Closure and Compactness in Frames

by

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Declaration

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Abstract

As an introduction to point-free topology, we will explicitly show the connection between topology and frames (locales) and introduce an abstract notion, which in the point-free setting, can be thought of as a subspace of a topological space. In this setting, we refer to this notion as a *sublocale* and we will show that there are at least four ways to represent sublocales.

By using the language of category theory, we proceed by investigating closure in the point-free setting by way of operators. We define what we mean by a *co-closure operator* in an abstract context and give two seemingly different examples of co-closure operators of **Frm**. These two examples are then proven to be the same.

Compactness is one of the most important notions in classical topology and therefore one will find a great number of results obtained on the subject. We will undertake a study into the interrelationship between three weaker compact notions, i.e. *feeble compactness*, *pseudocompactness* and *countable compactness*. This relationship has been established and is well understood in topology, but (to a degree) the same cannot be said for the point-free setting. We will give the frame interpretation of these weaker compact notions and establish a point-free connection. A potentially promising result will also be mentioned.

Uittreksel

As ’n inleiding tot punt-vrye topologie, sal ons eksplisiet die uiteensetting van hierdie benadering tot topologie weergee. Ons definieer ’n abstrakte konsep wat, in die punt-vrye konteks, ooreenstem met ’n subruimte van ’n topologiese ruimte. Daar sal verder vier voorstellings van hierdie konsep gegee word.

Afsluiting, deur middel van operatore, word in die puntvrye konteks ondersoek met behulp van kategorie teorie as taalmedium. Ons sal ’n spesifieke operator in ’n abstrakte konteks definieer en twee oënskynlik verskillende voorbeelde van hierdie operator verskaf. Daar word dan bewys dat hierdie twee operatore dieselfde is.

Kompaktheid is een van die mees belangrikste konsepte in klassieke topologie en as gevolg daarvan geniet dit groot belangstelling onder wiskundiges. ’n Studie in die verwantskap tussen drie swakker forme van kompaktheid word onderneem. Hierdie verwantskap is al in topologie bevestig en goed begryp onder wiskundiges. Dieselfde kan egter, tot ’n mate, nie van die puntvrye konteks gesê word nie. Ons sal die puntvrye formulering van hierdie swakker konsepte van kompaktheid en hul verbintenis, weergee. ’n Resultaat wat moontlik belowend kan wees, sal ook genoem word.

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I took the one less traveled by,
And that has made all the difference.
-Robert Frost

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Introduction

I hope that by giving a historical survey of the subject known as "pointless topology" I shall succeed in convincing the reader that it does after all have some point to it. - Peter Johnstone (1983)

Pointless or (as it is also known) point-free topology has been the focus of mathematicians since the early 1910's. Initial interest was sparked by the German mathematician Felix Hausdorff [14] who is believed to be the first to consider, instead of points in the space, the "notion of (open) set (or neighbourhood) as primitive..." [19]. Consequently, after 1914, it was common knowledge that a topological space gives rise to a lattice of open sets. A detailed outline of the history and development of point-free topology can be found in Johnstone [19].

The early part in the point-free development of topology can, to a large extent, be credited to the study of the connection between algebra (lattice theory) and topology by mathematicians. We mention the research by the American mathematician, Marshall Stone, who published papers on the topological representation of Boolean algebras [1936, 1934] and distributive lattices [1937]. Stone showed that the sets of prime ideals can be represented as open sets.

Another American mathematician, Henry Wallman, published a paper entitled *"Lattices and topological spaces"* [1938] where he applied lattice theory to define a compactification of a T_1 topological space. Authors such as the Polish mathematician Tarski, the American logician McKinsey and the Austrian Karl Menger, published articles [1944, 1940] that specifically focused on the algebra (lattice theory) of topology, and thereby narrowing the gap between these two fields of study. In addition, the first textbook, in which general topology was considered consistently from a lattice-theoretic point of view, was written by the German Georg Nöbeling [1954].

Articles published in the late 1950's and afterward, show that a fundamental shift came about the view mathematicians had of research in topology. Instead of inquiring about a set of points equipped with a topology, in the classical sense, an "indirect" lattice theoretic approach was considered, now known as frame theory (locale theory). Interest amongst authors grew rapidly after a few initial papers and consequently a good many results were produced.

The paper by Bénabou [6] contains some earlier work in frame theory and Isbell [16] published a paper showing that products behave better in this setting than in topology. Great emphasis was put on enriching the point-free setting by providing the frame (locale) counterpart of classical notions in topology, e.g. separation axioms, sums and products. Authors such as B. Banaschewski [1969], C.H. Dowker, D. Papert Strauss [1974, 1976] and J. Isbell [1981] deserve mentioning. *Stone spaces* by P.T. Johnstone [1982] is still a primary source of reference to point-free topology for students today.

Since the mid 1980's, extensive research has gone into frame and locale theory, with notions such as closure, compactness and completion being studied. Point-free structures including uniform, nearness and σ-frames have also enjoyed interest amongst authors and continue doing so.

Compactness is one of the most important notions in topology and consequently enjoyed keen interest amongst researchers with a large number of results produced. Compactness and other weaker properties which have been studied extensively in general topology include pseudocompactness, countable compactness, sequential compactness and Lindelöf. Given their importance, research into compactness properties in the point-free context is not a surprising consequence. Particularly, since in the point-free setting, proofs are generally constructive and require no choice principles. (See Johnstone [19] for a more detailed account.)

The subject of our focus is the interrelationship between three weaker notions of compactness, i.e. *feeble compactness*, *pseudocompactness* and *countable compactness* and a brief account of more recent developments will be given.

In topology, the interrelationship between pseudocompact and countable compact spaces has been established as it is well-known, see Engelking [13], that every countably compact Tychonoff space is pseudocompact and every pseudocompact normal space is countably compact. Also, in 1984, Porter and Woods showed that for Tychonoff spaces feeble compactness and pseudocompactness are equivalent notions. This result however stretches back to 1955, when Mardešić and Papić first proved it as a characterization of pseudocompactness by means of a cover property.

Our interest, however, lies in the point-free setting. Category theory provides the appropriate language for such a pursuit and the category of frames a suitable setting. We give a frame translation of three of the classical axioms of separation, i.e. regular, completely regular and normal, which is due to Dowker and Strauss [10].

In order to define the point-free counterpart of pseudocompactness, one needs to introduce the classical reals without referring to points. This is done by Johnstone [18] and referred to as the *frame of reals* $\mathcal{L}(\mathbb{R})$. Banaschewski and Gilmour [5] contains a more recent account. The translation from topology to frames is then immediate and is taken as our definition:

> A frame L is said to be *pseudocompact* if every $\varphi : \mathcal{L}(\mathbb{R}) \to L$ is bounded.

Depending on one's need, this definition might not be ideal as it characterizes pseudocompactness by means of an external property and not as an attribute of the frame itself. In 1996, however, Banaschewski and Gilmour (and a year later Banaschewski [4]) provided three characterizations for completely regular pseudocompact frames without referring to the reals. Two of these characterizations involve the notion of a cozero part of a frame and we preferred utilizing the third:

> A frame L is pseudocompact iff any sequence $a_1 \prec\prec a_2 \prec\prec \ldots$ such that $\bigvee a_i = 1$ in L terminates, that is, there is $a_k = 1$ for some k.

Since the topological definition of countable compactness (see Engelking [13]) rests upon the use of open sets, the point-free definition can be taken as the direct frame translation:

> A frame L is said to be *countably compact* if every countable cover of L has a finite subcover.

Likewise, the definition of feeble compactness is taken as the frame counterpart of the topological one. In 1984, Porter and Woods gave a topological definition and also provided a characterization of feeble compact spaces. The frame counterpart, as taken from Hlongwa [15], can readily be seen to be equivalent to

> A frame L is *feebly compact* if every countable cover of L has a finite subset the join of which is dense in L.

Feeble compactness has received some more recent interest amongst authors such as Hlongwa (2004) and Dube (2008). In his PhD dissertation, Hlongwa attempted to show an equivalence between all three weaker compact notions, for completely regular frames, without assuming any additional notion. However, a closer look at his proof revealed a flaw in his reasoning.

In 2007, Dube and Matutu established a connection between pseudocompactness and countable compactness for normal frames. They assumed an additional notion, i.e. paracompactness. In light of this and our knowledge of the connection between the three compact notions in topology, one can argue that to hope for an equivalence, assuming complete regularity alone, might seem too ambitious.

We conclude by giving a brief outline of the remaining chapters that the reader will find in this thesis:

Chapter 1 contains the necessary background content on lattice theory, topology and category theory respectively. Very little prior knowledge is assumed. For a detailed introduction to Order theory we recommend the book by [7]. The theory of general topology and related topics can be found in [13], and the book by [1] will provide the reader with the introductory concepts of category theory.

As our introduction to point-free topology, the reader will be shown the connection between topology and frames (locales) explicitly in chapter 2. We will give a motivation behind regarding the category of locales as a generalization of the category of topological spaces. This is a well-known result and can also be found in Picado and Pultr [25].

In chapter 3 we will introduce an abstract notion, which in the point-free setting, can be thought of as a subspace of a topological space. In this setting, we refer to this notion as a *sublocale* and we will show that there are at least four ways to represent sublocales. Some of the representations were already known by Johnstone [18], but a complete survey is given by Pultr [27].

By using the language of category theory, we investigate closure in the point-free setting by way of operators. Closure operators and their related properties are well set out in the book by Dikranjan and Tholen [8] and is our focus in chapter 4. We will define what we mean by a co-closure operator in an abstract context and give two seemingly different examples of co-closure operators of **Frm**. These two examples are then proven to be the same.

Chapter 1

Preliminaries

In this chapter the reader will find the essential background material, with little prior knowledge assumed, on the content in succeeding chapters. In addition, the aim is to agree on terminology and notation.

1.1 Order Theory

1.1.1 Partially ordered sets

Let B be a set. A *partial order* on B is a binary relation $\leq$ on B with the following properties: For all $a, b, c \,\epsilon\, B$,

(1) $a \leq a$ (reflexivity),

(2) $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity) and

(3) $a \leq b$ and $b \leq a$ imply $a = b$ (antisymmetry).

A set B with a partial order $\leq$ defined on it will be denoted by $(B, \leq)$ or, more often, by B. We will say B is a *partially ordered set* or a *poset*, for short. Let A be any subset of a poset B. An element

$b' \,\epsilon\, A$ is a *maximal* element of A if $b' \leq b \,\epsilon\, A$ implies $b' = b$.

$b^* \,\epsilon\, A$ is the *maximum* (or greatest) element of A if $b \leq b^*$ for all $b \,\epsilon\, A$.

Similar definitions can be made for a *minimal* and the *minimum* (or least) element of A, by interchanging $\leq$ with $\geq$.

Remark For a given poset B, we form a new poset B^{op} be defining $x \leq y$ in B^{op} if and only if $y \leq x$ in B. The poset B^{op} is called the *dual* or *opposite* of B.

Let S be any subset of a poset B. We define the following sets:

$$S^u = \{\, b \,\epsilon\, B \mid \forall\, s \,\epsilon\, \mathrm{S},\, b \;\geq\; s\} \text{ and } S^l = \{\, b \,\epsilon\, B \mid \forall\, s \,\epsilon\, S,\, b \;\leq\; s\}.$$

S^u and S^l are read as 'S upper' and 'S lower' respectively. An element of S^u is called an *upper bound* of S and an element of S^l is called an *lower bound* of S.

1.1.2 Supremum and infimum

An element $s' \,\epsilon\, B$ is called the *supremum* (if it exists) of $S \subseteq B$, B a poset, if s' is the minimum of S^u. Similarly, an element $s^* \,\epsilon\, B$ is called the *infimum* (if it exists) of $S \subseteq B$ if s^* is a lower bound of S and $s^* \geq s$ for all $s \,\epsilon\, S^l$. Commonly used notation for the supremum (resp. infimum) of a set $S \subseteq B$, is $supS$ (resp. $infS$).

If a non-empty poset B has a greatest element, we quite naturally call this element the *top* element of B. Dually, if B has a least element, it is called the *bottom* element of B. Notation is 1 and 0 respectively.

1.1.3 Lattices

Let B be a non-empty poset.

> If $sup\{a, b\}$ and $inf\{a, b\}$ exist for any two $a, b \,\epsilon\, B$, then B is called a *lattice.*
>
> If $supS$ and $infS$ exist for any $S \subseteq B$, then B is called a *complete lattice.*

For a complete lattice, the definition can be stated as a poset in which each subset has a supremum (or infimum). A complete lattice is necessarily bounded, i.e. it has a top 1 and a bottom 0.

Remark More often we will denote $\sup\{a, b\}$ (resp. $\inf\{a, b\}$) by $a \vee b$ (resp. $a \wedge b$). Similarly $supS$ and $infS$ will be denoted by $\bigvee S$ and $\bigwedge S$ respectively, which is read as "join S" and "meet S". We denote the empty set by $\emptyset$.

1.1.4 Distributive lattices, pseudocomplemented lattices and complements

A lattice L is *distributive* if for all $a, b, c \,\epsilon\, L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

This property is then also equivalent to the formula in which meet and join are interchanged.

Let L be a lattice with 0 and let $a \in L$. We say a^* is the *pseudocomplement* of a if

$$x \wedge a = 0 \iff x \leq a^*.$$

One can readily show that an element has at most one pseudocomplement and a lattice L with 0 is said to be *pseudocomplemented* if each element of L has a pseudocomplement. A few interesting and important rules in a pseudocomplemented lattice (The proofs for all rules and formulas in the following two subsections can be found in Pultr [27]):

(1) $0^* = 1$ and $1^* = 0$,

(2) $a \leq b \Rightarrow b^* \leq a^*$,

(3) $a \leq a^{**}$,

(4) $a^* = a^{***}$.

Let L be a bounded distributive lattice. An element $\overline{a} \in L$ satisfying

$$a \wedge \overline{a} = 0 \text{ and } a \vee \overline{a} = 1$$

is said to be the *complement* of $a \in L$. One can readily show that in a distributive lattice the complement (if it exists) of an element is unique. We say that $a \in L$ is *complemented* if $\overline{a}$ exists.

1.1.5 Heyting algebras

A *Heyting algebra* is a lattice L with an additional binary operation, $\rightarrow$, sometimes referred to as the *Heyting arrow*, that satisfies for all $a, b, c \in L$,

$$c \leq a \rightarrow b \iff a \wedge c \leq b.$$

In addition, if L is a complete lattice with the Heyting operation defined on it, then we say L is a *complete Heyting algebra*. Note that a (complete) Heyting algebra is necessarily distributive and pseudocomplemented. Some interesting and useful Heyting formulas:

(1) $a \rightarrow a = 1$ and $1 \rightarrow a = a$,

(2) $a \leq b$ iff $a \rightarrow b = 1$,

(3) $b \leq a \rightarrow b$,

(4) $a \wedge (a \rightarrow b) = a \wedge b$,

(5) $a \rightarrow \bigwedge b_i = \bigwedge (a \rightarrow b_i)$.

1.1.6 Galois connection

A map $f : L \to M$ between posets L and M is said to be *monotone* if for all $a, b \,\epsilon\, L$,

$$a \leq b \Rightarrow f(a) \leq f(b).$$

Let $f : L \to M$ and $g : M \to L$ be monotone maps between posets L and M. We say that the pair (f, g) is a *Galois connection* if for all $a \,\epsilon\, L$, $b \,\epsilon\, M$,

$$f(a) \leq b \Longleftrightarrow a \leq g(b).$$

The above condition can also be equivalently given as

$$fg(b) \leq b \text{ and } gf(a) \geq a, \quad \text{for all } a \,\epsilon\, L, b \,\epsilon\, M.$$

If such a situation exists, then f is said to be the *left adjoint* of g, and g the *right adjoint* of f. The following are well-known and useful facts:

- g is uniquely determined by f; which is also true the other way round,
- f preserves existing joins and g preserves existing meets,
- Each join preserving map $f : L \to M$ is a left adjoint and each meet preserving map $g : M \to L$ is a right adjoint, with L and M complete lattices.

1.2 Topology

1.2.1 Topological space

A *topology* on a set X is a collection of subsets of X, denoted by τ, which satisfies the following three conditions:

(1) $\emptyset \,\epsilon\, \tau$ and $X \,\epsilon\, \tau$,

(2) τ is closed under finite intersection (i.e. $A, B \,\epsilon\, \tau \Rightarrow A \cap B \,\epsilon\, \tau$) and

(3) τ is closed under arbitrary union (i.e. $\mathcal{A} \subseteq \tau \Rightarrow \bigcup \mathcal{A} \,\epsilon\, \tau$).

The pair (X, τ) is called a *topological space*. We will refer to the elements of τ as *open sets*. We will more often talk of the topological space X where it is to be understood that we refer to the set with the topology on it.

1.2.2 Subspace

For a topological space (X, τ) and arbitrary subset $A \subseteq X$, we define the following family of subsets of A:

$$\tau_A := \{A \cap U \,|\, U \,\epsilon\, \tau\}.$$

The family of sets τ_A satisfies the three conditions of a topological space, and consequently (A, τ_A) is a topological space. We say that (A, τ_A) is a *subspace* of (X, τ), or more concisely, A is a subspace of X. The topology itself is then referred to as the *subspace topology.*

1.2.3 Neighbourhood, Closure and Cover

Let (X, τ) be a topological space. $U \subseteq X$ is a *neighbourhood* of $x \,\epsilon\, X$ if there exists an open set A such that $x \,\epsilon\, A \subseteq U$. The collection of all neighbourhoods of x will be denoted by $\mathcal{U}_x$.

A subset $A \subseteq X$ is said to be *closed* iff its complement is open, i.e. $X \backslash A \,\epsilon\, \tau$. For any set $A \subseteq X$, we define the *closure* of A by

$$\overline{A} = \bigcap \{\, B \subseteq X \,|\, B \text{ closed}, \, A \subseteq B\}.$$

We can also define a closed set $A \subseteq X$ by:

$$A \text{ is closed iff } A = \overline{A}.$$

For a topological space (X, τ), a set $\{\, U_i \,|\, i \,\epsilon\, I\} \subseteq \tau$ with $\bigcup_{i \,\epsilon\, I} U_i = X$ is called an *open cover* of X. An open cover $\{U_i \,|\, i \,\epsilon\, I\}$ of X is said to have a finite *subcover* if there exists a finite family $\{i_1, i_2, ..., i_k\} \subseteq I$ such that $U_{i_1} \cup U_{i_2} \cup ... \cup U_{i_k} = X$.

1.2.4 Continuous map

We trust that the reader is familiar with the definitions of an injective, surjective and inverse map. A map $f : X \to Y$ is *continuous* at $x \,\epsilon\, X$ iff for any $A \subseteq X$

$$x \,\epsilon\, \overline{A} \Longrightarrow f(x) \,\epsilon\, \overline{f(A)}.$$

We say that f is *continuous* iff f is continuous at x for every $x \,\epsilon\, X$. The following three statements are equivalent for a map f:

(1) $f : X \to Y$ is continuous,

(2) A open in $Y \Rightarrow f^{-1}(A)$ open in X and

(3) B closed in $Y \Rightarrow f^{-1}(B)$ closed in X.

1.2.5 Axioms of separation

A topological space X is called a

(1) $T_0 - space$ if for every pair of distinct points $x_1, x_2 \epsilon X$ there exists an open set $U \subseteq X$ containing exactly one of these points.

(2) $T_1 - space$ if for every pair of distinct points $x_1, x_2 \epsilon X$ there exists an open set $U \subseteq X$ such that $x_1 \epsilon U$ and $x_2 \notin U$.

(3) $T_2 - space$, or a *Hausdorff space*, if for every pair of distinct points $x_1, x_2 \epsilon X$ there exists open sets U_1, U_2 such that $x_1 \epsilon U_1$, $x_2 \epsilon U_2$ and $U_1 \cap U_2 = \emptyset$.

(4) $T_3 - space$, or a *regular space*, if X is a $T_1 - space$ and for every $x \epsilon X$ and every closed set $A \subseteq X$ such that $x \notin A$ there exist open sets U_1, U_2 such that $x \epsilon U_1$, $A \subseteq U_2$ and $U_1 \cap U_2 = \emptyset$.

(5) $T_4 - space$, or a *normal space*, if X is a $T_1 - space$ and for every pair of disjoint closed sets $A, B \subseteq X$ there exist open sets U, V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

1.3 Category Theory

1.3.1 Category

Let $\mathcal{C}$ denote the quadruple $\{\mathcal{O}bj\mathcal{C}, hom, \circ, id\}$ where

(a) $\mathcal{O}bj\mathcal{C}$ is a family of *objects*;

(b) for every pair of objects $A, B \epsilon \mathcal{O}bj\mathcal{C}$, the sets $hom_{\mathcal{C}}(A, B)$, whose elements are called *morphisms* or *arrows* from A to B, are disjoint;

(c) for every $A, B, C \epsilon \mathcal{O}bj\mathcal{C}$ and every $f \epsilon hom_{\mathcal{C}}(A, B)$ and $g \epsilon hom_{\mathcal{C}}(B, C)$;

$$(f, g) \to g \circ f \epsilon hom_{\mathcal{C}}(A, C)$$

yields morphisms called *compositions*.

We say $\mathcal{C}$ is a *category* if it satisfies the following two conditions:

(1) *Associativity*: For every $A, B, C, D \epsilon \mathcal{O}bj\mathcal{C}$ and all $f \epsilon hom_{\mathcal{C}}(A, B)$, $g \epsilon hom_{\mathcal{C}}(B, C)$ and $h \epsilon hom_{\mathcal{C}}(C, D)$ we have

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

(2) *Identity*: For every object $A \epsilon \mathcal{O}bj\mathcal{C}$ there is a morphism $id_A \epsilon hom_{\mathcal{C}}(A, A)$, called the *identity*, such that if $f \epsilon hom_{\mathcal{C}}(A, B)$ and $g \epsilon hom_{\mathcal{C}}(C, A)$ we have

$$f \circ id_A = f \text{ and } id_A \circ g = g.$$

Remark Instead of writing $f \epsilon hom_{\mathcal{C}}(A,B)$ we will more often write $f : A \to B$. If $f : A \to B$, then A is the *domain* and B the *codomain* of f. $\mathcal{M}or\mathcal{C}$ will denote the family of all morphisms and is defined as the union of all the sets $hom_{\mathcal{C}}(A,B)$. For each object $A \epsilon \mathcal{O}bj\mathcal{C}$ it is also common to denote the identity morphism $id_A : A \to A$ by $1_A : A \to A$. $hom_{\mathcal{C}}(A,B)$ will be abbreviated by $hom(A,B)$ if it does not give rise to ambiguity and is sometimes referred to as a "hom-set".

We will often refer to the following examples of categories in chapters to come:

(1) **Set**: The category of sets and functions.

(2) **Top**: The category of topological spaces and continuous maps.

(3) **Frm**: The category of frames and frame homomorphisms.

(4) **Loc**: The category of locales and localic maps.

1.3.2 Duality

Let $\mathcal{C}$ be any category. We now construct the following category $\mathcal{C}^{op}$:

(a) The family of objects of $\mathcal{C}^{op}$ is the family of objects of $\mathcal{C}$,

(b) For every pair of objects $A, B \epsilon \mathcal{O}bj\mathcal{C}^{op}$, the set of morphisms is defined by:

$$hom_{\mathcal{C}^{op}}(A,B) := hom_{\mathcal{C}}(B,A)$$

(c) For every $f \epsilon hom_{\mathcal{C}^{op}}(A,B)$ and $g \epsilon hom_{\mathcal{C}^{op}}(B,C)$, we define composition

$$(f,g) \to f \circ g \,\epsilon\, hom_{\mathcal{C}^{op}}(A,C)$$

with $f \circ g$ to be formed in $\mathcal{C}$.

Verifying that this composition in $\mathcal{C}^{op}$ is associative and that the identities of $\mathcal{C}^{op}$ are also the identities in $\mathcal{C}$, we say that $\mathcal{C}^{op}$ is the *dual* or *opposite* category of $\mathcal{C}$. In essence the two categories have the same objects, but arrows are "turned around" in the dual category.

Remark Not only does duality apply to categories, but also to statements about categories. As a result we have the following *Duality Principle*:

(i) For each concept P, concerning a general category $\mathcal{C}$, the *dual concept* is obtained by applying this concept to the dual category $\mathcal{C}^{op}$.

(ii) For each valid theorem on categories the *dual theorem*, which is obtained be changing all the concepts in the original theorem to their duals, is also valid.

1.3.3 Functor

Covariant and Contravariant functor

Let $\mathcal{C}, \mathcal{D}$ be categories. Let F from $\mathcal{C}$ to $\mathcal{D}$ consist of

(1) a map that associates each $A \,\epsilon\, \mathcal{O}bj\mathcal{C}$ with $F(A) \,\epsilon\, \mathcal{O}bj\mathcal{D}$,

(2) a family of morphisms: for every $A, B \,\epsilon\, \mathcal{O}bj\mathcal{C}$ and each $f \,\epsilon\, hom_{\mathcal{C}}(A, B)$,

$$f \to F(f) \,\epsilon\, hom_{\mathcal{D}}(F(A), F(B)).$$

F is called a *covariant functor* if it satisfies the following two conditions:

(i) $F(1_A) = 1_{F(A)}$ for all $A \,\epsilon\, \mathcal{O}bj\mathcal{C}$,

(ii) For all $A, B, C \,\epsilon\, \mathcal{O}bj\mathcal{C}$ and every $f \,\epsilon\, hom_{\mathcal{C}}(B, C), g \,\epsilon\, hom_{\mathcal{C}}(A, B)$ we have

$$F(f \circ g) = F(f) \circ F(g).$$

An assignment F' from $\mathcal{C}$ to $\mathcal{D}$, where we replace, respectively, notions (2) and (ii) with the two notions below, will be referred to as a *contravariant functor.*

$(2')$ a family of morphisms: for every $A, B \,\epsilon\, \mathcal{O}bj\mathcal{C}$ and each $f \,\epsilon\, hom_{\mathcal{C}}(A, B)$,

$$f \to F(f) \,\epsilon\, hom_{\mathcal{D}}(F(B), F(A)).$$

(ii') For all $A, B, C \,\epsilon\, \mathcal{O}bj\mathcal{C}$ and every $f \,\epsilon\, hom_{\mathcal{C}}(B, C), g \,\epsilon\, hom_{\mathcal{C}}(A, B)$ we have

$$F(f \circ g) = F(g) \circ F(f).$$

Remark One can think of functors as "morphisms" between categories. We shall often write $F : \mathcal{A} \to \mathcal{B}$ if F is a functor from category $\mathcal{A}$ to category $\mathcal{B}$. Also, we will write FA and Ff instead of $F(A)$ and $F(f)$ respectively - that is if there is no ambiguity. A covariant functor is also often referred to as a 'functor'.

1.3.4 Special morphisms

Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. f is called a *monomorphism* if for all morphisms $u, v \,\epsilon\, \mathcal{M}or\mathcal{C}$

$$fu = fv \Rightarrow u = v.$$

We also say f is left cancellable. (Naturally, u and v must have the same domain and the same codomain X.) An *epimorphism* is defined as the dual notion of a monomorphism. That is, g is an *epimorphism* if for all morphisms $u, v \,\epsilon\, \mathcal{M}or\mathcal{C}$

$$ug = vg \Rightarrow u = v.$$

Dually, we would say g is right cancellable. (Naturally, u and v must have the same codomain and the same domain Y.) We say f is an *isomorphism* if there exists a morphism $g : Y \to X$ such that

$$fg = 1_Y \text{ and } gf = 1_X.$$

1.3.5 Special functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

(1) F is called *faithful* if for all $A, B \,\epsilon\, \mathcal{O}bj\mathcal{C}$ and $f, g \,\epsilon\, \mathcal{H}om(A, B)$

$$F(f) = F(g) \Longrightarrow f = g.$$

(2) F is called *full* if for all $f : FA \to FB$

$$\exists g : A \to B \text{ with } Fg = f.$$

(3) F is an *embedding* if and only if it is faithful and injective on objects.

Remark A faithful functor is injective on hom-sets and a full functor is surjective on hom-sets.

1.3.6 Natural transformations

Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors. A *natural transformation* τ from F to G (denoted by $\tau : F \to G$) is a map that assigns to each object $A \,\epsilon\, \mathcal{O}bj\mathcal{A}$ a morphism $\tau_A : FA \to GA$ in such a way that the following condition holds: for each morphism $f : A \to A'$, the square below commutes.

$$\begin{tikzcd}
FA \arrow[r, "\tau_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"'] \\
FA' \arrow[r, "\tau_{A'}"'] & GA'
\end{tikzcd}$$

Figure 1.1: $\tau_{A'} \circ Ff = Gf \circ \tau_A$

1.3.7 Adjoint situation

An *adjoint situation* $(\eta, \varepsilon) : F \dashv G : \mathcal{A} \to \mathcal{B}$ consists of two functors $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{A}$ and two natural transformations $\eta : id_{\mathcal{B}} \to GF$ (called the *unit*) and $\varepsilon : FG \to id_{\mathcal{A}}$ (called the *co-unit*) that satisfy the following conditions:

(1) $G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G = G \xrightarrow{id_G} G$,

(2) $F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F = F \xrightarrow{id_F} F$.

Remark We say F is a *left adjoint* for G and G is a *right adjoint* for F (in symbols: $F \dashv G$) if an adjoint situation, as above, exists. Note that a Galois connection (f, g) is an adjoint situation in which we regard a poset as a category and a monotone map as a functor (see 1.1.6).

1.3.8 Limits and Colimits

A *source* is a pair $(A, (f_i)_{i \in I})$ consisting of an object A in a category $\mathcal{C}$, and a family of morphisms $f_i : A \to A_i$ with domain A, indexed by some class I. A source is sometimes also referred to as a *cone*. The dual concept of a source will be referred to as a *sink* or a *co-cone*.

A *diagram* in a category $\mathcal{C}$ is a functor $D : \mathcal{I} \to \mathcal{C}$ with codomain $\mathcal{C}$. The domain, $\mathcal{I}$, is called the *scheme* of the diagram. A $\mathcal{C}$-source $(A, (f_i)_{i \in \mathcal{O}bj\mathcal{I}})$ is said to be *natural* for D provided that for each $\mathcal{I}$-morphism $d : i \to j$, the triangle below commutes, that is, $Dd \circ f_i = f_j$. We will write D_i for Di.

$$\begin{tikzcd}
D_i \arrow[rr, "Dd"] & & D_j \\
& A \arrow[ul, "f_i"] \arrow[ur, "f_j"'] &
\end{tikzcd}$$

Figure 1.2: A natural source for D.

A *limit* of a diagram $D : \mathcal{I} \to \mathcal{C}$ is a natural source $(L, (g_i)_{i \in \mathcal{O}bjI})$ for D with the property that each natural source $(A, (f_i)_{i \in \mathcal{O}bjI})$ for D uniquely factors through it: that is, for every such source there exist a unique morphism $f : A \to L$ with $f_i = g_i \circ f$ for each $i \in \mathcal{O}bjI$.

Examples Specific types of limits include (identities are omitted in schemes):

(1) *products*: products are limits of diagrams with discrete schemes,

(2) *equalizers*: equalizers are limits of diagrams with scheme $\bullet \rightrightarrows \bullet$,

(3) *pullbacks*: limits of diagrams with the scheme below, are called pullbacks.

$$\begin{tikzcd}
& \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

We call the dual concept of a limit, a *colimit*. The dual formulation for the limits mentioned above are *coproducts*, *co-equalizers* and *pushouts*, respectively.

Chapter 2

Spaces, frames and locales

In this chapter we will establish the connection between topological spaces and frames(locales). We will demonstrate how the category of locales can be considered as a generalization or extension of the category of topological spaces. In addition, we will make some elementary but significant observations and come across an adjoint situation.

Definition 2.1. A *frame*[1] is a complete lattice L that satisfies the following infinite distributive law: For all $a \in L$ and $\{b_i \mid i \in I\} \subseteq L$,

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i).$$

If we consider the collection of open sets of a topological space and order it by set inclusion, then the topology on X (see 1.2.1) can be viewed as a complete lattice[2]. In addition, the open sets satisfy the infinite distributive law in 2.1 and therefore a topological space forms a frame.

Consequently, we found an object which can serve as a generalization of a topological space. But, to form a category, we are still in need of suitable morphisms.

Definition 2.2. A map $h : L \longrightarrow M$ between frames L and M, is said to be a *frame homomorphism*[3] if it preserves finite meets (including the top 1) and arbitrary joins (including the bottom 0), that is, for any $a, b, c_i \in L$,

$$\begin{aligned} h(a \wedge b) &= h(a) \wedge h(b), \\ h(\bigvee_{i \in I} c_i) &= \bigvee_{i \in I} h(c_i). \end{aligned}$$

And now we make the connection more clear:

There exists a functor (see 1.3.3) $\Omega : \mathbf{Top} \longrightarrow \mathbf{Frm}$ such that a topological space is sent to its frame of open sets and a continuous map is sent to the inverse image map:

$$\begin{aligned} \Omega : \mathbf{Top} &\longrightarrow \mathbf{Frm}, \\ X &\longmapsto \Omega(X), \text{ and} \\ f : X \longrightarrow Y &\longmapsto \Omega(f) : \Omega(Y) \longrightarrow \Omega(X), \end{aligned}$$

where $\Omega(f)(U) = f^{-1}(U)$.[4]

For the very optimistic reader, it might seem that we have found the category which can serve as an extension for **Top**. Regrettably it is not so - the above-mentioned functor is contravariant. As a result, the category in which we are truly interested, is the dual (see 1.3.2) category $\mathbf{Frm}^{op}$. This category has locales as objects and localic (continuous) maps as morphisms. It is more commonly referred to as the category of *locales* and is denoted by **Loc**. Therefore we have a covariant functor

$$\Omega : \mathbf{Top} \longrightarrow \mathbf{Loc}.$$

Note that, as objects, frames and locales are the same objects. The morphisms only differ in the respective categories. Although turning arrows around is mathematically trivial, to think "backwards" is not always clear and straightforward. As [25] put it, "Morphisms[5] in **Loc** are, of course, frame homomorphisms taken backwards, which may obscure the intuition."

Now might be as good a time as any to note that the reader can approach point-free topology from two points of view: The first is the contravariant approach, where research is done in **Frm** but the intention is to acquire results in **Loc**. The other is to work covariantly, that is, do research in the category **Loc** itself. In his article, Johnstone [20] makes his choice clear, "frame theory is lattice theory applied to topology, whereas locale theory is topology itself." There is however no preferential category amongst authors; many choose to operate in **Frm** while others prefer **Loc**. Depending on their need, some authors operate in both categories.

We have seen that the category **Loc** can successfully represent topological spaces and continuous maps. But how well does **Top** represent locales (frames) and localic maps (frame homomorphisms)? Before we give the answer, we first need to define what we mean by a *sober space*:

Definition 2.3. A topological space X is said to be *sober* if every meet-irreducible[6] open set $A \neq X$ is $X \setminus \overline{\{x\}}$ for a unique $x \in X$.

We note that if a space is sober, then it is also T_0. In all propositions to follow, unless stated otherwise, the proofs have essentially been taken from Pultr [27].

Proposition 2.4. *Let Y be a sober space and X a general one. Then for each frame homomorphism $h : \Omega(Y) \longrightarrow \Omega(X)$ there is exactly one continuous map $f : X \longrightarrow Y$ such that $h = \Omega(f)$. Thus, the restriction $\Omega : \mathbf{Sob} \longrightarrow \mathbf{Loc}$ of Ω is a full embedding*[7].

Proof. Let $h : \Omega(Y) \to \Omega(X)$ be a frame homomorphism. For $x \,\epsilon\, X$ set

$$\mathcal{F}_x = \{U \,\epsilon\, \Omega(Y) \,|\, x \notin h(U)\} \text{ and } F_x = \bigcup \mathcal{F}_x.$$

Since h preserves arbitrary joins we have that $x \notin h(F_x)$, and hence, for $U \,\epsilon\, \Omega(Y)$,

$$x \notin h(U) \text{ if and only if } U \subseteq F_x.$$

Since $x \notin h(F_x)$, we have that $F_x \neq X$ and if $F_x = U \cap V$, then $x \notin h(U) \cap h(V)$, because h preserves finite meet. Therefore, say $x \notin h(U)$, $U \subseteq F_x$ and F_x is meet-irreducible.

By the sobriety of Y, $F_x = Y \setminus \overline{\{y\}}$ for a unique $y \,\epsilon\, Y$. If we choose such y for $f(x)$, we can write

$$x \notin h(U) \quad \text{iff} \quad U \subseteq Y \setminus \overline{\{y\}} \quad \text{iff} \quad f(x) \notin U,$$

since U is open and therefore

$$x \,\epsilon\, h(U) \;\; \text{iff} \;\; f(x) \,\epsilon\, U, \;\; \text{that is, } x \,\epsilon\, f^{-1}(U).$$

Hence, $f^{-1}(U) = h(U) \,\epsilon\, \Omega(X)$ and thus f is continuous and $h = \Omega(f)$. One can readily show that f is unique, since Y is sober and therefore a T_0-space. □

Remark There are many sober topological spaces, and those that are not sober can be replaced by its soberification[8]. Johnstone (1991) says that by pretending that all spaces are sober, little harm comes to topology. He adds that by replacing a space with its soberification the open-set lattice is left unchanged, and very little damage is done to its topological properties.

Consequently, spaces and continuous maps represent locales and localic maps to a considerable degree. This then justifies the remark that locales can be regarded as "generalized spaces". Nevertheless, there is a shortcoming - a complete Boolean algebra[9] is of the form $\Omega(\mathcal{X})$, that is *spatial*[10], if and only if it is atomic[11][27].

We conclude this chapter by showing how to reconstruct sober spaces and continuous maps. In addition, we will give a proof of an adjoint situation.

Definition 2.5. A *point* of a frame L is a frame homomorphism $h : L \to \mathbf{2}$.[12] We denote by ΣL the set of all points of L. For $a \,\epsilon\, L$, set

$$\Sigma_a = \{\, h : L \to \mathbf{2} \,|\, h(a) = 1 \,\}.$$

Lemma 2.6. *$(\Sigma L, \tau)$ is a topological space, where $\tau = \{\Sigma_a \,|\, a \,\epsilon\, L\}$.*

Proof. (1) $\Sigma_1 = \{h : L \to \mathbf{2} \,|\, h(a) = 1\} = \Sigma L \,\epsilon\, \tau$ and
$\Sigma_0 = \{h : L \to \mathbf{2} \,|\, h(0) = 1\} = \emptyset \,\epsilon\, \tau$.

(2) $\Sigma_a \cap \Sigma_b = \{h : L \to \mathbf{2} \,|\, h(a) = 1\} \cap \{h : L \to \mathbf{2} \,|\, h(b) = 1\}$
$= \{h : L \to \mathbf{2} \,|\, h(a) = 1 \text{ and } h(b) = 1\}$
$= \{h : L \to \mathbf{2} \,|\, h(a) \wedge h(b) = 1\}$
$= \{h : L \to \mathbf{2} \,|\, h(a \wedge b) = 1\}$
$= \Sigma_{a \wedge b} \,\epsilon\, \tau$.

(3) $\bigcup \Sigma_{a_i} = \bigcup \{h : L \to \mathbf{2} \,|\, h(a_i) = 1\}$
$= \{h : L \to \mathbf{2} \,|\, \bigvee h(a_i) = 1\}$
$= \{h : L \to \mathbf{2} \,|\, h(\bigvee a_i) = 1\}$
$= \Sigma_{\vee a_i} \,\epsilon\, \tau$. □

Remark As usual, we will more often talk about the topological space ΣL instead of $(\Sigma L, \tau)$. This space is also known as the *spectrum* of L. There are at least two more alternative descriptions of the spectrum: The first utilizes the correspondence between a point of a frame L and complete filters on L, and the other the correspondence between a point and the meet-irreducible elements of L.[13]

Definition 2.7. For a frame homomorphism $h : L \to M$, define the following mapping $\Sigma h : \Sigma M \to \Sigma L$ by $(\Sigma h)(\alpha) = \alpha \circ h$.

Lemma 2.8. *For each $a \,\epsilon\, L$, $(\Sigma h)^{-1}(\Sigma_a) = \Sigma_{h(a)}$.*

Proof. $\alpha \,\epsilon\, \Sigma_{h(a)}$
$\iff \alpha(h(a)) = 1$
$\iff (\alpha \circ h)(a) = 1$
$\iff \alpha \circ h \,\epsilon\, \Sigma_a$
$\iff (\Sigma h)(\alpha) \,\epsilon\, \Sigma_a$.
$\iff \alpha \,\epsilon\, (\Sigma h)^{-1}(\Sigma_a)$.
Therefore, $\Sigma_{h(a)} = (\Sigma h)^{-1}(\Sigma_a)$. □

Remark It can be shown that the map $\Sigma h : \Sigma M \to \Sigma L$ is continuous and consequently we have found a contravariant functor

$$\Sigma : \mathbf{Frm} \to \mathbf{Top}.$$

We have shown how to reconstruct topological spaces and continuous maps from given frames and frame homomorphisms. But are these reconstructed spaces truly sober? It can be proven, with necessary insight, that these spaces ΣL are indeed sober [14]. Ultimately, one might ask if there is any connection between the functors Ω and Σ. After all, we have shown that spaces and continuous maps are closely related to frames and frame homomorphisms. We have

Proposition 2.9. $\Sigma : \mathbf{Loc} \to \mathbf{Top}$ *is a right adjoint to* $\Omega : \mathbf{Top} \to \mathbf{Loc}$.

Proof. For a topological space X define $\eta_X : X \to \Sigma\Omega X$ by setting

$$\eta_X(x)(U) = 1 \text{ if and only if } x \in U$$

Verifying that each $\eta_X(x)$ is a frame homomorphism is straightforward, and we have that η_X is continuous, since

$$\eta_X^{-1}(\Sigma_U) = \{ x \mid \eta_X(x) \in \Sigma_U \} = U.$$

For a frame L define $\varepsilon_L : L \to \Omega\Sigma L$ by setting $\varepsilon_L(a) = \Sigma_a$. From Lemma 2.6 it follows that ε_L is a frame homomorphism. If $f : X \to Y$ is a continuous map, then we have $(\Sigma\Omega f(\eta_X(x)))(U) = \eta_X(x)(\Omega f(U)) = \eta_X(x)(f^{-1}(U)) = 1$ iff $x \in f^{-1}(U)$ iff $f(x) \in U$ iff $\eta_Y(f(x))(U) = 1$.
If $h : L \to M$ is a frame homomorphism, then we have $(\Omega\Sigma h(\varepsilon_L))(a) = \Omega\Sigma h(\Sigma_a)$ $= (\Sigma h)^{-1}(\Sigma_a) = \Sigma_{h(a)} = \varepsilon_M(h(a))$. Consequently we have natural transformations (see 1.3.6) $\eta : id \to \Sigma\Omega$ and $\varepsilon : id \to \Omega\Sigma$.

These natural transformations are adjunction units (see 1.3.7):

(1) $(\Sigma\varepsilon_L(\eta_{\Sigma L}(\alpha)))(U) = \eta_{\Sigma L}(\alpha)(\eta_L(U)) = 1$ iff $\alpha \in \Sigma_U$ iff $\alpha(U) = 1$, therefore $\Sigma\varepsilon_L \circ \eta_{\Sigma L} = id$, and

(2) $\Omega(\eta_X)(\varepsilon_{\Omega X}(U)) = \eta_X^{-1}(\Sigma_U) = U$, hence $\Omega\eta_X \circ \varepsilon_{\Omega X} = id$.

$\square$

Notes

[1]From an algebraic point of view, a frame is better known as a *complete Heyting algebra*. This equivalence is established by a Galois connection. If the reader favours a more algebraic approach, a brief outline can be considered in [25].

[2]Finite meet and arbitrary join is given by finite intersection and arbitrary union, respectively. The top is the set X and the bottom is given by $\emptyset$.

[3]Even though a frame and a complete Heyting algebra is the same object, a frame homomorphism is not a homomorphism between complete Heyting algebras. An additional property needs to be satisfied, that is, for all $a, b \in L$ and lattice homomorphism $h : L \to M$,

$$h(a \to b) = h(a) \to h(b).$$

[4]The inverse image map f^{-1} is a frame homomorphism as it preserves arbitrary joins, including 0, and finite meets, including 1.

[5]In 2008, Picado and Pultr published an article in which locales were treated by a covariant approach. Although it is not an approach the author pursued, we point out the distinction for sake of being thorough: Since frame homomorphisms preserve arbitrary joins, they have unique left adjoints - these maps are then the maps which run in the "proper" direction. *Localic maps* are subsequently defined as mappings $f : L \to M$ that have left adjoints f^* preserving meets.

[6]An open set $A \neq X$ is meet-irreducible if

$$A = B \cap C \Longrightarrow A = B \text{ or } A = C.$$

[7]A functor which is full and an embedding (see 1.3.5), we quite naturally call a *full embedding*.

[8]We recommend the book, *Stone spaces*, by Johnstone [18] for a closer look at *soberification*.

[9]A *Boolean algebra* is a distributive lattice with 0 and 1 in which every element has a complement.

[10]A frame L which is isomorphic to an $\Omega(X)$ is said to be *spatial*.

[11]An element b of a lattice L is called an *atom* if for any $c \in L$ it is true that

$$c \leq b \Longrightarrow c = 0 \text{ or } c = b.$$

L is said to be *atomic* if every element of L is the join of all atoms below it.

[12]We denote the two-element Boolean algebra $\{0, 1\}$ by $\mathbf{2}$.

[13]The reader will find more detail on the alternative descriptions of the spectrum in Pultr [27].

[14]For a proof, we advise the reader to look in Johnstone [18] or Pultr [27]. In his proof, Johnstone makes use of prime elements and proves that $\psi : X \to \Sigma(\Omega(X))$ is a bijection, while Pultr keeps to meet-irreducible elements and our definition of sober.

Chapter 3

Generalized subspaces

In this chapter we will define, from a categorical point of view, what is meant by a subobject in a category and subsequently focus our attention on subobjects of generalized spaces. We will prove that there are at least 4 ways of representing subobjects of generalized spaces.

Definition 3.1. Assume a category $\mathcal{C}$ has a fixed family of monomorphisms $\mathcal{M}$. We define a $\mathcal{M}$-*subobject* of an object X to be a morphism $m \,\epsilon\, \mathcal{M}$ with codomain X, that is,

$$m : M \longrightarrow X, \quad m \,\epsilon\, \mathcal{M}, \text{ and let}$$

$$\mathcal{M}/X = \{m \,\epsilon\, \mathcal{M} \,|\, \text{codomain } m = X\}.$$

If there is no ambiguity, we will rather refer to m as a *subobject* of an object X. In concrete[1] categories the monomorphisms are, roughly speaking, represented by the injective morphisms. Representing subobjects by monomorphisms is, on occasion, to hope for too much - there are many categories in which the monomorphisms fail at adequate representation of subobjects. We give two examples which will aid our understanding of a subobject:

(1) In the category **Frm**, one can prove that the family of monomorphisms $\mathcal{M}$ can be represented by the injective frame homomorphisms[2] $m : A \to B$ with A and B arbitrary frames. In this category we will refer to the subobjects m of a frame B as *subframes*. It is worth mentioning that A can also be considered as a frame contained in frame B: In our example, one can readily prove that $m(A) \subseteq B$ satisfies the frame definition (see 2.1), and consequently is a frame. Moreover, $m(A)$ is isomorphic to A and hence A can be considered to be contained in B and therefore appropriately referred to as a *subframe*.

(2) For every subspace A of a topological space X, the mapping $m_A : A \to X$ defined by $m_A(a) = a$ is continuous[3] and injective. This mapping is called the *embedding of the subspace A in the space X*. One can verify that the subspace topology coincides with the topology generated by the mapping m_A.

Note that the subspace topology is the *initial topology*[4] on the subspace. A map $m_A : A \to X$ is an *embedding* only if it is initial, injective and continuous. It turns out that the subspace embeddings $m : A \to X$ are exactly the extremal monomorphisms in **Top** and not, as per usual, represented by monomorphisms.[5]

This fact is supported by the following result:

Consider a subspace Y of a topological space X. Then the embedding

$$j : Y \hookrightarrow X \longmapsto \Omega(j) = (U \mapsto U \cap Y) : \Omega(X) \longrightarrow \Omega(Y)$$

is associated with a surjective frame homomorphism $\Omega(j)$, where

$$\Omega(Y) = \{U \cap Y \,|\, U \in \Omega(X)\}.$$

We will prove that the surjective frame homomorphisms are the extremal epimorphisms in **Frm**. Consequently they are extremal monomorphisms in the dual category **Loc**. Before long we will define (see 3.4) what we mean by a subobject of a locale. The association between embeddings, which are the subobjects in **Top**, and onto frame homomorphism, by way of the functor Ω above, will serve as our motivation.

To prove this result, we first need to introduce a new concept:

Definition 3.2. Let $\mathcal{C}$ be a category. For $e, m \in \mathcal{M}or\mathcal{C}$, we will write $e \perp m$, and say "e is *orthogonal* to m" if for every $u, v \in \mathcal{M}or\mathcal{C}$ with $m \circ u = v \circ e$, there exists a unique d with $m \circ d = v$ and $d \circ e = u$.

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dl, "d"'] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

Figure 3.1: e is orthogonal to m.

We say that $\mathcal{C}$ has a $(\mathcal{E}, \mathcal{M})$-factorization system if the following conditions hold:

(1) $(\mathcal{E}, \mathcal{M})$ is a pre-factorization system, that is,
$\mathcal{M}^{\uparrow} = \{e \,\epsilon\, \mathcal{M}or\mathcal{C} \,|\, e \perp m, \forall\, m \,\epsilon\, \mathcal{M}\} = \mathcal{E}$ and
$\mathcal{E}^{\downarrow} = \{m \,\epsilon\, \mathcal{M}or\mathcal{C} \,|\, e \perp m, \forall\, e \,\epsilon\, \mathcal{E}\} = \mathcal{M}$.

(2) $\mathcal{C}$ has $(\mathcal{E}, \mathcal{M})$-factorization of morphisms, that is,
each morphism f in $\mathcal{C}$ can be factorized, $f = m \circ e$ with $m \,\epsilon\, \mathcal{M}, e \,\epsilon\, \mathcal{E}$.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\ & C \arrow[ur, "m"'] & \end{tikzcd}$$

Figure 3.2: $(\mathcal{E}, \mathcal{M})$-factorization of a morphism

In chapter 4 we will consider factorization systems in more detail, but for now, when it comes to subobjects, our interest only truly lies in the family $\mathcal{M}$ of morphisms. In the two examples given earlier, we have seen that the morphisms in this family, be it monomorphisms or extremal monomorphisms, possess the necessary properties to preserve the structure of the object.

We can now proceed to prove our result:

Proposition 3.3. *Let $m : A \to B$ be a morphism between frames A and B in* **Frm**. *Then m is onto if and only if m is an extremal epimorphism*[6].

Proof. ($\Rightarrow$:) Assume m is onto frame homomorphism and $f \circ m = g \circ m$ for arbitrary $f, g \,\epsilon\, \mathcal{M}or\mathbf{Frm}$. By surjectivity, for every $b \,\epsilon\, B$ there exists an $a \,\epsilon\, A$ so that $m(a) = b$. Hence

$$\begin{aligned} f(b) &= f(m(a)) \\ &= (f \circ m)(a)) \\ &= (g \circ m)(a), \text{ by assumption }, \\ &= g(m(a)) = g(b). \end{aligned}$$

Therefore, $f = g$, since b was arbitrary and we have that m is an epimorphism. Assume $m = gh$ with $h : A \to C$, $g : C \to B$ and g a monomorphism. From the surjectivity of m it follows that g is surjective, and therefore a bijection. Hence there exists a frame homomorphism[7] $g^{-1} : B \to C$ so that $g \circ g^{-1} = 1_B$ and $g^{-1} \circ g = 1_C$. Therefore, g is an isomorphism.

(⇐:) Assume m is an extremal epimorphism. We can factorize m by $m = f \circ g$ where $g : A \to m(A)$ with $g(a) = m(a)$ and $f : m(A) \to B$ with $f(b) = b$ (see figure 3.3 below).

$$\begin{tikzcd}
A \arrow[rr, "m"] \arrow[dr, "g"'] & & B \\
& m(A) \arrow[ur, hook, "f"'] &
\end{tikzcd}$$

Figure 3.3: Factorization of a frame homomorphism

It is easy to see that f is an injective frame homomorphism and hence a monomorphism in **Frm**. By our assumption that m is an extremal epimorphism, it follows that f is an isomorphism. Therefore m is an onto frame homomorphism, since g is surjective. □

Remark It can be proven that **Frm** has a (extremal epi, mono)-factorization system. Therefore the dual category **Loc** has a (epi, extremal mono)-factorization system[8]. For a detailed introduction and an enlightening outline of factorization systems, we recommend the book *Concrete and abstract categories* by Adámek *et al.* [1].

We continue this chapter by formally introducing four ways to represent subobjects of locales. All proofs, if not explicitly stated otherwise, are the result of the author's own work.

Sublocale maps

Definition 3.4. We define a *sublocale (map)* of a locale L as an *onto frame homomorphism*

$$h : L \longrightarrow M.$$

The reason for such a definition has previously been motivated. Note that frames, locales and complete Heyting algebras are entirely synonymous when considered as objects. However, the difference is more evident when we refer to the objects with their morphisms in the respective categories.

The reader may be tempted to think of *subframes* and *sublocales* as synonymous as well. This, however, is not the case and as we continue this chapter we trust that the distinction will become more apparent.

The set of sublocales of L can be ordered by the pre-order

$$h \leq k \text{ if and only if there exists an } l \text{ such that } h = lk.$$

$$\begin{tikzcd}
L \arrow[rr, "h"] \arrow[dr, "k"'] & & M \\
& N \arrow[ur, "l"'] &
\end{tikzcd}$$

Figure 3.4: $h \leq k$

Two sublocales h, k with $h \leq k$ and $k \leq h$ are said to be *equivalent*.

Allowing this equivalence to mean equality, the set of sublocales of L give rise to a poset. Joins will be denoted by $h \sqcup k, \bigsqcup h_i$, and meets by $h \sqcap k, \bigsqcap h_i$. We will denote this complete lattice[9] by $\mathcal{S}(L)$.

Congruences

Definition 3.5. An equivalence relation E[10] on a frame L (subset of $L \times L$) that is closed under finite meet and arbitrary join (expressed below), is called a (frame) *congruence* on L.

$$\begin{aligned}
(a, b), (c, d) \,\epsilon\, E &\implies (a \wedge c, b \wedge d) \,\epsilon\, E, \\
(a_i, b_i) \,\epsilon\, E &\implies (\bigvee a_i, \bigvee b_i) \,\epsilon\, E.
\end{aligned}$$

Consequently, a *frame congruence* is a subset of $L \times L$ that is both an equivalence relation and a subframe. The set of all congruences on L, ordered by inclusion, will be denoted by $\mathcal{C}L$.

Proposition 3.6 (**Picado and Pultr 2008**). *There is an invertible correspondence between $\mathcal{S}(L)$ and $\mathcal{C}(L)$. This correspondence is given by*

$$\begin{aligned}
h &\longmapsto E_h = \{(x, y) \mid h(x) = h(y)\}, \\
E &\longmapsto h_E = \{x \mapsto xE\} : L \longrightarrow L/E.
\end{aligned}$$[11]

Proof. ($\Rightarrow$:) Let $h : L \to M$ be an arbitrary sublocale map. Define a congruence $E_h \subseteq L \times L$ by

$$E_h = \{(x, y) \mid h(x) = h(y)\}.$$

(1) Equivalence relation:

- reflexive: $h(x) = h(x) \Longrightarrow xE_hx$.
- transitive: Assume xE_hy and yE_hz. Then $h(x) = h(y)$ and $h(y) = h(z)$. Consequently, $h(x) = h(z)$, that is xE_hz.
- symmetric: Assume xE_hy. Then $h(x) = h(y)$, which implies that $h(y) = h(x)$. Therefore, yE_hx.

(2) Subframe: $(1, 1) \,\epsilon\, E_h$ and $(0, 0) \,\epsilon\, E_h$.

- $(x, y) \wedge (s, t) = (x \wedge s, y \wedge t)$:
  $h(x \wedge s) = h(x) \wedge h(s) = h(y) \wedge h(t) = h(y \wedge t)$
  $\Longrightarrow (x \wedge s, y \wedge t) \,\epsilon\, E_h$.
- $\bigvee(x_i, y_i) = (\bigvee x_i, \bigvee y_i)$:
  $h(\bigvee x_i) = \bigvee h(x_i) = \bigvee h(y_i) = h(\bigvee y_i)$.
  $\Longrightarrow (\bigvee x_i, \bigvee y_i) \,\epsilon\, E_h$.

($\Leftarrow$:) Let $E \subseteq L \times L$ be a congruence on frame L. Define a sublocale map by

$$\begin{aligned} h_E : L &\longrightarrow L/E, \\ x &\longmapsto xE. \end{aligned}$$

(1) Onto: Trivial.

(2) $h_E(0) = 0E = 0_{L/E}$ and $h_E(1) = 1E = 1_{L/E}$.

(3) $h_E(x \wedge y) = h_E(x) \wedge h_E(y)$:
$h_E(x \wedge y) = (x \wedge y)E = xE \wedge yE = h_E(x) \wedge h_E(y)$.

(4) $h_E(\bigvee x_i) = \bigvee h_E(x_i)$:
$h_E(\bigvee x_i) = (\bigvee x_i)E = \bigvee(x_iE) = \bigvee h_E(x_i)$.

Ultimately we have,

- $h \leq k$ if and only if $E_k \subseteq E_h$,
  (($\Leftarrow$:) If $h : L \to M$ and $k : L \to K$ are sublocales, define $l : K \to M$ as $l := h \circ k^{-1}$. One can verify that l is a frame homomorphism and $h = l \circ k$.)
- $h_{E_h}(x) = h(x)$ and $xE_{h_E}y \Longleftrightarrow xEy$.

It can be shown that the arbitrary intersections of frame congruences is again a frame congruence. Consequently, $\mathcal{C}L$ is a complete lattice. Therefore, $\mathcal{S}(L)$ is (complete lattice) isomorphic to $\mathcal{C}(L)^{op}$. $\square$

Nucleus

Definition 3.7. A *nucleus*[12] on a frame L is a map $\nu : L \longrightarrow L$ satisfying for all $x, y \,\epsilon\, L$:

(1) $x \leq \nu(x)$,

(2) $x \leq y \Longrightarrow \nu(x) \leq \nu(y)$,

(3) $\nu(\nu(x)) = \nu(x)$,

(4) $\nu(x \wedge y) = \nu(x) \wedge \nu(y)$.

The set of all nuclei on L endowed with the natural pointwise order will be denoted by $\mathcal{N}(L)$.

Lemma 3.8. *1. If $x_i = \nu(x_i)$ for all i, then $\bigwedge \nu(x_i) = \nu(\bigwedge x_i)$.*

2. $\forall x \,\epsilon\, L, \quad x \,\epsilon\, \nu(L) \Longleftrightarrow \nu(x) = x$.

Proof. 1. $[\leq:]$ $\bigwedge \nu(x_i) \leq \nu(\bigwedge \nu(x_i)) = \nu(\bigwedge x_i)$.
$[\geq:]$ $x_i \geq \bigwedge x_i$, for all $x_i \,\epsilon\, L$.

$$\begin{aligned} \Longrightarrow \nu(x_i) &\geq \nu(\bigwedge x_i), \text{ by the monotonicity of a nucleus,} \\ \Longrightarrow \bigwedge \nu(x_i) &\geq \nu(\bigwedge x_i). \end{aligned}$$

2. $(\Leftarrow:)$ Trivial.
$(\Rightarrow:)$ Assume $y \,\epsilon\, \nu(L)$. Then there exists $x \,\epsilon\, L$ so that $\nu(x) = y$. Therefore

$$\begin{aligned} \nu(b) &= \nu(\nu(a)), \\ &= \nu(a), \text{ by property (3) of a nucleus,} \\ &= b. \end{aligned}$$

□

We note that, in general, a nucleus $\nu : L \rightarrow L$ is not a frame homomorphism. However, we will show that the restriction $\nu : L \rightarrow \nu(L)$ is. The proof of the next proposition has essentially been taken form Pultr [27].

Proposition 3.9 (Pultr 2003). *The subset $\nu(L) \subseteq L$ is a frame with infima coinciding with those of L and the suprema given by $\bigvee' x_i = \nu(\bigvee x_i)$; the restriction $\nu : L \rightarrow \nu(L)$ is a (onto) frame homomorphism.*

Proof. By Lemma 3.8, $\nu(L)$ is closed under arbitrary meet. We have $x_j \leq \nu(\bigvee x_i)$ for all j, and if $x_j \leq y \,\epsilon\, \nu(L)$ for all j, $\bigvee x_i \leq y$ and $\bigvee' x_i = \nu(\bigvee x_i) \leq \nu(y) = y$. The mapping $\nu : L \rightarrow \nu(L)$ preserves

- finite meets since it is a nucleus,
- arbitrary joins: $\nu(\bigvee x_i) \leq \nu(\bigvee \nu(x_i)) = \bigvee' \nu(x_i)$, by monotonicity and $x_i \leq \nu(x_i)$ implies $\bigvee' x_i \geq \bigvee' \nu(x_i)$.

Ultimately, $\nu : L \to \nu(L)$ is clearly onto and as it preserves all joins and all finite meets, $\nu(L)$ satisfies the infinite distributive law. $\square$

Proposition 3.10 (Pultr 2003). *The correspondences*

$$\begin{array}{rcl} E & \longmapsto & \nu_E = \{x \mapsto \bigvee xE\} : L \longrightarrow L, \\ \nu & \longmapsto & E_\nu = \{(x,y) \mid \nu(x) = \nu(y)\}, \end{array}$$

establish an isomorphism of the posets $\mathcal{C}(L)$ and $\mathcal{N}(L)$.

Proof. ($\Rightarrow$:) Let E be an arbitrary congruence on L. Define a nucleus on L by

$$\begin{array}{rcl} \nu_E : L & \longrightarrow & L, \\ x & \longmapsto & \bigvee xE. \end{array}$$

(1) $x \leq \nu_E(x)$: Clearly $x \,\epsilon\, \{y \mid (x,y) \,\epsilon\, E\}$, since xEx. Therefore, $x \leq \bigvee\{y \mid (x,y) \,\epsilon\, E\}$, that is, $x \leq \nu_E(x)$.

(2) $\nu_E(\nu_E(x)) = \nu_E(x)$:
[$\geq$:] From (1) we have $x \leq \nu_E(x)$, for $x \,\epsilon\, L$. But $\nu_E(x) \,\epsilon\, L$, therefore $\nu_E(x) \leq \nu_E(\nu_E(x))$.
[$\leq$:] Consider $\{y \mid (x,y) \,\epsilon\, E\}$, with x fixed. Then $(x, \bigvee y) \,\epsilon\, E$, since E a congruence. Consequently, $\{z \mid (\bigvee y)Ez\} \subseteq \{y \mid xEy\}$, by symmetry and transitivity.
As a result we have

$$\begin{array}{rcl} \nu_E(\nu_E(x)) & = & \nu_E(\bigvee\{y \mid xEy\}) \\ & = & \bigvee\{z \mid (\bigvee\{y \mid xEy\})Ez\} \\ & \leq & \bigvee\{y \mid xEy\} \\ & = & \nu_E. \end{array}$$

(3) $x \leq y \implies \nu_E(x) \leq \nu_E(y)$: Assume $x \leq y$ and let $\nu_E(x) = \{t \mid xEt\}$ and $\nu_E(y) = \{s \mid yEs\}$. Then for every $t \,\epsilon\, \nu_E(x)$ and $s \,\epsilon\, \nu_E(y)$, we have that $yE(t \vee s)$, by assumption. Therefore $\bigvee\{t \mid xEt\} \leq \bigvee\{s \mid yEs\}$, that is, $\nu_E(x) \leq \nu_E(y)$.

(4) $\nu_E(x \wedge y) = \nu_E(x) \wedge \nu_E(y)$:
[$\leq$:] Follows from (3).

$$[\geq:] \qquad \begin{aligned} \nu_E(x) \wedge \nu_E(y) &= \bigvee\{t \mid xEt\} \wedge \bigvee\{s \mid yEs\} \\ &= \bigvee\{t \wedge s \mid xEt, yEs\} \\ &\leq \bigvee\{z \mid (x \wedge y)Ez\} \\ &= \nu_E(x \wedge y). \end{aligned}$$

($\Leftarrow$:) Let ν be any nucleus on L. Define a congruence $E_\nu \subseteq L \times L$ on L by

$$xE_\nu y \iff \nu(x) = \nu(y).$$

(1) Equivalence relation: See proof of Prop. 3.6.

(2) Subframe: Trivially, $(0,0) \in E_v$ and $(1,1) \in E_v$. Also, from Prop. 3.6 one readily sees that E_v is closed under taking finite meet.

- $\bigvee(x_i, y_i) = (\bigvee x_i, \bigvee y_i)$:
  $\nu(\bigvee x_i) \leq \nu(\bigvee \nu(x_i)) = \bigvee' \nu(x_i) = \bigvee' \nu(y_i) \leq \nu(\bigvee y_i)$.
  If we interchange x_i with y_i above, then we have $\nu(\bigvee x_i) = \nu(\bigvee y_i)$.

It is easy to see that the correspondence $E \mapsto \nu_E$ is monotone. Ultimately we have, and the reader can verify,

- $xE_{\nu_E}y$ iff $\bigvee xE = yE$ iff xEy,
- $\nu_{E_\nu}(x) = \bigvee xE_\nu = \bigvee\{y \mid \nu(y) = \nu(x)\} = \nu(x)$.

□

Remark We have noted in Proposition 3.6 that $\mathcal{C}(L)$ is a complete lattice. Consequently, $\mathcal{N}(L)$ is a complete lattice.

Sublocale sets

Definition 3.11. A subset S of a frame L is said to be a **sublocale set** if

(1) for each $A \subseteq S$, $\bigwedge A \in S$ (specifically, $1 = \bigwedge \emptyset \in S$),

(2) for each $x \in L$ and $y \in S$, $x \to y \in S$.

The family of all sublocale sets ordered by inclusion will be denoted by $\mathcal{S}'(L)$.

Proposition 3.12 (**Picado and Pultr 2008**). *There is an invertible one-one correspondence between $\mathcal{N}(L)$ and $\mathcal{S}'(L)$. This correspondence is given by*

$$\begin{aligned} \nu &\longmapsto S_\nu = \{\nu(x) \mid x \in L\} = \nu(L), \\ S &\longmapsto \nu_S = \{x \mapsto \bigwedge\{y \in S \mid x \leq y\}\} : L \longrightarrow L. \end{aligned}$$

Proof. ($\Rightarrow$:) Let $\nu : L \to L$ be an arbitrary nucleus on L. Define a sublocale set $S_\nu \subseteq L$ by

$$S_\nu = \{\nu(x) \,|\, x \,\epsilon\, L\} = \nu(L).$$

(1) Take an arbitrary $A \subseteq \nu(L)$. By Lemma 3.8, arbitrary meets exist in $\nu(L)$.

(2) Take any $x \,\epsilon\, L$ and $y \,\epsilon\, \nu(L)$. We use the following result:

$$a \,\epsilon\, \nu(L) \Longleftrightarrow \nu(a) = a.$$

Therefore we need to show that $\nu(x \to y) = x \to y$:
[$\geq$:] $\nu(x \to y) \geq x \to y$, since ν is a nucleus.
[$\leq$:] $\nu(x \to y) \leq x \to y \Longleftrightarrow x \wedge \nu(x \to y) \leq y$, by definition. By the first property of a nucleus, we have $x \leq \nu(x)$. Therefore

$$\begin{aligned} x \wedge \nu(x \to y) &\leq \nu(x) \wedge \nu(x \to y) \\ &= \nu(x \wedge (x \to y)), \text{ by the third property of a nucleus,} \\ &\leq \nu(y), \text{ by Heyting formula (5), see 1.1.5,} \\ &= y. \end{aligned}$$

($\Leftarrow$:) Given an arbitrary sublocale set $S \subseteq L$, define a nucleus on L by

$$\begin{aligned} \nu_S : L &\to L, \\ x &\mapsto \bigwedge \{y \,\epsilon\, S \,|\, x \leq y\}. \end{aligned}$$

(1) $x \leq \nu_S(x) : x \leq y$, for all $y \,\epsilon\, S$. Therefore $x \leq \bigwedge\{y \,\epsilon\, S \,|\, x \leq y\} = \nu_S(x)$.

(2) $\nu_S(x) = \nu_S(\nu_S(x))$:
[$\leq$:] Trivial, by (1).
[$\geq$:] Let $y' = \bigwedge\{y \,\epsilon\, S \,|\, x \leq y\}$. Then $y' \,\epsilon\, \{z \,\epsilon\, S \,|\, y' \leq z\}$. Consequently, $\bigwedge\{z \,\epsilon\, S \,|\, y' \leq z\} \leq y'$, that is, $\nu_S(\nu_S(x)) \leq \nu_S(x)$.

(3) $x \leq y \Longrightarrow \nu_S(x) \leq \nu_S(y)$:

$$\begin{aligned} \{z \,\epsilon\, S \,|\, x \leq z\} &\supseteq \{z' \,\epsilon\, S \,|\, y \leq z'\}, \text{ since } x \leq y. \\ \Longrightarrow \bigwedge\{z \,\epsilon\, S \,|\, x \leq z\} &\leq \bigwedge\{z' \,\epsilon\, S \,|\, y \leq z'\}, \\ \Longrightarrow \nu_S(x) &\leq \nu_S(y). \end{aligned}$$

(4) $\nu_S(x \wedge y) = \nu_S(x) \wedge \nu_S(y)$:
[$\leq$:] Trivial, since $\{z' \,\epsilon\, S \,|\, x \leq z'\} \subseteq \{z \,\epsilon\, S \,|\, x \wedge y \leq z\}$ and $\{z'' \,\epsilon\, S \,|\, y \leq z''\} \subseteq \{z \,\epsilon\, S \,|\, x \wedge y \leq z\}$.

[$\geq$:] Since ν is a nucleus, $x \wedge y \leq \nu_S(x \wedge y)$. By definition of the Heyting arrow, we then have $x \leq y \to \nu_S(x \wedge y)$. Since S is a sublocale set, we have that $y \to \nu_S(x \wedge y) \,\epsilon\, S$. By the third property of a nucleus, $\nu_S(x) \leq \nu_S(y \to \nu_S(x \wedge y)) = y \to \nu_S(x \wedge y)$. This in turn gives, $\nu_S(x) \wedge y \leq \nu_S(x \wedge y)$. And if we now repeat the process: $\nu_S(x) \wedge y \leq \nu_S(x \wedge y) \iff y \leq \nu_S(x) \to \nu_S(x \wedge y)$, and therefore $\nu_S(y) \leq \nu_S(\nu_S(x) \to \nu_S(x \wedge y))$. This gives $\nu_S(y) \leq \nu_S(x) \to \nu_S(x \wedge y)$. Ultimately, we have $\nu_S(x) \wedge \nu_S(y) \leq \nu_S(x \wedge y)$.

Ultimately we have that

- $S_1 \subseteq S_2 \implies \nu_{S_2} \leq \nu_{S_1}$,
- $x \,\epsilon\, S_{\nu_S} \iff x = \nu_S(x) \iff x \,\epsilon\, S$,
- $\nu_{S_\nu}(x) = \bigwedge\{y \,\epsilon\, \nu(L) \mid x \leq y\} = \nu(x)$,

and consequently $\mathcal{S}'(L)$ is isomorphic to $\mathcal{N}(L)^{op}$. □

Remark Even though it is not absolutely necessary, for sake of being thorough, we refer the reader to the notes at the end of this chapter for the three remaining correspondences.[13]

We have seen now that the posets $\mathcal{S}(L)$, $\mathcal{C}(L)$, $\mathcal{N}(L)$ and $\mathcal{S}'(L)$ give rise to complete lattices. To conclude this chapter, we will show that these lattices are also (co-)frames. First, however, we need:

Lemma 3.13. *For arbitrary $S_i \,\epsilon\, \mathcal{S}'(L), i \,\epsilon\, I$, we have*

(1) $\bigwedge_{i \,\epsilon\, I} S_i = \bigcap_{i \,\epsilon\, I} S_i$ *and*

(2) $\bigvee_{i \,\epsilon\, I} S_i = \{\bigwedge A \mid A \subseteq \bigcup_{i \,\epsilon\, I} S_i\}$.

Proof. (1) Trivial, since one can readily see that the intersection of sublocale sets is a sublocale set.
(2) Take arbitrary $B \subseteq S$ where $S = \{\bigwedge A \mid A \subseteq \bigcup_{i \,\epsilon\, I} S_i\}$. Then $B \subseteq \bigcup_{i \,\epsilon\, I} S_i$ and therefore $\bigwedge B \,\epsilon\, S$.
Take $a \,\epsilon\, L$ and $b \,\epsilon\, S$, then $a \to b \,\epsilon\, S \iff a \to b = \bigwedge A$ for some $A \subseteq \bigcup_{i \,\epsilon\, I} S_i$. But, for $A = \{b_j \mid j \,\epsilon\, J\}$, $a \to \bigwedge b_j = \bigwedge(a \to b_j)$, by formula (5) in 1.1.5, and $a \to b_j \,\epsilon\, \bigcup S_i$, since $a \to b_j \,\epsilon\, S_i$, for some $i \,\epsilon\, I$. □

Remark The reader can verify that in $\mathcal{S}'(L)$ the least element is given by $0 = \{1\}$ and L is the greatest element.

Proposition 3.14 (Pultr 2003). *$\mathcal{S}'(L)$ is a co-frame*[14].

Proof. We need to show that for $A_i, B \,\epsilon\, \mathcal{S}'(L)$,

$$(\bigcap_{i \,\epsilon\, I} A_i) \vee B = \bigcap_{i \,\epsilon\, I}(A_i \vee B).$$

[$\subseteq$:] Let $x \,\epsilon\, (\bigcap A_i) \vee B$. Then

$$\begin{aligned} x &\;\epsilon\; \{\bigwedge D \mid D \subseteq (\bigcap A_i \cup B)\}, \\ \Longrightarrow x &\;\epsilon\; \{\bigwedge D \mid D \subseteq (A_i \cup B)\}, \quad \text{for all } i, \\ \Longrightarrow x &\;\epsilon\; A_i \vee B, \quad \text{for all } i, \\ \Longrightarrow x &\;\epsilon\; \bigcap (A_i \vee B). \end{aligned}$$

[$\supseteq$:] We can assume $I \neq \emptyset$. Let $x \,\epsilon\, \bigcap(A_i \vee B)$, then $x = a_i \wedge b_i$ with $a_i \,\epsilon\, A_i$ and $b_i \,\epsilon\, B$ for each $i \,\epsilon\, I$, by definition. If we let $b = \bigwedge b_i$, then we have

$$x = (\bigwedge a_i) \wedge b \leq a_i \wedge b \leq a_i \wedge b_i = x,$$

and therefore $x = a_i \wedge b$ for all $i \,\epsilon\, I$. Hence we can write, by Heyting formula (4) in 1.1.5, $x = (b \to a_i) \wedge b$ and we see[15] that $b \to a_i = a$ does not depend on i. Therefore, $x = a \wedge b$ with $b \,\epsilon\, B$ and $a \,\epsilon\, \bigcap A_i$, since $a = b \to a_i \,\epsilon\, A_i$ for each i. $\square$

Remark Consequently, $\mathcal{S}(L)$ and $\mathcal{S}'(L)$ are co-frames, while $\mathcal{C}(L)$ and $\mathcal{N}(L)$ are frames.

Notes

[1]A category is said to be *concrete* if it is equipped with a faithful functor to the category of sets.

[2]The forward implication of the proof is trivial, for the reverse implication, however, one needs to define a suitable frame. It turns out to be the three element chain or, from a topological perspective, the frame formed by taking the Sierpinski topology, on a set with 2 elements, with set-inclusion (see Johnstone [18]).

[3]This mapping is continuous since $m_A^{-1}(U) = A \cap U$, for every open set U.

[4]If two topologies, τ_1 and τ_2, defined on a set X are ordered by inclusion, then we say that τ_1 is *coarser* than τ_2 if $\tau_1 \subseteq \tau_2$. A continuous map $f : X \to Y$ is *initial* if τ_X is the coarsest topology on X for which f is continuous.

[5]This result is taken from Dikranjan and Tholen [8].

[6]An epimorphism f is called an *extremal epimorphism* if

$$f = gh \text{ and } g \text{ monomorphism implies that } g \text{ isomorphism.}$$

[7]One can easily show that the inverse function $g^{-1} : B \to C$ preserves arbitrary join (including 0) and finite meet (including 1), and therefore is a frame homomorphism.

[8]Note that while **Top** has a (onto continuous maps, embeddings)-factorization system, the couple of families (onto localic maps, one-one localic maps) constitute a factorization system in **Loc**. We refer the reader to Picado and Pultr [25] for a proof.

[9]As taken from Pultr [27], the category **Frm** is complete: The *products* in **Frm** are given by the cartesian products

$$\prod_{i \in I} L_i = \{\, f : I \to \bigcup_{i \in I} L_i \mid f(i) \in L_i, \forall\, i \in I \,\}$$

with standard projections

$$\prod L_i \to L_j : f \mapsto f(j).$$

The order, and consequently meets and joins, are defined coordinatewise. If $h_1, h_2 : L \to M$ are frame homomorphisms, then it can be shown that

$K = \{x \mid h_1(x) = h_2(x)\}$ is a subframe of L and the embedding $j : K \subseteq L$ is the *equalizer* of h_1, h_2. Pultr also provides a construction of coproducts and coequalizers in **Frm** - consequently **Frm** is also cocomplete.

[10]An *equivalence relation* E on a set X is a binary relation on X which is reflexive, transitive and symmetric.

[11]We denote a *quotient frame* by $L/E := \{xE \mid x \in L\}$, where $xE = \{y \mid (x,y) \in E\} = \{y \mid xEy\}$. It is defined from an algebraic point of view:

$$xE \wedge yE := (x \wedge y)E \text{ and } \bigvee (x_i E) := (\bigvee x_i)E.$$

The reader can verify that these definitions are well-defined and that the infinite distributive law in 2.1 holds. Consequently, L/E is a frame with elements xE called *equivalence classes*.

[12]Some authors exclude condition (2) from their definition. This is done intentionally, as one can readily show that (2) is a consequence of conditions (1) and (4). Even though it might be less beneficial, it can be shown, see Johnstone [20], that a nucleus can be characterized by a single identity: it is the map $\nu : L \to L$ which for all $x, y \in L$ satisfies

$$x \to \nu(b) = \nu(x) \to \nu(y).$$

[13]The following correspondences can be found in Picado and Pultr [25]: The correspondence between sublocale maps and nuclei is given by

$$\begin{aligned} h \mapsto \nu_h &= (x \mapsto h_* h(x)) : L \to L \text{ with } h_* \text{ the corresponding right adjoint,} \\ \nu \mapsto h_\nu &= \text{ the restriction } \nu : L \to \nu(L). \end{aligned}$$

Given a sublocale set $S \subseteq L$ we can construct the congruence E_S by

$$xE_S y \iff (\forall s \in S, x \le s \iff y \le s)$$

and given a congruence E, we have the associated sublocale set

$$S_E = \{x_{E_{max}} \mid x \in L\} \text{ where } x_{E_{max}} = \bigvee \{y \mid yE_S x\}.$$

Ultimately, the translation between sublocale maps $h : L \to M$ and sublocale sets $S \subseteq L$ is given by

$$\begin{aligned} h &\mapsto h_*(M) \subseteq L, \\ j &\mapsto j^* : L \to S \text{ for } j : S \subseteq L \text{ and } j^* \text{ the corresponding left adjoint.} \end{aligned}$$

[14]That is, the dual poset $\mathcal{S}'(L)^{op}$ is a frame. The proof has essentially been taken from Pultr [27].

[15]Given a Heyting algebra L and $a, b, c \in L$, then

$$a \wedge b = a \wedge c \iff a \to b = a \to c.$$

Chapter 4

Closure and co-closure operators

In this chapter we seek a closure operator in the category of generalized spaces. We will therefore explore the notion of a co-closure operator in **Frm**. First, however, we will consider the theory of factorization structures (systems), as it lays the foundation for closure operators, and mention a few types of closure operators.

Factorization structures

In chapter 3 we gave a short introduction to factorization systems, we will now consider it in more detail. Our first step is to give an equivalent definition:

Definition 4.1. Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in a category $\mathcal{C}$. $\mathcal{C}$ is called $(\mathcal{E},\, \mathcal{M})$-*structured* provided that

(1) each of $\mathcal{E}$ and $\mathcal{M}$ is closed under composition with isomorphisms,[1]

(2) $\mathcal{C}$ has $(\mathcal{E}, \mathcal{M})$-factorization of morphisms,

(3) $\mathcal{C}$ has the *unique* $(\mathcal{E}, \mathcal{M})$-*diagonalization property*, that is, for $m \in \mathcal{M}$, $e \in \mathcal{E}$ and for every $u, v \in \mathcal{M}or\mathcal{C}$ with $m \circ u = v \circ e$, there exists a unique d with $m \circ d = v$ and $d \circ e = u$. (See also Def. 3.2.)

We also say that $(\mathcal{E},\, \mathcal{M})$ is a *factorization structure*[2] for morphisms in $\mathcal{C}$. The reader might already have suspected it - there is a duality principle: If a category $\mathcal{C}$ is $(\mathcal{E},\, \mathcal{M})$-*structured*, then the dual category $\mathcal{C}^{op}$ is $(\mathcal{M},\, \mathcal{E})$-*structured.* Specifically, if a property holds for $\mathcal{E}$, then the dual property holds for $\mathcal{M}$.

Examples Let $\mathcal{I}$so$\mathcal{C}$ denote the family of all isomorphisms in category $\mathcal{C}$. One can easily verify that

(i) for any category $\mathcal{C}$, $(\mathcal{I}so\mathcal{C}, \mathcal{M}or\mathcal{C})$ and $(\mathcal{M}or\mathcal{C}, \mathcal{I}so\mathcal{C})$ are (trivial) factorization structures for morphisms,

(ii) **Frm** is (extremal epi, mono)-structured, this we saw in chapter 3,

(iii) by the duality principle, **Loc** is (epi, extremal mono)-structured and

(iv) **Top** has a (surjection, embedding)-factorization structure.

The next two propositions reveal some of the interesting properties of factorization structures. The result with its proof has essentially been taken from Adámek *et al.* [1].

Proposition 4.2. *Let category* $\mathcal{C}$ *be* ($\mathcal{E}$, $\mathcal{M}$)-structured. *Then* $(\mathcal{E}, \mathcal{M})$*-factorizations of morphisms are, up to isomorphism, unique.*

Proof. (1) Given two factorizations of a morphisms $f = m_i \circ e_i$ with $f : A \to B$, $e_i = A \to C_i$ and $m_i = C_i \to B$, $i = 1, 2$. We can construct two commutative diagrams and by the diagonalization property we have two unique diagonals $d_1 : C_1 \to C_2$ and $d_2 : C_2 \to C_1$.

$$\begin{tikzcd} A \arrow[r, "e_1"] \arrow[d, "e_2"'] & C_1 \arrow[d, "m_1"] \arrow[dl, "d_1"'] \\ C_2 \arrow[r, "m_2"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "e_2"] \arrow[d, "e_1"'] & C_2 \arrow[d, "m_2"] \arrow[dl, "d_2"'] \\ C_1 \arrow[r, "m_1"] & B \end{tikzcd}$$

Figure 4.1: Two unique diagonals d_1 and d_2.

As a result, the following two diagrams commute,

$$\begin{tikzcd} A \arrow[r, "e_1"] \arrow[d, "e_1"'] & C_1 \arrow[d, "m_1"] \arrow[dl, "d_2 \circ d_1"'] \\ C_1 \arrow[r, "m_1"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "e_2"] \arrow[d, "e_2"'] & C_2 \arrow[d, "m_2"] \arrow[dl, "d_1 \circ d_2"'] \\ C_2 \arrow[r, "m_2"] & B \end{tikzcd}$$

Figure 4.2: Two unique diagonals $d_1 \circ d_2$ and $d_2 \circ d_1$.

and since $e_1 \perp m_1$ and $e_2 \perp m_2$, diagonals are unique. Consequently $d_1 \circ d_2 = id$ and $d_2 \circ d_1 = id$ and the factorization is unique. □

Remark Even though the factorization of morphisms is essentially unique for a given ($\mathcal{E}$, $\mathcal{M}$) factorization structure; there can however be many different factorization structures for a fixed category. For example, **Set** has, besides the two trivial factorization structures, a (epi, mono)-factorization structure for morphisms, as well. None of these structures coincide, since their morphisms are different.

Lemma 4.3. *Let $\mathcal{C}$ be $(\mathcal{E}, \mathcal{M})$-*structured *and let $e \in \mathcal{E}$ and $m \in \mathcal{M}$. If the diagram below commutes, then e is an isomorphism and $f \in \mathcal{M}$.*

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "id"'] & B \arrow[d, "m"] \arrow[dl, "d"'] \\
C \arrow[r, "f"] & D
\end{tikzcd}$$

Figure 4.3: e is an isomorphism and $f \in \mathcal{M}$.

Proof. The diagram below commutes for $g = id$ and for $g = e \circ d$, since by assumption $d \circ e = id$ and $m \circ (e \circ d) = m$. Therefore, by the uniqueness of the diagonal, we have $e \circ d = id$. Consequently, e is an isomorphism. By our assumption, $f = m \circ e$ and since e is an isomorphism and m is closed under composition with isomorphisms, $f \in \mathcal{M}$. □

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "e"'] & B \arrow[d, "m"] \arrow[dl, "g"'] \\
B \arrow[r, "m"] & D
\end{tikzcd}$$

Figure 4.4: $g = id$ and $g = e \circ d$.

Proposition 4.4. *If $\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$-*structured*, then the following hold:*

(1) $\mathcal{E} \cap \mathcal{M} = \mathcal{I}so\mathcal{C}$,

(2) each of $\mathcal{E}$ and $\mathcal{M}$ are closed under composition.

Proof. (1) [$\subseteq$:] Take arbitrary $f : A \to B$ with $f \in \mathcal{E} \cap \mathcal{M}$. We can construct

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "id"'] & B \arrow[d, "id"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Figure 4.5: There exists a unique diagonal $d : B \to A$.

a commutative diagram (see Figure 4.5) and hence there exists a unique diagonal $d : B \to A$ such that $d \circ f = id$ and $f \circ d = id$. Therefore, $f \,\epsilon\, \mathcal{I}so\mathcal{C}$.

[$\supseteq$:] Take arbitrary $g : A \to B$ with $g \,\epsilon\, \mathcal{I}so\mathcal{C}$. Since $\mathcal{C}$ has $(\mathcal{E}, \mathcal{M})$-factorization of morphisms, we can write $g = m \circ e$ for $e : A \to C$ and $m : C \to B$ where $e \,\epsilon\, \mathcal{E}$ and $m \,\epsilon\, \mathcal{M}$. The diagram

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "id"'] & C \arrow[d, "m"] \arrow[dl, "d"'] \\
A \arrow[r, "g"] & B
\end{tikzcd}$$

Figure 4.6: The diagram commutes for $d = g^{-1} \circ m$.

commutes for $d = g^{-1} \circ m$, g is an isomorphism and hence g^{-1} exists. Consequently, by Lemma 4.3, $g \,\epsilon\, \mathcal{M}$. By duality we have that $g \,\epsilon\, \mathcal{E}$.

(2) Let $m_1, m_2 \,\epsilon\, \mathcal{M}$ where $m_1 : A \to B$ and $m_2 : B \to C$. Since $\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$-structured, we can write $m_2 \circ m_1 = m \circ e$ with $e : A \to D$ in $\mathcal{E}$ and $m : D \to C$ in $\mathcal{M}$. Hence there exists two unique diagonals d_1 and d_2

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "m_1"'] & D \arrow[d, "m"] \arrow[dl, "d_1"'] \\
B \arrow[r, "m_2"] & C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "id"'] & D \arrow[d, "d_1"] \arrow[dl, "d_2"'] \\
A \arrow[r, "m_1"] & B
\end{tikzcd}$$

Figure 4.7: Diagonals d_1 and d_2 can be constructed.

such that the diagrams (see Figure 4.7) commute. As a result the diagram in Figure 4.8 commutes and, by Lemma 4.3, we have that $m_2 \circ m_1 \,\epsilon\, \mathcal{M}$. $\mathcal{E}$ is also closed under composition, by duality. $\square$

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "id"'] & D \arrow[d, "m"] \arrow[dl, "d_2"'] \\
A \arrow[r, "m_2 \circ m_1"] & C
\end{tikzcd}$$

Figure 4.8: $m_2 \circ m_1 \,\epsilon\, \mathcal{M}$.

The foundation for closure operators has now been set, we will however briefly mention, for the sake of interest, the relationship between factorization structures and limits. For a proof[3], the reader may want to consider Adámek *et al.* [1].

Proposition 4.5 (Adámek, Herrlich and Strecker 1990). *If $\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$-structured, then*

(1) $\mathcal{M}$ is closed under the formation of products and pullbacks,

(2) $\mathcal{M} \cap \mathcal{M}ono\mathcal{C}$ is closed under the formation of intersections[4].

Remark If $\mathcal{M}$ is closed under the formation of pullbacks, then we also say that $\mathcal{M}$ is *pullback stable*.

Closure operators

We will now introduce the concept of a closure operator from a category theoretic point of view. First, however, we recall the definition of a $\mathcal{M}$-subobject and define what we mean by the *image* and *inverse image of a subobject m under a morphism*:

Definition 4.6. Assume a category $\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$-structured with $\mathcal{M} \subseteq \mathcal{M}ono\mathcal{C}$[5]. We define a $\mathcal{M}$*-subobject* of an object X to be a morphism $m \,\epsilon\, \mathcal{M}$ with codomain X, that is

$$m : M \longrightarrow X, \quad m \,\epsilon\, \mathcal{M}, \text{ and let}$$

$$\mathcal{M}/X = \{m \,\epsilon\, \mathcal{M} \mid \text{codomain } m = X\}.$$

We can define a binary relation on $\mathcal{M}/X$ given by

$$m \leq n \Longleftrightarrow (\exists j) \text{ with } m = n \circ j.$$

$$\begin{tikzcd} M \arrow[rr, "m"] \arrow[dr, "j"'] & & X \\ & L \arrow[ur, "n"'] & \end{tikzcd}$$

Figure 4.9: $m \leq n \Longleftrightarrow (\exists j)$ with $m = n \circ j$.

Two $\mathcal{M}$-subobjects (or subobjects, for short) m, n with $m \leq n$ and $n \leq m$ are said to be *isomorphic*. Allowing this isomorphism to mean equality, the class of all subobjects $\mathcal{M}/X$ give rise to a partially ordered class. In fact, one can say more, $\mathcal{M}/X$ is a complete lattice[6] if intersections exist and are again in $\mathcal{M}$.

$$\begin{tikzcd}
M \arrow[rr, "f \circ m"] \arrow[dr, "e"'] & & Y \\
& f(M) \arrow[ur, "m'"'] &
\end{tikzcd}$$

Figure 4.10: $(\mathcal{E}, \mathcal{M})$-factorization of $f \circ m$.

Definition 4.7. Assume a category $\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$-structured. Then, for $f : X \to Y$ in $\mathcal{M}or\mathcal{C}$ and $m : M \to X$ in $\mathcal{M}$, the composite can be factorized $f \circ m = m' \circ e$ with $e \,\epsilon\, \mathcal{E}$ (see Figure 4.10) and $m' \,\epsilon\, \mathcal{M}$. We define the *image of m under f* by

$$\begin{aligned}
f(-) : \mathcal{M}/X &\longrightarrow \mathcal{M}/Y \\
m : M \longrightarrow X &\longmapsto m' = f(m) : f(M) \longrightarrow Y.
\end{aligned}$$

For every morphism $f : X \to Y$ and every $n \,\epsilon\, \mathcal{M}/Y$, if a pullback diagram

$$\begin{tikzcd}
M \arrow[r, "m"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
N \arrow[r, "n"] & Y
\end{tikzcd}$$

Figure 4.11: Inverse images are given by pullback.

exists in $\mathcal{C}$, then we say $\mathcal{C}$ has $\mathcal{M}$-pullbacks. We call m the *inverse image of n under f* and denote it by $f^{-1}(n) : f^{-1}(N) \to X$ where $f^{-1}(-) : \mathcal{M}/Y \to \mathcal{M}/X$. If for every family $(m_i)_{i \,\epsilon\, I}$ in $\mathcal{M}/X$ a multiple pullback

$$\begin{tikzcd}
M \arrow[rr, "m"] \arrow[dr, "j_i"'] & & X \\
& M_i \arrow[ur, "m_i"'] &
\end{tikzcd}$$

Figure 4.12: $\mathcal{C}$ has $\mathcal{M}$-intersections.

diagram exists in $\mathcal{C}$ with $m \,\epsilon\, \mathcal{M}/X$, then we say $\mathcal{C}$ has $\mathcal{M}$-intersections.

Remark One can show that $f(-)$ is left-adjoint to $f^{-1}(-)$ (see 1.1.6).

Examples

(i) For the categories **Frm**, with $\mathcal{M}$ the class of injective frame homomorphisms, and **Top**, choose $\mathcal{M}$ to be the class of embeddings, the image of a subobject[7] M of X, with $f : X \to Y$, is given by set-theoretic image $f(M)$ considered as a subobject of Y.

(ii) **Set**, with $\mathcal{M}$ the class of injective functions, has inverse-images of subobjects A of Y, with $f : X \to Y$, given by set-theoretic inverse-image $f^{-1}(A)$ considered as a subobject of X.

We can now state the definition of a closure operator. Throughout we assume two properties:

- $\mathcal{M} \subseteq \mathcal{M}ono\mathcal{C}$ and $\mathcal{C}$ has $\mathcal{M}$-pullbacks.

Definition 4.8. Assume a category $\mathcal{C}$ has a ($\mathcal{E}$, $\mathcal{M}$)-factorization structure. A *closure operator* C of the category $\mathcal{C}$ with respect to the class $\mathcal{M}$ of subobjects is given by a family $C = (c_X)_{X \,\epsilon\, \mathcal{O}bj\mathcal{C}}$ of maps $c_X : \mathcal{M}/X \to \mathcal{M}/X$ such that for every $X \,\epsilon\, \mathcal{O}bj\mathcal{C}$:

(1) (Extension) $m \leq c_X(m)$ for all $m \,\epsilon\, \mathcal{M}/X$,

(2) (Monotonicity) if $m \leq m^{'}$ in $\mathcal{M}/X$, then $c_X(m) \leq c_X(m^{'})$,

(3) (Continuity) $f(c_X(m)) \leq c_Y(f(m))$ for all $f : X \to Y$ in $\mathcal{M}or\mathcal{C}$ and $m \,\epsilon\, \mathcal{M}/X$.

For an $\mathcal{M}$-subobject $m : M \to X$, the domain of its closure $c_X(m)$ will be denoted by $c_X(M)$. It can be shown that conditions (2) and (3) combined, is equivalent to:

$$f(m) \leq n \text{ implies } f(c_X(m)) \leq c_Y(n)$$

for all $f : X \to Y, m \,\epsilon\, \mathcal{M}/X$ and $n \,\epsilon\, \mathcal{M}/Y$.

Examples Throughout we assume that the category $\mathcal{C}$ has a $(\mathcal{E}, \mathcal{M})$-factorization structure. One can verify that the following are examples of closure operators:

(i) $S = (s_X)_{X \,\epsilon\, \mathcal{O}bj\mathcal{C}}$ with $s_X(m) = m$ for all $m \,\epsilon\, \mathcal{M}/X$ (called the *discrete closure operator*),

(ii) $T = (t_X)_{X \,\epsilon\, \mathcal{O}bj\mathcal{C}}$ with $t_X(m) = 1_X$ for all $m \,\epsilon\, \mathcal{M}/X$ (called the *trivial closure operator*)

(iii) For a poset $(X, \leq)$, we call a function $c : X \to X$ a *closure operation* of X if for all $m, m^{'} \,\epsilon\, X$ we have

$$m \leq c(m) \text{ and } m \leq m^{'} \Longrightarrow c(m) \leq c(m^{'}).$$

If we consider $(X, \leq)$ as a category[8] $\mathcal{C}$, then every closure operation induces a closure operator $C = (c_x)_{x \,\epsilon\, X}$ of the category $\mathcal{C}$ w.r.t. $\mathcal{M}$ with

$$c_x(m) = c(m) \wedge x \quad \text{for all} \quad m \leq x \quad \text{in X} .$$

(iv) For a subset M of a topological space[9] X, we define the (Kuratowski) closure of M in X by

$$k_X(M) = \{\, x \,\epsilon\, X : U \cap M \neq \emptyset \ \text{ for every open set } \ U \ni x \,\}^{10} = \overline{M}.$$

If you choose $\mathcal{M}$ to be the class of embeddings, then this defines a closure operator $K = (k_X)_{X \,\epsilon\, \mathcal{O}bj\mathbf{Top}}$ of **Top**.

(v) For $\mathcal{M}/X$ thought of as the class of subframes $L \subseteq X$, we can define the closure of L in X by $c_X(L) = \bigcap\{\, Q \,\epsilon\, \mathcal{M}/X \mid L^{'} \subseteq Q \,\}$ where $L^{'} = \{\, a \,\epsilon\, X \mid a \,\epsilon\, L \text{ or } \overline{a} \,\epsilon\, L \,\}$. This then defines a closure operator $C = (c_X)_{X \,\epsilon\, \mathcal{O}bj\mathbf{Frm}}$ of **Frm**[11].

Remark A closure operator C is then a family of maps that takes a subobject, say of X, and maps it to another subobject of X such that certain properties hold. In **Set**, a subset A of a set B will then be mapped to an intermediate subset $c_X(A) \subseteq B$, and in **Frm**, a subframe A of a frame L will be mapped to a (larger) subframe $c_X(A) \subseteq L$.

There is a narrow relationship between a closure operator in **Top** and a topology. We will make this relationship clear, however, we are in need of:

Definition 4.9. An $\mathcal{M}$-subobject $m : M \to X$ is called *C-closed* (in X) if it is isomorphic to its *C-closure*, that is, if $j_m : M \to c_X(M)$ is an isomorphism. m is called *C-dense* in X if its C-closure is isomorphic to 1_X, that is, if $c_X(m) : c_X(M) \to X$ is an isomorphism.

$$\begin{tikzcd} M \arrow[rr, "m"] \arrow[dr, "j_m", "\cong"'] & & X \\ & c_X(M) \arrow[ur, "c_X(m)"'] & \end{tikzcd}$$

Figure 4.13: m is C-closed.

$$\begin{tikzcd} M \arrow[rr, "m"] \arrow[dr, "j_m"] & & X \\ & c_X(M) \arrow[ur, "\cong", "c_X(m)"'] & \end{tikzcd}$$

Figure 4.14: m is C-dense.

Remark One can verify that for subspace inclusion $M \hookrightarrow X$, K-closed and K-dense, w.r.t the Kuratowski closure operator K of **Top**, agree with the usual topological definitions of closed and dense[12].

We now introduce three types of closure operators:

Definition 4.10. Assume category $\mathcal{C}$ has finite $\mathcal{M}$-unions[13]. A closure operator C of category $\mathcal{C}$ is said to be

(1) *idempotent* if the C-closure of a $\mathcal{M}$-subobject of X is C-closed, that is, $c_X(c_X(m)) = c_X(m)$ for all $m : M \to X$ in $\mathcal{M}$,

(2) *grounded* (for $X \in \mathcal{O}bj\mathcal{C}$) if $c_X(0_X) = 0_X$[14],

(3) *additive* (for X) if $c_X(m \vee n) = c_X(m) \vee c_X(n)$ for $m, n \in \mathcal{M}/X$.

Remark One can show that the Kuratowski closure operator K of **Top**, w.r.t the class of embeddings, enjoys all of the above properties. We now make our point clear with respect to the relationship between a closure operator and a topology:

In topology, for $A \subseteq X$, a closure operator is defined as a map $c_\tau : \mathcal{P}(X) \to \mathcal{P}(X)$ with $c_\tau(A) = \overline{A}$, that is grounded, extensive, additive and idempotent. The Kuratowski closure operator K of **Top** is an example of such closure operator in topology. Hence we can use an idempotent, grounded and additive closure operator to characterize closed sets. In turn, we know a topology can be defined (equivalently) by closed sets, that is, $A \subseteq X$ closed $\iff X \setminus A$ open.

A logical consequence is that in **Top** an idempotent, grounded and additive closure operator should characterize a topology[15]. This is true and we can say more - the converse is also true: A topology $\tau^{'}$ can be used to define a closure operator $c_{\tau^{'}}$, where $\overline{A} = X \setminus \bigcup\{\, B \in \tau^{'} \mid A \subseteq X \setminus B \,\}$ with $c_{\tau^{'}}(A) = \overline{A}$.

We have presented the introductory theory to closure operators with a handful of examples - our aim however is to find a closure operator of **Loc**. As we restrict ourselves to the dual category **Frm** for the remainder of this chapter, a family of maps in this category should have properties dual to that of a closure operator of **Loc**. We will refer to this family of maps as a *co-closure operator* of **Frm**. Before we define it formally, we will need:

Definition 4.11. Let a be an element of frame L. We define sublocales

(1) $\widehat{a} = (x \mapsto a \wedge x) : L \to \downarrow a$ (referred to as an *open* sublocale),

(2) $\breve{a} = (x \mapsto a \vee x) : L \to \uparrow a$ (referred to as a *closed* sublocale).

One can readily show that the sublocale set associated with the closed sublocale is given by $\mathbf{c}(a) = \uparrow a$. We quite naturally call this the *closed* sublocale set[16].

Let $h : L \to M$ be a frame homomorphism and $\gamma : M \to N$ be a sublocale. The *image* of γ under h, denoted by $h(\gamma)$, is the projection homomorphism $(x \mapsto xE) : L \to L/E$ of the congruence

$$xEy \iff \gamma h(x) = \gamma h(y).$$

Assume a category $\mathcal{C}$ is $(\mathcal{E}, \mathcal{M})$-structured. Let $X/\mathcal{E}$ denote the class of all $\mathcal{E}$-morphisms with domain X, and define a binary relation on $X/\mathcal{E}$ given by

$$m \leq n \iff (\exists j) \text{ with } m = j \circ n.$$

$$\begin{tikzcd} X \arrow[rr, "m"] \arrow[dr, "n"'] & & M \\ & N \arrow[ur, "j"'] & \end{tikzcd}$$

Figure 4.15: $m \leq n \iff (\exists j)$ with $m = j \circ n$.

Co-closure operators

Definition 4.12. Assume a category $\mathcal{C}$ has a $(\mathcal{E},\, \mathcal{M})$-factorization structure. A *co-closure operator* C of the category $\mathcal{C}$ with respect to the class $\mathcal{E}$ is given by a family $C = (c_X)_{X \,\epsilon\, \mathcal{O}bj\mathcal{C}}$ of maps $c_X : X/\mathcal{E} \to X/\mathcal{E}$ such that for every $X \,\epsilon\, \mathcal{O}bj\mathcal{C}$:

(1) (Extension) $m \leq c_X(m)$ for all $m \,\epsilon\, X/\mathcal{E}$,

(2) (Monotonicity) if $m \leq m^{'}$ in $X/\mathcal{E}$, then $c_X(m) \leq c_X(m^{'})$,

(3) (Continuity) $f(c_X(m)) \leq c_Y(f(m))$ for all $f : Y \to X$ in $\mathcal{M}or\mathcal{C}$ and $m \,\epsilon\, X/\mathcal{E}$.

Remark Due to manner in which we defined a binary relation on $X/\mathcal{E}$, the formulation of the three conditions above is similar to that for a closure operator with $\leq$ not being interchanged with $\geq$. A co-closure operator is nothing but a closure operator of $\mathcal{C}^{op}$ with respect to $\mathcal{E}^{op}$.

Examples

(i) For a sublocale set $S \subseteq L$ of frame L, it can be shown that the least closed sublocale set containing S is given by $\overline{S} = \uparrow \bigwedge S$. Translating this to sublocales, the *closure*[17] of sublocale $h : L \to M$ that corresponds to the closure $\overline{S}$ is given by $\breve{c} : L \to \uparrow c$ with $c = \bigvee\{\, x \mid h(x) = 0 \,\}$.

Let $\mathcal{E}$ be the class of surjective frame homomorphisms (sublocales) in **Frm**. Then the family $C = (c_X)_{X \,\epsilon\, \mathcal{O}bj\mathcal{C}}$ of maps $c_X : X/\mathcal{E} \to X/\mathcal{E}$ with

$$c_X(m) = \breve{c} \quad \text{with} \quad c = \bigvee\{\, x \mid m(x) = 0 \,\}$$

defines a co-closure operator C of **Frm** with respect to $\mathcal{E}^{op}$.

(ii) If we define

$$c_X(m) := m_* m : X \to m_* m(X) \quad \text{for every} \quad X \,\epsilon\, \mathcal{O}bj\mathbf{Frm},$$

then the family of maps $c_X : X/\mathcal{E} \to X/\mathcal{E}$ also constitutes a co-closure operator $C' = (c_X)_{X \,\epsilon\, \mathcal{O}bj\mathbf{Frm}}$ of **Frm** with respect to $\mathcal{E}^{op}$.

Note that the right adjoint m_* is not necessarily a frame homomorphism, since it does not necessarily preserve existing joins. If $m : X \to M$ is a sublocale, then one can show that $m_* m : X \to X$ is a nucleus (see notes on chap. 3). Furthermore, by Prop. 3.9, $m_* m : X \to m_* m(X)$ is an onto frame homomorphism. We will more often write $\overline{m}$ instead of $c_X(m)$.

Proof. (1) $m \leq \overline{m}$: Let $m : X \to M \,\epsilon\, X/\mathcal{E}$. We need to find a frame homomorphism $m' : c_X(X) \to M$ such that $j \circ \overline{m} = m$. Define m' to be m restricted to $c_X(X) := \overline{m}(X)$. m' is clearly a frame homomorphism.

$$\begin{tikzcd}
X \arrow[rr, "m"] \arrow[dr, "\overline{m}"'] & & M \\
& \overline{m}(X) \arrow[ur, "m'"'] &
\end{tikzcd}$$

Figure 4.16: m' is m restricted to $\overline{m}(L)$.

Then, for any $a \,\epsilon\, X$,

$$\begin{aligned}
(m' \circ \overline{m})(a) &= m'(\overline{m}(a)) \\
&= m'(m_* m(a)) \\
&= m(m_* m(a)) \\
&= (m m_* m)(a) = m(a).
\end{aligned}$$

(2) $m \leq n \Longrightarrow \overline{m} \leq \overline{n}$: Let $m : X \to Y$, $n : X \to Z \,\epsilon\, X/\mathcal{E}$ and assume $m \leq n$. Define a frame homomorphism $k : \overline{n}(X) \to \overline{m}(X)$ (see Figure 4.17) by $k := \overline{m} \circ m_* \circ j \circ n' = m_* \circ j \circ n'$. For arbitrary $x \,\epsilon\, X$, we have

$$\begin{aligned}
(k \circ \overline{n})(x) &= ((m_* \circ j \circ n') \circ \overline{n})(x) \\
&= (m_* \circ j)(n'(\overline{n}(x))) \\
&= (m_* \circ j)(n(x)) \\
&= m_*(j(n(x)) \\
&= m_*(m(x)) \\
&= \overline{m}(x).
\end{aligned}$$

$$\begin{tikzcd}
\overline{m}(X) \arrow[rrrd, dashed, "m'"] & & & \\
& X \arrow[ul, "\overline{m}"'] \arrow[rr, "m"] \arrow[dl, "\overline{n}"] \arrow[dr, "n"] & & Y \\
\overline{n}(X) \arrow[uu, dashed, "k"] \arrow[rr, dashed, "n'"] & & Z \arrow[ur, dashed, "j"'] &
\end{tikzcd}$$

Figure 4.17: Define frame homomorphism $k := m_* \circ j \circ n^{'}$.

k preserves finite meet, since each map in the composite does, and we can write $a_i = \overline{n}(x_i) \,\epsilon\, \overline{n}(X), i \,\epsilon\, I$, since $\overline{n}$ is onto. Therefore we have $k(\bigvee a_i) = k(\bigvee \overline{n}(x_i)) = (k \circ \overline{n})(\bigvee x_i) = \overline{m}(\bigvee x_i) = \bigvee \overline{m}(x_i) = \bigvee k(a_i)$.

(3) $f(\overline{m}) \leq \overline{f(m)}$: Let $m : X \to M \,\epsilon\, X/\mathcal{E}$ and $f : Y \to X \,\epsilon\, \mathcal{M}or\mathcal{C}$. Then we have $f(m) : Y \to Y/E$ (respectively $f(\overline{m}) : Y \to Y/E^{'}$) given by the congruence $xEy \Longleftrightarrow mf(x) = mf(y)$ (respectively $xE^{'}y \Longleftrightarrow \overline{m}f(x) = \overline{m}f(y)$) and $\overline{f(m)} := f(m)_* f(m) : Y \to \overline{f(m)}(Y)$ (see Figure 4.18 below).

$$\begin{tikzcd}
M & X \arrow[l, "m"'] \arrow[dl, "\overline{m}"] & Y \arrow[l, "f"'] \arrow[r, "f"] \arrow[dl, "f(\overline{m})"'] \arrow[d, "\overline{f(m)}"] \arrow[drr, "f(m)"] & X \arrow[r, "m"] & M \\
\overline{m}(X) \arrow[u] & Y/E^{'} \arrow[l] & \overline{f(m)}(Y) \arrow[l, dashed, "h"] \arrow[rr] & & Y/E \arrow[u]
\end{tikzcd}$$

Figure 4.18: $f(\overline{m}) \leq \overline{f(m)}$

We need to find a frame homomorphism $h : \overline{f(m)}(Y) \to Y/E^{'}$. Define h to be $f(\overline{m})$ restricted to $\overline{f(m)}(Y)$. Clearly this h is a frame homomorphism. Now we show that $h \circ \overline{f(m)} = f(\overline{m})$: This will only be true if for arbitrary $y \,\epsilon\, Y$ we have

$$(\overline{f(m)}(y))E^{'} = yE^{'} \Longleftrightarrow (\overline{m}f)(y) = (\overline{m}f)(\overline{f(m)}(y)).$$

$$\begin{aligned}
m(\overline{m}f(\overline{f(m)}(y))) &= m\overline{m}(f\overline{f(m)}(y)) \\
&= m(f\overline{f(m)}(y)) \\
&= mf(\overline{f(m)}(y)) \\
&= jf(m)(\overline{f(m)}(y)) \\
&= j(f(m)\overline{f(m)}(y)) \\
&= j(f(m)(y)) = m(f(y))
\end{aligned}$$

Since m_* preserves order, we have $m_*(m(\overline{m}f(\overline{f(m)}(y)))) = m_*(m(f(y)))$. Consequently, $\overline{m}(\overline{m}f(\overline{f(m)}(y)))) = m_*(m(f(y)))$ and ultimately we have $\overline{m}(f(\overline{f(m)}(y))) = \overline{m}(f(y))$, since $\overline{m}$ is a nucleus.

$\square$

Remark Since we started out with an arbitrary sublocale m, any nucleus m_*m formed in this way will give rise to a co-closure operator.

We will conclude this chapter by proving that in fact we only have one example of a co-closure operator of **Frm** with respect to $\mathcal{E}$:

Lemma 4.13. *The frame homomorphism $\nu : L \to \nu(L)$, formed by the nucleus $\nu : L \to L$, has right-adjoint the inclusion $\nu(L) \to L$.*

Proof. ($\Rightarrow$:) For $x \in L$ and $y \in \nu(L)$,

$$x \leq y \Longrightarrow \nu(x) \leq \nu(y) = y.$$

($\Leftarrow$:) Trivially, if $\nu(x) \leq y$ then $x \leq y$. $\square$

Proposition 4.14. *Let $m : X \to L$ be in $X/\mathcal{E}$. Then the sublocales $m_*m : X \to m_*m(X)$ and $\check{c} : X \to \uparrow c$ with $c = \bigvee\{\, x \mid m(x) = 0 \,\}$, are equivalent.*

$$\begin{tikzcd} X \arrow[rr, "\check{c}"] \arrow[dr, "m_*m"'] & & \uparrow c \\ & m_*m(X) \arrow[ur, dashed, "k"'] & \end{tikzcd}$$

Figure 4.19: $\check{c} \leq m_*m$

$$\begin{tikzcd} X \arrow[rr, "m_*m"] \arrow[dr, "\check{c}"'] & & m_*m(X) \\ & \uparrow c \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Figure 4.20: $m_*m \leq \check{c}$

Proof. Define $k : m_*m(X) \to \uparrow c$ to be $\check{c}$ restricted to $m_*m(X)$. Clearly k is a frame homomorphism. Then for arbitrary $x \in X$ we have

$$(k \circ m_*m)(x) = k(m_*m(x)) = m_*m(x) \vee c = x \vee c.$$

(Trivially, $m_*m(x) \vee c \geq x \vee c$. Since $x \leq x \vee c$, by Lemma 4.13 we have $m_*m(x) \leq x \vee c$. Also $c \leq x \vee c$ and consequently $m_*m(x) \vee c \leq x \vee c$.)

Define $h : \uparrow c \to m_*m(X)$ to be m_*m restricted to $\uparrow c$. Trivially, h is a frame homomorphism. For arbitrary $x \in X$ we have

$$\begin{aligned} (h \circ \check{c})(x) &= h(\check{c}(x)) \\ &= h(x \vee c) \\ &= m_*m(x) \vee m_*m(c) \\ &= m_*m(x), \text{ since } m(x) \geq m(c). \end{aligned}$$

$\square$

Notes

[1]We can replace condition (1) with:

- if $e \in \mathcal{E}$ and $h \in \mathcal{I}so\mathcal{C}$, and $h \circ e$ exists, then $h \circ e \in \mathcal{E}$,
- if $m \in \mathcal{M}$ and $h \in \mathcal{I}so\mathcal{C}$, and $m \circ h$ exists, then $m \circ h \in \mathcal{M}$.

[2]Some authors prefer the terminology *factorization system* (as defined in chapter 3), while others use *factorization structure*. These are equivalent notions and can therefore be used interchangeably.

[3]The proof rests upon the use of $(\mathcal{E}, \mathcal{M})$-factorization of morphisms, the diagonalization property and Lemma 4.3.

[4]Let $\mathcal{A}$ be a family of subobjects (A_i, m_i) of an object B, indexed by a class I. A subobject (A, m) of B is called an *intersection of* $\mathcal{A}$ provided that the following two conditions are satisfied:

(1) m factors through each m_i, that is, for each i there exists an f_i with $m = m_i \circ f_i$,

(2) if a morphism $f : C \to B$ factors through each m_i, then it factors through m.

[5]$\mathcal{M}ono\mathcal{C}$ denotes the class of monomorphisms in a category $\mathcal{C}$.

[6]Arbitrary meet exists (and therefore arbitrary join) and is given by intersection in $\mathcal{M}/X$. The reader will find a detailed outline of the structure of $\mathcal{M}/X$ in Dikranjan and Tholen [8].

[7]We can think of a subobject $m : M \to X$ as $M \subseteq X$.

[8]The objects in $\mathcal{C}$ are all $x \in X$ and the morphisms are given by $f : x \to y \iff x \leq y$. One readily sees that there is only one such morphism and therefore every morphism in $\mathcal{M}or\mathcal{C}$ is a monomorphism. Consequently, a $(\mathcal{E}, \mathcal{M})$-structure for morphisms is given by $(\mathcal{I}so\mathcal{C}, \mathcal{M}or\mathcal{C})$.

[9]**Top** is $(\mathcal{E}, \mathcal{M})$-structured and has $\mathcal{M}$-pullbacks.

[10]The closure of M can also described as in 1.2.3.

[11]This category has a $(\mathcal{E}, \mathcal{M})$-structure for morphisms where $\mathcal{E}$ is the class of surjective frame homomorphisms and $\mathcal{M}$ represents the class of injective frame

homomorphisms. We note that since **Frm** is complete (this we saw in chapter 3) it has $\mathcal{M}$-pullbacks.

[12]A set $A \subseteq X$ is called *dense* in X if $\overline{A} = X$.

[13]Note that if a category $\mathcal{C}$ has $\mathcal{M}$-pullbacks and $\mathcal{M}$-intersections, then $\mathcal{C}$ has $\mathcal{M}$-unions.

[14]0_X denotes the least element of $\mathcal{M}/X$.

[15]One can prove that if $c : \mathcal{P}(X) \to \mathcal{P}(X)$ is an idempotent, grounded and additive closure operator in **Top**, then $\tau_c := \{\, X \setminus A \,|\, A = c(A) \,\}$ is a topology on X.

[16]The *open* sublocale set is given by $\mathbf{o}(a) = \{a \to x \,|\, x \,\epsilon\, L\}$. A proof can be found in Pultr [27].

[17]The closure of h is the smallest closed sublocale $\breve{a}$ such that $h \leq \breve{a}$.

Chapter 5

Compactness

In this final chapter we begin with a comparison between three weaker notions of compactness in the topological setting. We then proceed onto the point-free setting in which the frame translation of three of the separation axioms serve as introduction. The chapter concludes with a comparison between the weaker notions of compactness in this setting.

In chapter 4 our focus centered around closure - we now turn to compactness which is a very important notion in topology. This importance is reflected in the keen interest it enjoys amongst researchers and the good many results produced to date. A brief topological overview of a few well-known weaker notions of compactness will serve as introduction.

Definition 5.1. A topological space X is called a *Tychonoff*, $T_{3\frac{1}{2}}$ or *completely regular* space if X is a T_1-space and for every closed set $A \subseteq X$ and $x \notin A$ there is a continuous map $f : X \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$ for $y \,\epsilon\, A$ (see 1.2.5).

Remark Note that in the definition the codomain of the map f could also have been given by $[a,b]$ or $\mathbb{R}$. Also, we see that $T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$.

We recall that a topological space X is said to be *compact* if X is a Hausdorff space and every open cover of X has a finite subcover. We now introduce our first weaker notion of compactness, called *pseudocompactness*:

Definition 5.2. A topological space X is called *pseudocompact* if X is a Tychonoff space and every continuous real-valued function defined on X is bounded.

Proposition 5.3. *Let X be a Tychonoff space. Then X is pseudocompact if and only if every countable open cover contains a finite subcollection whose union is dense in X.*

Proof. ($\Leftarrow$:) Assume $f : X \to \mathbb{R}$ is continuous. Define $C_n := f^{-1}((-n, n))$ for all $n \,\epsilon\, \mathbb{N}$. Then C_n is an open cover of X and by assumption there exists a $k \,\epsilon\, \mathbb{N}$ such that $\overline{\bigcup_{i=1}^{k} C_i} = \overline{C_k} = X$. Therefore $f(X) = f(\overline{C_k}) \subseteq \overline{f(f^{-1}(-k,k))} \subseteq \overline{(-k,k)} = [-k, k]$ and f is bounded.

($\Rightarrow$:) Let $\mathcal{C} = \{\, C_n \,|\, n \,\epsilon\, \mathbb{N}\,\}$ be an open cover of X with $n \leq m \Rightarrow C_n \subseteq C_m$ and assume $\overline{C_n} \neq X$ for all $n \,\epsilon\, \mathbb{N}$. For each n there exists $x_n \,\epsilon\, X \setminus \overline{C_n}$ and therefore we have a separating map $f : X \to [0,1]$ so that $f(\overline{C_n}) \subseteq \{0\}$ and $f(x_n) = n$. Note that since $\mathcal{C}$ is a cover, for all $x \,\epsilon\, X$ there exists $n_x = min\{\, n \,|\, x \,\epsilon\, C_n \,\}$.

For $x \,\epsilon\, X$ define $F(x) := \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{n_x} f_n(x)$. Given $x \,\epsilon\, X$ and for all $n \,\epsilon\, \mathbb{N}$ such that $n \leq n_x$ we have that $f_n : X \to \mathbb{R}$ is continuous. Thus for $\varepsilon > 0$ and $n \leq n_x$ there exists a neighbourhood U_n of x so that for all $y \,\epsilon\, U_n$,

$$|\, f_n(x) - f_n(y)\,| < \frac{\varepsilon}{n_x}$$

.

Let $U := \bigcap_{n=1}^{n_x} U_n \cap C_{n_x}$. Then U is a neighbourhood of x such that for all $y \,\epsilon\, U$,

$$|\, F(x) - F(y)\,| = |\sum_{n=1}^{n_x} f_n(x) - \sum_{n=1}^{n_x} f_n(y)\,| < n_x \cdot \frac{\varepsilon}{n_x} = \varepsilon,$$

since $y \,\epsilon\, C_{n_x}$ implies that $f_n(y) = 0$, for all $n \geq n_x$. $\square$

One will find a handful of equivalent notions characterizing a pseudocompact space in Engelking [13]. These characterizations are, depending on one's need, more suitable since they utilize the internal properties of the space to define pseudocompactness - unlike our definition (see 5.2) earlier. We mention one in particular:

> A space X is pseudocompact if and only if for every countable family $\{U_i\}_{i=1}^{\infty}$ of open subsets which has the finite intersection property[1] the intersection $\bigcap_{i=1}^{\infty} \overline{V_i}$ is non-empty.

By using the contrapositive the statement can be rephrased:

> A space X is pseudocompact if and only if for every countable family $\{V_i\}_{i=1}^{\infty}$ of non-empty open subsets of X with $\bigcap_{i=1}^{\infty} \overline{V_i} = \emptyset$, there exists a finite set $\{i_1, i_2, ..., i_k\} \subseteq I$ such that $V_{i_1} \cap V_{i_2} \cap ... \cap V_{i_k} = \emptyset$.

We will now use this formulation to give an alternative proof to Prop. 5.3:

Proof. (⇐:) Assume every countable open cover of X contains a finite subcollection whose union is dense in X. Assume furthermore that $\bigcap_{i=1}^{\infty} \overline{V_i} = \emptyset$ for a countable family $\{ V_i \}_{i=1}^{\infty}$ of open subsets of X. Then

$$\bigcap_{i=1}^{\infty} \overline{V_i} = \emptyset \implies X \setminus \bigcap_{i=1}^{\infty} \overline{V_i} = X \implies \bigcup_{i=1}^{\infty} X \setminus \overline{V_i} = X.$$

$X \setminus \overline{V_i}$ is open, since $\overline{V_i}$ is closed. Hence $\{ W_i \}_{i=1}^{\infty} := \{ X \setminus \overline{V_i} \}_{i=1}^{\infty}$ is a countable open cover of X. Therefore, there exists a finite set, say $\{i_1, i_2, ..., i_k\}$ such that

$$\begin{aligned} \overline{\bigcup_{m=1}^{k} W_{i_m}} &= X \\ \implies \bigcup_{m=1}^{k} \overline{W_{i_m}} &= X \\ \implies \bigcap_{m=1}^{k} X \setminus \overline{X \setminus \overline{V_{i_m}}} &= \emptyset \\ \implies \bigcap_{m=1}^{k} (\overline{V_{i_m}})^{\circ} &= \emptyset. \end{aligned}$$

But $\bigcap_{m=1}^{k} (\overline{V_{i_m}})^{\circ} \supseteq \bigcap_{m=1}^{k} V_{i_m}$, since V_{i_m} is open and consequently $\bigcap_{m=1}^{k} V_{i_m} = \emptyset$.

(⇒:) Assume that for every countable family $\{V_i\}_{i=1}^{\infty}$ of non-empty open subsets of X with $\bigcap_{i=1}^{\infty} \overline{V_i} = \emptyset$, there exists a finite set $\{i_1, i_2, ..., i_k\} \subseteq I$ such that $V_{i_1} \cap V_{i_2} \cap ... \cap V_{i_k} = \emptyset$. Let $\{ W_i \}_{i=1}^{\infty}$ be an arbitrary open cover of X. Then

$$\bigcup_{i=1}^{\infty} W_i = X \implies \bigcap_{i=1}^{\infty} \overline{W_i} = X \implies \bigcap_{i=1}^{\infty} X \setminus \overline{W_i} = \emptyset.$$

Since $\overline{V_i}$ is closed, $X \setminus \overline{V_i}$ is open and hence $\{ V_i \}_{i=1}^{\infty} := \{ X \setminus \overline{W_i} \}_{i=1}^{\infty}$ is a countable family of open subsets of X.

We now show that $\bigcap_{i=1}^{\infty} \overline{V_i} = \emptyset$:

$\bigcup_{i=1}^{\infty} W_i \subseteq \bigcup_{i=1}^{\infty} (\overline{W_i})^{\circ}$, since W_i is open. But, by assumption, $\bigcup_{i=1}^{\infty} W_i = X$ and therefore

$$
\begin{aligned}
\bigcup_{i=1}^{\infty}(\overline{W_i})^{\circ} &= X \\
\Longrightarrow \bigcup_{i=1}^{\infty} X \setminus \overline{X \setminus \overline{W_i}} &= X \\
\Longrightarrow \bigcap_{i=1}^{\infty} \overline{X \setminus \overline{W_i}} &= \emptyset \\
\Longrightarrow \bigcap_{i=1}^{\infty} \overline{V_i} = \emptyset.
\end{aligned}
$$

Hence there exists a finite set $\{i_1, i_2, ..., i_k\} \subseteq I$ such that $V_{i_1} \cap V_{i_2} \cap ... \cap V_{i_k} = \emptyset$. Furthermore,

$$
\bigcap_{m=1}^{k} V_{i_m} = \emptyset \implies \bigcap_{i=1}^{k} \overline{W_i} = X \implies \overline{\bigcup_{m=1}^{k} W_{i_m}} = X.
$$

$\square$

Porter and Woods published a paper in 1984 wherein the definition of a *feebly compact* space is given as a (Hausdorff) space in which every locally finite family of open subsets of X is finite. This definition of feeble compactness is attributed to Mardešić and Papić [21] and Bagley *et al.* [2] called such a space *lightly compact.* Included in the paper by Porter and Woods are notions proven equivalent to feeble compactness - one of these will serve as our definition:

Definition 5.4. A space X is called *feebly compact* if every countable open cover of X contains a finite subcollection whose union is dense in X.

Remark If one momentarily recalls the previous proposition, then it is clear that for a Tychonoff space X, pseudocompactness and feeble compactness are equivalent notions.

We direct our attention to the third weaker notion of compactness in the topological setting:

Definition 5.5. A topological space X is called a *countably compact* space if X is a Hausdorff space and every countable open cover has a finite subcover.

Proposition 5.6. *Every countably compact space X is feebly compact. Consequently, a countably compact Tychonoff space is pseudocompact.*

Proof. Assume X is a countably compact space and let $\mathcal{C} = \{ C_n \mid n \in \mathbb{N} \}$ be a countable open cover of X. Then there exists a finite subcover, say $\mathcal{C}' = \{ C_{i_1}, C_{i_2}, ..., C_{i_k} \} \subseteq \mathcal{C}$, and hence $\overline{\bigcup_{j=1}^{k} C_{i_j}} = X$. $\square$

The following proposition shows that for a normal space, pseudocompactness implies countable compactness. The proof has been taken from Engelking [13].

Proposition 5.7. *Every pseudocompact normal space X is countably compact.*

Proof. Assume that X is a normal space which is not countably compact. Thus there exists a decreasing sequence[2] $F_1 \supseteq F_2 \supseteq F_3 \supseteq ...$ of non-empty closed subsets of X so that $\bigcap_{i=1}^{\infty} F_i = \emptyset$.

Choose $x_i \in F_i, x_i \neq x_j$ whenever $i \neq j$ and let $A = \{ x_1, x_2, ... \}$. We also have $A^d = \emptyset$[3].

Clearly A is a discrete closed subspace of X, and by the Tietze-Urysohn theorem there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x_i) = i$ for $i = 1, 2, 3, ...$ Since f is not bounded, the space X is not pseudocompact. $\square$

Remark Trivially, every compact space is countably compact and we have seen that for Tychonoff spaces, countably compactness implies pseudocompactness (feeble compactness). Therefore, a compact Tychonoff space is pseudocompact (feebly compact).

We now continue this chapter by introducing the above-mentioned compact notions in the point-free setting. We trust that this can be done, provided that we restrict ourselves to notions and formulations involving open sets. We begin this transition to the point-free setting by presenting the frame counterpart of three separation axioms of classical topology:

Definition 5.8. Let x, y be elements of a frame L. We will say x is *rather below* y in L and denote it by

$$x \prec y \text{ if } x^* \vee y = 1$$[4].

We refer the reader to Pultr [27] for a proof of the following Lemma:

Lemma 5.9. *(1)* $a \prec b \Longrightarrow a \leq b$ *and for any* a, $0 \prec a \prec 1$.

(2) $x \leq a \prec b \leq y \Longrightarrow x \prec y$.

(3) If $a \prec b$, *then* $b^* \prec a^*$.

(4) If $a \prec b$, *then* $a^{**} \prec b$.

(5) If $a_i \prec b_i$ *for* $i = 1, 2$ *then* $a_1 \vee a_2 \prec b_1 \vee b_2$ *and* $a_1 \wedge a_2 \prec b_1 \wedge b_2$.

Definition 5.10. A frame L is said to be *regular* if for each $a \in L$ we have

$$a = \bigvee \{\, b \mid b \prec a \,\}.$$

Lemma 5.11. *For* $U \in \Omega(X)$, *we have* $U^* = X \backslash \overline{U}$.

Proof. We will show that $V \subseteq X \backslash \overline{U} \Longleftrightarrow U \cap V = \emptyset$, for $V \in \Omega(X)$.

($\Rightarrow$:) Assume $V \subseteq X \backslash \overline{U}$. Since $U \subseteq \overline{U}$, we have $V \subseteq X \backslash \overline{U} \subseteq X \backslash U$ and hence $U \cap V = \emptyset$.

($\Leftarrow$:) Assume $U \cap V = \emptyset$. This implies that $U \subseteq X \backslash V$. We know that $X \backslash \overline{U} = X \backslash \bigcap \{\, B \subseteq X \mid B \text{ closed}, U \subseteq B \} = \bigcup \{\, X \backslash B \subseteq X \mid B \text{ closed}, U \subseteq B \}$. Then $V \in \{\, X \backslash B \subseteq X \mid B \text{ closed}, U \subseteq B \}$ and consequently $V \subseteq X \backslash \overline{U}$. □

Proposition 5.12. *Let* X *be a topological space. For arbitrary* $U, V \in \Omega(X)$

$$V \prec U \quad \text{if and only if} \quad \overline{V} \subseteq U.$$

Proof. Take arbitrary $U, V \in \Omega(X)$ with $V \prec U$,

$$\begin{aligned}
&\Longleftrightarrow && V^* \cup U = X, \\
&\Longleftrightarrow && X \backslash \overline{V} \cup U = X, \text{ by Lemma 5.11} \\
&\Longleftrightarrow && \overline{V} \cap X \backslash U = \emptyset, \\
&\Longleftrightarrow && \overline{V} \subseteq U.
\end{aligned}$$

□

Considering frame homomorphisms and the "rather below" notion, it can readily be seen that these homomorphisms are well behaved with respect to it:

Proposition 5.13. *For any frame homomorphism* $h : L \to M$, *we have that*

$$x \prec y \Longrightarrow h(x) \prec h(y).$$

Proof. The proof is an immediate consequence of $h(x^*) \leq h(x)^*$. □

Definition 5.14. Let x, y be elements of a frame L. We will say that x is *completely below* y in L and denote it by

$$x \prec\prec y$$

if there are $x_r \in L$ for r dyadic rational in the interval [0,1] such that $x_0 = x,\ x_1 = y$ and $x_r \prec x_s$ for $r < s$.

The proof of the following lemma follows directly from the definition of "completely below" and Lemma 5.9:

Lemma 5.15. *(1) $a \prec\prec b \Longrightarrow a \leq b$ and for any a, $0 \prec\prec a \prec\prec 1$.*

(2) $x \leq a \prec\prec b \leq y \Longrightarrow x \prec\prec y$.

(3) If $a \prec\prec b$, then $b^* \prec\prec a^*$.

(4) If $a \prec\prec b$, then $a^{**} \prec\prec b$.

(5) If $a_i \prec\prec b_i$ for $i = 1, 2$ then $a_1 \vee a_2 \prec\prec b_1 \vee b_2$ and $a_1 \wedge a_2 \prec\prec b_1 \wedge b_2$.

Definition 5.16. A frame L is said to be *completely regular* if

$$\text{for each} \quad a \in L,\ a = \bigvee \{\, b \mid b \prec\prec a \,\}.$$

Frame homomorphisms not only behave well with respect to the "rather below" notion, we have that:

Proposition 5.17. *For any frame homomorphism $h : L \to M$,*

$$x \prec\prec y \Longrightarrow h(x) \prec\prec h(y).$$

Proof. For $x, y \in L$ assume $x \prec\prec y$. By definition there are $x_r \in L$ for r dyadic rational in [0,1] with $x_r \prec x_s$ for $r < s$.

By Prop. 5.13 there are $h(x_r) \in M$ for r dyadic rational in [0,1] with $h(x_r) \prec h(x_s)$ for $r < s$. Therefore $h(x) \prec\prec h(y)$. □

Definition 5.18. We say that a frame L is *normal* if whenever $a \vee b = 1$ for $a, b \in L$, there exists $u, v \in L$ such that

$$u \wedge v = 0,\ u \vee b = 1 \ \text{ and } \ a \vee v = 1.$$

Remark We know from classical topology that if a space X is completely regular, then it is also regular. One would naturally like to know if this property is transfered to the point-free setting. It can readily be verified that it is indeed the case. However, contrary to our expectation from topology, normality in the point-free setting does not imply complete regularity.[5]

Definition 5.19. A relation R is said to be *interpolative* if

$$aRb \Longrightarrow \exists\, c,\ aRcRb.$$

The proof of the following Lemma was taken from Pultr [27]:

Lemma 5.20. *The relation $\prec$ in a normal frame is interpolative.*

Proof. Let $a \prec b$ in a normal frame L. Then there are u, v such that $u \leq v^*, u \vee b = 1$ and $a^* \vee v = 1$. Thus, $a \prec v$ and $v^* \vee b \geq u \vee b = 1$ so that also $v \prec b$. □

Remark As a consequence of Lemma 5.20 one has:

If a frame L is normal and regular, then it is completely regular[6].

A question lingers in the back of our minds: How well does the frame translation of the separation axioms relate to the topological notion? We would like this relation to be an equivalence, but cannot expect it for all the separation axioms: Pultr argues that since separation axioms T_1 and T_2 rely heavily on points, no exact translation to the point-free setting should be expected[7]. On the other hand, it is well-known that the notions of regularity, complete regularity and normality in frames, are equivalent to their topological counterpart.

We will prove the equivalence for complete regularity. Our proof contains an imitation of the standard construction utilized in the proof of the well-known Urysohn lemma.

Proposition 5.21. *A space X is completely regular if and only if the frame $\Omega(X)$ is completely regular.*

Proof. ($\Rightarrow$:) Assume X is completely regular. We must show that

$$A = \bigcup \{\, B \mid B \prec\prec A \,\}, \text{ for all } A \in \Omega(X).$$

Since X is completely regular, we have that for every $x \in X$ and every neighbourhood U of x, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in X \backslash U$.

Take arbitrary non-empty $A \in \Omega(X)$ and $x \in A$. Then A is a neighbourhood of x and hence we have a continuous real function $\widetilde{f} : X \to \mathbb{R}$ such that $\widetilde{f}(x) = 0$ and $\widetilde{f}(y) = 1$ for $y \in X \backslash A$.

Let $B_x := \widetilde{f}^{-1}([0, \frac{1}{2}))$. Then B_x is open and $B_x \subseteq A$. We also have $f(x) < \frac{1}{2}$ for all $x \in B_x$ and $f(y) \geq 1$ for all $x \notin A$. Therefore $B_x \prec\prec A$[8].

Since $x \in A$ was arbitrary, we have that

$$\bigcup \{ B_x \mid B_x \prec\prec A \} = A.$$

($\Leftarrow$:) Assume $\Omega(X)$ is completely regular. Take arbitrary $x \in X$ and neighbourhood U of x. We need to show that there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in X \backslash U$.

There exists an open $V \prec\prec U$ with $x \in V$, since by assumption $U = \bigcup \{ V \mid V \prec\prec U \}$. Hence we have the family

$$\{ V_i \mid i \in \{0, \frac{1}{2^n}, \frac{2}{2^n}, ..., \frac{2^n - 1}{2^n}, 1\}, n \in \mathbb{N} \}^9,$$

with $V_0 = V$, and $V_1 = U$. Also each V_i is open and $\overline{V_i} \subseteq V_j$ for all $i < j$.

Now define V_t for every $t \in [0,1]$ by

$$V_t := \bigcup_{r \in \mathcal{A}, r \leq t} V_r.$$

This V_t is well-defined and V_t is open for each $t \in [0,1]$ with the property

$$\overline{V_t} \subseteq V_{t'}, \ \ \forall t, t' \in [0,1], \ t < t'.$$

Now define a function $f : X \to [0,1]$ by

$$f(x) := \begin{cases} inf \{ t \in [0,1] \mid x \in V_t \} \\ 1 \text{ if } x \notin V_t, \ \forall t \in [0,1] \end{cases}$$

Since $x \in V_0 \subseteq V_t$ this implies that $f(x) = 0$. Also for $y \in X \backslash U$ we have $f(y) = 1$, since $V_t \subseteq U, \forall t \in [0,1]$.

We still have to show that f is continuous:

Pick $x \in X$ and put $t_x := f(x) \in [0,1]$. Let $\varepsilon > 0$. We must show that there exists $W \in \mathcal{U}_x$ such that $\forall y \in W$,

$$| t_x - f(y) | \leq \varepsilon.$$

Define W by $W := V_{t_x + \varepsilon} \cap (X \backslash \overline{V}_{t_x - \varepsilon})$ where we set

$$\begin{aligned} V_{t_x+\varepsilon} &= X \ \text{ if } \ t_x + \varepsilon > 1, \\ V_{t_x-\varepsilon} &= \emptyset \ \text{ if } \ t_x - \varepsilon < 0. \end{aligned}$$

Clearly W is open and we now show that $x \in W$:

(i) $x \,\epsilon\, V_{t_x+\varepsilon}$: This is clear if $t_x = 1$, since then $V_{1+\varepsilon} = X$. If $t_x \,\epsilon\, [0,1]$ then by definition of f there is a $t \,\epsilon\, [0,1]$ with $x \,\epsilon\, V_t$ and $t_x < t < t_x + \varepsilon$.

(ii) $x \notin V_{t_x-\varepsilon}$: This is clear if $t_x - \varepsilon < 0$, since then $V_{t_x-\varepsilon} = \emptyset$. If $t_x - \varepsilon \geq 0$, then by definition of f we have $x \notin V_{t_x-\frac{\varepsilon}{2}}$, but then $x \notin \overline{V}_{t_x-\varepsilon}$.

By definition of f, $y \epsilon V_{t_x+\varepsilon}$ implies that $f(y) \leq t_x + \varepsilon$ and $y \notin \overline{V}_{t_x-\varepsilon}$ implies that $f(y) \geq t_x - \varepsilon$. Consequently we have

$$|\, t_x - f(y)\,| \leq \varepsilon$$

for all $y \,\epsilon\, W$ and thus f is continuous. □

We can now introduce the weaker notions of compactness in the point-free setting. We start off with a frame translation of classical pseudocompactness (see 5.2) - it is therefore necessary to introduce the reals in a point-free way:

Definition 5.22. The *frame of reals* is the frame $\mathcal{L}(\mathbb{R})$ generated by all ordered pairs (p,q) where $p,q \,\epsilon\, \mathbb{Q}$ subject to the following relations:

(1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,

(2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \leq r < q \leq s$,

(3) $(p,q) = \bigvee\{\,(r,s) \,|\, p < r < s < q\,\}$,

(4) $1 = \bigvee\{\,(p,q) \,|\, p,q \,\epsilon\, \mathbb{Q}\,\}$.

Remark One notes that this bears a likeness to the interval topology of the real line.

Definition 5.23. For a frame L, a frame homomorphism $\varphi : \mathcal{L}(\mathbb{R}) \to L$ is said to be *bounded* if there exists $p,q \,\epsilon\, \mathbb{Q}$ such that $\varphi(p,q) = 1$. We say that L is *pseudocompact* whenever all these φ are bounded.

One might ask if this notion of pseudocompactness is an equivalent translation of the topological one. It is indeed[10]:

One can show that there is a one-one onto map

$$\mathbf{Frm}(\mathcal{L}(\mathbb{R}), \mathbf{\Omega}(\mathbf{X})) \to \mathbf{Top}(\mathbf{X}, \mathbb{R})$$

defined by taking φ to $\tilde{\varphi}$ such that

$$p < \tilde{\varphi}(x) < q \Longleftrightarrow x \,\epsilon\, \varphi(p,q).$$

Therefore φ is bounded ($\exists\, p,q$ with $\varphi(p,q) = X$) if and only if $\tilde{\varphi}$ is bounded (for some p,q, $p < \tilde{\varphi}(x) < q$ for all x). As a result we can immediately say

The space X is pseudocompact if and only if the completely regular frame $\Omega(X)$ is pseudocompact.

Definition 5.24. An element a of L is said to be *dense* if $a^* = 0$.

Proposition 5.25. *If every countable cover of a frame L contains a finite subset whose join is dense, then L is pseudocompact.*

Proof. Assume every countable cover of L contains a finite subset whose join is dense. Furthermore, assume there exist a frame homomorphism $\varphi : \mathcal{L}(\mathbb{R}) \to L$ which is not bounded.

Let $A = \{\, (-n, n) \,|\, n \,\epsilon\, \mathbb{N} \,\}$. Clearly A is a countable cover of $\mathcal{L}(\mathbb{R})$ and consequently $\varphi(\bigvee A) = \bigvee(\varphi(A)) = 1$, where $\varphi(A) = \{\, \varphi(-n, n) \,|\, n \,\epsilon\, \mathbb{N} \,\}$.

Therefore $\varphi(A)$ is a countable cover of L. By assumption there exists a finite subset $B \subseteq \varphi(A)$ such that $(\bigvee B)^* = 0$.

For $n \leq m$, where $n, m \,\epsilon\, \mathbb{N}$ we have that $(-n, n) \leq (-m, m)$ and therefore $\varphi(-n, n) \leq \varphi(-m, m)$, since φ preserves order. Therefore B has to attain a maximum, which we denote by $n_0 = max\{\, n \,|\, \varphi(-n, n) \,\epsilon\, B \,\}$.

Then $h(-n_0, n_0) = \bigvee B$ and consequently we have $h((-n_0, n_0)^*) \leq h(-n_0, n_0)^* = (\bigvee B)^* = 0$. As a result, $h((-n_0, n_0)^*) = 0$ and thus $h((-, -n_0) \vee (n_0, -)) = 0$, since for any $p < q$, $(p, q)^* = (-, p) \vee (q, -)$.[11]

But $h((-n_0 - 1, n_0 + 1) \vee a) = h(1) = 1$, where $a = (-, -n_0) \vee (n_0, -)$, and hence $h(-n_0 - 1, n_0 + 1) \vee h(a) = 1$. Since we have that $h(a) = 0$, this implies that $h(-n_0 - 1, n_0 + 1) = 1$. $\lightning$ □

Pseudocompactness has been researched to a great extent and consequently many results have been obtained. We mention that Gilmour and Banaschewski (1996) characterized pseudocompactness in terms of the co-zero part of a frame (Prop. 2), while in a more recent article by Dube and Matutu [12], one will find pseudocompactness characterized by both external (Prop. 3.5) and internal (Prop. 4.1) properties. We mention that, in both articles, these characterizations are shown for a general frame, complete regularity is not imposed.

For our purposes, an external characterization of pseudocompactness is not ideal and consequently we will more often favour the following internal characterization:

Proposition 5.26 (Banaschewski and Gilmour 1996). *A frame L is pseudocompact if and only if any sequence $a_1 \prec\!\prec a_2 \prec\!\prec \ldots$ such that $\bigvee a_i = 1$ in L terminates, that is, there is $a_k = 1$ for some k.*

One can now give an alternative proof of Prop. 5.25 by using this internal characterization:

Proof. Let $a_1 \prec\prec a_2 \prec\prec ...$ be an arbitrary sequence such that $\bigvee a_i = 1$ in L. Then $A = \{ a_i \,|\, i \,\epsilon\, \mathbb{N} \}$ is a countable cover of L. From our assumption, there exists a finite subset $B = \{ a_2, a_2, ..., a_k \} \subseteq A$ with $(\bigvee B)^* = 0$.

Since $a_i \prec\prec a_{i+1}$ for all i, we have that $a_i \leq a_{i+1}$ for all i and consequently $(\bigvee B)^* = (a_k)^* = 0$. But also $a_k \prec\prec a_{k+1}$ which implies that $a_k \prec a_{k+1}$ and hence $a_{k+1} = 1$. $\square$

Remark Note that we have not utilized the full potential of the "completely below" notion in the proof: If we substitute $\prec\prec$ with $\prec$, then the proof will still hold.

The frame translation of the topological definition of feeble compactness, as afforded by Mardešić and Papić [21], is given as: a frame L is feebly compact if every locally finite[12] subset of L is finite. For our purposes, a more useful characterization is given by:

Proposition 5.27 (Hlongwa 2004). *A frame L is feebly compact if and only if for every countable cover U of L there exists a finite $W \subseteq U$ such that $\bigwedge\{ w_i^* \,|\, w \,\epsilon\, W \} = 0$.*

Since $\bigwedge\{ w_i^* \,|\, w \,\epsilon\, W \} = (\bigvee W)^*$, one notices that the notion above is equivalent to:

Definition 5.28. A frame L is *feebly compact* if every countable cover of L has a finite subset the join of which is dense in L.

Remark Since the point-free definition of feeble compactness is taken as the direct translation of its topological counterpart, an immediate consequence is:

A space X is feebly compact if and only if $\Omega(X)$ is feebly compact.

Before introducing the third and final weaker notion of compactness in the point-free setting, we mention that a frame L is *compact* if every cover of L has a finite subcover.

Definition 5.29. A frame L is said to be *countably compact* if every countable cover of L has a finite subcover.

Trivially, every compact frame is countably compact. One would also expect the following result:

Proposition 5.30. *If a frame L is countably compact, then it is also feebly compact.*

Proof. Assume frame L is countably compact and take an arbitrary countable cover $A = \{ a_i \,|\, i \in \mathbb{N} \}$ of L. By our assumption there exists a finite $B = \{ a_1, a_2, ..., a_k \} \subseteq A$ with $\bigvee_{i=1}^{k} B = 1$. Hence $(\bigvee B)^* = 0$. $\square$

Remark We have shown that countable compactness implies feeble compactness, and by Proposition 5.25, feeble compactness in turn implies pseudocompactness. Consequently, a countably compact frame is pseudocompact.

For a Tychonoff space X, pseudocompactness and feeble compactness were shown to be equivalent notions. One might ask if this property is transferred to the point-free setting. The one direction is immediate and has been proven. In his Ph.D thesis, Hlongwa (2004) attempted the reverse direction, but unfortunately his proof (Prop. 5.2.7) fails.

Hlongwa assumed a completely regular frame L to be pseudocompact, but not feebly compact. His aim was then to show that the σ-frame $Coz(L)$ is not compact. To prove this, he defined a cover S of $Coz(L)$ as the countable union of sets S_i of co-zero elements. The error lies in the fact that since each S_i is not necessarily countable, S is not necessarily a countable set.

When considering the scheme below, one will note the strikingly similar representation between the definition of pseudocompactness (1) and feeble compactness (3), with an additional notion (2) (to make the point more clear) included in between.

$$\begin{array}{c}
a_1 \prec\!\prec a_2 \prec\!\prec .... \text{ with } \bigvee a_i = 1 \implies \exists\, a_k = 1 \text{ for some } k. \ (1) \\
\Uparrow \\
a_1 \prec a_2 \prec .... \text{ with } \bigvee a_i = 1 \implies \exists\, a_k = 1 \text{ for some } k. \ (2) \\
\Uparrow \\
a_1 \leq a_2 \leq .... \text{ with } \bigvee a_i = 1 \implies \exists\, (a_k)^* = 0 \text{ for some } k. \ (3)
\end{array}$$

This similarity then served as motivation to further consider the potential equivalence between pseudocompactness and feeble compactness in the category of completely regular frames. Through private communication it is believed to be true and the proof is in process of verification.

One might wonder if there is any equivalence between countable compactness and the other (weaker) compact notions. Hlongwa (2004) attempted to prove that for a completely regular frame, pseudocompactness implies countable compactness. Regrettably his proof (Prop. 5.3.4) is based on the same erroneous reasoning as before and therefore the conjecture still stands unproven.

Also, if one briefly recalls Prop. 5.7, then we can arguably say that to hope for an equivalence, assuming complete regularity alone, might seem too ambitious.

However, assuming an additional notion, i.e. *paracompactness*[13], an equivalence does exist:

Proposition 5.31 (Dube and Matutu 2007). *A completely regular frame is countably compact iff it is pseudocompact and countably paracompact.*

And without assuming any additional compact notion or imposing a separation axiom, one will find (as a Corollary in Dube and Matutu [12]) that the Boolean frames[14] are exactly the frames in which countable compactness and pseudocompactness are equivalent notions:

> **Corollary** A Boolean frame is pseudocompact if and only if it is countably compact.

Notes

[1]We say that a family $\mathcal{F}$ of subsets of a set X has the *finite intersection property* if $\mathcal{F} \neq \emptyset$ and $\bigcap \mathcal{F}' \neq \emptyset$ for every finite set $\mathcal{F}' \subseteq \mathcal{F}$.

[2]This condition is equivalent to a space X not being countably compact. See Engelking [13] for more details.

[3]A point x in a topological space X is called an *accumulation point* of a set $A \subseteq X$ if $x \,\epsilon\, \overline{A \backslash \{x\}}$; and we denote the set of all accumulation points of A by A^d.

[4]The definition can also be formulated, equivalently, as:

$$x \prec y \;\text{ if }\; x \wedge z = 0 \;\text{ and }\; y \vee z = 1 \;\text{ for some }\; z \,\epsilon\, L.$$

[5]An additional property, referred to as *subfitness*, is required. See Pultr [27] for more detail.

[6]Provided we assume the Axiom of Dependent Choice:
Given any relation R on a set E such that, for each $x \,\epsilon\, E, (x, y) \,\epsilon\, R$ for some $y \,\epsilon\, E$, there exists a sequence $x_0, x_1, ...$ in E such that $(x_n, x_{x+1}) \,\epsilon\, R$ for all $n = 0, 1, 2, ...$

[7]We refer the reader to Pultr [27] wherein point-free notions broadly related to T_1 and T_2 are considered.

[8]We made use of the following Corollary from Banaschewski [4]:

> **Corollary** For any space X, $U \prec\!\prec V$ in $\Omega(X)$ iff there exists a continuous real function f on X such that $f(x) < \frac{1}{2}$ for all $x \,\epsilon\, U$ and $f(x) \geq 1$ for all $x \notin V$.

[9]This follows from the frame definition for completely regular and also from Prop. 5.12.

[10]The reader will find a more detailed account of this fact in Banaschewski and Gilmour [5].

[11]We refer the reader to Banaschewski [4] for a proof of this result.

[12]A subset A of a frame L is *locally finite* if there exists a cover C of L such that each element of C meets only finitely many elements of A.

[13]A frame L is said to be *paracompact* if it is regular and if each cover of L has a locally finite refinement. One will find a detailed account of the theory of paracompactness in Pultr [27].

[14]A *Boolean frame* is a frame in which every element is regular, i.e. $a^{**} = a$.

Final remarks

In the earlier chapters the introductory ideas and concepts in the point-free setting were presented and the notion of a co-closure operator was introduced in our pursuit to better understand closure in this context. The benefit of such a pursuit was that closure can be studied in the comfort of the category of frames, rather than locales. In the final chapter the interrelationship between three weaker notions of compactness in frames were uncovered. As a whole, this thesis has lain the foundation for study plans to come.

Such future plans include exploring the option of finding co-closure operators, essentially different from those previously mentioned, in frames and studying the properties of these operators.

A closer look at the relationship between the three weaker (and other) compact notions will be taken on. Pseudocompactness and its connection to feeble compactness will receive a more comprehensive inquiry and how to use an appropriate notion of closure to express this as a type of compactness in the categorical sense.

Thereafter, we will proceed to seek "Kuratowski-Mrowka" type theorems (similarly to that done in Pultr and Tozzi [28]) for other well known compactness like properties - countable compactness, Lindelöf and sequentially compact being the first candidates. Providing a unified framework for better understanding of these compact properties in the category of frames, will be undertaken.

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