A UNIFIED APPROACH TO THE ECONOMIC ASPECTS OF STATISTICAL QUALITY CONTROL AND IMPROVEMENT

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Declaration

I, the undersigned, hereby declare that the work contained in this assignment is my own original work and that I have not previously in its entirety or in part submitted it at any University for a degree.

Abstract

The design of control charts refers to the selection of the parameters implied, including the sample size n, control limit width parameter k, and the sampling interval h. The design of the $\bar{X}$-control chart that is based on economic as well as statistical considerations is presently one of the more popular subjects of research. Two assumptions are considered in the development and use of the economic or economic statistical models. These assumptions are potentially critical. It is assumed that the time between process shifts can be modelled by means of the exponential distribution. It is further assumed that there is only one assignable cause. Based on these assumptions, economic or economic statistical models are derived using a total cost function per unit time as proposed by a unified approach of the Lorenzen and Vance model (1986). In this approach the relationship between the three control chart parameters as well as the three types of costs are expressed in the total cost function. The optimal parameters are usually obtained by the minimization of the expected total cost per unit time. Nevertheless, few practitioners have tried to optimize the design of their $\bar{X}$-control charts. One reason for this is that the cost models and their associated optimization techniques are often too complex and difficult for practitioners to understand and apply. However, a user-friendly Excel program has been developed in this paper and the numerical examples illustrated are executed on this program. The optimization procedure is easy-to-use, easy-to-understand, and easy-to-access. Moreover, the proposed procedure also obtains exact optimal design values in contrast to the approximate designs developed by Duncan (1956) and other subsequent researchers.

Numerical examples are presented of both the economic and the economic statistical designs of the $\bar{X}$-control chart in order to illustrate the working of the proposed Excel optimal procedure. Based on the Excel optimization procedure, the results of the economic statistical design are compared to those of a pure economic model. It is shown that the economic statistical designs lead to wider control limits and smaller sampling intervals than the economic designs. Furthermore, even if they are more costly than the economic design they do guarantee output of better quality, while keeping the number of false alarm searches at a minimum. It also leads to low process variability. These properties are the direct result of the requirement that the economic statistical design must assure a satisfactory statistical performance.

Additionally, extensive sensitivity studies are performed on the economic and economic statistical designs to investigate the effect of the input parameters and the effects of varying the

bounds on, α, $1-\beta$, the average time-to-signal, ATS as well as the expected shift size δ on the minimum expected cost loss as well as the three control chart decision variables. The analyses show that cost is relatively insensitive to improvement in the type I and type II error rates, but highly sensitive to changes in smaller bounds on ATS as well as extremely sensitive for smaller shift levels, δ.

Note: expressions like *economic design*, *economic statistical design*, *loss cost* and *assignable cause* may seen linguistically and syntactically strange, but are borrowed from and used according the known literature on the subject.

Opsomming

Die ontwerp van kontrolekaarte verwys na die seleksie van die parameters geïmpliseer, insluitende die steekproefgrootte n, kontrole limiete interval parameter k, en die steekproefinterval h. Die ontwerp van die $\bar{X}$-kontrolekaart, gebaseer op ekonomiese sowel as statistiese oorwegings, is tans een van die meer populêre onderwerpe van navorsing. Twee aannames word in ag geneem in die ontwikkeling en gebruik van die ekonomiese en ekonomies statistiese modelle. Hierdie aannames is potensieel krities. Dit word aanvaar dat die tyd tussen prosesverskuiwings deur die eksponensiaalverdeling gemodelleer kan word. Daar word ook verder aangeneem dat daar slegs een oorsaak kan wees vir 'n verskuiwing, of te wel 'n *aanwysbare oorsaak* (*assignable cause*). Gebaseer op hierdie aannames word ekonomies en ekonomies statistiese modelle afgelei deur gebruik te maak van 'n totale kostefunksie per tydseenheid soos voorgestel deur deur 'n verenigende (*unified*) benadering van die Lorenzen en Vance-model (1986). In hierdie benadering word die verband tussen die drie kontrole parameters sowel as die drie tipes koste in die totale kostefunksie uiteengesit. Die optimale parameters word gewoonlik gevind deur die minimering van die verwagte totale koste per tydseenheid. Desnieteenstaande het slegs 'n minderheid van praktisyns tot nou toe probeer om die ontwerp van hulle $\bar{X}$-kontrolekaarte te optimeer. Een rede hiervoor is dat die kostemodelle en hulle geassosieerde optimeringstegnieke té kompleks en moeilik is vir die praktisyns om te verstaan en toe te pas. 'n Gebruikersvriendelike Excelprogram is egter hier ontwikkel en die numeriese voorbeelde wat vir illustrasie doeleindes getoon word, is op hierdie program uitgevoer. Die optimeringsprosedure is maklik om te gebruik, maklik om te verstaan en die sagteware is geredelik beskikbaar. Wat meer is, is dat die voorgestelde prosedure eksakte optimale ontwerp waardes bereken in teenstelling tot die benaderde ontwerpe van Duncan (1956) en navorsers na hom.

Numeriese voorbeelde word verskaf van beide die ekonomiese en ekonomies statistiese ontwerpe vir die $\bar{X}$-kontrolekaart om die werking van die voorgestelde Excel optimale prosedure te illustreer. Die resultate van die ekonomies statistiese ontwerp word vergelyk met dié van die suiwer ekomomiese model met behulp van die Excel optimerings-prosedure. Daar word aangetoon dat die ekonomiese statistiese ontwerpe tot wyer kontrole limiete en kleiner steekproefintervalle lei as die ekonomiese ontwerpe. Al lei die ekonomies statistiese ontwerp tot ietwat hoër koste as die ekonomiese ontwerpe se oplossings, waarborg dit beter kwaliteit terwyl dit die aantal vals seine tot 'n minimum beperk. Hierbenewens lei dit ook tot kleiner

prosesvariasie. Hierdie eienskappe is die direkte resultaat van die vereiste dat die ekonomies statistiese ontwerp aan sekere statistiese vereistes moet voldoen.

Verder is uitgebreide sensitiwiteitsondersoeke op die ekonomies en ekonomies statistiese ontwerpe gedoen om die effek van die inset parameters sowel as van variërende grense op α, $1-\beta$, die gemiddelde tyd-tot-sein, ATS sowel as die verskuiwingsgrootte δ op die minimum verwagte kosteverlies sowel as die drie kontrolekaart besluitnemingsveranderlikes te bepaal. Die analises toon dat die totale koste relatief onsensitief is tot verbeterings in die tipe I en die tipe II fout koerse, maar dat dit hoogs sensitief is vir wysigings in die onderste grens op ATS sowel as besonder sensitief vir klein verskuiwingsvlakke, δ.

Let op: Die uitdrukkings *ekonomiese ontwerp* (*economic design*), *ekonomies statistiese ontwerp* (*economic statistical design*), *verlies kostefunksie* (*loss cost function*) en *aanwysbare oorsaak* (*assignable cause*) mag taalkundig en sintakties vreemd voordoen, maar is geleen uit, en word so gebruik in die bekende literatuur oor hierdie onderwerp.

Dedication

This paper is dedicated with love to my wife Abeba Tesfai, my mother, Hanna Fesahatsion, and my Sister, Ghenet Ghebrertensae, for their encouragement, inspiration and endless love.

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Chapter One

Introduction

1.1. Background and problem statement

Statistical quality control is one of the more useful and economically important applications in the field of industry. The purpose of statistical quality control is to ensure, in a cost efficient manner, that the products shipped to customers meet their specifications. One of the important tools in statistical quality control is the statistical control chart technique, which may be considered as a graphical display of statistical hypothesis testing. It was developed in the 1920s by Dr. Walter A. Shewhart as a statistical approach to the study of manufacturing process variation for the purpose of improving the economic effectiveness of the process. The major function of control charting is to detect the occurrence of assignable causes so that the necessary corrective action may be taken before a large number of nonconforming products are manufactured.

Saniga and Shirland (1977) indicated that the $\bar{X}$-chart is used more often than any other control chart technique when quality is measured on a continuous scale. The effective use of control charts is largely dependent upon their design, that is, selection of the decision variables such as sample size, n, sampling period or sampling interval, h, and control limit parameter, k, based on some subjective and or objective criteria.

The problem of control chart design has received much attention because the design of the chart has economic implications. It involves various expenses, such as the costs of sampling and testing, costs associated with investigating out-of-control signals and possibly correcting assignable causes and costs of allowing nonconforming units to reach the customer. Since all these costs are affected by the choice of the three control chart parameters, it is reasonable to consider the design of $\bar{X}$-control charts from an economic viewpoint.

Consequently, several general methodologies have been developed in order to improve on the design suggested by Shewhart. Economic design and economic statistical designs are the most important designs that affect the cost and statistical considerations. The concept of an economic design was first introduced by Girshick and Rubin (1952). Although the optimal control rules in their model are too complex to have practical value, their work provided the basis for most cost-based models in control chart designs. Duncan (1956) developed the first economic design model and applied it to an $\bar{X}$-control chart. In the economic design, the objective is to determine the control chart parameters, i.e. the sample size, n, sampling interval, h, and control limit parameter, k that minimize the expected loss cost accrued by a production process. A considerable amount of research has been done in the economic design of various control charts after Duncan's paper. In 1986, Lorenzen and Vance provided a unified approach to the economic design of process control charts. They considered various options regarding continuation of production during search for or repair of the assignable cause. The economic statistical model was first proposed by Saniga (1989). The objective is to minimize the expected total cost per unit time, as in the economic design, subject to constraints on the average run lengths, ARL (or equivalently type I and type II error probabilities or average time-to-signal, ATS).

With regard to the economic and economic statistical designs of the $\bar{X}$-control chart, it can be said that very few practitioners have adopted optimization procedures in designing their $\bar{X}$-control charts. The main reason is that the cost models and their associated optimization techniques are often seen as too complex and difficult for practitioners to apply. Duncan (1956), Gibra (1971), Chiu and Wetherill (1974) and Montgomery (1982) developed the optimization procedures for determining optimal parameters for the $\bar{X}$-control chart. The methodologies described in these papers were difficult to use in practice. The optimization procedure presented by Lorenzen and Vance (1986) employed Newton's method, the golden section search as well as Fibonacci search methods. This could be the main reason that limits the application of their method. However, in this paper, we propose an alternative optimization procedure, which is a modification to the preceding optimization procedures. We develop a user-friendly Excel program that can be used to calculate the optimal parameter values based on a unified approach of the Lorenzen and Vance model for both economic and economic statistical designs of the $\bar{X}$-control chart. Hypothetical examples are used to show that the proposed optimization procedure works well. Based on this optimization procedure, comprehensive comparisons of the economic and economic statistical designs of the $\bar{X}$-control chart are made. Extensive sensitivity analyses are performed to investigate the effects of the key input parameter as well as the effect of varying the bounds on the probability of the type I error, α, the power, $1-\beta$, the average time-to-signal, ATS, and the expected shift size, δ, on the minimum expected cost as well as the three control chart decision variables n, h, and k.

1.2. Objective of the study

The objective of the study is to investigate the relevance and applicability of the economic aspects of statistical quality control and improvement in industry, by deriving a general methodology for the minimization of the expected cost function. In this process the optimum sample size, n, sampling period, h, and control limit parameter, k, for an $\bar{X}$-control chart is determined. A user-friendly, Excel program is developed to search for the optimal values of the parameters by minimizing the total cost function in both economic and economic statistical designs of the $\bar{X}$-control chart. Hypothetical examples are used to show that the proposed procedures do work, while also giving the optimal values for the parameters. Comprehensive comparisons of the economic and economic statistical designs of the $\bar{X}$-control chart are made with cost as well as statistical performance as criteria. Effects of the bounds for statistical and performance measures, such as type I error rate, α, the power, $1-\beta$, the average time-to-signal, ATS and shift sizes, δ are extensively investigated. This study will focus on the expected cost and the three decision variables with different bounds in α, $1-\beta$, ATS and δ.

1.3. Scope of the study

The study aims to describe and understand several models that have been developed and applied to most of the major types of control charts. To achieve this, an extensive investigation was conducted into the economic aspect of statistical quality control and improvement literature, including the practical implementation thereof. Whilst the emphasis of the study is on the economic and economic statistical design of the $\bar{X}$-control chart, comparisons and analyses of major economic models have also been made in order to review the relevance in dealing with statistical quality control and improvement.

1.4. Research methodology

The study was conducted by way of an extensive literature investigation of secondary sources of information, including books, journal articles, academic and professional conference proceedings, internet sources, and other reliable documents. The sources have been gathered by accessing library and internet searches. These led to the delineation of a number of relevant assumptions. Based on these assumptions, Duncan's (1956) single assignable cause model as well as the unified approach of the Lorenzen and Vance (1986) model are considered. Hourly cost functions are derived. An Excel program is developed to calculate the optimal values of sample size n, sampling interval h, and control limit width parameter k as well as the corresponding value of the minimum expected cost. Hypothetical examples are used to show that the proposed optimization procedures do work. Comprehensive comparisons as well as detailed sensitivity analyses are performed on the economic and economic statistical designs of the $\bar{X}$-control charts.

1.5. Organization of the study

The study is organized and presented in six chapters. Chapter one provides an overview of the study, objective of the study, scope of the study and research methodology.

Chapter two briefly discusses the economic design of the $\bar{X}$-control chart. In an attempt to understand and describe the assumptions, the notation of the $\bar{X}$-control chart is explained. This chapter looks in depth at the economic model, the total quality control cost, and the derivation of the loss cost function.

Chapter three discusses numerical approximation techniques that can be used to minimize the loss cost function. Duncan's approximation is derived and an iterative search technique is used to obtain the minimum. Two examples of the $\bar{X}$-control chart illustrate the solution procedure. From these examples, we perform a brief sensitivity analysis to compare the cost parameters and process parameters.

Chapter four discusses the economic and economic statistical designs of a unified approach as it is applied on the $\bar{X}$-control chart for controlling the process mean. In this section, further assumptions and notations will be given. The expected cost functions of the economic and economic statistical models are derived based on the assumptions stated and using the Lorenzen and Vance unified approach methodology. An Excel program is developed to calculate the optimal economic and economic statistical designs of the $\bar{X}$-control chart. Finally, a brief discussion of the optimization procedures is given.

Chapter five shows numerical examples of the economic and economic statistical designs. The results of the economic statistical design are compared to those of the economic model. An extensive sensitivity study of the economic and economic statistical designs is conducted on the input variables and the statistical constraints such as the average run length (ARL) and the average time-to-signal (*ATS*) to determine which are crucial and a discussion of the results is provided. Finally, the economic versus the economic statistical design of control charts is studied.

In Chapter six, conclusions and remarks are presented.

Chapter Two

Economic design of the $\bar{X}$-control chart

2.1. Introduction

In 1924 W.A. Shewhart introduced a new method for controlling the quality of a production process, namely the control chart, more specifically, the $\bar{X}$-control chart. The most general control chart methodology consists of sampling from a process and evaluating the samples in order to find a signal that the considered production process is out-of-control. Whenever such a signal is observed the process of searching and removing the assignable causes takes place. Implementation of control charts requires a number of technical and behavioural decisions. One important technical decision comprises the design of the $\bar{X}$-control chart. Designing a control chart means making fundamental decisions about chart parameters such as the sample size, n, sampling interval, h, and control limit width, k.

Traditionally, control charts are often designed with respect to statistical criteria only. This usually involves selecting the sample size and control limits in such a way that the capability of the control chart to detect a particular shift in the quality characteristic and the type I error probability are equal to specified values. The frequency of sampling is rarely determined by analytic methods. The practitioner is advised to consider such factors as the production rate, the expected frequency of shifts to an out-of-control state, and the possible consequences of such process shifts in selecting the sampling interval. The use of statistical criteria such as these along with industrial experience has led to general guidelines and procedures for designing control charts. These procedures usually consider cost factors only in an implicit manner. Recently, however, interest has been aroused in examining control chart design from an economic point of view, considering all the costs explicitly (Montgomery, 2001).

Economic design, that is a design that is based on an economic criterion is one of the popular approaches in today's control chart design. The objective is to determine the control chart parameters i.e. the sample size, n, sampling interval, h, and control limit width, k that minimize the expected loss cost accrued by a production process. Duncan (1956) developed the first model and applied it to an $\bar{X}$-control chart. In this model Duncan assumed that one monitor the process to detect the occurrence of a single assignable cause that causes a fixed shift in the process, and then defines the relevant costs over a specified cycle.

This chapter is organized as follows. It begins by defining the results of previous research in section 2.2. Assumptions and notations will be given in section 2.3. In Section 2.4, the economic model of the control chart will be analysed. In section 2.5, the quality control cost will be defined and the formulae will be derived for the expected loss function in section 2.6.

2.2. Previous work

After the introduction of control charts in 1924 by W.A. Shewhart it took more than 25 years until the first approaches for determining the control chart parameters according to economic criteria appeared in statistical literature. The first who proposed such a procedure were the Americans, Aroian and Levene (1950). They noted that what matters is not the probability of a false alarm per sample, but the frequency of false alarms, which also depends on the time interval between samples. They assumed a process which operates in exactly two states, a desirable state and an undesirable state. Their aim was to minimize the number of product units produced in the undesirable state. In a first approximation this number can be regarded as a measure of costs connected with the production of nonconforming units. The new element in their approach was that instead of the false alarm probability, they chose the average time between two false alarms as a side condition. Through this, it was possible to determine the control limits as a function of the admissible frequency of false alarms and the time interval between samples. The inspection costs, however were not considered in this approach.

Weirler (1952) proposed a model to minimizes the average amount of inspection until discovery of a process shift of magnitude, $\delta\sigma$ with the sample size as the only decision criterion and σ being the process standard deviation. For example, in case of determining the control limits by using $\alpha = 0.01$, he obtained an optimal sample size of $n = 4.4\delta^2$. His approach totally neglects the fact that the time interval between samples and the probability of detecting a process shift directly influence the average run length of the process in an undesired state, and thus also the costs related to the production of nonconforming units.

The next progress was presented by Pfanzagl (1954). His model had - due to the ideas of Aroian and Levene - a lower limit for the time until the occurrence of a false alarm, but also an upper limit for the average run length in an undesirable state. What was still not considered were the costs related to a false alarm as well as the costs related to the production of defective units, which arise while the production process is in an undesirable state. He also ignored the effects of the frequency of shifts between the two process states (Mittag and Rinne, 1993).

Mittag and Rinne (1993) noted that a major breakthrough to a full consideration of all the previously mentioned factors came through the work of Duncan (1956). Duncan investigated models with only one undesirable state and searched for control strategies which aimed at minimizing the average total cost per time unit of the respective production and control system. Over this period many theoretical treatises on minimal cost process control, have been written. An overview of work up to 1980 is given by Montgomery (1980), whereas Lorenzen and Vance (1986) provide one unified approach to the economic design of process control charts. They considered a general process model that applied to all control charts, regardless of the statistics used. A by-product of their effort is a unification of the notation used. Their model included twelve cost and operating parameters, two indicator variables, which determine whether the manufacturing activities continue during the search or repair stage, and three control chart design parameters (subgroup size, sampling interval, and width of the control limits), which need to be optimized in order to minimize the hourly-based expected loss. Two assumptions were discussed. One was the use of the memoryless exponential distribution for time in control. The other was the assumption of a single assignable cause and a shift of a known amount. A numerical technique was presented to minimize the cost function. An example was given and a sensitivity analysis was conducted. Lorenzen and Vance (1986) found that the expected minimum loss per hour is sensitive to the change in magnitude of the process mean shift, δ, but that the sampling plan itself is not sensitive to the change, δ. Therefore, a crude approximation of the process parameters can be made to design a good sampling plan.

Collani (1988) also proposed a unified approach to the optimal design of process control charts. However, he adopted a different approach. The stated emphasis in the paper is on "simplicity" and "generality". In his model, the process is assumed to operate under one of two states. State-I represents "satisfactory", in which no corrective action is thought to be necessary. State-II represents "unsatisfactory", in which a corrective action is thought to be necessary. Three different policies (monitoring, inspection, and renewal/replacement) were defined and incorporated into his model. The model can easily be generalized to explicitly include further activities, for instance, repair actions. One example using the $\bar{X}$-chart was given. The objective was to find the optimal design parameters (the interval between sampling, the subgroup size, and the control limit) in order to maximize the net profit per item produced. Another example assumed that the state of the process was known at all times, making monitoring and inspection unnecessary. Thus the focus was on the renewal interval. Collani's approach unifies different theories, reduces the number of input variables, results in a simpler objective function and permits an approximation algorithm to be used.

2.3. Assumptions and notation

2.3.1. Assumptions

According to Montgomery (2001) to formulate an economic model for the design of a control chart, it is necessary to make certain assumptions about the behaviour of the process.

(A1) The distribution of the quality characteristics of the process output is normal. Applying the central limit theorem, this result is still approximately true for the $\bar{X}$-chart even if the underlying distribution is not normal.

(A2) The process is started in the in-control state with mean μ_0 and standard deviation σ. The occurrence of the assignable cause results in a shift in the process mean from μ_0 to $\mu_0 \pm \delta\sigma$, where $\delta > 0$. When the $\bar{X}$-control chart identifies the shift of the process mean, it is restored to the 'in-control' state by repairing and eliminating the cause.

(A3) The process is assumed to be characterized by a single in-control state and each out-of-control state is usually associated with a particular type of assignable cause.

(A4) It is a customary to assume that the assignable cause occurs during an interval of time according to the Poisson process. This implies that the length of time the process remains in the in-control state, given that it begins in the in-control state, is an exponential random variable with mean $\frac{1}{\lambda}$, i.e. λ is the mean number of shifts from the in-control to the out-of-control state per unit time. This implies a memoryless process.

(A5) The transition from the in-control state to the out-of-control state is irreversible. That is, once a transition to an out-of-control state has occurred, the process can be returned to the in-control condition only by management intervention.

(A6) The process is monitored by random samples of size n at time h, $2h$, $3h$, and so forth. The control limits of the $\bar{X}$-chart are set at $\mu_0 + \frac{k\sigma}{\sqrt{n}}$, where k is a multiple of the standard deviation of the sample mean. In contrast to k, δ, in assumption A2 is a multiple of the standard deviation of the process (Chiu and Huang, 1996).

2.3.2. Notation

In order to formulate the cost function, we should consider the following important notation. The parameters can be classified into three categories.

1) Design parameters

n sample size

k control limit width parameter

h sampling interval

2) Process parameters

δ magnitude of the process mean shift expressed in process standard deviation units

α type I error probability of the chart $= P($exceeding control limits | process in-control)

β type II error probability of the chart $= P($not exceeding control limits | process out-of-control)

$1-\beta = p$ power of the chart

3) Cost and time parameters

a the fixed sampling cost

b the variable sampling cost per sample unit

$a+nb$ the cost of taking and inspection of samples of size n, where a and b are the fixed and variable sampling costs, respectively

a_3 the cost of finding an assignable cause given a signal

a_4 the hourly penalty cost of operating out-of-control

a_5 the cost of investigating a false alarm

g the average sampling, inspection, evaluation and plotting time for each sample unit

gn the time used to test and interpret the results for a point $\bar{X}$ that falls outside a control limit

D the time needed to discover the assignable cause following an action signal

$E(I)$ the expected net income per cycle

A the expected net income per unit time

$E(C)$ the expected total cost incurred during a cycle

$E(T)$ the expected length of a cycle

$E(L)$ the expected cost of quality per unit time

2.4. Economical model

The usual approach to the economic design of the control chart is to specify a normal distribution for the manufacturing process, estimate the relevant process and cost parameters, and then minimize the expected cost per time unit, $E(L)$, which is derived from the process model and parameters. An economic model is generally formulated using a total cost function, which expresses the relationships between the control chart design parameters and the three types of costs (Montgomery, 2001).

The production, monitoring, and process adjustment with the help of a cost minimization $\bar{X}$-chart with sampling interval h can be seen as a series of independent renewal cycles. Each new cycle starts when the process switches back to production in the in-control state. The production then goes on until the control charts indicate that the process has shifted to the out-of-control state. This signal causes a process adjustment (readjustment to in-control state), that concludes the present cycle.

The length of the ith renewal cycle is a random variable and so are the total costs related to the ith cycle $(i = 1,2,3,\ldots.)$. Usually in this model, the cycle lengths as well as the total costs per cycle are defined to be independently, identically distributed and because of this, we can pick an arbitrary cycle for further analysis.

Let the length of the cycle be denoted by T and the related cost by C. Then the ratio $\frac{C}{T}$ can be interpreted as the total costs per time unit. The goal of economic process control models usually is to minimize the expectation, $E(\frac{C}{T})$ of this cost variable. In our model $E(\frac{C}{T})$ can be approximated by (Ross, 2000)

$$E(L) = \frac{E(C)}{E(T)}. \qquad (2.1)$$

Monitoring is carried out by taking successive samples at fixed sampling intervals. Corrective action is taken whenever a sample average falls outside the interval bounded by upper and lower control limits. The process starts in a state of statistical control and is allowed to continue in operation during the search for the assignable cause.

In order to formulate a realistic cost function we need to derive the following characteristics.

(1) The time from the start of the jth sample interval until the process goes out of control is denoted by τ. Duncan (1971) showed that the probability distribution of τ is

$$f_\tau(t) = \frac{\lambda \exp(-\lambda t)}{1-\exp(-\lambda h)}, \qquad 0 \le t \le h. \qquad (2.2)$$

Assume a fixed interval length and let the assignable cause occur in the jth interval. Then the average time of the occurrence of the assignable cause within the sample interval is

$$E(\tau) = \int_0^h \frac{t\lambda \exp(-\lambda t)dt}{1-\exp(-\lambda h)}$$

$$= \frac{1}{1-\exp(-\lambda h)} \int_0^h t \exp(-\lambda t)dt$$

If $u = -\lambda t$ and $du = -\lambda dt$ then this becomes

$$\frac{\lambda}{1-\exp(-\lambda h)} \int_0^h \frac{u \exp(u) du}{\lambda^2}$$

$$= \frac{1}{\lambda(1-\exp(-\lambda h))} \int_0^h u \exp(u) du$$

$$= \frac{1-(1+\lambda h)\exp(-\lambda h)}{\lambda(1-\exp(-\lambda h))}. \qquad (2.3)$$

(2) Montgomery (2001) stated α as the probability of a point falling outside the control limits when the process is in control. Then

$$\alpha = 2\Phi(-k)$$

$$= 2\int_k^\infty \phi(z)dz \text{ where } \phi(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}.$$

Similarly, when an assignable cause non-conforming event has occurred, the probability that it will be detected on the subsequent sample is

$$\begin{aligned}(1-\beta) &= \Phi(\delta\sqrt{n}-k)+\Phi(-\delta\sqrt{n}-k)\\ &= \int_{-\infty}^{-k-\delta\sqrt{n}}\phi(z)dz+\int_{k-\delta\sqrt{n}}^{\infty}\phi(z)dz\\ &\approx \int_{k-\delta\sqrt{n}}^{\infty}\phi(z)dz\end{aligned} \tag{2.4}$$

since the first integral is very close to zero where $1-\beta$ is the power of the test.

(3) After the occurrence of the assignable cause, the probability that it will be detected on the jth-inspected sample is given by $\beta^{j-1}(1-\beta)$, (Gibra, 1971). Therefore, the expected number of samples taken before the assignable cause is detected, is given by

$$\begin{aligned}&\sum_{j=1}^{\infty} j\beta^{j-1}(1-\beta)\\ &=\frac{1}{1-\beta},\end{aligned} \tag{2.5}$$

which is the expected value of the geometric distribution.

(4) As stated by Montgomery (2001), the expected number of the false alarms that will occur before a shift is α times the expected number of samples taken before the shift, i.e.

$$\alpha\sum_{j=0}^{\infty}\int_{jh}^{(j+1)h} j\lambda e^{-\lambda t}dt\,.$$

Let's first resolve the integral $\int_{jh}^{(j+1)h}\lambda e^{-\lambda t}dt$, i.e.

$$\begin{aligned}&\int_{jh}^{(j+1)h}\lambda e^{-\lambda t}dt\\ &=\left(e^{-j\lambda h}-e^{-(j+1)\lambda h}\right),\end{aligned}$$

therefore

$$\begin{aligned}&\alpha\sum_{j=0}^{\infty}\int_{jh}^{(j+1)h} j\lambda e^{-\lambda t}dt\\ &=\alpha\sum_{j=0}^{\infty} j(e^{-j\lambda h}-e^{-(j+1)\lambda h})\end{aligned}$$

$$=\alpha\left(1-e^{-\lambda h}\right)\sum_{j=0}^{\infty} je^{-j\lambda h}$$

$$=-\alpha\left(1-e^{-\lambda h}\right)\frac{\partial}{\partial\lambda}\frac{1}{h}\sum_{j=0}^{\infty} e^{-j\lambda h}.$$

Furthermore

$$\sum_{j=0}^{\infty} e^{-j\lambda h}$$

$$=1+e^{-\lambda h}+e^{-2\lambda h}+e^{-3\lambda h}+.......$$

$$=\frac{1}{1-e^{-\lambda h}}$$

from which follows that

$$\frac{\partial}{\partial\lambda}\frac{1}{h}\left(\frac{1}{1-e^{-\lambda h}}\right)$$

$$=\frac{-e^{-\lambda h}}{\left(1-e^{-\lambda h}\right)^2}.$$

Thus, putting it all together the expected number of false alarms as described above is given by

$$-\alpha(1-e^{-\lambda h})\frac{\partial}{\partial\lambda}\frac{1}{h}\sum_{j=0}^{\infty} e^{-j\lambda h}$$

$$=\frac{\alpha e^{-\lambda h}}{1-e^{-\lambda h}}$$

$$=\frac{\alpha}{e^{\lambda h}-1}. \tag{2.6}$$

2.5. Quality control cost

Montgomery (2001) explored the control chart design from an economic point of view, considering three categories of cost explicitly. These categories are the costs of sampling and testing, the costs associated with investigating out-of-control signals including the repair or correction action of any assignable causes found, and the costs associated with the production of nonconforming items.

(1) The cost of sampling and testing includes the out of pocket expenses of inspectors and technicians' salaries and wages. Montgomery (2001) indicated that the cost consists of both fixed and variable components, say a and b, respectively, such that the total cost of sampling and testing $a+bn$.

(2) The cost of investigating and possibly correcting the process following an out-of-control signal, has been modelled in several ways. Some authors have suggested that the cost of investigating false alarms will differ from the cost of correcting assignable causes and, consequently these two situations must be represented in the model by different cost coefficients (Montgomery, 2001). Furthermore, the cost of repairing or correcting the process could depend on the type of assignable cause present. Thus, in models having s out-of-control states, $s+1$ cost coefficients might be necessary to model the search and adjustment procedure associated with out of control signals. Usually, these cost coefficients would be chosen so that large process shifts incurred large costs of repair or adjustment. Other authors have argued that this level of modelling detail is unnecessary because in many cases small shifts are difficult to find but easy to correct, whereas large shifts are easy to find but difficult to correct. Thus, one would lose little accuracy by a single cost of investigating and possibly correcting the process following an action signal.

(3) The costs associated with the production of defective items consist of typical failure costs i.e. the costs of rework or scrap for internal failures, or the out of pocket costs of replacement or repair for units covered by warranties in the case of external failures (Montgomery, 2001). In the case of external failures there may also be secondary effects from production of non-conforming items if the customer's dissatisfaction with the product causes him to alter his pattern of purchasing the product or other products manufactured by the company. Finally, there may be losses resulting from product liability claims against the company. In most cases these costs are presented by a single average cost coefficient, expressed on either per unit time or per item basis.

2.6. Formulation of the loss cost function

Duncan (1956) assumed that the process starts under an in-control condition and is subject to random shifts in the process mean. Once a shift occurs, the process remains there until it is corrected. The cycle length is defined as the total period from when the process starts in-control, to when it shifts to an out-of-control condition, to when the out-of-control condition is detected, which results in the assignable cause being identified. These four time intervals are, respectively, the interval during which the process is in-control, the interval during which the process is out-of-control before the final sample of the detecting subgroup is taken, the interval used to sample, inspect, evaluate and plot the subgroup results, and the interval used to search for the assignable cause. When the average cycle length is determined, the cost components can be converted to a "per hour of operation" basis. Given associated cost and time parameters, the optimal values of the three decision parameters for the model can then be determined by using optimization techniques.

2.6.1. Duration of a production cycle

In the economic model of Montgomery (2001) the expected production cycle length for the model was derived from the following four time periods:

(1) Assuming that the process begins in the in-control state, the time interval during which the process remains in control is an exponential random variable with mean $\frac{1}{\lambda}$, which is the average process in-control time.

(2) When an assignable cause occurs, the probability that this out-of-control condition will be detected on any subsequent sample is $(1-\beta)$, which is the power of the control chart. Thus, the expected number of subgroups taken before a shift in the process mean is detected is $\frac{1}{1-\beta}$. The average time, $E(\tau)$ of occurrence of a shift within an interval between the jth and $(j+1)$st subgroups, given an occurrence of a shift in the interval between these subgroups, is given by

$$E(\tau) = \frac{\int_{jh}^{(j+1)h} \lambda(t-hj)e^{-\lambda t}\,dt}{\int_{jh}^{(j+1)h} \lambda e^{-\lambda t}\,dt}$$

$$= \frac{1-(1+\lambda h)\exp(-\lambda h)}{\lambda(1-\exp(-\lambda h))}.$$

Therefore, the expected length of the out-of-control period is

$$\frac{h}{1-\beta} - E(\tau).$$

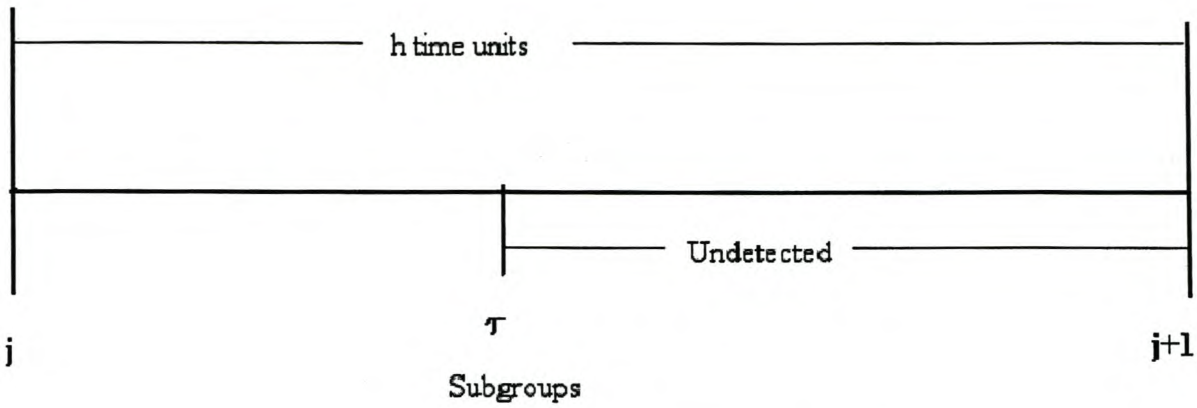

Figure 2.1: Average time of occurrence of a shift in the interval

(3) The average sampling, inspecting, evaluating, and plotting time for each sample is a constant g proportional to the sample size n, so that the delay in plotting a subgroup point on the $\bar{X}$-control chart is gn.

(4) The time needed to find the assignable cause following an action signal is a constant D.

Therefore, the expected length of a cycle, denoted by $E(T)$, is

$$E(T) = \frac{1}{\lambda} + \frac{h}{1-\beta} - E(\tau) + gn + D. \tag{2.7}$$

2.6.2. The expected net income

The net income per hour of operation in the in control state is defined as V_0, and the net income per hour of operation in the out-of-control state is V_1. The cost of taking a sample of size n is assumed to be of the form $a+bn$; i.e. a and b represent the fixed and variable components, respectively of the sampling cost. The expected number of samples taken within a cycle is the expected cycle length divided by the time interval between samples, $\frac{E(T)}{h}$. Montgomery (2001) denotes the cost of finding an assignable cause by a_3, and the cost of investigating a false alarm by a_5. The expected number of false alarms generated during the cycle is α times the expected number of samples taken before the shift, or

$$=\frac{\alpha\exp(-\lambda h)}{1-\exp(-\lambda h)}, \text{ from (2.6).}$$

Therefore, the net income per cycle is

$$E(I)=V_0\frac{1}{\lambda}+V_1\left(\frac{1}{\lambda}+\frac{h}{1-\beta}-E(\tau)+gn+D\right)-a_3-\frac{\alpha a_5\exp(-\lambda h)}{1-\exp(-\lambda h)}-(a+bn)\frac{E(T)}{h}. \qquad (2.8)$$

The expected net income per cycle for the process model of the above expression (2.8) consists of the net income when the process is in control and the net income when the process is out of control. It also includes the cost of finding of an assignable cause when it exists, the expected cost of examining false alarms per cycle, and the expected cost of taking n samples per cycle.

Chapter Three

Minimization of the loss cost function

3.1. The expected loss cost function

The expected net income per cycle, divided by the expected cycle length, is denoted by A and thus given by

$$A = \frac{E(I)}{E(T)}.$$

Applying the property of the renewal reward process (Ross, 2000), the expected net-income per hour is given by

$$\begin{aligned} A &= \frac{V_0 \frac{1}{\lambda} + V_1\left(\frac{h}{1-\beta} - E(\tau) + gn + D\right) - a_3 - \frac{a_5 \alpha \exp(-\lambda h)}{1-\exp(-\lambda h)} - E(T)\frac{(a+bn)}{h}}{\frac{1}{\lambda} + \frac{h}{1-\beta} - E(\tau) + gn + D} \\ &= \frac{V_0 \frac{1}{\lambda} + V_1\left(\frac{h}{1-\beta} - E(\tau) + gn + D\right) - a_3 - \frac{a_5 \alpha \exp(-\lambda h)}{1-\exp(-\lambda h)}}{\frac{1}{\lambda} + \frac{h}{1-\beta} - E(\tau) + gn + D} - \frac{(a+bn)}{h} \end{aligned} \tag{3.1}$$

Let $a_4 = V_0 - V_1$, i.e., a_4 represents the hourly penalty cost associated with production in the out-of-control state, and then equation (3.1) can be rewritten as:

$$= \frac{V_0 \frac{1}{\lambda} + (V_0 - a_4)\left(\frac{h}{1-\beta} - E(\tau) + gn + D\right) - a_3 - \frac{a_5 \alpha \exp(-\lambda h)}{1-\exp(-\lambda h)}}{\frac{1}{\lambda} + \frac{h}{1-\beta} - E(\tau) + gn + D} - \frac{(a+bn)}{h}$$

$$= \frac{V_0\left(\frac{1}{\lambda} + \frac{h}{1-\beta} - E(\tau) + gn + D\right) - a_4\left(\frac{h}{1-\beta} - E(\tau) + gn + D\right) - a_3 - \frac{a_5 \alpha \exp(-\lambda h)}{1-\exp(-\lambda h)}}{\frac{1}{\lambda} + \frac{h}{1-\beta} - E(\tau) + gn + D}$$

$$- \frac{a+bn}{h}$$

$$= V_0 + \frac{-a_4\left(\frac{h}{1-\beta} - E(\tau) + gn + D\right) - a_3 - \frac{a_5 \alpha \exp(-\lambda h)}{1-\exp(-\lambda h)}}{\frac{1}{\lambda} + \frac{h}{1-\beta} - E(\tau) + gn + D} - \frac{(a+bn)}{h}.$$

Furthermore $A = V_0 - E(L)$ where

$$E(L) = \frac{a_4\left(\frac{h}{1-\beta} - E(\tau) + gn + D\right) + a_3 + \frac{a_5 \alpha \exp(-\lambda h)}{1-\exp(-\lambda h)}}{\frac{1}{\lambda} + \frac{h}{1-\beta} - E(\tau) + gn + D} + \frac{(a+bn)}{h}, \tag{3.2}$$

i.e. $E(L)$ represents the expected loss per hour incurred by the process. $E(L)$ is a function of the control chart parameters n, h, and k. Clearly, maximizing the expected net income per hour is equivalent to minimization of $E(L)$. Montgomery (2001) noted that V_0 is independent of these variables.

3.2. Duncan's approximation

For mathematical simplicity and practical convenience Duncan (1956) introduced several approximations to develop an optimization procedure in the actual structure of the model. It can be shown that theorems 3.1 and 3.2 hold (see appendix (A.1), (A.2)) (Chung, 1995).

Theorem 3.1

$$0 < \frac{1}{\lambda h} - \frac{1}{e^{\lambda h} - 1} < 1, \qquad \text{for } \lambda > 0, \text{ and } h > 0$$

Theorem 3.2

$$\lim_{x \to 0^+}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \frac{1}{2}$$

Theorem 3.1 tells us that the difference between $\frac{1}{\lambda h}$ and $\frac{1}{e^{\lambda h} - 1}$ is within the range between 0 and 1 for any λh. Duncan (1956) showed that the expected number of samples taken during an in-control period is given by $\frac{1}{e^{\lambda h} - 1}$. So, from the point of view of the expected number of samples taken during the in-control period, the number, $\frac{1}{\lambda h}$ in general is a good approximation to $\frac{1}{e^{\lambda h}-1}$, as the error is less than one sample.

Thus as

$$E(L)=\frac{a_4\left(\frac{h}{1-\beta}-E(\tau)+gn+D\right)+a_3+\frac{a_5\alpha\exp(-\lambda h)}{1-\exp(-\lambda h)}}{\frac{1}{\lambda}+\frac{h}{1-\beta}-E(\tau)+gn+D}+\frac{(a+bn)}{h}.$$

Let $p=1-\beta$, and approximate $E(\tau)$ by $\frac{h}{2}-\frac{\lambda h^2}{12}$ (Duncan, 1956) as well as the expected number of false alarms by

$$\begin{aligned}&\frac{\exp(-\lambda h)}{1-\exp(-\lambda h)}\\&=\frac{1}{\exp(\lambda h)-1}\\&\approx\frac{1}{\lambda h}.\end{aligned}$$

Now it follows that $E(L)$ can be expressed as

$$E(L)=\frac{a_4\left(\frac{h}{p}-\frac{h}{2}+\frac{\lambda h^2}{12}+gn+D\right)+a_3+\frac{\alpha a_5}{\lambda h}}{\frac{1}{\lambda}+\frac{h}{p}-\frac{h}{2}+\frac{\lambda h^2}{12}+gn+D}+\frac{a}{h}+\frac{bn}{h} \tag{3.3}$$

where $B=Ch+gn+D$ and $C=\frac{1}{p}-\frac{1}{2}+\frac{\lambda h}{12}$.

After further simplification $E(L)$ becomes

$$E(L)=\frac{\lambda a_4 B+\frac{\alpha a_5}{h}+\lambda a_3}{1+\lambda B}+\frac{a}{h}+\frac{bn}{h}. \tag{3.4}$$

3.3. Finding an approximation to the optimal design

A numerical study by Duncan (1956) of the function $E(L)$ suggests that for realistic values of the parameters δ, λ, a, b, a_3, a_4, a_5, g and D, a local minimum does exist in the neighbourhood of values of n, h, and k.

First, let us note what relationships must exist between the optimum values of n, h, and k (Duncan, 1956). Setting equal to zero the partial derivatives of $E(L)$ with respect to n, h, and k (noting that p is a function of n and k, and α is a function of k and for the moment treating n as if it were continuous), we get (see appendix (A.3), (A.4), and (A.5))

$$\frac{\partial E(L)}{\partial n} = \lambda h \frac{\partial B}{\partial n}\left(a_4 - \frac{\alpha a_5}{h} - \lambda a_3\right) + b(1+\lambda B)^2 = 0 \tag{3.5}$$

with $\frac{\partial B}{\partial n} = -\frac{h\frac{\partial P}{\partial n}}{P^2} + g$,

$$\frac{\partial E(L)}{\partial h} = \lambda h^2 \frac{\partial B}{\partial h}\left(a_4 - \frac{\alpha a_5}{h} - \lambda a_3\right) - \alpha a_5(1+\lambda B) - (a+bn)(1+\lambda B)^2 = 0 \tag{3.6}$$

with $\frac{\partial B}{\partial h} = \left(\frac{1}{P} - \frac{1}{2} + \frac{\lambda h}{6}\right)$, and

$$\frac{\partial E(L)}{\partial k} = \lambda \frac{\partial B}{\partial k}\left(a_4 - \frac{\alpha a_5}{h} - \lambda a_3\right) + \frac{a_5}{h}\frac{\partial \alpha}{\partial k}(1+\lambda B) = 0 \tag{3.7}$$

with $\frac{\partial B}{\partial k} = -\frac{h\frac{\partial P}{\partial k}}{P^2}$.

Thus, equations (3.5), (3.6), (3.7) do not give us simple expressions for evaluating the optimum n, h, and k. We need to approximate the numerical values for the given parameters, e.g. for small α, say 0.003, and with h equal to 1 or 2, and keeping in mind that λ is small the terms like $\frac{\alpha a_5}{h}$ and λB may be neglected. Duncan (1956) has approximated equations (3.5)-(3.7), by assuming λ small and neglecting all terms in an equation of a smaller order of magnitude than the principal term. This gives us (see appendix A.6, A.7, and A.8)

$$-\frac{\lambda h^2 a_4 \frac{\partial P}{\partial n}}{P^2} + b \approx 0 \tag{3.5'}$$

$$\lambda h^2 a_4\left(\frac{1}{P} - \frac{1}{2}\right) - \alpha a_5 - a - bn \approx 0 \tag{3.6'}$$

$$-\frac{\lambda h^2 a_4 \dfrac{\partial P}{\partial k}}{P^2} + a_5 \frac{\partial \alpha}{\partial k} \approx 0. \tag{3.7'}$$

Equation (3.6') immediately gives us (see appendix A.9)

$$h \approx \sqrt{\frac{\alpha a_5 + a + bn}{\lambda a_4 \left(\dfrac{1}{P} - \dfrac{1}{2}\right)}}. \tag{3.8}$$

Using this approximate value of h in (3.5') we get after some rearrangement (see appendix A.10),

$$-n + \frac{P^2\left(\dfrac{1}{P} - \dfrac{1}{2}\right)}{\dfrac{\partial P}{\partial n}} \approx \frac{\alpha a_5 + a}{b}. \tag{3.9}$$

When equation (3.7') is combined with equation (3.5'), and by substituting (see appendix A.12, A.13, and A.14)

$$\frac{\partial P}{\partial k} = -\frac{e^{\frac{-(k-\delta\sqrt{n})^2}{2}}}{\sqrt{2\pi}}, \quad \frac{\partial P}{\partial n} = \frac{\delta}{2\sqrt{n}}\left(\frac{e^{\frac{-(k-\delta\sqrt{n})^2}{2}}}{\sqrt{2\pi}}\right) \text{ and } \frac{\partial \alpha}{\partial k} = -\frac{2e^{\frac{k^2}{2}}}{\sqrt{2\pi}}.$$

Finally, it is found that (see appendix A.15)

$$\frac{\partial \alpha}{\partial k} = -\frac{2b\sqrt{n}}{\delta a_5}$$

or

$$\frac{e^{\frac{-k^2}{2}}}{\sqrt{2\pi}} = \frac{b\sqrt{n}}{\delta a_5}. \tag{3.10}$$

$E(L)$ is a function of the process parameters (δ, λ, α), the cost parameters (a, b, a_3, a_4, a_5, g, D), and the sampling and charting parameters (n, h, k). This function must be minimized in order to pursue the economic goal, whereas the statistical objectives are reached by minimizing α, and maximizing $1-\beta$.

The optimization procedure suggested is based on solving for a numerical approximation to the system for the first partial derivatives of $E(L)$ with respect to n, h, and k. An iterative procedure is required to solve for the optimal n and k and a closed form solution for h is given using the optimal value of n and k (Duncan, 1956).

Montgomery (2001) noted that various authors have reported optimization methods for Duncan's model. Chiu and Wetherill (1974) have developed a simple, approximate procedure for optimization of Duncan's model. They noted that by constraining the power of the chart $1-\beta$ to a specified value (say 0.90 or 0.95) the optimal value of n and k can be approximated by the solution (see appendix (A.16) and (A.17))

$$\delta\sqrt{n}-k=z \tag{3.11}$$

$$\frac{z+k}{\phi(k)}=\frac{\delta^2 a_5}{b+\lambda a_4 g} \tag{3.12}$$

where z = 1.28264 if $(1-\beta)$ = 0.90 and z = 1.6449 if $(1-\beta)$ = 0.95 and $\phi(\mu)$ is the density function of a standard normal random variable. The program uses z = 1.28264 to solve (3.11) and (3.12). The resulting n, say n^*, from (3.11) is used to set lower and upper limits in the search for the optimal sample size.

It was also noted that $E(L)$ could be minimized by using an unconstrained optimization or search technique. This is the approach to optimization most subsequent researchers have taken. Pattern search and various modifications of the Fibonacci search approach have been used effectively.

3.4. An example and its solution

In this section, a hypothetical example is presented to demonstrate the proposed approach. The model parameters and the solution obtained are based on the assumption of normality.

Example: 3.1

To illustrate the solution procedure of the proposed model, the following industrial example whose process and cost parameters borrow directly from Chou, Chen, & Liu, (2000) is presented. A plant manufactures packed orange juice that has a "quantity of content" specification of 250 cc with a tolerance of ±0.3 cc. From past data, the process standard deviation is estimated as 0.1 cc. Process shifts occur at random with a frequency of about one every 20 hours of operation ($\lambda = 0.05$). The manufacturer uses an $\bar{X}$-chart to monitor the process. Based on an analysis of quality-control technicians' salaries and the costs of test equipment, it is estimated that the fixed cost of taking a sample is R1 (i.e., $a = 1$). The estimated variable cost of sampling is about R0.10 per quantity of content (i.e., $b = 0.10$) and it takes approximately 1 min (i.e., $g = 0.0167$) to measure and record the quantity of content of a bottle of orange juice. On average, when the process goes out of control, the magnitude of the shift is approximately two standard deviations ($\delta = 2.0$). The average time required to investigate an out-of-control signal is one hour (i.e., $D = 1$). The cost of investigating an action signal that results in the elimination of an assignable cause is R25 while the cost of investigating a false alarm is R50 (i.e., $a_3 = 25$ and $a_5 = 50$). The manufacturer estimates that the penalty cost of operating in the out-of-control state for 1 hour is R100 (i.e. $a_4 = 100$).

Table 3.1: Output solutions for example 3.1

n	***k***	***h***	***α***	***β***	**1-*β***	***E(L),R/hr***
1	2.5	0.4	0.012419	0.691459	0.308541	14.84325
2	2.5	0.6	0.012419	0.371294	0.628706	11.89537
3	2.7	0.7	0.006934	0.222403	0.777597	10.89046
4	2.8	0.8	0.005110	0.115070	0.884930	10.49857
5	**3.0**	**0.8**	**0.002700**	**0.070492**	**0.929508**	**10.37085**
6	3.1	0.9	0.001935	0.036011	0.963989	10.38797
7	3.1	0.9	0.001935	0.014208	0.985792	10.48236
8	3.1	0.9	0.001935	0.005281	0.994719	10.62949
9	3.1	1.0	0.001935	0.001866	0.998134	10.79204
10	3.1	1.0	0.001935	0.000631	0.999369	10.95704
11	3.1	1.0	0.001935	0.000205	0.999795	11.12535
12	3.1	1.0	0.001935	0.000065	0.999935	11.29476
13	3.1	1.1	0.001935	0.000020	0.999980	11.45956
14	3.1	1.1	0.001935	0.000006	0.999994	11.61987
15	3.1	1.1	0.001935	0.000002	0.999998	11.78011

The goal of the economic design of the control chart is to find n, h, and k that minimize $E(L)$. Using a hand calculator this is an extremely difficult if not impossible task. The appropriate level of computation is the personal computer. Therefore, using the spreadsheet for a certain combination of n, h, and k, the program calculates the corresponding α risk and power. The output from this program, using the values of the model parameters given in the example, is shown in Table 3.1. The program calculates the optimal control limit width k and sampling frequency h for various values of n, and the resulting value of the cost function in

Equation (3.2). The optimal control chart design can be found by inspecting the values of the cost function so as to find the minimum. From Table 3.1, we also note that the minimum cost is R10.37085 per hour, that the optimal $\bar{X}$-chart uses samples of size $n = 5$, that the control limits are located at $\bar{X} \pm k\sigma$, with $k = 3.00$, and that the samples are taken at intervals of $h = 0.80$ hour (approximately once every 48 minutes). The type I error probability of this design is $\alpha = 0.0027$, and the power of the chart is $p = (1-\beta) = 0.9295$.

Example: 3.2

This example is borrowed from Montgomery (2001) exercise 9.30. An $\bar{X}$-chart is used to maintain current control of a process. A single assignable cause of magnitude 2σ ($\delta = 2$) occurs, and the time that the process remains in control is an exponential random variable with mean 100 hours ($\lambda = 0.01$). Suppose that sampling costs are R0.50 per sample ($a = 0.5$) and R0.10 per unit ($b = 0.10$). It costs R5 to investigate a false alarm ($a_5 = 5$), R2.50 to find the assignable cause ($a_3 = 2.50$), and R100 is the penalty cost per hour ($a_4 = 100$) to operate in the out-of-control state. The time required to collect and evaluate a sample is 0.05 hours ($g = 0.05$), and it takes two hours to locate the assignable cause ($D = 2$). Assume operation carries on during searches for the assignable causes.

Table 3.2: Output solutions for example 3.2

n	*k*	*h*	*α*	*β*	*1-β*	*E(L),R/hr*
1	2.1	0.7	0.035729	0.539807	0.460193	4.249571
2	2.1	1.1	0.035729	0.233176	0.766824	3.717282
3	**2.2**	**1.3**	**0.027807**	**0.103097**	**0.896903**	**3.609813**
4	2.4	1.4	0.016395	0.054799	0.945201	3.620397
5	2.5	1.5	0.012419	0.024297	0.975703	3.680690
6	2.7	1.5	0.006934	0.013940	0.986060	3.761470
7	2.9	1.6	0.003732	0.008390	0.991610	3.854068
8	2.5	1.7	0.012419	0.000797	0.999203	3.975526
9	3.1	1.7	0.001935	0.001866	0.998134	4.053053
10	3.1	1.8	0.001935	0.000631	0.999369	4.154499
11	3.1	1.9	0.001935	0.000205	0.999795	4.255995
12	3.1	1.9	0.001935	0.000065	0.999935	4.355013
13	3.1	2.0	0.001935	0.000020	0.999980	4.453392
14	3.1	2.0	0.001935	0.000006	0.999994	4.549868
15	3.1	2.1	0.001935	0.000002	0.999998	4.645204

Using Excel, the program calculates the optimal control limit width k and sampling frequency h for several sample sizes and displays the corresponding value of the cost function according to equation (3.2). The α risk (false alarm probability) and power, $1-\beta$ for each combination n, h, and k are also provided. The optimal control chart design (n, h, k) is found by inspecting the output values of the function to find the minimum.

The output is shown in table 3.2. Note that the optimal design has $n=3$, $k=2.2$ and $h=1.3$ hours, with minimum cost R3.609813 per hour. The α risk for this control chart design is 0.027807 and the power of the chart is $1-\beta=0.896903$. Notice that there are several other designs that employ a sample size slightly different from the optimal value of $n=3$ that are close to the optimal in terms of minimum cost.

3.5. Sensitivity analysis

A study of the sensitivity to the magnitude and frequency of process shifts in order to determine the appropriate adjustment of control chart parameters for the eventual of improvement in the expected cost was made. Table 3.3 shows the effects of model parameters on the economic design of the $\bar{X}$-control chart. Some important conclusions can be drawn about the optimal economic design of the $\bar{X}$-control charts. Some of these conclusions are illustrated below.

(1) Increasing the fixed cost of sampling, a and variable cost of sampling, b increases the sampling interval, h. However, the control limits decrease slightly.

(2) Changes in the mean number of occurrences of the assignable cause per hour, mainly affect the interval between samples. Table 3.3 shows that with $\lambda=0.01$, the optimum sampling interval increases considerably to 4.8 hours. The optimum sample size increases slightly and the control limits also decrease slightly.

(3) The magnitude of the process mean shift, δ, has a significant effect on the design. A larger value of δ leads to a smaller sample size and a short sampling interval. Table 3.3 illustrates the solution with $\delta=10$. The optimal sample size decreases considerably to $n=1$. The optimal control limit multiple increases to $k=3.1$ and the optimal sampling interval slightly decreases to $h=0.7$.

(4) The cost of investigating an action signal, a_3, that results in the elimination of an assignable cause and the cost of investigating a false alarm, a_5, mainly affect the value of the control limit multiple, k. They also have a slight effect on the sample size.

(5) The penalty cost of operating in the out-of-control state, a_4, mainly affects the interval between samples, h. A large value of a_4 implies smaller values of h, (more frequent sampling), while a smaller value of a_4 implies large values of h (less frequent sampling). Table 3.3 shows that increasing penalty cost to $a_4 = 150$, the optimum sampling frequency decreases to $h = 0.7$.

Table 3.3: Effects of model parameters on the optimal design of the $\bar{X}$-chart

parameters	n	k	h	α	β	$1-\beta$	E(L),R/hr
a =0.1	5	3.1	0.5	0.001935	0.08501	0.914989	9.0092851
a =1	5	3.0	0.8	0.002700	0.07049	0.929508	10.3708471
a =10	7	2.9	2.3	0.003732	0.00839	0.991610	16.1932360
b=0.01	6	3.1	0.7	0.001935	0.03601	0.963989	9.6901249
b=0.1	5	3.0	0.8	0.002700	0.07049	0.929508	10.3708471
b=10	3	2.5	3.5	0.012419	0.16750	0.832503	25.1626522
a_3=25	5	3.0	0.8	0.002700	0.07049	0.929508	10.3708471
a_3=50	5	3.0	0.8	0.002700	0.07049	0.929508	11.5311102
a_3=100	5	3.0	0.8	0.002700	0.07049	0.929508	13.8516364
a_5=25	5	2.8	0.8	0.005110	0.04725	0.952751	10.2731858
a_5=50	5	3.0	0.8	0.002700	0.07049	0.929508	10.3708471
a_5=100	6	3.1	0.9	0.001935	0.03601	0.963989	10.4875851
λ=0.01	7	2.9	4.8	0.003732	0.00839	0.991610	5.9612644
λ=0.05	5	3.0	0.8	0.002700	0.07049	0.929508	10.3708471
λ=0.5	5	2.9	0.4	0.003732	0.05796	0.942040	51.2375484
δ=0.7	15	2.5	0.8	0.012419	0.41641	0.583591	14.9556028
δ=2	5	3.0	0.8	0.002700	0.07049	0.929508	10.3708471
δ=10	1	3.1	0.7	0.001935	0.00000	1.000000	9.2760910
g=0.001	6	3.1	0.9	0.001935	0.03601	0.963989	9.9874260
g=0.0167	5	3.0	0.8	0.002700	0.07049	0.929508	10.3708471
g=0.1	4	2.8	0.8	0.005110	0.11507	0.884930	11.8865026
D=0.1	5	3.0	0.8	0.002700	0.07049	0.929508	6.3821617
D=1	5	3.0	0.8	0.002700	0.07049	0.929508	10.3708471
D=10	5	2.9	1.2	0.003732	0.05796	0.942040	37.1512772
a_4=50	6	3.1	1.2	0.001935	0.0360109	0.963989	6.5820120
a_4=100	5	3.0	0.8	0.002700	0.0704921	0.929508	10.3708471
a_4=150	5	3.0	0.7	0.002700	0.0704921	0.929508	13.8770310

(6) The average sampling, inspection, evaluation and plotting time for each sample, g has an effect on the sample size and the control limit width.

(7) The time required to find the assignable cause, D affects the sampling interval, h. Large values of D correspond to infrequent sampling.

(8) Montgomery (1980) stated that the economic design is insensitive to changes in all parameters except the magnitude of the shift, δ, the in-control state, μ_0 and the standard deviation, σ.

Chapter Four

Economic and economic statistical design of the $\bar{X}$-control chart-a unified approach

4.1. Introduction

Statistical process control (SPC) concepts and methods have been successfully implemented in manufacturing industries for decades. As one of the primary SPC tools, control charts play a very important role in attaining process stability. The major function of control charting is to detect the occurrence of assignable causes so that the necessary corrective action may be taken before a large number of nonconforming products are manufactured. The control chart design has received much attention since the design has behavioural, economic, as well as quality implications (Saniga, 1992). As a result of that, several general methodologies have been developed to improve on the design suggested by Shewhart.

There are two general methods of designing control charts in use today, the statistical design and the economic design. With the statistical design, one considers statistical properties such as the type I and type II error and the average run length when selecting the parameters for the control chart (Saniga, 1991). In statistical design the objective is to have control charts signal shifts in the process quickly and accurately and to keep false signals to a minimum. Woodall (1985) addressed the issue of statistical design. As we have seen in the previous chapter, in the economic design, the objective is to determine the control chart parameters that minimize the expected loss cost occurring in a production process.

An alternative to statistical and economic designs has been proposed by Saniga (1989) and is known as the economic statistical design. The economic statistical design is a method in which statistical constraints are placed on economic models to yield a design that meets statistical requirements at minimum cost. This approach maintains the effectiveness of economic designs and simultaneously maintains the required statistical performance of the control chart (Montgomery *et. al.*, 1995).

In this chapter, we present two widely used designs of control charts. The first of these relates to the selection of decision variables n, h, and k such that the expected cost per unit time is minimized. This approach is the economic design. The second type is the method in which statistical constraints are placed on the economic model. This approach is called the economic statistical design. Finally, we present the derivation, and optimization, of the economic and economic statistical designs based on the unified Lorenzen and Vance (1986) single assignable cause $\bar{X}$-control chart model.

4.2. Design methods

A large number of design methods have been proposed in the literature. Although various design approaches exist for different types of control charts, they can be classified into four general categories: heuristic, statistical, economical and economic statistical (Saniga, 1989). Without doubt the most popular is Shewhart's heuristic approach, which is to take a sample of size four or five (for $\bar{X}$ and R charts), set three sigma control limits and sample perhaps once an hour. Although the costs associated with the Shewhart charts are implicitly considered by setting the sample size and sampling frequency, the resulting charts are not guaranteed always to be economically optimal. In addition, statistical properties are not always in line with management's desire to find process shifts promptly and correctly.

The lack of formal systematic criteria in the heuristic design of control charts led many researchers and practitioners to search for more structured methods. Statistically designed control charts (Woodall, 1985) form one such method. Saniga's (1991) statistical design of control charts refers to the selection of the control limit parameter as well as the sample size in such a way that certain statistical objectives can be achieved. In statistical designs, the type I error probability and power are usually pre-specified at desired levels. Thus, the sample size and control limits can be determined. The average run length (ARL) or average time to signal (ATS) can be used to find the sampling frequency. Saniga (1989) applied his method to a joint design of $\bar{X}$ and R charts.

The third method of designing control charts is based on economic criteria. The concept of an economic design was first introduced by Gershick and Rubin (1952). Although the optimal control rules in their model are too complex to have practical value, their work provided the basis for most cost based models in control chart designs. Duncan (1956) developed a complete economic design model of the $\bar{X}$-control chart. The decision variables, n, h, and k are selected in such a way that the expected net income per unit time is maximized or the expected cost per unit time is minimized. Following Duncan's paper, a considerable amount of research has been done in the economic design of various control charts. Lorenzen and Vance (1986) proposed a unified approach to the economic control chart design that had major influences on subsequent research.

The advantage of economic designs is that all of the factors and costs that are measurable are considered in achieving a design. Thus the design is optimal in at least an economic sense (Saniga, 1989). There are certain weaknesses related to both statistical designs and economic designs. Statistical designs do not explicitly consider the economic point of view, and the choice of control chart parameters does not take into account the costs associated with the operation of the control chart. Some problems with economic designs have been noted by Woodall (1986) and include the possibility of higher type I error probability, which implies a large number of false searches, something that the production manager will not tolerate. Woodall (1986) also indicated that the economic design can allow poor quality products even if the policy is economically optimal. In the context of total quality management this is possibly unacceptable. As far as the communication with (potential) clients is concerned, it is important to quantify consumer's risk by means of the average time to signal for a process, which is out-of-control. A client is interested in the statistical properties of detecting an assignable cause rather than the fact that the producer has minimized his total cost in monitoring the production process. From the viewpoint of clients the quality products are more essential than the fact that the producer has minimized his total cost.

An alternative to the preceding design methods is known as the economic statistical design. The economic statistical design was first proposed by Saniga (1989) in order to combine the benefits of both pure statistical and economic designs while minimizing their weaknesses. The economic statistical design is defined as a design in which the economic loss cost function is minimized subject to constraints in terms of the minimum value of power and the maximum value of the type I error probability, as well as on the average time-to-signal (ATS) of an expected shift in process parameters.

4.3. The method of economic statistical design

The economic statistical design is a method in which statistical constraints such as a minimum value (lower bound) on the in-control ARL (ARL_L) and maximum value (upper bound) on the out-of-control ARL (ARL_U) are placed in the pure economic model so as to yield a design that meets statistical requirements at which the loss cost function is minimized (Montgomery *et. al.*, 1995). The economical statistical design was proposed by Saniga (1989) in order to improve both the statistical properties and the economical properties of the resulting control charts (McWilliams, 1994).

Alternatively, the ATS, which expresses the average run length in units of time, can be used to replace ARL in the formulation of the design model (Montgomery *et. al.*, 1995). Linderman and Love (2000a) showed that on the basis of the selected statistical constraints, control charts are then designed to have long ARL_0 or ATS_0 values when the process is in control and small ARL_1 or ATS_1 values when the process is out of control.

In the following, optimal economic statistical design control charts are derived using ARL and ATS constraints. Let F be the loss cost function for an economic model. The model for an economic statistical design can be formulated as:

Minimize $F(n,h,k)$

Subject to $ARL_0 \geq ARL_L$

$ARL_1 \leq ARL_U$

where ARL_L and ARL_U are the desired bounds at the expected shift level. The solution to this model is an improvement on the pure statistical design because it has the required statistical properties and still minimizes the lost cost function. A solution without the constraints will give the optimal economic design. Montgomery *et. al.* (1995) showed that additional constraints could be added to the design model if sensitivity to shifts that are different from the expected shifts, is required.

Economic statistical designs are determined via non-linear constrained optimization techniques. The objective is to minimize the expected total cost per unit time, as in the economic design, subject to constraints on the type I error rate, power, and ATS (Montgomery *et. al.* 1995). Alternative and additional constraints can be specified depending on the designer's needs. Economic statistical designs are the constrained version of economic designs. If the constraints alone are used in determining design parameters, without considering the cost objectives, they become statistical designs. Zhang and Berardi (1997) showed that economic statistical designs are generally more costly than economic designs due to the added constraints. However, the tight limits on the statistical properties of the control charts can lead to low process variability that enhances output quality which leads to reduction in cost of comebacks and rewards.

4.4. Assumptions and notations of the unified approach

We make the following assumptions for the economic and economic statistical design of control charts:

(1) The process is subject to a single assignable cause.

(2) The process starts in a state of statistical control with mean μ_0 and standard deviation σ.

(3) The occurrence of the assignable cause results in a shift in the process mean from μ_0 to $\mu_0 + \delta\sigma$, where the shift size δ is known.

(4) The distribution of the time between occurrences of the assignable cause is exponential with a mean of θ occurrences per hour (thus $\frac{1}{\theta}$ hours is the mean time in the in-control state).

(5) If a single sample point falls outside the control limits, the process is assumed to be out-of-control and the search for the assignable cause is initiated.

(6) Once the process is out-of-control and a signal is triggered, human intervention is required.

(7) The economic and economic statistical designs of control charts assume a renewal reward process. In essence, the corrective actions are assumed to return the process to the initial state of statistical control.

Lorenzen and Vance (1986) provided a unified approach to the economic and economic statistical design of the $\bar{X}$-control chart and a unification of notation. The following notation will be used in the formulation of the cost function. The parameters can be classified into four categories. The definitions of these parameters are given below.

(1) Cost and operating parameters

$E(\tau) =$ The expected time of occurrence of the assignable cause between two samples while in-control

$s =$ Expected number of samples taken while in-control

$a =$ Fixed cost per sample

$b =$ Cost per unit sampled

$Y =$ Cost per false alarm

$W =$ Cost to locate and repair the assignable cause

$C_0 =$ Quality cost /hour while producing in control

$C_1 =$ Quality cost /hour while producing out of control

$g =$ Time to sample and chart one item

$T_0 =$ The expected search time when the signal is a false alarm

$T_1 =$ The expected time to discover the assignable cause

$T_2 =$ The expected time to repair the process

$C =$ Total cost per cycle

$L =$ Total cost per time unit (hour)

$ARL_0 =$ Average run length while in control

$ARL_1 =$ Average run length while out of control

$ARL_L =$ Lower bound of the Average run length while in control

$ARL_U =$ Upper bound of the Average run length while out of control

$ATS =$ Average time-to-signal

$ATS_U =$ Upper bound of the average time-to-signal

(2) Indicator variable

$\gamma_1 = 1$, if production continues during search

$\gamma_1 = 0$, if production ceases during search

$\gamma_2 = 1$, if production continues during repair

$\gamma_2 = 0$, if production ceases during repair

(3) Three control chart design parameters

$n =$ Sample size

$h =$ Sampling interval

$k =$ Width of the control limit

(4) Assignable cause $\equiv$ assignable cause event

4.5. Derivation of the economic and economic statistical models

The Lorenzen and Vance model provides practitioners with the most flexible of any of the widely known single assignable cause models available by using average run lengths instead of type I and type II errors in order to define an economic model (McWilliams, 1989). The authors also include indicator variables in the model to identify whether production ceases or continues during search and/or repair, so that any possible operation scenario can be appropriately modelled (Simpson and Keats, 1995). It is in this regard that Lorenzen and Vance's model is superior to that of Duncan. In the derivation of the economic or economic statistical model, there are two major elements in the loss function: (1) the estimated expected length of the production cycle and (2) the expected costs generated in a production cycle. After these elements have been determined, the hourly costs and resulting operating loss cost function can be determined. Based on the assumptions stated above, Lorenzen and Vance (1986) provided a unified approach to the economic or economic statistical models of the control chart. They considered a general process model that applies to all control charts (McWilliams, 1994).

4.5.1. The expected cycle of production time

The production cycle here is defined as the time length from the point in time when the process is started in the in-control state to when it shifts to the out-of-control state, and onwards to where in time the detection and elimination of the assignable cause takes place. The cycle time consists of the following five parts:

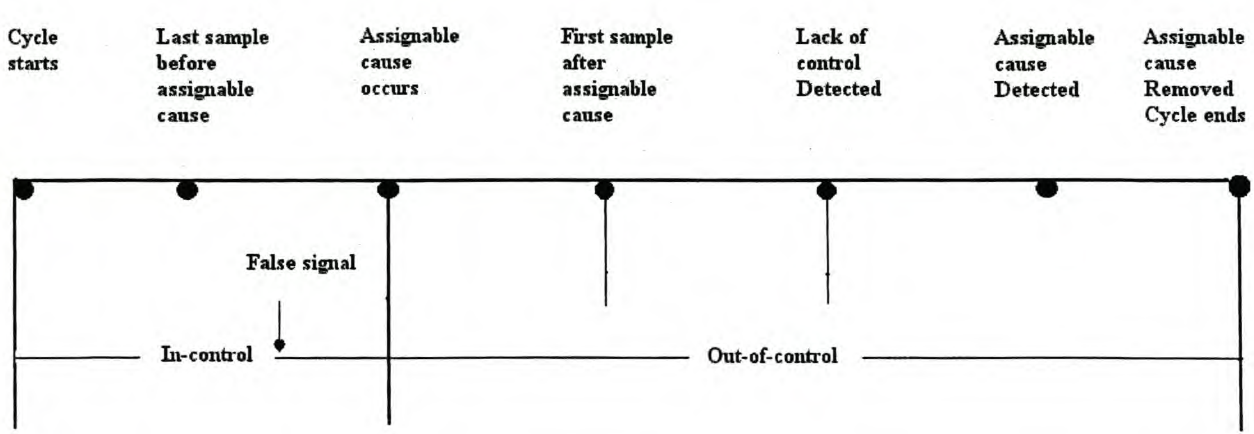

Figure 4.1: Diagram of in-control and out-of-control states of the process

(a) The time until the assignable cause occurs

Given that this is a memoryless process subject to random shocks, the time to occurrence of an assignable cause, is distributed as an exponential random variable with mean $\frac{1}{\theta}$.

If production continues during searches, the average time for occurrence of the assignable cause is simply $\frac{1}{\theta}$.

If production ceases during the search period, the average time until the assignable cause is $\frac{1}{\theta}$ plus the time spent searching due to false alarms. Let T_0 be the expected search time when the signal is a false alarm. Then, the expected time spent searching due to false alarms is T_0 times the expected number of false alarms i.e. $T_0\left(\frac{s}{ARL_0}\right)$, where ARL_0 is the average run length while in-control and s is the expected number of samples taken while in-control. This is the non-production time because of false alarms (Lorenzen and Vance, 1986).

Note that $s = \sum_{i=0}^{\infty} iP($ assignable cause occurs between the i th and $(i+1)$ st sample) (see appendix (A.18))

$$= \frac{1}{\exp(\theta h) - 1}$$

and $\qquad ARL_0 = \frac{1}{\alpha}$

where $\alpha = P($ exceeding control limits | process in control).

$$= 2\int_k^{\infty} \phi(z)dz$$

where $\phi(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$, so that

$$\alpha = 2\Phi(-k).$$

Note that ARL_0 depends only on the assumed underlying distribution and the control limit parameter k.

The way in which Lorenzen and Vance combines both conditions is as follows.

Let $\qquad \gamma_1 = 1$, if production continues during searches

$\gamma_1 = 0$, if production ceases during searches

Then, the expected time until the assignable cause occurs is:

$$\frac{1}{\theta} + (1-\gamma_1)s\frac{T_0}{ARL_0} \tag{4.1}$$

(b) The time until the next sample is taken

Given the occurrence of an assignable cause signal between the ith and $(i+1)$st sample, the expected time of occurrence within the interval, denoted by τ, is (see appendix (A.19))

$$E(\tau) = \frac{\int_{ih}^{(i+1)h} \theta(t-hi)e^{-\theta t}dt}{\int_{ih}^{(i+1)h} \theta e^{-\theta t}dt}$$
$$= \frac{1}{\theta} - \frac{h}{\exp(\theta h)-1},$$

which is independent of i. The expected time between the occurrence of the assignable cause and the next sample then equals

$$h - E(\tau). \tag{4.2}$$

(c) The time to analyze the sample and chart the result

Let g be the expected time to sample and chart one item. For a sample of n items, the time to analyze the sample and chart the result is given by

$$ng. \tag{4.3}$$

(d) The time until the chart gives an out-of-control signal

The expected time before an out-of-control signal occurs, is given by $h(ARL_1 - 1)$, where ARL_1 is the average run length when the process has shifted to an out-of-control state. If the sample statistics are independent, then $ARL_1 = \frac{1}{1-\beta}$, where $\beta = P$ (in-control signal | process is out of control), and

$$(1-\beta) = \Phi(\delta\sqrt{n}-k) + \Phi(-\delta\sqrt{n}-k)$$
$$= \int_{-\infty}^{-k-\delta\sqrt{n}} \phi(z)dz + \int_{k-\delta\sqrt{n}}^{\infty} \phi(z)dz.$$

Note that ARL_1 depends on the underlying distribution, the control limit width parameter k, the sample size n, and the extent of the shift δ when the assignable cause occurs.

(e) The time to discover the assignable cause and repair the process

Lorenzen and Vance (1986) indicated T_1 to be the expected time to discover the assignable cause and T_2 the expected time to repair the process. Then the expected time to detect a shift, discover the assignable cause, and repair the process equals

$$h(ARL_1 - 1) + T_1 + T_2 \tag{4.4}$$

Combining (4.1) through (4.4), we obtain the expected cycle time as

$$E(T) = \frac{1}{\theta} + (1-\gamma_1)s\frac{T_0}{ARL_0} - E(\tau) + ng + h(ARL_1) + T_1 + T_2. \tag{4.5}$$

4.5.2. The cost function

The costs per cycle are incurred by defective production while in-control as well as out-of-control, for false alarms, for locating and repair of the assignable causes, and also for sampling and inspection.

(a) Cost per cycle due to defective products

Lorenzen and Vance (1986) used C_0 as the cost of quality control per hour while production is in control and C_1 as the cost of quality control per hour while production is out of control where $(C_1 > C_0)$. Assume that production continues during both search and repair. Then the expected cost per cycle equals

$$\frac{C_0}{\theta} + C_1(-E(\tau) + ng + h(ARL_1) + T_1 + T_2).$$

If production ceases during repair only, the expected cost per cycle equals

$$\frac{C_0}{\theta} + C_1(-E(\tau) + ng + h(ARL_1) + T_1).$$

If production ceases during both search and repair, then the expected cost per cycle equals

$$\frac{C_0}{\theta}+C_1(-E(\tau)+ng+h(ARL_1)\,.$$

Define $\gamma_2=1$ if production continues during repair and $\gamma_2=0$ if production ceases during repair. Then the expected cost per cycle due to defective products can be written as (Lorenzen and Vance, 1986)

$$\frac{C_0}{\theta}+C_1(-E(\tau)+ng+h(ARL_1)+\gamma_1T_1+\gamma_2T_2)\,. \tag{4.6}$$

(b) Cost per cycle due to false alarms as well as locating and repair of assignable causes

Let Y be the cost per false alarm. This consists of the cost of search for the cause plus the cost of down time if production ceases during the search. Let W be the cost of locating and repairing the assignable cause when one exists. Again W includes any down time cost that is appropriate. Then, the expected cost for false alarms and locating and repairing the true assignable causes is given by

$$\frac{sY}{ARL_0}+W\,. \tag{4.7}$$

(c) Cost per cycle for sampling and inspection

Using Lorenzen and Vance's (1986) unified approach, if we let a be the fixed cost per sample and b be the cost per unit sampled, then the expected cost for sampling and inspection is given by

$$(a+bn)\left(\frac{production\ time}{h}\right).$$

The production time depends on whether or not production continues during search and/or repair. The expected cost per cycle for sampling equals

$$(a+bn)\left(\frac{\frac{1}{\theta}-E(\tau)+ng+h(ARL_1)+\gamma_1T_1+\gamma_2T_2}{h}\right). \tag{4.8}$$

Adding (4.6), (4.7), (4.8), we obtain the total expected quality cost per cycle as:

$$E(C)=\frac{C_0}{\theta}+C_1(-E(\tau)+ng+h(ARL_1)+\gamma_1T_1+\gamma_2T_2)+\frac{sY}{ARL_0}+W$$
$$+(a+bn)\left(\frac{\frac{1}{\theta}-E(\tau)+ng+h(ARL_1)+\gamma_1T_1+\gamma_2T_2}{h}\right) \tag{4.9}$$

Because the cycle length is variable and is a function of n, h, and k, we must express the cost function per unit of time (hour), not per cycle. Note that since this is a renewal reward process (Ross, 2000), the expected cost per hour is found by dividing the total quality cost per cycle, by the expected cycle length, (see equation (4.5) and equation (4.9)), resulting in:

$$E(L)=\frac{\frac{C_0}{\theta}+C_1\left(-E(\tau)+ng+h(ARL_1)+\gamma_1T_1+\gamma_2T_2\right)+\frac{sY}{ARL_0}+W}{\frac{1}{\theta}+(1-\gamma_1)s\frac{T_0}{ARL_0}-E(\tau)+ng+h(ARL_1)+T_1+T_2}$$
$$+\frac{a+bn}{h}\left(\frac{\frac{1}{\theta}-E(\tau)+ng+h(ARL_1)+\gamma_1T_1+\gamma_2T_2}{\frac{1}{\theta}+(1-\gamma_1)s\frac{T_0}{ARL_0}-E(\tau)+ng+h(ARL_1)+T_1+T_2}\right) \tag{4.10}$$

4.6. Optimal economic and economic statistical design of the control chart

In a given process, the function $E(L)$ represents the expected cost per hour for the present model and it should be noted that $E(L)$ is a function of the three quality control chart parameters the sample size, n, the sampling period, h, and the control limit width parameter, k. Note that α and $1-\beta$ are also functions of n and k. As a result of these, the function $E(L)$ is highly nonlinear in each of the three parameters.

The algorithm by Lorenzen and Vance (1986) to find the most economical design is somewhat complicated as it consists of Newton's method, the golden section search as well as the Fibonacci search method. This could be the main reason why it is not applied often. Montgomery (2001) also indicated that very few practitioners have implemented economic models for the design of control charts. There are at least two reasons for the lack of practical implementation of this methodology. First, the mathematical models and their associated optimization schemes are relatively complex and are often presented in a manner that is difficult for the practitioners to understand and use. A second problem is the difficulty in estimating costs and other model parameters.

During the present study, a user friendly Excel program was developed that can be used to determine an economic or economic statistical design for an $\bar{X}$-control chart. This program uses the model of Lorenzen and Vance and is configured to be applicable to most actual production situations.

Based on the model described in this chapter and the parameters just defined, Lorenzen and Vance (1986) showed that the expected loss cost per hour of operation can be expressed as in (4.10) and can be written as

$$E(L) = \frac{NUM_1}{DEN} + \frac{NUM_2}{DEN} \tag{4.11}$$

where

$$NUM_1 = \frac{C_0}{\theta} + C_1(-E(\tau) + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2) + \frac{sY}{ARL_0} + W,$$

$$NUM_2 = (a + bn)\left(\frac{\frac{1}{\theta} - E(\tau) + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2}{h}\right)$$

and

$$DEN = \frac{1}{\theta} + (1-\gamma_1)s\frac{T_0}{ARL_0} - E(\tau) + ng + h(ARL_1) + T_1 + T_2.$$

The expected number of samples, s taken while in-control is (e.g. (A.18))

$$s = \frac{1}{e^{\theta h} - 1} \text{ (compare section 3.2)}$$

and the average time of occurrence of the assignable cause between ith and $(i+1)$st interval is given by

$$
\begin{aligned}
E(\tau) &= \frac{1-(1+\theta h)\exp(-\theta h)}{\theta(1-\exp(-\theta h))} \qquad \text{(compare section 3.2)} \\
&= \frac{1}{\theta} - \frac{h}{\exp(\theta h)-1}.
\end{aligned}
$$

This is an exact solution in contrast to Duncan's approximate approach (see appendix (A.19)).

Note that the input parameters are θ, δ, a, b, Y, W, C_0, C_1, g, T_0, T_1, T_2, γ_1 and γ_2 and thus are entered as fixed values. Note further that n, h, and k are input as variables.
The program calculates the optimal control limit width parameter k and the sampling interval h for several sample sizes n, and displays the corresponding values of the expected cost function $E(L)$. It also calculates the corresponding α risk and power, $(1-\beta)$ (see appendix (A.20) and (A.21)). All combinations of results are tabulated and the optimum combination is easily obtained from the tables.

Using the Lorenzen and Vance model is fairly simple. The procedure consists of solving by searching for the optimal control limit k and the sampling interval h for several values of n and displaying the value of the cost function together with the associated in-control and out-of-control average run lengths. Furthermore, no terms are neglected or approximated in finding the optimum solution as in Duncan's and other approaches. Consequently, the solution is not only reliable, but also more appropriate to use for the economic and economic statistical design of the $\bar{X}$-control chart than Duncan's model.

In the optimization of the economic statistical design of the $\bar{X}$-control chart, one can determine the optimal design parameters for the $\bar{X}$-chart by introducing statistical constraints such as the in-control and out-of-control average run lengths. Saniga (1989) noticed that these two sets of in-control and out-of-control ARL bounds are ARL_L and ARL_U, respectively. In statistical designs the $ARL's$ are the main objects of interest, i.e. the minimization of ARL_1 and the maximization of ARL_0. Using these ideas, constraints are put on the $ARL's$ in the economical design. These constraints then give rise to the economic statistical design.

4.7. Summary

The general approaches for the design of control charts are the statistical, the economic and the economic statistical approaches. Statistically designed control charts are those in which the control limit width, which determines the type I error probability and the power, are presented. These then determine the sampling frequency (e.g. Woodall 1985). The other method of designing a control chart is based on an economic criterion. In economic design the objective is to find the sample size, the control limit width, and the sampling frequency that minimize the loss in profit accruing to the firm because of poor quality. This loss in profit is composed of the cost of producing products not within specifications, the cost of false alarms, and the cost of searching for and eliminating the assignable causes. Lorenzen and Vance (1986) developed a single assignable cause economic control chart model and applied it to the $\bar{X}$ -chart.

An alternative to statistical and economic designs has been proposed by Saniga (1989) and is known as the economic statistical design. The loss cost function of the process is minimized subject to:

- a constrained minimum value for the power $1-\beta$ of the control chart,
- a maximum value for the type I error probability α, and
- a maximum value on the average time-to-signal for a specified shift in the process parameters.

The economic statistical design has the advantages of improving the assurance of long term product quality as well as a reduction of the variance of the distribution of the quality characteristic. The disadvantage is that it yields a higher expected loss than the pure economic design.

Lorenzen and Vance (1986) derived an expected cost function applicable to any quality control chart of the form developed by Shewhart. This function depends on twelve cost and time parameters that describe the process, two indicator variables that show whether production continues during search or repair, and three design parameters that describe the charting procedure. The minimization of this function over the choice of the design parameters leads to the optimal economic and economic statistical $\bar{X}$ -control charts. In this thesis, a user-friendly Excel program was developed to determine the optimal values of the economic or the economic statistical design of an $\bar{X}$ -control chart. The program is based on an improved form of the Lorenzen and Vance model. The application of this optimization procedure will be illustrated in the next chapter. Numerical analyses and comparisons of both the economic and the economic statistical designs are presented. An extensive sensitivity analysis is also conducted.

Chapter Five
Numerical illustration and analysis

In this chapter, the optimal economic design is compared to the optimal economic statistical designs by means of examples. Based on the unified approach of the cost model developed by Lorenzen and Vance (1986), a more detailed comparison and analysis are made of the economic and economic statistical designs in order to investigate the effects on the loss function, of the input parameters and of adding constraints to the statistical performance measurements.

5.1. Numerical illustration

Three numerical examples are presented in order to demonstrate the solution procedure as well as to make some comparisons of the economic and economic statistical designs of the $\bar{X}$-control chart. The model parameters in these examples are taken from the statistical constrained economic EWMA control chart presented by Torng, Cochran, Montgomery, and Lawrence (1995).

5.1.1. Example of the economic design

Example 5.1

Torng, Cochran, Montgomery, and Lawrence (1995) provide an application of the single objective design of an $\bar{X}$-control chart based on the Lorenzen and Vance unified approach. Suppose that the fixed cost sampling is R0.50 (i.e., $a = 0.50$) and the variable cost of sampling is estimated to be R0.10 (i.e., $b = 0.10$). It takes approximately three minutes (i.e., $g = 0.05$ hours) to take and analyze each observation. The magnitude of the process shifts is one standard deviation ($\delta = 1$), and process shifts occur according to the exponential distribution with a mean frequency of about one every hundred hours of operation. Thus the $\theta = 0.01$. It takes two hours to investigate an action signal (i.e., $T_1 = 2$). The cost of investigating a false alarm is R50 (i.e., Y = R50), and a true action signal costs R25 to investigate (i.e., W = R25). The hourly costs for operating in the in-control state and in the out-of-control state are R10 (i.e., C_0 = R10) and R100 (i.e., C_1 = R100), respectively. The process continues operation during the search and repair periods of the assignable cause.

Table 5.1: Optimal economic design of the $\bar{X}$ -control chart

n	k	h	α	β	$(1-\beta)$	ARL_0	ARL_1	ATS_0	ATS_1	$E(L)$,R/hr	
1	2.1	0.7	0.035729	0.863366	0.136634	27.989	7.319	19.592	5.123	19.22080	
2	2.3	0.7	0.021448	0.812032	0.187968	46.624	5.320	32.637	3.724	17.35571	
3	2.3	0.9	0.021448	0.714938	0.285062	46.624	3.508	41.962	3.157	16.42810	
4	2.4	0.9	0.016395	0.655416	0.344584	60.994	2.902	54.895	2.612	15.87054	
5	2.4	1.1	0.016395	0.565106	0.434894	60.994	2.299	67.093	2.529	15.51280	
6	2.4	1.3	0.016395	0.480264	0.519736	60.994	1.924	79.292	2.501	15.27609	
7	2.5	1.3	0.012419	0.442059	0.557941	80.519	1.792	104.675	2.330	15.10723	
8	2.5	1.5	0.012419	0.371294	0.628706	80.519	1.591	120.779	2.386	14.99482	
9	2.5	1.6	0.012419	0.308538	0.691462	80.519	1.446	128.831	2.314	14.91908	
10	2.6	1.6	0.009322	0.286963	0.713037	107.268	1.402	171.629	2.244	14.87267	
11	2.6	1.7	0.009322	0.236803	0.763197	107.268	1.310	182.356	2.227	14.84646	
12	**2.6**	**1.9**	**0.009322**	**0.193766**	**0.806234**	**107.268**	**1.240**	**203.809**	**2.357**	**14.83830**	**MIN**
13	2.7	1.9	0.006934	0.182587	0.817413	144.216	1.223	274.010	2.324	14.84578	
14	2.7	2.0	0.006934	0.148785	0.851215	144.216	1.175	288.432	2.350	14.86075	
15	2.7	2.1	0.006934	0.120401	0.879599	144.216	1.137	302.853	2.387	14.88680	
16	2.7	2.2	0.006934	0.096801	0.903199	144.216	1.107	317.275	2.436	14.92200	
17	2.8	2.2	0.005110	0.092900	0.907100	195.680	1.102	430.496	2.425	14.96075	
18	2.8	2.3	0.005110	0.074561	0.925439	195.680	1.081	450.064	2.485	15.00574	
19	2.8	2.4	0.005110	0.059510	0.940490	195.680	1.063	469.632	2.552	15.05649	
20	2.9	2.4	0.003732	0.057960	0.942040	267.970	1.062	643.128	2.548	15.10868	

As mentioned in the previous chapter, Excel was used to search for an optimum. The optimal control limit, k and sampling interval, h were computed for several values of n. The values of the cost function together with the associated in-control and out-of-control average run lengths were calculated and are shown in table 5.1. This is the same approach used by Montgomery (2001), Alexander *et. al*. (1995) and Linderman and Love (2000b). Table 5.1 reveals that the optimal design has $n=12$, $k=2.6$, $h=1.9$ hours, with a minimum cost of R14.83830 per hour. The in-control and out-of-control average run lengths for this control chart design are 107.268 and 1.240, respectively. Note that the design at $n=13$ has minimum cost close to the optimum and also has slightly better statistical properties than the optimal design at $n=12$. From table 5.1 we see that the power is improved from 0.806234 at $n=12$ to 0.817413 at $n=13$. The improvement in statistical performance leads to a wider control limit parameter, i.e. from 2.6 to 2.7.

5.1.2. Example of the economic statistical design

In this problem we use the economic parameters from example 5.1, but apply some statistical constraints in terms of ARL_L, ARL_U and ATS. This illustrates the use of the approach for finding an economic statistical design. Table 5.1 shows that the probability that a single point falls outside the limits when the process is in-control, is 0.009322. That is, even if the process remains in-control, an out-of-control signal will be generated every 107 samples, on average. This large number of false alarms introduces extra variability into the process through over adjustment and destroys confidence in the control procedure. In other wards, the average run length while in-control, ARL_0, is equal to 107. It is desirable to have this value larger so that false alarms are avoided. Therefore, an economic statistical design should be investigated due to this high false alarm rate associated with the economic design.

5.1.2.1. Optimal economic statistical designs with *ARL* constraints

Example 5.2

This example illustrates the economic statistical design of the control chart with ARL constraints. The starting point for this example is the specified ARL bounds i.e. $ARL_L = 267$ and $ARL_U = 40$ for $\delta = 1$. Note that the reason for using the ARL bounds is to constrain the economic statistical design to an in-control ARL value of at least 267, while keeping the out-of-control ARL at a value of less than or equal to 40.

Table 5.2: Optimal economic statistical designs with ARL constraints

n	k	h	α	β	(1-β)	ARL_0	ARL_1	ATS_0	ATS_1	$E(L)$,R/hr	
1	2.9	0.2	0.003732	0.971235	0.028765	267.970	34.765	53.594	6.953	21.44304	
2	2.9	0.3	0.003732	0.931324	0.068676	267.970	14.561	80.391	4.368	18.50123	
3	2.9	0.4	0.003732	0.878585	0.121415	267.970	8.236	107.188	3.294	17.16481	
4	2.9	0.5	0.003732	0.815939	0.184061	267.970	5.433	133.985	2.716	16.40727	
5	2.9	0.6	0.003732	0.746633	0.253367	267.970	3.947	160.782	2.368	15.92921	
6	2.9	0.8	0.003732	0.673829	0.326171	267.970	3.066	214.376	2.453	15.59165	
7	2.9	0.9	0.003732	0.600348	0.399652	267.970	2.502	241.173	2.252	15.35989	
8	2.9	1.1	0.003732	0.528529	0.471471	267.970	2.121	294.767	2.333	15.19818	
9	2.9	1.2	0.003732	0.460172	0.539828	267.970	1.852	321.564	2.223	15.07857	
10	2.9	1.3	0.003732	0.396554	0.603446	267.970	1.657	348.361	2.154	14.99761	
11	2.9	1.5	0.003732	0.338476	0.661524	267.970	1.512	401.955	2.267	14.94475	
12	2.9	1.6	0.003732	0.286342	0.713658	267.970	1.401	428.752	2.242	14.91220	
13	**2.9**	**1.7**	**0.003732**	**0.240234**	**0.759766**	**267.970**	**1.316**	**455.549**	**2.238**	**14.89848**	MIN
14	2.9	1.8	0.003732	0.199990	0.800010	267.970	1.250	482.346	2.250	14.89989	
15	2.9	1.9	0.003732	0.165281	0.834719	267.970	1.198	509.143	2.276	14.91370	
16	2.9	2.0	0.003732	0.135666	0.864334	267.970	1.157	535.940	2.314	14.93782	
17	2.9	2.1	0.003732	0.110645	0.889355	267.970	1.124	562.737	2.361	14.97059	
18	2.9	2.2	0.003732	0.089694	0.910306	267.970	1.099	589.534	2.417	15.01070	
19	2.9	2.3	0.003732	0.072297	0.927703	267.970	1.078	616.331	2.479	15.05703	
20	2.9	2.4	0.003732	0.057960	0.942040	267.970	1.062	643.128	2.548	15.10868	

The first ARL_0 constraint is equivalent to $\alpha \le \frac{1}{267}$ = 0.003745, and the ARL_1 constraint is equivalent to $1-\beta \ge \frac{1}{40}$ = 0.025 when a one σ shift occurs. Thus, to obtain an economic statistical design, we add two constraints, i.e. $ARL_0 \ge ARL_L$ and $ARL_1 \le ARL_U$ with $ARL_0 \ge 267$ and $ARL_1 \le 40$.

The results are shown in the output table 5.2. The optimal design has $n=13$, $k=2.9$, $h=1.7$ hours, with a minimum cost of R14.89848 per hour. The in-control and out-of-control average run length for this control chart design are 267.970 and 1.316, respectively compared to 107.268 and 1.240 in the economic design. Note that the designs at $n=12$ and $n=14$ have minimum costs close to the optimum. A comparison between the pure economical design and the economic statistical design of the $\bar{X}$-control chart with an ARL constraint as illustrated above, shows that the economic statistical design with ARL constraints have wider control limits and smaller sampling intervals than the economic design. The ARL_L constraint of example 5.2 leads to a significant reduction in the frequency of false alarms, while the additional cost incurred by imposing the ARL_U and ARL_L constraints is minimal. Table 5.2 shows that it is not expensive to achieve the desired statistical properties. We have calculated the percentage increase in the cost of the economic statistical design over that of the economic design, and the increase in overall expected cost is only 0.41%, i.e. from R14.83830 to R14.89848. This may, in many situations be a relatively small price to pay in order to achieve the improved statistical performance of the control charts. The false alarm rate is also reduced from 0.009322 to 0.003732. Note that the output of the two examples agree very closely with the results of the two examples in the statistical constrained economic EWMA control chart presented by Torng, Cochran, Montgomery, and Lawrence (1995).

5.1.2.2. Optimal economic statistical designs with ATS constraints

Since it is sometimes more appropriate in process monitoring to express shift detection performance in time units, economic statistical designs for the $\bar{X}$-control chart also investigate average time-to-signal as the statistical constraint (Montgomery *et. al.*, 1995). The desired ATS bounds are then computed by multiplying the ARL by its corresponding sampling interval. The constraint here could be written as

$$ATS_1 = \frac{h}{1-\beta} \\ = h(ARL_1),$$

where each signal shows an out-of-control situation.

Example 5.3

In this example, we use the same input parameters as in the previous two examples. Adding the ATS constraint to the pure economic model of the $\bar{X}$-control chart, we have the economic statistical design of the $\bar{X}$-control chart with an ATS constraint. Suppose the following statistical constraint is added to the Torng, Cochran, Montgomery, and Lawrence (1995) example:

$$ATS_1 \leq 1.90.$$

Based on the optimization results, table 5.3 presents the optimal economic statistical design of the $\bar{X}$-control chart for the proposed model and ATS bound. Table 5.3 shows that the optimal design has $n = 12$, $k = 2.6$, $h = 1.5$ hours, with a minimum cost of R14.89331 per hour. The average time-to-signal is 1.861.

Table 5.3: Optimal economic statistical designs with ATS constraints

n	k	h	α	β	$(1-\beta)$	ARL_0	ARL_1	ATS_0	ATS_1	$E(L)$,R/hr	
1	2.2	0.2	0.027807	0.884243	0.115757	35.962	8.639	7.192	1.728	23.13212	
2	2.4	0.3	0.016395	0.837813	0.162187	60.994	6.166	18.298	1.850	18.49734	
3	2.5	0.4	0.012419	0.778730	0.221270	80.519	4.519	32.208	1.808	16.99375	
4	2.3	0.7	0.021448	0.617903	0.382097	46.624	2.617	32.637	1.832	16.19573	
5	2.4	0.8	0.016395	0.565106	0.434894	60.994	2.299	48.795	1.840	15.67825	
6	2.5	0.9	0.012419	0.520142	0.479858	80.519	2.084	72.468	1.876	15.35856	
7	2.5	1.0	0.012419	0.442059	0.557941	80.519	1.792	80.519	1.792	15.20093	
8	2.6	1.1	0.009322	0.409657	0.590343	107.268	1.694	117.995	1.863	15.05237	
9	2.5	1.3	0.012419	0.308538	0.691462	80.519	1.446	104.675	1.880	14.97145	
10	2.6	1.3	0.009322	0.286963	0.713037	107.268	1.402	139.448	1.823	14.92851	
11	2.6	1.4	0.009322	0.236803	0.763197	107.268	1.310	150.175	1.834	14.90232	
12	**2.6**	**1.5**	**0.009322**	**0.193766**	**0.806234**	**107.268**	**1.240**	**160.902**	**1.861**	**14.89331**	**MIN**
13	2.6	1.6	0.009322	0.157316	0.842684	107.268	1.187	171.629	1.899	14.89822	
14	2.7	1.6	0.006934	0.148785	0.851215	144.216	1.175	230.746	1.880	14.91511	
15	2.8	1.6	0.005110	0.141639	0.858361	195.680	1.165	313.088	1.864	14.95170	
16	2.7	1.7	0.006934	0.096801	0.903199	144.216	1.107	245.167	1.882	14.99450	
17	2.8	1.7	0.005110	0.092900	0.907100	195.680	1.102	332.656	1.874	15.03692	
18	2.9	1.7	0.003732	0.089694	0.910306	267.970	1.099	455.549	1.868	15.09308	
19	3.0	1.7	0.002700	0.087089	0.912911	370.379	1.095	629.645	1.862	15.16002	
20	2.8	1.8	0.005110	0.047249	0.952751	195.680	1.050	352.224	1.889	15.21943	

To illustrate the effect of the ATS constraint in the economic statistical design, we will compare it to the pure economic design. Table 5.3 also points out that the economic statistical design has a smaller sampling interval, i.e. $h = 1.5$ for the economic statistical model with ATS constraint and $h = 1.9$ for the pure economic model. The out-of-control ATS_1 for the economic statistical design is much better than the corresponding ATS_1 for the pure economic design, i.e. 1.861 against 2.357, resulting in a cost increase of only about 0.37%, i.e. from R14.83830 to R14.89331. Similar results were reported by Saniga (1989) and Montgomery *et. al*. (1995).

Note that the three examples above indicate that economic statistical designs are generally more expensive than the economic design due to the added constraints. However, the tighter limits on control chart statistical properties can guarantee long-term product or service quality and low process variability. This results directly from the requirement that the economic statistical design assures a satisfactory statistical performance.

5.2. Sensitivity analysis

In this section, the relationship between the twelve input parameters and the loss function is investigated. Furthermore, the effects of variation in the bounds on α, $1-\beta$, the average time to signal ATS, the expected shift size δ, on the minimum expected cost as well as on the three decision variables n, h, and k, are investigated. The sensitivity analysis is designed to provide insight into the effect of those inputs having significant effects in the Lorenzen and Vance model when the economic and economic statistical design control charts are employed.

5.2.1. Sensitivity on the economic design

When applying the Lorenzen and Vance (1986) model in the economic design of the $\bar{X}$-control chart, there are twelve input parameters related to the loss function. Each example in table 5.4 represents the economic design of the $\bar{X}$-control chart, changing the value of one parameter at a time. Therefore, the sensitivity analysis of the proposed model with regard to these input parameters is fairly straightforward.

Note that the centre option is the default in the parameters which are kept fixed, e.g. in the case of $a = 0.25$, we use $\theta = 0.01$, and $b = 0.1$ etc.

Table 5.4: Sensitivity analysis on the economic design of the $\bar{X}$-control chart

Parameters	n	k	h	α	β	$(1-\beta)$	ARL_0	ARL_1	$E(L)$,R/hr
θ=0.005	13	2.7	2.6	0.00693	0.18260	0.81741	144.22	1.22340	12.93897
θ =0.01	12	2.6	1.9	0.01000	0.19400	0.80623	107.27	1.24030	14.83830
θ =0.05	9	2.5	0.8	0.01242	0.30850	0.69146	80.52	1.44620	25.92872
a =0.25	11	2.6	1.6	0.00932	0.23680	0.7632	107.27	1.31030	14.69813
a =0.5	12	2.6	1.9	0.01000	0.19400	0.80623	107.27	1.24030	14.83830
a =1	13	2.6	2.2	0.00932	0.15730	0.84268	107.27	1.18670	15.08503
b=0.05	13	2.8	1.5	0.00511	0.21030	0.78975	195.68	1.26620	14.46620
b=0.1	12	2.6	1.9	0.01000	0.19400	0.80623	107.27	1.24030	14.83830
b=0.5	9	2.1	3.4	0.03573	0.18410	0.81594	27.99	1.22560	16.42690
Y=25	11	2.4	1.8	0.01640	0.17970	0.82033	60.99	1.21900	14.68092
Y=50	12	2.6	1.9	0.01000	0.19400	0.80623	107.27	1.24030	14.83830
Y=100	14	2.9	1.9	0.00373	0.20000	0.80001	267.97	1.25000	14.99482
W=10	12	2.6	1.9	0.00932	0.19380	0.80623	107.27	1.24030	14.69408
W=25	12	2.6	1.9	0.01000	0.19400	0.80623	107.27	1.24030	14.83830
W=50	12	2.6	1.9	0.00932	0.19380	0.80623	107.27	1.24030	15.07866
C_0=5	12	2.6	1.8	0.00932	0.19380	0.80623	107.27	1.24030	10.02888
C_0=10	12	2.6	1.9	0.01000	0.19400	0.80623	107.27	1.24030	14.83830
C_0=100	13	2.6	2.7	0.00932	0.15730	0.84268	107.27	1.18670	53.22659
C_1=50	13	2.6	3.0	0.00932	0.15730	0.84268	107.27	1.18670	12.78691
C_1=100	12	2.6	1.9	0.01000	0.19400	0.80623	107.27	1.24030	14.83830
C_1=200	11	2.6	1.2	0.00932	0.23680	0.7632	107.27	1.31030	18.41464
δ=0.5	20	2.1	2.0	0.03573	0.44588	0.55412	27.99	1.80465	17.10696
δ=1.0	12	2.6	1.9	0.01000	0.19400	0.80623	107.27	1.24033	14.83830
δ=2.0	5	3.1	1.5	0.00194	0.08501	0.91499	516.7	1.09291	13.71214
T_0=0	12	2.6	1.9	0.01000	0.194	0.80623	107.27	1.24030	14.83830
T_0=0.2	12	2.6	1.9	0.01000	0.194	0.80623	107.27	1.24030	14.83830
T_0=0.4	12	2.6	1.9	0.01000	0.194	0.80623	107.27	1.24030	14.83830
T_1=0	12	2.6	1.8	0.00932	0.1938	0.80623	107.27	1.24030	13.15016
T_1=2	12	2.6	1.9	0.01000	0.194	0.80623	107.27	1.24030	14.83830
T_1=1	12	2.6	1.9	0.00932	0.1938	0.80623	107.27	1.24030	14.00288
T_2=1	12	2.6	1.9	0.00932	0.1938	0.80623	107.27	1.24030	15.65781
T_2=0	12	2.6	1.9	0.01000	0.194	0.80623	107.27	1.24030	14.83830
T_2=2	12	2.6	1.9	0.00932	0.1938	0.80623	107.27	1.24030	16.46186
γ_1=0,γ_2=0	12	2.6	1.8	0.00932	0.19377	0.806234	107.268	1.24033	12.89712
γ_1=0,γ_2=1	12	2.6	1.8	0.00932	0.19377	0.806234	107.268	1.24033	12.89712
γ_1=1,γ_2=0	12	2.6	1.9	0.00932	0.19377	0.806234	107.268	1.24033	14.83830
γ_1=1,γ_2=1	12	2.6	1.9	0.00932	0.19377	0.806234	107.268	1.24033	14.83830

Based on the observations from table 5.4 we find that θ, which is the average occurrence rate of the assignable cause, has a significant effect on the optimum sampling interval as well as the cost. A smaller θ implies a larger optimal sampling interval and smaller cost. It can be seen that when the value of θ decreases by 80% (from 0.05 to 0.01) the value of h increases by approximately 137.5% (from 0.8 to 1.9) and the value of $E(L)$ decreases approximately 42.77% (from 25.92872 to 14.83830). Consider for example the cases where $\theta = 0.05$ and $\theta = 0.01$ in the table 5.4 the expected cost decreases from R25.92872 to R14.83830. However, the optimum control limit parameter k is rather robust to changes in θ.

Table 5.4 also describes the economic design for three different shift sizes ($\delta = 0.5$, 1.0, and 2.0). When the shift level δ is increased, we obtain a different set of optimal design parameters. As a result, the different shift sizes produce different hourly production costs. In general, as δ increases, the excepted cost, the sample size and sampling interval decrease. However, the control limit width increases.

Regarding the costs of sampling, when the fixed cost per sample a increases, there is an increase in the sample size n and the sampling interval h. The control limit width parameter, k, however, is unaffected. Increasing the variable cost per unit sampled, b, results in a significant increase in the sampling interval h, but a slight decrease in sample size n and control limit width parameter k. Moreover, the expected cost increases only slightly as the fixed and/or variable costs increase.

The analysis of the economic design of the $\bar{X}$-control chart based on the Lorenzen and Vance (1986) approach shows that the time to sample and chart one item, g, the expected time to discover the assignable cause, T_1, and the expected time to repair the process, T_2, all have little effect on the optimal values. However, there are still some trends that can be observed. For example, the larger T_1, the larger $E(L)$. However, T_0, the expected search time when the alarm is false, seems to have no effect on the optimal values.

Table 5.4 shows that whether the production ceases or continues during the search and also whether the production ceases or continues during repair affect the optimal values only slightly. When the production continues during search, the cost will be slightly larger than when the production ceases during search, regardless of whether the production ceases or continues during repair. Variations in the quality cost per hour when the process is in-control and out-of-control

(C_0 and C_1) can cause relatively large effects on the optimum values of h and $E(L)$. Surprisingly the cost per false alarm, Y, and the cost to locate and repair the assignable cause, W, only have a small effect on the optimum values.

Summarizing the results in the sensitivity analysis of the economic design of the $\bar{X}$-control chart shown above, it is clear that the optimum control limit parameter k is relatively robust to all the input parameters. Among all the factors, C_0, C_1, θ, and δ have more impact on the optimal cost and optimal sampling interval of the $\bar{X}$-control chart. The remaining eight input parameters have no significant effect. This result agrees with the finding of Simpson and Keats, (1995).

5.2.2. Sensitivity on the economic statistical design

In this section, an extensive sensitivity analysis is performed. The major purpose is to find how constraints on statistical performance measures such as α, $1-\beta$, and ATS affect the expected cost (Saniga, 1989). Results of the sensitivity analysis will be valuable to users of control charts, providing guidelines for making trade-off decisions between cost and statistical properties. In order to perform the sensitivity analysis experiments, the same example provided by Torng, Cochran, Montgomery, and Lawrence (1995) is used. The parameters are $\theta = 0.01$, δ=1, $a = 0.5$, b=0.1, $Y = 50$, $W = 25$, $C_0 = 10$, $C_1 = 100$, $g = 0.05$, $T_0 = 0$, $T_1 = 2$, $T_2 = 0$, $\gamma_1 = 1$ and $\gamma_2 = 1$. Here we investigate the effects of varying the bounds on α, $1-\beta$, the average time to signal ATS and the expected shift size δ on the minimum expected cost and the three decision variables n, h, and k, based on the unified Lorenzen and Vance approach when the economic statistical design control chart is employed. Note that in the analysis of the economic statistical design, the indicator variables, i.e. $\gamma_1 = 1$, and $\gamma_2 = 1$, indicate that production continues during the search and repair periods.

The type I error rate or false alarm rate is the probability of concluding that the process mean has shifted due to an assignable cause when in fact it has not. A high number of false alarms will quickly undermine an operator's confidence in the use of the control charts and acquire unnecessary search costs. In control chart terms, power is the probability of correctly identifying a shift in the process when one exists. Power gives a performance measure of the control chart's capability to detect undesirable shifts that may occur in the production process. ATS refers to the promptness in which a significant process shift is in fact identified. It is a measure of the control chart design's responsiveness in detecting process shifts and is especially important when producing defective products resulting in penalty costs as shown in Zhang and Berardi, (1997).

5.3. Discussions

An extensive sensitivity analysis is performed on a unified approach of the Lorenzen and Vance model of the economic statistical design of the $\bar{X}$-control chart in order to investigate the relevant effect of the bounds on the statistical measures, such as the type I error rate, the power, the ATS, and the shift size to be detected on the minimum expected cost.

Zhang and Berardi (1997) noted that when the sensitivity analysis is performed on the economic statistical design, there is no general rule governing the selection of bounds for the statistical constraints. They should be chosen based on the specific problem situation, the relevant cost information, as well as the economic and statistical consequences.

The values and ranges for the sensitivity analysis are chosen as follows. The upper bound on α, when fixed, equals 0.05 while the lower bound on $(1-\beta)$ is fixed at 0.95. The ATS upper bound is set to 4.0 as this is the economic design's actual value. The investigated value for each sensitivity variable is chosen to range from being relatively cost influential to high cost influential. The upper bound of α ranges from 0.002 to 0.0167 and the lower bound of $(1-\beta)$ ranges from 0.700 to 0.975, whereas the upper bound of ATS varies from 1.0 to 4.0. In line with the literature we also use shift sizes to be detected in the range 0.20 to 2.50. Figs 5.1-5.3 contain the effect of the expected cost per hour due to changing the bounds on α, $(1-\beta)$, and ATS, respectively. Figure 5.4 gives the δ sensitivity results. Each figure represents the effects on the expected cost, sample size, sampling interval and control limit parameter for varying bounds.

The α and $(1-\beta)$ sensitivity data in Fig 5.1 and 5.2 designate similar cost sensitivity over the investigated ranges. As the bounds are tightened, each shows the expected cost increase. Similar results were reported by Saniga (1989). The sample size, sampling interval, and control limit interactions account for these changes. The effects of the bounds on α and $(1-\beta)$ on the decision variables n, h, and k are also shown in the figures. The patterns of the effects in these figures are consistent with what one can expect. For example, the control limit parameter, k, is determined by α, but is not related to the power, $(1-\beta)$ as can be seen in fig. 5.2.(d).

Table 5.5: Effect of bounds on α on the optimal design of the $\bar{X}$-control chart

ARL_0	α_U	EXPECTED COST	SAMPLE SIZE	SAMPLING INTERVAL	CONT LIMIT PARAMETER k	ACTUAL α	ACTUAL $1-\beta$	ACTUAL ATS
60	0.0167	14.83830	12	1.9	2.6	0.00932	0.80623	2.35664
80	0.0125	14.83830	12	1.9	2.6	0.00932	0.80623	2.35664
100	0.0100	14.83830	12	1.9	2.6	0.00932	0.80623	2.35664
140	0.0071	14.84440	12	1.8	2.7	0.00693	0.77760	2.31482
190	0.0053	14.86360	13	1.8	2.8	0.00511	0.78975	2.27921
267	0.0038	14.89850	13	1.7	2.9	0.00373	0.75977	2.23753
500	0.0020	14.99650	15	1.8	3.1	0.00194	0.78023	2.30700

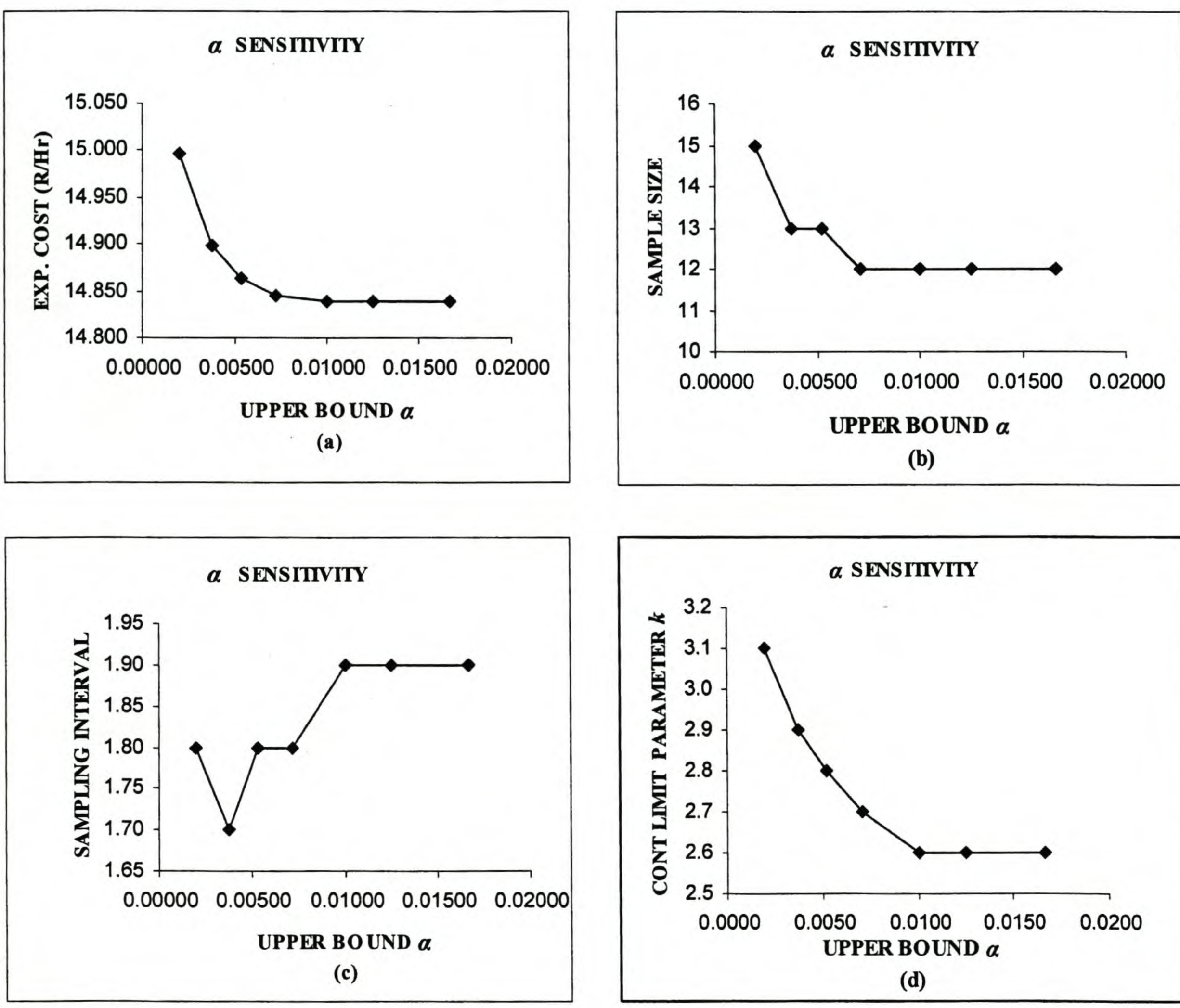

Figure5.1: α sensitivity analysis results

Table 5.6: Effect of bounds on $1-\beta$ on the optimal design of the $\bar{X}$-control chart

ARL_1	$(1-\beta)_L$	EXPECTED COST	SAMPLE SIZE	SAMPLING INTERVAL	CONT LIMIT PARAMETER k	ACTUAL α	ACTUAL $1-\beta$	ACTUAL ATS
1.0256	0.975	15.19497	20	2.7	2.5	0.01242	0.97570	2.76724
1.0526	0.950	15.06830	19	2.4	2.7	0.00693	0.95143	2.52251
1.0811	0.925	14.96862	16	2.4	2.5	0.01242	0.93319	2.57182
1.1111	0.900	14.92200	16	2.2	2.7	0.00693	0.90320	2.43579
1.1429	0.875	14.88680	15	2.1	2.7	0.00693	0.87960	2.38745
1.1765	0.850	14.86075	14	2.0	2.7	0.00693	0.85122	2.34958
1.2121	0.825	14.84588	13	2.0	2.6	0.00932	0.84268	2.37337
1.2500	0.800	14.83830	12	1.9	2.6	0.00932	0.80623	2.35664
1.4286	0.700	14.83830	12	1.9	2.6	0.00932	0.80623	2.35664

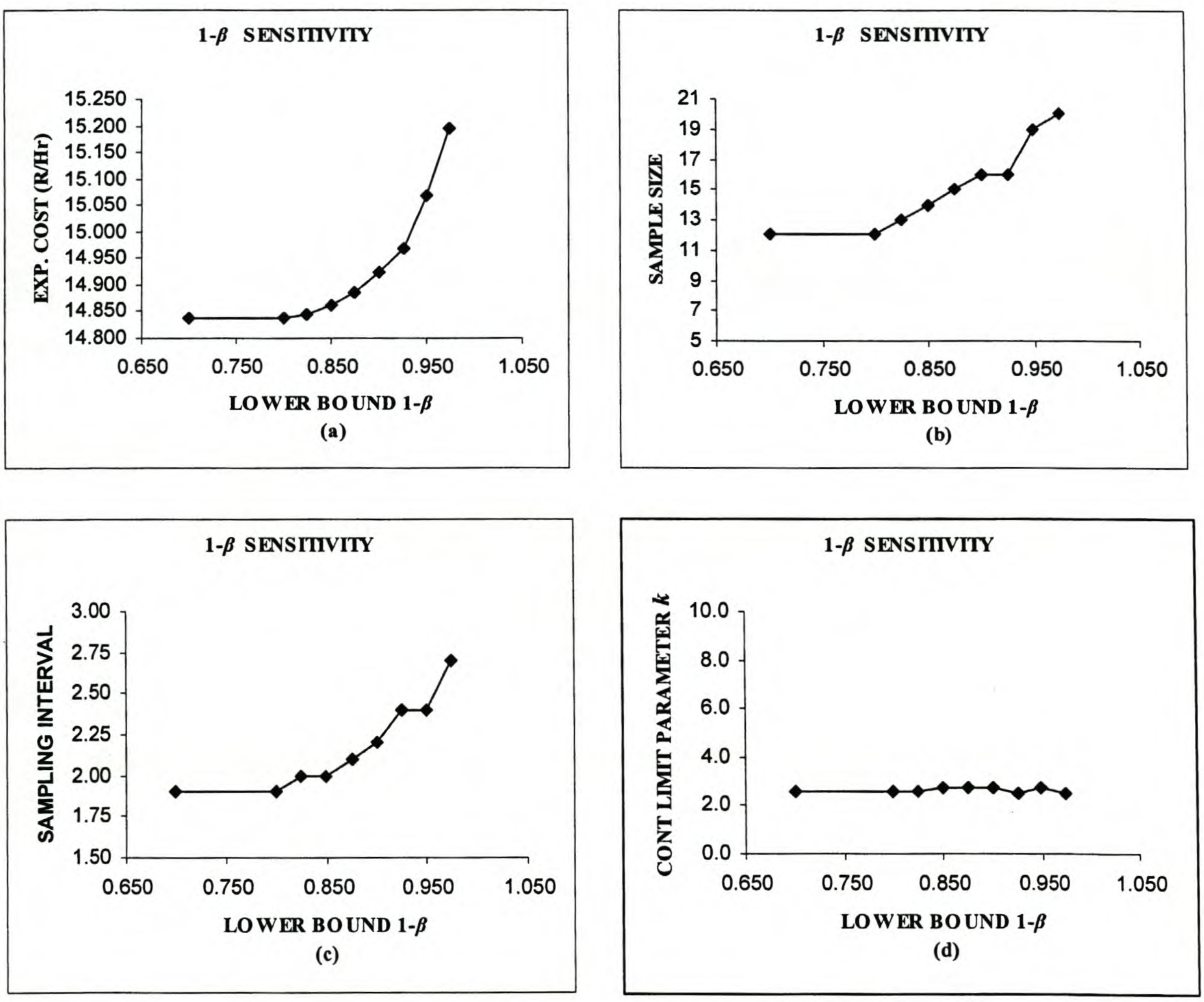

Figure 5.2: $1-\beta$ sensitivity analysis results

From figures 5.1 and 5.2 we observe that, as α_U decreases, the sampling size increases and the control limits become wider. If the upper bound of the α_U risk is greater than 0.01, the optimal design remains unchanged. The sampling interval is also affected by the value of α_U, but no specific tendency can be seen. If the lower bound of the power, $(1-\beta)_L$ is less than 0.80, the optimal design remains unchanged. As $(1-\beta)_L$ increases from 0.80, both the sample size and the sampling interval increase, and the change in the control limits, in general, show no pattern. Similar results were reported by Al-Oriani and Rahim (2002).

The sensitivity data in table 5.7 and Fig. 5.3 show increasing cost as the ATS upper bound decreases, but on a much larger scale than on α and $(1-\beta)$. When the ATS_U is below 2.0, the cost increases rapidly to a high of R15.714 at the upper bound ATS of 1.0. This suggests that an appropriate selection of the bound on the ATS may be important in terms of the economic consequence of low upper bounds on the ATS. The effect of the upper bounds on the ATS on the sampling size is quite interesting (fig 5.3 (b)). Above the ATS bound of 1.25, the sampling size is not sensitive to the bounds on ATS. From 1.00-1.25, the control limits first increase and then decrease with a maximum value of 2.7 at the ATS_U of 1.25. This phenomenon may result from the interaction effect of the bounds on α, $(1-\beta)$ and the ATS as suggested in the data table 5.7. However, in this example the effect of ATS_U on the control limit is not significant.

From Figs 5.1-5.3 as well as table 5.5-5.7, we find that in some instances the economic statistical design corresponds to a statistical design. A statistical design results when the type I error rate, α and the power are constrained at bounds which have an influence on the cost. This occurs in the α sensitivity analysis for α upper bounds of 0.01 and below. Above this α bound, $(1-\beta)$ remains fixed while the actual α relaxes from the strict equality constraint. For lower bounds of $(1-\beta)$ at 0.80 and above or for ATS upper bounds of 2.50 and below, the designs are also pure statistical designs.

The δ sensitivity analysis in Fig. 5.4 indicates a relative cost insensitivity for shift values of 1.0 and above with an extreme sensitivity for smaller values. The sample size is quite sensitive to the shift level with the highest value of 25 at $\delta = 0.2$ and a lowest value of 3 at $\delta = 2.5$.

Table 5.7: Effect of bounds on ATS on the optimal design of the $\bar{X}$-control chart

ATS_U	EXPECTED COST	SAMPLE SIZE	SAMPLING INTERVAL	CONT LIMIT PARAMETER k	ACTUAL α	ACTUAL $1-\beta$	ACTUAL ATS
4.00	14.83830	12	1.9	2.6	0.00930	0.80620	2.35664
3.00	14.83830	12	1.9	2.6	0.00930	0.80620	2.35664
2.50	14.83830	12	1.9	2.6	0.00930	0.80620	2.35664
2.00	14.86550	12	1.6	2.6	0.00930	0.80620	1.98454
1.75	14.93387	12	1.4	2.6	0.00930	0.80620	1.73647
1.50	15.06607	12	1.2	2.6	0.00930	0.80620	1.48840
1.25	15.29247	11	0.9	2.7	0.00690	0.73130	1.23075
1.00	15.71425	12	0.8	2.6	0.00930	0.80620	0.99227

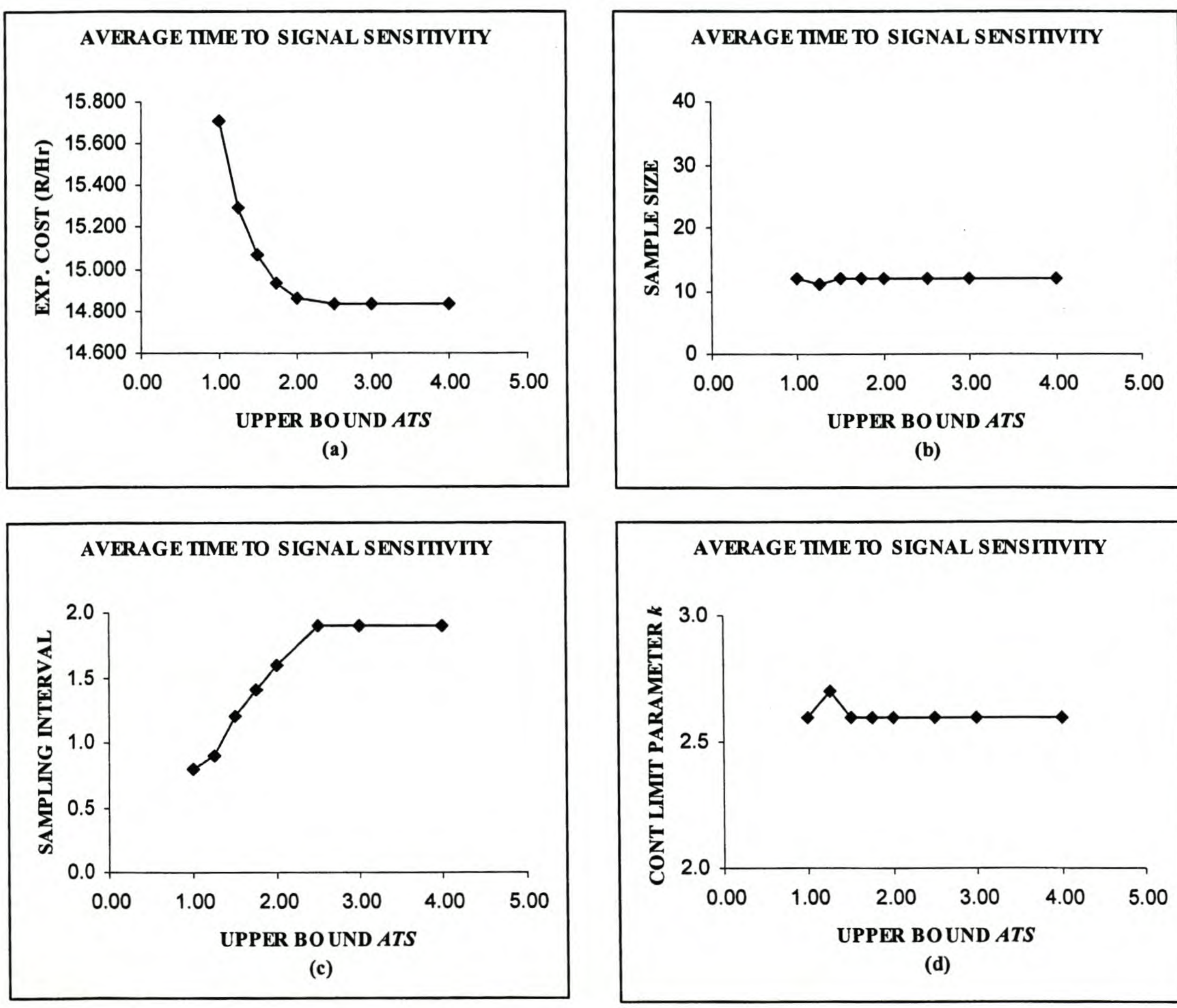

Figure5.3: ATS sensitivity analysis results

Table 5.8: Effect on δ on the optimal design of the $\bar{X}$-control chart

δ	EXPECTED COST	SAMPLE SIZE	SAMPLING INTERVAL	CONT LIMIT PARAMETER k	ACTUAL α	ACTUAL $1-\beta$	ACTUAL ATS
0.2	23.07564	25	1.0	2.1	0.03573	0.13663	7.31882
0.3	19.95052	25	1.4	2.1	0.03573	0.27441	5.10182
0.4	18.12762	25	2.0	2.1	0.03573	0.46019	4.34600
0.5	17.10696	20	2.0	2.1	0.03573	0.55412	3.60930
1.0	14.83830	12	1.9	2.6	0.00932	0.80623	2.35664
1.5	14.07837	7	1.6	2.9	0.00373	0.85738	1.86615
2.0	13.71214	5	1.5	3.1	0.00194	0.91499	1.63936
2.5	13.51826	3	1.3	3.1	0.00194	0.89068	1.45957

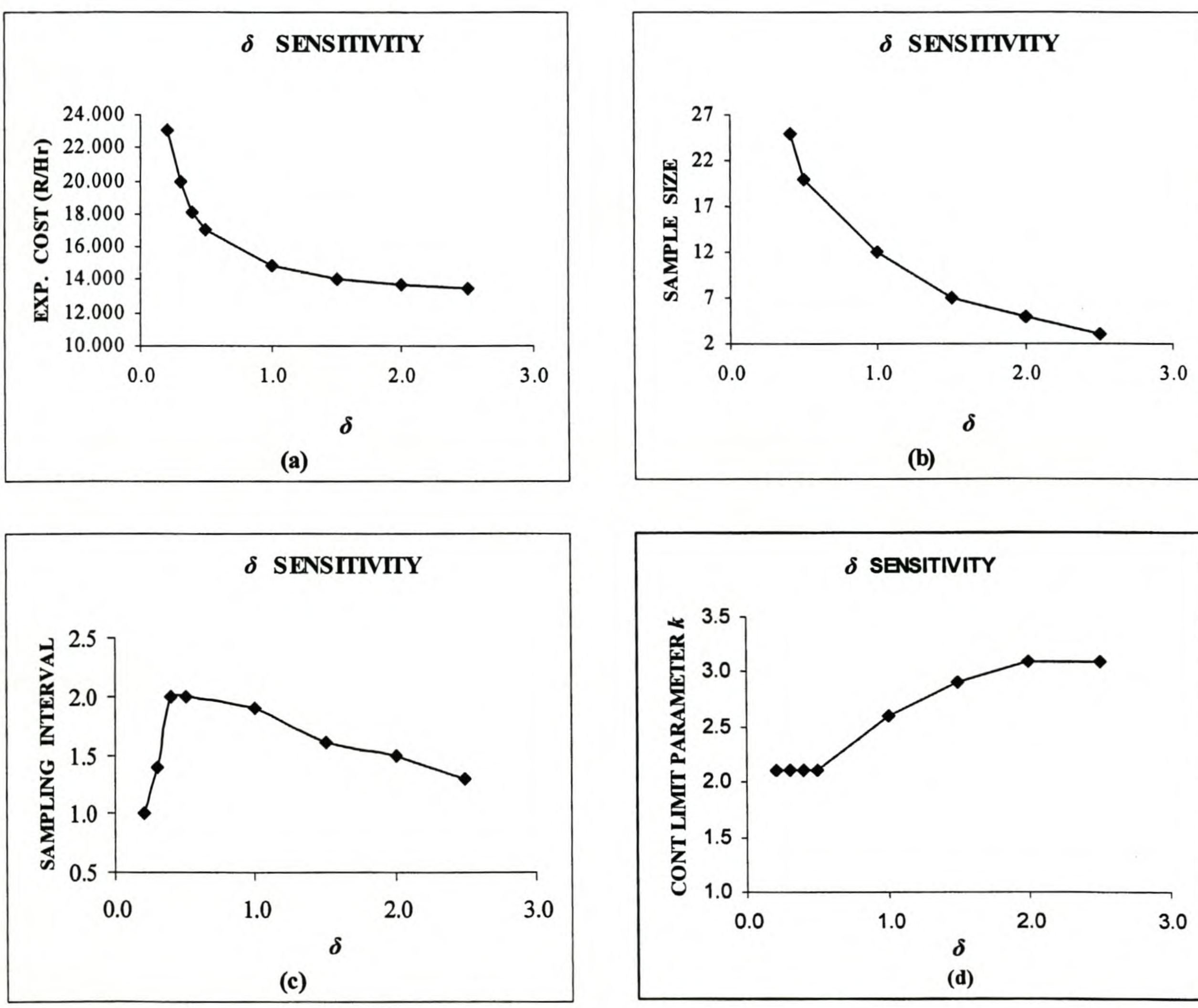

Figure 5.4: δ sensitivity analysis results

Note that the control limits are also affected by the shift size. We observe that an increase in the size of the shift, δ from 0.5 to 2.50 results in an increase in the control limit width parameter k from 2.1 to 3.1. However, if δ is less than 0.5, the control limit width parameter k remains unchanged. Furthermore, the sampling interval first increases and then decreases with a maximum value of 2.0 at the shift size, $\delta = 0.4$.

5.4. Economic statistical design versus economic design

As seen in the previous chapter, the objective of both economic designs and economic statistical designs is to minimize the expected total cost per unit time via a non-linear optimization procedure. The differences are that in the economic designs it is required that the user estimates a number of cost and system parameters, but yields a design that is economically optimal in that it minimizes the expected total cost (McWilliams and Saniga, 2001). Economic statistical designs on the other hand are subject to constraints on the type I error rate and power, or any other constraints according to the designer's needs (Al-Oraini and Rahim, 2002). In other words, the economic statistical designs are economic designs that are subject to constraints. Consequently, economic statistical designs are costlier than economic designs due to the added constraints. However, these added statistical constraints can guarantee long-term product or service quality, keeping the false alarm searches at a minimum, and lead to low process variability. This results directly from the requirement that the economic statistical design assures a satisfactory statistical performance.

According to the comparisons in the three numerical examples in section 5.1 of the minimum cost of the optimal economic design and optimal economic statistical designs with ARL as well as ATS constraints, the costs in the economic statistical designs increase by 0.41% and 0.37% for ARL and ATS constraints, respectively. This implies that the placing of statistical constraints results in a relatively small increase in the expected cost, while an improved statistical performance of the control chart has been achieved.

Perhaps the most important attribute of the economic statistical design is its flexibility. Saniga (1989) points out that with the appropriate choice of design constraints the user can choose a purely statistical design, a purely economic design or a design meeting any of the temporal requirements of the system to which it is to be applied.

An economical statistical design is at least as difficult to implement as an economic design since the same parameters must be estimated and more complex algorithms must be employed. Also, an economic statistical design can allow the process to operate out-of-control more often than an economic design. Fortunately, if caution is used in the selection of the design constraints, this problem can easily be remedied.

Chapter Six

Concluding remarks

6.1. Conclusion

According to Deming's hypothesis, customer satisfaction is gained through a quality product service, and it can be postulated that economic success is the measure of that satisfaction. The $\bar{X}$-control chart is the engineer's and statistician's most important tool for management of the quality of the firm's products. Thus, when research is done with respect to the economic aspects of quality control and improvement, it is clear that the design of the $\bar{X}$-control chart is substantive in any such research as it attaches directly to the way in which actual decisions are made if the firm wishes to remain alive.

Designing a control chart means making fundamental decisions about chart parameters such as the sample size n, sampling interval h and control limits width parameter k. The criteria used for developing rational designs are typically based on either statistical performance or economic considerations, or both. There are two important assumptions stated in the development and use of economic or economic statistical models, which are potentially critical. The assumptions of the exponential distribution as a model for the time between the process shifts, as well as the assumption of a single assignable cause. Moreover, three categories of costs were considered: the costs of sampling and testing, the costs associated with investigating an out-of-control signal and with repair or correction of any assignable causes found, and the costs associated with the production of defective items.

Economic or economic statistical models are generally derived using a total cost function per unit time, where the function expresses the relationship between the control chart parameters n, k and h as well as the three types of costs mentioned above. The cost function in the unified approach of the Lorenzen and Vance model depends on twelve cost and time parameters that describe the process, two indicator variables that show whether production continues during search or repair, and three design parameters that describe the charting procedure. The minimization of this function over the choice of design parameters leads to the most economic or economic statistical $\bar{X}$-control chart.

It is clear that few practitioners have adopted the economic modeling approach to design their control charts, because the cost models and their associated optimization techniques are often too complex and difficult for practitioners to apply. However, the numerical examples shown in this paper were executed on a user-friendly Excel program, and the proposed procedure is easy to use and easy to understand. Moreover, the proposed procedure can also obtain an exact optimal design rather than the approximate designs as derived by Duncan (1956) and other subsequent researchers. Thus, this procedure can be used to implement both economic and economic statistical designs of $\bar{X}$-control charts.

In the case of an economic design, the program finds the optimal sample size, control limit width and sampling interval by minimizing a total cost function. In the statistically constrained economic design, statistical constraints are put on some parameter such as the average run length (ARL) or the average time-to-signal (ATS) in the Lorenzen and Vance cost function. The program calculates the optimal values of k and h for several sample sizes and displays the corresponding values of the minimum cost for each value of n. The values of ARL_0 and ARL_1 are also provided for each combination of n, k and h. All combinations of results are tabulated and the optimum combination is easily obtained from the tables. Furthermore, no terms are neglected or approximated in finding the optimum solution as is done in Duncan's approach, so that the solution is deemed to be more reliable than that according to Duncan's approach. Hence, this approach is more appropriate to use for the economic and economic statistical design of $\bar{X}$-control chart.

A comprehensive comparison was made of the economic and the economic statistical designs of the $\bar{X}$-control chart using cost and statistical performance as the criteria. Results of this study point out that the economic statistical designs have several advantages of importance in today's industry when compared to the economic designs. Economic statistical designs have wider control limits and smaller sampling intervals than economic designs. In addition, while they are more costly than the economic designs, they have other advantages such as guaranteeing high output quality, keeping the number of false alarm searches at a minimum and low process variability. This results directly from the requirement that the economic statistical design must assure a satisfactory statistical performance.

An extensive sensitivity analysis was also performed to provide insight into the effect of the significant inputs on the proposed cost function when the economic and economic statistical designs of the $\bar{X}$-control chart were employed. The sensitivity study of the economic design of the $\bar{X}$-control chart has indicated that the four primary cost drivers such as the mean occurrence rate of the assignable cause, θ, the shift level, δ, the quality control cost while producing in control, C_0 and the quality control cost while producing out of control, C_1 are significant parameters while the remaining eight inputs are less significant.

Using the economic statistical design with an appropriate sensitivity analysis, one can readily observe the impact on cost, sample size, sampling interval, and control limits due to the constraints on the statistical error rates. The selections of bounds for the statistical constraints were based on the specific problem situation, the relevant cost information, as well as the possible economic and statistical consequences. The sensitivity analysis is useful to the designer in making these decisions. As mentioned above, the statistical performance of control charts can be improved significantly with only a slight increase in cost by using an economic statistical design instead of an economic design. The cost increase is further shown to be relatively insensitive to the improvement in the type I error and the power throughout the investigated range. This implies that it may not be the correct approach for practitioners to implement control charts with lower false alarm rates and with higher probability for detecting a process shift when one actually exists. On the other hand, the bound on ATS should not be set too low since the expected cost is highly sensitive to small ATS bounds. A reasonable bound on ATS may be found from the actual ATS in the economic design. Moreover, relatively large shift sizes, δ often result in relatively smaller cost sensitivity, but extreme sensitivity for smaller shift sizes, δ. The optimal sample size and control limit width parameter were also largely determined by the magnitude of the shift size, δ. This result agrees with the result of the analysis given by Saniga (1989).

As a result of this study one can state that the principles of the economic statistical design are fully consistent with the objectives of statistical quality control, i.e. simultaneously reducing costs, while maintaining high quality. The flexibility of this method in developing alternative designs is also illustrated and it was argued that this flexibility is of much importance in the context of the firm's wider decision-making. Therefore, whenever possible, a unified approach of the economic statistical model should be considered as a viable general method in the $\bar{X}$-control chart design.

6.2. Remarks

Based on the discussions and conclusions of this study, the following remarks can be made:

- A further detailed study needs to be conducted to develop systematic methods for parameter estimation from the process data in the economic or economic statistical design of the $\bar{X}$-chart. Moreover, sensitivity analyses should concentrate on identifying the parameter and data requirements for particular models that have the greatest effect on the economic or economic statistical designs of the $\bar{X}$-chart.
- According to the study, the underlying assumption with respect to the distribution of the process mechanism in the economic or the economic statistical designs of the $\bar{X}$-control charts is the exponential distribution. However, this assumption is not always appropriate in practice for the process in which machine wear occurs over time. This means that the assumption of an exponential distribution may not be appropriate when the process describes deterioration over time. The fact is that the appropriate assumption in the case for the economic or economic statistical designs of the $\bar{X}$-control chart with a varying sampling interval is the Weibull or Gamma failure mechanism. For instance, in the case of a process with an increasing failure rate, a more realistic approach is to shorten the sampling intervals since the process deteriorates further as time goes by. The gamma distribution often is an appropriate distribution in the quality control chart as well as reliability studies. For example, consider a standby redundant system having two components with a perfect switch. While component 1 is on, component 2 is off, and when component 1 fails, the switch turns component 2 on. If each component has a life time described by the exponential distribution with parameter λ then the system's life is gamma distributed with scale parameter λ and shape parameter $\gamma = 2$.
- Software developers should develop a simple, standard approach to economic and economic statistical designs by stating problems and solutions clearly, by requiring the estimation of only a few important parameters and providing a method to estimate these parameters, providing easy-to-use, easy-to-understand and easy-to-access software.
- The Excel optimal procedure which was developed in this study can be extended for other designs of control charts such as the p-chart, the c-chart, the cumulative sum ($CUSUM$) control chart, moving average (MA) control chart, and the exponentially weighted moving average ($EWMA$) control chart. A further study is also needed using

this Excel optimal procedure for comparing the performances of the $\bar{X}$-control chart, the *CUSUM* control chart, and the *EWMA* control chart. Here, for the case of the *EWMA* control chart the calculation of the in-control ARL_0 and out-of-control ARL_1 average run lengths are required to use the control limit parameter k, the values of the weighted factor λ, and the shift size δ. The calculation of the value of the weighted factor λ may not be straightforward since it considers Fredholm integral equations or numerical integration methods.

- Last but not the least, since the Excel optimal procedure developed in this study is simple-to-use, simple-to-understand, easy-to-access and performs better in finding the optimal solutions in the designs of both the economic and the economic statistical $\bar{X}$-control charts than previous approaches, practitioners and researchers using these optimal designs are encouraged to use this for both academic and industrial purposes.
- Moreover, it is recommended that a unified approach of the economic statistical design of the $\bar{X}$-control chart be used in practice since it considers indicator variables in the model to identify whether production ceases or continues during search and or repair, so that any possible operation scenario can be appropriately modelled. It gives also superior protection over a wider range of process shifts and also have statistical properties that are as good as control charts designed entirely from statistical considerations.

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Appendix

(A.1) Theorem 3.1

$$0<\frac{1}{\lambda h}-\frac{1}{e^{\lambda h}-1}<1, \text{ for } \lambda>0, \text{ and } h>0$$

Proof

To prove $0<\frac{1}{x}-\frac{1}{e^{x}-1}<1$, for all $x>0$

$$e^{x}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\frac{x^{4}}{4!}+\ldots\ldots$$

a) First we need to show that the above expression is greater than zero

$$\frac{1}{x}-\frac{1}{e^{x}-1}$$

$$=\frac{1}{x}-\frac{1}{x+\frac{x^{2}}{2!}+----}>0$$

b) In a similarly way it is also less than 1

$$\frac{1}{x}-\frac{1}{e^{x}-1}$$

$$=\frac{e^{x}-1-x}{x(e^{x}-1)}$$

$$=\frac{\frac{x}{2!}+\frac{x^{2}}{3!}+-----}{x+\frac{x^{2}}{2!}+-----}<1, \text{ for all } x>0$$

Therefore from (a) and (b), the proof of theorem 3.1 is completed.

Hence, $0<\frac{1}{\lambda h}-\frac{1}{e^{\lambda h}-1}<1$, for all $\lambda>0$ and $h>0$

(A.2) Theorem 3.2

$$\lim_{x\to 0^{+}}\left(\frac{1}{x}-\frac{1}{e^{x}-1}\right)=\frac{1}{2}$$

Proof

From (b) in the proof of theorem 3.1

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\frac{1}{2}$$

$$=\lim_{x\to 0^+}\frac{\frac{x}{2!}+\frac{x^2}{3!}+-----}{x+\frac{x^2}{2!}+-----}$$

$=\frac{1}{2}$ (using L'Hospital)

This completes the proof of theorem 3.2

(A.3) Derivation of equation (3.5)

$E(L)=\dfrac{\lambda a_4 B+\frac{\alpha a_5}{h}+\lambda a_3}{1+\lambda B}+\dfrac{a}{h}+\dfrac{bn}{h}$, setting equal to zero the partial derivative of $E(L)$ with respect to n, we get $\dfrac{\partial E(L)}{\partial n}=0$.

$$\therefore \frac{\frac{\partial B\lambda a_4}{\partial n}(1+\lambda B)-(\lambda a_4 B+\lambda a_3+\frac{\alpha a_5}{h})\frac{\partial B}{\partial n}\lambda}{(1+B\lambda)^2}+\frac{b}{h}=0$$

$$\therefore \frac{\frac{\partial B}{\partial n}\lambda a_4+\frac{\partial B}{\partial n}\lambda^2 Ba_4-\frac{\partial B}{\partial n}\lambda^2 Ba_4-\frac{\partial B}{\partial n}\lambda^2 a_3-\frac{\partial B}{\partial n}\frac{\lambda\alpha a_5}{h}}{(1+B\lambda)^2}+\frac{b}{h}=0$$

$$\therefore \frac{\frac{\partial B}{\partial n}(\lambda a_4+\lambda^2 Ba_4-\lambda^2 Ba_4-\lambda^2 a_3-\frac{\lambda\alpha a_5}{h})}{(1+B\lambda)^2}+\frac{b}{h}=0$$

$$\therefore \frac{\frac{\partial B}{\partial n}(\lambda a_4-\lambda^2 a_3-\frac{\lambda\alpha a_5}{h})}{(1+B\lambda)^2}+\frac{b}{h}=0$$

$\therefore \lambda h\dfrac{\partial B}{\partial n}\left(a_4-\dfrac{\alpha a_5}{h}-\lambda a_3\right)+b(1+\lambda B)^2=0$, where $\dfrac{\partial B}{\partial n}=-\dfrac{\partial p}{\partial n}\left(\dfrac{h}{P^2}\right)+g$.

(A.4) Derivation of equation (3.6)

$$E(L)=\frac{\lambda a_4 B+\frac{\alpha a_5}{h}+\lambda a_3}{1+\lambda B}+\frac{a}{h}+\frac{bn}{h}$$

$$\frac{\partial E(L)}{\partial h}=0$$

$$\therefore \frac{\left(\frac{\partial B}{\partial h}\lambda a_4 - \frac{\alpha a_5}{h^2}\right)(1+\lambda B) - \frac{\partial B}{\partial h}\lambda\left(\lambda a_4 B + \lambda a_3 + \frac{\alpha a_5}{h}\right)}{(1+B\lambda)^2} - (\frac{a+bn}{h^2}) = 0$$

$$\therefore \frac{\frac{\partial B}{\partial h}(\lambda a_4 + B\lambda^2 a_4) - \frac{\alpha a_5}{h^2}(1+\lambda B) - \frac{\partial B}{\partial h}\left(\lambda^2 a_4 B + \lambda^2 a_3 + \frac{\alpha\lambda a_5}{h}\right)}{(1+B\lambda)^2} - (\frac{a+bn}{h^2}) = 0$$

$$\therefore \frac{\frac{\partial B}{\partial h}(\lambda a_4 - \lambda^2 a_3 - \frac{\alpha\lambda a_5}{h}) - \frac{\alpha a_5}{h^2}(1+\lambda B)}{(1+B\lambda)^2} - (\frac{a+bn}{h^2}) = 0$$

$$\therefore \frac{h^2\frac{\partial B}{\partial h}(\lambda a_4 - \lambda^2 a_3 - \frac{\alpha\lambda a_5}{h}) - \alpha a_5(1+\lambda B)}{(1+B\lambda)^2} - (a+bn) = 0$$

$$\therefore h^2\frac{\partial B}{\partial h}\left(\lambda a_4 - \lambda^2 a_3 - \frac{\alpha\lambda a_5}{h}\right) - \alpha a_5(1+B\lambda) = (1+B\lambda)^2(a+bn)$$

$$\lambda h^2\frac{\partial B}{\partial h}\left(a_4 - \frac{\alpha a_5}{h} - \lambda a_3\right) - \alpha a_5(1+\lambda B) - (a+bn)(1+\lambda B)^2 = 0, \text{ where } \frac{\partial B}{\partial h} = \left(\frac{1}{P} - \frac{1}{2} + \frac{\lambda h}{6}\right)$$

(A.5) Derivation of equation (3.7)

$$E(L) = \frac{\lambda a_4 B + \frac{\alpha a_5}{h} + \lambda a_3}{1+\lambda B} + \frac{a}{h} + \frac{bn}{h}$$

$$\frac{\partial E(L)}{\partial k} = 0$$

$$\therefore \frac{\frac{\partial B}{\partial k}\lambda a_4 + \frac{\partial\alpha}{\partial k}\frac{a_5}{h}(1+\lambda B) - (\lambda a_4 B + \lambda a_3 + \frac{\alpha a_5}{h})(\frac{\partial B}{\partial k}\lambda)}{(1+B\lambda)^2} = 0$$

$$\therefore \frac{\frac{\partial B}{\partial k}\left(\lambda a_4 + \lambda^2 a_4 B - \lambda^2 a_4 B - \lambda^2 a_3 + \frac{\alpha\lambda a_5}{h}\right) + \frac{\partial\alpha}{\partial k}\frac{a_5}{h}(1+\lambda B)}{(1+B\lambda)^2} = 0$$

$\frac{\partial E(L)}{\partial k} = 0$ implies the numerator to be zero where $(1+B\lambda)^2 \neq 0$

$$\therefore \frac{\partial B}{\partial k}\lambda\left(a_4 - \lambda a_3 + \frac{\alpha a_5}{h}\right) + \frac{\partial\alpha}{\partial k}\frac{a_5}{h}(1+\lambda B) = 0, \text{ with } \frac{\partial B}{\partial k} = -\frac{h}{p^2}\left(\frac{\partial p}{\partial k}\right).$$

(A.6) For small α and λ, the terms like αa_3, $\frac{\alpha a_5}{h}$, and λB may be neglected. Duncan (1956) approximated equations (A.3)-(A.5) by assuming λ to be small and neglecting all terms in an equation of a smaller order of magnitude than the principal term. This gives us

$$\frac{\partial B}{\partial n}\lambda h\left(a_4-\frac{\alpha a_5}{h}-\lambda a_3\right)+b(1+\lambda B)^2=0$$

$$\frac{\partial B}{\partial n}\lambda h a_4+b\approx 0 \text{ and } \frac{\partial B}{\partial n}=-\frac{\partial p}{\partial n}\left(\frac{h}{P^2}\right)$$

$$-\frac{\lambda h^2 a_4\dfrac{\partial P}{\partial n}}{P^2}+b\approx 0$$

(A.7) $$\lambda h^2\frac{\partial B}{\partial h}\left(a_4-\frac{\alpha a_5}{h}-\lambda a_3\right)-\alpha a_5(1+\lambda B)-(a+bn)(1+\lambda B)^2=0$$

$$\therefore \lambda h^2\frac{\partial B}{\partial h}a_4-\alpha a_5-(a+bn)\approx 0 \text{ and } \frac{\partial B}{\partial h}=\left(\frac{1}{P}-\frac{1}{2}+\frac{\lambda h}{6}\right)$$

$$\therefore \lambda h^2 a_4\left(\frac{1}{P}-\frac{1}{2}\right)-\alpha a_5-a-bn\approx 0$$

(A.8) $$\frac{\partial B}{\partial k}\lambda\left(a_4-\lambda a_3+\frac{\alpha a_5}{h}\right)+\frac{\partial \alpha}{\partial k}\frac{a_5}{h}(1+\lambda B)=0$$

$$\therefore \frac{\partial B}{\partial k}\lambda a_4+\frac{\partial \alpha}{\partial k}\frac{a_5}{h}\approx 0 \text{ and } \frac{\partial B}{\partial k}=-\frac{h}{p^2}\left(\frac{\partial p}{\partial k}\right)$$

$$\therefore -\frac{h}{p^2}\left(\frac{\partial p}{\partial k}\right)\lambda a_4+\frac{\partial \alpha}{\partial k}\frac{a_5}{h}\approx 0$$

$$\therefore -\frac{\lambda h^2 a_4\dfrac{\partial P}{\partial k}}{P^2}+a_5\frac{\partial \alpha}{\partial k}\approx 0$$

(A.9) Equation (A.7) immediately gives us

$$\lambda h^2 a_4\left(\frac{1}{P}-\frac{1}{2}\right)-\alpha a_5-a-bn\approx 0$$

$$\therefore h^2\approx\frac{\alpha a_5+a+bn}{\lambda a_4\left(\frac{1}{P}-\frac{1}{2}\right)}$$

$$\therefore h\approx\sqrt{\frac{\alpha a_5+a+bn}{\lambda a_4\left(\frac{1}{P}-\frac{1}{2}\right)}}$$

(A.10) Using this approximate value of h as in (A.9) in (A.6) we get after some rearrangement,

$$-\frac{\lambda h^2 a_4 \frac{\partial P}{\partial n}}{P^2} + b \approx 0$$

$$\therefore \frac{-\frac{\partial p}{\partial n}\lambda\left(\frac{\alpha a_5 + a + bn}{\lambda a_4\left(\frac{1}{P}-\frac{1}{2}\right)}\right)a_4}{P^2} + b \approx 0$$

$$\frac{\partial p}{\partial n}\left(\frac{\alpha a_5 + a + bn}{p^2\left(\frac{1}{p}-\frac{1}{2}\right)}\right) \approx b$$

$$\therefore \frac{P^2\left(\frac{1}{P}-\frac{1}{2}\right)}{\frac{\partial P}{\partial n}} \approx \frac{\alpha a_5 + a + bn}{b}$$

$$\therefore -n + \frac{P^2\left(\frac{1}{P}-\frac{1}{2}\right)}{\frac{\partial P}{\partial n}} \approx \frac{\alpha a_5 + a}{b}$$

(A.11) $\beta = P$ (in-control signal | process is out-of-control).

and $(1-\beta) = \Phi(\delta\sqrt{n}-k) + \Phi(-\delta\sqrt{n}-k)$

$= \int_{-\infty}^{-k-\delta\sqrt{n}} \phi(z)dz + \int_{k-\delta\sqrt{n}}^{\infty} \phi(z)dz$ where $\phi(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$

$\cong \int_{k-\delta\sqrt{n}}^{\infty} \phi(z)dz$ since the first integral is very close to zero

$= \int_{-\infty}^{\delta\sqrt{n}-k} \phi(z)dz$

$= \Phi\left(\delta\sqrt{n}-k\right).$

(A.12) $\frac{\partial p}{\partial k} = \frac{\partial}{\partial k}\left(\int_{k-\delta\sqrt{n}}^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz\right)$

$$= \frac{-e^{-\frac{\left(k-\delta\sqrt{n}\right)^2}{2}}}{\sqrt{2\pi}}$$

(A.13) $\therefore \dfrac{\partial p}{\partial n} = \dfrac{\partial}{\partial n}\left(\displaystyle\int_{k-\delta\sqrt{n}}^{\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,dz\right)$

$$= \frac{\delta}{2\sqrt{n}}\left(\frac{e^{-\frac{\left(k-\delta\sqrt{n}\right)^2}{2}}}{\sqrt{2\pi}}\right)$$

(A.14) $\alpha = P(\text{exceeding control limits} \mid \text{process in control})$.

$$= 2\int_{k}^{\infty}\phi(z)dz\text{, where } \phi(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$$

$$= 2\int_{-\infty}^{-k}\phi(z)dz$$

$$= 2\Phi(-k)$$

$$\therefore \frac{\partial \alpha}{\partial k} = \frac{\partial}{\partial k}\left(2\int_{-\infty}^{-k}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,dz\right)$$

$$= -\frac{2e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}$$

(A.15) $\dfrac{\lambda h^2 a_4}{p^2} = \dfrac{b}{\frac{\partial p}{\partial n}}$, from (A.6) and combined with equation (A.8)

$$-\frac{\partial P}{\partial k}\left(\frac{b}{\frac{\partial p}{\partial n}}\right) + a_5\frac{\partial \alpha}{\partial k} \approx 0 \text{ and by substituting}$$

$$\frac{\partial P}{\partial k} = \frac{-e^{\frac{\left(-k-\delta\sqrt{n}\right)^2}{2}}}{\sqrt{2\pi}}\text{, from (A.12),}$$

$$\frac{\partial P}{\partial n}=\frac{\delta}{2\sqrt{n}}\left(\frac{e^{\frac{\left(-k-\delta\sqrt{n}\right)^2}{2}}}{\sqrt{2\pi}}\right)$$, from (A.13) and

$$\frac{\partial \alpha}{\partial k}=-\frac{2e^{-\frac{k^2}{2}}}{\sqrt{2\pi}}$$, from (A.14)

Finally, it is found that

$$\frac{\partial \alpha}{\partial k}=-\frac{2b\sqrt{n}}{\delta a_5},$$

or

$$\frac{e^{\frac{-k^2}{2}}}{\sqrt{2\pi}}=\frac{b\sqrt{n}}{\delta a_5}.$$

(A.16) Chiu and Wetherill (1974) noted that in practice, λ is a small quantity, say $\lambda = 0.01$, and hence λB is small compared with unity. Therefore, the term λB can be omitted from the first denominator of

$$E(L)=\frac{\lambda a_4 B+\frac{\alpha a_5}{h}+\lambda a_3}{1+\lambda B}+\frac{a}{h}+\frac{bn}{h}$$

so that

$$E(L^{'})=\lambda a_4 B+\frac{\alpha a_5}{h}+\lambda a_3+\frac{a}{h}+\frac{bn}{h}$$

Substituting the value of n from $\delta\sqrt{n}-k=z$ and $B=\left(\frac{1}{p}-\frac{1}{2}\right)h+gn+D$ in to $E(L^{'})$.

Since $\tau \approx \frac{h}{2}$ (theorem 3.1 and 3.2)

$$E(L^{'})=\lambda a_4\left((\frac{1}{p}-\frac{1}{2})h+\frac{g(z+k)^2}{\delta^2}+D\right)+\frac{\alpha a_5}{h}+\lambda a_3+\frac{a}{h}+\frac{b(z+k)^2}{h\delta^2}.$$

Equating the partial derivative, $\frac{\partial E(L^{'})}{\partial k}$ to zero we obtain

$\therefore 2\lambda a_4 g\frac{(z+k)}{\delta^2}+\frac{\partial \alpha a_5}{\partial kh}+2b\frac{(z+k)}{h\delta^2}=0$ since p is a constant (say 0.90 or 0.95).

$$\therefore 2(b+\lambda a_4 gh)(z+k)+\delta^2\frac{\partial \alpha}{\partial k}a_5=0.$$

(A.17) From equation (A.16), the term $\lambda a_4 gh$ is usually small because g is often small, and could have been omitted, as Duncan (1956) has done, but, Chiu and Wetherill (1974) showed that the effect of omitting this term may be serious if g happens to be moderately large, say $g = 0.3$. The presence of h in this term makes the equation in (A.16) intractable and complicated. For the sake of simplicity and practicality, they replaced $\lambda a_4 gh$ by $\lambda a_4 g$ by dropping the h only. Therefore, the optimal value of k can be approximated by the solution

$$\frac{z+k}{\phi(k)} = \frac{\delta^2 a_5}{b + \lambda a_4 g}$$

(A.18) $s = \sum_{i=0}^{\infty} iP($ assignable cause occurs between the i th and $(i+1)$ st sample)

$$= \sum_{i=0}^{\infty} i\left(e^{\theta hi} - e^{\theta h(i+1)}\right)$$

$$= \frac{e^{-\theta h}}{1 - e^{-\theta h}}$$

$$= \frac{1}{e^{\theta h} - 1}.$$

(A.19) The expected time of occurrence of the assignable cause within the interval the i th and $(i+1)$ st, denoted by $E(\tau)$, is

$$= \frac{\int_{ih}^{(i+1)h} \theta(t - hi)\exp(-\theta t)dt}{\int_{ih}^{(i+1)h} \theta \exp(-\theta t)dt}$$

$$= \frac{-\theta^{-1}\exp(-\theta t)(1+\theta t) + hi\exp(-\theta t)\Big|_{hi}^{h(i+1)}}{-\exp(-\theta t)\Big|_{hi}^{h(i+1)}}$$

$$= \frac{1-(1+\theta h)\exp(-\theta h)}{\theta(1-\exp(-\theta h))}$$

$$= \frac{1}{\theta(1-\exp(-\theta h))} - \frac{\exp(-\theta h)}{\theta(1-\exp(-\theta h))} - \frac{h\exp(-\theta h)}{(1-\exp(-\theta h))}$$

$$= \frac{1-\exp(-\theta h)}{\theta(1-\exp(-\theta h))} - \frac{h\exp(-\theta h)}{(1-\exp(-\theta h))}$$

$$= \frac{1}{\theta} - \frac{h}{\exp(\theta h) - 1}.$$ (Compare to Duncan's approximation.)

(A.20) Formula in Excel:

$$\alpha = 2\Phi(-k)$$
$$= 2 * NORMDIST\ (-k, 0, 1, TRUE)$$

(A.21) Formula in Excel:

$$\beta = \Phi(k - \delta\sqrt{n}) - \Phi(-k - \delta\sqrt{n})$$
$$= NORMDIST\ (k - \delta * SQRT(n), 0, 1, TRUE)$$
$$-NORMDIST\ (-k - \delta * SQRT(n), 0, 1, TRUE)$$

Computer code

$\alpha = 2 * NORMDIST\,(-B6, 0, 1, TRUE)$

$$\beta = NORMDIST\,(B6 - \$B\$2 * SQRT(A6), 0, 1, TRUE) \\ - NORMDIST\,(-B6 - \$B\$2 * SQRT(A6), 0, 1, TRUE)$$

$E(\tau) = 1/\$A\$2 - C6/((2.7182818)\text{^}(\$A\$2 * C6) - 1)$

$s = 1/((2.7182818)\text{^}(\$A\$2 * C6) - 1)$

$$NUM_1 = \$I\$2/\$A\$2 + \$G\$2 * (-G6 + A6 * \$H\$2 + C6 * N6 + \$M\$2 * \$K\$2 + \$N\$2 * \$L\$2) \\ + H6 * \$E\$2/L6 + \$F\$2$$

$$NUM_2 = ((\$C\$2/\$D\$2 * A6)/C6) \\ *(1/\$A\$2 - G6 + A6 * \$H\$2 + C6 * N6 + \$M\$2 * \$K\$2 + \$N\$2 * \$L\$2)$$

$DEN = 1/\$A\$2 + (1 - \$M\$2) * (H6 * \$J\$2)/L6 - G6 + A6 * \$H\$2 + C6 * N6 + \$K\$2 + \$L\$2)$

$E(L) = (I6 + J6)/O6$

$MIN = (P6 : P655)$ (for $n = 1$)

$MIN\ OF\ MIN = (P6 : P16555)$ (for $n = 1,2,3,.........20$)

Example 5.2: Optimal economic statistical designs with ARL constraints

$\alpha \le \frac{1}{267}$ $ARL_0 \ge 267$ $CONSTRAINT\ ARL_0 = IF(K6 > 267, K6, "1")$

$1 - \beta \ge \frac{1}{40}$ $ARL_1 \le 40$ $CONSTRAINT\ ARL_1 = IF(M6 <= 40, M6, "10000")$

Example 5.3: Optimal economic statistical designs with ATS constraints.

$\frac{h}{1-\beta} \le 1.90$ $ATS_1 \le 1.90$ $CONSTRAINT\ ATS_1 = IF(L6 < 1.90, L6, "10000")$

Sensitivity on the economic statistical design

(a) Effect of bounds on α on the optimal design of the $\bar{X}$-control chart.

$ARL_L = 60$ $CONSTRAINT\ ON\ ARL_0 = IF(K6 > 60, K6, "1")$

$ARL_L = 80$ $CONSTRAINT\ ON\ ARL_0 = IF(K6 > 80, K6, "1")$

$ARL_L = 100 \quad CONSTRAINT\ ON\ ARL_0 = IF(K6 > 100, K6, "1")$

$ARL_L = 140 \quad CONSTRAINT\ ON\ ARL_0 = IF(K6 > 140, K6, "1")$

$ARL_L = 190 \quad CONSTRAINT\ ON\ ARL_0 = IF(K6 > 190, K6, "1")$

$ARL_L = 267 \quad CONSTRAINT\ ON\ ARL_0 = IF(K6 > 267, K6, "1")$

$ARL_L = 500 \quad CONSTRAINT\ ON\ ARL_0 = IF(K6 > 500, K6, "1")$

(b) Effect of bounds on $1-\beta$ on the optimal design of the $\bar{X}$-control chart

$ARL_U = 1.0256 \quad CONSTRAINT\ ON\ ARL_1 = IF(M6 < 1.0256, M6, "1")$

$ARL_U = 1.0526 \quad CONSTRAINT\ ON\ ARL_1 = IF(M6 < 1.0526, M6, "1")$

$ARL_U = 1.0811 \quad CONSTRAINT\ ON\ ARL_1 = IF(M6 < 1.0811, M6, "1")$

$ARL_U = 1.1111 \quad CONSTRAINT\ ON\ ARL_1 = IF(M6 < 1.1111, M6, "1")$

$ARL_U = 1.1429 \quad CONSTRAINT\ ON\ ARL_1 = IF(M6 < 1.1429, M6, "1")$

$ARL_U = 1.1765 \quad CONSTRAINT\ ON\ ARL_1 = IF(M6 < 1.1765, M6, "1")$

$ARL_U = 1.2121 \quad CONSTRAINT\ ON\ ARL_1 = IF(M6 < 1.2121, M6, "1")$

$ARL_U = 1.2500 \quad CONSTRAINT\ ON\ ARL_1 = IF(M6 < 1.2500, M6, "1")$

$ARL_U = 1.4286 \quad CONSTRIANT\ ON\ ARL_1 = IF(M6 < 1.4286, M6, "1")$

(C) Effect of bounds on ATS on the optimal design of the $\bar{X}$-control chart

$ATS_U = 4.00 \quad CONSTRAINT\ ATS_1 = IF(L6 < 4.00, L6, "10000")$

$ATS_U = 3.00 \quad CONSTRAINT\ ATS_1 = IF(L6 < 3.00, L6, "10000")$

$ATS_U = 2.50 \quad CONSTRAINT\ ATS_1 = IF(L6 < 2.50, L6, "10000")$

$ATS_U = 2.00 \quad CONSTRAINT\ ATS_1 = IF(L6 < 2.00, L6, "10000")$

$ATS_U = 1.75 \quad CONSTRAINT\ ATS_1 = IF(L6 < 1.75, L6, "10000")$

$ATS_U = 1.50 \quad CONSTRAINT\ ATS_1 = IF(L6 < 1.50, L6, "10000")$

$$ATS_U = 1.25 \quad CONSTRAINT\ ATS_1 = IF(L6 < 1.25, L6, "10000")$$

$$ATS_U = 1.00 \quad CONSTRAINT\ ATS_1 = IF(L6 < 1.00, L6, "10000")$$