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Hyperconvex metric spaces

by

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Thesis presented in partial fulfilment
of the requirements for the degree of
Master of Science
at Stellenbosch University

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January 2010

Declaration

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Ando Désiré Razafindrakoto

Date: 17th February, 2010

Abstract

One of the early results that we encounter in Analysis is that every metric space admits a completion, that is a complete metric space in which it can be densely embedded. We present in this work a new construction which appears to be more general and yet has nice properties. These spaces subsequently called hyperconvex spaces allow one to extend nonexpansive mappings, that is mappings that do not increase distances, disregarding the properties of the spaces in which they are defined. In particular, theorems of Hahn-Banach type can be deduced for normed spaces and some subsidiary results such as fixed point theorems can be observed. Our main purpose is to look at the structures of this new type of "completion". We will see in particular that the class of hyperconvex spaces is as large as that of complete metric spaces.

Opsomming

Een van die eerste resultate wat in die Analise teegekom word is dat enige metriese ruimte 'n vervollediging het, oftewel dat daar 'n volledige metriese ruimte bestaan waarin die betrokke metriese ruimte dig bevat word. In hierdie werkstuk beskryf ons sogenaamde hiperkonvekse ruimtes. Dit gee 'n konstruksie wat blyk om meer algemeen te wees, maar steeds gunstige eienskappe het. Hiermee kan nie-uitbreidende, oftewel afbeeldings wat nie afstande rek nie, uitgebrei word sodanig dat die eienskappe van die ruimte waarop dit gedefinieer is nie 'n rol speel nie. In die besonder kan stellings van die Hahn-Banach-tipe afgelei word vir genormeerde ruimtes en sekere addisionele ressultate ondere vastepuntstellings kan bewys word. Ons hoofdoel is om hiperkonvekse ruimtes te ondersoek. In die besonder toon ons aan dat die klas van alle hiperkonvekse ruimtes net so groot soos die klas van alle metriese ruimtes is.

Dedication

To Geomira G. Sanga

Acknowledgments

I wish to convey my sincere gratitude to my supervisors Prof. Hans-Peter Künzi and Prof. David Holgate for their valuable comments and advices, and for giving me an entire freedom in writting this thesis. I also thank those who, during the seminars at Stellenbosch as well as at UCT in which I had to participate, have raised questions and comments that were helpful.

My collective thanks go to all my friends and colleagues for their support and concern. I am grateful to Walter and Retha for their willingness to translate the abstract. I owe a special debt to both Retha and Mino for their ceaseless encouragement in finishing off this writing. Finally, I heartily thank my family for their moral support and prayers.

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Introduction

The term "hyperconvex spaces" appeared first with Aronszajn and Panitchpakdi [AP56] to refer to those metric spaces that have the extension property for nonexpansive mappings. These spaces, called injective metric spaces, were fully characterized. As the term "hyperconvex" may suggest, this characterization involves a type of convexity which turns out to be more general than the algebraic convexity. Although this fact, and mainly the properties that are deduced from this concept, might tell us that since hyperconvex spaces enjoy nice properties there might be a few of them, Isbell [Isb64b] has shown that every metric space has an envelope which is hyperconvex. In particular, a metric space and its completion have exactly the same hyperconvex envelope.

Our work has [KK01] and [Nac50] as main references but does not however follow their general outline. We dedicate Chapter 1 to the study of extremal functions. These are distance functions, that is of the type $x \mapsto d(a, x)$, defined on a metric space. Also called *tight maps* [Dre84] they are crucial in investigating hyperconvex spaces. Indeed, if X is a metric space then its hyperconvex envelope consists of the set of all those extremal functions defined on X [KK01, Isb64b]. We will focus on their multiple and nice properties. Surprisingly, they provide a very simple and beautiful example of the type of extension property that we will investigate in Chapter 2. The properties of these particular maps will be useful to us throughout the rest of the work.

As the title indicates, Chapter 2 is devoted to hyperconvex spaces. Here, the hyperconvexity and the extension property are defined independently. We will first investigate the structures of the spaces that are hyperconvex. We will see that hyperconvexity involves two properties each of which are important. We will afterwards investigate those spaces that have the extension property and their equivalence to hyperconvex spaces shall be demonstrated. The important results of that chapter are the construction of the hyperconvex envelope (due to Isbell) and the study of the "minimality" of that envelope (independently due to Dress [Dre84]) which we will finally define via a functional approach.

As Euclidian spaces and generally Banach spaces form an important and interesting range of spaces and because of their intricate relations with metric spaces, we have chosen to devote Chapter 3 to their study. We will in particular look at those that are hyperconvex.

As hyperconvex spaces have the extension property and the hyperconvex envelope of a metric space is defined by considering only some particular maps (result from Chapter 2) we could define a "linear hyperconvex envelope" for Banach spaces. An important result in that chapter is that for a Banach space X, the hyperconvex envelope of its underlying metric space coincides with its linear hyperconvex envelope.

We have tried to give an axiomatic approach to our study. However our work is meant to be read with a basic knowledge of general topology and linear algebra. Throughout this text, an isometry means an injective function which preserves distances whether or not it is onto. It will become an isomorphism of metric spaces if the latter happens. The symbol $\cong$ is used to denote isomorphisms. A proof or a development that is left unreferenced is our own. This is also valid for theorems, remarks, examples or propositions. Nevertheless, even when some part is credited to be from some author, we notify the relative changes that we have made. This is also true for the approach. We have inserted notes at the end of each chapter for that purpose. We would like also to note a fact that the reader may find inconvenient: a lemma, as we may refer to it more than once, is numbered separately according to the order in which it appears in a section or a subsection.

Hyperconvexity is a very wide subject and hence writings about it are very diverse. We have chosen these topics since, as our task is to study the structures of hyperconvex spaces, they occur in a natural way. They provide a very basic and rich material to help understand the theories surrounding hyperconvexity. We note however that a more abstract approach to the subject can be done in a categorical setting [AHS] as well as in a setting more general than a metric space, for example a quasi-pseudometric space. The reader will undoubtedly find further developments and topics in the references.

1. Extremal functions

1.1 Function spaces

One of the important tools to investigate function spaces is the concept of uniformity which appears as a generalization of metric spaces and certain topological spaces that are endowed with algebraic structures such as topological groups. We shall point out few results from the study of uniformity and end the section by showing that it provides an economy of effort in proving some theorems for instance the Ascoli theorem. The latter will be useful in Section 2.

1.1.1 Uniformity

Definition 1.1.1. *A uniform space is a set X endowed with a collection $\mathcal{D}$ of subsets of the cartesian product $X \times X$ such that:*

- *For each $D \in \mathcal{D}$, $\Delta_X \subseteq D$ where Δ_X is the diagonal of X;*
- *For each $D \in \mathcal{D}$, there is $E \in \mathcal{D}$ such that $E \circ E \subseteq D$;*
- *For each $D \in \mathcal{D}$, there is $V \in \mathcal{D}$ such that $V^{-1} \in \mathcal{D}$ and $V^{-1} \subseteq D$, where $V^{-1} = \{(x, y) : (y, x) \in V\}$;*
- *$\mathcal{D}$ is a filter.*

$\mathcal{D}$ is called a uniform structure or a uniformity on X. When we have two uniformities $\mathcal{D}_1$ and $\mathcal{D}_2$ on a X such that $D_1 \subseteq \mathcal{D}_2$, then we say that D_2 is finer than $\mathcal{D}_1$ and $\mathcal{D}_1$ is coarser than $\mathcal{D}_2$.

The elements of $\mathcal{D}$ which are called entourage are reflexive relations. Hence the "$\circ$" is considered as a composition of relations. The uniform space $(X, \mathcal{D})$ is sometimes denoted by X when there is no possibility of confusion.

A map $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is said to be uniformly continuous if for each $E \in \mathcal{E}$ there is $D \in \mathcal{D}$ such that $(f \times f)(D) \subseteq E$. Uniformly continuous functions are the morphisms in the category of uniform spaces.

If (X, d) is a metric space, then the collection of subsets of the form $\{(x, y) \in X \times X : d(x, y) \leq \epsilon\}$ where $\epsilon > 0$ is a uniformity on X. In this case the notion of uniform continuity in analysis coincides with our definition.

We note that a uniform structure on a space X induces a topology on that space in a natural way by taking the collection $\{D[x] : D \in \mathcal{D}\}$ as neighborhood base at a point $x \in X$, where $D[x] = \{y : (x, y) \in D\}$. Theorems that we can find in the theory of metric spaces have their natural generalization within uniform spaces in particular precompactness.

Definition 1.1.2. *A uniform space $(X, \mathcal{D})$ is precompact or totally bounded if for any $D \in \mathcal{D}$, there is finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that $X = D[x_1] \bigcup D[x_2] \bigcup \cdots \bigcup D[x_n]$.*

Let X be a set and let $(X_i, \mathcal{D}_i)_{i \in I}$ be a family of uniform spaces such that for each $i \in I$ there is a map f_i from X to X_i. Let $\mathcal{D}$ be the collection of all subsets of the form $(f_{i_1} \times f_{i_1})^{-1}(D_{i_1}) \bigcap (f_{i_2} \times f_{i_2})^{-1}(D_{i_2}) \bigcap \cdots \bigcap (f_{i_n} \times f_{i_n})^{-1}(D_{i_n})$ where $\{i_1, i_2, \ldots, i_n\}$ is a finite subset of I and $D_{i_k} \in \mathcal{D}_{i_k}$ for each $i_k \in I$. Then $\mathcal{D}$ is a uniformity on X called the weak uniformity induced by the maps f_i. It is the coarsest uniformity on X making the f_i's uniformly continuous.

We note that for a subset A of X, if $\mathcal{D}$ is a uniformity on X then the collection $\mathcal{D}_A = \{(A \times A) \bigcap D : D \in \mathcal{D}\}$ is a uniformity on A. This uniformity is actually the weak uniformity on A induced by the natural injection $i : A \longrightarrow X$.

Example 1.1.3. *If $(X_i, \mathcal{D}_i)_{i \in I}$ is a family of uniform spaces, then the product uniform space $\prod_{i \in I}(X_i, \mathcal{D}_i)$ is the cartesian product $\prod_{i \in I} X_i$ endowed with the weak uniformity induced by the projection maps. It is the coarsest uniformity on the cartesian product making the projection maps uniformly continuous.*

Proposition 1.1.4. *[Gro73]*

(i) Let $(E, \mathcal{D})$ and $(F, \mathcal{D}')$ be uniform spaces and $f : E \longrightarrow F$ a uniformly continuous function. If $(E, \mathcal{D})$ is precompact then $(f(E), \mathcal{D}'_{f(E)})$ is precompact.

(ii) Let E be a set carrying the weak uniformity induced by the family of maps $(f_i)_{i \in I}$ to a family of uniform spaces $(E_i)_{i \in I}$. Then E is precompact if and only if for each $i \in I$, $f_i(E)$ is precompact in E_i.

Proof. (i) Let $D' \in \mathcal{D}'_{f(E)}$, there is $D \in \mathcal{D}$ such that $(f \times f)(D) \subseteq D'_{f(E)}$. Since E is precompact E is finite union of $D[x]$'s where $x \in E$. But then $f(E)$ is a finite union of $f(D[x])$'s. But $f(D[x]) \subseteq D'[f(x)]$ for each $x \in E$ so $f(E)$ is a finite union of $D'[f(x)]$'s.

(ii) The necessity is already done from (i). The sufficiency follows from the definition of weak uniformity.

$\square$

1.1.2 σ-convergence

Let E be a set and $(F, \mathcal{D})$ a uniform space. Denote by $\mathcal{F}(E, F)$ the set of all mappings from E to F. Let $A \subseteq E$ and $U \in \mathcal{D}$. Let $W(A, U) = \{(u, v) \in \mathcal{F}(E, F) \times \mathcal{F}(E, F) \ : (u(x), v(x)) \in U$ for all $x \in A\}$ and let us denote by $\mathcal{E}$ the collection of all such $W(A, U)$. Then $\mathcal{E}$ is a uniform structure on $\mathcal{F}(E, F)$ called the A-convergence uniform structure.

Now, let σ be a set of subsets of E and let $\mathcal{F}(E, F)$ carry the least upper bound of all A-convergence uniformity on $\mathcal{F}(E, F)$, where $A \in \sigma$. Then, it is easy to see that $\mathcal{F}(E, F)$ carries the weak uniformity induced by the maps $(u \longmapsto u_{|A})_{A\in\sigma}$ from $\mathcal{F}(E, F)$ to $(\mathcal{F}(A, F))_{A\in\sigma}$. This is called the *$\sigma-$convergence uniform structure* on $\mathcal{F}(E, F)$.

Remark 1.1.5. - *If $\sigma = \{\{x\} : x \in E\}$, then the $\sigma-$convergence uniform structure on $\mathcal{F}(E, F)$ is the uniformity of pointwise convergence whose topology is the topology of pointwise (or simple) convergence. $\mathcal{F}(E, F)$ is then the product uniform space F^E and denoted by $\mathcal{F}_s(E, F)$. This uniformity is the coarsest uniformity on F^E making the projection maps $u \longmapsto u(x)$ from F^E to F denoted by e_x uniformly continuous.*

- *If $\sigma = \{E\}$, then we obtain the uniformity of "uniform convergence" on $\mathcal{F}(E, F)$ whose topology is the topology of uniform convergence and the space will be denoted by $\mathcal{F}_u(E, F)$.*

In case we have to deal with a collection of continuous functions the symbol $\mathcal{F}$ in the term $\mathcal{F}(E, F)$ shall be replaced by $\mathcal{C}$.

1.1.3 Equicontinuity

Definition 1.1.6. *Let E be a topological space and $(F, \mathcal{D})$ a uniform space. Let $\mathcal{G} \subseteq \mathcal{F}(E, F)$. We say that $\mathcal{G}$ is equicontinuous at $x \in E$ if for every $U \in \mathcal{D}$ there is a neighborhood V of x such that for all $u \in \mathcal{G}$ and all $y \in V$, $(u(y), u(x)) \in U$.*

$\mathcal{G}$ is equicontinuous if it is equicontinuous at each point of E.

If $(E, \mathcal{E})$ is itself a uniform space then we say that $\mathcal{G}$ is uniformly equicontinuous if for every $U \in \mathcal{D}$ there exists $V \in \mathcal{E}$ such that for all $u \in \mathcal{G}$ we have $(u \times u)(V) \subseteq U$.

We are now able to state results with the Ascoli Theorem in view.

Proposition 1.1.7. *[Gro73]*

(i) If the topological space E is compact, than the uniformity of uniform convergence and the uniformity of pointwise convergence are identical on an equicontinuous subset $\mathcal{G}$ of $\mathcal{F}(E, F)$.

(ii) The projection maps e_x from $\mathcal{C}_u(E, F)$ to F are uniformly continuous.

(iii) Every precompact subset $\mathcal{G}$ of $\mathcal{C}_u(E, F)$ is equicontinuous.

Proof. [Gro73]

(i) We should show that for any entourage D' of F, there is an entourage D of F and a finite subset A of E such that $W(A, D) \bigcap (\mathcal{G} \times \mathcal{G}) \subseteq W(E, D') \bigcap (\mathcal{G} \times \mathcal{G})$. Let D be an entourage of F such that $D^3 \subseteq D'$. For each $x_0 \in E$, there is an open neighborhood V of x_0 such that for any $u \in \mathcal{G}$ and for any $x \in V$ we have $(u(x), u(x_0)) \in D$. Let $\{V_i : 1 \leq i \leq n\}$ be a finite sequence of such neighborhoods covering E. Let us choose $x_i \in V_i$ for each i such that $1 \leq i \leq n$ and let A be the union of such x_i's. Now, let $(u, v) \in W(A, D) \bigcap (\mathcal{G} \times \mathcal{G})$ and let $x \in E$. There is an index i_k such that $x \in V_{i_k}$. But then $(u(x_{i_k}), u(x)) \in D$, $(v(x_{i_k}), v(x)) \in D$ and $(u(x_{i_k}), v(x_{i_k})) \in D$. Hence $(u(x), v(x)) \in D^3 \subseteq D'$. Thus $(u, v) \in W(E, D') \bigcap (\mathcal{G} \times \mathcal{G})$.

(ii) The uniformity of uniform convergence is finer than the uniformity of pointwise convergence that makes the projection maps e_x uniformly continuous.

(iii) Let $x_0 \in E$ and let D' be an entourage of F. We must show that there is an open neighborhood V of x_0 such that for all $x \in V$ and for every $u \in \mathcal{G}$ we have $(u(x), u(x_0)) \in D'$. Let D be an entourage of F such that $D^3 \subseteq D'$. By precompactness, there is a finite subset $\{u_1, u_2, \ldots, u_p\}$ of $\mathcal{G}$ such that

$$\mathcal{G} \subseteq W(E, D)[u_1] \bigcup W(E, D)[u_2] \bigcup \cdots \bigcup W(E, D)[u_p].$$

Let $u \in \mathcal{G}$. There is $k \leq p$ such that $(u, u_k) \in W(E, D)$. Thus $(u(x), u_k(x)) \in D$ for all $x \in E$ and in particular we have $(u(x_0), u_k(x_0)) \in D$. Since $u_k \in \mathcal{G} \subseteq \mathcal{C}_u(E, F)$, there is an open neighborhood V of x_0 such that $u_k(V) \subseteq D[u_k(x_0)]$. Thus $(u_k(x), u_k(x_0)) \in D$ for all $x \in V$. Therefore $(u(x), u(x_0)) \in D^3 \subseteq D'$ for all $x \in V$. So $\mathcal{G}$ is equicontinuous.

□

Theorem 1.1.8. *[Gro73] (Ascoli) Let E be a compact topological space and F a uniform space. Then a subset $\mathcal{G}$ of $\mathcal{C}_u(E, F)$ is precompact if and only if $\mathcal{G}$ is equicontinuous and $\mathcal{G}(x) = \{u(x) : u \in \mathcal{G}\}$ is precompact for each $x \in E$.*

Proof. [Gro73] The necessity follows from Proposition 1.1.7 (ii), (iii) and Proposition 1.1.4 (i). The sufficiency follows from the Proposition 1.1.7 (i) and Proposition 1.1.4 (ii). □

The strength of this theorem relies on the fact that it applies for a wide class of spaces [1].

1.2 Basic properties of extremal functions

[KK01] Let (X, d) be a metric space. We motivate our study of extremal functions by considering for each $x \in X$ the positive real-valued function $f_x : X \longrightarrow [0; \infty)$ defined by:

$$f_x(y) = d(x, y) \text{ for all } y \in X.$$

The triangle inequality gives for any $x, y, a \in X$:

$$d(x, y) \leq f_a(x) + f_a(y),$$

and

$$|f_a(x) - f_a(y)| \leq d(x, y).$$

Let $f : X \longrightarrow [0 : \infty)$ such that for any $x, y \in X$:

$$d(x, y) \leq f(x) + f(y),$$

and such that for some $a \in X$ we have $f(x) \leq f_a(x)$ for all $x \in X$. Then $f = f_a$ since we have:

[1] See Notes 1.

$$f_a(x) = d(x, a) \leq f(x) + f(a) \leq f(x) + f_a(a) = f(x) \text{ for all } x \in X.$$

Definition 1.2.1. *[KK01] An extremal function on a subset A of X is a positive real-valued function f on A such that for any $x, y \in A$:*

$$d(x, y) \leq f(x) + f(y),$$

and that f is pointwise minimal: if $g : A \longrightarrow [0; \infty)$ is such that $g(x) \leq f(x)$ for all $x \in A$ and

$$d(x, y) \leq g(x) + g(y),$$

then it follows that $f = g$.

We will refer to the first property as superadditivity. The set of all extremal functions on A is denoted by $\varepsilon(A)$. In particular, for any $a \in A$ the function f_a, as introduced above, belongs to $\varepsilon(A)$ and if we endow $\varepsilon(A)$ with the metric defined by

$$d_\infty(f, g) = \sup\{d(f(x), g(x)) : x \in A\},$$

then it has a structure of a metric space. As we will see from the following proposition, the above quantity is finite.

Then the mapping $e : A \longrightarrow \varepsilon(A)$ defined by $e(a) = f_a$ for all $a \in A$ is clearly an isometry from A to $e(A)$.

Some basic properties of extremal functions on metric spaces are given in the following proposition. They are easy to settle and can be found for example in [KK01].

Proposition 1.2.2. *[KK01, EK01] Let (X, d) be a metric space and let $A \subseteq X$. Then we have:*

(i) For all $f \in \varepsilon(A)$, for all $x, y \in A$:

$$|f(x) - f(y)| \leq d(x, y),$$

and for all $x \in A$:

$$f(x) = d_\infty(f, e(x)).$$

(ii) For any $f \in \varepsilon(A)$, for any $\delta > 0$ and $x \in A$, there is some $y \in A$ such that:

$$f(x) + f(y) < d(x, y) + \delta.$$

(iii) $\varepsilon(A)$ is compact if A is compact.

(iv) If h is an extremal function on $\varepsilon(X)$ where X is a metric space and if $e : X \longrightarrow \varepsilon(X)$ is the isometry defined above, then the composition he is an extremal function on X.

Proof. [KK01, EK01] (i) Suppose that the statement is false: for some $x_0, y_0 \in A$,

$$d(x_0, y_0) + f(y_0) < f(x_0).$$

By defining, for any $x \in A$:

$$g(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ d(x_0, y_0) + f(y_0) & \text{if } x = x_0 \end{cases}$$

one can easily verify that $g(x) \leq f(x)$ for any $x \in A$ and that

$$d(x, y) \leq g(x) + g(y) \text{ for all } x, y \in A.$$

The pointwise minimality of f yields $f = g$, which is a contradiction since $g(x_0) < f(x_0)$. Therefore the statement is true.

Now for all $x, y \in A$ we have $|f(y) - f_x(y)| \leq f(x)$, and since the equality holds for $y = x$ we have

$$f(x) = \sup\{|f(y) - f_x(y)| : y \in A\} = d_\infty(f, e(x)).$$

(ii) Assume that the statement is not true: there are $x_0 \in A$ and $\delta > 0$ such that for any $x \in A$, we have

$$d(x_0, x) + \delta \leq f(x_0) + f(x).$$

Define

$$g(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ f(x_0) - \delta & \text{if } x = x_0 \end{cases}$$

Again, one can easily verify that $g(x) \leq f(x)$ for all $x \in A$ and

$$d(x, y) \leq g(x) + g(y) \text{ for all } x, y \in A.$$

Since f is pointwise minimal $f = g$. This is a contradiction since $g(x_0) < f(x_0)$. Therefore the statement is true.

(iii) Assume that A is compact, in view of the Theorem 1.1.8, we shall prove that $\varepsilon(A)$ is equicontinuous and that the set $\varepsilon(A)(x) = \{f(x) : f \in \varepsilon(A)\}$ is precompact for a fixed $x \in X$, in which case the completeness of $\varepsilon(A)$ would imply its compactness. Let us note that $\varepsilon(A) \subseteq \mathcal{C}_u(A, \mathbb{R})$ and that a pointwise limit of extremal functions is an extremal function. Therefore it is closed with respect to the topology of pointwise convergence and hence with the topology of uniform convergence. Since $\mathcal{C}_u(A, \mathbb{R})$ is complete, $\varepsilon(A)$ is complete as a closed subspace of $\mathcal{C}_u(A, \mathbb{R})$. Since the extremal functions are k-lipschitzian for $k \geq 1$, $\varepsilon(A)$ is equicontinuous. It remains then to prove that $\varepsilon(A)(x)$ is precompact.

Let $x \in A$ fixed and let $\delta > 0$, by the property (ii), for each $f \in \varepsilon(A)$ there is $y_f \in A$ such that

$$f(x) \leq f(x) + f(y) < d(x, y_f) + \delta.$$

Since A is compact it is precompact and then bounded. Hence there exists $M > 0$ such that $f(x) \leq M + \delta$ for all $f \in \varepsilon(A)$. Therefore $\varepsilon(A)(x) \subseteq [0; M + \delta]$ which implies the precompactness of $\varepsilon(A)(x)$ (since $\mathbb{R}$ is linearly ordered, boundedness implies precompactness). $\varepsilon(A)$ being complete and precompact is then compact.

(iv) Let $g : X \longrightarrow \mathbb{R}_+$ be a superadditive function such that $g \leq he$. Let $i : e(X) \longrightarrow X$ such that ei is the identity on $e(X)$ and ie the identity on X. Then we have $gi \leq h$ on $e(X)$ and for all $y, y' \in e(X)$:

$$d(y, y') = d(i(y), i(y')) \leq \; g(i(y)) + g(i(y')).$$

since h is still extremal on $e(X) \subseteq \varepsilon(X)$ we have $gi = h$ and $g = he$. So he is extremal on X. □

Now, given a metric space X and $f, g \in \varepsilon(X)$ we have:

$$d_\infty(f,g) \le d_\infty(f,e(x)) + d_\infty(e(x),g) = f(x) + g(x) \text{ for all } x \in X.$$

Since x is arbitrary in X, $d_\infty(f,g)$ is finite. We note that the above inequality is true not because d_∞ satisfies the triangle inequality, which is not assumed yet, but because of its definition and the fact that the absolute value in $\mathbb{R}$ is a norm.

The fact that an extremal function on a space Y is still extremal on any of its subsets is not true in general, it remains however superadditive. The above proof is then incomplete, but as we will see in Chapter 2, $\varepsilon(\varepsilon(X)) = \varepsilon(X)$, in other words, if h is extremal on $\varepsilon(X)$ then it is on X.

Remark 1.2.3. *The condition of compactness in* (iii) *in the previous proposition can still be strengthened as we will see in Chapter 2.*

It is possible to extend an extremal function defined on a subset A of a metric space X to a superadditive function defined on the whole space X:

Lemma 1.2.1. *Let* $A \subseteq X$*. If* $f : A \longrightarrow [0;+\infty)$ *is superadditive, then there exists a superadditive map* $g : X \longrightarrow [0;+\infty)$ *that extends* f*.*

For the sequel, we shall write $f \subseteq g$ to mean g extends f.

Proof. Let $\mathcal{F} = \{(G, f_G) : f_G : G \longrightarrow \mathbb{R}_+ \text{ superadditive },\ A \subseteq G \subseteq X,\ f \subseteq f_G\}$. $\mathcal{F}$ is nonempty since $(A, f) \in \mathcal{F}$. Let us define the following order on $\mathcal{F}$: $(G, f_G) \preceq (H, f_H)$ if and only if $G \subseteq H$ and $f_G \subseteq f_H$. Let $\mathcal{C}$ be a chain in $\mathcal{F}$. And let $B = \bigcup\{G : (G, f_G) \in \mathcal{C} \text{ for some } f_G\}$ and $f_B = \bigcup\{f_G : (G, f_G) \in \mathcal{C} \text{ for some } G\}$. It is clear that $(B, f_B) = \sup \mathcal{C}$ and that $(B, f_B) \in \mathcal{F}$. By Zorn's lemma $\mathcal{F}$ has a maximal element, say (F, f_F). We claim that $F = X$. For if $F \neq X$ then there is $x_0 \in X \setminus F$. Let $H = F \bigcup \{x_0\}$. For any $x, y \in F$ we have:

$$|d(x,x_0) - d(y,x_0)| \le d(x,y) \le f_F(x) + f_F(y).$$

Let us denote $\alpha_x = d(x, x_0)$ for all $x \in F$. It follows that

$$[\alpha_x - f_B(x); \alpha_x + f_B(x)] \bigcap [\alpha_y - f_B(y); \alpha_y + f_B(y)] \neq \emptyset \text{ for all } x, y \in F.$$

Thus, since a collection of closed intervals in $\mathbb{R}$ has nonempty intersection provided that each couple of them intersects, we have:

$$\bigcap\{[\alpha_x - f_B(x); \alpha_x + f_B(x)] : x \in F\} \neq \emptyset$$

Let α be a point in this intersection. Then $|d(x, x_0)| \leq \alpha + f_B(x)$ for all $x \in F$. We define $f_H : H \longrightarrow \mathbb{R}_+$ as follows: $f_F \subseteq f_H$ and $f_H(x_0) = \alpha$ (if $\alpha < 0$ we may choose $f_H(x_0) = -\alpha$). Hence $(H, f_H) \in \mathcal{F}$ and $(F, f_F) \prec (H, f_H)$. So it should be the case that $F = X$. We take $g = f_F$. $\square$

This lemma is pointed out in [KK01] with the additional conclusion that there is an extremal function h such that $h \leq g$. But one can expect such a result using Zorn's Lemma. However Dress [Dre84] showed that it is possible to construct such extremal function from a superadditive function without using the lemma of Zorn, hence without requiring the Axiom of Choice. Nachbin in [Nac50] used Zorn's lemma to find such extremal function, having a superadditive one, on a normed space. The term extremal functions and especially their properties being quite new if not unknown in the time during which the author wrote [Nac50], his result as well as his proof was heavy.

If we associate to each $f \in \varepsilon(X)$ its restriction $r(f)$ on A where $A \subseteq X$, then by the previous lemma and the remark above, for each $h \in \varepsilon(A)$ there is $g \in \varepsilon(X)$ such that $r(g) = h$. Moreover if A is dense in X then r is an isometry for the continuity of any functions f and g in $\varepsilon(X)$ together with the fact that the real line is Hausdorff imply that $d_\infty(f, g) = d_\infty(r(f), r(g))$. This fact together with the previous lemma show that $\varepsilon(X) = \varepsilon(A)$. In particular, if $\zeta(X)$ is the completion of X, then $\varepsilon(X) = \varepsilon(\zeta(X))$. Also if $f, g \in \varepsilon(X)$ and such that $f = g$ on A, then they must be equal on the whole space X.

By extension of a metric space X, we mean a metric space Y together with an isometry $\varphi : X \longrightarrow Y$. The definition of an extension of a normed space follows naturally.

1.3 Extremal functions on normed spaces

As a normed space $(X, \|.\|)$ enjoys more properties than the space X endowed with a metric not induced from a norm, some additional properties of extremal functions defined on a normed space can be still studied. In contrast with the previous section, these are closely related to the algebraic structures of the vector space X. We begin with a proposition.

Proposition 1.3.1. *Let $(X, \|.\|)$ be a normed space. Then:*

(i) If $f \in \varepsilon(X)$ then the function defined by $f_s(x) = f(x+s)$, where $s \in X$, is an element of $\varepsilon(X)$.

(ii) If $f \in \varepsilon(X)$ then the function g defined by $g(x) = |b|f(x/b)$ for any $x \in X$ where $b \in \mathbb{R} \setminus \{0\}$ is an element of $\varepsilon(X)$. In particular if $f \in \varepsilon(X)$ and if $g(x) = f(-x)$ for any $x \in X$, then $g \in \varepsilon(X)$.

(iii) Every element of $\varepsilon(X)$ is convex. Moreover if $f \in \varepsilon(X)$ then for each $s \in X$, the function h defined by $h(x) = f(x+s) - f(s)$ for any $x \in X$ is still convex.

Proof. (i) Let $x, y \in X$. Then:

$$\|x - y\| = \|x - s + s - y\| \leq f(x+s) + f(y+s).$$

Now, let $g : X \longrightarrow [0; \infty)$ such that for any $x, y \in X$

$$\|x - y\| \leq g(x) + g(y) \text{ and } g \leq f_s.$$

But then $g(y-s) \leq f(y)$ for any $y \in X$ and since f is extremal and since g_{-s} is still superadditive, $g(y-s) = f(y)$ for all $y \in X$. Therefore $g(x) = f(x+s)$ for all $x \in X$. So $f_s = g$ and $f_s \in \varepsilon(X)$.

(ii) Let $x, y \in X$. Since f is superadditive and $b \neq 0$ we have:

$$(1/|b|)d(x, y) = d(x/b, y/b) \leq f(x/b) + f(y/b).$$

Hence g is superadditive. Let h be a superadditive function on X such that $h \leq g$. But then for any $x \in X$, $(1/|b|h(x)) \leq f(x/b)$ and then $(1/|b|)h(by) \leq f(y)$ for any $y \in X$. We have equality since f is extremal and because $(1/|b|)h(b.)$ is superadditive by the above inequality.

(iii) Let $f \in \varepsilon(X)$ and suppose that f is not convex: there are $x_0, y_0 \in X$ and $\beta \in (0; 1)$ such that:

$$\beta f(x_0) + (1-\beta)f(y_0) < f(z_0) \text{ where } z_0 = \beta x_0 + (1-\beta)y_0.$$

Now let g the function defined by:

$$g(z) = \begin{cases} f(z) & \text{if } z \neq z_0 \\ \beta f(x_0) + (1-\beta)f(y_0) & \text{if } z = z_0 \end{cases}$$

It is true that $g \leq f$. Now consider $x \in X$ with $x \neq z_0$, we have:

$$\|x - z_0\| = \|z_0 - x\| = \|\beta(x_0 - x) + (1-\beta)(y_0 - x)\| \leq \beta\|x_0 - x\| + (1-\beta)\|y_0 - x\|.$$

But $\|x_0 - x\| \leq f(x_0) + f(x)$ and $\|y_0 - x\| \leq f(y_0) + f(x)$. So $\|x - z_0\| \leq g(x) + g(z_0)$.

That is

$$\|x - y\| \leq g(x) + g(y) \text{ for all } x, y \in X.$$

Since f is pointwise minimal we would have $f = g$. But $g(z_0) \neq f(z_0)$. So it should be the case that f is convex.

Now, let $f \in \varepsilon(X)$ and denote $h(x) = f(x+s) - f(s)$. Let $\lambda \in (0;1)$. We know that $f_s \in \varepsilon(X)$ by (i) and that f_s is convex by the previous proof. Then, for all $x, y \in X$ we have:

$$h(\lambda x + (1-\lambda)y) \leq \lambda f(x+s) + (1-\lambda)f(y+s) - f(s).$$

But

$$\lambda f(x+s) + (1-\lambda)f(y+s) - f(s) = \lambda(f(x+s) - f(s)) + (1-\lambda)(f(y+s) - f(s))$$

Therefore $h(\lambda x + (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y)$. □

Following the terminology of [Nac50], an extension $e : X \longrightarrow Y$ in normed space is an immediate extension if Y is minimal in a vector sense (X is a hyperplane). It is shown that if $f \in \epsilon(X)$ and $f(x) > 0$ for all $x \in X$, then there is an immediate extension Y of X and $y \in Y \setminus X$ such that $f = \|y - .\|$. In fact, the condition that the value of f should be strictly positive at each point ensures that $y \notin X$. Indeed we have the following theorem [Nac50]:

Theorem 1.3.2. *[Nac50] Let $(X, \|.\|)$ be a normed space and let $f \in \varepsilon(X)$ such that $f(x) > 0$ for all $x \in X$. Then there exists an immediate extension $e : (X, \|.\|) \longrightarrow (Y, p)$ and a point $y \in Y \setminus X$ such that for all $x \in X$, $f(x) = p(x - y)$.*

Extremal functions are also called tight maps. These will be useful in investigating the concept of hyperconvexity which will be the subject of the following chapter.

We find useful to repeat some important properties of these functions in normed spaces. If $(X, \|.\|)$ is a normed space, then for each $f \in \varepsilon(X)$ we have the following:

(i) f is superadditive.

(ii) For all $x, y \in X$, $|f(x) - f(y)| \leq \|x - y\|$.

(iii) f is a convex function on X.

Finally we end this section with a theorem of Hahn-Banach type which anticipates what will be discussed in the next chapter.

Theorem 1.3.3. *Let X be a real normed space and Y a linear subspace of X. Let f be a continuous linear functional defined on Y. Then there exists a continuous linear functional F extending f on X and such that $\|F\| = \|f\|$.*

Proof. Let $x_0 \in X \setminus Y$. It is sufficient to extend f to the vector space C spanned by Y and x_0 since by the transfinite induction or Zorn's lemma, it allows us easily to extend f to the whole space X. Let us set $M = \|f\| = \sup\{|f(x)| : x \in Y\}$. We want to define a linear functional F on C such that $F(y + \alpha x_0) = F(y) + \alpha F(x_0)$, where $f(y) = F(y)$, and $F(y + \alpha x_0) \leq M$ for any $y \in Y$ and $\alpha \in \mathbb{R}$. We need to find a point $k = F(x_0) \in \mathbb{R}$ such that $f(y) + \alpha k \leq M$ for any $y \in Y$ and $\alpha \in \mathbb{R}$. This can be still written as follow: $|f(y)| + \alpha k \leq M$ for any $y \in Y$ and $\alpha \in \mathbb{R}$. Let $\beta > 0$ such that $\alpha = \beta$ if $\alpha > 0$ and $\alpha = -\beta$ if $\alpha < 0$. In order to have k we should have:

$$-1/\beta\ (M - |f(y)|) \leq k \leq 1/\beta\ (M - |f(y)|) \text{ for all } y \in Y \text{ with } \beta > 0.$$

In other words it must be the case that:

$$\bigcap\{[-1/\beta\ (M - |f(y)|);\ 1/\beta\ (M - |f(y)|)] : y \in Y\} \neq \emptyset.$$

Using the translation $x \mapsto \beta x$ and by the fact that $\mathbb{R}$ is a topological vector space, it is reduced to the intersection $\bigcap\{I_y : y \in Y\}$ where $I_y = [-(M - |f(y)|); M - |f(y)|]$. Because of the property of the real line that we have mentioned earlier it suffices to check that any two of them intersect for the whole family to intersect. Let $y, y' \in Y$. By the definition of M, there exists $y_0 \in Y$ such that $\sup\{|f(y)|, |f(y')|\} \leq |f(y_0)| \leq M$. Thus we have:

$$0 \leq M - |f(y_0)| \leq M - |f(y)|, M - |f(y')|.$$

Therefore $I_{y_0} \subseteq I_y \bigcap I_{y'}$ and the whole family has a nonempty intersection. Any point of this intersection can be chosen as $k = F(x_0)$. Since $f \leq F$ on Y it is clear that $M = \|f\| \leq \|F\|$. □

It is interesting to compare this theorem with the Lemma 1.2.1. They both rely on the property that a collection of closed intervals of $\mathbb{R}$ has nonempty intersection provided that any two of them intersect. This property will be studied in the next chapter.

This theorem is a particular case of the usual form of the Hahn-Banach theorem [EK01]:

Theorem 1.3.4. *[EK01] Let X be a real vector space and Y a linear subspace of X.Let f be a linear functional defined on Y such that $f(y) \leq \ \rho(y)$ for all $y \in Y$, where ρ is a seminorm on X. Then there exists a linear functional g defined on X such that g extends f and $g(x) \leq \ \rho(x)$ for all $x \in X$.*

Proof. The proof [EK01] is essentially the same as in Theorem 1.3.3. □

Indeed the Theorem 1.3.3 may be obtained by setting the function $x \mapsto M\|x\|$ defined on $(X, \|.\|)$ where $M = \|f\|$. It is a seminorm on X that satisfies the conditions in the Theorem 1.3.4.

1.4 Notes

1. The concept of uniformity was first introduced by André Weil as an attempt to define general settings for some properties such as uniform continuity and total boundedness without using distance. The idea lies in the fact that two points are "close" to each other if the pair formed by these two points in the cartesian product of the space to which they belong are "close" to the diagonal. Thus a uniformity is to a uniform space what a filter is to a non-metrizable topological space. However another equivalent approach using collection of covers was introduced by John Tukey. We chose for Section 1 the approach using the diagonal (hence the letter $\mathcal{D}$), since it is more suitable for function spaces.

Although a more particular version of the Ascoli theorem could be easily found in the literature, we prefered to give a general version of it. We especially chose [Gro73] for it provides a quick way to prove this theorem from axiomatic definitions.

2. We gave in the Proposition 1.1.7 (i) and (iii) detailed proofs.

3. The steps of the Proposition 1.2.2 (iii), in particular the equicontinuity of $\varepsilon(A)$, are that of [KK01]. However we have stretched the proof by giving more precision. We gave our own proof for the precompactness of $\varepsilon(A)(x)$.

4. All the statements in the Proposition 1.3.1 are implicitly used in [CP97] except the quite obvious part (i) which was stated clearly.

5. The Theorem 1.3.3 as well as its proof are inspired from [EK01]. However, such a version of Hahn-Banach theorem may be easily found in the literature, for instance [NB85]. We wrote this theorem to emphasize the extension property that we will define in the next chapter. The changes that we have made are for that purpose.

2. Hyperconvex spaces

It is known that the compactness of a Hausdorff topological space X is equivalent to the fact that any nonempty family of closed subsets of X has nonempty intersection provided it has the finite intersection property. We would like to define in this chapter a weaker property that has a compactness character using this definition. Precisely, a collection $\mathcal{F}$ has the *n-ary intersection property* if $\bigcap \mathcal{F} \neq \emptyset$ provided that every n-uple of members of $\mathcal{F}$ intersects. One has to be careful with this definition for it might create some confusion with the finite intersection property. For $n = 2$ we talk about the *binary intersection property.* We are in particular interested in the latter property for this chapter and we shall use the term **BIP** for the sequel to refer to it. If for a metric space X any family of closed balls has the **BIP** then we say that X has the **BIP**. The notation of the metric (resp. norm) will be the same for any space, when confusion might occur we will denote by d_X (resp. $\|.\|_X$) the metric (resp. norm) on a space X. Every ball that we shall consider is assumed to be closed since we deal mainly with a property that has a compactness character.

A map f between two metric spaces X and Y is said to be *nonexpansive* if

$$d(f(x), f(y)) \leq d(x, y) \text{ for all } x, y \in X.$$

We will say that f is a *contraction* when the inequality is strict. We say that a metric space X has the extension property if for any metric space Y and an extension $e : Y \longrightarrow Z$, if there is a nonexpansive mapping $f : Y \longrightarrow X$ then a nonexpansive map $F : Z \longrightarrow X$ can be found such that the following diagram commutes (meaning $f = Fe$):

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "e"'] & X \\ Z \arrow[ur, "F"'] & \end{tikzcd}$$

A normed space X has the extension property if for any normed space Y and an extension $e : Y \longrightarrow Z$, if there is a continuous linear mapping $f : Y \longrightarrow X$ then a continuous linear map $F : Z \longrightarrow X$ can be found such that the above diagram commutes (meaning $f = Fe$

and $\|f\| = \|F\|$). We will sometimes denote the extension $e : Y \longrightarrow Z$ by $Y \subseteq Z$ and the extension $f = Fe$ by $f \subseteq F$ depending on the context and the need of such notations. These notations will be mainly used in the first two sections of this chapter. We will emphasize the difference between subset and subspace when the need occurs.

Hyperconvexity is a concept that is strongly related to these two main properties. In a normed space this concept is equivalent to the **BIP** and thus establishes an equivalence between the latter and the extension property (the maps being linear and continuous). As we have already seen in Chapter 1, the extension of extremal functions was possible because the real line has the **BIP**. Without involving the maps and their domain, the hyperconvexity of the space where these maps take their value furnishes an extension of Hahn-Banach type (see[Nac50]). The term "hyperconvexity" itself suggests an idea of convexity. The kind of convexity that shall be used here is that of Karl Menger. It is a crucial link between **BIP** and hyperconvexity in a metric space. This shall be introduced in Section 2. Section 1 deals mainly with the **BIP** and in Section 3 we shall discuss the fact that any metric space has an extension which is hyperconvex. Section 4 will be devoted to the extension property and some of its possible consequences.

2.1 Binary intersection property

We begin by stating a theorem known as Helly's theorem:

Theorem 2.1.1. *[KK01] Let $\mathcal{F}$ be a family of bounded and closed convex sets in the Euclidian space $\mathbb{R}^n$. Then $\mathcal{F}$ has the $(n+1)$-ary property.*

The special case when $n = 1$ tells us that the real line has the **BIP** since a closed convex and bounded subset of the real line is a closed interval, that is a closed ball. We have already used in some sense this theorem in Chapter 1 while trying to extend extremal functions and continuous linear mappings.

Proposition 2.1.2. *[KK01] A metric space X that has the **BIP** is complete.*

Proof. Let $\{F_n : n \in \mathbb{N}\}$ be a decreasing sequence of closed and bounded subsets of X. For each $n \in \mathbb{N}$ let δ_n be the diameter of F_n and assume that $\delta_n \to 0$. Choose $x_n \in F_n$ for each $n \in \mathbb{N}$ and let us set the family $\mathcal{B} = \{B(x_n; \delta_n) : n \in \mathbb{N}\}$. Each two members of $\mathcal{B}$

intersect and since X has the **BIP**, we have $\bigcap \mathcal{B} \neq \emptyset$. Since $\delta_n \to 0$, this intersection is reduced to one point x_0. Then $\{x_0\} = \bigcap \mathcal{B} \subseteq \bigcap F_n$. □

The same proof is given in [KK01] with the difference that the space being hyperconvex, a stronger property that we will define later on, was directly used. We point out that we made use of a countable family of balls in the latter proof which means that the **BIP** is still stronger to imply the completeness of a metric space. In [EK01], the proof of this proposition involves directly Cauchy sequences but the essence of the proof remains the same. The converse of this theorem is not true. Consider for example the Euclidian space $\mathbb{R}^2$ with the usual metric. Consider the following three balls: $B((0,0);1)$, $B((2,0);1)$ and $B((1,\sqrt{3});1)$.

Each two of them intersect on their respective boundaries, however we have

$$B((0,0);1) \bigcap B((2,0);1) \bigcap B((1,\sqrt{3});1) = \emptyset.$$

Therefore $\mathbb{R}^2$, and in general $\mathbb{R}^n$ with the usual metric, does not have the **BIP**.

The following theorem from [Nac50] emphasizes the importance of the **BIP** and illustrates what we have already anticipated in the beginning:

Theorem 2.1.3. *[Nac50] A normed space $(X, \|.\|)$ has the extension property if and only if it has the **BIP**.*

Proof. [Nac50] Suppose that $(X, \|.\|)$ has the extension property. We will prove that X has the **BIP** by contraposition. Let $\mathcal{B} = \{B(x_i; r_i) : i \in I\}$ a family of balls and suppose that any two members of this collection has nonempty intersection. Suppose that $\bigcap \mathcal{B} = \emptyset$. Let $A = \{x_i : i \in I\}$. If $A \neq X$ then we consider the collection $\mathcal{C}_0 = \{B(x, \|x - x_{i_0}\| + r_{i_0}) : x \in X \setminus A\}$ where $i_0 \in I$ and set $\mathcal{C} = \mathcal{B} \bigcup \mathcal{C}_0$. $\mathcal{C}$ has the same property as $\mathcal{B}$ so we can assume that $A = X$. We define the function $r : A \longrightarrow [0; \infty)$ by $r(x_i) = r_i$. Since any two members of $\mathcal{B}$ intersect each other, r is superadditive. Using Zorn's lemma (or from the results in Chapter 1) one can find an extremal function g such that $g \leq r$. Now, for any $y \in X$, we cannot have the following inequalities:

$$\|x - y\| \leq g(x) \leq r(x) \text{ for all } x \in X.$$

This would imply that $\mathcal{B}$ has a nonempty intersection in X. Thus $g(x) > 0$ for all $x \in X$ since in the contrary we would have $g(y) = 0$ for some $y \in X$ and then we would have the above inequalities. By virtue of Theorem 1.3.2, we have an immediate extension (Y, p) of X and $y \in Y \setminus X$ such that $g(x) = p(x - y)$ for all $x \in X$. Now consider the identity linear map i on X. We have $\|i\| = 1$. If there was a linear continuous map $f : (Y, p) \longrightarrow (X, \|.\|)$ extending i, then $\|f\| = 1$. But we would then have:

$$\|f(x) - f(y)\| = \|f(f(x) - y)\| \leq p(f(x) - y) = g(f(x)) \leq r(f(x)) \text{ for all } x \in Y.$$

But this would mean that $f(y) \in \bigcap \mathcal{B}$ since $f(Y) = X$. Therefore there is no linear extension of i and X does not have the extension property.

Conversely, suppose that $(X, \|.\|)$ has the **BIP**. Let (Y, p) and (Z, q) be normed spaces, $f : Y \longrightarrow X$ a continuous linear mapping and $Y \subseteq Z$. Let $\mathcal{F}$ be the collection of all the couples (g, W) such that W is a normed space with $Y \subseteq W \subseteq Z$ and $g : W \longrightarrow X$ linear continuous with $f \subseteq g$ and $\|f\| = \|g\|$. Then $\mathcal{F} \neq \emptyset$ since $(f, Y) \in \mathcal{F}$. We order $\mathcal{F}$ as follow : $(g_1, W_1) \preceq (g_2, W_2)$ if and only if $W_1 \subseteq W_2$ and $g_1 \subseteq g_2$ for all $(g_1, W_1), (g_2, W_2) \in \mathcal{F}$. Let $\mathcal{C}$ be a chain in $\mathcal{F}$. Let $h = \bigcup\{g : (g, W) \in \mathcal{C} \text{ for some } W\}$ and $R = \bigcup\{W : (g, W) \in \mathcal{C} \text{ for some } g\}$. Then $(h, R) \in \mathcal{F}$ and $(g, W) \preceq (h, R)$ for all $(g, W) \in \mathcal{C}$. So $\mathcal{F}$ is inductive and by Zorn's lemma, $\mathcal{F}$ has a maximal element, say (F, W). We claim that $W = Z$. Suppose not, let $z \in Z \setminus W$. Let $M = \|f\|$. We have by the triangle inequality:

$$\|F(x) - F(y)\| \leq Mq(x - z) + Mq(z - y) \text{ for all } x, y \in W.$$

Let $\mathcal{B} = \{B(F(x); Mq(x - z)) : x \in W\}$. Since X has the **BIP**, $\bigcap \mathcal{B} \neq \emptyset$. So there is $x_0 \in X$ such that

$$\|F(x) - x_0\| \leq Mq(x - z) \text{ for all } x \in W.$$

Let V be the vector subspace spanned by $W \bigcup \{z\}$. Any element v of V is expressed as $w + \lambda z$ where $w \in W$ and $\lambda \in \mathbb{R}$. We define the following map $G : V \longrightarrow X$ by $G(w + \lambda z) = F(w) + \lambda x_0$. G is linear and G is continuous since by the previous inequality, we have:

$$\|F(-w/\lambda) - x_0\| \leq Mq((-w/\lambda) - z) \text{ for all } w \in W.$$

Furthermore, $\|G\| \leq M$ and since $F \subseteq G$, $M \leq \|G\|$. But then the couple (G, V) would belong to $\mathcal{F}$ with $(F, W) \prec (G, V)$. So $W = Z$. □

It is worth pointing out that a normed space that has an n-dimensional underlying vector space has the **BIP** if and only if it is isomorphic to $\mathbb{R}^n$ with the norm $\|.\|_\infty$ in the norm sense. That means when the balls are cubic. We will come to this fact in Chapter 3. A consequence of that is that $\mathbb{R}^n$ endowed with the sum distance does not satisfy the **BIP** for $n > 2$. A fact that we have already said earlier. Since $\mathbb{R}^2$ with the usual metric and the complex plane $\mathbb{C}$ are isomorphic in the metric sense, $\mathbb{C}$ does not have the **BIP**. The **BIP** is not enough for a metric space to have the extension property, it needs some "convexity" structure as we will see in the following section.

2.2 Hyperconvexity

In a metric space, the natural way of generalizing the convexity in the algebraic sense is to "put points between" any two distinct points. This is formalized in the following definition.

Definition 2.2.1. *[KK01] A metric space (X, d) is metrically convex if for any two distinct points $x, y \in X$ there is $z \in X$ with $z \neq x$ and $z \neq y$ such that the following holds:*

$$d(x, y) = d(x, z) + d(z, y).$$

The point z is also said to be metrically between x and y.

In [EK01] the metric convexity is defined as follow;

Definition 2.2.2. *[EK01] Let X be a metric space. We say that X is metrically convex if for any points $x, y \in X$ and positive numbers α and β such that $d(x, y) \leq \alpha + \beta$, there exists $z \in X$ such that $d(x, z) \leq \alpha$ and $d(z, y) \leq \beta$, or equivalently $B(x; \alpha) \bigcap B(y; \beta) \neq \emptyset$.*

It is clear that the second definition implies the first one. It suffices to take $\alpha = \beta = 1/2\ d(x, y)$. However, the first definition does imply the second only if in addition the metric space involved is complete. Consider for example the rational line $\mathbb{Q}$ endowed with the metric induced from the real line $\mathbb{R}$. $\mathbb{Q}$ is metrically convex in the sense of the first definition. However if we consider the points $x = 1$ and $y = 2$ together with the numbers $\alpha = \sqrt{2} - 1$ and $\beta = 2 - \sqrt{2}$, then we have:

$$d(x, y) = |x - y| = 1 = \alpha + \beta.$$

But

$$[B(x;\alpha)\bigcap B(y;\beta)]\bigcap\mathbb{Q}=(\ [-\sqrt{2};\sqrt{2}]\bigcap[\sqrt{2};4-\sqrt{2}])\bigcap\mathbb{Q}=\emptyset.$$

Therefore, according to second definition the rational line is not metrically convex. As we will see, if the metric space is complete, the two definitions happen to be the same. For the sequel, we will refer to the first definition when we talk about metric convexity since it seems more natural to us.

This type of convexity was introduced by Menger and is also called “Menger convexity”. It is easy to verify that normed spaces are metrically convex. In fact for two distinct points x and y in a normed space, it suffices to take $z=\lambda x+(1-\lambda)y$ where $\lambda\in(0;1)$. As one can expect, the rational line is metrically convex without being convex in the algebraic sense. Thus metric convexity is more general and one can easily see that some of the basic properties of the convexity in the algebraic sense are not satisfied. For instance, an intersection of any two metrically convex subsets is not always metrically convex. The reason of this failure is apparently the fact that for any two distinct points, there might be more than one point that lie metrically between them.

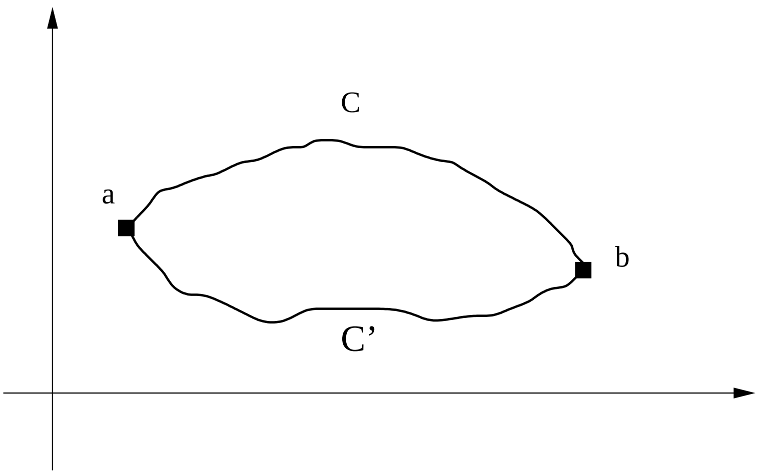

FIG. 2.1. A curve is a metrically convex set.

Consider for example the Figure 2.1. If we assume that the distance between two points is the length of the shortest path between them, then the closed curve is metrically convex. Also the two parts C and C' are metrically convex. However $C\bigcap C'=\{a,b\}$ which is not metrically convex.

Definition 2.2.3. *[KK01] Let (X,d) be a metric space. A subset A of X is called a metric* *segment with endpoints x and y if there are a closed interval $[a;b] \subseteq \mathbb{R}$ and an isometry $\varphi : [a;b] \longrightarrow X$ such that $\varphi(a) = x$ and $\varphi(b) = y$.*

The following theorem due to Menger is a sufficient condition for a metric space to have an isomorphic copy of a closed interval as a subset.

Theorem 2.2.4. *[KK01] Let (X,d) be a metric space and let $A \subseteq X$. If A is complete and metrically convex, then each two points of A are the endpoints of at least one metric segment.*

A proof of this theorem is left to Section 4, Theorem 2.4.5. We come now to the definition of a hyperconvex space.

Definition 2.2.5. *A metric space (X,d) is called hyperconvex if it is metrically convex and has the **BIP**.*

Hence a hyperconvex space is complete and any two points of it are the endpoints of a metric segment. Furthermore, hyperconvex spaces have the extension property as the following proposition states.

Proposition 2.2.6. *[KK01] Let X, Y and Z be metric spaces such that $Y \subseteq Z$ and $f : Y \longrightarrow X$ a nonexpansive mapping. Assume that X is hyperconvex. Then there exists a nonexpansive mapping $F : Z \longrightarrow X$ such that $f \subseteq F$.*

Proof. The proof is similar to that given in [KK01]. It is sufficient to extend the map f to a point not in Y. Indeed, having such an extension we can extend f to any subspace of Z containing Y. The lemma of Zorn ensures that among such subspaces of Z, there is a maximal one. Our goal is to prove that Z is this maximal element. We assume that there is a metric space A with $Y \subseteq A \subseteq Z$ and a nonexpansive mapping $g : A \longrightarrow X$ such that $f \subseteq g$. The couple (g, A) is assumed to be maximal with that property. We will show that $A = Z$. Suppose that there is $z \in Z \setminus A$. Let $B = A \bigcup \{w\}$. We want a nonexpansive mapping $h : B \longrightarrow X$ such that $g \subseteq h$. For that, we need a point $h(z) = x_0 \in X$ such that for all $x \in A$:

$$d(g(x), x_0) \leq d(x, z).$$

This is equivalent to saying that $\bigcap \{B(g(x); d(x,z)) : x \in A\} \neq \emptyset$. It can be done if any two members of this collection intersect since X has the **BIP**. It suffices to check that for any $x, y \in A$ we have the following:

$$B(g(x); d(x,z)) \bigcap B(g(y); d(y,z)) \neq \emptyset.$$

We know that

$$d(g(x), g(y)) \leq d(x,y) \leq d(x,z) + d(y,z) \text{ for all } x, y \in A.$$

Since X is complete and metrically convex, there is an isomorphic copy S of $[0, d(g(x), g(y))]$ joining $g(x)$ and $g(y)$. There is a point $s \in S$ such that $d(g(x), s) \leq d(x,z)$ and $d(g(y), s)) \leq d(y,z)$ since for any $r \in S$:

$$d(g(x), r) + d(r, g(y)) \leq d(x,z) + d(y,z).$$

Thus any two balls of the collection intersect and by choosing $x_0 = h(z)$ we have $A \subseteq B$ with $A \neq B$ and $g \subseteq h$ with $g \neq h$ and h nonexpansive. So it must be the case that $A = Z$. □

As we will see later on, the converse of this proposition is also true (Theorem 2.4.6). The difference between the proof and the one given in [KK01] resides in the definition of hyperconvex spaces. As we have seen in the proof, if X is hyperconvex and $\mathcal{B} = \{B(x_i; r_i) : i \in I\}$ a family of balls such that for any $i, j \in I$: $d(x_i, x_j) \leq r_i + r_j$, then it is the case that $\bigcap \mathcal{B} \neq \emptyset$. The converse is also true as we can see in [KK01]. Assume that any family of balls satisfies such a property in a metric space X. Let $x, y \in X$ with $x \neq y$. Let $p = \alpha d(x,y)$ and $q = (1-\alpha)d(x,y)$ where $\alpha \in (0;1)$. We have $d(x,y) = p + q$, therefore $B(x,p) \bigcap B(y,q) \neq \emptyset$. For any point z that belongs to this intersection, we have $d(x,z) \leq p$ and $d(y,z) \leq q$. Since $p + q = d(x,y) \leq d(x,z) + d(y,z)$ we have equalities. So X is metrically convex. One can easily see that X has the **BIP**. We have consequently the following equivalent characterization of a hyperconvex space.

Theorem 2.2.7. *[KK01] A metric space X is hyperconvex if for any family of balls $\mathcal{B} = \{B(x_i; r_i) : i \in I\}$ such that $d(x_i, x_j) \leq r_i + r_j$ for all $i, j \in I$, it is necessarily the case that $\bigcap \mathcal{B} \neq \emptyset$.*

This theorem is given as the definition of a hyperconvex space in the literature. One can see that it is shorter and simpler . Any finite product of hyperconvex spaces need not be hyperconvex (for example $\mathbb{R}^2$ with the usual metric). However, if we endow a finite product of hyperconvex spaces with the metric defined as the supremum of all the distances in the given spaces, then it remains hyperconvex as is the case with the Euclidian spaces mentioned in the previous section. We have the following proposition:

Proposition 2.2.8. *[KK01] Let M and N be two metric spaces such that M is hyperconvex. Then the cartesian product $M \times N$ endowed with the sup metric d_∞ is hyperconvex if and only if N is hyperconvex.*

Proof. It is clear that if N is hyperconvex, then $(M \times N, d_\infty)$ is hyperconvex. Conversely, suppose that $(M \times N, d_\infty)$ is hyperconvex. Let $\{(x_i, r_i) : i \in I\} \subseteq N \times \mathbb{R}_+$ such that $d(x_i, x_j) \leq r_i + r_j$ for all $i, j \in I$. Consider the points (x, x_i) in $M \times N$ where $x \in M$ is arbitrary. Thus we have:

$$d_\infty((x, x_i), (x, x_j)) = d_N(x_i, x_j) \leq r_i + r_j \text{ for all } i \in I.$$

By hypothesis there is a point $(a, b) \in M \times N$ such that $d_\infty((a, b), (x, x_i)) \leq r_i$ for all $i \in I$. But then $b \in \bigcap\{B_N(x_i; r_i) : i \in I\}$. $\square$

For an infinite product of metric spaces we have the following result:

Proposition 2.2.9. *[EK01] Let $\{X_i : i \in I\}$ be a family of hyperconvex spaces and let X be their cartesian product. For any $x = (x_i)_{i \in I} \in X$, we consider the following subset of X:*

$$M_x = \{(y_i)_{i \in I} : \sup\{d(x_i, y_i) : i \in I\} < \infty\}.$$

Then M_x endowed with the sup metric is hyperconvex.

Proof. [EK01] Let $\{y^j : j \in J\} \subseteq M_x$ and $\{r^j : j \in J\} \subseteq \mathbb{R}_+$ with $d(y^j, y^k) \leq r_j + r_k$ for all $j, k \in J$. It is easy to see that $B(y^j, r_j) = \prod_{i \in I} B(y^j_i, r_j)$ for each $j \in J$. Since $d_{X_i}(y^j_i, y^k_i) \leq r_j + r_k$ for all $j \in J$, we have $\bigcap\{B(y^j_i, r_j) : j \in J\} \neq \emptyset$ for all $i \in I$. But then $\bigcap\{B(y^j, r_j) : j \in J\} \neq \emptyset$. It is clear that by the definition of M_x, any point in this intersection belongs to M_x. $\square$

Before going further into the properties of hyperconvex spaces we will introduce few notations from [KK01] and [EK01] which will be helpful in understanding the structures of these spaces. Let X be a metric space and A a nonempty subset of X. we define the following:

$r_x(A) = \sup\{d(x, y) : y \in A\};$

$r(A) = \inf\{r_x(A) : x \in X\};$

$r_A(A) = \inf\{r_x(A) : x \in A\};$

$c(A) = \{x \in X : r_x(A) = r(A)\};$

$c_A(A) = c(A) \bigcap A;$

$cov(A) = \bigcap\{B : B \text{ is a ball and } A \subseteq B\}.$

$r(A)$ is called the *radius* of A relative to X, $r_A(A)$ is called the *Chebyshev radius* of A, $c(A)$ is called the *center* of A, $c_A(A)$ is called the *Chebyshev center* of A and $cov(A)$ is called the cover of A. A is called *admissible* if A is nonempty and $A = cov(A)$, we will denote by $\mathcal{A}(X)$ all such subsets in a space X. In particular, if $B \subseteq X$ is a ball, then $B \in \mathcal{A}(X)$.

It would be more natural to define $cov(A)$ as the ball closure of A. We will however keep the definition from [EK01].

Few consequences of these definitions follow:

Proposition 2.2.10. *[KK01] Let X be a metric space and A a nonempty subset of X. Then:*

(a) $cov(A) = \bigcap\{B(x; r_x(A)) : x \in X\}$*;*

(b) $r_x(A) = r_x(cov(A))$ *for any* $x \in A$*;*

(c) $r_A(cov(A)) \leq r_A(A)$*;*

(d) $\delta(cov(A)) = \delta(A)$*.*

$\delta(A)$ is the diameter of A. Furthermore if X is hyperconvex and A is bounded then:

Lemma 2.2.1. *[EK01] [KK01]*

(i) $r(cov(A)) = r(A)$*;*

(ii) $r(A) = (1/2)\delta(A)$*;*

(iii) If $A \in \mathcal{A}(X)$ *then* $r_A(cov(A)) = r_A(A)$*;*

Proof. [EK01] [KK01]

(i) Follows from (b) in the previous proposition and from the definition of r_x.

(ii) Let us set $\delta = \delta(A)$. For any $a, b \in A$ we have $d(a,b) \leq \delta/2 + \delta/2$. By hyperconvexity of X:

$$\bigcap\{B(a;\delta/2) : a \in A\} \neq \emptyset.$$

Let x be a point in this intersection, we have $r(A) \leq r_x(A) \leq \delta/2$. On the other hand if $y \in X$, then for any $a, b \in A$, $d(a,b) \leq d(a,y) + d(b,y)$. Thus $\delta \leq 2r_y(A)$. Since y is arbitrary $\delta \leq 2r(A)$.

(iii) From (ii) we have $\delta/2 \leq r(A) \leq r_A(A)$ where $\delta = \delta(A)$ and $\bigcap\{B(a;\delta/2) : a \in A\} \neq \emptyset$. If x is a point in this intersection then for all $a \in A$:

$$d(x,a) \leq \delta/2 \leq r(A) \leq r_y(A) \text{ for all } y \in X.$$

From (a) in the previous proposition and the fact that A is admissible $x \in cov(A) = A$. Therefore $x \in A \bigcap (\bigcap\{B(a;\delta/2) : a \in A\})$. Then $r_x(A) \leq \delta/2$ and:

$$\delta \leq 2r(A) \leq 2r_A(A) \leq 2r_x(A) \leq d.$$

□

Remark 2.2.11. *From the proof of (iii) in the Lemma 2.2.1 above, we can conclude that every element A of $\mathcal{A}(X)$ is hyperconvex if X is a hyperconvex metric space and also that [EK01]:*

$$c(A) = (\bigcap\{B(a;r(A)) : a \in A\}) \bigcap A \in \mathcal{A}(X)$$

We recall that if $x \in c(A)$ then $r_x(A) = r(A)$. The above implies that $\delta(c(A)) \leq r(A) = \delta/2$.

As we have stated earlier in the beginning of this section, any intersection of metrically convex subsets of a given metric space may fail badly to be metrically convex. The importance of the previous definitions and results lies in the fact that they provide a suitable approach while dealing with the intersection of hyperconvex spaces. In particular they are of interest in fixed point theory. Indeed every descending chain of bounded hyperconvex metric spaces has a nonempty and hyperconvex intersection.

Theorem 2.2.12. *[EK01] [KK01] Let X be a metric space. Let $\{H_i : i \in I\}$ be a decreasing family of nonempty and bounded hyperconvex subsets of X. Then $\bigcap\{H_i : i \in I\}$ is nonempty and hyperconvex.*

We note that $\{H_i : i \in I\}$ is decreasing if I is linearly ordered and $i \leq j$ in I if and only if $H_j \subseteq H_i$.

Proof. [EK01] Let us set the following collection:

$$\mathcal{F} = \{\prod_{i\in I} A_i : A_i \in \mathcal{A}(H_i) \text{ for each } i \text{ and } \{A_i : i \in I\} \text{ decreasing } \}.$$

$\mathcal{F} \neq \emptyset$ since $\prod_{i\in I} H_i \in \mathcal{F}$. We order $\mathcal{F}$ by inclusion. If $\mathcal{C}$ is a chain in $\mathcal{F}$, then $\bigcap \mathcal{C} \neq \emptyset$ since each H_i is hyperconvex and each element of $\mathcal{A}(H_i)$ is an intersection of balls. Therefore $\mathcal{F}$ is inductive. By virtue of Zorn's lemma, $\mathcal{F}$ has a minimal element, say $A = \prod_{i\in I} A_i$. We will show that there is $i_0 \in I$ such that $\delta(A_i) = 0$ whenever $i \geq i_0$.

Let $j \in I$ be fixed. For any $B \subseteq X$, we define

$$cov_i(B) = \bigcap\{B(x; r_x(B)) : x \in H_i\} \text{ for any } i \in I.$$

Let us consider $A' = \prod_{i\in I} A'_i$ where $A'_i = cov_j(A_j)\bigcap A_i$ if $i \leq j$ and $A'_i = A_i$ otherwise. It is clear that the family $\{A'_i : i \in I\}$ is decreasing. On the other hand $A'_i \in \mathcal{A}(H_i)$ since $cov_j(A_j)\bigcap A_i \in \mathcal{A}(H_i)$ ($H_j \subseteq H_i$ if and only if $i \leq j$). Thus $A' \in \mathcal{F}$. By the minimality of A, $A = A'$ from which we may write

$$A'_i = A_i = cov_j(A_j)\bigcap A_i \text{ for any } i \leq j.$$

Now let $x \in H_j$ and $i \leq j$. Since $A_j \subseteq A_i$ and by the definition of r_x, $r_x(A_j) \leq r_x(A_i)$. By the definition of cov_i, $cov_j(A_j) \subseteq B(x; r_x(A_j))$ and then $r_x(cov_j(A_j)) \leq r_x(A_j)$. On the other hand , $A_i = A'_i \subseteq cov_j(A_j)\bigcap A_i$, so:

$$r_x(A_j) \leq r_x(A_i) \leq r_x(cov_j(A_j)) \leq r_x(A_j).$$

Therefore $r_x(A_j) = r_x(A_i)$ for all $x \in H_j$.

By the definition of r and since $A_j \subseteq A_i$ we have $r(A_i) \leq r(A_j)$. Let $a \in A_i$ and let us set $k = r_a(A_i)$. Again since $A_i \subseteq cov_j(A_j)\bigcap A_i$, $a \in \bigcap(\{B(x; k) : x \in A_j\})\bigcap cov_j(A_j)$. Since H_j is hyperconvex we have

$$H_j \bigcap (\{B(x;k) : x \in A_j\}) \bigcap cov_j(A_j)$$

Let y be a point in this intersection. We have $r_y(A_j) \leq k$ since for any $x \in A_j$, $d(x,y) \leq k$. But then

$$r(A_j) \leq r_y(A_j) \leq k = r_a(A_i) \text{ for all } a \in A_i.$$

In particular $r(A_j) \leq r_{A_i}(A_i)$. From the proof of Lemma 2.2.1 (iii) there is a point b in A_i with the property that

$$\delta(A_i) \leq 2r(A_i) \leq 2r_{A_i}A_i \leq 2r_b(A_i) \leq \delta(A_i).$$

Therefore $r(A_j) \leq r_{A_i}(A_i) \leq r(A_i)$. Hence $r(A_i) = r(A_j)$ for every $i, j \in I$.

Now suppose that $\delta(A_j) > 0$ for all $j \in I$. Let us set $C_i = c(A_i)$ for any $i \in I$. If $i \leq j$ in I and $x \in C_j$, then $r_x(A_j) = r(A_j) = r(A_i)$. Since $x \in A_j \subseteq H_j$ we have $r_x(A_j) = r_x(A_i)$ so $x \in C_i$. Hence the family $\{C_i : i \in I\}$ is decreasing. In light of the Remark 2.2.11, $\prod_{i \in I} C_i \in \mathcal{F}$. Since A is minimal we have $A_i = C_i = c(A_i)$ for all $i \in I$. Again according to the Remark 2.2.11 it is impossible. So there is $i_0 \in I$ such that $\delta(A_i) = 0$ whenever $i \geq i_0$. Thus for some $a \in X$, $A_i = \{a\}$ whenever $i \geq i_0$. It is clear that $a \in \bigcap\{H_i : i \in I\}$.

Let us denote by H this intersection. Let $\mathcal{B} = \{B(x^\alpha, r_\alpha) : \alpha \in \Gamma\}$ where $x^\alpha \in H$ for any $\alpha \in \Gamma$ and such that $d(x^\alpha, x^\beta) \leq r_\alpha + r_\beta$ for any $\alpha, \beta \in \Gamma$. Since each H_i is hyperconvex, it is the case that $B_i = (\bigcap \mathcal{B}) \bigcap H_i \neq \emptyset$ for each $i \in I$. Each B_i is hyperconvex since $B_i \in \mathcal{A}(H_i)$. Since the H_i's are decreasing, the family $\{B_i : i \in I\}$ is decreasing and because of what has been shown above $\bigcap\{B_i : i \in I\} \neq \emptyset$. Therefore H is hyperconvex. $\square$

2.3 Retractions and extensions

In this section, we mainly discuss the hyperconvex hull of a given metric space, that means the possibility for a metric space to have an extension which is hyperconvex and such that this extension is "minimal". The construction that we will present here is that of John Isbell [Isb64b]. The term "extension" being already known, we shall define what is a retraction.

2.3.1 Retractions

Definition 2.3.1. *Let X, Y be metric spaces. A map $f : X \longrightarrow Y$ is called a retraction if f is onto, nonexpansive and there is an isometry $e : Y \longrightarrow X$ such that fe is the identity on Y.*

For a given metric space X, we define the ϵ-neighbourhood of a subset A as follow:

$$N(A, \epsilon) = \bigcup \{B(a, \epsilon) : a \in A\}.$$

It is easy to check that if X is hyperconvex and $A \in \mathcal{A}(X)$, then

$$N(A, \epsilon) = \bigcap \{B(x_i; r_i + \epsilon) : i \in I\},$$

where $A = \bigcap \{B(x_i; r_i) : i \in I\}$. This means that for any $\epsilon > 0$ whenever $A \in \mathcal{A}(X)$ it is the case that $N(A, \epsilon) \in \mathcal{A}(X)$. We recall that each member of $\mathcal{A}(X)$ is hyperconvex if X is hyperconvex (each member of $\mathcal{A}(X)$ is an intersection of family of balls).

We have the following result.

Theorem 2.3.2. *[KK01] Let X be a hyperconvex space and $\epsilon > 0$. If $A \in \mathcal{A}(X)$ then there is a nonexpansive retraction $r : N(A, \epsilon) \longrightarrow A$ such that $d(x, r(x)) \leq \epsilon$ for any $x \in N(A, \epsilon)$.*

Proof. [KK01] Let $\mathcal{F}$ be the family of all couples (f, B) such that $A \subseteq B \subseteq N(A, \epsilon)$ and $f : B \longrightarrow A$ is a nonexpansive retraction with $d(x, f(x)) \leq \epsilon$ for all $x \in B$. $\mathcal{F}$ is nonempty since $(i, A) \in \mathcal{F}$ where i is the identity mapping on A. We order $\mathcal{F}$ as follows: $(f, B) \preceq (g, D)$ if and only if $f \subseteq g$ and $B \subseteq D$ for any $(f, B), (g, D) \in \mathcal{F}$. With this ordering $\mathcal{F}$ is inductive and by Zorn's lemma, $\mathcal{F}$ has a maximal element, say (R, D). Our goal is to prove that $D = N(A, \epsilon)$. Suppose that there exists $x \in N(A, \epsilon) \setminus D$. Let $B = D \bigcup \{x\}$. We want a nonexpansive retraction $h : B \longrightarrow A$ such that for all $y \in B$ we have $d(y, h(y)) \leq \epsilon$. We would have in that case $(R, D) \preceq (h, B)$ with $(R, D) \neq (h, B)$ and $(h, B) \in \mathcal{F}$ so it should be true that $D = N(A, \epsilon)$. We define h with the condition that $R \subseteq h$. We need to find an appropriate point $x_0 = h(x) \in A$ so that $(h, B) \in \mathcal{F}$. This is possible if the following subset S of A is nonempty:

$$S = (\bigcap \{B(R(y); d(x, y)) : y \in D\}) \bigcap A \bigcap B(x; \epsilon).$$

Any point $s \in S$ satisfy the conditions that we need. Since $A \in \mathcal{A}(X)$ there is a family $\{B(x_i; r_i) : i \in I\}$ such that $A = \bigcap\{B(x_i; r_i) : i \in I\}$. It is then sufficient to prove that any two members of the family of balls used in defining the subset S intersect and because X has the **BIP** it should be the case that $S \neq \emptyset$. For any $y_1, y_2 \in D$ since r is nonexpansive and by the triangle inequality we have:

$$d(R(y_1), R(y_2)) \leq d(y_1, x) + d(x, y_2).$$

And since X is hyperconvex, we have:

$$B(R(y_1); d(x, y_1)) \bigcap B(R(y_2); d(x, y_2)) \neq \emptyset.$$

If $y \in D$ then $R(y) \in A = \bigcap\{B(x_i; r_i) : i \in I\}$ so that:

$$B(R(y); d(x, y)) \bigcap B(x_i; r_i) \neq \emptyset \text{ for all } y \in D \text{ and } i \in I.$$

Now, since $x \in B \subseteq N(A, \epsilon) = \bigcap\{B(x_i; r_i + \epsilon) : i \in I\}$ we have $d(x, x_i) \leq \ r_i \ + \ \epsilon$ and $B(x, \epsilon) \bigcap \mathcal{B}(x_i, r_i) \neq \emptyset$ for all $i \in I$. And for all $y \in D$ we have:

$$d(R(y), x) \leq d(R(y), y) + d(y, x) \leq \epsilon + d(y, x).$$

So that $B(R(y), d(y, x)) \bigcap B(x, \epsilon) \neq \emptyset$ for all $y \in D$. Thus every pair of the balls that we used intersect. □

Now let $f : X \longrightarrow X$ be a mapping, the ϵ-*fixed points* $F(f, \epsilon)$ of f is defined as follow:

$$F(f, \epsilon) = \{x \in X : d(x, f(x)) \leq \epsilon\}.$$

It is shown in [KK01] that if X is hyperconvex and f nonexpansive, then for any $\epsilon > 0$, $F(f, \epsilon)$ is nonempty and hyperconvex. One can observe that if $\epsilon_1 < \epsilon_2$, then $F(f, \epsilon_1) \subseteq F(f, \epsilon_2)$. Then when X is bounded, because of the Theorem 2.2.12, f would have a fixed point[1].

It is true that any retract of a hyperconvex space is still hyperconvex:

[1]See Notes 4.

Proposition 2.3.3. *Let X be a hyperconvex space and $f : X \longrightarrow Y$ a retraction. Then Y is hyperconvex.*

Proof. Let $\{y_i : i \in I\} \subseteq Y$ such that $d(y_i, y_j) \leq r_i + r_j$ for all $i, j \in I$ and for some $\{r_i : i \in I\} \subseteq \mathbb{R}_+$. Let $e : Y \longrightarrow X$ be an isometry such that fe is the identity on Y. Since X is hyperconvex we have $\bigcap\{B(e(y_i); r_i) : i \in I\} \neq \emptyset$. Let a be a point in this intersection, since f is a retraction,

$$d(f(a), y_i) \leq d(f(a), f(e(y_i))) = d(a, e(y_i)) \leq r_i \text{ for all } i \in i.$$

Therefore $\bigcap\{B(y_i; r_i) : i \in I\} \neq \emptyset$. □

Furthermore, if X and Y are metric spaces such that X is hyperconvex, and if there is an isometry $e : X \longrightarrow Y$, then there is a retraction $r : Y \longrightarrow X$ such that re is the identity on X since the inverse of e defined on $e(X)$ is nonexpansive and X has the extension property as we have shown earlier. In particular, if S is a metric segment in a metric space X then there is a retraction $r : X \longrightarrow S$ because S is an isomorphic copy [KK01] of a closed interval in the real line.

2.3.2 Extensions

In this section, extensions that are closely related to hyperconvexity are developed. For a given metric space X, there exists an extension $e : X \longrightarrow Y$ such that Y is hyperconvex. Furthermore, we can find Y such that this extension is minimal. This is called the hyperconvex hull of X in analog to the convex hull in analysis. The construction of Isbell [Isb64b, KK01] of this hyperconvex hull will be discussed in this section. The following proposition shows the existence of such an envelope.

Proposition 2.3.4. *[EK01] [KK01] Given a metric space X, there exists an extension $e : X \longrightarrow \iota(X)$ such that $\iota(X)$ is hyperconvex and no proper subset of $\iota(X)$ that extends X is hyperconvex.*

Proof. [EK01] Let us set $\mathcal{B}(X) = \{f \in \mathbb{R}^X : \sup\{|f(x)| : x \in X\} < \infty\}$. We endow $\mathcal{B}(X)$ with the sup metric d_∞. Since the constant function $f_0 : X \longrightarrow \{0\}$ belongs to $\mathcal{B}(X)$, it is easy to see that $\mathcal{B}(X) = M_{f_0}$ as defined in the Proposition 2.2.9. Therefore $(\mathcal{B}(X), d_\infty)$ is hyperconvex. Now, let us fix a point $b \in X$, we define the function $I_b(x) = d(x, .) - d(b, .)$ for all $x \in X$, from X to $\mathcal{B}(X)$. Indeed

$$d_\infty(I_b(x), f_0) \leq\ d(x,b) < \infty \text{ for any } x \in X.$$

so $I_b(x) \in \mathcal{B}(X)$. Now at the point $y = b$ we have $I_b(x)(b) = d(x,b)$, so $d_\infty(I_b(x), f_0) = d(x,b)$. On the other hand, for any y, y' and $z \in X$, we have:

$$(d(y,z) - d(b,z)) - (d(y',z) - d(b,z)) = d(y,z) - d(y',z) \leq\ d(y,y').$$

So $|I_b(y) - I_b(y')| \leq\ d(y,y')$ for any $y, y' \in X$. Together with the previous statements this implies that $d_\infty(I_b(y), I_b(y')) = d(y,y')$. Therefore I_b is an isometry. Now let $\mathcal{F}$ be the collection $\{Z : I_b(X) \subseteq Z \subseteq \mathcal{B}(X)$ and Z is hyperconvex $\}$. Let $\mathcal{C}$ be a chain in $\mathcal{F}$ ordered by inclusion. By Theorem 2.2.12 we have $\bigcap \mathcal{C} \neq \emptyset$ and $\bigcap \mathcal{C} \in \mathcal{F}$. Because of Zorn's Lemma, $\mathcal{F}$ has a minimal element W. We take $W = \iota(X)$. $\square$

As we will see later on, two hyperconvex hulls of a metric space are unique up to isomorphism. The construction that we made in the proof is relatively close to Isbell's construction since the embedding that we have used involves function distances.

Definition 2.3.5. *[Her92, AHS] Let X be a metric space and $e : X \longrightarrow Y$ an extension,*

(a) e is called tight if for any metric d on Y, which satisfies $d \leq\ d_Y$ and $d(e(x), e(x')) = d(x,x')$ for all $x, x' \in X$, it must be the case that $d = d_Y$.

(b) e is called essential if for any nonexpansive map $f : Y \longrightarrow Z$ for which $fe : X \longrightarrow Z$ is an isometry , it must be the case that f is an isometry.

Proposition 2.3.6. *[Her92] An extension of a metric space X is essential if and only if it is tight.*

Proof. Let Y be a metric space and $e : X \longrightarrow Y$ an extension. Suppose that e is essential. Let d be a metric on Y that satisfies the conditions of the part (a) of the previous definition. The identity map $i : (Y, d_Y) \longrightarrow (Y, d)$ is nonexpansive and since e is an isometry ie is an isometry from (X, d_X) to (Y, d). Therefore, i is an isometry and $d = d_Y$. So e is tight.

Conversely, suppose that e is tight and let $f : Y \longrightarrow Z$ be a nonexpansive mapping such that fe is an isometry. We define the following metric d on Y.

$$d(y,y') = kd_Y(y,y') + (1-k)d_Z(f(y), f(y')) \text{ for all } y, y' \in Y.$$

Where $k \in (0;1) \subseteq \mathbb{R}$.

Since f is nonexpansive $d \leq\ d_Y$. On the other hand since fe and e are isometries we have:

$$d(e(x), e(x')) = kd_Y(e(x), e(x')) + (1-k)d(e(x), e(x')) \text{ for all } x, x' \in X.$$

Therefore $d(e(x), e(x')) = d_Y(e(x), e(x')) = d_X(x, x')$ for all $x, x' \in X$. Because e is tight we have $d = d_Y$ on Y. But then $d_Z(f(y), f(y')) = d_Y(y, y')$ for all $y, y' \in Y$. So f is an isometry. □

Proposition 2.3.7. *[KK01] Let X be a metric space, then $\varepsilon(X)$ is the hyperconvex hull of X, up to isomorphism.*

Proof. [KK01] We prove first that $\varepsilon(X)$ is hyperconvex. Let $\mathcal{B} = \{B(f_i; r_i) : i \in I\}$ such that $f_i \in \varepsilon(X)$ for each $i \in I$ and such that for any $i, j \in I$:

$$d(f_i, f_j) \leq r_i + r_j.$$

Let us denote $Y = \{f_i : i \in I\}$ and let us define $r : Y \longrightarrow [0; +\infty)$ by $r(f_i) = r_i$ for each $i \in I$. r is superadditive and by Lemma 1.2.1 we can extend r to $\varepsilon(X)$. We can assume that r is extremal on $\varepsilon(X)$. Now by virtue of the Proposition 1.2.2 (iv), re is extremal on X. For any $f \in \varepsilon(X)$, since r is extremal and by virtue of the Proposition 1.2.2 (i) we have

$$|r(e(x)) - f(x)| = |r(e(x)) - d_\infty(f, e(x))| \leq r(f) \text{ for all } x \in X.$$

Therefore $d_\infty(re, f) \leq r(f)$ for all $f \in \varepsilon(X)$. But this means that

$$r.e \in \bigcap\{B(f, r(f)) : f \in \varepsilon(X)\} \subseteq \bigcap \mathcal{B}.$$

Thus $\varepsilon(X)$ is hyperconvex. Now suppose that there exists a hyperconvex space H such that $X \subseteq H \subseteq \varepsilon(X)$. Since H is hyperconvex, there is a nonexpansive retraction $R : \varepsilon(X) \longrightarrow H$ which is a retraction extending the identity on H. For any $f \in \varepsilon(X)$ we have:

$$d_\infty(R(f), e(x)) = R(f)(x) \leq d_\infty(f, e(x)) = f(x) \text{ for any } x \in X.$$

Indeed since $R(f) \in H \subseteq \varepsilon(X)$, $d_\infty(R(f), e(x)) = R(f)(x)$.

Since f is extremal and $d_\infty(R(f), .)$ superadditive, it should be the case that $R(f) = f$. Therefore $H = \varepsilon(X)$. □

We note that taking the intersection of all the hyperconvex spaces that contain X metrically does not give a hyperconvex hull. The reason is that, as we have seen with the metric convexity, any intersection of metrically convex spaces need not be metrically convex. The last theorem states that the class of hyperconvex spaces is as "big" as the class of metric spaces. However, hyperconvex hulls do not behave as completion of metric spaces. The unicity of $\iota(X)$ up to isomorphism will be demonstrated later on.

We also note that for a given metric space X, the previous theorem assures us the existence of an extension $i : X \longrightarrow Y$ such that Y is pathwise connected. We recall that because of the hyperconvexity of Y, each two points of Y are the endpoints of a metric segment.

Corollary 2.3.8. *Given a metric space X and a tight extension $i : X \longrightarrow Y$, there exists a unique extension $j : Y \longrightarrow \varepsilon(X)$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "i"'] & \varepsilon(X) \\ Y \arrow[ur, "j"'] & \end{tikzcd}$$

Proof. The existence of the map j and the commutativity of the diagram are assured by the fact that $\varepsilon(X)$ is hyperconvex. Now, since ji which is equal to e is an isometry and since i is tight, that is essential, j is an isometry. Let $j' : Y \longrightarrow \varepsilon(X)$ be an isometry such that $j'i = e$. Let $y \in Y$, for any $x \in X$ we have the following:

$$j(y)(x) = d_\infty(j(y), e(x)) = d_Y(y, i(x)) = d_Y(j'(y), e(x)) = j'(y)(x).$$

Therefore $j(y) = j'(y)$. Since y is arbitrary, it follows that $j = j'$ □

In fact, we can even have an expression for the extension j. Indeed, let $y \in Y$ and $x \in X$. We have the following:

$$j(y)(x) = d_\infty(j(y), e(x)) = d_Y(y, i(x))$$

Thus, for any $y \in Y$, $j(y) = d(y, i(.))$ which we can write $d(y, .)_{|X}$.

Therefore, any tight extension of X can be embedded into $\varepsilon(X)$. In fact the extension $e : X \longrightarrow \varepsilon(X)$ is tight [Dre84] and consequently it is the maximal tight extension of X. In order to show that it is really a tight extension, we need to prove two lemmas.

Lemma 2.3.1. *[Dre84, Her92] Let X be a metric space. An extension $i : X \longrightarrow Y$ of X is tight if for any $y, y' \in Y$:*

$$d_Y(y, y') = \sup\{d_X(x, x') - d_Y(y, i(x)) - d_Y(y', i(x')) : x, x' \in X\}.$$

Proof. Let d be a metric on Y such that $d \leq d_Y$ and $d(i(x), i(x')) = d_X(x, x')$ for all $x, x' \in X$. We need to show that $d_Y \leq d$ in order to prove that they are equal. Given $y, y' \in Y$ we have:

$$d_X(x, x') = d_Y(i(x), i(x')) \leq d(i(x), y) + d(y, y') + d(y', i(x')) \text{ for all } x, x' \in X.$$

Since $d \leq d_Y$, it implies:

$$d_X(x, x') \leq d_Y(i(x), y) + d(y, y') + d_Y(y', i(x')) \text{ for all } x, x' \in X.$$

Which can be still written as follows:

$$d_X(x, x') - d_Y(i(x), y) - d_Y(y', i(x')) \leq d(y, y') \text{ for all } x, x' \in X.$$

Therefore by hypothesis $d_Y(y, y') \leq d(y, y')$. So $d = d_Y$ □

The proof can be found in [Dre84] where it is shown that the converse of the lemma is also true. Thus the lemma may be taken as an equivalent definition of a tight extension. We are concerned for now with the necessary condition. The next lemma which involves extremal functions will be crucial for our proof.

Lemma 2.3.2. *Let X be a metric space. For any $f, g \in \varepsilon(X)$ we have:*

$$d_\infty(f, g) = \sup\{f(x) - g(x) : x \in X\}$$

Proof. For any extremal function f and for any x and y in X we have the inequality:

$$d(x, y) - f(y) \leq f(x).$$

We want first to prove that $f(x) = \sup\{d(x,y) - f(y) : y \in X\}$. This is the same as writing $f(x) = \sup\{d(x,y) - f(y) : y \in X \text{ and } y \neq x\}$ since $d(x,x) - f(x) = -f(x) < 0$. Suppose that it is not the case. Let us consider the function $g : X \longrightarrow \mathbb{R}_+$ defined as follows:

$$g(z) = \begin{cases} f(z) & \text{if } z \neq x \\ \sup\{d(x,y) - f(y) : y \in X \text{ and } y \neq x\} & \text{if } z = x \end{cases}$$

For any $z \in X$ such that $z \neq x$ we have:

$$d(x,z) - f(z) \leq \sup\{d(x,y) - f(y) : y \in X \text{ and } y \neq x\}.$$

Hence $d(x,z) \leq g(x) + f(z) = g(x) + g(z)$. Therefore g is superadditive and $g \leq f$ with $g(x) \neq f(x)$. Since f is extremal, it should be the case that:

$$f(x) = \sup\{d(x,y) - f(y) : y \in X \text{ and } y \neq x\} = \sup\{d(x,y) - f(y) : y \in X\}.$$

Now, for any $x, y \in X$ and for any $g \in \varepsilon(X)$ we have:

$$d(x,y) - f(y) - g(x) \leq f(x) - g(x).$$

Since $d(x,y) - f(y) \leq f(x)$. This can be still written as

$$d(x,y) - g(x) - f(y) \leq f(x) - g(x).$$

Thus

$$\sup\{d(x,y) - g(x) - f(y) : x \in X\} \leq \sup\{f(x) - g(x) : x \in X\}.$$

Since $g \in \varepsilon(X)$ and because of what is already shown, we have:

$$\sup\{d(x,y) - g(x) - f(y) : x \in X\} = g(y) - f(y).$$

and

$$g(y) - f(y) \leq \sup\{f(x) - g(x) : x \in X\} \text{ for all } y \in X.$$

Hence

$$\sup\{g(y) - f(y) : y \in X\} \leq \sup\{f(x) - g(x) : x \in X\}.$$

Since f and g are arbitrary, we have equality. □

Therefore we can avoid the absolute value. We have shown in the proof that for any function f which is extremal on a metric space X we have:

$$f(x) = \sup\{d(x, y) - f(y) : y \in X\} \text{ for any } x \in X.$$

In fact, in [Dre84] extremal functions which are then called tight maps are defined as such. Indeed, a function that satisfies the above property is necessarily an extremal function. To see this, let f be a real-valued function defined on a metric space X having that property. Then f is automatically superadditive. Now, let g be a superadditive function defined on X such that $g \leq f$. Thus

$$d(x, y) - f(y) \leq d(x, y) - g(y) \text{ for any } x, y \in X.$$

But then

$$f(x) = \sup\{d(x, y) - f(y) : y \in X\} \leq \sup\{d(x, y) - g(y) : y \in X\} \leq g(x) \text{ for all } x \in X$$

Therefore $g = f$ and f is pointwise minimal.

Having proven these two lemmas, we can show the following theorem:

Theorem 2.3.9. *[Dre84] For any metric space X, the extension $e : X \longrightarrow \varepsilon(X)$ is a maximal tight extension.*

Proof. It suffices to prove that the extension is tight since we have already shown that any tight extension of X is contained in $\varepsilon(X)$.

Let f and g be elements of $\varepsilon(X)$. By Lemma 2.3.2 we have:

$$d_\infty(f, g) = \sup\{f(x) - g(x) : x \in X\}.$$

And from the proof of the same lemma, it can be written as:

$$d_\infty(f,g) = \sup\{\sup\{d(x,y) - f(y) : y \in Y\} - g(x) : x \in X\} = \sup\{d(x,y) - f(y) - g(x) : x, y \in X\}.$$

On the other hand $f(y) = d_\infty(f, e(y))$ and $g(x) = d_\infty(g, e(x))$. So

$$d_\infty(f,g) = \sup\{d(x,y) - d_\infty(f, e(y) - d_\infty(g, e(x)) : x, y \in X\}$$

By Lemma 2.3.1 the extension $e : X \longrightarrow \varepsilon(X)$ is tight. □

In other words since a composition of tight extensions is a tight extension, $\varepsilon(X)$ does not have a proper tight extension. Indeed a proper tight extension of $\varepsilon(X)$ would be a tight extension of X since a composition of tight extensions is tight. This leads us to the following corollary:

Corollary 2.3.10. *Any extremal function on a metric space X is a function distance, that is if $f \in \varepsilon(X)$ then f can be identified with $d(y, .)$ for some y not necessarily in X.*

Proof. Since f is extremal on X, hence on a subset of $\varepsilon(X)$, we can extend f to an extremal function $\tilde{f}$ on $\varepsilon(X)$. But since the extension $e : \varepsilon(X) \longrightarrow \varepsilon(\varepsilon(X))$ is an isomorphism, there exists $y \in \varepsilon(X)$ such that $\tilde{f} = e(y) = d_\infty(y, .)$. Therefore $f = d_\infty(y, .)_{|X}$. □

This corollary shows that if a metric space X is hyperconvex and if f is an extremal function defined on X, then $f(y) = 0$ for some $y \in X$. Indeed, $f = d_\infty(y, .)$ for some $y \in \varepsilon(X)$. Since the extension $e : X \longrightarrow \varepsilon(X)$ is an isomorphism, we may say that $y \in X$ and hence $f(y) = 0$. We note that this last result and the corollary might be obtained by direct computation [Her92, CP97] using the properties of extremal functions and the equivalent definition of hyperconvexity given in Theorem 2.2.7.

We point out that the result of the corollary is not really new to us since for any extremal function f defined on a metric space X we have:

$$f(x) = d_\infty(f, e(x)) \text{ for any } x \in X.$$

Since x can be identified with $e(x)$, the above equation really shows that f is the restriction of $d_\infty(f,.)$ to $e(X)$. This also shows Theorem 1.3.2 stated in Chapter 1. As we will see in Chapter 3, if X is a real normed space, then $\varepsilon(X)$ is real Banach space.

For a given metric space X the minimality of the extension $i : X \longrightarrow \zeta(X)$ where $\zeta(X)$ is the completion of X is expressed in the fact that $i(X)$ is dense in $\zeta(X)$. In fact this extension is tight since for any $y, y' \in \zeta(X)$ there are respective sequences $\{y_n : n \in \mathbb{N}\}$ and $\{y'_n : n \in \mathbb{N}\}$ in X such that they converge respectively to y and y'. But then we have

$$d(y,y') = \lim_{n\to\infty}(d(y_n,y'_n) - d(y,y_n) - d(y',y'_n)).$$

Therefore

$$d(y,y') = \sup\{d(x,x') - d_Y(y,x) - d_Y(y',x') : x, x' \in X\}.$$

We have also shown in the previous chapter that $\varepsilon(\zeta(X)) \cong \varepsilon(X)$. We will see that in general if an extension $i : X \longrightarrow Y$ is tight then $\varepsilon(Y) \cong \varepsilon(X)$.

[Dre84] Let us denote by $\sigma(X)$ the set of all superadditive functions defined on a metric space X. It is clear that $\varepsilon(X) \subseteq \sigma(X)$ and since $\varepsilon(X)$ is hyperconvex, there exists a retraction $\rho : \sigma(X) \longrightarrow \varepsilon(X)$. Now, if $i : X \longrightarrow Y$ is a tight extension of X and if $f \in \sigma(Y)$ then $f \in \sigma(i(X))$, that is $fi \in \sigma(X)$. Since we do not consider the difference between X and $i(X)$ we can write $f \in \sigma(X)$. In particular if $f \in \varepsilon(X)$ then $f_{|X}$ (or precisely $f_{|i(X)}$ or $(fi)_{|X}$) belongs to $\sigma(X)$ (which is the same as $\sigma(i(X))$).

Thus we have defined a map $k : \varepsilon(Y) \longrightarrow \varepsilon(X)$ such that $k(f) = f_{|X}$ for any $f \in \varepsilon(Y)$. Precisely, k is the composition of the isometry $\tilde{i} : \sigma(i(X)) \longrightarrow \sigma(X)$ defined by $\tilde{i}(f) = fi$ for any $f \in \sigma(i(X))$ and the restriction map $(.)_{|X} : \varepsilon(Y) \longrightarrow \sigma(i(X))$ defined by $(.)_{|X}(f) = f_{|X}$. This is resumed in the following diagram:

$$\begin{tikzcd}
\varepsilon(Y) \arrow[r, "k"] \arrow[d, "(.)_{|X}"'] & \sigma(X) \\
\varepsilon(i(X)) \arrow[ur, "\tilde{i}"'] &
\end{tikzcd}$$

Now, since $d_\infty(f_{|X}, g_{|X}) \leq d_\infty(f,g)$ for any f and g in $\varepsilon(Y)$, k is a nonexpansive map. Finally we have a nonexpansive mapping $k\rho : \varepsilon(Y) \longrightarrow \varepsilon(X)$. Let us now consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "i"'] & \varepsilon(X) \\ Y \arrow[r, "e'"'] & \varepsilon(Y) \arrow[u, "k\rho"'] \end{tikzcd}$$

If the above commutes, then the uniqueness of the extension j from the Corollary 2.3.8 will imply that $k\rho e'$ is an isometry and hence $k\rho$ is an isometry. Let us take $x \in X$, $e'i(x) = d(i(x), .)$ and $(k\rho)(d(i(x), .)) = d(i(x), .)_{|X} = d(x, .) = e(x)$. Thus $k\rho e' i = e$ and the above diagram commutes. What remains to show is that $k\rho$ is surjective. As a retraction, ρ is surjective so it suffices to verify that k is surjective. Let $f \in \sigma(X)$, precisely $fi \in \sigma(X)$. By Lemma 1.2.1 and the remark that follows, we can have $F \in \varepsilon(Y)$ such that $F_{|X} = f$, or precisely $F_{|i(X)} = fi$. Thus $k(F) = f$. Hence k is surjective and therefore $k\rho$ is an isomorphism[2].

This long construction may be avoided if we only consider the fact that the extension $i : X \longrightarrow Y$ is tight and that $e : X \longrightarrow \varepsilon(X)$ is a maximal tight extension:

Theorem 2.3.11. *Let X be a metric space and $i : X \longrightarrow Y$ a tight extension of X. Then $\varepsilon(X) \cong \varepsilon(Y)$.*

Proof. We will denote by e' the extension $Y \longrightarrow \varepsilon(Y)$. Since the extension $i : X \longrightarrow Y$ is tight, there exists a unique extension $j : Y \longrightarrow \varepsilon(X)$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "i"'] & \varepsilon(X) \\ Y \arrow[ur, "j"'] & \end{tikzcd}$$

Since $e = ji$ is tight, j is tight and then there is a unique tight extension $f : \varepsilon(X) \longrightarrow \varepsilon(Y)$ by the maximality of e' such that the following diagram commutes:

[2]See Notes 6.

$$\begin{tikzcd} Y \arrow[r, "e'"] \arrow[d, "j"'] & \varepsilon(Y) \\ \varepsilon(X) \arrow[ur, "f"'] & \end{tikzcd}$$

Again since $e' = fj$ is tight, f is tight. Since $\varepsilon(X)$ does not have any proper tight extension, f is automatically an isomorphism. $\square$

As we have shown in the previous chapter, if X is compact, then $\varepsilon(X)$ is compact. We have also stated that this result can be still strengthened. A consequence of the previous theorem follows:

Corollary 2.3.12. *Given a metric space X, its hyperconvex hull is compact if and only if it is precompact.*

Proof. If X is precompact then its completion $\zeta(X)$ being also precompact is compact. By Proposition 1.2.2 *(iii)*, $\varepsilon(\zeta(X))$ is compact. Since $\varepsilon(X) \cong \varepsilon(\zeta(X))$ by the previous theorem, $\varepsilon(X)$ is compact. Conversely, assume that $\varepsilon(X)$ is compact. Since $\zeta(X)$ is closed in $\varepsilon(\zeta(X))$, it is compact. The precompactness of $\zeta(X)$ implies that of X. $\square$

We note that this result is also spotted in [Dre84].

Corollary 2.3.13. *Given a metric space X, its maximal tight extension is unique up to isomorphism.*

Proof. Let Y be another maximal tight extension of X. By the last theorem $\varepsilon(X) \cong \varepsilon(Y)$. Since Y does not have a proper tight extension $\varepsilon(Y) \cong Y$ and the result follows. $\square$

Now assume that $\iota(X)$ is a hyperconvex hull of metric space X. Let us denote by i the extension $X \longrightarrow \iota(X)$. By the hyperconvexity of $\iota(X)$, there is a unique extension $j : \varepsilon(X) \longrightarrow \iota(X)$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "e"'] & \iota(X) \\ \varepsilon(X) \arrow[ur, "j"'] & \end{tikzcd}$$

Since $\varepsilon(X)$ is hyperconvex and contains X, j is an isomorphism. Hence $\iota(X)$ is unique up to isomorphism. It follows that if an extension $i : X \longrightarrow Y$ is tight, where Y is hyperconvex, then Y is a hyperconvex hull. These remark motivate the following definition:

Definition 2.3.14. *[Her92] Let X be a metric space. An extension $i : X \longrightarrow Y$ is a hyperconvex hull if i is tight and if Y is hyperconvex.*

We have already shown that a hyperconvex space is complete. We will give here a second proof which is a direct consequence of the previous theorem.

Corollary 2.3.15. *If X is a hyperconvex space, then X is complete.*

Proof. Since X is hyperconvex and since the extension of completion $X \longrightarrow \zeta(X)$ is tight, we have $X \cong \varepsilon(X) \cong \varepsilon(\zeta(X))$. Hence we have a tight extension $i : \varepsilon(\zeta(X)) \longrightarrow \zeta(X)$. But $\varepsilon(\zeta(X))$ does not have a proper tight extension, therefore i is an isomorphism and $X \cong \zeta(X)$. □

As a hyperconvex space enjoys the extension property, it is interesting to look at the possible relations between tight extensions and those spaces that have the extension property.

2.4 Injectivity

In this section, we look at some properties of the injective metric spaces.

Definition 2.4.1. *A metric space X is called injective if it is nonempty and has the extension property. That is, for any metric spaces Y, a nonexpansive mapping $f : Y \longrightarrow X$ and an extension $e : Y \longrightarrow Z$ there is a nonexpansive mapping $F : Z \longrightarrow X$ such that the following diagram commutes:*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "e"'] & X \\ Z \arrow[ur, "F"'] & \end{tikzcd}$$

Then every hyperconvex space is injective. As we will see later on, the converse is also true. As a first consequence of this definition, we have the following proposition.

Proposition 2.4.2. *An injective metric space is complete.*

Proof. Let us denote by i the extension $X \longrightarrow \zeta(X)$ where $\zeta(X)$ is the completion of X as usual and let us denote by id the identity map on X. The injectivity of X implies the existence of a nonexpansive mapping $f : \zeta(X) \longrightarrow X$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, "i"'] & X \\ \zeta(X) \arrow[ur, "f"'] & \end{tikzcd}$$

Since i is tight and $fi = id$, f is an extension. Since id is onto, f is onto, therefore f is an isomorphism. □

Proposition 2.4.3. *Let X be an injective metric space. Then either $|X| = 1$ or $|X| > n$ for any $n \in \mathbb{N}$, where $|X|$ is the cardinal of the underlying set of X.*

Proof. Suppose that $|X| \neq 1$. Suppose that $X = \{a, b\}$ with $d(a, b) = k$. By defining $f(0) = a$ and $f(k) = b$, we have a nonexpansive map from $\{0, k\} \subseteq \mathbb{R}$ to X. And since X is injective we can extend f to $\mathbb{R}$. But then the extension being nonexpansive is continuous and $\mathbb{R}$ would not be connected. This is enough to assume that for any $n \in \mathbb{N}$, $|X| > n$. □

In fact, as we will see in the following theorem, $|X| > \omega$ where $\omega = |\mathbb{N}|$.

Proposition 2.4.4. *If X is injective. Then X is metrically convex.*

Proof. [KK01] We assume that X is not finite. Let $x, y \in X$ such that $x \neq y$. The identity mapping $i : \{x, y\} \longrightarrow X$ is nonexpansive. Take $w \in X$ with $w \neq y$ and $w \neq x$. Let $Z = \{x, y, w\}$ and let d' be the metric on Z such that $d(x, y) = d'(x, y)$ and $d'(x, w) = d'(y, w) = (1/2)d(x, y)$. It is easy to check that (Z, d') is a metric space. Since X is injective, i can be extended to a nonexpansive map $I : Z \longrightarrow X$. But then, using the triangle inequality, one can get:

$$d(I(w), x) = d(I(w), y) = (1/2)d(x, y).$$

Therefore X is metrically convex. □

This result is proved in [KK01] as a part of a another proof.

Furthermore, we have:

Theorem 2.4.5. *If X is injective, then each two distinct pairs x, y in X is the endpoints of at least one metric segment.*

Proof. Let $x, y \in X$. Let $s_{1/2} \in X$ such that $d(x, y) = d(x, s_{1/2}) + d(s_{1/2}, y)$. There are $s_{1/4}, s_{3/4} \in X$ such that $d(x, s_{1/2}) = d(x, s_{1/4}) + d(s_{1/4}, s_{1/2})$ and $d(s_{1/2}, y) = d(s_{1/2}, s_{3/4}) + d(s_{3/4}, y)$. Continuing the process, we have a sequence $S = \{s_d : d \in \mathcal{D}\} \subseteq X$ and such that $|d(x, s_{d_1}) - d(x, s_{d_2})| = d(s_{d_1}, s_{d_2})$ for any $d_1, d_2 \in \mathcal{D}$ where $\mathcal{D}$ is the set of dyadic numbers between in the interval $[0; 1]$.

We note that for each $d \in \mathcal{D}$, there is $n \in \mathbb{N}$ and $k < 2^n$ such that $d = k/2^n$. Then if $d_1, d_2 \in \mathcal{D}$, then there is $n \in \mathbb{N}$ such that $d_1, d_2 \in \mathcal{D}_n$. Suppose that $d(x, s_{d_1}) < d(x, s_{d_2}) + d(s_{d_1}, s_{d_2})$. By the above process, we have

$$d(x, y) = d(x, s_{1/2^n}) + d(s_{(2^n-1)/2^n}, y) + \textstyle\sum_{k=1}^{2^n-2} d(s_{k/2^n}, s_{(k+1)/2^n}).$$

But then we would have

$$d(x, y) \leq d(x, s_{d_1}) + d(s_{d_1}, y) < d(x, s_{d_2}) + d(s_{d_2}, s_{d_1}) + d(s_{d_1}, y).$$

And by the triangle inequality the latter term is less than

$$d(x, s_{1/2^n}) + d(s_{(2^n-1)/2^n}, y) + \textstyle\sum_{k=1}^{2^n-2} d(s_{k/2^n}, s_{(k+1)/2^n}) = d(x, y).$$

So it should be the case that $|d(x, s_{d_1}) - d(x, s_{d_2})| = d(s_{d_1}, s_{d_2})$.

Now let $\varphi : S \bigcup \{x, y\} \longrightarrow [0; d(x, y)]$ with $\varphi(z) = d(x, z)$. By the previous calculations φ is an isometry. Moreover, if we order S by $s \preceq s'$ if and only if $\varphi(s) \leq \varphi(s')$ then φ is a isomorphism of lattices. Let $R = \varphi(S \bigcup \{x, y\})$. Let $\mathcal{F}$ be all the collections of pairs (A, f) such that $S \subseteq A$ and A is countable, and $f : A \bigcup \{x, y\} \longrightarrow [0; d(x, y)]$ is an isometry such that $\varphi \subseteq f$. $\mathcal{F}$ is clearly nonempty. We order $\mathcal{F}$ as follows: $(A_1, f_1) \preceq (A_2, f_2)$ if and only if $A_1 \subseteq A_2$ and $f_1 \subseteq f_2$. Let $\mathcal{C}$ be a chain in $\mathcal{F}$. Let $B = \bigcup\{A : (A, f) \in \mathcal{C}$ for some $f \subseteq X \times X\}$ and $f' = \bigcup\{f : (A, f) \in \mathcal{C}$ for some $A \subseteq X\}$. B is countable. If $x, y \in B$ then, $x \in A_1$ and $y \in A_2$ for some A_1 and A_2. Since $\mathcal{C}$ is a chain, $f_1 \subseteq f_2$ or $f_2 \subseteq f_1$. But then $d(x, y) = |f'(x) - f'(y)|$ since $f_1, f_2 \subseteq f'$. Hence $(B, f') \in \mathcal{F}$ and $(A, f) \preceq (B, f')$ for all $(A, f) \in \mathcal{C}$. Thus $\mathcal{F}$ is inductive and by Zorn's lemma there exists a maximal element (S, φ) in $\mathcal{F}$. We claim that the closure $cl(R)$ is $[0; d(x, y)]$ where $R = \varphi(S \bigcup \{x, y\})$.

Assume that the closure $cl(R)$ is not $[0; d(x, y)]$. Let $(a; b) \subseteq [0; d(x, y))]$ such that $(a; b) \bigcap R = \emptyset$. Let $\mathcal{G}$ be the collection of all subsets (c, d) such that $(a; b) \subseteq (c; d)$ and $(c; d) \bigcap R = \emptyset$. It is clear that $\mathcal{G}$ is nonempty. Let $\mathcal{C}$ be a chain in $\mathcal{G}$ and let $(\alpha; \beta) = \bigcup \mathcal{C}$. $\mathcal{C}$ is bounded above by $(\alpha; \beta)$ and $(\alpha; \beta) \bigcap R = \emptyset$. By Zorn's lemma $\mathcal{G}$ has a maximal element $(p; q)$. Since $(p; q)$ is maximal, for any $\epsilon > 0$ we have $(p - \epsilon; q) \bigcap R \neq \emptyset$ and

$(p; q+\epsilon) \bigcap R \neq \emptyset$. By the definition of the closure $p, q \in cl(R)$. There are then sequences $\{p_n : n \in \mathbb{N}\} \subseteq R$ and $\{q_n : n \in \mathbb{N}\} \subseteq R$ such that $\lim(p_n) = p$ and $\lim(q_n) = q$. But then there are sequences $\{x_n : n \in \mathbb{N}\} \subseteq X$ and $\{y_n : n \in \mathbb{N}\} \subseteq X$ such that there are $p^* = \lim(x_n) \in X$ and $q^* = \lim(y_n) \in X$ with $x_n = \varphi^{-1}(p_n)$ and $y_n = \varphi^{-1}(q_n)$ for each $n \in \mathbb{N}$ since φ is an isometry and (X, d) complete. Let $Z = S \bigcup \{p^*, r, q^*\}$ where $d(p^*, q^*) = d(p^*, r) + d(r, q^*)$ and g such that $\varphi = g$ on $S \bigcup \{x, y\}$ and $g(p^*) = p$, $g(q^*) = q$ and $g(r) = d(x, r)$. Then $(Z, g) \in \mathcal{F}$ and $(R, \varphi) \prec (Z, g)$ which is impossible. Therefore $cl(R) = [0; d(x, y)]$.

Let $\psi : [0; d(x, y)] \longrightarrow X$ defined by $\psi(a) = \varphi^{-1}(a)$ if $a \in R$ and $\psi(a) = \lim(r_n)$ for $a \notin R$ where $\{r_n : n \in \mathbb{N}\} \subseteq R$ and $\lim(r_n) = a$. $\psi(a) \in X$ since X is complete. Now, let $a, b \in R$ and let $\{r_n, r'_m : n, m \in \mathbb{N}\}$ such that $\lim(r_n) = a$ and $\lim(r'_m) = b$. We have:

$$d(\psi(a), \psi(b)) = d(\lim \psi(r_n), \lim \psi(r'_m)) = \lim d(\psi(r_n), \psi(r'_m))) = \lim d(r_n, r'_m)) = d(a, b).$$

So ψ is an isometry from $[0; d(x, y)]$ to X with $\psi(0) = x$ and $\psi(d(x, y)) = y$. □

One may see that this theorem is an immediate consequence of the Theorem 2.2.4 and Propositions 2.4.2 and 2.4.4. However we have proved this theorem directly since Theorem 2.2.4, which was first obtained by Menger, was left without proof in Section 2. The previous proof, however, can be easily adapted to prove Theorem 2.2.4 [3]. The following theorem establishes the equivalence between hyperconvex spaces and injective metric spaces. This equivalence was first established by Aronszajn and Panitchpakdi in [AP56]. This characterizes all the metric spaces that have the extension property.

Theorem 2.4.6. *[KK01] If X is an injective metric space, then X is hyperconvex.*

Proof. [KK01] Since X is already metrically convex, it suffices to show that X has the **BIP**. Let $\mathcal{B} = \{B(x_i; r_i) : i \in I\}$ be a family of balls such that each two members of them intersect. Let $A = \{x_i : i \in I\}$ and let denote by i the identity map from A to X. Let w be a point not in A (w is not necessarily in X). Let us set $A^* = A \bigcup \{w\}$. We define on A^* a new metric d' such that d' is on A the same as the metric induced from X and for all $i \in I$:

$$d'(w, x_i) = \inf\{r : B(x_j; r_j) \subseteq B(x_i; r) \text{ for some } j \in I\}.$$

It is clear that $d'(w, x_i) \leq r_i$ for any $i \in I$. Therefore, we have $d'(w, x_i) = r_i$ or $B(x_j; r_j) \subseteq B(x_i; d'(w, x_i))$ for some $j \neq i$. Thus for any $i, j \in I$ there are $m, n \in I$ such that:

[3]See Notes 2.

$$B(x_n; r_n) \subseteq B(x_i; d'(w, x_i)) \text{ and } B(x_m; r_m) \subseteq B(x_j; d'(w, x_j)).$$

Since each two members of $\mathcal{B}$ intersect it is the case that:

$$B(x_i; d'(w, x_i)) \bigcap B(x_j; d'(w, x_j)) \neq \emptyset.$$

But then we have $d(x_i, x_j) \leq d'(x_i, w) + d'(w, x_j)$. So (A^*, d') is a metric space. Since i is nonexpansive and X injective, there is a nonexpansive map I extending i on A^*. Hence for any $i \in I$:

$$d(I(w), x_i) = d(I(w), I(x_i)) \leq d'(w, x_i) \leq r_i.$$

Therefore $\bigcap \mathcal{B} \neq \emptyset$. □

Therefore a metric space is hyperconvex if and only if it is injective [KK01, Her92, EK01], this is a consequence of the previous Theorem and Propsition 2.2.6. A direct proof using the definition of hyperconvexity for the sufficient condition and which is different from the previous one is given in [EK01]. Since the proof is shorter, we find it convenient to reproduce it here.

Given an injective metric space X and a family $A = \{x_i : i \in I\} \subseteq X$ with the condition that $d(x_i, x_j) \leq r_i + r_j$ for all $i, j \in I$ for some $\{r_i : i \in I\} \subseteq \mathbb{R}_+$, we define the map $r : A \longrightarrow \mathbb{R}_+ : x_i \mapsto r_i$. It is clear that $r \in \sigma(A)$. Let $f = \rho(r)$ where ρ is the retraction from $\sigma(X)$ to $\varepsilon(X)$. By the properties of the extremal functions we have $f(x_i) \leq d(x_i, x_j) + f(x_j)$ for all $i, j \in I$.

Let $A^* = A \bigcup \{w\}$ where w is not a point in A (not necessarily in X). We define the metric d' on A^* such that d' is on A the metric induced from X and $d'(w, x_i) = f(x_i)$ for all $i \in I$. (A^*, d') is clearly a metric space. Let i be the natural isometry from A to X. Since X is injective, there is a nonexpansive map $I : A^* \longrightarrow X$ extending i. Then we have:

$$d_X(I(w), I(x_i)) \leq d'(w, x_i) \leq f(x_i) \leq r_i \text{ for all } i \in I.$$

Therefore $\bigcap \{B(x_i; r_i) : i \in I\} \neq \emptyset$ in X.

This last proof shows why the second definition allows easily the use of nonexpansive mappings and extremal functions. The proof that we have given is divided in two parts for two reasons: the first one because we wanted to give a proof of the existence of an isomorphic copy of an interval in the real line in a complete and metrically convex space at the same time; the second reason is that, by giving different proofs, we wanted to emphasize the fact that the extension property is powerful enough to imply other properties that are not less beautiful nor less useful.

From this equivalence, it is then natural to say that any metric space has an injective envelope or injective hull, that is the minimal (up to isomorphism) injective metric space in which it can be embedded. Precisely the metric injective hull of a metric space X is a tight extension $i : X \longrightarrow Y$ where Y is injective. We resume this chapter with a proposition that follows naturally from the last theorem.

Proposition 2.4.7. *[Her92] Let X be a metric space and $i : X \longrightarrow Y$ an extension. The following statements are equivalent:*

(i) i is tight and Y hyperconvex;

(ii) i is tight (essential) and Y injective;

(iii) $i : X \longrightarrow Y$ is a maximal tight (essential) extension.

We note that these statements are essentially the same. However we wrote these statements to emphasize the following facts: (i) is more internal while (ii) and (iii) are external. That is, the characterization (i) is defined with the subsets of Y and metrics on X and Y while the latter two characterizations are defined with focus on the behaviour of X and Y with respect to other metric spaces.

2.5 Notes

1. The necessity of the Theorem 2.1.3 has been shortened thanks to the Theorem 1.3.2 in Chapter 1 and the fact that an extremal function could be always found having an superadditive one. We have also largely simplified the sufficiency of the theorem.

2. The Theorem 2.2.4 is stated in [KK01] without a proof. Instead, the authors refer to [Blu53] and [GK90] which were not available to us. We gave our own proof in Section 4 in the context of injectivity.

[NB85] The investigation of the notion of convexity began with H. Minkowski and H. Brum. As we could see in this chapter, there are at least two types of convexity. Indeed, various notions of convexity have been defined in various settings [Kle63].

3. In the proof of the Theorem 2.2.12, the steps from the result: $r_x(A_j) = r_x(A_i)$ for all $x \in H_j$ to the result $r(A_i) = r(A_j)$ for every $i, j \in I$ have been written here with more details.

4. This statement is stated in [KK01] and proved directly without using the the sets ϵ-fixed points $F(f, \epsilon)$.

5. The result in the Lemma 2.3.2 is stated in [CP97] without a proof. As Dress in [Dre84] has defined extremal functions in another way by considering directly their properties, this lemma is unnecessary in his paper. The expression

$$f(x) = \sup\{d(x, y) - f(y) : y \in X\} \text{ for any } x \in X.$$

should not be surprising since it is the same as writting: $f(x) = d_\infty(e(x), f)$ for any $x \in X$, e being the extension $X \longrightarrow \varepsilon(X)$.

6. The difference between this construction and that of Dress in [Dre84] is that the property

of hyperconvexity is not used. For example, the retraction $\rho : \sigma(X) \longrightarrow \varepsilon(X)$ which seems obvious to us is literally computed in [Dre84]. The surjection of the maps $k : \varepsilon(Y) \longrightarrow \sigma(X)$ is also given in Chapter 1. Here again, the crucial point is that the real line is hyperconvex. We shall quote Dress' note in [Dre84] on page 329: " *As I learned in the meantime, the construction* $X \rightsquigarrow T_X$ *has already been studied by* J. R. Isbell (Comment. Math. Helv. **39**) (1964-1965), 65-74), [...]". The construction $X \rightsquigarrow T_X$ is supposed to be the hyperconvex hull of X.

3. A few examples

As the extension property involves only maps (morphisms) one can define it in various settings, for example in the context of fields or partially ordered sets and even in a categorical setting [AHS]. From the last section of Chapter 2, the concept of hyperconvex hull has been brought to a point where it can be defined using only the extension property and the tightness of extensions. The latter, as we have defined it, can be characterized by using embeddings and nonexpansive mappings. Therefore, having defined and characterized what would be a “hyperconvex” object or injective object in a particular category, the question whether or not any object has an injective hull may be raised. For example, in the category of sets with functions, every non-initial object (nonempty set) is injective hence every object is its injective hull. However, in the category of groups with group homomorphisms only the terminal object (trivial group) is injective (this is also the case for lattices and rings) [AHS], so no nontrivial group has an injective hull.

We will not treat all these examples in this chapter for it could be investigated in a categorical way [AHS] which is not the main purpose of our task. We have chosen to look at the real Banach spaces. The reason is that the concept of hyperconvexity is closely related to the **BIP** which appears to be the key of the extension property in metric spaces and real normed spaces. In particular as we have seen in Chapter 1, the Hahn-Banach theorem is a particular consequence of the **BIP**. But the intimate relationship between metric spaces and real normed spaces goes beyond these observations: for a real normed space, the hyperconvex hull of its underlying metric space coincides with its injective hull in the category of real normed space. This will be treated in Section 2. We also point out that we have a version of the Krein-Milman theorem in a hyperconvex metric space. We devote Section 1 to Euclidian space for it is of interest to us in explaining the structures of finite dimensional normed spaces that are hyperconvex. In Section 3 we will cite few examples of hyperconvex spaces and hyperconvex hulls, we will in particular talk about a consequence of the property of hyperconvexity that leads to the Brouwer's fixed point theorem.

3.1 Euclidian space

As we have seen in the previous chapter, if two metric spaces X and Y are hyperconvex and if their cartesian product $X \times Y$ is endowed with the sup metric d_∞, then this product is hyperconvex. The questions arises whether or not this is the only metric that makes the product hyperconvex. A counterexample may be constructed in more particular settings. Consider the cartesian product $\mathbb{R}^2$ with two metrics: the sup metric d_∞ and the sum metric that we will denote by d. Consider the map $f : (\mathbb{R}^2, d) \longrightarrow (\mathbb{R}^2, d_\infty)$ [Her92] defined by $f(x, y) = (x + y, x - y)$ for all $(x, y) \in (\mathbb{R}^2, d)$. f is bijective and for all $(x, y) \in \mathbb{R}^2$:

$$d(x, y) = |x| + |y| = \sup\{|x + y|, |x - y|\} = d_\infty(f(x), f(y)).$$

Therefore f is an isometry, hence it is an isomorphism of metric spaces (it is even an automorphism of the vector space $\mathbb{R}^2$). The construction of f indicates that f is not the only such isomorphism. Thus $(\mathbb{R}^2, d)$ is hyperconvex since an isomorphism of metric spaces is but a special case of a retraction. Hence d_∞ is not the only metric that makes $\mathbb{R}^2$ hyperconvex. However, we do not know if it is still the case for general metric spaces which do not have a vector space structure. Indeed, although the metric d and d_∞ on $\mathbb{R}^2$ are not the same, they are "isomorphic" in some sense since their balls are the same up to rotation. In general, a finite-dimensional vector space is hyperconvex [Nac50] if and only if the balls defined by the norm are cubic. The aim of this section is to give a proof of this statement and investigate the behaviour of such norms.

We recall that for a normed space X, hyperconvexity is equivalent to the binary intersection property which we have already denoted by **BIP**. Also X has the extension property if and only if X has the **BIP** (Theorem 2.1.3). We begin with some definitions which will be used throughout the section.

Definition 3.1.1. *A set X with an order relation $\leq$ is a lattice if any nonempty finite subset A of X has an infimum and a supremum in X.*

A lattice X is complete if any nonempty subset A of X has a supremum denoted by $\sup A$ *in X.*

The rational line $\mathbb{Q}$ with its usual order is an example of a lattice which is not complete.

Definition 3.1.2. *Let $(X, \leq)$ be an ordered set. For any $x, y \in X$, the subset $\{z : x \leq z \leq y\}$ is called segment and denoted by $[x; y]$.*

*A ordered set $(X, \leq)$ has the **BIP** if for any collection $\mathcal{F}$ of segments in X such that each two members of $\mathcal{F}$ intersect, then $\bigcap \mathcal{F} \neq \emptyset$.*

We use here the same "**BIP**" terminology as in a metric space since the two properties are essentially the same. Only the subsets involved in the family considered differ, they depend on the structures with which the set is endowed. However we will always make clear in the sequel which of the two "**BIP**" we are considering.

Remark 3.1.3. *[Nac50] If X is a vector space, then a segment in a vector sense is a subset of the form $\{\lambda x + (1 - \lambda)y : \lambda \in [0; 1]\}$ where $x, y \in X$. In our case, we will not use this definition for it might create some confusion, we will rather use the term convex span of x and y denoted by C_{xy} when it comes to refer to this subset.*

Also , we say that a point e is an extreme point of a subset A of X if for any $x, y \in A$ such that $e \in C_{xy}$ then necessarily $e = x$ or $e = y$.

Definition 3.1.4. *[Nac50]*

An ordered set $(X, \leq)$ has the interpolatory property if for any $\{x, x', y, y'\} \subseteq X$ such that $x, x' \leq y, y'$ there is $z \in X$ such that $x, x' \leq z \leq y, y'$.

This notion was introduced by Riesz [Rie37] ([Nac50]) in the context of ordered groups.

Now, we have the following lemma:

Lemma 3.1.1. *[Nac50] An ordered set $(X, \leq)$ is a complete lattice if and only if it has the **BIP** in the order sense and the interpolatory property.*

Proof. [Nac50] Assume that X is a complete lattice. Let $A = \{x, x', y, y'\} \subseteq X$ such that $x, x' \leq y, y'$. Since X is a lattice, we have:

$$x, x' \leq \sup\{x, x'\} \leq \inf\{y, y'\} \leq y, y'.$$

Now, let $\mathcal{I} = \{I_i : i \in J\}$ be family of segments in X such that $I_i \bigcap I_k \neq \emptyset$ for all $i, k \in J$ with $I_i = [x_i; y_i]$ for any $i \in J$. Since X is complete $x = \sup\{x_i : i \in J\}$ belongs to X and $x_i \leq x \leq y_j$ for any $i, j \in J$. Hence $\bigcap \mathcal{F} \neq \emptyset$. Conversely, suppose that X satisfies the two properties. Let A be a nonempty subset of X. Let $B = \{x \in X : a \leq x \text{ for all } a \in A\}$ and let $\mathcal{F}$ be the collection of all segments $[a; b]$ where $a \in A$ and $b \in B$. The interpolatory property implies that any two members of $\mathcal{F}$ intersect. Since X has the **BIP**, $\bigcap \mathcal{F} \neq \emptyset$. If $c \in \bigcap \mathcal{F}$, then c is the supremum of A. $\square$

It is easy to see that if X is a linearly ordered set then the interpolatory property would be a redundant condition. Consider for example the real line $\mathbb{R}$ with its usual order. $\mathbb{R}$ is complete lattice and it has the **BIP** in a sense of the above definition.

Definition 3.1.5. *Let X be a vector space. X is an ordered vector space if there is an order $\leq$ in X such that $\leq$ is preserved under translation and multiplication by positive scalars.*

The real line $\mathbb{R}$ with its usual order is an obvious example of ordered vector space.

Definition 3.1.6. *[Nac50] Let X be a real vector space. X is called a complete vector lattice if the following holds:*

- *X is an ordered vector space.*
- *X with its order is a complete lattice.*

Furthermore, X is called a complete vector lattice with unit if there is an element $e \in X$ called order unity such that $e > 0$ and for any $x \in X$, there exists $\lambda \in \mathbb{R}$ such that $-\lambda e \leq x \leq \lambda e$.

We note that Nachbin in [Nac50] has defined a complete vector lattice as a complete vector lattice with unit. We shall use this terminology for the sequel for it is shorter and since we will always consider complete vector lattice with unit.

[Nac50] Now, suppose that X is a complete vector lattice and let us consider the function $N : X \longrightarrow \mathbb{R}_+$ defined by $N(x) = \inf\{\lambda : -\lambda e \leq x \leq \lambda e\}$. N defines a norm on X [1]. Hence a complete vector lattice is a normed space. Since $e \leq e$, $N(e) \leq 1$ and we have equality since there is no λ with $0 < \lambda < 1$ such that $e \leq \lambda e \leq e$. Indeed this would imply that $e = \lambda e + (1 - \lambda)e \leq \lambda e$ which is the same as writting $(1 - \lambda)e \leq 0$. Since X is an ordered vector space, the last inequality cannot happen. We also note that e is an extreme point of the unit ball $B_N(0; 1)$, that is e does not belong to any convex span of any elements x and y in $B_N(0; 1)$ [Nac50]. Suppose that it is not true, assume that $e = \lambda x + (1 - \lambda)y$ for some $x, y \in B_N(0; 1)$ with $x \neq e$ and $y \neq e$. Since $N(y), N(x) \leq 1$ we have $-e \leq x \leq e$ and $-e \leq y \leq e$. Since X is an ordered vector space $e = \lambda x + (1 - \lambda y) < \lambda e + (1 - \lambda)e = e$. Therefore e is an extreme point of $B_N(0; 1)$. It is also easy to see that $B_N(0; 1) = [-e; e]$

[1] See notes 2.

Hence if X is a complete vector lattice then (X, N), where N is the norm defined above, is a normed space having the **BIP** in the metric sense (or the norm sense) and such that its order unity is an extreme point of its unit ball.

For example the real line $\mathbb{R}$ is a complete vector lattice with 1 as order unity and the norm deduced from the order relation and 1 is the absolute value on $\mathbb{R}$. Therefore all the segments in $(\mathbb{R}, \leq)$ coincide with all the balls in $(\mathbb{R}, |.|)$. As we have stated many times, $\mathbb{R}$ has the **BIP** in the metric sense and hence it is hyperconvex. This is true because it is a complete vector lattice and by the previous lemma it has the **BIP** in the order sense and hence in the metric sense. It is this completeness in the lattice sense that we want to generalize in higher dimension.

[Nac50] The converse is also true, that is if a normed space X has the **BIP** in the metric sense and such that its unit ball has an extreme point e then there is a unique order $\leq$ on X such that $(X, \leq)$ is a complete vector lattice with e as an order unity and such that the norm N deduced from the order relation and e is identical to the initial norm.

Before proving the converse, we need to show the following proposition.

Proposition 3.1.7. *[Nac50] Let X be a normed space such that it has the **BIP** in the metric sense and its unit ball B has an extreme point, say e. Then there is a unique order in X such that B is equal to the segment $[-e; e]$.*

Proof. [Nac50] We need to find an order relation R on X such that $R+R \subseteq R$ and $\lambda R \subseteq R$ for any $\lambda > 0$. Let P be the set of elements of the form $\lambda(u+e)$ where $\lambda > 0$ and $u \in B$. Clearly $\lambda P \subseteq P$. If $x = \lambda(u+e)$ and $y = \mu(v+e)$ are elements of P then:

$$x + y = (\lambda + \mu)(\lambda/(\lambda + \mu)u + \mu/(\lambda + \mu)v + e).$$

Since $u, v \in B$, the convex combination $\lambda/(\lambda+\mu)u + \mu/(\lambda+\mu)v$ belongs to B. Therefore $x + y \in P$ and $P + P \subseteq P$. We define the relation R as follow:

$$(x, y) \in R \text{ if and only if } y - x \in P.$$

We note that $0 = \lambda(-e+e) \in P$ where $\lambda > 0$, so R is reflexive. It is clear that $R+R \subseteq R$ and $\lambda R \subseteq R$ for any $\lambda > 0$. It remains to verify that R is antisymmetric since it is clearly transitive. It is enough to show that $P \bigcap (-P) = \{0\}$ since if $(x, y) \in R$ and $(y, x) \in R$, it is equivalent to $y - x \in P \bigcap (-P)$. Let x be a point in this intersection. For some $u, v \in B$

and for some $\lambda, \mu \in \mathbb{R}_+^*$ we have $x = \lambda(u+e)$ and $-x = \mu(v+e)$ (since $-x \in P$). Then $\lambda u + \mu v + (\lambda+\mu)e = 0$. If $x \neq 0$ then $u \neq -e$ and $v \neq -e$, also $u \neq -v$ and $\lambda \neq -\mu$. Therefore

$$e = \lambda/(\lambda+\mu)(-u) + \mu/(\lambda+\mu)(-v).$$

Since $u, v \in B$, $-u, -v \in B$ and e would not be an extreme point of B. Therefore $x = 0$ and $P \bigcap (-P) = \{0\}$. So the relation R is antisymmetric. From now on we will denote R by $\leq$ and the expression $(x, y) \in R$ will be denoted by $x \leq y$ or $y \geq x$. We also note that if $x \in P$ then $x \geq 0$ since $x = x - 0$ and conversely if $x \geq 0$ then $x \in P$ since $x - 0 \geq 0 - 0 = 0$ ($(0, x) \in R$). Since $x \leq y$ if and only if $0 \leq x - y$, the relation $\leq$ is determined uniquely by the set of its positive elements, but then $\leq$ is uniquely determined by P. Now we claim that

$$[-e; e] = (-e+P) \bigcap (e-P).$$

Where $(-e+P) = \{-e+p : p \in P\}$ (and so for $(e-P)$).

If $x \in [-e; e]$ then $-e \leq x \leq e$, thus $x + e \in P$ and $e - x \in P$. Hence $x \in (-e+P)$ and $x \in (e-P)$. Conversely, let $x \in (-e+P) \bigcap (e-P)$, then $x = -e + p_1$ and $x = e - p_2$ for some $p_1, p_2 \in P$. But then $x + e = p_1 \in P$ and $e - x = p_2 \in P$, thus by the definition of the order $-e \leq x$ and $x \leq e$. So $x \in [-e; e]$.

Now let $b \in B$. We have $b = e - (-b+e)$ and $b = -e + (b+e)$ with $1(-b+e) \in P$ and $1(b+e) \in P$. Thus $b \in (e-P) \bigcap (-e+P) = [-e; e]$ and $B \subseteq [-e; e]$. Conversely, let $x \in [-e; e]$. For some $\alpha > 0$ and $\beta > 0$ and for some $v, w \in B$, we have:

$$x = e - \alpha(v+e) = -e + \beta(w+e).$$

We note that if $\alpha \leq 1$ then x would belong to B since in that case $x = (1-\alpha)e + \alpha(-v)$ and since $e, -v \in B$. We then assume that $\alpha > 1$, and similarly $\beta > 1$. Let f be the map defined by $f(z) = x + (\alpha - 1)(z+e)$ for any $z \in X$ and let g be the map defined by $g(z) = x + (\beta - 1)(z - e)$ for any $z \in X$. Now, since X is an ordered vector space

$$f([-e; e]) = [x; x + 2(\alpha-1)e] \text{ and } g([-e; e]) = [x - 2(\beta-1)e; x].$$

Let $A = f(B)$ and $C = g(B)$. Since $B \subseteq [-e; e]$:

$$A \bigcap C \subseteq f([-e; e]) \bigcap g([-e; e]) = \{x\}.$$

On the other hand $x = e - \alpha(v+e)$, so $v = -x + (1-\alpha)e + (1-\alpha)v$ and $-v = x + (\alpha-1)(e+v) \in A$. So $-v = f(v)$ and $-v \in A \bigcap B$. Similarly, since $x = -e + \beta(w+e)$, $w = x + (\beta - 1)(-w - e) \in C$, so $w = g(-w)$ and $w \in B \bigcap C$. Since X has the **BIP**:

$$\emptyset \neq A \bigcap B \bigcap C \subseteq A \bigcap C = \{x\}.$$

Therefore $A \bigcap B \bigcap C = \{x\}$ and $x \in B$. Thus $B = [-e; e]$. $\square$

We will now proceed to show completely that X, endowed with such an order is a complete vector lattice with e as an order unity.

Now if $x \in X$ then we have:

$$B(x;r) = x + rB = x + r[-e;e] = [x - re; x + re],$$

where B is the unit ball of X. Therefore, any ball can be expressed as a segment. The converse, that is every segment is a ball, is not true. As we will see later on, in $\mathbb{R}^2$ with the sup metric d_∞, this statement fails. The reason is that any intersection of balls need not be a ball. However, a segment is always the intersection of two balls [Nac50].

[Nac50] Indeed let $x, y \in X$ such that $x \leq y$ and let r be such that $2r = \|x - y\|$. Let us denote by x^* the vector $y - 2re$ and by y^* the vector $x + 2re$. Let a and b be such that $2a = x^* + y$ and $2b = x + y^*$. Since $y - x \in B(0; 2r) = [-2re; 2re]$, $y - x \leq 2re$, so $x^* = y - 2re \leq x$ and $y \leq x + 2re = y^*$. So $[x^*; y] \bigcap [x; y^*] = [x; y]$.

On the other hand

$$[x^*; y] = [a - re; a + re] = B(a; r) \text{ and } [x; y^*] = [b - re; b + re] = B(b, r).$$

So $[x; y] = B(a; r) \bigcap B(b; r)$.

Now since any segment is the intersection of two balls and since X has the **BIP** in the metric sense, X has the **BIP** in the order sense. Furthermore if $x \in X$ and $\lambda \geq \|x\|$ then $x \in B(0; \lambda) = [-\lambda e; \lambda e]$, so $-\lambda e \leq x \leq \lambda e$ and e is an order unity on X. It remains to show that X has the interpolatory property so that, by virtue of Lemma 3.1.1 X can be a complete vector lattice.

Let $\{x_1, x_2, y_1, y_2\} \subseteq X$ with $x_1, x_2 \leq y_1, y_2$. Let $M_i = [x_i^*; y_i]$ and $N_i = [x_i; y_i^*]$ where $x_i^* = y_i - 2r_i e$ and $y_i^* = x_i + 2r_i e$ for $i = 1, 2$. As above $M_i \bigcap N_i = [x_i; y_i]$ for $i = 1, 2$. If any of the balls M_i and N_i for $i = 1, 2$ intersect, then we are done since any point in the intersection could be our required point.

We already know that M_i and N_i intersect for $i = 1, 2$. For N_1 and N_2 (and similarly for M_1 and M_2), it suffices to prove that the distance between their respective centers $(x_1 + y_1^*)/2$ and $(x_2 + y_2^*)/2$ are less than the sum of their radii r_1 and r_2. Indeed, since X has the **BIP** in the metric sense, that is X is hyperconvex, we would have $N_1 \bigcap N_2 \neq \emptyset$. This is equivalent to writting:

$$|(x_1 + y_1^*)/2 - (x_2 + y_2^*)/2| \leq r_1 + r_2.$$

This means $|x_1 - x_2 + (r_1 - r_2)e| \leq r_1 + r_2$ which is the same as writting:

$$-(r_1 + r_2)e \leq x_1 - x_2 + (r_1 - r_2) \leq (r_1 + r_2)e.$$

But then $-2r_1 e \leq x_1 - x_2 \leq 2r_2 e$. Now since $y_1 - x_1 \leq 2r_1 e$, $-2r_1 e \leq x_1 - y_1$ and since $x_2 \leq y_1$, we have:

$$-2r_1 e \leq x_1 - y_1 \leq (x_1 - y_1) + (y_1 - x_2) = x_1 - x_2.$$

On the other hand $x_1 \leq y_2$ and $y_2 - x_2 \leq 2r_2 e$, so

$$x_1 - x_2 = (x_1 - y_2) + (y_2 - x_2) \leq y_2 - x_2 \leq 2r_2 e.$$

Therefore the two balls N_1 and N_2 intersect. With the same calculations, M_1 and M_2 also intersect. Hence all the balls M_i and N_i for $i = 1, 2$ intersect each other. Now $M_1 \bigcap M_2 \bigcap N_1 \bigcap N_2 \neq \emptyset$ and if z is a point in this intersection then $z \in M_i \bigcap N_i = [x_i; y_i]$ for $i = 1, 2$, that is $x_1, x_2 \leq z \leq y_1, y_2$.

[Nac50] Thus if X is a normed space having the **BIP** in the metric sense and such that its unit ball B has an extreme point e, then there is a unique order $\leq$ on X such that $(X, \leq)$ is a complete vector lattice with e as an order unity.

As we have already stated, the real line $\mathbb{R}$ is a complete lattice, or precisely it is Dedekind complete. It is also complete in the metric sense, that is every Cauchy net (or every Cauchy sequence) in $\mathbb{R}$ converges in $\mathbb{R}$. In both cases, $\mathbb{R}$ appears as the completion of the rational line $\mathbb{Q}$. In fact these two completions are the same. Indeed, the completeness in the metric sense is equivalent to the **BIP** in the metric sense in this case and the completeness in the lattice sense, as we have proved in the Lemma 3.1.1, is equivalent to the **BIP** in the order sense, the interpolatory property being obvious in $\mathbb{R}$ (the rational line $\mathbb{Q}$ is dense in the order sense in the real line). But any segment in $\mathbb{R}$ is a ball and any ball is a segment. Therefore, the **BIP** in the order sense is the same as the **BIP** in the metric sense and the two completions are equivalent. In particular, these two types of completion of the rational line coincide to its hyperconvex hull.

Now consider the cartesian product $\mathbb{R}^2$ with the following order:

$$(x, y) \leq (w, z) \text{ if and only if } x \leq w \text{ and } y \leq z \text{ for any } \{x, y, w, z\} \subseteq \mathbb{R}.$$

The point $e = (1, 1)$ is an order unity in $\mathbb{R}^2$, take $m = \max\{|x|, |y|\}$ from which $-me \leq (x, y) \leq me$. Thus $(\mathbb{R}^2, \leq)$ is a complete vector lattice and if we endow $\mathbb{R}^2$ with the norm N induced from e then it is hyperconvex. Now, it is easy to see that:

$$N((x, y)) = \inf\{m > 0 : -me \leq (x, y) \leq me\} = \sup\{|x|, |y|\}.$$

Therefore N is the sup metric on $\mathbb{R}^2$.

As we can see in the Figure 3.1 any segment $[(x, y); (w, z)]$ is not a ball.

In general for any $n \in \mathbb{N}$ if we consider on $\mathbb{R}^n$ the following order: for any $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$

$$x \leq y \text{ if and only if } x_i \leq y_i \text{ for } i = 1, 2, \ldots, n.$$

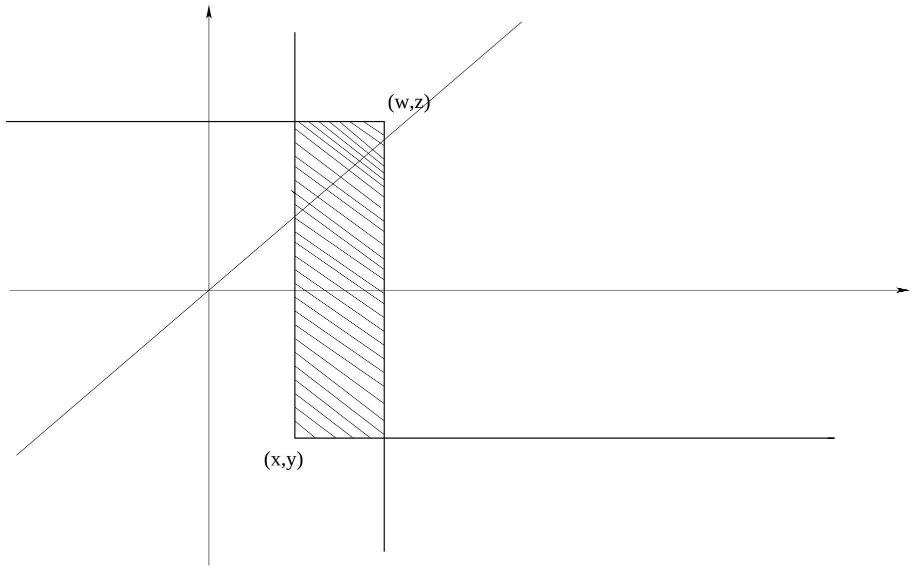

FIG. 3.1. A segment in $(\mathbb{R}^2, d_\infty)$.

Then $(\mathbb{R}^n, \leq)$ will be a complete vector lattice with $e = (1, 1, \ldots, 1)$ as an order unity. The norm on $\mathbb{R}^n$ will be then the sup norm (or sup metric).

Remark 3.1.8. *As one could see the order that we have defined on $\mathbb{R}^n$ has a certain componentwise character [Nac50]. That is, a vector is positive if and only if each of its components is positive. In general, a vector is positive if and only if each of the scalars involved in the combination that defines the vector is positive, is also true for a finite dimensional complete vector lattice. This result is spotted in [Nac50] and we shall recall it.*

Lemma 3.1.2. *[Nag45, You39] Let X be a complete vector lattice with dimension $n \in \mathbb{N}$. On X there is a basis $\{e_1, e_2, \ldots, e_n\}$ such that $x = \sum_{i=1}^{n} \lambda_i e_i \geq 0$ if and only if $\lambda_i \geq 0$ for $i = 1, 2, \ldots, n$.*

We have the following proposition.

Proposition 3.1.9. *If X is a finite dimensional vector space, then there is a norm N on X such that (X, N) has the **BIP** in the metric sense.*

Proof. Let $\{a_1, a_2, \ldots, a_n\}$ be a basis of X where n is the dimension of X. Let $f : \mathbb{R}^n \longrightarrow X$ be a linear map such that $f(e_i) = a_i$ for $i = 1, 2, \ldots, n$ where the components of e_i but the i-th place are zero. Then f is a isomorphism of vector spaces. We define on X the following order:

$$f(x) \leq f(y) \text{ if and only if } x \leq y \text{ in } \mathbb{R}^n.$$

$(X, \leq)$ is a complete vector lattice with $a = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} f(e_i)$ as a order unity. If N is the norm induced from a on X then (X, N) has the **BIP** in the metric sense. □

This proposition is quite obvious since it is clear that if a finite dimensional normed space X is endowed with the sup norm, then it has the **BIP** in the metric sense, that is it is hyperconvex. The converse is not true as we have already seen: $\mathbb{R}^2$ with the sum metric is hyperconvex. However if X is hyperconvex, then X is isomorphic to $\mathbb{R}^n$ in the norm sense, where n is the dimension of X. We will proceed to show that statement.

Let X be a finite dimensional normed space such that X is hyperconvex. The existence of an extreme point in the unit ball of X is assured by the popular Krein-Milman theorem [NB85, KM40, Ral07] or precisely by a weaker version of it that we shall prove here. We will however recall the Krein-Milman theorem since we want to point out a similarity with a result [Her92] obtained for general bounded hyperconvex metric spaces. As one may expect these results rely somehow on the Hahn-Banach theorem. We shall first recall some definitions and properties [NB85, Ral07] from analysis.

Definition 3.1.10. *A subspace M of a vector space X is maximal if M is not contained in any proper subspace of X.*

If $f : X \longrightarrow X$ is a translation and M maximal in X, then $f(M)$ is called a hyperplane and for any subspace N, $f(N)$ is called a linear variety.

As a few consequences of these definitions we have:

Lemma 3.1.3. *(i) M is a maximal subspace of X if and only if there is a nontrivial linear form f such that $M = \ker f$.*

(ii) H is a hyperplane if and only if $H = \{x \in X : f(x) = a\}$ for some linear form f and some scalar a.

Proof. [NB85] We note that a subspace M of X is maximal if and only if the quotient X/M has dimension 1.

(i) Suppose that M is a maximal subspace of X. Let $x \in X \setminus M$. We have $X = M + \mathbb{K}x$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $M \bigcap \mathbb{K} = \{0\}$. If $y \in X$, then $y = m + ax$ where $m \in M$ and $a \in \mathbb{K}$ and they are unique. We define $f(y) = a$ for any $y \in X$. f is linear and $\ker f = M$. Conversely if f is a linear form on X with $M = \ker f$ then $X/M \cong \mathbb{K}$. Thus X/M has dimension 1 and M is a maximal subspace of X.

(ii) Suppose that H is an hyperplane, then $H = x + M$ for some maximal subspace M and some $x \in X \setminus M$. By (i) there is a linear form f on X such that $M = \ker f$. If $a = f(x)$ then $H = \{x \in X : f(x) = a\}$. Conversely if $H = \{x \in X : f(x) = a\}$ where f is a linear form and a a scalar, then $H - x = \ker f$ is a maximal subspace and $H = x + (H - x)$. $\square$

We are now able to formulate the "geometric" form of the Hahn-Banach theorem [2].

Theorem 3.1.11. *(Hahn-Banach) Let X be a real normed space. Given an open convex subset C of X and a linear variety M of X such that $C \bigcap M = \emptyset$, there is a closed hyperplane H such that $M \subseteq H$ and $H \bigcap C = \emptyset$.*

Originally, X is a topological vector space, that is a vector space endowed with a topology and such that the addition and the multiplication by scalar are continuous with respect to the topology. That is, for any open subset O of X there is an open subset U of X such that $U + U \subseteq O$ and there are $\lambda > 0$ and open subset V of X such that $\lambda V \subseteq O$. The morphisms considered in the category of topological vector spaces are continuous linear mappings. The real line is a trivial example of a topological vector space. A normed space is in fact a topological vector space.

We note that a linear form on X is continuous if and only if $\ker f$ is closed [NB85, Ral07]. The necessity condition is obvious since $\ker f = f^{-1}(\{0\})$.

As a consequence of the Hahn-Banach theorem and thanks to the Lemma 3.1.3 for any $x, y \in X$ such that $x \neq y$, there is a continuous linear form f such that $f(x) \neq f(y)$. That is f separates x and y. This will be of interest to us in proving the existence of extreme points in the unit ball of X. Indeed we have:

Proposition 3.1.12. *[NB85] Let X be a normed space. Given a subspace M and a nonempty open and convex subset C such that $C \bigcap M = \emptyset$, there is a continuous linear form f on X such that $f(M) = \{0\}$ and either $f(C) > 0$ or $f(C) < 0$.*

Proof. [NB85] We note that by $f(C) > 0$ or $f(C) < 0$ we mean $f(c) > 0$ or $f(c) < 0$ for any $c \in C$.

There is a closed hyperplane H such that $C \bigcap H = \emptyset$ and $M \subseteq H$. $H = \ker f$ for some linear form f thanks to the Lemma 3.1.3 and f is continuous. Thus $f(M) \subseteq \{0\}$. Since $C \bigcap H = \emptyset$ we have $f(C) < 0$ or $f(C) > 0$. We claim that either $f(C) > 0$ or $f(C) < 0$.

[2]See Notes 4.

Suppose that $f(x) = a > 0$ and $f(y) = b < 0$ for some $x, y \in C$. We can assume without loss of generality that $b < -a$. Let $c = -a/b > 1$. Let us consider the following vector:

$$w = 1/(1+c)x + c/(1+c)y.$$

Since C is convex $w \in C$. But

$$f(w) = 1/(1+c)f(x) + c/(1+c)f(y) = a/(1+c) + cb/(1+c) = ba/(b-a) - ba/(b-a) = 0.$$

Thus we would have $w \in \ker f = H$. It is impossible since $C \bigcap H = \emptyset$. □

In the result of the proposition, it is enough to say that $f(C) > 0$. Indeed if $f(C) < 0$ then we can consider the map $g = -f$. g is a continuous linear form with $\ker g = \ker f$ and $g(C) > 0$.

Again, this result holds for general topological vector spaces [NB85].

Now if M is a closed subspace of X and $x \in X \setminus M$ then there is an open ball C, which is convex such that $x \in C$ and $C \bigcap M \neq \emptyset$. By the preceding proposition, there is a continuous linear form f such that $f(x) \neq 0$ and $f(M) = 0$. If $x, y \in X$ with $x \neq y$, then by taking M as the subset $\{0\}$ which is closed. One can see that $x - y \notin \{0\}$. Then there is continuous linear form f such that $f(x - y) \neq 0$. That is $f(x) \neq f(y)$ [3].

Definition 3.1.13. *[NB85] Let C be a nonempty convex set in a normed space X. A nonempty subset F of C is called a face provided that for any $a \in F$ and for any $x, y \in C \setminus \{a\}$ such that $a \in C_{xy}$ then $x, y \in F$.*

In Figure 3.2 is an example of face F of a convex subset C in $\mathbb{R}^2$

As a direct consequence of this definition we have:

Proposition 3.1.14. *[NB85] Let C be a convex subset of a normed space X and f a linear form on X. If $F = \{x : f(x) = \inf f(C)\}$ or $F = \{x : f(x) = \sup f(C)\}$, then F is a convex face.*

Proof. [NB85] It is sufficient to prove for the supremum. For any $x, y \in F$ and $t \in [0; 1]$:

[3] In a given TVS, it is required for any point x to have a neighborhood base consisting of convex sets and since any singleton need not be closed, the closure of the subset $\{0\}$ may be considered instead of the subset itself.

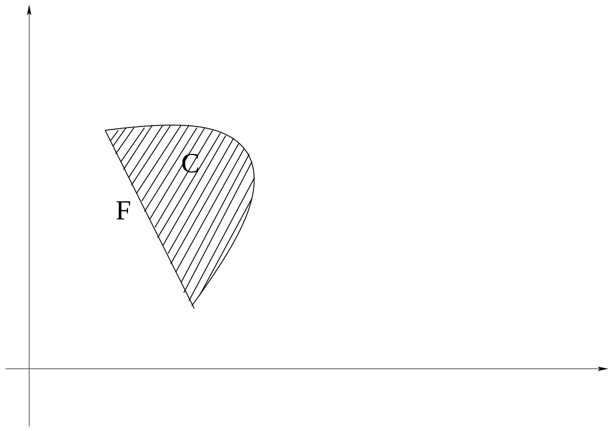

FIG. 3.2. Face in the plane.

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y) = t\sup f(C) + (1-t)\sup f(C) = \sup f(C).$$

So F is convex. Now let $a \in F$ and $x, y \in C \setminus \{a\}$ such that $a \in C_{xy}$. Then $f(C) \subseteq (-\infty; f(a)]$ and for any $t \in [0; 1]$ we have $f(a) = tf(x) + (1-t)f(y)$. Since $f(a)$ is an extreme point of $(-\infty; f(a)]$, $f(x) = f(a)$ or $f(y) = f(a)$, in each case $f(x) = f(a) = f(y)$. Therefore $x, y \in F$. $\square$

We come now to the weaker version of the Krein-Milman theorem.

Theorem 3.1.15. *[NB85] Let C be a compact and convex subset of a normed space X, then C has at least one extreme point.*

Proof. [NB85] We will first show that C has a minimal face F and that F is reduced to one point. It is clear from the definitions that if a point is a face then it is an extreme point.

Let f be a linear continuous form on X. The compactness of C implies that the set $F = f^{-1}(\sup f(C))$ is nonempty. F is closed since f is continuous. By the previous proposition F is a convex face. Hence the collection $\mathcal{F}$ of all convex and closed faces of C is nonempty. We order $\mathcal{F}$ by inclusion and let $\mathcal{C}$ be a chain in $\mathcal{F}$. $\bigcap \mathcal{C}$ is nonempty since C is compact and it is closed. $\bigcap \mathcal{C}$ is a convex face because since every $F \in \mathcal{F}$ is a convex face. Therefore $\bigcap \mathcal{C} \in \mathcal{F}$ and $\mathcal{C}$ is bounded below. By Zorn's lemma, $\mathcal{F}$ has a minimal element say G. If G is reduced to one point then we are done. Suppose that G is not reduced to one point and let $x, y \in G$ with $x \neq y$. Now there is a continuous linear form g such that $g(x) \neq g(y)$. Let $I = g^{-1}(\sup g(G))$. Since $G \in \mathcal{F}$, G is closed and hence compact since $G \subseteq C$, thus $I \neq \emptyset$ and $I \in \mathcal{F}$. Since $g(x) \neq g(y)$, $G \setminus I \neq \emptyset$. This is impossible, therefore G is reduced to a single point [4]. $\square$

Thus if X is a finite dimensional normed space, then its unit ball has an extreme point e since it is convex and compact. We recall that in this particular case compactness is equivalent to boundedness and closedness.

[Nac50] Now, we set up an order relation $\leq$ on X. Thanks to Lemma 3.1.2, we may assume that there is a basis $\{e_1, e_2, \ldots, e_n\} \subseteq X$ such that for any $x = \sum_{i=1}^{n} x_i e_i \in X$:

$$x \geq 0 \text{ if and only if } x_i \geq 0 \text{ for } i = 1, 2, \ldots, n.$$

Where n is the dimension of X. We may also assume that $\|e_i\| = 1$ for $i = 1, 2, \ldots, n$. Otherwise we take the basis $\{e_1/\|e_1\|, e_2/\|e_2\|, \ldots, e_n/\|e_n\|\}$ which does not affect the component character of the positiveness of the relation $\leq$. We claim that $e = \sum_{i=1}^{n} e_i$. From our construction of the order $e > 0$ therefore if $e = \sum_{i=1}^{n} \lambda_i e_i$ for some scalars λ_i, $i = 1, 2, \ldots, n$ then $\lambda_i \geq 0$ for $i = 1, 2, \ldots, n$. We have:

$$e_i \leq \sum_{i=1}^{n} \lambda_i e_i \text{ for } i = 1, 2, \ldots, n.$$

Thus $(\lambda_i - 1)e_i + \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_{i-1} e_{i-1} + \lambda_{i+2} e_{i+2} + \cdots + \lambda_n e_n \geq 0$ for $i = 1, 2, \ldots, n$.

By Lemma 3.1.2, $\lambda_j \geq 0$ for $j \neq i$ and $\lambda_i \geq 1$, therefore $\lambda_i \geq 1$ for $i = 1, 2, \ldots, n$.

Now let $p \in \mathbb{N}$, we have $(1 - 1/p)\lambda_i \leq 1$ for $i = 1, 2, \ldots, n$ otherwise we would have:

$$e_i \leq \sum_{i=1}^{n} (1 - 1/p)\lambda_i e_i = (1 - 1/p)e \text{ for } i = 1, 2, \ldots, n.$$

Since $\|e_i\| = 1$ and $(1 - 1/p) < 1$, this cannot be the case. Thus $(1 - 1/p)\lambda_i \leq 1$ for $i = 1, 2, \ldots, n$. Since p is arbitrary $\lambda_i \leq 1$ for $i = 1, 2, \ldots, n$ and $\lambda_i = 1$ for $i = 1, 2, \ldots, n$. Thus $e = \sum_{i=1}^{n} e_i$.

Let $x = \sum_{i=1}^{n} x_i e_i \in X$. Assume that $-\lambda e \leq x \leq \lambda e$. Then for $i = 1, 2, \ldots, n$:

[4] Again the result holds for a Hausdorff TVS each of its points has a neighborhood base consisting of convex sets [NB85, Ral07].

$$(\lambda - x_i) \geq 0 \text{ and } x_i + \lambda \geq 0.$$

This is the same as writting $-\lambda \leq x_i \leq \lambda$ for $i = 1, 2, \ldots, n$. Conversely if $-\lambda \leq x_i \leq \lambda$ for $i = 1, 2, \ldots, n$, then $-\lambda e_i \leq x_i e_i \leq \lambda e_i$. By summing from 1 to n we have

$$-\lambda e \leq \sum_{i=1}^{n} x_i e_i \leq \lambda e.$$

So $-\lambda e \leq x \leq \lambda e$. But then we have:

$$\|x\| = \inf \{\lambda : -\lambda e \leq x \leq \lambda e\} = \sup \{|x_i| : i = 1, 2, \ldots, n\}.$$

This establishes an isomorphism of normed spaces between X and $\mathbb{R}^n$ with the sup norm. We can say that the balls on X are cubic. The following theorem resumes the result of this section:

Theorem 3.1.16. *[Nac50] Given a finite dimensional normed space X, it is injective if and only if it is isomorphic in the norm sense to $\mathbb{R}^n$ endowed with the sup norm, where n is the dimension of X.*

Now we shall point out an analog of the Krein-Milman theorem in the context of metric spaces. We will first recall the theorem itself.

Definition 3.1.17. *If X is a TVS, then the convex hull of A denoted by $conv(A)$ is the smallest convex set containing A.*

Theorem 3.1.18. *(Krein-Milman) Let X be a Hausdorff TVS such that each of its point admits a neighborhood base consisting of convex sets and let C be a nonempty compact and convex subset of X. Then C is the closure of the convex hull of its extreme points.*

The Theorem 3.1.15 is necessary to prove this theorem. As we have stated earlier, the hyperconvex hull has been defined as an analog to the convex hull. Also the latter theorem has an analog in a general metric space [Dre84, Her92]. However, there is still a slight difference. Indeed, what would be the “extreme points” in a bounded metric space are not involved anymore.

Definition 3.1.19. *[Her92] Let X be a bounded metric space. An element $x \in X$ is an endpoint if there is $y \in X$ such that for any $z \in X$, if $d(y,z) = d(y,x) + d(x,z)$ then $z = x$.*

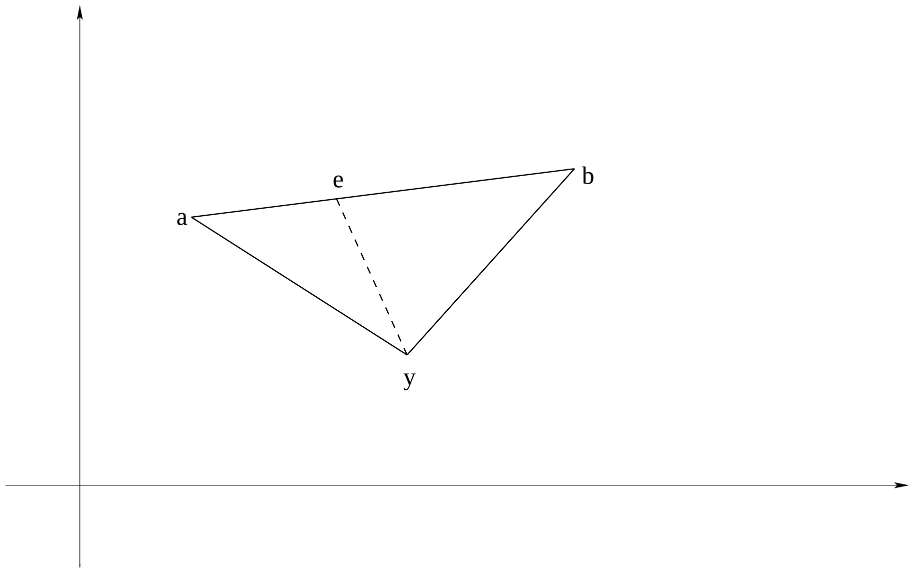

FIG. 3.3. Endpoint in the plane.

If the conditions $d(y,z) = d(y,x) + d(x,z)$ holds for x, y and z in X then we will say that (y,x,z) (or (z,x,y)) is aligned. And generally a family $\{x_i : i \in I\}$ where I is linearly ordered is aligned if $d(x_i, x_k) = d(x_i, x_j) + d(x_j, x_k)$ provided that $i \leq j \leq k$ in I. From the proof of the Theorem 2.4.5 we can conclude that if (x,y,z) and (x,z,u) are both aligned for a subset $\{x,y,z,u\}$ then (x,y,z,u) is aligned.

In fact in a metric space we can define an extreme point of subset A as a point which is not metrically between any two points of A. With that definition, an endpoint is not necessarily an extreme point as we can see in the Figure 3.3:

By considering the usual euclidian metric in the plane $\mathbb{R}^2$ (see Figure 3.3) we see that e lies metrically between a and b but e is an endpoint of the triangle since there is no $z \neq x$ in the triangle such that (y,e,z) is aligned. The set of endpoints are then "face" rather than "extreme points" in a metric space.

This remark motivates the following definition:

Definition 3.1.20. *If A is a bounded subset of a metric space X, then the set of its endpoints is called faces of A and denoted by $\xi(A)$.*

We can now state the result.

Theorem 3.1.21. *[Her92] If X is a bounded hyperconvex space, then $X = \varepsilon(\xi(X))$.*

Proof. [Her92] For any $x_1, x_2 \in X$ we need to find $e_1, e_2 \in X$ such that (e_1, x_1, x_2, e_2) is aligned and e_1, e_2 are endpoints for it will imply:

$$d(x_1, x_2) = d(e_1, e_2) - d(x_1, e_1) - d_(x_2, e_2).$$

And by the definition of tighness of an extension, this will show that the extension $e : \xi(X) \longrightarrow X$ is tight. Since X is hyperconvex, we will have $\varepsilon(\xi(X)) = X$.

Let $x_1 \in X$. For any $x \in X$, let $A(x)$ be the set of all $y \in X$ such that (x_1, x, y) is aligned. We define $s(x) = \sup\{d(x, y) : y \in A(x)\}$. For any $n \in \mathbb{N} \setminus \{1\}$ we choose $x_n \in A(x_{n-1})$ in the following way: we take x^* in the closure of $A(x_{n-1})$ such that:

$$s(x_{n-1}) = d(x_{n-1}, x^*).$$

Given $\delta > 0$ there is $y \in A(x_{n-1})$ such that $d(y, x^*) \leq \delta$. We have

$$d(x_{n-1}, x^*) \leq d(x_{n-1}, y) + d(y, x^*) \leq d(x_{n-1}, y) + \delta.$$

By choosing $\delta < s(x_{n-1})/k$ where $k \geq 2$, we have $s(x_{n-1}) \leq kd(x_{n-1}, y)$. We set y for x_n. Continuing the process we have a sequence $\{x_n : n \in \mathbb{N}\} \subseteq X$ such that if $n < p < m$ in $\mathbb{N}$ then (x_n, x_p, x_m) is aligned. Now for any $n \in \mathbb{N} \setminus \{1\}$:

$$\sum_{i=1}^{n} d(x_i, x_{i+1}) = d(x_2, x_{i+1}) \leq s(x_2).$$

Therefore the series $\sum_{i=1}^{\infty} d(x_i, x_{i+1})$ converges. Since $d(x_n, x_m) \leq \sum_{i=1}^{m} d(x_i, x_{i+1}) - \sum_{i=1}^{n} d(x_i, x_{i+1})$ if $m > n$ in N, the sequence $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence in X. Thus it converges to a point $e_2 \in X$.

Since (x_1, x_n, x_m) is aligned if $n < m$, that is $d(x_1, x_m) = d(x_1, x_n) + d(x_n, x_m)$, by taking the limit on m, we have: $d(x_1, e_2) = d(x_1, x_n) + d(x_n, e_2)$. Therefore $e_2 \in A(x_n)$ for any $n \in \mathbb{N}$, in particular $(x_1, x_2, \ldots, x_n, \ldots, e_2)$ is aligned. We claim that e_2 is an endpoint. Let $z \in X$ such that $z \in A(e_2)$. But then $z \in A(x_n)$ for any $n \in N$ since (x_1, x_n, e_2) is already aligned for any $n \in \mathbb{N}$. Hence

$$d(e_2, z) \leq d(x_n, z) \leq s(x_n) \leq k.d(x_n, x_{n+1}) \text{ for } n \in \mathbb{N} \setminus \{1\}.$$

But then since $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence $e_2 = z$. Thus e_2 is an endpoint of X and clearly (x_1, x_2, e_2) (or (e_2, x_2, x_1)) is aligned. By repeating the process for x_1 and e_2 we get another endpoint e_1 such that (e_1, x_1, e_2) (or (e_2, x_1, e_1)) is aligned. But then (e_1, x_1, x_2, e_2) is aligned. $\square$

We end this section with some observations. We have stated in Chapter 1 that a real normed space X has the extension property if and only if it has the **BIP**. We would like to emphasize some aspects of the normed space $(\mathbb{R}^n, d_\infty)$ where $n \in \mathbb{N}$. This space may be considered as the set of all real-valued continuous functions defined on $\mathbb{N}_n = \{1, 2, \ldots, n\}$ denoted by $\mathcal{C}(\mathbb{N}_n, \mathbb{R})$ endowed with the sup norm. $\mathbb{N}_n$ has some characteristic properties, namely it is compact, Hausdorff and extremally disconnected (meaning the closure of any open subset is open) with the discrete topology which is the most natural topology that we could put on $\mathbb{N}_n$. These properties can be also observed for a general normed space X having the extension property:

[NB85] Indeed, Nachbin [Nac50] and Goodner [Goo50] independently have established the following: if a normed space X has the extension property and if its unit ball has an extreme point, then it can be characterized as follow:

(i) X has the **BIP** and its unit ball has an extreme point.

(ii) $X \cong \mathcal{C}(Y, \mathbb{R})$ for some compact Hausdorff and extremally disconnected space Y.

We have already seen the characterization (i) in Chapter 2 and earlier in this section. However, Kelley [Kel52] as well as Aronszajn and Panitchpakdi [AP56]([NB85]) showed that we can omit the condition of existence of an extreme point. In particular, Kelley showed that:

Theorem 3.1.22. *[Kel52, AHS] A Banach space has the extension property if and only if it is equivalent to the space $\mathcal{C}(X, \mathbb{R})$ of real-valued functions defined on some compact Hausdorff and extremally disconnected space X.*

We note that indeed, Nachbin [Nac50] already conjectured that a normed space that has the **BIP** has an extreme point in its unit ball.

Unlike the previous theorems that we have stated so far, the last one is still valid for complex Banach spaces [Has58] ([NB85]). However, we would like to point out that there is no complex Banach space that is hyperconvex. [Rao93] Indeed let X be a nontrivial complex Banach space. Let $x \in X$ and let us consider the three balls $B(0, \|x\|/2)$, $B(x, \|x\|/2)$ and $B(\lambda x, \|x\|/2)$ where $\lambda = (1 + i\sqrt{3})/2$. The distance between any two centers of these balls is exactly $\|x\| = \|x\|/2 + \|x\|/2$. Since X is metrically convex (since it is convex) and complete, these balls intersect each other. However they have no point in common. Without loss of generality, we can assume that $\|x\| = 1$. By the Hahn-Banach theorem, there is a linear form f such that $f(x) = \|x\| = 1$ and $\|f\| = 1$. Hence if these balls have a common intersection point a, then

$$\|f(a)\| \leq 1/2, \ \|f(a) - 1\| \leq 1/2 \text{ and } \|f(a) - \lambda\| \leq 1/2.$$

Thus $f(a) = 1/2$ from the first two inequalities and we would have $\|1/2 - \lambda\| \leq 1/2$ which is not true. Thus the three balls do not have a point in common.

3.2 Injective real Banach spaces

Since we have already shown that a metric space that has the **BIP** is complete and since a metric space and its completion are closely related in terms of extension property, we will consider Banach spaces instead of normed spaces in this section. We shall recall here what it is for a Banach space X to have the extension property.

A Banach space X has the extension property, and we say that X is injective if for any continous linear mapping $f : Y \longrightarrow X$ and for any linear isometry $e : Y \longrightarrow Z$, where Y and Z are Banach spaces, there is a continuous linear mapping $F : Z \longrightarrow X$ such that $\|f\| = \|F\|$ and such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "e"'] & X \\ Z \arrow[ur, "F"'] & \end{tikzcd}$$

In other words, F extends f. We note that it is sufficient to consider nonexpansive linear mappings. Indeed if f is a continuous linear mapping then $(1/\|f\|)f$ is a nonexpansive linear mapping and if F is the extension of $(1/\|f\|)f$ then $\|f\|F$ will be the extension of f (here we assume that $f \neq 0$). From now on then, we shall consider only nonexpansive linear mappings.

We thus define a linear tight extension as follow: given two Banach spaces X and Y, an extension $e : X \longrightarrow Y$ is tight if for any Banach space Z and a nonexpansive linear mapping $f : Y \longrightarrow Z$, if the composition fe is a linear isometry, then f is necessarily an isometry.

The existence of injective Banach spaces has been answered in the previous section, namely the space $(\mathbb{R}^n, d_\infty)$ is a trivial example. Given a Banach space X, the question whether or not a linear injective envelope $\varepsilon_l(X)$ of X exists is raised, that is if there exists a Banach space Y and a linear extension $e : X \longrightarrow Y$ such that no proper subspace of Y containing $e(X)$ is injective in the norm sense. This problem was suggested by Isbell and Cohen [Coh64] gave a construction of this envelope. Before going further, we would like to make some observation about a possible linear injective envelope of a given Banach space X:

Lemma 3.2.1. *A linear extension $e : X \longrightarrow Y$ between two Banach spaces is a linear injective hull (that is Y is a linear injective envelope of X) if and only if e is tight and Y is injective.*

Proof. Suppose that e is tight and Y is injective. Let Z be a subspace of Y such that $e(X) \subseteq Z$ and Z is injective. Let i be the extension $X \longrightarrow Z$ (which is the composition of the natural injection $e(X) \longrightarrow Z$ and the extension e). Since i is linear and nonexpansive, there is a nonexpansive linear mapping $j : Y \longrightarrow Z$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "e"'] & Z \\ Y \arrow[ur, "j"'] & \end{tikzcd}$$

Since e is tight and $je = i$, j is an isometry. Therefore j is an isomorphism since $Z \subseteq Y$. Conversely suppose that Y is a linear injective envelope of X. Let Z be a Banach space and $f : Y \longrightarrow Z$ a nonexpansive linear mapping such that $fe : X \longrightarrow Z$ is a linear isometry. Since Y is injective, there exists a nonexpansive linear mapping $g : Z \longrightarrow Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "fe"'] & Y \\ Z \arrow[ur, "g"'] & \end{tikzcd}$$

Hence since $gfe = e$, gf is a linear isometry from Y to itself which is then onto. Therefore f is a linear isometry. If not, for some points $x, y \in Y$ we would have:

$$\|f(x-y)\| \; < \; \|x-y\|.$$

Since g is nonexpansive and linear on Z we have:

$$\|g(f(x)) - g(f(y))\| \leq \|f(x) - f(y)\| = \|f(x-y)\| \; < \; \|x-y\|.$$

The last inequality is not true since gf is a linear isometry. Therefore e is tight. $\square$

It is clear from this lemma that a linear injective envelope is unique up to isomorphism (of Banach spaces) and that for a given Banach space X, $\varepsilon_l(X)$ is its maximal linear tight extension.

Now if X is a Banach space we will denote by UX its underlying metric space. The motivation of this section is to find a relation between $\varepsilon(UX)$ and $\varepsilon_l(X)$. We have two injective hulls: $e : UX \longrightarrow \varepsilon(UX)$ and $e' : X \longrightarrow \varepsilon_l(X)$. Since $U\varepsilon_l(X)$ is hyperconvex (since it has the **BIP**), we have a nonexpansive map $\psi : \varepsilon(UX) \longrightarrow U\varepsilon_l(X)$ such that the following diagram commutes:

$$\begin{tikzcd} UX \arrow[r, "e'"] \arrow[d, "e"'] & U\varepsilon_l(X) \\ \varepsilon(UX) \arrow[ur, "\psi"'] & \end{tikzcd}$$

Since e is tight in the metric sense, ψ is an extension in the metric sense. Isbell [Isb64b] has shown that ψ is in fact onto and that $\varepsilon(UX)$ has a structure of normed space [Isb64a]([Rao92]).

The proof of Isbell in [Isb64b] has a small gap [Rao92]. Rao in the paper [Rao92] improved the result by establishing the following:

Theorem 3.2.1. *[Rao92] $\varepsilon(UX)$ has a structure of a normed space such that the extension $e : UX \longrightarrow \varepsilon(UX)$ is a linear extension.*

This theorem will imply that $\varepsilon(UX)$ and $\varepsilon_l(X)$ are isomorphic in the normed sense. Indeed since $\varepsilon_l(X)$ has the extension property, there is a nonexpansive linear mapping $\phi : \varepsilon(UX) \longrightarrow \varepsilon_l(X)$ such that the following diagram commutes:

$$\begin{tikzcd} UX = X \arrow[r, "e'"] \arrow[d, "e"'] & \varepsilon_l(X) \\ \varepsilon(UX) \arrow[ur, "\phi"'] & \end{tikzcd}$$

Since $\phi e = e'$ and e is tight in the metric sense, ϕ is an isometry. Hence ϕ is a linear isometry and e is a linear tight extension. By the definition of the injective hull of a Banach space and its uniqueness $\varepsilon(UX) \cong \varepsilon_l(X)$ in the norm sense.

However the linearity of e and the norm structure on $\varepsilon(UX)$ were not explicit in [Rao92] although Isbell [Isb69] raised the question of a possible explicit construction ([CP97]).

In the paper [CP97], Cianciaruso and De Pascale defined explicitly the algebraic operations on $\varepsilon(UX)$. This section is devoted to these operations and with the tools that we have developed so far, we will eplicitly show that the extension $e : UX \longrightarrow \varepsilon(UX)$ is a linear injective hull of the Banach space X.

We will first begin with an intuitive argument which we will formalize in a theorem. Our development follows that of [CP97].

First of all, since we want e to be linear, we need to know the expression of $e(x + y)$ and $e(\lambda y)$ for any $x, y \in X$ and $\lambda \in \mathbb{R}$. We recall that for any $x \in X$, $e(x) = \|x - .\|$. Let $x_1, x_2 \in X$ and $y \in X$. For any $z \in X$ we have:

$$e(x_1 + x_2)(y) \leq \|x_1 - z\| + \|x_2 - (y - z)\|.$$

And for $z = x_1$ we have equality, therefore:

$$e(x_1 + x_2)(y) = \inf\{d(x_1, z) + d(x_2, y - z) : z \in X\}.$$

Where d is the metric induced by the norm $\|.\|$.

Therefore, if $\oplus$ is the operation of addition in $\varepsilon(UX) \times \varepsilon(UX)$ that we are looking for, then we have [CP97]:

$$[e(x_1) \oplus e(x_2)](y) = \inf\{d(x_1, z) + d(x_2, y - z) : z \in X\}.$$

We know from Chapter 1, Proposition 1.3.1 (iii) that extremal maps are convex on a normed space, hence $e(x_1)$ and $e(x_2)$ are convex and the operation defined above is the infimal convolution of convex functions [Roc70]. Precisely, the infimal convolution f of a finite sequence $\{f_1, f_2, \ldots, f_p\}$ of convex functions defined on a normed space X is defined as follow [Roc70]:

$$f(x) = \inf\{f_1(x_1) + f_2(x_2) + \cdots + f_p(x_p) : x_1 + x_2 + \cdots + x_p = x\}.$$

The function f is usually denoted by $f = f_1 \Box f_2 \Box \ldots \Box f_p$ [Roc70].

Since $x_1 + x_2 \in X$, $e(x_1 + x_2) \in \varepsilon(UX)$, hence $e(x_1) \oplus e(x_2) \in \varepsilon(UX)$.

Now if $x, y \in X$ and $\lambda \in \mathbb{R}$, then:

$$e(\lambda x)(y) = \|\lambda x - y\| = \begin{cases} |\lambda| \|x - y/\lambda\| & \text{if } \lambda \neq 0 \\ \|y\| & \text{if } \lambda = 0 \end{cases}$$

Therefore, if $\circ$ is the scalar multiplication defined on $\mathbb{R} \times \varepsilon(UX)$ then we have:

$$(\lambda \circ e(x))(y) = \begin{cases} |\lambda| d(x, y/\lambda) & \text{if } \lambda \neq 0 \\ \|y\| & \text{if } \lambda = 0 \end{cases}$$

Since $\lambda x \in X$, it is clear that $e(\lambda x) \in \varepsilon(UX)$. So $\lambda \circ e(x) \in \varepsilon(UX)$.

We want now to define these operations on all elements of $\varepsilon(UX)$. Before doing so, we would like to point out that since e must be linear, $e(0)$ has to be the neutral element for $\oplus$. Hence for any $f \in \varepsilon(UX)$:

$$\|f\|_\infty = d_\infty(f, e(0)) = f(0).$$

A straightforward calculation shows that $e(0) \oplus f = f \oplus e(0) = f$ if $f \in \varepsilon(UX)$.

Thanks to the Corollary 2.3.10 in Chapter 2, we will identify x with $e(x)$ for any $x \in X$ and $\tilde{f} = d_\infty(f, .)$ with f on X. Thus for any $f \in \varepsilon(UX)$ and $\lambda \in \mathbb{R}$ we have:

$$(\lambda \circ f)(y) = (\lambda \circ \tilde{f})(e(y)) = |\lambda| d_\infty(f, \lambda^{-1} \circ e(y)) = |\lambda| d_\infty(f, e(\lambda^{-1} y)) = |\lambda| f(y/\lambda).$$

Hence $\lambda \circ f \in \varepsilon(UX)$ and in particular the inverse $\ominus f = -1 \circ f$ of f defined by $\ominus f(x) = f(-x)$ for any $x \in X$ belongs to $\varepsilon(UX)$.

For the addition $\oplus$, we have the following: for any $f, g \in \varepsilon(UX)$ and $y \in X$

$$(f \oplus g)(y) = (\tilde{f} \oplus \tilde{g})(e(y)) = \inf\{d_\infty(f, h) + d_\infty(g, e(y) \ominus h) : h \in \varepsilon(UX)\}.$$

We note that $e(y) \ominus h = e(y) \oplus (\ominus h)$. We need to simplify the above equation and for that we need to exhibit a "meaningful" expression of $e(y) \ominus h$. We will for now assume that $f \oplus g \in \varepsilon(UX)$ whenever $f, g \in \varepsilon(UX)$.

For any $z \in X$ we have:

$$[e(y) \ominus h](z) = \inf\{d_\infty(e(y), k) + d_\infty(\ominus h, e(z) \ominus k) : k \in \varepsilon(UX)\}.$$

For $k = e(y)$ we have:

$$[e(y) \ominus h](z) \le d_\infty(\ominus h, e(z) \ominus e(y)) = d_\infty(\ominus h, e(z - y)).$$

We note that $e(z) \ominus e(y) = e(z) \oplus (-1 \circ e(y)) = e(z - y)$. Since $\ominus h \in \varepsilon(UX)$,

$$d_\infty(\ominus h, e(z - y)) = \ominus h(z - y) = h(y - z).$$

But $h \in \varepsilon(UX)$ and then $h(y - .) \in \varepsilon(UX)$ thanks to the properties of extremal functions on a normed space. Indeed if $f \in \varepsilon(UX)$ then we have seen that $f(-.) \in \varepsilon(UX)$ and by considering the translation $z \mapsto x + z$ we see that $f(z - .) \in \varepsilon(UX)$.

On the other hand, since we have assumed that $e(y) \ominus h \in \varepsilon(UX)$ [5], we have:

$$[e(y) \ominus h](z) = h(y - z).$$

Now, substituting h to f in our main equation, we have:

$$(f \oplus g)(y) \leq d_\infty(g, e(y) \ominus f) = \tilde{g}(e(y) \ominus f).$$

Here assuming again that the operation $\oplus$ is well-defined, $\tilde{g}(. \ominus f)$ would be extremal by the argument that we have shown above and then we have equality in the last equation. We would then have:

$$(f \oplus g)(y) = d_\infty(e(y) \ominus f, g) = \sup\{(e(y) \ominus f)(z) - g(z) : z \in X\} = \sup\{f(y-z) - g(z) : z \in X\}$$

We note that if $f = e(x_1)$ and if $g = e(x_2)$ for some $x_1, x_2 \in X$, then the above expression is reduced to the following one:

$$[e(x_1) \oplus e(x_2)](y) = \inf\{e(x_1)(y - z) + e(x_2)(z) : z \in X\}.$$

Indeed, for any $z \in X$, we have;

$$\|x_1 - (y - z)\| - \|x_2 - z\| \leq \|x_1 - (y - z)\| + \|x_2 - z\|.$$

And for $z = x_2$, we have equality, therefore:

$$\sup\{\|x_1 - (y - z)\| - \|x_2 - z\| : z \in X\} = \inf\{\|x_1 - (y - z)\| + \|x_2 - z\| : z \in X\}.$$

[5]We note also that $\tilde{g} = d_\infty(g, .)$ which has $e(y) \ominus h$ as argument is defined on $\varepsilon(UX)$

The assertion that we have assumed to be true so far is that $\oplus$ is well-defined. We come now to the theorem that formalizes what we have written so far.

Theorem 3.2.2. *[CP97]*

The metric space $\varepsilon(UX)$ endowed with the norm $\|.\|$ defined by:

$$\|f\| = f(0) \textit{ for any } f \in \varepsilon(UX).$$

and with the operations:

$$(f \oplus g)(x) = \sup\{f(x-w) - g(w) : w \in X\} \textit{ for any } f, g \in \varepsilon(UX) \textit{ and } x \in X.$$

$$(\lambda \circ f)(x) = \begin{cases} |\lambda| f(x/\lambda) & \textit{if } \lambda \neq 0 \\ \|x\| & \textit{if } \lambda = 0 \end{cases} \quad \textit{for any } \lambda \in \mathbb{R}, f \in \varepsilon(UX) \textit{ and } x \in X.$$

is a real Banach space. Furthermore, the extension $e : UX \longrightarrow \varepsilon(UX)$ is a linear injective hull.

Proof. [CP97]We have already shown that the operation of multiplication by a scalar is well-defined. The development of the proof is as follow: we will first prove that if $f, g \in \varepsilon(UX)$ then $f \oplus g \in \sigma(UX)$ ($f \oplus g$ is superadditive). In the second step, we will simultaneously show that $\oplus$ is associative and that $f \oplus g \in \varepsilon(UX)$. Finally, we shall verify some axioms for a vector space, the remaining ones being obvious.

1. Let $g \in \varepsilon(UX)$. We consider the function ϕ defined as follow:

$$\phi(x) = \sup\{g(x+w) - g(w) : w \in X\} \text{ for any } x \in X.$$

As we have already shown in Chapter 1, $g(.+w) - g(w)$ is convex. Hence ϕ is convex [Roc70]. Furthermore, if $x, y \in X$, then:

$$\phi(x+y) \leq \sup\{g(x+y+w) - g(y+w) : w \in X\} + \sup\{g(y+w) - g(w) : w \in X\} = \phi(x) + \phi(y).$$

So ϕ is superadditive. On the other hand $\phi(0) = 0$. We claim that $\phi(px) = p\phi(x)$ for any $p > 0$ and $x \in X$ (that is ϕ is homogenuous). Let $a \in \mathbb{R}$ such that $0 < a < 1$. We have:

$$\phi(ax + (1-a)0) \leq a\phi(z) + (1-a)\phi(0) = a\phi(x).$$

Now for any $p > 0$, there is $n \in \mathbb{N}$ such that $1.n > p$. Hence for any $x \in X$:

$$\phi(px) = \phi(n.(p/n)x) \leq n\phi((p/n)x) \leq n.(p/n)\phi(x) = p\phi(x).$$

And then this is an equality since:

$$\phi(x) = \phi((1/p).px) \leq (1/p)\phi(px) \leq (1/p).p\phi(x) = \phi(x).$$

Since $g \in \varepsilon(UX) \subseteq \sigma(UX)$ we have for $w = 0$ and for any $x \in X$:

$$\phi(x) \geq g(x) - g(0) = g(x) + g(0) - 2g(0) \geq \|x\| - 2g(0).$$

Thus $\phi(x/p) \geq \|x/p\| - 2g(0)$ and $\phi(x) \geq \|x\| - 2pg(0)$ for any $p > 0$. Hence $\phi(x) \geq \|x\|$. And since g is extremal, $g(x+w) - g(w) \leq \|x\|$ for any $w \in X$. Therefore $\phi(x) \leq \|x\|$ and these imply that $\phi(x) = \|x\|$ for any $x \in X$.

Let us denote by $\psi(x)$ the quantity $\sup\{f(x-w) - g(w) : s \in X\}$. For any $w, w' \in X$ and for any $x, y \in X$:

$$\psi(x) + \psi(y) \geq (f(x-w) - g(w)) + (f(y-w') - g(w')) \geq \|x - y + w' - w\| - g(w) - g(w').$$

This is true since $f \in \varepsilon(UX) \subseteq \sigma(UX)$. But

$$\sup\{\|x - y + w' - w\| - g(w) : w \in X\} = g(x - y + w').$$

hence $\psi(x) + \psi(y) \geq g(x - y + w') - g(w')$. By considering the supremum with respect to w' we have:

$$\psi(x) + \psi(y) \geq \phi(x - y) = \|x - y\|.$$

Therefore, $f \oplus g \in \sigma(UX)$.

2. To show that $\oplus$ is associative, we will first assume that $f \oplus g \in \varepsilon(UX)$ whenever $f, g \in \varepsilon(UX)$. Let f, g and h be elements of $\varepsilon(UX)$ and $x \in X$. We have the folowing:

$$\begin{aligned}[(f \oplus g) \oplus h](x) &= \sup\{(f \oplus g)(x - w) - h(w) : w \in X\} \\ &= \sup\{\sup\{g(x - w - u) - f(u) : u \in X\} - h(w) : w \in X\}\end{aligned}$$

On the other hand, since $\oplus$ is commutative (from Chapter 2, Lemma 2.3.2):

$$\begin{aligned}[f \oplus (g \oplus h)](x) = [(g \oplus h) \oplus f](x) &= \sup\{(g \oplus h)(x-w) - f(w) : w \in X\} \\ &= \sup\{\sup\{g(x-w-u) - h(u) : u \in X\} - f(w) : w \in X\}\end{aligned}$$

Therefore, we have equality since (this is also valid for the first expression):

$$\sup\{\sup\{g(x-w-u)-h(u) : u \in X\}-f(w) : w \in X\} = \sup\{g(x-w-u)-h(u)-f(w) : u, w \in X\}.$$

Now, let $h \in \varepsilon(UX)$ such that $h \leq g \oplus f$. We have;

$$\begin{aligned}(h \ominus g)(x) &= \sup\{h(x-u) - g(-u) : u \in X\} \\ &= \sup\{h(x+w) - g(w) : w \in X\} \\ &\leq \sup\{(f \oplus g)(x+w) - g(w) : w \in X\}\end{aligned}$$

But $(f \oplus g)(x+w) = \sup\{g(x+w-u) - f(u) : u \in X\}$ hence

$$(h \oplus g)(x) \leq \sup\{\sup\{g(x+w-u) - g(w) : w \in X\} - f(u) : u \in X\}.$$

Therefore

$$(h \oplus g)(x) \leq \{\|x-u\| - f(u) : u \in X\} = f(x).$$

We have shown that $h \ominus g \in \sigma(UX)$ in the first part of the proof. Since f is extremal we have $(h \ominus g)(x) = f(x)$ so $h \ominus g = f$. But then $h \ominus g \in \varepsilon(UX)$. On the other hand $(\ominus g \oplus g) = e(0) \in \varepsilon(UX)$. Thus we have:

$$f \oplus g = (h \ominus g) \oplus g = h \oplus (\ominus g \oplus g) = h \oplus e(0) = h.$$

Therefore $f \oplus g \in \varepsilon(UX)$ and $\oplus$ is associative.

3. We will now check three properties of vector space namely if $a, b \in \mathbb{R} \setminus \{0\}$ and if $f, g \in \varepsilon(UX)$ then:

(i) $(a+b)\circ f = (a\circ f)\oplus(b\circ f)$;

(ii) $a\circ(b\circ f) = (ab)\circ f$;

(iii) $a\circ(f\oplus g) = (a\circ f)\oplus(a\circ g)$.

Let us begin with (ii). Let $x\in X$:

$$[a\circ(b\circ f)](x) = |a|(b\circ f(x/a)) = |a|(|b|f(x/ab)) = |ab|f(x/ab) = [(ab)\circ f](x).$$

(iii) For any $x\in X$:

$$\begin{aligned}[a\circ(f\oplus g)](x) &= |a|(f\oplus g)(x/b)\\ &= |a|\sup\{f(x/a-u)-g(u) : u\in X\}\\ &= |a|\sup\{f((x-w)/a)-g(w/a) : w/a\in X\}\\ &= \sup\{|a|f((x-w)/a)-|a|g(w/a) : w\in X\}\\ &= [(a\circ f)\oplus(a\circ g)](x)\end{aligned}$$

(i) Suppose that $ab<0$. Without loss of generality, we can assume that $b<0$ and $|b|<a$. Indeed, if $a<0$ and $|a|>b>0$, then $-a>0$ and $|-b|<a$ and by the previous calculations $[-1(-a-b)]\circ f = (a+b)\circ f$. Furthermore, if the equation holds for the assumption, then by (ii):

$$[-(-a-b)\circ f] = [-1\circ((-a-b)\circ f)] = [-1\circ(-a\circ f)]\oplus[-1\circ(-b\circ f)] = (a\circ f)\oplus(b\circ f).$$

Now we have $0<-b/a<1$. Since $f\in\varepsilon(UX)$, f is convex, hence:

$$\begin{aligned}f((x-u)/a)) &= f[\left(\frac{a+b}{a}\right)\frac{x}{a+b} - \frac{b}{a}\frac{u}{b}]\\ &\le \frac{a+b}{a}f\left(\frac{x}{a+b}\right) - \frac{b}{a}f\left(\frac{u}{b}\right).\end{aligned}$$

hence

$$af((x-u)/a) - |b|f(u/b) \le (a+b)f(x/a+b)).$$

By taking the supremum with respect to u at the left, we have $[(a \circ f) \oplus (b \circ f)](x) \leq [(a+b) \circ f](x)$. Since both of them are extremal, we have equality. If $ab > 0$, then:

$$
\begin{aligned}
a \circ f = (a+b-b) \circ f &= [(a+b) \circ f] \oplus [-b \circ g] \\
&= [(a+b) \circ f] \oplus [-1 \circ (b \circ g)] \\
&= [(a+b) \circ f] \ominus (b \circ g)
\end{aligned}
$$

Hence $(a \circ f) \oplus (b \circ f) = (a+b) \circ f$.

Since $\varepsilon(UX)$ has the **BIP**, it is clear that $\varepsilon(UX)$ is an injective Banach space. As we have shown in the beginning of this section, the extension $e : UX \longrightarrow \varepsilon(UX)$ is also tight in the linear sense. □

Therefore for a real normed space its hyperconvex hull and its linear injective hull coincide. We end this chapter with some concrete examples.

3.3 Concrete examples

In this section, we will cite some interesting and concrete examples of hyperconvex spaces and hyperconvex hulls. For that purpose and because of the fact that some of the cited examples are already well-developed by their authors we are not going into much details as in previous sections and chapters.

1. [KK01] Consider the space $l_\infty(\mathbb{R})$ of bounded and real-valued sequences endowed with the sup norm d_∞ and generally the space $\mathcal{B}([0;1], \mathbb{R})$ of bounded real-valued functions on $[0;1]$ again with the sup norm. Then both of these spaces are hyperconvex.

If $\mathcal{C}_0$ is the set of sequences which converge to $0 \in \mathbb{R}$, then $\mathcal{C}_0 \subseteq l_\infty(\mathbb{R})$ and $\varepsilon(\mathcal{C}_0) = l_\infty$. Indeed each point in $l_\infty(\mathbb{R})$ can be expressed as an intersection of balls centered at some points in $\mathcal{C}_0$ [Her92].

[Her92]If d is the sum metric on $\mathbb{R}^n$ then we have $\varepsilon(\mathbb{R}^n, d) = (\mathbb{R}^{2^{n-1}}, d_\infty)$. If $n = 2$, we see that we have an isomorphism.

2. We have already seen that the real line $\mathbb{R}$ is hyperconvex. All the subsets of $\mathbb{R}$ which are hyperconvex are of the form $[a;b]$, $[a;+\infty)$ and $(-\infty;a]$, where $a, b \in \mathbb{R}$ with $a < b$. Indeed if a subset A of $\mathbb{R}$ is hyperconvex, then A has to be convex, since any two points of A must be joined by a metric segment, and complete. Thus if $X \subseteq \mathbb{R}$, then $\varepsilon(X)$ coincides with the intersection of all intervals (bounded or not) that contain X. Hence if X is unbounded, then $\varepsilon(X) = \mathbb{R}$. In particular $\varepsilon(\mathbb{Q}) = \varepsilon(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$. This shows that even if $\varepsilon(X) = \varepsilon(Y)$ for two given metric spaces X and Y, then we do not have any specific relation between the structures of X and Y.

We note also that the hyperconvex hull does not behave in the same way as an algebraic closure operator does. Consider for example two points a and b in the real line. We have $\varepsilon(\{a\}) = \{a\}$ and $\varepsilon(\{b\}) = \{b\}$ but $\varepsilon(\{a, b\}) = [a; b]$. Hence $\varepsilon(\{a\}) \bigcup \varepsilon(\{b\}) \neq \varepsilon(\{a, b\})$.

We have also the following observation:

Proposition 3.3.1. *Let f be a k-Lipschitz function defined on a bounded interval $[a;b]$ in $\mathbb{R}$. If $|f(a)-f(b)| = k|b-a|$, then there is $\beta \in \mathbb{R}$ such that for any $x \in [a;b]$, $f(x) = kx+\beta$.*

Proof. Let us consider the function $g = (1/k)f$. g is nonexpansive and the extension $e : \{a, b\} \longrightarrow [a; b]$ is tight since $\varepsilon(\{a, b\}) = [a; b]$. Since ge is an isometry by hypothesis, g is an isometry. Hence there is $\alpha \in \mathbb{R}$ such that for any $x \in [a, b]$, $g(x) = x + \alpha$. □

An intuitive proof of that proposition is the fact that the derivative of g should never exceed 1. The hypothesis ensures that it never goes below 1.

Hyperconvex hulls of finite metric spaces are developed in [Dre84]. Dress provided an algorithm to determine the hyperconvex hulls of such spaces, at least for those whose cardinals are less that 8. We will describe here the shape of such hyperconvex hulls for the case where the cardinal of the space is 3.

[Dre84] Let $X = \{a, b, c\}$. Let $\theta = (a\ b\ c)$ be a permutation of X [6]. Let $f_0 : X \longrightarrow \mathbb{R}$ be a function defined by:

[6] We use here the notation used in group theory, this small detail about permutation is not used in [Dre84] .

$$f_0(x) = [d(x, \theta(x)) + d(x, \theta^{-1}(x)) - d(\theta(x), \theta^{-1}(x))]/2 \text{ for any } x \in X.$$

Then f_0 is the unique element of $\varepsilon(X)$ such that $f(x) + f(y) = d(x, y)$ for any $x, y \in X$. If $H_x = \{f \in \varepsilon(X) : f(x) < f_0(x)\}$ for $x \in X$ then $\varepsilon(X) = H_a \bigcup H_b \bigcup H_c$ where the union is disjoint. Furthermore, for any $x, y \in X$ with $x \neq y$, the function ψ from $H_x \bigcup \{f_0\} \bigcup H_y$ to $[0; d(x, y)]$ defined by $\psi(f) = f(x)$ is an isomorphism (of metric spaces) and that $H_x \bigcup \{f_0\} \cong [e(x); f_0]$. The shape of $\varepsilon(X)$ is then as shown in the Figure 3.4.

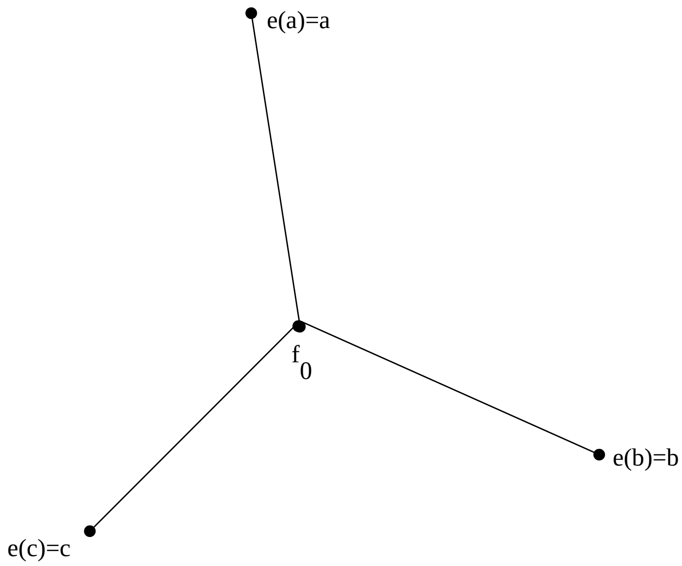

FIG. 3.4. Hyperconvex hull of a 3 points space.

We note that the distance between two points is the length of the path between them. We also recall that for any $x \in X$: $d(f_0, x) \cong d_\infty(f_0, e(x)) = f_0(x)$.

3. Finally we end this chapter with some observations related to the Euclidian space $\mathbb{R}^n$, in particular we will discuss the Brouwer's fixed point theorem. This theorem is linked to the fact that in $\mathbb{R}^n$ with the usual topology, there is no retraction from a nontrivial ball (closed) B to its boundary [ES92, KK01]. Since a retract of a hyperconvex metric space is hyperconvex (Proposition 2.3.3), we will use that fact to prove the latter statement.

We start with Brouwer's theorem:

Theorem 3.3.2. *[KK01] Let $f : B \longrightarrow B$ be a continuous mapping where B is a closed ball in $\mathbb{R}^n$, $n \in \mathbb{N}$. Then f has at least one fixed point.*

Equivalent to this theorem is the following one [KK01]:

Theorem 3.3.3. *[KK01, ES92] Let B be a closed ball in $\mathbb{R}^n$, $n \in \mathbb{N}$. Then there is no retraction from B to its boundary sphere S.*

We will only discuss the sufficient condition: suppose that there exists a continuous mapping f from B onto itself such that f does not have a fixed point. Without loss of generality we can assume that the metric on $\mathbb{R}^n$ is the Euclidian metric or the usual metric. [ES92, KK01] We consider the half-line L starting from $f(x)$ and passing through x. Let $s(x)$ be the point $L \bigcap S$. Then the map $x \mapsto s(x)$ is a retraction from B to S. Since there is no such retraction, f must have a fixed point.

We will now give a proof of the last theorem, we may consider the sup metric d_∞ on $\mathbb{R}^n$. Without loss of generality we may also assume that B is the unit ball since any two balls are the same up to translation and multiplication by positive scalars.

Consider the family $\mathcal{B} = \{B_s : s \in S\}$ of balls such that each member B_s of $\mathcal{B}$ has radius 1 and centered at a point s in S. It is easy to see that for any $s \in S$, $0 = (0, 0, \ldots, 0) \in B_s$. For any $s, t \in S$ we have:

$$d_\infty(s,t) \leq d_\infty(s,0) + d_\infty(0,t) = 1 + 1.$$

Now the ball B_1 with center $1 = (1, 1, \ldots, 1)$ and the ball B_{-1} with center $-1 = (-1, -1, \ldots, -1)$ have only 0 as intersection. Indeed if $a = (a_1, a_2, \ldots, a_n) \in B_1 \bigcap B_{-1}$ then:

$$|a_i - 1| \leq 1 \text{ and } |a_i + 1| \leq 1 \text{ for } i = 1, 2, \ldots, n.$$

Hence $a_i = 0$ for $i = 1, 2, \ldots, n$ since $a_i \in [-1; 0] \bigcap [0; 1]$ for $i = 1, 2, \ldots, n$.

Thus $\bigcap \mathcal{B} = \{0\}$ and $(\bigcap \mathcal{B}) \bigcap S = \emptyset$. So S is not hyperconvex. But from Chapter 2, we know that every admissible subset (subset that can be expressed as intersection of balls), hence a ball, of a hyperconvex space is hyperconvex and that the retract of a hyperconvex space is hyperconvex. Therefore a retraction from B to S could not exist.

3.4 Notes

1. If we consider the Lemma 3.1.1 and if we consider the interpolatory property as a type of convexity property, then we see that the conditions of the lemma have some similarity with our first definition of hyperconvexity.

2. This way of generating a norm or a seminorm from a particular subset of a vector space is well-known in functional analysis. Also the Minkowski functional (or gauge) [NB85, Ral07] is generated in a similar way: let B be a subset of a topological vector space X having the following properties:

(i) $aX \subseteq B$ whenever $|a| \leq 1$;

(ii) B is convex;

(iii) For any $x \in X$, there is $r > 0$ such that $ax \in B$ whenever $|a| \leq r$.

Now, the function $g : X \longrightarrow \mathbb{R}_+$ defined by $g(x) = \inf\{a > 0 : x \in aB\}$ is a seminorm on X. We note that (iii) is equivalent to the following: for any $x \in X$, there is $a > 0$ such that $x \in aX$ whenever $|a| \geq r$.

3. In the proof of the Proposition 3.1.7, we were more explicit with the properties of the order relation R, namely the antisymmetry and reflexivity were clearly treated. For the second half of the proof, the maps f and g were not used as explicitly as we did by Nachbin [Nac50]. Since these maps are composition of translations and multiplication by positive scalars, we could use the fact that R is preserved under such operations, thus reducing largely the length of the proof.

We note that the set P is called *cone* in functional analysis. The letter P was chosen for it is the set of positive elements.

4. [NB85] This result was first proved by Mazur in 1933 for normed spaces (Normed spaces have just been defined independently by Hahn and Banach respectively in 1922 and 1923). It was the group Bourbaki who then called this theorem the "geometric form of the Hahn-Banach theorem". We also note that there are versions of the Hahn-Banach theorem that do not rely on the Axiom of Choice, see for example [Bri79]([NB85]).

5. We gave more details in the proof of the Theorem 3.1.21, in particular in the construction of the sequence $\{x_n : n \in \mathbb{N}\}$ which where skipped in [Her92].

Similar results have been obtained by Dress [Dre84] and Isbell [Isb64b] for compact hyperconvex spaces.

6. In the proof of the Theorem 3.2.2, we gave a detailed proof of the homogeneity of the function ϕ in the first step. We have also added the verification of some axioms of a vector space, namely (ii) and (iii) and we have used these to explicitly reduce the axiom (i) to only one case.

Every detail concerning the properties of extremal functions has been already proven in previous chapters for different contexts.

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