

# Rank one and finite rank elements of Banach algebras

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## Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

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## Abstract

Let  $A$  be a *unital complex Banach algebra*, which we shall simply refer to as a *Banach algebra*. An element  $u$  in  $A$  is *single* if  $xuy = 0$ , where  $x, y \in A$ , implies that  $xu = 0$  or  $uy = 0$ . We say that  $u$  *acts compactly* on  $A$  if the operator  $x \mapsto uxu$  is compact. For an element  $x \in A$  the set  $\text{Sp}(x) = \{\lambda \in \mathbb{C} : \lambda - x \text{ is not invertible in } A\}$  is called the *spectrum* of  $x$  in  $A$ . The notation  $\#\text{Sp}(x)$  indicates the number of points in  $\text{Sp}(x)$  and  $\#\text{Sp}'(x)$  denotes the number of non-zero points in  $\text{Sp}(x)$ .

In 1978 J. Puhl introduced and studied *rank one elements* of *semiprime Banach algebras*. He gave the following definition for a rank one element: *A non-zero element  $u$  in a semiprime Banach algebra  $A$  is a rank one element if there exists a linear functional  $f_u$  on  $A$  such that  $uxu = f_u(x)u$  for all  $x \in A$ .* In the same paper he defined *finite rank elements* as the finite sums of the rank one elements in the preceding definition, together with 0. At about the same time, J.A. Erdos, S. Giotopoulos and M.S. Lambrou introduced another definition of rank one elements in semiprime Banach algebras, for which the following is an equivalent formulation: *A non-zero element  $u$  in a semiprime Banach algebra  $A$  is rank one if and only if  $u$  is single and acts compactly on  $A$ .* Since then various other authors have contributed to the topics of rank one and finite rank elements, yielding several characterizations and another definition of rank one elements: *An element  $u$  in a semiprime Banach algebra  $A$  is a rank one element if  $\#\text{Sp}'(xu) \leq 1$  for all  $x \in A$ .* This led to another definition of a finite rank element: *An element  $u$  in a semiprime Banach algebra  $A$  is a finite rank element if there exists a positive integer  $n$  such that  $\#\text{Sp}'(xu) \leq n$  for all  $x \in A$ .*

The purpose of this thesis is to study the relationship among the three notions of rank one and the relationship between the two concepts of finite rank. Some consequences of these relationships will be discussed. An application of rank one elements to a perturbation result of B. Aupetit will also be included.

## Opsomming

Laat  $A$  'n komplekse unitêre Banach algebra wees, waarna ons bloot as 'n Banach algebra sal verwys. 'n Element  $u \in A$  is enkel as  $xuy = 0$ , waar  $x, y \in A$ , impliseer dat  $xu = 0$  of  $uy = 0$ . Ons sê dat  $u$  kompak op  $A$  inwerk as die operator  $x \mapsto uxu$  kompak is. Vir 'n element  $x \in A$  word die versameling  $\text{Sp}(x) = \{\lambda \in \mathbb{C} : \lambda - x \text{ nie inverteerbaar in } A \text{ is nie}\}$  die spektrum van  $x$  in  $A$  genoem. Die notasie  $\#\text{Sp}(x)$  dui die aantal punte in  $\text{Sp}(x)$  aan en  $\#\text{Sp}'(x)$  dui die aantal nienul punte in  $\text{Sp}(x)$  aan.

In 1978 het J. Puhl die rang-een-elemente van semipriem Banach algebras gedefinieer en bestudeer. Hy het die volgende definisie vir 'n rang-een-element gegee: 'n Nienul element  $u$  in 'n semipriem Banach algebra  $A$  is 'n rang-een-element as daar 'n lineêre funksionaal  $f_u$  op  $A$  bestaan sodat  $uxu = f_u(x)u$  vir alle  $x \in A$ . In dieselfde artikel het hy eindige-rang-elemente as die eindige somme van die rang-een-elemente in die voorafgaande definisie, tesame met 0 gedefinieer. Op min of meer dieselfde tyd het J.A. Erdos, S. Giotopoulos en M.S. Lambrou 'n ander definisie van rang-een-elemente in semipriem Banach algebras geskep, waarvan die volgende 'n ekwivalente formulering is: 'n Nienul element  $u$  in 'n semipriem Banach algebra  $A$  is 'n rang-een-element as en slegs as  $u$  enkel is en kompak op  $A$  inwerk. Sedertdien het 'n aantal ander outeurs bygedra tot die onderwerpe van rang-een- en eindige-rang-elemente, wat verskeie karakteriserings tot gevolg gehad het, asook nog 'n definisie van rang-een-elemente: 'n Element  $u$  in 'n semipriem Banach algebra  $A$  is 'n rang-een-element as  $\#\text{Sp}'(xu) \leq 1$  vir alle  $x \in A$ . Dit het tot nog 'n definisie van 'n eindige-rang-element gelei: 'n Element  $u$  in 'n semipriem Banach algebra  $A$  is 'n eindige-rang-element as daar 'n positiewe heelgetal  $n$  bestaan sodat  $\#\text{Sp}'(xu) \leq n$  vir alle  $x \in A$ .

Die doel van hierdie tesis is om die verwantskap tussen die drie konsepte van rang een en die verwantskap tussen die twee konsepte van eindige rang te bestudeer. Sommige gevolge van hierdie verwantskappe sal bespreek word. Verder sal 'n toepassing van rang-een-elemente op 'n versteuringsresultaat van B. Aupetit ingesluit word.

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# Introduction

Let  $A$  be a *unital complex Banach algebra*, which we will simply call a *Banach algebra*. The set of all  $x$  such that  $1 - zx$  is invertible for all  $z \in A$  is called the *radical* of  $A$ , denoted by  $\text{Rad}(A)$ . If  $\text{Rad}(A) = \{0\}$  then  $A$  is called a *semisimple* Banach algebra. An example of a semisimple Banach algebra is the algebra  $\mathbb{C}$  of complex numbers. A Banach algebra is said to be *semiprime* if  $I = \{0\}$  is the only two-sided ideal of  $A$  with the property  $I^2 = \{0\}$ . Semisimple Banach algebras are semiprime, but the converse is not true in general. If the sum of the minimal left ideals coincides with the sum of the minimal right ideals, it is called the *socle* of  $A$ , denoted by  $\text{Soc}(A)$ . The socle is a two-sided ideal. For an idempotent  $p$  in a Banach algebra  $A$ , the set  $pAp$  is an algebra. An idempotent  $p$  in a Banach algebra  $A$  is a *minimal idempotent* if  $pAp$  is division algebra, where a division algebra is an algebra in which every non-zero element is invertible. The *spectrum* of an element  $x$  in a Banach algebra  $A$  is the set  $\text{Sp}(x) = \{\lambda \in \mathbb{C} : \lambda - x \text{ is not invertible in } A\}$ , where we write  $\lambda$  for the element  $\lambda 1 \in A$ . The *polynomially convex hull* of  $\text{Sp}(x)$ , denoted by  $\sigma(x)$ , is the union of  $\text{Sp}(x)$  and the holes of  $\text{Sp}(x)$ , where a hole of  $\text{Sp}(x)$  is a bounded component of  $\mathbb{C} \setminus \text{Sp}(x)$ . The notation  $\#\text{Sp}(x)$  indicates the number of points in  $\text{Sp}(x)$ , while  $\#\text{Sp}'(x)$  denotes the number of non-zero points in  $\text{Sp}(x)$ . An element  $u$  in a Banach algebra  $A$  is said to be *single* if  $xuy = 0$ , where  $x, y \in A$ , implies that  $xu = 0$  or  $uy = 0$ . We say that an element  $u$  in a Banach algebra  $A$  *acts compactly* on  $A$  if the operator  $x \mapsto uxu$  is compact.

We call a non-zero element  $u$  in a semiprime Banach algebra  $A$  a *compactly rank one element* of  $A$  if  $u$  is single and acts compactly on  $A$ . The set of compactly rank one elements of  $A$  is denoted by  $\mathcal{E}_1$ . A non-zero element  $u$  in a semiprime Banach algebra  $A$  is called a *spatially rank one element* of  $A$  if there exists a linear functional  $f_u$  on  $A$  such that  $uxu = f_u(x)u$  for all  $x \in A$ . We will denote the spatially rank one elements by  $\mathcal{F}_1$ . An element  $u$  in a semiprime Banach algebra  $A$  is a *spatially finite rank* element if  $u = 0$  or a finite sum of spatially rank one elements. The notation  $\mathcal{F}$  indicates the set of spatially finite rank elements. We call a non-zero element  $u$  in a semiprime Banach algebra  $A$  a *spectrally rank one element* of  $A$  if  $\#\text{Sp}'(xu) \leq 1$  for all  $x \in A$ . The spectrally rank one elements will be denoted by  $\mathcal{G}_1$ . If  $u$  is an element in a semiprime Banach algebra  $A$ , we say that  $u$  is *spectrally of finite rank* if there exists a positive integer  $n$  such that  $\#\text{Sp}'(xu) \leq n$  for all



$x \in A$ . We will use the notation  $\mathcal{G}$  to denote the set of spectrally finite rank elements.

In 1978 J. Puhl showed that every minimal idempotent in a semiprime Banach algebra  $A$  is an element of  $\mathcal{F}_1$  (see Lemma 2.2.1). In the same paper, he showed that the set  $\mathcal{F}_1$  absorbs all non-zero products with elements of  $A$  (see Theorem 2.1.18). At about the same time J.A. Erdos, S. Giotopoulos and M.S. Lambrou proved the following result: If  $u$  is an element in a semisimple Banach algebra  $A$  and if  $u \in \mathcal{E}_1$ , then there exists a minimal idempotent  $p \in A$  such that  $u = up$  (see Theorem 4.3.1). What this result means, in the light of the previously mentioned results, is that in semisimple Banach algebras, the inclusion  $\mathcal{E}_1 \subset \mathcal{F}_1$  is true. Puhl also proved that in a semiprime Banach algebra, the spectrum of every element in  $\mathcal{F}_1$  contains at most one non-zero point (see Theorem 2.3.1). Combining this with the fact that  $\mathcal{F}_1$  absorbs all non-zero products with the elements of  $A$ , we get the inclusion  $\mathcal{F}_1 \subset \mathcal{G}_1$ . R.M. Brits, L. Lindeboom and H. Raubenheimer showed that this inclusion may in general be strict (see Corollary 3.1.5). In 1995 R. Harte proved that in semisimple Banach algebras,  $\mathcal{G}_1 \subset \mathcal{F}_1$  ([14], Theorem 4). In an earlier paper, T. Mouton and H. Raubenheimer gave a spectral characterization of spatially rank one elements. The proof of their result actually implies Harte's result ([19], Theorem 2.2). In 2003 Brits, Lindeboom and Raubenheimer obtained the following result relating the three sets of rank one elements in semisimple Banach algebras: *If  $A$  is a semisimple Banach algebra, then  $\mathcal{E}_1 = \mathcal{F}_1 = \mathcal{G}_1$*  (see Theorem 4.3.2). In 1993 Mouton and Raubenheimer gave a spectral characterization of elements of  $\mathcal{F}$  in semisimple Banach algebras. In their proof, they implicitly showed that  $\mathcal{F} = \mathcal{G}$ .

In this thesis we study some properties of rank one and finite rank elements and we examine the relationships among the various versions of rank one and finite rank elements. Some consequences of these relationships will be given. We also include an application of rank one elements to a perturbation result of B. Aupetit.

This thesis is organized into five chapters. Chapter 1 reviews some Banach algebra theory that will be needed in the rest of the document. As such, proofs of well known results that can be found in standard Banach algebra theory texts will generally be omitted. Proofs of results from papers will generally be included, as some may not be easily accessible to the reader.

In Chapter 2 we develop a theory on spatially rank one and spatially finite rank elements of semiprime Banach algebras. We start this chapter by

giving some basic properties of spatially rank one elements. The main results here are Example 2.1.14 and Example 2.1.17. Example 2.1.14 says that in the Banach algebra  $A = \mathcal{L}(X)$  of bounded linear operators on a Banach space  $X$ , the spatially rank one elements are the one-dimensional operators. Example 2.1.17 says that the spatially rank one elements in the Banach algebra  $M_2(\mathbb{C})$  of complex two by two matrices are the non-invertible elements.

In Section 2.2 we study the relationship between spatially rank one elements, minimal idempotents and minimal left ideals. According to Lemma 2.2.1, minimal idempotents in a semiprime Banach algebra are contained in  $\mathcal{F}_1$ . Theorem 2.2.3 tells us that if  $L$  is any minimal left ideal of  $A$ , then  $L \setminus \{0\} \subset \mathcal{F}_1$ .

Section 2.3 gives the spectral properties of spatially rank one elements. The main results in this section are Theorem 2.3.1 and Theorem 2.3.6. Theorem 2.3.1, due to J. Puhl, says that the spectrum of a spatially rank one element contains at most one non-zero point. We will state and prove this result slightly differently from the way it was originally stated and proved by Puhl. Theorem 2.3.6, by T. Mouton and H. Raubenheimer, states that a spatially rank one element  $u$  in a semiprime Banach algebra  $A$  satisfies the property  $\text{Sp}(b + s_0u) \cap \text{Sp}(b + s_1u) \subset \text{Sp}(b)$  for all  $b \in A$  and for any pair of distinct non-zero scalars  $s_0$  and  $s_1$ .

In Section 2.4 we study the relationship between spatially rank one elements and one-dimensional operators. The most important result in this section is Theorem 2.4.4, done by Puhl. It says that a non-zero element  $u$  in a semiprime Banach algebra  $A$  is in  $\mathcal{F}_1$  if and only if the operator  $x \mapsto xux$  is one-dimensional.

Section 2.5 gives an important topological property of the set  $\mathcal{F}_1$ . Theorem 2.5.3 and Theorem 2.5.9, both due to R.M. Brits, L. Lindeboom and H. Raubenheimer, are the main results in this section. Theorem 2.5.3 says that if  $A$  is a semiprime Banach algebra with  $\mathcal{F}_1 \neq \emptyset$  and  $u \in \mathcal{F}_1$ , then the connected component of  $\mathcal{F}_1$  containing  $u$  is the set  $\text{Exp}(A)u\text{Exp}(A)$ . In Theorem 2.2.3 we get that every minimal left ideal  $J$  of a semiprime Banach algebra  $A$  is of the form  $J = Au$ , where  $u \in \mathcal{F}_1$ . Theorem 2.5.9, which is a consequence of Theorem 2.5.3, gives a simplification of this characterization of minimal left ideals. It says that every minimal left ideal  $J$  of a semiprime Banach algebra  $A$  is of the form  $J = \text{Exp}(A)u \cup \{0\}$ .

We conclude Chapter 2 with Section 2.6, where we discuss the relationship between spatially finite rank elements and finite-dimensional operators. The

main result here is Theorem 2.6.9, due to Puhl: *An element  $u$  in a semiprime Banach algebra  $A$  is in  $\mathcal{F}$  if and only if the operator  $x \mapsto uxu$  is finite-dimensional.*

In Chapter 3 we study spectrally rank one and spectrally finite rank elements of Banach algebras. Section 3.1 gives some important properties of spectrally rank one elements. There are two main results in this section. Theorem 3.1.3 characterizes semisimple Banach algebras containing invertible spectrally rank one elements as the Banach algebras isomorphic to  $\mathbb{C}$ . Theorem 3.1.4 says that in semiprime Banach algebras,  $\mathcal{F}_1 \subset \mathcal{G}_1$ .

We end Chapter 3 with Section 3.2, which gives us some important properties of spectrally finite rank elements. The main result here is Corollary 3.2.4, which gives the following characterization of spectrally finite rank elements: *An element  $u$  in a semiprime Banach algebra  $A$  is in  $\mathcal{G}$  if and only if there exists a positive integer  $n$  such that  $\bigcap_{i=0}^n \text{Sp}(x + s_i u) \subset \text{Sp}(x)$  for every  $x \in A$  and for any set of distinct non-zero scalars  $\{s_i : i = 0, 1, \dots, n\}$ .* T. Mouton and H. Raubenheimer proved a similar result with  $\mathcal{F}$  in place of  $\mathcal{G}$ . As such, the proof of Corollary 3.2.4 is slightly different from theirs, although it follows much along the same lines.

Chapter 4 is about relationships among the various concepts of rank one and finite rank elements. It opens with Section 4.1, which gives the relationship between the sets  $\mathcal{F}_1$  and  $\mathcal{G}_1$  in semisimple Banach algebras. According to Theorem 4.1.4, in a semisimple Banach algebra,  $\mathcal{F}_1 = \mathcal{G}_1$ . We prove this result by using the fact that  $\mathcal{F}_1 \subset \mathcal{G}_1$  from Theorem 3.1.4, and then obtain the inclusion  $\mathcal{G}_1 \subset \mathcal{F}_1$  from the proof of ([19], Theorem 2.2) by T. Mouton and H. Raubenheimer. This result relies on the scarcity lemma of B. Aupetit ([2], Theorem 3.4.25). In 1995 R. Harte gave an alternative proof which does not use the scarcity lemma. This is Theorem 4.1.5, which relies on Theorem 4.1.2. We give a more elegant proof of Theorem 4.1.2 than the one given by Harte.

In Section 4.2 we study the relationship between  $\mathcal{F}_1$  and  $\mathcal{G}_1$  in semiprime Banach algebras which are not semisimple. The work on this was done by R.M. Brits, L. Lindeboom and H. Raubenheimer.

Section 4.3 gives the relationship among all three sets of rank one elements in semisimple Banach algebras. The main result in this section is Theorem 4.3.2, due to Brits, Lindeboom and Raubenheimer. It says that in semisimple Banach algebras,  $\mathcal{E}_1 = \mathcal{F}_1 = \mathcal{G}_1$ .

We conclude Chapter 4 with Section 4.4, which gives the relationship between the sets  $\mathcal{F}$  and  $\mathcal{G}$ . The main results here are Theorem 4.4.1 and Theorem 4.4.5. Theorem 4.4.1 says that in a semisimple Banach algebra,  $\mathcal{F} = \mathcal{G}$ . We prove this result using part of the proof of ([19], Theorem 3.1) by Mouton and Raubenheimer. Theorem 4.4.5, done by Aupetit and Mouton, characterizes spatially finite rank elements in terms of the polynomially convex hull of the spectrum.

Chapter 5 is about an application of rank one elements to Aupetit's perturbation theorem. In 1994 T. Mouton gave an alternative proof of Aupetit's perturbation result. His proof uses more elementary tools than the original proof by Aupetit. Rank one elements are the cornerstone of Mouton's proof. In this chapter we give the main result that Mouton used to prove Aupetit's theorem. We adopt the version in [14]. We begin this chapter by stating a result from complex analysis that is used in the proof of Mouton's result. We then move on to Section 5.2, where we prove the result: *If  $a$  is any element in a semiprime Banach algebra  $A$  and if  $u \in \mathcal{F}_1$  then  $\text{acc Sp}(a+u) \subset \sigma(a)$  and  $\text{acc Sp}(a) \subset \sigma(a+u)$ , where  $\text{acc Sp}(a+u)$  is the set of all limit points of  $\text{Sp}(a+u)$ .*

In this thesis definitions, theorems and other results are numbered successively. By Theorem 2.3.1 we mean result 1 of Section 3 of Chapter 2. The symbol  $\square$  indicates the end of a proof. We use the notation  $A \subsetneq B$  to indicate that set  $A$  is properly contained in set  $B$ . By  $cl_A(B)$  we mean the closure of the set  $B$  relative to the set  $A$ .

# Chapter 1

## Preliminaries

This chapter is intended to review some basic Banach algebra theory that will be needed in the remainder of the text. As such, proofs of results that can be found in standard texts on Banach algebra theory will generally be omitted. The proofs of results coming from papers will be given, as the reader may not easily have access to some of the papers. It will be assumed that the reader has knowledge of basics in real analysis, complex analysis, functional analysis, ring theory and topology.

### 1.1 Introducing Banach algebras

**Definition 1.1.1** *A complex algebra  $A$  is a vector space over  $\mathbb{C}$  such that for each ordered pair  $x, y \in A$ , a unique product  $xy \in A$  is defined such that for all  $x, y, z \in A$  and for  $\lambda \in \mathbb{C}$ , the following properties are satisfied:*

$$(i) \quad x(yz) = (xy)z,$$

$$(ii) \quad (x + y)z = xz + yz,$$

$$(iii) \quad x(y + z) = xy + xz,$$

$$(iv) \quad \lambda(xy) = (\lambda x)y = x(\lambda y).$$

We say that  $A$  is a *unital complex Banach algebra* if  $A$  is a complex algebra and a Banach space with a norm  $\|\cdot\|$  which satisfies  $\|xy\| \leq \|x\|\|y\|$  for all  $x, y \in A$  and  $\|1\| = 1$ , where  $1$  is the unit of  $A$ . In this text, we will be dealing

only with unital complex Banach algebras. These will simply be referred to as Banach algebras throughout the text.

**Definition 1.1.2** *A Banach algebra  $A$  is said to be commutative if  $xy = yx$  for all  $x, y \in A$ . Otherwise it is non-commutative.*

The following are examples of Banach algebras.

**Example 1.1.3** ([2], Example 5, p.32) *Let  $X$  be a Banach space. Then, with the usual operator norm, the algebra  $\mathcal{L}(X)$  of bounded linear operators on  $X$  is a Banach algebra. If  $\dim(X) > 1$ , then  $\mathcal{L}(X)$  is a non-commutative Banach algebra.*

**Example 1.1.4** ([2], p.32) *Let  $M_2(\mathbb{C})$  be the algebra of all complex two by two matrices. Then, with the norm  $\|M\| = \sup\{\|Mx\| : x \in \mathbb{C}^2, \|x\| \leq 1\}$ , where the norm in  $\mathbb{C}^2$  is the Euclidean norm,  $M_2(\mathbb{C})$  is a non-commutative Banach algebra.*

The simplest commutative Banach algebra is  $\mathbb{C}$ .

**Definition 1.1.5** *A subset  $B$  of a Banach algebra  $A$  is called a subalgebra of  $A$  if  $B$  is closed under addition, multiplication and scalar multiplication.*

The following lemma will be needed in Chapter 2.

**Lemma 1.1.6** *If  $A$  is a Banach algebra and if  $a \in A$  is such that  $\dim(aAa) = n < \infty$ , then  $\dim(axAax) \leq n$  for all  $x \in A$ .*

**Proof.** Let  $a \in A$  such that  $\dim(aAa) = n < \infty$ . Then for any  $x \in A$  we have that  $axAa \subset aAa$ , so that  $\dim(axAa) \leq n$ . If  $\{y_1, \dots, y_m\}$  is a basis for  $axAa$ , then  $\{y_1x, \dots, y_mx\}$  spans  $axAax$ , so that  $\dim(axAax) \leq n$ .  $\square$

An element  $x$  in a Banach algebra  $A$  is said to be *invertible* in  $A$  if there exists a  $y \in A$  such that  $yx = xy = 1$ . The set of invertible elements of  $A$  will be denoted by  $A^{-1}$ . The set  $A^{-1}$  is open.

**Proposition 1.1.7** ([2], Theorem 3.2.1) *Suppose that  $A$  is a Banach algebra,  $x \in A$  and  $\|x\| < 1$ . Then  $1 - x$  is invertible and  $(1 - x)^{-1} = \sum_{k=0}^{\infty} x^k$ .*

**Definition 1.1.8** An element  $p$  in a Banach algebra  $A$  is called an idempotent if  $p^2 = p$ .

**Definition 1.1.9** Two idempotents  $p$  and  $q$  in a Banach algebra  $A$  are called orthogonal idempotents if  $pq = qp = 0$ . A set  $\{p_i : i \in I\}$  of idempotents in  $A$  is said to be orthogonal if  $p_i p_j = p_j p_i = 0$  for all  $i \neq j$ .

**Definition 1.1.10** Let  $A$  be a Banach algebra. The centre of  $A$  is the set  $Z(A) = \{y \in A : yx = xy \text{ for all } x \in A\}$ .

**Definition 1.1.11** A Banach algebra  $A$  is said to be a division algebra if every non-zero element of  $A$  is invertible.

If  $p$  is an idempotent in a Banach algebra  $A$ , then  $pAp$  is an algebra with unit  $p$ .

**Definition 1.1.12** An idempotent  $p$  in a Banach algebra  $A$  is called a minimal idempotent if  $p \neq 0$  and  $pAp$  is a division algebra.

**Definition 1.1.13** An element  $x$  in a Banach algebra  $A$  is said to be a divisor of zero if there exists  $0 \neq y \in A$  such that  $yx = 0$  or  $xy = 0$ .

**Proposition 1.1.14** ([19], p.213) Suppose that  $A$  is a Banach algebra with no non-zero divisors of zero. Then the only idempotents in  $A$  are 0 and 1.

**Proof.** If  $e$  is an idempotent in  $A$ , then  $e(e-1) = 0$ . Since  $A$  has no non-zero divisors of zero, it follows that either  $e = 0$  or  $e = 1$ . ■

**Lemma 1.1.15** ([13], Lemma 2) If  $A$  is a finite-dimensional Banach algebra with no non-zero divisors of zero, then  $A$  is a division algebra.

**Proof.** Let  $x \neq 0$  be any element of  $A$ . Define  $L_x : A \rightarrow A$  by  $L_x y = xy$ . Clearly,  $L_x$  is a linear operator. Also,  $\|L_x y\| = \|xy\| \leq \|x\| \|y\|$  for all  $y \in A$ , so that  $L_x$  is bounded. Therefore  $L_x \in \mathcal{L}(A)$ . Since  $A$  is finite-dimensional, we have that  $\mathcal{L}(A)$  is finite-dimensional, say  $\dim(\mathcal{L}(A)) = n$ . This implies that the set  $\{I = L_x^0, L_x, \dots, L_x^n\}$  is linearly dependent, where  $I$  is the identity in  $\mathcal{L}(A)$ . This means that there exist scalars  $\alpha_k$ , not all zero, such that  $\sum_{k=0}^n \alpha_k L_x^k = 0$ . Hence there exists a minimal polynomial  $q$  such that  $q(L_x) = 0$ .

We show that  $q$  has non-zero constant term. To this end suppose that  $q$  has zero constant term. Then  $q$  can be written as

$$q(L_x) = L_x q_0(L_x), \quad (1.1.16)$$

where  $q_0$  is a polynomial of degree less than that of  $q$ . Since  $q$  is a minimal polynomial, this implies that  $q_0(L_x) \neq 0$ . Hence there exists a  $y \in A$  such that  $q_0(L_x)y \neq 0$ . But from (1.1.16) we have that  $xq_0(L_x)y = L_x q_0(L_x)y = q(L_x)y = 0$  for all  $y \in A$ . This contradicts the fact that  $A$  has no non-zero divisors of zero. Therefore  $q$  has a non-zero constant term, say  $\beta_0 I + \beta_1 L_x + \cdots + \beta_m L_x^m = q(L_x) = 0$ , where  $\beta_0 \neq 0$ . Therefore

$$I = -\frac{1}{\beta_0}(\beta_1 I + \beta_2 L_x + \cdots + \beta_m L_x^{m-1})L_x = p_0(L_x)L_x = L_x p_0(L_x),$$

where  $p_0(L_x) = -\frac{1}{\beta_0}(\beta_1 I + \beta_2 L_x + \cdots + \beta_m L_x^{m-1})$ . Hence  $1 = I1 = (p_0(L_x)L_x)1 = p_0(L_x)x$  and from

$$\begin{aligned} p_0(L_x)x &= -\frac{1}{\beta_0}(\beta_1 I + \beta_2 L_x + \cdots + \beta_m L_x^{m-1})x \\ &= -\frac{1}{\beta_0}(\beta_1 x + \beta_2 L_x x + \cdots + \beta_m L_x^{m-1}x) \\ &= -\frac{1}{\beta_0}(\beta_1 x + \beta_2 x^2 + \cdots + \beta_m x^m) \\ &= -\frac{1}{\beta_0}(\beta_1 + \beta_2 x + \cdots + \beta_m x^{m-1})x \\ &= p_0(x)x \\ &= xp_0(x), \end{aligned}$$

we have that  $1 = xp_0(x) = p_0(x)x$ , so that  $x$  is invertible with inverse  $p_0(x)$ . Since  $x$  is an arbitrary non-zero element of  $A$ , this implies that  $A$  is a division algebra.  $\square$

**Definition 1.1.17** *An element  $s$  of a Banach algebra  $A$  is called a single element of  $A$  if  $xsy = 0$ , where  $x, y \in A$ , implies that  $xs = 0$  or  $sy = 0$ .*

**Definition 1.1.18** *Let  $a$  be an element of a Banach algebra  $A$ . We say that  $a$  acts compactly on  $A$  if the map  $x \mapsto axa$  is compact.*



**Lemma 1.1.19** ([13], Lemma 1) *Let  $a$  be any element of a Banach algebra  $A$ .*

(i) *If  $s$  is a single element of  $A$ , then  $as$  and  $sa$  are also single elements of  $A$ .*

(ii) *If  $s$  acts compactly on  $A$ , then  $as$  and  $sa$  also act compactly on  $A$ .*

**Proof.** (i) Suppose that  $x(as)y = 0$  for some  $x, y \in A$ . Then, since  $s$  is single, we have that  $xas = 0$  or  $sy = 0$ . This implies that  $xas = 0$  or  $asy = 0$ , so that  $as$  is single. A similar argument can be used to show that  $sa$  is single.

(ii) Define the map  $f : A \rightarrow A$  by  $x \mapsto asxas$ . Then this is the composition of the maps  $x \xrightarrow{g} xa \xrightarrow{h} sxas \xrightarrow{k} asxas$ . Since  $\|g(x)\| = \|xa\| \leq \|x\|\|a\|$ , we have that  $g$  is bounded. Similarly,  $k$  is bounded. The map  $h$  is compact because  $s$  acts compactly on  $A$ . Since  $f = k \circ h \circ g$ , we have that  $f$  is compact. Therefore  $as$  acts compactly on  $A$ . Similarly,  $sa$  acts compactly on  $A$ .  $\square$

In Chapter 2, we will need to use Lemma 1.1.21. In order to prove it we need the following result.

**Proposition 1.1.20** *Let  $A$  be a Banach algebra. If  $B$  is a subset of  $A$  which is closed under addition and scalar multiplication, then  $B \setminus \{0\}$  is pathwise connected.*

**Proof.** Let  $a, b \in B \setminus \{0\}$ , with  $a \neq b$ , and let  $L_{a,b} = \{\alpha a + (1 - \alpha)b : \alpha \in [0, 1]\}$ . If  $0 \neq \alpha a + (1 - \alpha)b$  for all  $\alpha \in \mathbb{R}$ , then  $L_{a,b} \subset B \setminus \{0\}$ . Define  $\phi : [0, 1] \rightarrow B \setminus \{0\}$  by  $\phi(\alpha) = \alpha a + (1 - \alpha)b$ . Then  $\phi$  is a continuous function connecting  $a$  and  $b$ , with  $\phi([0, 1]) = L_{a,b} \subset B \setminus \{0\}$ . This means that  $B \setminus \{0\}$  is pathwise connected.

Now suppose that  $0 = \beta a + (1 - \beta)b$  for some  $\beta \in \mathbb{R}$ . Choose a  $c \in B$  such that  $c \neq \alpha a + (1 - \alpha)b$  for all  $\alpha \in \mathbb{R}$ . To show that such a  $c$  exists define  $c = ia + (1 - i)b$ , where  $i^2 = -1$ . If there is an  $\alpha_0 \in \mathbb{R}$  such that  $c = \alpha_0 a + (1 - \alpha_0)b$ , then  $(i - \alpha_0)a = (1 - \alpha_0 - (1 - i))b = (i - \alpha_0)b$ , so that  $a = b$ , which is a contradiction. Now, we have that  $L_{a,c}, L_{c,b} \subset B$ . We show that  $L_{a,c} \cup L_{c,b} \subset B \setminus \{0\}$ . If  $0 \in L_{a,c}$  then there exists an  $\alpha_0 \in [0, 1]$  such that  $0 = \alpha_0 a + (1 - \alpha_0)c$ . From  $0 = \beta a + (1 - \beta)b$ , it follows that  $\beta a + (1 - \beta)b = \alpha_0 a + (1 - \alpha_0)c$ . This implies that  $c = \left(\frac{\beta - \alpha_0}{1 - \alpha_0}\right)a + \left(\frac{1 - \beta}{1 - \alpha_0}\right)b$ . Let  $\gamma = \frac{\beta - \alpha_0}{1 - \alpha_0}$ . Then  $1 - \gamma = \frac{1 - \beta}{1 - \alpha_0}$ . Therefore  $c = \gamma a + (1 - \gamma)b$ , which contradicts

the fact that  $c \neq \alpha a + (1 - \alpha)b$  for all  $\alpha \in \mathbb{R}$ . Hence  $0 \notin L_{a,c}$ . Similarly,  $0 \notin L_{c,b}$ . This means that  $L_{a,c} \cup L_{c,b} \subset B \setminus \{0\}$ . Define  $\phi_1(\alpha) = \alpha a + (1 - \alpha)c$  and  $\phi_2(\alpha) = \alpha c + (1 - \alpha)b$ , where  $\alpha \in [0, 1]$ . Then  $\phi_1$  and  $\phi_2$  are continuous functions connecting  $a$  with  $c$ , and  $c$  with  $b$  respectively. Also,  $\phi_1([0, 1]) = L_{a,c}$  and  $\phi_2([0, 1]) = L_{c,b}$ . Let  $\phi = \phi_1 \cup \phi_2$ . Then  $\phi$  is a path in  $B \setminus \{0\}$  connecting  $a$  and  $b$ . Hence  $B \setminus \{0\}$  is pathwise connected.  $\square$

**Lemma 1.1.21** *Let  $a$  be a non-zero element in a Banach algebra  $A$ . Then the set  $AaA \setminus \{0\}$  is pathwise connected.*

**Proof.** Let  $aub, xuy \in AuA \setminus \{0\}$ . Then  $au \neq 0$  and  $xu \neq 0$ . Let  $B = auA$ . Clearly,  $B$  is closed under addition and scalar multiplication. It follows from Proposition 1.1.20 that there exists a path  $\phi_1$  in  $B \setminus \{0\} \subset AuA \setminus \{0\}$  connecting  $aub$  and  $au$ . Similarly there is a path  $\phi_2$  in  $Au \setminus \{0\} \subset AuA \setminus \{0\}$  connecting  $au$  and  $xu$ . Also, there is a path  $\phi_3$  in  $xuA \setminus \{0\} \subset AuA \setminus \{0\}$  connecting  $xu$  and  $xuy$ . Let  $\phi = \phi_1 \cup \phi_2 \cup \phi_3$ . Then  $\phi$  is a path in  $AuA \setminus \{0\}$  connecting  $aub$  and  $xuy$ . This implies that  $AuA \setminus \{0\}$  is pathwise connected.  $\square$

## 1.2 Ideals of a Banach algebra

**Definition 1.2.1** *A non-empty subset  $I$  of a Banach algebra  $A$  is said to be a left ideal of  $A$  if  $I$  satisfies the following properties:*

- (i) *If  $a, b \in I$ , then  $a + b \in I$ ,*
- (ii) *If  $a \in I$  and  $\lambda \in \mathbb{C}$ , then  $\lambda a \in I$ ,*
- (iii) *If  $a \in I$  and  $x \in A$ , then,  $xa \in I$ .*

*A non-empty subset  $I$  of a Banach algebra  $A$  is a right ideal of  $A$  if  $I$  satisfies the following properties:*

- (i) *If  $a, b \in I$ , then  $a + b \in I$ ,*
- (ii) *If  $a \in I$  and  $\lambda \in \mathbb{C}$ , then  $\lambda a \in I$ ,*
- (iii) *If  $a \in I$  and  $x \in A$ , then,  $ax \in I$ .*

We call  $I$  a *two-sided ideal* of  $A$  if  $I$  is both a left and a right ideal of  $A$ .

**Definition 1.2.2** *Let  $A$  be a Banach algebra. A proper left ideal  $M$  of  $A$  is called a maximal left ideal of  $A$  if there is no proper left ideal  $I$  of  $A$  such that  $M \subsetneq I \subsetneq A$ . A non-zero left ideal  $M$  of  $A$  is called a minimal left ideal of  $A$  if there is no left ideal  $I$  of  $A$  such that  $\{0\} \subsetneq I \subsetneq M$ .*

*Maximal right ideals and minimal right ideals are defined in a similar manner. The same applies for maximal two-sided ideals and minimal two-sided ideals.*

**Proposition 1.2.3** ([2], Lemma 3.1.1) *Let  $A$  be a Banach algebra. Then every left (right) ideal of  $A$  is contained in a maximal left (right) ideal of  $A$ .*

**Theorem 1.2.4** ([2], Theorem 3.1.3) *Let  $A$  be a Banach algebra. Then the following sets are identical:*

- (i) *the intersection of all maximal left ideals of  $A$ ,*
- (ii) *the intersection of all maximal right ideals of  $A$ ,*
- (iii) *the set of all  $x$  such that  $1 - zx$  is invertible in  $A$ , for all  $z \in A$ ,*
- (iv) *the set of all  $x$  such that  $1 - xz$  is invertible in  $A$ , for all  $z \in A$ .*

The set having the properties (i)–(iv) is called the *radical* of  $A$ , denoted by  $\text{Rad}(A)$ . It is clear that  $\text{Rad}(A)$  is a two-sided ideal of  $A$ .

**Proposition 1.2.5** ([2], Corollary 3.2.2) *If  $A$  is a Banach algebra, then  $\text{Rad}(A)$  is closed in  $A$ .*

**Definition 1.2.6** *A Banach algebra  $A$  is called semisimple if  $\text{Rad}(A) = \{0\}$ .*

An example of a semisimple Banach algebra is  $\mathcal{L}(X)$ , the algebra of bounded linear operators on a Banach space  $X$ .

**Definition 1.2.7** *Let  $A$  be a Banach algebra. If  $A$  has minimal left ideals, the sum of all the minimal left ideals is called the left socle of  $A$ . The right socle of  $A$  is defined in a similar way. If  $A$  has no minimal left (right) ideals, then the left (right) socle of  $A$  is zero. If the left socle of  $A$  and the right socle of  $A$  coincide, it is called the socle of  $A$ , denoted by  $\text{Soc}(A)$ . Then we say that  $\text{Soc}(A)$  exists.*

**Proposition 1.2.8** ([21], p.659) *Let  $A$  be a Banach algebra. If  $\text{Soc}(A)$  exists, then it is a two-sided ideal of  $A$ .*

**Proof.** Suppose that  $\text{Soc}(A)$  exists. If  $t \in \text{Soc}(A)$ , then  $t = \sum_{i=1}^n t_i$ , where each  $t_i$  lies in a minimal left ideal  $L_i$  of  $A$ . For any  $x \in A$ , we have that  $xt = \sum_{i=1}^n xt_i$ . Since  $L_i$  is a left ideal,  $xt_i \in L_i$  for all  $i$ . This implies that  $xt \in \text{Soc}(A)$ , so that  $\text{Soc}(A)$  is a left ideal. Since  $\text{Soc}(A)$  is also the sum of all minimal right ideals, a similar argument can be used to show that  $\text{Soc}(A)$  is a right ideal.  $\square$

### 1.3 Semiprime Banach algebras, compactly rank one elements

**Definition 1.3.1** *A Banach algebra  $A$  is called a semiprime Banach algebra if  $I = \{0\}$  is the only two-sided ideal of  $A$  with the property  $I^2 = \{0\}$ .*

Theorem 1.3.3 gives another characterization of semiprime Banach algebras. In order to prove it, we shall need the following lemma.

**Lemma 1.3.2** ([9], Lemma 4, p.155) *Let  $A$  be a semiprime Banach algebra. If  $I$  is a left ideal of  $A$  such that  $I^2 = \{0\}$ , then  $I = \{0\}$ .*

**Theorem 1.3.3** ([21], 2.1) *Let  $A$  be a Banach algebra and  $a \in A$ . Then  $A$  is semiprime if and only if  $aAa = \{0\}$  implies that  $a = 0$ .*

**Proof.** Suppose that  $A$  is semiprime. If  $aAa = \{0\}$ , then  $AaAa = \{0\}$ . But the set  $Aa$  is a left ideal of  $A$ : If  $x \in Aa$ , then there exists  $y \in A$  such that  $x = ya$ , so that for all  $z \in A$ , we have that  $zx = (zy)a \in Aa$ . Since  $Aa$  is a left ideal, it follows from Lemma 1.3.2 that  $Aa = \{0\}$ , so that  $a = 0$ .

Conversely, suppose that  $aAa = \{0\}$  implies that  $a = 0$ . Let  $I$  be a two-sided ideal of  $A$  such that  $I^2 = \{0\}$ . If  $x \in I$ , then  $xI \subset I$ , so that  $xIx \subset I^2 = \{0\}$ . Therefore  $x = 0$ . Since  $x$  is an arbitrary element of  $I$ , this implies that  $I = \{0\}$ , so that  $A$  is semiprime.  $\square$

**Theorem 1.3.4** ([9], Proposition 5, p.155) *A semisimple Banach algebra is semiprime.*

Since the Banach algebras  $\mathcal{L}(X)$ , where  $X$  is a Banach space, and  $M_2(\mathbb{C})$  are semisimple, they are semiprime.

Next we define *compactly rank one elements*. The concept was introduced by J.A. Erdos, S. Giotopoulos and M.S. Lambrou in 1977, but the terminology is our own.

**Definition 1.3.5** *An element  $0 \neq u$  of a semiprime Banach algebra  $A$  is called a compactly rank one element of  $A$  if  $u$  is single and acts compactly on  $A$ .*

The set of compactly rank one elements of  $A$  will be denoted by  $\mathcal{E}_1$ .

## 1.4 Spectrum of an element in a Banach algebra

**Definition 1.4.1** *The spectrum of an element  $x$  in a Banach algebra  $A$  is the set  $\text{Sp}(x) = \{\lambda \in \mathbb{C} : \lambda - x \text{ is not invertible in } A\}$ , where we write  $\lambda$  for the element  $\lambda 1 \in A$ . The spectral radius of  $x$  is the real number  $\rho(x) = \sup\{|\lambda| : \lambda \in \text{Sp}(x)\}$ .*

If there is danger of confusion in a given situation, the spectrum of  $x$  in  $A$  will be written as  $\text{Sp}(x, A)$ . Similarly, the spectral radius will be written as  $\rho(x, A)$ .

In subsequent chapters, the set  $\text{Sp}(x) \setminus \{0\}$  will be frequently used. For brevity, this will be denoted by  $\text{Sp}'(x)$ .

**Definition 1.4.2** *Let  $x$  be an element of a Banach algebra  $A$ . The polynomially convex hull (or full spectrum) of  $\text{Sp}(x)$ , denoted by  $\sigma(x)$ , is the union of  $\text{Sp}(x)$  and the holes of  $\text{Sp}(x)$ , where a hole of  $\text{Sp}(x)$  is a bounded component of  $\mathbb{C} \setminus \text{Sp}(x)$ .*

**Lemma 1.4.3** ([2], Corollary 3.2.10) *Let  $A$  be a Banach algebra and suppose that  $x, y \in A$  satisfy  $xy = yx$ . Then  $\rho(x + y) \leq \rho(x) + \rho(y)$  and  $\rho(xy) \leq \rho(x)\rho(y)$ .*

**Theorem 1.4.4** ([2], Theorem 3.2.8) (I.M. Gelfand) *Let  $A$  be a Banach algebra and let  $x \in A$ . Then*

(i) the map  $\lambda \mapsto (\lambda - x)^{-1}$  is analytic on  $\mathbb{C} \setminus \text{Sp}(x)$ ,

(ii)  $\text{Sp}(x)$  is compact and non-empty,

(iii)  $\rho(x) = \lim_{n \rightarrow \infty} \|x^n\|^{1/n}$ .

**Corollary 1.4.5** ([2], Corollary 3.2.9) (I.M. Gelfand and S. Mazur) *If  $A$  is a Banach algebra in which every non-zero element is invertible, then  $A$  is isomorphic to  $\mathbb{C}$ .*

The following lemma will be required in Chapter 5.

**Lemma 1.4.6** *Let  $a$  be an element in a Banach algebra  $A$ . Then the set  $\mathbb{C} \setminus \sigma(a)$  is open and connected.*

**Proof.** To show that  $\mathbb{C} \setminus \sigma(a)$  is open we show that  $\sigma(a)$  is closed in  $\mathbb{C}$ . By definition of  $\sigma(a)$ , we have that  $\partial\sigma(a) \subset \partial\text{Sp}(a)$ , where  $\partial\sigma(a)$  and  $\partial\text{Sp}(a)$  denote the boundaries of  $\sigma(a)$  and  $\text{Sp}(a)$  respectively. Since  $\text{Sp}(a)$  is closed, from a standard result in topology, we get that  $\sigma(a)$  is closed.

To show that  $\mathbb{C} \setminus \sigma(a)$  is connected, note that  $\mathbb{C} \setminus \text{Sp}(a)$  consists of the holes of  $\text{Sp}(a)$  and one unbounded component. Hence  $\mathbb{C} \setminus \sigma(a)$  consists of one unbounded component, which means that  $\mathbb{C} \setminus \sigma(a)$  is connected.  $\square$

**Lemma 1.4.7** ([2], Lemma 3.1.2) (N. Jacobson) *Let  $A$  be a Banach algebra and  $x, y \in A$ . Also let  $0 \neq \lambda \in \mathbb{C}$ . Then  $\lambda - xy$  is invertible in  $A$  if and only if  $\lambda - yx$  is invertible in  $A$ . Hence  $\text{Sp}(xy) \cup \{0\} = \text{Sp}(yx) \cup \{0\}$ .*

**Corollary 1.4.8** *Let  $A$  be a Banach algebra and let  $x, y \in A$ . Then  $\rho(xy) = \rho(yx)$ .*

In [19], T. Mouton and H. Raubenheimer proved that if  $A$  is a Banach algebra and  $B$  a subalgebra of  $A$  such that  $\text{Sp}(x, A)$  consists of zero and possibly one other point for all  $x \in B$ , then there are no non-zero orthogonal idempotents in  $B$  ([19], Lemma 2.8). In Chapter 3 of this thesis, we will need the following generalized form of this result.

**Lemma 1.4.9** *Let  $A$  be a Banach algebra and  $B$  a subalgebra of  $A$  such that  $\text{Sp}(x, A)$  consists of zero and possibly  $n$  other distinct points for all  $x \in B$  and some  $n \in \mathbb{N}$ . Then there are at most  $n$  non-zero orthogonal idempotents in  $B$ .*

**Proof.** Suppose that  $\{p_i : i = 1, 2, \dots, n+1\}$  is a set of non-zero orthogonal idempotents in  $B$ . Since  $B$  is an algebra, the element  $p = p_1 + 2p_2 + \dots + (n+1)p_{n+1}$  is in  $B$ . By the hypothesis, this means that  $\text{Sp}(p, A)$  consists of zero and possibly  $n$  other distinct points. Now consider the element  $p_1 + 2p_2 + \dots + (n+1)p_{n+1} - 1$ . Since the  $p_i$  are orthogonal, we have that  $(p_1 + 2p_2 + \dots + (n+1)p_{n+1} - 1)p_1 = 0$ . Therefore if  $p_1 + 2p_2 + \dots + (n+1)p_{n+1} - 1$  is invertible, then

$$p_1 = (p_1 + 2p_2 + \dots + (n+1)p_{n+1} - 1)^{-1}(p_1 + 2p_2 + \dots + (n+1)p_{n+1} - 1)p_1 = 0,$$

which is a contradiction. Hence  $p_1 + 2p_2 + \dots + (n+1)p_{n+1} - 1$  is not invertible, so that  $1 \in \text{Sp}(p, A)$ . Applying the same argument to  $p_1 + 2p_2 + \dots + (n+1)p_{n+1} - 2$  and  $p_2$ , we obtain that  $2 \in \text{Sp}(p, A)$ . Using similar arguments, we have that  $\{1, 2, \dots, n+1\} \subset \text{Sp}(p, A)$ , which is a contradiction since  $\text{Sp}(p, A)$  has at most  $n$  distinct non-zero points.  $\square$

**Lemma 1.4.10** ([19], Lemma 3.3) *Let  $A$  be a Banach algebra and  $0 \neq a \in A$ . If there exists a positive integer  $n$  such that  $a$  satisfies  $\bigcap_{i=0}^n \text{Sp}(x + s_i a) \subset \text{Sp}(x)$  for all  $x \in A$  and for every set  $\{s_i : i = 0, 1, \dots, n\}$  of distinct non-zero scalars, then every element of  $A^{-1}a$  has at most  $n$  distinct non-zero points in its spectrum.*

**Proof.** Let  $0 \neq a \in A$  and  $x \in A^{-1}$ . Then by the hypothesis there exists a positive integer  $n$  such that  $a$  satisfies  $\bigcap_{i=1}^n \text{Sp}(x + s_i a) \subset \text{Sp}(x)$  for any set of distinct non-zero scalars  $\{s_i : i = 0, 1, \dots, n\}$ . Now, suppose that  $\{\lambda_i : i = 0, 1, \dots, n\}$  is a set of distinct non-zero points in  $\text{Sp}(x^{-1}a)$ . Take  $s_i = -\frac{1}{\lambda_i}$ . Since  $x \in A^{-1}$ , we have that  $0 \notin \text{Sp}(x)$ . It follows that  $0 \notin \text{Sp}(x + s_j a)$  for at least one  $j \in \{0, 1, \dots, n\}$ . Since  $-s_j x(-\frac{1}{s_j} - x^{-1}a) = (x + s_j a) \in A^{-1}$ , this implies that  $-\frac{1}{s_j} - x^{-1}a \in A^{-1}$ , so that  $-\frac{1}{s_j} \notin \text{Sp}(x^{-1}a)$  for at least one  $j$  in  $\{0, 1, \dots, n\}$ . This implies that  $\lambda_j \notin \text{Sp}(x^{-1}a)$  for some  $j \in \{0, 1, \dots, n\}$ , which is a contradiction. Therefore every element of  $A^{-1}a$  has at most  $n$  distinct non-zero points in its spectrum.  $\square$

**Definition 1.4.11** *An element  $x$  in a Banach algebra  $A$  is a quasinilpotent element of  $A$  if  $\text{Sp}(x) = \{0\}$ .*

The set of quasinilpotent elements of  $A$  will be denoted by  $\text{QN}(A)$ .

**Theorem 1.4.12** ([2], p.36) *Suppose that  $A$  is a Banach algebra. Then*

$$(i) \text{Rad}(A) = \{x \in A : Ax \subset \text{QN}(A)\},$$

$$(ii) \text{Rad}(A) = \{x \in A : xA \subset \text{QN}(A)\}.$$

Hence  $\text{Rad}(A) \subset \text{QN}(A)$ .

In the characterization of the radical as in Theorem 1.4.12 (i), the condition  $Ax \subset \text{QN}(A)$  can be weakened to  $A^{-1}x \subset \text{QN}(A)$ . We prove this in Corollary 1.4.16. The proof of Corollary 1.4.16 relies on Corollary 1.4.14, which follows from the following result.

**Lemma 1.4.13** *If  $A$  is a Banach algebra, then  $A = A^{-1} + A^{-1}$ .*

**Proof.** Let  $x \in A$  and let  $\lambda \in \mathbb{C}$  be such that  $\rho(x) < |\lambda|$ . Then  $x - \lambda \in A^{-1}$  and since  $\lambda \in A^{-1}$ , we have that  $x = (x - \lambda) + \lambda \in A^{-1} + A^{-1}$ . Therefore  $A \subset A^{-1} + A^{-1}$ . The inclusion  $A^{-1} + A^{-1} \subset A$  is obvious since  $A$  is a Banach algebra and  $A^{-1} \subset A$ .  $\square$

**Corollary 1.4.14** ([14], p.73) *Suppose that  $A$  is a Banach algebra and that  $a \in A$ . Then  $a \in \text{Rad}(A)$  if and only if  $1 - A^{-1}a \subset A^{-1}$ .*

**Proof.** Let  $a \in \text{Rad}(A)$ . Then  $1 - Aa \subset A^{-1}$ , and since  $A^{-1} \subset A$ , it follows that  $1 - A^{-1}a \subset A^{-1}$ .

Conversely, suppose that  $a$  is an element in  $A$  such that  $1 - A^{-1}a \subset A^{-1}$ . Let  $x \in A^{-1}$ . Then  $x - a = x(1 - x^{-1}a)$ . Since  $1 - A^{-1}a \subset A^{-1}$ , we have that  $1 - x^{-1}a \in A^{-1}$ . Therefore  $x - a = x(1 - x^{-1}a) \in A^{-1}$ , so that

$$A^{-1} - a \subset A^{-1}. \quad (1.4.15)$$

Now let  $x$  be an arbitrary element of  $A$ . From Lemma 1.4.13, there exist  $y, z \in A^{-1}$  such that  $x = y + z$ . Therefore  $1 - xa = 1 - ya - za = z(z^{-1}(1 - ya) - a)$ . Since  $1 - A^{-1}a \subset A^{-1}$  we have that  $1 - ya \in A^{-1}$ , so that  $z^{-1}(1 - ya) \in A^{-1}$ . From equation (1.4.15), this implies that  $z^{-1}(1 - ya) - a \in A^{-1}$ , so that  $1 - xa = z(z^{-1}(1 - ya) - a) \in A^{-1}$ . Since  $x$  is an arbitrary element of  $A$ , this means that  $a \in \text{Rad}(A)$  by Theorem 1.2.4.  $\blacksquare$

**Corollary 1.4.16** ([14], p.73) *Let  $A$  be a Banach algebra and let  $a \in A$ . Then  $a \in \text{Rad}(A)$  if and only if  $\text{Sp}(xa) = \{0\}$  for all  $x \in A^{-1}$ .*



**Proof.** Let  $a \in \text{Rad}(A)$ . Then from Corollary 1.4.14, we have that  $1 - xa \in A^{-1}$  for all  $x \in A^{-1}$ . Let  $0 \neq \lambda \in \mathbb{C}$ . Then  $\frac{x}{\lambda} \in A^{-1}$ , so that  $\lambda - xa = \lambda(1 - \frac{xa}{\lambda}) \in A^{-1}$ , for all  $x \in A^{-1}$ . This implies that  $\lambda \notin \text{Sp}(xa)$  for all  $x \in A^{-1}$ . Since the spectrum is non-empty, this means that  $\text{Sp}(xa) = \{0\}$  for all  $x \in A^{-1}$ .

Conversely, suppose that  $\text{Sp}(xa) = \{0\}$  for all  $x \in A^{-1}$ . If  $\lambda \neq 0$ , then  $\lambda \notin \text{Sp}(xa)$ , i.e.  $\lambda - xa \in A^{-1}$  for all  $x \in A^{-1}$ . Take  $\lambda = 1$ . Then  $1 - xa \in A^{-1}$  for all  $x \in A^{-1}$ . It follows from Corollary 1.4.14 that  $a \in \text{Rad}(A)$ . ■

**Theorem 1.4.17** ([4], Lemma 2.5) *Let  $A$  be a semisimple Banach algebra and  $p \in A$  an idempotent element. Then*

(i)  $pAp$  is a closed subalgebra of  $A$  with identity  $p$ ,

(ii)  $pAp$  is semisimple,

(iii)  $\text{Sp}(pxp, pAp) \subset \text{Sp}(pxp, A)$  for every  $x \in A$ .

**Proof.** (i) Let  $a, b \in A$ . Then  $(pap)(pbp) = papbp \in pAp$ . Also,  $pap + pbp = p(a + b)p \in pAp$  and if  $\lambda \in \mathbb{C}$ , then  $\lambda(pap) = p(\lambda a)p \in pAp$ . Therefore  $pAp$  is a subalgebra of  $A$ . To show that  $pAp$  has identity  $p$ , let  $a \in A$ . Then  $(pap)p = pap$  and  $p(pap) = pap$ , so that  $p$  is the identity of  $pAp$ . We now show that  $pAp$  is closed. Let  $px_n p \in pAp$  such that  $px_n p \rightarrow x$ , where  $x \in A$ . Then  $p(px_n p)p \rightarrow pxp$ , so that  $px_n p \rightarrow pxp$ . By uniqueness of limits,  $x \in pAp$ , so that  $pAp$  is closed in  $A$ .

(ii) If  $a \in pAp$  then  $\rho(a, pAp) = \lim_{n \rightarrow \infty} \|a^n\|^{1/n} = \rho(a, A)$ , by Theorem 1.4.4. So we will write  $\rho(a)$  without ambiguity. Let  $a \in \text{Rad}(pAp)$ . Then there exists  $b \in A$  such that  $a = pbp$ , so that  $pap = p(pbp)p = pbp = a$ . Also, we have from Theorem 1.4.12 that  $\rho(apxp) = 0$  for all  $x \in A$ . It follows from Corollary 1.4.8 that  $\rho(ax) = \rho(papx) = \rho(apxp) = 0$  for all  $x \in A$ . This implies that  $ax \in \text{QN}(A)$  for all  $x \in A$ , so that  $a \in \text{Rad}(A)$  by Theorem 1.4.12. Since  $A$  is semisimple,  $a = 0$ . Hence  $\text{Rad}(pAp) = \{0\}$ , so that  $pAp$  is semisimple.

(iii) For any  $x \in A$ , suppose that  $\lambda \notin \text{Sp}(pxp, A)$ . Then  $pxp - \lambda$  is invertible in  $A$ , with inverse, say,  $q$ . Therefore  $(pxp - \lambda)q = q(pxp - \lambda) = 1$ . Hence

$(pxp - \lambda p)pqp = pqp(pxp - \lambda p) = p$ , which means that  $pxp - \lambda p$  is invertible in  $pAp$ . Therefore  $\lambda \notin \text{Sp}(pxp, pAp)$ . This implies that  $\text{Sp}(pxp, pAp) \subset \text{Sp}(pxp, A)$ .  $\blacksquare$

**Theorem 1.4.18** ([2], Theorem 3.2.13(ii)) *Let  $A$  be a Banach algebra and  $B$  a closed subalgebra of  $A$  containing 1. If  $x \in B$ , then  $\text{Sp}(x, B)$  is the union of  $\text{Sp}(x, A)$  and a (possibly empty) collection of bounded components of  $\mathbb{C} \setminus \text{Sp}(x, A)$ , and  $\partial\text{Sp}(x, B) \subset \partial\text{Sp}(x, A)$ , where  $\partial\text{Sp}(x, B)$  and  $\partial\text{Sp}(x, A)$  denote the boundaries of  $\text{Sp}(x, B)$  and  $\text{Sp}(x, A)$  respectively.*

**Corollary 1.4.19** ([2], Theorem 3.1.5) *Let  $A$  be a Banach algebra. Then  $A/\text{Rad}(A)$  is semisimple and  $\text{Sp}(x, A) = \text{Sp}(x + \text{Rad}(A), A/\text{Rad}(A))$  for all  $x \in A$ .*

In Chapter 3 the following lemma will be necessary.

**Lemma 1.4.20** *Let  $A$  be a Banach algebra which is not semisimple. If  $0 \neq a \in \text{Rad}(A)$ , then  $\text{Sp}(x + a) = \text{Sp}(x)$  for all  $x \in A$ .*

**Proof.** From Corollary 1.4.19 we have that  $\text{Sp}(x + a, A) = \text{Sp}(x + a + \text{Rad}(A), A/\text{Rad}(A))$ . Now since  $a \in \text{Rad}(A)$ , we get that  $x + a + \text{Rad}(A) = x + \text{Rad}(A)$ , so that  $\text{Sp}(x + a + \text{Rad}(A), A/\text{Rad}(A)) = \text{Sp}(x + \text{Rad}(A), A/\text{Rad}(A))$ . Applying Corollary 1.4.19 to  $\text{Sp}(x + \text{Rad}(A), A/\text{Rad}(A))$ , we obtain that  $\text{Sp}(x + a, A) = \text{Sp}(x + \text{Rad}(A), A/\text{Rad}(A)) = \text{Sp}(x, A)$ .  $\blacksquare$

**Definition 1.4.21** *A two-sided ideal  $I$  of a Banach algebra  $A$  is called an inessential ideal of  $A$  if  $\text{Sp}(x)$  is finite or a sequence converging to zero for all  $x \in I$ .*

Let  $I$  be a fixed two-sided ideal of a Banach algebra  $A$  and let  $x \in A$ . Let  $\mathcal{R}(x) = \{\lambda : \lambda \text{ is an isolated point of } \text{Sp}(x) \text{ and the spectral idempotent associated with } x \text{ and } \lambda \text{ is in } I\}$ . We define the set  $D(x)$  by  $D(x) = \text{Sp}(x) \setminus \mathcal{R}(x)$ , and the set  $(D(x))^\circ$  by  $(D(x))^\circ = D(x) \cup \{\text{the holes of } D(x)\}$ .

**Theorem 1.4.22** ([2], Theorem 5.7.4(ii)) *Let  $A$  be a Banach algebra and  $I$  an inessential ideal of  $A$ . If  $x \in A$  and  $y \in I$ , then the unbounded components of  $\mathbb{C} \setminus D(x)$  and  $\mathbb{C} \setminus D(x + y)$  coincide, that is,  $(D(x))^\circ = (D(x + y))^\circ$ .*

**Definition 1.4.23** *Let  $A$  be a Banach algebra. A linear functional  $\chi$  on  $A$  is said to be multiplicative if  $\chi(xy) = \chi(x)\chi(y)$  for all  $x, y \in A$ . We call  $\chi$  a character of  $A$  if  $\chi$  is multiplicative and  $\chi \neq 0$ .*

The set of characters on  $A$  will be denoted by  $\mathcal{M}(A)$ .

Corollary 1.4.25 will be required in Chapter 2. In order to prove it, we first state the following result.

**Theorem 1.4.24** ([2], Theorem 4.1.2) (I.M. Gelfand) *Let  $A$  be a commutative Banach algebra. Then  $\text{Sp}(x) = \{\chi(x) : \chi \in \mathcal{M}(A)\}$  for all  $x \in A$ .*

**Corollary 1.4.25** *Let  $A$  be a commutative Banach algebra. If  $x, y \in A$  then  $\text{Sp}(x + y) \subset \text{Sp}(x) + \text{Sp}(y)$ .*

**Proof.** Since  $A$  is commutative, we have from Theorem 1.4.24 that

$$\begin{aligned} \text{Sp}(x + y) &= \{\chi(x + y) : \chi \in \mathcal{M}(A)\} \\ &= \{\chi(x) + \chi(y) : \chi \in \mathcal{M}(A)\} \\ &\subset \{\chi(x) : \chi \in \mathcal{M}(A)\} + \{\chi(y) : \chi \in \mathcal{M}(A)\} \\ &= \text{Sp}(x) + \text{Sp}(y), \end{aligned}$$

and the result follows.  $\square$

## 1.5 The holomorphic functional calculus

The notation  $H(\Omega)$  will be used to denote the algebra of holomorphic functions on an open set  $\Omega$ .

**Theorem 1.5.1** ([2], p.42-43) *Suppose that  $A$  is a Banach algebra and  $x \in A$ . Then*

(i) *if  $p(\lambda) = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_n\lambda^n$  is a polynomial with  $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{C}$ , then*

$$p(x) = \alpha_0 + \alpha_1x + \cdots + \alpha_nx^n \in A,$$

(ii) *if  $r(\lambda) = \frac{p(\lambda)}{q(\lambda)}$  is a rational function with poles in  $\mathbb{C} \setminus \text{Sp}(x)$ , then*

$$r(x) = p(x)(q(x))^{-1} \in A,$$

(iii) if  $f : B(0, R) \rightarrow \mathbb{C}$  is analytic, say  $f(\lambda) = \sum_{k=0}^{\infty} \alpha_k \lambda^k$  for  $\lambda$  in the open ball  $B(0, R)$ , and  $\|x\| < R$ , then

$$f(x) = \sum_{k=0}^{\infty} \alpha_k x^k \in A,$$

(iv) if  $f \in H(\Omega)$ , where  $\text{Sp}(x) \subset \Omega$ , and  $\Gamma$  is a smooth contour in  $\Omega \setminus \text{Sp}(x)$  surrounding  $\text{Sp}(x)$ , then

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda - x)^{-1} d\lambda \in A.$$

**Theorem 1.5.2** ([2], Theorem 3.2.6) (**Spectral Mapping Theorem**) Let  $A$  be a Banach algebra and let  $x \in A$ . Then for every non-constant polynomial  $p$  with complex coefficients, we have that  $\text{Sp}(p(x)) = p(\text{Sp}(x))$ .

Theorem 1.5.2 leads to the following proposition, which will be required in subsequent chapters.

**Proposition 1.5.3** Let  $x$  be an element of a Banach algebra  $A$  and let  $\lambda \in \mathbb{C}$  be such that  $|\lambda| > \rho(x)$ . Then  $0 \notin \sigma(\lambda - x)$ .

**Proof.** Since  $|\lambda| > \rho(x)$  we have that  $\lambda \notin \sigma(x)$ . Consider  $\sigma(\lambda - x)$ . By the spectral mapping theorem,  $\sigma(\lambda - x)$  is the union of  $\lambda - \text{Sp}(x)$  and the holes of  $\lambda - \text{Sp}(x)$ . This implies that  $\sigma(\lambda - x) = \lambda - \sigma(x)$ . If  $0 \in \sigma(\lambda - x)$ , then  $\lambda \in \sigma(x)$ . This is a contradiction.  $\blacksquare$

**Theorem 1.5.4** ([2], Theorem 3.3.3) (**Holomorphic Functional Calculus**) Let  $A$  be a Banach algebra and let  $x \in A$ . Suppose that  $\Omega$  is an open set containing  $\text{Sp}(x)$  and that  $\Gamma$  is an arbitrary smooth contour included in  $\Omega$  and surrounding  $\text{Sp}(x)$ . Then the map  $x \mapsto f(x) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda 1 - x)^{-1} d\lambda$  from  $H(\Omega)$  into  $A$  has the following properties:

(i)  $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ ,

(ii)  $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) = f_2(x) \cdot f_1(x)$ ,

(iii)  $1(x) = 1$  and  $I(x) = x$ , (where  $I(\lambda) = \lambda$ )

(iv) if  $(f_n)$  converges to  $f$  uniformly on compact subsets of  $\Omega$ , then  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ ,

(v)  $\text{Sp}(f(x)) = f(\text{Sp}(x))$ .

Property (v) is the spectral mapping theorem for analytic functions. The following result, which will be needed in Chapter 4, is a consequence of the spectral mapping theorem for analytic functions.

**Lemma 1.5.5** *Let  $x$  be an element of a Banach algebra  $A$  and let  $\lambda \in \mathbb{C}$  be such that  $2\rho(x) < |\lambda|$ . If  $\mu = -\frac{1}{\lambda}$  and if  $y = \mu - (x - \lambda)^{-1}$ , then  $\rho(y) < |\mu|$ .*

**Proof.** By the spectral mapping theorem,  $\text{Sp}(y) = \{\mu - \frac{1}{\gamma - \lambda} : \gamma \in \text{Sp}(x)\} = \{-\frac{\gamma}{\lambda(\gamma - \lambda)} : \gamma \in \text{Sp}(x)\}$ . Therefore  $\rho(y) = \sup\{|\frac{\gamma}{\lambda(\gamma - \lambda)}| : \gamma \in \text{Sp}(x)\}$ . Now from  $2\rho(x) < |\lambda|$ , we obtain that  $\rho(x) < |\lambda| - \rho(x)$ , so that  $\rho(x) < |\lambda| - |\gamma| \leq |\lambda - \gamma|$  for all  $\gamma \in \text{Sp}(x)$ . This implies that  $|\frac{\gamma}{\lambda(\gamma - \lambda)}| < |\frac{\gamma}{\lambda\rho(x)}|$ . Since  $|\gamma| \leq \rho(x)$  for all  $\gamma \in \text{Sp}(x)$ , it follows that  $|\frac{\gamma}{\lambda(\gamma - \lambda)}| < |\frac{\gamma}{\lambda\gamma}| = |\frac{1}{\lambda}| = |\mu|$ . This implies that  $\rho(y) < |\mu|$ , as required.  $\square$

A subset  $S$  of  $\mathbb{C}$  is said to be *totally disconnected* if the only connected subsets of  $S$  are the one-point sets.

**Theorem 1.5.6** ([2], Theorem 3.3.4) *Let  $A$  be a Banach algebra and suppose that an element  $x \in A$  has a disconnected spectrum. Let  $U_0$  and  $U_1$  be two disjoint open sets such that  $\text{Sp}(x) \subset U_0 \cup U_1$ ,  $\text{Sp}(x) \cap U_0 \neq \emptyset$  and  $\text{Sp}(x) \cap U_1 \neq \emptyset$ . Then there exists a non-trivial idempotent  $p$  commuting with  $x$ , such that  $\text{Sp}(px) = (\text{Sp}(x) \cap U_1) \cup \{0\}$  and  $\text{Sp}(x - px) = (\text{Sp}(x) \cap U_0) \cup \{0\}$ .*

Suppose that  $\alpha$  is an isolated point of  $\text{Sp}(x)$ . Then, in the case where  $\text{Sp}(x)$  is the disconnected set  $\{\alpha\} \cup \text{Sp}(x) \setminus \{\alpha\}$ , the above theorem is of special interest. If  $\Gamma$  is a circle centered at  $\alpha$  and separating  $\alpha$  from the rest of  $\text{Sp}(x)$ , then the idempotent  $p = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - x)^{-1} d\lambda$  is called the *spectral idempotent* associated with  $x$  and  $\alpha$ .

**Corollary 1.5.7** ([19], Proposition 2.4) *Let  $A$  be a Banach algebra,  $x \in A$  and suppose that  $\alpha$  is a non-zero isolated point of  $\text{Sp}(x)$ . If  $p$  is the spectral idempotent associated with  $x$  and  $\alpha$ , then there exists a  $y \in A$  such that  $p = xy = yx$ .*

**Proof.** Let  $x \in A$  and suppose that  $\alpha$  is a non-zero isolated point of  $\text{Sp}(x)$ . Also let  $\Gamma$  be a circle centered at  $\alpha$  and separating  $\alpha$  from 0 and from the rest of  $\text{Sp}(x)$ . If  $\lambda \in \Gamma$ , then from  $(\lambda - x)(\lambda - x)^{-1} = 1$ , we get that  $\lambda(\lambda - x)^{-1} = 1 + x(\lambda - x)^{-1}$ . This implies that  $(\lambda - x)^{-1} = \frac{1}{\lambda} + \frac{1}{\lambda}x(\lambda - x)^{-1}$ . Therefore

$$p = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - x)^{-1} d\lambda = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\lambda} d\lambda + \frac{x}{2\pi i} \int_{\Gamma} \frac{1}{\lambda} (\lambda - x)^{-1} d\lambda.$$

By Cauchy's theorem,  $\int_{\Gamma} \frac{1}{\lambda} d\lambda = 0$ , which implies that

$$p = \frac{x}{2\pi i} \int_{\Gamma} \frac{1}{\lambda} (\lambda - x)^{-1} d\lambda = \left( \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\lambda} (\lambda - x)^{-1} d\lambda \right) x.$$

Let  $y = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\lambda} (\lambda - x)^{-1} d\lambda$ . Then  $p = xy = yx$ .  $\blacksquare$

Let  $A$  be a Banach algebra. We define the set of *exponentials* of  $A$  by  $\exp(A) = \{e^x : x \in A\}$ . The set of *generalized exponentials* of  $A$  is defined by  $\text{Exp}(A) = \left\{ \prod_{i=1}^n e^{x_i} : x_i \in A \text{ and } n \in \mathbb{N} \right\}$ . If  $A$  is not commutative, then it is possible ([20], p.230-231) that  $\exp(A) \neq \text{Exp}(A)$ . The set  $\text{Exp}(A)$  is the connected component of  $A^{-1}$  containing the unit 1 ([2], Theorem 3.3.7).

The following lemma will be required in Chapter 2.

**Lemma 1.5.8** *Let  $a$  be any element in a Banach algebra  $A$ . Then the set  $B = \text{Exp}(A)a\text{Exp}(A)$  is connected.*

**Proof.** Let  $b$  and  $c$  be arbitrary elements in  $B$ , with  $b = \prod_{i=1}^j e^{x_i} a \prod_{i=1}^k e^{y_i}$  and  $c = \prod_{i=1}^m e^{v_i} a \prod_{i=1}^n e^{w_i}$ . Without loss of generality suppose that  $m \geq j$  and  $n \geq k$ .

Let  $g : [0, 1] \rightarrow B$  be defined by  $g(t) = \prod_{i=1}^m e^{tx_i + (1-t)v_i} a \prod_{i=1}^n e^{ty_i + (1-t)w_i}$ , where  $x_i = 0$  for  $i > j$  and  $y_i = 0$  for  $i > k$ . Then  $g$  is a path in  $B$  connecting  $b$  and  $c$ , so that  $B$  is pathwise connected. Hence  $B$  is connected.  $\blacksquare$

**Theorem 1.5.9** ([2], Theorem 3.3.6) *Let  $A$  be a Banach algebra. Suppose that an element  $x \in A$  has a spectrum which does not separate 0 from infinity. Then there exists  $y \in A$  such that  $x = e^y$ . In particular, for any integer  $n \geq 1$  there exists  $z \in A$  such that  $z^n = x$ .*

The following lemma, which will be used in Chapter 2, follows from Theorem 1.5.9.

**Lemma 1.5.10** *Let  $A$  be a Banach algebra. Then  $A = \text{Exp}(A) + \text{Exp}(A)$ .*

**Proof.** Let  $a \in A$ . Since the spectrum is compact, there exists a  $\lambda \in \mathbb{C}$  such that  $|\lambda| > \rho(a)$ . It follows from Proposition 1.5.3 that  $0 \notin \sigma(-\lambda + a)$ . From Theorem 1.5.9 we have that  $-\lambda + a \in \text{Exp}(A)$ . Also, since  $\lambda$  is a non-zero complex number,  $\lambda \in \text{Exp}(A)$ . Therefore  $a = (-\lambda + a) + \lambda \in \text{Exp}(A) + \text{Exp}(A)$ .  $\square$

**Lemma 1.5.11** ([24], Lemma 3.1) *Let  $A$  be a Banach algebra. Suppose that  $e$  and  $f$  are idempotents in  $A$  such that  $\rho(e - f) < 1$ . Then there exists an element  $u \in \text{exp}(A)$  such that  $f = u^{-1}eu$ .*

**Proof.** Since  $\rho(e - f) < 1$ , we have from the spectral mapping theorem that  $\rho((e - f)^2) = (\rho(e - f))^2 < 1$ . Therefore  $1 \notin \sigma((e - f)^2)$ . It follows from Proposition 1.5.3 that  $0 \notin \sigma(1 - (e - f)^2)$ . By the spectral mapping theorem, this implies that  $0 \notin \sigma((1 - (e - f)^2)^{-1})$ . It follows from Theorem 1.5.9 that there exists  $z \in A$  such that  $z^2 = (1 - (e - f)^2)^{-1}$ . From the proof of Theorem 1.5.9, we have that  $z \in \text{exp}(A)$ .

Let  $z^2 = (1 - x)^{-1}$ , where  $x = (e - f)^2$ . We must show that  $e$  commutes with  $z$ . We begin by showing that  $e$  commutes with  $x$ . We have that  $ex = e(e - f)^2 = e - ef - efe + ef = e(1 - fe)$ . Also,  $xe = (e - f)^2e = e - efe - fe + fe = e(1 - fe)$ . Therefore  $ex = xe$ . Now let  $z_1^2 = 1 - (e - f)^2$ . Then  $z = z_1^{-1}$ . By the proof of Theorem 1.5.9, the element  $z_1$  commutes with every element that commutes with  $1 - x$ , so that  $z_1e = ez_1$ , and hence  $ze = ez$ . Similarly,  $zf = fz$ .

Let  $u = z(e + f - 1)$ . Then

$$\begin{aligned} u^2 &= z^2(e + f - 1)^2 \\ &= (1 - (e - f)^2)^{-1}(e + f - 1)^2 \\ &= (1 - e + ef + fe - f)^{-1}(1 - e - f + ef + fe) \\ &= 1. \end{aligned}$$

Therefore  $u$  is invertible and  $u^{-1} = u$ . Also,  $\text{Sp}(u^2) = \{1\}$ , so that by the spectral mapping theorem,  $\text{Sp}(u) \subset \{-1, 1\}$ . It follows from Theorem 1.5.9 that  $u \in \text{exp}(A)$ . Now since  $ez = ze$ , we have that  $eu = ez(e + f - 1) = z(e + ef - e) = zef$ . Also,  $uf = z(e + f - 1)f = z(ef + f - f) = zef$ , so that

$eu = uf$ . Recalling that  $u^{-1} = u$ , it follows that  $u^{-1}eu = ueu = u(uf) = u^2f = f$ , as required.  $\blacksquare$

## 1.6 Continuity Properties of the Spectrum

In order to measure the continuity of the spectrum, we introduce a distance on the set of compact subsets of  $\mathbb{C}$ , called the *Hausdorff distance* and defined by

$$\Delta(K_1, K_2) = \max \left( \sup_{z \in K_2} \text{dist}(z, K_1), \sup_{z \in K_1} \text{dist}(z, K_2) \right),$$

where  $K_1$  and  $K_2$  are compact subsets of  $\mathbb{C}$ . Let  $r > 0$  and  $K$  be a compact subset of  $\mathbb{C}$ . We will use the notation  $K + r$  to denote  $\{z : \text{dist}(z, K) \leq r\}$ .

**Definition 1.6.1** *Let  $A$  be a Banach algebra and let  $x \in A$ . The map  $x \mapsto \text{Sp}(x)$  is continuous at  $a \in A$  if, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $\|x - a\| < \delta$  implies that  $\Delta(\text{Sp}(x), \text{Sp}(a)) < \epsilon$ . If, for a given  $\epsilon > 0$ , the number  $\delta > 0$  is independent of  $a$ , we say that  $x \mapsto \text{Sp}(x)$  is uniformly continuous at  $a$ .*

**Theorem 1.6.2** ([2], Theorem 3.4.5) (**J.D. Newburgh**) *Let  $A$  be a Banach algebra and  $a \in A$ . If  $\text{Sp}(a)$  is totally disconnected, then the spectrum function  $x \mapsto \text{Sp}(x)$  is continuous at  $a$ .*

## 1.7 Elements with finite spectrum

**Definition 1.7.1** *Let  $A$  be a Banach algebra. An element  $x \in A$  is said to be algebraic if there exists a non-zero polynomial  $p$  such that  $p(x) = 0$ . It is algebraic of degree  $n$  if the polynomial  $p$  is of degree  $n$  and  $q(x) \neq 0$  for any polynomial  $q$  of degree less than  $n$ .*

**Theorem 1.7.2** ([2], p.96) *Let  $A$  be a Banach algebra. Then*

- (i) *if  $\dim(A) = n < \infty$ , then  $x$  is algebraic of degree  $\leq n$  for all  $x \in A$ ,*
- (ii) *if  $x$  is algebraic of degree  $n$ , then  $\text{Sp}(x)$  is finite and  $\#\text{Sp}(x) \leq n$ ,*
- (iii) *if  $\dim(A/\text{Rad}(A)) < \infty$ , then  $\text{Sp}(x)$  is finite for all  $x \in A$ .*



**Proof.** (i) Suppose that  $\dim(A) = n < \infty$ . If  $x \in A$  then the set of elements  $\{1, x, x^2, \dots, x^n\}$  is linearly dependent, say  $\sum_{k=0}^n s_k x^k = 0$ , with not all the  $s_k$  zero. Let  $p(z) = \sum_{k=0}^n s_k z^k$ . Then  $p \neq 0$  and  $\deg(p) \leq n$ . Also  $p(x) = 0$ . Hence  $x$  is algebraic of degree  $\leq n$ .

(ii) Suppose that  $x$  is algebraic of degree  $n$ . Let  $p$  be a polynomial of degree  $n$  such that  $p(x) = 0$ . By the spectral mapping theorem,  $p(\text{Sp}(x)) = \text{Sp}(p(x)) = \{0\}$ . So if  $\lambda \in \text{Sp}(x)$ , then  $p(\lambda) = 0$ , i.e.  $\lambda$  is a zero of the polynomial  $p$ . Since  $\deg(p) = n$ , we get that  $p$  has at most  $n$  distinct zeros. This implies that  $\#\text{Sp}(x) \leq n$ .

(iii) Let  $x$  be an arbitrary element in  $A$ . Then by (i) and (ii), we get that  $\#\text{Sp}(x + \text{Rad}(A), A/\text{Rad}(A)) \leq n$ . It follows from Corollary 1.4.19 that  $\#\text{Sp}(x, A) \leq n$ .  $\square$

**Corollary 1.7.3** *If  $x$  is an element of a Banach algebra  $A$  with  $\dim(xAx) = n$ , then  $x$  is algebraic and  $\#\text{Sp}(x) \leq n + 2$ .*

**Proof.** Suppose that  $\dim(xAx) = n$ . Then the set  $\{x^2, x^3, \dots, x^{n+2}\}$  is linearly dependent. Therefore  $\sum_{k=2}^{n+2} s_k x^k = 0$ , with not all  $s_k = 0$ . Let  $p$  be a polynomial defined by  $p(z) = \sum_{k=2}^{n+2} s_k z^k$  ( $z \in \mathbb{C}$ ). Then  $p(x) = 0$  and  $\deg(p) \leq n + 2$ , so that  $x$  is algebraic of degree  $\leq n + 2$ . It follows from Theorem 1.7.2 that  $\#\text{Sp}(x) \leq n + 2$ .  $\square$

**Definition 1.7.4** *A set  $U$  in a Banach algebra  $A$  is called an absorbing set if there exists an element  $a \in U$  with the following property: If  $x \in A$ , then there exists an  $r > 0$  such that  $a + \lambda x \in U$  for all real  $\lambda$  such that  $-r \leq \lambda \leq r$ .*

An open set is absorbing, but not vice versa.

The concept of *capacity* can be defined for Borel sets. It is a difficult concept. But for our purposes, it suffices to know that the capacity of a set is an indication of the size of the set. Closed disks and closed line segments have non-zero capacities. A discrete set has zero capacity. If a set  $S$  has zero capacity, then  $S$  is small in some sense.

**Theorem 1.7.5** ([2], Theorem 3.4.25)(**Scarcity Lemma**) *Suppose that  $A$  is a Banach algebra. Let  $f : D \rightarrow A$  be analytic, where  $D$  is a domain in  $\mathbb{C}$ . Then either the set of  $\lambda \in D$  such that  $\text{Sp}(f(\lambda))$  is finite is a Borel set having zero capacity, or there exists an  $n \geq 1$  and a closed discrete subset  $E$  of  $D$  such that  $\#\text{Sp}(f(\lambda)) = n$  for all  $\lambda \in D \setminus E$  and  $\#\text{Sp}(f(\lambda)) < n$  for all  $\lambda \in E$ .*

**Corollary 1.7.6** *Let  $A$  be a Banach algebra. Then*

(i) *if  $D$  is a domain in  $\mathbb{C}$ , the map  $f : D \rightarrow A$  is analytic and  $\#\text{Sp}(f(\lambda)) \leq n$  for all  $\lambda$  in a subset of  $D$  with non-zero capacity, then  $\#\text{Sp}(f(\lambda)) \leq n$  for all  $\lambda \in D$ ,*

(ii) *if  $\#\text{Sp}(xa) \leq n$  for all  $x$  in an absorbing set  $U$  of  $A$  and for any fixed element  $a$  of  $A$ , then  $\#\text{Sp}(xa) \leq n$  for all  $x \in A$ .*

**Proof.** (i) Suppose that  $\#\text{Sp}(f(\lambda)) \leq n$  for all  $\lambda \in S$ , where  $S$  is a subset of  $D$  with non-zero capacity. Then by Theorem 1.7.5, there exists an  $m \in \mathbb{N}$  and a closed discrete subset  $E$  of  $D$  such that  $\#\text{Sp}(f(\lambda)) = m$  for all  $\lambda \in D \setminus E$  and  $\#\text{Sp}(f(\lambda)) < m$  for all  $\lambda \in E$ . If  $m > n$  then  $\#\text{Sp}(f(\lambda)) \leq n < m$  for all  $\lambda \in S$ . But since  $E$  is discrete,  $E$  has zero capacity, so that  $\#\text{Sp}(f(\lambda)) < m$  only on a set with zero capacity. This is a contradiction. Hence  $m \leq n$  for all  $\lambda \in D$ .

(ii) Suppose that  $\#\text{Sp}(xa) \leq n$  for all  $x \in U$  and for a fixed  $a \in A$ , where  $U$  is an absorbing subset of  $A$ . Let  $b \in U$  such that if  $x \in A$ , there exists  $r_x > 0$ , with  $b + \lambda x \in U$  for all  $\lambda$  such that  $-r_x \leq \lambda \leq r_x$ . Let  $x \in A$  and let  $f : \mathbb{C} \rightarrow A$  be defined by  $f(\lambda) = [b + \lambda(x - b)]a$ . Then  $f$  is analytic on  $\mathbb{C}$ . Also, if  $\lambda \in \mathbb{R}$  and  $-r_{x-b} \leq \lambda \leq r_{x-b}$ , then  $b + \lambda(x - b) \in U$ . Hence  $\#\text{Sp}(f(\lambda)) \leq n$  for all  $\lambda \in [-r_{x-b}, r_{x-b}]$ , a set with non-zero capacity. By (i), we have that  $\#\text{Sp}(f(\lambda)) \leq n$  for all  $\lambda \in \mathbb{C}$ . Take  $\lambda = 1$ . Then  $f(\lambda) = [b + \lambda(x - b)]a = xa$ , so that  $\#\text{Sp}(xa) \leq n$  for all  $x \in A$ .  $\blacksquare$

Since an open set is absorbing, the following result follows from Corollary 1.7.6(ii).

**Corollary 1.7.7** ([19], Lemma 2.7) *Let  $A$  be a Banach algebra and  $x \in A$ . If  $\#\text{Sp}(yx) \leq n$  for all  $y \in A^{-1}$ , then  $\#\text{Sp}(yx) \leq n$  for all  $y \in A$ .*

Theorem 1.7.8 is also an application of the scarcity lemma. We will use it to prove several results in subsequent chapters.

**Theorem 1.7.8** ([2], Theorem 5.4.2) *Let  $A$  be a Banach algebra. Then*

(i) *if  $A$  contains an absorbing set  $U$  such that  $\text{Sp}(x)$  is finite for all  $x \in U$ , then  $A/\text{Rad}(A)$  is finite-dimensional,*

(ii) *if  $A$  contains an absorbing set  $U$  such that  $\#\text{Sp}(x) \leq n$  for all  $x \in U$  and some fixed  $n \in \mathbb{N}$ , then  $\dim(A/\text{Rad}(A)) \leq n^4$ .*

Hence if  $A$  is semisimple and if  $\#\text{Sp}(x) \leq n$  for all  $x$  in an absorbing set  $U$  of  $A$ , then  $\dim(A) \leq n^4$ .

**Corollary 1.7.9** ([2], Exercise 4, p.114) *Let  $A$  be a semisimple Banach algebra. If  $U$  is a non-empty open subset of  $A$  such that  $\text{Sp}(x)$  consists of one point for every  $x \in U$ , then  $A$  is isomorphic to  $\mathbb{C}$ .*

**Proof.** Since  $U$  is open,  $U$  is an absorbing set. So  $\#\text{Sp}(x) \leq 1$  for all  $x$  in an absorbing subset  $U$  of  $A$ . Since  $A$  is semisimple, it follows from Theorem 1.7.8 that  $\dim(A) \leq 1^4 = 1$ , so that  $\dim(A) = 1$ . Therefore  $A = \text{span}\{1\}$ . So  $A = \{\lambda \cdot 1 : \lambda \in \mathbb{C}\}$ . Every non-zero element in  $A$  is invertible: If  $\lambda \neq 0$ , then  $(\lambda \cdot 1)(\frac{1}{\lambda} \cdot 1) = 1 = (\frac{1}{\lambda} \cdot 1)(\lambda \cdot 1)$ . Hence  $A$  is isomorphic to  $\mathbb{C}$ , by Corollary 1.4.5.  $\square$

The following lemma will be required in Chapter 3.

**Lemma 1.7.10** *Let  $a$  be an element of a Banach algebra  $A$  and suppose that  $\#\text{Sp}(xa) < \infty$  for all  $x \in A$ . If  $k$  is a fixed positive integer, then the set defined by  $A_k = \{x \in A : \#\text{Sp}(xa) \leq k\}$  is closed in  $A$ .*

**Proof.** Let  $(x_n)$  be a sequence in  $A_k$  such that  $x_n \rightarrow x$ , where  $x \in A$ . Suppose that  $\#\text{Sp}(xa) > k$ , and  $\lambda_1, \lambda_2, \dots, \lambda_{k+1} \in \text{Sp}(xa)$ . Let  $\epsilon < \frac{1}{2} \min\{|\lambda_j - \lambda_i| : i \neq j; i, j = 1, 2, \dots, k+1\}$ .

Define the map  $f : A \rightarrow K(\mathbb{C})$  by  $x \mapsto \text{Sp}(xa)$ , where  $K(\mathbb{C})$  is the set of all compact subsets of  $\mathbb{C}$  with the Hausdorff metric. Since  $\text{Sp}(xa)$  is finite, and hence totally disconnected for all  $x \in A$ , the map  $xa \mapsto \text{Sp}(xa)$  is continuous by Theorem 1.6.2. Since  $f$  is the composition of  $x \mapsto xa$  and  $xa \mapsto \text{Sp}(xa)$ , this implies that  $f$  is continuous on  $A$ . Hence  $\lim_{n \rightarrow \infty} \text{Sp}(x_n a) = \lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n) = f(x) = \text{Sp}(xa)$ , so that given any  $\epsilon > 0$  there exists an  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $\Delta(\text{Sp}(x_n a), \text{Sp}(xa)) < \epsilon$ , i.e.

$$\max \left( \sup_{z \in \text{Sp}(xa)} \text{dist}(z, \text{Sp}(x_n a)), \sup_{z \in \text{Sp}(x_n a)} \text{dist}(z, \text{Sp}(xa)) \right) < \epsilon.$$

Hence  $\text{dist}(\lambda_j, \text{Sp}(x_n a)) < \epsilon$  for all  $j = 1, 2, \dots, k + 1$ , and for all  $n \geq N$ . Now,  $\#\text{Sp}(x_n a) \leq k$ , say  $\text{Sp}(x_n a) = \{\mu_1, \mu_2, \dots, \mu_p\}$  with  $p \leq k$ . Suppose that  $\text{dist}(\lambda_j, \text{Sp}(x_n a)) = |\lambda_j - \mu_{i_j}| < \epsilon$ , for  $j = 1, 2, \dots, k + 1$ . Then since  $p \leq k < k + 1$ , there exist  $1 \leq i < j \leq k + 1$  and  $1 \leq t \leq p$  such that  $|\lambda_i - \mu_t| < \epsilon$  and  $|\lambda_j - \mu_t| < \epsilon$ . Therefore  $|\lambda_i - \lambda_j| < 2\epsilon$ , which is a contradiction. Hence  $\#\text{Sp}(xa) \leq k$ , so that  $x \in A_k$ .  $\square$

## Chapter 2

# Spatially rank one and finite rank elements

In Chapter 1 we defined a compactly rank one element as follows: A non-zero element  $u$  in a semisimple Banach algebra  $A$  is a compactly rank one element of  $A$  if  $u$  is single and  $u$  acts compactly on  $A$ . In 1978 J. Puhl gave the following definition for what he called one-dimensional elements: An non-zero element  $u$  in a semiprime Banach algebra  $A$  is called one-dimensional if there exists a linear functional  $f_u$  on  $A$  such that  $uxu = f_u(x)u$  for all  $x \in A$ . Various authors such as in [11] and [14] have since studied the elements with this property and have called these elements spatially rank one elements. In this chapter we develop a theory on the spatially rank one and spatially finite rank elements. The spatially finite rank elements are the finite sums of the spatially rank one elements.

### 2.1 Basic properties of spatially rank one elements

In this section we present the basic properties of spatially rank one elements. The main results are Example 2.1.14, Example 2.1.17 and Theorem 2.1.18.

We begin with the definition of a spatially rank one element. This concept was introduced by J. Puhl in 1978, and the terminology was introduced by R. Harte in 1995.

**Definition 2.1.1** *Let  $A$  be a semiprime Banach algebra and  $u$  a non-zero*

element of  $A$ . We say that  $u$  is a spatially rank one element of  $A$  if there exists a linear functional  $f_u$  on  $A$  such that  $uxu = f_u(x)u$  for all  $x \in A$ .

One motivation for this definition is that for the Banach algebra  $A = \mathcal{L}(X)$  of bounded linear operators on a Banach space  $X$ , the spatially rank one elements coincide with the one-dimensional operators. We will prove this shortly.

The set of spatially rank one elements of  $A$  will be denoted by  $\mathcal{F}_1$ .

**Proposition 2.1.2** *Let  $A$  be a semiprime Banach algebra and let  $u \in \mathcal{F}_1$ . Then  $uAu = \mathbb{C}u$ .*

**Proof.** From the definition of spatially rank one,  $uAu \subset \mathbb{C}u$ . To show the other inclusion, let  $\lambda u \in \mathbb{C}u$ . Now since  $A$  is semiprime and  $u \in \mathcal{F}_1$ , there exists a  $y \in A$  such that  $uyu \neq 0$  and  $uyu = u$ . Therefore  $\lambda u = u(\lambda y)u \in uAu$ . Hence  $\mathbb{C}u \subset uAu$ .  $\square$

We prove in the next proposition that the linear functional in the definition of spatially rank one elements is unique and bounded (hence it is continuous).

**Proposition 2.1.3** *Let  $A$  be a semiprime Banach algebra and let  $u \in \mathcal{F}_1$ . Then the linear functional  $f_u$  in Definition 2.1.1 is unique and bounded (hence it is continuous).*

**Proof.** Suppose that  $f'_u$  is another linear functional satisfying  $uxu = f'_u(x)u$  for all  $x \in A$ . Then we have that  $f_u(x)u = f'_u(x)u$  for all  $x \in A$ , i.e.  $(f_u(x) - f'_u(x))u = 0$  for all  $x \in A$ . Since  $u \neq 0$ , it follows that  $f_u = f'_u$ , proving uniqueness.

To show that  $f_u$  is bounded, we have from  $uxu = f_u(x)u$  for all  $x \in A$  that  $\|f_u(x)u\| = |f_u(x)|\|u\| = \|uxu\| \leq \|u\|\|x\|\|u\|$ . Since  $u$  is non-zero,  $\|u\| \neq 0$ , so that  $|f_u(x)| \leq \|u\|\|x\|$ . Taking  $C = \|u\|$ , we obtain that  $|f_u(x)| \leq C\|x\|$ , so that  $f_u$  is bounded.  $\square$

We have the following simple propositions.

**Proposition 2.1.4** *Every non-zero element of the Banach algebra  $\mathbb{C}$  is a spatially rank one element of  $\mathbb{C}$ .*

**Proof.** Let  $\lambda$  be any non-zero point in  $\mathbb{C}$ . For any point  $\mu \in \mathbb{C}$ , we can always find a complex number  $\alpha(\mu)$  such that  $\lambda\mu\lambda = \alpha(\mu)\lambda$ . Take  $f_\lambda(\mu) = \alpha(\mu)$ . ■

**Proposition 2.1.5** *If  $A$  is a one-dimensional Banach algebra, then every non-zero element of  $A$  is a spatially rank one element of  $A$ .*

**Proof.** Let  $a$  be a non-zero element of  $A$ . Since  $A$  is one dimensional, we have that  $A = \mathbb{C} \cdot 1$ . This implies that  $ax = \lambda(x) \cdot 1$  for all  $x \in A$  and some  $\lambda(x) \in \mathbb{C}$ . Hence  $axa = \lambda(x)a$  for all  $x \in \mathbb{C}$ . Take  $f_a(x) = \lambda(x)$ . ■

Following is the definition of the *trace* of a spatially rank one element. This plays a vital role in determining some of the properties of spatially rank one elements.

**Definition 2.1.6** *Let  $A$  be a semiprime Banach algebra and let  $u \in \mathcal{F}_1$ . The complex number  $tr(u)$  defined by  $u^2 = tr(u)u$  is called the trace of  $u$ .*

**Proposition 2.1.7** *Suppose that  $A$  is a semiprime Banach algebra and that  $u \in \mathcal{F}_1$ . Then  $tr(u)$  is unique and  $tr(u) = f_u(1)$ .*

**Proof.** Let  $u \in \mathcal{F}_1$  and suppose that  $tr'(u)$  also satisfies  $u^2 = tr'(u)u$ . Then  $tr(u)u = tr'(u)u$ , so that  $(tr(u) - tr'(u))u = 0$ . Since  $u \neq 0$ , we have that  $tr(u) = tr'(u)$ .

To show that  $tr(u) = f_u(1)$ , we note that since  $u \in \mathcal{F}_1$ , we have that  $uxu = f_u(x)u$  for all  $x \in A$ . Taking  $x = 1$ , it follows that  $u^2 = f_u(1)u$ . By uniqueness of  $tr(u)$ , we obtain that  $f_u(1) = tr(u)$ . ■

In the case of commutative semiprime Banach algebras, the trace enables us to get the following simplification for the definition of spatially rank one elements.

**Proposition 2.1.8** ([21], Remark 2.3) *Let  $A$  be a commutative semiprime Banach algebra and let  $0 \neq u \in A$ . Then  $u \in \mathcal{F}_1$  if and only if there exists a linear functional  $g_u$  on  $A$  such that  $ux = g_u(x)u$  for all  $x \in A$ .*

**Proof.** Suppose that  $u \in \mathcal{F}_1$ . Then there exists a linear functional  $f_u$  such that  $uxu = f_u(x)u$  for all  $x \in A$ . Since  $A$  is commutative, it follows that  $u^2x = f_u(x)u$ . Since  $u^2 = f_u(1)u$  by Proposition 2.1.7, this implies that  $f_u(1)ux = f_u(x)u$ . If  $f_u(1) = 0$ , then  $uxu = 0$  for all  $x \in A$ . Since  $A$

is semiprime, this implies that  $u = 0$ , which is a contradiction. Therefore  $f_u(1) \neq 0$ , so that  $ux = \frac{f_u(x)u}{f_u(1)}$ . Let  $g_u(x) = \frac{f_u(x)}{f_u(1)}$ . Then  $g_u$  is a linear functional on  $A$ , and  $ux = g_u(x)u$  for all  $x \in A$ .

Conversely, suppose that  $g_u$  is a linear functional on  $A$  such that  $ux = g_u(x)u$  for all  $x \in A$ . Then  $u(xu) = uxu = g_u(xu)u$  for all  $x \in A$ . Let  $f_u(x) = g_u(xu)$ . Then  $f_u$  is a linear functional on  $A$ , and  $uxu = f_u(x)u$  for all  $x \in A$ . This means that  $u \in \mathcal{F}_1$ .  $\square$

The following lemma will be needed to prove several results in this and subsequent chapters.

**Lemma 2.1.9** *Let  $A$  be a semiprime Banach algebra and let  $u \in \mathcal{F}_1$  be non-nilpotent. Then  $\frac{u}{\text{tr}(u)}$  is a non-zero idempotent in  $\mathcal{F}_1$ .*

**Proof.** Since  $u$  is non-nilpotent, we have that  $u^2 \neq 0$ . It follows from  $u^2 = \text{tr}(u)u$  that  $\text{tr}(u) \neq 0$ . Therefore  $(\frac{u}{\text{tr}(u)})^2 = \frac{u^2}{(\text{tr}(u))^2} = \frac{\text{tr}(u)u}{(\text{tr}(u))^2} = \frac{u}{\text{tr}(u)}$ , so that  $\frac{u}{\text{tr}(u)}$  is an idempotent. Clearly,  $\frac{u}{\text{tr}(u)} \neq 0$ .  $\square$

**Corollary 2.1.10** *Let  $A$  be a semiprime Banach algebra and let  $u \in \mathcal{F}_1$  be non-nilpotent. Then  $w^{-1}\frac{u}{\text{tr}(u)}w$  is an idempotent for all  $w \in \text{Exp}(A)$ .*

**Proof.** Since  $\frac{u}{\text{tr}(u)}$  is an idempotent by Lemma 2.1.9, we have that

$$\begin{aligned} \left(w^{-1}\frac{u}{\text{tr}(u)}w\right)^2 &= \left(w^{-1}\frac{u}{\text{tr}(u)}w\right)\left(w^{-1}\frac{u}{\text{tr}(u)}w\right) \\ &= w^{-1}\frac{u^2}{(\text{tr}(u))^2}w \\ &= w^{-1}\frac{u}{\text{tr}(u)}w, \end{aligned}$$

as required.  $\square$

The trace also enables us to get the following result, which we will need later in this chapter.

**Lemma 2.1.11** *Let  $A$  be a semiprime Banach algebra and let  $u \in \mathcal{F}_1$ . If  $0 \neq \alpha \in \mathbb{C}$  and  $\alpha u$  is an idempotent, then  $\alpha = \frac{1}{\text{tr}(u)}$ .*



**Proof.** Since  $\alpha u$  is an idempotent,  $(\alpha u)^2 = \alpha u$ . But also,  $(\alpha u)^2 = \alpha^2 u^2 = \alpha^2 \text{tr}(u)u$ . This implies that  $\alpha \text{tr}(u) = 1$ , so that  $\alpha = \frac{1}{\text{tr}(u)}$ . ■

Using the trace, we get the following simplification for nilpotent elements that are spatially of rank one. This result will be used in the proof of Lemma 2.5.1.

**Lemma 2.1.12** *Let  $A$  be a semiprime Banach algebra and let  $u \in \mathcal{F}_1$ . Then  $u$  is nilpotent if and only if  $u^2 = 0$ .*

**Proof.** Suppose that  $u$  is nilpotent. Then there exists a smallest integer  $n \geq 2$  such that  $u^n = 0$ . Now since  $u \in \mathcal{F}_1$ , we have that  $u^2 = \text{tr}(u)u$ . If  $\text{tr}(u) = 0$  then  $u^2 = 0$ . If  $\text{tr}(u) \neq 0$  then from  $u^n = 0$  and  $u^2 = \text{tr}(u)u$ , we get that  $\text{tr}(u)u^{n-1} = u^2 u^{n-2} = u^n = 0$ . Since  $\text{tr}(u) \neq 0$ , it follows that  $u^{n-1} = 0$ , which is a contradiction. Hence  $\text{tr}(u)$  is always 0, and using the same argument as before,  $u^2 = 0$ .

Conversely, if  $u^2 = 0$  then  $u$  is nilpotent by definition. ■

Recall that a Banach algebra  $A$  is semisimple if  $\text{Rad}(A) = \{0\}$ . Since  $0 \notin \mathcal{F}_1$ , this means that in semisimple Banach algebras,  $\mathcal{F}_1$  and  $\text{Rad}(A)$  are disjoint. In the following theorem we prove that in semiprime Banach algebras which are not semisimple,  $\mathcal{F}_1$  and  $\text{Rad}(A)$  are still disjoint.

**Theorem 2.1.13** *([11], p.298) Let  $A$  be a semiprime Banach algebra which is not semisimple. Then  $\mathcal{F}_1 \cap \text{Rad}(A) = \emptyset$ .*

**Proof.** Since  $A$  is not semisimple,  $\text{Rad}(A) \neq \{0\}$ . So let  $0 \neq a \in \text{Rad}(A)$ . If  $a \in \mathcal{F}_1$  then  $axa = f_a(x)a$  for all  $x \in A$ . Since  $A$  is semiprime there exists a  $y \in A$  such that  $aya = f_a(y)a \neq 0$ , so that  $f_a(y) \neq 0$ . Now, the element  $\frac{ay}{f_a(y)}$  is an idempotent since  $\left(\frac{ay}{f_a(y)}\right)^2 = \frac{ayay}{(f_a(y))^2} = \frac{f_a(y)ay}{(f_a(y))^2} = \frac{ay}{f_a(y)}$ . Since  $a \in \text{Rad}(A)$  and  $\text{Rad}(A)$  is a two-sided ideal,  $\frac{ay}{f_a(y)} \in \text{Rad}(A) \subset \text{QN}(A)$ , so that  $\text{Sp}\left(\frac{ay}{f_a(y)}\right) = \{0\}$ . If  $\frac{ay}{f_a(y)} = 0$  then  $\frac{aya}{f_a(y)} = a = 0$ , which is a contradiction. Therefore  $\frac{ay}{f_a(y)} \neq 0$ , so that  $1 \in \text{Sp}\left(\frac{ay}{f_a(y)}\right)$ . This contradicts  $\frac{ay}{f_a(y)} \in \text{Rad}(A)$ . Therefore  $a \notin \mathcal{F}_1$ . ■

Recall that in the comment following the definition of spatially rank one elements, we remarked that one motivation for the definition is that in the

Banach algebra  $\mathcal{L}(X)$  of bounded linear operators on a Banach space  $X$ , the spatially rank one elements coincide with the one-dimensional operators. In [21] J. Puhl stated this, although he did not prove it. We prove this fact in the next example. In order to do so, first note that  $\mathcal{L}(X)$  is semisimple ([2], Theorem 3.1.4).

**Example 2.1.14** ([21], Proposition 2.6) *Let  $X$  be a Banach space and let  $A = \mathcal{L}(X)$ . Then the spatially rank one elements of  $A$  are the one-dimensional operators.*

**Proof.** Since  $A$  is a semisimple, it follows from Theorem 1.3.4 that  $A$  is semiprime. Now suppose that  $T \in \mathcal{F}_1$ . Then there exists a linear functional  $f_T$  on  $A$  such that  $TST = f_T(S)T$  for all  $S \in A$ . Let  $M_T : A \rightarrow A$  be the map defined by  $M_T(S) = TST$  for all  $S \in A$ . Then  $M_T(S) = f_T(S)T$  for all  $S \in A$ . This implies that  $M_T(A) = \text{span}\{T\}$ , so that

$$\dim(M_T(A)) = 1. \quad (2.1.15)$$

Suppose that  $Tx$  and  $Ty$  are linearly independent for some  $x, y \in X$ . Let  $E = \text{span}\{Tx, Ty\}$  and define  $f(Tx) = 1, f(Ty) = 0, g(Tx) = 0, g(Ty) = 1$ . Then  $f$  and  $g$  can be extended linearly and continuously to  $X$  by the Hahn-Banach Theorem. Now consider the equation

$$\lambda TS_1T + \mu TS_2T = 0, \quad (2.1.16)$$

with  $S_1 = f \otimes x$  and  $S_2 = g \otimes y$ , where  $S_1z = f(z)x$  and  $S_2z = g(z)y$ . Then we have that  $(\lambda TS_1T + \mu TS_2T)x = 0$ . Since  $S_1(Tx) = f(Tx)x$  and  $S_2(Tx) = g(Tx)y$ , it follows from  $f(Tx) = 1$  and  $g(Tx) = 0$  that  $S_1(Tx) = x$  and  $S_2(Tx) = 0$ . This implies that  $\lambda Tx = 0$ . Similarly,  $(\lambda TS_1T + \mu TS_2T)y = 0$  yields  $\mu Ty = 0$ . So we have that  $\lambda Tx + \mu Ty = 0$ . Since  $Tx$  and  $Ty$  are linearly independent, it follows that  $\lambda = 0 = \mu$ . From the fact that (2.1.16) implies that  $\lambda = 0 = \mu$ , it follows that  $TS_1T$  and  $TS_2T$  are linearly independent elements of  $M_T(A)$ . This contradicts (2.1.15). Hence  $Tx$  and  $Ty$  are linearly dependent, so that  $\dim(T(X)) \leq 1$ . Since  $T \neq 0$ , it follows that  $\dim(T(X)) = 1$ . So we have shown that if  $T \in \mathcal{F}_1$  then  $T$  is a one-dimensional operator.

Conversely, suppose that  $T \in A$  is a one-dimensional operator. Then  $\dim(T(X)) = 1$ . This means that there exists  $x_0 \in X$  such that  $Tx =$

$\lambda(x)Tx_0$  for all  $x \in X$  and for  $\lambda(x) \in \mathbb{C}$ . Hence

$$\begin{aligned} (TST)x &= T(STx) \\ &= T(\lambda(x)STx_0) \\ &= \lambda(x)T(STx_0) \\ &= \lambda(x)\lambda(STx_0)Tx_0 \\ &= \lambda(STx_0)Tx. \end{aligned}$$

Let  $f_T(S) = \lambda(STx_0)$ . Then  $f_T$  is a linear functional on  $A$  and we have that  $TST = f_T(S)T$ , so that  $T$  is a spatially rank one element of  $A$ .  $\square$

One of the most important Banach algebras is  $M_2(\mathbb{C})$ , the Banach algebra of complex two by two matrices. Having seen the nature of the spatially rank one elements in the Banach algebra  $\mathcal{L}(X)$  of bounded linear operators on Banach space  $X$ , it is natural to seek to know the nature of the spatially rank one elements in  $M_2(\mathbb{C})$ . The following example shows that in this algebra, the spatially rank one elements are the non-invertible elements.

**Example 2.1.17** ([7], Example 3.4.3) Let  $A = M_2(\mathbb{C})$ , with standard matrix addition, scalar multiplication, multiplication and with norm

$$\|M\| = \sup\{\|Mx\| : x \in \mathbb{C}^2, \|x\| \leq 1\},$$

where the norm in  $\mathbb{C}^2$  is the Euclidean norm. Then the spatially rank one elements of  $A$  are the non-invertible elements.

**Proof.** We first show that  $A$  is semiprime. Let  $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$  and  $y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$  be arbitrary elements of  $A$ , with  $xyx = 0$ . Taking  $y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ , we have that  $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ . This gives us that  $\begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_3 & x_2x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ , which yields  $x_1 = 0$ . Repeating this procedure using  $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ ,  $y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and  $y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , we get that  $x_2 = 0$ ,  $x_3 =$

0 and  $x_4 = 0$  respectively. This implies that  $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ . It follows from Theorem 1.3.3 that  $A$  is semiprime.

Now suppose that  $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$  is a non-invertible element of  $A$ . Then  $\det(x) = x_1x_4 - x_2x_3 = 0$ , so that  $x_1x_4 = x_2x_3$ . So if  $y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$  is any element in  $A$ , then we have that

$$\begin{aligned} xyx &= \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \\ &= \begin{pmatrix} (x_1y_1 + x_2y_3 + x_3y_2 + x_4y_4)x_1 & (x_1y_1 + x_2y_3 + x_3y_2 + x_4y_4)x_2 \\ (x_1y_1 + x_2y_3 + x_3y_2 + x_4y_4)x_3 & (x_1y_1 + x_2y_3 + x_3y_2 + x_4y_4)x_4 \end{pmatrix} \\ &= (x_1y_1 + x_2y_3 + x_3y_2 + x_4y_4) \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \\ &= \lambda_x(y)x, \end{aligned}$$

where  $\lambda_x(y) = x_1y_1 + x_2y_3 + x_3y_2 + x_4y_4$ . Take  $f_x(y) = \lambda_x(y)$ . Then  $xyx = \lambda_x(y)x = f_x(y)x$ , which means that  $x$  is a spatially rank one element of  $A$ .

Conversely, suppose that  $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$  is a spatially rank one element of  $A$ . Then there exists a linear functional  $f_x$  on  $A$  such that  $xyx = f_x(y)x$  for all  $y \in A$ . Take  $y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . Then

$$f_x(y) \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_3 & x_2x_3 \end{pmatrix}.$$

Therefore

$$\det \begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_3 & x_2x_3 \end{pmatrix} = \det f_x(y) \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = f_x(y) \det \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

But  $\det \begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_3 & x_2x_3 \end{pmatrix} = x_1^2x_2x_3 - x_1^2x_2x_3 = 0$ . Therefore

$$f_x(y) \det \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = f_x(y)(x_1x_4 - x_2x_3) = 0.$$

If  $f_x(y) = 0$  for all  $y \in A$ , then  $xyx = f_x(y)x = 0$  and from Theorem 1.3.3 we have that  $x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ , which is non-invertible. If  $f_x(y) \neq 0$  for some  $y \in A$ , then  $\det(x) = x_1x_4 - x_2x_3 = 0$ , so that  $x$  is non-invertible. Therefore spatially rank one elements of  $A$  are non-invertible.  $\blacksquare$

Having seen the nature of the spatially rank one elements in two of the most important Banach algebras  $\mathcal{L}(X)$  of bounded linear operators on a Banach space  $X$ , and  $M_2(\mathbb{C})$  of complex two by two matrices, we proceed to see how we can use a spatially rank one element to construct other spatially rank one elements of a semiprime Banach algebra. The following result due to J. Puhl shows that all non-zero products of spatially rank one elements and other elements in the Banach algebra are spatially of rank one.

**Theorem 2.1.18** ([21], Lemma 2.7) *Let  $A$  be a semiprime Banach algebra and let  $u \in \mathcal{F}_1$ . If  $x, y \in A$  such that  $xvy \neq 0$ , then  $xuy \in \mathcal{F}_1$ .*

**Proof.** Suppose that  $u \in \mathcal{F}_1$  and  $x, y \in A$  with  $xvy \neq 0$ . For any  $t \in A$ , we have that  $(xuy)t(xuy) = xu(ytx)uy$ . Since  $u \in \mathcal{F}_1$ , it follows that  $xu(ytx)uy = f_u(ytx)xuy$ . Taking  $f_{xuy}(t) = f_u(ytx)$ , the result follows.  $\blacksquare$

**Corollary 2.1.19** *Let  $A$  be a semiprime Banach algebra. If  $0 \neq \alpha \in \mathbb{C}$  and  $u \in \mathcal{F}_1$ , then  $\alpha u \in \mathcal{F}_1$  and  $\text{tr}(\alpha u) = \alpha \text{tr}(u)$ .*

**Proof.** Since  $\alpha \neq 0$  and  $u \neq 0$ , we have that  $\alpha u \neq 0$ . It follows from Theorem 2.1.18 that  $\alpha u \in \mathcal{F}_1$ . Therefore  $(\alpha u)^2 = \text{tr}(\alpha u)\alpha u = \alpha \text{tr}(\alpha u)u$ . Also,  $(\alpha u)^2 = \alpha^2 u^2 = \alpha^2 \text{tr}(u)u$ . This implies that  $\alpha^2 \text{tr}(u) = \alpha \text{tr}(\alpha u)$ , so that  $\alpha \text{tr}(u) = \text{tr}(\alpha u)$ .  $\blacksquare$

The next result also follows from Theorem 2.1.18.

**Corollary 2.1.20** *Let  $A$  be a semiprime Banach algebra and let  $u \in \mathcal{F}_1$ . If  $a$  is any element in  $A$  such that  $au \neq 0$ , then  $\text{tr}(au) = f_u(a)$ . A similar statement holds for  $ua$ .*

**Proof.** Let  $u \in \mathcal{F}_1$  and  $a \in A$  such that  $au \neq 0$ . Then from Theorem 2.1.18 we obtain that  $au \in \mathcal{F}_1$ . By definition of  $\text{tr}(au)$  and from  $u \in \mathcal{F}_1$ , we have that  $\text{tr}(au)au = (au)^2 = a(ua) = f_u(a)au$ . This implies that  $\text{tr}(au) = f_u(a)$ .  $\blacksquare$

## 2.2 Spatially rank one elements and minimal idempotents

This section is aimed at introducing a special class of spatially rank one elements, the minimal idempotents. We then go on to show how minimal idempotents relate spatially rank one elements to minimal left ideals. The main result of the section is Theorem 2.2.3.

In the following result we prove that minimal idempotents are spatially rank one elements.

**Lemma 2.2.1** ([21], Remark 2.4) *If  $p$  is a minimal idempotent in a semiprime Banach algebra  $A$ , then  $p \in \mathcal{F}_1$ .*

**Proof.** Suppose that  $p \in A$  is a minimal idempotent. Then by definition  $pAp$  is a division algebra. It follows from Theorem 1.4.5 that  $pAp = \mathbb{C}p$ . Hence there exists a linear functional  $f_p$  on  $A$  such that  $pxp = f_p(x)p$  for all  $x \in A$ .  $\square$

The converse of this result is generally not true. However, if an element is both an idempotent and spatially of rank one, then it is a minimal idempotent. This is the next result.

**Lemma 2.2.2** *Let  $A$  be a semiprime Banach algebra and let  $p \in A$ . If  $p$  is an idempotent and  $p \in \mathcal{F}_1$ , then  $p$  is a minimal idempotent in  $A$ .*

**Proof.** Suppose that  $p$  is an idempotent in  $A$  and that  $p \in \mathcal{F}_1$ . Let  $0 \neq pxp \in pAp$ . Then  $pxp = f_p(x)p$  with  $f_p(x) \neq 0$ . Therefore

$$(p \frac{1}{f_p(x)} p)(pxp) = (pxp)(p \frac{1}{f_p(x)} p) = f_p(x)(p \frac{1}{f_p(x)} p)p = p^3 = p,$$

so that  $pxp$  is invertible in  $pAp$ .  $\square$

As we remarked at the beginning of this section, minimal idempotents provide a vital link between the set of spatially rank one elements and the minimal left ideals. In 1978 J. Puhl showed this link in the proof of ([21], Remark 2.5). Stated more explicitly, his result says that: Every minimal left ideal of a semiprime Banach algebra contains a minimal idempotent. Also, every spatially rank one element of a semiprime Banach algebra is contained

in a minimal left ideal of the Banach algebra. The first part of this result was actually proved by C.E. Rickart earlier in 1960. Following is this result as proved by J. Puhl.

**Theorem 2.2.3** ([21], Remark 2.5) *Let  $A$  be a semiprime Banach algebra. Then we have the following:*

(1) *If  $L$  is a minimal left ideal of  $A$  then  $L$  contains a minimal idempotent  $p$  such that  $L = Ap$ .*

(2) *If  $u \in \mathcal{F}_1$  then  $L = Au$  is a minimal left ideal of  $A$ .*

**Proof.** (1) Let  $\{0\} \neq L$  be a minimal left ideal of  $A$ . Since  $A$  is semiprime,  $L^2 \neq \{0\}$ . This means that there exists  $z \in L$  such that  $Lz \neq \{0\}$ . Obviously,  $Lz$  is a left ideal. Since  $z \in L$ , we have that  $Lz \subset L$ . Since  $L$  is a minimal left ideal, it follows that  $Lz = L$ . This implies that there exists  $p \in L$  such that

$$pz = z. \quad (2.2.4)$$

Clearly,  $L(p-1)$  is also a left ideal contained in  $L$ . So either  $L(p-1) = \{0\}$  or  $L(p-1) = L$ . If  $L(p-1) = L$  then there exists  $f \in L$  such that  $fp - f = p$ . From (2.2.4) we have that  $(p-1)z = 0$ . This implies that  $z = pz = f(p-1)z = 0$ . Hence  $Lz = \{0\}$ , which is a contradiction. Therefore  $L(p-1) = \{0\}$ . It follows that  $p(p-1) = 0$ , so that  $p$  is an idempotent. If  $p = 0$  then  $L(p-1) = \{0\} = L(-1)$ , which implies that  $L = \{0\}$ . This is a contradiction. Therefore  $p \neq 0$  and since  $p \in L$ , we obtain that  $\{0\} \neq Ap \subset L$ . Since  $L$  is a minimal left ideal, this implies that  $L = Ap$ . We have shown that  $L$  contains an idempotent  $p$  such that  $L = Ap$ .

We show that  $p$  is a minimal idempotent. From Theorem 1.4.17 we have that  $pAp$  is an algebra with unit  $p$ . So there exists  $d \in A$  such that  $pdp \neq 0$ . Now  $pdp = p(pdp) \in Apdp$ , so that  $Apdp \neq \{0\}$ . Since  $pdp \in L$ , we have that  $Apdp \subset L$ . Since  $L$  is a minimal left ideal, this implies that  $Apdp = L$ . Hence there exists  $g \in A$  such that  $gpdp = p$ . Therefore  $pgpdp = p^2 = p$ , i.e.  $(pgp)(pdp) = p$ . Hence every non-zero element  $pdp$  of  $pAp$  has a left inverse. This implies that  $pAp$  is a division algebra.

(2) Suppose that  $u \in \mathcal{F}_1$ . Let  $L = Au$  and suppose that  $\{0\} \neq J$  is a left ideal of  $A$  such that  $J \subset L$ . Then each non-zero element of  $J$  is of the

form  $yu$ . Since  $A$  is semiprime, there exists  $z \in A$  such that

$$yuzyu \neq 0. \quad (2.2.5)$$

This implies that  $zy \neq 0$ . Since  $u \in \mathcal{F}_1$ , for any  $x \in A$  we have that  $f_u(zy)xu = xuzyu$ . If  $f_u(zy) = 0$ , then  $xuzyu = 0$  for all  $x \in A$ . If we take  $x = 1$ , we have that  $uzyu = 0$ , which implies that  $yuzyu = 0$ . This contradicts (2.2.5). Hence  $f_u(zy) \neq 0$ , so that  $xu = \frac{1}{f_u(zy)}xuzyu \in J$ . Since  $x$  is an arbitrary element of  $A$ , we get that  $L = Au \subset J$ . This proves that  $L$  is a minimal left ideal.  $\square$

## 2.3 Spectrum of a spatially rank one element

One of the key concepts relating to Banach algebra elements is the concept of the spectrum, introduced in Chapter 1. This section is devoted to the study of the spectrum of a spatially rank one element. The main results here are Theorem 2.3.1 and Theorem 2.3.6.

In 1978 J. Puhl proved a result that characterizes the spectrum of a spatially rank one element ([21], Lemma 2.8). His proof relies on a property of elements of Banach algebras called *quasi-inverses*. In the following theorem, we prove Puhl's result. However, here it is stated and proved slightly differently.

**Theorem 2.3.1** ([21], Lemma 2.8) *Let  $A$  be a semiprime Banach algebra and let  $u \in \mathcal{F}_1$ . Then we have the following:*

- (1)  $\text{Sp}(u) \subset \{0, \text{tr}(u)\}$ ,
- (2)  $\dim(A) = 1 \Rightarrow \text{Sp}(u) = \{\text{tr}(u)\}$ ,
- (3)  $\dim(A) \geq 2 \Rightarrow \text{Sp}(u) = \{0, \text{tr}(u)\}$ .

**Proof.** (1) Let  $u \in \mathcal{F}_1$ . Then  $u^2 - \text{tr}(u)u = 0$ . It follows from the spectral mapping theorem that  $\{0\} = \text{Sp}(u^2 - \text{tr}(u)u) = \{\lambda^2 - \text{tr}(u)\lambda : \lambda \in \text{Sp}(u)\}$ . So if  $\lambda \in \text{Sp}(u)$ , then  $\lambda^2 - \text{tr}(u)\lambda = 0$ , so that  $\lambda = 0$  or  $\lambda = \text{tr}(u)$ . This implies that  $\text{Sp}(u) \subset \{0, \text{tr}(u)\}$ .

(2) Let  $u \in \mathcal{F}_1$  and suppose that  $\dim(A) = 1$ . Then there exists  $0 \neq \alpha \in \mathbb{C}$



such that  $u = \alpha 1$ . This implies that  $\text{Sp}(u) = \alpha \text{Sp}(1) = \{\alpha\}$ . From  $u^2 = \text{tr}(u)u$  and  $u = \alpha 1$ , we have  $\alpha = \text{tr}(u)$ , so that  $\text{Sp}(u) = \{\alpha\} = \{\text{tr}(u)\}$ .

(3) Let  $u \in \mathcal{F}_1$ . Then we have from (1) that  $\text{Sp}(u) \subset \{0, \text{tr}(u)\}$ . We prove the inclusion  $\{0, \text{tr}(u)\} \subset \text{Sp}(u)$ . Since  $u \in \mathcal{F}_1$ , there exists a linear functional  $f_u$  such that  $uxu = f_u(x)u$  for all  $x \in A$ . If  $0 \notin \text{Sp}(u)$  then  $u$  is invertible. This means that  $u^2u^{-1} = \text{tr}(u)uu^{-1}$ , so that  $u = \text{tr}(u) \cdot 1$ . It follows that  $\text{tr}(u)x\text{tr}(u) = f_u(x)\text{tr}(u)$ . Since  $u \neq 0$  we have that  $\text{tr}(u) \neq 0$ . Therefore  $x = \frac{f_u(x) \cdot 1}{\text{tr}(u)}$  for all  $x \in A$ . This implies that  $\dim(A) = 1$ , which is a contradiction. Hence  $0 \in \text{Sp}(u)$ .

Now suppose that  $\text{tr}(u) \notin \text{Sp}(u)$ . Then  $u - \text{tr}(u) \cdot 1$  is invertible. From  $u^2 = \text{tr}(u)u$  it follows that  $u = (u - \text{tr}(u) \cdot 1)^{-1}(u - \text{tr}(u) \cdot 1)u = 0$ , which is a contradiction. This together with  $0 \in \text{Sp}(u)$  means that  $\{0, \text{tr}(u)\} \subset \text{Sp}(u)$ , so that  $\text{Sp}(u) = \{0, \text{tr}(u)\}$ .  $\square$

**Lemma 2.3.2** *The trace of a spatially rank one element of a semiprime Banach algebra  $A$  is a continuous function.*

**Proof.** Let  $u \in \mathcal{F}_1$ . Then  $\text{Sp}(u) = \{0, \text{tr}(u)\}$  by Theorem 2.3.1. It follows from Theorem 1.6.2 that the function  $u \mapsto \text{Sp}(u)$  is continuous on  $\mathcal{F}_1$ . Therefore if  $u_0$  is any element in  $\mathcal{F}_1$ , then given any  $\epsilon > 0$  there exists a  $\delta > 0$  such that  $\|u - u_0\| < \delta$  with  $u \in \mathcal{F}_1$  implies that  $\Delta(\text{Sp}(u), \text{Sp}(u_0)) < \frac{\epsilon}{2}$ , i.e.  $\max \left( \sup_{z \in \text{Sp}(u_0)} \text{dist}(z, \text{Sp}(u)), \sup_{z \in \text{Sp}(u)} \text{dist}(z, \text{Sp}(u_0)) \right) < \frac{\epsilon}{2}$ . Now from  $\text{Sp}(u_0) = \{0, \text{tr}(u_0)\}$  and  $\text{Sp}(u) = \{0, \text{tr}(u)\}$  we have that

$$\begin{aligned} \sup_{z \in \text{Sp}(u_0)} \text{dist}(z, \text{Sp}(u)) &= \sup_{z \in \text{Sp}(u_0)} \text{dist}(z, \{0, \text{tr}(u)\}) \\ &= \sup\{\inf\{0, |\text{tr}(u)|\}, \inf\{|\text{tr}(u_0)|, |\text{tr}(u) - \text{tr}(u_0)|\}\} \\ &= \sup\{0, \inf\{|\text{tr}(u_0)|, |\text{tr}(u) - \text{tr}(u_0)|\}\} \\ &= \inf\{|\text{tr}(u_0)|, |\text{tr}(u) - \text{tr}(u_0)|\}. \end{aligned}$$

Similarly,  $\sup_{z \in \text{Sp}(u)} \text{dist}(z, \text{Sp}(u_0)) = \inf\{|\text{tr}(u)|, |\text{tr}(u_0) - \text{tr}(u)|\}$ . Therefore

$\Delta(\text{Sp}(u), \text{Sp}(u_0)) = \max(\inf\{|tr(u)|, |tr(u) - tr(u_0)|\}, \inf\{|tr(u_0)|, |tr(u_0) - tr(u)|\}) < \frac{\epsilon}{2}$  for  $u \in \mathcal{F}_1$  such that  $\|u - u_0\| < \delta$ .

There are four cases:

(i) If  $\inf\{|tr(u)|, |tr(u) - tr(u_0)|\} = |tr(u)|$  and  $\inf\{|tr(u_0)|, |tr(u_0) - tr(u)|\} = |tr(u_0)|$ , then  $|tr(u_0)| < \frac{\epsilon}{2}$  and  $|tr(u)| < \frac{\epsilon}{2}$ , so that  $|tr(u) - tr(u_0)| \leq |tr(u)| + |tr(u_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ .

(ii) If  $\inf\{|tr(u)|, |tr(u) - tr(u_0)|\} = |tr(u)|$  and  $\inf\{|tr(u_0)|, |tr(u_0) - tr(u)|\} = |tr(u_0) - tr(u)|$  then  $|tr(u) - tr(u_0)| = |tr(u_0) - tr(u)| < \frac{\epsilon}{2} < \epsilon$ .

(iii) If  $\inf\{|tr(u)|, |tr(u) - tr(u_0)|\} = |tr(u) - tr(u_0)|$  and  $\inf\{|tr(u_0)|, |tr(u_0) - tr(u)|\} = |tr(u_0)|$  then  $|tr(u) - tr(u_0)| < \frac{\epsilon}{2} < \epsilon$ .

(iv) If  $\inf\{|tr(u)|, |tr(u) - tr(u_0)|\} = |tr(u) - tr(u_0)|$  and  $\inf\{|tr(u_0)|, |tr(u_0) - tr(u)|\} = |tr(u_0) - tr(u)|$  then  $|tr(u) - tr(u_0)| < \frac{\epsilon}{2} < \epsilon$ .

Therefore in all cases, we have that  $|tr(u) - tr(u_0)| < \epsilon$ . So we have shown that given any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $\|u - u_0\| < \delta$  with  $u \in \mathcal{F}_1$  implies that  $|tr(u) - tr(u_0)| < \epsilon$ . This implies that the trace function is continuous on  $\mathcal{F}_1$ .  $\square$

Note that a key step to prove Lemma 2.3.2 is the fact that the spectrum function is continuous on  $\mathcal{F}_1$ . One can avoid this and prove directly that the trace function is continuous on  $\mathcal{F}_1$  by using sequences and the definition of the trace. The details follow.

**Alternative proof of Lemma 2.3.2:** Let  $(u_n)$  be a sequence in  $\mathcal{F}_1$  with  $u_n \rightarrow u \in \mathcal{F}_1$ . Recall from Definition 2.1.1 that  $u \neq 0$ . We show that  $tr(u_n) \rightarrow tr(u)$ . Firstly suppose that  $u$  is nilpotent. Then  $u^2 = 0$ , so that  $tr(u) = 0$ . We have that  $tr(u_n)u_n = u_n^2 \rightarrow u^2 = 0$ . Also,  $|tr(u_n)||u_n| = \|u_n^2\| \leq \|u_n\|\|u_n\|$ . Since  $\|u_n\| \neq 0$  for all  $n \in \mathbb{N}$ , we get that  $|tr(u_n)| \leq \|u_n\| \leq K$ , where  $K = \sup\{\|u_n\| : n \in \mathbb{N}\}$ . This means that  $tr(u_n)$  is bounded. Let  $(tr(u_{n_k}))$  be a convergent subsequence of  $tr(u_n)$ , say  $tr(u_{n_k}) \rightarrow l$ : The existence of such a subsequence is guaranteed by the boundedness of  $(tr(u_n))$  and the Bolzano-Weierstrass Theorem. We have that  $tr(u_{n_k})u_{n_k} \rightarrow lu$ . Since  $(tr(u_{n_k})u_{n_k})$  is a subsequence of  $(tr(u_n)u_n)$  and  $tr(u_n)u_n \rightarrow 0$ , it follows from uniqueness of limits that  $lu = 0$ . Since  $u \neq 0$ , we obtain that

$l = 0$ . Since  $(tr(u_{n_k}))$  is an arbitrary subsequence of  $(tr(u_n))$ , it follows that  $tr(u_n) \rightarrow 0 = tr(u)$ .

Now suppose that  $u$  is non-nilpotent. In view of Lemma 2.1.9, we can suppose without loss of generality that  $u$  is an idempotent. Then  $u = u^2 = tr(u)u$ , so that  $tr(u) = 1$ . Also  $u_n^2 \rightarrow u^2 = u$ . This implies that  $(tr(u_n) - 1)u_n = u_n^2 - u_n \rightarrow 0$ . Using an argument similar to the one in the first part, we obtain that  $tr(u_n) - 1 \rightarrow 0$ . This implies that  $tr(u_n) \rightarrow 1 = tr(u)$ .  $\square$

In the first section of this chapter, we identified the spatially rank one elements in the Banach algebras  $\mathcal{L}(X)$  of bounded linear operators on a Banach space  $X$  and  $M_2(\mathbb{C})$  of complex two by two matrices (see Example 2.1.14 and Example 2.1.17). An interesting question to ask is whether there are any Banach algebras without spatially rank one elements. We answer this question in Theorem 2.3.3. In 1993 T. Mouton and H. Raubenheimer gave a proof of this result ([19], p.213). Our proof follows along the same lines and relies on Theorem 2.3.1.

**Theorem 2.3.3** *If  $A$  is a semiprime Banach algebra with no non-zero divisors of zero and if  $\dim(A) \geq 2$  then  $\mathcal{F}_1 = \emptyset$ .*

**Proof.** Suppose that there exists a  $u \in \mathcal{F}_1$ . Then  $uxu = f_u(x)u$  for all  $x \in A$ . If  $f_u(1) = 0$  then  $u^2 = 0$ , which means that  $u$  is a non-zero divisor of zero. This is a contradiction since  $A$  has no nonzero divisors of zero. If  $f_u(1) \neq 0$  then  $\text{Sp}(u) = \{0, f_u(1)\}$  by Theorem 2.3.1. It follows from Theorem 1.5.6 that there exists a non-trivial idempotent  $p \in A$ . This is a contradiction since by Proposition 1.1.14, the only idempotents in  $A$  are 0 and 1. Hence  $\mathcal{F}_1 = \emptyset$ .  $\square$

Observe that most of the common Banach algebras including  $\mathcal{L}(X)$ ,  $M_2(\mathbb{C})$ ,  $l^2, l^\infty$  and the function algebras, have non-zero divisors of zero. This makes the condition of having no non-zero divisors of zero in Theorem 2.3.3 very restrictive. As such, the class of Banach algebras without spatially rank one elements is very small.

In 1993 Mouton and Raubenheimer proved that spatially rank one elements satisfy a certain spectral property ([19], Proposition 2.1(3)). Following along similar lines, we prove this result in Theorem 2.3.6. The following lemma will be required.

**Lemma 2.3.4** ([19], Proposition 2.1(2)) *Suppose that  $A$  is a Banach algebra and that  $a \in A$ . If  $b \in A^{-1}$ , then  $b + a \notin A^{-1}$  if and only if  $-1 \in \text{Sp}(b^{-1}a)$ .*

**Proof.** Let  $a \in A$  and  $b \in A^{-1}$ . Suppose that  $b + a \notin A^{-1}$ . Then  $b + a = -b(-1 - b^{-1}a) \notin A^{-1}$ , i.e.  $-1 - b^{-1}a \notin A^{-1}$ . This implies that  $-1 \in \text{Sp}(b^{-1}a)$ .

Conversely, suppose that  $-1 \in \text{Sp}(b^{-1}a)$ . Then  $-1 - b^{-1}a \notin A^{-1}$ , which implies that  $-b(-1 - b^{-1}a) = b + a \notin A^{-1}$  since  $b \in A^{-1}$ .  $\square$

**Corollary 2.3.5** *Let  $A$  be a semiprime Banach algebra and let  $u \in \mathcal{F}_1$ . If  $x \in A^{-1}$  then  $x + u \notin A^{-1}$  if and only if  $f_u(x^{-1}) = -1$ .*

**Proof.** Let  $u \in \mathcal{F}_1$  and  $x \in A^{-1}$ . Then we have from Lemma 2.3.4 that  $x + u \notin A^{-1}$  if and only if  $-1 \in \text{Sp}(x^{-1}u)$ . From Theorem 2.1.18, Theorem 2.3.1(1) and Corollary 2.1.20 it follows that  $x + u \notin A^{-1}$  if and only if  $f_u(x^{-1}) = -1$ .  $\square$

**Theorem 2.3.6** ([19], Proposition 2.1(3)) *Let  $A$  be a semiprime Banach algebra and let  $u \in \mathcal{F}_1$ . Then  $u$  satisfies  $\text{Sp}(x + s_0u) \cap \text{Sp}(x + s_1u) \subset \text{Sp}(x)$  for all  $x \in A$  and for any pair of distinct non-zero scalars  $\{s_0, s_1\}$ .*

**Proof.** Let  $u \in \mathcal{F}_1$  and let  $\{s_0, s_1\}$  be any set of two distinct non-zero scalars. Also let  $x \in A$  and suppose that  $\lambda \notin \text{Sp}(x)$ . Then  $\lambda - x \in A^{-1}$ . So if  $\lambda \in \text{Sp}(x + s_0u)$ , then  $(\lambda - x) - s_0u \notin A^{-1}$ . It follows from Lemma 2.3.5 that  $f_{(-s_0u)}((\lambda - x)^{-1}) = -1$ . From Corollary 2.1.20 and Corollary 2.1.19, we have that

$$\begin{aligned} f_{(-s_0u)}((\lambda - x)^{-1}) &= \text{tr}(-s_0u(\lambda - x)^{-1}) \\ &= -s_0 \text{tr}(u(\lambda - x)^{-1}) \\ &= -s_0 f_u((\lambda - x)^{-1}). \end{aligned}$$

Therefore  $-s_0 f_u((\lambda - x)^{-1}) = -1$ . Similarly, if  $\lambda \in \text{Sp}(x + s_1u)$ , then  $-s_1 f_u((\lambda - x)^{-1}) = -1$ . This implies that  $-s_0 f_u((\lambda - x)^{-1}) = -s_1 f_u((\lambda - x)^{-1}) = -1$ , so that  $s_0 = s_1$ . This is a contradiction. Hence  $\lambda \notin \text{Sp}(x + s_0u) \cap \text{Sp}(x + s_1u)$ , so that  $\text{Sp}(x + s_0u) \cap \text{Sp}(x + s_1u) \subset \text{Sp}(x)$ .  $\blacksquare$

In 1986 A.A. Jafarian and A.R. Sourour showed that if  $A = \mathcal{L}(X)$  of bounded linear operators on a Banach space  $X$ , then the condition  $\text{Sp}(x+s_0u) \cap \text{Sp}(x+s_1u) \subset \text{Sp}(x)$  is sufficient for  $u \in \mathcal{F}_1$  ([16], Theorem 1). In 1993 Mouton and Raubenheimer showed that this result is not true in general. In Theorem 2.3.7 we prove the result of Mouton and Raubenheimer.

**Theorem 2.3.7** ([19], p.214) *Let  $A$  be a commutative semiprime Banach algebra which is not semisimple. Also let  $x \in A$  and  $\{s_0, s_1\}$  any set of two distinct non-zero scalars. If  $0 \neq u$  is an element of  $\text{Rad}(A)$  then  $u$  satisfies  $\text{Sp}(x+s_0u) \cap \text{Sp}(x+s_1u) \subset \text{Sp}(x)$ , but  $u \notin \mathcal{F}_1$ .*

**Proof.** Since  $A$  is commutative, we have from Corollary 1.4.25 that  $\text{Sp}(x+s_0u) \subset \text{Sp}(x) + \text{Sp}(s_0u)$ . Since  $u \in \text{Rad}(A) \subset \text{QN}(A)$ , we have that  $\text{Sp}(s_0u) = \{0\}$ . This implies that  $\text{Sp}(x+s_0u) \subset \text{Sp}(x)$ . Similarly,  $\text{Sp}(x+s_1u) \subset \text{Sp}(x)$ . Hence  $\text{Sp}(x+s_0u) \cap \text{Sp}(x+s_1u) \subset \text{Sp}(x)$ .

Now suppose that  $u \in \mathcal{F}_1$ . Then  $u^2 = f_u(1)u$  and from Theorem 2.3.1, we have that  $\text{Sp}(u) \subset \{0, f_u(1)\} = \{0\}$ . This implies that  $f_u(1) = 0$ , so that  $u^2 = 0$ . Since  $A$  is commutative, this means that  $AuAu = Au^2 = \{0\}$ . But  $Au \neq \{0\}$  as  $u \neq 0$ . By Lemma 1.3.2, this is a contradiction. Hence  $u \notin \mathcal{F}_1$ , as required.  $\square$

## 2.4 One-dimensional operators and spatially rank one elements

In the previous section spectral properties of spatially rank one elements were given, yielding among other things a spectral property satisfied by spatially rank one elements (Theorem 2.3.6). The aim of this section is to give a characterization of spatially rank one elements in terms of one-dimensional operators. The main results of the section are Theorem 2.4.4 and Theorem 2.4.7.

Theorem 2.4.4 due to J. Puhl, gives the one-dimensional operator characterization of spatially rank one elements. The proof of this result relies on the following lemma.

**Lemma 2.4.1** ([21], Lemma 3.2) *Let  $A$  be a semiprime Banach algebra and let  $0 \neq u \in A$  be such that  $\dim(uAu) < \infty$ . Then there exists a minimal idempotent  $p$  in  $Au$  (respectively  $uA$ ).*

**Proof.** Let  $0 \neq u \in A$  and  $0 \neq v \in uAu$ . Then  $v = uxu$  for some  $x \in A$ , and so  $vAv = u(xuAux)u \subset uAu$ . Therefore  $\dim(vAv) \leq \dim(uAu)$ . Let  $v$  be chosen so that  $\dim(vAv)$  is as small as possible. We show that if  $y \in A$  and  $vyv \neq 0$ , then there exists a  $z \in A$  such that  $v = vzvyv$ . To this end let  $y \in A$ , with  $vyv \neq 0$ . Since  $A$  is semiprime, we have that  $\{0\} \neq (vyv)A(vyv) \subset vAv$ . Since  $\dim(vAv)$  is as small as possible, this implies that  $(vyv)A(vyv) = vAv$ . So there is a  $z \in A$  such that

$$vyv = (vyv)z(vyv). \quad (2.4.2)$$

Clearly  $(vzvvyv - v)A(vzvvyv - v) \subset vAv$ . If  $(vzvvyv - v)A(vzvvyv - v) = vAv$ , then

$$\begin{aligned} (vyvzvvyv - vyv)A(vzvvyv - vyv) &= vy(vzvvyv - v)A(vzvvyv - v)vy \\ &= vyvAvvyv. \end{aligned}$$

It follows from (2.4.2) that  $vyvAvvyv = \{0\}$ , which is a contradiction since  $A$  is semiprime. Since  $\dim(vAv)$  is as small as possible, this implies that  $(vzvvyv - v)A(vzvvyv - v) = \{0\}$ . It follows from Theorem 1.3.3 that

$$v = vzvvyv. \quad (2.4.3)$$

Now consider  $Av$ . We want to show that  $Av$  is a minimal left ideal. So suppose that  $\{0\} \neq J$  is a left ideal such that  $J \subset Av$ . Let  $0 \neq yv \in J$ . Since  $A$  is semiprime, there exists an  $x_0 \in A$  such that  $yx_0yv \neq 0$ . This implies that  $vx_0yv \neq 0$ . By (2.4.3) there exists  $z \in A$  such that  $v = vzvx_0yv$ . This implies that  $Av = Avzvx_0yv \subset J$ , so that  $Av$  is a minimal left ideal. By Theorem 2.2.3 (1), it follows that there exists a minimal idempotent  $p \in Av$ . Since  $v \in uAu$  we get that  $p \in Av \subset Au$ .  $\blacksquare$

**Theorem 2.4.4** ([21], Corollary 3.3) *Let  $A$  be a semiprime Banach algebra and let  $0 \neq u \in A$ . Then  $u \in \mathcal{F}_1$  if and only if  $\dim(uAu) = 1$ .*

**Proof.** Let  $u \in \mathcal{F}_1$ . Then  $uxu = f_u(x)u$  for all  $x \in A$ . This implies that  $uAu = \text{span}\{u\}$ , so that  $\dim(uAu) = 1$ .

Conversely, suppose that  $\dim(uAu) = 1$ . Then from Lemma 2.4.1 we have that there exists a minimal idempotent  $p \in Au$ . Therefore  $(u - up)A(u -$

$up) \subset uAu$ . If  $(u - up)A(u - up) = uAu$  then  $(u - up)A(u - up)p = \{0\} = uAup$ . This means that  $AuAup = \{0\}$ . Since  $p \in Au$ , this implies  $p^3 = p = 0$ , which is a contradiction. So  $(u - up)A(u - up) = \{0\}$ , and since  $A$  is semiprime,  $u = up$ . It follows from Theorem 2.1.18 and from Lemma 2.2.1 that  $u = up \in \mathcal{F}_1$ .  $\square$

Theorem 2.4.7, also due to J. Puhl, is another result relating spatially rank one elements to one-dimensional operators. It says that every pair of spatially rank one elements that satisfy a certain equivalence relation determine a one-dimensional operator. The details follow.

**Definition 2.4.5** *Let  $A$  be a semiprime Banach algebra and let  $u, v \in \mathcal{F}_1$ . Then  $u \sim v$  if there exists  $x_0 \in A$  such that  $ux_0v \neq 0$ .*

Puhl went on to show that this is an equivalence relation on  $\mathcal{F}_1$ .

**Proposition 2.4.6** *([21], Lemma 4.1) Let  $A$  be a semiprime Banach algebra. Then the relation  $\sim$  is an equivalence relation on  $\mathcal{F}_1$ .*

**Proof.** Reflexive: Let  $u \in \mathcal{F}_1$ . Since  $A$  is semiprime, there exists  $x_0 \in A$  such that  $ux_0u \neq 0$ , so that  $u \sim u$ .

Symmetric: Let  $u, v \in \mathcal{F}_1$  such that  $u \sim v$ . Then there exists an  $x_0 \in A$  with  $ux_0v \neq 0$ . Since  $A$  is semiprime, there exists a  $y_0 \in A$  such that  $(ux_0v)y_0(ux_0v) \neq 0$ . This implies that  $vy_0u \neq 0$ , so that  $v \sim u$ .

Transitive: Let  $u, v, w \in \mathcal{F}_1$  with  $u \sim v$  and  $v \sim w$ . Then there exist  $x_0, x_1 \in A$  such that  $ux_0v \neq 0$  and  $vx_1w \neq 0$ . Since  $A$  is semiprime, there exists a  $y_0 \in A$  such that  $(ux_0v)y_0(ux_0v) \neq 0$ . By Lemma 2.1.18, we have that  $ux_0v \in \mathcal{F}_1$ . This means that

$$0 \neq (ux_0v)y_0(ux_0v) = (ux_0)v(y_0ux_0)v = (ux_0)f_v(y_0ux_0)v = f_v(y_0ux_0)ux_0v.$$

Hence  $f_v(y_0ux_0) \neq 0$ . Therefore  $vy_0ux_0vx_1w = f_v(y_0ux_0)vx_1w$ , so that  $vy_0ux_0vx_1w \neq 0$ . It follows that  $ux_0vx_1w \neq 0$ , and so  $u \sim w$ .  $\blacksquare$

**Theorem 2.4.7** *([21], Lemma 4.2) Let  $A$  be a semiprime Banach algebra. If  $u, v \in \mathcal{F}_1$  such that  $u \sim v$ , then the operator  $D_{u,v} : A \rightarrow A$  defined by  $D_{u,v}x = uxv$  for all  $x \in A$ , is one-dimensional.*

**Proof.** Let  $u, v \in \mathcal{F}_1$  such that  $u \sim v$ . Then there exists an  $x_0 \in A$  for which  $ux_0v \neq 0$ . Since  $A$  is semiprime, there is a  $y_0 \in A$  such that  $0 \neq (ux_0v)y_0(ux_0v)$ . It follows from Theorem 2.1.18 that  $0 \neq (ux_0v)y_0(ux_0v) = f_v(y_0ux_0)ux_0v$ . Hence  $f_v(y_0ux_0) \neq 0$ . Since  $v \in \mathcal{F}_1$ , we have that  $vy_0ux_0v = f_v(y_0ux_0)v$ , so that  $v = \frac{vy_0ux_0v}{f_v(y_0ux_0)}$ . Therefore  $D_{u,v}x = \frac{uxvy_0ux_0v}{f_v(y_0ux_0)}$ . Also from  $u \in \mathcal{F}_1$  we obtain that  $uxvy_0u = f_u(xvy_0)u$ . This implies that  $D_{u,v}x = \frac{f_u(xvy_0)ux_0v}{f_v(y_0ux_0)}$ , so that  $D_{u,v}(A) = \text{span}\{ux_0v\}$ . Hence  $\dim(D_{u,v}(A)) = 1$ .  $\square$

## 2.5 Connected components of the set of spatially rank one elements.

So far, most of the results we have on spatially rank one elements are from the 1978 paper [21] by J. Puhl and from the 1993 paper [19] by T. Mouton and H. Raubenheimer. R.M. Brits, L. Lindeboom and H. Raubenheimer have also done some extensive work on spatially rank one elements. Their publications include the 2003 paper [11], in which they bring out some key topological aspects of the set  $\mathcal{F}_1$ . In this section, our aim is to discuss some of the key results of this paper that characterize spatially rank one elements. Theorem 2.5.3 and Theorem 2.5.9 are the main results in this section.

We start with the following: Let  $A$  be a semiprime Banach algebra with  $\mathcal{F}_1 \neq \emptyset$ . We denote the set of nilpotent elements of  $\mathcal{F}_1$  by  $\mathcal{F}_1^0$ . The set of non-nilpotent elements of  $\mathcal{F}_1$  will be denoted by  $\mathcal{F}_1^1$ . The sets  $\mathcal{F}_1^0$  and  $\mathcal{F}_1^1$  partition  $\mathcal{F}_1$ .

For an element  $u \in \mathcal{F}_1$  in a semiprime Banach algebra  $A$ , Brits, Lindeboom and Raubenheimer in 2003 described the nature of the connected component of  $\mathcal{F}_1$  containing  $u$ . In Theorem 2.5.3, we give the proof of this result. We will need the following lemma.

**Lemma 2.5.1** ([11], Lemma 2.1(ii)) *Let  $A$  be a semiprime Banach algebra with  $\mathcal{F}_1 \neq \emptyset$ . If  $u \in \mathcal{F}_1^0$  then there exists  $v \in \exp(A)$  such that  $uv, vu \in \mathcal{F}_1^1$ .*

**Proof.** Since  $u \in \mathcal{F}_1$  we have that  $uxu = f_u(x)u$  for all  $x \in A$ . Since  $A$  is semiprime, there exists a  $z \in A$  such that  $uzu = f_u(z)u \neq 0$ , so that  $f_u(z) \neq 0$ . Therefore  $u \frac{z}{f_u(z)} u = u$ . Let  $y = \frac{z}{f_u(z)}$ . Then

$$uyu = u. \tag{2.5.2}$$



Since the spectrum is compact by Theorem 1.4.4, there exists a  $\lambda \in \mathbb{C}$  such that  $|\lambda| > \rho(y)$ . By Proposition 1.5.3 we have that  $0 \notin \sigma(-\lambda + y)$ . It follows from Theorem 1.5.9 that  $-\lambda + y \in \exp(A)$ . Since  $-\lambda + y \in \exp(A) \subset A^{-1}$  we have that  $u(-\lambda + y) \neq 0$  for if not, then  $u = u(-\lambda + y)(-\lambda + y)^{-1} = 0$ , which is a contradiction. It follows from Theorem 2.1.18 that  $u(-\lambda + y) \in \mathcal{F}_1$ .

We show that  $u(-\lambda + y) \in \mathcal{F}_1^1$ . Since  $u \in \mathcal{F}_1^0$ , from Lemma 2.1.12 we have that  $u^2 = 0$ . Using equation (2.5.2), it follows that

$$(u(-\lambda + y))^2 = (-\lambda u + uy)^2 = \lambda^2 u^2 - \lambda u^2 y - \lambda(uyu) + (uyu)y = u(-\lambda + y).$$

Since  $u(-\lambda + y) \neq 0$ , this implies that  $(u(-\lambda + y))^2 \neq 0$ . It follows from Lemma 2.1.12 that  $u(-\lambda + y)$  is not nilpotent. Therefore  $uv \in \mathcal{F}_1^1$ , with  $v = -\lambda + y \in \exp(A)$ . Similarly,  $vu \in \mathcal{F}_1^1$ .  $\square$

Let  $A$  be a semiprime Banach algebra with  $\mathcal{F}_1 \neq \emptyset$ . For  $u \in \mathcal{F}_1$ , we use the notation  $K_u^{\mathcal{F}_1}$  to denote the connected component of  $\mathcal{F}_1$  that contains  $u$ . We have

**Theorem 2.5.3** ([11], Theorem 2.2) *Let  $A$  be a semiprime Banach algebra with  $\mathcal{F}_1 \neq \emptyset$ . If  $u \in \mathcal{F}_1$ , then the set  $B = \text{Exp}(A)u\text{Exp}(A)$  is the connected component of  $\mathcal{F}_1$  containing  $u$ . Hence  $K_u^{\mathcal{F}_1} = \text{Exp}(A)u\text{Exp}(A)$ .*

**Proof.** By Lemma 1.5.8, the set  $B$  is connected. Since  $1 \in \text{Exp}(A)$ , we get that  $u \in B$ . Also  $B \subset \mathcal{F}_1$  by Theorem 2.1.18. It follows that  $B \subset K_u^{\mathcal{F}_1}$ .

We show that  $B$  is closed and open in  $K_u^{\mathcal{F}_1}$ . To show that  $B$  is closed, let  $(r_n u s_n)$  be a sequence in  $B$  such that  $r_n u s_n \rightarrow v \in K_u^{\mathcal{F}_1}$ . If  $v \in \mathcal{F}_1^1$  then  $0 \neq v^2 = \text{tr}(v)v$  by Lemma 2.1.12, so that  $\text{tr}(v) \neq 0$ . Now, since the trace is continuous by Lemma 2.3.2, we have that  $\lim_{n \rightarrow \infty} \text{tr}(r_n u s_n) = \text{tr}(v) \neq 0$ . For  $n$  sufficiently large, we may therefore assume that  $r_n u s_n \in \mathcal{F}_1^1$ . From Lemma 2.1.9 we then have that  $\frac{r_n u s_n}{\text{tr}(r_n u s_n)}$  and  $\frac{v}{\text{tr}(v)}$  are non-zero idempotents. Also since  $r_n u s_n \rightarrow v$ , we have that  $\left\| \frac{r_n u s_n}{\text{tr}(r_n u s_n)} - \frac{v}{\text{tr}(v)} \right\| \rightarrow 0$ . Hence for  $n$  sufficiently large,  $\rho\left(\frac{r_n u s_n}{\text{tr}(r_n u s_n)} - \frac{v}{\text{tr}(v)}\right) < 1$ . It follows from Lemma 1.5.11 that there exists  $z_n \in \exp(A)$  such that  $\frac{v}{\text{tr}(v)} = z_n^{-1} \frac{r_n u s_n}{\text{tr}(r_n u s_n)} z_n$ . Therefore  $v = \alpha z_n^{-1} r_n u s_n z_n$ , where  $\alpha = \frac{\text{tr}(v)}{\text{tr}(r_n u s_n)}$ . Hence  $v \in B$ .

On the other hand, if  $v \in \mathcal{F}_1^0$ , then we have from Lemma 2.5.1 that there exists  $w \in \exp(A)$  such that  $vw \in \mathcal{F}_1^1$ . Therefore  $r_n u s_n w \in B$  and  $r_n u s_n w \rightarrow vw$ . Using the previous argument, we get that  $vw \in B$ . Since

$w \in \exp(A)$ , we have that  $w^{-1} \in \exp(A)$ . Therefore  $v = vww^{-1} \in B$ . This proves that  $B$  is closed in  $K_u^{\mathcal{F}_1}$ .

To show that  $B$  is open in  $K_u^{\mathcal{F}_1}$ , we will show that  $K_u^{\mathcal{F}_1} \setminus B$  is closed in  $K_u^{\mathcal{F}_1}$ . To this end suppose that  $(v_n)$  is a sequence in  $K_u^{\mathcal{F}_1} \setminus B$  but  $v_n \rightarrow rus \in B$ . Then  $r, s \in \exp(A)$ , so that  $r, s \in A^{-1}$ . Therefore  $r^{-1}v_n s^{-1} \rightarrow r^{-1}rus s^{-1} = u$ . As in the previous argument, we have that  $\rho\left(\frac{u}{\text{tr}(u)} - \frac{r^{-1}v_n s^{-1}}{\text{tr}(r^{-1}v_n s^{-1})}\right) < 1$  for  $n$  sufficiently large. Applying Lemma 1.5.11 as done previously,  $r^{-1}v_n s^{-1} = \beta z^{-1}uz$  for some scalar  $\beta$  and some  $z \in \exp(A)$ . This implies that  $v_n = \beta r z^{-1}uzs \in B$ , which is a contradiction. Therefore  $K_u^{\mathcal{F}_1} \setminus B$  is closed, so that  $B$  is open. Since  $B \neq \emptyset$ , with  $B$  both closed and open in  $K_u^{\mathcal{F}_1}$  and with  $K_u^{\mathcal{F}_1}$  connected,  $K_u^{\mathcal{F}_1} = B$ .  $\square$

If  $u \in \mathcal{F}_1$  is also in the centre of  $A$ , Brits, Lindeboom and Raubenheimer obtained a simplified form for the connected component of  $\mathcal{F}_1$  containing  $u$ . With minor changes to their proof, we prove this result in Theorem 2.5.7. Lemma 2.5.4 is crucial in the proof of this result.

**Lemma 2.5.4** ([11], Lemma 2.1 (i)) *Let  $A$  be a semiprime Banach algebra with  $\mathcal{F}_1 \neq \emptyset$ . If  $u \in \mathcal{F}_1^1$  and  $v \in A$  with  $uv \in \mathcal{F}_1^1$ , then there exists  $0 \neq \alpha \in \mathbb{C}$  and  $x \in A$  such that  $uv = \alpha e^x u e^{-x}$ . A similar statement holds for  $vu \in \mathcal{F}_1^1$ .*

**Proof.** Let  $u \in \mathcal{F}_1^1$  and  $v \in A$  such that  $uv \in \mathcal{F}_1^1$ . Then  $\frac{u}{\text{tr}(u)}$  and  $\frac{uv}{\text{tr}(uv)}$  are idempotents by Lemma 2.1.9. From  $(uv)^2 = \text{tr}(uv)uv$ , we obtain that  $(uvu)v = \text{tr}(uv)uv$ . Now since  $u \in \mathcal{F}_1$ , we have from Corollary 2.1.20 that  $f_u(v) = \text{tr}(uv)$ . Therefore

$$\frac{uvu}{\text{tr}(u)\text{tr}(uv)} = \frac{u}{\text{tr}(u)}. \quad (2.5.5)$$

Also,

$$\frac{u^2v}{\text{tr}(u)\text{tr}(uv)} = \frac{\text{tr}(u)uv}{\text{tr}(u)\text{tr}(uv)} = \frac{uv}{\text{tr}(uv)}. \quad (2.5.6)$$

From equations (2.5.5) and (2.5.6) it follows that

$$\begin{aligned} \left(\frac{u}{\text{tr}(u)} - \frac{uv}{\text{tr}(uv)}\right)^2 &= \left(\frac{u}{\text{tr}(u)}\right)^2 - \frac{u^2v}{\text{tr}(u)\text{tr}(uv)} - \frac{uvu}{\text{tr}(u)\text{tr}(uv)} + \left(\frac{uv}{\text{tr}(uv)}\right)^2 \\ &= \frac{u}{\text{tr}(u)} - \frac{uv}{\text{tr}(uv)} - \frac{u}{\text{tr}(u)} + \frac{uv}{\text{tr}(uv)} \\ &= 0. \end{aligned}$$

This implies that  $\rho\left(\frac{u}{\text{tr}(u)} - \frac{uv}{\text{tr}(uv)}\right)^2 = 0$ . It follows from the spectral mapping theorem that  $\rho\left(\frac{u}{\text{tr}(u)} - \frac{uv}{\text{tr}(uv)}\right) = 0$ . From Lemma 1.5.11 this implies that  $\frac{uv}{\text{tr}(uv)} = e^x \frac{u}{\text{tr}(u)} e^{-x}$ , where  $x \in A$ . Therefore  $uv = \alpha e^x u e^{-x}$ , where  $\alpha = \frac{\text{tr}(uv)}{\text{tr}(u)}$ , as required.  $\blacksquare$

**Theorem 2.5.7** ([11], Theorem 2.2 (ii)) *Let  $A$  be a semiprime Banach algebra with  $\mathcal{F}_1 \neq \emptyset$ . If  $u \in \mathcal{F}_1$  then  $u \in Z(A)$  if and only if  $K_u^{\mathcal{F}_1} = \mathbb{C}u \setminus \{0\}$ .*

**Proof.** Let  $u \in Z(A)$ . Then from Theorem 2.5.3, we have that  $K_u^{\mathcal{F}_1} = \text{Exp}(A)u\text{Exp}(A) = u\text{Exp}(A)$ . Now, if  $u \in \mathcal{F}_1^0$  then  $uAu = Au^2 = \{0\}$  by Lemma 2.1.12. This is a contradiction since  $A$  is semiprime. Therefore  $u \in \mathcal{F}_1^1$ . Let  $v \in \text{Exp}(A)$  and suppose that  $uv \in \mathcal{F}_1^0$ . Then  $(uv)^2 = u^2v^2 = 0$ . Since  $v \in \text{Exp}(A) \subset A^{-1}$  we get that  $u^2 = u^2v^2(v^{-1})^2 = 0$ , so that  $u \in \mathcal{F}_1^0$ . This is a contradiction, and so  $uv \in \mathcal{F}_1^1$ . It follows from Lemma 2.5.4 that  $uv = \alpha e^x u e^{-x}$  for some  $0 \neq \alpha \in \mathbb{C}$  and some  $x \in A$ . Since  $u \in Z(A)$  this implies that  $uv = \alpha u$ . This implies that  $K_u^{\mathcal{F}_1} = u\text{Exp}(A) \subset \mathbb{C}u \setminus \{0\}$ . Also, since every scalar in  $\mathbb{C}$  is an element of  $\text{Exp}(A)$ , we have that  $\mathbb{C}u \setminus \{0\} \subset u\text{Exp}(A) = K_u^{\mathcal{F}_1}$ . Therefore  $K_u^{\mathcal{F}_1} = \mathbb{C}u \setminus \{0\}$ .

Conversely, suppose that  $K_u^{\mathcal{F}_1} = \mathbb{C}u \setminus \{0\}$ . If  $u \in \mathcal{F}_1^0$  then by Lemma 2.5.1 there exists  $v \in \text{exp}(A)$  such that  $uv \in \mathcal{F}_1^1$ . From the fact that  $K_u^{\mathcal{F}_1} = \text{Exp}(A)u\text{Exp}(A)$  by Theorem 2.5.3, we get that  $uv \in K_u^{\mathcal{F}_1}$ . From  $K_u^{\mathcal{F}_1} = \mathbb{C}u \setminus \{0\}$ , it follows that  $uv = \alpha u$ , for some  $0 \neq \alpha \in \mathbb{C}$ . This is a contradiction since  $u \in \mathcal{F}_1^0$  leads to  $(uv)^2 = \alpha^2 u^2 = 0$ , and this means that  $uv \in \mathcal{F}_1^0$ . So we must have that  $u \in \mathcal{F}_1^1$ . Therefore  $u^2 \neq 0$ , and from  $u^2 = \text{tr}(u)u$ , it follows that  $\text{tr}(u) \neq 0$ .

Now, since  $K_u^{\mathcal{F}_1} = \text{Exp}(A)u\text{Exp}(A)$  by Theorem 2.5.3, we have that  $w^{-1} \frac{u}{\text{tr}(u)} w \in K_u^{\mathcal{F}_1}$  for all  $w \in \text{Exp}(A)$ . It follows from  $K_u^{\mathcal{F}_1} = \mathbb{C}u \setminus \{0\}$  that  $w^{-1} \frac{u}{\text{tr}(u)} w = \alpha u$ , for some  $0 \neq \alpha \in \mathbb{C}$ . Since  $w^{-1} \frac{u}{\text{tr}(u)} w$  is an idempotent by Corollary 2.1.10, it follows that  $\alpha u$  is an idempotent. From Lemma 2.1.11, this implies that  $\alpha = \frac{1}{\text{tr}(u)}$ . This means that  $w^{-1} \frac{u}{\text{tr}(u)} w = \frac{u}{\text{tr}(u)}$  for all  $w \in \text{Exp}(A)$ . It follows that  $\frac{u}{\text{tr}(u)} w = w \frac{u}{\text{tr}(u)}$  for all  $w \in \text{Exp}(A)$ . From Proposition 1.5.10, for any  $x \in A$  there exists a  $y, z \in \text{Exp}(A)$  such that  $x = y + z$ . From this we have that  $z = x - y$ . Therefore  $z \frac{u}{\text{tr}(u)} = \frac{u}{\text{tr}(u)} z = \frac{u}{\text{tr}(u)} (x - y) = (x - y) \frac{u}{\text{tr}(u)}$ , from which we have that  $\frac{u}{\text{tr}(u)} x - \frac{u}{\text{tr}(u)} y = x \frac{u}{\text{tr}(u)} - y \frac{u}{\text{tr}(u)}$ . Since  $y \in \text{Exp}(A)$ , we have that  $y \frac{u}{\text{tr}(u)} = \frac{u}{\text{tr}(u)} y$ , so that  $x \frac{u}{\text{tr}(u)} = \frac{u}{\text{tr}(u)} x$ . Since  $x$  is an arbitrary element in  $A$ , it follows that  $\frac{u}{\text{tr}(u)} \in Z(A)$ . Hence  $u \in Z(A)$ .  $\blacksquare$

In Theorem 2.5.3, we saw that for a spatially rank one element  $u$  in a semiprime Banach algebra  $A$ , the connected component of  $\mathcal{F}_1$  containing  $u$  is the set  $K_u^{\mathcal{F}_1} = \text{Exp}(A)u\text{Exp}(A)$ . In the case in which  $u$  is in the set  $\mathcal{F}_1^1$ , Brits, Lindeboom and Raubenheimer obtained the following result for the connected component of  $\mathcal{F}_1$  containing  $u$ .

**Theorem 2.5.8** ([11], Proposition 2.6) *Let  $A$  be a semiprime Banach algebra with  $\mathcal{F}_1 \neq \emptyset$  and let  $u \in \mathcal{F}_1^1$ . Then  $K_u^{\mathcal{F}_1} = \text{cl}_B(\text{exp}(A)u\text{exp}(A)) \setminus \{0\}$ , where  $B = K_u^{\mathcal{F}_1}$ . In particular if  $K_u^{\mathcal{F}_1}$  contains no nilpotent elements then  $K_u^{\mathcal{F}_1} = \text{exp}(A)u\text{exp}(A)$ .*

**Proof.** Since  $u \in \mathcal{F}_1^1$ , we have from Lemma 2.1.9 that  $\frac{u}{\text{tr}(u)}$  is an idempotent. Therefore, without loss of generality, we may assume that  $u$  is an idempotent. Let  $a, b \in \text{Exp}(A)$ . Then  $aub \in K_u^{\mathcal{F}_1}$  by Theorem 2.5.3. We have four cases:

(i) Let  $au \in \mathcal{F}_1^1$  and  $ub \in \mathcal{F}_1^1$ . Then from Lemma 2.5.4 we get that  $au = \alpha e^x u e^{-x}$  and  $ub = \beta e^y u e^{-y}$ , for some  $\alpha, \beta \in \mathbb{C}$  and some  $x, y \in A$ . Therefore

$$aub = (au)(ub) = (\alpha e^x u e^{-x})(\beta e^y u e^{-y}) = \alpha \beta e^x (uzu) e^{-y},$$

where  $z = e^{-x} e^y$ . Since  $u \in \mathcal{F}_1$ , we have that  $uzu = \mu u$  for some  $\mu \in \mathbb{C}$ . Therefore  $aub = \gamma e^x u e^{-y}$ , where  $\gamma = \alpha \beta \mu$ . Hence  $aub = e^{x+\frac{\lambda}{2}} u e^{-y+\frac{\lambda}{2}}$ , with  $e^\lambda = \gamma$ . This implies that  $aub \in \text{exp}(A)u\text{exp}(A)$ .

(ii) Suppose that  $au \in \mathcal{F}_1^0$  and  $ub \in \mathcal{F}_1^0$ . Then  $(au)^2 = 0$ , so that  $uau = a^{-1}(auau) = 0$ . Let  $(\lambda_n)$  be a sequence in  $\mathbb{C}$  such that  $\lambda_n \neq 0$  and  $\lambda_n \rightarrow 0$ . Then for every  $n \in \mathbb{N}$ , we have that  $((\lambda_n + a)u)^2 = \lambda_n^2 u^2 + \lambda_n uau + au\lambda_n u + (au)^2 = \lambda_n^2 u + \lambda_n uau + \lambda_n au^2 + (au)^2 = \lambda_n^2 u + \lambda_n au$ .

Note that  $\lambda_n^2 u + \lambda_n au \neq 0$  for if  $\lambda_n^2 u + \lambda_n au = 0$ , then  $\lambda_n u + au = 0$ . This implies that  $au(\lambda_n u + au) = \lambda_n au + auau = 0$ , so that  $\lambda_n au = 0$ . Since  $\lambda_n \neq 0$  for all  $n$ , we get that  $au = 0$  and so  $u = a^{-1}au = 0$ , which is a contradiction. Hence  $\lambda_n^2 u + \lambda_n au \neq 0$  and from  $((\lambda_n + a)u)^2 = \lambda_n^2 u + \lambda_n au$ , we conclude that  $(\lambda_n + a)u \in \mathcal{F}_1^1$ . Therefore  $(\lambda_n + a)u \rightarrow au$  with each  $(\lambda_n + a)u$  in  $\mathcal{F}_1^1$ . It follows from Lemma 2.5.4 that  $(\lambda_n + a)u = \alpha_n e^{x_n} u e^{-x_n}$ , with  $\alpha_n \in \mathbb{C}$  and  $x_n \in A$ .

Using a similar argument, we also get that  $u(b + \lambda_n) = \beta_n e^{y_n} u e^{-y_n}$  for  $\beta_n \in \mathbb{C}$  and  $y_n \in A$ . Therefore  $aub = \lim_{n \rightarrow \infty} (\alpha_n e^{x_n} u e^{-x_n})(\beta_n e^{y_n} u e^{-y_n}) =$

$\lim_{n \rightarrow \infty} \alpha_n \beta_n e^{x_n} (uz_n u) e^{-y_n}$ , where  $z_n = e^{-x_n} e^{y_n}$ . Since  $u \in \mathcal{F}_1$ , we have that  $uz_n u = \mu_n u$  for  $\mu_n \in \mathbb{C}$ . Therefore  $aub = \lim_{n \rightarrow \infty} \gamma_n e^{x_n} u e^{-y_n}$ , where  $\gamma_n = \alpha_n \beta_n \mu_n$ . This implies that  $aub = \lim_{n \rightarrow \infty} e^{x_n + \frac{\lambda_n}{2}} u e^{-y_n + \frac{\lambda_n}{2}}$ , where  $e^{\lambda_n} = \gamma_n$ , so that  $aub \in cl_B(\exp(A)u\exp(A)) \setminus \{0\}$ .

(iii) Suppose that  $au \in \mathcal{F}_1^0$  and  $ub \in \mathcal{F}_1^1$ . Using the same argument as that in (ii), we have that  $(\lambda_n + a)u = \alpha_n e^{x_n} u e^{-x_n}$ , where  $(\lambda_n)$  is a sequence in  $\mathbb{C} \setminus \{0\}$  with  $\lambda_n \rightarrow 0$ , and  $\alpha_n \in \mathbb{C}$  and  $x_n \in A$ . Also, using the same argument as in (i), we have that  $ub = \beta e^y u e^{-y}$ , for some  $\beta \in \mathbb{C}$  and some  $y \in A$ . Therefore  $aub = \lim_{n \rightarrow \infty} (\alpha_n e^{x_n} u e^{-x_n})(\beta e^y u e^{-y}) = \lim_{n \rightarrow \infty} \alpha_n \beta e^{x_n} (uz_n u) e^{-y}$ , where  $z_n = e^{-x_n} e^y$ . Since  $u \in \mathcal{F}_1$  there exists a sequence  $\mu_n \in \mathbb{C}$  such that  $uz_n u = \mu_n u$  for all  $n \in \mathbb{N}$ . Therefore  $aub = \lim_{n \rightarrow \infty} \gamma_n e^{x_n} u e^{-y}$ , where  $\gamma_n = \alpha_n \beta \mu_n$ . Hence  $aub = \lim_{n \rightarrow \infty} e^{x_n + \frac{\lambda_n}{2}} u e^{-y + \frac{\lambda_n}{2}}$ , where  $e^{\lambda_n} = \gamma_n$ . This implies that  $aub \in cl_B(\exp(A)u\exp(A)) \setminus \{0\}$ .

(iv) If  $au \in \mathcal{F}_1^1$  and  $ub \in \mathcal{F}_1^0$ , then an argument similar to that in case (iii) yields  $aub \in cl_B(\exp(A)u\exp(A)) \setminus \{0\}$ .

For the second part of the theorem, note that if  $K_u^{\mathcal{F}_1}$  has no nilpotent elements, then cases (ii), (iii) and (iv) do not apply. It follows from case (i) that  $K_u^{\mathcal{F}_1} = \exp(A)u\exp(A)$ .  $\square$

Recall from Theorem 2.2.3 that a left ideal  $J$  of a Banach algebra  $A$  is a minimal left ideal if and only if there exists a  $u \in \mathcal{F}_1$  such that  $J = Au$ . Using Theorem 2.5.3, Brits, Lindeboom and Raubenheimer obtained the following simplification for this characterization of minimal left ideals.

**Theorem 2.5.9** ([11], Corollary 2.3) *Let  $A$  be a semiprime Banach algebra with  $\mathcal{F}_1 \neq \emptyset$ . Then every minimal left ideal  $J$  of  $A$  has the form  $J = \text{Exp}(A)u \cup \{0\}$ , where  $u \in \mathcal{F}_1$ .*

**Proof.** Since  $J$  is a minimal left ideal of  $A$  if and only if there exists a  $u \in \mathcal{F}_1$  such that  $J = Au$ , it suffices to show that  $Au = \text{Exp}(A)u \cup \{0\}$  for all  $u \in \mathcal{F}_1$ .

Let  $u$  be any element in  $\mathcal{F}_1$ . By Lemma 1.1.21, we have that  $AuA \setminus \{0\}$  is a connected set. Now from Theorem 2.1.18, we get the inclusion  $AuA \setminus \{0\} \subset \mathcal{F}_1$ . Hence  $AuA \setminus \{0\}$  is a connected subset of  $\mathcal{F}_1$ . By Theorem 2.5.3, this implies that  $AuA \setminus \{0\} \subset \text{Exp}(A)u\text{Exp}(A)$ . Since we also have the inclusion

$\text{Exp}(A)u\text{Exp}(A) \subset AuA \setminus \{0\}$ , this means that  $\text{Exp}(A)u\text{Exp}(A) = AuA \setminus \{0\}$ . Therefore

$$\text{Exp}(A)u\text{Exp}(A)u \cup \{0\} = AuAu. \quad (2.5.10)$$

We show that  $u\text{Exp}(A)u \cup \{0\} = uAu$ . Since  $u \in \mathcal{F}_1$  we have from Proposition 2.1.2 that  $uAu = Cu$ . So to prove that  $uAu \subset u\text{Exp}(A)u \cup \{0\}$ , we will show that  $Cu \subset u\text{Exp}(A)u \cup \{0\}$ . Since for every  $x \in \text{Exp}(A)$  we have that  $\lambda x \in \text{Exp}(A)$  for any  $0 \neq \lambda \in \mathbb{C}$ , it suffices to show that  $u \in u\text{Exp}(A)u \cup \{0\}$ . To this end since  $u \in \mathcal{F}_1$ , we obtain that  $uxu = f_u(x)u$  for all  $x \in \text{Exp}(A)$ . Suppose that  $f_u(x) = 0$  for all  $x \in \text{Exp}(A)$ . Let  $a \in A$ . Then by Proposition 1.5.10 there exist  $b, c \in \text{Exp}(A)$  such that  $a = b + c$ . This implies that  $f_u(a) = f_u(b + c) = f_u(b) + f_u(c) = 0$ . This means that  $uau = 0$  for all  $a \in A$ . This is a contradiction since  $A$  is semiprime. So there exists a  $y \in \text{Exp}(A)$  such that  $u = uyu$ , and hence  $u \in u\text{Exp}(A)u \cup \{0\}$ . Therefore  $Cu \subset u\text{Exp}(A)u \cup \{0\}$ , so that  $uAu \subset u\text{Exp}(A)u \cup \{0\}$ . Since we also have the inclusion  $u\text{Exp}(A)u \cup \{0\} \subset uAu$ , this implies that

$$u\text{Exp}(A)u \cup \{0\} = uAu. \quad (2.5.11)$$

Substituting the right hand side of (2.5.11) into the left hand side of (2.5.10), we obtain that

$$\text{Exp}(A)uAu = AuAu. \quad (2.5.12)$$

Since  $u \in \mathcal{F}_1$ , we have that  $uAu = Cu$ . From (2.5.12), this implies that  $\text{Exp}(A)Cu = ACu$ . Since  $\text{Exp}(A)Cu = \text{Exp}(A)u \cup \{0\}$  and  $ACu = Au$ , it follows that  $Au = \text{Exp}(A)u \cup \{0\}$ .  $\square$

## 2.6 Spatially finite rank elements of Banach algebras

So far in this chapter, we have defined spatially rank one elements and have discussed the theory of these elements. In this section we study spatially finite rank elements, which are the finite sums of the spatially rank one elements. In [21] J. Puhl called them finite elements. The main results of this section are Theorem 2.6.3 and Theorem 2.6.9.

**Definition 2.6.1** *Let  $A$  be a semiprime Banach algebra. An element  $u \in A$  is called a spatially finite rank element of  $A$  if  $u = 0$  or  $u$  has the form*

$u = \sum_{i=1}^n u_i$ , where each  $u_i$  is a spatially rank one element of  $A$ .

The set of spatially finite rank elements of  $A$  will be denoted by  $\mathcal{F}$ .

Note that by this definition, spatially rank one elements are necessarily spatially finite rank elements.

We have the following simple proposition.

**Proposition 2.6.2** ([21], p.659) *Suppose that  $A$  is a semiprime Banach algebra. Then  $\mathcal{F}$  is a two-sided ideal of  $A$ .*

**Proof.** Let  $u \in \mathcal{F}$ . Then  $u = \sum_{i=1}^n u_i$  with  $u_i \in \mathcal{F}_1$  for all  $i$ . Let  $x, y \in A$ .

From Theorem 2.1.18, we obtain that  $xuy = \sum_{i=1}^n xu_iy \in \mathcal{F}$ .  $\square$

Recall from Chapter 1 that the socle of a Banach algebra  $A$  is the sum of all the minimal left (right) ideals of  $A$ . In 1978 J. Puhl showed that the socle of a semiprime Banach algebra  $A$  coincides with the set of spatially finite rank elements of  $A$ . This is the next result.

**Theorem 2.6.3** ([21], p.659) *Let  $A$  be a semiprime Banach algebra. Then  $\mathcal{F} = \text{Soc}(A)$ .*

**Proof.** Let  $u \in \mathcal{F}$ . Then  $u = \sum_{i=1}^n u_i$  with  $u_i \in \mathcal{F}_1$  for all  $i$ . Now, by Theorem 2.2.3 (2), the set  $Au_i$  is a minimal left ideal for all  $i$ . By definition of the socle, this means that  $Au_i \subset \text{Soc}(A)$  for all  $i$ . Therefore  $u_i = 1 \cdot u_i \in Au_i \subset \text{soc}(A)$ . Consequently,  $u = \sum_{i=1}^n u_i \in \text{Soc}(A)$ .

Conversely, let  $t \in \text{Soc}(A)$ . Then  $t = \sum_{i=1}^m t_i$ , where  $t_i$  is an element of a minimal left ideal  $L_i$  for all  $i$ . By Theorem 2.2.3 (1), we have that  $L_i$  contains a minimal idempotent  $p_i$  such that  $L_i = Ap_i$ . This means that there exists  $x \in A$  such that  $t_i = xp_i$  for all  $i$ . By Theorem 2.2.1 and Theorem 2.1.18, it follows that  $t_i = xp_i \in \mathcal{F}$ . Since  $\mathcal{F}$  is a two-sided ideal,  $t = \sum_{i=1}^m t_i \in \mathcal{F}$ .  $\square$

Theorem 2.4.4 gives a characterization of spatially rank one elements in terms of one-dimensional operators. In [21] J. Puhl gave a result which also characterizes spatially finite rank elements in terms of operators. These

operators are finite dimensional. With minor modifications to the original proof of Puhl, we prove this result in Theorem 2.6.9. The proof uses Theorem 2.6.8, which in turn relies on Lemma 2.6.4, Lemma 2.6.5 and Lemma 2.6.7.

**Lemma 2.6.4** *Let  $A$  be a Banach algebra and let  $a$  be a non-zero element of  $A$  such that  $\dim(aAa) < \infty$ . Then every subset of non-zero orthogonal idempotents in  $Aa$  is finite.*

**Proof.** Suppose that  $\{p_n : n \in \mathbb{N}\}$  is an infinite set of non-zero orthogonal idempotents in  $Aa$ . Since  $p_n \in Aa$ , there exists  $x_n \in A$  such that  $p_n = x_n a$ . If  $x_n = 0$ , then  $p_n = 0$ . So since  $p_n \neq 0$ , we have that  $x_n \neq 0$ . Hence  $0 < \|x_n\|$  for all  $n$ . Since  $0 < \frac{1}{2\|x_1\|}$ , we can find  $\lambda_1 \in \mathbb{C}$  such that  $0 < |\lambda_1| \leq \frac{1}{2\|x_1\|}$ . Let  $\lambda_2 \neq 0$  be chosen so that  $|\lambda_2| < \min\{\frac{1}{2^2\|x_2\|}, |\lambda_1|\}$ . Then  $\lambda_1 \neq \lambda_2$ . Continuing in this way, we produce a sequence  $(\lambda_n)$  of distinct points such that  $|\lambda_n| \leq 2^{-n}\|x_n\|^{-1}$  for all  $n$ . This implies that  $0 \leq \|\lambda_n x_n\| \leq \frac{1}{2^n}$  for all  $n$ . Since  $\sum \frac{1}{2^n} < \infty$ , it follows that  $\sum \|\lambda_n x_n\| < \infty$ . This implies that  $\sum \lambda_n x_n$  is absolutely convergent. Since  $A$  is a Banach algebra,  $\sum \lambda_n x_n$  is convergent. Hence  $x = \sum \lambda_n x_n \in A$ . Therefore  $xa = (\sum \lambda_n x_n)a = \sum \lambda_n x_n a = \sum \lambda_n p_n \in Aa$ .

We show that  $\lambda_i \in \text{Sp}(xa)$  for all  $i \in \mathbb{N}$ . Let  $p_i \in \{p_n : n \in \mathbb{N}\}$ . Then  $(\lambda_i - xa)p_i = (\lambda_i - \sum \lambda_n p_n)p_i = \lambda_i p_i - \lambda_i p_i^2 = 0$ . So if  $(\lambda_i - xa)$  is invertible, with inverse say,  $v_i$ , then  $p_i = v_i(\lambda_i - xa)p_i = 0$ , which is a contradiction. This implies that  $\lambda_i \in \text{Sp}(xa)$  for all  $i \in \mathbb{N}$ . Now, by Lemma 1.1.6 we have that  $\dim(xaAxa) < \infty$ . It follows from Corollary 1.7.3 that  $\#\text{Sp}(xa) < \infty$ . Since  $\{\lambda_n : n \in \mathbb{N}\}$  is an infinite set,  $\lambda_n \in \text{Sp}(xa)$  for all  $n$  is not possible. So there is no infinite set of non-zero orthogonal idempotents in  $Aa$ .  $\square$

**Lemma 2.6.5** *Let  $A$  be a semiprime Banach algebra and let  $a$  be a non-zero element of  $A$  such that  $\dim(aAa) < \infty$ . Then there exists a maximal set of orthogonal minimal idempotents in  $Aa$ . A similar statement is true for  $aA$ .*

**Proof.** Since  $\dim(aAa) < \infty$ , by Lemma 2.4.1, there exists a minimal idempotent  $p$  in  $Aa$ . The set  $\{p\}$  is a set of orthogonal minimal idempotents in  $Aa$ . So if  $M = \{P_j : j \in I\}$  is the family of all sets of orthogonal minimal idempotents in  $Aa$ , then  $M$  is non-empty. Also, from Lemma 2.6.4, all the sets in  $M$  are finite.

Now, the set  $M$  is partially ordered under set inclusion: If  $P_{j_0} \in M$  then  $P_{j_0} \subset P_{j_1}$ , proving reflexivity. If  $P_{j_0}, P_{j_1} \in M$  with  $P_{j_0} \subset P_{j_1}$  and



$P_{j_1} \subset P_{j_0}$ , then  $P_{j_0} = P_{j_1}$ , showing antisymmetry. If  $P_{j_0}, P_{j_1}, P_{j_2} \in M$  with  $P_{j_0} \subset P_{j_1}$  and  $P_{j_1} \subset P_{j_2}$ , then  $P_{j_0} \subset P_{j_2}$ , which proves transitivity. Every chain  $C \subset M$  has an upper bound: Let  $C = \{P_{j_n} : n \in \mathbb{N}\}$  be a chain in  $M$ . Take  $P^* = \bigcup_n P_{j_n}$ . Then for every  $P_{j_n} \in C$ , we have that  $P_{j_n} \subset P^*$ , so that  $P^*$  is an upper bound of  $C$ . By Zorn's Lemma, there exists a maximal orthogonal set of minimal idempotents in  $Aa$ .  $\square$

In order to prove Lemma 2.6.7 we need the following result.

**Lemma 2.6.6** *Let  $A$  be semiprime Banach algebra and let  $0 \neq a \in A$ . If  $p$  is an idempotent in  $Aa$  and if  $A(ap - a)$  contains a minimal idempotent  $q$ , then there exists a minimal idempotent  $w \in Aa$  such that  $wp = pw = 0$ .*

**Proof.** Let  $0 \neq a \in A$  and let  $p$  be an idempotent in  $Aa$ . Suppose that  $q$  is a minimal idempotent in  $A(ap - a)$ . Then there exists an  $x \in A$  such that  $q = x(ap - a)$ . This means that  $qp = x(ap - a)p = x(ap - ap) = 0$ . Now let  $w = q - pq = (1 - p)q$ . Then  $qw = q(q - pq) = q^2 - qpq = q$ , so that  $w \neq 0$ . Since  $q$  is a minimal idempotent, it follows from Lemma 2.2.1 and Theorem 2.1.18 that  $w \in \mathcal{F}_1$ . Also,

$$w^2 = (q - pq)(q - pq) = q^2 - qpq - pq^2 + pqpq = q - pq = w,$$

which means that  $w$  is an idempotent. By Lemma 2.2.2 this implies that  $w$  is a minimal idempotent. Furthermore,  $pw = p(q - pq) = pq - pq = 0$ . Also, since  $qp = 0$ , we have that  $wp = (1 - p)qp = 0$ . From  $q \in A(ap - a) \subset Aa$  and from  $w = (1 - p)q$ , it is clear that  $w \in Aa$ .  $\square$

**Lemma 2.6.7** *Let  $A$  be a semiprime Banach algebra and let  $a$  be a non-zero element of  $A$  such that  $\dim(aAa) < \infty$ . If for some  $k \in \mathbb{N}$ , the set  $\mathcal{P}_0 = \{p_i : i = 1, 2, \dots, k\}$  is a maximal set of orthogonal minimal idempotents in  $Aa$ , then there exists an idempotent  $p \in \mathcal{F} \cap Aa$  such that  $ap = a$ . A similar statement holds for  $aA$ .*

**Proof.** Let  $p = \sum_{i=1}^k p_i$ . Then  $p^2 = p_1^2 + \dots + p_k^2 = p$ , so that  $p$  is an idempotent. Also, since  $p_i \in \mathcal{F}_1 \cap Aa$  for all  $i$ , it follows that  $p \in \mathcal{F} \cap Aa$ .

We show that  $ap = a$ . If  $ap - a \neq 0$  then  $(ap - a)A(ap - a) \neq \{0\}$ . Also  $(ap - a)A(ap - a) \subset aAa$ . Since  $\dim(aAa) < \infty$ , it follows that  $\dim((ap - a)A(ap - a)) < \infty$ . By Lemma 2.4.1, this implies that there exists a minimal

idempotent  $q \in A(ap - a)$ . It follows from Lemma 2.6.6 that there exists a minimal idempotent  $w \in Aa$  such that  $wp = pw = 0$ . Therefore  $wp_1 = w(p_1 + p_2 + \cdots + p_k)p_1 = wpp_1 = 0$ . Similarly,  $p_1w = 0$ . Proceeding this way we obtain that  $wp_i = p_iw = 0$  for all  $i = 1, 2, \dots, k$ . Also  $w \notin \{p_i : i = 1, 2, \dots, k\}$ , since  $wp = pw = 0$ . This contradicts the fact that the set  $\{p_i : i = 1, 2, \dots, k\}$  contains all the orthogonal minimal idempotents in  $Aa$ . Therefore  $ap = a$ .  $\square$

**Theorem 2.6.8** ([21], Theorem 3.4) *Let  $A$  be a semiprime Banach algebra. If  $u$  is a non-zero element of  $A$  such that  $\dim(uAu) < \infty$ , then there exists an idempotent  $p \in \mathcal{F} \cap Au$ , with  $up = u$ .*

**Proof.** By Lemma 2.6.4, every subset of non-zero orthogonal idempotents in  $Au$  is finite. Therefore by Lemma 2.6.5, there exists a finite maximal set of orthogonal minimal idempotents in  $Au$ . It follows from Lemma 2.6.7 that there exists an idempotent  $p \in \mathcal{F} \cap Au$  with  $up = u$ .  $\square$

**Theorem 2.6.9** ([21], Corollary 3.5) *Let  $A$  be a semiprime Banach algebra and  $0 \neq u \in A$ . Then the operator  $D_u : A \rightarrow A$ , defined by  $D_u x = uxu$  for all  $x \in A$ , is finite-dimensional if and only if  $u \in \mathcal{F}$ .*

**Proof.** Let  $u \in \mathcal{F}$ . Then  $u = \sum_{i=1}^n u_i$  with  $u_i \in \mathcal{F}_1$  for all  $i$ . Therefore

$$D_u x = uxu = \left( \sum_{i=1}^n u_i \right) x \left( \sum_{i=1}^n u_i \right) = u_1 x u_1 + u_1 x u_2 + \cdots + u_n x u_n.$$

Observe that  $u_i x u_j = D_{u_i, u_j} x$ . If  $u_i \sim u_j$ , then by Theorem 2.4.7, we have that  $\dim(D_{u_i, u_j}) = 1$ , say  $u_i x u_j \in \text{span}\{v_{i,j}\}$  for all  $x \in A$ . Hence  $\dim(D_u(A)) < \infty$ .

Conversely, suppose that  $\dim(D_u(A)) = \dim(uAu) < \infty$ . Then it follows from Theorem 2.6.8 that there exists an idempotent  $p \in \mathcal{F} \cap Au$  such that  $up = u$ . Since  $\mathcal{F}$  is a left ideal,  $u = up \in \mathcal{F}$ .  $\square$

Recall from Theorem 2.6.3 that the socle of  $A$  coincides with the set of spatially finite rank elements of  $A$ . This means that the characterization of spatially finite rank elements in terms of finite-dimensional operators is the same in the case of the socle.

## Chapter 3

# Spectrally rank one and finite rank elements

In the first two chapters two types of rank one elements have been discussed. These are the compactly rank one elements and the spatially rank one elements. From the spatially rank one elements, the finite sums, which we called spatially finite rank elements, were defined and some of their properties studied. In this chapter we will study yet another type of rank one elements, called the spectrally rank one elements. This will also lead us to another class of finite rank elements, the spectrally finite rank elements.

### 3.1 Spectrally rank one elements of Banach algebras

The aim of this section is to define spectrally rank one elements and give some of the properties of these elements. Theorem 3.1.3, Theorem 3.1.4 and Theorem 3.1.6 are the main results of this section.

Following is the definition of a spectrally rank one element. This concept was introduced by B. Aupetit and T. Mouton in 1994, and the terminology by R. Harte in 1995.

**Definition 3.1.1** *Let  $A$  be a semiprime Banach algebra. An element  $a \in A$  is a spectrally rank one element of  $A$  if  $a \neq 0$  and  $\#\text{Sp}'(xa) \leq 1$  for all  $x \in A$ .*

The set of spectrally rank one elements of  $A$  will be denoted by  $\mathcal{G}_1$ .

We have the following simple proposition.

**Proposition 3.1.2** *Every non-zero element in the Banach algebra  $\mathbb{C}$  is a spectrally rank one element of  $\mathbb{C}$ .*

**Proof.** Recall that the only non-invertible element of  $\mathbb{C}$  is 0. So for any  $\alpha \in \mathbb{C}$  we get that  $\text{Sp}(\alpha) = \{\alpha\}$  by definition of the spectrum. So if  $0 \neq \lambda \in \mathbb{C}$  then  $\text{Sp}(\alpha\lambda) = \{\alpha\lambda\}$  for all  $\alpha \in \mathbb{C}$ . This implies that  $\lambda \in \mathcal{G}_1$ .  $\square$

In 1994 B. Aupetit and T. Mouton gave the following characterization of semisimple Banach algebras that have invertible spectrally rank one elements.

**Theorem 3.1.3** ([5], p.95) *A semiprime Banach algebra  $A$  contains invertible spectrally rank one elements if and only if it is isomorphic to  $\mathbb{C}$ .*

**Proof.** Let  $a$  be an invertible spectrally rank one element of  $A$ . Note that for  $x \in A$  and  $\mu \in \mathbb{C}$  such that  $\mu \notin \text{Sp}(x)$ , we have that  $\#\text{Sp}((\mu - x)^{-1}a) = 1$ . Let  $y$  be an arbitrary element of  $A$  and let  $\lambda \in \mathbb{C}$  be such that  $|\lambda| > 2\rho(y)$ . Let  $x = \mu - (y - \lambda)^{-1}$ , where  $\mu = -\frac{1}{\lambda}$ . Then from Lemma 1.5.5, we have that  $\rho(x) < |\mu|$ , so that  $\mu \notin \text{Sp}(x)$ . Therefore  $\mu - x \in A^{-1}$  and from  $x = \mu - (y - \lambda)^{-1}$ , we obtain that  $y - \lambda = (\mu - x)^{-1}$ . Therefore  $\text{Sp}((y - \lambda)a) = \text{Sp}((\mu - x)^{-1}a)$ . Since  $a \in A^{-1} \cap \mathcal{G}_1$ , we have that  $(\mu - x)^{-1}a \in A^{-1}$  and  $\text{Sp}((\mu - x)^{-1}a)$  consists of one non-zero point. This implies that  $\#\text{Sp}((y - \lambda)a) = 1$  for all  $\lambda$  in the set  $\{\lambda \in \mathbb{C} : |\lambda| > 2\rho(y)\}$ . Since this set has non-zero capacity, it follows from Corollary 1.7.6 (i) that  $\#\text{Sp}((y - \lambda)a) = 1$  for all  $\lambda \in \mathbb{C}$ . Take  $\lambda = 0$ . Then  $\#\text{Sp}(ya) = 1$  for all  $y \in A$ . Since  $a \in A^{-1}$  we have that  $Aa = A$  as  $y = (ya^{-1})a \in Aa$  for all  $y \in A$ . This implies that  $\#\text{Sp}(y) = 1$  for all  $y \in A$ . Since  $A$  is open, it follows from Corollary 1.7.9 that  $A$  is isomorphic to  $\mathbb{C}$ .

Conversely, suppose that  $A$  is isomorphic to  $\mathbb{C}$ . Then every non-zero element of  $A$  is invertible. Also, every non-zero element of  $A$  is spectrally of rank one by Proposition 3.1.2.  $\square$

The following result shows that the set of spatially rank one elements of a semiprime Banach algebra is contained in the set of spectrally rank one elements of the Banach algebra.

**Theorem 3.1.4** ([11], p.298) *If  $A$  is a semiprime Banach algebra then  $\mathcal{F}_1 \subset \mathcal{G}_1$ .*

**Proof.** Let  $a \in \mathcal{F}_1$  and let  $x$  be an arbitrary element of  $A$ . If  $xa = 0$ , then  $\text{Sp}(xa) = \{0\}$ . If  $xa \neq 0$ , then  $xa \in \mathcal{F}_1$  by Theorem 2.1.18. It follows from Theorem 2.3.1 that  $\text{Sp}(xa) \subset \{0, \text{tr}(xa)\}$ . Therefore  $\#\text{Sp}'(xa) \leq 1$  for all  $x \in A$ , so that  $a \in \mathcal{G}_1$ .  $\square$

R.M. Brits, L. Lindeboom and H. Raubenheimer showed that the inclusion  $\mathcal{F}_1 \subset \mathcal{G}_1$  may in general be strict. We prove this fact in the following corollary.

**Corollary 3.1.5** ([11], p.298) *If  $A$  is a semiprime Banach algebra which is not semisimple, then  $\text{Rad}(A) \setminus \{0\} \subset \mathcal{G}_1$  and  $\mathcal{F}_1$  is properly contained in  $\mathcal{G}_1$ .*

**Proof.** Since  $A$  is not semisimple,  $\text{Rad}(A) \neq \{0\}$ . So let  $0 \neq a \in \text{Rad}(A)$ . By Theorem 1.4.12, we have that  $Aa \subset \text{QN}(A)$ . It follows that  $\text{Sp}(xa) = \{0\}$  for all  $x \in A$ . This implies that  $a \in \mathcal{G}_1$ , so that  $\text{Rad}(A) \setminus \{0\} \subset \mathcal{G}_1$ . Since  $\mathcal{F}_1 \cap \text{Rad}(A) = \emptyset$  by Corollary 2.1.13, we have that  $a \notin \mathcal{F}_1$ . It follows from Theorem 3.1.4 that  $\mathcal{F}_1 \subsetneq \mathcal{G}_1$ .  $\square$

In Theorem 2.5.3 we saw that if  $u$  is a spatially rank one element in a semiprime Banach algebra  $A$ , then the connected component of  $\mathcal{F}_1$  containing  $u$  is the set  $\text{Exp}(A)u\text{Exp}(A)$ . For the situation where the Banach algebra is semiprime but not semisimple and the set involved is  $\mathcal{G}_1$ , Brits, Lindeboom and Raubenheimer got the following result.

**Theorem 3.1.6** ([11], Proposition 2.9) *Let  $A$  be a semiprime Banach algebra which is not semisimple and let  $u \in \mathcal{G}_1$ . If  $K_u^{\mathcal{G}_1}$  is the connected component of  $\mathcal{G}_1$  containing  $u$ , then  $K_u^{\mathcal{G}_1} = \mathcal{G}_1$ .*

**Proof.** Since  $A$  is not semisimple,  $\text{Rad}(A) \neq \{0\}$ . So let  $0 \neq a \in \text{Rad}(A)$ . For any  $x \in A$  consider  $x(\alpha u + (1 - \alpha)a) = \alpha xu + (1 - \alpha)xa$ , where  $\alpha \in [0, 1]$ . Since  $\text{Rad}(A)$  is a two-sided ideal,  $(1 - \alpha)xa \in \text{Rad}(A)$ . It follows from Lemma 1.4.20 that  $\text{Sp}(x(\alpha u + (1 - \alpha)a)) \subset \text{Sp}(\alpha xu)$ . Since  $u \in \mathcal{G}_1$ , we have that  $\#\text{Sp}'(\alpha xu) \leq 1$  for all  $x \in A$ . This implies that  $\#\text{Sp}'(x(\alpha u + (1 - \alpha)a)) \leq 1$  for all  $x \in A$ , so that the line segment joining  $u$  and  $a$  is a path in  $\mathcal{G}_1$ . This implies that any two elements in  $\mathcal{G}_1$  can be joined by a polygonal path passing

through  $a$ . Therefore  $\mathcal{G}_1$  is pathwise connected, so that  $\mathcal{G}_1$  is connected. Since  $\mathcal{G}_1$  is connected and contains  $u$ , it follows that  $\mathcal{G}_1 \subset K_u^{\mathcal{G}_1}$ . Since we also have the inclusion  $K_u^{\mathcal{G}_1} \subset \mathcal{G}_1$ , we obtain that  $K_u^{\mathcal{G}_1} = \mathcal{G}_1$ .  $\blacksquare$

## 3.2 Spectrally finite rank elements of Banach algebras

Having discussed spectrally rank one elements in the previous section, we now move on to the spectrally finite rank elements. The main results in this section are Theorem 3.2.2, Theorem 3.2.3, Corollary 3.2.4 and Corollary 3.2.6.

**Definition 3.2.1** *An element  $a$  in a semiprime Banach algebra  $A$  is a spectrally finite rank element of  $A$  if  $\#\text{Sp}'(xa) \leq n$  for all  $x \in A$  and for some  $n \in \mathbb{N}$ .*

The set of spectrally finite rank elements of  $A$  will be denoted by  $\mathcal{G}$ .

In 1996 B. Aupetit and T. Mouton proved the following result. It says that for an element  $u$  in a semisimple Banach algebra  $A$  to be in  $\mathcal{G}$ , the requirement  $\#\text{Sp}'(xa) \leq n$  for all  $x \in A$  and for some fixed  $n \in \mathbb{N}$  can be relaxed to the condition  $\#\text{Sp}'(xa) < \infty$  for all  $x \in A$ . This is the next result.

**Theorem 3.2.2** ([6], p.117) *Suppose that  $A$  is a semiprime Banach algebra and that  $a \in A$ . Then  $a \in \mathcal{G}$  if and only if  $\#\text{Sp}(xa) < \infty$  for all  $x \in A$ .*

**Proof.** Let  $a \in \mathcal{G}$ . Then there exists a positive integer  $n$  such that  $\#\text{Sp}'(xa) \leq n$  for all  $x \in A$ . Hence  $\#\text{Sp}(xa) < \infty$  for all  $x \in A$ .

Conversely, suppose that  $\#\text{Sp}(xa) < \infty$  for all  $x \in A$ . Then  $A = \bigcup_{k \in \mathbb{N}} A_k$ , where the  $A_k$  are the sets defined by  $A_k = \{b \in A : \#\text{Sp}(ba) \leq k\}$ . By Lemma 1.7.10, every  $A_k$  is closed. It follows from Baire's theorem that there exists an open set  $U$  in at least one of the sets  $A_k$ . Let  $k_0$  be the smallest integer such that  $A_{k_0}$  contains the open set  $U$ . Then  $\#\text{Sp}(xa) \leq k_0$  for all  $x \in U$ . Since  $U$  is open,  $U$  is an absorbing set. By Corollary 1.7.6 (ii), this implies that  $\#\text{Sp}'(xa) \leq k_0$  for all  $x \in A$ , so that  $a \in \mathcal{G}$ .  $\blacksquare$

Theorem 2.3.6 says that every spatially rank one element  $u$  of a semiprime Banach algebra  $A$  satisfies the spectral property  $\text{Sp}(x + s_0u) \cap \text{Sp}(x + s_1u) \subset \text{Sp}(x)$  for all  $x \in A$  and for any two distinct non-zero scalars  $s_0$  and  $s_1$ . In the following result we prove that if an element  $u$  in a semiprime Banach algebra  $A$  satisfies a similar spectral property, then  $u$  is a spectrally finite rank element of  $A$ .

**Theorem 3.2.3** *Let  $A$  be a semiprime Banach algebra and let  $a \in A$ . If there exists an  $n \in \mathbb{N}$  such that  $a$  satisfies  $\bigcap_{i=0}^n \text{Sp}(x + s_i a) \subset \text{Sp}(x)$  for every  $x \in A$  and for any set of distinct non-zero scalars  $\{s_i : i = 0, 1, \dots, n\}$ , then  $\#\text{Sp}'(xa) \leq n$  for all  $x \in A$ . Hence  $a \in \mathcal{G}$ .*

**Proof.** Since  $a$  satisfies  $\bigcap_{i=0}^n \text{Sp}(x + s_i a) \subset \text{Sp}(x)$  for every  $x \in A$  and for any set of distinct non-zero scalars  $\{s_i : i = 0, 1, \dots, n\}$ , we have from Lemma 1.4.10 that the spectrum of every element of  $A^{-1}a$  has at most  $n + 1$  distinct points. It follows from Corollary 1.7.7 that the spectrum of every element in  $Aa$  has at most  $n + 1$  distinct points.

Now consider an arbitrary element  $xa$  in  $Aa$ . If  $x \in A^{-1}$  then from Lemma 1.4.10 we have that  $\#\text{Sp}'(xa) \leq n$ . If  $x \notin A^{-1}$  then we have two cases:

(i) If  $a \in A^{-1}$  then  $xa \notin A^{-1}$ , so that  $0 \in \text{Sp}(xa)$ . Since  $\text{Sp}(xa)$  has at most  $n + 1$  distinct points, this implies that  $\#\text{Sp}'(xa) \leq n$ .

(ii) If  $a \notin A^{-1}$  then  $Aa \neq A$  or  $aA \neq A$ . We will only take the case  $Aa \neq A$ , as the other case can be dealt with in a similar way. Since  $Aa \neq A$ , no element in  $Aa$  is invertible. Therefore  $xa \notin A^{-1}$ , so that  $0 \in \text{Sp}(xa)$ . As in case (i) we conclude that  $\#\text{Sp}'(xa) \leq n$ .

So we have shown that  $\#\text{Sp}'(xa) \leq n$  for all  $x \in A$ . Hence  $a \in \mathcal{G}$ .  $\square$

T. Mouton and H. Raubenheimer proved that an element  $a$  in a semisimple Banach algebra  $A$  is spatially of finite rank if and only if there exists an  $n \in \mathbb{N}$  such that  $a$  satisfies  $\bigcap_{i=0}^n \text{Sp}(x + s_i a) \subset \text{Sp}(x)$  for every  $x \in A$  and for any set of distinct non-zero scalars  $\{s_i : i = 0, 1, \dots, n\}$  ([19], Theorem 3.1). As a consequence of Theorem 3.2.3, we prove in the next corollary that this

result still holds when we replace  $\mathcal{F}$  by  $\mathcal{G}$ . It is to be noted that the proof follows essentially along the lines of part of the proof of ([19], Theorem 3.1).

**Corollary 3.2.4** *Let  $A$  be a semiprime Banach algebra. Then an element  $a \in A$  is a spectrally finite rank element of  $A$  if and only if there exists an  $n \in \mathbb{N}$  such that  $a$  satisfies  $\bigcap_{i=0}^n \text{Sp}(x + s_i a) \subset \text{Sp}(x)$  for every  $x \in A$  and for any set of distinct non-zero scalars  $\{s_i : i = 0, 1, \dots, n\}$ .*

**Proof.** Suppose that  $a \in \mathcal{G}$ . Then  $\#\text{Sp}'(xa) \leq n$  for all  $x \in A$  and some  $n \in \mathbb{N}$ . Let  $x \in A$  and let  $\{s_i : i = 0, 1, \dots, n\}$  be any set of distinct non-zero scalars. Suppose that  $\lambda \notin \text{Sp}(x)$ . Then  $\lambda - x \in A^{-1}$ . If  $\lambda \in \bigcap_{i=0}^n \text{Sp}(x + s_i a)$ , then  $\lambda - x - s_i a \notin A^{-1}$  for all  $i \in \{0, 1, \dots, n\}$ . It follows from Lemma 2.3.4 that  $\frac{1}{s_i} \in \text{Sp}((\lambda - x)^{-1}a)$  for all  $i \in \{0, 1, \dots, n\}$ . Since there are  $n + 1$  distinct non-zero points in the set  $\{\frac{1}{s_i} : i = 0, 1, \dots, n\} \subset \text{Sp}((\lambda - x)^{-1}a)$ , this contradicts  $\#\text{Sp}'(xa) \leq n$  for all  $x \in A$ . Therefore  $\lambda \notin \bigcap_{i=0}^n \text{Sp}(x + s_i a)$ , so

that  $\bigcap_{i=0}^n \text{Sp}(x + s_i a) \subset \text{Sp}(x)$ .

Conversely, suppose that there exists an  $n \in \mathbb{N}$  such that  $a \in A$  satisfies  $\bigcap_{i=0}^n \text{Sp}(x + s_i a) \subset \text{Sp}(x)$  for all  $x \in A$  and for any set of distinct non-zero scalars  $\{s_i : i = 0, 1, \dots, n\}$ . Then we have from Theorem 3.2.3 that  $a \in \mathcal{G}$ . ■

**Remark 3.2.5** *Observe that Theorem 3.2.3 and the proof of Corollary 3.2.4 combined lead us to the following conclusion: For a fixed  $n \in \mathbb{N}$ , an element  $a$  in a semiprime Banach algebra  $A$  satisfies  $\bigcap_{i=0}^n \text{Sp}(x + s_i a) \subset \text{Sp}(x)$  for every  $x \in A$  and for any set of distinct non-zero scalars  $\{s_i : i = 0, 1, \dots, n\}$ , if and only if  $\#\text{Sp}'(xa) \leq n$  for all  $x \in A$ .*

From this remark we obtain the following result. T. Mouton and H. Raubenheimer obtained a similar result for the spatially rank one elements ([19], Theorem 2.2).

**Corollary 3.2.6** *Suppose that  $A$  is a semiprime Banach algebra and  $a \in A$ . Then  $a \in \mathcal{G}_1$  if and only if  $a$  satisfies  $\text{Sp}(x + s_0 a) \cap \text{Sp}(x + s_1 a) \subset \text{Sp}(x)$  for all  $x \in A$  and for any two-point set of distinct non-zero scalars  $\{s_0, s_1\}$ .*



In 1996 B. Aupetit and T. Mouton obtained the following equivalent formulations for spectrally finite rank elements.

**Theorem 3.2.7** ([6], Theorem 2.1) *Let  $A$  be a semiprime Banach algebra. If  $a \in A$  and  $n \geq 0$  is an integer, then the following properties are equivalent:*

(i)  $\#\text{Sp}'(ba) \leq n$  for every  $b \in A$ ,

(ii)  $\#\{s \in \mathbb{C} : 0 \in \text{Sp}(b + sa)\} \leq n$  for every invertible  $b$  in  $A$ ,

(iii)  $\bigcap_{i=0}^n \text{Sp}(b + s_i a) \subset \text{Sp}(b)$  for every  $b \in A$  and for every set  $\{s_i : i = 0, 1, \dots, n\}$  of distinct non-zero scalars.

**Proof.** (i)  $\Rightarrow$  (ii): Let  $s \in \mathbb{C}$  and suppose that  $0 \in \text{Sp}(b + sa)$  with  $b$  invertible. Then  $b + sa$  is not invertible. Since  $b + sa = sb(\frac{1}{s} + b^{-1}a)$ , this implies that  $\frac{1}{s} + b^{-1}a \notin A^{-1}$ , so that  $-\frac{1}{s} \in \text{Sp}(b^{-1}a)$ . We have proved that  $0 \in \text{Sp}(b + sa) \Rightarrow -\frac{1}{s} \in \text{Sp}(b^{-1}a)$ . Since (i) gives us that  $\#\text{Sp}'(b^{-1}a) \leq n$ , this implies that  $\#\{s \in \mathbb{C} : 0 \in \text{Sp}(b + sa)\} \leq n$ .

(ii)  $\Rightarrow$  (iii): Suppose that  $\lambda \notin \text{Sp}(b)$ . Then  $b - \lambda$  is invertible. If  $\lambda \in \bigcap_{i=0}^n \text{Sp}(b + s_i a)$ , then  $b - \lambda + s_i a$  is not invertible for all  $i = 0, 1, \dots, n$ . Therefore  $0 \in \bigcap_{i=0}^n \text{Sp}(b - \lambda + s_i a)$ . This contradicts (ii) applied to  $b - \lambda$ , since there are  $n + 1$  scalars  $\{s_i : i = 0, 1, \dots, n\}$ . Hence  $\lambda \notin \bigcap_{i=1}^n \text{Sp}(b + s_i a)$ , so that

$$\bigcap_{i=0}^n \text{Sp}(b + s_i a) \subset \text{Sp}(b).$$

(iii)  $\Rightarrow$  (i): Let  $\bigcap_{i=0}^n \text{Sp}(b + s_i a) \subset \text{Sp}(b)$ . It follows from Theorem 3.2.3 that  $a$  satisfies  $\#\text{Sp}'(ba) \leq n$  for all  $b \in A$ .  $\blacksquare$

## Chapter 4

# Relationships among the various concepts of rank one and finite rank

In Chapter 1, we introduced the compactly rank one elements of Banach algebras. The spatially rank one and finite rank elements were discussed in Chapter 2, while Chapter 3 studied the spectrally rank one and finite rank elements. The aim of this chapter is to study the relationships among the various concepts of rank one and finite rank.

### 4.1 Spatially rank one and spectrally rank one elements in semisimple Banach algebras

Theorem 3.1.4 says that in semiprime Banach algebras, the inclusion  $\mathcal{F}_1 \subset \mathcal{G}_1$  holds. It was shown in Corollary 3.1.5 that this inclusion may in general be strict, notably in semiprime Banach algebras which are not semisimple. The aim of this section is to give the relationship between  $\mathcal{F}_1$  and  $\mathcal{G}_1$  in semisimple Banach algebras. Theorem 4.1.4 (and Theorem 4.1.5) are the main results of this section.

In 1995 R. Harte proved that in a semisimple Banach algebra, all the idempotents of  $\mathcal{G}_1$  are contained in  $\mathcal{F}_1$  ([14], Lemma 3). His proof relies on Theorem 1.4.18. We prove this fact in Theorem 4.1.2 in a more elegant way

using Theorem 1.4.17. The following lemma will be required.

**Lemma 4.1.1** ([14], Lemma 1) *Let  $A$  be a semisimple Banach algebra. If  $\#\text{Sp}(a) = 1$  for every invertible element  $a \in A$ , then  $A$  is one-dimensional.*

**Proof.** Suppose that  $\#\text{Sp}(a) = 1$  for all  $a \in A^{-1}$ . We want to show that  $\#\text{Sp}(x) = 1$  for all  $x \in A$ . So let  $x$  be an arbitrary element of  $A$  and let  $\lambda \in \mathbb{C}$  such that  $|\lambda| > \|x\|$ . Then  $x - \lambda \in A^{-1}$  by Proposition 1.1.7. By assumption,  $\text{Sp}(x - \lambda) = \{\alpha\}$ , for some  $\alpha \in \mathbb{C}$ . It follows from the spectral mapping theorem that  $\text{Sp}(x) = \{\lambda + \alpha\}$ , so that  $\#\text{Sp}(x) = 1$  for all  $x \in A$ .

Now, if  $a$  is an arbitrary element of  $A$ , then  $\text{Sp}(a) = \{\mu\}$ , for some  $\mu \in \mathbb{C}$ . This means that  $a - \mu \notin A^{-1}$ . It follows that  $b(a - \mu) \notin A^{-1}$  for all  $b \in A^{-1}$ , so that  $0 \in \text{Sp}(b(a - \mu))$ . Since  $\#\text{Sp}(x) = 1$  for all  $x \in A$ , this implies that  $\text{Sp}(b(a - \mu)) = \{0\}$ . It follows from Corollary 1.4.16 that  $a - \mu \in \text{Rad}(A)$ . Since  $A$  is semisimple,  $a - \mu = 0$ , so that  $a = \mu \cdot 1$ . Since  $a$  is an arbitrary element of  $A$ , we have that  $\dim(A) = 1$ .  $\square$

**Theorem 4.1.2** ([14], Lemma 3) *Let  $A$  be a semisimple Banach algebra and let  $p$  be an idempotent element in  $A$ . If  $p \in \mathcal{G}_1$  then  $p \in \mathcal{F}_1$ .*

**Proof.** Let  $p$  be a spectrally rank one idempotent element of  $A$ . Then  $\#\text{Sp}'(xp, A) \leq 1$  for all  $x \in A$ . If  $pyp$  is an arbitrary element in  $pAp$ , then we have from Theorem 1.4.17 (iii) that  $\text{Sp}'(pyp, pAp) \subset \text{Sp}'(pyp, A)$ . Since  $p$  is a spectrally rank one element and  $py \in A$ , we have that  $\#\text{Sp}'(pyp, A) \leq 1$ . This implies that  $\#\text{Sp}'(pyp, pAp) \leq 1$  for all  $pyp \in pAp$ , so that  $\#\text{Sp}(pzp, pAp) = 1$  for all invertible elements  $pzp \in pAp$ . Since  $pAp$  is a semisimple Banach algebra by Theorem 1.4.17 (i) and (ii), it follows from Lemma 4.1.1 that  $pAp = \mathbb{C}p$ . Therefore  $pyp = \lambda(y)p$  for all  $y \in A$  and some  $\lambda(y) \in \mathbb{C}$ . Hence  $p$  is a spatially rank one element of  $A$ .  $\square$

Theorem 4.1.4 gives us a stronger relationship between  $\mathcal{G}_1$  and  $\mathcal{F}_1$  in a semisimple Banach algebra. It tells us that the sets  $\mathcal{G}_1$  and  $\mathcal{F}_1$  are actually equal. The proof relies on the arguments used in the proof of ([19], Theorem 2.2) and on the following lemma.

**Lemma 4.1.3** ([19], Lemma 3.4) *Let  $A$  be a semisimple Banach algebra and let  $0 \neq x \in A$ . If the spectrum of every element of  $Ax$  consists of 0 and at most  $n$  other distinct non-zero points for some  $n \in \mathbb{N}$ , then there exists a minimal idempotent  $p \in Ax$ .*

**Proof.** Let  $0 \neq x \in A$  such that for every  $y \in A$ , the set  $\text{Sp}(yx)$  consists of 0 and at most  $n$  other distinct non-zero points. Suppose that  $Ax \subset \text{QN}(A)$ . Then  $x \in \text{Rad}(A)$  by Theorem 1.4.12. Since  $A$  is semisimple, this implies that  $x = 0$ . This is a contradiction. So there exists  $ax$  in  $Ax$  with a non-zero isolated point in its spectrum. Let  $q$  be the spectral idempotent associated with this point. Then by Corollary 1.5.7, there exists  $z \in A$  such that  $q = zax \in Ax$ . Since  $q \in Ax$ , there exists  $b \in A$  such that  $q = bx$ . Hence  $qAq = (bxAb)x \subset Ax$ . From the hypothesis, it follows that every element of  $qAq$  has a finite spectrum. Since  $qAq$  is a semisimple Banach algebra by Theorem 1.4.17 (i) and (ii), it follows from Theorem 1.7.8 (i) that  $qAq$  is finite dimensional. By Lemma 2.4.1, there exists a minimal idempotent  $p \in Aq \subset Ax$ .  $\square$

The proof of the following theorem is essentially part of the proof of ([19], Theorem 2.2) by T. Mouton and H. Raubenheimer.

**Theorem 4.1.4** *If  $A$  is a semisimple Banach algebra then  $\mathcal{F}_1 = \mathcal{G}_1$ .*

**Proof.** Since  $A$  is semisimple,  $A$  is semiprime by Theorem 1.3.4. It follows from Theorem 3.1.4 that  $\mathcal{F}_1 \subset \mathcal{G}_1$ .

We prove that  $\mathcal{G}_1 \subset \mathcal{F}_1$ . Let  $a \in \mathcal{G}_1$ . If  $a \in A^{-1}$  then we have from Theorem 3.1.3 that  $A = \mathbb{C} \cdot 1$ . By Proposition 2.1.5, this implies that  $a \in \mathcal{F}_1$ . If  $a \notin A^{-1}$  then  $Aa \neq A$  or  $aA \neq A$ . We will only consider the case  $Aa \neq A$  as the other case can be treated in a similar way. Since  $Aa \neq A$ , no element of  $Aa$  is invertible. Since  $a \in \mathcal{G}_1$ , this means that the spectrum of every element in  $Aa$  consists of 0 and possibly one other point. By Lemma 4.1.3, this implies that there exists a minimal idempotent  $p \in Aa$ .

We show that  $ap = a$ . If  $ap \neq a$  then  $ap - a \neq 0$ , so that  $\{0\} \neq A(ap - a) \subset Aa$ . So by Lemma 4.1.3 there exists a minimal idempotent  $q \in A(ap - a)$ . It follows from Lemma 2.6.6 that there exists an idempotent  $w = q - pq$  in  $Aa$  such that  $wp = pw = 0$ . Since the spectrum of every element of  $Aa$  consists of 0 and possibly one other point, this is a contradiction according to Lemma 1.4.9. Therefore  $ap = a$  and by Theorem 2.1.18, we have that  $a \in \mathcal{F}_1$ .  $\square$

Note that this result relies on the scarcity lemma, as it uses Theorem 3.1.3 and Lemma 4.1.3. To prove Theorem 3.1.3 we use Corollary 1.7.6 and Corollary 1.7.9, which are both applications of the scarcity lemma. Lemma 4.1.3 relies on Theorem 1.7.8, an application of the scarcity lemma. Harte proved Theorem 4.1.4 without the use of the scarcity lemma. With minor changes,

we give this result as proved by Harte.

**Theorem 4.1.5** ([14], Theorem 4) *If  $A$  is a semisimple Banach algebra then  $\mathcal{F}_1 = \mathcal{G}_1$ .*

**Proof.** Since  $A$  is semisimple,  $A$  is semiprime. By Theorem 3.1.4, this implies that  $\mathcal{F}_1 \subset \mathcal{G}_1$ .

Conversely, let  $a \in \mathcal{G}_1$ . Then  $\#\text{Sp}'(xa) \leq 1$  for all  $x \in A$ . If  $\text{Sp}(xa) = \{0\}$  for all  $x \in A$ , then  $Aa \subset \text{QN}(A)$ . It follows from Theorem 1.4.12 that  $a \in \text{Rad}(A)$ . Since  $A$  is semisimple, this implies that  $a = 0$ , which is a contradiction. So there exists some  $x \in A$  for which  $\text{Sp}(xa) = \{\lambda\}$  or  $\text{Sp}(xa) = \{0, \lambda\}$ , for some  $\lambda \neq 0$ . We consider the two cases separately.

Suppose that  $\text{Sp}(xa) = \{\lambda\}$  for some  $x \in A$ . Then  $xa \in A^{-1}$ . Let  $y$  be any element in  $A^{-1}$ . Since  $a \in \mathcal{G}_1$ , we have that  $\#\text{Sp}'(y) = \#\text{Sp}'(y(xa)^{-1}xa) \leq 1$ . This implies that  $\#\text{Sp}(y) = 1$ . It follows from Lemma 4.1.1 that  $A = \mathbb{C} \cdot 1$ . By Proposition 2.1.5, we have that  $a \in \mathcal{F}_1$ .

Now suppose that  $\text{Sp}(xa) = \{0, \lambda\}$  for some  $x \in A$ . It follows from Corollary 1.5.7 that  $p = yxa$ , where  $p$  is the spectral idempotent associated with  $xa$  and  $\lambda$ , and  $y$  is some element in  $A$ . So for all  $z \in A$ , we have that  $\#\text{Sp}'(zp) = \#\text{Sp}'(zyxa)$ . Since  $a \in \mathcal{G}_1$ , this implies that  $\#\text{Sp}'(zp) \leq 1$ , so that  $p \in \mathcal{G}_1$ . It follows from Theorem 4.1.2 that  $p \in \mathcal{F}_1$ .

We prove that  $a = ap$ . To do this we need to show that  $Aa(1-p) \subset \text{QN}(A)$ . To this end, suppose that there is a  $t \in Aa(1-p)$  with a non-zero point  $\beta$  in its spectrum. By Corollary 1.5.7, this means that we have a spectral idempotent  $q$  associated with  $t$  and  $\beta$  such that  $q = st$ , for some  $s \in A$ . Recall that  $p \in Aa$ . From this we get that  $t \in Aa(1-p) \subset Aa$ , so that  $q \in Aa(1-p) \subset Aa$ . This implies that  $qp = 0$  and that  $\lambda p + \mu q \in Aa$  for all  $\lambda, \mu \in \mathbb{C}$ . Now, we have that  $\lambda \in \text{Sp}(\lambda p + \mu q)$  for if not, then  $p = (\lambda - (\lambda p + \mu q))^{-1}(\lambda - (\lambda p + \mu q))p = 0$ , which is a contradiction. Similarly,  $\mu \in \text{Sp}(\lambda p + \mu q)$ . Hence  $\{\lambda, \mu\} \subset \text{Sp}(\lambda p + \mu q)$ . This is a contradiction since  $\lambda p + \mu q \in Aa$  and  $a \in \mathcal{G}_1$ . So we have that  $Aa(1-p) \subset \text{QN}(A)$ . By Theorem 1.4.12, this implies that  $a(1-p) \in \text{Rad}(A)$ . Since  $A$  is semisimple,  $a(1-p) = 0$ , so that  $a = ap$ . The fact that  $a \in \mathcal{F}_1$  follows from Theorem 2.1.18.  $\square$

In [19] T. Mouton and H. Raubenheimer gave a spectral characterization of spatially rank one elements. This result now becomes an immediate consequence of Corollary 3.2.6 and Theorem 4.1.4.

**Corollary 4.1.6** ([19], Theorem 2.2) *Let  $A$  be a semisimple Banach algebra and  $0 \neq a \in A$ . Then  $a \in \mathcal{F}_1$  if and only if  $a$  satisfies  $\text{Sp}(x + s_0a) \cap \text{Sp}(x + s_1a) \subset \text{Sp}(x)$  for all  $x \in A$  and for any two-point set of distinct non-zero scalars  $\{s_0, s_1\}$ .*

## 4.2 Spatially rank one and spectrally rank one elements in semiprime Banach algebras which are not semisimple

In the previous section we saw from Theorem 4.1.4 that in semisimple Banach algebras, the sets  $\mathcal{F}_1$  and  $\mathcal{G}_1$  are equal. Corollary 3.1.5 says that if the Banach algebra  $A$  is semiprime but not semisimple, then  $\mathcal{F}_1$  is properly contained in  $\mathcal{G}_1$ . Furthermore, the proof of this result pointed out that some of the elements that are in  $\mathcal{G}_1$  but not in  $\mathcal{F}_1$  are those that lie in the set  $\text{Rad}(A) \setminus \{0\}$ . The question that follows is whether there are other elements of  $\mathcal{G}_1$  that are not in either  $\mathcal{F}_1$  or  $\text{Rad}(A) \setminus \{0\}$ . A follow up question is how large this set is. In this section we will be addressing these questions. The main result of this section is Theorem 4.2.3.

In 2003 R.M. Brits, L. Lindeboom and H. Raubenheimer proved a result that shows the nature of the elements of  $\mathcal{G}_1$  that are not in  $\mathcal{F}_1 \cup \text{Rad}(A) \setminus \{0\}$  and indicated how large this set may be ([11], Theorem 2.10). We prove this result as Theorem 4.2.3. The following lemma will be used.

**Lemma 4.2.1** ([11], Lemma 2.8) *Let  $A$  be a semiprime Banach algebra such that  $A$  is not semisimple and suppose that  $a \notin \text{Rad}(A)$ . Then  $a \in \mathcal{G}_1$  if and only if given any  $x \in A$ , there is a unique  $\lambda_x \in \mathbb{C}$  such that  $axa - \lambda_x a \in \text{Rad}(A)$ .*

**Proof.** Let  $a \in \mathcal{G}_1$  with  $a \notin \text{Rad}(A)$ . Also let  $B = A/\text{Rad}(A)$ . By Corollary 1.4.19 we have that  $\text{Sp}(xa, A) = \text{Sp}(xa + \text{Rad}(A), B)$  for all  $x \in A$ . Since  $a \in \mathcal{G}_1$  we have that  $\#\text{Sp}'(xa, A) \leq 1$  for all  $x \in A$ . It follows that  $\#\text{Sp}'(xa + \text{Rad}(A), B) \leq 1$  for all  $x \in A$ . This implies that  $a + \text{Rad}(A)$  is spectrally of rank one in  $B$ . Since  $B$  is semisimple by Corollary 1.4.19, it follows from Theorem 4.1.4 that  $a + \text{Rad}(A)$  is spatially of rank one in  $B$ . This means that for all  $x \in A$  there exist  $\lambda_x \in \mathbb{C}$  such that  $axa + \text{Rad}(A) = \lambda_x a + \text{Rad}(A)$ , so that  $axa - \lambda_x a \in \text{Rad}(A)$ . To show the uniqueness of  $\lambda_x$  suppose that

$\lambda_x'$  is another scalar satisfying  $axa + \text{Rad}(A) = \lambda_x'a + \text{Rad}(A)$  for all  $x \in A$ . Then  $axa - \lambda_x'a \in \text{Rad}(A)$ . Since the radical is a two-sided ideal, this means that  $(\lambda_x' - \lambda_x)a = axa - \lambda_x a - (axa - \lambda_x'a) \in \text{Rad}(A)$ . This implies that  $a \in \text{Rad}(A)$ , which is a contradiction. Therefore  $\lambda_x = \lambda_x'$ .

Conversely, suppose that given any  $x \in A$  there is a unique  $\lambda_x \in \mathbb{C}$  such that  $axa - \lambda_x a \in \text{Rad}(A)$ . Then since  $\text{Rad}(A)$  is a two-sided ideal, we have that  $(xa)^2 - \lambda_x xa = x(axa - \lambda_x a) \in \text{Rad}(A)$ . It follows from Theorem 1.4.12 that  $\text{Sp}((xa)^2 - \lambda_x xa) = \{0\}$ . So if  $\alpha \in \text{Sp}(xa)$  then by the spectral mapping theorem,  $\alpha$  satisfies  $\alpha^2 - \lambda_x \alpha = 0$ , so that  $\alpha = 0$  or  $\alpha = \lambda_x$ . This implies that  $\#\text{Sp}'(xa) \leq 1$  for all  $x \in A$ , so that  $a \in \mathcal{G}_1$ .  $\square$

It has been mentioned that in semiprime Banach algebras which are not semisimple, there may be elements of  $\mathcal{G}_1$  that belong to neither  $\mathcal{F}_1$  nor  $\text{Rad}(A) \setminus \{0\}$ . The set of these elements will be denoted by  $\mathcal{H}_1$  and will be called the set of *quasipatially* rank one elements of  $A$ . As we will prove in Theorem 4.2.3, the set  $\mathcal{H}_1$  is dense in  $\mathcal{G}_1$ , which means that it is a large set.

**Remark 4.2.2** ([11], p.306) *Let  $A$  be a semiprime Banach algebra which is not semisimple. By definition of spatially rank one, an element  $a$  is not in  $\mathcal{F}_1$  if and only if there is at least one  $x_0$  for which  $ax_0a \neq \lambda a$  for all  $\lambda \in \mathbb{C}$ . Combining this statement with the statement of Lemma 4.2.1, we conclude that an element  $a$  is in  $\mathcal{H}_1$  if and only if there exists for each  $x \in A$  a unique  $\lambda_x$  such that  $axa - \lambda_x a \in \text{Rad}(A)$  and there exists at least one  $x_0$  and  $\lambda_0$  such that  $ax_0a - \lambda_0 a \neq 0$ .*

With this remark and Lemma 4.2.1, we are in a position to prove Theorem 4.2.3.

**Theorem 4.2.3** ([11], Theorem 2.10) *Let  $A$  be a semiprime Banach algebra such that  $A$  is not semisimple and suppose that  $\mathcal{F}_1 \neq \emptyset$ . Then*

$$(i) \text{cl}_A(\mathcal{F}_1) \cap \text{Rad}(A) = \{0\},$$

$$(ii) \text{cl}_A(\mathcal{F}_1) \cdot \text{Rad}(A) = \text{Rad}(A) \cdot \text{cl}_A(\mathcal{F}_1) = \{0\},$$

$$(iii) \mathcal{F}_1 \text{ and } \text{Rad}(A) \setminus \{0\} \text{ are closed in } \mathcal{G}_1 \text{ and hence } \mathcal{H}_1 \text{ is open in } \mathcal{G}_1 \text{ with } \mathcal{H}_1 \neq \emptyset,$$

(iv)  $\text{Rad}(A) \setminus \{0\} + \mathcal{F}_1 \subset \mathcal{H}_1$ ,

(v)  $\mathcal{H}_1$  is dense in  $\mathcal{G}_1$ .

**Proof.** (i) Suppose that  $(u_n)$  is a sequence in  $\mathcal{F}_1$  converging to an element  $a \in \text{Rad}(A)$ . Take  $x \in A^{-1}$ . Then  $xu_n \neq 0$  for if not, then  $u_n = x^{-1}xu_n = 0$ , which is a contradiction. Since  $u_n \in \mathcal{F}_1$ , it follows from Corollary 2.1.20 that  $u_nxu_n = f_{u_n}(x)u_n = \text{tr}(xu_n)u_n$ . Since  $\text{tr}(xu_n) \in \text{Sp}(xu_n)$  and  $a \in \text{Rad}(A)$ , we have that  $|\text{tr}(xu_n)| \leq \rho(xu_n) \rightarrow \rho(xa) = 0$ . Hence  $u_nxu_n \rightarrow 0$ , so that  $axa = 0$  for all  $x \in A^{-1}$ . Let  $y$  be an arbitrary element in  $A$ . Then from Lemma 1.4.13, there exist  $b, c \in A^{-1}$  such that  $y = b + c$ . This implies that  $aya = a(b + c)a = aba + aca = 0$ . It follows from Theorem 1.3.3 that  $a = 0$ , so that  $cl_A(\mathcal{F}_1) \cap \text{Rad}(A) = \{0\}$ .

(ii) Let  $u \in cl_A(\mathcal{F}_1)$  and let  $0 \neq a \in \text{Rad}(A)$ . If  $u \in \mathcal{F}_1$  and  $ua \neq 0$  then  $ua \in \mathcal{F}_1$  by Theorem 2.1.18. Also, since  $\text{Rad}(A)$  is a two-sided ideal,  $ua \in \text{Rad}(A)$ . This implies that  $\mathcal{F}_1 \cap \text{Rad}(A) \neq \emptyset$ . Since  $A$  is semiprime but not semisimple, this is a contradiction by Corollary 2.1.13. Therefore  $ua = 0$ .

If  $u$  is a limit of a sequence  $(u_n)$  in  $\mathcal{F}_1$ , then we have that  $u_n \rightarrow u$ , so that  $u_n a \rightarrow ua$ . From the first part of this proof, we have that  $u_n a = 0$  for all  $n \in \mathbb{N}$ , since  $(u_n)$  is a sequence in  $\mathcal{F}_1$ . This implies that  $ua = 0$ . Therefore  $cl_A(\mathcal{F}_1) \cdot \text{Rad}(A) = \{0\}$ . A similar argument can be used to show that  $\text{Rad}(A) \cdot cl_A(\mathcal{F}_1) = \{0\}$ .

(iii) Let  $(a_n)$  be a sequence in  $\text{Rad}(A) \setminus \{0\}$  such that  $a_n \rightarrow a$ , for  $a \in \mathcal{G}_1$ . Since  $\text{Rad}(A) \setminus \{0\} \subset \text{Rad}(A)$  and  $\text{Rad}(A)$  is closed in  $A$  by Proposition 1.2.5, we have that  $a \in \text{Rad}(A) \setminus \{0\}$ , so that  $\text{Rad}(A) \setminus \{0\}$  is closed in  $\mathcal{G}_1$ .

To show that  $\mathcal{F}_1$  is closed in  $\mathcal{G}_1$ , let  $(u_n)$  be a sequence in  $\mathcal{F}_1$  such that  $u_n \rightarrow u \in \mathcal{G}_1$ . Since  $u \neq 0$  it follows from (i) that  $u \notin \text{Rad}(A)$ . So if  $u \notin \mathcal{F}_1$  then  $u \in \mathcal{H}_1$ . By Remark 4.2.2, there exists an  $x_0 \in A$  and a  $\lambda_0 \in \mathbb{C}$  such that  $ux_0u - \lambda_0u = r \in \text{Rad}(A)$ ,  $r \neq 0$ . Now, since  $(u_n)$  is a sequence in  $\mathcal{F}_1$ , we have that  $u_nx_0u_n = \lambda_nu_n$  for all  $n \in \mathbb{N}$ , and where  $(\lambda_n)$  is a sequence in  $\mathbb{C}$ . Therefore  $|\lambda_n|||u_n|| = ||\lambda_nu_n|| = ||u_nx_0u_n|| \leq ||u_n|||x_0|||u_n||$ . This implies that  $|\lambda_n| \leq ||u_n|||x_0||$ , so that the sequence  $(\lambda_n)$  is bounded. From the Bolzano Weierstrass theorem,  $(\lambda_n)$  has a convergent subsequence  $(\lambda_{n_k})$ , say  $\lambda_{n_k} \rightarrow \lambda$ . Now since  $u_n \rightarrow u$ , we obtain that  $u_{n_k} \rightarrow u$ . Therefore



$\lambda_{n_k} u_{n_k} \rightarrow \lambda u$  and  $u_{n_k} x_0 u_{n_k} \rightarrow u x_0 u$ . By uniqueness of limits,  $u x_0 u = \lambda u$ . Substituting  $\lambda u$  for  $u x_0 u$  in  $r = u x_0 u - \lambda_0 u$ , we have that  $r = \lambda u - \lambda_0 u$ . Since  $r \neq 0$  it follows that  $\lambda \neq \lambda_0$ , so that  $(\lambda - \lambda_0) \neq 0$ . Since  $\text{Rad}(A)$  is a two-sided ideal, this means that  $u \in \text{Rad}(A)$ , which is a contradiction. Therefore  $u \in \mathcal{F}_1$ , so that  $\mathcal{F}_1$  is closed in  $\mathcal{G}_1$ .

Since  $\mathcal{H}_1$  is the complement of  $\mathcal{F}_1 \cup \text{Rad}(A) \setminus \{0\}$  in  $\mathcal{G}_1$ , and both  $\mathcal{F}_1$  and  $\text{Rad}(A) \setminus \{0\}$  are closed in  $\mathcal{G}_1$ , we have that  $\mathcal{H}_1$  is open in  $\mathcal{G}_1$ .

We show that  $\mathcal{H}_1 \neq \emptyset$ . By assumption,  $\mathcal{F}_1 \neq \emptyset$ . Also, since  $A$  is not semisimple,  $\text{Rad}(A) \setminus \{0\} \neq \emptyset$ . From (i) we have that  $cl_A(\mathcal{F}_1) \cap \text{Rad}(A) \setminus \{0\} = \emptyset$ . Since  $cl_{\mathcal{G}_1}(\mathcal{F}_1) \subset cl_A(\mathcal{F}_1)$ , we have that  $cl_{\mathcal{G}_1}(\mathcal{F}_1) \cap \text{Rad}(A) \setminus \{0\} = \emptyset$ . Also, since  $\text{Rad}(A) \setminus \{0\}$  is closed in  $\mathcal{G}_1$ , we get from (i) that  $cl_{\mathcal{G}_1}(\text{Rad}(A) \setminus \{0\}) \cap \mathcal{F}_1 = \emptyset$ . Therefore if  $\mathcal{H}_1 = \emptyset$ , then  $\mathcal{G}_1 = \mathcal{F}_1 \cup \text{Rad}(A) \setminus \{0\}$ , with  $\mathcal{F}_1$  and  $\text{Rad}(A) \setminus \{0\}$  separating  $\mathcal{G}_1$ . This implies that  $\mathcal{G}_1$  is not connected, which is a contradiction by Theorem 3.1.6. Therefore  $\mathcal{H}_1 \neq \emptyset$ .

(iv) Let  $r \in \text{Rad}(A) \setminus \{0\}$  and  $u \in \mathcal{F}_1$ . It follows from Lemma 1.4.20 and the fact that  $\text{Rad}(A)$  is a two-sided ideal that  $\text{Sp}(x(u+r)) \subset \text{Sp}(xu)$  for all  $x \in A$ . Since  $u \in \mathcal{G}_1$  by Corollary 3.1.5, we have that  $\#\text{Sp}'(xu) \leq 1$  for all  $x \in A$ . This implies that  $\#\text{Sp}'(x(u+r)) \leq 1$  for all  $x \in A$ , so that  $u+r \in \mathcal{G}_1$ .

If  $u+r \in \text{Rad}(A)$ , then since  $\text{Rad}(A)$  is a two-sided ideal, we have that  $u \in \text{Rad}(A)$ . This is a contradiction by (i). Therefore  $u+r \notin \text{Rad}(A) \setminus \{0\}$ . Suppose that  $u+r \in \mathcal{F}_1$ . Then for all  $x \in A$  there exist  $\mu_x \in \mathbb{C}$  such that  $(u+r)x(u+r) = \mu_x(u+r)$ . Therefore  $uxu + rxu + uxr + rxr = \mu_x u + \mu_x r$ . If  $xu = 0$  then  $rxu = 0$ . If  $xu \neq 0$  then  $xu \in \mathcal{F}_1$  by Theorem 2.1.18. This implies that  $rxu = 0$  by (ii). Similarly,  $uxr = 0$ . Hence

$$uxu + rxr = \mu_x u + \mu_x r. \quad (4.2.4)$$

But since  $u \in \mathcal{F}_1$ , there exists  $\lambda_x$  with  $uxu = \lambda_x u$  for all  $x \in A$ . Substituting  $\lambda_x u$  for  $uxu$  in equation (4.2.4), we have that  $(\lambda_x - \mu_x)u = \mu_x r - rxr \in \text{Rad}(A)$ . If  $\lambda_x \neq \mu_x$  then  $u \in \text{Rad}(A)$ , which is a contradiction. Hence  $\lambda_x = \mu_x$ , so that  $(\lambda_x - \mu_x)u = 0$ . This implies that  $rxr = \mu_x r$  for all  $x \in A$ , which means that  $r \in \mathcal{F}_1$ . This is a contradiction by (i). Therefore  $u+r \notin \mathcal{F}_1$ . This implies that  $u+r \in \mathcal{H}_1$ , so that  $\mathcal{F}_1 + \text{Rad}(A) \setminus \{0\} \subset \mathcal{H}_1$ .

(v) To prove that  $\mathcal{H}_1$  is dense in  $\mathcal{G}_1$  we show that  $cl_{\mathcal{G}_1}(\mathcal{H}_1) = \mathcal{G}_1$ . Obviously,  $cl_{\mathcal{G}_1}(\mathcal{H}_1) \subset \mathcal{G}_1$ .

We prove the inclusion  $\mathcal{G}_1 \subset cl_{\mathcal{G}_1}(\mathcal{H}_1)$ . Let  $a \in \mathcal{G}_1$ . If  $a \in \mathcal{F}_1$  then by (iv) for any  $r \in \text{Rad}(A) \setminus \{0\}$ , we get that  $\{\alpha a + (1 - \alpha)r : \alpha \in (0, 1)\} \subset \mathcal{H}_1$ . This implies that every neighbourhood  $\delta_a$  of  $a$  contains an element of  $\mathcal{H}_1$ . Therefore  $a$  is a limit point of  $\mathcal{H}_1$ , so that  $a \in cl_{\mathcal{G}_1}(\mathcal{H}_1)$ . If  $a \in \text{Rad} \setminus \{0\}$ , a similar argument can be used to show that  $a \in cl_{\mathcal{G}_1}(\mathcal{H}_1)$ . Therefore  $\mathcal{G}_1 \subset cl_{\mathcal{G}_1}(\mathcal{H}_1)$ , so that  $cl_{\mathcal{G}_1}(\mathcal{H}_1) = \mathcal{G}_1$ .  $\square$

### 4.3 Relationship among compactly rank one, spatially rank one and spectrally rank one elements in semisimple Banach algebras

Recall from Definition 1.3.5 that an element  $u$  in a semiprime Banach algebra  $A$  is a compactly rank one element of  $A$  if  $u$  is single and  $u$  acts compactly on  $A$ . Also recall that  $\mathcal{E}_1$  denotes the set of compactly rank one elements of  $A$ . In Theorem 4.1.4 and Theorem 4.1.5 we established that in semisimple Banach algebras,  $\mathcal{F}_1 = \mathcal{G}_1$ . This section is aimed at establishing a similar relationship among the sets  $\mathcal{E}_1$ ,  $\mathcal{F}_1$  and  $\mathcal{G}_1$ .

In 1977 J.A. Erdos, S. Giotopoulos and M.S. Lambrou proved that if  $s$  is a non-zero compactly acting single element of a semisimple Banach algebra  $A$ , then there is a minimal idempotent  $e$  such that  $s = se$  ([13], Theorem 4). This result implies the following.

**Theorem 4.3.1** ([13], Theorem 4) *Let  $A$  be a semisimple Banach algebra and let  $s \in \mathcal{E}_1$ . Then there exists a minimal idempotent  $e$  in  $A$  such that  $s = se$ . Hence  $\mathcal{E}_1 \subset \mathcal{F}_1$ .*

**Proof.** Consider the left ideal  $As$  of  $A$ . If  $xs$  is quasinilpotent for all  $x \in A$ , then  $s \in \text{Rad}(A)$ . Since  $A$  is semisimple, this means that  $s = 0$ , which is a contradiction. Therefore there exists an element  $v = ys$  which is not quasinilpotent. This implies that  $\rho(v) > 0$ , so that  $\|v\| > 0$ . Take  $u = \frac{v}{\|v\|}$  and  $x = \frac{y}{\|v\|}$ . Then  $u = xs$ , with  $u$  non-quasinilpotent and of unit norm. Consider the operator  $D_u a = uau$  for all  $a \in A$ . Since  $u$  acts compactly on  $A$  by Lemma 1.1.19, the operator  $D_u$  is compact.

Now, by definition of  $D_u$  we have that  $D_u u = u^3$  and  $D_u^2 u = D_u D_u u = D_u u^3 = u^5$ . Proceeding this way, we obtain that  $D_u^n u = u^{2n+1}$ . Note that since  $u$  is not quasinilpotent,  $\rho(u) > 0$ . Also since  $\|u\| = 1$ , we have from

$D_u^n u = u^{2n+1}$  that  $\|D_u^n\| = \sup_{\substack{x \in A \\ x \neq 0}} \frac{\|D_u^n x\|}{\|x\|} \geq \frac{\|D_u^n u\|}{\|u\|} = \|u^{2n+1}\|$ . Therefore  $\|D_u^n\|^{\frac{1}{n}} \geq \|u^{2n+1}\|^{\frac{1}{n}} = \left(\|u^{2n+1}\|^{\frac{1}{2n+1}}\right)^{\frac{2n+1}{n}} = \left(\|u^{2n+1}\|^{\frac{1}{2n+1}}\right)^2 \left(\|u^{2n+1}\|^{\frac{1}{2n+1}}\right)^{\frac{1}{n}}$ . Therefore  $\lim_{n \rightarrow \infty} \|D_u^n\|^{\frac{1}{n}} \geq \left(\lim_{n \rightarrow \infty} \|u^{2n+1}\|^{\frac{1}{2n+1}}\right)^2 \left(\lim_{n \rightarrow \infty} \|u^{2n+1}\|^{\frac{1}{2n+1}}\right)^{\frac{1}{n}}$ . Let  $f(n) = \|u^{2n+1}\|^{\frac{1}{2n+1}}$ . Then  $f(n) \rightarrow \rho(u)$ . Let  $y_n = \ln(f(n))^{\frac{1}{n}} = \frac{1}{n} \ln f(n)$ . Since  $\frac{1}{n} \rightarrow 0$  and  $f(n) \rightarrow \rho(u)$ , we have that  $y_n \rightarrow 0$ . This implies that  $\left(\lim_{n \rightarrow \infty} \|u^{2n+1}\|^{\frac{1}{2n+1}}\right)^{\frac{1}{n}} = 1$ . It follows from

$$\lim_{n \rightarrow \infty} \|D_u^n\|^{\frac{1}{n}} \geq \left(\lim_{n \rightarrow \infty} \|u^{2n+1}\|^{\frac{1}{2n+1}}\right)^2 \left(\lim_{n \rightarrow \infty} \|u^{2n+1}\|^{\frac{1}{2n+1}}\right)^{\frac{1}{n}}$$

that  $\rho(D_u) \geq (\rho(u))^2 > 0$ . Hence  $\text{Sp}(D_u) \neq \{0\}$ .

Since every non-zero point in the spectrum of a compact operator is an eigenvalue, this means that  $D_u$  has a non-zero eigenvalue  $\lambda$  and a corresponding finite-dimensional eigenspace  $\mathcal{N}(D_u - \lambda I)$ . Let  $l$  be a non-zero element of  $\mathcal{N}(D_u - \lambda I)$ . Then  $ulu = D_u l = \lambda l$ . So for any positive integer  $n$ , we have that  $D_u l u^n = ulu^{n+1} = \lambda u^n$ . This means that  $lu^n \in \mathcal{N}(D_u - \lambda I)$  for all  $n \in \mathbb{N}$ . Since  $\mathcal{N}(D_u - \lambda I)$  is finite-dimensional, the set  $\{lu^n : n \in \mathbb{N}\}$  is linearly dependent and so there exists a polynomial  $p$  of minimal degree such that  $lup(u) = 0$ . Now, since  $u = xs$  and  $s$  is single, we have from Lemma 1.1.19 (i) that  $u$  single. This implies that  $lu = 0$  or  $up(u) = 0$ . If  $lu = 0$  then  $ulu = 0$ , which contradicts the fact that  $ulu = D_u l = \lambda l \neq 0$ . So  $up(u) = 0$ . If  $p$  can be factored as a product of non-constant polynomials  $p = p_1 \cdot p_2$ , then  $p_1(u)up_2(u) = up(u) = 0$ . Since  $u$  is single, this implies that  $up_1(u) = 0$  or  $up_2(u) = 0$ , so that  $lup_1(u) = 0$  or  $lup_2(u) = 0$ , which contradicts the minimality of  $p$ . Therefore  $p$  is of degree one and hence  $up(u) = u(u - k) = 0$  for some scalar  $k \neq 0$ . From this we get that  $u = \frac{u^2}{k}$ , so that  $\left(\frac{u}{k}\right)^2 = \frac{u^2}{k} \frac{1}{k} = \frac{u}{k}$ . This means that  $e = \frac{u}{k}$  is an idempotent.

We show that  $e$  is a minimal idempotent. From Lemma 1.1.19, we have that  $e$  is single and acts compactly on  $A$ . Let  $i : A \rightarrow A$  be the map defined by  $i(a) = eae$ . Then  $i(eae) = e(eae)e = eae$  for all  $a \in A$ . This implies that  $i$  is the identity map on  $eAe$ , and since  $i$  is compact, we have that  $eAe$  is finite-dimensional. Now let  $eae$  and  $ebe$  be elements in  $eAe$  such that  $eae \cdot ebe = eaebe = 0$ . Since  $e$  is single,  $eae = 0$  or  $ebe = 0$ . This shows that  $eAe$  has no non-zero divisors of zero. It follows from Lemma 1.1.15 that  $eAe$  is a division algebra, so that  $e$  is a minimal idempotent.

To show that  $s = se$ , let  $a$  be any element of  $A$  and recall that  $u = xs$  and  $e = \frac{u}{k}$ . Then  $xs(a - ea) = ke(a - ea) = 0$ . Since  $s$  is single,  $xs = 0$  or  $s(a - ea) = 0$ . From the relations  $u = xs$  and  $e = \frac{u}{k}$ , we obtain that  $xs = ke \neq 0$ . So we must have  $s(a - ea) = (s - se)a = 0$ . Therefore  $(s - se)a \in \text{QN}(A)$ . Since  $a$  is an arbitrary element of  $A$ , we have from Theorem 1.4.12 that  $s - se \in \text{Rad}(A)$ . Since  $A$  is semisimple, it follows that  $s - se = 0$ , so that  $s = se$ . The fact that  $s \in \mathcal{F}_1$  follows from Lemma 2.2.1 and from Theorem 2.1.18.  $\square$

In 2003 R.M. Brits, L. Lindeboom and H. Raubenheimer established a stronger relationship between  $\mathcal{E}_1$  and  $\mathcal{F}_1$ . They proved that for semisimple Banach algebras, the sets  $\mathcal{E}_1$  and  $\mathcal{F}_1$  are actually equal ([11], Corollary 2.4). Recall that Theorem 4.1.4 established that for semisimple Banach algebras,  $\mathcal{G}_1 = \mathcal{F}_1$ . These facts lead us to the following result.

**Theorem 4.3.2** ([11], Corollary 2.4) *Let  $A$  be a semisimple Banach algebra. Then  $\mathcal{E}_1 = \mathcal{F}_1$ . Hence  $\mathcal{E}_1 = \mathcal{F}_1 = \mathcal{G}_1$ .*

**Proof.** If  $u \in \mathcal{E}_1$  then from Theorem 4.3.1 we get that  $u \in \mathcal{F}_1$ , so that  $\mathcal{E}_1 \subset \mathcal{F}_1$ .

Conversely, suppose that  $u \in \mathcal{F}_1$ . Let  $(x_n)$  be a bounded sequence in  $A$ . Then the image of  $(x_n)$  under the map  $a \mapsto uau$  is the sequence  $(ux_nu)$ . Since  $u \in \mathcal{F}_1$ , we have that  $ux_nu = \lambda_n(x_n)u$  for all  $n \in \mathbb{N}$  and for some  $\lambda_n(x_n) \in \mathbb{C}$ . Therefore  $|\lambda_n(x_n)|\|u\| \leq \|u\|\|x_n\|\|u\|$ , so that  $|\lambda_n(x_n)| \leq \|u\|\|x_n\|$  for all  $n \in \mathbb{N}$ . Since  $(x_n)$  is bounded, this implies that the sequence  $(\lambda_n(x_n))$  is bounded. It follows from the Bolzano-Weierstrass Theorem that  $(\lambda_n(x_n))$  has a convergent subsequence. This implies that  $(ux_nu)$  has a convergent subsequence. Hence the operator  $a \mapsto uau$  is compact, so that  $u$  acts compactly on  $A$ .

Now suppose that  $aub = 0$  for some  $a, b \in A$ . If  $au \neq 0$ , then Lemma 2.5.9 implies that  $au = vu$  for some  $v \in \text{Exp}(A)$ . This means that  $vub = 0$ . Since  $v \in \text{Exp}(A)$ , we have that  $v \in A^{-1}$ . Therefore  $ub = v^{-1}vub = 0$ , so that  $u$  is a single element. Hence  $u \in \mathcal{E}_1$ , so that  $\mathcal{F}_1 \subset \mathcal{E}_1$ . The last part of the theorem follows from Theorem 4.1.4.  $\square$

## 4.4 Spatially finite rank and spectrally finite rank elements in semisimple Banach algebras

We have established in Theorem 4.1.4 that in semisimple Banach algebras,  $\mathcal{F}_1 = \mathcal{G}_1$ . This result led to Theorem 4.3.2, which says that the sets  $\mathcal{E}_1$ ,  $\mathcal{F}_1$  and  $\mathcal{G}_1$  are all equal. Our aim in this section is to prove that for semisimple Banach algebras, the sets  $\mathcal{F}$  and  $\mathcal{G}$  also coincide. This equality has some important consequences, some of which we will show. The main results of this section are Theorem 4.4.1 and Theorem 4.4.5.

In 1993 T. Mouton and H. Raubenheimer obtained a spectral characterization of spatially finite rank elements in semisimple Banach algebras ([19], Theorem 3.1). Using part of their proof, we prove the following result.

**Theorem 4.4.1** *Let  $A$  be a semisimple Banach algebra. Then  $\mathcal{F} = \mathcal{G}$ .*

**Proof.** Let  $a \in \mathcal{G}$ . Then there exists an  $n \in \mathbb{N}$  such that  $\#\text{Sp}'(xa) \leq n$  for all  $x \in A$ , so that every element in  $Aa$  has finite spectrum.

(i) If  $a \in A^{-1}$  then  $Aa = A$ . This implies that every element of  $A$  has finite spectrum. Since  $A$  is semisimple, we get from Theorem 1.7.8 (i) that  $\dim(A) < \infty$ , say  $A = \text{span}\{x_1, x_2, \dots, x_m\}$  for some  $m \in \mathbb{N}$ . This means that  $\{ax_1a, ax_2a, \dots, ax_ma\}$  generates  $aAa$ , so that  $\dim(aAa) < \infty$ . It follows from Theorem 2.6.9 that  $a \in \mathcal{F}$ .

(ii) If  $a \notin A^{-1}$  then  $Aa \neq A$  or  $aA \neq A$ . We will only consider the case  $Aa \neq A$  as the other case can be treated in a similar way. Since  $Aa \neq A$ , every element of  $Aa$  is non-invertible. This implies that  $0 \in \text{Sp}(xa)$  for all  $x \in A$ . Also since  $a \in \mathcal{G}$ , we have that  $\#\text{Sp}'(xa) \leq n$  for all  $x \in A$ . It follows from Lemma 4.1.3 that there exists a minimal idempotent  $e \in Aa$ . Now since  $0 \in \text{Sp}(xa)$  for all  $x \in A$  and  $\#\text{Sp}'(xa) \leq n$  for all  $x \in A$ , we conclude from Lemma 1.4.9 that there can be at most  $n$  distinct orthogonal minimal idempotents in  $Aa$ . Let  $k \leq n$  be the integer such that  $\{p_i : i = 1, 2, \dots, k\}$  is the set of all the orthogonal minimal idempotents in  $Aa$ . Take  $p = \sum_{i=1}^k p_i$ . Then  $p^2 = (p_1 + p_2 + \dots + p_k)(p_1 + p_2 + \dots + p_k) = p_1^2 + p_2^2 + \dots + p_k^2 = p_1 + p_2 + \dots + p_k = p$ , so that  $p$  is an idempotent in  $Aa$ .

We show that  $ap = a$ . If  $ap \neq a$  then  $A(ap - a) \neq \{0\}$ . Since  $A(ap - a) \subset Aa$  and the spectrum of every element in  $Aa$  contains 0 and at most  $n$  other distinct points, we have from Lemma 4.1.3 that there exists a minimal idempotent  $q$  in  $A(ap - a)$ . It follows from Lemma 2.6.6 that there exists a minimal idempotent  $w = q - pq$  in  $Aa$  such that  $wp = pw = 0$ . Therefore  $wp_1 = w(p_1 + p_2 + \cdots + p_k)p_1 = wpp_1 = 0$ . Similarly,  $p_1w = 0$ . Proceeding this way we obtain that  $wp_i = p_iw = 0$  for all  $i = 1, 2, \dots, k$ . Also  $w \notin \{p_i : i = 1, 2, \dots, k\}$ , since  $wp = pw = 0$ . This contradicts the fact that the set  $\{p_i : i = 1, 2, \dots, k\}$  contains all the orthogonal minimal idempotents in  $Aa$ . Hence  $ap = a$  and since  $p$  is a sum of minimal idempotents, we have from Lemma 2.2.1 and Proposition 2.6.2 that  $a = ap \in \mathcal{F}$ .

Conversely, suppose that  $a \in \mathcal{F}$ . Then by Theorem 2.6.9, we have that  $\dim(aAa) = n < \infty$ . It follows from Lemma 1.1.6 that  $\dim(xaAxa) \leq n$  for all  $x \in A$ . By Corollary 1.7.3, this implies that  $\#\text{Sp}'(xa) \leq n + 2$  for all  $x \in A$ . Hence  $a \in \mathcal{G}$ .  $\square$

In 1994 B. Aupetit and T. Mouton proved that if  $A$  is a semisimple Banach algebra and  $a \in A$ , then  $a \in \text{Soc}(A)$  if and only if  $\text{Sp}(xa)$  is finite for all  $x \in A$ . In view of the fact that in semiprime Banach algebras,  $\mathcal{F} = \text{Soc}(A)$  by Theorem 2.6.3, the following is an equivalent result. It follows from Theorem 3.2.2 and Theorem 4.4.1.

**Corollary 4.4.2** ([5], Theorem 2.1(1)) *Let  $A$  be a semisimple Banach algebra and let  $u \in A$ . Then  $u \in \mathcal{F}$  if and only if  $\text{Sp}(xa)$  is finite for all  $x \in A$ .*

From the spectral characterization of  $\mathcal{G}$  in Theorem 3.2.4 and from the equality of  $\mathcal{F}$  and  $\mathcal{G}$  in Theorem 4.4.1, the characterization result of Mouton and Raubenheimer for spatially finite rank elements becomes the following corollary.

**Corollary 4.4.3** ([19], Theorem 3.1) *Let  $A$  be a semisimple Banach algebra and  $a \in A$ . Then  $a \in \mathcal{F}$  if and only if there exists  $n \in \mathbb{N}$  such that  $a$  satisfies*

$$\bigcap_{i=0}^n \text{Sp}(x + s_i a) \subset \text{Sp}(x) \text{ for all } x \in A \text{ and for any set of distinct non-zero scalars } \{s_i : i = 0, 1, \dots, n\}.$$

In Theorem 4.4.5, we will prove that this characterization extends to the polynomially convex hull of the spectrum. In 1994 Aupetit and Mouton

proved this characterization for  $\mathcal{F}$  ([5], Theorem 2.2(2)). Our proof is slightly different from theirs, as it refers to Theorem 4.4.1 while they do not. The following proposition will be required.

**Proposition 4.4.4** *Let  $A$  be a semiprime Banach algebra. Then  $\text{Soc}(A)$  is an inessential ideal of  $A$ .*

**Proof.** Let  $a \in \text{Soc}(A)$ . Then  $a \in \mathcal{F}$  by Theorem 2.6.3. It follows from Theorem 2.6.9 that  $aAa$  is finite-dimensional, say  $\dim(aAa) = n$ . By Corollary 1.7.3, this implies that  $a$  is algebraic and  $\text{Sp}(a)$  is finite.  $\square$

**Theorem 4.4.5** ([5], Theorem 2.2(2)) *Let  $A$  be a semisimple Banach algebra and let  $a \in A$ . Then  $a \in \mathcal{F}$  if and only if there exists an  $n \in \mathbb{N}$  such that  $\bigcap_{i=0}^n \sigma(b + s_i a) \subset \sigma(b)$  for every  $b \in A$  and for any set of distinct non-zero scalars  $\{s_i : i = 1, 2, \dots, n\}$ .*

**Proof.** Let  $a \in \mathcal{F}$ . Then from Corollary 4.4.3 there exists an  $n \in \mathbb{N}$  such that  $\bigcap_{i=0}^n \text{Sp}(b + s_i a) \subset \text{Sp}(b)$  for every  $b \in A$  and for any set of distinct non-zero scalars  $\{s_i : i = 0, 1, \dots, n\}$ . Now since  $a \in \mathcal{F}$ , we have from Theorem 2.6.3 that  $a \in \text{Soc}(A)$ . Since  $\text{Soc}(A)$  is a two sided ideal by Proposition 1.2.8, we have that  $s_i a \in \text{Soc}(A)$  for all  $i = 0, 1, \dots, n$ . Since  $\text{Soc}(A)$  is an inessential ideal of  $A$  by Proposition 4.4.4, Theorem 1.4.22 implies that  $\sigma(b + s_i a)$  and  $\sigma(b)$  differ at most by isolated points of  $\text{Sp}(b + s_i a)$  and  $\text{Sp}(b)$  for all  $i = 0, 1, \dots, n$ . So if  $\mu \notin \sigma(b)$  but  $\mu \in \bigcap_{i=0}^n \sigma(b + s_i a)$  then  $\mu$  is an isolated

point of  $\text{Sp}(b + s_i a)$  for all  $i = 0, 1, \dots, n$ . This implies that  $\mu \in \bigcap_{i=0}^n \text{Sp}(b + s_i a)$ .

From  $\bigcap_{i=0}^n \text{Sp}(b + s_i a) \subset \text{Sp}(b)$ , this means that  $\mu \in \text{Sp}(b)$ , so that  $\mu \in \sigma(b)$ .

This is a contradiction. Hence  $\mu \notin \bigcap_{i=0}^n \sigma(b + s_i a)$ , so that  $\bigcap_{i=0}^n \sigma(b + s_i a) \subset \sigma(b)$ .

Conversely, suppose that there exists an  $n \in \mathbb{N}$  such that  $\bigcap_{i=0}^n \sigma(b + s_i a) \subset \sigma(b)$  for all  $b \in A$  and for any set of distinct non-zero scalars  $\{s_i : i = 0, 1, \dots, n\}$ . Let  $\mu \notin \sigma(b)$ . Then  $\mu - b$  is invertible and  $\mu \notin \text{Sp}(b + sa)$

for some  $s \in \{s_i : i = 0, 1, \dots, n\}$ . It follows from Lemma 2.3.4 that  $-\frac{1}{s} \notin \text{Sp}((\mu - b)^{-1}a)$ . Therefore  $-\frac{1}{s} \in \mathbb{C} \setminus \text{Sp}((\mu - b)^{-1}a)$ . This means that every subset  $\{s_i : i = 0, 1, \dots, n\}$  of  $\mathbb{C} \setminus \{0\}$  with  $n + 1$  distinct points has at least one point in  $\mathbb{C} \setminus \text{Sp}((\mu - b)^{-1}a)$ . This implies that  $\#\text{Sp}'((\mu - b)^{-1}a) \leq n$ . Let  $y \in A$  and let  $\lambda \in \mathbb{C}$  be such that  $2\rho(y) < |\lambda|$ . Take  $b = \mu - (y - \lambda)^{-1}$ , where  $\mu = -\frac{1}{\lambda}$ . From Lemma 1.5.5 we have that  $\rho(b) < |\mu|$ , so that  $\mu - b$  is invertible. Hence  $\text{Sp}((\mu - b)^{-1}a) = \text{Sp}(y - \lambda)a$ .

(i) If  $a \in A^{-1}$  then  $\#\text{Sp}(y - \lambda)a \leq n$ . Let  $D = \{\lambda \in \mathbb{C} : 2\rho(y) < |\lambda|\}$  and let  $f : D \rightarrow A$  be defined by  $f(\lambda) = (\lambda - y)a$ . Then  $f$  is analytic on  $D$  and  $\#\text{Sp}(f(\lambda)) = \#\text{Sp}((\lambda - y)a) \leq n$  for all  $\lambda \in D$ . It follows from Theorem 1.7.6 (i) that  $\#\text{Sp}((\lambda - y)a) \leq n$  for all  $\lambda \in \mathbb{C}$ . Take  $\lambda = 0$ . Then  $\#\text{Sp}(-ya) = \#\text{Sp}(ya) \leq n$  for all  $y \in A$ . This means that  $a \in \mathcal{G}$ , so that  $a \in \mathcal{F}$  by Theorem 4.4.1.

(ii) If  $a$  is not invertible then  $\#\text{Sp}(y - \lambda)a \leq n + 1$ . Using the same argument as above, we get that  $\#\text{Sp}(ya) \leq n + 1$  for all  $y \in A$ . This means that  $a \in \mathcal{G}$ , so that  $a \in \mathcal{F}$  by Theorem 4.4.1  $\square$ .

**Corollary 4.4.6** *Let  $A$  be a semisimple Banach algebra and let  $a \in A$ . Then  $a \in \mathcal{G}$  if and only if there exists an  $n \in \mathbb{N}$  such that  $\bigcap_{i=0}^n \sigma(b + s_i a) \subset \sigma(b)$  for every  $b \in A$  and for any set of distinct non-zero scalars  $\{s_i : i = 1, 2, \dots, n\}$ .*

In [5] Aupetit and Mouton also obtained a characterization of  $\mathcal{G}_1$  in terms of the polynomially convex hull of the spectrum ([5], Theorem 2.2(1)). The proof of this result is similar to the proof of ([5], Theorem 2.2(2)). We state this result as the following corollary.

**Corollary 4.4.7** *([5], Theorem 2.2(1)) Let  $A$  be a semisimple Banach algebra and let  $a \in A$ . Then  $a \in \mathcal{G}_1$  if and only if  $\sigma(b + s_0 a) \cap \sigma(b + s_1 a) \subset \sigma(b)$  for every  $b \in A$  and for any two-point set of distinct non-zero scalars  $\{s_0, s_1\}$ .*

From this corollary and from Theorem 4.1.4 we obtain the characterization of  $\mathcal{F}_1$  in terms of the polynomially convex hull of the spectrum as a simple corollary.



**Corollary 4.4.8** *Let  $A$  be a semisimple Banach algebra and let  $a \in A$ . Then  $a \in \mathcal{F}_1$  if and only if  $\sigma(b + s_0a) \cap \sigma(b + s_1a) \subset \sigma(b)$  for every  $b \in A$  and for any two-point set of distinct non-zero scalars  $\{s_0, s_1\}$ .*

## Chapter 5

# Application of rank one elements to Aupetit's perturbation theorem

In a 1986 paper [3] B. Aupetit studied the perturbation of elements of a Banach algebra  $A$  by elements of an inessential ideal  $I$  of  $A$ . The main result of his paper, ([3], Theorem 2.4) called Aupetit's perturbation theorem, is based on a lemma ([3], Theorem 1.1) obtained by the use of subharmonic methods and analytic multivalued functions. Aupetit later included this result in [2] (see [2], Theorem 3.4.26). In 1994 T. Mouton proved Aupetit's perturbation result using more elementary methods. Rank one elements are the cornerstone of Mouton's proof. In this chapter we will prove the main result leading to Mouton's proof of Aupetit's perturbation theorem.

### 5.1 A useful result from complex analysis

The following theorem will be required in the proof of Theorem 5.2.2, the main result of this chapter.

**Theorem 5.1.1** ([12], Corollary 3.8, p.79) *If  $f$  and  $g$  are two analytic functions on an open connected set  $G$  then  $f = g$  if and only if the set  $\{\lambda \in G : f(\lambda) = g(\lambda)\}$  has a limit point in  $G$ .*

## 5.2 A key result leading to Mouton's proof of Aupetit's perturbation theorem

Before we can state Theorem 5.2.2, we need the following notation: Let  $S$  be a non-empty subset of  $\mathbb{C}$ . We use the notation  $\text{acc } S$  to denote the set of limit points of  $S$ , and  $\text{iso } S$  for the set of isolated points of  $S$ .

In order to prove Theorem 5.2.2, the following Lemma 5.2.1 will be required.

**Lemma 5.2.1** *Let  $A$  be a Banach algebra and let  $a, u \in A$ . If  $\mathbb{C} \setminus \sigma(a) \cap \text{Sp}(a+u) \subset \text{iso } \text{Sp}(a+u)$ , then  $\text{acc } \text{Sp}(a+u) \subset \sigma(a)$ .*

**Proof.** Suppose that  $\mathbb{C} \setminus \sigma(a) \cap \text{Sp}(a+u) \subset \text{iso } \text{Sp}(a+u)$ . Let  $\lambda \in \text{acc } \text{Sp}(a+u)$ . Since  $\text{Sp}(a+u)$  is closed, we have that  $\lambda \in \text{Sp}(a+u)$ . Since  $\mathbb{C} \setminus \sigma(a) \cap \text{Sp}(a+u) \subset \text{iso } \text{Sp}(a+u)$ , this implies that  $\lambda \notin \mathbb{C} \setminus \sigma(a)$ . Hence  $\lambda \in \sigma(a)$ , so that  $\text{acc } \text{Sp}(a+u) \subset \sigma(a)$ .  $\square$

Following is the key result leading to Mouton's proof of Aupetit's perturbation theorem. We will adopt the version presented by R. Harte in [14]. Harte's proof is essentially the proof of ([18], Theorem 2.4) by Mouton. Therefore we will adopt the original proof by Mouton.

**Theorem 5.2.2** ([14], Theorem 5) *Let  $A$  be a semiprime Banach algebra and let  $a \in A$ . If  $u \in \mathcal{F}_1$  then  $\text{acc } \text{Sp}(a+u) \subset \sigma(a)$  and  $\text{acc } \text{Sp}(a) \subset \sigma(a+u)$ .*

**Proof.** Let  $G = \mathbb{C} \setminus \sigma(a)$ . From Lemma 1.4.6 we have that  $G$  is open and connected. Also let  $K = \text{Sp}(a+u)$ . Define  $f : G \rightarrow \mathbb{C}$  by  $f(\lambda) = f_u((\lambda - a)^{-1})$ , where  $f_u$  is the linear functional on  $A$  such that  $uxu = f_u(x)u$  for all  $x \in A$ . Let  $g : G \rightarrow \mathbb{C}$  be defined by  $g(\lambda) = -1$ . Then  $f$  and  $g$  are analytic on  $G$ . Now we have from Corollary 2.3.5 that  $\lambda \in \text{Sp}(a+u)$  if and only if  $f_u((\lambda - a)^{-1}) = -1$ . This implies that  $f(\lambda) = g(\lambda)$  for all  $\lambda \in G \cap K$ . Note that  $G \cap K \subsetneq G$  since  $K$  is bounded and  $G$  is unbounded. This implies that  $f \neq g$ . By Theorem 5.1.1, this implies that  $G \cap K \subset \text{iso } \text{Sp}(a+u)$ . It follows from Lemma 5.2.1 that  $\text{acc } \text{Sp}(a+u) \subset \sigma(a)$ . Since  $\text{Sp}(a) = \text{Sp}((a+u) - u)$ , we have from  $\text{acc } \text{Sp}(a+u) \subset \sigma(a)$  that  $\text{acc } \text{Sp}(a) \subset \sigma(a+u)$ .  $\square$

Recall from Corollary 3.1.5 that for semiprime Banach algebras, the inclusion  $\mathcal{F}_1 \subset \mathcal{G}_1$  may in general be strict. In this light R.M. Brits, L. Lindeboom

and H. Raubenheimer proved a stronger version of Theorem 5.2.2. This is the next result.

**Corollary 5.2.3** ([11], Proposition 2.11) *Let  $A$  be a semiprime Banach algebra. If  $a \in A$  and  $d \in \mathcal{G}_1$ , then  $\text{acc Sp}(a + d) \subset \sigma(a)$ .*

**Proof.** Consider  $d + \text{Rad}(A)$ . Since  $d \in \mathcal{G}_1$  we have that  $\#\text{Sp}'(xd + \text{Rad}(A), A/\text{Rad}(A)) \leq 1$  for all  $x \in A$ . Therefore  $d + \text{Rad}(A)$  is a spectrally rank one element of  $A/\text{Rad}(A)$ . Since  $A/\text{Rad}(A)$  is semisimple by Corollary 1.4.19 we have from Theorem 4.1.4 that  $d + \text{Rad}(A)$  is a spatially rank one element of  $A/\text{Rad}(A)$ . It follows from Corollary 1.4.19 and Theorem 5.2.2 that  $\text{acc Sp}(a + d, A) = \text{acc Sp}((a + d) + \text{Rad}, A/\text{Rad}(A)) \subset \sigma(a + \text{Rad}(A), A/\text{Rad}(A)) = \sigma(a, A)$ . Hence  $\text{acc Sp}(a + d) \subset \sigma(a)$ .  $\square$

## Bibliography

- [1] T.M. Apostol, *Mathematical analysis*, Second edition, Addison-Wesley Pub. Co., Reading, Mass., 1974.
- [2] B. Aupetit, *A primer on spectral theory*, Springer-Verlag, New York, 1991.
- [3] B. Aupetit, Inessential elements in Banach algebras, *Bull. London Math. Soc.* **18** (1986), 493-497.
- [4] B. Aupetit, Spectrum-preserving linear mappings between Banach algebras or Jordan-Banach algebras, *J. London Math. Soc.* **62** (2000), 917-924.
- [5] B. Aupetit and H. du. T. Mouton, Spectrum preserving linear mappings in Banach algebras, *Studia Math.* **109**(1) (1994), 91-100.
- [6] B. Aupetit and H. du. T. Mouton, Trace and determinant in Banach algebras, *Studia Math.* **121**(2) (1996), 115-136.
- [7] D.R. Behrendt, *On Inessential elements in Banach algebras*, Ph.D. Thesis, University of Johannesburg, 2005.
- [8] S.K. Berberian, *Fundamentals of real analysis*, Springer-Verlag, New York, 1999.
- [9] F.F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, Berlin-Heidelberg-New York, 1973.
- [10] D.M. Burton, *A first course in rings and ideals*, Addison-Wesley, Phillipines, 1970.
- [11] R.M. Brits, L. Lindeboom, H. Raubenheimer, On the structure of rank one elements in Banach algebras, *Extracta Math.* **18** (2003), 297-309.

- [12] J.B. Conway, *Functions of one complex variable*, second edition, Springer-Verlag, New York, 1978.
- [13] J.A. Erdos, S. Giotopoulos, M.S. Lambrou, Rank one elements of Banach algebras, *Mathematika* **24** (1977), 178-181.
- [14] R. Harte, On rank one elements, *Studia Math.* **117**(1) (1995), 73-77.
- [15] R.A. Hirschfeld and B.E. Johnson, Spectral characterization of finite-dimensional algebras, *Indag. Math.* **34** (1972), 19-23.
- [16] A.A. Jafarian and A.R. Sourour, Spectrum-preserving linear maps, *J. Funct. Anal.* **66** (1986), 255-261.
- [17] E. Kreyszig, *Introductory functional analysis with applications*, John Wiley & Sons, New York, 1978.
- [18] T. Mouton, On Inessential ideals in Banach algebras, *Quaestiones Math.* **17** (1994), 59-66.
- [19] T. Mouton and H. Raubenheimer, On rank one and finite elements of Banach algebras, *Studia Math.* **104**(3) (1993), 211-219.
- [20] G.J. Murphy, The index group, the exponential spectrum, and some spectral containment theorems, *Proc. Roy. Irish Acad.* **92**(2) (1992), 229-238.
- [21] J. Puhl, The trace of finite and nuclear elements in Banach algebras, *Czechoslovak Math. J.* **28** (1978), 656-676.
- [22] C.E. Rickart, *General theory of Banach algebras*, Van Nostrand-New York, 1960.
- [23] W.A. Sutherland, *Introduction to metric and topological spaces*, Oxford University Press, New York, 1975.
- [24] J. Zemánek, Idempotents in Banach algebras, *Bull. London Math. Soc.* **11** (1979), 177-183.

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