A Multiresolutional Approach to the Construction of Spline Wavelets

by

Birgit Rohwer

Thesis presented in partial fulfilment of the requirements for the degree of Master of Science at the University of Stellenbosch.

Supervisor : Prof. J.M. de Villiers

March, 2000.

Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

Summary

In this thesis we study a wavelet construction procedure based on a multiresolutional method, before specializing to the case of spline wavelets.

First, we introduce and analyze the concepts of scaling functions and their duals, after which we analyze the multiresolutional analysis (MRA) which they generate. The advantages of orthonormality in scaling functions are pointed out and discussed. Following the methods which were introduced in two standard texts of Chui, we next show how a minimally supported wavelet and its dual can be explicitly constructed from a given MRA, thereby yielding an orthogonal decomposition of the space of square-(Lebesgue)integrable functions on the real line. We show that our method applied to orthonormal scaling functions also yields orthonormal wavelets, including as a special case the Daubechies wavelet. General decomposition and reconstruction algorithms are explicitly formulated, and the importance of the vanishing moments of a wavelet in practical applications is shown.

We next introduce and analyze cardinal B-splines, in particular showing that these functions are refinable, and that they satisfy the criteria of Riesz stability. Thus the cardinal B-spline is an admissible choice for a scaling function, so that the previously developed wavelet construction procedure based on a MRA yields an explicit formula for the minimally supported B-spline wavelet. The corresponding vanishing moment order is calculated, and the resulting ability of the B-spline wavelet to detect singularities in a given function is demonstrated by means of a numerical example. Finally, we develop an explicit procedure for the construction of minimally supported B-spline wavelets on a bounded interval. This method, as developed in work by de Villiers and Chui, is then compared with a previous boundary wavelet construction method introduced in work by Chui and Quak.

Opsomming

In hierdie tesis bestudeer ons 'n golfie konstruksieprosedure wat gebaseer is op 'n multiresolusiemetode, voordat ons spesialiseer na die geval van latfunksie-golfies.

Eerstens word die konsepte van skaalfunksies en hulle duale bekendgestel en geanaliseer, waarna ons die multiresolusie analise (MRA) wat sodoende gegenereer word, analiseer. Die voordeel van ortonormaliteit by skaalfunksies word uitgewys en bespreek. Deur die metodes te volg wat bekendgestel is in twee standaardtekste van Chui, wys ons vervolgens hoe 'n minimaal-gesteunde golfie en die duaal daarvan eksplisiet gekonstrueer kan word vanuit 'n gegewe MRA, en daarmee 'n ortogonale dekomposisie van die ruimte van kwadraties-(Lebesgue)integreerbare funksies op die reële lyn lewer. Ons wys dat ons metode toegepas op ortonormale skaalfunksies ook ortonormale golfies oplewer, insluitende as 'n spesiale geval die Daubechies golfie. Algemene dekomposisie en rekonstruksie algoritmes word eksplisiet geformuleer, en die belangrikheid in praktiese toepassings van 'n golfie met die nulmomenteienskap word aangetoon.

Vervolgens word kardinale B-latfunksies bekendgestel, en word daar in die besonder aangetoon dat hierdie funksies verfynbaar is, en dat hulle aan die Riesz-stabiliteit vereiste voldoen. Dus is die kardinale B-latfunksie 'n toelaatbare keuse vir 'n skaalfunksie, sodat die golfie konstruksieprosedure gebaseer op 'n MRA, soos vantevore ontwikkel, 'n eksplisiete formule vir die minimaal-gesteunde B-latfunksiegolfie oplewer. Die ooreenkomstige nulmomentorde word bereken, en die gevolglike vermoë van 'n B-latfunksiegolfie om singulariteite in 'n gegewe funksie raak te sien en uit te wys word gedemonstreer deur middel van 'n numeriese voorbeeld. Laastens ontwikkel ons 'n eksplisiete prosedure vir die konstruksie van minimaal-gesteunde B-latfunksiegolfies op 'n begrensde interval. Hierdie metode, soos ontwikkel in werk deur de Villiers en Chui, word dan vergelyk met 'n vorige randgolfie konstruksie wat bekendgestel is in werk deur Chui en Quak.

Acknowledgements

I would like to thank my supervisor, Prof. J.M. de Villiers, for the many hours sacrificed and the tremendous help provided throughout the writing of this thesis. I would also like to thank my father, Dr. C.H. Rohwer, for his interest and much appreciated advice in my career, my mother, Marguerite Rohwer, for her sympathy and support in times of stress, the rest of my family and friends for listening and understanding, and of course the FRD for the study funds provided.

Contents

Chapter 1

Introduction

The term 'wavelets' is used broadly to describe families of basis functions for $L^2(\Re)$, generated by the translations and dilations of one 'mother' function, or wavelet, as well as families of basis functions for the bounded interval $[a, b]$, consisting of inner and outer (or boundary) wavelets.

A characteristic property of wavelets is their locality in time and frequency, enabling them to be used, with much success, as tools for time-frequency analysis, digital data compression (particularly on audio signals), as well as feature detection and extraction. Wavelets are thus valuable mathematical tools in the fields of electrical engineering, computer science, seismology, medical science, etc..

Our approach to wavelet construction is through multiresolutional analysis, the key idea being the segmentation of a Hilbert space into different scales of frequency. We shall concentrate on the construction of spline wavelets, which are special in the sense that they can be explicitly formulated, are computationally easy to work with, symmetric or anti-symmetric, and smooth. We shall also, in this thesis, study the case of spline wavelets on the bounded interval $[a, b]$, as the need for wavelets restricted to a bounded interval arises in many physical problems, including those of image processing and numerical solutions of integral equations.

Interest in the subject of this thesis was stimulated by our involvement in an investigation of the use of low-order B-spline wavelets as a tool with which to analyse missile trajectories. Since the trajectory data obtained from sensors such as radar contain valuable information in both time and frequency, wavelets are a natural choice of mathematical tool to aid in the transmission, storage and evaluation of this data.

In the aforementioned investigation, missile trajectories were simulated on the realistic assumption that they are made up of a quadratic spline function with locally more or less equidistant knots, superimposed with at least three fixed frequency pieces of harmonic vibrations of small amplitude. Quadratic B-spline wavelets were then used to localise the superimposed vibrations in time and frequency and to extract the basic path of the missile. This proved successful.

Overview

In this thesis we shall concentrate on the particular method of wavelet construction by means of a multiresolutional analysis (MRA), with the eventual aim of constructing dyadic B-spline wavelets in $L^2(\Re)$, as well as B-spline wavelets with arbitrary knots restricted to the bounded interval $[a, b]$.

First, in Chapter 2, we introduce the notation which is used throughout the thesis, as well as some results from Fourier Analysis.

In Chapter 3, we introduce scaling functions, dual scaling functions and the multiresolutional analyses which they generate. We also show how to construct orthonormal scaling functions, a procedure which ensures self-dual, but not necessarily finitely supported, scaling functions.

We then proceed, in Chapter 4, to develop a general theory of a wavelet construction method based on a multiresolutional analysis, before specializing to the case of compactly supported biorthogonal wavelets and their duals in $L^2(\Re)$. We show that in the case of orthonormal scaling functions, the construction procedure yields orthonormal wavelets. We then explicitly construct the decomposition algorithm used to compute the wavelet decomposition of a function $f \in L^2(\Re)$, as well as the reconstruction algorithm used to recover f. We also define the vanishing moments of a wavelet and discuss their use in singularity detection.

In Chapter 5, we introduce the concept of cardinal splines, and we show that the cardinal B-spline is indeed a scaling function, before applying the multiresolutional method of Chapter 4 to explicitly construct a minimally supported cardinal B-wavelet. A numerical example, which illustrates the power of wavelet decomposition in the detection of a singularity in a given function, is included.

Finally, in Chapter 6, we employ, following the method introduced in [4], some of the notions of Chapter 4 to construct semi-orthogonal spline wavelets in the finite dimensional setting of spline functions with arbitrary knots on a bounded interval $[a, b]$. We then compare this method to an alternative one introduced by Chui and Quak in [5] and we point out some significant differences between the two respective types of boundary wavelets.

Chapter 2

Preliminaries

In this thesis, the symbol $\Re$ represents the real line, while $\mathbf{C}$ represents the complex plane. We employ the symbol $\mathbf{Z}$ for the set of integers, whereas $\mathbf{N}$ denotes the natural numbers. For $p \in [1, \infty)$, we write $L^p(\Re)$ for the Banach space of measurable (complex-valued) functions f on $\Re$ which are such that the (Lebesgue) integral defined by $\int_{-\infty}^{\infty} |f(t)|^p \, \mathrm{d}t$ is finite, and with corresponding norm

$$\|f\|_p := \left(\int_{-\infty}^{\infty} |f(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}}, \quad f \in L^p(\Re). \tag{2.1}$$

Then $L^2(\Re)$ is a Hilbert space with inner product

$$\langle f, g \rangle := \int_{-\infty}^{\infty} f(t)\overline{g(t)} \, \mathrm{d}t, \quad f, g \in L^2(\Re). \tag{2.2}$$

When f is an analog signal, we call its domain of definition $\Re$ the continuous time-domain.

We shall employ the notation $\{\alpha_k\}$ to denote a bi-infinite sequence $\{\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots\}$. If $\{\alpha_k\} \subset \mathbf{C}$, we write $\sum_k \alpha_k := \sum_{k=-\infty}^{\infty} \alpha_k$ for the resulting bi-infinite series.

For $p \in [1, \infty)$, we write ℓ^p for the Banach space of bi-infinite sequences $\{\alpha_k\} \subset \mathbf{C}$, which are such that the series $\sum_k |\alpha_k|^p$ converges, and with corresponding norm

$$\|\{\alpha_k\}\|_{\ell^p} := \left(\sum_k |\alpha_k|^p\right)^{\frac{1}{p}}, \quad \{\alpha_k\} \in \ell^p. \tag{2.3}$$

The following results from Fourier Analysis will be needed in our work. Although less general than the presentation in [2, Chapter 2], our Fourier Analysis section below will suffice for our more restricted aims than those in [2].

The *Fourier transform* $\mathcal{F}$ is defined by

$$(\mathcal{F}f)(\omega) := \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,\mathrm{d}t, \quad \omega \in \Re, \quad f \in L^1(\Re). \tag{2.4}$$

We also use the notation

$$\widehat{f} := \mathcal{F}f, \quad f \in L^1(\Re).$$

The domain of definition of the Fourier transform is again $\Re$, but is called the *frequency domain.*

We use the symbol $C(\Re)$ to denote the space of (complex-valued) functions which are continuous on $\Re$, whereas the symbol $C_u(\Re)$ denotes the space of (complex-valued) functions which are uniformly continuous on $\Re$.

Now define the linear space $\mathcal{A}$ by

$$\mathcal{A} := \left\{ f \,|\, f \in L^1(\Re) \cap C_u(\Re);\ \widehat{f} \in L^1(\Re) \right\}. \tag{2.5}$$

We claim that then

$$\mathcal{A} \subset L^2(\Re), \tag{2.6}$$

for suppose $f \in \mathcal{A}$, so that $f \in L^1(\Re) \cap C_u(\Re)$, and thus (see e.g. [17, Ex.8(a), p.345]) we have $f(t) \to 0$, $|t| \to \infty$, whence $f \in L^2(\Re)$, by virtue also of the fact that $|f(t)|^2 \leq |f(t)|$ for all $t \in \Re$ in the set $\{t : |f(t)| \leq 1\}$.

According to [2, Theorem 2.5] a function $f \in \mathcal{A}$ can be recovered from its Fourier transform $\widehat{f}$ by means of the inversion formula

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\omega} \widehat{f}(\omega)\,\mathrm{d}\omega, \quad t \in \Re, \quad f \in \mathcal{A}. \tag{2.7}$$

Also, as proved in [2, Theorem 2.13], we have

$$\langle f,g\rangle = \frac{1}{2\pi}\left\langle \widehat{f},\widehat{g}\right\rangle, \quad f,g \in \mathcal{A}, \tag{2.8}$$

having noted also from (2.5) and (2.6) that the inclusion $\mathcal{A} \subset L^1(\Re) \cap L^2(\Re)$ is satisfied. The identity (2.8) is known as the *Parseval identity.* Observe in particular from (2.8) that

$$\|f\|_2 = \frac{1}{\sqrt{2\pi}}\|\widehat{f}\|_2, \quad f \in \mathcal{A}. \tag{2.9}$$

The following result can now be proved.

Theorem 2.1 *The Fourier transform operator $\mathcal{F}$ is a one-one map of $\mathcal{A}$ onto itself, with inverse Fourier transform $\mathcal{F}^{-1}$ defined by*

$$(\mathcal{F}^{-1}f)(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\omega} f(\omega)\,\mathrm{d}\omega, \quad t \in \Re, \quad f \in \mathcal{A}.$$

Proof. First, we show that $\mathcal{F}$ maps $\mathcal{A}$ into itself. Since, as proved in [2, Theorem 2.2(ii)], $f \in \mathcal{A}$ implies $\widehat{f} \in C_u(\Re)$, we have to show that

$$\mathcal{F}\widehat{f} \in L^1(\Re), \quad f \in \mathcal{A}. \tag{2.10}$$

Suppose therefore $f \in \mathcal{A}$, and use (2.4) and (2.7) to deduce that

$$(\mathcal{F}\widehat{f})(t) = \int_{-\infty}^{\infty} e^{-it\omega}\widehat{f}(\omega)\,\mathrm{d}\omega = 2\pi f(-t), \quad t \in \Re,$$

whence

$$\int_{-\infty}^{\infty}\left|(\mathcal{F}\widehat{f})(t)\right|\,\mathrm{d}t = 2\pi\int_{-\infty}^{\infty}|f(-t)|\,\mathrm{d}t = 2\pi\int_{-\infty}^{\infty}|f(t)|\,\mathrm{d}t,$$

and it follows that (2.10) holds, since $f \in \mathcal{A} \subset L^1(\Re)$. Hence $\mathcal{F}$ maps $\mathcal{A}$ into itself.

The result of our theorem will therefore follow from (2.7) if we can show that, for a given $g \in \mathcal{A}$, there exists a unique $f \in \mathcal{A}$ such that $\widehat{f} = g$. To this end, we define the function g_- by $g_-(t) := g(-t)$, $t \in \Re$. Then, since $g \in \mathcal{A}$, and since, as can easily verified from (2.4), $\widehat{g_-}(\omega) = \widehat{g}(-\omega)$, $\omega \in \Re$, we also have $g_- \in \mathcal{A}$. Hence, if we define $f := \frac{1}{2\pi}\widehat{g}_-$, we have $f \in \mathcal{A}$. Moreover, using (2.4) and (2.7), we get

$$\widehat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\omega}\widehat{g}_-(\omega)\,\mathrm{d}\omega = g_-(-t) = g(t), \quad t \in \Re,$$

and thus $\widehat{f} = g$. It remains to prove the uniqueness of f. Suppose therefore $h \in \mathcal{A}$ is such that $\widehat{h} = g$. Then, with the function v defined by $v := f - h$, we have $v \in \mathcal{A}$ and $\widehat{v} = 0$. But then also $\|\widehat{v}\|_2 = 0$, so that (2.9) implies $\|v\|_2 = 0$ and thus $v = 0$, i.e. $f = h$. ■

The following Fourier series result is proved in [1].

Theorem 2.2 *Suppose f is a measurable (complex-valued) function on $\Re$, such that f is 2π-periodic on $\Re$:*

$$f(t + 2\pi) = f(t), \quad t \in \Re,$$

and such that the (Lebesgue) integral

$$\int_0^{2\pi} |f(t)|^2\,\mathrm{d}t < \infty.$$

Then f has the Fourier series representation

$$f(t) = \sum_k c_k e^{ikt}, \quad t \in \Re, \tag{2.11}$$

with c_k denoting the Fourier coefficients

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\,\mathrm{d}t, \quad k \in \mathbf{Z},$$

and where (2.11) should be interpreted as

$$\int_0^{2\pi}\left|f(t) - \sum_{k=-M}^{N} c_k e^{ikt}\right|^2 \mathrm{d}t \to 0, \quad M, N \to \infty.$$

The next result has to do with a periodization of a given function f. For the proof we refer to [2, Lemma 2.24, Theorem 2.25 and Corollary 2.26].

Theorem 2.3 *Suppose $f \in \mathcal{A}$ is such that*

$$f(t), \widehat{f}(t) = \mathcal{O}\left(\frac{1}{1+|t|^\alpha}\right), \quad t \in \Re, \tag{2.12}$$

for some $\alpha > 1$. Then the series $\sum_k f(t+2\pi k)$ converges for all $t \in \Re$ to a 2π-periodic function $g \in \mathbf{C}(\Re)$:

$$g(t) := \sum_k f(t+2\pi k), \quad t \in \Re.$$

Moreover,

$$\sum_k f(t+2\pi k) = \frac{1}{2\pi}\sum_k \widehat{f}(k)e^{ikt}, \quad t \in \Re, \tag{2.13}$$

or, equivalently,

$$\sum_k f(t+k) = \sum_k \widehat{f}(2\pi k)e^{i2\pi kt}, \quad t \in \Re, \tag{2.14}$$

with the right hand sides of (2.13) and (2.14) denoting the Fourier series of the respective functions of t in the left hand sides.

The identities (2.13) and (2.14) are known as the *Poisson summation formula.*

We define the *Kronecker delta* function as

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad i, j \in \mathbf{Z}\,.$$

A function $f \in L^2(\Re)$ is called *orthonormal* if

$$\langle f(\cdot - k), f(\cdot - l)\rangle = \delta_{k,l}, \quad k, l \in \mathbf{Z}. \tag{2.15}$$

We then have the following alternative formulation of (2.15), as proved in [2, Theorem 3.23].

Theorem 2.4 *If $f \in \mathcal{A}$, and (2.12) holds, then (2.15) has the equivalent Fourier transform formulation*

$$\sum_k \left|\widehat{f}(t+2\pi k)\right|^2 = 1, \quad t \in \Re.$$

The convolution g of two real-valued functions f and h in $L^1(\Re)$ is defined by

$$g(t) = (f * h)(t) := \int_{-\infty}^{\infty} f(t-x)h(x)\,\mathrm{d}x, \quad t \in \Re, \tag{2.16}$$

and is also described as *linear analog filtering* of f with the filter function h. In the frequency domain, filtering is done by multiplication of $\widehat{f}$ with the so-called *transfer* function $H = \widehat{h}$, i.e.

$$\widehat{g} = \widehat{f * h} = \widehat{h}\widehat{f} = H\widehat{f}, \tag{2.17}$$

as is obtained by noting from (2.16) that, for $\omega \in \Re$,

$$\int_{-\infty}^{\infty} e^{-i\omega t} g(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} e^{-i\omega t} \int_{-\infty}^{\infty} f(t-x)h(x)\,\mathrm{d}x\,\mathrm{d}t,$$

and thus, by (2.4),

$$\begin{aligned}
\widehat{g}(\omega) &= \int_{-\infty}^{\infty} h(x) \int_{-\infty}^{\infty} e^{-i\omega(t+x)} f(t)\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} h(x)e^{-i\omega x} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t\,\mathrm{d}x \\
&= \widehat{h}(\omega)\widehat{f}(\omega).
\end{aligned}$$

It can also be shown from (2.16) that convolution is associative in the sense that

$$(f * g) * h = f * (g * h). \tag{2.18}$$

A function $f : \Re \to \mathbf{C}$ has *finite support* (or *compact support*) if there exists a bounded interval $[a, b] \subset \Re$ such that

$$f(t) = 0, \quad t \notin [a, b]. \tag{2.19}$$

If $[a, b]$ is the smallest closed interval for which (2.19) holds, we call $[a, b]$ the *support* of f and we write supp $f = [a, b]$. Observe that if $f \in C(\Re)$, and f has finite support, then $f \in L^p(\Re)$, $1 \leq p < \infty$.

We shall say that a sequence $\{\alpha_k\} \in \ell^2$ has finite support if there exist integers M and N, with $M \leq N$, such that $\alpha_k = 0$ if $k \notin \{M, \ldots, N\}$. Observe that if $\{\alpha_k\}$ is a finitely supported sequence, then $\{\alpha_k\} \in \ell^p$, $1 \leq p < \infty$.

For a sequence $\{f_k\}$ of functions we use the notation

$$\operatorname{span} \langle f_k : \ k \in \mathbf{Z} \rangle := \left\{ \sum_k c_k f_k : \ \ \{c_k\} \subset \mathbf{C}; \ \ \{c_k\} \text{ is finitely supported} \right\},$$

i.e. the set of all finite linear combinations of $\{f_k\}$. For a linear subspace $V \subset L^2(\Re)$ defined by

$$V = \operatorname{span} \langle f_k : \ k \in \mathbf{Z} \rangle ,$$

where $\{f_k\} \subset L^2(\Re)$, the notation $\overline{V}$ will denote the closure of V with respect to the L^2 norm $\|\cdot\|_2$, which is the set of all possible finite linear combinations of $\{f_k\}$, together with all their limits under the L^2-norm, and we write

$$\overline{V} = \operatorname{clos}_{L^2} \langle f_k : \ k \in \mathbf{Z} \rangle .$$

We shall write Π_n for the space of all polynomials of degree $\leq n$.

A *Hurwitz polynomial* is a polynomial for which all the zeros lie in the open left half plane of $\mathbf{C}$.

Chapter 3

Scaling functions in $L^2(\Re)$

In this chapter, we shall define a scaling function ϕ, and show how it generates a multiresolutional analysis (MRA).

3.1 The Scaling Function

The generation of a MRA is dependent on the existence of a so-called scaling function. Our definition below of a scaling function admits a smaller class of functions than the more general approach taken in references like [2], [3] and [9]. However our approach, which is essentially based on the one followed in [2] and [3], suffices for the types of wavelets considered here, and allows for some simplifications of the general theory.

Definition 3.1 *Suppose the function $\phi \in \mathcal{A}$ satisfies the following conditions:*

(a)

$$\phi(t), \widehat{\phi}(t) = \mathcal{O}\left(\frac{1}{1+|t|^{\alpha}}\right), \quad t \in \Re, \tag{3.1}$$

for some $\alpha > 1$;

(b) there exists a sequence $\{p_k\} \in \ell^1 \cap \ell^2$ such that

$$\phi(t) = \sum_k p_k \phi(2t - k), \ t \in \Re; \tag{3.2}$$

(c) ϕ is Riesz stable, in the sense that there exist real numbers A and B, with $0 < A \leq B$, such that

$$A \left\|\{c_k\}\right\|_{\ell^2}^2 \leq \left\| \sum_k c_k \phi(\cdot - k) \right\|_2^2 \leq B \left\|\{c_k\}\right\|_{\ell^2}^2, \ \{c_k\} \in \ell^2; \tag{3.3}$$

(d)

$$\sum_k \phi(t - k) = 1, \ t \in \Re. \tag{3.4}$$

Then we call ϕ a scaling function.

In general, a function ϕ satisfying (3.2), for a sequence $\{p_k\} \in \ell^1 \cap \ell^2$, is called *refinable*, and the equation (3.2) is called the *two-scale relation* (or *refinement equation*) for ϕ. The sequence $\{p_k\}$ in (3.2) is called the corresponding *two-scale sequence* (or *refinement mask*) for ϕ.

The following uniqueness result holds for scaling functions.

Theorem 3.1 *Suppose ϕ is a scaling function. Then $\{p_k\}$ is the unique two-scale sequence in $\ell^1 \cap \ell^2$ for which the two-scale relation (3.2) holds.*

Proof. Suppose $\{q_k\} \in \ell^1 \cap \ell^2$ is such that

$$\phi(t) = \sum_k q_k \phi(2t - k), \ t \in \Re,$$

which, together with (3.2), yields

$$\sum_k (p_k - q_k) \phi(2t - k) = 0, \quad t \in \Re,$$

and thus

$$\sum_k (p_k - q_k) \phi(t - k) = 0, \quad t \in \Re.$$

But then (3.3) gives

$$\|\{p_k\} - \{q_k\}\|_{\ell^2} \leq \frac{1}{\sqrt{A}} \left\| \sum_k (p_k - q_k)\phi(\cdot - k) \right\|_2 = 0,$$

whence $\{p_k\} = \{q_k\}$. ■

A simple example of a refinable function is the *Haar function* ϕ_H, otherwise known as the box function or, as we shall see later, the cardinal B-spline of order 1. The Haar function is given by

$$\phi_H(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 0, & \text{elsewhere}, \end{cases} \tag{3.5}$$

for which clearly (see also Fig.3.1)

$$\phi_H(t) = \phi_H(2t) + \phi_H(2t-1), \; t \in \Re,$$

and thus the two-scale sequence $\{p_k\}$ of ϕ_H is given by

$$p_k = \begin{cases} 1, & \text{if } k = 0, 1, \\ 0, & \text{otherwise}. \end{cases} \tag{3.6}$$

Note that ϕ_H is not an element of $\mathcal{A}$, as it is not a continuous function.

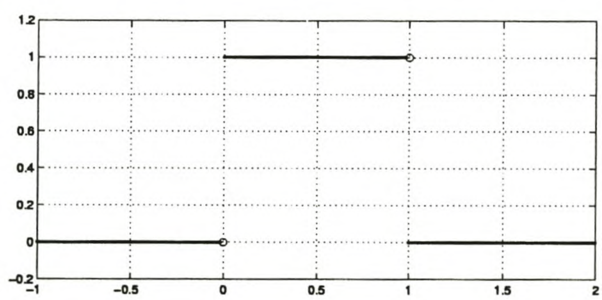


Figure 3.1: The Haar function

The condition (3.3) is called the *Riesz stability condition*, and A and B are called the *Riesz bounds*. The Haar function ϕ_H satisfies (3.3) with equality, and with Riesz bounds $A = B = 1$, as is obtained by using (3.5) to find, for $\{c_k\} \in \ell^2$,

$$\begin{aligned}
\int_{-\infty}^{\infty} \left|\sum_k c_k \phi_H(t-k)\right|^2 \mathrm{d}t &= \sum_l \int_l^{l+1} \left|\sum_k c_k \phi_H(t-k)\right|^2 \mathrm{d}t \\
&= \sum_l \int_l^{l+1} |c_l|^2 \mathrm{d}t \\
&= \sum_l |c_l|^2 .
\end{aligned}$$

The condition (3.4) is called the *partition of unity* property. Note that, since (3.5) implies

$$\phi_H(t-k) = \begin{cases} 1, & k \leq t < k+1, \\ 0, & \text{elsewhere,} \end{cases}$$

we have $\sum_k \phi_H(t-k) = 1$ for all $t \in \Re$, and thus ϕ_H satisfies the partition of unity property (3.4).

For a given scaling function ϕ, we define the sequence $\{\phi_{j,k} : \ j,k \in \mathbf{Z}\} \subset \mathcal{A}$ by

$$\phi_{j,k}(t) := 2^{j/2}\phi(2^j t - k), \quad t \in \Re, \quad j,k \in \mathbf{Z}. \tag{3.7}$$

It can then be verified by means of (2.1) that

$$\|\phi_{j,k}\|_2 = \|\phi\|_2, \quad j,k \in \mathbf{Z}.$$

Next, we define the sequence $\{V_j\}$ of linear spaces by

$$V_j := \left\{\sum_k c_k \phi_{j,k} : \ \{c_k\} \in \ell^2\right\}, \quad j \in \mathbf{Z}. \tag{3.8}$$

The following fundamental properties of V_j then hold.

Theorem 3.2 *For a given scaling function ϕ, suppose that the sequence V_j of linear spaces are defined by (3.8) and (3.7). Then $\{V_j\}$ is a nested sequence of closed linear subspaces of $L^2(\Re)$, with*

$$(a) \quad \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots; \tag{3.9}$$

(b) $$\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\Re); \tag{3.10}$$

(c) $$\bigcap_{j\in\mathbf{Z}} V_j = \{0\}. \tag{3.11}$$

Moreover,

$$V_j = \text{clos}_{L^2}\left\langle \phi_{j,k} : k \in \mathbf{Z} \right\rangle, \quad j \in \mathbf{Z}. \tag{3.12}$$

Proof. First, we observe that, for a fixed $j \in \mathbf{Z}$, and $\{c_k\} \in \ell^2$, we have, from (2.1),

$$\begin{aligned}
\left\| \sum_k c_k \phi_{j,k} \right\|_2^2 &= \int_{-\infty}^{\infty} \left[\sum_k c_k 2^{j/2} \phi(2^j t - k) \right] \left[\sum_l \bar{c}_l 2^{j/2} \overline{\phi(2^j t - l)} \right] \mathrm{d}t \\
&= \int_{-\infty}^{\infty} \left[\sum_k c_k \phi(t - k) \right] \left[\sum_l \bar{c}_l \overline{\phi(t - l)} \right] \mathrm{d}t \\
&= \left\| \sum_k c_k \phi(\cdot - k) \right\|_2^2,
\end{aligned}$$

which, together with (3.3), yield the condition

$$A \left\| \{c_k\} \right\|_{\ell^2}^2 \leq \left\| \sum_k c_k \phi_{j,k} \right\|_2^2 \leq B \left\| \{c_k\} \right\|_{\ell^2}^2, \quad \{c_k\} \in \ell^2, \quad j \in \mathbf{Z}. \tag{3.13}$$

It follows from the inequality (3.13) that $V_j \subset L^2(\Re)$, $j \in \mathbf{Z}$.

To prove that V_j is closed, we must show that if, for a fixed $j \in \mathbf{Z}$, the sequence $\{f_n : n \in \mathbf{N}\} \subset V_j$ and the function $f \in L^2(\Re)$ is such that

$$\|f - f_n\|_2 \to 0, \quad n \to \infty, \tag{3.14}$$

then $f \in V_j$.

Since $f_n \in V_j$, $n \in \mathbf{N}$, there exist sequences $\{c_k^{(n)}\} \in \ell^2$, $n \in \mathbf{N}$, such that

$$f_n = \sum_k c_k^{(n)} \phi_{j,k}, \quad n \in \mathbf{N}. \tag{3.15}$$

Since $\{f_n : n \in \mathbf{N}\}$ is a convergent sequence, it is also a Cauchy sequence, i.e. $\|f_m - f_n\|_2 \to 0,\ m,n \to \infty$, and thus, from (3.15),

$$\left\| \sum_k (c_k^{(m)} - c_k^{(n)})\phi_{j,k} \right\|_2 \to 0, \quad m,n \to \infty. \tag{3.16}$$

But the first inequality in (3.13) gives

$$\left\| \{c_k^{(m)}\} - \{c_k^{(n)}\} \right\|_{\ell^2} \leq \frac{1}{\sqrt{A}} \left\| \sum_k (c_k^{(m)} - c_k^{(n)})\phi_{j,k} \right\|_2, \quad m,n \in \mathbf{N}, \tag{3.17}$$

and it follows from (3.16) and (3.17) that $\{\{c_k^{(n)}\} : n \in \mathbf{N}\}$ is a Cauchy sequence in ℓ^2. By the completeness of the Hilbert space ℓ^2, we know that there exists a sequence $\{c_k\} \in \ell^2$ such that

$$\left\| \{c_k\} - \{c_k^{(n)}\} \right\|_{\ell^2} \to 0, \quad n \to \infty. \tag{3.18}$$

But then, using also (3.15), we get

$$\begin{aligned} \left\| f - \sum_k c_k \phi_{j,k} \right\|_2 &\leq \|f - f_n\|_2 + \left\| \sum_k (c_k^{(n)} - c_k)\phi_{j,k} \right\|_2 \\ &\leq \|f - f_n\|_2 + \sqrt{B} \left\| \{c_k\} - \{c_k^{(n)}\} \right\|_{\ell^2}, \end{aligned} \tag{3.19}$$

from the second inequality in (3.13). Combining (3.19), (3.14) and (3.18) then yields $f = \sum_k c_k \phi_{j,k}$, i.e. $f \in V_j$.

To prove property (a), we fix $j \in \mathbf{Z}$ and suppose $f \in V_j$, i.e. there exists a sequence $\{c_k\} \in \ell^2$ such that $f = \sum_k c_k \phi_{j,k}$. Using also (3.2) and (3.7), it follows that, for $t \in \Re$, and with $\{p_k\} \in \ell^1 \cap \ell^2$ denoting the two-scale sequence in (3.2), we have

$$\begin{aligned} f(t) &= \sum_k c_k 2^{j/2} \phi(2^j t - k) \\ &= 2^{j/2} \sum_k c_k \sum_l p_l \phi(2^{j+1} t - 2k - l) \\ &= 2^{j/2} \sum_k c_k \sum_l p_{l-2k} \phi(2^{j+1} t - l) \\ &= 2^{j/2} \sum_l \alpha_l \phi(2^{j+1} t - l), \end{aligned} \tag{3.20}$$

with

$$\alpha_l := \sum_k c_k p_{l-2k}, \quad l \in \mathbf{Z}. \tag{3.21}$$

We claim that $\alpha_l \in \ell^2$. Indeed, from (2.3) and (3.21), we have

$$\begin{aligned}
\|\{\alpha_l\}\|_{\ell^2}^2 &= \sum_l \left|\sum_k c_k p_{l-2k}\right|^2 \\
&= \sum_l \left|\sum_k c_k p_{2l-2k}\right|^2 + \sum_l \left|\sum_k c_k p_{2l+1-2k}\right|^2 \\
&= \sum_l \left|\sum_k p_{2k} c_{l-k}\right|^2 + \sum_l \left|\sum_k p_{2k+1} c_{l-k}\right|^2 .
\end{aligned}$$

Here, employing also the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\sum_l \left|\sum_k p_{2k} c_{l-k}\right|^2 &= \sum_l \sum_k p_{2k} c_{l-k} \sum_j \overline{p}_{2j} \overline{c}_{l-j} \\
&= \sum_k p_{2k} \sum_j \overline{p}_{2j} \sum_l c_{l-k} \overline{c}_{l-j} \\
&= \left|\sum_k p_{2k} \sum_j \overline{p}_{2j} \sum_l c_{l-k} \overline{c}_{l-j}\right| \\
&\leq \sum_k |p_{2k}| \sum_j \left|\overline{p}_{2j}\right| \left|\sum_l c_{l-k} \overline{c}_{l-j}\right| \\
&\leq \sum_k |p_{2k}| \sum_j \left|\overline{p}_{2j}\right| \sqrt{\sum_l |c_{l-k}|^2} \sqrt{\sum_l |\overline{c}_{l-j}|^2} \\
&= \sum_k |p_{2k}| \sum_j \left|\overline{p}_{2j}\right| \sqrt{\sum_l |c_l|^2} \sqrt{\sum_l |\overline{c}_l|^2} \\
&= \|\{c_k\}\|_{\ell^2}^2 \sum_k |p_{2k}| \sum_j |\overline{p}_{2j}| \\
&\leq \|\{c_k\}\|_{\ell^2}^2 \sum_k |p_k| \sum_j |\overline{p}_j| \\
&= \|\{c_k\}\|_{\ell^2}^2 \; \|\{p_k\}\|_{\ell^1}^2 .
\end{aligned}$$

Similarly,

$$\sum_l \left|\sum_k p_{2k+1} c_{l-k}\right|^2 \leq \|\{c_k\}\|_{\ell_2}^2 \; \|\{p_k\}\|_{\ell^1}^2 .$$

It follows, from (3.21), that $\{\alpha_l\} \in \ell^2$, so that (3.20) gives $f \in V_{j+1}$. Hence $V_j \subset V_{j+1}$, $j \in \mathbf{Z}$, thereby yielding property (a).

For the proofs of properties (b) and (c), we refer to [9, Propositions 5.3.1 and 5.3.2]. Observe also that the condition $\widehat{\phi}(0) \neq 0$, which is required in [9, Proposition 5.3.2], holds by virtue of our Theorem 3.3(c) below.

Lastly, we must show that (3.12) holds. The inclusion $V_j \subset \overline{\text{span}\,\langle \phi_{j,k} : k \in \mathbf{Z} \rangle}$ follows from (3.8) and by noting that, for $\{c_k\} \in \ell^2$,

$$\sum_k c_k \phi_{j,k} = \lim_{M,N\to\infty} \sum_{-M}^{N} c_k \phi_{j,k}.$$

Suppose next $f \in \overline{\text{span}\,\langle \phi_{j,k} : k \in \mathbf{Z} \rangle}$, i.e. $f \in L^2(\Re)$ is such that there exists a sequence $\left\{\{c_k^{(n)}\},\ n \in \mathbf{N}\right\}$ of finitely supported sequences for which it holds that

$$\left\| f - \sum_k c_k^{(n)} \phi_{j,k} \right\|_2 \to 0, \quad n \to \infty.$$

Now use the fact that $\{c_k^{(n)}\} \in \ell^2$, $n \in \mathbf{N}$, together with the fact that the space V_j is closed, to deduce that $f \in V_j$. Hence $\overline{\text{span}\,\langle \phi_{j,k} : k \in \mathbf{Z} \rangle} \subset V_j$. ∎

We call $\{V_j\}$ the *multiresolutional analysis* (MRA) generated by the scaling function ϕ. Equation (3.12) is often used (see e.g. [2], [3] and [9]) as the defining property for the sequence V_j.

Note that properties (a), (b) and (c) of Theorem 3.2 together imply that $\{V_j\}$ is a nested sequence of closed linear subspaces of $L^2(\Re)$ such that

$$\{0\} \leftarrow \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \to L^2(\Re). \tag{3.22}$$

This implies that any function $f \in L^2(\Re)$ can be approximated arbitrarily well in a function space V_j by choosing j large enough, i.e. given a function $f \in L^2(\Re)$ and $\varepsilon > 0$ there exists an integer $n \in \mathbf{N}$ such that we can approximate f by a function $f_n \in V_n$ with $\|f - f_n\|_2 < \varepsilon$.

The properties (b), (c) and (d) in Definition 3.1 of a scaling function all have equivalent Fourier transform formulations, as given in Theorem 3.3 below, which often prove convenient.

We shall denote by P the Laurent series

$$P(z) := \frac{1}{2}\sum_k p_k z^k, \quad z \in \mathbf{C} \setminus \{0\}, \tag{3.23}$$

which we call the *two-scale symbol* associated with the two-scale sequence $\{p_k\}$.

Theorem 3.3 *In Definition 3.1 of a scaling function ϕ, the conditions (b), (c) and (d) can each, respectively, be replaced by the following equivalent Fourier transform formulations:*

(a) the two-scale relation (3.2) is equivalent to the functional equation

$$\widehat{\phi}(\omega) = P(e^{-i\omega/2})\widehat{\phi}\left(\frac{\omega}{2}\right), \quad \omega \in \Re; \tag{3.24}$$

(b) the Riesz stability condition (3.3) has the equivalent formulation

$$A \leq \sum_k \left|\widehat{\phi}(\omega + 2\pi k)\right|^2 \leq B, \quad \omega \in \Re, \tag{3.25}$$

with A and B denoting the same positive numbers as in (3.3);

(c) the partition of unity condition (3.4) is equivalent to the condition

$$\widehat{\phi}(0) = 1. \tag{3.26}$$

Proof. To prove the equivalence (a), we use the fact that if (3.2) holds, then (2.4) gives, for $\omega \in \Re$,

$$\begin{aligned}
\widehat{\phi}(\omega) &= \sum_k p_k \int_{-\infty}^{\infty} e^{-i\omega t}\phi(2t-k)\, \mathrm{d}t \\
&= \frac{1}{2}\sum_k p_k \int_{-\infty}^{\infty} e^{-i\omega(t+k)/2}\phi(t)\, \mathrm{d}t \\
&= \frac{1}{2}\sum_k p_k e^{-i\omega k/2} \int_{-\infty}^{\infty} e^{-i(\omega/2)t}\phi(t)\, \mathrm{d}t \\
&= \frac{1}{2}\sum_k p_k (e^{-i\omega/2})^k \widehat{\phi}\left(\frac{\omega}{2}\right),
\end{aligned}$$

having also used the fact that the condition $\{p_k\} \in \ell^1$ ensures that P is a continuous function on the unit circle $|z| = 1$. Conversely, if (3.24) holds, we can appeal to Theorem 2.1, together with the fact that $\phi \in \mathcal{A}$, and (3.23), to deduce that, for $t \in \Re$,

$$\begin{aligned}
\phi(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\omega}\widehat{\phi}(\omega)\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\omega}\left[\frac{1}{2}\sum_k p_k e^{-i(\omega/2)k}\right]\widehat{\phi}\left(\frac{\omega}{2}\right)\,\mathrm{d}\omega \\
&= \frac{1}{2}\sum_k p_k \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-k/2)}\widehat{\phi}\left(\frac{\omega}{2}\right)\,\mathrm{d}\omega \\
&= \sum_k p_k \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(2t-k)}\widehat{\phi}(\omega)\,\mathrm{d}\omega \\
&= \sum_k p_k \phi(2t-k).
\end{aligned}$$

For the proof of the equivalence between (3.3) and (3.25), we refer to [2, Theorem 3.24].

To prove the equivalence between (3.4) and (3.26), we first recall, from Theorem 2.3, the Poisson summation formula in the form (2.14), i.e.

$$\sum_k \phi(t-k) = \sum_k \widehat{\phi}(2\pi k)e^{i2\pi kt}, \quad t \in \Re. \tag{3.27}$$

Suppose now (3.4) holds, so that (3.27) gives

$$\sum_k \widehat{\phi}(2\pi k)e^{i2\pi kt} = 1, \quad t \in \Re,$$

and thus

$$\frac{1}{2\pi}\int_0^{2\pi}\sum_k \widehat{\phi}(2\pi k)e^{i2\pi(k-j)t}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} e^{-i2\pi jt}\,\mathrm{d}t, \quad j \in \mathbf{Z},$$

so that

$$\sum_k \widehat{\phi}(2\pi k)\frac{1}{2\pi}\int_0^{2\pi} e^{i2\pi(k-j)t}\,\mathrm{d}t = \delta_{j,0}, \quad j \in \mathbf{Z},$$

yielding

$$\widehat{\phi}(2\pi j) = \delta_{j,0}, \quad j \in \mathbf{Z}. \tag{3.28}$$

Hence (3.26) holds.

Next, suppose (3.26) holds. Setting $\omega = 0$ in the Riesz condition (3.25) yields $\sum_k |\widehat{\phi}(2\pi k)|^2 \leq B$, whence $\sum_k |\widehat{\phi}(2\pi k)|^2$ is a convergent series and it follows that

$$\widehat{\phi}(2\pi k) \to 0, \ |k| \to \infty. \tag{3.29}$$

We claim that indeed

$$\widehat{\phi}(2\pi k) = 0, \ \ k \in \mathbf{Z} \setminus \{0\}, \tag{3.30}$$

for if not, then there exists a $k_0 \in \mathbf{Z}$ such that $\widehat{\phi}(2\pi k_0) \neq 0$. Also, from (3.24),

$$\widehat{\phi}(2^{n+1}\pi k_0) = P(1)\widehat{\phi}(2^n \pi k_0) = \ldots = [P(1)]^n \widehat{\phi}(2\pi k_0), \ \ n \in \mathbf{N}.$$

But, if we set $\omega = 0$ in (3.24) and use (3.26), we get $P(1) = 1$ and thus

$$\widehat{\phi}(2^n \pi k_0) = \widehat{\phi}(2\pi k_0) \neq 0, \ \ n \in \mathbf{N},$$

which contradicts (3.29). Hence (3.30) holds, which, together with (3.27) and (3.26), then yield the desired condition (3.4). ∎

Remark. Observe from (2.4) that the condition (3.26) is equivalent to the condition

$$\int_{-\infty}^{\infty} \phi(t) \, \mathrm{d}t = 1. \tag{3.31}$$

The following result follows immediately from Theorem 3.3(c) and (3.28).

Corollary 3.4 *Suppose ϕ is a scaling function. Then*

$$\widehat{\phi}(2\pi k) = \delta_{k,0}, \ \ k \in \mathbf{Z}. \tag{3.32}$$

Next, we derive necessary conditions on the two-scale sequence $\{p_k\}$ for a scaling function ϕ.

Theorem 3.5 *Suppose ϕ is a scaling function. Then the two-scale sequence $\{p_k\}$ in (3.2) satisfies*

$$\sum_k p_{2k} = \sum_k p_{2k+1} = 1, \tag{3.33}$$

or, equivalently, in terms of the corresponding two-scale symbol P, as given by (3.23),

$$P(1) = 1, \quad P(-1) = 0. \tag{3.34}$$

Proof. Setting $\omega = 0$ in (3.24), and using (3.26), yields the value $P(1) = 1$, and thus, from (3.23),

$$\sum_k p_k = 2. \tag{3.35}$$

Now set $\omega = \pi$ in (3.25) to deduce that

$$\sum_k |\widehat{\phi}((2k+1)\pi)|^2 \geq A > 0,$$

and it follows that there exists a $k_0 \in \mathbf{Z}$ such that

$$\widehat{\phi}((2k_0+1)\pi) \neq 0. \tag{3.36}$$

If we now set $\omega = 2(2k_0+1)\pi$ in (3.24), it follows from (3.32) that

$$0 = \widehat{\phi}(2(2k_0+1)\pi) = P(-1)\widehat{\phi}((2k_0+1)\pi), \tag{3.37}$$

so that (3.37) and (3.36) yield $P(-1) = 0$, i.e. by (3.23), we have

$$\sum_k p_{2k} = \sum_k p_{2k+1}. \tag{3.38}$$

Combining (3.35) and (3.38) then yields (3.33). ∎

It should be pointed out that the Haar function ϕ_H, as given in (3.5), is not a scaling function according to our Definition 3.1, since $\phi_H \notin C(\Re)$, and thus $\phi_H \notin \mathcal{A}$. Also, as obtained from (2.4) and (3.5),

$$\widehat{\phi}_H(\omega) = \begin{cases} e^{-i\omega/2}\dfrac{\sin(\omega/2)}{\omega/2}, & \omega \in \Re \setminus \{0\}, \\ 1, & \omega = 0, \end{cases}$$

so that $\widehat{\phi}_H \notin L^1(\Re)$. However, as shown in [9, Section 1.3.3], if we choose $\phi = \phi_H$ in (3.7) and (3.8), the resulting sequence $\{V_j\}$ is indeed a nested sequence of closed linear subspaces of $L^2(\Re)$ satisfying (3.9), (3.10) and (3.11), and therefore (3.22), i.e. $\{V_j\}$ can be interpreted as the (Haar) MRA generated by ϕ_H. In this thesis however, we shall, apart from $\phi = \phi_H$, only consider scaling functions according to our Definition 3.1.

Next, we give a set of sufficient conditions on a sequence $\{p_k\}$ for the existence of a continuous solution ϕ of the refinement equation (3.2).

As proved in [14, Theorem 2.5 (second proof, pp.78-81), Corollary 2.6, Proposition 2.1 and pp 96-97], the following result holds.

Theorem 3.6 *Let N denote an integer with $N \geq 2$, and suppose $\{p_k\}$ is a finitely supported real-valued sequence such that*

(a) $p_k = 0, \quad k \notin \{0, 1, \ldots, N\};$ (3.39)

(b) $p_k > 0, \quad k = 0, 1, \ldots, N;$

(c) the condition (3.33) holds;

(d) the polynomial P defined by (3.23) is a Hurwitz polynomial.

Then there exists a unique function $\phi \in C(\Re)$ such that

(i) ϕ is finitely supported, with

$$\phi(t) = 0, \quad t \notin [0, N]; \tag{3.40}$$

(ii) $\phi(t) > 0, \ \ t \in (0, N)$;

(iii) ϕ *is refinable in the sense of (3.2);*

(iv) ϕ *satisfies the Riesz stability condition (3.3);*

(v) ϕ *satisfies the partition of unity condition (3.4).*

Moreover, the sequence $\{V_j\}$ *defined by (3.7) and (3.8) is a nested sequence of closed linear subspaces of* $L^2(\Re)$ *which satisfies (3.9), (3.10) and (3.11).*

An efficient method for the numerical evaluation of a scaling function ϕ is provided by the *cascade algorithm*, according to which the sequence $\{\phi_k : k = 0, 1, 2, \ldots\}$ is generated recursively by means of the formulas

$$\phi_0(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ 2 - t, & 1 < t \leq 2, \\ 0, & otherwise, \end{cases} \tag{3.41}$$

and

$$\phi_n(t) = \sum_k p_k \phi_{n-1}(2t - k), \ \ t \in \Re, \ \ n = 1, 2, \ldots. \tag{3.42}$$

One set of sufficient conditions on the two-scale sequence $\{p_k\}$ for the convergence of the cascade algorithm follows from the method of proof in [14, Theorem 2.5, second proof, pp.78-81]. The result is as follows.

Theorem 3.7 *For an integer* N *with* $N \geq 2$*, suppose* $\{p_k\}$ *is a finitely supported sequence in* $\Re$ *satisfying the conditions (a), (b) and (c) of Theorem 3.6. Then the cascade algorithm (3.41), (3.42) generates a sequence* $\{\phi_n : n \in \mathbf{N}\} \subset C(\Re)$ *of compactly supported functions that converges uniformly on* $\Re$ *to the refinable function* ϕ *of Theorem 3.6.*

It can be shown as in [2, p.132], that the support property (3.40) holds in general for the cascade algorithm in the following sense:

Theorem 3.8 *Suppose ϕ is a refinable function in the sense that (3.2) holds, with two-scale sequence $\{p_k\}$ satisfying (3.39) with $p_0, p_N \neq 0$, and suppose the cascade algorithm (3.41), (3.42) converges uniformly to a solution $\phi \in C(\Re)$ of (3.2). Then ϕ is compactly supported, with*

$$\text{supp } \phi = [0, N].$$

Proof. Using mathematical induction, we shall show that for the sequence $\{\phi_n : n \in \mathbf{N}\}$, as generated by the cascade algorithm (3.41), (3.42), we have

$$\text{supp } \phi_n = \left[0, \frac{2 + (2^n - 1)N}{2^n}\right], \quad n \in \mathbf{N}. \tag{3.43}$$

From (3.41) it is clear that

$$\text{supp } \phi_0 = [0, 2],$$

and thus (3.43) holds for $n = 0$. Suppose (3.43) is true for $n = l$, i.e.

$$\text{supp } \phi_l = \left[0, \frac{2 + (2^l - 1)N}{2^l}\right],$$

then, using (3.42), we have, for $t \in \Re$ and $n = l + 1$,

$$\begin{aligned} \phi_{l+1}(t) &= \sum_{k=0}^{N} p_k \phi_l(2t - k) \\ &= p_0 \phi_l(2t) + \ldots + p_N \phi_l(2t - N), \end{aligned}$$

so that the left endpoint of the support interval of ϕ_{l+1} is given by the left endpoint of the support interval of $\phi_l(2\cdot)$, which is 0, whereas the right endpoint of the support interval of ϕ_{l+1} is given by the right endpoint of the support interval of $\phi_l(2 \cdot - N)$, which is where

$$2t - N = \frac{2 + (2^l - 1)N}{2^l},$$

and thus

$$\begin{aligned} t &= \frac{2+(2^l-1)N}{2^{l+1}} + \frac{N}{2} \\ &= \frac{2+(2^{l+1}-1)N}{2^{l+1}}. \end{aligned}$$

Hence

$$\text{supp } \phi_{l+1} = \left[0, \frac{2+(2^{l+1}-1)N}{2^{l+1}}\right],$$

so that (3.43) is true for $n = l+1$, and therefore by induction true for all $n \in \mathbf{N}$. Note that

$$\frac{2+(2^n-1)N}{2^n} = N - \frac{N-2}{2^n} \leq N, \quad n = 0, 1, \ldots$$

because $N \geq 2$, so that supp $\phi_n \subseteq [0, N]$, $n \in \mathbf{N}$. Lastly, since the cascade algorithm converges uniformly, we may take the limit of (3.43) as $n \to \infty$ to obtain the support of ϕ:

$$\begin{aligned} \text{supp } \phi = \text{supp } \lim_{n\to\infty} \phi_n &= \lim_{n\to\infty} \text{supp } \phi_n \\ &= \left[0, \lim_{n\to\infty} \frac{2+(2^n-1)N}{2^n}\right] = [0, N]. \end{aligned}$$

■

Finally, we show that the property of orthonormality in the context of scaling functions has the significant advantage that the Riesz stability property is then automatically satisfied.

Theorem 3.9 *Suppose the function $\phi \in \mathcal{A}$ satisfies the condition (a) of Definition 3.1 of a scaling function. Then the orthonormality condition*

$$\langle \phi(\cdot - k), \phi(\cdot - l) \rangle = \delta_{k,l}, \quad k, l \in \mathbf{Z}, \tag{3.44}$$

is satisfied if and only if ϕ is Riesz stable in the sense of (3.3) with Riesz bounds A=B=1.

Proof. The result is an immediate consequence of Theorem 2.4 and Theorem 3.3(b). ∎

We shall say that a scaling function ϕ is an *orthonormal scaling function* if the condition (3.44) holds. The property of orthonormality of a scaling function ϕ is useful for finding the coefficients $\{c_k\} \in \ell^2$ such that, for a given $f \in V_n$,

$$f = \sum_k c_k \phi_{n,k}.$$

This is done by taking the inner product of f with $\phi_{n,l}$ in order to find c_l, $l \in \mathbf{Z}$:

$$\begin{aligned} \langle f, \phi_{n,l} \rangle &= \left\langle \sum_k c_k \phi_{n,k}, \phi_{n,l} \right\rangle \\ &= \sum_k c_k \langle \phi_{n,k}, \phi_{n,l} \rangle \\ &= \sum_k c_k \delta_{l,k} \\ &= c_l, \end{aligned}$$

so that we have a formula for c_l.

We shall see later that the spline functions of order $m > 1$ are not orthonormal, whereas it is easy to verify from (3.5) that the spline of order $m = 1$, i.e. the Haar function, is orthonormal.

Ingrid Daubechies (see [9, Chapter 2]) constructed finitely supported orthonormal scaling functions $\phi = \phi_{D,m}$ of order $m = 1, 2, \ldots$, where the Daubechies scaling function of order 1 is the Haar scaling function, i.e. $\phi_{D,1} = \phi_H$. The scaling functions $\phi_{D,m}$ for $m > 1$ can not be explicitly formulated and are not symmetric. However, the finite two-scale sequences $\{p_{m,k}\}$ of $\phi_{D,m}$, for $m = 2, \ldots, 10$, are given in [9, Table 6.1, p.195], so that we have, by example, for order $m = 2$,

$$\phi_{D,2}(t) = \sum_{k=0}^{3} p_{2,k} \phi_{D,2}(2t - k), \quad t \in \Re,$$

with

$$\left.\begin{aligned} p_{2,0} &= \tfrac{1}{4}(1+\sqrt{3}), \\ p_{2,1} &= \tfrac{1}{4}(3+\sqrt{3}), \\ p_{2,2} &= \tfrac{1}{4}(3-\sqrt{3}), \\ p_{2,3} &= \tfrac{1}{4}(1-\sqrt{3}). \end{aligned}\right\} \tag{3.45}$$

Also, from [9], $\phi_{D,2} \in C(\Re) \setminus C^1(\Re)$, with supp $\phi_{D,2} = [0,3]$, as illustrated by the graph in Fig.3.2, which was generated by means of the cascade algorithm (3.45), (3.41), (3.42).

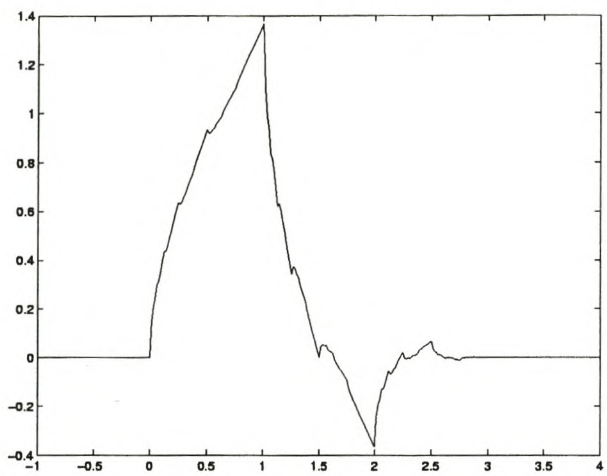


Figure 3.2: The Daubechies scaling function $\phi_{D,2}$ of order 2

3.2 The Dual Scaling Function

Before proceeding to the construction of wavelets, we first introduce in this section the concept of the dual $\tilde{\phi}$ of a scaling function ϕ. For the sake of simplicity, we shall restrict attention to the case where ϕ is compactly supported.

For a given scaling function ϕ, we define the *auto-correlation function* Φ associated with ϕ by

$$\Phi(t) := \langle \phi, \phi(\cdot + t) \rangle = \int_{-\infty}^{\infty} \phi(x)\overline{\phi(x+t)}\, \mathrm{d}x, \quad t \in \Re. \tag{3.46}$$

Observe that the function Φ is then also of finite support.

Theorem 3.10 *Suppose ϕ is a compactly supported scaling function, and denote by Φ the corresponding autocorrelation function, as given by (3.46). Then*

$$\sum_k |\widehat{\phi}(\omega + 2\pi k)|^2 = \sum_k \Phi(k) e^{i\omega k}, \quad \omega \in \Re. \tag{3.47}$$

Proof. From (2.2), (3.46) and (2.8), we have, for $l \in \mathbf{Z}$, and with the function $u \in \mathcal{A}$ defined by $u(t) := \phi(t + l), t \in \Re$, whence $\widehat{u}(\omega) = e^{i\omega l}\widehat{\phi}(\omega)$, $\omega \in \Re$,that

$$\begin{aligned}
\Phi(l) &= \langle \phi, u \rangle \\
&= \frac{1}{2\pi} \left\langle \widehat{\phi}, \widehat{u} \right\rangle \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\phi}(\omega)\overline{\widehat{\phi}(\omega)} e^{-i\omega l}\, \mathrm{d}\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\widehat{\phi}(\omega)|^2 e^{-i\omega l}\, \mathrm{d}\omega \\
&= \frac{1}{2\pi} \sum_k \int_{2\pi k}^{2\pi(k+1)} |\widehat{\phi}(\omega)|^2 e^{-i\omega l}\, \mathrm{d}\omega \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_k |\widehat{\phi}(\omega + 2\pi k)|^2 e^{-i\omega l}\, \mathrm{d}\omega.
\end{aligned} \tag{3.48}$$

But, since $\phi \in \mathcal{A}$, and since (3.1) holds for some $\alpha > 1$, we can appeal to Theorem 2.3 to deduce that the function G defined by

$$G(\omega) := \sum_k |\widehat{\phi}(\omega + 2\pi k)|^2, \quad \omega \in \Re, \tag{3.49}$$

is a real-valued continuous function on $\Re$ which is 2π-periodic:

$$G(\omega + 2\pi) = G(\omega), \quad \omega \in \Re,$$

and which, by Theorem 2.2, is such that the Fourier series of G converges everywhere to G:

$$G(\omega) = \sum_k \left[\frac{1}{2\pi} \int_0^{2\pi} G(t) e^{-ikt}\, \mathrm{d}t \right] e^{i\omega k}, \quad \omega \in \Re. \tag{3.50}$$

Combining (3.48), (3.49) and (3.50) then yields the desired identity (3.47). ∎

Observe in particular that, since Φ is a finitely supported function, the sequence $\{\Phi(k) : k \in \mathbf{Z}\}$ is also finitely supported. If we now define the Laurent polynomial

$$E(z) := \sum_k \Phi(k) z^k, \quad z \in \mathbf{C} \setminus \{0\}, \tag{3.51}$$

which is called the *Euler-Frobenius Laurent polynomial* associated with ϕ, we can combine (3.25) and (3.47) to deduce that the following result holds on the unit circle $|z| = 1$.

Theorem 3.11 *Suppose ϕ is a scaling function with corresponding Euler-Frobenius Laurent polynomial E defined by (3.51), (3.46). Then*

$$E(e^{i\omega}) = \sum_k |\widehat{\phi}(\omega + 2\pi k)|^2, \quad \omega \in \Re, \tag{3.52}$$

with

$$0 < A \leq E(e^{i\omega}) \leq B, \quad \omega \in \Re, \tag{3.53}$$

and with A and B denoting the Riesz bounds in (3.3).

The following fundamental result then holds.

Theorem 3.12 *Suppose ϕ is a finitely supported scaling function which generates the MRA $\{V_j\}$. Then the function $\widetilde{\phi}$ defined by*

$$\widetilde{\phi}(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \left[\frac{\widehat{\phi}(\omega)}{E(e^{i\omega})} \right] \mathrm{d}\omega, \quad t \in \Re, \tag{3.54}$$

is a scaling function which also generates the MRA $\{V_j\}$. Moreover,

$$\widehat{\widetilde{\phi}}(\omega) = \frac{\widehat{\phi}(\omega)}{E(e^{i\omega})}, \quad \omega \in \Re, \tag{3.55}$$

and

$$\left\langle \phi(\cdot - k), \widetilde{\phi}(\cdot - l) \right\rangle = \delta_{k,l}, \quad k, l \in \mathbf{Z}. \tag{3.56}$$

Proof. With the function g defined by

$$g(\omega) := \frac{\widehat{\phi}(\omega)}{E(e^{i\omega})}, \quad \omega \in \Re, \tag{3.57}$$

it follows from the fact that $\phi \in \mathcal{A}$, and thus, from Theorem 2.1, $\widehat{\phi} \in \mathcal{A} \subset L^1(\Re) \cap C_u(\Re)$, and the fact that the Laurent polynomial E is a continuous function on the unit circle $|z| = 1$ which satisfies (3.53), that $g \in L^1(\Re) \cap C_u(\Re)$.

Now use the fact that, according to (3.53), the function h defined by

$$h(z) = \frac{1}{E(z)}, \quad z \in \mathbf{C} \setminus \{0\}, \tag{3.58}$$

is analytic in a neighborhood of the unit circle $|z| = 1$; and thus, appealing to standard theory in complex analysis (see e.g. [8, Theorem 59]), that there exists a sequence $\{h_k\}$ with exponential decay for $|k| \to \infty$, so that also $\{h_k\} \in \ell^1 \cap \ell^2$, such that

$$h(z) = \sum_k h_k z^k, \quad |z| = 1. \tag{3.59}$$

It follows from (3.58), (3.59) and (3.57) that

$$g(\omega) = h(e^{i\omega})\widehat{\phi}(\omega), \quad \omega \in \Re. \tag{3.60}$$

Now use (2.4), (3.59) and (3.60) to obtain, for $\omega \in \Re$,

$$\begin{aligned}
\widehat{g}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} \sum_k h_k e^{itk} \widehat{\phi}(t) \, \mathrm{d}t \\
&= \sum_k h_k \int_{-\infty}^{\infty} e^{i(-\omega+k)t} \widehat{\phi}(t) \, \mathrm{d}t \\
&= 2\pi \sum_k h_k \phi(-\omega + k),
\end{aligned}$$

by virtue of Theorem 2.1, together with the fact that $\phi, \widehat{\phi} \in \mathcal{A}$. Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} |\widehat{g}(\omega)| \, \mathrm{d}\omega &\leq 2\pi \sum_k |h_k| \int_{-\infty}^{\infty} |\phi(-\omega + k)| \, \mathrm{d}\omega \\
&= 2\pi \left[\sum_k |h_k| \right] \left[\int_{-\infty}^{\infty} |\phi(t)| \, \mathrm{d}t \right] < \infty,
\end{aligned} \tag{3.61}$$

since $\phi \in L^1(\Re)$, and since $\sum_k |h_k|$ is a convergent series by virtue of the fact that the sequence $\{h_k\}$ has exponential decay for $|k| \to \infty$. Hence also $\widehat{g} \in L^1(\Re)$, and it follows that $g \in \mathcal{A}$. But then, according to (3.54) and Theorem 2.1, we have $\phi \in \mathcal{A}$ and $\widehat{\widetilde{\phi}} = g$, so that (3.55) follows from (3.57).

To prove that $\widetilde{\phi}$ is a scaling function, we first use a similar technique to the one employed in the proof of Theorem 3.3(a), to find, recalling also the fact that $\widehat{\widetilde{\phi}} = g$, that (3.60) is equivalent to

$$\widetilde{\phi}(t) = \sum_k h_k \phi(t+k), \quad t \in \Re. \tag{3.62}$$

Since ϕ is compactly supported, and the sequence $\{h_k\}$ has exponential decay for $|k| \to \infty$, it follows from (3.62) that the function $\widetilde{\phi}$ has exponential decay for $|t| \to \infty$. Recalling also (3.55), (3.53), and the fact that $\widehat{\phi}$ satisfies (3.1), we deduce that $\widetilde{\phi}$ and $\widehat{\widetilde{\phi}}$ both satisfy (3.1), with ϕ replaced by $\widetilde{\phi}$, for some $\alpha > 1$.

Next, we observe from (3.55), (3.58) and (3.24), that

$$\widehat{\widetilde{\phi}}(\omega) = A(e^{-i\omega/2})\widehat{\widetilde{\phi}}\left(\frac{\omega}{2}\right), \quad \omega \in \Re, \tag{3.63}$$

where

$$A(z) := h\left(\frac{1}{z^2}\right) E\left(\frac{1}{z}\right) P(z), \quad z \in \mathbf{C} \setminus \{0\}. \tag{3.64}$$

But the sequence $\{h_k\}$, the finitely supported sequence $\{\Phi(k) : k \in \mathbf{Z}\}$, as well as the two-scale sequence $\{p_k\}$, are all in $\ell^1 \cap \ell^2$. It can then be shown from (3.64), as in the proof of Theorem 3.2(a), that there exists a sequence $\{a_k\} \in \ell^1 \cap \ell^2$ such that

$$A(z) = \frac{1}{2} \sum_k a_k z^k, \quad |z| = 1. \tag{3.65}$$

Next, observe from (3.55) that

$$\sum_k |\widehat{\widetilde{\phi}}(\omega + 2\pi k)|^2 = \frac{1}{[E(e^{i\omega})]^2} \sum_k |\widehat{\phi}(\omega + 2\pi k)|^2, \quad \omega \in \Re,$$

which, together with (3.53) and (3.25), gives

$$\frac{A}{B^2} \leq \sum_k |\widehat{\widetilde{\phi}}(\omega + 2\pi k)|^2 \leq \frac{B}{A^2}, \quad \omega \in \Re. \tag{3.66}$$

Next, we use (3.55), (3.52), (3.26) and (3.32) to deduce that

$$\widehat{\widetilde{\phi}}(0) = \frac{\widehat{\phi}(0)}{E(1)} = \frac{\widehat{\phi}(0)}{\sum_k \widehat{\phi}(2\pi k)} = 1. \tag{3.67}$$

Hence, recalling also (3.63), (3.66) and (3.67), we see that $\widetilde{\phi} \in \mathcal{A}$, that $\widetilde{\phi}$ and $\widehat{\widetilde{\phi}}$ both satisfy (3.1), with ϕ replaced by $\widetilde{\phi}$, and also that $\widetilde{\phi}$ satisfies (3.24), (3.25) and (3.26) with ϕ replaced by $\widetilde{\phi}$, with P replaced by A as given in (3.65) with $\{a_k\} \in \ell^1 \cap \ell^2$, and with lower and upper Riesz bounds given by, respectively, $\frac{A}{B^2}$ and $\frac{B}{A^2}$. We can therefore appeal to Theorem 3.3 to deduce that $\widetilde{\phi}$ is indeed a scaling function.

Now define the sequence $\{\widetilde{\phi}_{j,k} : j, k \in \mathbf{Z}\} \subset \mathcal{A}$ by

$$\widetilde{\phi}_{j,k} := 2^{j/2}\widetilde{\phi}(2^j t - k), \quad t \in \Re, \quad j, k \in \mathbf{Z},$$

so that the sequence $\{\widetilde{V}_j\}$ of linear spaces defined by

$$\widetilde{V}_j := \left\{ \sum_k c_k \widetilde{\phi}_{j,k} : \{c_k\} \in \ell^2 \right\}, \quad j \in \mathbf{Z}, \tag{3.68}$$

is the MRA generated by the scaling function $\widetilde{\phi}$. We claim that

$$\widetilde{V}_j = V_j, \quad j \in \mathbf{Z}, \tag{3.69}$$

i.e. the two scaling functions ϕ and $\widetilde{\phi}$ generate the same MRA $\{V_j\}$.

To prove (3.69), we need to show that $\widetilde{V}_j \subset V_j$ and $V_j \subset \widetilde{V}_j$, for all $j \in \mathbf{Z}$. But these statements can be proved by using the same technique as was used in the proof of Theorem 3.2(a), provided that we can find sequences $\{\alpha_k\}$, $\{\beta_k\} \in \ell^1 \cap \ell^2$ such that

$$\widetilde{\phi}(t) = \sum_k \alpha_k \phi(t - k), \quad t \in \Re, \tag{3.70}$$

and

$$\phi(t) = \sum_k \beta_k \widetilde{\phi}(t-k), \quad t \in \Re. \tag{3.71}$$

Arguing as in the proof of Theorem 3.3(a), we can moreover show that (3.70) and (3.71) have, respectively, the equivalent Fourier transform formulations

$$\widehat{\widetilde{\phi}}(\omega) = \widetilde{A}(e^{-i\omega})\widehat{\phi}(\omega), \quad \omega \in \Re, \tag{3.72}$$

and

$$\widehat{\phi}(\omega) = \widetilde{B}(e^{-i\omega})\widehat{\widetilde{\phi}}(\omega), \quad \omega \in \Re, \tag{3.73}$$

with the Laurent series $\widetilde{A}$ and $\widetilde{B}$ defined by

$$\widetilde{A}(z) := \sum_k \alpha_k z^k, \quad z \in \mathbf{C} \setminus \{0\},$$

and

$$\widetilde{B}(z) := \sum_k \beta_k z^k, \quad z \in \mathbf{C} \setminus \{0\}.$$

Hence, if we can prove that there exist sequences $\{\alpha_k\}$, $\{\beta_k\} \in \ell^1 \cap \ell^2$ such that (3.72), (3.73) hold, then (3.69) would follow. But (3.55), (3.60) and the fact that $\widehat{\widetilde{\phi}} = g$, show that (3.72) holds with $\widetilde{A}$ defined by $\widetilde{A}(z) = h\left(\frac{1}{z}\right)$, $z \in \mathbf{C} \setminus \{0\}$, whereas (3.73) holds with $\widetilde{B} = E$, a Laurent polynomial, thereby establishing (3.72) and (3.73).

It remains to prove condition (3.56). Using the Parseval identity (2.8), together with (3.55) and (3.52), we find, for $k, l \in \mathbf{Z}$, and with the functions v and w defined by $v(t) := \phi(t-k)$, $t \in \Re$ and $w(t) := \widetilde{\phi}(t-l)$, $t \in \Re$, that, for $k, l \in \mathbf{Z}$,

$$\begin{aligned}
\left\langle \phi(\cdot - k), \widetilde{\phi}(\cdot - l) \right\rangle &= \langle v, w \rangle \\
&= \frac{1}{2\pi} \langle \widehat{v}, \widehat{w} \rangle \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(l-k)} \frac{|\widehat{\phi}(\omega)|^2}{E(e^{i\omega})} \, \mathrm{d}\omega \\
&= \frac{1}{2\pi} \sum_n \int_{2\pi n}^{2\pi(n+1)} e^{i\omega(l-k)} \frac{|\widehat{\phi}(\omega)|^2}{E(e^{i\omega})} \, \mathrm{d}\omega
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\sum_n \int_0^{2\pi} e^{i\omega(l-k)} \frac{|\widehat{\phi}(\omega+2\pi n)|^2}{E(e^{i\omega})}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i\omega(l-k)} \frac{\sum_n |\widehat{\phi}(\omega+2\pi n)|^2}{E(e^{i\omega})}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{i\omega(l-k)}\,\mathrm{d}\omega \\
&= \delta_{k,l}.
\end{aligned}
$$

■

The scaling function $\tilde{\phi}$, as obtained from ϕ in Theorem 3.12, is called the *dual scaling function* of ϕ, and will be used in our wavelet construction procedure of Section 4.1 below.

One important advantage of the availability of $\tilde{\phi}$ is evident from our next result.

Theorem 3.13 *Let ϕ and $\tilde{\phi}$ be as in Theorem 3.12, and suppose, for a given $j \in \mathbf{Z}$, that $f \in V_j$. Then*

$$f(t) = \sum_k \left\langle f, \tilde{\phi}(\cdot - k)\right\rangle \phi(t-k), \quad t \in \Re, \tag{3.74}$$

and

$$f(t) = \sum_k \langle f, \phi(\cdot - k)\rangle\, \tilde{\phi}(t-k), \quad t \in \Re. \tag{3.75}$$

Proof. Since $f \in V_j$, there exists a sequence $\{c_k\} \in \ell^2$ such that

$$f(t) = \sum_k c_k \phi(t-k), \quad t \in \Re. \tag{3.76}$$

But then (3.56) gives, for $l \in \mathbf{Z}$,

$$
\begin{aligned}
\left\langle f, \tilde{\phi}(\cdot - l)\right\rangle &= \left\langle \sum_k c_k \phi(\cdot - k), \tilde{\phi}(\cdot - l)\right\rangle \\
&= \sum_k c_k \left\langle \phi(\cdot - k), \tilde{\phi}(\cdot - l)\right\rangle \\
&= \sum_k c_k \delta_{k,l} = c_l,
\end{aligned}
$$

which, together with (3.76), yields (3.74).

Also, from (3.69) and (3.68), there exists a sequence $\{d_k\} \in \ell^2$ such that

$$f(t) = \sum_k d_k \widetilde{\phi}(t-k), \quad t \in \Re. \tag{3.77}$$

Then, from (3.56), with $l \in \mathbf{Z}$,

$$\begin{aligned}
\langle f, \phi(\cdot - l)\rangle &= \left\langle \sum_k d_k \widetilde{\phi}(\cdot - k), \phi(\cdot - l)\right\rangle \\
&= \sum_k d_k \overline{\left\langle \phi(\cdot - l), \widetilde{\phi}(\cdot - k)\right\rangle} \\
&= \sum_k d_k \delta_{k,l} = d_l,
\end{aligned}$$

which, together with (3.77), yields (3.75). ■

Regarding the uniqueness of a dual scaling function, we can now prove the following result.

Theorem 3.14 *Suppose ϕ and $\widetilde{\phi}$ are as in Theorem 3.12, and suppose $\widetilde{\phi}_0$ is a scaling function generating the same MRA $\{V_j\}$ as ϕ, with*

$$\left\langle \phi(\cdot - k), \widetilde{\phi}_0(\cdot - l)\right\rangle = \delta_{k,l}, \quad k, l \in \mathbf{Z}. \tag{3.78}$$

Then $\widetilde{\phi}_0 = \widetilde{\phi}$.

Proof. Define $\widetilde{\phi}_1 := \widetilde{\phi} - \widetilde{\phi}_0$; then, from (3.56) and (3.78),

$$\left\langle \phi(\cdot - k), \widetilde{\phi}_1\right\rangle = 0, \quad k \in \mathbf{Z}. \tag{3.79}$$

By the linearity of V_0 we have $\widetilde{\phi}_1 \in V_0$ and thus, from (3.75), and for $t \in \Re$,

$$\begin{aligned}
\widetilde{\phi}_1(t) &= \sum_k \left\langle \widetilde{\phi}_1, \phi(\cdot - k)\right\rangle \widetilde{\phi}(t-k) \\
&= \sum_k \overline{\left\langle \phi(\cdot - k), \widetilde{\phi}_1\right\rangle} \widetilde{\phi}(t-k),
\end{aligned}$$

which, together with (3.79), yield $\widetilde{\phi}_1 = 0$, i.e. $\widetilde{\phi} = \widetilde{\phi}_0$. ∎

One advantage of a scaling function ϕ being orthonormal in the sense of (3.44) is already evident from Theorem 3.9.

Another advantage is shown in the following theorem.

Theorem 3.15 *Suppose ϕ is a finitely supported orthonormal scaling function. Then ϕ is self-dual, i.e. $\widetilde{\phi} = \phi$.*

Proof. Observe from (3.46) and (3.44) that, in this case,

$$\Phi(k) = \langle \phi, \phi(\cdot + k) \rangle = \delta_{k,0}, \quad k \in \mathbf{Z},$$

so that (3.51) gives

$$E(z) = 1, \quad z \in \mathbf{C}.$$

But then (3.55), together with Theorem 2.1, implies that $\widehat{\widetilde{\phi}} = \widehat{\phi}$, and thus $\widetilde{\phi} = \phi$. ∎

If a scaling function ϕ is not orthonormal, the following orthonormalization procedure can be used to construct an orthonormal scaling function.

Theorem 3.16 *Suppose ϕ is a finitely supported scaling function which generates the MRA $\{V_j\}$. Then the function $\phi^{\perp}$ defined by*

$$\phi^{\perp}(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\omega} \left[\frac{\widehat{\phi}(\omega)}{\{E(e^{i\omega})\}^{1/2}} \right] d\omega, \quad t \in \Re, \tag{3.80}$$

is an orthonormal scaling function which also generates the MRA $\{V_j\}$. Moreover,

$$\widehat{\phi}^{\perp}(\omega) = \frac{\widehat{\phi}(\omega)}{\{E(e^{i\omega})\}^{1/2}}, \quad \omega \in \Re. \tag{3.81}$$

Proof. The fact that $\phi^{\perp}$ is a scaling function which generates the MRA $\{V_j\}$, and the formula (3.81) for $\widehat{\phi}^{\perp}$ is proved similarly to the analogous result for $\widetilde{\phi}$ in

Theorem 3.12. The orthonormality condition (3.44) is proved by using (3.81) to deduce that, for $\omega \in \Re$,

$$\begin{aligned}\sum_k |\widehat{\phi}^{\perp}(\omega + 2\pi k)|^2 &= \frac{\sum_k |\widehat{\phi}(\omega + 2\pi k)|^2}{E(e^{i\omega})} \\ &= 1,\end{aligned}$$

by virtue of (3.52), and (3.44) follows from Theorem 3.9. ■

Hence, given a finitely supported scaling function ϕ, the formula (3.80) yields a construction for an orthonormal scaling function $\phi^{\perp}$ generated by ϕ. Note, however, that $\phi^{\perp}$ is no longer necessarily finitely supported.

We have therefore shown in this section that, for a given finitely supported scaling function ϕ, there exists a unique dual scaling function $\widetilde{\phi}$, as given by the formula (3.54), such that $\widetilde{\phi}$ generates the same MRA $\{V_j\}$ as does ϕ, and for which the property (3.56) holds. An orthonormal scaling function ϕ has the advantage of self-duality, i.e. $\widetilde{\phi} = \phi$, whereas, if ϕ is not orthonormal, we have available the option to construct from ϕ the (possibly non-finitely supported) orthonormal scaling function $\phi^{\perp}$.

Chapter 4

Wavelets in $L^2(\Re)$

In this chapter we shall demonstrate how the concepts of a scaling function ϕ and its dual $\widetilde{\phi}$, and the MRA $\{V_j\}$ thus generated, can be employed for the construction of wavelets. Once again, we essentially follow the approach of [2] and [3].

4.1 A Construction Technique Based on a Multiresolutional Analysis

Suppose ϕ is a scaling function which generates the MRA $\{V_j\}$. Noting from Theorem 3.2(a) that

$$V_j \subset V_{j+1}, \quad j \in \mathbf{Z},$$

we define

$$W_j = V_j^{\perp} := \{f \in V_{j+1} : \langle f, g \rangle = 0, \ g \in V_j\}, \quad j \in \mathbf{Z}, \tag{4.1}$$

i.e. W_j is the orthogonal complement of V_j with respect to V_{j+1}.

In general, if W and X are linear subspaces of a linear space V, with $W \perp X$, i.e. W and X are mutually orthogonal, then the orthogonal decomposition notation

$$V = W \oplus X$$

will be used to denote that, for each $f \in V$, there exist elements $g \in W$ and $h \in X$, with g and h uniquely determined by f, such that

$$f = g + h.$$

The following basic result then holds.

Theorem 4.1 *Suppose ϕ is a scaling function which generates the MRA $\{V_j\}$. Then*

$$V_{j+1} = V_j \oplus W_j, \quad j \in \mathbf{Z}, \tag{4.2}$$

where the sequence $\{W_j\}$ is defined by (4.1).

Proof. Suppose $f \in V_{j+1}$. Then, according to Theorem 3.2, $f \in L^2(\Re)$, a Hilbert space, and V_j is a closed linear subspace of $L^2(\Re)$. Hence we may appeal to the projection theorem [12, Theorem 6.2.2], to deduce that (4.2) does indeed hold. ■

The following properties can now immediately be shown.

Theorem 4.2 *In Theorem 4.1, we have*

$$V_j \cap W_j = \{0\}, \quad j \in \mathbf{Z}, \tag{4.3}$$

$$W_k \cap W_j = \{0\}, \quad k \neq j, \quad k, j \in \mathbf{Z}, \tag{4.4}$$

$$W_k \perp W_j, \quad k \neq j, \quad k, j \in \mathbf{Z}. \tag{4.5}$$

Proof. Suppose $f \in V_j \cap W_j$. But then, from the definition (4.1) of W_j, we have $\langle f, f \rangle = 0$ and thus $f = 0$, thereby proving (4.3).

To prove (4.4), suppose, for given $k, j \in \mathbf{Z}$ with $k < j$, we have $f \in W_k \cap W_j$. Then, since $W_k \subset V_{k+1} \subseteq V_j$ from (4.2), it follows that $f \in V_j \cap W_j$, and thus, from (4.3), $f = 0$. Hence

$$W_k \cap W_j = \{0\}, \quad k, j \in \mathbf{Z},$$

thereby proving (4.4).

To prove (4.5), suppose, for given $k, j \in \mathbf{Z}$ with $k < j$, we have $f \in W_k$, $g \in W_j$. But (4.2) gives $f \in V_{k+1} \subseteq V_j$, and thus, from the definition (4.1), we have $\langle f, g \rangle = 0$. Hence

$$W_k \perp W_j, \quad k, j \in \mathbf{Z},$$

thereby proving (4.5). ■

In view of (3.22) and (4.2), we see that

$$L^2(\Re) = \bigoplus_{j=-\infty}^{\infty} W_j := \ldots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \ldots, \tag{4.6}$$

i.e. the sequence $\{W_j\}$ provides an orthogonal decomposition of $L^2(\Re)$.

A wavelet in $L^2(\Re)$ is now defined as follows.

Definition 4.1 *A function $\psi \in L^2(\Re)$ is called a wavelet in $L^2(\Re)$ if the following conditions are satisfied:*

(a) the sequence $\{W_j\}$ of linear spaces defined by

$$W_j := \left\{ \sum_k c_k \psi_{j,k} : \{c_k\} \in \ell^2 \right\}, \quad j \in \mathbf{Z}, \tag{4.7}$$

where the sequence $\{\psi_{j,k} : j, k \in \mathbf{Z}\} \subset L^2(\Re)$ is given by

$$\psi_{j,k}(t) := 2^{j/2}\psi(2^j t - k), \quad t \in \Re, \quad j, k \in \mathbf{Z}, \tag{4.8}$$

is an orthogonal decomposition of $L^2(\Re)$ in the sense of (4.6);

(b) ψ is Riesz-stable in the sense that there exist real numbers α and β, with $0 < \alpha \leq \beta$, such that

$$\alpha \|\{c_k\}\|_{\ell^2}^2 \leq \left\| \sum_k c_k \psi(\cdot - k) \right\|_2^2 \leq \beta \|\{c_k\}\|_{\ell^2}^2, \quad \{c_k\} \in \ell^2. \tag{4.9}$$

The linear spaces $\{W_j\}$ are then called the wavelet spaces generated by the wavelet ψ.

The following wavelet expansion result then immediately follows.

Theorem 4.3 *Suppose ψ is a wavelet in $L^2(\Re)$, and suppose $f \in L^2(\Re)$. Then there exists a unique coefficient sequence $\{d_{j,k} : j, k \in \mathbf{Z}\} \subset \mathbf{C}$ such that*

$$f = \sum_{j,k} d_{j,k} \psi_{j,k}, \tag{4.10}$$

with the sequence $\{\psi_{j,k}\} \subset L^2(\Re)$ defined as in (4.8).

Proof. Since (4.6) holds, we see that there exists, for a given $f \in L^2(\Re)$, a unique sequence $\{g_j\} \subset L^2(\Re)$, with $g_j \in W_j$, $j \in \mathbf{Z}$, such that

$$f = \sum_j g_j.$$

But from the definition (4.7), there exists, for each $j \in \mathbf{Z}$, a sequence $\{c_k^{(j)}\} \in \ell^2$ such that

$$g_j = \sum_k c_k^{(j)} \psi_{j,k}. \tag{4.11}$$

Moreover, similar to the proof of (3.13), we can show, from (4.9), that the condition

$$\alpha \|\{c_k\}\|_{\ell^2}^2 \leq \left\| \sum_k c_k \psi_{j,k} \right\|_2^2 \leq \beta \|\{c_k\}\|_{\ell^2}^2, \quad \{c_k\} \in \ell^2, \tag{4.12}$$

holds. Using (4.12), we can then show that $\{c_k^{(j)}\}$ is the unique sequence in ℓ^2 for which (4.11) holds. The result (4.10) then follows by defining

$$d_{j,k} := c_k^{(j)}, \quad j,k \in \mathbf{Z}.$$

■

The expansion (4.10) is called the *wavelet expansion* of f with respect to the wavelet ψ. The elements of the sequence $\{g_j\}$ in (4.11) are called the *wavelet components* of f with respect to the wavelet spaces $\{W_j\}$. The coefficients $\{d_{j,k} : j,k \in \mathbf{Z}\}$ in (4.10) are called the *wavelet coefficients* of f with respect to the wavelet ψ.

Observe from (4.6) and (4.7) that, since (4.5) holds, we have

$$\langle \psi_{k,m}, \psi_{l,n} \rangle = 0, \quad k \neq l, \quad k,l,m,n \in \mathbf{Z}. \tag{4.13}$$

A wavelet ψ is called *orthonormal* if the condition

$$\langle \psi(\cdot - k), \psi(\cdot - l) \rangle = \delta_{k,l}, \quad k,l \in \mathbf{Z}, \tag{4.14}$$

is satisfied. Hence, from (4.13) and (4.14), if ψ is an orthonormal wavelet, then

$$\langle \psi_{k,l}, \psi_{m,n} \rangle = \delta_{k,m}\delta_{l,n}, \quad k,l,m,n \in \mathbf{Z}.$$

If (4.14) does not hold (but of course (4.13) still does), we call ψ a *biorthogonal wavelet*.

In this thesis we shall focus on the following construction method for a wavelet ψ:

(a) Construct a scaling function ϕ.

(b) Define the MRA $\{V_j\}$ generated by ϕ by means of (3.8) and (3.7).

(c) Define the sequence $\{W_j\}$ by means of (4.1), for which the orthogonal decomposition result (4.6) then holds.

(d) Construct a function $\psi \in L^2(\Re)$ such that, with the definition (4.8), the right hand side of (4.7) is identical to W_j for all $j \in \mathbf{Z}$.

(e) Verify that ψ satisfies the Riesz stability condition (4.9).

The five steps above constitutes the MRA method for the construction of a wavelet ψ, which we proceed to pursue in Section 4.2.

4.2 Minimally Supported Wavelets and their Duals

We show here how a method based on the MRA $\{V_j\}$ generated by a scaling function ϕ, the five steps of which were given at the end of Section 4.1, can be used to construct a wavelet ψ.

For the sake of simplicity, we shall restrict attention to the case where ϕ is a finitely supported real-valued scaling function, from which we shall then construct a finitely supported real-valued wavelet ψ.

Suppose therefore ϕ is a finitely supported real-valued scaling function which generates the MRA $\{V_j\}$, and define the sequence $\{W_j\}$ by means of (4.1).

Moreover, on the basis of Theorems 3.5, 3.6 and 3.7, we shall henceforth assume that the two-scale sequence $\{p_k\}$ in (3.2) is real-valued and finitely supported, and that there exists an integer $N \geq 2$ such that

$$\left.\begin{array}{l} p_k = 0, \quad k \notin \{0, 1, \ldots, N\}, \\ p_0 \neq 0, \quad p_N \neq 0, \end{array}\right\} \tag{4.15}$$

and

$$\phi(t) = 0, \quad t \notin [0, N]. \tag{4.16}$$

Observe that the two-scale symbol P, as given by (3.23), is then a polynomial of degree N with real coefficients, and satisfying

$$P(z) = \frac{1}{2}\sum_{k=0}^{N} p_k z^k, \quad z \in \mathbf{C}. \tag{4.17}$$

Note also from (3.46) that the autocorrelation function Φ then is real-valued, and thus, from (3.51) and (3.52), we see that

$$E(e^{i\omega}) = \overline{E(e^{i\omega})} = E(\overline{e^{i\omega}}) = E(e^{-i\omega}), \quad \omega \in \Re. \tag{4.18}$$

Observing from (4.2) that $W_0 \subset V_1$, we shall attempt, according to (3.8), (3.7), (4.7) and (4.8), to find a finitely supported real-valued sequence $\{q_k\}$ such that the function ψ defined by

$$\psi(t) = \sum_k q_k \phi(2t-k), \quad t \in \Re, \tag{4.19}$$

satisfies the properties of Definition 4.1.

Similar to the proof of Theorem 3.3(a), it can be shown that (4.19) has the equivalent Fourier transform formulation

$$\widehat{\psi}(\omega) = R(e^{-i\omega/2})\widehat{\phi}\left(\frac{\omega}{2}\right), \quad \omega \in \Re, \tag{4.20}$$

with the Laurent polynomial R defined by

$$R(z) := \frac{1}{2}\sum_k q_k z^k, \quad z \in \mathbf{C} \setminus \{0\}. \tag{4.21}$$

Our following result characterizes the class of Laurent polynomials R for which $\psi \in W_0$.

Theorem 4.4 *For a given integer $N \geq 2$, suppose ϕ is a finitely supported real-valued scaling function such that (4.15) and (4.16) hold, and which generates the MRA $\{V_j\}$. Let $\{W_j\}$ be the sequence defined by (4.1). Then ψ is a finitely* *supported function in W_0 if and only if ψ is given by (4.19), with the Laurent polynomial R, as defined by (4.21), satisfying the identity*

$$P(z)E(z)\overline{R(z)} + P(-z)E(-z)\overline{R(-z)} = 0, \quad |z| = 1, \tag{4.22}$$

where P denotes the two-scale polynomial (4.17), and where E denotes the Euler-Frobenius Laurent polynomial (3.51), (3.46).

Proof. Using, successively, the Parseval identity (2.8), the Fourier transform formulations (3.24) and (4.20), and the formulas (3.52) and (4.18), we find, for $l \in \mathbf{Z}$, and using the notation $z = e^{-i\omega/2}$, $\omega \in \Re$, that

$$\begin{aligned}
\langle \phi(\cdot - l), \psi \rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega l}\widehat{\phi}(\omega)\overline{\widehat{\psi}(\omega)}\, \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega l}P(z)\widehat{\phi}\left(\frac{\omega}{2}\right)\overline{R(z)\widehat{\phi}\left(\frac{\omega}{2}\right)}\, \mathrm{d}\omega \\
&= \frac{1}{2\pi}\sum_k \int_{4\pi k}^{4\pi(k+1)} P(z)\left|\widehat{\phi}\left(\frac{\omega}{2}\right)\right|^2 \overline{R(z)}e^{-i\omega l}\, \mathrm{d}\omega \\
&= \frac{1}{2\pi}\sum_k \int_{0}^{4\pi} P(z)\left|\widehat{\phi}\left(\frac{\omega}{2} + 2\pi k\right)\right|^2 \overline{R(z)}e^{-i\omega l}\, \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{0}^{4\pi} P(z)E(z)\overline{R(z)}e^{-i\omega l}\, \mathrm{d}\omega \\
&= \frac{1}{2\pi}\left[\int_{0}^{2\pi} P(z)E(z)\overline{R(z)}e^{-i\omega l}\, \mathrm{d}\omega + \int_{2\pi}^{4\pi} P(z)E(z)\overline{R(z)}e^{-i\omega l}\, \mathrm{d}\omega\right] \\
&= \frac{1}{2\pi}\left[\int_{0}^{2\pi} \left\{P(z)E(z)\overline{R(z)} + P(-z)E(-z)\overline{R(-z)}\right\} e^{-i\omega l}\, \mathrm{d}\omega\right].
\end{aligned} \tag{4.23}$$

If we now define the function H by

$$H(\omega) := P(e^{-i\omega/2})E(e^{-i\omega/2})\overline{R(e^{-i\omega/2})} + P(-e^{-i\omega/2})E(-e^{-i\omega/2})\overline{R(-e^{-i\omega/2})}, \quad \omega \in \Re, \tag{4.24}$$

we have, for $\omega \in \Re$,

$$\begin{aligned}
H(\omega + 2\pi) &= P(-e^{-i\omega/2})E(-e^{-i\omega/2})\overline{R(-e^{-i\omega/2})} + P(e^{-i\omega/2})E(e^{-i\omega/2})\overline{R(e^{-i\omega/2})} \\
&= H(\omega),
\end{aligned}$$

i.e. H is 2π-periodic. Hence, using also (4.23), and observing that $P(z)E(z)\overline{R}(z)$ is a Laurent polynomial, the identity

$$H(\omega) = \sum_{l} \langle \phi(\cdot - l), \psi \rangle e^{i\omega l}, \quad \omega \in \Re, \tag{4.25}$$

follows from Theorem 2.2. It is then clear from (4.24) and (4.25) that (4.22) holds if and only if

$$\langle \phi(\cdot - l), \psi \rangle = 0, \quad l \in \mathbf{Z}. \tag{4.26}$$

But (4.26) is equivalent to the statement $\psi \perp V_0$, since, if (4.26) holds and if $f = \sum_k c_k \phi(\cdot - k) \in V_0$, then

$$\langle f, \psi \rangle = \left\langle \sum_k c_k \phi(\cdot - k), \psi \right\rangle = \sum_k c_k \langle \phi(\cdot - k), \psi \rangle = 0,$$

whereas, if $\psi \perp V_0$ then, since $\phi(\cdot - l) \in V_0$, $l \in \mathbf{Z}$, we have (4.26). Hence $\psi \in W_0$ if and only if (4.22) holds. ■

Hence, according to Definition 4.1 and Theorem 4.4, a necessary condition for ψ to be a finitely supported wavelet in $L^2(\Re)$ is that ψ has the form (4.19), and, with the Laurent polynomial R defined by (4.21), that R satisfies the identity (4.22) on the unit circle $|z| = 1$. We therefore proceed to solve for R from (4.22).

According to [3, pg.100], the general Laurent polynomial solution R of the identity (4.22) is given by

$$R(z) = zK(z^2)P\left(-\frac{1}{z}\right)E(-z), \quad |z| = 1, \tag{4.27}$$

with K denoting an arbitrary Laurent polynomial. To verify that R then does indeed satisfy the identity (4.22), we use the fact that the polynomial P has real coefficients, and the fact that the Laurent polynomial E is real-valued on $|z| = 1$ according to (3.52), to obtain, for $z = e^{i\omega}$, $\omega \in \Re$,

$$P(z)E(z)\overline{R(z)} = P(z)E(z)\overline{zK(z^2)P\left(-\frac{1}{z}\right)E(-z)}$$

$$= P(e^{i\omega})E(e^{i\omega})e^{-i\omega}\overline{K(z^2)}P(-e^{i\omega})E(-e^{i\omega}), \qquad (4.28)$$

whereas

$$\begin{aligned} P(-z)E(-z)\overline{R(-z)} &= -P(-z)E(-z)\overline{zK(z^2)P\left(\frac{1}{z}\right)E(z)} \\ &= -P(-e^{i\omega})E(-e^{i\omega})e^{-i\omega}\overline{K(z^2)}P(e^{i\omega})E(e^{i\omega}), \end{aligned} \qquad (4.29)$$

and it follows from (4.28) and (4.29) that (4.22) holds if R is defined by (4.27).

It is clear from (4.19) and (4.16) that the function ψ will be minimally supported if we choose the arbitrary factor K in (4.27) to be a monomial, i.e. $K(z^2) = cz^{2M}$, with $c \in \Re \setminus \{0\}$, for some $M \in \mathbf{Z}$. Hence we have the following result.

Theorem 4.5 *Let ϕ, $\{V_j\}$ and $\{W_j\}$ be as in Theorem 4.4. Then the minimally supported function ψ in W_0 is given by the formula*

$$\psi(t) = \sum_k q_k \phi(2t - k), \quad t \in \Re, \qquad (4.30)$$

where, with the Laurent polynomial Q defined by

$$Q(z) := \frac{1}{2}\sum_k q_k z^k, \quad z \in \mathbf{C} \setminus \{0\}, \qquad (4.31)$$

we have

$$Q(z) = cz^{2M+1}P\left(-\frac{1}{z}\right)E(-z), \quad |z| = 1, \qquad (4.32)$$

where $c \in \Re \setminus \{0\}$ and $M \in \mathbf{Z}$.

Next, we observe that, under the conditions of Theorem 4.4, it follows from (3.46) that, for $t \in \Re$,

$$\begin{aligned} \Phi(-t) &= \int_{-\infty}^{\infty} \phi(x)\phi(x-t)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \phi(x+t)\phi(x)\,\mathrm{d}x, \end{aligned}$$

and thus

$$\Phi(t) = \Phi(-t), \quad t \in \Re. \tag{4.33}$$

Moreover, using (4.16) and (3.46), we see that Φ is a finitely supported function, with

$$\Phi(t) = 0, \quad t \notin [-N, N]. \tag{4.34}$$

Now define

$$n := \text{the largest integer such that } \Phi(n) \neq 0, \tag{4.35}$$

whence, from (3.51), (4.34), (4.33) and (4.35), we have

$$E(z) = \sum_{k=-n}^{n} \Phi(k) z^k, \quad z \in \mathbf{C} \setminus \{0\}.$$

Hence, with Q denoting the Laurent polynomial of Theorem 4.5, we use (4.32), (3.23), (3.51), (4.15) and (4.35), to obtain, for $|z| = 1$,

$$\begin{aligned}
Q(z) &= cz^{2M+1}\frac{1}{2}\sum_k(-1)^k p_k z^{-k}\sum_l(-1)^l\Phi(l)z^l \\
&= \frac{c}{2}\sum_k(-1)^k p_k\sum_l(-1)^l\Phi(l)z^{2M+1-k+l} \\
&= \frac{c}{2}\sum_k(-1)^k p_k\sum_l(-1)^{l-2M-1+k}\Phi(l-2M-1+k)z^l \\
&= -\frac{c}{2}\sum_k p_k\sum_l(-1)^l\Phi(l-2M-1+k)z^l \\
&= -\frac{c}{2}\sum_l(-1)^l\left[\sum_k p_k\Phi(l-2M-1+k)\right]z^l \\
&= -\frac{c}{2}\sum_{l=(2M+1)-(n+N)}^{(2M+1)+n}(-1)^l\left[\sum_{k=0}^{N} p_k\Phi(l-2M-1+k)\right]z^l. \qquad (4.36)
\end{aligned}$$

Using the notation $\lfloor x \rfloor$ for the largest integer $\leq x$, we now choose, in (4.36), $c = -1$ and

$$M = \left\lfloor \frac{n+N}{2} \right\rfloor. \tag{4.37}$$

Inserting these values into (4.36) then yields, for $|z| = 1$,

$$Q(z) = \begin{cases} \frac{1}{2}\sum_{l=0}^{2n+N}(-1)^l\left[\sum_{k=0}^{N}p_k\Phi(l-n-N+k)\right]z^l, & \text{if } n+N \text{ is odd,} \\ \frac{1}{2}\sum_{l=1}^{2n+N+1}(-1)^l\left[\sum_{k=0}^{N}p_k\Phi(l-n-N-1+k)\right]z^l, & \text{if } n+N \text{ is even.} \end{cases} \tag{4.38}$$

This choice of the Laurent polynomial Q gives, according to (4.30), (4.31) and (4.32),

$$\psi(t) = \begin{cases} \sum_{k=0}^{2n+N}(-1)^k\left[\sum_{l=0}^{N}p_l\Phi(k-n-N+l)\right]\phi(2t-k), \quad t \in \Re, \\ \qquad \text{if } n+N \text{ is odd,} \\ \sum_{k=1}^{2n+N+1}(-1)^k\left[\sum_{l=0}^{N}p_l\Phi(k-n-N-1+l)\right]\phi(2t-k), \quad t \in \Re, \\ \qquad \text{if } n+N \text{ is even.} \end{cases} \tag{4.39}$$

The following result therefore holds.

Theorem 4.6 *Let ϕ, $\{V_j\}$ and $\{W_j\}$ be given as in Theorem 4.4, and let the positive integers N and n be given by (4.15) and (4.35). Then the function ψ defined by (4.39) is a minimally supported function in W_0. Moreover, the Laurent polynomial Q satisfying (4.30), (4.31) and (4.32) is then given by*

$$Q(z) = -z^{2M+1}P\left(-\frac{1}{z}\right)E(-z), \tag{4.40}$$

with M given by (4.37). Also,

$$\left.\begin{array}{ll} \psi(t) = 0, \quad t \notin [0, n+N] & \textit{if } n+N \textit{ is odd,} \\ \textit{and} & \\ \psi(t) = 0, \quad t \notin \left[\frac{1}{2}, n+N+\frac{1}{2}\right] & \textit{if } n+N \textit{ is even.} \end{array}\right\} \tag{4.41}$$

We proceed to show that the function ψ of Theorem 4.5 is a wavelet in $L^2(\Re)$. First, we shall prove that ψ is Riesz stable in the sense of (4.9). To this end, we prove the following identities.

Theorem 4.7 *In Theorem 4.4, we have*

$$E(z^2) = |P(z)|^2E(z) + |P(-z)|^2E(-z), \quad |z| = 1, \tag{4.42}$$

whereas, in Theorem 4.6, the identities

$$\sum_k |\widehat{\psi}(\omega + 2\pi k)|^2 = |Q(e^{-i\omega/2})|^2E(e^{-i\omega/2}) + |Q(-e^{-i\omega/2})|^2E(-e^{-i\omega/2}), \quad \omega \in \Re, \tag{4.43}$$

and

$$\sum_k |\widehat{\psi}(\omega + 2\pi k)|^2 = E(e^{-i\omega/2})E(e^{-i\omega})E(-e^{-i\omega/2}), \quad \omega \in \Re, \tag{4.44}$$

hold.

Proof. First, with the notation $z = e^{-i\omega/2}$, $\omega \in \Re$, we use (4.18), (3.52) and (3.24) to deduce that

$$\begin{aligned}
E(z^2) &= E(e^{-i\omega}) = E(e^{i\omega}) \\
&= \sum_k |\widehat{\phi}(\omega + 2\pi k)|^2 \\
&= \sum_k |\widehat{\phi}(\omega + 4\pi k)|^2 + \sum_k |\widehat{\phi}(\omega + 2\pi(2k+1))|^2 \\
&= \sum_k \left|P(e^{-i\omega/2})\widehat{\phi}\left(\frac{\omega}{2} + 2\pi k\right)\right|^2 + \sum_k \left|P(-e^{-i\omega/2})\widehat{\phi}\left(\frac{\omega}{2} + (2k+1)\pi\right)\right|^2 \\
&= |P(z)|^2 \sum_k \left|\widehat{\phi}\left(\frac{\omega}{2} + 2\pi k\right)\right|^2 + |P(-z)|^2 \sum_k \left|\widehat{\phi}\left(\frac{\omega}{2} + \pi + 2\pi k\right)\right|^2 \\
&= |P(z)|^2E(z) + |P(-z)|^2E(-z),
\end{aligned}$$

thereby proving (4.42).

Next, to prove (4.43), we use the fact, as obtained by choosing $R = Q$ in (4.20), that (4.30) has the equivalent Fourier transform formulation

$$\widehat{\psi}(\omega) = Q(e^{-i\omega/2})\widehat{\phi}\left(\frac{\omega}{2}\right), \quad \omega \in \Re, \tag{4.45}$$

to deduce, using also (3.52), that, for $\omega \in \Re$,

$$\begin{aligned}
\sum_k |\widehat{\psi}(\omega + 2\pi k)|^2 &= \sum |\widehat{\psi}(\omega + 4\pi k)|^2 + \sum |\widehat{\psi}(\omega + 2(2k+1)\pi)|^2 \\
&= |Q(e^{-i\omega/2})|^2 \sum \left|\widehat{\phi}\left(\frac{\omega}{2} + 2\pi k\right)\right|^2 \\
&\quad + |Q(-e^{-i\omega/2})|^2 \sum \left|\widehat{\psi}\left(\frac{\omega}{2} + (2k+1)\pi\right)\right|^2 \\
&= |Q(e^{-i\omega/2})|^2 E(e^{i\omega/2}) + |Q(-e^{-i\omega/2})|^2 E(-e^{i\omega/2}) \\
&= |Q(e^{-i\omega/2})|^2 \overline{E(e^{i\omega/2})} + |Q(-e^{-i\omega/2})|^2 \overline{E(-e^{i\omega/2})} \\
&= |Q(e^{-i\omega/2})|^2 E(e^{-i\omega/2}) + |Q(-e^{-i\omega/2})|^2 E(-e^{-i\omega/2}),
\end{aligned}$$

thereby establishing (4.43). Next, we use (4.40) and (4.43) to find, with $z = e^{-i\omega/2}$, $\omega \in \Re$, and using the fact that (3.52) implies $E(z) \in \Re$, $|z| = 1$, that

$$\begin{aligned}
\sum_k |\widehat{\psi}(\omega + 2\pi k)|^2 &= \left|P\left(-\frac{1}{z}\right)\right|^2 [E(-z)]^2 E(z) + \left|P\left(\frac{1}{z}\right)\right|^2 [E(z)]^2 E(-z) \\
&= E(z)E(-z)\left[\left|P\left(-\frac{1}{z}\right)\right|^2 E(-z) + \left|P\left(\frac{1}{z}\right)\right|^2 E(z)\right].
\end{aligned} \tag{4.46}$$

But, for $|z| = 1$, i.e. $z = \dfrac{1}{\overline{z}}$,

$$\begin{aligned}
\left|P\left(\frac{1}{z}\right)\right|^2 &= P\left(\frac{1}{z}\right)\overline{P\left(\frac{1}{z}\right)} \\
&= P(\overline{z})P\left(\frac{1}{\overline{z}}\right) \\
&= P(\overline{z})P(z) \\
&= \overline{P(z)}P(z) = |P(z)|^2,
\end{aligned}$$

and, similarly

$$\left|P\left(-\frac{1}{z}\right)\right|^2 = |P(-z)|^2, \tag{4.47}$$

so that, from (4.46),

$$\sum_k |\widehat{\psi}(\omega + 2\pi k)|^2 = E(z)E(-z)\left[|P(-z)|^2 E(-z) + |P(z)|^2 E(z)\right], \tag{4.48}$$

and (4.44) follows by combining (4.48) and (4.42). ■

The following result can now be proved.

Theorem 4.8 *The function ψ of Theorem 4.6 is Riesz stable in the sense of (4.9), with Riesz bounds $\alpha = A^3$ and $\beta = B^3$, where A and B are the Riesz bounds in (3.3).*

Proof. The result follows if we combine (4.44) and (3.53) with the fact that, analogous to Theorem 3.3(b), the Riesz stability condition (4.9) has the equivalent Fourier transform formulation

$$0 < \alpha \leq \sum_k |\widehat{\psi}(\omega + 2\pi k)|^2 \leq \beta, \quad \omega \in \Re. \tag{4.49}$$

■

It remains to show that, with the definition (4.8), and for each $j \in \mathbf{Z}$, the linear space defined by the righthand side of (4.7) is indeed equal to W_j.

The following result, which is analogous to the result of Theorem 3.12 for scaling functions, will be instrumental in our proof of this fact.

Theorem 4.9 *Let the finitely supported function ψ be defined as in Theorem 4.6. Then, with the function $\widetilde{\psi}$ defined by*

$$\widetilde{\psi}(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\left[\frac{\widehat{\psi}(\omega)}{E(e^{-i\omega/2})E(e^{-i\omega})E(-e^{-i\omega/2})}\right] d\omega, \quad t \in \Re, \tag{4.50}$$

we have the following:

(a) $$\widehat{\psi} \in \mathcal{A};$$

(b) $$\widehat{\widetilde{\psi}}(\omega) = \frac{\widehat{\psi}(\omega)}{E(e^{-i\omega/2})E(e^{-i\omega})E(-e^{-i\omega/2})}, \quad \omega \in \Re; \tag{4.51}$$

(c) $$\left\langle \psi(\cdot - k), \widetilde{\psi}(\cdot - l)\right\rangle = \delta_{k,l}, \quad k, l \in \mathbf{Z}. \tag{4.52}$$

Proof. Since ϕ is a scaling function, we have $\phi \in \mathcal{A}$, and thus Theorem 2.1 implies $\widehat{\phi} \in \mathcal{A} \subset L^1(\Re) \cap C_u(\Re)$. Then using (4.45), and the fact that Q is a Laurent polynomial, we get that $\widehat{\psi} \in L^1(\Re) \cap C_u(\Re)$.

Now observe from (4.45), (4.31), (2.4) and Theorem 2.1 that, for $\omega \in \Re$,

$$\begin{aligned}
(\mathcal{F}\widehat{\psi})(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t}\left[\sum_k q_k e^{-i(t/2)k}\right]\widehat{\phi}\left(\frac{t}{2}\right)\,\mathrm{d}t \\
&= \sum_k q_k \int_{-\infty}^{\infty} e^{i(-\omega-k/2)t}\widehat{\phi}\left(\frac{t}{2}\right)\,\mathrm{d}t \\
&= 4\pi\sum_k q_k \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(-2\omega-k)t}\widehat{\phi}(t)\,\mathrm{d}t \\
&= 4\pi\sum_k q_k \phi(-2\omega-k),
\end{aligned}$$

and thus

$$\begin{aligned}
\int_{-\infty}^{\infty} |(\mathcal{F}\widehat{\psi})(t)|\,\mathrm{d}t &= 4\pi\sum_k |q_k| \int_{-\infty}^{\infty} |\phi(-2\omega-k)|\,\mathrm{d}\omega \\
&= 2\pi\left[\sum_k |q_k|\right]\left[\int_{-\infty}^{\infty} |\phi(t)|\,\mathrm{d}t\right],
\end{aligned}$$

whence $\mathcal{F}\widehat{\psi} \in L^1(\Re)$, since $\{q_k\}$ is a finitely supported sequence, and since $\phi \in \mathcal{A} \subset L^1(\Re)$. Hence $\widehat{\psi} \in \mathcal{A}$.

Similarly to the proof in Theorem 3.12 of the fact that $g \in \mathcal{A}$, we can now show, by virtue of the fact that E is a Laurent polynomial satisfying (3.53), that the function G defined by

$$G(\omega) := \frac{\widehat{\psi}(\omega)}{E(e^{-i\omega/2})E(e^{-i\omega})E(-e^{-i\omega/2})}, \quad \omega \in \Re,$$

also satisfies $G \in \mathcal{A}$. Appealing again to Theorem 2.1, we deduce from (4.50) that (4.51) holds.

Next, we employ the Parseval identity (2.8), together with (4.44), and the fact that, for $\omega \in \Re$ and $k \in \mathbf{Z}$,

$$E(e^{-\frac{i(\omega+2\pi k)}{2}})E(-e^{-\frac{i(\omega+2\pi k)}{2}}) = E(e^{-i\omega/2}e^{-i\pi k})E(-e^{-i\omega/2}e^{-i\pi k}) = E(e^{-i\omega/2})E(-e^{-i\omega/2}),$$

to find, for $\omega \in \Re$ and $k, l \in \mathbf{Z}$, that

$$\begin{aligned}
&\left\langle \psi(\cdot - k), \widetilde{\psi}(\cdot - l) \right\rangle \\
= \;& \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(k-l)\omega} \frac{|\widehat{\psi}(\omega)|^2}{E(e^{-i\omega/2})E(e^{-i\omega})E(-e^{-i\omega/2})} \, \mathrm{d}\omega \\
= \;& \frac{1}{2\pi} \sum_k \int_{2\pi k}^{2\pi(k+1)} e^{-i(k-l)\omega} \frac{|\widehat{\psi}(\omega)|^2}{E(e^{-i\omega/2})E(e^{-i\omega})E(-e^{-i\omega/2})} \, \mathrm{d}\omega \\
= \;& \frac{1}{2\pi} \sum_k \int_0^{2\pi} e^{-i(k-l)\omega} \frac{|\widehat{\psi}(\omega + 2\pi k)|^2}{E(e^{-i\omega/2})E(e^{-i\omega})E(-e^{-i\omega/2})} \, \mathrm{d}\omega \\
= \;& \frac{1}{2\pi} \int_0^{2\pi} e^{-i(k-l)\omega} \frac{\sum_k |\widehat{\psi}(\omega + 2\pi k)|^2}{\sum_k |\widehat{\psi}(\omega + 2\pi k)|^2} \, \mathrm{d}\omega \\
= \;& \delta_{k,l},
\end{aligned}$$

thereby establishing (4.52). ∎

We now define the linear spaces

$$\widetilde{W}_j := \left\{ \sum_k c_k \widetilde{\psi}_{j,k} : \{c_k\} \in \ell^2 \right\}, \quad j \in \mathbf{Z}, \tag{4.53}$$

where

$$\widetilde{\psi}_{j,k}(t) := 2^{j/2} \widetilde{\psi}(2^j t - k), \quad t \in \Re, \quad j, k \in \mathbf{Z}. \tag{4.54}$$

Theorem 4.10 *In the notation of Theorem 4.6 and Theorem 4.9, and with the sequence $\{\widetilde{W}_j\}$ of linear spaces defined by (4.53) and (4.54), we have*

$$W_j = \widetilde{W}_j, \quad j \in \mathbf{Z}. \tag{4.55}$$

Proof. First, we prove that there exists a Laurent polynomial $\widetilde{E}$ such that

$$E(z)E(-z)E(z^2) = \widetilde{E}(z^2), \quad z \in \mathbf{C} \setminus \{0\}, \tag{4.56}$$

with the Laurent polynomial E given by (3.51). Indeed, from (3.51), we have, for $z \in \mathbf{C} \setminus \{0\}$,

$$\begin{aligned}
E(z)E(-z)E(z^2) &= \left[\sum_k \Phi(k)z^k\right]\left[\sum_k (-1)^k\Phi(k)z^k\right]E(z^2) \\
&= \left[\sum_k \Phi(2k)z^{2k} + \sum_k \Phi(2k+1)z^{2k+1}\right] \\
&\qquad \left[\sum_k \Phi(2k)z^{2k} - \sum_k \Phi(2k+1)z^{2k+1}\right]E(z^2) \\
&= \left[\sum_k \Phi(2k)z^{2k} + z\sum_k \Phi(2k+1)z^{2k}\right] \\
&\qquad \left[\sum_k \Phi(2k)z^{2k} - z\sum_k \Phi(2k+1)z^{2k}\right]E(z^2) \\
&= \left[\left(\sum_k \Phi(2k)z^{2k}\right)^2 - \left(z\sum_k \Phi(2k+1)z^{2k}\right)^2\right]E(z^2) \\
&= \left[\left(\sum_k \Phi(2k)(z^2)^k\right)^2 - z^2\left(\sum_k \Phi(2k+1)(z^2)^k\right)^2\right]E(z^2),
\end{aligned}$$

thereby establishing the existence of a Laurent polynomial $\widetilde{E}$ for which (4.56) holds.

Hence, combining (4.51) and (4.56), we get

$$\widehat{\widetilde{\psi}}(\omega) = \frac{\widehat{\psi}(\omega)}{\widetilde{E}(e^{-i\omega})}, \quad \omega \in \Re,$$

where, according to (4.56) and (3.53), we also have

$$0 < A^3 \leq \widetilde{E}(e^{-i\omega}) \leq B^3, \quad \omega \in \Re.$$

It follows, as in the proof of the analogous result (3.69) in Theorem 3.12, and as can be shown by the method followed in Theorem 3.12, that there exist sequences $\{\mu_k\}$ and $\{\nu_k\}$, with $\{\mu_k\}$ having exponential decay for $|k| \to \infty$, and with $\{\nu_k\}$ having finite support, i.e. $\{\mu_k\}$, $\{\nu_k\} \in \ell^1 \cap \ell^2$, such that

$$\widetilde{\psi}(t) = \sum_k \mu_k \psi(t-k), \quad t \in \Re, \tag{4.57}$$

and

$$\psi(t) = \sum_k \nu_k \widetilde{\psi}(t-k), \quad t \in \Re. \tag{4.58}$$

The required inclusions $\widetilde{W}_j \subset W_j$, $j \in \mathbf{Z}$, and $W_j \subset \widetilde{W}_j$, $j \in \mathbf{Z}$, are then proved by using (4.57), (4.58), and the method followed in the proof of Theorem 3.12. ■

We proceed to establish the following identities with regard to the function $\widetilde{\phi}$ and $\widetilde{\psi}$.

Theorem 4.11 *Suppose, in Theorem 4.4, that $\widetilde{\phi}$ is the dual scaling function, as defined by (3.54), of ϕ, and let $\widetilde{\psi}$ be given as in Theorem 4.9. Then*

(a)

$$\hat{\widetilde{\phi}}(\omega) = A(e^{-i\omega/2})\hat{\widetilde{\phi}}\left(\frac{\omega}{2}\right), \quad \omega \in \Re, \tag{4.59}$$

where A denotes the Laurent series

$$A(z) := \frac{1}{2}\sum_k a_k z^k, \quad z \in \mathbf{C} \setminus \{0\}, \tag{4.60}$$

with $\{a_k\} \in \ell^1 \cap \ell^2$, and where

$$A(z) = \frac{E(z)}{E(z^2)}P(z), \quad |z| = 1; \tag{4.61}$$

(b)

$$\hat{\widetilde{\psi}}(\omega) = B(e^{-i\omega/2})\hat{\widetilde{\phi}}\left(\frac{\omega}{2}\right), \quad \omega \in \Re, \tag{4.62}$$

where B denotes the Laurent series

$$B(z) = \frac{1}{2}\sum_k b_k z^k, \quad z \in \mathbf{C} \setminus \{0\}, \tag{4.63}$$

with $\{b_k\} \in \ell^1 \cap \ell^2$, and where

$$B(z) = \frac{-z^{2M+1}P\left(-\frac{1}{z}\right)}{E(z^2)}, \quad |z| = 1, \tag{4.64}$$

with M defined by (4.37).

Proof.

(a) Noting that, from (4.18), we have

$$E\left(\frac{1}{z}\right) = E(z), \quad |z| = 1,$$

and

$$E\left(\frac{1}{z^2}\right) = E(z^2), \quad |z| = 1,$$

we observe that the results (4.59), (4.60) and (4.61), with $\{a_k\} \in \ell^1 \cap \ell^2$, have already been noted in (3.58), (3.63), (3.64) and (3.65) in the proof of Theorem 3.12.

(b) Using (4.51) and (4.45), we get

$$\widehat{\tilde{\psi}}(\omega) = \frac{Q(e^{-i\omega/2})}{E(e^{-i\omega/2})E(-e^{-i\omega/2})E(e^{-i\omega})}\widehat{\phi}\left(\frac{\omega}{2}\right), \quad \omega \in \Re. \tag{4.65}$$

Now recall from the proof of Theorem 3.12 that $\widetilde{B} = E$ in (3.73), which, together with (4.65), gives

$$\widehat{\tilde{\psi}}(\omega) = \frac{Q(e^{-i\omega/2})}{E(-e^{-i\omega/2})E(e^{-i\omega})}\widehat{\tilde{\phi}}\left(\frac{\omega}{2}\right), \quad \omega \in \Re. \tag{4.66}$$

Inserting (4.40) into (4.66) then gives (4.62), with the Laurent series B given by (4.64), and M defined by (4.37). The fact that B has the form (4.63), with $\{b_k\} \in \ell^1 \cap \ell^2$, follows from (4.64), (3.53) and the fact that P is a polynomial and E is a Laurent polynomial.

■

The following identities can now be proven.

Theorem 4.12 *In Theorem 4.11, we have*

$$P(z)\overline{A(z)} + Q(z)\overline{B(z)} = 1, \quad |z| = 1, \tag{4.67}$$

and

$$P(z)\overline{A(-z)} + Q(z)\overline{B(-z)} = 0, \quad |z| = 1. \tag{4.68}$$

Proof. From (4.61), (4.40), (4.64), and (4.47) and the fact that (3.52) implies $E(z) \in \Re$ for $|z| = 1$, we obtain, on the unit circle $|z| = 1$,

$$\begin{aligned}
&P(z)\overline{A(z)} + Q(z)\overline{B(z)} \\
= \;& P(z)\frac{E(z)}{E(z^2)}\overline{P(z)} - \frac{z^{2M+1}P\left(-\frac{1}{z}\right)E(-z)(-\overline{z}^{2M+1})\overline{P\left(-\frac{1}{z}\right)}}{E(z^2)} \\
= \;& \frac{|P(z)|^2E(z)}{E(z^2)} + \frac{|z|^{2M+1}E(-z)}{E(z^2)}\left|P\left(-\frac{1}{z}\right)\right|^2 \\
= \;& \frac{|P(z)|^2E(z) + |P(-z)|^2E(-z)}{E(z^2)} = 1,
\end{aligned}$$

by virtue also of (4.42).

Similarly, for $|z| = 1$, i.e. $z = \dfrac{1}{\overline{z}}$, we get

$$\begin{aligned}
&P(z)\overline{A(-z)} + Q(z)\overline{B(-z)} \\
= \;& P(z)\frac{E(-z)\overline{P(-z)}}{E(z^2)} - \frac{z^{2M+1}P\left(-\frac{1}{z}\right)E(-z)\overline{z}^{2M+1}\overline{P\left(\frac{1}{z}\right)}}{E(z^2)} \\
= \;& \frac{E(-z)}{E(z^2)}\left[P(z)\overline{P(-z)} - P(-\overline{z})\overline{P(\overline{z})}\right] \\
= \;& \frac{E(-z)}{E(z^2)}\left[P(z)\overline{P(-z)} - \overline{P(-z)}P(z)\right] = 0,
\end{aligned}$$

having used also the fact that the polynomial P has real coefficients. ■

The following fundamental decomposition result then holds.

Theorem 4.13 *For $l \in \mathbf{Z}$, there exist sequences $\{\alpha_{l,k}\}$ and $\{\beta_{l,k}\}$ in $\ell^1 \cap \ell^2$ such that, in Theorem 4.6,*

$$\phi(2t - l) = \sum_k \alpha_{l,k}\phi(t - k) + \sum_k \beta_{l,k}\psi(t - k), \quad t \in \Re. \tag{4.69}$$

Moreover,

$$\left.\begin{aligned}\alpha_{l,k} &= \tfrac{1}{2}a_{l-2k},\\ \beta_{l,k} &= \tfrac{1}{2}b_{l-2k},\end{aligned}\right\} \quad l,k \in \mathbf{Z}, \tag{4.70}$$

with $\{a_k\}$ and $\{b_k\}$ the sequences in $\ell^1 \cap \ell^2$ which are defined by means of (4.60), (4.63), (4.61) and (4.64).

Proof. Similar to the proof of Theorem 3.3(a), we can show that (4.69) has the equivalent Fourier transform formulation

$$\frac{1}{2}e^{-\frac{i\omega}{2}l}\widehat{\phi}\left(\frac{\omega}{2}\right) = \left[\sum_k \alpha_{l,k}e^{-i\omega k}\right]\widehat{\phi}(\omega) + \left[\sum_k \beta_{l,k}e^{-i\omega k}\right]\widehat{\psi}(\omega), \quad \omega \in \Re,$$

and thus, in the notation $z = e^{-i\omega/2}$, $\omega \in \Re$, and using (3.24) and (4.45),

$$\frac{1}{2}z^l\widehat{\phi}\left(\frac{\omega}{2}\right) = \left[\sum_k \alpha_{l,k}z^{2k}\right]P(z)\widehat{\phi}\left(\frac{\omega}{2}\right) + \left[\sum_k \beta_{l,k}z^{2k}\right]Q(z)\widehat{\phi}\left(\frac{\omega}{2}\right), \quad \omega \in \Re,$$

which is equivalent to the identity

$$\frac{1}{2}z^l = P(z)\left[\sum_k \alpha_{l,k}z^{2k}\right] + Q(z)\left[\sum_k \beta_{l,k}z^{2k}\right], \quad |z| = 1.$$

Now observe from (4.67) and (4.68) that

$$\frac{1}{2}z^l = \frac{1}{2}\left[P(z)\left(\overline{A(z)+A(-z)}\right) + Q(z)\left(\overline{B(z)+B(-z)}\right)\right]z^l, \quad |z| = 1, \tag{4.71}$$

and

$$\frac{1}{2}z^l = \frac{1}{2}\left[P(z)\left(\overline{A(z)-A(-z)}\right) + Q(z)\left(\overline{B(z)-B(-z)}\right)\right]z^l, \quad |z| = 1. \tag{4.72}$$

But, from (4.60) and (4.63),

$$\overline{A(z)+A(-z)} = \sum_k a_{2k}z^{-2k}, \quad |z| = 1, \tag{4.73}$$

$$\overline{A(z)-A(-z)} = \sum_k a_{2k+1}z^{-2k-1}, \quad |z| = 1, \tag{4.74}$$

$$\overline{B(z)+B(-z)} = \sum_k b_{2k}z^{-2k}, \quad |z| = 1, \tag{4.75}$$

and

$$\overline{B(z) - B(-z)} = \sum_k b_{2k+1} z^{-2k-1}, \quad |z| = 1. \tag{4.76}$$

Also, note that the choice (4.70) yields, for $z \in \mathbf{C} \setminus \{0\}$,

$$\begin{aligned} \sum_k \alpha_{l,k} z^{2k} &= \frac{1}{2} \sum_k a_{l-2k} z^{2k} \\ &= \frac{1}{2} \sum_k a_{l+2k} z^{-2k} \\ &= \begin{cases} \frac{1}{2} z^l \sum_k a_{2k} z^{-2k} & \text{if } l \text{ is even,} \\ \frac{1}{2} z^l \sum_k a_{2k+1} z^{-2k-1} & \text{if } l \text{ is odd,} \end{cases} \end{aligned} \tag{4.77}$$

and similarly, for $z \in \mathbf{C} \setminus \{0\}$,

$$\sum_k \beta_{l,k} z^{2k} = \begin{cases} \frac{1}{2} z^l \sum_k b_{2k} z^{-2k} & \text{if } l \text{ is even,} \\ \frac{1}{2} z^l \sum_k b_{2k+1} z^{-2k-1} & \text{if } l \text{ is odd,} \end{cases} \tag{4.78}$$

The fact that (4.69) holds with the choices (4.70) then follows from (4.71), ... , (4.78). ■

Our final step is to prove the following result.

Theorem 4.14 *If we define the sequence $\{X_j\}$ of linear spaces by*

$$X_j := \left\{ \sum_k c_k \psi_{j,k} : \{c_k\} \in \ell^2 \right\}, \quad j \in \mathbf{Z},$$

with ψ given as in Theorem 4.6, and with the sequence $\{\psi_{j,k} : j, k \in \mathbf{Z}\}$ defined by (4.8), then

$$X_j = W_j, \quad j \in \mathbf{Z}, \tag{4.79}$$

with the sequence $\{W_j\}$ defined as in Theorem 4.4.

Proof. Suppose $f \in X_j$, i.e. there exists a sequence $\{c_k\} \in \ell^2$ such that $f = \sum_k c_k \psi_{j,k}$, and suppose $g \in V_j$, i.e. there exists a sequence $\{d_k\} \in \ell^2$ such that $g = \sum_k d_k \phi_{j,k} \in V_j$. Then

$$\begin{aligned}
\langle g, f\rangle &= \left\langle \sum_k d_k \phi_{j,k}, \sum_l c_l \psi_{j,l} \right\rangle \\
&= \sum_k d_k \sum_l \overline{c_l} \int_{-\infty}^{\infty} 2^{j/2}\phi(2^j t - k) 2^{j/2} \overline{\psi(2^j t - l)}\, \mathrm{d}t \\
&= \sum_k d_k \sum_l \overline{c_l} \int_{-\infty}^{\infty} \phi(t-k)\psi(t-l)\, \mathrm{d}t \\
&= \sum_k d_k \sum_l \overline{c_l} \int_{-\infty}^{\infty} \phi(t-(k-l))\psi(t)\, \mathrm{d}t \\
&= 0,
\end{aligned}$$

since $\psi \in W_0 = V_0^{\perp}$. Hence $f \in W_j$, and thus $X_j \subset W_j$.

Suppose next $f \in W_j$. Then, since $W_j \subset V_{j+1}$ from (4.2), we also have $f \in V_{j+1}$, i.e. there exists a sequence $\{c_k\} \in \ell^2$ such that

$$f(t) = \sum_k c_k \phi(2^{j+1}t - k), \quad t \in \Re. \tag{4.80}$$

Now use (4.80), (4.69) and (4.70) to deduce that, for $t \in \Re$,

$$\begin{aligned}
f(t) &= \sum_k c_k \left\{ \sum_l \alpha_{k,l}\phi(2^j t - l) + \sum_l \beta_{k,l}\psi(2^j t - l) \right\} \\
&= \sum_l \mu_l \phi(2^j t - l) + \sum_l \nu_l \psi(2^j t - l),
\end{aligned} \tag{4.81}$$

with

$$\left.\begin{aligned}
\mu_l &= \frac{1}{2}\sum_k c_k a_{l-2k}, \\
\\
\nu_l &= \frac{1}{2}\sum_k c_k b_{l-2k},
\end{aligned}\right\} \quad l, k \in \mathbf{Z}.$$

Exploiting the fact that $\{a_k\}$ and $\{b_k\}$ are in $\ell^1 \cap \ell^2$, we can then show, as in the proof of Theorem 3.2(a), that $\{\mu_k\}$ and $\{\nu_k\}$ are in ℓ^2.

Now recall the dual scaling function $\tilde{\phi}$ of Theorem 3.12, and use (4.81) to obtain, for $m \in \mathbf{Z}$,

$$\begin{aligned}
&\left\langle f, \tilde{\phi}(2^j \cdot -m)\right\rangle \\
= &\left\langle \sum_l \mu_l \phi(2^j \cdot -l), \tilde{\phi}(2^j \cdot -m)\right\rangle + \left\langle \sum_l \nu_l \psi(2^j \cdot -l), \tilde{\phi}(2^j \cdot -m)\right\rangle \\
= &\sum_l \mu_l \int_{-\infty}^{\infty} \phi(2^j t - l)\overline{\tilde{\phi}(2^j t - m)}\,\mathrm{d}t + \sum_l \nu_l \int_{-\infty}^{\infty} \psi(2^j t - l)\overline{\tilde{\phi}(2^j t - m)}\,\mathrm{d}t \\
= &\sum_l \mu_l \frac{1}{2^j}\left\langle \phi(\cdot - l), \tilde{\phi}(\cdot - m)\right\rangle + \sum_l \nu_l \frac{1}{2^j}\left\langle \psi(\cdot - l), \tilde{\phi}(\cdot - m)\right\rangle. \qquad (4.82)
\end{aligned}$$

But, for $l, m \in \mathbf{Z}$

$$\begin{aligned}
\left\langle \psi(\cdot - l), \tilde{\phi}(\cdot - m)\right\rangle &= \overline{\left\langle \tilde{\phi}(\cdot - m), \psi(\cdot - l)\right\rangle} \\
&= \int_{-\infty}^{\infty} \tilde{\phi}(t - m)\overline{\psi(t - l)}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \tilde{\phi}(t - (m - l))\overline{\psi(t)}\,\mathrm{d}t \\
&= \left\langle \tilde{\phi}(\cdot - (m - l)), \psi\right\rangle \\
&= 0, \qquad (4.83)
\end{aligned}$$

since Theorem 3.12 implies that $\tilde{\phi} \in V_0$, whereas, according to Theorem 4.6, $\psi \in W_0 = V_0^T$.

Combining (4.82), (4.83) and (3.56) then gives

$$\mu_k = 2^j \left\langle f, \tilde{\phi}(2^j \cdot -k)\right\rangle, \quad k \in \mathbf{Z}. \qquad (4.84)$$

But, from Theorem 3.12, we have $\tilde{\phi}(2^j \cdot -k) \in V_j,\ j \in \mathbf{Z}$. Then, since $f \in W_j = V_j^T$, it follows from (4.84) that $\mu_k = 0,\ k \in \mathbf{Z}$, and then, using also (4.81),

$$f(t) = \sum_l \nu_l \psi(2^j t - l), \quad t \in \Re,$$

which shows, since $\{\nu_k\} \in \ell^2$, that $f \in X_j$. Hence $W_j \subset X_j$, thereby proving (4.79). ■

Hence, we have now shown, according to the results of Theorem 4.4, Theorem 4.5, Theorem 4.6, Theorem 4.8 and Theorem 4.14, that the finitely supported function ψ defined by (4.30), (4.31), (4.40), (4.37), is indeed a wavelet.

Our wavelet construction method by means of MRA can be summarized as follows.

(a) Find a real sequence $\{p_k\}$ and a real-valued continuous function ϕ such that, for an integer $N \geq 2$, we have

$$p_k = 0, \quad k \notin \{0, 1, \ldots, N\},$$

$$p_0 \neq 0, \quad p_N \neq 0,$$

$$\phi(t) = 0, \quad t \notin [0, N],$$

$$\phi(t) = \sum_{k=0}^{N} p_k \phi(2t - k), \quad t \in \Re.$$

(b) Verify that the Fourier transform $\widehat{\phi}$ satisfies the conditions $\widehat{\phi} \in L^1(\Re)$ and (3.1).

(c) Verify that the Riesz stability condition (3.3) as well as the partition of unity condition (3.4) (or, equivalently, the integral condition (3.31)) is satisfied by ϕ.

(d) Define the sequence $\{\Phi(l); \ l = -N, \ldots, N\}$ by

$$\Phi(l) = \int_{-\infty}^{\infty} \phi(t)\phi(t + l) \, \mathrm{d}t, \quad l = -N, \ldots, N.$$

(e) Let

$$n := \text{the largest integer such that } \Phi(n) \neq 0.$$

(f) Define

$$\psi(t)=\begin{cases}\displaystyle\sum_{k=0}^{2n+N}(-1)^k\left[\sum_{l=0}^{N}p_l\Phi(k-n-N+l)\right]\phi(2t-k), & t\in\Re,\\ \qquad\text{if } n+N \text{ is odd,}\\ \displaystyle\sum_{k=1}^{2n+N+1}(-1)^k\left[\sum_{l=0}^{N}p_l\Phi(k-n-N-1+l)\right]\phi(2t-k), & t\in\Re,\\ \qquad\text{if } n+N \text{ is even,}\end{cases}$$

(g) Then ψ is a real-valued continuous wavelet, with

$$\begin{cases}\psi(t)=0, \quad t\notin[0,n+N], & \text{if } n+N \text{ is odd,}\\ \psi(t)=0, \quad t\notin\left[\frac{1}{2},n+N+\frac{1}{2}\right], & \text{if } n+N \text{ is even.}\end{cases}$$

The function $\widetilde{\psi}$ defined by (4.50) is called the *dual wavelet* of ψ, and provides an explicit formulation of the wavelet coefficients $\{d_k : j,k\in\mathbf{Z}\}$ in (4.10), as given in the following result.

Theorem 4.15 *Suppose $f\in L^2(\Re)$, and let ψ and $\widetilde{\psi}$ be as in, respectively, Theorem 4.6 and Theorem 4.9. Then*

$$f=\sum_{j,k}\left\langle f,\widetilde{\psi}_{j,k}\right\rangle\psi_{j,k}, \tag{4.85}$$

and

$$f=\sum_{j,k}\left\langle f,\psi_{j,k}\right\rangle\widetilde{\psi}_{j,k}, \tag{4.86}$$

with the sequences $\{\psi_{j,k};j,k\in\mathbf{Z}\}$ and $\{\widetilde{\psi}_{j,k};j,k\in\mathbf{Z}\}$ defined by (4.8) and (4.54).

Proof. According to the representation (4.10) in Theorem 4.3, we have, for $m,n\in\mathbf{Z}$,

$$\begin{aligned}\left\langle f,\widetilde{\psi}_{m,n}\right\rangle &= \left\langle\sum_{j,k}d_{j,k}\psi_{j,k},\widetilde{\psi}_{m,n}\right\rangle\\ &= \sum_{j,k}d_{j,k}\left\langle\psi_{j,k},\widetilde{\psi}_{m,n}\right\rangle.\end{aligned} \tag{4.87}$$

But, from (4.7) and (4.53), we have $\psi_{j,k} \in W_j$, $j \in \mathbf{Z}$, and $\widetilde{\psi}_{m,n} \in \widetilde{W}_m = W_m$, $m \in \mathbf{Z}$, by virtue of (4.55). Hence (4.5) and (4.87) imply that, for $m, n \in \mathbf{Z}$,

$$\begin{aligned}
\left\langle f, \widetilde{\psi}_{m,n} \right\rangle &= \sum_k d_{m,k} \left\langle \psi_{m,k}, \widetilde{\psi}_{m,n} \right\rangle \\
&= \sum_k d_{m,k} \int_{-\infty}^{\infty} 2^{m/2}\psi(2^m t - k) 2^{m/2} \widetilde{\psi}(2^m t - n)\, \mathrm{d}t \\
&= \sum_k d_{m,k} \int_{-\infty}^{\infty} \psi(t-k)\widetilde{\psi}(t-n)\, \mathrm{d}t \\
&= \sum_k d_{m,k} \left\langle \psi(\cdot - k) \widetilde{\psi}(\cdot - n) \right\rangle \\
&= d_{m,n},
\end{aligned} \tag{4.88}$$

by virtue of (4.52). The desired result (4.85) then follows by substituting (4.88) into (4.10).

To prove (4.86), we first note from (4.55) and (4.6) that $L^2(\Re) = \bigoplus_{j=-\infty}^{\infty} \widetilde{W}_j$, whence, similarly to the situation in Theorem 4.3, there exists a unique sequence $\widetilde{d}_{j,k} : j, k \in \mathbf{Z}$ such that $f = \sum_{j,k} \widetilde{d}_{j,k} \widetilde{\psi}_{j,k}$. The representation (4.86) is then proved analogously to the steps which led to (4.87) and (4.88) above. ∎

4.3 Orthonormal Wavelets.

Next, we show that if the scaling function ϕ is also orthonormal, then the construction procedure of Section 4.1 yields an orthonormal wavelet ψ.

Theorem 4.16 *Suppose the scaling function ϕ of Theorem 4.4 is also orthonormal in the sense of (3.44). Then the compactly supported wavelet ψ defined by (4.30), (4.31) and (4.32), with $c = -1$ and $M = 0$, is orthonormal, and given by the formula*

$$\psi(t) = \sum_k (-1)^k p_{1-k} \phi(2t - k), \quad t \in \Re. \tag{4.89}$$

Also,

$$\psi(t) = 0,\ t \notin \left[-\frac{N-1}{2}, \frac{N+1}{2}\right].$$

Proof. Since (3.44) and (3.46) imply

$$\Phi(k) = \delta_{k,0}, \quad k \in \mathbf{Z},$$

it follows from (3.51) that then

$$E(z) = 1, \quad z \in \mathbf{C}. \tag{4.90}$$

Combining (4.90) and (4.44) then gives

$$\sum_k |\widehat{\psi}(\omega + 2\pi k)|^2 = 1, \quad \omega \in \Re, \tag{4.91}$$

Now combine (4.90) and (4.32), with $c = -1$ and $M = 0$, to obtain

$$Q(z) = -zP\left(-\frac{1}{z}\right), \tag{4.92}$$

which, together with (4.45), yield

$$\widehat{\psi}(\omega) = -e^{-i\omega/2}P(-e^{i\omega/2})\widehat{\phi}\left(\frac{\omega}{2}\right), \quad \omega \in \Re. \tag{4.93}$$

Since ϕ is a scaling function, we know that $\phi \in \mathcal{A}$, and that $\widehat{\phi}$ and ϕ satisfy (3.1). Hence, since P is a polynomial, it can be shown (cf. the proofs of Theorem 3.12 and 4.9) from (4.93) and Theorem 2.1 that $\psi, \widehat{\psi} \in \mathcal{A}$, and, by virtue of a similar argument to the one used for $\widetilde{\phi}$ and $\widehat{\widetilde{\phi}}$ in the proof of Theorem 3.12, that ψ and $\widehat{\psi}$ satisfy (2.12) with f replaced by ψ. We can therefore appeal to Theorem 2.4 to deduce, from (4.91), that (4.14) holds, i.e. ψ is an orthonormal wavelet. Next we use (4.92) to find

$$\begin{aligned} Q(z) &= -z\sum_{k=0}^{N}(-1)^k p_k z^{-k} \\ &= -z\sum_{k=-N}^{0}(-1)^k p_{-k} z^k \end{aligned}$$

$$= \sum_{k=-N}^{0} (-1)^{k+1} p_{-k} z^{k+1}$$
$$= \sum_{k=-N+1}^{1} (-1)^{k} p_{1-k} z^{k}. \tag{4.94}$$

Hence, from (4.30), (4.31), and (4.94),

$$\psi(t) = \sum_{k} (-1)^k p_{1-k} \phi(2t-k), \quad t \in \Re,$$

whereas (4.15) and (4.16) yield

$$\psi(t) = 0, \quad t \notin \left[-\frac{N-1}{2}, \frac{N+1}{2}\right].$$

■

Observe from (4.51), (4.90) and Theorem 2.1 that, for an orthonormal wavelet ψ, we have $\widetilde{\psi} = \psi$, i.e. ψ is self-dual.

A significant advantage of a wavelet ψ being orthonormal is provided by the following result.

Theorem 4.17 *Suppose ψ is an orthonormal wavelet, and suppose $f \in L^2(\Re)$. Then*

$$f = \sum_{j,k} \langle f, \psi_{j,k} \rangle \psi_{j,k}.$$

Proof. The desired result is a direct consequence of Theorem 4.15 and the fact that ψ is self-dual. ■

Moreover, it should be noted from (4.91), and the equivalent Fourier transform formulation (4.49) of (4.9), that, for an orthonormal wavelet ψ, the Riesz stability condition (4.9) automatically holds, with equality, and with Riesz bounds $\alpha = \beta = 1$.

Although the Haar scaling function ϕ_H does not strictly satisfy our conditions for a scaling function in Definition 3.1, if we use the formula (4.89) with $\phi = \phi_H$ and the sequence $\{p_k\}$ given by (3.6), we obtain the orthonormal Haar wavelet ψ_H given by

$$\psi_H(t) = \phi_H(2t) - \phi_H(2t-1), \quad t \in \Re,$$

with the sequence $\{q_k\}$ given by

$$q_k = \begin{cases} 1, & k = 0, \\ -1, & k = 1, \\ 0, & \text{elsewhere.} \end{cases}$$

We illustrate ψ_H in Fig.4.1.

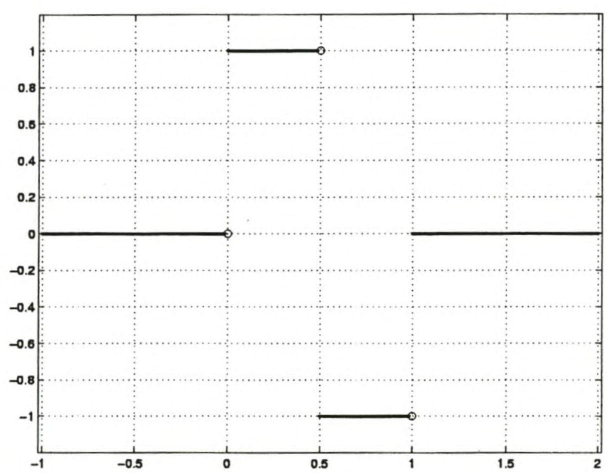


Figure 4.1: The Haar wavelet

It was proved by Daubechies in [10] that the sequence $\{W_j\} \subset L^2(\Re)$ defined by (4.7), (4.8), and with the choice $\psi = \psi_H$, does indeed satisfy (4.6). Moreover, ψ_H was shown to be the only compactly supported orthonormal wavelet that is symmetric.

We now use the formula (4.89), with the Daubechies sequence $\{p_k\} = \{p_{2,k}\}$ given in (3.45), to obtain the sequence

$$q_{2,k} = \begin{cases} \frac{1}{4}(1-\sqrt{3}), & k=-2, \\ -\frac{1}{4}(3-\sqrt{3}), & k=-1, \\ \frac{1}{4}(3+\sqrt{3}), & k=0, \\ \frac{1}{4}(1+\sqrt{3}), & k=1, \\ 0, & k \notin \{-2,\dots,1\}. \end{cases},$$

The orthonormal Daubechies wavelet $\psi_{D,2}$ of order 2 is then given, according to (4.30), by the formula

$$\psi_{D,2}(t) = \sum_{k=-2}^{1} q_{2,k}\phi_{D,2}(2t-k), \quad t \in \Re.$$

A graph of $\psi_{D,2}$ is shown in Fig.4.2.

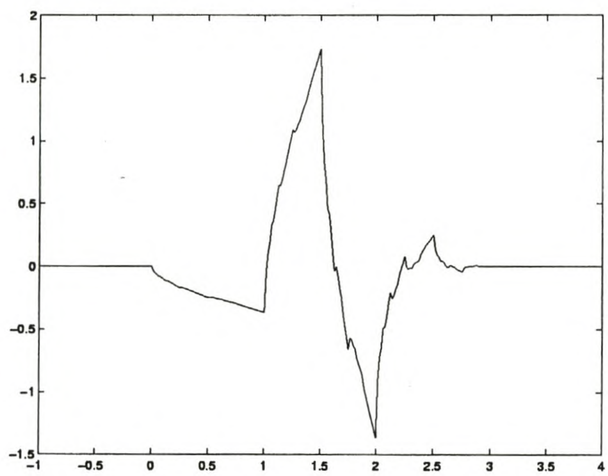


Figure 4.2: The Daubechies wavelet $\psi_{D,2}$ of order 2

4.4 Reconstruction and Decomposition Algorithms

We now investigate the reconstruction and decomposition algorithms associated with a given scaling function ϕ and corresponding wavelet ψ. The sequences $\{p_k\}$

and $\{q_k\}$ in $\ell^1 \cap \ell^2$ governing the relations

$$\phi(t) = \sum_k p_k \phi(2t-k), \quad t \in \Re,$$

and

$$\psi(t) = \sum_k q_k \phi(2t-k), \quad t \in \Re,$$

are called the *reconstruction sequences.* According to our wavelet construction procedure described in Section 4.1, we may obtain the reconstruction sequence $\{q_k\}$, as was done in (4.38) by means of the formula

$$Q(z) = -z^{2M+1} P\left(-\frac{1}{z}\right) E(-z) = \frac{1}{2}\sum_k q_k z^k, \tag{4.95}$$

where

$$P(z) = \frac{1}{2}\sum_k p_k z^k \tag{4.96}$$

and E is given by (3.51) and (3.46).

The sequences $\{a_k\}$ and $\{b_k\}$ in $\ell^1 \cap \ell^2$ governing the relation

$$\phi(2t-l) = \frac{1}{2}\sum_k a_{l-2k}\phi(t-k) + \frac{1}{2}\sum_k b_{l-2k}\psi(t-k), \quad t \in \Re \tag{4.97}$$

are called the *decomposition sequences.* According to Theorem 4.11 we may obtain these sequences by the formulas

$$A(z) = \frac{E(z)}{E(z^2)}P(z) = \frac{1}{2}\sum_k a_k z^k, \quad |z| = 1,$$

and

$$B(z) = \frac{-z^{2M+1}P\left(-\frac{1}{z}\right)}{E(z^2)} = \frac{1}{2}\sum_k b_k z^k, \quad |z| = 1.$$

Note that, if our scaling function ϕ is orthonormal, i.e. $E(z) = 1$, $z \in \mathbf{C}$, then $A(z) = P(z)$ and, by (4.95), $B(z) = Q(z)$. The decomposition relation of the Haar scaling function ϕ_H is thus given by

$$\phi_H(2t-l) = \frac{1}{2}\phi_H(t-l) + \frac{1}{2}\psi_H(t-l)$$

We have seen by Theorem 4.15 that, for a given $f \in L^2(\Re)$, we have wavelet decompositions given by

$$f(t) = \sum_{j,k} \left\langle f, \widetilde{\psi}_{j,k} \right\rangle \psi_{j,k}(t), \quad t \in \Re,$$

or

$$f(t) = \sum_{j,k} \langle f, \psi_{j,k} \rangle \, \widetilde{\psi}_{j,k}(t), \quad t \in \Re,$$

i.e. the role of the wavelet and the dual wavelet may be interchanged. Observe in Theorem 4.12 that the roles of the pair of two-scale symbols $\{P, Q\}$ associated with the pair $\{\phi, \psi\}$, and the pair of two-scale symbols $\{A, B\}$ associated with the pair $\{\widetilde{\phi}, \widetilde{\psi}\}$ may also be interchanged. This principle is called the *duality principle*.

The above method of finding the wavelet series of a function f is a cumbersome method as it involves calculating possibly difficult integrals and furthermore is only possible if we have an analog signal $f \in L^2(\Re)$ and $f \in V_j$ for some $j \in \mathbf{Z}$. In most practical problems we have only a digital sample of $f \in L^2(\Re)$, which, since $\lim_{j\to\infty} V_j = L^2(\Re)$, we can approximate arbitrarily well with a function $f_n \in V_n$ by choosing n sufficiently large, i.e. we can compute a coefficient sequence $\{c_{n,k}\}$ such that

$$f(t) \approx f_n(t) = \sum_k c_{n,k}\phi(2^n t - k), \quad t \in \Re.$$

Then, since $V_n = V_{n-1} \oplus W_{n-1}$, there exist unique functions $f_{n-1} \in V_{n-1}$ and $g_{n-1} \in W_{n-1}$ such that $f_n = f_{n-1} + g_{n-1}$. Repeated use of this procedure yields the decomposition

$$f_n = f_{n-m} + g_{n-m} + \ldots + g_{n-2} + g_{n-1},$$

where m is chosen large enough for f_{n-m} to be sufficiently 'blurred', i.e. $\|f_{n-m}\|_2 < \epsilon$ for some chosen $\epsilon > 0$, so that we may discard this term without loss of detail. This is possible because $V_n \to \{0\}$ as $n \to \infty$. This fact also means that as we decompose, the 'detail' of f_n is 'picked up' by the wavelets. In fact, on each level a different 'frequency' of detail is picked up due to the dilations of the wavelets on each level. The above procedure for wavelet decomposition is represented graphically in Fig.4.3.

$$\begin{array}{ccccccccc} & & g_{n-1} & & g_{n-2} & & & & g_{n-m} \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\ f_n & \rightarrow & f_{n-1} & \rightarrow & f_{n-2} & \rightarrow & \dots & \rightarrow & f_{n-m} \end{array}$$

Figure 4.3: The wavelet decomposition of f_n.

We proceed to deduce formulas for the decomposition

$$f_{j+1} = f_j + g_j, \tag{4.98}$$

with $f_j \in V_j$ and $g_j \in W_j$ at each index $j \in \{n-1, \dots, n-m\}$. Thus there exist unique sequences $\{c_{j,k}\}$ and $\{d_{j,k}\}$ in ℓ^2, such that

$$f_j(t) = \sum_k c_{j,k}\phi(2^j t - k) \tag{4.99}$$

and

$$g_j(t) = \sum_k d_{j,k}\psi(2^j t - k). \tag{4.100}$$

Next, we use (4.97) and (4.99) to deduce that

$$\begin{aligned} f_{j+1}(t) &= \sum_k c_{j+1,k}\left[\frac{1}{2}\sum_l a_{k-2l}\phi(2^j t - l) + \frac{1}{2}\sum_l b_{k-2l}\psi(2^j t - l)\right] \\ &= \frac{1}{2}\sum_l \left[\sum_k a_{k-2l}c_{j+1,k}\right]\phi(2^j t - l) + \frac{1}{2}\sum_l\left[\sum_k b_{k-2l}c_{j+1,k}\right]\psi(2^j t - l). \end{aligned} \tag{4.101}$$

Also, from (4.98), (4.99) and (4.100),

$$f_{j+1}(t) = \sum_k c_{j,k}\phi(2^j t - k) + \sum_k d_{j,k}\psi(2^j t - k), \quad t \in \Re, \tag{4.102}$$

so that, from (4.101), (4.102) and the uniqueness of the decomposition (4.98), we deduce the formulas

$$\left.\begin{aligned} c_{j,k} &= \frac{1}{2}\sum_l a_{l-2k}c_{j+1,l} \\ d_{j,k} &= \frac{1}{2}\sum_l b_{l-2k}c_{j+1,l}. \end{aligned}\right\} \qquad (4.103)$$

Note that the algorithm given by (4.103) is simply a filtering of the original coefficients $\{c_{n,k}\}$ with downsamples of the decomposition sequences $\{a_k\}$ and $\{b_k\}$. During this filtering process we discard all but the last $\{c_{j,k}\}$ sequence, but keep the $\{d_{j,k}\}$ sequences, which we shall use to reconstruct f_n again. The recursive *pyramid* algorithm given by (4.103), as illustrated graphically in Fig.4.4, is called the decomposition algorithm, and can be used to evaluate the wavelet decomposition

$$f_n(t) = \sum_{j,k} d_{j,k}\psi(2^j t - k),$$

of a given function $f_n \in V_n$.

$$\begin{array}{ccccccccc} & & \{d_{n-1,k}\} & & \{d_{n-2,k}\} & & & & \{d_{n-m,k}\} \\ & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\ \{c_{n,k}\} & \rightarrow & \{c_{n-1,k}\} & \rightarrow & \{c_{n-2,k}\} & \rightarrow & \dots & \rightarrow & \{c_{n-m,k}\} \end{array}$$

Figure 4.4: The decomposition algorithm

Suppose we have the wavelet coefficients $\{d_{j,k}\}$, $j \in \{n-m, \dots, n-1\}$ and $\{c_{n-m,k}\}$ of a function $f_n \in V_n$. To reconstruct f_n in terms of the basis functions of V_n we begin by adding the functions f_{n-m} and g_{n-m} to get f_{n-m+1} and carry on with this procedure until we have f_n as is shown graphically in Fig.4.5.

We now deduce formulas for the reconstruction $f_j + g_j = f_{j+1}$. Substituting (4.99), (4.100) into (4.98) and then using (4.95) and (4.96), we find

$$\sum_k c_{j+1,k}\phi(2^{j+1}t - k) \quad = \quad f_{j+1}(t) = f_j(t) + g_j(t)$$

$$\boxed{\begin{array}{ccccccccc} g_{n-m} & & g_{n-m+1} & & & & g_{n-1} & & \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ f_{n-m} & \rightarrow & f_{n-m+1} & \rightarrow & \dots & \rightarrow & f_{n-1} & \rightarrow & f_n \end{array}}$$

Figure 4.5: The reconstruction of f_n

$$\begin{aligned} &= \sum_k c_{j,k}\phi(2^j t - k) + \sum_k d_{j,k}\psi(2^j t - k) \\ &= \sum_k c_{j,k}\left[\sum_l p_l \phi(2^{j+1}t - 2k - l)\right] \\ &\qquad + \sum_k d_{j,k}\left[\sum_l q_l \phi(2^{j+1}t - 2k - l)\right] \\ &= \sum_l \left[\sum_k (c_{j,k}p_{l-2k} + d_{j,k}q_{l-2k})\right]\phi(2^{j+1}t - l), \end{aligned}$$

so that

$$c_{j+1,k} = \sum_l (c_{j,l}p_{k-2l} + d_{j,l}q_{k-2l}). \tag{4.104}$$

The recursive algorithm given by (4.104) and illustrated in Fig.4.6, is called the *reconstruction algorithm*, and is used to reconstruct f_n perfectly.

$$\boxed{\begin{array}{ccccccccc} \{d_{n-m,k}\} & & \{d_{n-m+1,k}\} & & & & \{d_{n-1,k}\} & & \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ \{c_{n-m,k}\} & \rightarrow & \{c_{n-m+1,k}\} & \rightarrow & \dots & \rightarrow & \{c_{n-1,k}\} & \rightarrow & \{c_{n,k}\} \end{array}}$$

Figure 4.6: The reconstruction algorithm

4.5 Vanishing Moments

The ability of a wavelet ψ to pick up 'detail' (different levels of frequency) is measured by its so-called *vanishing moments*, which we define next.

Definition 4.2 *We say a function f has vanishing moments of order m if*

$$\int_{-\infty}^{\infty} t^k f(t)\, \mathrm{d}t = 0, \quad k = 0, 1, \ldots, m-1. \tag{4.105}$$

The following result then holds.

Theorem 4.18 *Suppose ψ is a compactly supported wavelet obtained from Theorem 4.5. Then ψ has vanishing moments of order $m \geq 1$, with*

$$\int_{-\infty}^{\infty} \psi(t)\, \mathrm{d}t = 0. \tag{4.106}$$

Proof. First, observe from (2.4) that

$$\int_{-\infty}^{\infty} \psi(t)\, \mathrm{d}t = \widehat{\psi}(0). \tag{4.107}$$

But, from (4.45) and (4.32), we have, with $z = e^{-i\omega/2}$, $\omega \in \Re$,

$$\widehat{\psi}(\omega) = cz^{2M+1} P(-z^{-1}) E_{\phi}(-z) \widehat{\phi}\left(\frac{\omega}{2}\right), \quad \omega \in \Re,$$

and thus

$$\widehat{\psi}(0) = cP(-1)E(-1)\widehat{\phi}(0) = 0, \tag{4.108}$$

from (3.34). The desired result (4.106) is then obtained by combining (4.107) and (4.108). ∎

Note that the identity (4.106) implies that the function ψ must behave in a wave-like fashion along the t-axis, hence the name *wavelet*.

The vanishing moment condition (4.105) has, for compactly supported functions f, the following equivalent Fourier transform formulation, which is often useful.

Theorem 4.19 *Suppose $f \in \mathcal{A}$, and suppose f is a compactly supported function. Then f has vanishing moments of order m in the sense of (4.105) if and only if the condition*

$$\widehat{f}^{(l)}(0) = 0, \quad l = 0, 1, \ldots, m-1, \tag{4.109}$$

holds.

Proof. Since f has compact support, we may differentiate the formula (2.4) l-times, to obtain

$$\widehat{f}^{(l)}(\omega) = \int_{-\infty}^{\infty} (-it)^l e^{-i\omega t} f(t)\, \mathrm{d}t, \quad \omega \in \Re,$$

and thus

$$\widehat{f}^{(l)}(0) = (-i)^l \int_{-\infty}^{\infty} t^l f(t)\, \mathrm{d}t. \tag{4.110}$$

The result of the theorem then follows from (4.109), (4.110) and (4.105). ■

Next, we show that the property of vanishing moments of a wavelet ψ is inherited by its dual $\widetilde{\psi}$.

Theorem 4.20 *Let $\widetilde{\psi}$ denote the dual wavelet, as defined by (4.50), of the wavelet ψ in Theorem 4.18, and suppose ψ has vanishing moments of order m, i.e.*

$$\int_{-\infty}^{\infty} t^l \psi(t)\, \mathrm{d}t = 0, \quad l = 0, 1, \ldots, m-1. \tag{4.111}$$

Then $\widetilde{\psi}$ also has vanishing moments of order m, i.e.

$$\int_{-\infty}^{\infty} t^l \widetilde{\psi}(t)\, \mathrm{d}t = 0, \quad l = 0, 1, \ldots, m-1. \tag{4.112}$$

Moreover

$$\int_{-\infty}^{\infty} t^l \widetilde{\psi}_{j,k}(t)\, \mathrm{d}t = 0, \quad l = 0, 1, \ldots, m-1, \quad j, k \in \mathbf{Z}, \tag{4.113}$$

with the sequence $\{\widetilde{\psi}_{j,k} : j, k \in \mathbf{Z}\}$ defined as in (4.54).

Proof. First observe that, for $l \in \{0, 1, \ldots, m-1\}$ and $k \in \mathbf{Z}$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} t^l \psi(t-k)\,\mathrm{d}t &= \int_{-\infty}^{\infty} (t+k)^l \psi(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \sum_{r=0}^{l} \binom{l}{r} k^{l-r} t^r \psi(t)\,\mathrm{d}t \\
&= \sum_{r=0}^{l} \binom{l}{r} k^{l-r} \int_{-\infty}^{\infty} t^r \psi(t)\,\mathrm{d}t \\
&= 0, \qquad (4.114)
\end{aligned}$$

by virtue of (4.111). Thus, since the sequence $\{\mu_k\}$ in (4.57) has exponential decay for $|k| \to \infty$, we see, from (4.57) and the compact support property (4.41) of ψ, that the function $\tilde{\psi}$ has exponential decay for $|t| \to \infty$.

Hence, from (4.57) and (4.114), and for $l \in \{0, 1, \ldots, m-1\}$, we have

$$\begin{aligned}
0 &= \sum_k \mu_k \left[\int_{-\infty}^{\infty} t^l \psi(t-k)\,\mathrm{d}t\right] \\
&= \int_{-\infty}^{\infty} t^l \left[\sum_k \mu_k \psi(t-k)\right] \mathrm{d}t \\
&= \int_{-\infty}^{\infty} t^l \tilde{\psi}(t)\,\mathrm{d}t,
\end{aligned}$$

thereby yielding (4.112).

To prove (4.113) we note from (4.54) that, for $l \in \{0, 1, \ldots, m-1\}$ and $j, k \in \mathbf{Z}$,

$$\begin{aligned}
\int_{-\infty}^{\infty} t^l \tilde{\psi}_{j,k}(t)\,\mathrm{d}t &= 2^{-j(l+1/2)} \int_{-\infty}^{\infty} (t+k)^l \tilde{\psi}(t)\,\mathrm{d}t \\
&= 2^{-j(l+1/2)} \sum_{r=0}^{l} \binom{l}{r} k^{l-r} \int_{-\infty}^{\infty} t^r \tilde{\psi}(t)\,\mathrm{d}t
\end{aligned}$$

which, together with (4.112), then yield (4.113). ■

The vanishing moments of order m of a wavelet enables the wavelet decomposition procedure to detect local non-polynomial-like (or non-smooth) behaviour (of order $< m$) of a function $f \in L^2(\Re)$ in the following manner.

Suppose a function $f \in L^2(\Re)$ has smooth polynomial-like behaviour locally on an interval $I = [t_0 - \delta, t_0 + \delta]$, with $t_0, \delta \in \Re$, in the sense that f has a Taylor-expansion on I given by

$$f(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(t_0)}{k!}(t - t_0)^k + R_m(t), \quad t \in I, \tag{4.115}$$

with

$$\|R_m\|_\infty := \max_{t \in I} |R_m(t)|$$

'small' in some sense. Suppose moreover that, for a given resolution level $j \in \mathbf{Z}$, and with ψ denoting a compactly supported wavelet based on Theorem 4.5, we define M_j as the largest index set in $\mathbf{Z}$ with the property that supp $\psi_{j,k} \cap I = \emptyset,\ k \notin M_j$.

Then, from (4.85), we have

$$f(t) = \sum_j \sum_{k \in M_j} \left\langle f, \widetilde{\psi}_{j,k} \right\rangle \psi_{j,k}(t), \quad t \in I, \tag{4.116}$$

But, from (4.115), we have, for $k \in M_j$, that

$$\begin{aligned} \left\langle f, \widetilde{\psi}_{j,k} \right\rangle &= \left\langle \sum_{k=0}^{m-1} \frac{f^{(k)}(t_0)}{k!}(t - t_0)^k + R_m, \widetilde{\psi}_{j,k} \right\rangle \\ &= \sum_{k=0}^{m-1} \frac{f^{(k)}(t_0)}{k!} \int_{-\infty}^{\infty} (t - t_0)^k \widetilde{\psi}_{j,k}(t)\ \mathrm{d}t + \left\langle R_m, \widetilde{\psi}_{j,k} \right\rangle . \end{aligned} \tag{4.117}$$

Here

$$\begin{aligned} &\int_{-\infty}^{\infty} (t - t_0)^k \widetilde{\psi}_{j,k}(t)\ \mathrm{d}t \\ &= \sum_{r=0}^{k} \binom{k}{r} (-t_0)^{k-r} \int_{-\infty}^{\infty} t^k \widetilde{\psi}_{j,k}(t)\ \mathrm{d}t = 0, \end{aligned} \tag{4.118}$$

by virtue of (4.113) in Theorem 4.20. Hence, from (4.118) and (4.117), together with the Cauchy-Schwarz inequality, we have for $k \in M_j$ that

$$\begin{aligned} |\left\langle f, \widetilde{\psi}_{j,k} \right\rangle| &\leq |\left\langle R_m, \widetilde{\psi}_{j,k} \right\rangle| \\ &\leq \|R_m\|_2 \|\widetilde{\psi}_{j,k}\|_2 \\ &\leq \sqrt{2\delta} \|R_m\|_\infty \|\widetilde{\psi}\|_2, \end{aligned} \tag{4.119}$$

having used also (4.54).

We see therefore from (4.116) and (4.119) that, if a function $f \in L^2(\Re)$ has locally m-th order smooth or polynomial-like behaviour on an interval I, then the corresponding wavelet coefficients $d_{j,k} = \left\langle f, \tilde{\psi}_{j,k} \right\rangle$ are expected to be small in the above sense.

In practical applications therefore, we compare, for a given resolution level $j \in \mathbf{Z}$, the relative sizes of the wavelet coefficients $d_{j,k}$ for increasing values of $k \in \mathbf{Z}$, signifying movement along the time domain from left to right according to the support intervals of the function $\{\psi_{j,k} : k \in \mathbf{Z}\}$.

Relatively small values of $|d_{j,k}|$ then indicate smooth polynomial-like behaviour of f, whereas a sudden increase in $|d_{j,k}|$ identifies a position of non-smoothness in f.

A numerical example will be given in Section 5.5 below.

Chapter 5

Cardinal Spline Wavelets

In this chapter, we show that the cardinal B-spline of order m is an admissible choice for a scaling function ϕ according to Definition 3.1. We then proceed to use the results of Chapter 4 to explicitly construct the corresponding B-spline wavelet, i.e. the minimally supported cardinal spline wavelet.

5.1 Cardinal Spline Functions

For $m \in \mathbf{N}$, $j \in \mathbf{Z}$, we define the linear space $S_m^{(j)} = S_m(\mathbf{Z}/2^j)$ as the space of functions $f : \Re \to \Re$ which are such that

$$\left.\begin{aligned} & f\,|_{[\frac{k}{2^j},\frac{k+1}{2^j}]} = p_k \in \Pi_{m-1}, \quad k \in \mathbf{Z}, \\ \text{with} & \\ & f \in C^{m-2}(\Re). \end{aligned}\right\} \tag{5.1}$$

Here we have used the notation $C^k(\Re)$ for the space of functions f for which $f, f', f'', \ldots, f^{(k)} \in C(\Re)$, so that $C^0(\Re) = C(\Re)$, and with $C^{-1}(\Re)$ denoting the space of piecewise constant functions on $\Re$. The space $S_m := S_m^{(0)} = S_m(\mathbf{Z})$ is called the *cardinal spline space*, and functions in S_m are called *cardinal spline functions*.

Observe from (5.1) that $\{S_m^{(j)} : j \in \mathbf{Z}\}$ is a nested sequence:

$$\ldots S_m^{(-1)} \subset S_m^{(0)} \subset S_m^{(1)} \subset \ldots.$$

We define the sequence $\{N_m : m \in \mathbf{N}\}$ of real-valued functions on $\Re$ recursively by means of the definition

$$N_m(t) = \int_0^1 N_{m-1}(t-x)\,\mathrm{d}x, \quad t \in \Re, \quad m = 2, 3, \ldots, \tag{5.2}$$

where

$$N_1(t) = \begin{cases} 1, & t \in [0,1), \\ 0, & t \notin [0,1), \end{cases} \tag{5.3}$$

i.e. recalling also (3.5), $N_1 = \phi_H$, the Haar (or box) function. Observe in particular from (5.2) and (5.3) and the definition (2.16) of the convolution of two real-valued functions, that the recursion formula (5.2) has the equivalent convolution formulation

$$N_m = N_{m-1} * N_1, \quad m = 2, 3, \ldots. \tag{5.4}$$

We now refer to [2, Theorem 4.3] for the proof of the following properties of the sequence generated by (5.2) and (5.3).

Theorem 5.1 *The sequence $\{N_m : m \in \mathbf{N}\}$, as defined by (5.2) and (5.3), satisfies the following properties:*

(i)
$$N_m(t) > 0, \quad t \in (0, m), \quad m \in \mathbf{N}; \tag{5.5}$$

(ii)
$$\operatorname{supp} N_m = [0, m], \quad m \in \mathbf{N}; \tag{5.6}$$

(iii)
$$\sum_k N_m(t-k) = 1, \quad t \in \Re, \quad m \in \mathbf{N}; \tag{5.7}$$

(iv)
$$N_m(\cdot - k) \in S_m, \quad k \in \mathbf{Z}, \quad m \in \mathbf{N}; \tag{5.8}$$

$$\textit{(v)}\quad N'_m(t) = N_{m-1}(t) - N_{m-1}(t-1), \;\; t \in \Re, \;\; m = 2,3,\ldots; \tag{5.9}$$

$$\textit{(vi)}\qquad N_m(m-t) = N_m(t), \;\; t \in \Re, \;\; m = 2,3,\ldots; \tag{5.10}$$

$$\textit{(vii)}\qquad N_m(t) = \tfrac{t}{m-1}N_{m-1}(t) + \tfrac{m-t}{m-1}N_{m-1}(t-1), \;\; t \in \Re,$$
$$m = 2,3,\ldots. \tag{5.11}$$

Remark. Observe from (5.8) and (5.1) that

$$N_m(2^j \cdot -k) \in S_m^{(j)}, \;\; j,k \in \mathbf{Z}. \tag{5.12}$$

We next state the following result, the proof of which is found in [14, Theorem 2.1].

Theorem 5.2 *The sequence $\{N_m(\cdot - k) : k \in \mathbf{Z}\}$ of spline functions is a basis of S_m in the sense that for each $f \in S_m$ there exists a unique sequence $\{c_k\} \subset \Re$ such that $f = \sum\limits_k c_k N(\cdot - k)$.*

The function N_m is called the *cardinal B-spline of order m.*

Next we show that N_m is a refinable function with an explicitly known two-scale sequence, as implied by the last result of the next theorem.

Theorem 5.3 *In Theorem 5.1, we also have, for $m \in \mathbf{N}$,*

$$\textit{(a)}\quad \widehat{N}_m(\omega) = \begin{cases} \left(\dfrac{1-e^{-i\omega}}{i\omega}\right)^m = e^{-im\omega/2}\left(\dfrac{\sin(\omega/2)}{\omega/2}\right)^m, & \omega \in \Re\setminus\{0\}, \\ 1, \;\; \omega = 0; & \end{cases} \tag{5.13}$$

$$\textit{(b)}\qquad \widehat{N}_m(2\pi j) = \delta_{j,0}, \;\; j \in \mathbf{Z}; \tag{5.14}$$

(c) $$\int_{-\infty}^{\infty} N_m(t)\, \mathrm{d}t = 1; \tag{5.15}$$

(d) $$N_m(t) = \frac{1}{2^{m-1}} \sum_{k=0}^{m} \binom{m}{k} N_m(2t-k), \quad t \in \Re. \tag{5.16}$$

Proof. First, observe from the definition (2.4) of the Fourier transform, and (5.3), that

$$\widehat{N_1}(\omega) = \begin{cases} \dfrac{1-e^{-i\omega}}{i\omega} = e^{-i\omega/2}\dfrac{\sin(\omega/2)}{\omega/2}, & \omega \in \Re \setminus \{0\}, \\ 1, \quad \omega = 0. \end{cases} \tag{5.17}$$

But (5.4), and the properties (2.17) and (2.18) of the convolution $*$ give

$$N_m = N_1 * N_1 * \ldots * N_1, \quad (m \text{ times}). \tag{5.18}$$

and thus

$$\widehat{N_m} = (\widehat{N_1})^m, \tag{5.19}$$

and (5.13) follows from (5.19) and (5.17). The results (5.14) and (5.15) are immediate consequences of (5.13) and (2.4).

To prove (5.16), we first note from (5.17) that

$$\widehat{N_1}(\omega) = \left(\frac{1+e^{-i\omega/2}}{2}\right) \widehat{N_1}\left(\frac{\omega}{2}\right), \quad \omega \in \Re, \tag{5.20}$$

and thus, combining (5.20) and (5.19),

$$\widehat{N_m}(\omega) = \left(\frac{1+e^{-i\omega/2}}{2}\right)^m \widehat{N_m}\left(\frac{\omega}{2}\right), \quad \omega \in \Re. \tag{5.21}$$

Noting that, from (5.3),

$$N_1(t) = N_1(2t) + N_1(2t-1), \quad t \in \Re,$$

yielding (5.16) for $m = 1$.

Suppose next $m \geq 2$, so that (5.13) implies $\widehat{N}_m \in L^1(\Re)$. Also, (5.1), (5.6) and (5.8) show that $N_m \in L^1(\Re) \cap C_u(\Re)$, whence $N_m \in \mathcal{A}$. Hence we may appeal to Theorem 2.1 to deduce from (5.21) that, for $t \in \Re$,

$$\begin{aligned}
N_m(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\widehat{N}_m(\omega)\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\left(\frac{1+e^{-i\omega/2}}{2}\right)^m \widehat{N}_m\left(\frac{\omega}{2}\right)\,\mathrm{d}\omega \\
&= \frac{1}{2^{m-1}}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{2i\omega t}\left(1+e^{-i\omega}\right)^m \widehat{N}_m(\omega)\,\mathrm{d}\omega \\
&= \frac{1}{2^{m-1}}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{2i\omega t}\sum_{k=0}^{m}\binom{m}{k} e^{-ik\omega}\widehat{N}_m(\omega)\,\mathrm{d}\omega \\
&= \frac{1}{2^{m-1}}\sum_{k=0}^{m}\binom{m}{k}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(2t-k)\omega}\widehat{N}_m(\omega)\,\mathrm{d}\omega \\
&= \frac{1}{2^{m-1}}\sum_{k=0}^{m}\binom{m}{k} N_m(2t-k),
\end{aligned}$$

thereby proving (5.16). ■

According to the definitions (3.46) and (3.51), we now define the autocorrelation function $\Phi = \Phi_m$ and the Euler-Frobenius Laurent polynomial $E = E_m$ corresponding to the cardinal B-spline N_m by

$$\Phi_m(t) = \int_{-\infty}^{\infty} N_m(x)N_m(x+t)\,\mathrm{d}x, \quad t \in \Re, \tag{5.22}$$

and

$$E_m(t) = \sum_k \Phi_m(k)z^k, \quad z \in \mathbf{C}\setminus\{0\}. \tag{5.23}$$

Now observe from (5.22) and (5.10), and the definition (2.16) of the convolution operator, that, for $l \in \mathbf{Z}$,

$$\Phi_m(l) = \int_{-\infty}^{\infty} N_m(m-x)N_m(x+l)\,\mathrm{d}x$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} N_m(m+l-x)N_m(x)\,\mathrm{d}x \\ &= (N_m * N_m)(m+l). \end{aligned} \tag{5.24}$$

But, from (5.18), and the associative property (2.18) of a convolution, we have

$$N_m * N_m = N_1 * N_1 * \ldots * N_1 \ \ (2m \text{ times}) = N_{2m},$$

which, together with (5.24), yield the formula

$$\Phi_m(l) = N_{2m}(m+l), \ \ l \in \mathbf{Z}. \tag{5.25}$$

Moreover, (5.23) and (5.25) then give

$$E_m(z) = \sum_k N_{2m}(m+k)z^k, \ \ z \in \mathbf{C} \setminus \{0\}, \tag{5.26}$$

and thus, from (5.6) and (5.8), since $N_m(t) = 0,\ t \notin (0,m)$ for $m \geq 2$,

$$E_m(z) = \sum_{k=-m+1}^{m-1} N_{2m}(m+k)z^k, \ \ z \in \mathbf{C} \setminus \{0\}.$$

Observe that (5.26) can also be written in the form

$$E_m(z) = z^{1-m}\widetilde{N}_{2m}(z), \ \ z \in \mathbf{C} \setminus \{0\},$$

with the polynomial $\widetilde{N}_m \in \Pi_{m-2}$ defined for $m \geq 2$ by the formula

$$\widetilde{N}_m(z) = \sum_{k=0}^{m-2} N_m(k+1)z^k, \ \ z \in \mathbf{C}. \tag{5.27}$$

The polynomial $\widetilde{N}_m$ is called the *Euler-Frobenius polynomial* of order m.

The following properties of $\widetilde{N}_m$ will be instrumental in proving the Riesz stability of N_m. Our result is an extension of the analysis given in [2, Section 6.4], in which only cardinal B-splines of even order were considered. The proof below is joint work with my fellow masters student Karin M. Goosen, and may also be found in her thesis [13].

Theorem 5.4 *For an integer $m \geq 2$, the Euler-Frobenius polynomial $\widetilde{N}_m$ is a symmetric polynomial of degree $m-2$. Moreover, we have*

$$\widetilde{N}_2(z) = 1, \quad z \in \mathbf{C}, \tag{5.28}$$

$$\widetilde{N}_3(z) = \frac{1}{2}(z+1), \quad z \in \mathbf{C}, \tag{5.29}$$

whereas, for $m \geq 4$, there exists a sequence $\left\{\lambda_{m,j} : j = 1, 2, \ldots, \left\lfloor \frac{m-2}{2} \right\rfloor\right\} \subset \Re$, such that

$$-1 < \lambda_{m,j} < 0, \qquad j = 1, 2, \ldots, \left\lfloor \frac{m-2}{2} \right\rfloor, \tag{5.30}$$

and

$$\widetilde{N}_m(z) = \frac{1}{(m-1)!} \left\{ \begin{array}{lll} P_m(z), & z \in \mathbf{C}, & \text{if } m \text{ is even}, \\ (z+1)P_m(z), & z \in \mathbf{C}, & \text{if } m \text{ is odd}, \end{array} \right\} \tag{5.31}$$

where the polynomial P_m is given by the formula

$$P_m(z) = \prod_{j=1}^{\left\lfloor \frac{m-2}{2} \right\rfloor} (z - \lambda_{m,j}) \left(z - \frac{1}{\lambda_{m,j}}\right), \quad z \in \mathbf{C}. \tag{5.32}$$

Proof. The fact that $\widetilde{N}_m$ has degree $m-2$ is clear from (5.27) and (5.5). Moreover, using (5.10), we find for $z \in \mathbf{C}$ that

$$\begin{aligned} z^{m-2}\widetilde{N}_m\left(\frac{1}{z}\right) &= z^{m-2}\sum_{k=0}^{m-2} N_m(k+1)z^{-k} \\ &= \sum_{k=0}^{m-2} N_m(k+1)z^{m-2-k} \\ &= \sum_{k=0}^{m-2} N_m(m-(k+1))z^k \\ &= \sum_{k=0}^{m-2} N_m(k+1)z^k = \widetilde{N}_m(z), \end{aligned}$$

i.e. $\widetilde{N}_m$ is a symmetric polynomial.

Next, we observe that the recursion formula (5.11) yields

$$N_{m+1}(k+1) = \frac{k+1}{m}N_m(k+1) + \frac{m-k}{m}N_m(k), \quad k \in \mathbf{Z}, \quad m = 1, 2, \ldots, \tag{5.33}$$

which, together with (5.3) and (5.27), give the formulas (5.28) and (5.29),

Next, suppose $m \geq 4$, and define the sequence $\{e_n : n = 0, 1, \ldots\}$ by

$$e_n(\omega) = e^{-in\omega/2}\widetilde{N}_{n+2}(e^{i\omega}), \quad \omega \in \Re, \quad n = 0, 1, \ldots \ , \tag{5.34}$$

which, together with (5.28) and (5.29), give

$$e_0(\omega) = 1, \quad \omega \in \Re, \tag{5.35}$$

and

$$e_1(\omega) = \cos(\omega/2), \quad \omega \in \Re. \tag{5.36}$$

We claim that the recursion formula

$$e_{n+1}(\omega) = \cos(\omega/2)e_n(\omega) - \frac{2}{n+2}\sin(\omega/2)e_n'(\omega), \quad \omega \in \Re, \quad n = 0, 1, \ldots, \tag{5.37}$$

holds. Observe from (5.35) and (5.36) that (5.37) holds for $n = 0$. In general, for $n \in \{0, 1, \ldots\}$, we find from (5.34) and (5.27), for $\omega \in \Re$,

$$\begin{aligned}
&\cos(\omega/2)e_n(\omega) - \frac{2}{n+2}\sin(\omega/2)e_n'(\omega) \\
= \ &\frac{e^{i\omega/2} + e^{-i\omega/2}}{2}e^{-in\omega/2}\sum_{k=0}^{n} N_{n+2}(k+1)e^{i\omega k} \\
&\quad - \frac{2}{n+2}\frac{e^{i\omega/2} - e^{-i\omega/2}}{2i}\sum_{k=0}^{n} N_{n+2}(k+1)i\left(k - \frac{n}{2}\right)e^{i\left(k-\frac{n}{2}\right)\omega} \\
= \ &\frac{e^{i(1-n)\omega/2} + e^{-i(1+n)\omega/2}}{2}\sum_{k=0}^{n} N_{n+2}(k+1)e^{i\omega k} \\
&\quad - \frac{e^{i(1-n)\omega/2} - e^{-i(1+n)\omega/2}}{2(n+2)}\sum_{k=0}^{n}(2k-n)N_{n+2}(k+1)e^{i\omega k} \\
= \ &\frac{e^{i(1-n)\omega/2}}{n+2}\sum_{k=0}^{n}(n+1-k)N_{n+2}(k+1)e^{i\omega k} + \frac{e^{-i(1+n)\omega/2}}{n+2}\sum_{k=0}^{n}(k+1)N_{n+2}(k+1)e^{i\omega k}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{i(1-n)\omega/2}}{n+2}\sum_{k=1}^{n+1}(n+2-k)N_{n+2}(k)e^{i\omega(k-1)} + \frac{e^{-i(1+n)\omega/2}}{n+2}\sum_{k=0}^{n}(k+1)N_{n+2}(k+1)e^{i\omega k} \\
&= e^{-i(n+1)\omega/2}\sum_{k=0}^{n+1}\left[\frac{k+1}{n+2}N_{n+2}(k+1) + \frac{n+2-k}{n+2}N_{n+2}(k)\right]e^{i\omega k} \\
&= e^{-i(n+1)\omega/2}\sum_{k=0}^{n+1}N_{n+3}(k+1)e^{i\omega k} \\
&= e_{n+1}(\omega),
\end{aligned}
$$

by virtue also of (5.33), (5.6) and (5.34). Hence the recursion formula (5.37) is satisfied.

Our next step is to show that there exists a sequence $\{\alpha_{n,k} : k = 0, 1, \ldots, n\} \subset \Re$ such that

$$
e_n(\omega) = \sum_{k=0}^{n}\alpha_{n,k}(\cos\omega/2)^k, \quad \omega \in \Re, \quad n = 0, 1, \ldots, \tag{5.38}
$$

and where

$$
\alpha_{n,n} = \frac{2^n}{(n+1)!}, \quad n = 0, 1, \ldots. \tag{5.39}
$$

Observe from (5.35) and (5.36) that (5.38), (5.39) hold for $n = 0$ and $n = 1$. The proof that (5.38), (5.39) hold for all $n \in \{0, 1, \ldots\}$ is by means of induction, the inductive step from n to $n+1$ being given, from (5.37), by

$$
\begin{aligned}
e_{n+1}(\omega) &= \cos(\omega/2)\sum_{k=0}^{n}\alpha_{n,k}(\cos\omega/2)^k \\
&\qquad -\frac{2}{n+2}\sin(\omega/2)\sum_{k=1}^{n}k\alpha_{n,k}(\cos\omega/2)^{k-1}\left(-\frac{1}{2}\sin(\omega/2)\right) \\
&= \sum_{k=1}^{n+1}\alpha_{n,k-1}(\cos\omega/2)^k \\
&\qquad +\frac{1}{n+2}(1-\cos^2(\omega/2))\sum_{k=0}^{n-1}(k+1)\alpha_{n,k+1}(\cos\omega/2)^k,
\end{aligned}
$$

from which we see that there exists a sequence $\{\alpha_{n+1,k} : k = 0, 1, \ldots, n+1\} \subset \Re$ such that

$$
e_{n+1}(\omega) = \sum_{k=0}^{n+1}\alpha_{n+1,k}(\cos\omega/2)^k, \quad \omega \in \Re,
$$

with

$$\alpha_{n+1,n+1} = \frac{2}{n+2}\alpha_{n,n}. \tag{5.40}$$

Hence (5.38), (5.39) hold for all $n \in \{0,1,\ldots\}$, where the formula (5.39) is easily proved by means of (5.40) and the fact that $\alpha_{0,0} = 1$, $\alpha_{1,1} = 1$ by virtue of (5.35) and (5.36).

Now define the polynomial sequence $\{U_n : n = 0,1,\ldots\}$ by means of

$$U_n(z) = \sum_{k=0}^{n} \alpha_{n,k} z^k, \quad z \in \mathbf{C}, \quad n = 0,1,\ldots, \tag{5.41}$$

whence, from (5.38), we have

$$e_n(\omega) = U_n(\cos\omega/2), \quad \omega \in \Re, \quad n = 0,1,\ldots. \tag{5.42}$$

Observe from (5.35), (5.36) and (5.42) that

$$U_0(z) = 1, \quad z \in \mathbf{C}, \tag{5.43}$$

and

$$U_1(z) = z, \quad z \in \mathbf{C}. \tag{5.44}$$

Now observe from (5.42) that, for $\omega \in \Re$,

$$e_n'(\omega) = U_n'(\cos\omega/2)\left(-\frac{1}{2}\sin\omega/2\right),$$

which, together with (5.37) and (5.42), imply that, for $n \in \{0,1,\ldots\}$ and $\omega \in \Re$,

$$U_{n+1}(\cos\omega/2) = (\cos\omega/2)U_n(\cos\omega/2) + \frac{1-\cos^2(\omega/2)}{n+2}U_n'(\cos\omega/2),$$

and thus, since U_{n+1} and U_n are polynomials,and since if a polynomial is identically zero on $[-1,1]$ then it is identically zero on all of $\mathbf{C}$, we see that the sequence $\{U_n : n = 0,1,\ldots\}$ satisfies the recursive formula

$$U_{n+1}(z) = zU_n(z) + \frac{1-z^2}{n+2}U_n'(z), \quad z \in \mathbf{C}, \quad n = 0,1,\ldots, \tag{5.45}$$

with U_0 given by (5.43). Observe also from (5.41), (5.39) that U_n is a polynomial of degree n for $n = 0, 1, \ldots$.

Now define the sequence $\{u_n : n = 0, 1, \ldots\}$ by

$$u_n(z) = \frac{U_n(iz)}{i^n}, \quad z \in \mathbf{C}, \quad n = 0, 1, \ldots, \tag{5.46}$$

so that, using (5.46), (5.43) and (5.44), we have

$$u_0(z) = 1, \quad z \in \mathbf{C}, \tag{5.47}$$

and

$$u_1(z) = z, \quad z \in \mathbf{C}. \tag{5.48}$$

Since (5.46) implies

$$u_n'(z) = \frac{U_n'(iz)}{i^{n-1}}, \quad z \in \mathbf{C}, \quad n = 0, 1, \ldots, \tag{5.49}$$

it follows from replacing z by iz in (5.45), and using (5.49), that the sequence $\{u_n : n = 0, 1, \ldots\}$ satisfies the recursion formula

$$u_{n+1}(z) = zu_n(z) - \frac{1+z^2}{n+2} u_n'(z), \quad z \in \mathbf{C}, \quad n = 0, 1, \ldots, \tag{5.50}$$

and with u_0 given by (5.47).

It is clear from (5.50) and (5.47) that, for each $n \in \{0, 1, \ldots\}$, u_n is a polynomial with real coefficients, i.e. there exists a sequence $\{\beta_{n,k} : k = 0, 1, \ldots, n\} \subset \Re$ such that

$$u_n(z) = \sum_{k=0}^{n} \beta_{n,k} z^k, \quad z \in \mathbf{C}, \quad n = 0, 1, \ldots, \tag{5.51}$$

with, from (5.46), (5.41) and (5.39),

$$\beta_{n,n} = \alpha_{n,n} = \frac{2^n}{(n+1)!}, \quad n = 0, 1, \ldots, \tag{5.52}$$

so that the polynomial u_n has degree n for $n = 0, 1, \ldots$.

We claim that

$$\left.\begin{array}{l} u_{2k} \text{ is an even function} \\ u_{2k+1} \text{ is an odd function} \end{array}\right\} k = 0, 1, \ldots. \tag{5.53}$$

Noting, from (5.47) and (5.48) that (5.53) holds for $k = 0, 1$, we proceed to prove (5.53) inductively with the inductive step from k to $k+1$ proceeding as follows. From (5.50) we have, for $z \in \mathbf{C}$,

$$\begin{aligned} u_{2k+2}(-z) &= -zu_{2k+1}(-z) - \frac{1+z^2}{n+2}u'_{2k+1}(-z) \\ &= zu_{2k+1}(z) - \frac{1+z^2}{n+2}u'_{2k+1}(z), \end{aligned} \tag{5.54}$$

since $u_{2k+1}(-z) = -u_{2k+1}(z)$, $z \in \mathbf{C}$, and thus $u'_{2k+1}(-z) = u'_{2k+1}(z)$, $z \in \mathbf{C}$, from the inductive hypothesis. Hence (5.54) and (5.50) show that $u_{2k+2}(z) = u_{2k+2}(-z)$, $z \in \mathbf{C}$, i.e. u_{2k+2} is an even function. Next, we use (5.50) again to obtain, for $z \in \mathbf{C}$,

$$\begin{aligned} u_{2k+3}(-z) &= -zu_{2k+2}(-z) - \frac{1+z^2}{n+2}u'_{2k+2}(-z) \\ &= -\left[zu_{2k+2}(z) - \frac{1+z^2}{n+2}u'_{2k+2}(z)\right], \end{aligned} \tag{5.55}$$

since u_{2k+2} is an even function, i.e. $u_{2k+2}(z) = u_{2k+2}(-z)$, $z \in \mathbf{C}$, and thus $u'_{2k+2}(z) = -u'_{2k+2}(-z)$, $z \in \mathbf{C}$. It follows from (5.55) and (5.50) that $u_{2k+3}(z) = -u_{2k+3}(-z)$, $z \in \mathbf{C}$, i.e. u_{2k+3} is an odd function, so that we have advanced the inductive hypothesis from k to $k+1$, and it follows that (5.53) is true.

We prove next the statement:

$$u_n \text{ has } n \text{ real simple zeros for } n = 1, 2, \ldots. \tag{5.56}$$

Using (5.50), we find that

$$u_2(z) = \frac{1}{3}(2z^2 - 1), \quad z \in \mathbf{C}, \tag{5.57}$$

and thus, from (5.48) and (5.57), the statement (5.56) holds for $n = 1$ and $n = 2$. We proceed to show that (5.56) holds for all $n \in \mathbf{N}$ by means of induction.

Suppose therefore, for a given $k \in \mathbf{N}$, the polynomial u_{2k} has $2k$ real simple zeros.

Observe also that $u_{2k}(0) \neq 0$, for if $z = 0$ is a zero of u_{2k}, then (5.53) implies that $z = 0$ is a zero of order ≥ 2 of u_{2k}, contradicting the inductive hypothesis that all the zeros of u_{2k} are simple. Hence, using also (5.53), we deduce that there exists a sequence $\{\xi_j : j = 1, 2, \ldots, k\} \subset \Re$, with

$$0 < \xi_1 < \xi_2 < \ldots < \xi_k,$$

such that

$$u_{2k}(\pm\xi_j) = 0, \quad j = 1, 2, \ldots, k. \tag{5.58}$$

Moreover, since the zeros of u_{2k} are all simple, and since $\lim_{t\to\infty} u_{2k}(t) = \infty$ from (5.51), (5.52), we also have

$$(-1)^{k+j} u'_{2k}(\xi_j) > 0, \quad j = 1, 2, \ldots, k. \tag{5.59}$$

Now use (5.50) and (5.58) to obtain

$$u_{2k+1}(\xi_j) = -\frac{1+\xi_j^2}{2k+2} u'_{2k}(\xi_j), \quad j = 1, 2, \ldots, k,$$

which, together with (5.59), yield

$$(-1)^{k+1+j} u_{2k+1}(\xi_j) > 0, \quad j = 1, 2, \ldots, k. \tag{5.60}$$

It follows from the intermediate value theorem, together with (5.60), that there exists, if $k \geq 2$, a sequence $\{\eta_j : j = 1, \ldots, k-1\} \subset \Re$ such that the interlacing property

$$\xi_1 < \eta_1 < \xi_2 < \eta_2 < \ldots < \xi_{k-1} < \eta_{k-1} < \xi_k, \tag{5.61}$$

holds, and

$$u(\eta_j) = 0, \quad j = 1, 2, \ldots, k-1. \tag{5.62}$$

Observing from (5.60) that $u_{2k+1}(\xi_k) < 0$, and using the fact that $\lim_{t\to\infty} u_{2k+1}(t) = \infty$ from (5.51), (5.52), it similarly follows that there exists a real number $\eta_k \in (\xi_k, \infty)$ such that $u_{2k+1}(\eta_k) = 0$. Hence, recalling also (5.61) and (5.62), and exploiting the fact that u_{2k+1} is an odd function as obtained from (5.53), whence $u_{2k+1}(0) = 0$, it follows that, with the choice $\eta_0 = 0$, we have established the existence of a strictly increasing sequence $\{-\eta_k, -\eta_{k-1}, \dots, -\eta_1, \eta_0, \eta_1, \dots, \eta_k\} \subset \Re$ such that

$$u_{2k+1}(-\eta_j) = 0,\ u_{2k+1}(\eta_0) = 0,\ u_{2k+1}(\eta_j) = 0,\quad j = 1, \dots, k.$$

Since u_{2k+1} is a polynomial of degree $2k+1$, it follows that u_{2k+1} has $2k+1$ real simple zeros, with

$$(-1)^{k+j} u'_{2k+1}(\eta_j) > 0,\quad j = 0, 1, \dots, k, \tag{5.63}$$

having recalled once again also the fact that $\lim_{t\to\infty} u_{2k+1}(t) = \infty$.

Next, we use (5.50) to deduce that

$$u_{2k+2}(\eta_j) = -\frac{1+\eta_j^2}{2k+3} u'_{2k+1}(\eta_j),\quad j = 0, 1, \dots, k,$$

which, together with (5.63), yields

$$(-1)^{k+1+j} u_{2k+2}(\eta_j) > 0,\quad j = 0, 1, \dots, k. \tag{5.64}$$

As before, the intermediate value theorem, together with (5.64), implies the existence of a sequence $\{\zeta_j : j = 0, 1, \dots, k-1\} \subset \Re$, with

$$0 = \eta_0 < \zeta_0 < \eta_1 < \zeta_1 < \dots < \zeta_{k-1} < \eta_k, \tag{5.65}$$

such that

$$u_{2k+2}(\zeta_j) = 0,\quad j = 0, 1, \dots, k-1. \tag{5.66}$$

Moreover, (5.64) gives $u_{2k+2}(\eta_k) < 0$, and thus, since (5.51) and (5.52) imply $\lim_{t\to\infty} u_{2k+2}(t) = \infty$, there exists a real number $\zeta_k \in (\eta_k, \infty)$ such that $u_{2k+2}(\zeta_k) = 0$.

Since u_{2k+2} is an even function from (5.53), it follows, by recalling also (5.65), (5.66), that there exists a sequence $\{\zeta_j : j = 0, 1, \ldots, k\}$, with

$$0 < \zeta_0 < \zeta_1 < \ldots < \zeta_k,$$

such that

$$u_{2k+2}(\pm\zeta_j) = 0, \quad j = 0, 1, \ldots, k.$$

Hence u_{2k+2} has $2k+2$ real simple zeros, thereby concluding our inductive proof of the statement (5.56).

Combining the results (5.51), (5.52), (5.53) and (5.56), we deduce that there exists, for each integer $n \geq 2$, a sequence $\left\{\alpha_{n,j} : j = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\} \subset \Re$, with

$$0 < \alpha_{n,1} < \alpha_{n,2} < \ldots < \alpha_{n,\left\lfloor \frac{n}{2} \right\rfloor}, \tag{5.67}$$

such that

$$u_n(z) = \frac{2^n}{(n+1)!} \begin{cases} Q_n(z), & z \in \mathbf{C}, \quad \text{if } n \text{ is even}, \\ zQ_n(z), & z \in \mathbf{C}, \quad \text{if } n \text{ is odd}, \end{cases} \tag{5.68}$$

where the polynomial Q_n is defined by

$$Q_n(z) = \prod_{j=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (z^2 - \alpha_{n,j}^2), \quad z \in \mathbf{C}. \tag{5.69}$$

Now observe that (5.46) implies

$$U_n(z) = i^n u_n(-iz), \quad z \in \mathbf{C}, \quad n = 0, 1, \ldots. \tag{5.70}$$

Substituting (5.68) and (5.69) into (5.70) then gives, for $n \geq 2$,

$$U_n(z) = \frac{2^n}{(n+1)!} \begin{cases} R_n(z), & z \in \mathbf{C}, \quad \text{if } n \text{ is even}, \\ zR_n(z), & z \in \mathbf{C}, \quad \text{if } n \text{ is odd}, \end{cases} \tag{5.71}$$

where the polynomial R_n is defined by

$$R_n(z) = \prod_{j=1}^{\left\lfloor \frac{n}{2} \right\rfloor} (z^2 + \alpha_{n,j}^2), \quad z \in \mathbf{C}. \tag{5.72}$$

Now recall (5.42) to deduce from (5.71) that, for $n \geq 2$,

$$e_n(\omega) = \frac{2^n}{(n+1)!}\begin{cases} R_n(\cos\omega/2), & \omega \in \Re, \text{ if } n \text{ is even}, \\ \cos(\omega/2)R_n(\cos\omega/2), & \omega \in \Re, \text{ if } n \text{ is odd}, \end{cases} \tag{5.73}$$

and with the polynomial R_n given as in (5.72). Setting $z = \cos(\omega/2)$ in (5.72), we see that, for $\omega \in \Re$,

$$\begin{aligned} z^2 + \alpha_{n,j}^2 &= \cos^2(\omega/2) + \alpha_{n,j}^2 \\ &= \left[\frac{e^{i\omega/2} + e^{-i\omega/2}}{2}\right]^2 + \alpha_{n,j}^2 \\ &= \frac{1}{4}\left[e^{i\omega} + (2 + 4\alpha_{n,j}^2) + e^{-i\omega}\right] \\ &= \frac{1}{4e^{i\omega}}\left[e^{2i\omega} + (2 + 4\alpha_{n,j}^2)e^{i\omega} + 1\right] \\ &= \frac{1}{4e^{i\omega}}\left[(e^{i\omega})^2 + 2(1 + 2\alpha_{n,j}^2)e^{i\omega} + 1\right] \\ &= \frac{e^{-i\omega}}{4}(e^{i\omega} - \mu_{n,j})\left(e^{i\omega} - \frac{1}{\mu_{n,j}}\right), \end{aligned} \tag{5.74}$$

where

$$\mu_{n,j} = -1 + 2\alpha_{n,j}(\sqrt{1 + \alpha_{n,j}^2} - \alpha_{n,j}), \tag{5.75}$$

with clearly

$$-1 < \mu_{n,j} = -1 + \frac{2}{\sqrt{1 + \frac{1}{\alpha_{n,j}^2}} + 1} < 0, \tag{5.76}$$

having noted also (5.67).

Finally, we combine (5.34), (5.73), (5.72), (5.74) and (5.75) to deduce that, for $m \geq 4$, m even, and $\omega \in \Re$,

$$\begin{aligned} \widetilde{N}_m(e^{i\omega}) &= e^{i\frac{m-2}{2}\omega}\frac{2^{m-2}}{(m-1)!}\prod_{j=1}^{\frac{m-2}{2}}\frac{1}{4}e^{-i\omega}(e^{i\omega} - \mu_{m-2,j})\left(e^{i\omega} - \frac{1}{\mu_{m-2,j}}\right) \\ &= \frac{1}{(m-1)!}\prod_{j=1}^{\frac{m-2}{2}}(e^{i\omega} - \mu_{m-2,j})\left(e^{i\omega} - \frac{1}{\mu_{m-2,j}}\right), \end{aligned} \tag{5.77}$$

whereas, for $m \geq 5$, m odd, and $\omega \in \Re$,

$$\begin{aligned}\widetilde{N}_m(e^{i\omega}) &= e^{i\frac{m-2}{2}\omega}\frac{2^{m-2}}{(m-1)!}\frac{e^{i\omega/2}+e^{-i\omega/2}}{2}\prod_{j=1}^{\frac{m-3}{2}}\frac{e^{-i\omega}}{4}(e^{i\omega}-\mu_{m-2,j})\left(e^{i\omega}-\frac{1}{\mu_{m-2,j}}\right)\\ &= \frac{1}{(m-1)!}(e^{i\omega}+1)\prod_{j=1}^{\frac{m-3}{2}}(e^{i\omega}-\mu_{m-2,j})\left(e^{i\omega}-\frac{1}{\mu_{m-2,j}}\right). \qquad (5.78)\end{aligned}$$

If we now define, for $m \geq 4$, the sequence $\{\lambda_{m,j} : j = 1, 2, \ldots, \lfloor\frac{m-2}{2}\rfloor\} \subset \Re$ by

$$\lambda_{m,j} := \mu_{m-2,j}, \quad j = 1, 2, \ldots, \left\lfloor\frac{m-2}{2}\right\rfloor, \qquad (5.79)$$

we see from (5.78), (5.79) and (5.77) that the result (5.30), (5.31), (5.32) holds on the unit circle $|z| = 1$. Since $\widetilde{N}_m$ and P_m are polynomials, it follows that (5.31) and (5.32) hold for all $z \in \mathbf{C}$. ■

Using Theorem 5.4, we can now prove the following stability result for the cardinal B-spline N_m.

Theorem 5.5 *For $m \in \mathbf{N}$, there exist real numbers A_m and B_m, with $0 < A_m \leq B_m$, such that*

$$A_m\|\{c_k\}\|_{\ell^2}^2 \leq \left\|\sum_k c_k N_m(\cdot - k)\right\|_2^2 \leq B_m\|\{c_k\}\|_{\ell^2}^2, \quad \{c_k\} \in \ell^2. \qquad (5.80)$$

Proof. If $m = 1$, i.e. $N_1 = \phi_H$, it has already been noted before in Section 3.1 that (5.80) holds with equality, and with Riesz bounds $A_1 = B_1 = 1$.

Suppose therefore $m \geq 2$. But then $N_m \in \mathcal{A}$, and

$$N_m(t),\ \widehat{N}_m(t) = \mathcal{O}\left(\frac{1}{1+|t|^m}\right), \quad t \in \Re,$$

and it follows from Theorem 3.3 that (5.80) is equivalent to the condition

$$A_m \leq \sum_k \left|\widehat{N}_m(\omega + 2\pi k)\right|^2 \leq B_m, \quad \omega \in \Re. \qquad (5.81)$$

But, as in the proof of Theorem 3.11, we can show that, with the Euler-Frobenius Laurent polynomial E_m defined by (5.23) and (5.22), we have

$$\begin{aligned}\sum_k \left|\widehat{N}_m(\omega+2\pi k)\right|^2 &= E_m(e^{i\omega}) \\ &= \sum_k \Phi_m(k)e^{i\omega k} \\ &= \sum_k N_{2m}(m+k)e^{i\omega k} \\ &= \sum_k N_{2m}(k+1)e^{i\omega(k+1-m)} \\ &= e^{i\omega(1-m)}\sum_k N_{2m}(k+1)e^{i\omega k}, \end{aligned} \tag{5.82}$$

and thus

$$\sum_k \left|\widehat{N}_m(\omega+2\pi k)\right|^2 = e^{i\omega(1-m)}\widetilde{N}_{2m}(e^{i\omega}), \quad \omega \in \Re, \tag{5.83}$$

from (5.6) and (5.27).

Now combine (5.83), (5.31) and (5.32) to obtain, for $\omega \in \Re$,

$$\begin{aligned}\sum_k \left|\widehat{N}_m(\omega+2\pi k)\right|^2 &= \frac{e^{i\omega(1-m)}}{(m-1)!}\prod_{j=1}^{m-1}\left(e^{i\omega}-\lambda_{2m,j}\right)\left(e^{i\omega}-\frac{1}{\lambda_{2m,j}}\right) \\ &= \left|\frac{e^{i\omega(1-m)}}{(m-1)!}\prod_{j=1}^{m-1}\left(e^{i\omega}-\lambda_{2m,j}\right)\left(e^{i\omega}-\frac{1}{\lambda_{2m,j}}\right)\right| \\ &= \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\left|e^{i\omega}-\lambda_{2m,j}\right|\left|e^{i\omega}-\frac{1}{\lambda_{2m,j}}\right| \\ &= \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\left|1-\lambda_{2m,j}e^{-i\omega}\right|\left|e^{i\omega}-\frac{1}{\lambda_{2m,j}}\right| \\ &= \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\left|\left(1-\lambda_{2m,j}e^{-i\omega}\right)\left(e^{i\omega}-\frac{1}{\lambda_{2m,j}}\right)\right| \\ &= \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\left|e^{i\omega}+e^{-i\omega}-\left(\lambda_{2m,j}+\frac{1}{\lambda_{2m,j}}\right)\right| \\ &= \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\left|2\cos\omega-\left(\lambda_{2m,j}+\frac{1}{\lambda_{2m,j}}\right)\right| \\ &= \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\left|\frac{\lambda_{2m,j}^2-2\lambda_{2m,j}\cos\omega+1}{\lambda_{2m,j}}\right| \end{aligned}$$

$$= \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\frac{\left|\lambda_{2m,j}^2 - 2\lambda_{2m,j}\cos\omega + 1\right|}{|\lambda_{2m,j}|}. \tag{5.84}$$

But, since (5.30) gives

$$-1 < \lambda_{2m,j} < 0, \qquad j = 1, 2, ..., m-1, \tag{5.85}$$

we see that, for $\omega \in \Re$,

$$\begin{aligned} \lambda_{2m,j}^2 - 2\lambda_{2m,j}\cos\omega + 1 &\geq \lambda_{2m,j}^2 + 2\lambda_{2m,j} + 1 \\ &= (\lambda_{2m,j}+1)^2, \end{aligned} \tag{5.86}$$

which, together with (5.84), yield, for $\omega \in \Re$,

$$\sum_k \left|\widehat{N}_m(\omega + 2\pi k)\right|^2 \geq \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\frac{(\lambda_{2m,j}+1)^2}{|\lambda_{2m,j}|}, \tag{5.87}$$

and thus

$$\sum_k \left|\widehat{N}_m(\omega + 2\pi k)\right|^2 \geq \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\frac{(\lambda_{2m,j}+1)^2}{-\lambda_{2m,j}}, \quad \omega \in \Re, \tag{5.88}$$

by virtue of (5.85) and (5.87).

Now observe from (5.82) that, for $\omega \in \Re$,

$$\begin{aligned} \sum_k \left|\widehat{N}_m(\omega + 2\pi k)\right|^2 &= \left|e^{i\omega(1-m)}\sum_{k=0}^{m-2} N_{2m}(k+1)e^{i\omega k}\right| \\ &= \left|\sum_{k=0}^{m-2} N_{2m}(k+1)e^{i\omega k}\right| \\ &\leq \sum_{k=0}^{m-2} N_{2m}(k+1) \\ &\leq \sum_k N_{2m}(k+1) = 1, \end{aligned} \tag{5.89}$$

from (5.5) and (5.7).

Hence, combining (5.88) and (5.89), we see that the condition (5.81) is satisfied, with

$$A_m = \frac{1}{(m-1)!}\prod_{j=1}^{m-1}\frac{(\lambda_{2m,j}+1)^2}{-\lambda_{2m,j}}, \tag{5.90}$$

and

$$B_m = 1, \tag{5.91}$$

and therefore the Riesz stability condition holds for $m \geq 2$ with A_m and B_m as given by (5.90) and (5.91), having recalled also the fact that $\lambda_{m,j} < 0$ from (5.85). ■

5.2 The Cardinal B-spline Wavelet

We can now easily verify that, for $m \geq 2$, $\phi = N_m$ is an admissible choice in the Definition 3.1 of a scaling function.

Indeed, (5.8), (5.6), (5.1) and (5.13) show that $N_m \in \mathcal{A}$. Moreover, (5.8), (5.6) and (5.13) show that N_m and $\widehat{N_m}$ satisfy

$$N_m(t), \widehat{N}_m(t) = \mathcal{O}\left(\frac{1}{1+t^m}\right), \quad t \in \Re,$$

so that the condition (3.1) in Definition 3.1 holds, with $\alpha = m \geq 2$, if we choose $\phi = N_m$.

Moreover, the choice $\phi = N_m$ satisfies properties (b), (c) and (d) of Definition 3.1 by virtue of, respectively, Theorem 5.3(d), Theorem 5.5 and Theorem 5.1(iii). Hence we have the following result.

Theorem 5.6 *The cardinal B-spline N_m of order m is a scaling function, with two-scale relation given by*

$$N_m(t) = \sum_k p_{m,k} N_m(t-k), \quad t \in \Re,$$

where

$$p_{m,k} = \frac{1}{2^{m-1}} \binom{m}{k}, \quad k = 0, 1, \ldots, m. \tag{5.92}$$

Observe in particular from (5.92) that

$$p_{m,k} = 0, \quad k \notin \{0, 1, \ldots, m\}.$$

Moreover, it is clear from (5.25) and (5.6), and the fact that $N_m \in S_m \subset C(\Re)$ for $m \geq 2$, that the largest integer n for which $\Phi_m(n) \neq 0$ is given by $n = m - 1$.

Hence, according to (4.34) and (4.35) we have in Theorem 4.6 that $N = m$ and $n = m-1$, so that $N+n = 2m-1$, which is an odd integer. We can therefore use the formula (4.39), together with the formula (5.25), to explicitly construct the minimally supported wavelet $\psi = \psi_m$ which is generated by the scaling function $\phi = N_m$. We shall call ψ_m the *cardinal B-spline wavelet* (or merely the *cardinal B-wavelet*).

Indeed, from formula (4.39), we can now explicitly formulate ψ_m as follows.

Theorem 5.7 *The cardinal B-spline wavelet $\psi = \psi_m$ which is generated, according to Theorem 4.6, by the scaling function $\phi = N_m$, is given by the formula*

$$\psi_m(t) = \sum_{k=0}^{3m-2} q_{m,k} N_m(2t-k), \quad t \in \Re, \tag{5.93}$$

where

$$q_{m,k} = (-1)^k \sum_{l=0}^{k} \frac{1}{2^{m-1}} \binom{m}{l} N_{2m}(k-l+1), \quad k = 0, 1, \ldots, 3m-2, \tag{5.94}$$

and where

$$\psi_m(t) = 0, \quad t \notin [0, 2m-1]. \tag{5.95}$$

Proof. Using the formula (4.39) with $n = m-1$ and $N = m$, we get, setting $\psi = \psi_m$, the formula

$$\psi_m(t) = \frac{1}{2^{m-1}} \sum_{k=0}^{3m-2} (-1)^k \left[\sum_{l=0}^{m} \binom{m}{l} \Phi(k+l-2m+1) \right] N_m(2t-k), \quad t \in \Re. \tag{5.96}$$

But, from (5.25),

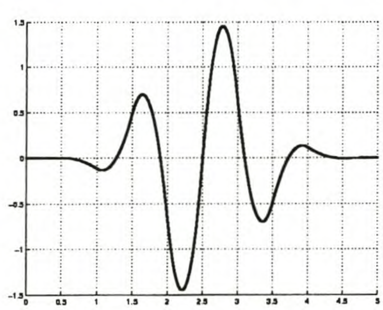
$$\begin{aligned}
\sum_{l=0}^{m} \binom{m}{l} \Phi_m(k+l-2m+1) &= \sum_{l=0}^{m} \binom{m}{l} N_{2m}(k+l-m+1) \\
&= \sum_{l=0}^{m} \binom{m}{m-l} N_{2m}(k+l-m+1) \\
&= \sum_{l=0}^{m} \binom{m}{l} N_{2m}(k-l+1) \\
&= \sum_{l=0}^{k} \binom{m}{l} N_{2m}(k-l+1), \qquad (5.97)
\end{aligned}$$

since $N_{2m}(j) = 0$ for $j \leq 0$ by virtue of (5.6). The formula (5.93) and (5.94) then follows by combining (5.96) and (5.97).

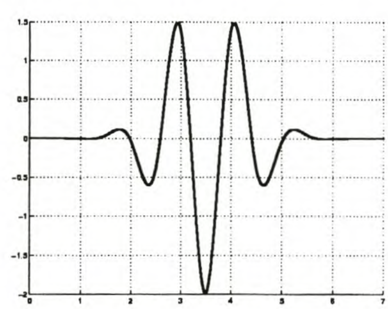
Finally, observe that (5.95) is a consequence of (4.41) and the fact that here $n + N = 2m - 1$, an odd integer. ■

The explicit formulation of ψ_m, as given by (5.93) and (5.94), was first obtained by Chui and Wang in [7].

The quadratic B-wavelet ψ_3 and cubic B-wavelet ψ_4 are drawn in Fig.5.1 below. It should also be observed here, from (5.93), (5.94), and as illustrated in Fig.5.1,



(a)



(b)

Figure 5.1: (a) the quadratic B-wavelet ψ_3 and (b) the cubic B-wavelet ψ_4

that ψ_3 is an antisymmetric wavelet, and ψ_4 a symmetric wavelet.

Hence, although the spline B-wavelet ψ_m lacks the orthonormality property of the Daubechies wavelets $\psi_{D,m}$, $m = 1, 2, \ldots$, we see that ψ_m have symmetry properties which are not posessed by $\psi_{D,m}$, $m = 1, 2, \ldots$.

5.3 An Alternative Approach to Cardinal B-wavelet Construction

We consider an alternative formulation of the cardinal B-spline wavelet ψ_m, which will be valuable when investigating, in Chapter 6 below, the construction of B-spline wavelets on a bounded interval. The result is as follows.

Theorem 5.8 *For $m \geq 2$, define the function Ψ_m by*

$$\Psi_m(t) = \sum_{k=0}^{m-2} (-1)^k N_m(k+1) N_m(2t-k), \quad t \in \Re. \tag{5.98}$$

Then Ψ_m satisfies the interpolation property

$$\Psi_m(j) = 0, \quad j \in \mathbf{Z}, \tag{5.99}$$

as well as the compact support property

$$\Psi_m(t) = 0, \quad t \notin [0, m-1]. \tag{5.100}$$

Moreover,

$$\Psi_{2m}^{(m)} = 2^m \psi_m, \quad m = 2, 3, \ldots, \tag{5.101}$$

with ψ_m denoting the cardinal B-spline wavelet of Theorem 5.7.

Proof. From (5.98) and (5.6), we have, for $j \in \mathbf{Z}$, that

$$\begin{aligned}
\Psi_m(j) &= \sum_k (-1)^k N_m(k+1) N_m(2j-k) \\
&= \sum_k (-1)^{2j-k-1} N_m(2j-k) N_m(k+1) \\
&= -\sum_k (-1)^k N_m(k+1) N_m(2j-k) \\
&= -\Psi_m(j),
\end{aligned}$$

thereby proving (5.99).

The compact support property (5.100) is an immediate consequence of (5.98) and (5.6).

To prove (5.101), suppose $\{V_j^{(m)}\}$ is the MRA generated by the scaling function $\phi = N_m$, and let

$$W_j^{(m)} := (V_j^{(m)})^{\perp}, \quad j \in \mathbf{Z}.$$

We claim that the function ψ_m^* defined by

$$\psi_m^* := \Psi_{2m}^{(m)}, \tag{5.102}$$

satisfies

$$\psi_m^* \in W_0^{(m)}. \tag{5.103}$$

To prove (5.103), we first observe from (5.98), (5.12), (5.8) and (5.10) that

$$\Psi_{2m}^{(l)} \in S_{2m-l}^{(1)} \subset C^{2m-2-l}(\Re) \subset C(\Re), \quad l = 0, 1, \ldots, m, \tag{5.104}$$

since $m \geq 2$; also

$$\Psi_{2m}^{(l)}(t) = 0, \quad t \notin [0, 2m-1], \quad l = 0, 1, \ldots, m, \tag{5.105}$$

by virtue of (5.100) and (5.104). Similarly, using (5.8) and (5.9), we have

$$N_m^{(l)} \in S_{m-l} \subset C^{m-l-2}(\Re) \subset C(\Re), \quad l = 0, 1, \ldots, m-2,$$

whereas, from (5.1), $N_m^{(m-1)} \in C^{-1}(\Re)$, and there exists a real sequence $\{\mu_0, \mu_1, \ldots, \mu_{m-1}\}$ such that

$$N_m^{(m-1)}(t) = \begin{cases} \mu_j, & t \in [j, j+1), \quad j = 0, 1, \ldots, m-1, \\ 0, & t \notin [0, m]. \end{cases} \tag{5.106}$$

Moreover, from (5.6),

$$N_m^{(l)}(t) = 0, \quad t \notin [0, m], \quad l = 0, 1, \ldots, m-2. \tag{5.107}$$

Now let $f \in V_0^{(m)}$, i.e. there exists a sequence $\{c_k\} \in \ell^2$ such that $f = \sum_k c_k N_m(\cdot - k)$, and thus

$$\langle f, \psi_m^* \rangle = \left\langle \sum_k c_k N_m(\cdot - k), \psi_m^* \right\rangle = \sum_k c_k \left\langle N_m(\cdot - k), \psi_m^* \right\rangle. \tag{5.108}$$

Now use (5.102), (5.104), (5.105), (5.106) and (5.107) to obtain, for $k \in \mathbf{Z}$, and from integration by parts,

$$\begin{aligned}
\langle N_m(\cdot - k), \psi_m^* \rangle &= \int_{-\infty}^{\infty} N_m(t-k)\Psi_{2m}^{(m)}(t)\, \mathrm{d}t \\
&= -\int_{-\infty}^{\infty} N_m'(t-k)\Psi_{2m}^{(m-1)}(t)\, \mathrm{d}t \\
&= \ldots \\
&= (-1)^{m-1}\int_{-\infty}^{\infty} N_m^{(m-1)}(t-k)\Psi_{2m}'(t)\, \mathrm{d}t \\
&= (-1)^{m-1}\int_{-\infty}^{\infty} N_m^{(m-1)}(t)\Psi_{2m}'(t+k)\, \mathrm{d}t \\
&= (-1)^{m-1}\sum_l \int_l^{l+1} N_m^{(m-1)}(t)\Psi_{2m}'(t+k)\, \mathrm{d}t \\
&= (-1)^{m-1}\sum_{l=0}^{m-1} \int_l^{l+1} \mu_l \Psi_{2m}'(t+k)\, \mathrm{d}t \\
&= (-1)^{m-1}\sum_{l=0}^{m-1} \mu_l \left[\Psi_{2m}(l+k+1) - \Psi_{2m}(l+k)\right] \\
&= 0,
\end{aligned} \tag{5.109}$$

by virtue of (5.99). Combining (5.108) and (5.109) then yields $\langle f, \psi_m^* \rangle = 0$, i.e. $\psi_m^* \perp V_0^{(m)}$, so that (5.103) does indeed hold.

Also, (5.102) and (5.105) imply

$$\psi_m^*(t) = 0, \quad t \notin [0, 2m-1]. \tag{5.110}$$

It therefore follows from (5.95) and (5.110) that ψ_m and ψ_m^* are both minimally supported elements of $W_0^{(m)}$, based on the same value of $M (= m-1)$ in Theorem 4.5. Hence, from (4.32), we deduce that

$$\psi_m^* = \alpha \psi_m, \tag{5.111}$$

for some value of $\alpha \in \Re \setminus \{0\}$. It remains to show that $\alpha = 2^m$.

First, we use induction to show that, if $m \geq 2$, then

$$N_m^{(l)}(t) = \sum_{j=0}^{l} (-1)^j \binom{l}{j} N_{m-l}(t-j), \quad t \in \Re, \quad l = 0, 1, \ldots, m-2. \tag{5.112}$$

Clearly (5.112) holds for $l = 0$, whereas the inductive step from l to $l+1$ is given by using (5.9) to deduce, for $t \in \Re$,

$$\begin{aligned}
N_m^{(l+1)}(t) &= \sum_{j=0}^{l} (-1)^j \binom{l}{j} N'_{m-l}(t-j) \\
&= \sum_{j} (-1)^j \binom{l}{j} N'_{m-l}(t-j) \\
&= \sum_{j} (-1)^j \binom{l}{j} [N_{m-l-1}(t-j) - N_{m-l-1}(t-j-1)] \\
&= \sum_{j} (-1)^j \binom{l}{j} N_{m-l-1}(t-j) - \sum_{j} (-1)^j \binom{l}{j} N_{m-l-1}(t-j-1) \\
&= \sum_{j} (-1)^j \binom{l}{j} N_{m-l-1}(t-j) + \sum_{j} (-1)^j \binom{l}{j-1} N_{m-l-1}(t-j) \\
&= \sum_{j} (-1)^j \left[\binom{l}{j} + \binom{l}{j-1}\right] N_{m-l-1}(t-j) \\
&= \sum_{j} (-1)^j \binom{l+1}{j} N_{m-l-1}(t-j) \\
&= \sum_{j=0}^{l+1} (-1)^j \binom{l+1}{j} N_{m-(l+1)}(t-j).
\end{aligned}$$

Now use (5.102), (5.98), (5.112) and (5.6) to deduce that, for $t \in \Re$,

$$\begin{aligned}
\psi_m^*(t) &= 2^m \sum_{k=0}^{2m-2} (-1)^k N_{2m}(k+1) N_{2m}^{(m)}(2t-k) \\
&= 2^m \sum_{k=0}^{2m-2} (-1)^k N_{2m}(k+1) \sum_{l=0}^{m} (-1)^l \binom{m}{l} N_m(2t-k-l) \\
&= 2^m \sum_{k} (-1)^k N_{2m}(k+1) \sum_{l} (-1)^l \binom{m}{l} N_m(2t-k-l)
\end{aligned}$$

$$
\begin{aligned}
&= 2^m \sum_k (-1)^k N_{2m}(k+1) \sum_l (-1)^{l-k} \binom{m}{l-k} N_m(2t-l) \\
&= 2^m \sum_l (-1)^l \left[\sum_k \binom{m}{l-k} N_{2m}(k+1) \right] N_m(2t-l) \\
&= 2^m \sum_l (-1)^l \left[\sum_k \binom{m}{k} N_{2m}(l-k+1) \right] N_m(2t-l) \\
&= 2^m \sum_k (-1)^k \left[\sum_l \binom{m}{l} N_{2m}(k-l+1) \right] N_m(2t-k) \\
&= 2^m \sum_k (-1)^k \left[\sum_{l=0}^{m} \binom{m}{l} N_{2m}(k-l+1) \right] N_m(2t-k). \qquad (5.113)
\end{aligned}
$$

Since (5.6) implies $N_{2m}(j) = 0$ for $j \notin \{1, 2, \ldots, 2m-1\}$, we find from (5.113) that, for $t \in \Re$,

$$
\begin{aligned}
\psi_m^*(t) &= 2^m \sum_k (-1)^k \left[\sum_{l=0}^{k} \binom{m}{l} N_{2m}(k-l+1) \right] N_m(2t-k) \\
&= 2^m \sum_{k=0}^{3m-2} (-1)^k \left[\sum_{l=0}^{k} \binom{m}{l} N_{2m}(k-l+1) \right] N_m(2t-k) \\
&= 2^m \psi_m(t),
\end{aligned}
$$

from (5.93), (5.94), which, together with (5.111), give the desired value $\alpha = 2^m$. ■

5.4 Vanishing Moments for ψ_m

The significance of a wavelet ψ having vanishing moments of a certain order has been pointed out in Section 4.4. The following vanishing moment property holds for ψ_m.

Theorem 5.9 *The m-th order cardinal B-spline wavelet ψ_m has vanishing moments of order m:*

$$\int_{-\infty}^{\infty} t^l \psi_m(t)\, dt = 0, \quad l = 0, 1, \ldots, m-1. \qquad (5.114)$$

Proof. Since the two-scale sequence $\{p_k\} = \{p_{m,k}\}$ is given by the formula (5.92), the corresponding cardinal spline two-scale symbol $P = P_m$ is the polynomial

$$P_m(z) = \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} z^k = \left(\frac{1+z}{2}\right)^m, \quad z \in \mathbf{C}. \tag{5.115}$$

But then, according to the definition (4.40) of the symbol $Q = Q_m$, and recalling from the proof of Theorem 5.7 that here $M = m - 1$, we have, for $z \in \mathbf{C}$,

$$\begin{aligned} Q_m(z) &= -z^{2m-1} P_m\left(-\frac{1}{z}\right) E_m(-z) \\ &= -\frac{1}{2^m} z^{m-1}(z-1)^m E_m(-z) \end{aligned} \tag{5.116}$$

by virtue of (5.115).

Now use (4.45) and (5.116) to deduce that, with the scaling function $\phi = N_m$, we have

$$\begin{aligned} \widehat{\psi}(\omega) &= Q_m(e^{-i\omega/2})\widehat{N}_m\left(\frac{\omega}{2}\right) \\ &= -\frac{1}{2^m} e^{-(m-1)i\omega/2}(e^{-i\omega/2} - 1)^m E_m(-e^{-i\omega/2})\widehat{N}_m\left(\frac{\omega}{2}\right). \end{aligned} \tag{5.117}$$

From (5.117) we then see that

$$\widehat{\psi}^{(l)}(0) = 0, \quad l = 0, 1, \ldots, m-1.$$

The desired vanishing moment property (5.114) then follows from Theorem 4.19. ■

5.5 Detection of a Singularity.

As previously discussed in Section 4.4, wavelet decomposition can be employed for the detection of a singularity in a given function. For example, the function G (see

Fig.5.2) defined by

$$G(t) = \begin{cases} \dfrac{1}{1+t^2}, & 0 \leq t \leq 1, \\ \\ \dfrac{1}{2}t(t-2)^2, & 1 < t \leq 2, \\ \\ 0, & t > 2, \end{cases} \tag{5.118}$$

$$G(t) = G(-t), \;\; t \in \Re,$$

satisfies $G \in C^1(\Re) \setminus C^2(\Re)$, with discontinuities in the second derivative G'' at $t = -2, -1, 1$ and 2.

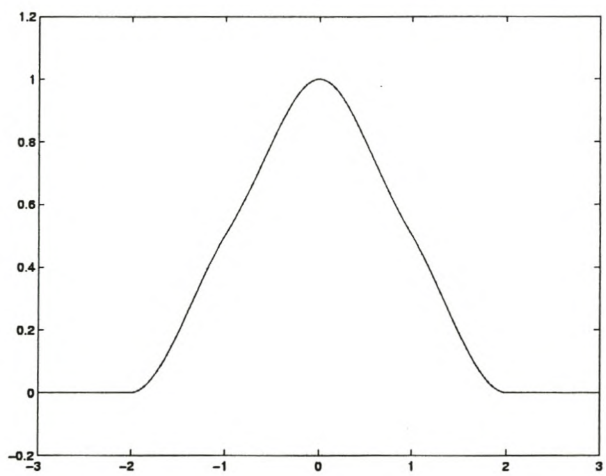


Figure 5.2: The function G

According to the wavelet decomposition technique described in Section 4.4, our first step is to find, for a given tolerance $\epsilon > 0$, a suitable value for N and a function $f_N \in V_N$ such that $||G - f_N|| < \epsilon$, with respect to some appropriate norm $||\cdot||$.

To this end we introduce, for $j \in \mathbf{Z}$, the spline approximation operator T_j :

$C(\Re) \to S_m^{(j)}$ as defined by

$$(T_j f)(t) = \sum_k f\left(\frac{k}{2^j}\right) N_m(2^j t - k), \quad t \in \Re, \quad j \in \mathbf{Z}, \quad f \in C(\Re). \tag{5.119}$$

The following estimate can now be proved.

Theorem 5.10 *For $j \in \mathbf{Z}$, let the approximation operator $T_j : C(\Re) \to S_m^{(j)}$ be defined by (5.119). Then, if $f \in C(\Re)$, with f and f' uniformly bounded on $\Re$, the spline approximation $T_j f$ is also uniformly bounded on $\Re$, and*

$$\sup_{t \in \Re} |f(t) - (T_j f)(t)| \leq \frac{m}{2^j} \sup_{t \in \Re} |f'(t)|. \tag{5.120}$$

Proof. Choose $t \in \Re$. Then, using (5.119), (5.7) and (5.5), we get, for $j \in \mathbf{Z}$,

$$\begin{aligned} |(T_j f)(t)| &= \left| \sum_k f\left(\frac{k}{2^j}\right) N_m(2^j t - k) \right| \\ &\leq \sup_t |f(t)| \sum_k N_m(2^j t - k) \\ &= \sup_t |f(t)|. \end{aligned}$$

Hence,

$$\sup_j |(T_j f)(t)| \leq \sup_t |f(t)|,$$

so that, since f is uniformly bounded, $T_j f$ is also uniformly bounded. Now, using (5.7), (5.5) and (5.6), we have

$$\begin{aligned} & \left| f(t) - \sum_k f\left(\frac{k}{2^j}\right) N_m(2^j t - k) \right| \\ = & \left| f(t) \sum_k N_m(2^j t - k) - \sum_k f\left(\frac{k}{2^j}\right) N_m(2^j t - k) \right| \\ = & \left| \sum_k \left[f(t) - f\left(\frac{k}{2^j}\right) \right] N_m(2^j t - k) \right| \\ \leq & \sum_k \left| f(t) - f\left(\frac{k}{2^j}\right) \right| N_m(2^j t - k) \\ \leq & \|f'\|_\infty \sum_{k \in I} \left| t - \frac{k}{2^j} \right| N_m(2^j t - k), \end{aligned} \tag{5.121}$$

where

$$I := \left\{k : 0 \leq 2^j t - k \leq m\right\}, \tag{5.122}$$

where we have also employed the mean value theorem. But, from (5.122),

$$\left|t - \frac{k}{2^j}\right| \leq \frac{m}{2^j}, \;\; k \in I,$$

which, together with (5.121) and (5.7), gives

$$\left|f(t) - \sum_k f\left(\frac{k}{2^j}\right) N_m(2^j t - k)\right| \leq \|f'\|_\infty \frac{m}{2^j}, \tag{5.123}$$

thereby proving the desired estimate (5.120). ∎

A standard calculus procedure applied to the function G defined by (5.118) now shows that

$$\sup_{t\in\Re} |G(t)| = G(0) = 1, \;\; \sup_{t\in\Re} |G'(t)| = G'\left(\frac{4}{3}\right) = 2,$$

whence, according to Theorem 5.10, we have

$$\sup_{t\in\Re} |G(t) - (T_j G)(t)| \leq \frac{m}{2^{j-1}}, \;\; j \in \mathbf{Z}.$$

Hence, if we denote by $\{V_j^{(m)}\}$ the MRA generated by the B-spline scaling function $\phi = N_m$, we have from (5.123) that, for a given tolerance $\varepsilon > 0$, if we choose the integer N such that

$$N > \frac{\ln\left(\frac{m}{\varepsilon}\right)}{\ln 2} + 1, \tag{5.124}$$

then, with the choice $f_N := T_N G$, we have

$$\|f - f_N\|_\infty := \sup_{t\in\Re} |G(t) - f_N(t)| < \epsilon. \tag{5.125}$$

We shall now demonstrate how a 3rd order (quadratic) B-spline wavelet decomposition of G identifies the discontinuities in the second derivative G'' at the points $t = -2, -1, 1$ and 2. We choose the tolerance $\epsilon = 0.025$ in (5.125), so that, by (5.124), the approximation f_N with $N = 8$ satisfies the required inequality (5.125). The wavelet decomposition

$$\boxed{\begin{array}{ccccccc} & & g_{N-1} & & g_{N-2} & & g_{N-3} \\ & \nearrow & & \nearrow & & \nearrow & \\ f_N & \rightarrow & f_{N-1} & \rightarrow & f_{N-2} & \rightarrow & f_{N-3}, \end{array}}$$

is displayed in Fig.5.5. Note the comparitively large values of the wavelet components g_{N-1}, g_{N-2} and g_{N-3} at the points $t = -2, -1, 1$ and 2, where we know there are discontinuities in G''. Note also that as the level of decomposition increases the influence of the discontinuities 'spreads' due to the increasing support of the wavelets.

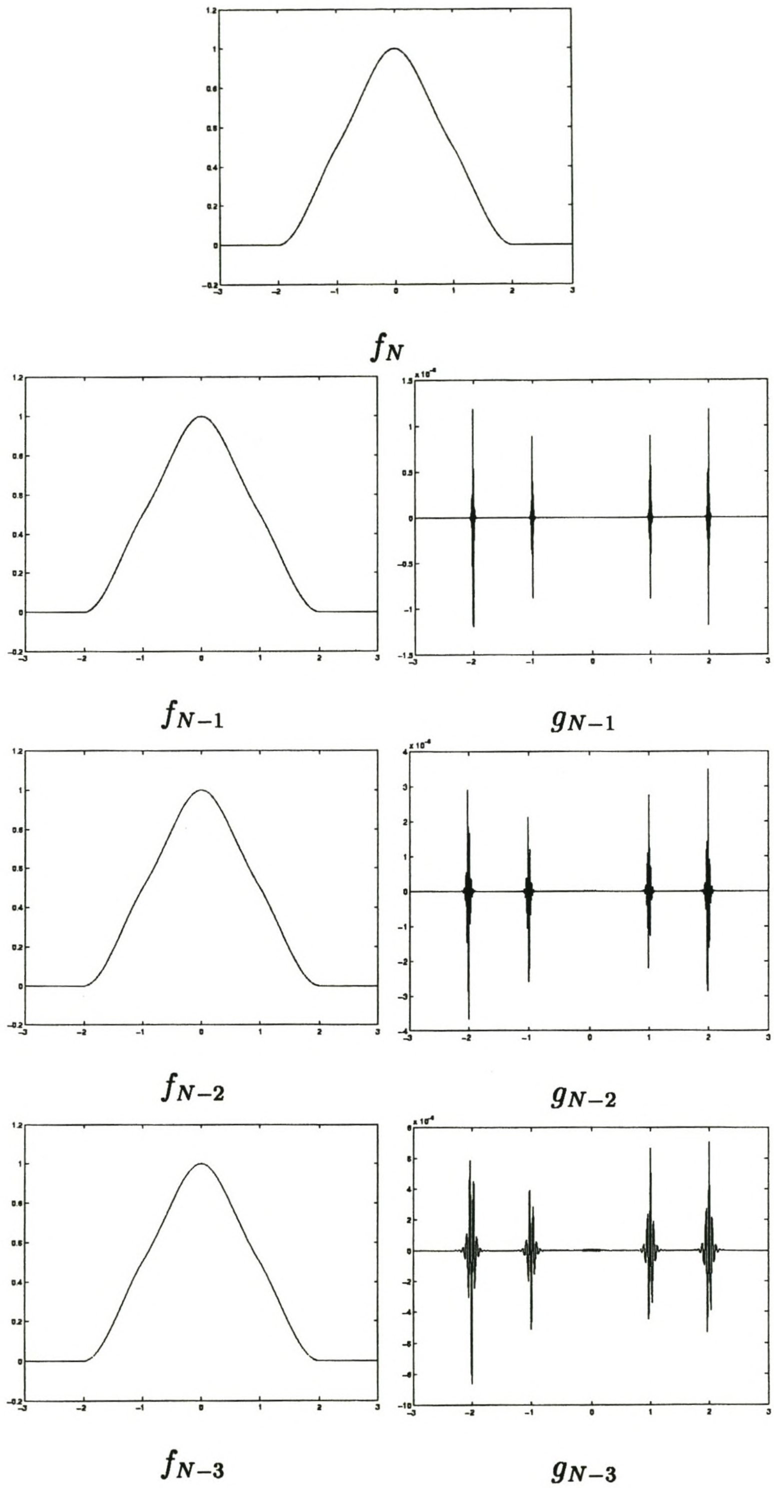


Figure 5.3: The wavelet decomposition of G

Chapter 6

Spline Wavelets on a Bounded Interval

In Chapter 5 we constructed a minimally supported cardinal B-spline wavelet with respect to the function space $L^2(\Re)$. In this chapter we investigate the complications which arise when we consider as domain of definition for the function f to be analyzed, not the whole real line $\Re$, but instead a bounded interval in $\Re$, as is often desirable in practical applications. We show how our wavelet construction method of Chapter 5 can be adapted to yield B-spline wavelets on an interval.

6.1 Spline Functions on an Interval.

Suppose that we are given a bounded interval $[a, b]$ on $\Re$, positive integers k and σ, and a finite knot sequence $\mathbf{y}$ such that

$$\begin{aligned} &\mathbf{y} = \mathbf{y}_{k,\sigma} : \\ &y_{-k+1} = \ldots = y_{-1} = a = y_0 < y_1 < \ldots < y_\sigma = b = y_{\sigma+1} = \ldots = y_{\sigma+k-1}. \end{aligned} \tag{6.1}$$

We then write $\mathcal{S}_{k,\mathbf{y}}$ for the polynomial spline space of order k and with knot sequence $\mathbf{y}$, i.e. $f \in \mathcal{S}_{k,\mathbf{y}}$ if and only if

$$\left.\begin{aligned} & f|_{[y_i,y_{i+1})} = p_i \in \Pi_{k-1}, \quad i = 0,\ldots,\sigma-1, \\ \text{with} \quad & \\ & f \in C^{k-2}[a,b]. \end{aligned}\right\} \tag{6.2}$$

Observe in particular that we allow here for the knots $\{y_1,\ldots,y_{\sigma-1}\}$ to be placed arbitrarily in (a,b), as opposed to the situation for the cardinal spline spaces $S_m^{(j)}$, $j = 0,1,\ldots$, of Section 5.1 which were based on uniform interval lengths.

The truncated power function is defined, for a given $c \in \Re$, by

$$(x-c)_+^k = \begin{cases} (x-c)^k & \text{if } x \geq c, \\ 0 & \text{if } x < c, \end{cases}$$

with the convention that $0^0 = 1$.

Let

$$\mathbf{y} : \ldots \leq y_0 \leq y_1 \leq y_2 \leq \ldots,$$

be a sequence of nondecreasing real numbers. We then define the divided difference of a (sufficiently smooth) function f as follows: The divided difference of order 0 at $t = y_i$ is given by

$$[y_i]f := f(y_i), \quad i \in \mathbf{Z},$$

while the k-th order divided difference, for all $k > 0$, is defined recursively by

$$[y_i,\ldots,y_{i+k}]f := \begin{cases} \dfrac{f^{(k)}(y_i)}{k!} & \text{if } y_i = \ldots = y_{i+k}, \\[2ex] \dfrac{[y_{i+1},\ldots,y_{i+k}]f - [y_i,\ldots,y_{i+k-1}]f}{y_{i+k} - y_i} & \text{if } y_i \neq y_{i+k}. \end{cases}$$

Recall from basic spline theory (see e.g. de Boor [11], Nürnberger [15] or Powell [16]) that $\mathcal{S}_{k,\mathbf{y}}$ has dimension $\sigma + k - 1$, and that a basis for this space is given by the set of normalized B-splines $\{N_{k,\mathbf{y},i} : i = -k+1, \ldots, \sigma-1\}$ defined by

$$N_{k,\mathbf{y},i}(t) := (y_{i+k} - y_i)[y_i, \ldots, y_{i+k}](-1)^k(t - \cdot)_+^{k-1}, \quad t \in \Re. \tag{6.3}$$

It can also be shown that (see [11], [15], [16]) these B-splines $N_{k,\mathbf{y},i}$ have the following properties:

(a)

$$\operatorname{supp} N_{k,\mathbf{y},i} = [y_i, y_{i+k}]; \tag{6.4}$$

(b)

$$N_{k,\mathbf{y},i}(t) > 0, \ t \in (y_i, y_{i+k}); \tag{6.5}$$

(c)

$$\left[\begin{array}{l} N_{k,\mathbf{y},i}^{(r)}(a) \left\{ \begin{array}{ll} = 0, & r = 0, \ldots, k-2+i\,, \\ \neq 0, & r = k-1+i\,, \end{array} \right\} i = -k+1, \ldots, 0\,, \\ \\ N_{k,\mathbf{y},i}^{(r)}(b) \left\{ \begin{array}{ll} = 0, & r = 0, \ldots, \sigma-2-i\,, \\ \neq 0, & r = \sigma-1-i\,, \end{array} \right\} i = \sigma-k, \ldots, \sigma-1, \end{array}\right. \tag{6.6}$$

where, as throughout this chapter, derivatives at a and b are taken in the sense of, respectively, right and left derivatives, whenever neccesary;

(d) the B-splines can be evaluated recursively on $[a, b]$ by means of

$$N_{k,\mathbf{y},i}(t) = \begin{cases} \dfrac{y_1 - t}{y_1 - y_0} N_{k-1,\mathbf{y},-k+2}(t), \ i = -k+1, \\ \\ \dfrac{t - y_i}{y_{i+k-1} - y_i} N_{k-1,\mathbf{y},i}(t) + \dfrac{y_{i+k} - t}{y_{i+k} - y_{i+1}} N_{k-1,\mathbf{y},i+1}(t), \\ \\ \qquad\qquad\qquad\qquad i = -k+2, \ldots, \sigma-2, \\ \\ \dfrac{t - y_{\sigma-1}}{y_\sigma - y_{\sigma-1}} N_{k-1,\mathbf{y},\sigma-1}(t), \ i = \sigma-1, \end{cases} \tag{6.7}$$

for $t \in [a,b]$ and $k = 2, 3, \ldots$, with

$$N_{1,\mathbf{y},i}(t) := \left\{ \begin{array}{ll} 1, & y_i \leq t < y_{i+1}, \\ 0, & \text{elsewhere on } [a,b], \end{array} \right\} i = 0, \ldots, \sigma - 1; \tag{6.8}$$

(e) for a given coefficient sequence $\{\alpha_j : j = -k+1, \ldots, \sigma-1\} \subset \Re$ we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^r \left[\sum_{j=-k+1}^{\sigma-1} \alpha_j N_{k,\mathbf{y},j}(t)\right] = \sum_{j=r-k+1}^{\sigma-1} \alpha_j^{(r)} N_{k-r,\mathbf{y},j}(t), \quad t \in [a,b], \quad r = 1, \ldots, k-2, \tag{6.9}$$

where the coefficients $\alpha_j^{(r)}$ can be generated recursively by means of

$$\alpha_j^{(r)} = \left\{ \begin{array}{ll} \alpha_j, & r = 0, \\ (k-r)\dfrac{\alpha_j^{(r-1)} - \alpha_{j-1}^{(r-1)}}{y_{j+k-r} - y_j}, & r = 1, \ldots, k-2. \end{array} \right. \tag{6.10}$$

Observe from (6.3) and (6.6) that

$$N_{k,\mathbf{y},i} \in C^{k-2+i}(\Re), \quad i = -k+1, \ldots, -1, \tag{6.11}$$

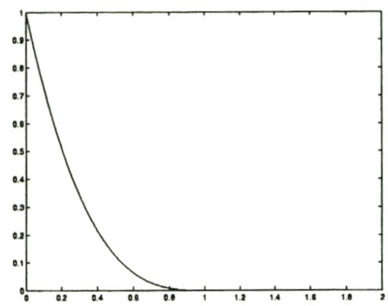
$$N_{k,\mathbf{y},i} \in C^{k-2}(\Re), \quad i = 0, \ldots, \sigma - k, \tag{6.12}$$

$$N_{k,\mathbf{y},i} \in C^{\sigma-2-i}(\Re), \quad i = \sigma - k + 1, \ldots, \sigma - 1. \tag{6.13}$$

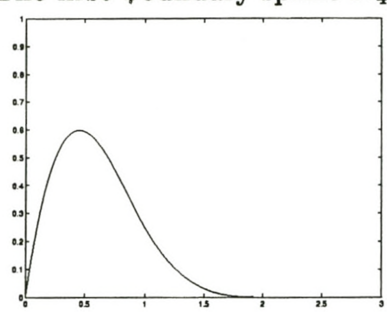
Examples for the case $k = 4$ on the integer knot sequence $\mathbf{y}$ given by (6.1), with $[a,b] = [0,4]$ and $y_i = i$ for $i = 0, \ldots, 4$, illustrating (6.6), (6.11), (6.12) and (6.13) are shown in Fig.6.1(a)-(d).

We therefore see that, for a given order k, there are $\sigma - k + 1$ *interior* splines unaffected by the boundary points a and b, while the other $(\sigma+k-1)-(\sigma-k+1) = 2(k-1)$ basis functions are made up of $k-1$ *boundary* splines at each end of the interval $[a,b]$.

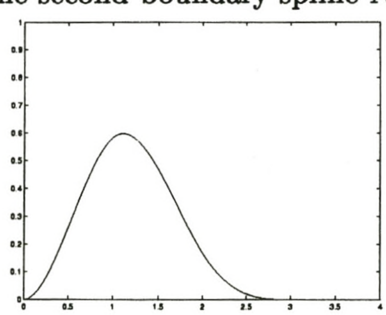
We proceed to consider the concept of multiresolutional analysis in the setting of spline functions on a bounded interval.



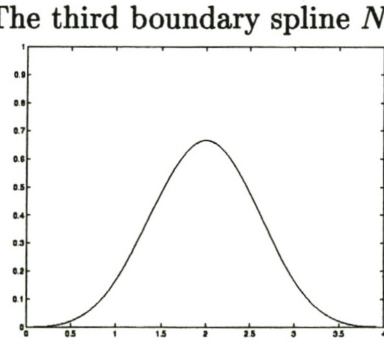
(a) The first boundary spline $N_{4,\mathbf{y},-3}$.



(b) The second boundary spline $N_{4,\mathbf{y},-2}$.



(c) The third boundary spline $N_{4,\mathbf{y},-1}$.



(d) The first inner spline $N_{4,\mathbf{y},0}$.

Figure 6.1: The splines $N_{4,\mathbf{y},i}$ for $i = -3, \ldots, 0$

6.2 A Wavelet Construction Technique.

Following [4, Section 3], we now introduce the concept of spline wavelets on a bounded interval $[a, b]$ as follows. Suppose m and n are positive integers such that $n \geq 2m - 1$. We define the finite knot sequences

$$\begin{aligned}&\mathbf{t} = \mathbf{t}_{m,n} : \\ &t_{-m+1} = \ldots = t_{-1} = a = t_0 < t_1 < \ldots < t_{2n} = b = t_{2n+1} = \ldots = t_{2n+m-1},\end{aligned} \tag{6.14}$$

and

$$\left.\begin{aligned}&\mathbf{x} = \mathbf{x}_{m,n} : \\ &x_{-m+1} = \ldots = x_{-1} = a = x_0 < x_1 < \ldots < x_n = b = x_{n+1} = \ldots = x_{n+m-1}, \\ &\text{where } x_i = t_{2i}, \quad i = 0, 1, \ldots, n.\end{aligned}\right\} \tag{6.15}$$

Then, since $\mathbf{x} \subset \mathbf{t}$, we have from (6.2), the proper inclusion $\mathcal{S}_{m,\mathbf{x}} \subset \mathcal{S}_{m,\mathbf{t}}$, and it follows that $\mathcal{S}_{m,\mathbf{x}}$ is a proper linear subspace of $\mathcal{S}_{m,\mathbf{t}}$.

Note in particular from (6.14) and (6.15) that $t_i = x_i$ for $i = -m + 1, \ldots, -1$ and $t_{n+i} = x_i$ for $i = n + 1, \ldots, n + m - 1$, which together with the fact that $t_{2i} = x_i$ for $i = 0, \ldots, n$, gives the relation

$$\left.\begin{aligned}&t_{\max\{i,2i\}} = x_i \quad && i = -m + 1, \ldots, n, \\ &t_{\min\{2i,n+i\}} = x_i \quad && i = 0, \ldots, n + m - 1.\end{aligned}\right\} \tag{6.16}$$

In this chapter we shall use the inner product

$$\langle f, g \rangle := \int_a^b f(t)\overline{g(t)}\, \mathrm{d}t, \quad f, g \in C[a, b].$$

Now define $W_{m,\mathbf{t},\mathbf{x}}$ as the orthogonal complement of $\mathcal{S}_{m,\mathbf{x}}$ with respect to $\mathcal{S}_{m,\mathbf{t}}$, i.e.

$$W_{m,\mathbf{t},\mathbf{x}} := \left\{ f \in \mathcal{S}_{m,\mathbf{t}} : \langle f, g \rangle = 0, \ g \in \mathcal{S}_{m,\mathbf{x}} \right\}.$$

Then $W_{m,\mathbf{t},\mathbf{x}}$ is a linear subspace of the finite dimensional linear space $\mathcal{S}_{m,\mathbf{t}}$. Hence $W_{m,\mathbf{t},\mathbf{x}}$ is also a finite dimensional linear space and thus, from the standard result [12, Theorem 4.3.2], $W_{m,\mathbf{t},\mathbf{x}}$ is a closed linear subspace of $\mathcal{S}_{m,\mathbf{t}}$. We can therefore appeal to the projection theorem [12, Theorem 6.2.2] to deduce that the orthogonal decomposition result

$$\mathcal{S}_{m,\mathbf{t}} = \mathcal{S}_{m,\mathbf{x}} \oplus W_{m,\mathbf{t},\mathbf{x}}. \tag{6.17}$$

holds.

Moreover, recalling also that the definitions (6.14) and (6.15) of the knot sequences $\mathbf{t} = \mathbf{t}_{m,n}$ and $\mathbf{x} = \mathbf{x}_{m,n}$ yield the dimensions

$$\dim \mathcal{S}_{m,\mathbf{t}} = m + 2n - 1, \tag{6.18}$$

and

$$\dim \mathcal{S}_{m,\mathbf{x}} = m + n - 1, \tag{6.19}$$

it follows from (6.17) that

$$\dim \mathcal{S}_{m,\mathbf{t}} = \dim \mathcal{S}_{m,\mathbf{x}} + \dim W_{m,\mathbf{t},\mathbf{x}},$$

and thus, using (6.18) and (6.19),

$$\dim W_{m,\mathbf{t},\mathbf{x}} = n. \tag{6.20}$$

Observe in particular that the dimension of the space $W_{m,\mathbf{t},\mathbf{x}}$ is independent of the order m of the underlying spline spaces.

In the spirit of Chapter 5, the concept of a spline wavelet on $[a, b]$ is now defined as follows.

Definition 6.1 *Let the knot sequences $\mathbf{t} = \mathbf{t}_{m,n}$ and $\mathbf{x} = \mathbf{x}_{m,n}$ on $[a, b]$ be given by (6.14) and (6.15). We shall call the n elements of the sequence $\{\psi_{m,i} = \psi_{m,\mathbf{t},\mathbf{x},i} : i = -m + 1, \ldots, n - m\} \subset \mathcal{S}_{m,\mathbf{t}}$ m-th order spline wavelets with knot sequence $\mathbf{t}$*

relative to the knot sequence $\mathbf{x}$ *if and only if* $\{\psi_{m,i} : i = -m+1, \ldots, n-m\}$ *is a basis of* $W_{m,\mathbf{t},\mathbf{x}}$. *The space* $W_{m,\mathbf{t},\mathbf{x}}$ *is called the corresponding* m*-th order spline wavelet space.*

We proceed to construct, as in [4, Section 3], a sequence of (minimally supported) B-spline wavelets with knot sequence $\mathbf{t}$ relative to the knot sequence $\mathbf{x}$.

We shall use the following notation. If, for a given non-negative integer μ, we are given a set $\{f_0, f_1, \ldots, f_\mu\}$ of functions defined on $[a, b]$, and a set of points $\{y_0, y_1, \ldots, y_\mu\} \subset [a, b]$, we define the determinant

$$D\begin{pmatrix} f_0, \ldots, f_\mu \\ y_0, \ldots, y_\mu \end{pmatrix} := \begin{vmatrix} f_0(y_0) & f_1(y_0) & \cdots & f_\mu(y_0) \\ f_0(y_1) & f_1(y_1) & \cdots & f_\mu(y_1) \\ \vdots & \vdots & & \vdots \\ f_0(y_\mu) & f_1(y_\mu) & \cdots & f_\mu(y_\mu) \end{vmatrix}. \tag{6.21}$$

We now adapt the idea of Theorem 5.8 to a bounded interval as follows.

Theorem 6.1 *For positive integers* m *and* n, *with* $n \geq 2m-1$, *let the knot sequences* $\mathbf{t} = \mathbf{t}_{m,n}$ *and* $\mathbf{x} = \mathbf{x}_{m,n}$ *on* $[a, b]$ *be defined by (6.14) and (6.15). Then the elements of the sequence* $\{\psi_{m,i} = \psi_{m,\mathbf{t},\mathbf{x},i} : i = -m+1, \ldots, n-m\}$ *defined by*

$$\psi_{m,i} = \Psi_{2m,i}^{(m)}, \quad i = -m+1, \ldots, n-m, \tag{6.22}$$

where

$$\Psi_{2m,i}(t) = \Psi_{2m,\mathbf{t},\mathbf{x},i}(t) = D\begin{pmatrix} N_{2m,\mathbf{t},\max\{i,2i\}}, \ldots, N_{2m,\mathbf{t},\min\{2i+2m-2,i+n-1\}} \\ t, x_{\max\{1,i+1\}}, \ldots, x_{\min\{i+2m-2,n-1\}} \end{pmatrix},$$

$$t \in [a, b], \quad i = -m+1, \ldots, n-m, \tag{6.23}$$

are compactly supported m*-th order spline wavelets with knot sequence* $\mathbf{t}$ *relative to the knot sequence* $\mathbf{x}$. *Moreover,*

$$\operatorname{supp} \psi_{m,i} \subseteq [x_i, x_{i+2m-1}], \quad i = -m+1, \ldots, n-m+1, \tag{6.24}$$

and

$$\left[\begin{array}{l} \psi_{m,i}^{(r)}(a)\left\{\begin{array}{ll} =0, & r=0,\ldots,m-2+i, \\ \neq 0, & r=m-1+i, \end{array}\right\} i=-m+1,\ldots,0, \\ \\ \psi_{m,i}^{(r)}(b)\left\{\begin{array}{ll} =0, & r=0,\ldots,n-m-1-i, \\ \neq 0, & r=n-m-i, \end{array}\right\} i=n-2m+1,\ldots,n-m. \end{array}\right. \tag{6.25}$$

Proof. First, observe from (6.22), (6.23), (6.21), (6.9) and (6.10) that

$$\Psi_{2m,i} \in \mathcal{S}_{2m,\mathbf{t}}, \quad \psi_{m,i} \in \mathcal{S}_{m,\mathbf{t}}, \quad i=-m+1,\ldots,n-m.$$

Next we note that the determinant in (6.23) gives $\Psi_{2m,i}$ as a linear combination of $2m$-th order B-splines on the knot sequence $\mathbf{t}$, that is

$$\Psi_{2m,i}(t) = \sum_{j=\max\{i,2i\}}^{\min\{2i+2m-2,i+n-1\}} c_{m,i,j} N_{2m,\mathbf{t},j}(t),$$

for some coefficients $c_{m,i,j} \in \Re$, from which it is clear that

$$\text{supp } \Psi_{2m,i} \subseteq \left[t_{\max\{i,2i\}}, t_{\min\{2i+4m-2,n+i+2m-1\}}\right] = \left[x_i, x_{i+2m-1}\right], \quad i=-m+1,\ldots,n-m, \tag{6.26}$$

using (6.16), and (6.24) follows from (6.26) and (6.22).

Also note that the determinant in (6.23) equals zero if $t = x_j$ for $j = \max\{1, i+1\},\ldots,\min\{i+2m-2, n-1\}$, due to the fact that there are then two identical rows in the determinant, and zero if $t = x_j$ for all other $j \in \{0,\ldots,n\}$, due to the fact that x then does not lie in any of the supports of $N_{2m,\mathbf{t},k}$, $k = \max\{i, 2i\},\ldots,\min\{2i+2m-2, i+n-1\}$, so that $\Psi_{2m,i}$ has the property

$$\Psi_{2m,i}(x_j) = 0, \quad j=0,\ldots,n. \tag{6.27}$$

Now fix $i \in \{0, \ldots, n-2m+1\}$ and $t \in [x_i, t_{2i+1}]$, so that the definition (6.23) becomes

$$\Psi_{2m,i}(t) = N_{2m,\mathbf{t},2i}(t) D \begin{pmatrix} N_{2m,\mathbf{t},2i+1}, \ldots, N_{2m,\mathbf{t},2i+2m-2} \\ x_{i+1}, \ldots, x_{i+2m-2} \end{pmatrix}, \tag{6.28}$$

due to the fact that t does not lie in the support of $N_{2m,\mathbf{t},k}$, $k = 2i+1, \ldots, 2i+2m-2$. Using the recursive algorithm given by (6.7), together with the support property (6.4) of B-splines, we obtain

$$\begin{aligned}
N_{2m,\mathbf{t},2i}(t) &= \frac{t - t_{2i}}{t_{2i+2m-1} - t_{2i}} N_{2m-1,\mathbf{t},2i}(t) + \frac{t_{2i+2m} - t}{t_{2i+2m} - t_{2i+1}} N_{2m-1,\mathbf{t},2i+1}(t) \\
&= \frac{t - t_{2i}}{t_{2i+2m-1} - t_{2i}} \left[\frac{t - t_{2i}}{t_{2i+2m-2} - t_{2i}} N_{2m-2,\mathbf{t},2i}(t) \right] + 0 \\
&\ \vdots \\
&= \frac{(t - t_{2i})^{2m-1}}{(t_{2i+2m-1} - t_{2i}) \ldots (t_{2i+1} - t_{2i})} N_{1,\mathbf{t},2i}(t) \\
&= \frac{(t - t_{2i})^{2m-1}}{(t_{2i+2m-1} - t_{2i}) \ldots (t_{2i+1} - t_{2i})},
\end{aligned}$$

so that we may now take the m-th derivative on both sides to obtain

$$\begin{aligned}
N^{(m)}_{2m,\mathbf{t},2i}(t) &= \frac{(2m-1) \ldots (m)(t - t_{2i})^{m-1}}{(t_{2i+2m-1} - t_{2i}) \ldots (t_{2i+1} - t_{2i})} \\
&= \frac{(2m-1)!}{(m-1)!} \frac{(t - x_i)^{m-1}}{(t_{2i+2m-1} - t_{2i}) \ldots (t_{2i+1} - t_{2i})}.
\end{aligned} \tag{6.29}$$

Then the m-th derivative of $\Psi_{2m,i}$ at t is given, according to (6.28) and (6.29), by

$$\begin{aligned}
\Psi^{(m)}_{2m,i}(t) &= N^{(m)}_{2m,\mathbf{t},2i}(t) D \begin{pmatrix} N_{2m,\mathbf{t},2i+1}, \ldots, N_{2m,\mathbf{t},2i+2m-2} \\ x_{i+1}, \ldots, x_{i+2m-2} \end{pmatrix} \\
&= \frac{(2m-1)!}{(m-1)!} \frac{(t - x_i)^{m-1}}{(t_{2i+2m-1} - t_{2i}) \ldots (t_{2i+1} - t_{2i})} D \begin{pmatrix} N_{2m,\mathbf{t},2i+1}, \ldots, N_{2m,\mathbf{t},2i+2m-2} \\ x_{i+1}, \ldots, x_{i+2m-2} \end{pmatrix}
\end{aligned} \tag{6.30}$$

which together with the fact that the determinant in (6.30) is nonvanishing, by virtue of the Schoenberg-Whitney theorem (see e.g. [15, Theorem 3.7]), implies that

$$\Psi^{(m)}_{2m,i}(t) \neq 0, \quad t \in (x_i, t_{2i+1}), \quad i = 0, \ldots, n-2m+1. \tag{6.31}$$

Observe that, for $i \in \{-m+1, \ldots, 0\}$ and $r \in \{1, \ldots, 2m-2+i\}$, we have, by differentiation of (6.23),

$$\Psi^{(r)}_{2m,i}(a) = \left(\frac{d}{dx}\right)^r D\left(\begin{matrix} N_{2m,\mathbf{t},i}, \ldots, N_{2m,\mathbf{t},2i+2m-2} \\ x, x_1, \ldots, x_{i+2m-2} \end{matrix}\right)\Bigg|_{x=a} = 0, \tag{6.32}$$

where the top line of (6.6) was used. For $r = 2m-1+i$ we have

$$\Psi^{(2m-1+i)}_{2m,i}(a) = N^{(2m-1+i)}_{2m,\mathbf{t},i}(a) D\left(\begin{matrix} N_{2m,\mathbf{t},i+1}, \ldots, N_{2m,\mathbf{t},2i+2m-2} \\ x_1, \ldots, x_{i+2m-2} \end{matrix}\right)\Bigg|_{x=a} \neq 0, \tag{6.33}$$

from the top line of (6.6) and the already noted nonvanishing of the determinant. Combining (6.32) and (6.33) then yields the result

$$\Psi^{(r)}_{2m,i}(a)\left\{\begin{matrix} = 0, & r = 1, \ldots, 2m-2+i, \\ \neq 0, & r = 2m-1+i \end{matrix}\right\} \quad i = -m+1, \ldots, 0. \tag{6.34}$$

Similarly we can prove that

$$\Psi^{(r)}_{2m,i}(b)\left\{\begin{matrix} = 0, & r = 1, \ldots, n-1-i, \\ \neq 0, & r = n-i \end{matrix}\right\} \quad i = n-2m+1, \ldots, n-m. \tag{6.35}$$

The result (6.25) is then a consequence of (6.22), (6.34) and (6.35).

We know that $\psi_{m,i} \in \mathcal{S}_{m,\mathbf{t}}$ for all $i = -m+1, \ldots, n-m$ and thus to show that $\psi_{m,i} \in W_{m,\mathbf{t},\mathbf{x}}$ it suffices to show that $\psi_{m,i} \perp \mathcal{S}_{m,\mathbf{x}}$. We fix an $i \in \{-m+1, \ldots, n-m\}$ and a $j \in \{-m+1, \ldots, n-1\}$ and then obtain, from (6.22), by repeated integration by parts, and using the properties (6.34) and (6.35), we find (cf. also the steps that led to (5.109))

$$\begin{aligned} \langle \psi_{m,i}, N_{m,\mathbf{x},j}\rangle &= \int_a^b \Psi^{(m)}_{2m,i}(t) N_{m,\mathbf{x},j}(t)\, \mathrm{d}t \\ &= \Psi^{(m-1)}_{2m,i}(t) N_{m,\mathbf{x},j}(t)\big|_a^b - \int_a^b \Psi^{(m-1)}_{2m,i}(t) N'_{m,\mathbf{x},j}(t)\, \mathrm{d}t \end{aligned}$$

$$\begin{aligned}
&= 0 - \left[\Psi_{2m,i}^{(m-2)}(t)N'_{m,\mathbf{x},j}(t)\Big|_a^b - \int_a^b \Psi_{2m,i}^{(m-2)}(t)N''_{m,\mathbf{x},j}(t)\,\mathrm{d}t\right] \\
&= \int_a^b \Psi_{2m,i}^{(m-2)}(t)N''_{m,\mathbf{x},j}(t)\,\mathrm{d}t \\
&= \ldots \\
&= (-1)^{m-1}\int_a^b \Psi'_{2m,i}(t)N_{m,\mathbf{x},j}^{(m-1)}(t)\,\mathrm{d}t.
\end{aligned}$$

Since (6.9) shows that $N_{m,\mathbf{x},j}^{(m-1)} \in \mathcal{S}_{1,\mathbf{x}}$, which is the set of piecewise constant splines on the knot sequence $\mathbf{x}$, we may suppose that

$$N_{m,\mathbf{x},j}^{(m-1)}(t) = c_{m,j,k}, \quad t \in [x_k, x_{k+1}), \quad k = 0, \ldots, n-1,$$

for some constants $c_{m,j,k} \in \Re$, and thus

$$\begin{aligned}
\langle \psi_{m,i}, N_{m,\mathbf{x},j}\rangle &= (-1)^{m-1}\sum_{k=0}^{n-1} c_{m,j,k}\int_{x_k}^{x_{k+1}} \Psi'_{2m,i}(t)\,\mathrm{d}t \\
&= (-1)^{m-1}\sum_{k=0}^{n-1} c_{m,j,k}\; \Psi_{2m,i}(t)\Big|_{x_k}^{x_{k+1}} \\
&= 0,
\end{aligned}$$

by virtue of property (6.27).

Since we have a set of n functions $\{\psi_{m,i} :\ i = -m+1, \ldots, n-m\}$ in $W_{m,\mathbf{t},\mathbf{x}}$, it follows from (6.20), that this set will be a basis for $W_{m,\mathbf{t},\mathbf{x}}$ if and only if it is a linear independent set. Suppose therefore that

$$\sum_{i=-m+1}^{n-m} k_i\psi_{m,i}(t) = 0, \quad t \in [a,b], \tag{6.36}$$

for some $k_{-m+1}, \ldots, k_{n-m} \in \Re$. Then by setting $t = a$ in (6.36) and using (6.25), we get

$$0 = k_{-m+1}\psi_{m,-m+1}(a),$$

and thus $k_{-m+1} = 0$. Similarly by setting $t = b$ and using (6.25), we find that $k_{n-m} = 0$. Now we differentiate (6.36) to obtain

$$\sum_{i=-m+2}^{n-m-1} k_i\psi'_{m,i}(t) = 0, \quad t \in [a,b]. \tag{6.37}$$

Again, by setting $t = a$ and $t = b$ successively in (6.37), and using (6.25) we find that $k_{-m+2} = 0 = k_{n-m-1}$. This procedure can be repeated until we have differentiated (6.36) $(m-2)$ times, by which time we have $k_{-m+1} = \ldots = k_{-1} = 0 = k_{n-2m+2} = \ldots = k_{n-m}$, which, together with (6.36), gives

$$\sum_{i=0}^{n-2m+1} k_i \psi_{m,i}(t) = 0,$$

and thus, from (6.31),

$$\sum_{i=0}^{n-2m+1} k_i \Psi_{2m,i}^{(m)}(t) = 0, \quad t \in [a, b]. \tag{6.38}$$

Now choose $t \in (a, t_1)$ whence, from (6.38), (6.31) and (6.26), we get $k_0 = 0$. Hence (6.38) and (6.26) give

$$\sum_{i=1}^{n-2m+1} k_i \Psi_{2m,i}^{(m)}(t) = 0, \quad t \in [x_1, b].$$

By choosing t successively in the intervals (x_j, t_{2j+1}), $j = 1, \ldots, n-2m$, and using (6.31), we find that also $k_1 = k_2 = \ldots = k_{n-2m+1} = 0$. We have thus proved the linear independence of the set $\{\psi_{m,i} : \; i = -m+1, \ldots, n-m\}$, which completes our proof. ■

Since the spline functions $\{\psi_{m,i} : i = -m+1, \ldots, n-m\}$ as defined by (6.22), are the minimally supported spline wavelets with knot sequence $\mathbf{t}$ relative to the knot sequence $\mathbf{x}$, we shall call them the m-th order B-spline wavelets, or merely B-wavelets.

6.3 Uniform Interior Knot Spacings.

An interesting special case of Theorem 6.1 is provided by the case where the knot sequence $\mathbf{t}$ provides a subdivision into intervals of equal length of the interval $[a, b]$, i.e.

$$t_j = a + \frac{b-a}{2n} j, \qquad j = 0, 1, \ldots, 2n, \tag{6.39}$$

in (6.14), so that

$$x_j = a + \frac{b-a}{n} j, \qquad j = 0, 1, \ldots, n, \tag{6.40}$$

in (6.15). Observe from (6.7), (6.8), (6.39) and (6.40) that then

$$N_{m,\mathbf{t},j}(t) = N_m\left(\frac{2n(t-a)}{b-a} - j\right), \quad t \in [a,b], \quad j = 0, 1, \ldots, 2n-m,$$

whereas

$$N_{m,\mathbf{x},j}(t) = N_m\left(\frac{n(t-a)}{b-a} - j\right), \quad t \in [a,b], \quad j = 0, 1, \ldots, n-m.$$

Moreover, if we define the sequence $\{\Psi^*_{2m,j} : j = 0, 1, \ldots, n-2m+1\}$ by

$$\Psi^*_{2m,j}(t) = \Psi_{2m}\left(\frac{n(t-a)}{b-a} - j\right), \quad t \in [a,b], \quad j = 0, 1, \ldots, n-2m+1, \tag{6.41}$$

with Ψ_{2m} defined by the formula (5.98) of Theorem 5.8, then, from (6.41), we have for $j \in \{0, 1, \ldots, n-2m+1\}$ and $t \in [a,b]$ that

$$\begin{aligned} \Psi^*_{2m,j}(t) &= \sum_k (-1)^k N_{2m}(k+1) N_{2m}\left(\frac{2n(t-a)}{b-a} - 2j - k\right) \\ &= \sum_k (-1)^k N_{2m}(k-2j+1) N_{2m}\left(\frac{2n(t-a)}{b-a} - k\right) \end{aligned}$$

and thus

$$\begin{aligned} \Psi^*_{2m,j}(t) &= \sum_{k=2j}^{2j+2m-2} (-1)^k N_{2m}(k-2j+1) N_{2m}\left(\frac{2n(t-a)}{b-a} - k\right), \\ & \qquad t \in [a,b], \quad j = 0, 1, \ldots, n-2m+1. \end{aligned} \tag{6.42}$$

It follows from (6.7), (6.9), (6.40) and (6.42) that

$$\text{supp } \Psi^*_{2m,j} = [x_j, x_{j+2m-1}], \quad j = 0, 1, \ldots, n-2m+1. \tag{6.43}$$

Moreover, using (6.41), (6.40) and (5.99), we get, for $j \in \{0, 1, \ldots, n-2m+1\}$ and $l \in \{0, 1, \ldots, n\}$ that

$$\begin{aligned} \Psi^*_{2m,j}(x_l) &= \Psi_{2m}\left(\frac{n}{b-a}\left(\frac{b-a}{n} l\right) - j\right) \\ &= \Psi_{2m}(l-j) = 0, \end{aligned}$$

which, together with (6.43), shows that the sequence $\{\Psi^*_{2m,j} : j = 0, 1, \ldots, n - 2m + 1\}$ satisfies the conditions (6.26), (6.31), (6.27), (6.34) and (6.35). Since the elements of the sequence $\{\Psi_{2m,j} : j = -m+1, \ldots, n-m\}$ in Theorem 6.1 are minimally supported, we must therefore have

$$\Psi_{2m,j} = c\Psi^*_{2m,j}, \quad j = 0, 1, \ldots, n - 2m + 1,$$

for some $c \in \Re \setminus \{0\}$, so that, using also (6.22), (6.41) and (5.101), we get the result

$$\psi_{m,j}(t) = k\psi_m\left(\frac{n(t-a)}{b-a} - j\right), \quad t \in [a, b], \quad j = 0, 1, \ldots, n - 2m + 1, \qquad (6.44)$$

for some $k \in \Re \setminus \{0\}$.

Hence we can combine (6.44) and Theorem 6.1 to deduce the following result.

Theorem 6.2 *Let m and n be positive integers with $n \geq 2m - 1$, and suppose the knot sequence $\mathbf{t}$ and $\mathbf{x}$ in (6.14) and (6.15) satisfy (6.39) and (6.40). Then the m-th order B-spline wavelets $\{\psi_{m,j} : j = -m+1, \ldots, n-m\}$ of Theorem 6.1 are given by*

$$\psi_{m,j}(t) = \begin{cases} \left(\dfrac{d}{dt}\right)^m D\begin{pmatrix} N_{2m,\mathbf{t},j}, \ldots, N_{2m,\mathbf{t},2j+2m-2} \\ t, x_1, \ldots, x_{j+2m-2} \end{pmatrix}, & j = -m+1, \ldots, -1, \\[2ex] k\psi_m\left(\dfrac{n(t-a)}{b-a} - j\right), & j = 0, 1, \ldots, n-2m+1, \\[2ex] \left(\dfrac{d}{dt}\right)^m D\begin{pmatrix} N_{2m,\mathbf{t},2j}, \ldots, N_{2m,\mathbf{t},j+n-1} \\ t, x_{j+1}, \ldots, x_{n-1} \end{pmatrix}, & j = n-2m+2, \ldots, n-m, \end{cases} \qquad (6.45)$$

for $t \in [a, b]$, for some $k \in \Re \setminus \{0\}$, and with ψ_m defined by (5.93) and (5.94).

We shall henceforth refer to the wavelets described by the middle line of (6.45) as *inner spline wavelets*, whereas those described by the first and last lines of (6.45) will be called *boundary spline wavelets.*

We can now calculate the following special cases of (6.45), in which we also choose, for simplicity, $k = 1$, $[a, b] = [0, 1]$ and $n = 2m - 1$.

The linear case $m = 2$: (and thus $n = 3$).
We see from (6.19) and (6.20) that $\dim \mathcal{S}_{2,\mathbf{x}} = 4$ and $\dim W_{2,\mathbf{t},\mathbf{x}} = 3$. We illustrate the spline basis of $\mathcal{S}_{2,\mathbf{x}}$ in Fig.6.2.

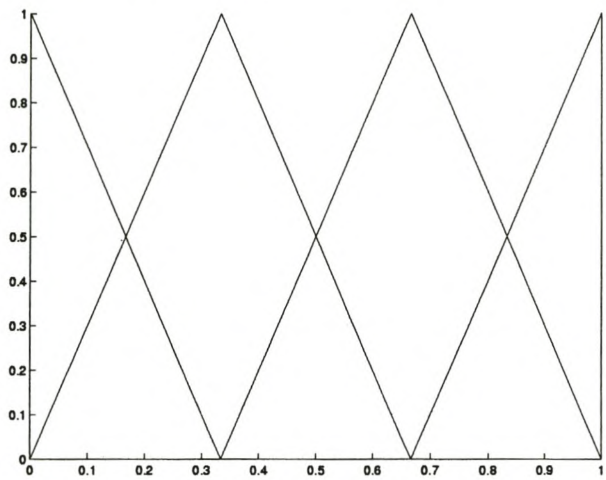


Figure 6.2: The spline basis of $S_{2,\mathbf{x}}$

From (6.45) and (6.40) we have here, for $t \in [0, 1]$,

$$\begin{aligned} \psi_{2,-1}(t) &= \left(\frac{d}{dt}\right)^2 \begin{vmatrix} N_{4,\mathbf{t},-1}(t) & N_{4,\mathbf{t},0}(t) \\ N_{4,\mathbf{t},-1}\left(\frac{1}{3}\right) & N_{4,\mathbf{t},0}\left(\frac{1}{3}\right) \end{vmatrix} \\ &= N_{4,\mathbf{t},0}\left(\frac{1}{3}\right) N''_{4,\mathbf{t},-1}(t) - N_{4,\mathbf{t},-1}\left(\frac{1}{3}\right) N''_{4,\mathbf{t},0}(t). \end{aligned} \tag{6.46}$$

We may use the recursion formula (6.7) and (6.8) to obtain

$$N_{4,\mathbf{t},0}\left(\frac{1}{3}\right) = \frac{2}{3},$$

and

$$N_{4,\mathbf{t},-1}\left(\frac{1}{3}\right) = \frac{1}{6},$$

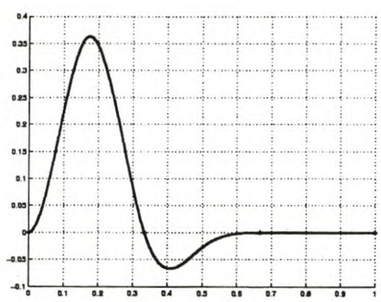
so that by (6.46) and the derivative formulas (6.9) and (6.10) we have

$$\psi_{2,-1}(t) = 6 \sum_{j=-1}^{2} q_{2,-1,j} N_{2,\mathbf{t},j}(t), \quad t \in [0,1], \tag{6.47}$$

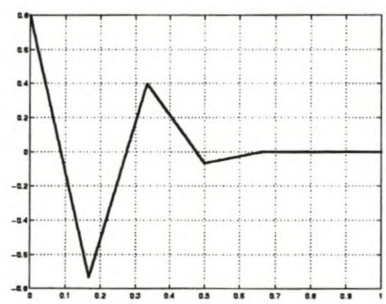
where

$$q_{2,-1,j} = \begin{cases} 12, & j=-1, \\ -11, & j=0, \\ 6, & j=1, \\ -1, & j=2. \end{cases} \tag{6.48}$$

The recursion formula (6.7) and (6.8) may now be used to evaluate the B-splines and consequently the B-wavelets. We illustrate $\Psi_{4,-1}$ and $\psi_{2,-1}$ in Fig.6.3(a) and (b) respectively.



(a)



(b)

Figure 6.3: (a) $\Psi_{4,-1}$ and (b) the boundary wavelet $\psi_{2,-1}$

Similarly we have, for $t \in [0,1]$,

$$\begin{aligned} \psi_{2,1}(t) &= \left(\frac{d}{dt}\right)^2 \begin{vmatrix} N_{4,\mathbf{t},2}(t) & N_{4,\mathbf{t},3}(t) \\ N_{4,\mathbf{t},2}\left(\frac{2}{3}\right) & N_{4,\mathbf{t},3}\left(\frac{2}{3}\right) \end{vmatrix} \\ &= N_{4,\mathbf{t},3}\left(\frac{2}{3}\right) N''_{4,\mathbf{t},2}(t) - N_{4,\mathbf{t},2}\left(\frac{2}{3}\right) N''_{4,\mathbf{t},3}(t) \\ &= \frac{1}{6} N''_{4,\mathbf{t},2}(t) - \frac{2}{3} N''_{4,\mathbf{t},3}(t), \end{aligned} \tag{6.49}$$

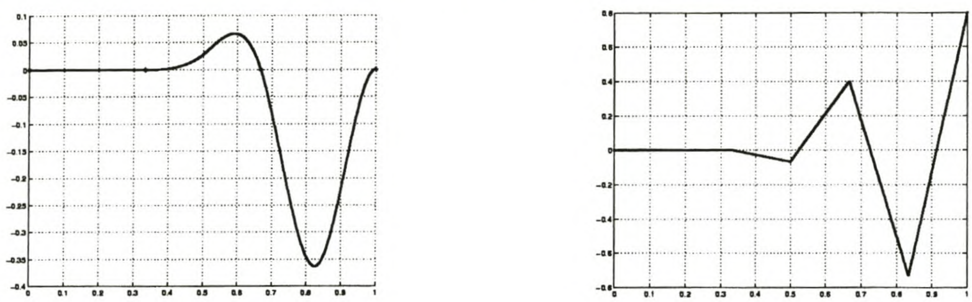
so that by the derivative formulas (6.9) and (6.10) we have

$$\psi_{2,1}(t) = 6 \sum_{j=2}^{5} q_{2,1,j} N_{2,\mathbf{t},j}(t), \quad t \in [0,1],$$

where

$$q_{2,1,j} = \begin{cases} 1, & j = 2, \\ -6, & j = 3, \\ 11, & j = 4, \\ -12, & j = 5. \end{cases}$$

We illustrate $\Psi_{4,1}$ and $\psi_{2,1}$ in Fig.6.4(a) and (b) respectively.



(a) (b)

Figure 6.4: (a) $\Psi_{4,1}$ and (b) the boundary wavelet $\psi_{2,1}$

To find the interior wavelet $\psi_{2,0}$ we employ the middle line of (6.45), and (5.94) to obtain

$$\psi_{2,0}(t) = \sum_{j=0}^{4} q_{2,0,j} N_2(6t - j), \quad t \in [0, 1],$$

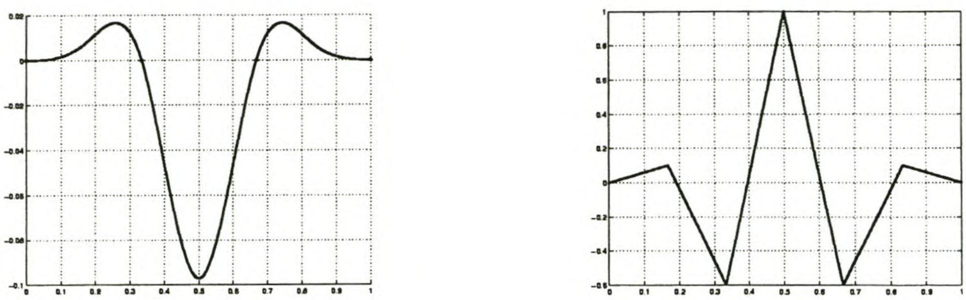
where

$$q_{2,0,j} = \begin{cases} \frac{3}{2}, & j = 0, \\ -9, & j = 1, \\ 15, & j = 2, \\ -9, & j = 3, \\ \frac{3}{2}, & j = 4. \end{cases}$$

We illustrate $\Psi_{4,0}$ and $\psi_{2,0}$ in Fig.6.5(a) and (b) respectively.

The quadratic case $m = 3$: (and thus $n = 5$).

Again we note from (6.19) and (6.20) that $\dim \mathcal{S}_{3,\mathbf{x}} = 6$ and $\dim W_{3,\mathbf{t},\mathbf{x}} = 5$. We



(a) (b)

Figure 6.5: (a) $\Psi_{4,0}$ and (b) the inner wavelet $\psi_{2,0}$

illustrate the spline basis of $\mathcal{S}_{3,\mathbf{x}}$ in Fig.6.6. Now we use (6.45) and (6.40) to obtain

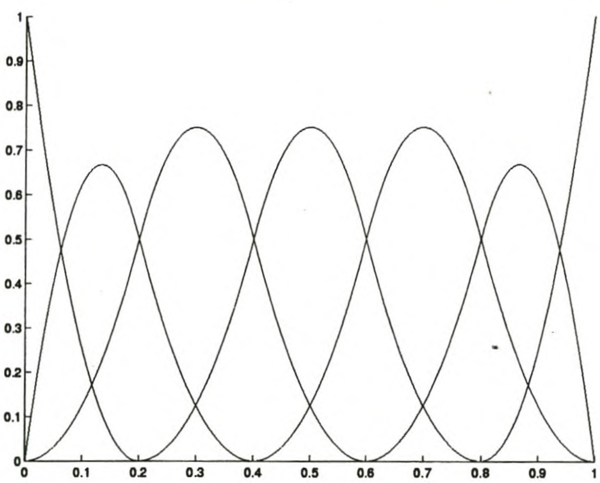


Figure 6.6: The spline basis of $\mathcal{S}_{3,\mathbf{x}}$

the boundary wavelets

$$\psi_{3,-2}(t) = \left(\frac{d}{dt}\right)^3 \begin{vmatrix} N_{6,\mathbf{t},-2}(t) & N_{6,\mathbf{t},-1}(t) & N_{6,\mathbf{t},0}(t) \\ N_{6,\mathbf{t},-2}\left(\frac{1}{5}\right) & N_{6,\mathbf{t},-1}\left(\frac{1}{5}\right) & N_{6,\mathbf{t},0}\left(\frac{1}{5}\right) \\ N_{6,\mathbf{t},-2}\left(\frac{2}{5}\right) & N_{6,\mathbf{t},-1}\left(\frac{2}{5}\right) & N_{6,\mathbf{t},0}\left(\frac{2}{5}\right) \end{vmatrix},$$

$$\psi_{3,-1}(t) = \left(\frac{d}{dt}\right)^3 \begin{vmatrix} N_{6,\mathbf{t},-1}(t) & N_{6,\mathbf{t},0}(t) & N_{6,\mathbf{t},1}(t) & N_{6,\mathbf{t},2}(t) \\ N_{6,\mathbf{t},-1}\left(\frac{1}{5}\right) & N_{6,\mathbf{t},0}\left(\frac{1}{5}\right) & N_{6,\mathbf{t},1}\left(\frac{1}{5}\right) & N_{6,\mathbf{t},2}\left(\frac{1}{5}\right) \\ N_{6,\mathbf{t},-1}\left(\frac{2}{5}\right) & N_{6,\mathbf{t},0}\left(\frac{2}{5}\right) & N_{6,\mathbf{t},1}\left(\frac{2}{5}\right) & N_{6,\mathbf{t},2}\left(\frac{2}{5}\right) \\ N_{6,\mathbf{t},-1}\left(\frac{3}{5}\right) & N_{6,\mathbf{t},0}\left(\frac{3}{5}\right) & N_{6,\mathbf{t},1}\left(\frac{3}{5}\right) & N_{6,\mathbf{t},2}\left(\frac{3}{5}\right) \end{vmatrix},$$

$$\psi_{3,1}(t) = \left(\frac{d}{dt}\right)^3 \begin{vmatrix} N_{6,\mathbf{t},2}(t) & N_{6,\mathbf{t},3}(t) & N_{6,\mathbf{t},4}(t) & N_{6,\mathbf{t},5}(t) \\ N_{6,\mathbf{t},2}\left(\frac{2}{5}\right) & N_{6,\mathbf{t},3}\left(\frac{2}{5}\right) & N_{6,\mathbf{t},4}\left(\frac{2}{5}\right) & N_{6,\mathbf{t},5}\left(\frac{2}{5}\right) \\ N_{6,\mathbf{t},2}\left(\frac{3}{5}\right) & N_{6,\mathbf{t},3}\left(\frac{3}{5}\right) & N_{6,\mathbf{t},4}\left(\frac{3}{5}\right) & N_{6,\mathbf{t},5}\left(\frac{3}{5}\right) \\ N_{6,\mathbf{t},2}\left(\frac{4}{5}\right) & N_{6,\mathbf{t},3}\left(\frac{4}{5}\right) & N_{6,\mathbf{t},4}\left(\frac{4}{5}\right) & N_{6,\mathbf{t},5}\left(\frac{4}{5}\right) \end{vmatrix},$$

and

$$\psi_{3,2}(t) = \left(\frac{d}{dt}\right)^3 \begin{vmatrix} N_{6,\mathbf{t},4}(t) & N_{6,\mathbf{t},5}(t) & N_{6,\mathbf{t},6}(t) \\ N_{6,\mathbf{t},4}\left(\frac{3}{5}\right) & N_{6,\mathbf{t},5}\left(\frac{3}{5}\right) & N_{6,\mathbf{t},6}\left(\frac{3}{5}\right) \\ N_{6,\mathbf{t},4}\left(\frac{4}{5}\right) & N_{6,\mathbf{t},5}\left(\frac{4}{5}\right) & N_{6,\mathbf{t},6}\left(\frac{4}{5}\right) \end{vmatrix},$$

so that by using the recursion formula (6.7) and (6.8) to evaluate the B-splines we obtain

$$\psi_{3,-2}(t) = \left(\frac{d}{dt}\right)^3 \left[\frac{143}{1350}N_{6,\mathbf{t},-2}(t) - \frac{91}{1620}N_{6,\mathbf{t},-1}(t) + \frac{7}{3240}N_{6,\mathbf{t},0}(t)\right], \tag{6.50}$$

$$\psi_{3,-1}(t) = \left(\frac{d}{dt}\right)^3 \left[\frac{169}{6750}N_{6,\mathbf{t},-1}(t) - \frac{709}{12153}N_{6,\mathbf{t},0}(t) + \frac{369}{16078}N_{6,\mathbf{t},1}(t) - \frac{15}{16993}N_{6,\mathbf{t},2}(t)\right], \tag{6.51}$$

$$\psi_{3,1}(t) = \left(\frac{d}{dt}\right)^3 \left[\frac{15}{16993}N_{6,\mathbf{t},2}(t) - \frac{369}{16078}N_{6,\mathbf{t},3}(t) + \frac{709}{12153}N_{6,\mathbf{t},4}(t) - \frac{169}{6750}N_{6,\mathbf{t},5}(t)\right], \tag{6.52}$$

and

$$\psi_{3,2}(t) = \left(\frac{d}{dt}\right)^3 \left[\frac{7}{3240}N_{6,\mathbf{t},4}(t) - \frac{91}{1620}N_{6,\mathbf{t},5}(t) + \frac{143}{1350}N_{6,\mathbf{t},6}(t)\right]. \tag{6.53}$$

Using the derivative formula (6.9) and (6.10) in (6.50), (6.51), (6.52) and (6.53) yields

$$\psi_{3,-2}(t) = \frac{28600}{27}\sum_{j=-2}^{3} q_{3,-2,j}N_{3,\mathbf{t},j}(t), \tag{6.54}$$

where

$$q_{3,-2,j} = \begin{cases} 1, & j=-2, \\ -\frac{107}{88}, & j=-1, \\ \frac{885}{1223}, & j=0, \\ -\frac{989}{3259}, & j=1, \\ \frac{203}{3432}, & j=2, \\ -\frac{7}{3432}, & j=3, \end{cases} \tag{6.55}$$

$$\psi_{3,-1}(t) = \frac{14872}{81} \sum_{j=-1}^{5} q_{3,-1,j} N_{3,\mathbf{t},j}(t), \tag{6.56}$$

where

$$q_{3,-1,j} = \begin{cases} \frac{15}{44}, & j=-1, \\ -\frac{1949}{2288}, & j=0, \\ \frac{817}{537}, & j=1, \\ -\frac{1681}{1144}, & j=2, \\ \frac{809}{1144}, & j=3, \\ -\frac{29}{208}, & j=4, \\ \frac{1}{208}, & j=5, \end{cases} \tag{6.57}$$

$$\psi_{3,1}(t) = \frac{14872}{81} \sum_{j=2}^{8} q_{3,1,j} N_{3,\mathbf{t},j}(t),$$

where

$$q_{3,1,j} = \begin{cases} \frac{1}{208}, & j=2, \\ -\frac{29}{208}, & j=3, \\ \frac{809}{1144}, & j=4, \\ -\frac{1681}{1144} & j=5, \\ \frac{817}{537}, & j=6, \\ -\frac{1949}{2288}, & j=7, \\ \frac{15}{44}, & j=8, \end{cases}$$

$$\psi_{3,2}(t) = \frac{28600}{27} \sum_{j=4}^{9} q_{3,2,j} N_{3,\mathbf{t},j}(t),$$

where

$$q_{3,2,j} = \begin{cases} \frac{7}{3432}, & j = 4, \\ -\frac{203}{3432}, & j = 5, \\ \frac{989}{3259}, & j = 6, \\ -\frac{885}{1223}, & j = 7, \\ \frac{107}{88}, & j = 8, \\ -1 & j = 9. \end{cases}$$

We illustrate $\Psi_{6,j}$ and $\psi_{3,j}$, for $j = -2, -1, 1, 2$, in Fig.6.7, 6.8, 6.9 and 6.10 respectively.

To find the interior wavelet $\psi_{3,0}$ we employ the middle line of (6.45), and (5.94) to obtain

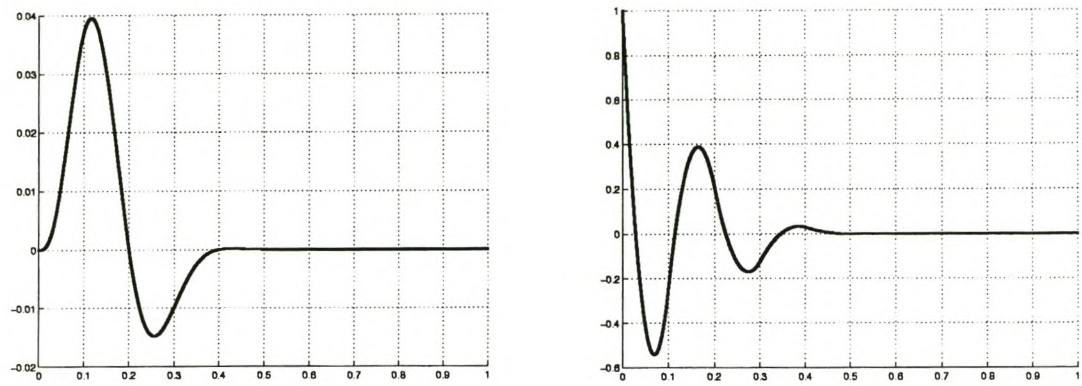
$$\psi_{3,0}(t) = \sum_{j=0}^{7} q_{3,0,j} N_3(10t - j), \quad t \in [0, 1],$$

where

$$q_{3,0,j} = \begin{cases} -\frac{1}{208}, & j = 0, \\ \frac{29}{208}, & j = 1, \\ \frac{-147}{208}, & j = 2, \\ \frac{303}{208}, & j = 3, \\ -\frac{303}{208}, & j = 4, \\ \frac{147}{208}, & j = 5, \\ -\frac{29}{208}, & j = 6, \\ \frac{1}{208}, & j = 7. \end{cases}$$

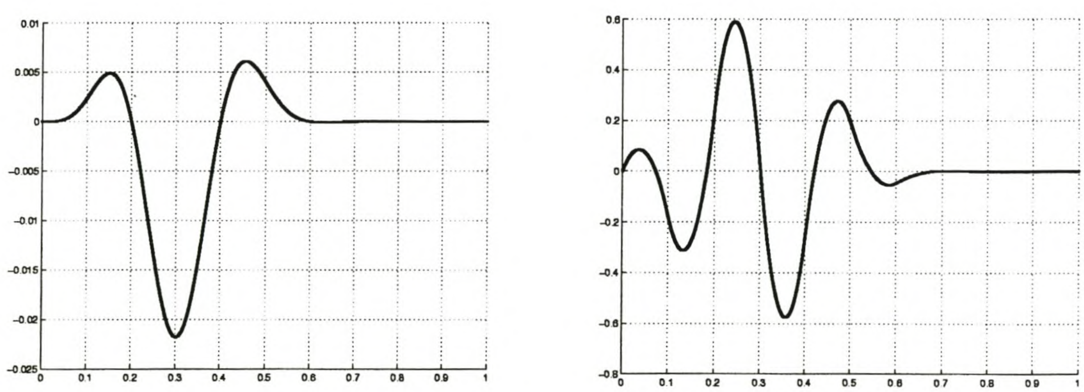
We illustrate $\Psi_{6,0}$ and $\psi_{3,0}$ in Fig.6.11(a) and (b) respectively.

We shall not pursue, analogous to Section 4.4, the decomposition and reconstruction algorithms for the bounded interval, but we refer to [4, Section 5] for a study of the matrices governing the decomposition and reconstruction algorithms.



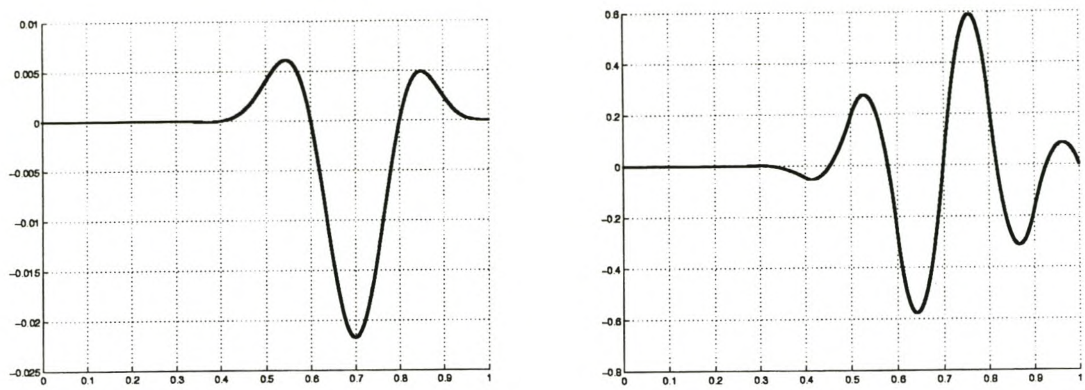
(a) (b)

Figure 6.7: (a) $\Psi_{6,-2}$ and (b) $\psi_{3,-2}$



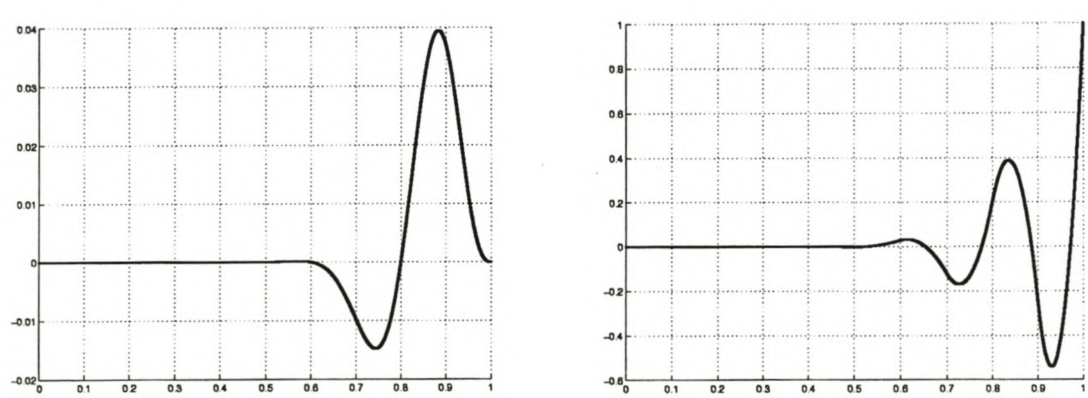
(a) (b)

Figure 6.8: (a) $\Psi_{6,-1}$ and (b) $\psi_{3,-1}$



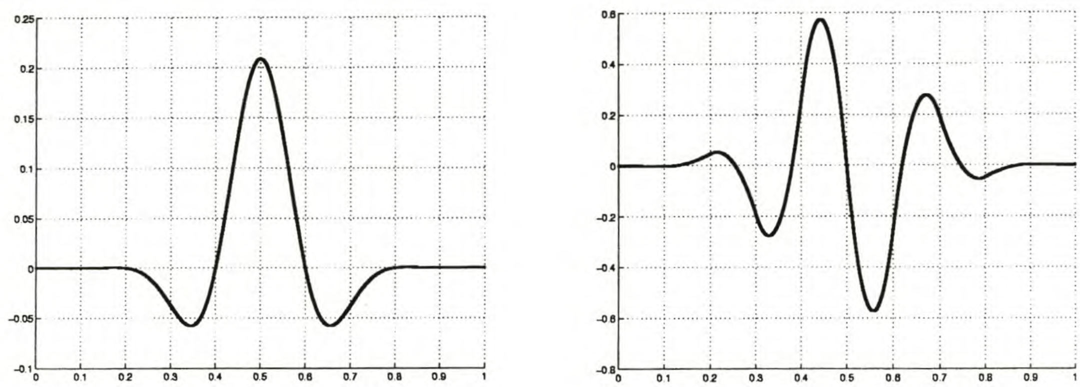
(a) (b)

Figure 6.9: (a) $\Psi_{6,1}$ and (b) $\psi_{3,1}$



(a) (b)

Figure 6.10: (a) $\Psi_{6,2}$ and (b) $\psi_{3,2}$



(a) (b)

Figure 6.11: (a) $\Psi_{6,0}$ and (b) $\psi_{3,0}$

6.4 The Chui-Quak Method.

In [5, Section 3] the authors used the following alternative approach to construct m-th order spline wavelets for uniformly spaced interior knot sequence $\mathbf{t}$ and $\mathbf{x}$ as given by (6.14), (6.15), (6.39) and (6.40), and with the choice $[a, b] = [0, 1]$.

First, in [5, Lemma 3.5], they constructed essentially the same interior spline wavelets as in our (6.44), but with the choice $k = \frac{1}{2^{2m-1}}$.

Then they constructed a sequence $\{\Psi^*_{2m,j} : j = -m+1, \ldots, -1, n-2m+2, \ldots, n-m\} \subset \mathcal{S}_{2m,\mathbf{t}}$ as follows: To obtain the boundary spline wavelets $\{\psi_{m,j} : j = -m+1, \ldots, -1\}$ they set

$$\begin{aligned} \Psi^*_{2m,j}(t) &= \sum_{l=-m+1}^{-1} \alpha_{m,j,l} N_{2m,\mathbf{t},l}(t) \\ &\qquad + \sum_{l=0}^{2m-2+2j} (-1)^l N_{2m}(l+1-2j) N_{2m}(2nt-l), \\ &\qquad\qquad t \in [0,1], \quad j = -m+1, \ldots, -1, \end{aligned} \tag{6.58}$$

with $\{\alpha_{m,j,l} : l = -m+1, \ldots, -1\}$ denoting a coefficient sequence in $\Re$, so that, from (6.5), (6.14), (6.15), (6.39) and (6.40),

$$\operatorname{supp} \Psi^*_{2m,j} = [x_j, x_{j+2m-1}] = [0, x_{j+2m-1}], \quad j = -m+1, \ldots, -1. \tag{6.59}$$

Since, from (6.4), we have

$$N_{2m,\mathbf{t},l}(x_k) = 0, \quad k \geq m, \quad l \in \{-m+1, \ldots, -1\},$$

it follows from (6.58), (6.59) and (6.40) that, for $j \in \{-m+1, \ldots, -1\}$, $k \geq m$,

$$\begin{aligned}
\Psi^*_{2m,j}(x_k) &= \sum_{l=0}^{2m-2+2j} (-1)^l N_{2m}(l+1-2j) N_{2m}(2k-l), \\
&= \sum_l (-1)^l N_{2m}(l+1-2j) N_{2m}(2k-l), \\
&= \sum_l (-1)^{2k+2j-1-l} N_{2m}(2k-l) N_{2m}(l+1-2j), \\
&= -\sum_l (-1)^l N_{2m}(l+1-2j) N_{2m}(2k-l), \\
&= -\sum_{l=0}^{2m-2+2j} (-1)^l N_{2m}(l+1-2j) N_{2m}(2k-l),
\end{aligned}$$

and thus

$$\Psi^*_{2m,j}(x_k) = -\Psi^*_{2m,j}(x_k),$$

whence

$$\Psi^*_{2m,j}(x_k) = 0, \quad j \in \{-m+1, \ldots, -1\}, \quad k \geq m. \tag{6.60}$$

The coefficient sequence $\{\alpha_{m,j,l} : l = -m+1, \ldots, -1\}$ was then chosen to satisfy the $(m-1) \times (m-1)$ linear system

$$\begin{aligned}
&\sum_{l=-m+1}^{-1} N_{2m,\mathbf{t},l}(x_k) \alpha_{m,j,l} \\
&= -\sum_{l=0}^{2m-2+2j} (-1)^l N_{2m}(l+1-2j) N_{2m}(2nx_k - l), \\
&\qquad k = 1, 2, \ldots, m-1,
\end{aligned} \tag{6.61}$$

which, according to the Schoenberg-Whitney Theorem [15, Theorem 3.7], has a unique solution $\{\alpha_{m,j,l} : l = -m+1, \ldots, -1\}$.

Combining (6.60) and (6.61), we see that, with $\{\alpha_{m,j,l}\}$ chosen to satisfy (6.61), we have

$$\Psi^*_{2m,j}(x_l) = 0, \quad l = 0, 1, \dots, j + 2m - 2, \quad j = -m + 1, \dots, -1.$$

Similarly, the sequence $\{\Psi^*_{2m,j} : j = n - 2m + 2, \dots, n - m\}$ was constructed to achieve

$$\left.\begin{array}{l} \text{supp } \Psi^*_{2m,j} = [x_j, x_{j+2m-1}], \quad j = n - 2m + 2, \dots, n - m \\ \Psi^*_{2m,j}(x_k) = 0, \quad j \in \{n - 2m + 2, \dots, n - m\}, \quad k \leq n - m. \end{array}\right\}$$

According to [5, Theorem 3.7], the sequence $\{\psi^*_{m,j} : j = -m + 1, \dots, n - m\}$ defined by

$$\psi^*_{m,j} = \Psi^{*(m)}_{2m,j}, \quad j = -m + 1, \dots, n - m, \tag{6.62}$$

are m-th order B-spline wavelets with respect to the knot sequence (6.14), (6.15), (6.39), (6.40) on $[0, 1]$.

We shall now compute the special cases $\psi^*_{2,j}$, $\psi^*_{3,j}$ explicitly, with the choice $n = 3$ and $n = 5$ respectively, in order to compare these wavelets with those of Section 6.3.

The linear case $m = 2$: (with $n = 3$).

We first use the formula (6.58) to obtain

$$\Psi^*_{4,-1}(t) = \alpha_{2,-1,-1} N_{4,\mathbf{t},-1}(t) + N_4(3) N_4(6t), \tag{6.63}$$

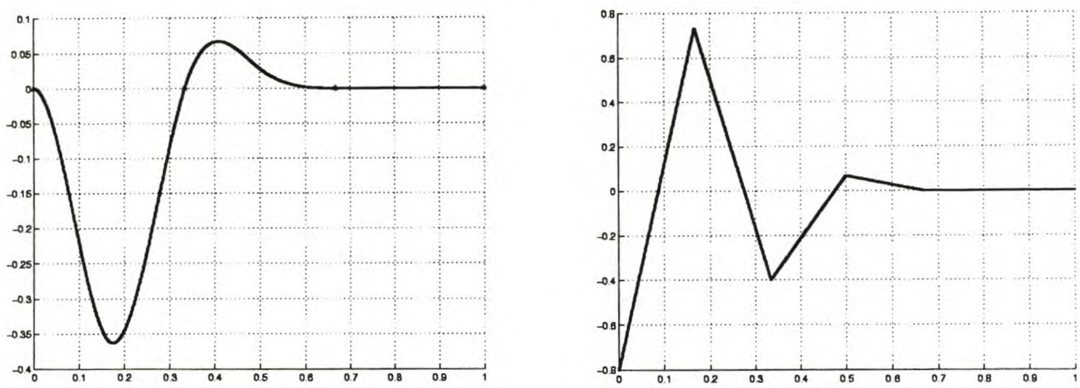
where, by (6.61), $\alpha_{2,-1,-1}$ must satisfy

$$\alpha_{2,-1,-1} N_{4,\mathbf{t},-1}\left(\frac{1}{3}\right) = -N_4(3) N_4(2),$$

so that $\alpha_{2,-1,-1} = -\frac{2}{3}$. Hence, from (6.63) and (6.62), we have the boundary wavelet

$$\begin{aligned} \psi^*_{2,-1}(t) &= \left(\frac{d}{dt}\right)^2 \left[-\frac{2}{3} N_{4,\mathbf{t},-1}(t) + \frac{1}{6} N_{4,\mathbf{t},0}(t)\right] \\ &= -72 N_{2,\mathbf{t},-1}(t) + 66 N_{2,\mathbf{t},-1}(t) - 36 N_{2,\mathbf{t},-1}(t) + 6 N_{2,\mathbf{t},-1}(t), \end{aligned} \tag{6.64}$$

which we illustrate, together with $\Psi^*_{4,-1}$, in Fig.6.12.



(a) (b)

Figure 6.12: (a) $\Psi^*_{4,-1}$ and (b) $\psi^*_{2,-1}$

Comparing $\psi^*_{2,-1}$ given by (6.64) to the the boundary wavelet $\psi_{2,-1}$ of Section 6.3 given by (6.47) and (6.48), we see that they differ by the multiplicative constant -1. This can also be seen by comparing Fig.6.3 and Fig.6.12.

Due to symmetry, the boundary wavelet at 1 is given by $\psi^*_{2,1} = -\psi_{2,1}$. The inner wavelet, as mentioned before, is given by $\psi^*_{2,0} = \frac{1}{8}\psi_{2,0}$.

The quadratic case $m = 3$: (with $n = 5$).

We use the formula (6.58) to obtain

$$\Psi^*_{6,-2}(t) = \sum_{l=-2}^{-1} \alpha_{3,-2,l} N_{6,\mathbf{t},l}(t) + N_6(5)N_6(10t), \tag{6.65}$$

where, by (6.61), $\alpha_{3,-2,-2}$ and $\alpha_{3,-2,-2}$ must satisfy the system of equations

$$\left.\begin{aligned} \alpha_{3,-2,-2}N_{6,\mathbf{t},-2}\left(\tfrac{1}{5}\right) + \alpha_{3,-2,-1}N_{6,\mathbf{t},-1}\left(\tfrac{1}{5}\right) &= -N_6(5)N_6(2), \\ \alpha_{3,-2,-2}N_{6,\mathbf{t},-2}\left(\tfrac{2}{5}\right) + \alpha_{3,-2,-1}N_{6,\mathbf{t},-1}\left(\tfrac{2}{5}\right) &= -N_6(5)N_6(4). \end{aligned}\right\}$$

Solving this system gives $\alpha_{3,-2,-2} = \frac{143}{350}$ and $\alpha_{3,-2,-1} = -\frac{13}{60}$, so that, from (6.65) and (6.62), we have the boundary wavelet

$$\begin{aligned}\psi^*_{3,-2}(t) &= \left(\frac{d}{dt}\right)^3 \left[\frac{143}{350}N_{6,\mathbf{t},-2}(t) - \frac{13}{60}N_{6,\mathbf{t},-1}(t) + \frac{1}{120}N_{6,\mathbf{t},0}(t)\right] \\ &= \frac{28600}{7}\sum_{j=-2}^{3} q_{3,-2,j}N_{3,\mathbf{t},j}(t), \end{aligned} \tag{6.66}$$

where

$$q_{3,-2,j} = \begin{cases} 1, & j=-2, \\ -\frac{107}{88}, & j=-1, \\ \frac{885}{1223}, & j=0, \\ -\frac{989}{3259}, & j=1, \\ \frac{203}{3432}, & j=2, \\ -\frac{7}{3432}, & j=3, \end{cases} \tag{6.67}$$

Observe that by comparing (6.54), (6.55) with (6.66), (6.67), we have $\psi^*_{3,-2} = \frac{27}{7}\psi_{3,2}$.

Again we use the formula (6.58) to obtain

$$\Psi^*_{6,-1}(t) = \sum_{l=-2}^{-1} \alpha_{3,-1,l}N_{6,\mathbf{t},l}(t) + \sum_{l=0}^{2}(-1)^l N_6(l+3)N_6(10t-l), \tag{6.68}$$

where, by (6.61), $\alpha_{3,-1,-2}$ and $\alpha_{3,-1,-2}$ must satisfy the system of equations

$$\left.\begin{aligned} \alpha_{3,-1,-2}N_{6,\mathbf{t},-2}\left(\tfrac{1}{5}\right) + \alpha_{3,-1,-1}N_{6,\mathbf{t},-1}\left(\tfrac{1}{5}\right) &= -N_6(3)N_6(2) + N_6(4)N_6(1) - N_6(5)N_6(0), \\ \alpha_{3,-1,-2}N_{6,\mathbf{t},-2}\left(\tfrac{2}{5}\right) + \alpha_{3,-1,-1}N_{6,\mathbf{t},-1}\left(\tfrac{2}{5}\right) &= -N_6(3)N_6(4) + N_6(4)N_6(3) - N_6(5)N_6(2). \end{aligned}\right\}$$

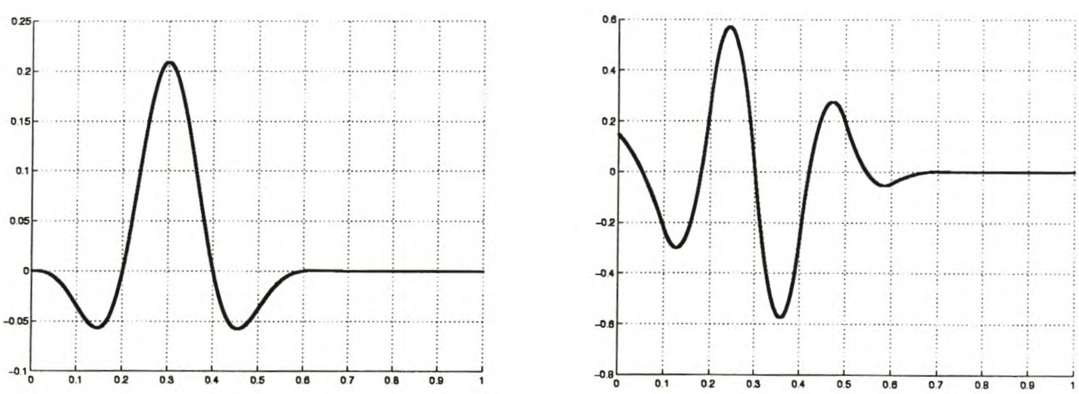
Solving this system gives $\alpha_{3,-1,-2} = -\frac{13}{350}$ and $\alpha_{3,-1,-1} = -\frac{13}{60}$, so that, from (6.68) and (6.62), we have the boundary wavelet

$$\begin{aligned}\psi^*_{3,-1}(t) &= \left(\frac{d}{dt}\right)^3 \left[\frac{13}{350}N_{6,\mathbf{t},-2}(t) - \frac{13}{60}N_{6,\mathbf{t},-1}(t) + \frac{11}{20}N_{6,\mathbf{t},0}(t) \right. \\ &\qquad \left. -\frac{13}{60}N_{6,\mathbf{t},1}(t) + \frac{1}{120}N_{6,\mathbf{t},2}(t)\right] \\ &= -\frac{5200}{3}\sum_{j=-2}^{5} q_{3,-1,j}N_{3,\mathbf{t},j}(t), \end{aligned} \tag{6.69}$$

where

$$q_{3,-1,j} = \begin{cases} \frac{3}{14}, & j=-2, \\ \frac{9}{112}, & j=-1, \\ -\frac{2029}{2912}, & j=0, \\ -\frac{935}{642}, & j=1, \\ -\frac{303}{208}, & j=2, \\ \frac{147}{208}, & j=3, \\ -\frac{29}{208}, & j=4, \\ \frac{1}{208}, & j=5 \end{cases} \tag{6.70}$$

Now by comparing (6.56), (6.57) with (6.69), (6.70), we see that $\psi^*_{3,-1}$ differs from $\psi_{3,2}$ mainly in the fact that the coefficient of $N_{3,\mathbf{t},-2}$ is not zero in (6.70) as it is in (6.57). This means that, since (6.6) implies $N_{3,\mathbf{t},-2}(0) \neq 0$, we have $\psi^*_{3,-1}(0) \neq 0$, whereas, by (6.25), $\psi_{3,-1}(0) = 0$. See Fig.6.13 for the graphs of $\Psi^*_{6,-1}$ and $\psi^*_{3,-1}$.



(a) (b)

Figure 6.13: (a) $\Psi^*_{6,-1}$ and (b) $\psi^*_{3,-1}$

It seems that, whereas the computation of the boundary wavelets, as done in [4], yields structured endpoint behaviour (see (6.25)), the boundary wavelets $\psi^*_{m,j}$, computed in [5], do not have this property for $j = -m+2, \ldots, -1$ and $j = n-m+2, \ldots, n-1$, when $m > 2$.

Structured endpoint behaviour of wavelets is useful in the solution of numerical differential equations, as it simplifies the implementation of boundary conditions.

Once again, we do not study here the decomposition and reconstruction algorithms for the Chui-Quak wavelets, as was obtained in the paper [6].

References

[1] L. CARLESON, *On convergence and growth of partial sums of Fourier series,* Acta Math., **116**, 135-157 (1966).

[2] CHARLES K. CHUI, *An Introduction to Wavelets*, Academic Press, Boston, (1992).

[3] CHARLES K. CHUI, *Wavelets: A Mathematical Tool for Signal Analysis*, Society for Industrial and Applied Mathematics, Philadelphia, (1997).

[4] CHARLES K. CHUI and JOHAN M. DE VILLIERS, *Spline-wavelets with arbitrary knots on a bounded interval: orthogonal decomposition and computational algorithms*, Communications in Applied Analysis **2** (1995), no. 4, 457-486.

[5] CHARLES K. CHUI and EWALD QUAK, *Wavelets on a bounded interval,* Numerical Methods of Approximation Theory, Vol. 6, D. Braess and L. L. Schumaker, eds., Birkhäuser-Verlag, Basel, 53-75 (1992).

[6] EWALD QUAK and NORMAN WEYRICH, *Decomposition and reconstruction algorithms for spline wavelets on a bounded interval*, Applied and Computational Harmonic Analysis **1**, 217-231 (1994).

[7] CHARLES K. CHUI and J.Z. WANG, *On compactly supported spline wavelets and a duality principle*, Trans. Amer. Soc., 330, 903-915 (1992).

[8] RUEL V. CHURCHILL, *Introduction to Complex Variables and Applications*, McGraw-Hill Book Company, Inc., (1948).

[9] INGRID DAUBECHIES, *Ten Lectures on Wavelets*, CBMS Series 61, Society for Industrial and Applied Mathematics, Philadelphia, (1992).

[10] INGRID DAUBECHIES, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math., **41**, 909-996 (1988).

[11] C. DE BOOR, *On calculating with B-splines*, J. Approx. Theory, **6**, 50-62 (1972).

[12] AVNER FRIEDMAN, *Foundations of Modern Analysis*, Holt, Rinehart and Winston, Inc., (1970).

[13] KARIN M. GOOSEN, *Subdivision, Interpolation and Splines*, Master's thesis, University of Stellenbosch, Department of Mathematics, (2000).

[14] CHARLES A. MICHELLI, *Mathematical Aspects of Geometric Modeling*, Society for Industrial and Applied Mathematics, Philadelphia, (1995).

[15] G. NÜRNBERGER, *Approximation by Spline Functions*, Springer-Verlag, Berlin, (1989).

[16] M.J.D. POWELL, *Approximation Theory and Methods*, Cambridge University Press, (1981).

[17] KARL R. STROMBERG, *Introduction to Classical Real Analysis*, Wadsworth International Groups, Belmont, California, (1981).