then there exists a bicolouring of the edges of K_c which does not contain a monochromatic $K_{n\times l}$ as subgraph. But $K_{\lfloor c/j \rfloor \times j} \subseteq K_c$ hence there also exists no monochromatic $K_{n\times l}$ in this particular bicolouring of $K_{\lfloor c/j \rfloor \times j}$. Consequently $\lfloor c/j \rfloor < M_j(n,l)$.

2. If $M_j(n,l) \leq t$ then there is a monochromatic $K_{n\times l}$ in any bicolouring of the edges of $K_{t\times j}$, but $K_{t\times j} \subseteq K_{tj}$. Hence there is a monochromatic $K_{n\times l}$ in any bicolouring of the edges of K_{tj}. Consequently $M_1(n,l) \leq tj$. □

Explicit bounds for $M_j(n,l)$ have now been established. However, these bounds are weak and hence of no practical use. It will be shown in the next section that not all set size multipartite Ramsey numbers exist (for example, it is clear that $m_1(n,l) = \infty$ if $n > 1$).

2.1.2 Set size multipartite Ramsey numbers

The following theorem establishes the partial existence of set size multipartite Ramsey numbers in terms of the known existence of set count multipartite Ramsey numbers and vice versa.

Theorem 2.2
1. If $a < M_j(n,l) \leq b$, *then* $m_a(n,l) > j$ *and* $m_b(n,l) \leq j$.
2. If $c < m_k(n,l) \leq d$, *then* $M_c(n,l) > k$ *and* $M_d(n,l) \leq k$.

Proof. 1. If $M_j(n,l) > a$, then an arbitrary bicolouring of the edges of $K_{a\times j}$ does not necessarily contain a monochromatic $K_{n\times l}$ as subgraph, so that $m_a(n,l) > j$. If $M_j(n,l) \leq b$, then any bicolouring of the edges of $K_{b\times j}$ necessarily contains a monochromatic $K_{n\times l}$ as subgraph, so that $m_b(n,l) \leq j$.

2. The proof of this result is similar to that of part 1. □

2.2 Basic properties

The following relationships just state that the set size and set count Ramsey numbers increase as the order of the subgraph to be forced increases, and decrease as the order of the graph whose edges are arbitrarily bicoloured, increases.

Proposition 2.5
1. $M_j(s,t) \leq M_j(n,l)$ *if* $s \leq n$ *and* $t \leq l$.
2. $m_k(s,t) \leq m_k(n,l)$ *if* $s \leq n$ *and* $t \leq l$.
3. $M_j(n,l) \leq M_a(n,l)$ *if* $a \leq j$.
4. $m_k(n,l) \leq m_b(n,l)$ *if* $b \leq k$.

Proof. 1. If $M_j(n,l) = r$ then an arbitrary bicolouring of the edges of $K_{r\times j}$ necessarily contains a monochromatic subgraph $K_{n\times l}$ as subgraph. Since $K_{s\times t} \subseteq K_{n\times l}$ if $s \leq n$ and $t \leq l$, there must also be a monochromatic $K_{s\times t}$ in the arbitrary bicolouring of the edges of $K_{r\times j}$, and hence $M_j(s,t) \leq r$. Consequently $M_j(s,t) \leq M_j(n,l)$ if $s \leq n$ and $t \leq l$.

2. The proof of this result is similar to that of part 1.

3. If $M_a(n,l) = \tau$ then an arbitrary bicolouring of the edges of $K_{\tau\times a}$ necessarily contains a monochromatic $K_{n\times l}$ as subgraph. But, since $K_{\tau\times a} \subseteq K_{\tau\times j}$ if $a \leq j$, there must also be a monochromatic subgraph, $K_{n\times l}$, in an arbitrary bicolouring of the edges of $K_{\tau\times j}$, and hence $M_j(n,l) \leq \tau$. Consequently $M_j(n,l) \leq M_a(n,l)$ if $a \leq j$.

4. The proof of this result is similar to that of part 3. □

Note that there are similar results to those of Proposition 2.5(1) and (2) for classical Ramsey numbers. The following theorem gives a lower bound for the set size multipartite Ramsey number $m_k(n,l)$.

Theorem 2.3 $m_k(n,l) \geq \lceil nl/k \rceil$.

Proof. The graph $K_{n\times l}$ has nl vertices. Hence there must be at least $\lceil nl/k \rceil$ vertices per partite set in a multipartite graph G with k partite sets in order for G to possibly contain $K_{n\times l}$ as subgraph. □

The following proposition provides values for the classes of $(1,l)$ and $(n,1)$ multipartite Ramsey numbers.

Proposition 2.6
1. $M_j(1,l) = \lceil l/j \rceil$ *for all* $j,k,l \geq 1$.
2. $M_j(n,1) = r(n,n)$ *for all* $j,n \geq 1$.
3. $m_k(1,l) = 1$ *for all* $k \geq r(n,n)$.
4. $m_k(n,l) = \infty$ *for all* $l \geq 1$ *and* $k < r(n,n)$.

Proof. 1. There exists a subset $V^\star \subseteq V(K_{\lceil l/j \rceil \times j})$ consisting of l vertices. Now $(V^\star, \emptyset)$ constitutes the "monochromatic" graph $K_{1\times j}$. Therefore $M_j(1,l) \leq \lceil l/j \rceil$, but $M_j(1,l) \geq \lceil l/j \rceil$ by Theorem 2.1. Consequently $M_j(1,l) = \lceil l/j \rceil$ for all $j, l \geq 1$. It can be shown in a similar way that $m_k(1,l) = \lceil l/k \rceil$ for all $k, l \geq 1$.

2. $M_j(n,l) \geq r(n,n)$ for all $j,n,l \geq 1$ by Theorem 2.1, but $M_j(n,1) \leq M_1(n,1) = r(n,n)$ by Proposition 2.5(3). Consequently $M_j(n,1) = r(n,n)$ for all $j, n \geq 1$.

3. From classical Ramsey theory it is known that $r(n,n) = t$ (say) partite sets (or more) is sufficient to force a monochromatic $K_{n\times 1}$ as subgraph of any bicolouring of the edges of $K_{t\times 1}$. Therefore $m_k(n,1) \leq 1$ for all $k \geq t$. But then it follows from Definition 1.6 that $m_k(n,1) = 1$ for all $k \geq t$.

4. If $k < r(n,n)$, then there exists a bicolouring of the edges of K_k that does not contain a monochromatic K_n as subgraph. But, since $K_n \subseteq K_{n\times l}$ for any $l \geq 1$, the expansive colouring of $K_{k\times j}$ induced by this specific colouring of K_k also does not contain a monochromatic $K_{n\times l}$ as subgraph, no matter how large we choose $j \geq 1$. It is concluded that $m_k(n,1) = \infty$ for all $k < r(n,n)$. □

2.3 The class $(2,2)$ multipartite Ramsey numbers

The following theorem contains known results for the class of $(2,2)$ set size multipartite Ramsey numbers, as was already mentioned in §1.4.1.

Theorem 2.4
1. $m_1(2,2) = \infty$. *2.* $m_2(2,2) = 5$. *3.* $m_3(2,2) = 3$. *4.* $m_4(2,2) = 2$. *5.* $m_5(2,2) = 2$. *6.* $m_k(2,2) = 1$ *for all* $k \geq 6$.

Proof. Part 1 holds by Proposition 2.6(4), since $r(2,2) = 2 > 1$. Part 2 is due to Beineke & Schwenk [1], while parts 3–6 are due to Day *et al.* [32]. □

The class of $(2,2)$ set count multipartite Ramsey numbers are now established in a trivial manner from Theorem 2.4.

Proposition 2.7
1. $M_1(2,2) = 6$. *2.* $M_2(2,2) = 4$. *3.* $M_3(2,2) = 3$. *4.* $M_4(2,2) = 3$. *5.* $M_j(2,2) = 2$ *for all* $j \geq 5$.

Proof. 1. This result is due to Chvátal & Harary [24].

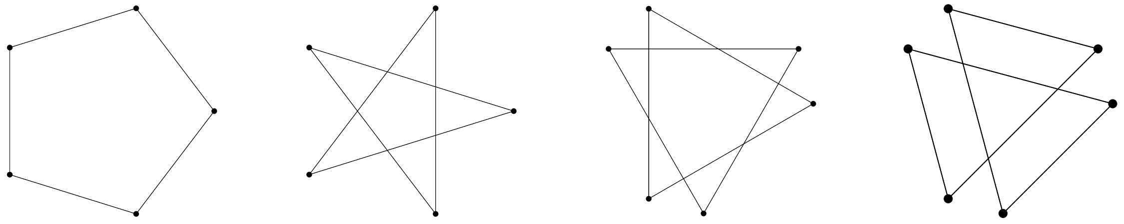
2. By Theorem 2.4(4), $m_4(2,2) \leq 2$. Therefore it follows by Theorem 2.2(2) that $M_2(2,2) \leq 4$. Also, by Theorem 2.4(3), $m_3(2,2) > 2$, so that $M_2(2,2) > 3$ by Theorem 2.2(2).

3. By Theorem 2.4(3), $m_3(2,2) \leq 3$. Hence by Theorem 2.2 $M_3(2,2) \leq 3$. By Theorem 2.4(3), $m_2(2,2) > 4$. Hence by Theorem 2.2(2), $M_4(2,2) > 2$. Therefore by Proposition 2.5(3) $M_3(2,2) > 2$.

4. $M_4(2,2) \leq M_3(2,2) = 3$ by Proposition 2.5(3) and Theorem 2.7(3). Also, by Theorem 2.4(2), $m_2(2,2) > 4$, so that $M_4(2,2) > 2$ by Theorem 2.2(2).

5. By Theorem 2.4(2), $m_2(2,2) \leq 5$, so that $M_5(2,2) \leq 2$ by Theorem 2.2(2). But then $M_j(2,2) \leq 2$ for all $j \geq 5$ by Proposition 2.5(3). Furthermore $M_j(2,2) \neq 1$, because the edge set of $K_{1\times j}$ is empty for all $j \geq 1$. □

It is also possible to establish the lower bounds of Theorem 2.4 and Proposition 2.7 by presenting circulant bicolourings of the relevant graphs. Consider the examples $M_1(2,2) > 5$ and $M_2(2,2) > 3$. A bicolouring of $K_{5\times1}$ by means of the circulant graphs $C_{5\times1}\langle 1\rangle$ and $C_{5\times1}\langle 2\rangle$ given in Figures 2.2(a) and (b), does not contain a monochromatic $K_{2\times2}$ and hence $M_1(2,2) > 5$. A bicolouring of the multipartite graph, $K_{3\times2}$ by means of the circulant graphs $C_{3\times2}\langle 2\rangle$ and $C_{3\times2}\langle 3\rangle$ given in Figures 2.2(c) and (d), does not contain a monochromatic $K_{2\times2}$ and hence $M_2(2,2) > 3$.



(a) $C_{5\times1}\langle 1\rangle$ (b) $C_{5\times1}\langle 2\rangle$ (c) $C_{3\times2}\langle 2\rangle$ (d) $C_{3\times2}\langle 1,3\rangle$

Figure 2.2: Circulant graphs representing the multipartite lower bounds $M_1(2,2) > 5$ and $M_2(2,2) > 3$.

The question of the existence of set count and set size multipartite Ramsey numbers was settled in this chapter and, in addition, some weak lower bounds were determined for these numbers. The use of circulant edge colourings for establishing multipartite Ramsey number lower bounds was illustrated. In the next chapter we shall set out to improve these weak lower bounds.

Chapter 3

Lower bounds

In this chapter lower bounds for multipartite Ramsey numbers are obtained by making use of analytical methods as well as computer searches. In the first section, properties concerning circulants that will assist in finding lower bounds in sections 3 and 4 are established. Section 2 introduces an algorithm for searching for $K_{n\times l}$ as subgraph in a given graph G and the worst order complexity of this algorithm is discussed. The algorithm is implemented in Section 3 and uses the properties of circulants established in Section 1 to find lower bounds. Some of the lower bounds established in Section 3 are improved by performing pseudo-random and random circulant computer searches in Section 4.

3.1 Circulants

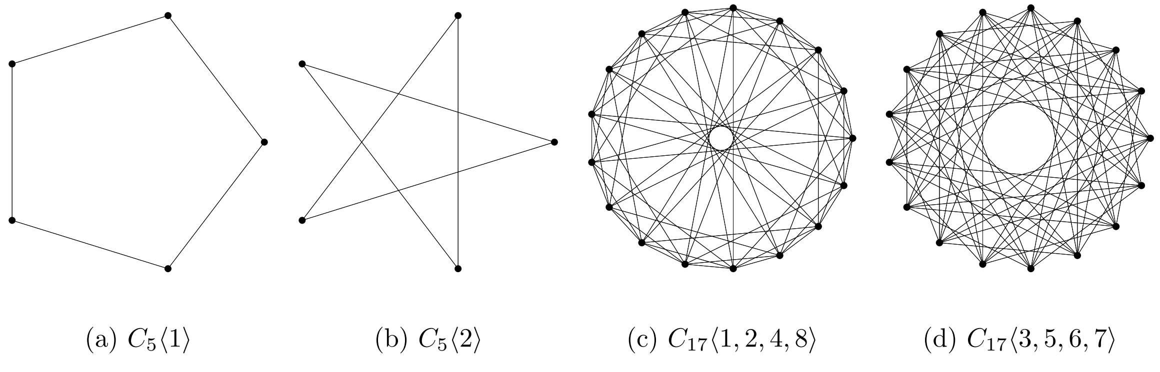

(a) $C_5\langle 1\rangle$ (b) $C_5\langle 2\rangle$ (c) $C_{17}\langle 1,2,4,8\rangle$ (d) $C_{17}\langle 3,5,6,7\rangle$

Figure 3.1: A bicolouring of the edges of K_5 corresponding to the disjoint circulants $C_5\langle 1\rangle$ and $C_5\langle 2\rangle$, that does not contain a monochromatic K_3 as subgraph. Thus $r(3,3) > 5$. A bicolouring of the edges of K_{17} corresponding to the circulants $C_{17}\langle 1,2,4,8\rangle$ and $C_{17}\langle 3,5,6,7\rangle$, that does not contain a monochromatic K_4 as subgraph. Thus $r(4,4) > 17$. Taken from Mynhardt [104].

The class of circulant graphs exhibits a high degree of symmetry and very elegant structure, making it ideal for establishing lower bounds for Ramsey numbers. The notion of a circulant graph was defined formally in Definition 1.1. As early as 1965 circulants were constructed to obtain lower bounds for classical Ramsey numbers, as seen in the survey by Chung and Grinstead [22]. Radziszowski and Kreher [111] used circulants to establish lower bounds for classical 2-colour Ramsey numbers in 1988 and Calkin *et al.* [15] found new classical Ramsey number bounds by implicit enumeration of cyclic 2-colourings in 1996 and showed that circulants of prime order are good candidates for classical Ramsey number lower bounds.

Lower bounds for the multipartite Ramsey numbers $M_j(n,l)$ and $m_k(n,l)$ will be found in this chapter by partitioning the edges of $K_{k\times j}$ into red and blue circulant, multipartite complement graphs. The graphs in Figures 3.1 (a),(b),(c) and (d) represent best lower bounds for classical Ramsey numbers and are also circulants. The strategy in §3.3 will be to determine which circulant graphs do not contain the complete, balanced multipartite subgraph $K_{n\times l}$ and to use exactly these circulants when colouring the edges of $K_{k\times j}$ in order to find lower bounds for $M_j(n,l)$ and $m_k(n,l)$. Some basic properties of elementary circulants will first be given. We start by stating a result from group theory that will be necessary to prove these basic circulant properties.

If $(G,*)$ is a group and $a \in G$, then

$$H = \{a^n | n \in Z\}$$

is a subgroup of $(G,*)$ and this group is called the **cyclic subgroup** $\langle a \rangle$ **of** $(G,*)$ **generated by** a. Here a^n denotes $a * a * \ldots * a$ ($n-1$ binary operations). Furthermore, if the cyclic subgroup is finite, then let the number of elements of $\langle a \rangle$ be called the **order of** $\langle a \rangle$ denoted by $|\langle a \rangle|$. A well known theorem of group theory states that if G is a cyclic group with T elements and generated by a, and $b = a^s \in (G,*)$, then b generates a cyclic subgroup $(H,*)$ of $(G,*)$ containing T/d elements, where d is the greatest common divisor of T and s.

Proposition 3.1 *Let $C_T\langle i \rangle$ be an elementary circulant. Then*

1. $C_T\langle i \rangle = C_T\langle i + yT \rangle$ *for all natural numbers i and y.*
2. $C_T\langle i \rangle = C_T\langle T - i \rangle$ *for all $i = 1, \ldots, T-1$.*
3. $E(C_T\langle i_1 \rangle) \cap E(C_T\langle i_2 \rangle) = \emptyset$ *for all* $i_1 \neq i_2 \pmod{T}$.
4. $deg(v_s) = \begin{cases} 1 & \textit{for all } v_s \in V(C_T\langle i \rangle) \textit{ if } T \textit{ is even and } i = T/2, \\ 2 & \textit{otherwise.} \end{cases}$
5. $q(C_T\langle i \rangle) = \begin{cases} T/2 & \textit{if } T \textit{ is even and } i = T/2, \\ T & \textit{otherwise.} \end{cases}$
6. $C_{kj}\langle i \rangle \subseteq K_{k\times j}$ *if and only if* $i \geq j \pmod{T}$.
7. $C_T\langle i \rangle$ *consists of one cycle of length T if i and T are relatively prime and i distinct cycles of length T/i if $i|T$.*

Proof. 1. It follows from the definition of a circulant that $V(C_T\langle i\rangle) = V(C_T\langle i + yT\rangle)$. The function $\phi_y : E(C_T\langle i\rangle) \to E(C_T\langle i+yT\rangle)$ defined by $\phi_y(v_r v_{r+i}) := v_r v_{r+i+yT \text{ (mod } T)}$ is an automorphism, since $r + i \equiv r + i + yT \pmod{T}$. Hence $E(C_T\langle i\rangle) = E(C_T\langle i + yT\rangle)$.

2. It follows from the definition of a circulant that $V(C_T\langle i\rangle) = V(C_T\langle T-i\rangle)$. The function $\phi_y : E(C_T\langle i\rangle) \to E(C_T\langle T - i\rangle)$ defined by $\phi_y(v_r v_{r+i \text{ (mod } T)}) = v_{r+i} v_{r+i+(T-i)} \pmod{T}$ is an automorphism since $r + i + (T - i) \equiv r \pmod{T}$ and $v_r v_{r+i} = v_{r+i} v_r$. Hence $E(C_T\langle i\rangle) = E(C_T\langle T - i\rangle)$.

3. Assume $v_r v_{r+i_1 \text{ (mod } T)} \in E(C_T\langle i_1\rangle) \cap E(C_T\langle i_2\rangle)$ and $i_1 \neq i_2$. Then $v_r v_{r+i_1 \text{ (mod } T)} \equiv v_r v_{r+i_2 \text{ (mod } T)}$ and therefore $i_1 \equiv i_2 \pmod{T}$ which contradicts our assumption.

4. The degree of every vertex v_r of a non-singular circulant $C_T\langle i\rangle$ is 2 since every vertex v_r is connected by an edge to vertices $v_{r+i \text{ (mod } T)}$ and $v_{r-i \text{ (mod } T)}$. However, when T is even and $i = T/2$ then the vertices $v_{r+i \text{ (mod } T)}$ and $v_{r-i \text{ (mod } T)}$ are in fact the same vertex.

5. From the fundamental theorem of graph theory $q(C_T\langle i\rangle) = \frac{1}{2}\sum_{v_r \in V(C_T\langle i\rangle)} \deg(v_r)$. Since there are T vertices in $C_T\langle i\rangle$, and the degree of every vertex was established in (4), the result follows.

6. Assume that the partite sets of $K_{k\times j}$ are given by

$$V_1 = \{v_0, \ldots, v_{j-1}\}, V_2 = \{v_j, \ldots, v_{2j-1}\}, \ldots, V_k = \{v_{(k-1)j}, \ldots, v_{kj-1}\}$$

and denote $C_{kj}\langle i_s\rangle \cap K_{k\times j}$ by $C_{k\times j}\langle i_s\rangle$. It is clear that $C_{k\times j}\langle i_s\rangle \subseteq C_{kj}\langle i_s\rangle$ if $i_s \geq j$. Conversely, if $i_s < j$ then $v_0 v_i$ is and inter-partite edge in C_{kj} and $C_{kj} \not\subseteq C_{k\times j}$.

7. The vertices of C_T are placed on the edge of an imaginary circle and ordered in increasing order as the circle is traversed in a clockwise direction. An n-cycle will be constructed by starting at vertex v_0 and visiting every i-th clockwise vertex on the edge of the imaginary circle, stopping as soon as a vertex that has been visited before, is reached. Let v_s be the penultimate vertex visited in this fashion and, for any $r \leq s$, let the number of vertices from v_r to v_s in a clockwise direction of the edge of the circle be denoted by $d(v_r, v_s)$. Then $d(v_0, v_r) = ni$ for some $n \in \mathcal{N}$ and $d(v_s, v_r) = i$ and $1 \leq i \leq \lfloor T - 1\rfloor$. Since v_s and v_0 are distinct, $d(v_s, v_0) > 0$.

Now, the set of integers $\{0, 1, 2, \ldots, T-1\}$ together with the operator, $+(\text{ mod } T)$ forms the finite group Z_T, which is cyclic and is generated by $\langle 1\rangle$. Hence there exists a positive integer s such that any element, i, of $(Z_T, +)$ can be expressed as $i = 1^s$. Since 1^s represents s additions of size 1 we know that $s = i$. Hence if i and T are relatively prime, then i is also a generator of T, and if not, then $\langle i\rangle$ is a cyclic subgroup of Z_T of order $T/\gcd(i, T)$. Therefore, starting at vertex v_0 would either generate a single cycle of length T inducing the whole circulant, C_T (if $\gcd(i, T) = 1$), or generate a single cycle of length $p := T/\gcd(i, T)$. Since an elementary circulant is constructed by joining every two vertices, v_α and $v_{\alpha+\beta(\text{mod } T)}$ for all values of $\alpha \in \{1, \ldots, T\}$, all vertices will be visited. Hence there will exist one cycle of length T if $\gcd(i, T) =$ or $T/p = \gcd(i, T)$ cycles of length $T/\gcd(i, T)$ otherwise. □

The relationship between elementary and composite circulants is established by the following proposition.

Proposition 3.2 *Let* $C_T\langle i_1,\ldots,i_z\rangle$ *be a composite circulant. Then*

$$C_T\langle i_1,\ldots,i_z\rangle = \bigcup_{s=1}^{z} C_T\langle i_s\rangle.$$

Proof. Since an edge $v_r v_{r+i_s \ (\mathrm{mod}\ T)} \in C_T\langle i_s\rangle$ and $v_r v_{r+i_s \ (\mathrm{mod}\ T)} \notin C_T\langle i_w\rangle$ for all $i_w \neq i_s$, all singular circulants, $C_T\langle i_1\rangle,\ldots,C_T\langle i_z\rangle$ are needed to construct $C_T\langle i_1,\ldots,i_z\rangle$. $\square$

It follows from Proposition 3.1(1), 3.1(2), 3.1(5) together with Proposition 3.2 that if the composite circulant $C_T\langle i_1,\ldots,i_z\rangle$ is a subgraph of $K_{k\times j}$ we may assume without loss of generality that $z \leq \lfloor T/2\rfloor - j + 1$ and that $j \leq i_s \leq \lfloor T/2\rfloor$ for all $s = 1,\ldots,z$. We shall henceforth assume these inequalities to hold throughout this chapter, without stating it explicitly. The size of the composite circulant, $C_T\langle i_1,\ldots,i_z\rangle$ as well as other basic properties of composite circulants are settled by the following corollary.

Corollary 3.1

1. Let $C_T\langle i_1,\ldots,i_z\rangle$ *be a composite circulant. Then*

$$q(C_T\langle i_1,\ldots,i_z\rangle) = \begin{cases} (2z-1)T/2 & \text{if } T \text{ is even and } i_s = T/2 \text{ for some } s \in \{1,\ldots,z\}. \\ zT & \text{otherwise.} \end{cases}$$

2. $C_T\langle 1,2,\ldots,\lfloor T/2\rfloor\rangle \simeq K_T$.

3. $C_{kj}\langle i_1,\ldots,i_z\rangle \subseteq K_{k\times j}$ *if and only if* $i_s \geq j \ (\mathrm{mod}\ T)$ *for all* $1 \leq s \leq z$.

Proof. 1. The result follows directly from Proposition 3.1(3), 3.1(4) and 3.2. There are z disjoint circulants, each comprising of T edges, except for the singular circulant with $T/2$ edges.

2. If T is odd then there are $(T-1)/2$ different elementary circulants on the vertices $\{v_0,\ldots,v_{T-1}\}$, each with T edges on these vertices. If T is even then there are $T/2-1$ different circulants on the vertices $\{v_0,\ldots,v_{T-1}\}$, each with T edges and one circulant with $T/2$ edges on these vertices. In both instances there are $T(T-1)/2$ edges in $C_T\langle 1,\ldots,\lfloor T/2\rfloor\rangle$ and since all the edges are different by Proposition 3.1 the result follows.

3. Assume that the partite sets of $K_{k\times j}$ are given by

$$V_1 = \{v_0,\ldots,v_{j-1}\}, V_2 = \{v_j,\ldots,v_{2j-1}\},\ldots,V_k = \{v_{(k-1)j},\ldots,v_{kj-1}\}.$$

If $C_{kj}\langle i_1,\ldots,i_z\rangle \nsubseteq K_{k\times j}$ then there must be an edge, v_a, v_{a+i_s} in $E(C_{k\times j}\langle i_1,\ldots,i_z\rangle)$ that is not in $E(K_{k\times j})$. Since, by Proposition 3.2, $C_{kj}\langle i_1,\ldots,i_z\rangle = \bigcup_{s=1}^{z} C_{kj}\langle i_s\rangle$ and by Proposition 3.1, $C_{kj}\langle i\rangle \subseteq K_{k\times j}$ if $i \geq j \ (\mathrm{mod}\ T)$, such an edge can only exist if $i_s < j$. Conversely, if $i_s < j$ for any $1 \leq s \leq z$, then $v_0 v_i$ is an inter-partite set edge in $C_{kj}\langle i_1,\ldots,i_z\rangle$ and hence $C_{kj}\langle i_1,\ldots,i_z\rangle \nsubseteq K_{k\times j}$.

The edges within $E(C_{k\times j}\langle j, j+1,\ldots,\lfloor kj/2\rfloor\rangle)$ are denoted **ordinary edges** of $K_{k\times j}$. There are also edges in $E(K_{k\times j})$ which are not ordinary edges if $j > 1$. These edges are called the **closure edges** of $K_{k\times j}$. The total number of edges in $K_{k\times j}$ is given by

$$q(K_{k\times j}) = kj^2(k-1)/2 = \underbrace{kj[(k-2)j+1]/2}_{\text{ordinary edges}} + \underbrace{kj(j-1)/2}_{\text{closure edges}}.$$

We now establish a method of generating isomorphic circulants from a given circulant. This will save a significant amount of computation time when searching all possible circulants (up to isomorphism) for a copy of $K_{n\times l}$ as subgraph in attempts at establishing multipartite Ramsey number lower bounds. There are $\binom{\lfloor T/2 \rfloor}{z}$ different z-tuples, $\langle i_1,\ldots,i_z\rangle$ from the set $Z_T=\{0,\ldots,T-1\}$, but many of the circulants corresponding to these z-tuples are isomorphic. The set $M_T=\{m\in Z_T : \gcd(m,T)=1\}$ is called the set of invertible elements of Z_T. It is well known that M_T forms a group under multiplication modulo T. Define the permutation

$$\sigma_m = \begin{pmatrix} 0 & 1 & 2 & \cdots & (T-1) \\ 0 & m & 2m & \cdots & m(T-1) \end{pmatrix}$$

and let S_T be the set of all permutations of Z_T. Therefore σ_m is one of the $T!$ elements of S_T. Let $N_T=\{\sigma_m : m\in M_T\}$. The group $(M_T,\cdot)$ is isomorphic to the subgroup $(N_T,\cdot)$ of $(S_T,\cdot)$, since the mapping $f: M_T\rightarrow N_T$ such that $f(m)=\sigma_m$ is an isomorphism. From now on $(M_T,\cdot)$ will be considered a subgroup of $(S_T,\cdot)$ and hence M_T will be identified with N_T.

Consider the circulant $C_T\langle i_1,\ldots,i_z\rangle$ and let $\sigma\in S_T$. Let the circulant $C_T\langle \sigma i_1,\ldots,\sigma i_z\rangle$ be denoted σC_T. It is clear that $C_T\simeq \sigma C_T$ for some $\sigma\in S_T$. If $\sigma\in S_T$ then the graph $\sigma C_T\langle i_1,\ldots,i_z\rangle$ does not necessarily have distinct, non-zero entries and hence cannot be considered a circulant. However, if $m\in M_T$, then $mC_T\langle i_1,\ldots,i_z\rangle$ is the circulant $C_T\langle mi_1,\ldots,mi_z\rangle$; see Figure 3.2 for an example. In this manner $|M_T|$ circulants that are isomorphic to $C_T\langle i_1,\ldots,i_z\rangle$ can be generated, by taking multiples $mC_T\langle i_1,\ldots,i_z\rangle$, $m\in M_T$. It might still happen that the entries in the generated circulants do not satisfy $i_s\leq\lfloor T/2\rfloor$ for $s=1,\ldots,z$. To remedy this, let i_s be assigned to $T-i_s$ if $i_s>\lfloor T/2\rfloor$, since it was shown in Proposition 3.1(2) that $C_T\langle i\rangle = C_T\langle T-i\rangle$ for all $i=1,\ldots,T-1$. This argument may be summarized in the following theorem [8].

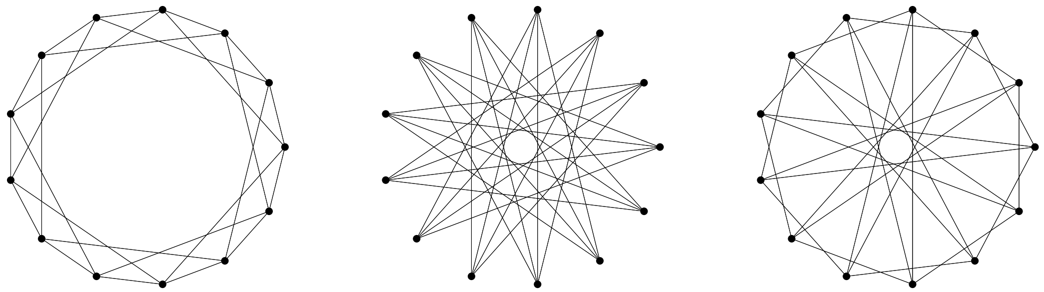
Theorem 3.1 *The multiples $mC_T\langle i_1,\ldots,i_z\rangle$, where $m\in M_T$, generate $\phi(T)/2$ circulants isomorphic to $C_T\langle i_1,\ldots,i_z\rangle$.*

Proof. Since the size of M_T is $\phi(T)$, where ϕ is the well known Euler function, the method described above can be used to generate $\phi(T)$ circulants isomorphic to any given circulant, $C_T\langle i_1,\ldots,i_z\rangle$, if it is allowed that a circulant entry, $i_s>\lfloor T/2\rfloor$. Since every entry $i_s>\lfloor T/2\rfloor$ has an equivalent entry, $i_s^\star = T-i_s$ according to Proposition 3.1(2) with $i_s^\star\leq\lfloor T/2\rfloor$, $\phi(T)$ is exactly double the number of isomorphic circulants, which concludes the proof. $\square$

The plan in the sections to come will be to partition $K_{k\times j}$ repeatedly into two disjoint, composite circulants and to search for a partition such that no copy of $K_{n\times l}$ can be found.

3.2 Algorithms

Four algorithms will be considered in this section. The first provides a means by which it can be determined whether a relatively small multipartite graph is a subgraph of a graph



(a) $C_{13}\langle 1,3\rangle$ (b) $C_{13}\langle 6,5\rangle$ (c) $C_{13}\langle 2,6\rangle$

Figure 3.2: $C_{13}\langle 1,3\rangle \simeq 6C_{13}\langle 1,3\rangle = C_{13}\langle 6,5\rangle \simeq 11C_{13}\langle 1,3\rangle = C_{13}\langle 2,6\rangle$

G. The second, third and fourth algorithms are useful for searching through all possible circulants of a given order with the hope of not obtaining a copy of the multipartite graph, $K_{n\times l}$ as subgraph. The worst order complexity of searching all circulant edge partitions is investigated and the improvement gained in worst order complexity by employing Theorem 3.1 is quantified in this section.

3.2.1 Subgraph Algorithm

When seeking to establish lower bounds for Ramsey numbers in multipartite graphs, specific bicolourings of the edges of certain multipartite graphs which do not contain certain other monochromatic multipartite graphs as subgraphs, must be produced. In order to find these specific bicolourings, one of three approaches may be used: one may attempt to produce random bicolourings of the edges of a multipartite graph and hope to find a specific colouring which does not contain a required monochromatic multipartite graph as subgraph. An alternative approach would be to partition the edges of the graph being searched into two circulants and hoping that neither circulant contains a copy of $K_{n\times l}$ as subgraph. In §3.3 this will be done for all possible non-isomorphic partitions of small multipartite graphs $K_{k\times j}$ into two disjoint circulants. A third possibility would be to *randomly* partition G into two disjoint circulants, which will prove particularly useful for larger values of k and j. This approach will be followed in §3.4.

All three approaches require a method of deciding whether a multipartite graph occurs as a subgraph of a given graph, G. Unfortunately the task of determining whether $K_{n\times l}$ occurs as a subgraph of a graph G is an NP-complete problem [55]. However, Algorithm 3.1 is practical for establishing whether $K_{n\times l}$ is a subgraph of a (p,q)-graph G if p, q, n and l are relatively small.

A graph G might not contain the multipartite graph $K_{n\times l}$ for any of the following reasons:

1. G may have too few vertices or too few edges to contain $K_{n\times l}$ as subgraph.

2. G may have enough vertices and edges, but may not contain at least n sets of l vertices per set, where every set has a large enough mutual neighbourhood set.

3. G might contain n sets of at least l vertices with large enough mutual neighbourhood sets, but these n sets might not be mutually disjoint.

4. G might contain n mutually disjoint sets of at least l vertices (with large enough neighbourhood sets), but the intersection of all the l-sets together with their mutual neighbourhood sets might not contain the required nl vertices.

The four criteria mentioned above are not mutually exclusive and were ordered in increasing order of computational determinacy. The strategy used in the algorithm is to attempt to conclude that $K_{n\times l} \nsubseteq G$ with the minimal amount of computation or else to acknowledge that indeed $K_{n\times l} \subseteq G$. Therefore each criterion mentioned above will be tested in the given order. Hence it is first tested whether the graph G contains the required nl vertices and $nl^2(n-1)/2$ edges. If this test fails, then $K_{n\times l} \nsubseteq G$.

Secondly, if this first test succeeds then the mutual neighbourhood set of every possible set of l-tuple of is computed. Every l-tuple that has a mutual neighbourhood set size of greater than or equal to $(n-1)l$, is stored in a set labelled $\mathbf{E}$, while the other l-tuples are discarded. If there are fewer than n l-tuples in $\mathbf{E}$ after all possible l-tuples have been considered, then $K_{n\times l} \nsubseteq G$.

Thirdly, if the second test is passed, then every possible set of n l-tuples is selected from $\mathbf{E}$ and the intersection of these n l-tuples is computed. If the size of such an intersection set is less than nl, then the particular n-set does not consist of mutually disjoint l-tuples and another n-set must be considered. However, if an n set of disjoint l-tuples is found, it qualifies for the fourth test, which entails the following: for this particular n-set, the union of every l-tuple with its mutual neighbourhood set is stored in a set denoted $E_w \cup F_w$ (w being the counter with an initial value of 0). If the intersection of all n $(E_w \cup F_w)$-sets has size greater than or equal to n, then a copy of $K_{n\times l}$ has been found in G. Otherwise another n-set is considered. If none of the n-sets satisfy both the third and fourth tests mentioned then, it can be concluded that there is no copy of $K_{n\times l}$ in G.

Algorithm 3.1 *To determine whether* $K_{n\times l} \subseteq G$.

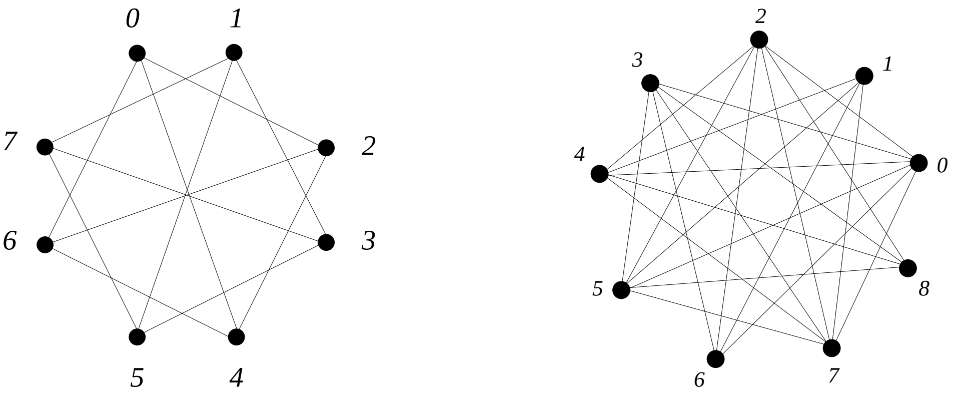
Input: The order p, size q and set of neighbourhood sets, $\{N_1, \dots, N_p\}$ of the graph G; integers n and l.
Output: Boolean value `SUBGRAPH` which takes the value `true` if $K_{n\times l} \subseteq G$ and `false` otherwise.

1. If $p < nl$ or $q < nl^2(n-1)l/2$, then `SUBGRAPH:=false`. Stop.

2. Let $\mathbf{N} = \{N_i : |N_i| \geq (n-1)l\}$. If $|N| < nl$ then `SUBGRAPH:=false`. Stop.

3. Let $\alpha := \binom{t}{l}$ and let $\{E_1, \dots, E_\alpha\}$ be all possible sets of size l of vertices from $V(G)$, where $E_d = \{V_{d_1}, V_{d_2}, \dots, V_{d_l}\}$ for all $d = 1, 2, \dots, \alpha$. Let $F_d := \cup_{s=1}^{l} N_{d_s}, \mathbf{F} := \{F_d : |F_d| \geq (n-1)l\}$ and $\mathbf{E} := \{E_d : |F_d| \geq (n-1)l\}$. If $|\mathbf{F}| < n$, then `SUBGRAPH:=false`. Stop.

4. If there exists n sets from $\mathbf{E}$, say $\{E_{\theta_1}, \dots, E_{\theta_n}\}$ with corresponding mutual neighbourhood sets $F_{s_1}, \dots, F_{s_n}$ that satisfies both the following two properties,



(a) The graph G in Example 3.1

(b) The graph G in Example 3.2

Figure 3.3: Examples for illustrating Algorithm 3.1

Property 4.1

$$\left| \bigcup_{w:=\theta_1}^{\theta_n} E_w \right| \geq nl.$$

Property 4.2

$$\left| \bigcap_{w:=\theta_1}^{\theta_n} E_w \cup F_w \right| \geq nl.$$

then `SUBGRAPH`:=`true`, otherwise `SUBGRAPH`:=`false`. □

Two examples will be used to illustrate the working of Algorithm 3.1. In the first example the graph $C_{4\times 2}\langle 2,4\rangle$ is shown to contain a copy of $K_{2\times 2}$ and in the second example the graph G given in Figure 3.3(b) is shown not to contain a copy of $K_{3\times 2}$.

Example 3.1 *To determine whether* $K_{2\times 2} \subseteq C_{4\times 2}\langle 2,4\rangle$.

A graphical representation of the graph $C_{4\times 2}\langle 2,4\rangle$ is given in Figure 3.3(a). As can be seen from the graphical representation, the neighbourhood sets are

$$\mathbf{N} := \left\{ \overbrace{\{2,4,6\}}^{N_0}, \overbrace{\{3,5,7\}}^{N_1}, \overbrace{\{0,4,6\}}^{N_2}, \overbrace{\{1,5,7\}}^{N_3}, \overbrace{\{0,2,6\}}^{N_4}, \overbrace{\{1,3,7\}}^{N_5}, \overbrace{\{0,2,4\}}^{N_6}, \overbrace{\{1,3,5\}}^{N_7} \right\},$$

while $p = 8, q = 12, n = 2$ and $l = 2$.

The test in Step 1 of Algorithm 3.1 is passed since $p = 8 \geq 4 = nl$ and $q = 12 \geq 4 = nl^2(n-1)/2$ and we cannot be certain whether $K_{2\times 2} \subseteq C_{4\times 2}\langle 2,4\rangle$ at this stage.

Since every vertex has a neighbourhood set of size greater than or equal to 2, $\mathbf{N}$ in Step 2 of Algorithm 3.1 is given by all the neighbourhood sets listed above. Since $|N| = 8 \geq$

w	E_w	F_w	Qualify	w	E_w	F_w	Qualify
0	0,1	$\emptyset$	No	14	2,4	0,6	Yes
1	0,2	4,6	Yes	15	2,5	$\emptyset$	No
2	0,3	$\emptyset$	No	16	2,6	0,4	Yes
3	0,4	2,6	Yes	17	2,7	$\emptyset$	No
4	0,5	$\emptyset$	No	18	3,4	$\emptyset$	No
5	0,6	2,4	Yes	19	3,5	1,7	Yes
6	0,7	$\emptyset$	No	20	3,6	$\emptyset$	No
7	1,2	$\emptyset$	No	21	3,7	1,5	Yes
8	1,3	5,7	Yes	22	4,5	$\emptyset$	No
9	1,4	$\emptyset$	No	23	4,6	0,2	Yes
10	1,5	3,7	Yes	24	4,7	$\emptyset$	No
11	1,6	$\emptyset$	No	25	5,6	$\emptyset$	No
12	1,7	3,5	Yes	26	5,7	1,3	Yes
13	2,3	$\emptyset$	No	27	6,7	$\emptyset$	No

Table 3.1: Test 3 of Algorithm 3.1 applied to Example 3.1. The qualification column states which of the sets E_w and F_w qualify for the sets $\mathbf{E}$ and $\mathbf{F}$ respectively.

$4 = nl$, the test in Step 2 of Algorithm 3.1 is passed and we cannot be certain whether $K_{2\times2} \subseteq C_{4\times2}\langle 2,4\rangle$ at this stage.

In Step 3 of Algorithm 3.1, $\alpha := \binom{8}{2} = 28$ and Table 3.1 shows all values of E_i and F_i. $\mathbf{E}$ and $\mathbf{F}$ are as follows:

$$\mathbf{E} := \left\{ \overbrace{\{0,2\}}^{E_1}, \overbrace{\{0,4\}}^{E_3}, \overbrace{\{0,6\}}^{E_5}, \overbrace{\{1,3\}}^{E_8}, \overbrace{\{1,5\}}^{E_{10}}, \overbrace{\{1,7\}}^{E_{12}}, \overbrace{\{2,4\}}^{E_{14}}, \overbrace{\{2,6\}}^{E_{16}}, \overbrace{\{3,5\}}^{E_{19}}, \overbrace{\{3,7\}}^{E_{21}}, \overbrace{\{4,6\}}^{E_{23}}, \overbrace{\{5,7\}}^{E_{25}}, \right\}$$

$$\mathbf{F} := \left\{ \overbrace{\{4,6\}}^{F_1}, \overbrace{\{2,6\}}^{F_3}, \overbrace{\{2,4\}}^{F_5}, \overbrace{\{5,7\}}^{F_8}, \overbrace{\{3,7\}}^{F_{10}}, \overbrace{\{3,5\}}^{F_{12}}, \overbrace{\{0,6\}}^{F_{14}}, \overbrace{\{0,4\}}^{F_{16}}, \overbrace{\{1,7\}}^{F_{19}}, \overbrace{\{1,5\}}^{F_{21}}, \overbrace{\{0,2\}}^{F_{23}}, \overbrace{\{1,3\}}^{F_{25}}, \right\}$$

Since $|F| = 12 \geq 8 = n$, the test in Step 3 of Algorithm 3.1 is passed and we cannot be certain whether $K_{2\times2} \subseteq C_{4\times2}\langle 2,4\rangle$ at this stage.

In Step 4 of Algorithm 3.1, the first choice for a n-set (with $n = 2$) is $E_1 \cup E_3$. The union, $E_1 \cup E_3 = \{0,2,4\}$ in Step 4.1 is not large enough. The next union, $E_1 \cup E_5$ is also not large enough.The third union, $E_1 \cup E_8$ is large enough, but the intersection $(E_1 \cup F_1) \cap (E_8 \cup F_8) = \{0,2,4,6\} \cap \{1,3,5,7\}$ is empty. In this way every choice for 2-set fails either the test in 4.1 or the test in 4.2, until the sets $E_1 = \{0,2\}$ and $E_{23} = \{4,6\}$ are considered. Hence the union in Step 4.1 gives $E_1 \cup E_{23} = \{0,2,4,6\}$. The intersection in Step 4.2 gives $(E_1 \cup F_1) \cap (E_9 \cup F_9) = \{0,2,4,6\} \cap \{0,2,4,6\} = \{0,2,4,6\}$. Hence `SUBGRAPH`:=`true` and no further 2-sets are considered. $\Box$

w	E_w	F_w	Qualify	w	E_w	F_w	Qualify
0	0,1	4,5,6,7	Yes	18	2,6	0	No
1	0,2	4,5,6,7	Yes	19	2,7	0,5	No
2	0,3	5,6,7	No	20	2,8	4,5	No
3	0,4	2,7	No	21	3,4	7,8	No
4	0,5	2,3,7	No	22	3,5	7,8	No
5	0,6	2,3	No	23	3,6	0	No
6	0,7	2,3,4,5	Yes	24	3,7	0,5	No
7	0,8	2,3,4,5	Yes	25	3,8	5	No
8	1,2	4,5,6,7	Yes	26	4,5	0,1,2,7,8	Yes
9	1,3	5,6,7	No	27	4,6	0,1,2	No
10	1,4	7	No	28	4,7	0,1,2	No
11	1,5	7	No	29	4,8	2	No
12	1,6	$\emptyset$	No	30	5,6	0,1,2,3	Yes
13	1,7	4,5	No	31	5,7	0,1,2,3	Yes
14	1,8	4,5	No	32	5,8	2,3	No
15	2,3	0,5,6,7,8	Yes	33	6,7	0,1,2,3	Yes
16	2,4	0,7,8	No	34	6,8	2,3	No
17	2,5	0,7,8	No	35	7,8	2,3,4,5	Yes

Table 3.2: Test 3 of Algorithm 3.1 applied to Example 3.2. The qualification column states which of the sets E_w and F_w qualify for the sets **E** and **F** respectively.

Example 3.2 *To determine whether* $K_{3\times 2} \subseteq G$ *given in Figure 3.3(b).*

As can be seen from the graphical representation the neighbourhood sets are:

$$\mathbf{N} := \left\{ \overbrace{\{2,3,4,5,6,7\}}^{N_0}, \overbrace{\{4,5,6,7\}}^{N_1}, \overbrace{\{0,4,5,6,7,8\}}^{N_2}, \overbrace{\{0,5,6,7,8\}}^{N_3}, \overbrace{\{0,1,2,7,8\}}^{N_4}, \right.$$

$$\left. \overbrace{\{0,1,2,3,7,8\}}^{N_5}, \overbrace{\{0,1,2,3\}}^{N_6}, \overbrace{\{0,1,2,3,4,5\}}^{N_7}, \overbrace{\{2,3,4,5\}}^{N_8} \right\}$$

and $p = 9, q = 23, n = 3$ and $l = 2$. The test in Step 1 of Algorithm 3.1 is passed since $p = 9 \geq 6 = nl$ and $q = 23 \geq 12 = nl^2(n-1)/2$ and we cannot be certain whether $K_{3\times 2} \subseteq G$ at this stage.

Since all the vertices have neighbourhood set sizes of greater than or equal to 4 the set **N** in Step 2 of Algorithm 3.1 is just all the sets $N_0, \dots, N_8$ given above. Since $|\mathbf{N}| = 9 \geq 6 = n$ the test in Step 2 is passed and we cannot be certain whether $K_{3\times 2} \subseteq G$ at this stage.

In Step 3, $\alpha := \binom{9}{2} = 36$ and Table 3.2 shows all values of E_i and F_i. **E** and **F** are as follows:

$$\mathbf{E} := \left\{ \overbrace{\{0,1\}}^{E_0}, \overbrace{\{0,2\}}^{E_1}, \overbrace{\{0,7\}}^{E_6}, \overbrace{\{0,8\}}^{E_7}, \overbrace{\{1,2\}}^{E_8}, \overbrace{\{2,3\}}^{E_{15}}, \overbrace{\{4,5\}}^{E_{26}}, \overbrace{\{5,6\}}^{E_{30}}, \overbrace{\{5,7\}}^{E_{31}}, \overbrace{\{6,7\}}^{E_{33}}, \overbrace{\{7,8\}}^{E_{35}}, \right\}$$

$\cup_{d=i_1}^{i_3} E_d$	$\cap_{d=i_1}^{i_3} (E_d \cup F_d)$	$\cup_{d=i_1}^{i_3} E_d$	$\cap_{d=i_1}^{i_3} (E_d \cup F_d)$
0,1,2,3,4,5	0,5,7	0,2,3,5,6,8	0,2,3,5
0,1,2,3,5,6	0,5,6	0,1,2,5,7,8	2,5
0,1,2,4,5,7	2,4,5,7	0,2,3,5,7,8	0,2,3,5
0,2,3,4,5,7	0,2,5,7	0,1,2,6,7,8	2
0,1,2,5,6,7	2,5	0,2,3,6,7,8	0,2,3
0,2,3,5,6,7	0,2,3,5	0,4,5,6,7,8	0,2
0,1,2,3,5,7	0,5,7	0,1,2,3,7,8	5,7
0,1,2,3,6,7	0,6,7	0,1,4,5,7,8	4,5,7
0,1,4,5,6,7	0,1,7	0,2,4,5,7,8	2,4,5,7
0,2,4,5,6,7	0,2,7	1,2,4,5,7,8	2,4,5,7
1,2,4,5,6,7	1,2,7	2,3,4,5,7,8	2,5,7,8
2,3,4,5,6,7	0,2,7	0,1,5,6,7,8	5
0,1,2,4,5,8	2,4,5	0,2,5,6,7,8	2,5
0,2,3,4,5,8	0,2,5,8	1,2,5,6,7,8	2,5
0,1,2,5,6,8	2,5	2,3,5,6,7,8	2,3,5

Table 3.3: Test 4.2 of Algorithm 3.1 applied to Example 3.2

$$\mathbf{F} := \left\{ \overbrace{\{4,5,6,7\}}^{F_0}, \overbrace{\{4,5,6,7\}}^{F_1}, \overbrace{\{2,3,4,5\}}^{F_6}, \overbrace{\{2,3,4,5\}}^{F_7}, \overbrace{\{4,5,6,7\}}^{F_8}, \overbrace{\{0,5,6,7,8\}}^{F_{15}}, \overbrace{\{0,1,2,7,8\}}^{F_{26}}, \right.$$

$$\left. \overbrace{\{0,1,2,3\}}^{F_{30}}, \overbrace{\{0,1,2,3\}}^{F_{31}}, \overbrace{\{0,1,2,3\}}^{F_{33}}, \overbrace{\{2,3,4,5\}}^{F_{35}}, \right\}$$

Since $|F| = 11 \geq 3 = n$, the test in Step 3 of Algorithm 3.1 is passed and we cannot be certain whether $K_{3\times 2} \subseteq G$ at this stage.

Of these 11 mutual neighbourhood sets, there are $\binom{11}{3} = 165$ choices for 3-sets (since $n = 3$) in Step 4 of Algorithm 3.1. Since 165 is considered too many to list only the 30, 3-sets that satisfy the test in Step 4.1 are listed in Table 3.3. However, none of the 30 candidates listed satisfy the test in Step 4.2 which requires a set size of $nl = 6$. Therefore `SUBGRAPH`:=`false`. $\square$

Next the correctness of Algorithm 3.1 will be proven, and then the worst order complexity of Algorithm 3.1 will be investigated.

Theorem 3.2 *The boolean variable in Algorithm 3.1 satisfies* `SUBGRAPH`=`true` *if and only if* $K_{n\times l} \subseteq G$.

Proof. Suppose `SUBGRAPH`=`true`, then there exist n sets from the set $\mathbf{E}$ in Algorithm 3.1, say $\mathbf{S} := \{E_{\theta_1}, \ldots, E_{\theta_n}\}$ with corresponding mutual neighbourhood sets $F_{s_1}, \ldots, F_{s_n}$ that satisfy both the following two properties, Property 4.1

$$\left| \bigcup_{w:=\theta_1}^{\theta_n} E_w \right| \geq nl,$$

Property 4.2

$$\left| \bigcap_{w:=\theta_1}^{\theta_n} E_w \cup F_w \right| \geq nl,$$

Furthermore, since `SUBGRAPH=true`, it is known that any E_{θ_i} from the set $\mathbf{E}$ has a corresponding set F_{θ_i} with $|F_{\theta_i}| \geq (n-1)l$ in Step 3 of Algorithm 3.1. Also, since `SUBGRAPH=true`, the set S contains vertices with large enough neighbourhood set to satisfy the test in Step 2 of Algorithm 3.1 and Step 1 must have been passed, guaranteeing sufficient vertices and edges for the set S to represent a $K_{n\times l}$. Hence the set S contains n mutually disjoint sets of at least l vertices (with large enough neighbourhood sets), and the intersection of all the l-sets together with their mutual neighbourhood sets contains nl vertices and must contain a copy of $K_{n\times l}$ by definition.

Conversely, suppose $K_{n\times l}$ occurs as a subgraph of the (p, q) graph G and number the vertices of G such that $P_i = \{v_{(i-1)l+1}, v_{(i-1)l+2}, \ldots, v_{il}\}, i = 1, \ldots, n$ are the partite sets of the occurrence of $K_{n\times l}$. Then $p \geq nl$ and $q \geq nl^2(n-1)/2$ in Step 1 of the algorithm, and since each vertex $v_{(i-1)l+j} \in P_i$ is adjacent to the $(n-1)l$ vertices in the partite sets $P_k (k \neq i)$, the set $\mathbf{N}$ in Step 2 contains at least nl entries $(i-1)l + j$ $(i = 1, \ldots, n)$ and $(j = 1, \ldots, l)$, so that $t \geq nl$. Furthermore, in Step 3 of Algorithm 3.1, when $E_i = P_i$ then $|F_i| \geq (n-1)l$, since the cardinality of the mutual neighbourhood set of a partite set, P_i is exactly $(n-1)l$. Since there are n partite sets, $\{P_1, \ldots, P_n\}$, $|\mathbf{F}| \geq n$. Hence consider Step 4. When the n partite sets are chosen as candidates for $\{E_{\theta_1}, \ldots, E_{\theta_n}\}$, their corresponding neighbourhood sets will be $\{P\backslash P_1, \ldots, P\backslash P_n\}$. The test in 4.1 will be satisfied since all partite sets are disjoint. In the test in Step 4.2, every union will yield $E_w \cup F_w = P$, where P is the set of all the vertices in $V(K_{n\times l})$. Hence the final intersection in Step 4.2 will intersect n copies of P which will yield P again. Since $P \geq nl$ `SUBGRAPH:=true`. $\square$

The worst order complexity of Algorithm 3.1 will be stated in terms of basic operations.

Definition 3.1 *A basic operation is defined as a single primitive operation* $(+, -, \times, \div)$*, comparing two numbers or the assignment of a value to a single variable in a data structure. Incrementing or decrementing loop counters are ignored.* $\square$

The values nl, $(n-1)l$ and $(n-1)l^2nl/2$ are calculated and stored. The first and third are then compare to p and q respectively. This requires six basic operations, since each of the following must be calculated and stored: $n-1, nl, (n-1)l, (n-1)l^2, (n-1)lnl, (n-1)lnl/2$. To calculate the sizes of all the neighbourhood sets, N_i, $1 \leq i \leq p$ and compare the size of each N_i to the stored value $(n-1)l$ and finally to compare $\mathbf{N}$ to nl requires at most $p(p+1)+1$ basic operations, since the maximum size of a neighbourhood set is p.

The maximum value for t in Step 2 is p, hence the maximum for α in Step 3 is $\binom{p}{l}$ and the maximum for of n-sets to be chosen from $\mathbf{E}$ in Step 4 of Algorithm 3.1 is $\binom{\binom{p}{l}}{n}$. To form a set F_d in Step 3, l ordered sets of size p must be intersected. This require at most lp^2 basic operations. Since a set F_d represents the mutual neighbourhood of l vertices, it can have a size of at most $p-l$. To calculate the size of $\mathbf{F}$ and then compare it to n requires $p+1$ basic operations.

In Step 4.1 it is required to find the union of n ordered sets of size l in order to calculate $\cup_{d=i_1}^{i_n} E_d$ and then to compare this to the stored value nl. This requires at most nl^2+1 basic operations. In step 4.2 it is required to first find the union of the two disjoint sets E_d and F_d (which requires p basic operations) and then to find the intersection of n such $E_d \cup F_d$ sets in order to calculate the set $\cap_{d=i_1}^{i_n}(E_d \cup F_d)$ and to compare its size to the stored value nl. Since the size of $E_d \cup F_d$ is at most p this requires at most $n(p+p^2)+1$ basic operations (where p represents the disjoint union and p^2 represents the intersection). Therefore to conclude that $K_{n\times l}$ is not a subset of the (p,q)-graph G, requires at most w basic operations, where the worst order complexity w is given by:

$$\begin{aligned} w &= 7+p(p+1)+\binom{p}{l}lp^2+(p+1)+\binom{\binom{p}{l}}{n}(nl^2+1+n(p+p^2)+1) \\ &= 7+(p+1)^2+\binom{p}{l}lp^2+\binom{\binom{p}{l}}{n}\left(n(p+p^2+l^2)+2\right) \qquad (3.1) \end{aligned}$$

3.2.2 Other Algorithms

Three algorithms will be considered in this section. First an algorithm for generating a composite circulant together with its multipartite graph complement will be given. This will be followed by discussion of an algorithm that generates all possible circulants on T vertices will be discussed, and finally an algorithm that employs Theorem 3.1 to identify non-isomorphic circulant classes will be considered. The algorithms will not be given explicitly, but will be discussed in some detail, after which the worst order complexity of these algorithms will be estimated at each step (this will be stated in terms of basic operations, as was defined in Definition 3.1).

A $kj \times kj$ matrix representation A of the bicoloured graph $K_{k\times j}$ is stored in a static data structure. $A[i,j]=0$ if i and j are not connected by an edge. $A[i,j]=1$ if i and j are connected by means of a red edge, while $A[i,j]=2$ if i and j are connected by means of a blue edge. It is anticipated that bicolourings of $K_{k\times j}$ such that the red and blue degrees of every vertex differ by at most one would be most suitable for establishing lower bounds for Ramsey numbers. The circulant $C_{k\times j}\langle 1,2,\ldots,\lfloor kj/2\rfloor\rangle$ is isomorphic to $K_{k\times j}$ according to Corollary 3.1(2). Therefore the plan is to repeatedly partition the set $\{1,2,\ldots,\lfloor kj/2\rfloor\}$ into two sets of size $\lfloor kj/4\rfloor$ (or one set of size $\lfloor kj/4\rfloor$ and one set of size $\lceil kj/4\rceil+1$ if $\lfloor kj/2\rfloor$ is odd). The edges of the graph $K_{k\times j}$ will then be partitioned into the circulants corresponding to the partitions until a circulant is found such that neither the circulant itself, nor its multipartite graph complement contains $K_{n\times l}$ as subgraph, or until all such circulant edge partitions were considered.

Algorithm 3.2 *Generating a composite circulant, $C_{k\times j}\langle i_1,\dots,i_z\rangle$, and its multipartite complement.*

Input: The number of partite sets, k, cardinality of all partite sets, l, and the circulant z-tuple, $\langle i_1,\dots,i_z\rangle$.
Output: Adjacency matrix of the circulant $C_{k\times j}\langle i_1,\dots,i_z\rangle$ and its multipartite complement.

A circulant $C_{k\times j}\langle i_1, i_2,\dots,i_z\rangle$ is generated as follows. The adjacency matrix, $A[i,j]$ is initialized with zeroes. The adjacency matrix, $A[i,j]$ is then filled with ones corresponding to each entry of the circulant. To generate an elementary circulant, $C_{k\times j}\langle i\rangle$ involves calculating $u = bi + a \pmod{kj} + 1$ and $v = (b+1)i + a \pmod{kj} + 1$ for all $u, v \in \{1,\dots,kj-1\}$. Next, for each such pair (u,v), the assignment $A[u,v] := 1$ is made. Therefore $(kj)^2$ basic operations are needed for the generation of each elementary circulant. Every composite circulant consists of at most $\lceil kj/4\rceil$ elementary circulants. Hence the worst order complexity of creating the composite circulant $C_{kj}\langle i_1, i_2,\dots,i_{\lfloor kj/4\rfloor}\rangle$ is $(kj)^3/4$. To convert $C_{kj}\langle i_1,\dots,i_z\rangle$ to $C_{k\times j}\langle i_1,\dots,i_z\rangle$, the inter-partite edges must be removed. This requires another $(kj)^2$ basic operations, i.e. $(kj)^3/2+(kj)^2$ basic operations for each of the two composite circulants. □

Algorithm 3.3 *Generating all possible variable instances $i_1,\dots,i_H$ of the circulant $C_T\langle i_1,\dots,i_H\rangle$, where $Z = \lfloor T/2\rfloor$, $H = \lfloor Z/2\rfloor$ and $i_s \in \{1,2,\dots,Z\}$.*

Input: The circulant order T.
Output: All possible variable instances $i_1,\dots,i_h$ where $i_s \in \{1,2,\dots,Z\}$.

All instances of the variables $i_1,\dots,i_h$ are created by counting in base H, while keeping $i_1,\dots,i_h$ an increasing sequence. Hence i_1 is the least significant digit and i_h is the most significant digit. Therefore, the first variable instance will be $1,2,\dots,H$, the next variable instance will be $1,2,\dots,(H+1)$ and finally $(H+1),(H+2),\dots,Z$ will be reached. □

Before the improvement due to Theorem 3.1 all circulants $C_{k\times j}\langle i_1,\dots,i_h\rangle$ corresponding to instances $i_1,\dots,i_h$ must be searched for a copy of $K_{n\times l}$ using Algorithm 3.1. There are $\binom{\lfloor kj/2\rfloor}{\lfloor kj/4\rfloor}$ different variable instances to check, giving a grand total of $\binom{\lfloor kj/2\rfloor}{\lfloor kj/4\rfloor}2((kj)^3/2 + (kj)^2)$ basic operations needed to generate all circulant couples and another w basic operations to search for a copy of $K_{n\times l}$ as a subgraph in a particular circulant, where w is the worst order complexity of Algorithm 3.1 (Equation (3.1)) and $p = kj$.

After being able to generate a class of isomorphic circulants from a given circulant according to Theorem 3.1, not all circulants corresponding to the variable instances $i_1,\dots,i_H$ must searched for a copy of $K_{n\times l}$. Since counting is performed in base H in Algorithm 3.3, it can be determined whether the circulant corresponding to a newly generated variable instance is isomorphic to one that has been generated before. If this is the case, then this particular variable instance need not be searched for a copy of $K_{n\times l}$, saving computing time.

Algorithm 3.4 *Determining whether the circulant* $C_{k\times j}\langle i_1,\dots,i_h\rangle$ *should be searched for a copy of* $K_{n\times j}$*.*

Input: Adjacency matrix of the circulant $C_{k\times j}\langle i_1,\dots,i_h\rangle$.
Output: The boolean variable `SEARCH` denoting whether the given circulant should be searched for a copy of $K_{n\times l}$.

All the positive integers not greater than and coprime to kj are found and hence $\phi(kj)$ is calculated. Let $R := \{r_1,\dots,r_{kj-\phi(kj)}\}$ be this set of numbers. This consists of $c(kj)^2$ basic operations where $0 < c \leq \frac{1}{2}$. Each variable instance, $i_1,\dots,i_h$ (other than the first one generated) is multiplied by an element of $r_s \in R \pmod{kj}$, $1 \leq s \leq kj$ to generate a new variable instance $x_1,\dots,x_h$ that has a corresponding circulant isomorphic to the circulant of the instance $i_1,\dots,i_h$. For $i_t \in \{i_1,\dots,i_h\}$, if $r_s i_t > \lfloor kj/2 \rfloor$ then $x_t := kj - r_s i_t$, since we know that $C_T\langle i\rangle = C_T\langle T-i\rangle$ for all $i = 1,\dots,T-1$ from Proposition 3.1(2). The new variable instance, $x_1,\dots,x_h$ is then sorted (since instance are kept in an increasing order in Algorithm 3.3) and labelled $y_1,\dots,y_h$. If the number (in base H) represented by $y_1,\dots,y_h$ is smaller than the number (in base H) represented by $i_1,\dots,y_h$ then `SEARCH:=false` since a circulant isomorphic to $C_{k\times j}\langle i_1,\dots,i_h\rangle$ has been searched before.

An insertion sort comprising h^2 basic operations is done. There are $\binom{z}{h}$ possible variable instances, $i_1,\dots,i_h$ but for each such instance there are $\phi(kj)$ instances with isomorphic corresponding circulant. Therefore $\binom{z}{h/\phi(kj)}$ circulant generations and searches are done instead of the full $\binom{z}{h}$, with the extra cost of doing $\phi(kj)$ insertions sorts (h^2 basic operations) for each of the $\binom{z}{h}$ variable instances. □

Hence, searching all possible circulants has a worst order complexity of $\binom{z}{h}w$ and only searching non-isomorphic circulants by performing the check in Algorithm 3.4 has a worst order complexity of $c(kj)^2 + \binom{z}{h}h^2 + \binom{z}{h}/\phi(kj)w$ where $z = \lfloor kj/2 \rfloor$, $h = \lceil kj/4 \rceil$, w is the worst order complexity of Algorithm 3.1 (Equation 3.1), $p = kj$ and $0 < c \leq 0.5$.

In the next section we employ Algorithm 3.1 to search for $K_{n\times l}$ in circulant edge partitions of $K_{k\times j}$ in the hope of finding lower bounds for Ramsey numbers in multipartite graphs.

3.3 Lower bounds using all possible circulants

In this section lower bounds for Ramsey numbers in multipartite graphs are established by performing all possible non-isomorphic circulant edge partitions of the relevant complete multipartite graph with the use of Theorem 3.1.

3.3.1 Lower bounds for classes $(2,3)$, $(3,2)$,$(2,4)$ and $(4,2)$

Lower bounds for set count multipartite Ramsey numbers of the classes $(2,3)$, $(3,2)$,$(2,4)$ and $(4,2)$ will be established in this section.

The following theorem provides lower bounds for the class of $(2,3)$-Ramsey numbers by giving the relevant circulant partition. The proof consists of giving a circulant partition which, together with its multipartite graph complement, does not contain a copy of $K_{2\times 3}$ as was ascertained by Algorithm 3.1.

Theorem 3.3
1. $M_1(2,3) > 17$ *and* $m_{17}(2,3) > 1$.
2. $M_2(2,3) > 7$ *and* $m_7(2,3) > 2$.
3. $M_3(2,3) > 5$ *and* $m_5(2,3) > 3$.
4. $M_4(2,3) > 3$ *and* $m_3(2,3) > 4$.

Proof. Graphical representations of the relevant circulant edge partitions are given in Figure 3.4.

1. $K_{2\times 3} \not\subseteq C_{17\times 1}\langle 2,6,7,8\rangle$ and $K_{2\times 3} \not\subseteq C_{17\times 1}\langle 1,3,4,5\rangle \Rightarrow M_1(2,3) > 17$ and $m_{17}(2,3) > 1$.
2. $K_{2\times 3} \not\subseteq C_{7\times 2}\langle 2,4,5\rangle$ and $K_{2\times 3} \not\subseteq C_{7\times 2}\langle 1,3,6,7\rangle \Rightarrow M_2(2,3) > 7$ and $m_7(2,3) > 2$.
3. $K_{2\times 3} \not\subseteq C_{5\times 3}\langle 3,4,6\rangle$ and $K_{2\times 3} \not\subseteq C_{5\times 3}\langle 1,2,5,7\rangle \Rightarrow M_3(2,3) > 5$ and $m_5(2,3) > 3$.
4. $K_{2\times 3} \not\subseteq C_{3\times 4}\langle 1,2,4\rangle$ and $K_{2\times 3} \not\subseteq C_{3\times 4}\langle 2,6,7,8\rangle \Rightarrow M_4(2,3) > 3$ and $m_3(2,3) > 4$. □

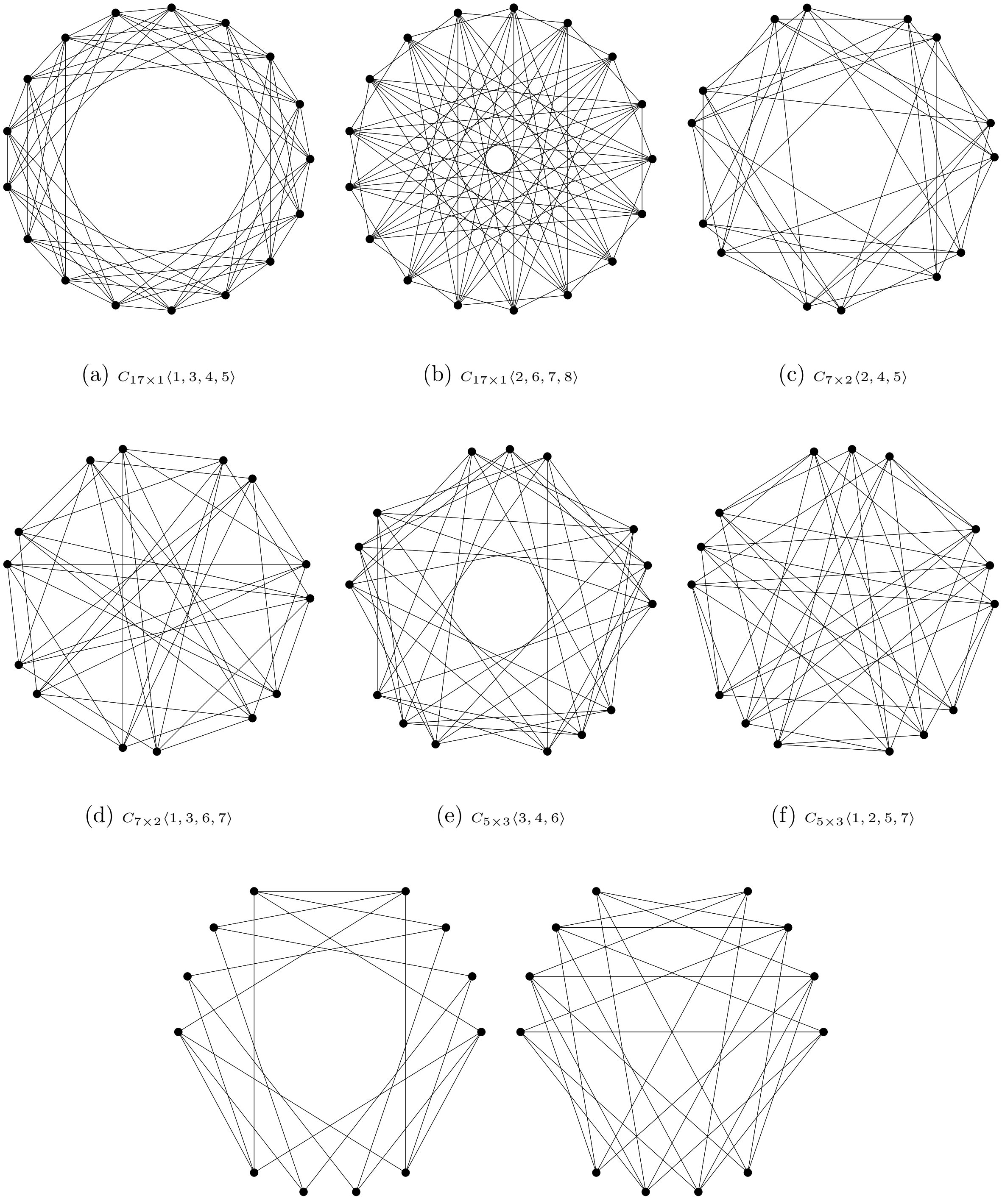
The following theorem provides lower bounds for the class of $(3,2)$-Ramsey numbers by giving the relevant circulant partition. The proof consists of giving a circulant partition which, together with its multipartite graph complement, does not contain a copy of $K_{3\times 2}$ as was ascertained by Algorithm 3.1.

Theorem 3.4
1. $M_1(3,2) > 29$ *and* $m_{29}(3,2) > 1$.
2. $M_2(3,2) > 11$ *and* $m_{11}(3,2) > 2$.
3. $M_3(3,2) > 8$ *and* $m_8(3,2) > 3$.
4. $M_4(3,2) > 5$ *and* $m_5(3,2) > 4$.

Proof. Graphical representations of the relevant circulant edge partitions are given in Figure 3.5.

1. $K_{3\times 2} \not\subseteq C_{29\times 1}\langle 1,4,5,6,7,9,13\rangle$ and $K_{3\times 2} \not\subseteq C_{29\times 1}\langle 2,3,8,10,11,12,14\rangle$ $\Rightarrow M_1(3,2) > 29$ and $m_{29}(3,2) > 1$.
2. $K_{3\times 2} \not\subseteq C_{11\times 2}\langle 1,2,3,6,8\rangle$ and $K_{3\times 2} \not\subseteq C_{11\times 2}\langle 4,5,7,9,10,11\rangle \Rightarrow M_2(3,2) > 11$ and $m_{11}(3,2) > 2$.
3. $K_{3\times 2} \not\subseteq C_{8\times 3}\langle 3,4,5,6,7,12\rangle$ and $K_{3\times 2} \not\subseteq C_{8\times 3}\langle 1,2,8,9,10,11\rangle \Rightarrow M_3(3,2) > 8$ and $m_8(3,2) > 3$.
4. $K_{3\times 2} \not\subseteq C_{5\times 4}\langle 1,2,3,4,5\rangle$ and $K_{3\times 2} \not\subseteq C_{5\times 4}\langle 6,7,8,9,10\rangle$ $\Rightarrow M_4(3,2) > 5$ and $m_5(3,2) > 4$. □

The following theorem provides lower bounds for the class of $(2,4)$-Ramsey numbers by giving the relevant circulant partition. The proof consists of giving a circulant partition



(a) $C_{17\times1}\langle 1,3,4,5\rangle$ (b) $C_{17\times1}\langle 2,6,7,8\rangle$ (c) $C_{7\times2}\langle 2,4,5\rangle$

(d) $C_{7\times2}\langle 1,3,6,7\rangle$ (e) $C_{5\times3}\langle 3,4,6\rangle$ (f) $C_{5\times3}\langle 1,2,5,7\rangle$

(g) $C_{3\times4}\langle 1,2,4\rangle$ (h) $C_{3\times4}\langle 2,6,7,8\rangle$

Figure 3.4: Lower bound circular edge partitionings for the class of $(2,3)$ multipartite Ramsey numbers of Theorem 3.3.

which, together with its multipartite graph complement, does not contain a copy of $K_{2\times 4}$ as was ascertained by Algorithm 3.1.

Theorem 3.5
1. $M_1(2,4) > 29$ *and* $m_{29}(2,4) > 1$.
2. $M_2(2,4) > 13$ *and* $m_{13}(2,4) > 2$.
3. $M_3(2,4) > 9$ *and* $m_9(2,4) > 3$.
4. $M_4(2,4) > 6$ *and* $m_6(2,4) > 4$.

Proof. Graphical representations of the relevant circulant edge partitions are given in Figure 3.6.

1. $K_{2\times 4} \not\subseteq C_{29\times 1}\langle 1,4,5,6,7,9,13\rangle$ and $K_{2\times 4} \not\subseteq C_{29\times 1}\langle 2,3,8,10,11,12,14\rangle$ $\Rightarrow M_1(2,4) > 29$ and $m_{29}(2,4) > 1$.

2. $K_{2\times 4} \not\subseteq C_{13\times 2}\langle 3,4,5,6,8,11\rangle$ and $K_{2\times 4} \not\subseteq C_{13\times 2}\langle 1,2,7,9,10,12,13\rangle \Rightarrow M_2(2,4) > 13$ and $m_{13}(2,4) > 2$.

3. $K_{2\times 4} \not\subseteq C_{9\times 3}\langle 3,4,6,8,9,10\rangle$ and $K_{2\times 4} \not\subseteq C_{9\times 3}\langle 1,2,5,7,11,12,13\rangle \Rightarrow M_3(2,4) > 9$ and $m_9(2,4) > 3$.

4. $K_{2\times 4} \not\subseteq C_{6\times 4}\langle 1,2,4,7,8,9\rangle$ and $K_{2\times 4} \not\subseteq C_{6\times 4}\langle 3,5,6,10,11,12\rangle \Rightarrow M_4(2,4) > 6$ and $m_6(2,4) > 4$. □

The following theorem provides lower bounds for the class of $(4,2)$-Ramsey numbers by giving the relevant circulant partition. The proof consists of giving a circulant partition which, together with its multipartite graph complement, does not contain a copy of $K_{4\times 2}$ as was ascertained by Algorithm 3.1.

Theorem 3.6
1. $M_1(4,2) > 29$ *and* $m_{29}(4,2) > 1$.
2. $M_2(4,2) > 16$ *and* $m_{16}(4,2) > 2$.
3. $M_3(4,2) > 9$ *and* $m_9(4,2) > 3$.
4. $M_4(4,2) > 8$ *and* $m_8(4,2) > 4$.

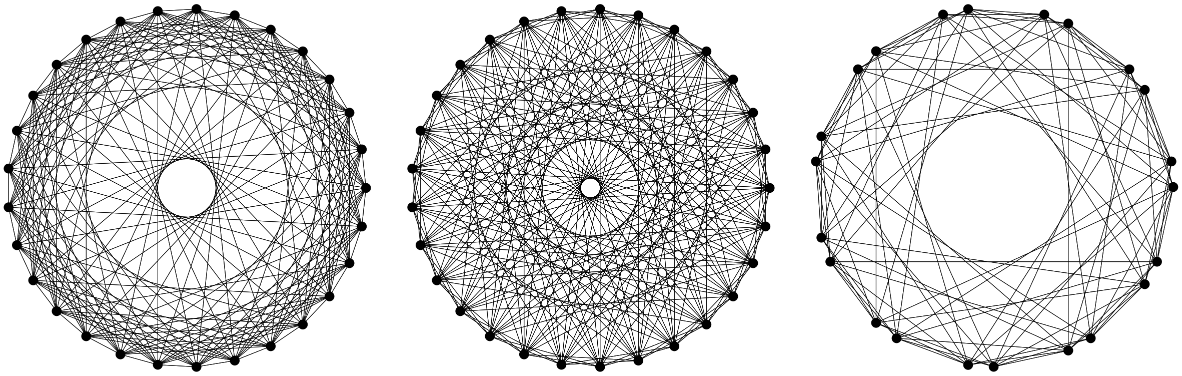
Proof. Graphical representations of the relevant circulant edge partitions are given in Figure 3.7.

1. $K_{4\times 2} \not\subseteq C_{29\times 1}\langle 1,2,4,5,6,7,8\rangle$ and $K_{4\times 2} \not\subseteq C_{29\times 1}\langle 3,9,10,11,12,13,14\rangle$ $\Rightarrow M_1(4,2) > 29$ and $m_{29}(4,2) > 1$.

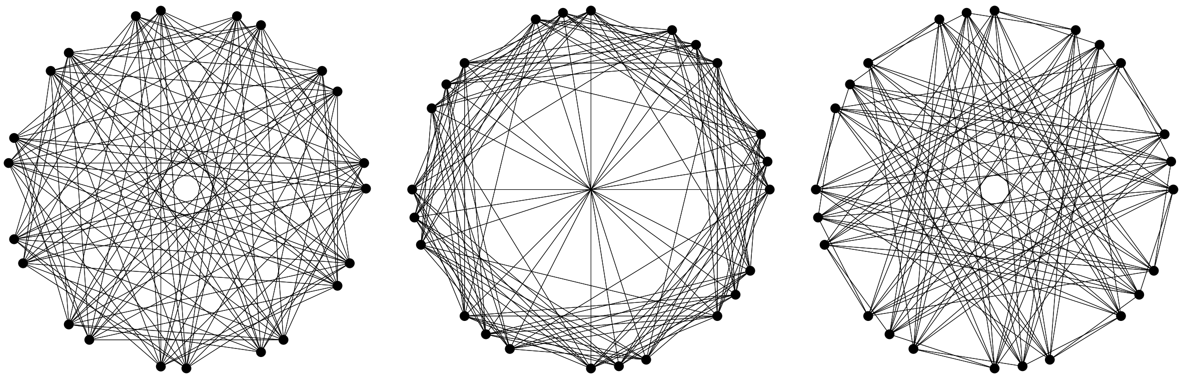
2. $K_{4\times 2} \not\subseteq C_{16\times 2}\langle 1,2,3,4,5,7,8,9\rangle$ and $K_{4\times 2} \not\subseteq C_{16\times 2}\langle 6,10,11,12,13,14,15,16\rangle$ $\Rightarrow M_2(4,2) > 16$ and $m_{16}(4,2) > 2$.

3. $K_{4\times 2} \not\subseteq C_{10\times 3}\langle 1,5,6,10,11,13,15\rangle$ and $K_{4\times 2} \not\subseteq C_{10\times 3}\langle 2,3,4,7,8,9,12,14\rangle$ $\Rightarrow M_3(4,2) > 10$ and $m_{10}(4,2) > 3$.

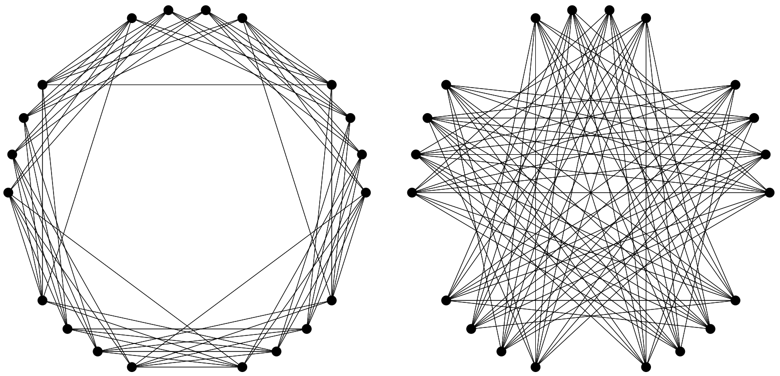
4. $K_{4\times 2} \not\subseteq C_{8\times 4}\langle 2,3,4,7,8,9,12,14\rangle$ and $K_{4\times 2} \not\subseteq C_{8\times 4}\langle 8,10,11,12,13,14,15,16\rangle \Rightarrow M_4(4,2) > 8$ and $m_8(4,2) > 4$. □



(a) $C_{29\times 1}\langle 1, 4, 5, 6, 7, 9, 13\rangle$ (b) $C_{29\times 1}\langle 2, 3, 8, 10, 11, 12, 14\rangle$ (c) $C_{11\times 2}\langle 1, 2, 3, 6, 8\rangle$

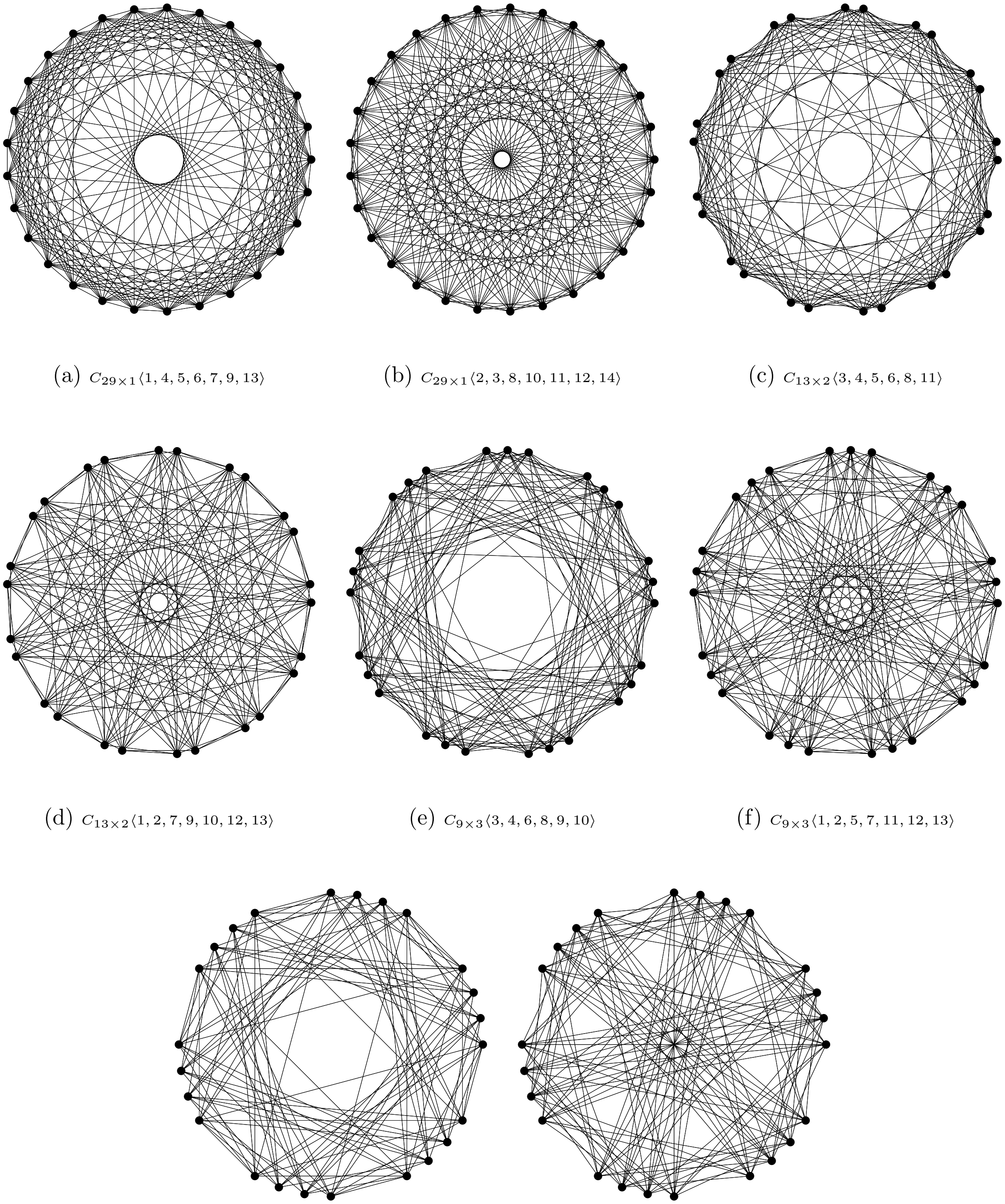


(d) $C_{11\times 2}\langle 4, 5, 7, 9, 10, 11\rangle$ (e) $C_{8\times 3}\langle 3, 4, 5, 6, 7, 12\rangle$ (f) $C_{8\times 3}\langle 1, 2, 8, 9, 10, 11\rangle$



(g) $C_{5\times 4}\langle 1, 2, 3, 4, 5\rangle$ (h) $C_{5\times 4}\langle 6, 7, 8, 9, 10\rangle$

Figure 3.5: Lower bound circular edge partitionings for the class of $(3, 2)$ multipartite Ramsey numbers of Theorem 3.4.

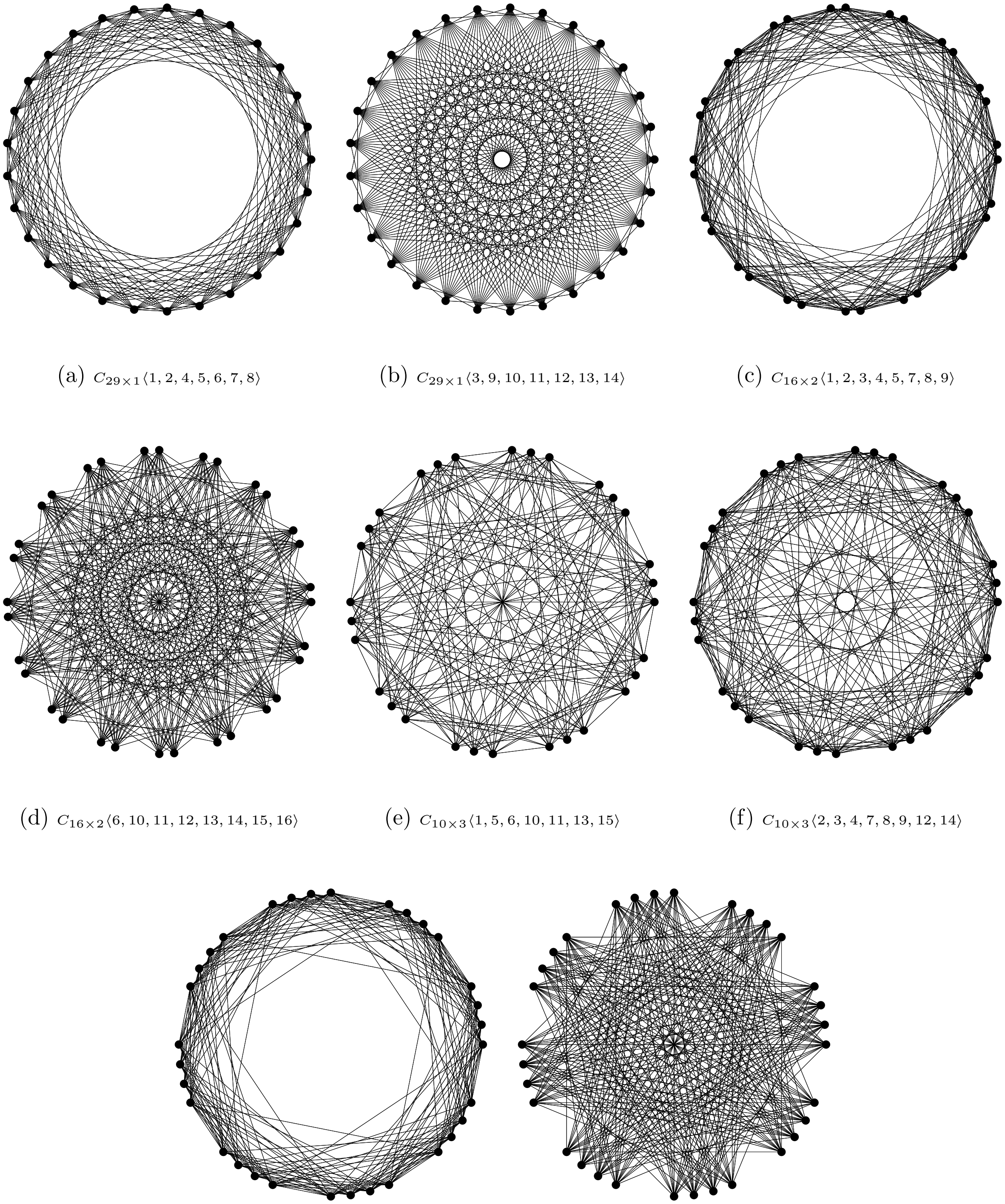


(a) $C_{29\times 1}\langle 1,4,5,6,7,9,13\rangle$ (b) $C_{29\times 1}\langle 2,3,8,10,11,12,14\rangle$ (c) $C_{13\times 2}\langle 3,4,5,6,8,11\rangle$

(d) $C_{13\times 2}\langle 1,2,7,9,10,12,13\rangle$ (e) $C_{9\times 3}\langle 3,4,6,8,9,10\rangle$ (f) $C_{9\times 3}\langle 1,2,5,7,11,12,13\rangle$

(g) $C_{6\times 4}\langle 1,2,4,7,8,9\rangle$ (h) $C_{6\times 4}\langle 3,5,6,10,11,12\rangle$

Figure 3.6: Lower bound circular edge partitionings for the class of $(2,4)$ multipartite Ramsey numbers of Theorem 3.5.



(a) $C_{29\times 1}\langle 1,2,4,5,6,7,8\rangle$

(b) $C_{29\times 1}\langle 3,9,10,11,12,13,14\rangle$

(c) $C_{16\times 2}\langle 1,2,3,4,5,7,8,9\rangle$

(d) $C_{16\times 2}\langle 6,10,11,12,13,14,15,16\rangle$

(e) $C_{10\times 3}\langle 1,5,6,10,11,13,15\rangle$

(f) $C_{10\times 3}\langle 2,3,4,7,8,9,12,14\rangle$

(g) $C_{8\times 4}\langle 1,2,3,4,5,6,7,9\rangle$

(h) $C_{8\times 4}\langle 8,10,11,12,13,14,15,16\rangle$

Figure 3.7: Lower bound circular edge partitionings for the class of $(4,2)$ multipartite Ramsey numbers of Theorem 3.6.

3.3.2 Set size lower bounds

In this section lower bounds for set size multipartite Ramsey numbers are established by performing all possible non-isomorphic circulant edge partitions of the relevant complete multipartite graph with the use of Theorem 3.1. Lower bounds for five set size multipartite Ramsey numbers are established in the next theorem.

Theorem 3.7
1. $m_3(2,3) > 8$ *and* $M_8(2,3) > 3$.
2. $m_4(2,3) > 5$ *and* $M_5(2,3) > 4$.
3. $m_2(2,4) > 17$ *and* $M_{17}(2,4) > 2$.
4. $m_3(2,4) > 12$ *and* $M_{12}(2,4) > 3$.
5. $m_4(2,4) > 8$ *and* $M_8(2,4) > 4$.

Proof. Graphical representations of the relevant circulant edge partitions are given in Figures 3.8 and 3.9.

1. $K_{2\times3} \not\subseteq C_{3\times8}\langle 1,2,6,7,9,10\rangle$ and $K_{2\times3} \not\subseteq C_{3\times8}\langle 3,4,5,8,11,12\rangle \Rightarrow M_3(2,3) > 8$ and $m_8(2,3) > 3$.

2. $K_{2\times3} \not\subseteq C_{4\times5}\langle 2,3,6,7,9\rangle$ and $K_{2\times3} \not\subseteq C_{4\times5}\langle 1,4,5,8,10\rangle \Rightarrow M_4(2,3) > 5$ and $m_5(2,3) > 4$.

3. $K_{2\times4} \not\subseteq C_{2\times17}\langle 3,6,7,8,9,10,11,14\rangle$ and $K_{2\times4} \not\subseteq C_{2\times17}\langle 1,2,4,5,12,13,15,16,17\rangle$ $\Rightarrow M_2(2,4) > 17$ and $m_{17}(2,4) > 2$.

4. $K_{2\times4} \not\subseteq C_{3\times12}\langle 1,2,3,7,9,10,11,12,15\rangle$ and $K_{2\times4} \not\subseteq C_{3\times12}\langle 4,5,6,8,13,14,16,17,18\rangle$ $\Rightarrow M_3(2,4) > 12$ and $m_{12}(2,4) > 3$.

5. $K_{2\times4} \not\subseteq C_{4\times8}\langle 1,3,5,6,9,10,12,14\rangle$ and $K_{2\times4} \not\subseteq C_{4\times8}\langle 2,4,7,8,11,13,15,16\rangle$ $\Rightarrow M_4(2,4) > 8$ and $m_8(2,4) > 4$. □

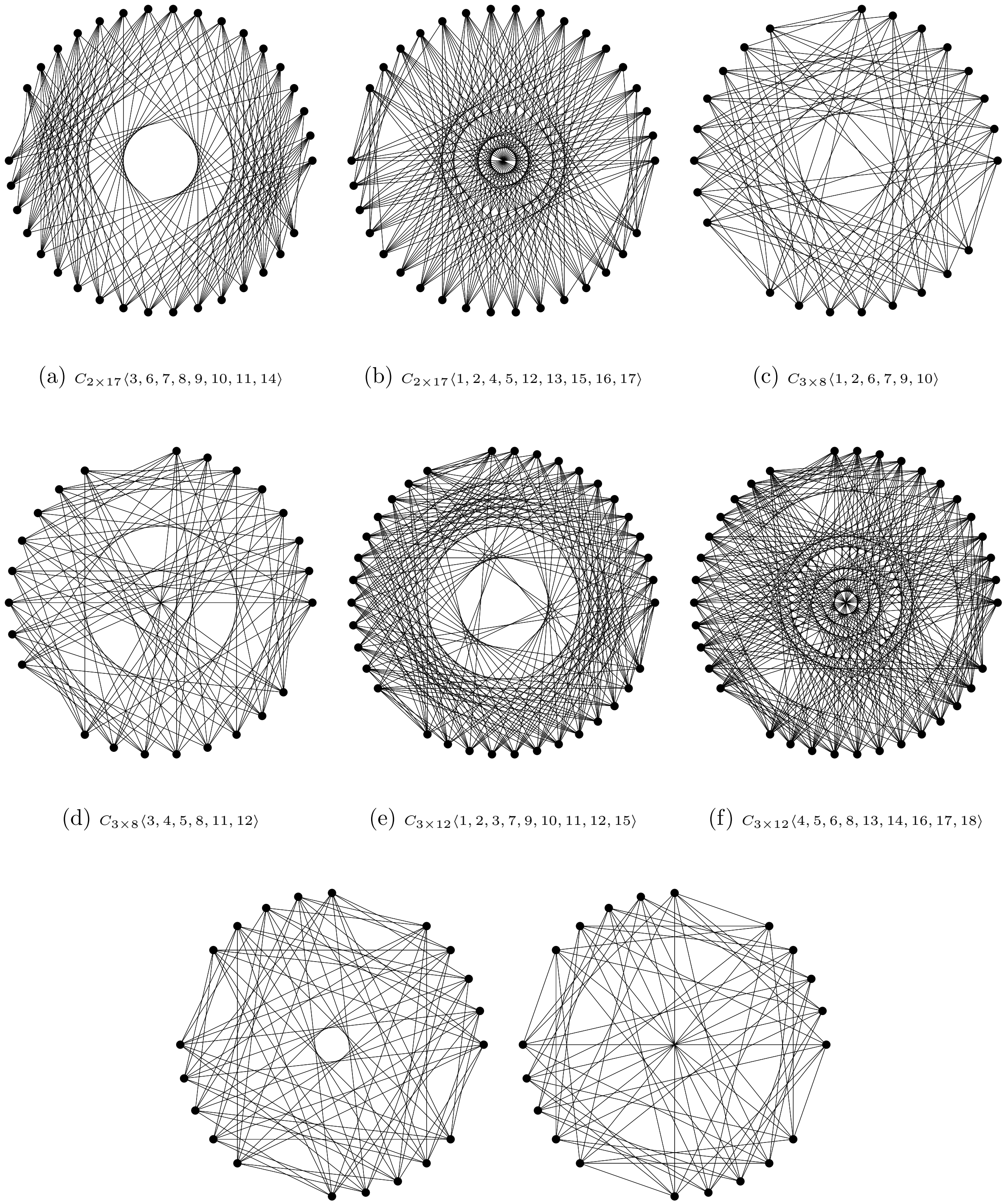
3.4 Lower bounds using random colourings

Since the time taken to search through all circulant bipartitions of the edges of $K_{k\times j}$ when the subgraph to be forced is class $(3,3)$, $(3,4)$, $(4,3)$ or $(4,4)$ becomes unrealistic, it was decided to search for a copy of $K_{n\times l}$ in random circulant bipartitions of the edges of $K_{k\times j}$ or total pseudo-random bipartitions of these edges.

3.4.1 Pseudo-random circulant edge colourings

Circulants of the form $C^{\text{red}}_{k\times j}\langle i_1,\ldots,i^*_z\rangle$ and $C^{\text{blue}}_{k\times j}\langle j_1,\ldots,j^*_z\rangle$ were used as edge colourings of $K_{k\times j}$, where z^* was taken as close as possible to $\lfloor kj/4\rfloor$, while still ensuring that the resulting circulants remain multipartite graph complements. Algorithm 3.1 was then applied to each of these edge-colourings in search of lower bounds.

Lower bounds for classes $(3,3), (3,4)$, $(4,3)$ and $(4,4)$ presented in Theorems 3.8, 3.9, 3.10 and 3.11 were found by employing random circulant edge colourings. Some of these

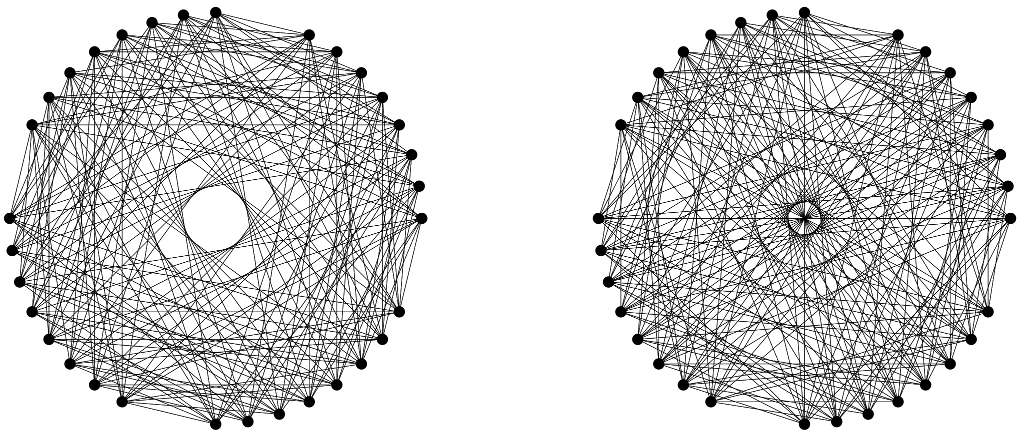


(a) $C_{2\times 17}\langle 3,6,7,8,9,10,11,14\rangle$ (b) $C_{2\times 17}\langle 1,2,4,5,12,13,15,16,17\rangle$ (c) $C_{3\times 8}\langle 1,2,6,7,9,10\rangle$

(d) $C_{3\times 8}\langle 3,4,5,8,11,12\rangle$ (e) $C_{3\times 12}\langle 1,2,3,7,9,10,11,12,15\rangle$ (f) $C_{3\times 12}\langle 4,5,6,8,13,14,16,17,18\rangle$

(g) $C_{4\times 5}\langle 2,3,6,7,9\rangle$ (h) $C_{4\times 5}\langle 1,4,5,8,10\rangle$

Figure 3.8: Lower bound circular edge partitionings for the set size multipartite Ramsey numbers of Theorem 3.7.



(a) $C_{4\times 8}\langle 1,3,5,6,9,10,12,14\rangle$ (b) $C_{4\times 8}\langle 2,4,7,8,11,13,15,16\rangle$

Figure 3.9: Lower bound circular edge partitionings for the multipartite Ramsey numbers of Theorem 3.7.

bounds will be improved by using total pseudo-random colourings, as will be discussed in the next section. Graphical representations of these circulants need to be very large so as to be discernable and are given in Appendix A, rather than in the main body of the text.

Theorem 3.8

1. $M_1(3,3) > 37$ *and* $m_{37}(3,3) > 1$.
2. $M_2(3,3) > 16$ *and* $m_{16}(3,3) > 2$.
3. $M_3(3,3) > 12$ *and* $m_{12}(3,3) > 3$.
4. $M_4(3,3) > \ 8$ *and* $m_8(3,3) \ > 4$.

Proof. Graphical representations of the relevant circulant edge colourings are given in Figures A.1, A.2, A.3 and A.4 in Appendix A.

1. $K_{3\times 3} \not\subseteq C_{37\times 1}\langle 1,2,4,5,7,10,11,12,18\rangle$ & $K_{3\times 3} \not\subseteq C_{37\times 1}\langle 3,6,8,9,13,14,15,16,17\rangle$
$\Rightarrow M_1(3,3) > 37$ and $m_{37}(3,3) > 1$.

2. $K_{3\times 3} \not\subseteq C_{16\times 2}\langle 1,2,3,4,5,6,8,9\rangle$ & $K_{3\times 3} \not\subseteq C_{16\times 2}\langle 6,11,12,13,14,15,16,17,18\rangle$
$\Rightarrow M_2(3,3) > 16$ and $m_{16}(3,3) > 2$.

3. $K_{3\times 3} \not\subseteq C_{12\times 3}\langle 1,2,3,4,5,7,8,9,10\rangle$ & $K_{3\times 3} \not\subseteq C_{12\times 3}\langle 6,11,12,13,14,15,16,17,18\rangle$
$\Rightarrow M_3(3,3) > 12$ and $m_{12}(3,3) > 3$.

4. $K_{3\times 3} \not\subseteq C_{8\times 4}\langle 1,2,3,4,5,6,7,9\rangle$ & $K_{3\times 3} \not\subseteq C_{8\times 4}\langle 8,10,11,12,13,14,15,16\rangle$
$\Rightarrow M_4(3,3) > 8$ and $m_8(3,3) > 4$. □

Theorem 3.9

1. $M_1(3,4) > 53$ *and* $m_{53}(3,4) > 1$.

2. $M_2(3,4) > 26$ *and* $m_{26}(3,4) > 2$.
3. $M_3(3,4) > 17$ *and* $m_{17}(3,4) > 3$.
4. $M_4(3,4) > 10$ *and* $m_{10}(3,4) > 4$.

Proof. Graphical representations of the relevant circulant edge colourings are given in Figures A.5, A.6, A.7 and A.8 in Appendix A.

1. $K_{3\times 4} \not\subseteq C_{53\times 1}\langle 2,4,5,6,8,13,15,16,21,22,24,25,26\rangle$ &
$K_{3\times 4} \not\subseteq C_{53\times 1}\langle 1,3,7,9,10,11,12,14,17,18,19,20,23\rangle \Rightarrow M_1(3,4) > 53$ and $m_{53}(3,4) > 1$.

2. $K_{3\times 4} \not\subseteq C_{26\times 2}\langle 1,5,6,7,9,11,13,14,16,19,21,22,25\rangle$ &
$K_{3\times 4} \not\subseteq C_{26\times 2}\langle 2,3,4,8,10,12,15,17,18,20,23,24,26\rangle \Rightarrow M_2(3,4) > 18$ and $m_{18}(3,4) > 2$.

3. $K_{3\times 4} \not\subseteq C_{17\times 3}\langle 2,6,8,9,10,12,13,16,19,21,24,25\rangle$ &
$K_{3\times 4} \not\subseteq C_{17\times 3}\langle 1,3,4,5,7,11,14,15,17,18,20,22,23\rangle \Rightarrow M_3(3,4) > 17$ and $m_{17}(3,4) > 3$.

4. $K_{3\times 4} \not\subseteq C_{10\times 4}\langle 2,3,5,6,7,8,11,12,14,15\rangle$ &
$K_{3\times 4} \not\subseteq C_{10\times 4}\langle 1,4,9,10,13,16,17,18,19,20\rangle \Rightarrow M_4(3,4) > 10$ and $m_{10}(3,4) > 4$. □

Theorem 3.10
1. $M_1(4,3) > 37$ *and* $m_{37}(4,3) > 1$.
2. $M_2(4,3) > 20$ *and* $m_{20}(4,3) > 2$.
3. $M_3(4,3) > 14$ *and* $m_{14}(4,3) > 3$.
4. $M_4(4,3) > 11$ *and* $m_{11}(4,3) > 4$.

Proof. Graphical representations of the relevant circulant edge colourings are given in Figures A.9, A.10, A.11 and A.12 in Appendix A.

1. $K_{4\times 3} \not\subseteq C_{37\times 1}\langle 1,3,4,5,7,10,11,12,18\rangle$ &
$K_{4\times 3} \not\subseteq C_{37\times 1}\langle 2,6,8,9,13,14,15,16,17\rangle \Rightarrow M_1(4,3) > 37$ and $m_{37}(4,3) > 1$.

2. $K_{4\times 3} \not\subseteq C_{20\times 2}\langle 1,3,4,5,6,8,9,11,12,19\rangle$ &
$K_{4\times 3} \not\subseteq C_{20\times 2}\langle 2,7,10,13,14,15,16,17,18,20\rangle \Rightarrow M_2(4,3) > 20$ and $m_{20}(4,3) > 2$.

3. $K_{4\times 3} \not\subseteq C_{14\times 3}\langle 3,5,6,8,9,10,11,14,15,18\rangle$ &
$K_{4\times 3} \not\subseteq C_{14\times 3}\langle 1,2,4,7,12,13,16,17,19,20,21\rangle \Rightarrow M_3(4,3) > 14$ and $m_{14}(4,3) > 3$.

4. $K_{4\times 3} \not\subseteq C_{11\times 4}\langle 1,3,4,7,8,10,11,13,15,16,17\rangle$ &
$K_{4\times 3} \not\subseteq C_{11\times 4}\langle 2,5,6,9,12,14,18,19,20,21,22\rangle \Rightarrow M_4(4,3) > 11$ and $m_{11}(4,3) > 4$. □

Theorem 3.11
1. $M_1(4,4) > 47$ *and* $m_{47}(4,4) > 1$.
2. $M_2(4,4) > 34$ *and* $m_{34}(4,4) > 2$.
3. $M_3(4,4) > 17$ *and* $m_{17}(4,4) > 3$.
4. $M_4(4,4) > 13$ *and* $m_{13}(4,4) > 4$.

Proof. Graphical representations of the relevant circulant edge colourings are given in Figures A.13, A.14, A.15 and A.16 in Appendix A.

1. $K_{4\times4} \not\subseteq C_{47\times1}\langle 2,3,4,5,6,8,9,10,11,19,21\rangle$ &
$K_{4\times4} \not\subseteq C_{47\times1}\langle 1,7,12,13,14,15,16,17,18,20,22,23\rangle \Rightarrow M_1(4,4) > 47$
and $m_{47}(4,4) > 1$.

2. $K_{4\times4} \not\subseteq C_{34\times2}\langle 1,2,4,5,6,7,8,13,14,15,17,19,25,28,29,31,33\rangle$ &
$K_{4\times4} \not\subseteq C_{34\times2}\langle 3,9,10,11,12,16,18,20,21,22,23,24,26,27,30,32,34\rangle \Rightarrow M_2(4,4) > 34$
and $m_{34}(4,4) > 2$.

3. $K_{4\times4} \not\subseteq C_{17\times3}\langle 2,3,4,5,6,7,9,12,14,17,21,22\rangle$ &
$K_{4\times4} \not\subseteq C_{17\times3}\langle 1,8,10,11,13,15,16,18,19,20,23,24,25\rangle \Rightarrow M_3(4,4) > 17$
and $m_{17}(4,4) > 3$.

4. $K_{4\times4} \not\subseteq C_{13\times4}\langle 2,3,6,7,8,9,10,11,12,13,19,24,25\rangle$ &
$K_{4\times4} \not\subseteq C_{13\times4}\langle 1,4,5,14,15,16,17,18,20,21,22,23,26\rangle \Rightarrow M_4(4,4) > 13$
and $m_{13}(4,4) > 4$. □

3.4.2 Total Pseudo-random colourings

To establish with total certainty that $M_j(n,l) \leq k$, all bicolourings of $K_{k\times j}$ must be considered. On the other hand, establishing lower bounds for the multipartite Ramsey numbers, $M_j(n,l)$ and $m_k(n,l)$ requires finding only one bicolouring of the edges of the edges of $K_{k\times j}$ that does not contain a monochromatic $K_{n\times l}$. Since each of the $q = nl(n-1)l/2$ edges in $K_{k\times j}$ can either be red or blue, there are 2^q possible bicolourings of $K_{k\times j}$.

Algorithm 3.5 performs the initial pseudo-random bicolouring of $K_{k\times j}$ in such a way that the red and blue degrees of every vertex differ by at most one. This hopefully maximizes the probability of not finding a copy of $K_{n\times l}$ in any of the bicolour induced subgraphs, G_{red} and G_{blue}. Furthermore, if a copy of $K_{n\times l}$ is found in G_{red} or G_{blue} then a red edge is randomly selected from the particular red $K_{n\times l}$ say G_{red} found, and swapped with a randomly selected edge from G_{blue}. A monochromatic $K_{n\times l}$ is only sought within $K_{k\times j}$ if it is in fact possible for $K_{n\times l}$ to be a subgraph of $K_{k\times j}$. Whether this is possible is settled by Theorem 2.2. A $kj \times kj$ matrix representation A of the bicoloured graph $K_{k\times j}$ is stored in a static data structure. Again $A[i,j] = 0$ if vertices i and j are not connected by an edge. $A[i,j] = 1$ if vertices i and j are connected by means of a red edge and $A[i,j] = 2$ if vertices i and j are connected by a blue edge.

Algorithm 3.5 *Initial pseudo-random colouring.*

Input: An empty matrix A.
Output: A $kj \times kj$ matrix A with pseudo-random entries $A[i,j]$.

The variable `colour` is assigned a value of 1 or 2. $A[i,j] := 3$ for all vertices i and j that are connected and $A[i,j] := 0$ for all vertices i and j that are not connected. The following is done for every row r of the matrix A. The list S of column numbers, c with $A[r,c] = 3$ is formed. A number i in the range $[1,|S|]$ is randomly chosen. $A[r,S[i]] :=$ COLOUR. The variable COLOUR is assigned the other value. Element i is removed from

k	q	c	f	k	q	c	f
19	171	1.8×10^{050}	1.4×10^{041}	30	435	3.4×10^{129}	7.2×10^{111}
20	190	9.1×10^{055}	1.3×10^{046}	31	465	3.5×10^{138}	1.1×10^{120}
21	210	9.0×10^{061}	2.4×10^{051}	32	496	7.3×10^{147}	3.4×10^{128}
22	231	1.8×10^{068}	8.3×10^{056}	33	528	3.0×10^{157}	2.0×10^{137}
23	253	7.2×10^{074}	5.9×10^{062}	34	561	2.5×10^{167}	2.4×10^{146}
24	276	5.8×10^{081}	7.9×10^{068}	35	595	4.2×10^{177}	5.5×10^{155}
25	300	9.4×10^{088}	2.1×10^{075}	36	630	1.4×10^{188}	2.5×10^{165}
26	325	3.0×10^{096}	1.1×10^{082}	37	666	9.5×10^{198}	2.3×10^{175}
27	351	1.9×10^{104}	1.2×10^{089}	38	703	1.3×10^{210}	4.0×10^{185}
28	378	2.5×10^{112}	2.3×10^{096}	39	741	3.4×10^{221}	1.4×10^{196}
29	406	6.5×10^{120}	9.3×10^{103}	40	780	1.8×10^{233}	9.8×10^{206}

Table 3.4: A comparison between the number of possible bicolourings of different order complete graphs, K_k. The first column is the order of K_k and the second column is the size of K_k. The third and forth columns represent the total number of different bicolourings of K_k in which half the total number of edges are monochromatic and half the total number of edges incident to each vertex are monochromatic, respectively.

the list S and the size of S is decremented. This process is continued until the set S is empty. □

The bicolour induced subgraphs G_{red} and G_{blue} are both stored as lists of vertices together with their neighbourhood sets. The same holds for G_{blue}. Algorithm 3.1 is then used to determine whether $K_{n\times l}$ is a subgraph of G_{red} or G_{blue} for a specific random colouring. If $K_{n\times l}$ is not a subgraph of G_{red} or G_{blue} then $M_k(n,l) > j$ and $m_j(n,l) > k$. We would like to have an idea of what fraction of the total number of bicolourings can be done in a realistic time span. The number of different ways to bicolour the edges of $K_{k\times j}$ so that the total number of red and blue edges in $E(K_{k\times j})$ differ by at most one is $c = \binom{q}{\lfloor q/2 \rfloor}$, where $q = nl(n-1)l/2$ is the number of edges of $K_{k\times j}$. The number of different bicolourings of $K_{k\times j}$ when specifying that the red and blue degree of *every vertex* can differ by at most one is given by

$$f = \prod_{a=2}^{k} \prod_{b=1}^{j} \binom{(a-1)j}{\lfloor (a-1)j/2 \rfloor}.$$

Table 3.4.2 contains a few instances of k, q, c and f. The results presented in Theorem 3.12 were obtained after performing a maximum of 10000 pseudo-random colourings. Only colourings that improved on the lower bounds generated by considering all circulant edge bicolourings or random circulant edge bicolourings are given.

Theorem 3.12
1. $M_1(4,3) > 41$ *and* $m_{41}(4,3) > 1$.
2. $M_1(4,4) > 72$ *and* $m_{72}(4,4) > 1$.
3. $M_2(4,4) > 38$ *and* $m_{38}(4,4) > 2$.
4. $M_3(4,4) > 25$ *and* $m_{25}(4,4) > 3$.
5. $M_4(4,4) > 19$ *and* $m_{19}(4,4) > 3$.

Proof. The adjacency matrices of the appropriate pseudo-random colourings are given in Appendix B, because the graphs are too large to present graphically.

Chapter 4

Upper bounds

> "The Pigeon-Hole principle: If m pigeons roost in n holes and $m > n$ then at least two pigeons must share a hole" [58]

In this chapter upper bounds for the classes $M_j(2,l)$, $m_j(2,l)$ and general recursive upper bounds for $M_j(n,l)$ will be established. The proofs utilize general edge colourings, the Pigeon-Hole principle and some combinatorial and recursive arguments.

4.1 Bounds for the class of $(2,l)$-Ramsey numbers

A general form of the Pigeon-Hole principle is central to establishing the upper bounds in this section. The principle states that when $ab+1$ objects are assigned to b pigeon-holes, there will be at least one pigeon-hole with $a+1$ objects assigned to it, and can be proved by a simple contradiction argument. The concept of a red- and blue connection will be necessary in the proof of Theorem 4.1. Let G be a graph and $S,T \subseteq V(G)$ with the property that there is an edge st between $s \in S$ and $t \in T$ for all $s \in S$ and $t \in T$. A set of l vertices in T that are all connected to a vertex in S by means of blue edges will be referred to as a *blue l-connection.* A *red l-connection* is defined similarly.

Theorem 4.1
1.

$$M_j(2,l) \leq \left\lceil \frac{2l-1}{j} \right\rceil + \left\lceil \frac{2(l-1)\binom{2l-1}{l}+1}{j} \right\rceil \text{ for all } j,l \geq 1.$$

2.

$$m_k(2,l) \leq \max\left\{ 2l-1, \left\lceil \frac{2(l-1)\binom{2l-1}{l}+1}{k-1} \right\rceil \right\} \text{ for all } k \geq 2 \text{ and } l \geq 1.$$

Proof. 1. Let

$$c = \left\lceil \frac{2l-1}{j} \right\rceil + \left\lceil \frac{2(l-1)\binom{2l-1}{l}+1}{j} \right\rceil .$$

Consider the graph $K_{c\times j}$ and let (S,T) be a bipartition of the partite sets of $K_{c\times j}$ with S consisting of p_S partite sets and T consisting of p_T partite sets, where

$$p_S = \left\lceil \frac{2(l-1)\binom{2l-1}{l}+1}{j} \right\rceil \text{ and } p_T = \left\lceil \frac{2l-1}{j} \right\rceil .$$

Hence there are at least $2(l-1)\binom{2l-1}{l}+1$ vertices in S and at least $2l-1$ vertices in T. Let U be any $2(l-1)\binom{2l-1}{l}+1$ vertices in S and W be any $2l-1$ vertices in T. Since any two vertices $u \in U$ and $w \in W$ are from different partite sets, there is an edge joining them in $K_{c\times j}$. Consider only the edges between vertices in U and vertices in W. Then the degree of every vertex in U is at least $2l-1$ and either the red or blue degree of every vertex in U is greater than or equal to l. Hence any vertex in U is connected to a set of l vertices in W by means of monochromatic edges. There are $\binom{2l-1}{l}$ sets of l vertices in W.

By the Pigeon-Hole principle, when $2(l-1)\binom{2l-1}{l}+1$ l-connections are assigned to $\binom{2l-1}{l}$ sets, there will be at least one set with $2(l-1)+1$ l-connections assigned to it. Since these $2(l-1)+1$ l-connections are either red or blue, l of them must be the same colour. Thus there is at least one set of l vertices in U with l monochromatic l-connections assigned to it, representing a monochromatic $K_{2\times l}$.

2. We consider the graph $K_{k\times c}$ where

$$c = \max\left\{ 2l-1, \left\lceil \frac{2(l-1)\binom{2l-1}{l}+1}{k-1} \right\rceil \right\}.$$

Let (S,T) be a bipartition of the partite sets of $K_{k\times c}$ where S consists of $k-1$ partite sets and T is a single partite set of $K_{k\times c}$. Since

$$c \geq \left\lceil \frac{2(l-1)\binom{2l-1}{l}+1}{k-1} \right\rceil \quad \text{and} \quad c \geq 2l-1,$$

S contains at least $2(l-1)\binom{2l-1}{l}+1$ vertices and T contains at least $2l-1$ vertices. Let U' be any $2(l-1)\binom{2l-1}{l}+1$ vertices in S and W' be any $2l-1$ vertices in T.

The rest of the proof is similar to that of part 1, where the argument is repeated with U' instead of U and W' in stead of W. □

The next chapter concludes this thesis by summarizing the methods used and results obtained.

Chapter 5

Conclusion

In Chapter 1 a brief history of Ramsey numbers was given, followed by some basic graph theoretic definitions and a concise survey of literature of Ramsey theory within the sphere of graph theory. The notion of a classical Ramsey number was generalized by searching in a complete, balanced, multipartite graph for a complete, balanced, multipartite graph. The generalization was made by either fixing the cardinality of the partite sets of the larger graph, or fixing the number of partite sets in the larger graph. The existence of these generalized Ramsey numbers, together with some basic properties, was established. The class of $(2, 2)$ set size Ramsey numbers were completely established prior to this study and hence the class of $(2, 2)$ set count Ramsey numbers could be derived in a trivial manner in Proposition 2.7.

A special class of graphs called circulants were introduced in Chapter 3. A new algorithm for determining whether a multipartite graph $K_{n\times l}$ is a subgraph of a general graph, G, was presented and its validity was established. Lower bounds for multipartite Ramsey numbers were established by employing computer searches using this algorithm to produce bicolourings of the larger multipartite graph which do not contain the desired, smaller, multipartite graphs. A multipartite graph was bi-coloured by partitioning it into either two circulants of roughly the same size, or two pseudo-randomly generated graphs of roughly the same size. All possible non-isomorphic circulants for classes $(2, 2)$, $(2, 3)$, $(3, 2)$, $(2, 4)$ and $(4, 2)$ are searched for a copy of $K_{n\times l}$. This was done by generating all possible variable instances $i_1, \ldots, i_h$ in a particular order and only searching the circulant corresponding to a newly generated variable instance if no circulant known to be isomorphic to it according to Theorem 3.1 has already been searched. All searches were done on a pentium III 500MHz Unix workstation. Searching times varied from two minutes for class-$(2, 3)$ Ramsey numbers to 5 days for class-$(4, 4)$ Ramsey numbers.

Upper bounds were established in Chapter 4 by performing general edge colourings, using the Pigeon-Hole principle and by employing some combinatorial and recursive techniques. Our best known lower and upper bounds for relatively small set count and set size multipartite Ramsey numbers are given in Table 5.1 and Table 5.2.

$M_j(n,l)$		$l=1$	$l=2$	$l=3$	$l=4$
$j=1$	$n=1$	**1**[1]	**2**[1]	**3**[1]	**4**[1]
	$n=2$	**2**[2]	**6**[3]	**18**[4]	(30[5]:218[a])
	$n=3$	**6**[6]	(30[7]:165[b])	(38[8]:6588[b])	(54[9]:*[d])
	$n=4$	**18**[10]	(30[11]:1870[b])	(42[12]:*[d])	(73[13]:*[d])
$j=2$	$n=1$	**1**[1]	**1**[1]	**2**[1]	**2**[1]
	$n=2$	**2**[14]	**4**[3]	(8[15]:18[c])	(14[16]:110[a])
	$n=3$	**6**[14]	(15[17]:165[c])	(19[17]:6588[c])	(27[18]:*[c])
	$n=4$	**18**[14]	(18[19]:1870[c])	(22[17]:*[c])	(39[20]:*[c])
$j=3$	$n=1$	**1**[1]	**1**[1]	**1**[1]	**2**[1]
	$n=2$	**2**[14]	**3**[3]	(6[21]:16[a])	(10[22]:74[a])
	$n=3$	**6**[14]	(7[17]:165[c])	(13[17]:6588[c])	(18[23]:*[c])
	$n=4$	**18**[14]	(18[19]:1870[c])	(18[19]:*[c])	(26[24]:*[c])
$j=4$	$n=1$	**1**[1]	**1**[1]	**1**[1]	**1**[1]
	$n=2$	**2**[14]	**3**[3]	(5[25]:13[a])	(8[17]:55[a])
	$n=3$	**6**[14]	(6[17]:165[c])	(9[17]:6588[c])	(14[17]:*[c])
	$n=4$	**18**[14]	(18[19]:1870[c])	(18[19]:*[c])	(20[26]:*[c])

Table 5.1: Bounds for symmetric set count multipartite Ramsey numbers $M_j(n,l)$ for $n,l,j = 1,2,3,4$. The known Ramsey numbers are typeset in bold, while bounds are given in the format (lower bound: upper bound) in cases where Ramsey numbers are not known. Lower bounds are motivated as follows: [1]By Proposition 2.4(1). [2]Classical Ramsey number (see Chartrand & Oellermann, [19]). [3]Chvátal & Harary, [24] established $r(C_4, C_4) = 6$. [4]By Harborth and Mengersen [68]. [5]Circulant edge colouring, Theorem 3.5(1). [6]Classical Ramsey number due to Greenwood & Gleason, [60]. [7]Circulant edge colouring, Theorem 3.4(1). [8]Circulant edge colouring, Theorem 3.8(1). [9]Circulant edge colouring, Theorem 3.9(1). [10]Classical Ramsey number due to Greenwood & Gleason, [60]. [11]Circulant edge colouring, Theorem 3.6(1). [12]Pseudo-random colouring, Theorem 3.12(1). [13]Pseudo-random colouring, Theorem 3.12(2). [14]By Proposition 2.4(2). [15]Circulant edge colouring, Theorem 3.3(2). [16]Circulant edge colouring, Theorem 3.5(2). [17]By Proposition 2.4. [18]Circulant edge colouring, Theorem 3.9(2). [19]Using Proposition 2.5. [20]Pseudo-random colouring, Theorem 3.12(3). [21]Circulant edge colouring, Theorem 3.3(3). [22]Circulant edge colouring, Theorem 3.5(3). [23]Circulant edge colouring, Theorem 3.9(3). [24]Pseudo-random colouring, Theorem 3.12(4). [25]Circulant edge colouring, Theorem 3.3(4). [26]Pseudo-random colouring, Theorem 3.12(5). The upper bounds are motivated as follows: [a]By Theorem 4.1(1). [b]By Mackey [96] [c]By Proposition 2.6(3).
[d]Best known bound is $r(nl, nl) \leq \binom{2nl-2}{nl-1}$ from the existence theorem for classical Ramsey numbers by Erdös and Szekeres [37].

$m_k(n,l)$		$l=1$	$l=2$	$l=3$	$l=4$
$k=1$	$n=1$	$\mathbf{1}^1$	$\mathbf{2}^1$	$\mathbf{3}^1$	$\mathbf{4}^1$
	$n=2$	∞^2	∞^2	∞^2	∞^2
	$n=3$	∞^2	∞^2	∞^2	∞^2
	$n=4$	∞^2	∞^2	∞^2	∞^2
$k=2$	$n=1$	$\mathbf{1}^1$	$\mathbf{1}^1$	$\mathbf{2}^1$	$\mathbf{2}^1$
	$n=2$	$\mathbf{1}^3$	$\mathbf{5}^4$	$\mathbf{17}^4$	$(18^5{:}70^a)$
	$n=3$	∞^2	∞^2	∞^2	∞^2
	$n=4$	∞^2	∞^2	∞^2	∞^2
$k=3$	$n=1$	$\mathbf{1}^1$	$\mathbf{1}^1$	$\mathbf{1}^1$	$\mathbf{2}^1$
	$n=2$	$\mathbf{1}^3$	$\mathbf{3}^6$	$(9^7{:}17^b)$	$(13^8{:}70^b)$
	$n=3$	∞^2	∞^2	∞^2	∞^2
	$n=4$	∞^2	∞^2	∞^2	∞^2
$k=4$	$n=1$	$\mathbf{1}^1$	$\mathbf{1}^1$	$\mathbf{1}^1$	$\mathbf{1}^1$
	$n=2$	$\mathbf{1}^3$	$\mathbf{2}^6$	$(6^9{:}14^c)$	$(9^{10}{:}70^b)$
	$n=3$	∞^2	∞^2	∞^2	∞^2
	$n=4$	∞^2	∞^2	∞^2	∞^2

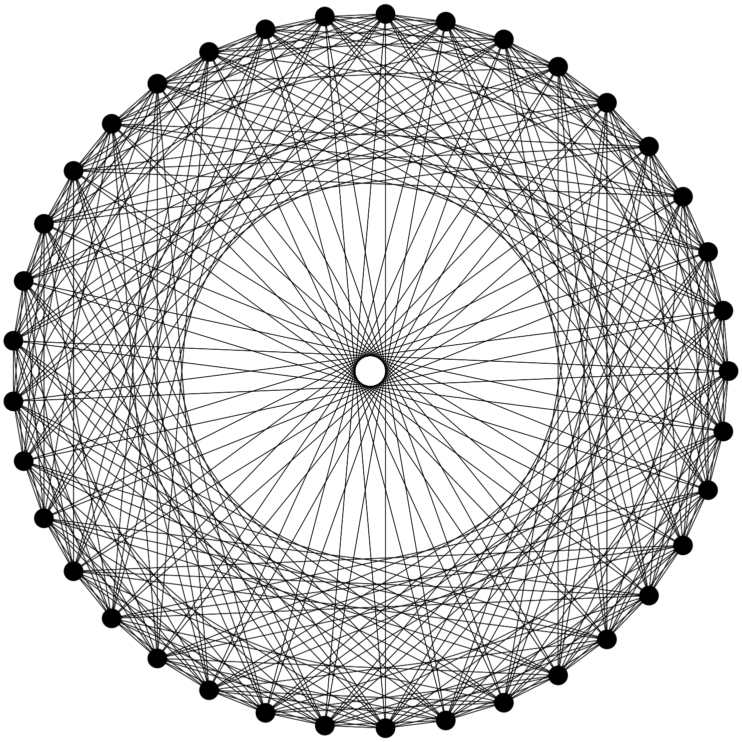
Table 5.2: Bounds for symmetric set size multipartite Ramsey numbers $m_k(n,l)$ for $k,n,l = 1,2,3,4$. The known Ramsey numbers and lower bounds are motivated as follows: [1]By Proposition 2.4(1). [2]By Proposition 2.4(4). [3]By Proposition 2.4(3). [4]By a theorem of Beineke & Schwenk, [2]. [5]By Theorem 3.7(1). [6]Due to Day *et al.*, [32]. [7]By Theorem 3.7(2). [8]By Theorem 3.7(3). [9]By Theorem 3.7(4). [10]By Theorem 3.7(5). The upper bounds are motivated as follows: [a]By Theorem 2.5. [b]By Proposition 2.6(4). [c]By Theorem 4.1(2).

Appendix A

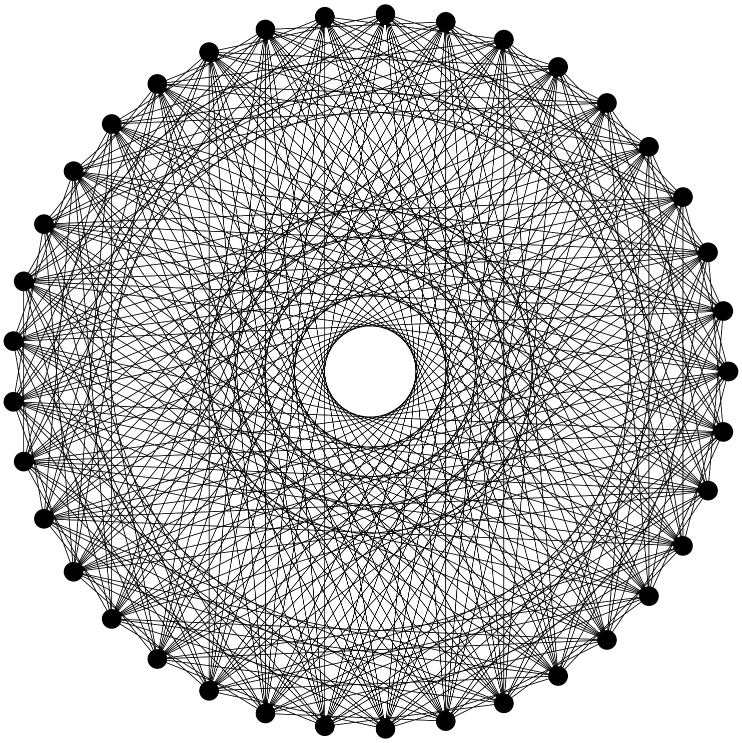
Large circulant graphs

Lower bounds for the classes of $(3,3)$, $(3,4)$, $(4,3)$ and $(4,4)$ Ramsey numbers are given by graphically presenting the relevant circulant edge partitions.

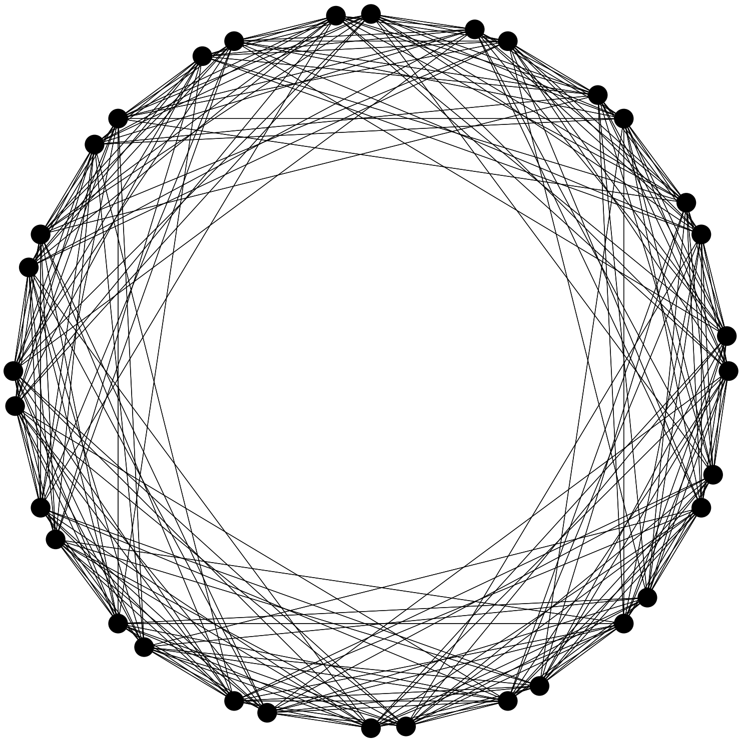
Result	Graphical Representation
Theorem 3.8(1)	Figure A.1(a)&(b)
Theorem 3.8(2)	Figure A.2(a)&(b)
Theorem 3.8(3)	Figure A.3(a)&(b)
Theorem 3.8(4)	Figure A.4(a)&(b)
Theorem 3.9(1)	Figure A.5(a)&(b)
Theorem 3.9(2)	Figure A.6(a)&(b)
Theorem 3.9(3)	Figure A.7(a)&(b)
Theorem 3.9(4)	Figure A.8(a)&(b)
Theorem 3.10(1)	Figure A.9(a)&(b)
Theorem 3.10(2)	Figure A.10(a)&(b)
Theorem 3.10(3)	Figure A.11(a)&(b)
Theorem 3.10(4)	Figure A.12(a)&(b)
Theorem 3.11(1)	Figure A.13(a)&(b)
Theorem 3.11(2)	Figure A.14(a)&(b)
Theorem 3.11(3)	Figure A.15(a)&(b)
Theorem 3.11(4)	Figure A.16(a)&(b)



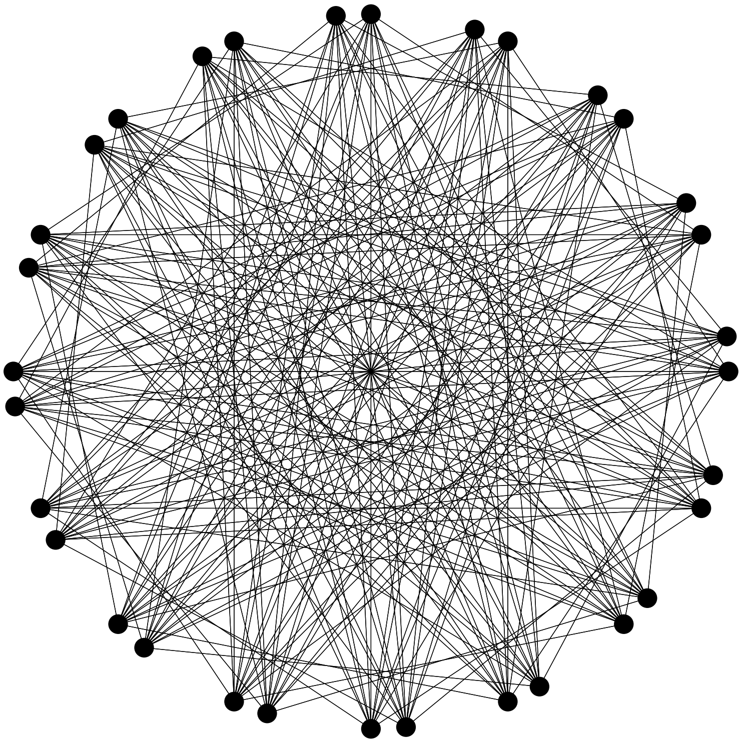
(a) $K_{3\times3} \not\subseteq C_{37\times1}\langle 1,2,4,5,7,10,11,12,18\rangle$



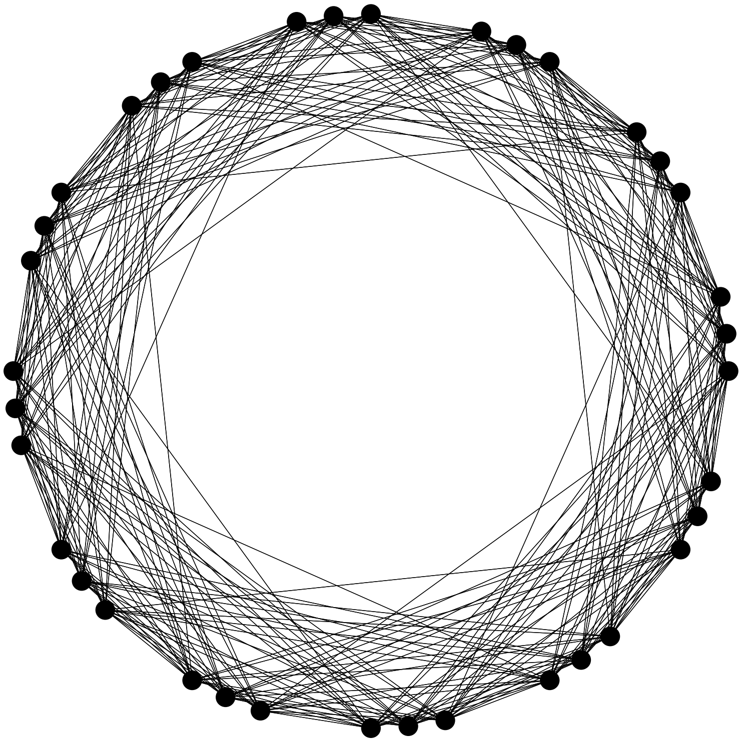
(b) $K_{3\times3} \not\subseteq C_{37\times1}\langle 3,6,8,9,13,14,15,16,17\rangle$



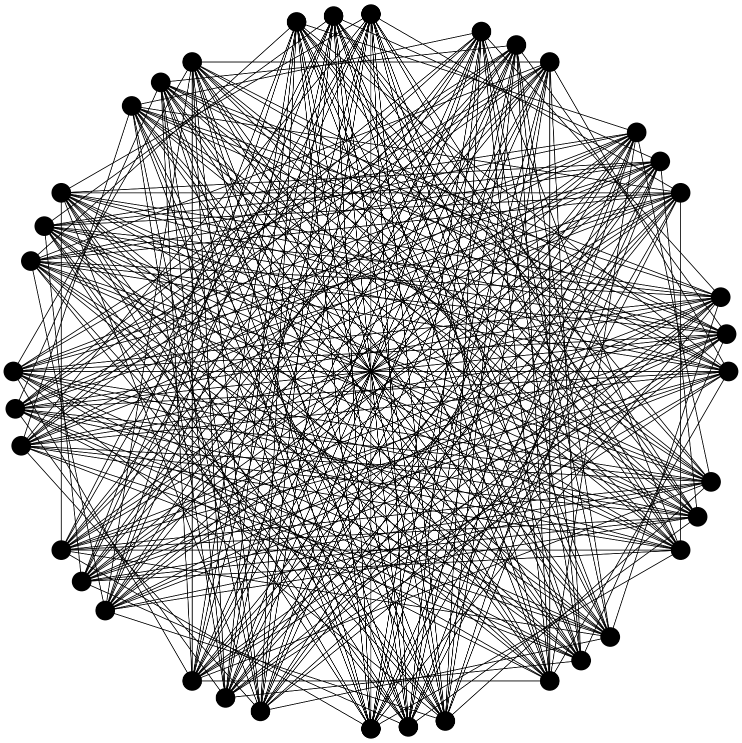
(a) $K_{3\times 3} \not\subseteq C_{16\times 2}\langle 1,2,3,4,5,6,8,9\rangle$



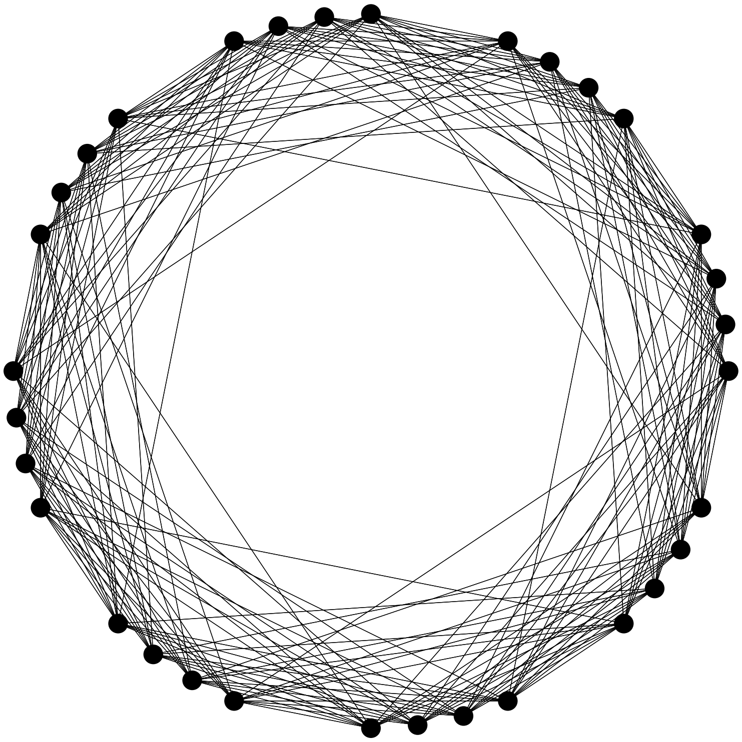
(b) $K_{3\times 3} \not\subseteq C_{16\times 2}\langle 6,11,12,13,14,15,16,17,18\rangle$



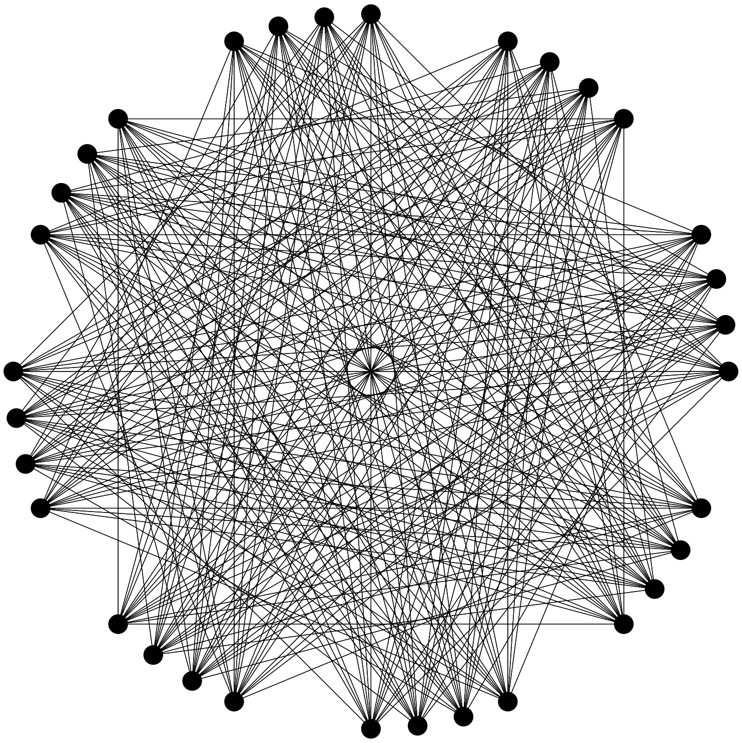
(a) $K_{3\times 3} \not\subseteq C_{12\times 3}\langle 1, 2, 3, 4, 5, 7, 8, 9, 10\rangle$



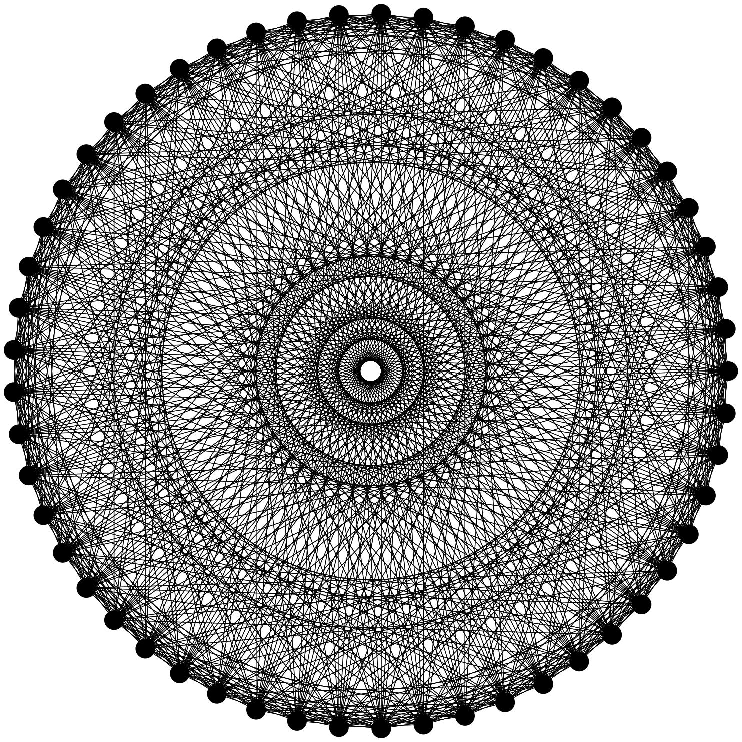
(b) $K_{3\times 3} \not\subseteq C_{12\times 3}\langle 6, 11, 12, 13, 14, 15, 16, 17, 18\rangle$



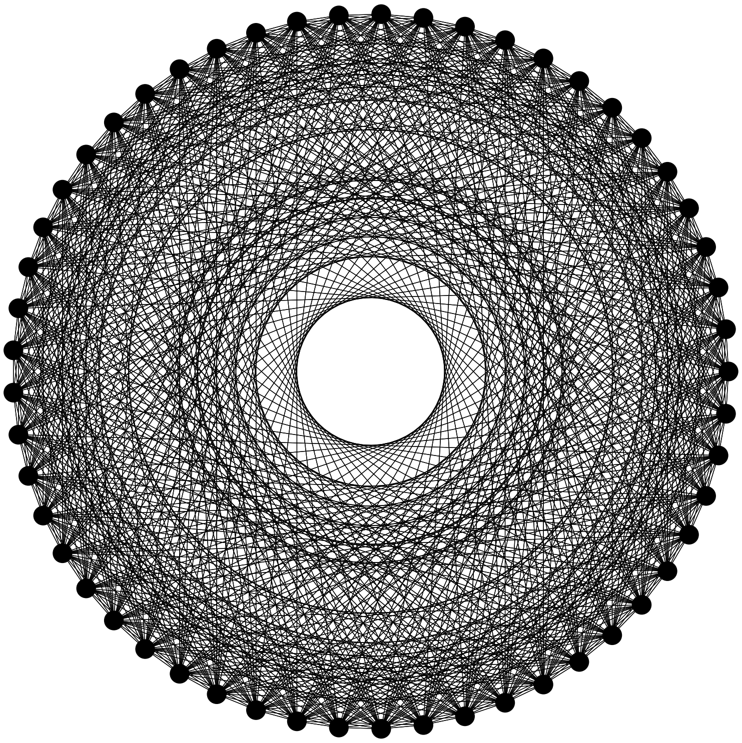
(a) $K_{3\times 3} \not\subseteq C_{8\times 4}\langle 1,2,3,4,5,6,7,9\rangle$



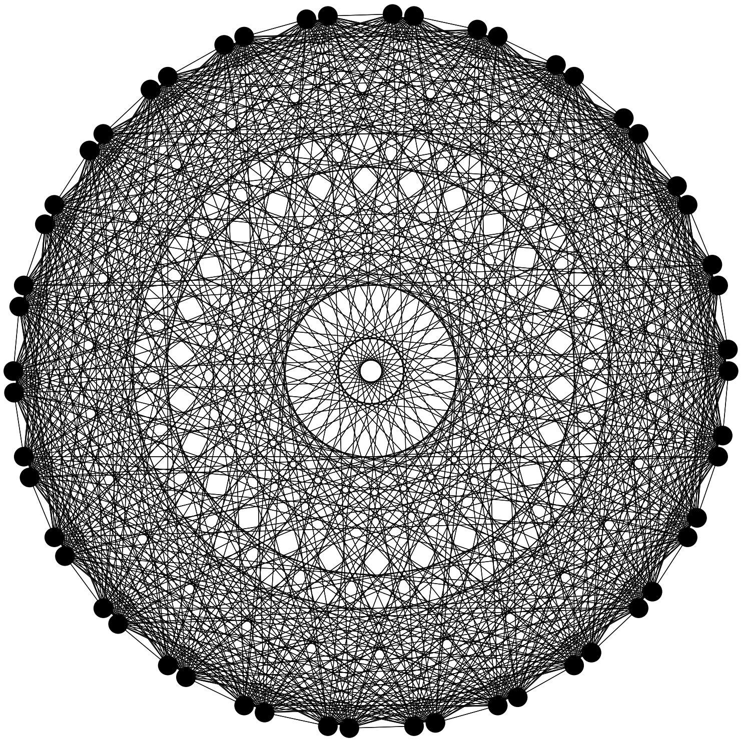
(b) $K_{3\times 3} \not\subseteq C_{8\times 4}\langle 8,10,11,12,13,14,15,16\rangle$



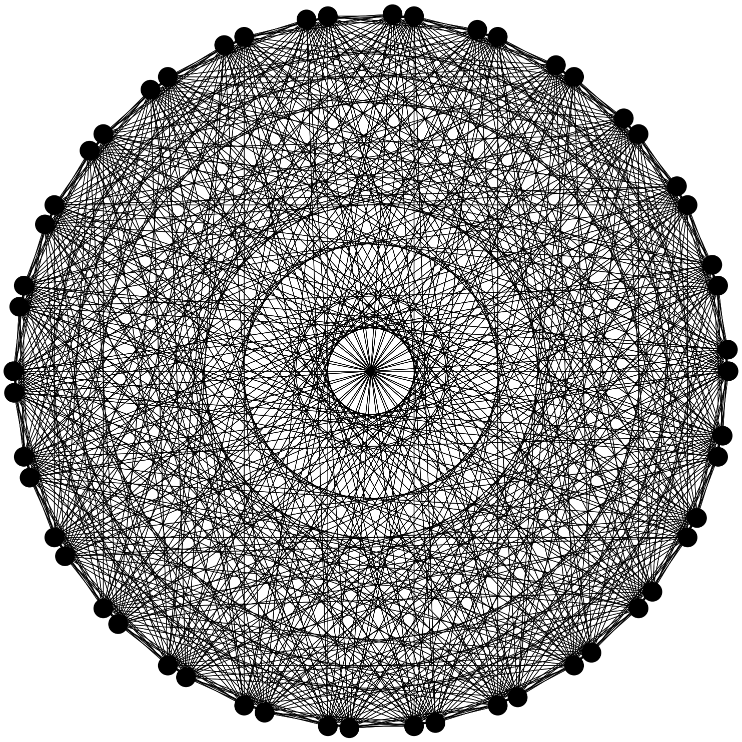
(a) $K_{3\times 4} \not\subseteq C_{53\times 1}\langle 2,4,5,6,8,13,15,16,21,22,24,25,26\rangle$



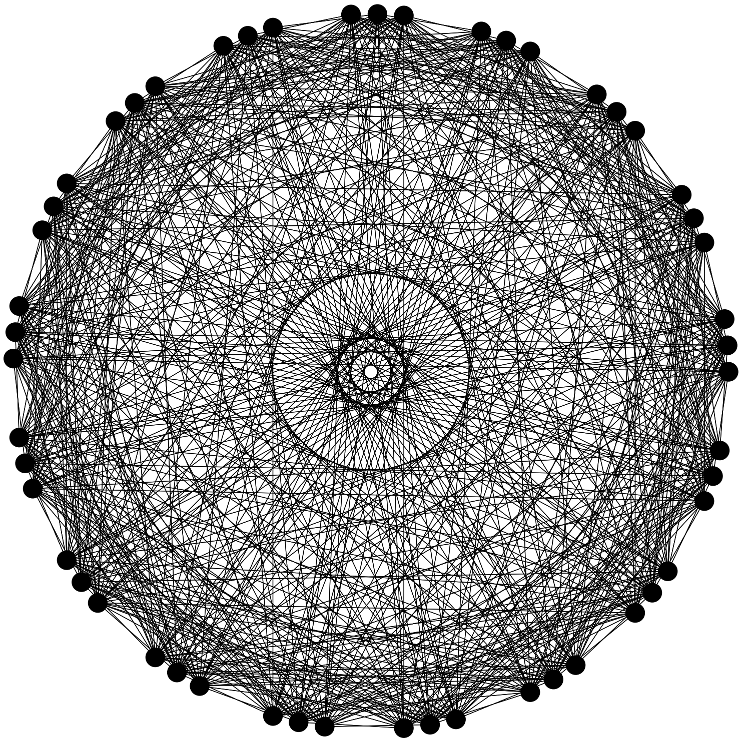
(b) $K_{3\times 4} \not\subseteq C_{53\times 1}\langle 1,3,7,9,10,11,12,14,17,18,19,20,21,23,27\rangle$



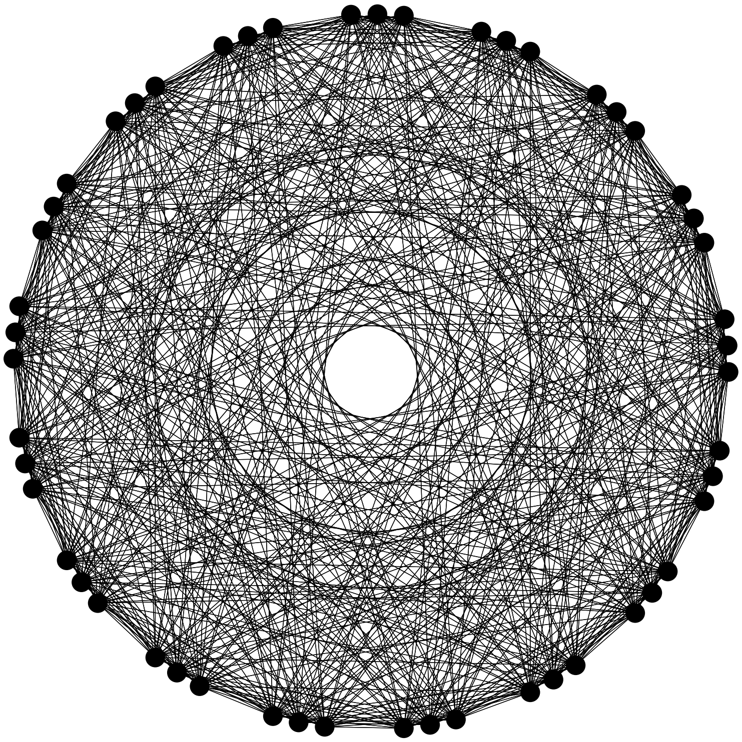
(a) $K_{3\times4} \not\subseteq C_{26\times2}\langle 1,5,6,7,9,11,13,14,16,19,21,22,25\rangle$



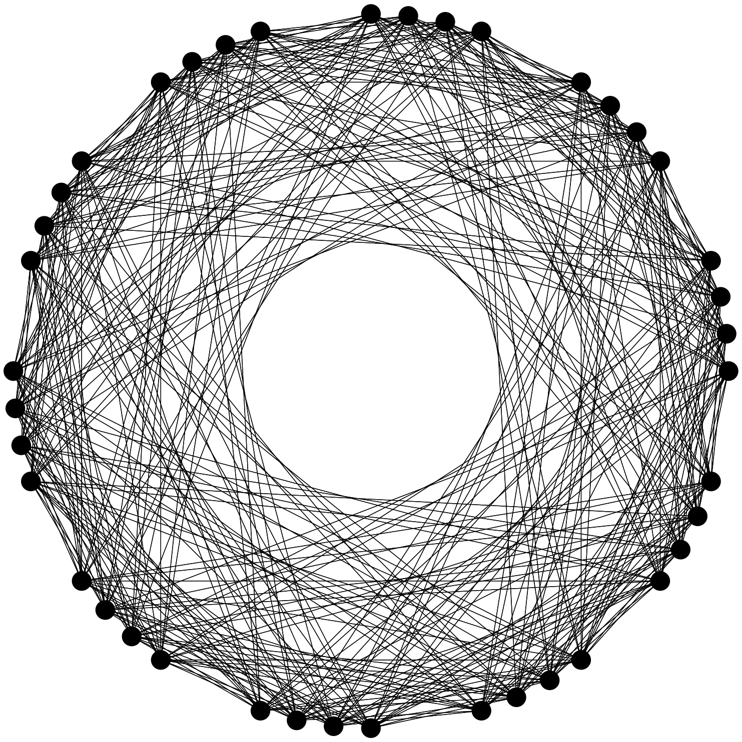
(b) $K_{3\times4} \not\subseteq C_{26\times2}\langle 2,3,4,8,10,12,15,17,18,20,23,24,26\rangle$



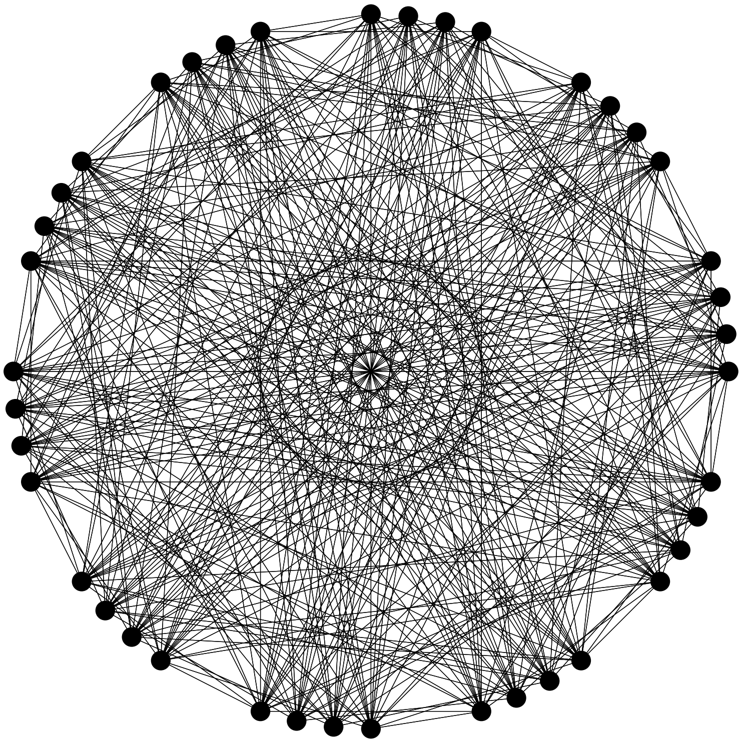
(a) $K_{3\times4} \nsubseteq C_{17\times3}\langle 2,6,8,9,10,12,13,16,19,21,24,25\rangle$



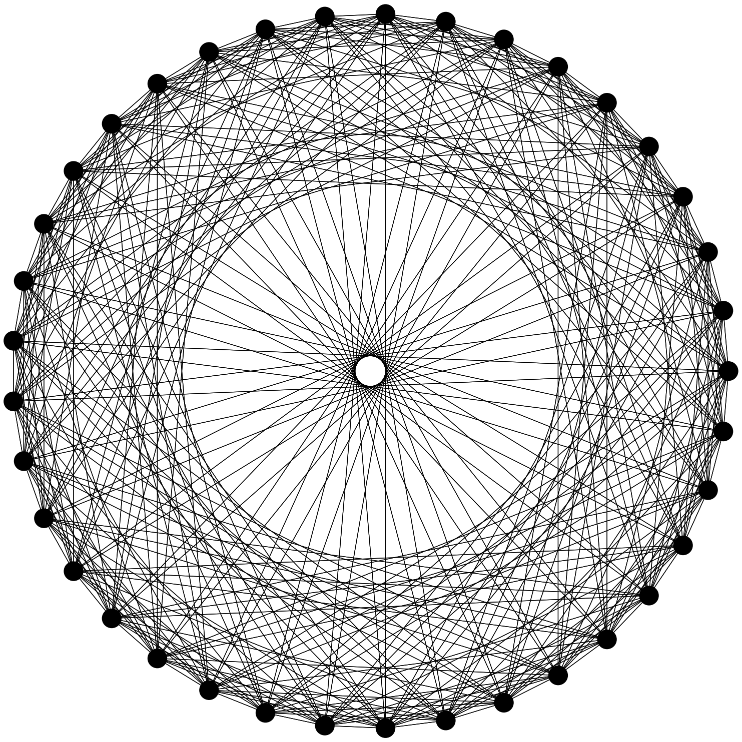
(b) $K_{3\times4} \nsubseteq C_{17\times3}\langle 1,3,4,5,7,11,14,15,17,18,20,22,23\rangle$



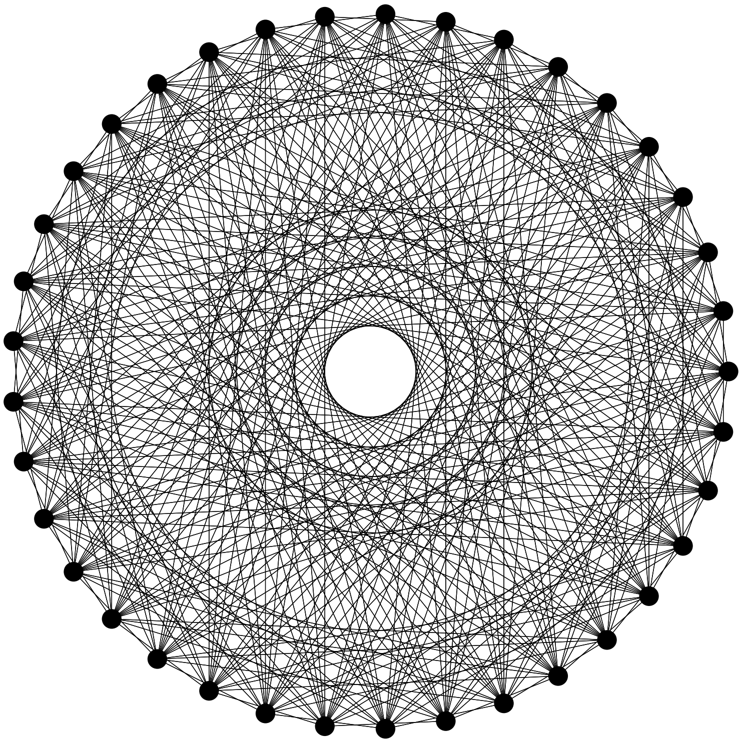
(a) $K_{3\times4} \not\subseteq C_{10\times4}\langle 2,3,5,6,7,8,11,12,14,15\rangle$



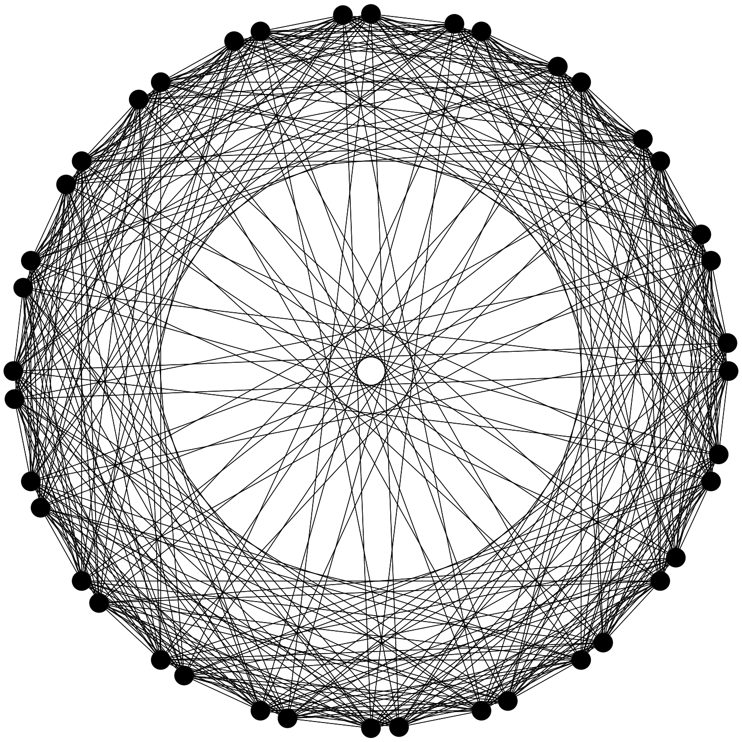
(b) $K_{3\times4} \not\subseteq C_{10\times4}\langle 1,4,9,10,13,16,17,18,19,20\rangle$



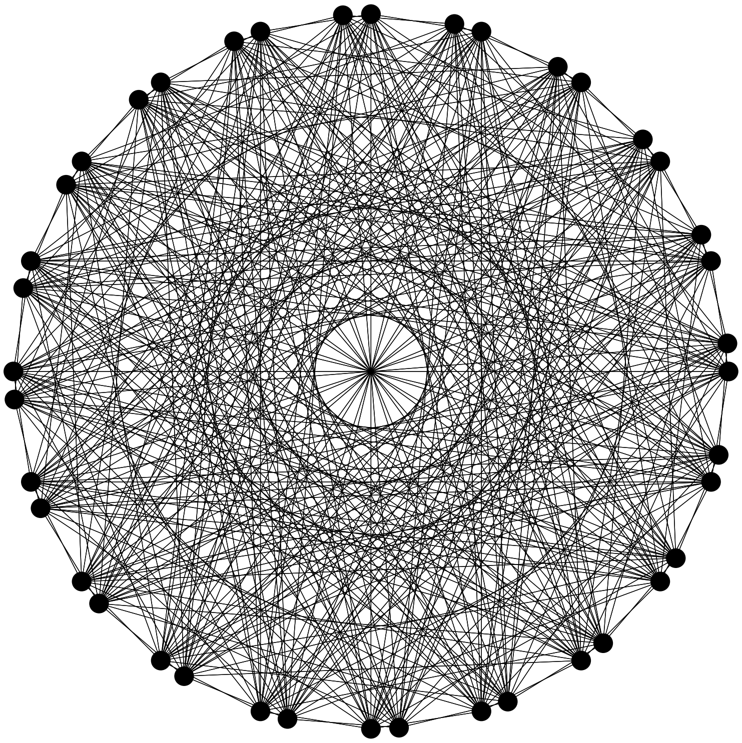
(a) $K_{4\times 3} \not\subseteq C_{37\times 1}\langle 1,3,4,5,7,10,11,12,18\rangle$



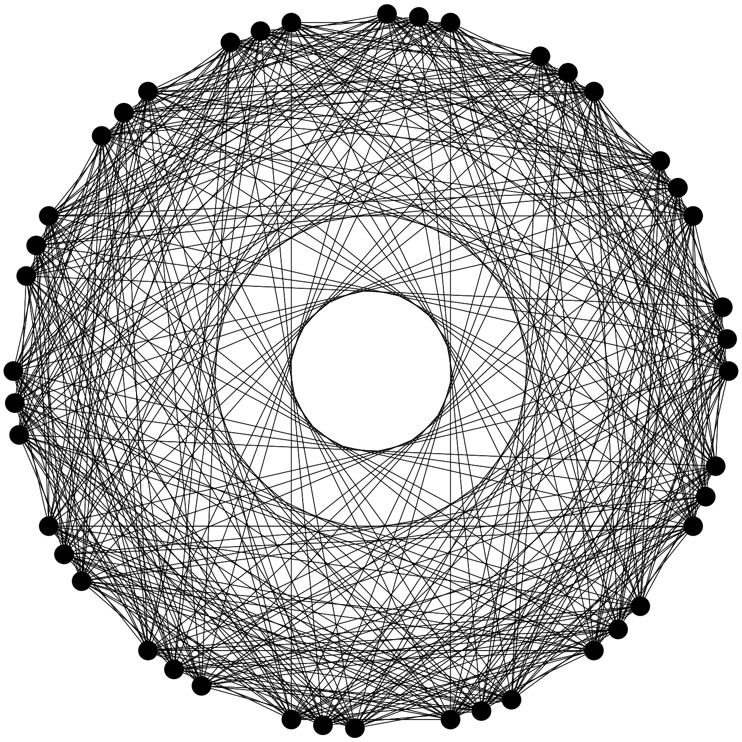
(b) $K_{4\times 3} \not\subseteq C_{37\times 1}\langle 2,6,8,9,13,14,15,16,17\rangle$



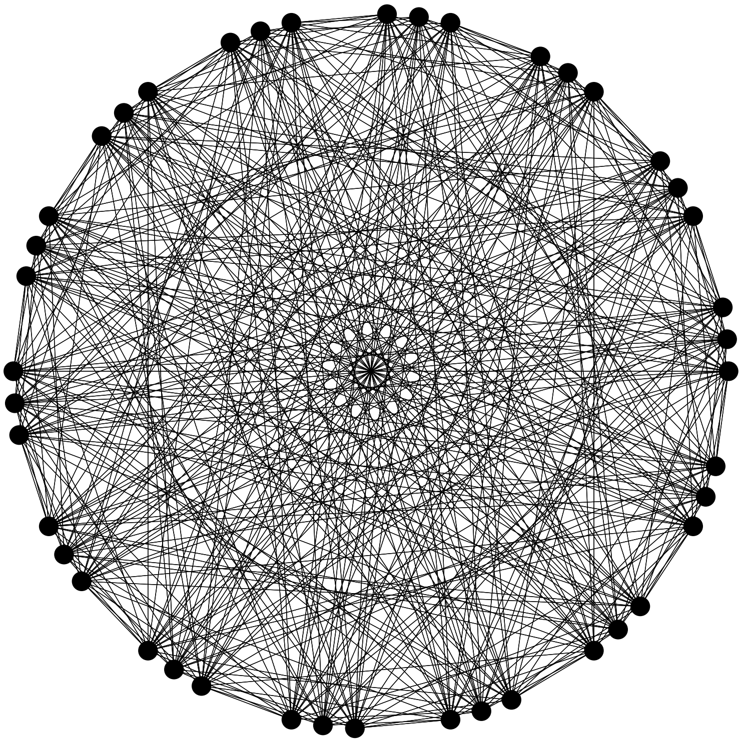
(a) $K_{4\times3} \not\subseteq C_{20\times2}\langle 1,3,4,5,6,8,9,11,12,19\rangle$



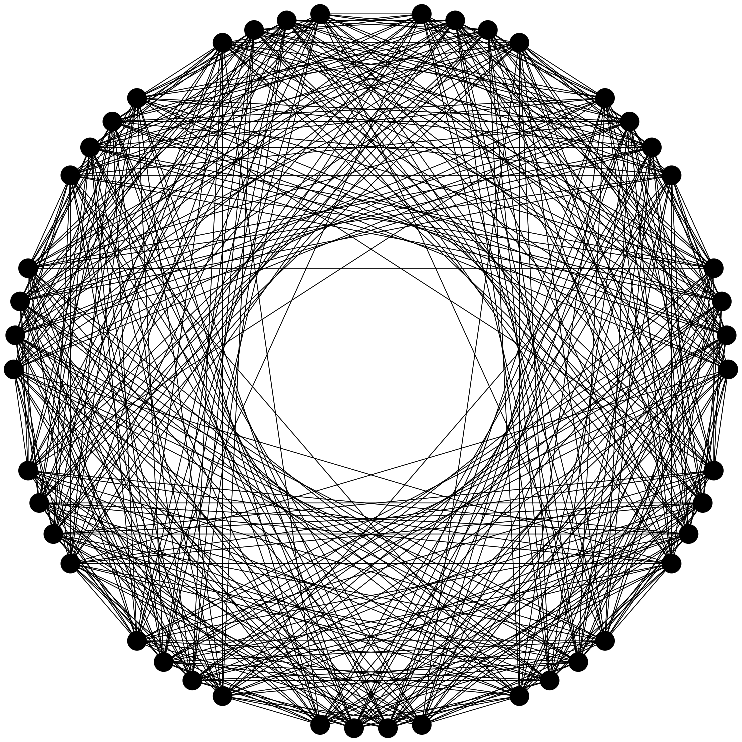
(b) $K_{4\times3} \not\subseteq C_{20\times2}\langle 2,7,10,13,14,15,16,17,18,20\rangle$



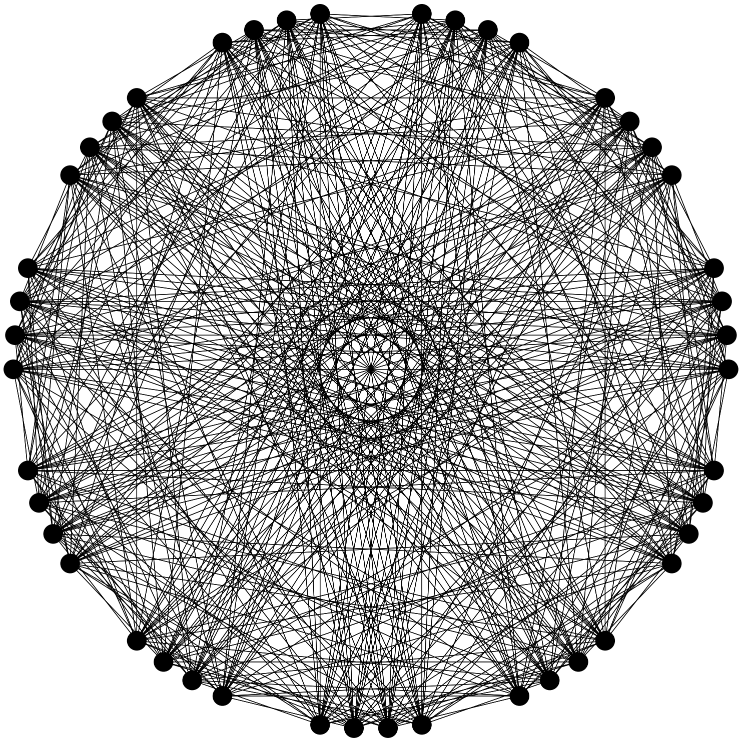
(a) $K_{4\times 3} \not\subseteq C_{14\times 3}\langle 3,5,6,8,9,10,11,14,15,18\rangle$



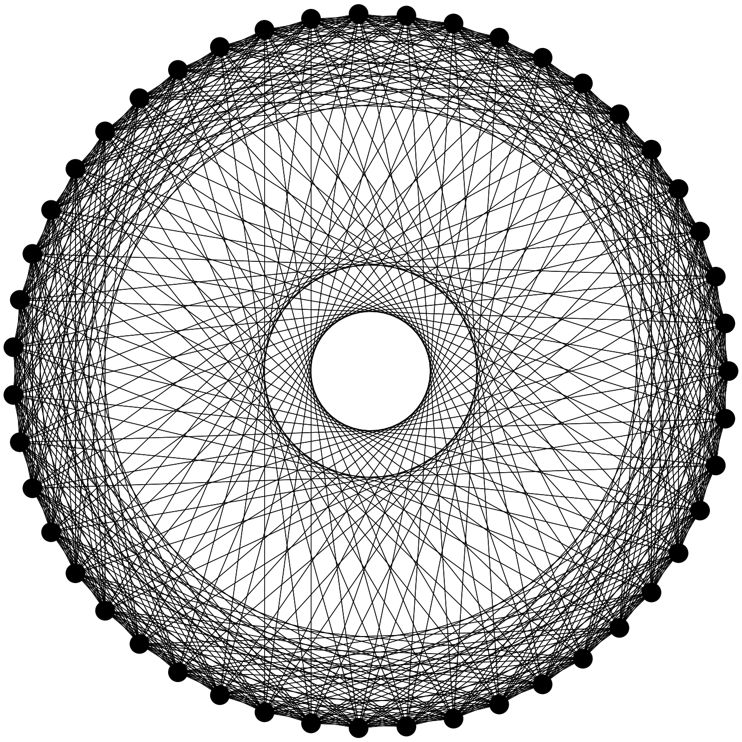
(b) $K_{4\times 3} \not\subseteq C_{14\times 3}\langle 1,2,4,7,12,13,16,17,19,20,21\rangle$



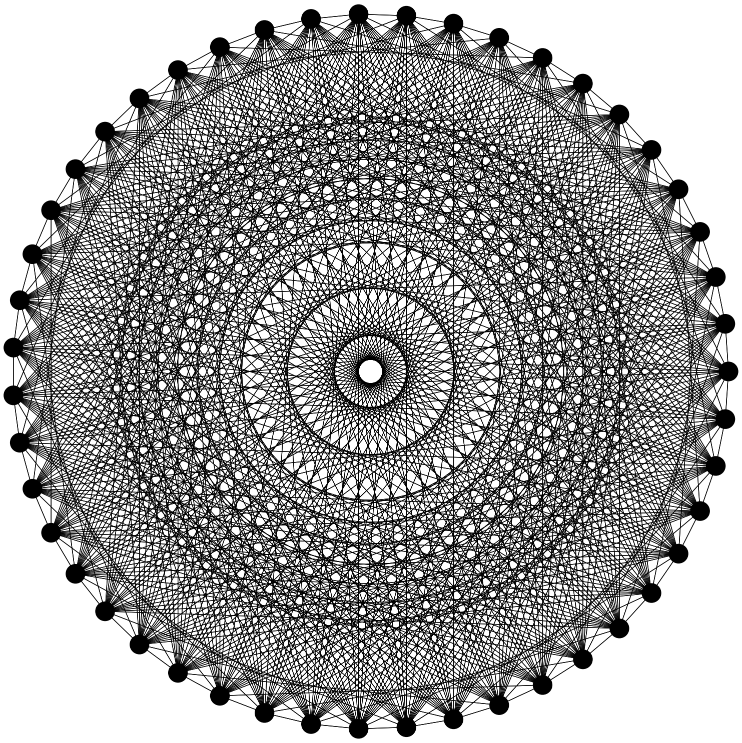
(a) $K_{4\times 3} \not\subseteq C_{11\times 4}\langle 1,3,4,7,8,10,11,13,15,16,17\rangle$



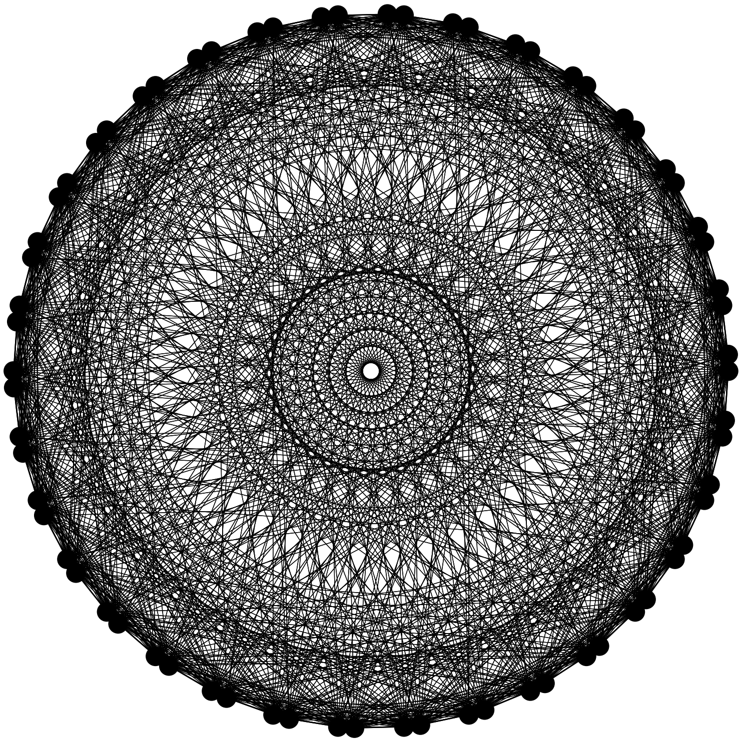
(b) $K_{4\times 3} \not\subseteq C_{11\times 4}\langle 2,5,6,9,12,14,18,19,20,21,22\rangle$



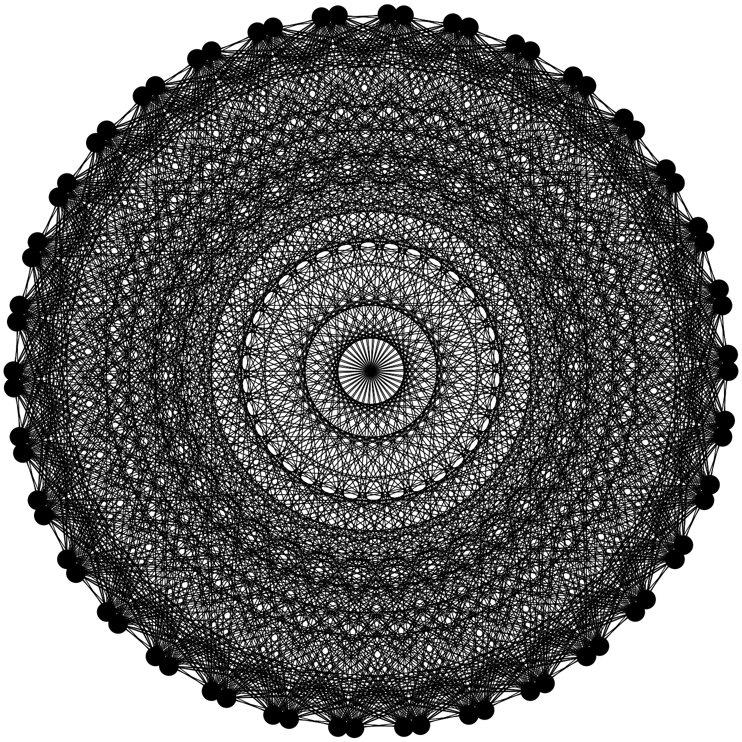
(a) $K_{4\times 4} \nsubseteq C_{47\times 1}\langle 2,3,4,5,6,8,9,10,11,19,21\rangle$



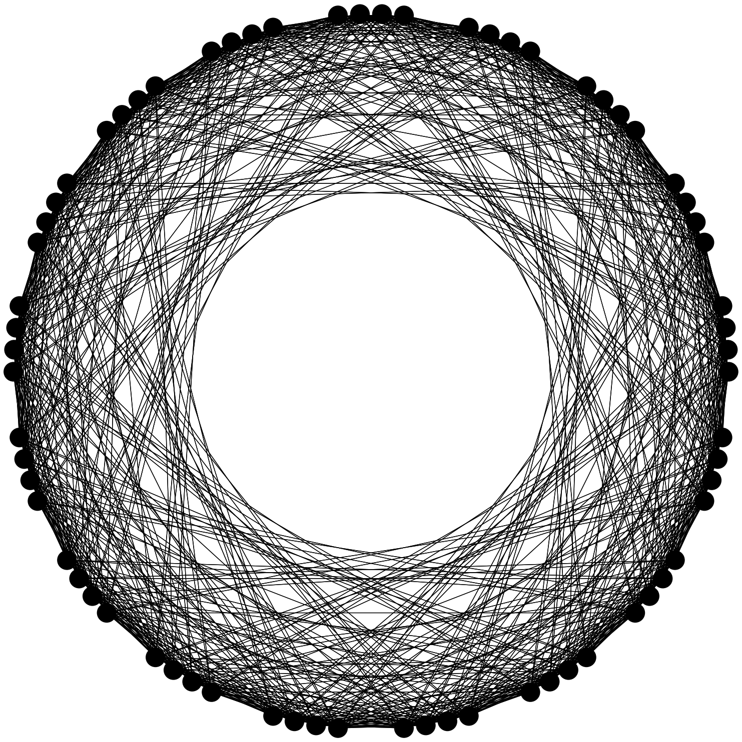
(b) $K_{4\times 4} \nsubseteq C_{47\times 1}\langle 1,7,12,13,14,15,16,17,18,20,22,23\rangle$



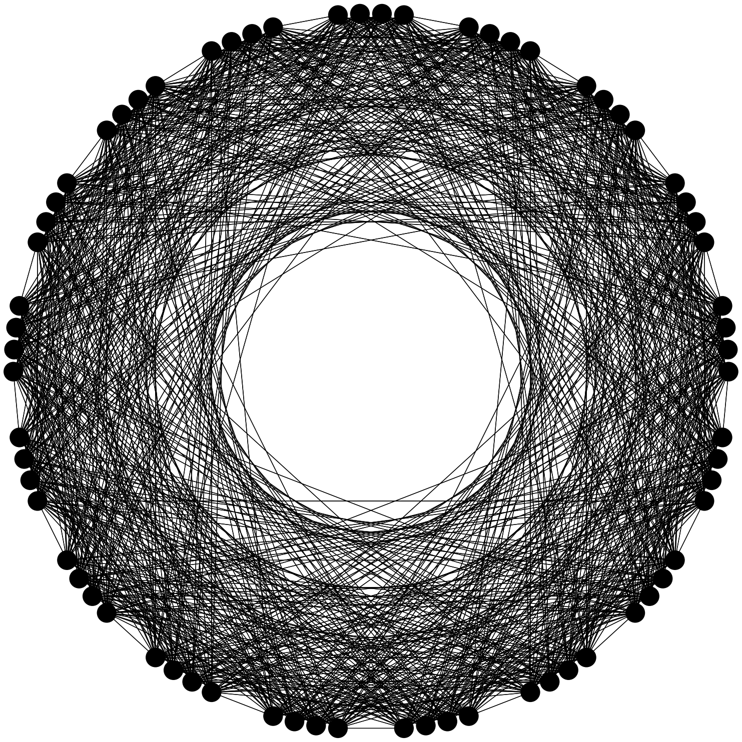
(a) $K_{4\times 4} \nsubseteq C_{34\times 2}\langle 1,2,4,5,6,7,8,13,14,15,17,19,25,28,29,31,33\rangle$



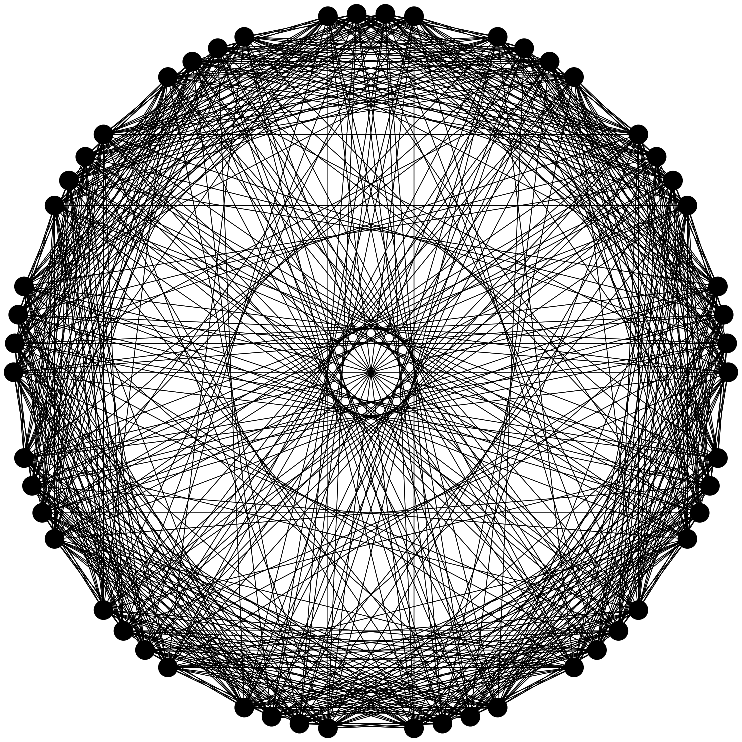
(b) $K_{4\times 4} \nsubseteq C_{34\times 2}\langle 3,9,10,11,12,16,18,20,21,22,23,24,26,27,30,32,34\rangle$



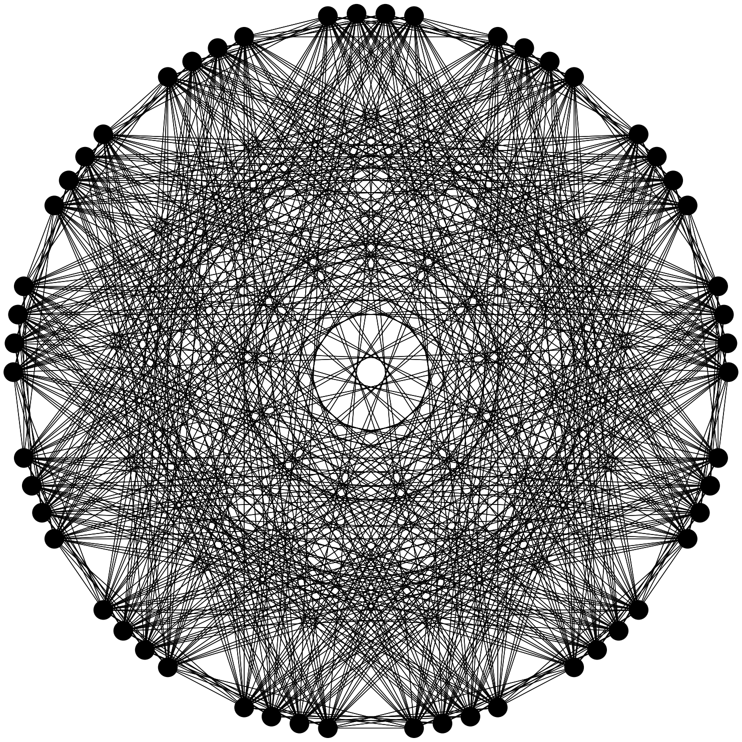
(a) $K_{4\times4} \nsubseteq C_{17\times4}\langle 2,3,4,5,6,7,9,12,14,17,21,22\rangle$



(b) $K_{4\times4} \nsubseteq C_{17\times4}\langle 1,8,10,11,13,15,16,18,19,20,23,24,25\rangle$



(a) $K_{4\times4} \not\subseteq C_{13\times4}\langle 2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 19, 24, 25\rangle$



(b) $K_{4\times4} \not\subseteq C_{13\times4}\langle 1, 4, 5, 14, 15, 16, 17, 18, 20, 21, 22, 23, 26\rangle$

Appendix B

Adjacency matrices for pseudo-random lowerbounds

Lower bounds for class $(3,3)$, $(3,4)$, $(4,3)$ and $(4,4)$ Ramsey numbers are given by presenting the relevant adjacency matrix. A $kj \times kj$ matrix representation A of the bi-coloured graph $K_{k\times j}$ is presented. $A[i,j] = 0$ if i and j are not connected by an edge. $A[i,j] = 1$ if i and j are connected with a red edge and $A[i,j] = 2$ if i and j are connected by a blue edge.

Result	Graphical Representation
Theorem 3.12(1)	Figure B.1(a)&(b)
Theorem 3.12(2)	Figure B.2(a)&(b)
Theorem 3.12(3)	Figure B.3(a)&(b)
Theorem 3.12(4)	Figure B.4(a)&(b)
Theorem 3.12(5)	Figure B.5(a)&(b)

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02111112222122121112221222122212121211111
20221221212111112122212111112212212212112
12021111121222122121122221211121121121222
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12212221221111111211212111212111211122120
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Figure B.1: $M_1(4,3) > 41$ and $m_{41}(4,3) > 1$

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012122211222111122211211222211121222211111122122121212112211112222212111
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```

Figure B.2: $M_1(4,4) > 72$ and $m_{72}(4,4) > 1$

```
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111121111121111112112212222122111122212221212222221222112221222221211100
```

Figure B.3: $M_2(4,4) > 36$ and $m_{36}(4,4) > 2$

```
000122212112121212222112212122121111111212111211221212211222212211112122212
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```

Figure B.4: $M_3(4,4) > 25$ and $m_{25}(4,4) > 3$

```
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1111222211121122000022211121121121121221212111121221112222122122112222221 1211
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2222221112212211000011212121111221212121211222221211121122121212212222121
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1121122122122222211121211111211111222121221121111211212221222112221222122 10000
```

Figure B.5: $M_4(4,4) > 19$ and $m_{19}(4,4) > 4$

Appendix C

Reserved Symbols

$K_{n\times l}$: complete, balanced, n-partite graph with l vertices per partite set.

$M_j(n,l)$: Symmetric set count multipartite Ramsey number. Graph to be bicoloured has j vertices per partite set. Monochromatic subgraph to be forced is $K_{n\times l}$.

$m_k(n,l)$: Symmetric set size multipartite Ramsey number. Graph to be bicoloured is k-partite. Monochromatic subgraph to be forced is $K_{n\times l}$.

$C_{k\times j}\langle i_1, i_2, \ldots, i_z\rangle$: Circulant graph with k partite sets and j vertices per partite set and an edge between every pair of vertices that are at distance i_n on the edge of the imaginary circle, $1 \le n \le z$, excluding inter-partite set connections.

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