On Subnormal Subgroups in Factorized Groups

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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

Abstract

In this thesis we give a survey of research done on a problem on subnormal subgroups in factorized groups $G = AB$, where A and B are two subgroups of G with H a subgroup of $A \cap B$ which is subnormal in both A and B. It is of interest to know whether or not such a subgroup H will also be subnormal in G.

During the past twenty five to thirty years some positive results were obtained in the case where G is a finite group. This was mainly due to work done by Maier and Wielandt, with results by Sidki and Casolo following shortly afterwards.

Counterexamples in the case of infinite groups seemed to be extremely hard to construct. For the infinite group case, some positive results were obtained through contributions by amongst others Stonehewer, Franciosi, de Giovanni and Sysak. Most recently some alternative proofs were given by Fransman.

Opsomming

In hierdie tesis poog ons om 'n oorsig te gee van navorsing uitgevoer oor 'n probleem rakende subnormale ondergroepe van 'n groep $G = AB$ wat uitgedruk kan word as 'n produk van twee ondergroepe A en B. Daar word gepoog om te bepaal vir watter klasse van groepe dit volg dat as die ondergroep H van A se deursnede met B subnormaal is in beide A en B, sal dit impliseer dat H ook subnormaal in die groep G sal wees.

Gedurende die afgelope vyf-en-twintig na dertig jaar is positiewe resultate bewys vir eindige sodanige groepe deur veral outeurs soos Maier en Wielandt, gevolg deur Sidki en Casolo.

Dit blyk dat dit nie maklik is om teenvoorbeelde te vind vir die oneindige geval nie. Daar is wel positiewe resultate gelewer vanweë bydraes deur onder andere Stonehewer, Franciosi, de Giovanni en Sysak. Meer onlangs is ook alternatiewe bewyse gegee deur Fransman.

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Contents

Chapter 1

Introduction

The concept of a subnormal subgroup of a group is derived naturally from that of a normal subgroup due to the fact that normality is not a transitive relation. If we consider the dihedral group

$$G = \langle a, b |\ a^4 = 1 = b^2, \quad b^{-1}ab = a^{-1} \rangle$$

of order 8, then it is easily seen that $S = \langle b \rangle$ is normal in $T = \langle a^2, b \rangle$, which in turn is normal in G, but S itself fails to be normal in G.

One of the most famous group theorists of all times, Philip Hall, considered the subnormal subgroups to be the bare bones or 'skeleton' of a group, providing the framework for all the other structures (see [9]).

Already the significance of subnormal subgroups in finite groups is apparent since they are precisely those subgroups which occur as terms of composition series, the factors of which are of major importance in describing a group's structure. As far as infinite groups are concerned, it is well-known that every subgroup of a nilpotent group is subnormal in the group.

Let us now consider a group $G = AB$ which is the product of two of its subgroups A and B, and let H be a subgroup of their intersection $A \cap B$. Then it is obvious that if H is normal in both A and B, it must be normal in G. On the other hand, if H is normal in A but subnormal in B, then its normal closure in G, namely

$$H^G = H^{AB} = H^B,$$

is of course contained in B. This implies that H is subnormal in H^G, and consequently H is subnormal in G.

Our main objective with this thesis is to give a survey of developments on this topic for the past about twenty five to thirty years. More precisely, we investigate the following general problem.

Let the group $G = AB$ be factorized by two subgroups A and B, and let H be a subgroup of $A \cap B$. If H is subnormal in both A and B, for which classes of groups will this imply that H is also subnormal in G?

In Chapter 2 we will introduce the necessary preliminary definitions, lemmas and elementary properties that we require in the work to follow.

The first breakthrough, on the problem stated above, was achieved by Maier [10] in 1977 with a promising result on finite soluble groups. Using this, Wielandt [18] extended Maier's result to the finite case in 1981. In the literature this theorem is now attributed to both these authors (see for instance [1; Theorem 7.5.7]). Some rather interesting conjectures were posed by Wielandt in [18] about the possibility of improving their results. Before embarking on them, let us first scrutinize the following example constructed by him in [18].

Example 1.1 (see [18; Beispiel 3.1.) Let p be an odd prime and consider the subgroups $H = \langle h \rangle$, $A = \langle h, a \rangle$, $B = \langle h, b \rangle$ of the general linear group $GL(3, p)$, where

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

One quickly verifies that $[h, a] = 1$ and hence H lies in $Z(A)$, the centre of A. A further calculation shows that H is subnormal in B. We now claim that H cannot be subnormal in $\langle A, B \rangle$. Suppose, on the contrary, that H is subnormal in $\langle A, B \rangle$. Let $K = b^{-1}Hb$. Then it is easily verified that $K = \langle k \rangle$, where

$$k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now it can be deduced that K is subnormal in $\langle A, B \rangle$ and hence K has to be subnormal in $\langle k, a \rangle$. However, it is a straightforward exercise to check that the element a has order p and that it is inverted under conjugation by k.

This contradiction shows that the assumption H is subnormal in $\langle A, B \rangle$ is invalid. $\square$

It therefore came as no surprize that in [18] Wielandt made the following conjectures for the way forward in this area.

Conjecture 1.2 *Suppose that H is a subgroup of the finite group $G = AB$, with A and B subgroups of G. Then*

(a) *H is subnormal in G if $HH^x = H^xH$ for all $x \in A \cup B$;*

(b) *H is subnormal in G if H is subnormal in $\langle H, H^x \rangle$ for all $x \in A \cup B$;*

(c) *H is subnormal in G if H is subnormal in $\langle H, x \rangle$ for all $x \in A \cup B$.* □

The investigation in Chapter 3 will primarily be devoted to the case where the factorized group $G = AB$ is finite. In particular, the contributions of Maier [10] and Wielandt [18] will be discussed in detail. Moreover, as far as Conjecture 1.2(b) is concerned, we shall present the work of Maier and Sidki [11] as well as the improvement on their result by Casolo [2].

In Chapter 4 we proceed to the infinite group case by introducing the research of Stonehewer [15]. He also remarked in [15] that it might not be an easy exercise to find counterexamples to the Maier-Wielandt Theorem for the infinite case. An example by Fransman [6] is discussed enforcing the suggestions by Stonehewer where to look for possible counterexamples. Indeed, we shall state with proofs, the positive results obtained for the following classes of infinite groups, namely metabelian, abelian-by-finite, nilpotent-by-abelian due to Stonehewer [15], as well as the nilpotent-by-finite case (see [7]).

Finally, in Chapter 5 we focus our attention on periodic nilpotent-by-abelian-by-finite factorized groups studied by Stonehewer [15]. A positive result was obtained in this case. However, the author asked the question as to whether the periodicity condition really was needed, in view of some earlier results, of his. A completely affirmative answer to Stonehewer's question was given in an interesting paper by Franciosi, de Giovanni and Sysak [5] some years later. In order to prove their result, they developed some nice group ring-theoretic methods. As a consequence to their theorems, they were able to derive the cases where the factorized group $G = AB$ is a soluble-by-linear group, as well as the soluble-by-finite group G which is also nilpotent-by-minimax.

Furthermore, we also include an alternative proof of their main result due to Fransman [7].

Chapter 2

Preliminaries

In this chapter we will introduce and develop the necessary material that will enable us to give a comprehensive account of the theme of this thesis. With this in mind, we begin our discussion by collecting some well-known definitions, properties and preliminary results that will be needed in the chapters to follow. Our notation will mostly be standard and the reader is referred to the textbooks [1], [9] and [14].

2.1 Basic properties

The first lemma is basic (see [14; 1.3.11]).

Lemma 2.1 *Let H and K be subgroups of a group G. Then*

$$|HK|\ |H \cap K| = |H|\ |K|, \text{ so that } |H : H \cap K| = \frac{|HK|}{|K|} \text{ if } H \text{ and } K \text{ are finite.} \qquad \square$$

The second lemma, called Dedekind's Modular Law, is very useful and will be needed throughout.

Lemma 2.2 (Dedekind's Modular Law) *Let H, K, L be subgroups of a group G and assume that K is contained in L. Then $HK \cap L = (H \cap L)K$.*

Proof. It is clear that $(H \cap L)K \leq HK$ and $(H \cap L)K \leq LK = L$. Hence

$$(H \cap L)K \leq HK \cap L.$$

For the converse inclusion we let $x \in (HK) \cap L$. Write $x = hk$ where $h \in H$ and $k \in K$. Then it is immediate that

$$h = xk^{-1} \in LK = L,$$

so that $h \in H \cap L$. Hence

$$x \in (H \cap L)K. \qquad \square$$

2.2 Commutators

In order to simplify computations it is always useful to have commutator properties at your disposal.

Definition 2.3 Let x_1 and x_2 be elements of a group G. Recall that the *commutator* of x_1 and x_2 is defined as

$$[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2.$$

A *simple commutator of weight* $n \geq 2$ is defined recursively by the rule

$$[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n],$$

where by convention $[x_1] = x_1$. For an integer $n \geq 0$, a useful shorthand notation is expressed as

$$[x, \ _n y] = [x, \underbrace{y, \ldots, y}_{n}],$$

where of course $[x, \ _0 y] = [x] = x$. □

Some elementary commutator identities are summarized in the next lemma. Recall that the *conjugate* $h^g = g^{-1}hg$ where $h, g \in G$.

Lemma 2.4 *Let x, y and z be elements of a group G. Then the following commutator identities hold.*

(i) $[x, y] = [y, x]^{-1}$.
(ii) $[xy, z] = [x, z]^y[y, z]$ and $[x, yz] = [x, z][x, y]^z$.

Proof. This is straightforward (see [14; 5.1.5]). □

Definition 2.5 Let X_1 and X_2 be subsets of a group G. The *commutator subgroup* of X_1 and X_2 is defined to be

$$[X_1, X_2] = \langle [x_1, x_2] \mid x_1 \in X_1, \ x_2 \in X_2 \rangle.$$

More generally, for subsets $X_1, \ldots, X_n$

$$[X_1, \ldots, X_n] = [[X_1, \ldots, X_{n-1}], X_n]$$

where $n \geq 2$. Observe that $[X_1, X_2] = [X_2, X_1]$ in view of Lemma 2.4(i).

For an integer $n \geq 0$ it is sometimes convenient to write

$$[X, {}_nY] \quad \text{for} \quad [X, \underbrace{Y, \dots, Y}_{n}],$$

where $[X, {}_0Y] = [X] = X$.

In particular, the *commutator subgroup* of a group G is denoted by $G', [G, G]$ or $\gamma_2(G)$. For any integer $n > 0$, the terms of the *lower central series* of G will be denoted by $\gamma_n(G)$, where by convention $\gamma_1(G) = G$, and $\gamma_n(G) = [\gamma_{n-1}(G), G]$. □

Definition 2.6 A group G is said to be *nilpotent of class* c if $\gamma_{c+1}(G) = 1$, but $\gamma_c(G) \neq 1$. Note that if the class of G is 1, then G is abelian. □

2.3 Definitions and properties on subnormality

We clearly require some background knowledge concerning subnormal subgroups. We mainly make use of the book by Lennox and Stonehewer [9] to present the following.

Definition 2.7 A subgroup H of a group G is said to be *subnormal in* G if there exists a non-negative integer m and a series $H = H_m \lhd H_{m-1} \lhd \dots \lhd H_0 = G$ of subgroups in G.

This is usually written as

$$H \quad \text{is subnormal in} \quad G \quad \text{or} \quad H \quad \text{sn} \quad G$$

or symbolically as

$$H \lhd^m G.$$ □

Definition 2.8 If $H \lhd^m G$, then the smallest such value of m is called the *defect* of the subnormal subgroup H of G. □

Definition 2.9 Let H be any subgroup of a group G. Then the *normal closure* of H in G, denoted by H^G, is the smallest normal subgroup of G containing H. So

$$H^G = \langle h^g | \ h \in H, \quad g \in G \rangle.$$

If we put $H_0 = G$, then H^G becomes the first of an inductively defined descending series of subgroups, namely H_1, where for all finite $i \geq 0$

$$H_{i+1} = H^{H_i}.$$

Therefore

$$H \leq \cdots \lhd H_{i+1} \lhd H_i \lhd \cdots \lhd H_1 \lhd H_0 = G, \tag{2.9.1}$$

and this is the most rapidly descending series of subgroups, containing H, with each normal in the one above. For, if

$$H \leq \cdots \lhd K_{i+1} \lhd K_i \lhd \cdot \lhd K_1 \lhd K_0 = G \tag{2.9.2}$$

is such a series and if $H_i \leq K_i$ for some i, then

$$H_{i+1} = H^{H_i} \leq H^{K_i} \leq K_{i+1}.$$

Hence our claim follows by induction on i. We call (2.9.1) the *normal closure series* of H in G and we call H_i the *i-th normal closure* of H in G. $\square$

Lemma 2.10 *Let H be a subgroup of a group G. Then*

(i) *the i-th normal closure of H in G is $H[G,\ _iH]$.*
(ii) *$H \lhd^m G$ if and only if H coincides with its m-th normal closure in G.*

Proof. (i) The statement is clear when $i = 0$ and 1. Let the i-th normal closure of H in G be denoted by H_i and suppose that

$$H_i = H[G,\ _iH]$$

for some integer $i \geq 1$. Then

$$H_{i+1} = H^{H_i} = H^{[G,\ _iH]} = H[G,\ _{i+1}H],$$

and the result follows by induction.

(ii) If $H \lhd^m G$, then there is a series

$$H \leq \ldots \lhd K_{i+1} \lhd K_i \lhd \ldots \lhd K_1 \lhd K_0 = G$$

with $H = K_m$ by (2.9.2) above. Thus

$$H_m \leq K_m \quad \text{implies} \quad H = H_m.$$

The converse is clear and the proof is complete. $\square$

Lemma 2.11 (i) *Let $H \lhd^m G$ and K be a subgroup of G. Then $H \cap K \lhd^m K$. In particular, $H \lhd^m L$ whenever L is a subgroup of G containing H.*

(ii) *If $H_\lambda \lhd^m G$ for all $\lambda \in \Lambda$ (Λ an indexing set), where m is independent of λ, then $\bigcap_\lambda H_\lambda \lhd^m G$.*

Proof. (i) There is a series

$$H = H_m \lhd H_{m-1} \lhd \ldots \lhd H_1 \lhd H_0 = G.$$

Thus

$$H \cap K = H_m \cap K \lhd H_{m-1} \cap K \lhd \ldots \lhd H_1 \cap K \lhd H_0 \cap K = K.$$

(ii) For any $\lambda \in \Lambda$ there are series

$$H_\lambda = H_{\lambda,m} \lhd \ldots \lhd H_{\lambda,1} \lhd H_{\lambda,0} = G.$$

So the result follows since

$$\bigcap_\lambda H_{\lambda,i+1} \lhd \bigcap_\lambda H_{\lambda,i}. \qquad \square$$

Nevertheless, the intersection of an arbitrary collection of subnormal subgroups may fail to be subnormal. The first example will illustrate this fact.

Example 2.12 Consider the infinite dihedral group

$$G = \langle x, a |\ a^x = a^{-1}, x^2 = 1\rangle$$

and set

$$H_i = \langle x, a^{2^i}\rangle,$$

where i is a positive integer. Since

$$[a^{2^i}, x] = a^{-2^{i+1}},$$

it follows that

$$H_{i+1} \lhd H_i.$$

In particular, $H_1 \lhd H_0 = G$ and consequently H_i is subnormal in G.

Let $T = \bigcap_i H_i$. Then it is obvious that

$$T = \langle x \rangle.$$

We now claim that

$$T = N_G(T).$$

So suppose, to the contrary, that T is distinct form $N_G(T)$. Then there exists an element $g \in G$ such that

$$T^g = T,$$

but $g \notin T$. So g has the form $g = xa^r$ or $g = a^r$, for some integer $r \neq 0$. Hence

$$x^g = x \quad \text{or} \quad x^g = 1.$$

If $x^g = x$, then

$$x^{xa^r} = x \quad \text{or} \quad x^{a^r} = 1.$$

Case 1: $\quad x^{xa^r} = x.$

Then $a^{-r}x^{-1}xxa^r = x$ and so

$$\begin{aligned} a^r &= x^{-1}a^r x \\ &= (x^{-1}ax)^r \\ &= a^{-r} \end{aligned}$$

which implies that $a^{2r} = 1$, and so a has finite order. This is a contradiction.

Case 2: $\quad x^{a^r} = 1.$

Here $a^{-r}xa^r = 1$ and so $x = 1$, which is also a contradiction.

Case 3: $\quad x^{xa^r} = 1.$

Now $a^{-r}x^{-1}xxa^r = 1$, and so $x = 1$, another contradiction.

Case 4: $\quad x^{a^r} = x.$

Clearly $a^{-r}xa^r = x$ and therefore, as in Case 1 above we conclude that $a^{2r} = 1$, which gives a final contradiction. This completes the proof of the claim. Hence T fails to be subnormal in G. □

2.4 Factorized groups

Some definitions and lemmas concerning products of groups will now follow. These are found in Amberg, Franciosi and de Giovanni [1], which will serve as our standard reference on factorized groups.

Definition 2.13 A group G is the product of two subgroups A and B if $G = AB$, where

$$AB = \{ab|\ a \in A,\ b \in B\}.$$

In this case we say that G is *factorized* by A and B. $\square$

Clearly, if $G = AB$, then every homomorphic image $\frac{G}{N} = \left(\frac{AN}{N}\right)\left(\frac{BN}{N}\right)$ of G is also factorized by the homomorphic images $\frac{AN}{N}$ of A and $\frac{BN}{N}$ of B.

Observe that whereas every factor group of a factorized group is always factorized, this property is not in general inherited by subgroups of factorized groups. Even for finite groups examples are easily available, since products of subgroups of a group may fail to be a subgroup. We have the following definition due to Wielandt (see [1; Lemma 1.1.1]).

Definition 2.14 A *subgroup* S of a factorized group $G = AB$ is said to be *factorized* with respect to A and B if it satisfies one of the equivalent conditions.

(i) If ab belongs to S, with a in A and b in B, then a belongs to S.
(ii) $S = (A \cap S)(B \cap S)$ and $A \cap B \leq S$. $\square$

The following two lemmas give further information on properties of factorized subgroups.

Lemma 2.15 *Let the group $G = AB$ be the product of two subgroups A and B. Then the following hold:*

(i) *The intersection of arbitrarily many factorized subgroups of G is factorized.*
(ii) *The subgroup generated by arbitrarily many factorized normal subgroups of G is factorized.*
(iii) *If N is a normal subgroup of G, a subgroup $\frac{S}{N}$ of the factor group $\frac{G}{N} = \left(\frac{AN}{N}\right)\left(\frac{BN}{N}\right)$ is factorized if and only if S is a factorized subgroup of G.*

Proof. (i) This is obvious.

(ii) Let $(S_i)_{i \in J}$, J an indexing set, be a system of factorized normal subgroups of G. Put

$$S = \langle S_i|\ i \in J\rangle.$$

For $x \in S$, there exist finitely many indices $i_1, \ldots, i_t$ in J such that x belongs to

$$\begin{aligned} S_{i_1} \ldots S_{i_t} &= (A \cap S_{i_1})(B \cap S_{i_1})S_{i_2}, \ldots S_{i_t} \\ &= (A \cap S_{i_1})S_{i_2}(B \cap S_{i_1})S_{i_3} \ldots, S_{i_t}, \end{aligned}$$

$$
\begin{aligned}
&= (A \cap S_{i_1})(A \cap S_{i_2})(B \cap S_{i_2})(B \cap S_{i_1})S_{i_3} \dots S_{i_t} \\
&= \dots \\
&= (A \cap S_{i_1}) \dots (A \cap S_{i_t})(B \cap S_{i_t}) \dots (B \cap S_{i_1}) \\
&\leq (A \cap S)(B \cap S).
\end{aligned}
$$

Thus $S = (A \cap S)(B \cap S)$ and clearly $A \cap B$ is contained in S, which implies that S itself is factorized.

(iii) Let S be a factorized subgroup of G containing N. Consider any $x \in S$, $a \in A$ and $b \in B$. If

$$xN = abN$$

is an element of $\frac{S}{N}$, then

$$x = aby$$

where y is in $N \leq S$. Thus

$$ab = xy^{-1}$$

belongs to S and so a is in S. Therefore $\frac{S}{N}$ is a factorized subgroup of $\frac{G}{N}$.

On the other hand, suppose that the subgroup $\frac{S}{N}$ is factorized in $\frac{G}{N}$. Let $x = ab$ be an element of S with $a \in A$ and $b \in B$. Since $xN = abN$, it follows that aN belongs to $\frac{S}{N}$. Thus $a \in S$ and so S is factorized. $\square$

Lemma 2.16 *Let the group $G = AB$ be the product of two subgroups A and B. If a subgroup S of G is factorized, then $S = AS \cap BS$.*

Proof. Consider any $x \in (AS \cap BS)$. Then

$$x = au = bv,$$

where $a \in A, b \in B$ and $u, v \in S$. From this it follows that

$$a^{-1}b = uv^{-1}$$

is in S. Hence a is in S which forces x to be an element of S. Therefore

$$S = AS \cap BS,$$

which completes the proof of the lemma. $\square$

This suggests the following definition.

Definition 2.17 Let $X(S)$ be the intersection of all the factorized subgroups of $G = AB$ containing the subgroup S. Then $X(S)$ is called the *factorizer* of S in $G = AB$. (Note that $X(S)$ is the smallest factorized subgroup of G containing S). □

The factorizer $X(N)$, where N is a normal subgroup of G, has an interesting factorization as shown in the next lemma.

Lemma 2.18 *Let the group $G = AB$ be the product of two subgroups A and B and let N be a normal subgroup of G. Then*

(i) $X(N) = AN \cap BN$.

(ii) $X(N) = (A \cap BN)N = (B \cap AN)N = (A \cap BN)(B \cap AN)$.

Proof. (i) Since the subgroups AN and BN contain A and B, respectively, they are factorized. Hence

$$X(N) \quad \text{lies in} \quad AN \cap BN.$$

If S is a factorized subgroup of G containing N, then by Lemma 2.16

$$AN \cap BN \leq AS \cap BS = S.$$

Thus $AN \cap BN$ is contained in $X(N)$ so that

$$X(N) = AN \cap BN.$$

(ii) By Lemma 2.2 we have that

$$\begin{aligned} X(N) &= AN \cap BN \\ &= N(A \cap BN) \\ &= N(B \cap AN). \end{aligned}$$

Now from (i) it follows that

$$\begin{aligned} X(N) &= (A \cap X(N))(B \cap X(N)) \\ &= (A \cap BN)(B \cap AN). \end{aligned}$$

This completes the proof. □

From Lemma 2.18 we see that factorized groups may have a so-called triple factorization. For a clear and concise account on this topic we refer the reader to [1]. Groups with this structure will play a major role in our present investigation. The following example turns out to be of special interest to our situation (see [6]).

Example 2.19 Let S be a non-zero subring of the real field $\mathbb{R}$ under the usual operations of addition and multiplication. Consider the infinite group

$$G = G(S) = \left\{ \begin{pmatrix} u & x \\ 0 & v \end{pmatrix} \mid u, v \in \langle -1 \rangle,\ x \in S \right\},$$

of 2×2 upper triangular matrices with respect to matrix multiplication. Here $\langle -1 \rangle$ means the multiplicative group $\{-1, 1\}$. (To simplify matters we will suppress the subring S in our notation for such groups). Define the following subgroups of G.

$$\begin{aligned} A &= \left\{ \begin{pmatrix} u & x \\ 0 & u \end{pmatrix} \mid u \in \langle -1 \rangle,\ x \in S \right\}; \\ B &= \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \mid u, v \in \langle -1 \rangle \right\}; \\ C &= \left\{ \begin{pmatrix} 1 & x \\ 0 & u \end{pmatrix} \mid u \in \langle -1 \rangle,\ x \in S \right\}; \\ U &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in S \right\}. \end{aligned}$$

It is readily checked that G admits the triple factorization

$$G = AB = AC = BC.$$

We observe the group G has the following properties.

(i) The subgroup A of G factorizes itself as

$$A = (A \cap B)(A \cap C),$$

(ii) the commutator subgroup $G' = U$, and

(iii) the centre $Z(G) = A \cap B = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$. □

2.5 Preliminary lemmas

A very useful lemma to have at hand is stated below.

Lemma 2.20 *If $G = HA$ with A an abelian normal subgroup of G, then $H \cap A \lhd G$.*

Proof. Clearly $H \cap A \lhd H$ and $H \cap A$ is an abelian subgroup of A. To see that $H \cap A$ is normal in G we let $g = ha \in G$, with $h \in H, a \in A$ and let $x \in H \cap A$. Then by Lemma 2.4 we have that

$$\begin{aligned} [g, x] &= [ha, x] \\ &= [h, x][h, x, a][a, x] \\ &= [h, x] \in H' \cap A. \end{aligned}$$

Hence $[G, H \cap A]$ is contained in $H \cap A$ and so $H \cap A \lhd G$. □

This already enables us to obtain the next lemma (see [1; Lemma 7.5.1]).

Lemma 2.21 *Let the group $G = AB$ be the product of two subgroups A and B, and let H be a subgroup of $A \cap B$ which is subnormal in A and B. If there exists an abelian normal subgroup K of G such that $H(A \cap K)(B \cap K)$ is subnormal in G, then H is subnormal in G.*

Proof. Since K is abelian, the subgroups $A \cap K$ and $B \cap K$ are normal in AK and BK, respectively (by Lemma 2.20). Now H is subnormal in $H(A \cap K)$, so that $H(B \cap K)$ is subnormal in $H(A \cap K)(B \cap K)$. But also H is subnormal in $H(B \cap K)$. This means that H is subnormal in $H(A \cap K)(B \cap K)$, and hence also in G. □

We shall frequently refer to the following lemma of [7].

Lemma 2.22 *Let the factorized group $G = AB$ be the product of two subgroups A and B. Suppose that $K \lhd G$ and K is nilpotent of class c. Then the following statements hold.*

(i) *The factorizer G_1 of $\gamma_c(K)$ in $G = AB$ has the triple factorization*

$$G_1 = A_1 B_1 = A_1 \gamma_c(K) = B_1 \gamma_c(K),$$

where

$$A_1 = A \cap B\gamma_c(K) \quad \textit{and} \quad B_1 = B \cap A\gamma_c(K).$$

(ii) $N = (A \cap \gamma_c(K))(B \cap \gamma_c(K))$ *is an abelian normal subgroup of G_1.*

(iii) $\frac{K \cap G_1}{N}$ *is a normal subgroup of* $\frac{G_1}{N}$ and $\frac{K \cap G_1}{N}$ is nilpotent of class at most $c - 1$.

Proof. (i) This follows directly from Lemma 2.18.

(ii) Since $\gamma_c(K)$ is an abelian normal subgroup of the group

$$G_1 = A_1\gamma_c(K) = B_1\gamma_c(K),$$

we can invoke Lemma 2.20 to obtain that $A_1 \cap \gamma_c(K)$ and $B_1 \cap \gamma_c(K)$ are abelian normal subgroups of G_1. However, it is easily checked that $A_1 \cap \gamma_c(K) = A \cap \gamma_c(K)$. Likewise $B_1 \cap \gamma_c(K) = B \cap \gamma_c(K)$, and hence we conclude that $N \lhd G_1$. It is clear that N is also abelian.

(iii) Obviously $K \cap G_1 \lhd G_1$. But by observing that

$$K \cap G_1 = \gamma_c(K)(A_1 \cap K),$$

a direct computation, using the commutator identities in Lemma 2.4(ii), ensures us that $\frac{K\cap G_1}{N}$ is nilpotent of class at most $c-1$. □

2.6 $\mathcal{X}$-by-$\mathcal{Y}$ groups and π-subgroups

For classes $\mathcal{X}$ and $\mathcal{Y}$ of groups, the concept of an $\mathcal{X}$-by-$\mathcal{Y}$ factorized group will form the basis of our discussion in chapters 4 and 5. Firstly we have a well-known definition followed by a lemma on some elementary properties of these groups.

Definition 2.23 Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of group. A group G is called an $\mathcal{X} - by - \mathcal{Y}$ group if it has a normal $\mathcal{X}$-subgroup N such that $\frac{G}{N}$ is a $\mathcal{Y}$-group. □

Note for instance that if G is nilpotent-by-finite, then G has a nilpotent normal subgroup N such that G/N is finite.

Lemma 2.24 *Let G be a group and let S be any subgroup of G with T any normal subgroup of G. If G is*

(i) *nilpotent-by-abelian, or*
(ii) *nilpotent-by-finite, or*
(iii) *metabelian-by-finite, or*
(iv) *nilpotent-by-abelian-by-finite,*

then so are S and $\frac{G}{T}$.

Proof. (i) Suppose G is nilpotent-by-abelian. Then G has a normal nilpotent subgroup N such that $\frac{G}{N}$ is abelian. Let S be any subgroup of G and T be any normal subgroup of G. Since

$$\frac{S}{S\cap N} \cong \frac{SN}{N} \leq \frac{G}{N},$$

it is clear that $S \cap N \lhd S$, $S \cap N$ is nilpotent and $\frac{S}{S\cap N}$ is abelian. Hence S is nilpotent-by-abelian. Furthermore, since

$$\frac{N}{T\cap N} \cong \frac{TN}{T} \lhd \frac{G}{T},$$

and

$$\frac{G/T}{TN/T} \cong \frac{G}{TN} \cong \frac{G/N}{TN/N},$$

it follows that $\frac{G}{T}$ is nilpotent-by-abelian.

The proofs of (ii), (iii) and (iv) are similar. □

We conclude this chapter with some basic facts about π-subgroups of a group G, where π is a non-empty set of primes (see for instance [14]).

Definition 2.25 For a finite group G and a non-empty set π of primes we define

(i) a π-subgroup H of G to be *Hall π-subgroup* if $|G : H|$ is a π'-number,
(ii) $O_\pi(G)$ to be the unique largest normal π-subgroup of G,
(iii) $O^\pi(G)$ to be the unique smallest normal subgroup of G such that the factor group $\frac{G}{O_\pi(G)}$ is a π-group. □

The following lemma on π-groups is well-known (see [14; 9.1.1]).

Lemma 2.26 *Let G be a finite group and π a set of primes.*

(i) *If H is a subnormal π-subgroup of G, then H is contained in $O_\pi(G)$.*
(ii) *$O_\pi(G)$ is the intersection of all Sylow π-subgroups of G.*
(iii) *$O_\pi(G)$ is contained in every Hall π-subgroup of G.* □

The final lemma is needed in Chapter 3.

Lemma 2.27 *Let the finite group $G = AB$ be the product of two subgroups A and B. Then for each prime p there exist Sylow p-subgroups A_0 of A and B_0 of B such that A_0B_0 is a Sylow p-subgroup of G.*

Proof. See [1; Corollary 1.3.3]. □

The following well-known definitions are required in Theorem 3.11.

Definition 2.28 If a prime number can be written in the form $2^p - 1$ for some prime p, it is called a *Mersenne prime.*

Definition 2.29 Primes of the form $F_k = 2^{2^k} + 1$, where k is an integer, are called *Fermat primes.*

Chapter 3

Finite factorized groups

3.1 Background

Our discussion begins with the well-known fact that for any subgroup H of a group G there is always a unique largest subgroup of G containing H as a normal subgroup, namely its normalizer $N_G(H)$. As a first step in our investigation on subnormal subgroups of factorized groups, we observe that the same is not always true in general when normal is replaced by subnormal.

Indeed, this is most beautifully demonstrated by the first example of this section.

Example 3.1 Consider the group $G = S_5$, the symmetric group of degree 5. Let A and B be the following subgroups of G.

$$A = \langle (12)(34), (13)(24), (14)(23), (12) \rangle$$

and

$$B = \langle (23)(45), (24)(35), (25)(34), (25) \rangle.$$

It is easily shown that A and B are subgroups of order 8. Hence they are both nilpotent. Let $H = \langle (34) \rangle$. Then H is of order 2 and H is contained in both A and B. Consequently H is subnormal in both A and B. By Appendix A.1 we see that the order of $\langle A, B \rangle$ is 120 and hence

$$G = \langle A, B \rangle.$$

Again by Appendix A.1 we have that

$$H \quad \text{is not subnormal in} \quad \langle A, B \rangle = G. \qquad \square$$

Furthermore, there are even examples, due to Wielandt [18], of finite soluble groups G generated by subgroups A and B with A abelian and B dihedral of order 8 containing a common subgroup H of order 2, necessarily subnormal in their intersection, but failing to be subnormal in G. We delay a closer look at some of these examples to Section 3.3 later on.

In view of the preceding observations it is therefore not at all surprising that one of the many problems on subnormal subgroups of factorized groups can therefore be formulated as follows:

Let the group $G = AB$ be factorized by two subgroups A and B, and let H be a subgroup of $A \cap B$ such that H is subnormal in both A and B. For which classes of groups G will H be subnormal in G?

Promising results in this direction were obtained by Maier [10] for finite soluble groups and shortly thereafter by Wielandt [18] for finite groups in general. This will be discussed in the next section, but we shall firstly introduce some preliminary lemmas in order to simplify matters.

Lemma 3.2 *Let H be a subgroup of a finite group G. If H is subnormal in $\langle H, H^x \rangle$ for every element x of G, then H is subnormal in G.*

Proof. Assume the result is false and consider a counterexample G of minimal order. Let S be the set of all subgroups L of G which are generated by conjugates of H and such that H is subnormal in L. Then $\langle H, H^x \rangle$ belong to S for every element x of G.

Now since H is not subnormal in G, by assumption, its normal closure H^G cannot be in S. So we can choose a pair (M_1, M_2) of distinct maximal elements of S such that the join K of all conjugates of H which are contained in $M_1 \cap M_2$ is maximal. By the minimality of the order of G, it follows that K is subnormal in M_1 and M_2. Next we consider the two subnormal series

$$K \lhd \cdots \lhd M_1 \tag{3.1.1}$$

and

$$K \lhd \cdots \lhd M_2 \tag{3.1.2}$$

where K_1 and K_2 are chosen to be the smallest terms properly containing K of the series (3.1.1) and (3.1.2), respectively.

Since K is normal in $\langle K_1, K_2 \rangle$, the subgroup H is subnormal in $\langle K_1, K_2 \rangle$ and hence

$\langle K_1, K_2\rangle$ belongs to S. If M is a maximal element of S containing $\langle K_1, K_2\rangle$, then K_1 lies in $M \cap M_1$. However, K is properly contained in K_1, and this contradicts the choice of (M_1, M_2). This completes the proof of the lemma. □

We derive the first corollary from Lemma 3.2.

Corollary 3.3 *Let H be a subgroup of the finite group G such that an element $x \in G$ belongs to H whenever it is in $\langle H, H^x\rangle$. Then H is subnormal in G.*

Proof. We may clearly assume that H is a proper subgroup of G. So let $x \in G$. Then $\langle H, H^x\rangle$ is properly contained in G. It follows by induction on the order of G that H is subnormal in $\langle H, H^x\rangle$ for every $x \in G$. The conclusion is now an immediate consequence of Lemma 3.2, which completes the proof of the corollary. □

The following two results will be needed in the proof of the main theorem in this section. They characterize subgroups normalizing every subnormal subgroup of a finite group.

Lemma 3.4 *Let N be a minimal normal subgroup of the finite group G. Then N normalizes every subnormal subgroup of G.*

Proof. Let H be a subnormal subgroup of G. The proof proceeds by induction on the defect c of H in G. Clearly if $c = 1$, then $H \lhd G$ and we are done. So suppose that $c > 1$. If $H^G \cap N \neq 1$, then by the minimality of N we have that $H^G \cap N = N$ and hence N is contained in H^G. Let M be a minimal normal subgroup of H^G contained in N. Then every conjugate of M in G is also a minimal normal subgroup of H^G. But since H has defect $c - 1 > 0$ in H^G, by induction on c we have that each conjugate of M normalizes H. Consequently $N = M^G$ normalizes H.

On the other hand, if $H^G \cap N = 1$, then

$$[H, N] \leq [H^G, N] \leq H^G \cap N = 1$$

and so N centralizes H. In particular, N normalizes H, which completes the proof of the lemma. □

Lemma 3.5 *Let L be a simple non-abelian subnormal subgroup of G. Then L normalizes every subnormal subgroup of G.*

Proof. Let H be a subnormal subgroup of G. We may without loss of generality assume that $G = \langle H, L \rangle$. If L is normal in G, then $[L, H]$ is contained in L. Hence either

$$[L, H] = 1 \quad \text{or} \quad [L, H] = L.$$

Clearly, if $[L, H] = 1$, then L centralizes H and we are done.

On the other hand, assume that $[L, H] = L$.

If H is subnormal of defect n in G, then

$$L = [L, H] = [L, {}_nH] \leq H.$$

and so L normalizes H. Suppose next that L has defect $c > 1$ in G. Let $x \in H$ such that $L^x \neq L$. Now L and L^x both have defect $c - 1$ in L^G. So by the induction hypothesis they are both normal in $\langle L, L^x \rangle$. Hence

$$[L, L^x] \leq L \cap L^x = 1.$$

Let y and z be elements of L. Then using the commutator identities of Lemma 2.4, it is easily checked that

$$\begin{aligned} 1 &= [y, z^x] \\ &= [y, z[z, x]] \\ &= [y, [z, x]][y, z]^{[z,x]} \end{aligned}$$

and so

$$\begin{aligned} [y, z] &= [z, x][y, [z, x]]^{-1}[z, x]^{-1} \\ &= y^{-1}[z, x]y[z, x]^{-1}. \end{aligned}$$

Consequently $[y, z]$ belongs to the normal subgroup $[L, H]$ of $\langle H, L \rangle = G$. This implies that $L = L'$ is contained in $[L, H]$. As was seen earlier, there exists a positive integer n such that

$$L \leq [L, {}_nH] \leq H,$$

in which case L normalizes H. This completes the proof of the lemma. □

3.2 The main theorem

At this stage we have done the necessary groundwork thereby enabling us to state and prove the principal theorem of this section. This is due to Wielandt [18] and Maier [10], as is evident from [1; Theorem 7.5.7].

Theorem 3.6 *Let the finite group $G = AB$ be the product of two subgroups A and B. If H is a subgroup of $A \cap B$ which is subnormal in both A and B, then H is subnormal in G.*

Proof. Let us suppose that the theorem is false. Amongst the counterexamples with minimal order we choose one group $G = AB$ such that the sum

$$|G : A| + |H|$$

is minimal. Let U be a proper subgroup of G containing A. Then by Lemma 2.2 it follows that

$$U = A(U \cap B).$$

Now H is subnormal in A and H is subnormal in $U \cap B$ and so by the induction hypothesis

$$H \quad \text{is subnormal in} \quad U.$$

However, since

$$G = UB,$$

we can invoke Lemma 2.1 to obtain

$$|G : U| = |G : A|.$$

But since U is a proper subgroup of G we have that $U = A$ and so A is a maximal subgroup of G.

Now if L is a proper non-trivial subnormal subgroup of H, then L is subnormal in G. Let N be a minimal normal subgroup of G such that N is contained in the normal closure L^G of L. The fact that the theorem obviously holds for the factor group $\frac{G}{N}$ guarantees us that HN is subnormal in G and so H cannot be subnormal in HN. In particular, it follows that N is not contained in A. This forces

$$G = AN.$$

On the other hand, N normalizes L in view of Lemma 3.4 above. This implies that

$$N \leq L^G = L^{NA} = L^A \leq A,$$

and so $G = A$, an obvious contradiction. We therefore conclude that H has no proper non-trivial subnormal subgroups and so it must be simple.

Let us now assume that H has prime order p. Then the normal closures H^A and H^B are p-groups. We infer from Lemma 2.27 that there are Sylow p-subgroups A_p of A and B_p of B such that

$$A_pB_p \quad \text{is a Sylow} \quad p-\text{subgroup of} \quad G.$$

But since $\langle H^A, H^B\rangle$ is contained in A_pB_p, it follows that

$$\langle H^A, H^B\rangle \quad \text{is a} \quad p-\text{group}.$$

Now let $x = ab \in G$ with $a \in A$ and $b \in B$. Then

$$\langle H, H^x\rangle = \langle H^{b^{-1}}, H^a\rangle^b \leq \langle H^B, H^A\rangle^b.$$

Consequently $\langle H, H^x\rangle$ is also a p-group, and hence it is nilpotent. Therefore

$$H \quad \text{is subnormal in} \quad \langle H, H^x\rangle$$

for every $x \in G$. By invoking Lemma 3.2 we conclude that H is subnormal in G. This contradiction means that H is a simple non-abelian group.

Finally we let W be the set of all elements g of G such that H^g is subnormal in both A and B. Then $1 \in W$, and hence H lies in the subgroup

$$V = \langle H^g |\ g \in W\rangle.$$

Since V is contained in $A \cap B$, it is clear that H is subnormal in V. Thus V cannot be subnormal in G. By Corollary 3.3 there exists an element y in G, with $y \notin V$, such that

$$y \in \langle V, V^y\rangle.$$

Write $y = a_1b_1$, with $a_1 \in A$ and $b_1 \in B$. Then

$$b_1a_1 = b_1yb_1^{-1}$$

is an element of $\langle V^{b_1^{-1}}, V^{a_1}\rangle$. However, since H is a simple non-abelian subnormal subgroup of A, it is evident from Lemma 3.5 that V^{a_1} normalizes every subnormal subgroup of A. Similarly $V^{b_1^{-1}}$ normalizes every subnormal subgroup of B. In particular, b_1a_1 normalizes V, so that

$$V^{b_1} = V^{a_1^{-1}}.$$

Let g be an element of W. Then H^g is subnormal in B, from which it follows that H^{gb_1} is also subnormal in B. Moreover,

$$H^{gb_1} \quad \text{is subnormal in} \quad V^{b_1} = V^{a_1^{-1}}.$$

Thus H^{gb_1} is subnormal in A. It follows that gb_1 belongs to W and therefore

$$Wb_1 = W.$$

But then we have that

$$V^{b_1} = V,$$

so that a_1 and b_1 both normalize V. Hence $V^y = V$, a final contradiction.

The proof of the theorem is now complete. $\square$

3.3 On a conjecture of Wielandt

In an effort to obtain further extensions of Theorem 3.6, we will in this section embark on one of Wielandt's conjectures mentioned in the introduction, namely Conjecture 1.2(b). This is stated as follows.

Conjecture 3.7 *Suppose that $G = AB$ is a finite group, with A and B subgroups of G. Let H be a subgroup of G such that H is subnormal in $\langle H, H^g \rangle$ for all $g \in A \cup B$, then H is subnormal in G.* $\square$

As an initial effort Maier and Sidki [11] proved the case where G is soluble and H is a p-subgroup of G. Since it involves the subnormalizer of a subgroup H in a group G, we state the definition first.

Definition 3.8 If H is a subgroup of G then the *subnormalizer* of H in G, denoted by $S_G(H)$ is the set

$$S_G(H) = \{g |\ H \text{ is subnormal in } \langle H, H^g \rangle,\ g \in G\}. \qquad \square$$

Although $N_G(H)$ is always a subgroup, it is not always true, even for finite groups, that $S_G(H)$ is a subgroup of G. This is illustrated by extending Example 3.1 at the beginning of this section.

Example 3.9 Consider $G = S_5$, the symmetric group of degree 5. Let

$$A = \langle (12)(24, (13)(24), (14)(23), (12) \rangle$$

and

$$B = \langle (23)(45), (24)(35), (25)(34), (25) \rangle.$$

Then A and B are subgroups of $G = S_5$ of order 8. If $H = \langle (34) \rangle$, then it is clear that H is of order 2. But A and B are nilpotent subgroups of G and hence H is subnormal in both A and B. Thus both A and B are contained in $S_G(H)$. If $S_G(H)$ were a subgroup of S_5, then we would have $S_G(H) = G$ and so H is subnormal in G which is obviously not the case (by Appendix A.1). Hence in this case $S_G(H)$ is not a subgroup of $G = S_5$. □

The authors of [11] considered a theorem for soluble factorized groups, that describes a more general condition for the subnormality of H in the subgroup generated by $S_G(H)$.

Before stating their main theorem, we recall the following definition which will be referred to in the proof.

Definition 3.10 A group G is called *monolithic* if it has a unique minimal normal subgroup. □

Theorem 3.11 *Let G be a soluble finite group factorized as a product $G = AB$ of two subgroups A, B. Let H be a subgroup of G of prime order p such that*

$$S_G(H) \quad \textit{contains} \quad A \cup B.$$

Then H is subnormal in G, provided one of the additional conditions below holds:

(i) *A, B are nilpotent,*
(ii) $(|A|, |B|) = 1$,
(iii) *if $|G|$ is even, then $|G|$ is not divisible by Fermat or Mersenne primes,*
(iv) *$H \leq A$ and A is nilpotent.* □

The following lemma is required in the proof of Theorem 3.11.

Lemma 3.12 (Projection Lemma) *Let G be a finite group factorized as a product $G = AB$ of two of its subgroups A and B. Let $\mathbf{P}(A), \mathbf{P}(B)$ be the power sets of A and B, respectively. Define*

$$\pi_A : G \to \mathbf{P}(A), \quad \pi_B : G \to \mathbf{P}(B)$$

to be the projections

$$\pi_A(g) = \{a \in A |\ a^{-1}g \in B\}, \quad \pi_B(g) = \{b \in B |\ gb^{-1} \in A\}.$$

If N is a normal subgroup of G, then

$$\pi_A(N) = \bigcup_{g \in N} \pi_A(g), \quad \pi_B(N) = \bigcup_{g \in N} \pi_B(g),$$

are subgroups of G, and

$$N\pi_A(N) = N\pi_B(N) = \pi_A(N)\pi_B(N).$$

Proof. Using the normality of N in G, the proof becomes a straightforward exercise. □

We have now enough material available for the following proof.

Proof of Theorem 3.11 Let G be a minimal counterexample to the theorem. Consider the case where H is of prime order p. Then

$$\langle H, H^y \rangle \text{ is a } p-\text{group for all } y \in A \cup B.$$

The structure of G is reduced in several steps.

Step 1. $O_p(G) = 1$.

This is clear.

Step 2. $|G| = p^\alpha q^\beta$ *for some prime* $p \neq q$ *and* $\alpha, \beta > 0$:

From Step 1, we may assume that G is not a p-group. So there exists $q \in \pi(G)$, the set of prime divisors of the order of G, such that $O_q(G) \neq 1$. Consequently, it follows that H is contained in $O_{q,p}(G)$. Then there exists a Hall $\{p,q\}$-subgroup H^* of G (see Definition 2.25) which is factorized as

$$H^* = (H^* \cap A)(H^* \cap B),$$

and

$$H \leq O_{q,p}(G) \leq H^*.$$

If $H^* \neq G$, then $H \leq O_p(H^*)$ and H centralizes $O_q(H^*)$ and thus $O_q(G)$ as well. This contradicts the fact that $C_G(O_q(G))$ is contained in $O_q(G)$.

Step 3. G *is monolithic*:

Suppose V_1, V_2 are nontrivial minimal normal subgroups in G and $V_1 \neq V_2$. Then

$$H \leq O_p(\tfrac{G}{V_2}) = \tfrac{K}{V_2}$$

and

$$[H, V_1] \leq K \cap V_1.$$

If $[H, V_1] \neq 1$, then $V_1 \leq K$ and as $V_1 \cap V_2 = 1$, V_1 is a p-group which leads to a contradiction. Thus,

$$[H, V_1] = 1 = [H, V_2].$$

Then H^G is a p-group since

$$H \leq C_K(V_2) \lhd G$$

and $\frac{K}{V_2}$ is a p-group, which is another contradiction. Hence G is monolithic.

So let V be the unique nontrivial minimal normal subgroup of G.

Step 4. *The Frattini subgroup* $\Phi(G) = 1$:

This follows directly from the fact that

$$\frac{\mathrm{Fit(G)}}{\Phi(G)} = \mathrm{Fit}(\frac{G}{\Phi(G)}).$$

Step 5. $V = O_q(G)$:

Since $\Phi(G) = 1$, it is clear that $O_q(G)$ is elementary abelian. So there exists a maximal subgroup M of G, which is complement for V, such that $G = VM$. Now $V \leq O_q(G)$ so that

$$O_q(G) = V(O_q(G) \cap M).$$

Now since $O_q(G) \cap M$ is normal in M, it is centralized by $O_q(G)$ and hence by V too. Thus $O_q(G) \cap M \lhd G$. By the uniqueness of V we have that $O_q(G) \cap M = 1$ and this implies that $V = O_q(G)$.

Step 6. $V \cap A = V \cap B = 1:$

Suppose $V_A = V \cap A \neq 1$. Then it follows that

$$L = N_G(V_A) \quad \text{contains both} \quad V \quad \text{and} \quad A.$$

Consequently

$$L = A(L \cap B).$$

Let $v \in V_A$. Then since $v \in A$ by hypothesis, we have that

$$\langle H, H^v \rangle = [v, H]H$$

is a p-group. Yet $[v, H]$, which is contained in V, is a q-group. Therefore $[v, H] = 1$ and so H centralizes V_A. Hence $V_A \neq V$ and $L \neq G$. Since H is properly contained in L, by induction, $H \leq O_p(L)$. Therefore as V is properly contained in L, H centralizes V. This is a contradiction.

Proof of Part (i): Since A and B are nilpotent we know by a result of Pennington [13] that $\mathrm{Fit}(G)$ can be expressed as

$$\mathrm{Fit}(G) = (\mathrm{Fit}(G) \cap A)(\mathrm{Fit}(G) \cap B).$$

Since $V = O_q(G) = \mathrm{Fit}(G)$, it follows from Step 6 that $V = 1$. This is of course another contradiction. Let Q be a factorized Sylow q-subgroup of G. Then

$$Q = (Q \cap A)(Q \cap B).$$

Now since V lies in Q we say by the projection lemma (Lemma 3.12) that

$$S = \pi_{Q \cap A}(V) \quad \text{and} \quad T = \pi_{Q \cap B}(V)$$

are q-groups. Moreover, it follows that

$$VS = VT = ST.$$

Step 7. $V \cap S = V \cap T,\ S \cong T,\ |V| \leq |S|$:

This follows directly from Step 6 together with Part (i) above.

Proof of Part (ii): This follows directly from Step 7.

Proof of Part (iii): Let $\overline{G} = \frac{G}{V}$, $O_q(\overline{G}) = \frac{K}{V}$, $K = VP_0, P_0$ a p-group. Then $C_{\overline{G}}(\frac{K}{V}) \leq \frac{K}{V}$. Consider $L = (VP_0)S$. Then by "the other Burnside Theorem" (see [3]),

$$|V| > |P_0| > |S|,$$

contradicting Step 7.

Step 8. *Assume that* $H \leq A$ *and let* $C_V(H) = W$. *Then* $G = VA$, $A = H^A S$ *and* T *is contained in* WSW*:*

Since H is contained in A, it follows that $P_0 = H^A$ is a p-group. Now put $A_0 = P_0 S$. Then we see that

$$\begin{aligned} TA_0 &= T(P_0S) \\ &= (TS)P_0 \end{aligned}$$

$$\begin{aligned} &= (VS)P_0 \\ &= VA_0 \\ &= A_0V \\ &= A_0(ST) \\ &= A_0T, \end{aligned}$$

and hence A_0T is a subgroup of G. But since H and V are contained in A_0T, we have by the minimality of G that

$$G = A_0T = A_0 \quad \text{and} \quad A = A_0.$$

Let now $t \in T$. It follows that

$$\langle H, H^t \rangle \subseteq VP_0,$$

and there exists $v \in V$ such that

$$\langle H, H^t \rangle \subseteq P_0^v.$$

Thus

$$H^{v^{-1}} \in P_0, \ [H, v] \leq P_0 \cap V = 1$$

and

$$v \in W, \ \langle H, H^{bv^{-1}} \rangle \subseteq P_0.$$

Now since $T \subseteq SV$, there exists $s \in S$ and $v' \in V$ such that $t = sv'$. However, since

$$H^s \quad \text{and} \quad H^{tv^{-1}} \quad \text{are contained in} \quad P_0,$$

it follows that

$$[H^s, v'v^{-1}] \leq P_0 \cap V = 1.$$

Hence

$$v'v^{-1} \in C_V(H^s) = W^s,$$

and so

$$v' \in W^s v \subseteq W^s W$$

and

$$v = sv' \in WsW.$$

The proof of Step 8 is now complete.

Proof of Part (iv): Since A is nilpotent, it follows that

$$A = P_0 \oplus S, \quad S \quad \text{centralizes} \quad H$$

and so S leaves W invariant. It follows that

$$WSW = WS.$$

Moreover, since T is contained in WSW we conclude that

$$TS(= VS) \subseteq WS,$$

and hence $V = W$, giving us the final contradiction.

We have completed the proof of the theorem. □

The next example is that of a q-group Q_0 admitting a triple factorization

$$Q_0 = VS = VT = ST,$$

and which will elucidate Step 7 above.

Example 3.13 Let q be an odd prime and let $SL(3, q)$ be the special linear group on the vector space $\frac{V}{\mathbb{Z}_q}$ with basis $\{v_1, v_2, v_3\}$. Consider the elements

$$s_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

in $SL(3, q)$. It is easily seen that

$$S = \langle s_1, s_2 \rangle \quad \text{has order} \quad q^3.$$

Let $Q_0 = VS$ be the semi-direct product of V by S under the above action. Consider

$$t_1 = (\beta_1 v_1 + \beta_2 v_2)s_1, \quad t_2 = (\beta_1' v_1 + \beta_2' v_2)s_2$$

where $\beta_1, \beta_2, \beta_1', \beta_2' \in \mathbb{Z}_q$, and

$$T = \langle t_1, t_2 \rangle.$$

A direct calculation shows that

$$Q_0 = VS = VT = ST$$

if and only if $\beta_1 = 0$ and the order of $\{0, \beta_2, \beta_1'\} = 3$. □

3.4 A further extension of Wielandt's theorem

We find a further development in the direction of Conjecture 3.7 above, in the work of Casolo [2]. He was able to completely remove the restriction that H should be a p-group as imposed by Maier and Sidki [11].

Theorem 3.14 *Suppose that $G = AB$ is a finite soluble group, with A and B subgroups. Let H be a subgroup of G such that H is subnormal in $\langle H, H^g \rangle$ for all $g \in A \cup B$, then H is subnormal in G.*

Proof. Maier and Sidki [11] have proved the case where H is a p-group for some prime p. Thus we assume that H is not a p-group and proceed by induction on $|G| + |H|$.

Suppose that S is a proper subnormal subgroup of H. Clearly S is subnormal in $\langle S, S^g \rangle$ for all $g \in A \cup B$. So by the inductive hypothesis, S is subnormal in G. In particular, if H has two distinct maximal normal subgroups M_1 and M_2, then both are subnormal in G so that $H = M_1 M_2$ is subnormal in G. We can henceforth assume that H has a unique maximal normal subgroup. Thus for a prime $p, N := O^p(H) \neq H$ and H is p'-perfect, which means that $O^{p'}(H) = H$. By the inductive hypothesis, N is subnormal in G.

Let us assume that $H \leq A$. Then according to a criterion of Wielandt [17] H is subnormal in A. If $A = G$, then there is nothing to prove. If not, let X be a maximal subgroup of G containing A. Then we infer from Lemma 2.2 that

$$(X \cap B)A = X \cap BA = X$$

and so by the inductive hypothesis, H is subnormal in X. Now we also have that $XB = G$. Let $K = X_G$ be the normal core of X in G. If $K \neq 1$, then by the inductive hypothesis applied to $\frac{G}{K}$, we have that $\frac{HK}{K}$ is subnormal in $\frac{G}{K}$ so that HK is subnormal in G. Since $HK \leq X$ we have H is subnormal in HK and since HK is subnormal in G, it is clear that H is subnormal in G. We may consequently assume that $K = 1$. Let M be a minimal normal subgroup of G. Then $M \cap X = 1$ and $MX = G$. In particular $M \cap N = 1$. Now N is subnormal in G so that if $F = \mathrm{Fit}(N)$, then $F \leq \mathrm{Fit}(G)$. Thus F centralizes every minimal normal subgroup of G. In particular,

$$F \leq C_X(M) \leq X_G = 1.$$

Thus $F = 1$, which in turn yields $N = 1$, since N is soluble. It follows that H is a p-group. This possibility has been excluded at the beginning. Thus the theorem is true for $H \leq A$.

Next we assume that H is not contained in A. Write $A_0 = \langle A, H \rangle$. Again we have by Lemma 2.2 that $(A_0 \cap B)A = A_0 \cap BA = A_0$ and so $A_0 B = G$.

If $A_0 \neq G$, then by the inductive hypothesis H is subnormal in A_0. So in particular H is subnormal in $\langle H, H^x \rangle$ for all $x \in A_0 \cap B$. From the discussion above it follows that H is subnormal in G.

We are now left with the case $\langle A, H \rangle = G$. Since H is not a p-group, $N = O^p(H) \neq 1$ and so $F = \text{Fit}(N) \neq 1$. Let q be a prime divisor of $|F|$ and let $R = O_q(G)$. Since F is subnormal in G, $1 \neq O_q(N) \leq R$. By the inductive hypothesis it follows that $\frac{RH}{R}$ is subnormal in $\frac{G}{R}$ and so RH is subnormal in G.

Suppose $q = p$, then because N is p-perfect it is normalized by R, so that N is a normal subgroup of RH and $\frac{RH}{N}$ is a p-group. So in particular $\frac{H}{N}$ is subnormal in $\frac{RH}{N}$ so that H is subnormal in RH, which in turn is subnormal in G. Hence H is subnormal in G.

Now let $q \neq p$ and let $V = R \cap N$, which of course coincides with $O_q(N)$. Then $V \neq 1$ and V is subnormal in H. Let $g \in A$. Then H is subnormal in $\langle H, H^g \rangle$. In particular H is subnormal in $\langle H, V^g \rangle$. Now H is q-perfect and $V^g \leq R$ so that $V^g \leq N_R(H)$. It follows that $V^A = \langle V^g | \ g \in A \rangle$ is contained in $N_G(H)$ and so

$$\begin{aligned} \lfloor V^A, H \rfloor &\leq [R \cap N_G(H), H] \\ &\leq R \cap H = R \cap N \\ &= V \leq V^A. \end{aligned}$$

In particular, V^A is normalized by H. Since $G = \langle A, H \rangle$, we have that V^A is normal in G. Now $V^A \neq 1$, because $V \neq 1$. By the inductive hypothesis, HV^A is subnormal in G. But H is normal in HV^A so that H is subnormal in G, which finally completes the proof of the theorem. □

Chapter 4

Metabelian factorized groups

4.1 Introduction and statement of the main results

In this chapter we proceed to the infinite case. More specifically we discuss the important contributions of Stonehewer [15]. It will soon become apparent that the classes of metabelian and abelian-by-finite groups will form the basis of his arguments. We also include the nilpotent-by-abelian case by Fransman [7], which extends the abelian-by-finite case of Stonehewer [15].

To begin with, it seems to be most unlikely that Theorem 3.6 will still be valid when G is any infinite group as remarked by Stonehewer [15]. However, a counterexample in this regard appears to be extremely difficult to find. Clearly, since every finite p-group is nilpotent, it is evident that Theorem 3.6 trivially holds for all finite p-groups, where p is a prime. In fact Wielandt [17] exploited this fact in his proof. If, on the other hand, G is an infinite p-group, then already there are difficulties in deciding whether or not H is subnormal in G. Of course, since it is well-known that every subgroup of a nilpotent (finite or infinite) group is subnormal in the group, there is no point in considering this class of groups. So the class of soluble groups is the next obvious class for investigation.

To this end we return for the moment to Example 2.19 introduced earlier in Chapter 2.

Example 4.1 Specializing to the case $S = \mathbb{R}$ in Example 2.19 we now consider the group

$$G = G(\mathbb{R}) = \left\{ \begin{pmatrix} u & x \\ 0 & v \end{pmatrix} \mid u, v \in \langle -1 \rangle, \quad x \in \mathbb{R} \right\}$$

of 2×2 upper triangular matrices under matrix multiplication. For

$$A = \left\{ \begin{pmatrix} u & x \\ 0 & u \end{pmatrix} \mid u \in \langle -1 \rangle, \quad x \in \mathbb{R} \right\}$$

and

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

it is abundantly clear that A and B are abelian. In fact, B is isomorphic to the Klein 4-group. Furthermore, $G = AB$ and

$$H = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

is an abelian subgroup of $A \cap B$. Indeed, it is obvious that

$$H = Z(G),$$

the centre of G. Hence H is (sub)normal in G as well as in both A and B. We also observe that A is an abelian normal subgroup of G and hence $\frac{G}{A}$ is isomorphic to a cyclic group of order 2. This forces the commutator subgroup G' to be abelian, which in turn implies that G is metabelian. □

In view of Example 4.1 we can draw the following conclusions.

Remark 4.2 One quickly checks that the infinite group $G = AB$ in Example 4.1 above has the following additional properties.

(i) abelian-by-finite, and hence also
(ii) nilpotent-by-finite,
(iii) nilpotent-by-abelian, as well as
(iv) metabelian-by-finite. □

More specifically, this enables us to record the following two results due to Stonehewer [15].

Theorem 4.3 *Let the metabelian group $G = AB$ be the product of two subgroups A and B, and let H be a subgroup of $A \cap B$. If H is subnormal in both A and B, then H is subnormal in G.* □

Theorem 4.4 *Let the abelian-by-finite group $G = AB$ be the product of two subgroups A and B, and let H be a subgroup of $A \cap B$. If H is subnormal in both A and B, then H is subnormal in G.* □

Furthermore, Stonehewer [15] also formulated an extension of Theorem 3.6 to the class of nilpotent-by-abelian groups.

Theorem 4.5 *Let the nilpotent-by-abelian group $G = AB$ be the product of two subgroups A and B, and let H be a subgroup of $A \cap B$. If H is subnormal in both A and B, then H is subnormal in G.* □

However, it seems as if no formal proof of the above theorem of Stonehewer appears in literature. Indeed, we discovered this through a private communication (see Appendix A.2). We will, therefore supply a proof of it, due to an unpublished result of Fransman [7]. Furthermore, we are in position to also state the following result of [7].

Theorem 4.6 *Let the nilpotent-by-finite group $G = AB$ be the product of two subgroups A and B, and let H be a subgroup of $A \cap B$. If H is subnormal in both A and B, then H is subnormal in G.* □

4.2 Preliminary results

In order to prove corresponding results on infinite factorized groups, we need some well-known lemmas of Stonehewer [15] on subnormal subgroups. The first one plays an important role in abelian-by-finite groups.

Lemma 4.7 *Let $G = AB = HK$ with K and abelian normal subgroup of G. If $H \lhd^m A$ and $H \lhd^m B$, then*

$$H \lhd^{2m} G.$$

Proof. Put $N = A \cap K$. It follows by Lemma 2.20 that $N \lhd G$. Moreover, it follows by Lemma 2.2 that

$$\begin{aligned} HN &= H(A \cap K) \\ &= A \cap HK \\ &= A \cap G \\ &= A. \end{aligned}$$

By hypothesis we have that

$$H \lhd^m A$$

and consequently it is also true that

$$H \lhd^m HN. \tag{4.1.1}$$

To conclude the proof, we distinguish two cases.

Case 1. Assume that $N = A \cap K = 1$. Then by Lemma 2.2 it follows that

$$A = A \cap HK$$

$$\begin{aligned} &= H(A \cap K) \\ &= H \leq B \end{aligned}$$

and hence

$$G = B,$$

which implies that

$$H \lhd^{2m} G.$$

Case 2. Assume that $N = A \cap K \neq 1$. Then

$$\begin{aligned} \tfrac{G}{N} &= \tfrac{G}{A\cap K} \\ &= (\tfrac{A}{A\cap K})(\tfrac{B(A\cap K)}{A\cap K}) \\ &= (\tfrac{H(A\cap K)}{A\cap K})(\tfrac{K}{A\cap K}), \end{aligned}$$

with

$$\tfrac{K}{A\cap K} \quad \text{an abelian normal subgroup of} \quad \tfrac{G}{A\cap K}.$$

A further application of Lemma 2.2 yields

$$\begin{aligned} &\tfrac{K}{A\cap K} \cap \tfrac{H(A\cap K)}{A\cap K} \\ &= \tfrac{(K\cap H)(A\cap K)}{A\cap K} \\ &= 1. \end{aligned}$$

It is also clear that

$$\tfrac{HN}{N} \quad \text{is subnormal in both} \quad \tfrac{A}{N} \quad \text{and} \quad \tfrac{BN}{N}.$$

So we infer from Case 1 above that

$$\tfrac{HN}{N} \lhd^{m} \tfrac{G}{N}.$$

Therefore

$$HN \lhd^{m} G. \tag{4.1.2}$$

Finally, by combining (4.1.1) and (4.1.2), it follows that

$$H \lhd^{2m} G,$$

as required. This completes the proof of the lemma. □

The second lemma of Stonehewer [15] will be used in the proofs of the main theorems in this section.

Lemma 4.8 *Let $G = AB$ be the product of two subgroups A and B. Suppose $H \lhd^m A$ and $H \lhd^m B$, and let K be a normal subgroup of G. Write*

$$G_1 = AK \cap BK,$$

and put

$$A_1 = A \cap KB, \quad B_1 = B \cap KA.$$

Then the following hold;

(i) $G_1 = A_1B_1 = A_1K = B_1K$, *with* $H \lhd^m A_1$ *and* $H \lhd^m B_1$.

Now suppose that K is abelian and let $N = (A \cap K)(B \cap K)$. Then

(ii) $N \lhd G_1$.

Let bars denote subgroups of G_1 modulo N. Then

(iii) $\overline{G}_1 = \overline{A}_1\overline{B}_1 = \overline{A}_1\overline{K} = \overline{B}_1\overline{K}$, *with* $\overline{H} \lhd^m \overline{A}_1$ *and* $\overline{H} \lhd^m \overline{B}_1$. *Also $\overline{K}$ is an abelian normal subgroup of $\overline{G}_1$ and*

$$\overline{A}_1 \cap \overline{K} = \overline{B}_1 \cap \overline{K} = 1.$$

Hence $\overline{A}_1$ and $\overline{B}_1$ are both embeddable in $\frac{G}{K}$.

Finally, suppose as a further hypothesis that

$$KH \lhd^{m_1} G \tag{4.1.3}$$

and

$$\overline{H} \lhd^{m_2} \overline{G}_1. \tag{4.1.4}$$

Then

(iv) $H \lhd^{2m+m_1+m_2} G$.

Proof. (i) We first show that $K \cap A_1 = K \cap A$ and $K \cap B_1 = K \cap A$. To see this, it is clear that

$$K \cap A_1 \;=\; K \cap (A \cap KB)$$

$$\begin{aligned} &= K \cap A \cap KB \\ &= K \cap KB \cap A \\ &= K \cap A. \end{aligned}$$

Similarly $K \cap B_1 = K \cap B$. Using Lemma 2.2 we have that

$$\begin{aligned} G_1 = G_1 \cap G &= KA \cap KB \cap AB \\ &= KA \cap AB \cap KB \\ &= A(KA \cap B) \cap KB \\ &= AB_1 \cap KB \\ &= (A \cap KB)B_1 = A_1B_1. \end{aligned}$$

Also

$$\begin{aligned} KA_1 &= K(A \cap KB) \\ &= KA \cap KB = G_1. \end{aligned}$$

Similarly $KB_1 = G_1$, so that

$$G_1 = A_1B_1 = A_1K = B_1K.$$

Furthermore, since $H \leq A_1 \leq A$ and $H \leq B_1 \leq B$, it is evident that

$$H \lhd^m A_1 \quad \text{and} \quad H \lhd^m B_1.$$

(ii) It follows from Lemma 2.20 that

$$K \cap A = K \cap A_1 \lhd KA_1 = G_1$$

and similarly $K \cap B \lhd G_1$. Hence $N \lhd G_1$.

(iii) From (i) we have that

$$G_1 = A_1B_1 = KA_1 = KB_1$$

with

$$H \lhd^m A_1 \quad \text{and} \quad H \lhd^m B_1.$$

Thus it follows trivially that

$$\overline{G}_1 = \overline{A}_1\overline{B}_1 = \overline{A}_1\overline{K} = \overline{B}_1\overline{K},$$

and also that

$$\overline{H} \lhd^m \overline{A}_1 \quad \text{and} \quad \overline{H} \lhd^m \overline{B}_1.$$

Clearly, $\overline{K}$ is an abelian normal subgroup of $\overline{G}_1$. By Lemma 2.2 we know that

$$K \cap NA_1 = N(K \cap A) = N,$$

and so

$$\overline{K} \cap \overline{A}_1 = 1.$$

Similarly

$$\overline{K} \cap \overline{B}_1 = 1.$$

This implies that

$$\overline{A}_1 \cong \frac{\overline{G}_1}{\overline{K}} \cong \frac{G_1}{K} \leq \frac{G}{K}.$$

Likewise one sees that

$$\overline{B}_1 \quad \text{embeds in} \quad \frac{G}{K}.$$

(iv) By the hypothesis $NH \lhd^{m_2} G_1$ and so

$$NH \lhd^{m_2} KH \lhd^{m_1} G.$$

Therefore we conclude that $NH \lhd^{m_1+m_2} G$.

Now it is trivially true that

$$NH = (K \cap A)(K \cap B)H.$$

Thus if

$$A_2 = (K \cap A)H \quad \text{and} \quad B_2 = (K \cap B)K,$$

then

$$NH = A_2B_2$$

and consequently it follows that

$$H \lhd^m A_2 \quad \text{and} \quad H \lhd^m B_2.$$

Therefore by Lemma 4.7, $H \lhd^{2m} NH$ and thus

$$H \lhd^{2m} NH \lhd^{m_1+m_2} G.$$

This in turn implies that

$$H \lhd^{2m+m_1+m_2} G,$$

as required. The proof of the lemma is finally complete. □

4.3 Proofs of the main theorems

We have now developed enough tools to prove Theorem 4.3 as well as Theorem 4.4*, which is a reformulation of Theorem 4.4.

Proof of Theorem 4.3 By appealing to Lemma 2.18, if follows that the factorizer $X = X(G')$ of the commutator subgroup G' of G admits the triple factorization

$$X = A_1B_1 = A_1G' = B_1G',$$

where

$$A_1 = A \cap BG' \quad \text{and} \quad B_1 = B \cap AG'.$$

Now G metabelian forces G' to be an abelian normal subgroup of G. The application of Lemma 4.8 now guarantees that

$$N = (A \cap G')(B \cap G')$$

is an abelian normal subgroup of X. The factor groups

$$\tfrac{A_1N}{N} \quad \text{and} \quad \tfrac{B_1N}{N}$$

are abelian and consequently we conclude that

$$\tfrac{A_1N}{N} \cap \tfrac{B_1N}{N} \leq Z(\tfrac{X}{N}),$$

the centre of $\frac{X}{N}$. So in particular we have that HN is subnormal in X and hence also subnormal in G. By again invoking Lemma 4.8, it is clear that H is subnormal in G. □

Since we also need the alternative versions of these results in Chapter 5, we now rephrase them differently.

Theorem 4.3* *Let the metabelian group $G = AB$ be the product of two subgroups A and B, and let H be a subgroup of $A \cap B$. If $H \lhd^m A$ and $H \lhd^m B$, then*

$$H \lhd^{2m} G.$$

Proof. Let K be an abelian normal subgroup G such that $\frac{G}{K}$ is abelian. It follows that

$$\tfrac{HK}{K} \quad \text{is a normal subgroup of} \quad \tfrac{G}{K}.$$

Hence HK is normal in G. So in the notation of Lemma 4.8 we have that $m_1 = 1$. Also by putting $N = (A \cap K)(B \cap K)$, we have that

$$\overline{G}_1 = \overline{A}_1\overline{B}_1 \quad \text{with} \quad \overline{A}_1 \quad \text{and} \quad \overline{B}_1 \quad \text{abelian.}$$

Therefore

$$\overline{H} \quad \text{lies in the centre of} \quad \overline{G}_1.$$

and hence $m_2 = 1$. By virtue of Lemma 4.8 (iv) we conclude that

$$H \lhd^{2(m+1)} G,$$

which completes the proof of the theorem. □

Theorem 4.4* *Let the abelian-by-finite group $G = AB$ be the product of two subgroups A and B and let H be a subgroup of $A \cap B$. Suppose that K is an abelian normal subgroup of G such that $|\frac{G}{K}| = n$ (finite). If*

$$H \lhd^m A \quad \textit{and} \quad H \lhd^m B,$$

then

$$H \lhd^\ell G,$$

where

$$\ell = 2m + n + n^2.$$

Proof. By Theorem 3.6 we see that HK is subnormal in G. Therefore in the notation of Lemma 4.8 we can take $m_1 = n$. Also $\overline{G}_1 = \overline{A}_1\overline{B}_1$ is finite and of order $\leq n^2$. Another application of Theorem 3.6 yields $\overline{H}$ subnormal in $\overline{G}_1$. We can therefore take $m_2 = n^2$. It is now clear that the theorem becomes a direct consequence of Lemma 4.8, as desired. □

We are now in position to supply a proof for Theorem 4.5, as promised.

Proof of Theorem 4.5 Let K be a nilpotent normal subgroup of G such that $\frac{G}{K}$ is abelian. Suppose further that K is nilpotent of class c. We proceed the proof by induction on c. If $c = 1$, then K is abelian and so G is metabelian. The result is therefore evident by Theorem 4.3. Assume next that $c > 1$ and that the result holds for all pairs of groups (G^*, K^*) where K^* is a normal subgroup of G^* such that K^* is nilpotent of class $c-1 > 0$, $\frac{G^*}{K^*}$ is abelian and G^* satisfies the hypothesis of the theorem.

Now it is of course clear by Lemma 2.24 that

$$\frac{G}{\gamma_c(K)} \quad \text{is nilpotent-by-abelian.}$$

Also $\frac{H\gamma_c(K)}{\gamma_c(K)}$ is obviously subnormal in both

$$\frac{A\gamma_c(K)}{\gamma_c(K)} \quad \text{and} \quad \frac{B\gamma_c(K)}{\gamma_c(K)}.$$

Moreover, the normal subgroup $\frac{K}{\gamma_c(K)}$ of $\frac{G}{\gamma_c(K)}$ is nilpotent of class $c-1>0$. So the pair

$$\left(\frac{G}{\gamma_c(K)}, \frac{K}{\gamma_c(K)}\right)$$

is certainly of the form (G^*, K^*) above. It follows therefore by the induction hypothesis that

$$\frac{H\gamma_c(K)}{\gamma_c(K)} \quad \text{is subnormal in} \quad \frac{G}{\gamma_c(K)}.$$

Hence

$$H\gamma_c(K) \quad \text{is subnormal in} \quad K. \tag{4.1.5}$$

Next we consider the factorizer G_1 of the abelian normal subgroup $\gamma_c(K)$ of G. Then G_1 has the triple factorization

$$G_1 = A_1B_1 = A_1\gamma_c(K) = B_1\gamma_c(K),$$

where

$$A_1 = A \cap B\gamma_c(K) \quad \text{and} \quad B_1 = B \cap A\gamma_c(K).$$

Put

$$N = (A \cap \gamma_c(K))(B \cap \gamma_c(K)).$$

Denoting subgroups of G_1 modulo N by bars, we have that

$$\overline{G}_1 = \overline{A}_1\overline{B}_1 = \overline{A}_1\overline{\gamma_c(K)} = \overline{B}_1\overline{\gamma_c(K)},$$

where $\overline{\gamma_c(K)}$ is an abelian normal subgroup of $\overline{G}_1$ and with

$$\overline{H} \quad \text{subnormal in both} \quad \overline{A}_1 \quad \text{and} \quad \overline{B}_1.$$

An application of Lemma 2.22 now yields that $\overline{K \cap G_1}$ is a normal subgroup of $\overline{G}_1$ and $\overline{K \cap G_1}$ is nilpotent of class at most $c-1$. Moreover, we quickly compute that

$$\frac{\overline{G}_1}{\overline{K \cap G_1}} \cong \frac{G_1}{K \cap G_1} \cong \frac{KG_1}{K} \leq \frac{G}{K},$$

which is abelian. Hence $(\overline{G_1}, \overline{K \cap G_1})$ is also a pair of the forem (G^*, K^*) above. We therefore infer from Theorem 4.3 that

$$\overline{H} \quad \text{is subnormal in} \quad \overline{G}_1. \tag{4.1.6}$$

In view of (4.1.5) and (4.1.6) together with Lemma 4.8(iv) we conclude that

$$H \quad \text{is subnormal in} \quad G,$$

which completes the proof of the theorem. $\square$

We conclude this chapter with a proof of Theorem 4.6, which basically extends Theorem 4.4 of Stonehewer [15].

Proof of Theorem 4.6 Let K be a nilpotent normal subgroup of G with $\frac{G}{K}$ finite. Assume that the nilpotency class of K is c. The proof is by induction on c. If $c = 1$, then K is abelian and the result follows by Theorem 4.4.

Assume next that $c > 1$ and that the result holds for all pairs of groups (G^*, K^*) where K^* is a normal subgroup of G^* such that K^* is nilpotent of class $c - 1 > 0$, $\frac{G^*}{K^*}$ is abelian and G^* satisfies the hypothesis of the theorem.

In view of Lemma 2.24 it is well-known that

$$\frac{G}{\gamma_c(K)} \quad \text{is nilpotent-by-finite.}$$

Now $\frac{H\gamma_c(K)}{\gamma_c(K)}$ is subnormal in both

$$\frac{A\gamma_c(K)}{\gamma_c(K}) \quad \text{and} \quad \frac{B\gamma_c(K)}{\gamma_c(K)}.$$

Since the normal subgroup $\frac{K}{\gamma_c(K)}$ of $\frac{G}{\gamma_c(K)}$ is nilpotent of class $c-1 > 0$, it follows by the induction hypothesis that

$$\frac{H\gamma_c(K)}{\gamma_c(K)} \quad \text{is subnormal in} \quad \frac{G}{\gamma_c(K)}.$$

This in turn implies that

$$H\gamma_c(K) \quad \text{is subnormal in} \quad K. \tag{4.1.7}$$

Just as in Theorem 4.4 above, the factorizer G_1 of the abelian normal subgroup $\gamma_c(K)$ of G has the triple factorization

$$G_1 = A_1B_1 = A_1\gamma_c(K) = B_1\gamma_c(K),$$

where

$$A_1 = A \cap B\gamma_c(K) \quad \text{and} \quad B_1 = B \cap A\gamma_c(K).$$

Also

$$N = (A \cap \gamma_c(K))(B \cap \gamma_c(K))$$

is an abelian normal subgroup of G_1. So by denoting subgroups of G_1 modulo N by bars, we have that

$$\overline{G}_1 = \overline{A}_1\overline{B}_1 = \overline{A}_1\overline{\gamma_c(K)} = \overline{B}_1\overline{\gamma_c(K)},$$

where $\overline{\gamma_c(K)}$ is an abelian normal subgroup of $\overline{G}_1$ and with

$$\overline{H} \quad \text{subnormal in both} \quad \overline{A}_1 \quad \text{and} \quad \overline{B}_1.$$

Lemma 2.22 now yields that $\overline{K \cap G_1}$ is a normal subgroup of $\overline{G}_1$ and it is nilpotent of class at most $c-1$. Evidently

$$\tfrac{\overline{G}_1}{\overline{K \cap G_1}} \cong \tfrac{G_1}{K \cap G_1} \cong \tfrac{KG_1}{K} \leq \tfrac{G}{K},$$

is finite. Clearly the pair $(\overline{G}_1, \overline{K \cap G_1})$ is of the form (G^*, K^*) above. An application of Theorem 4.4 ensures us that

$$\overline{H} \quad \text{is subnormal in} \quad \overline{G}_1. \tag{4.1.8}$$

Using (4.1.7), (4.1.8) and Lemma 4.8(iv), it follows that

$$H \quad \text{is subnormal in} \quad G.$$

This completes the proof of the theorem. $\square$

Chapter 5

Nilpotent-by-abelian-by-finite factorized groups

5.1 Statement of the main theorem

In a further effort to extend Theorem 3.6 to the infinite case, it seems easier in general to proceed from a factor group $\frac{G}{N}$, for which the result is known to hold, and with N suitably restricted, rather than from a subgroup. One obvious reason for this is the fact that factor groups inherit the factorization from G, whereas subgroups do not in general. This approach was pursued by Stonehewer [15] with some success resulting in amongst others the following theorem.

Theorem 5.1 *Let the periodic nilpotent-by-abelian-by-finite group $G = AB$ be factorized by two subgroups A and B, and let H be a subgroup of $A \cap B$. If H is subnormal in both A and B, then H is subnormal in G.* □

Recall that a group G is *periodic* if every element of G has finite order. He was also able to obtain a bound on the defect of H in G. The methods used involved some ring-theoretic descriptions of triply factorized groups $G = AB = AK = BK$, with A, B and K abelian, due to Sysak. (An extensive account of such constructions of triply factorized groups can be found in Section 6.1 of the textbook [1] of Amberg, Franciosi and de Giovanni). In [15] Stonehewer also posed the question as to whether or not the periodicity condition could be avoided. Some years later the answer to this question turned out to be in the affirmative due to a most remarkable paper by S. Franciosi, F. de Giovanni and Y.P. Sysak [5].

This final chapter is therefore mainly devoted to their contributions. In particular, they obtained the following result, which is regarded as the main theorem of this chapter.

Theorem 5.2 *Let the nilpotent-by-abelian-by-finite group $G = AB$ be the product of two subgroups A and B and let H be a subgroup of $A \cap B$. If H is subnormal in both A and B, then H is subnormal in G. Moreover, if G contains a nilpotent normal subgroup N of class c and a normal subgroup L such that $\frac{L}{N}$ is abelian and $\frac{G}{L}$ is finite of order n, and if the defect of H in A and B is at most m, then the defect of H in G is bounded by a function $f(c, m, n)$.* $\square$

In order to prove this result, an in depth discussion on group ring-theoretical considerations is required. For an extensive account on the theory of group rings, the reader is referred to the books of Passman [12] and Robinson [14].

5.2 Group ring methods

The proof of Theorem 5.2 depends on the behaviour of a subnormal subgroup H of a group G in the integral group ring $\mathbb{Z}G$. This requires some ring-theoretic methods. Thus the required results are developed here. We begin with some definitions and preliminary lemmas before stating and proving the important Theorem 5.15, which is essential in order to prove Theorem 5.2.

Our discussion continues with the concept of an augmentation ideal.

Definition 5.3 If G is a group and R is a commutative ring with identity, the *augmentation ideal* $\triangle_R(G)$ of the group ring RG is the ideal of RG consisting of elements of the form $\sum\limits_{1 \neq g \in G} r_g(g-1)$ where $r_g \in R$ for all $g \in G$, with $g \neq 1$. $\square$

It follows that $\triangle_R(G)$ is the kernel of the natural ring homomorphism $RG \to R$.

Definition 5.4 If H is a subgroup of G and I is a right deal of RG, the *centralizer* $C_I(H)$ is the set of all elements r of I such that $hr = rh$ for all $h \in H$. $\square$

Lemma 5.5 *Let G be a group, A a normal subgroup of G, R a commutative ring with identity and I a right ideal of the group ring RG such that $\triangle_R(G) = \triangle_R(A) + I$. If I_0 is the largest ideal of RG contained in I, then the centralizer $C_I(A)$ is contained in I_0.*

Proof. Clearly $RG = RA + I$ and thus

$$(RG)C_I(A) \quad = \quad (RA + I)C_I(A)$$

$$\begin{aligned} &\leq (RA)C_I(A) + I \\ &= C_I(A)(RA) + I \\ &\leq I. \end{aligned}$$

It follows that the ideal $(RG)C_I(A)(RG)$ is contained in $I(RG) = I$ and so also in I_0. In particular $C_I(A)$ is contained in I_0. □

Lemma 5.6 *Let G be a group and R a commutative ring with 1. If A is a normal subgroup of G and I is a right ideal of the group ring RG, then the set*

$$L = (1 + \triangle_R(A) + I) \cap G$$

is a subsemigroup of G.

Proof. Let x, y be elements of L and write

$$x = 1 + u + r \quad \text{and} \quad y = 1 + v + s,$$

where u, v are elements of $\triangle_R(A)$ and r, s belong to I. Then

$$\begin{aligned} xy &= (1 + u + r)y \\ &= y + uy + ry \\ &= y(1 + y^{-1}uy) + ry \\ &= (1 + v + s)(1 + y^{-1}uy) + ry \\ &= (1 + v)(1 + y^{-1}uy) + s(1 + y^{-1}uy) + ry \\ &= 1 + (v + y^{-1}uy + vy^{-1}uy) + (s(1 + y^{-1}uy) + ry). \end{aligned}$$

But $v + y^{-1}uy + vy^{-1}uy$ is an element of $\triangle_R(A)$ and $s(1 + y^{-1}uy) + ry$ belongs to I, so that xy lies in L, and so L is a subsemigroup of G. □

We recall the following definition.

Definition 5.7 A ring R is called *radical* if it coincides with its Jacobson radical. □

This is equivalent to the property that if R is embedded in the usual way in a ring $\tilde{R}$ with identity, then the set $1 + R$ is a subgroup of the group of units of $\tilde{R}$.

The following lemma has some information about a radical factor ring.

Lemma 5.8 *Let G be a group, R a commutative ring with identity and I a right ideal of the group ring RG such that $\triangle_R(G) = G - 1 + I$. If A is an abelian normal subgroup of G such that $(1 + \triangle_R(A) + I) \cap G$ is a subgroup of G, then the factor ring $\frac{\triangle_R(A)}{(I \cap \triangle_R(A))}$ is radical.*

Proof. Let x be any element of $\triangle_R(A)$. By hypothesis there exists $g \in G$ such that $x = g - 1 - r$ for some $r \in I$. So $g = 1 + x + r$ is an element of $L = (1 + \triangle_R(A) + I) \cap G$. It follows that g^{-1} belongs to L, and so $g^{-1} = 1 + y + s$, where $y \in \triangle_R(A)$ and $s \in I$. Since A is a normal subgroup of G, also $z = g^{-1}yg$ is an element of $\triangle_R(A)$ and so

$$\begin{aligned}
1 &= g^{-1}g \\
&= (1 + y + s)g \\
&= g + yg + sg \\
&= g(1 + g^{-1}yg) + sg \\
&= (1 + x + r)(1 + z) + sg \\
&= (1 + x)(1 + z) + r(1 + z) + sg.
\end{aligned}$$

It follows that $(1 + x)(1 + z) = 1 - r(1 + z) - sg$ belongs to the set $1 + I$. Thus for each element x of $\triangle_R(A)$, there exists $z \in \triangle_R(A)$ such that $(1+x)(1+z)$ lies in I. This means that the factor ring $\frac{\triangle_R(A)}{I \cap \triangle_R(A)}$ is radical. □

The following result on the augmentation ideal of a finite group with prime-power order is needed later on. We state it without proof (see [12; Lemma 3.1.6]).

Lemma 5.9 *Let p be a prime number and let G be a finite p-group of order n. If k is the field with p elements, then $\triangle_k(G)^n = 0$.* □

We recall that for a group G and a prime number p, $O_{p'}(G)$ denotes the largest periodic normal p'-subgroup of G.

The next lemma is needed in the proof of a lemma concerning the augmentation ideal of the integral group ring of an abelian-by-finite group.

Lemma 5.10 *Let G be a group and A an abelian normal subgroup of G with finite index n. Let k be the field with p elements and I a right ideal of the group algebra kG such that*

$$\triangle_k(G) = \triangle_k(A) + I = G - 1 + I.$$

If H is a subnormal subgroup of G contained in $G \cap (1 + I)$, then $\triangle_k(G)\triangle_k(H)^n$ lies in I.

Proof. Let $Y = G \cap (1+I)$ and suppose first that $C_Y(A) = 1$. Then we have in particular that

$$C_H(A) = H \cap A = 1.$$

Thus H is finite and its order divides n. Since H is subnormal in G, there exists a positive integer t such that

$$[A, \underbrace{H, \ldots H}_{t}] = 1.$$

Thus H stabilizes a series of finite length of A. It follows from a well-known result of P. Hall [8; Lemma 3]) that H is nilpotent. Asssume that A contains an element a of prime order $q \neq p$, and consider the element

$$c = 1 - \tfrac{1}{q}(1 + a + \cdots + a^{q-1})$$

of $\triangle_k(G)$. By hypothesis there exist elements g of G and r of I such that $-c = g - 1 + r$. Then

$$(g + r)a = (1 - c)a = 1 - c = g + r.$$

Hence $ga - g = r - ra$ belongs to I. So the element $gag^{-1} - 1$ is in I and thus

$$gag^{-1} \in A \cap (1 + I) = A \cap Y = 1.$$

This contradiction shows that A does not contain elements of prime order $q \neq p$, so that $A \cap O_{p'}(G) = 1$ and $O_{p'}(G)$ is contained in $C_G(A)$. On the other hand, $O_{p'}(H)$ lies in $O_{p'}(G)$ and hence

$$O_{p'}(H) \leq C_H(A) = 1.$$

Thus H is a p-group. Since k is the field of order p and $|H| \leq n$, it follows that

$$\triangle_k(H)^n = 0$$

by Lemma 5.9, whence

$$\triangle_k(G)\triangle_k(H)^n = 0.$$

This proves the lemma when $C_Y(A) = 1$.

In the general case, let I_0 be the largest ideal of kG contained in I. Put $N = G \cap (1 + I_0)$. Then N is a normal subgroup of G contained in Y. If M is any normal subgroup of G contained in Y, then $\triangle_k(M)kG$ is an ideal of kG contained in I, so that $\triangle_k(M)kG$ lies in I_0 and M is contained in N. Thus N is the largest normal subgroup of G contained in Y. If $\overline{G} = \frac{G}{N}$, the natural group homomorphism $G \to \overline{G}$ induces a group algebra homomorphism $kG \to k\overline{G}$ whose kernel $\triangle_k(N)kG$ is contained in I_0. Thus the group $\overline{G}$ and the group $k\overline{G}$ inherit the hypotheses of the lemma.

The centralizer $C_{\overline{I}}(A)$ is contained in $\overline{I}_0$ by Lemma 5.5. Hence $C_{\overline{Y}}(\overline{A}) = 1$. It follows from the first part of the proof that $\triangle_k(\overline{G})\triangle_k(\overline{H})^n$ lies in $\overline{I}$. Hence $\triangle_k(G)\triangle_k(H)^n$ is contained in I, and the proof is complete. □

Note: Let m be any positive integer. Then we define $\tau(m)$ by the position $\tau(1) = 0$ and $\tau(m) = \alpha_1 + \ldots + \alpha_t$ if $m > 1$ and $m = p_1^{\alpha_1} \ldots p_t^{\alpha_t}$, where $p_1, \ldots p_t$ are prime numbers.

The next three lemmas give more information about the augmentation ideal of the integral group ring of an abelian-by-finite group.

Lemma 5.11 *Let G be a group, A an abelian normal subgroup of G with finite index n, and I a right ideal of the group ring $\mathbb{Z}G$ such that*

$$\triangle_{\mathbb{Z}}(G) = \triangle_{\mathbb{Z}}(A) + I = G - 1 + I.$$

If H is a subnormal subgroup of G contained in $G \cap (1 + I)$, then $\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^{n\tau(m)}$ lies in $I + m\triangle_{\mathbb{Z}}(G)$ for each positive integer m.

Proof. The statement is evident for $m = 1$. Suppose then that $m > 1$ and let $m = ph$, where p is a prime number. By induction we may suppose that $\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^{n\tau(h)}$ is contained in $I + h\triangle_{\mathbb{Z}}(G)$. If k is the field $\frac{\mathbb{Z}}{p\mathbb{Z}}$, the factor ring $\frac{\mathbb{Z}G}{p\mathbb{Z}G}$ is naturally isomorphic with the group algebra kG. It follows from Lemma 5.10 that $\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^n$ is contained in $I + p\mathbb{Z}G$. Since $\frac{\mathbb{Z}G}{\triangle_{\mathbb{Z}}(G)}$ is isomorphic with $\mathbb{Z}$, we have that

$$p\mathbb{Z}G \cap \triangle_{\mathbb{Z}}(G) = p\triangle_{\mathbb{Z}}(G).$$

Hence $\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^n$ is contained in $I + p\triangle_{\mathbb{Z}}(G)$. Clearly $\tau(m) = \tau(h) + 1$, which implies that

$$\begin{aligned}
\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^{n\tau(m)} &= (\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^{n\tau(h)})\triangle_{\mathbb{Z}}(H)^n \\
&\leq (I + h\triangle_{\mathbb{Z}}(G))\triangle_{\mathbb{Z}}(H)^n \\
&\leq I + h\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^n \\
&\leq I + h(I + p\triangle_{\mathbb{Z}}(G)) \\
&= I + m\triangle_{\mathbb{Z}}(G).
\end{aligned}$$

The proof of the lemma is complete. □

Lemma 5.12 *Let G be a group, A an abelian normal subgroup of G with finite index n and I a right ideal of the group ring $\mathbb{Z}G$ such that*

$$\triangle_{\mathbb{Z}}(G) = \triangle_{\mathbb{Z}}(A) + I = G - 1 + I.$$

If H is a subnormal subgroup of defect m of G which is contained in $G \cap (1+I)$, then $\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^h$ lies in I for some positive integer $h \leq n.16^{m+n} + 2m + 2n$.

Proof. From Lemma 5.5 we have that $I \cap \triangle_{\mathbb{Z}}(A)$ is contained in the largest ideal I_0 of $\mathbb{Z}G$ contained in I. By Lemma 5.8 we have that the factor ring

$$\frac{\triangle_{\mathbb{Z}}(A)}{I \cap \triangle_{\mathbb{Z}}(A)}$$

is radical. Consider the ring $\overline{\mathbb{Z}G} = \frac{\mathbb{Z}G}{I_0}$ and denote by $\overline{S}$ the image of any subset S of $\mathbb{Z}G$ under the natural homomorphism of $\mathbb{Z}G$ onto $\overline{\mathbb{Z}G}$. Clearly

$$\overline{I} \cap \overline{\triangle_{\mathbb{Z}}(A)} = 0$$

so that the commutative ring $\overline{\triangle_{\mathbb{Z}}(A)}$ is isomorphic with

$$\frac{\triangle_{\mathbb{Z}}(A)}{I \cap \triangle_{\mathbb{Z}}(A)}$$

and so it is a commutative radical subring of $\overline{\mathbb{Z}G}$. Thus the set

$$U = 1 + \overline{\triangle_{\mathbb{Z}}(A)}$$

is an abelian subgroup of the group $\overline{\mathbb{Z}G^*}$ of units of $\overline{\mathbb{Z}G}$. Now A is normal in G and so the subgroup U is normalized by $\overline{G}$. Consequently $X = U\overline{G}$ is a subgroup of $\overline{\mathbb{Z}G^*}$ and U is normal in X. But $\overline{A}$ is contained in $U \cap \overline{G}$ and so the group $\frac{X}{U}$ is finite with order dividing n. Consider the subgroup

$$Y = X \cap (1 + \overline{I})$$

of X. Then Y contains $\overline{H}$, and so since

$$\overline{I} \cap \overline{\triangle_{\mathbb{Z}}(A)} = 0,$$

it follows that $Y \cap U = 1$. In particular, Y is finite. Clearly U is contained in $I + \overline{\triangle_{\mathbb{Z}}(G)}$, so that $X = U\overline{G}$ lies in

$$(I + \overline{\triangle_{\mathbb{Z}}(G)})\overline{G} = 1 + \overline{\triangle_{\mathbb{Z}}(G)}.$$

But by hypothesis we have also that

$$1 + \overline{\triangle_{\mathbb{Z}}(G)} = U + \overline{I} = \overline{G} + \overline{I}.$$

Let x be any element of X, and write

$$x = u + r = g + s,$$

where $u \in U, g \in \overline{G}$ and r, s are elements of $\overline{I}$. It follows that

$$x = (1 + ru^{-1})u = (1 + sg^{-1})g,$$

with ru^{-1} and sg^{-1} in $\overline{I}$. But on the other hand,

$$1 + ru^{-1} = xu^{-1}$$

belongs to

$$X \cap (1 + \overline{I}) = Y$$

so that $x \in YU$ and $YU = X$. Similarly,

$$1 + sg^{-1} = xg^{-1}$$

lies in Y so that x belongs to $Y\overline{G}$, and $Y\overline{G} = X$. Therefore the group X has the triple factorization

$$X = U\overline{G} = UY = Y\overline{G}.$$

Since $U\overline{H}$ is subnormal in X with defect at most m and

$$U\overline{H} \cap Y = \overline{H}(U \cap Y) = \overline{H},$$

by Lemma 2.2, the subgroup $\overline{H}$ is subnormal in Y with defect at most m. Hence $\overline{H}$ is also subnormal in X with defect

$$c \leq 2m + n + \tau(n^2)$$

according to Theorem 4.4*. But then we have that

$$c \leq 2m + 2n,$$

because $\tau(n^2) \leq n$. However, since

$$\overline{H} \cap U = 1,$$

we have that $\overline{H}$ stabilizes a finite series of U with length at most c. Furthermore, an application of [15; Lemma 5] yields that $\overline{H}$ stabilizes a series of length at most c of the additive group $t\overline{\triangle_{\mathbb{Z}}(A)}$ where

$$t = 2^{2^{c-1}-1} \prod_{i=1}^{2^{c-1}-1} (2^i - 1) \leq 2^{4^c}.$$

Thus

$$t\overline{\triangle_{\mathbb{Z}}(A)}\triangle_{\mathbb{Z}}(\overline{H})^c = 0,$$

which means that

$$t\triangle_{\mathbb{Z}}(A)\triangle_{\mathbb{Z}}(H)^c$$

is contained in I_0 and so also in I. Now by Lemma 5.11 it is clear that

$$\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^{n\tau(t)} \leq I + t\triangle_{\mathbb{Z}}(G)$$

and so

$$\begin{aligned} \triangle_{\mathbb{Z}}(G))\triangle_{\mathbb{Z}}(H)^{n\tau(t)+c} &\leq I\triangle_{\mathbb{Z}}(H)^c + t\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^c \\ &= I\triangle_{\mathbb{Z}}(H)^c + (t\triangle_{\mathbb{Z}}(A) + tI)\triangle_{\mathbb{Z}}(H)^c \\ &\leq I. \end{aligned}$$

If we finally put

$$h = n\tau(t) + c \leq n.4^c + c \leq n.16^{m+n} + 2m + 2n,$$

the conclusion of the lemma follows immediately. □

Lemma 5.13 *Let G be a group and let I be a right ideal of the group ring $\mathbb{Z}G$ such that $\triangle_{\mathbb{Z}}(G) = G - 1 + I$. If A is an abelian normal subgroup of G with finite index n, then the additive subgroup $\triangle_{\mathbb{Z}}(A) + I$ of $\triangle_{\mathbb{Z}}(G)$ contains a right ideal J of $\mathbb{Z}G$ such that $I \leq J$ and the additive factor group $\frac{\triangle_{\mathbb{Z}}(G)}{J}$ is finite with order at most n^n.*

Proof. Let $\{g_1, \ldots, g_n\}$ be a transversal to A in G. For every $i \leq n$, consider the additive subgroup

$$S_i = (\triangle_{\mathbb{Z}}(A) + I)g_i = \triangle_{\mathbb{Z}}(A)g_i + I$$

of $\triangle_{\mathbb{Z}}(G)$. If x is any element of $\triangle_{\mathbb{Z}}(G)$, there exist $g \in G$ and $r \in I$ such that

$$x = g - 1 + r.$$

But $g = ag_i$ with $a \in A$ and $i \leq n$, so that

$$\begin{aligned} x &= g - 1 + r \\ &= (a-1)g_i + r + (g_i - 1) \end{aligned}$$

belongs to $S_i + g_i - 1$. Thus $\triangle_{\mathbb{Z}}(G)$ is the set-theoretic union of the cosets

$$S_i + g_i - 1, \text{ where } 1 \leq i \leq n.$$

Hence by [12; Lemma 2.2 of Chapter 5] there exists $i \leq n$ such that S_i has index at most n in $\triangle_{\mathbb{Z}}(G)$. It follows that each of the subgroups $S_i, \ldots, S_n$ has index at most n in $\triangle_{\mathbb{Z}}(G)$

so that

$$J = \bigcap_{i=1}^{n} S_i$$

is an additive subgroup of $\triangle_{\mathbb{Z}}(G)$, with index at most n^n. If g is any element of G we have

$$\begin{aligned} S_i g &= \triangle_{\mathbb{Z}}(A) g_i g + I \\ &= \triangle_{\mathbb{Z}}(A) g_j + I \\ &= S_j \end{aligned}$$

for some $j \leq n$ and hence $Jg \leq J$ is a right ideal of $\triangle_{\mathbb{Z}}(G)$. $\square$

Lemma 5.14 *Let G be a group and let I be a right ideal of the group ring $\mathbb{Z}G$ such that $\triangle_{\mathbb{Z}}(G) = G - 1 + I$. If the additive factor group $\frac{\triangle_{\mathbb{Z}}(G)}{I}$ has finite order n and H is a subnormal subgroup of G which is contained in $G \cap (1 + I)$, then $\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^n$ is contained in I.*

Proof. Consider the right $\mathbb{Z}G$-module $M = \frac{\triangle_{\mathbb{Z}}(G)}{I}$ and the natural semidirect product $Y^* = G \ltimes M$. There exists a subgroup L of Y^* such that

$$Y^* = L \ltimes M = GL$$

and

$$G \cap L = G \cap (1 + I),$$

(see [16; Theorem 1]). The centralizer $C_G(M)$ is normal in Y^* and the factor group $\frac{Y^*}{C_G(M)}$ is finite. Now,

$$H = HM \cap L$$

is subnormal in L since HM is subnormal in Y^* and it follows from Theorem 3.6 that the subgroup $HC_G(M)$ is subnormal in Y^*. Thus H is subnormal in Y^* and in particular it is subnormal in HM with defect at most n. Then

$$[M, \underbrace{H, \ldots, H}_{n}] = 1$$

and hence

$$M\triangle_{\mathbb{Z}}(H)^n = 0.$$

This means that $\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^n$ is contained in I. $\square$

We can now proceed to prove the following theorem which is required to prove the main theorem, viz. Theorem 5.2, stated in section 5.1.

Theorem 5.15 *Let G be a group containing an abelian normal subgroup of finite index n, and let I be a right ideal of the group ring $\mathbb{Z}G$ such that $\triangle_{\mathbb{Z}}(G) = G - 1 + I$. If H is a subnormal subgroup of defect m of G which is contained in $G \cap (1+I)$, then $\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^h$ is contained in I for some positive integer $h = h(m,n)$.*

Proof. Let A be an abelian normal subgroup of G such that $\frac{G}{A}$ is finite of order n. From Lemma 5.6 we have that the set

$$L = (1 + \triangle_{\mathbb{Z}}(A) + I) \cap G$$

is a subsemigroup of G. Clearly A is contained in L and thus L is a subgroup of G. Now,

$$\triangle_{\mathbb{Z}}(A) + I = L - 1 + I$$

and

$$G \cap (1 + I) = L \cap (1 + I),$$

so that in particular the subnormal subgroup H is contained in

$$L \cap (1 + I).$$

Consider the right ideal $I^* = \triangle_{\mathbb{Z}}(L) \cap I$ of the group ring $\mathbb{Z}L$. It follows that H is contained in

$$L \cap (1 + I^*)$$

and

$$\triangle_{\mathbb{Z}}(L) = \triangle_{\mathbb{Z}}(A) + I^* = L - 1 + I^*.$$

Now it follows from Lemma 5.12 that

$$\triangle_{\mathbb{Z}}(L)\triangle_{\mathbb{Z}}(H)^t$$

is contained in I^* and so also in I, for some positive integer

$$t \leq n.16^{m+n} + 2m + 2n.$$

By applying Lemma 5.13 we see that there exists a right ideal J of $\mathbb{Z}G$ such that

$$I \leq J \leq \triangle_{\mathbb{Z}}(A) + I$$

and the additive factor group $\frac{\triangle_{\mathbb{Z}}(G)}{J}$ is finite with order at most n^n. We have that

$$\triangle_{\mathbb{Z}}(G) = G - 1 + J,$$

because $\triangle_{\mathbb{Z}}(G) = G - 1 + I$. So by Lemma 5.14 the product

$$\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^s$$

is contained in J for some positive integer $s \leq n^n$. Now if

$$h = t + s \leq n.16^{m+n} + 2m + 2n + n^n$$

we have

$$\begin{aligned}
\triangle_{\mathbb{Z}}(G)\triangle_{\mathbb{Z}}(H)^h &\leq J\triangle_{\mathbb{Z}}(H)^t \\
&\leq (\triangle_{\mathbb{Z}}(A) + I))\triangle_{\mathbb{Z}}(H)^t \\
&\leq \triangle_{\mathbb{Z}}(A)\triangle_{\mathbb{Z}}(H)^t + I \\
&\leq \triangle_{\mathbb{Z}}(L)\triangle_{\mathbb{Z}}(H)^t + I \\
&= I
\end{aligned}$$

and the theorem is proved. □

The following group-theoretic interpretation of Theorem 5.15 will be useful in the work that follows.

Corollary 5.16 *Let the group $G = AB = AM = BM$ be the product of two abelian-by-finite subgroups A and B and an abelian normal subgroup M such that $A \cap M = B \cap M = 1$. If H is a subgroup of $A \cap B$ which is subnormal in both A and B, then $[M, \underbrace{H, \ldots, H}_{t}] = 1$ for some positive integer t. Moreover, if A contains an abelian normal subgroup of finite index n and H has defect m in A, then t can be chosen bounded by a function $h(m, n)$.*

Proof. Consider M as a right $\mathbb{Z}A$-module. Then for the group ring $\mathbb{Z}A$, there exists a right ideal I such that

$$\triangle_{\mathbb{Z}}(A) = A - 1 + I$$

and M is isomorphic with the right $\mathbb{Z}A$-module $\frac{\triangle_{\mathbb{Z}}(A)}{I}$. Moreover,

$$A \cap (1 + I) = A \cap B,$$

(see [16; Theorem 1]). Utilizing Theorem 5.15 guarantees us that

$$\triangle_{\mathbb{Z}}(A)\triangle_{\mathbb{Z}}(H)^t$$

is contained in I for some positive integer t bounded by a function $h(m, n)$. This means that

$$M\triangle_{\mathbb{Z}}(H)^t = 0$$

and thus

$$[M, \underbrace{H, \dots H}_{t}] = 1.$$

The corollary is proved. $\square$

5.3 The proof of the main theorem

The following two lemmas are used in the proof of the main theorem.

Lemma 5.17 *Let the metabelian-by-finite group $G = AB$ be the product of two subgroups A and B, and let H be a subgroup of $A \cap B$ which is subnormal in both A and B. Then H is subnormal in G. Moreover, if G contains a metabelian normal subgroup of finite index n and the defect of H in A and B is at most m, then the defect of H in G is bounded by a function $f_1(m, n)$.*

Proof. Let K be a metabelian normal subgroup of G with finite index n, and let $N = K'$ be the commutator subgroup of K. Then the factor group $\frac{G}{N}$ contains an abelian normal subgroup of finite index n, and hence HN is subnormal in G with defect at most

$$2m + n + \tau(n^2) \leq 2m + 2n$$

according to Theorem 4.4*. The factorizer $X = X(N)$ of N in G has the triple factorization

$$X = A^*B^* = A^*N = B^*N$$

where $A^* = A \cap BN$ and $B^* = B \cap AN$. Suppose first that

$$A^* \cap N = B^* \cap N = 1$$

so that in particular A^* contains an abelian normal subgroup of index at most n. Since H is subnormal in A^* and B^* with defect at most m, it follows from Corollary 5.16 that $[N, \underbrace{H, \dots, H}_{t}] = 1$ for some positive integer $t \leq h(m, n)$, so that H is subnormal in HN with defect at most $h(m, n)$. In the general case, as N is abelian, the subgroups $A^* \cap N$ and $B^* \cap N$ are normal in X (see Lemma 2.20 in Chapter 2), so that

$$L = (A^* \cap N)(B^* \cap N)$$

is a normal subgroup of X. Replacing X by the factor group $\frac{X}{L}$ we obtain from the above argument that HL is subnormal in X with defect at most $h(m, n)$. However, H is subnormal in HL with defect at most $2m$, so H is subnormal in G with defect at most

$$f_1(m,n) = h(m,n) + 4m + n + \tau(n^2) \leq h(m,n) + 4m + 2n.$$

The lemma is proved. $\square$

The following subnormality criterion is required.

Lemma 5.18 *Let the group $G = KH$ be the product of a nilpotent normal subgroup K of class c and a subgroup H such that HK' is subnormal in G with defect m. Then H is subnormal in G with defect at most $\frac{mc^2(c+1)}{2}$.*

Proof. Suppose first that $H \cap K = 1$, so that in particular

$$HK' \cap K = K'.$$

As HK' is subnormal in G with defect m, the subgroup H stabilizes a finite series of $\frac{K}{K'}$ with length at most m. It follows that, for each positive integer i, H also stabilizes a finite series of $\frac{\gamma_i(K)}{\gamma_{i+1}(K)}$ with length at most im (see [14; 5.2.5 and 5.2.11]) so that H stabilizes a finite series of K with length at most md, where

$$d = \tfrac{c(c+1)}{2}.$$

Thus H is subnormal in $G = KH$ with defect at most md. In the general case, put

$$K_0 = H \cap K \quad \text{and} \quad K_{i+1} = N_K(K_i)$$

for each non-negative integer i. Then $K_c = K$ and H normalizes every K_i. Let $n \leq c$ be the smallest non-negative integer such that $K_n = K$, so that in particular the subgroup $N = K_{n-1}$ is normal in G. As $H \cap K$ is contained in N, it follows from the first part of the proof that HN is subnormal in G with defect at most md. On the other hand, we may assume by induction on n that H is subnormal in HN with defect at most $(n-1)md$, so that H is subnormal in G with defect at most

$$(n-1)md + md = nmd \leq mcd.$$

This completes the proof of the lemma. $\square$

We are finally ready for the proof of Theorem 5.2.

Proof of Theorem 5.2. The factor group $\frac{G}{N'}$ is metabelian-by-finite, thus it follows from Lemma 5.17 that the subgroup HN' is subnormal in G with defect at most $f_1(m,n)$. In particular HN' is subnormal in HN, so that by Lemma 5.18, H is subnormal in HN with defect at most

$$\frac{c^2(c+1)f_1(m,n)}{2}.$$

Therefore H is subnormal in G, and its defect is bounded by

$$f(c,m,n) = f_1(m,n) + \frac{c^2(c+1)f_1(m,n)}{2}$$

The theorem is finally proved. □

Since it is well-known that every soluble linear group is nilpotent-by-abelian-by-finite, the following consequence of Theorem 5.2 is immediate.

Corollary 5.19 *Let the soluble-by-finite linear group $G = AB$ be the product of two subgroups A and B, and let H be a subgroup of $A \cap B$. If H is subnormal in both A and B, then H is subnormal in G.* □

5.4 An alternative proof of the main theorem

Recently Fransman [7] was able to provide an alternative proof for Theorem 5.2, which is presented below.

Proof of Theorem 5.2 Let K be a normal subgroup of G, K nilpotent of class c and $\frac{G}{K}$ abelian-by-finite. We perform an induction argument on c. If $c = 1$, then G is metabelian-by-finite and the result is evident by Lemma 5.17. So assume that $c > 1$. An application of Lemma 2.24 ensures us that factorized group

$$\frac{G}{\gamma_c(K)} = \frac{A\gamma_c(K)}{\gamma_c(K)} \frac{B\gamma_c(K)}{\gamma_c(K)}$$

is nilpotent-by-abelian-by-finite. Moreover, since $\frac{K}{\gamma_c(K)}$ is nilpotent of class $c-1 > 0$, it follows by the induction hypothesis that

$$H\gamma_c(K) \quad \text{is subnormal in} \quad G. \tag{5.1.1}$$

Now the factorizer G_1 of $\gamma_c(K)$ in $G = AB$ can be triply factorized as

$$G_1 = A_1B_1 = A_1\gamma_c(K) = B_1\gamma_c(K)$$

where

$$A_1 = A \cap B\gamma_c(K) \quad \text{and} \quad B_1 = B \cap A\gamma_c(K).$$

Now by Lemma 2.22 we have that

$$N = (A \cap \gamma_c(K))(B \cap \gamma_c(K))$$

is an abelian normal subgroup of G_1. As usual, by expressing subgroups of G_1 modulo N by bars, it is evident that

$$\overline{G}_1 = \overline{A}_1\overline{B}_1 = \overline{A_1}\ \ \overline{\gamma_c(K)} = \overline{B_1}\ \ \overline{\gamma_c(K)},$$

where $\overline{\gamma_c(K)}$ is an abelian normal subgroup of $\overline{G}_1$. It is clear that

$$\overline{H} \quad \text{is subnormal in both} \quad \overline{A}_1 \quad \text{and} \quad \overline{B}_1.$$

So by invoking Lemma 2.22 it follows that

$$\overline{G_1 \cap K} \quad \text{is a normal subgroup of} \quad \overline{G}_1,$$

and $\overline{G_1 \cap K}$ is nilpotent of class at most $c-1$.

It is clear that the pair $(\overline{G}_1, \overline{G_1 \cap K})$ satisfies the hypothesis of the theorem. So another application of Lemma 5.17 yields that

$$\overline{H} \quad \text{is subnormal in} \quad \overline{G}_1. \tag{5.1.2}$$

The subnormality of H in G is now a direct consequence of Lemma 4.8(iv). This concludes the proof of the theorem. □

5.5 Nilpotent-by-minimax factorized groups

We conclude our discussion with an extension of Theorem 3.6 to the case of the infinite case of soluble minimax groups. The final definition will be helpful in this regard.

Definition 5.20. A group is said to be *minimax* if it has a series of finite length whose factors either satisfy the minimal or maximal condition on subgroups.

Theorem 5.21 *Let the soluble-by-finite group $G = AB$ be the product of two groups A and B and let H be a subgroup of $A \cap B$. If G is nilpotent-by-minimax and H is subnormal in both A and B, then H is subnormal in G.*

Proof. Let N be a nilpotent subnormal subgroup of G such that the factor group $\frac{G}{N}$ is minimax. Then the subgroup HN is subnormal in G (see [4; Theorem 2]). So it suffices to show that H is subnormal in HN. By Lemma 5.18 this is true provided that HN' is subnormal in HN, so that replacing G by $\frac{G}{N'}$ it can be assumed that the subgroup N is abelian. The factorizer $X = X(N)$ of N in G has the triple factorization

$$X = A^*B^* = A^*N = B^*N$$

where $A^* = A \cap BN$ and $B^* = B \cap AN$. In view of Lemma 2.20 the subgroups $A^* \cap N$ and $B^* \cap N$ are clearly normal in X. Now H is subnormal in $H(A^* \cap N)$, so that $H(B^* \cap N)$ is subnormal in

$$H(A^* \cap N)(B^* \cap N).$$

On the other hand, H is also subnormal in $H(B^* \cap N)$, and hence it is subnormal in

$$H(A^* \cap N)(B^* \cap N).$$

Replacing X by the factor group

$$\frac{X}{(A^* \cap N)(B^* \cap N)},$$

we may suppose that

$$A^* \cap N = B^* \cap N = 1.$$

Then A^* and B^* are minimax groups, so that also X is minimax (see [19; Theorem A1]) and H is subnormal in X. In particular, H is subnormal in HN, which completes the proof of the theorem. □

Appendix A.1

File: C\group.txt 11/21/2003, 11:40:16AM

```
> G := Sym(5);
> H := PermutationGroup< 5 | (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,2) >;
> H;
Permutation group H acting on a set of cardinalilty 5
        (1,2)(3,4)
        (1,3)(2,4)
        (1,4)(2,3)
        (1,2)
> Order(H);
8
> Elements(H);
>> Elements(H);
             ^
Runtime error in 'Elements': Bad argument types
Argument types given: GrpPerm
> K := PermutationGroup< 5 | (2,3)(4,5), (2,4)(3,5), (2,5)(3,4), (2,5) >;
> Order(K);
8
> X := Permutationgroup< 5 | (3,4) >;
> Order(X);
2
> IsSubNormal (G,X);
>> IsSubNormal (G,X);
    ^
Use error: Identifier 'IsSubNormal' has not been declared or assigned
> IsSubnormal (G,X);
false
> IsSubnormal (G,H);
false
> Is Subnormal (G,K);
false
> IsSubnormal (K,X);
true
> IsSubnormal (H,X);
true
> HK := PermutationGroup< 5 | (2,3)(4,5), (2,4)(3,5), (2,5)(3,4), (2,5), (1,2)(\
3,4), (1,3)(2,4), (1,4)(2,3), (1,2) >;
> Order(HK);
120
> SetLogFile("c/group2.txt");
```

Appendix A.2

- - - - - - - -Forwarded message follows- - - - - - -

Date set:	Mon, 23 Jun 2003 11:24:41 +0100
T:	Ria@ma2.sun.ac.za
From:	Yvonne Collins <yvonne@maths.warwick.ac.uk>
Subject:	Stewart Stonehewers Paper ....

Dear Ria -

I have heard from Stewart who said....

"I never got round to publishing the paper referred to. I still have all the notes in my files. but I lost enthusiasm, because I was trying to prove a better result at the time. Then when I didn't make much progress, it all got put to one side, where it has remained. Francesco
de Giovanni in Naples pestered me for ages to send him a copy & I never did. In the end, he proved the results himself & maybe he published them. Perhaps the best thing would be for Oberholzer to contact Giovanni".

Hope this helps.

- -

Regards

Yvonne Collins (yvonne@maths.warwick.ac.uk)
PA to Professor John Jones
Room 174
Mathematics Institute
University of Warwick
Coventry CV4 7AL

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Fax: 024 76 5 28969

- - - - - - - End of forwarded message - - - - - - -

- - - - - - - End of forwarded message - - - - - - -

Notation

$\mathbb{R}$	field of real numbers
$\mathbb{Z}$	ring of integers
$\frac{\mathbb{Z}}{p\mathbb{Z}}$	finite field for p a prime
RG	group ring of G over the ring R
I	ideal of a ring
$\triangle_R(G)$	augmentation ideal
$H \subseteq G$	H is a subset of G
$H \leq G$	H is a subgroup of G
$H \lhd G$	H is a normal subgroup of G
$H \lhd^m G$	H is subnormal of defect at most m in G
H sn G	H is a subnormal subgroup of G
$[x, y]$	$x^{-1}y^{-1}xy$
$[H, K]$	$\langle [h, k] \mid h \in H,\ k \in K \rangle$
$[G, G]$	commutator subgroup of group G
$\gamma_n(G)$	n-th term of the lower central series of G
$\left.\begin{array}{l}[A,\ {}_nH] \\ [A, \underbrace{H, \ldots, H}_{n}]\end{array}\right\}$	commutator subgroup of weight $n+1$

$\langle X \mid \ldots \rangle$	group or subgroup generated by X such that ...
$\langle X, Y \rangle$	the join of the subgroups X and Y of G
$\|G\|$	cardinality of G
$\|G : H\|$	index of H in G
$Z(G)$	centre of G
$\Phi(G)$	Frattini subgroup of G
$\mathrm{Fit}(G)$	Fitting subgroup of G
$O_p(G)$	maximal normal p-subgroup of G
$C_G(x)$	centralizer of the element x in G
$C_G(H)$	centralizer of the subgroup H in G
$N_G(H)$	normalizer of H in G
$S_G(H)$	subnormalizer of H in G
H^G	normal closure of H in G
$\mathbf{P}(G)$	power set
$SL(p, q)$	special linear group
$GL(p, q)$	general linear group
$P \oplus S$	direct sum of group P and group S
$H \ltimes K$	semidirect product of the normal subgroup K and the subgroup H

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