Aspects of Some Exotic Options

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Declaration

I, the undersigned, hereby declare that the work contained in this assignment is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

Signature: ________________

Date: ________________

Summary

The use of options on various stock markets over the world has introduced a unique opportunity for investors to hedge, speculate, create synthetic financial instruments and reduce funding and other costs in their trading strategies.

The power of options lies in their versatility. They enable an investor to adapt or adjust her position according to any situation that arises. Another benefit of using options is that they provide leverage. Since options cost less than stock, they provide a high-leverage approach to trading that can significantly limit the overall risk of a trade, or provide additional income. This versatility and leverage, however, come at a price. Options are complex securities and can be extremely risky.

In this document several aspects of trading and valuing some exotic options are investigated. The aim is to give insight into their uses and the risks involved in their trading. Two volatility-dependent derivatives, namely compound and chooser options; two path-dependent derivatives, namely barrier and Asian options; and lastly binary options, are discussed in detail.

The purpose of this study is to provide a reference that contains both the mathematical derivations and detail in valuating these exotic options, as well as an overview of their applicability and use for students and other interested parties.

Opsomming

Die gebruik van opsies in verskeie aandelemarkte reg oor die wêreld bied aan beleggers ‘n unieke geleentheid om te verskans, te spekuleer, sintetiese finansiële produkte te skep, en befondsing en ander kostes in hul verhandelstrategieë te verminder.

Die mag van opsies lê in hul veelsydigheid. Opsies stel ‘n belegger in staat om haar posisie op enige manier aan te pas of te manipuleer soos die situasie verander. Nog ‘n voordeel van die gebruik van opsies is dat hulle hefboomkrag verskaf. Aangesien opsies minder kos as aandele bied hulle ‘n hoë-hefboomkrag benadering tot verhandeling, wat die algehele risiko van ‘n verhandeling aansienlik kan beperk of addisionele inkomste kan verskaf. Hierdie veelsydigheid en hefboomkrag kom egter teen ‘n prys. Opsies is komplekse instrumente wat uiters riskant kan wees.

In hierdie werkstuk word verskeie aspekte van die verhandeling en prysing van ‘n aantal eksotiese opsies ondersoek. Die doel is om insig te bied in die gebruik van opsies en die risikos verbonde aan die verhandeling daarvan. Twee-volatiliteit afhanklike afgeleide instrumente, te wete saamgestelde- en keuse opsies; twee pad-afhanklike instrumente, te wete sper- en Asiatiese opsies; en laastens binêre opsies, word in diepte bespreek.

Die doel van hierdie studie is om ‘n dokument te verskaf wat beide die wiskundige afleidings en detail van die prysing van bogenoemde eksotiese opsies bevat, sowel as om ‘n oorsig van hul toepaslikheid en nut, aan studente en ander belangstellendes te bied.

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Contents

1.

Introduction and Overview

1.1 Introduction

The use of options on various stock markets over the world has introduced a unique opportunity for investors to hedge, speculate, create synthetic financial instruments and reduce funding and other costs in their trading strategies. As explained on www.Investopedia.com the power of options lies in their versatility. They enable an investor to adapt or adjust her position according to any situation that arises. Options can be as speculative or as conservative as preferred. This means everything from protecting a position from a decline, to outright betting on the movement of a market or index, can be implemented. Another benefit of using options is that they provide leverage. www.yahoo.com argues that since options cost less than stock they provide a high-leverage approach to trading, which can significantly limit the overall risk of a trade, or provide additional income. When a large number of shares is controlled by one contract, it does not take much of a price movement to generate large profits.

This versatility and leverage, however, does come at a price. Options are complex securities and can be extremely risky. This is why, when trading options, a disclaimer like the following is common:

"Options involve risks and are not suitable for everyone. Option trading can be speculative in nature and carry substantial risk of loss. Only invest with risk capital."

Being ignorant of any type of investment places an investor in a weak position. In this document several aspects of trading and valuing exotic options will be investigated. The aim is to give insight into their uses and the risks involved in their trading. Two volatility-dependent derivatives, namely compound and chooser options; two path-dependent derivatives, namely barrier and Asian options; and lastly binary options, are discussed in detail.

The purpose of this study is to provide a reference that contains both the mathematical derivations and detail in valuating some exotic options as well as an overview of their applicability and use for students and other interested parties.

1.2 Overview

The rest of this document consists of five chapters. In Chapter 2, a summary of well-known results on standard options is provided. This is included as background for chapters 3, 4 and 5, and for the sake of completeness. The material given there is expanded where necessary for each exotic option discussed in the subsequent chapters.

Chapter 3 explores the value of volatility-dependent derivatives. These derivatives depend in an important way on the level of future volatility. Two of the most common forms, compound and chooser options, are described.

The focus then turns to pricing certain path-dependent derivatives in Chapter 4. Two of the most common types of path-dependent options, barrier and Asian options, are described. These have in common the fact that the payoff of each is determined by the complete path taken by the underlying price, rather than its final value only.

Finally, in Chapter 5, it is illustrated that binary options are options with discontinuous payoffs. Three forms of this type of option are discussed, namely cash-or-nothing binary options, asset-or-nothing binary options and American-style cash-or-nothing binary options.

For each exotic option the option is first defined, before an overview of its applicability and use is given and compared to standard options. The option valuation is then derived in detail. This is followed by a discussion on notable aspects of that option. Where applicable, the arbitrage bounds on valuation of the options are given. These are the limits within which the price of an option should stay, because outside these bounds a risk-free arbitrage would be possible. They constrain an option price to a limited range and do not require any assumptions about whether the asset price is normally or otherwise distributed. Lastly, the sensitivities or Greeks of the options are given. Each Greek letter measures a different dimension of the risk in an option position, and the aim of a trader is to manage the Greeks so that all risks are acceptable.

1.3 Glossary of Notation

c	Price of an European call option
C	Price of an American call option
d	The size of the downward movement of the underlying asset in a binomial tree
D	Cash dividend
E	Expectation operator
H	Rand knock-out or knock-in barrier (only used for barrier options)
K	Predetermined cash payoff or strike price
L	Lower barrier in a barrier option
M	Current Rand minimum or maximum price of the underlying asset experienced so far during the life of an option (only needed for barrier options)
n	Number of steps in a binomial tree
N(.)	Area under the standard normal distribution function
N_2(.)	Area under the standard bivariate normal distribution function
p	Price of an European put option
	Up probability in tree model
P	Price of an American put option
PV_t()	Present Value at time t of the quantity in brackets
r	Risk-free interest rate
S	Current price of underlying asset
S_T	Price of an underlying asset at the expiration time of an option
T	Time to expiration of an option in number of years
T*	Time to expiration of the underlying option in number of years (only needed for compound options)
X	Predetermined payoff from an a all-or-nothing option
u	The size of the up movement of the underlying asset in a binomial tree
U	Upper barrier in barrier option
μ	Drift of underlying asset
π	The constant Pi≈3.14159265357

ρ Correlation coefficient

σ Volatility of the relative price change of the underlying asset

2.

Valuation of Standard Options

In this chapter well-known results on standard options are summarized; it is concerned with the theory of option pricing and its application to stock options. It is included as background for chapters 3, 4 and 5, and for the sake of completeness. The results are essential to an understanding of the later chapters on exotic options. The material given here is expanded where necessary for each exotic option that is discussed in the later chapters.

2.1 Standard Options

2.1.1 What Are Options?

An option is a contract that gives the buyer the right, but not the obligation, to buy (call option) or sell (put option) an underlying asset at a specific price (strike price) on or before a certain date (expiration date). The party selling the contract (writer) has an obligation to honour the terms of the agreement and is therefore paid a premium. The buyer has a 'long' position, and the seller a 'short' position.

The underlying asset is usually a bond, stock, commodity, interest rate, index or exchange rate. Throughout this paper a reference to one of these underlying assets is also a reference to any of the others, and the terms are therefore used interchangeably. Because this is a contract, the value of which is derived from an underlying asset and other variables, it is classified as a derivative. It is also a binding contract with strictly defined terms and properties.

Once an investor owns an option, there are three methods that can be used to make a profit or avoid a loss; exercise it, offset it with another option, or let it expire worthlessly. By exercising an option she has bought, an investor is choosing to take delivery of (call) or to sell (put) the underlying asset at the option's strike price. Only option buyers have the choice to exercise an option. Option sellers have to honour the agreement if the options they sold are exercised by the option holders.

Offsetting is a method of reversing the original transaction to exit the trade. This means that an investor holds two option positions with exactly opposite payoffs, leaving her in a risk-neutral position. If she bought a call, she would have to sell the call with the same strike price and expiration date. If she sold a call, she would have to buy a call with the same strike price and expiration date. If she bought a put, she would have to sell a put with the same strike price and expiration date. If she sold a put, she would have to buy a put with the same strike price and expiration date. If an investor does not offset her position, she has not officially exited the trade.

If an option has not been offset or exercised by expiration, the option expires worthlessly. The option buyer then loses the premium she paid to invest in the option. If the investor is the seller of an option she would want it to expire worthlessly, because then she gets to keep the option premium she received. Since an option seller wants an option to expire worthlessly, the passage of time is an option seller's friend and an option buyer's enemy. If the investor bought an option the premium is non-refundable, even if she lets the option expire worthlessly. As an option gets closer to expiration, it decreases in value.

The style of the option determines when the buyer may exercise the option. Generall,y the contract will either be American style, European style or Bermudan style. American style options can be exercised at any point in time, up to the expiration date. European style options can only be exercise on the expiration date. Bermudan style options may be exercised on several specific dates up to the expiration date. It is interesting to note that Bermuda lies halfway between America and Europe.

2.1.2 Types of Options

- ♦ A **call** option is a contract that gives the buyer the right, but not the obligation, to buy an underlying asset at a specific price on or before a certain date.
- If a call option is exercised at some future time, the payoff will be the amount by which the underlying asset price exceeds the strike price.

- It is only worth exercising the option if the current market price of the underlying asset is greater than the strike price.
- Breakeven point for exercising a call option equals the strike price plus a premium.
- The value of the option to the buyer of a call will increase as the underlying asset price increases within the expiration period.

♦ A **put** option is a contract that gives the buyer the right, but not the obligation, to sell an underlying asset at a specific price on or before a certain date.

- If a put option is exercised at some future time, the payoff will be the amount by which the strike price exceeds the underlying asset price.
- Call writers keep the full premium, unless the underlying asset price rises above the strike price.
- Breakeven point is the strike price plus a premium.
- The value of the option to the buyer of a put will increase as the underlying asset price decreases within the expiration period.

2.1.3 Participants in the Options Market

The two types of options lead to four possible types of positions in options markets:

1. Buyers of calls : long call position
2. Sellers of calls : short call position
3. Buyers of puts : long put position
4. Sellers of puts : short put position

These trades can be used directly for speculation. If they are combined with other positions they can also be used in hedging.

2.1.4 Valuation

The total cost of an option is called the option premium. This price for an option contract is ultimately determined by supply and demand, but is influenced by five principal factors:

- The current price of the underlying security (S).
- The strike price (K).
    - The intrinsic value element of the option premium is the value that the buyer can get from exercising the option immediately. For a call option this is $\max(S-K,0)$, and for a put option $\max(K-S,0)$. This means that for call options, the option is in-the-money if the share price is above the strike price. A put option is in-the-money when the share price is below the strike price. The amount by which an option is in-the-money is its intrinsic value. Options at-the-money or out-of-the-money has an intrinsic value of zero.
- The cumulative cost required to hold a position in the security, including the risk-free interest rate (r) and dividends (D) expected during the life of the option.
- The time to expiration (T).
    - The time value element of the premium is the chance that an option will move into the money during the time to its expiration date. It therefore decreases to zero at its expiration date and is dependent on the style of the option.
- The estimate of the future volatility of the security's price (σ).

The effect of these factors on the prices of both call and put options is explained by Reilly and Brown (2006) in *Investment Analysis and Portfolio Management* and is summarised as follows:

	Will Cause an increase / decrease in:	
An Increase in:	Call Value	Put Value
1. S	↑	↓
2. K	↓	↑
3. T	↑	↑ / ↓
4. r	↑	↓
5. σ	↑	↑

Call option:

1. An increase in S increases the call's intrinsic value and therefore also the value of the call option.
2. An increase in K decreases the call's intrinsic value and therefore also the value of the call option.
3. If T increases it means that the option has more time until expiration, which increases the value of the time premium component, because greater opportunity exists for the contract to finish in-the-money. The value of the call option increases.
4. As the value of r increases, it reduces the present value of K. The value of K is an expense for the call holder, who must pay it at expiration to exercise the contract. Since it is decreased, it will lead to an increase in the value of the option.
5. An increase in σ increases the probability that the option will be deeper in-the-money at expiration. The option becomes more valuable.

Put option:

1. An increase in S decreases the put's intrinsic value and therefore also the value of the put option.
2. An increase in K increases the put's intrinsic value and therefore also the value of the put option.
3. If T increases, there is a trade-off between the longer time over which the security price could move in the desired direction and the reduced present value of the exercise price received by the seller at expiration.
4. As the value of r increases, it reduces the present value of K. This hurts the holder of the put, who receives the strike price if the contract is exercised.

5. An increase in σ increases the probability that the option will be deeper in-the-money at expiration. The option becomes more valuable.

There are two basic methods of determining the price of an option using these factors; the Black-Scholes pricing model and the Binomial pricing model.

2.2 Arbitrage Bounds on Valuation

Arbitrage bounds define the bounds wherein an option should trade to exclude the possibility of arbitrage opportunities in the market. From Gemmill (1993), Hull (2006), and Reilly and Brown (2006) the following summary was constructed.

2.2.1 Arbitrage Bounds on Call Prices

► Upper bound

Both an American and European call option gives the holder the right, but not the obligation, to buy one unit of the underlying asset for a certain price at some future date. Therefore, where c is the European call value and C is the American call value

$$c \leq S \text{ and } C \leq S .$$

► American-style and European-style Call options

It is important that an American put or call has to be at least as valuable as its corresponding European style contract:

$$c \leq C .$$

► Lower bound

Any option, call or put, cannot be worth less than zero:

$$c \geq 0 \text{ and } C \geq 0 .$$

► Lower bound for American Calls on Non-Dividend-Paying Stocks

The minimum value for an American call option that can be exercised immediately is the current underlying asset price minus the strike price:

$$C \geq S - K \,.$$

► Lower bound for European Calls on Non-Dividend-Paying Assets

$$c + Ke^{-rT} \geq S$$

or
$$c \geq S - Ke^{-rT} \,.$$

It is never optimal to exercise an American call option on a non-dividend-paying asset before the expiration date. Since the lower bound for a European call option $(S - Ke^{-rT})$ lies above the intrinsic value bound $(S - K)$, as applicable to the American call option, the second is redundant. This is because $(S - Ke^{-rT}) \geq (S - K)$. This means that for an underlying asset which does not pay dividends, C and c will be equal to one another.

In summary, the arbitrage bounds for call options are:

$$0 \leq \max[0, S - K] \leq \max[0, S - Ke^{-rT}] \leq c \leq C \leq S \,.$$

This expression says that

1. the American call is at least as valuable as the European contract;
2. neither call can be more valuable than the underlying stock, and
3. both contracts are at least as valuable as their intrinsic values, expressed on both a nominal and discounted basis.

2.2.2 Arbitrage Bounds on Put Prices

► Upper bound

Both an American and European put option gives the holder the right, but not the obligation, to sell one unit of the underlying asset for the strike price K at some future date. No matter how low the stock price becomes, the option can never be worth more than K. Hence, where p is the European put value and P is the American put value,

$$p \leq K \text{ and } P \leq K .$$

For the European option, we know that the option cannot be worth more than K at maturity. It follows that it cannot be worth more than the present value of K today:

$$p \leq Ke^{-rT} .$$

► American-style and European-style Put options

An American put has to be at least as valuable as its corresponding European style contract:

$$p \leq P .$$

► Lower bound

Any option, call or put, cannot be worth less than zero:

$$p \geq 0 \text{ and } P \geq 0 .$$

► Lower bound for American Puts on Non-Dividend-Paying Assets

The minimum value for an American put option that can be exercised immediately is the current strike price minus the underlying asset price:

$$P \geq K - S .$$

► Lower bound for European Puts on Non-Dividend-Paying Assets

$$S \geq -p + Ke^{-rT}$$

or
$$p \geq Ke^{-rT} - S\,.$$

The American lower bound to the put price lies above the European bound since $(Ke^{-rT} - S) \leq (K - S)$. It can be optimal to exercise an American put option on a non-dividend-paying underlying asset before the expiration date.

Similarly to a call option, a put option can be seen as providing insurance. A put option, when held in conjunction with the stock, insures the holder against the price falling below a certain level. However, a put option is different from a call option in that it may be optimal for an investor to forego this insurance and exercise early in order to realise the strike price immediately.

In summary, the arbitrage bounds for put options are:

$$0 \leq \max[0, Ke^{-rT} - S] \leq p \leq P \leq K.$$

This expression says that

1. the American put is at least as valuable as the European contract;
2. neither put can be more valuable than the strike price, and
3. both contracts are at least as valuable as the intrinsic value expressed on a discounted basis.

2.2.3 Put-Call Parity

Put-Call Parity for European Options on Non-Dividend Paying Assets

There exists an important relationship between European put and call prices in efficient capital markets. Put-call parity depends on the assumption that markets are free from arbitrage opportunities. This relationship is given by

$$c + Ke^{-rT} = p + S. \tag{2.2.1}$$

It shows that the value of a European call, with a specific strike price and maturity date, can be deduced from the value of a European put with the same strike price and maturity date, and vice versa.

This relationship is useful in practice for two reasons. Firstly, if there does not exist the desired put or call position in the market, an investor can replicate the cashflow pattern of the put or call by using interrelated assets in the correct format. By rearranging (2.2.1) it follows that

$$c = p + S - Ke^{-rT} \tag{2.2.2}$$

and

$$p = c - S + Ke^{-rT}. \tag{2.2.3}$$

Secondly, it is useful in identifying arbitrage opportunities in the market. A relative statement of the prices of puts and calls can be made if they are compared to one another. If a call is overpriced relative to the put, the call can be sold and the put bought to make a riskless profit, and vice versa.

Put-Call Parity for American Options on Non-Dividend Paying Stock

The put-call parity relationship for American calls and puts on non-dividend-paying stock is given by

$$S - Ke^{-rT} \geq C - P \geq S - K.$$

Adjusting Arbitrage Bounds for Dividends

Assume the stock pays a dividend D(T) immediately before its expiration date at time T. Also assume that when a dividend is paid, the share price will fall by the full amount of the dividend. The present value of the fall is D(0). An important assumption that we are making is that the dividend payment is known at time 0, when the option contract is entered into. This is reasonable assumption, since in practise the dividends payable during the life of the option can usually be predicted with reasonable accuracy.

To adjust the previously derived bounds for dividends we can simple adjust the stock price downwards for the present value of the dividends. This means that we substitute $S - D(0)$ for S to get:

- Lower bounds for European calls on dividend-paying asset

$$c \geq S - D(0) - Ke^{-rT}. \tag{2.2.4}$$

When the underlying asset pays dividends it is no longer true that the American call option and European call option have exactly the same value. Then the argument that an American option must be at least as valuable as its European counterpart because it allows more choice, becomes relevant again. This choice can be used to preserve value when the European contract cannot.

- Lower bounds for European puts on dividend-paying asset

$$p \geq D(0) + Ke^{-rT} - S. \tag{2.2.5}$$

Deciding to exercise a put before expiration does not depend on the presence of dividends. It is known that a dividend payment increases the value of a put option by reducing the value of the underlying stock without an offset in the strike price. This is irrelevant in determining whether to exercise early when compared to the liability of the stock itself. This means that having the choice to exercise early and receive the intrinsic value immediately is the only deterministic factor.

- Put-Call Parity for European options on dividend-paying asset

$$c + D(0) + Ke^{-rT} = p + S.$$

- Put-Call Parity for American options on dividend-paying asset

$$S - Ke^{-rT} \geq C - P \geq S - D(0) - K \,.$$

2.3 Binomial Tree

If options are correctly priced in the market, it should not be possible to make definite profits by creating portfolios of long and short position in options and their underlying stocks. We therefore price options using risk-neutral valuation. In a risk-neutral world, all securities have an expected return equal to the risk-free interest rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cashflows is the risk-free interest rate. As shown by Gemmil (1993), Hull (2006) and Reilly *et al.* (2006), the Binomial Tree method can be used to find the 'fair' value for options and shares. A number of simplifying assumptions are made:

1. The underlying asset price follows a binomial random process over time.
2. The distribution of share prices is multiplicative binomial.
3. The upward (u) and downward (d) multipliers are the same in all periods.
4. There are no transaction costs, so that a riskless hedge can be constructed for each period between the option and the asset at no extra cost.
5. Interest rates are constant.
6. At first we assume that early exercise is not possible.
7. There are no dividends.
8. No riskless arbitrage opportunities exist.

2.3.1 The One-Step Binomial Model

To derive the value for an option we set up a hedged position with both an option and its underlying share. This creates a riskless position that must pay the risk-free rate. Suppose the option expires at the end of the next period of length T. Let S be the initial share price, which in the next period will either rise by an upward factor u to uS or fall by a downward factor d to dS, where u >1 and d < 1. The corresponding pay-offs to the option is f_u and f_d.

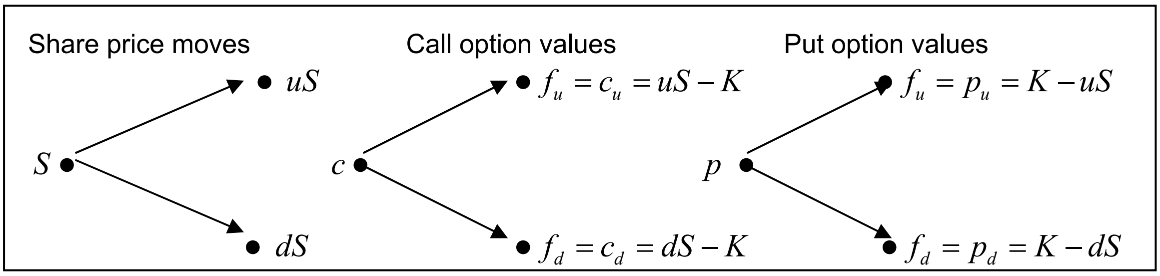

Figure 1: Stock and option prices in a general one-step tree

The value of the option is given by

$$f = e^{-rT}\left[pf_u + (1-p)f_d\right] \tag{2.3.1}$$

where

$$p = \frac{e^{-rT} - d}{u - d}. \tag{2.3.2}$$

In Eq. (2.3.1) the value of the option is given by the present value of the weighted average of the pay-offs to higher and lower share prices. The weights *p* and *(1-p)* are interpreted as the implicit probabilities of an up movement in the stock price and a down movement in the stock price respectively. The value of the option then simply becomes the present value of the probability weighted pay-offs. Therefore

$$f = PV\left[E\left(pay - off\right)\right].$$

2.3.2 The Binomial Model for Many Periods

Consider the two-period tree in Fig. 2 below, where the objective is to calculate the option price at the initial node of the tree. Using the same assumptions as before, with the length of each time step set equal to δt years, we apply our binomial formula Eq. (2.3.1) to the top two branches of the tree which gives

$$f_u = e^{-rT}\left[pf_{uu} + (1-p)f_{ud}\right]. \tag{2.3.3}$$

Repeating this procedure for the bottom two branches leads to

$$f_d = e^{-r\delta t}\left[pf_{ud} + (1-p)f_{dd}\right]. \quad (2.3.4)$$

Solve f by substituting Eq. (2.3.3) and Eq. (2.3.4) into

$$f = e^{-r\delta t}\left[pf_u + (1-p)f_d\right].$$

Hence

$$f = e^{-2r\delta t}\left[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}\right]. \quad (2.3.5)$$

The option price is again equal to its expected pay-off in a risk-neutral world discounted at the risk-free rate. This remains true as we add more steps to the binomial tree.

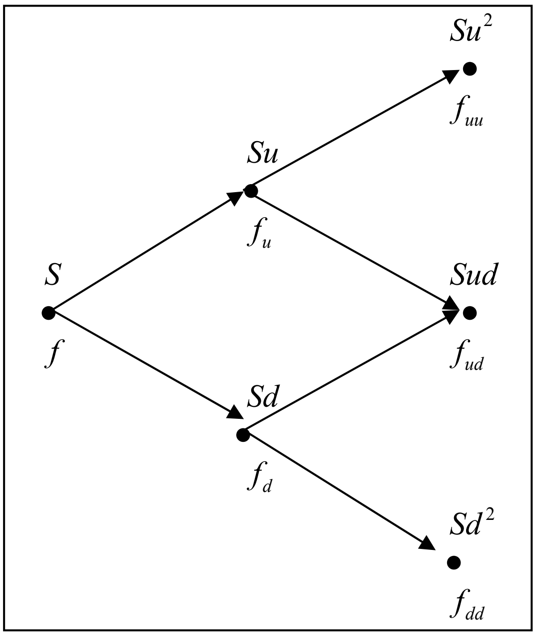

Figure 2: Stock and option prices in a general two-step tree

To calculate the value of an option in terms of the price of the underlying stock a tree is constructed that comprises of many successive two-branch segments. Valuation begins with the known final pay-off and works backwards step by step until the present time is reached. The method can be extended to options that have any number of discrete time periods to maturity. This allows an investor to make each period arbitrarily short by dividing the time to maturity into enough time steps in order to obtain reasonably accurate results.

The many-period binomial options-pricing formula is obtained for a call option as

$$c = e^{-rn\delta t} \sum_{k=0}^{n} \left[\binom{n}{k} p^k (1-p)^{n-k} \max\left\{u^k d^{n-k} S - K, 0\right\} \right]$$

and for a put option as (2.3.6)

$$p = e^{-rn\delta t} \sum_{k=0}^{n} \left[\binom{n}{k} p^k (1-p)^{n-k} \max\left\{K - u^k d^{n-k} S, 0\right\} \right].$$

This shows that European options can be valued by

1. calculating for each possible path the payoff at expiration (after *n* time steps);
2. weighting this by the risk neutral probability of the path,
3. adding the resulting terms, and
4. discounting this back to the present at the risk-free rate of interest.

2.3.3 The Binomial Model for American Options

To value American options the possibility of early exercise has to be considered. The option will only be exercised early if the pay-off from early exercise exceeds the value of the equivalent European value at a specific node. Therefore, work back through the tree from the end to the beginning in the same way as for the European options, but test at each node whether early exercise is optimal.

The value of the option at the final node is the same as for European options. At earlier nodes the value of the option is the greater of

1.) The value given by $f = e^{-rT}\left[pf_u + (1-p)f_d\right]$.

2.) The pay-off from early exercise given by the intrinsic value.

2.3.4 The Binomial Model for Options on Dividend-paying Stock

The Binomial model can be used for dividend-paying stock. When a dividend is paid, the price of a share will fall by the amount of the dividend. If it is less than the dividend, the trader could buy the share just prior to the ex-dividend date, capture the dividend, and sell the share immediately after it has fallen. It is therefore assumed that the fall in share price is equal to the full dividend amount. Both the cases where the dividend is a known Rand amount or a known dividend yield are considered

If it is assumed that the Rand amount of the dividend is known in advance, the share-price binomial tree will be knocked side-ways at the ex-dividend date. If a given dividend is paid in Rand, the tree will start to have branches that do not recombine. This means that the number of nodes that have to be evaluated, particularly if there are several dividends, becomes large. As shown in Fig. 3 a single dividend of size D results in a new, separately developing tree being formed for each node that existed at the time of the dividend payment. This process is computationally slow.

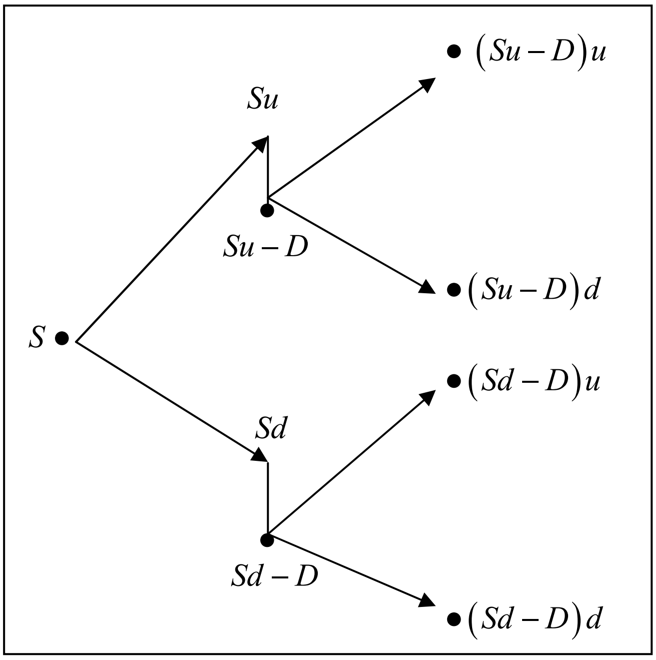

Figure 3: Two-period stock-price tree with a dividend after one period

To avoid this problem the assumption is made that the dividend is some proportion δ of the share price at that point in the tree. It is therefore assumed that the dividend yield is known. The share price after the dividend payment will then either increase to

$Su(1-\delta)$ or decrease to $Sd(1-\delta)$ after one step. The whole tree is again a geometric process and the nodes recombine.

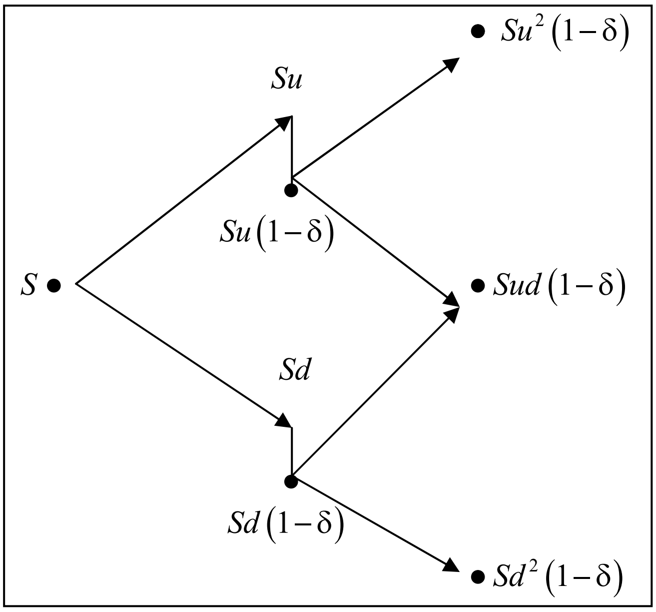

Figure 4: Two-period stock-price tree with a dividend-yield payment after one period

2.3.5 Determination of p, u and d

It is necessary to construct a binomial tree to represent the movements in a general stock price in the market. This is done by choosing the parameters u and d to match the volatility of the stock price and making it consistent with normally distributed returns.

The Binomial tree of share price, as described, is both symmetrical and recombines in the sense that an up movement followed by a down movement leads to the same stock price as a down movement followed by an up movement. In order for this to hold we choose the down multiplier (d) as the inverse of the up multiplier (u). This means that if $u = \frac{1}{d}$, then the returns to holding the asset will be symmetrical.

The width of the binomial tree is related to the size of u, the up multiplier per step, and the number of steps that have occurred. The equivalent assumption for an asset that has normally distributed returns is that the variance is constant per period. If the variance for a time step of δt years is given by $\sigma^2\delta t$, then the standard deviation or the volatility of the asset is equal to $\sigma\sqrt{\delta t}$. If we assume that prices are lognormally distributed, we can imagine the distribution widening as time goes by, just as the binomial tree widens at successive branches.

The actual values to use for the up and down multipliers in a binomial tree should be consistent with normally distributed returns. Let μ be the expected return on a stock and σ the volatility in the real world. Imagine a one step binomial tree with a step of length δt. The binomial process for asset prices gives normally distributed returns in the limit if

$$u = e^{\sigma\sqrt{\delta t}}$$

and

$$d = \frac{1}{u} = e^{-\sigma\sqrt{\delta t}}.$$

After a large number of steps this choice of *u* and *d* leads to a variance of $\sigma^2\delta t$.

Assume that the expected return of an up movement in the real world is *q*. In order to match the stock price volatility with the tree's parameters, the following equation must be satisfied

$$\sigma^2\delta t = e^{\mu\delta t}(u+d) - ud - e^{2\mu\delta t}.$$

One solution to this equation is

$$u = e^{\sigma\sqrt{\delta t}}$$

$$d = e^{-\sigma\sqrt{\delta t}}.$$

2.4 The Black-Scholes Pricing Formula

2.4.1 From Discrete to Continuous Time

The binomial formulas derived for the multi-period model is the discrete-time version of the continuous-time Black-Scholes formula. From the two-step binomial tree it can be shown that if it is assumed that the underlying stock prices are lognormally distributed, and u and d are defined in order to be consistent with the volatility of the stock price returns, the Binomial option values will converge to the Black Scholes values as $n \to \infty$. This is explained by Gemmill (1993), who carries on to show the similarity between the Binomial and Black Scholes models.

If we assume that prices are lognormal, its distribution widens as time goes by, just as the binomial tree widens at successive branches. Beginning at a share price S at time zero, the distribution widens until a part of it exceeds the strike price, K. At maturity, the pay-off to the option is the shaded area above the strike price for a call option. The Black-Sholes value of the option today is the present value of this shaded area.

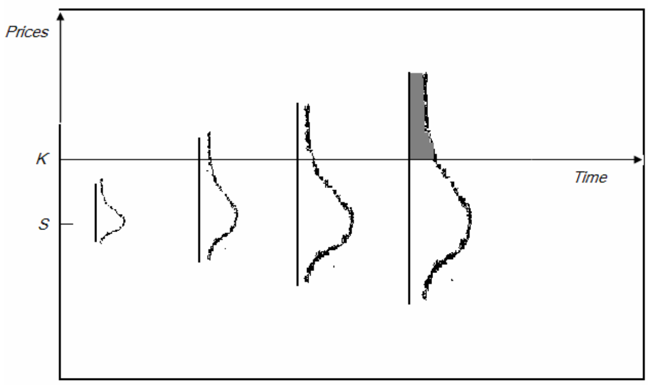

Figure 5: Call price rising as the price distribution widens over time

2.4.2 Derivation of the Black-Scholes Equation

The Black-Scholes equation is derived as shown in Gemmill (1993), Hull (2006), Smith (1976) and Chappel (1992). The derivation of the Black-Scholes equation consists of two parts. Firstly, it is shown that a riskless hedge can be constructed when the stochastic process for the underlying asset price is lognormal. This is done by setting up a portfolio containing stock and European call options. In the absence of arbitrage opportunities, the return from this portfolio must be the risk-free rate. The reason for this is that the sources of change in the value of the portfolio must be the prices, since it affects the value of both the stock itself and the derivative in the portfolio. This follows also from the fact that at a point in time the quantities of the assets are fixed. If the call price is a function of the stock price and the time to maturity, then changes in the call price can be expressed as a function of the changes in the stock price and changes in the time to maturity of the option. Thus, in a short period of time, the price of the derivative is directly correlated with the price of the underlying stock. Therefore, at any point in time, the portfolio can be made into a riskless hedge by choosing an appropriate portfolio of the stock and the derivative to offset any uncertainty. If quantities of the stock and option in the hedge portfolio are continuously adjusted in the appropriate manner as the asset price changes over time, then the return to the hedge portfolio becomes riskless and the portfolio must earn the risk-free rate. Secondly, it shows that the call option price is determined by a second order partial differential equation.

The Black-Scholes equation makes exactly the same assumptions as the binomial approach, plus one additional one; it is also assumed that the underlying asset price follows a lognormal distribution for which the variance is proportional to time. The assumptions used to derive the Black-Scholes equation are as follows:

1. The stock price follows a geometric Brownian motion, with μ and σ constant. Therefore the distribution of possible stock prices at the end of any finite interval is lognormal and the log returns are normally distributed.
2. Short selling is allowed and no penalties imposed.
3. There are no transaction costs, so that a riskless hedge can be constructed for each period between the option and the asset at no extra cost.
4. The risk-free interest rate is constant and the same for all maturities.
5. There are no dividends.
6. No riskless arbitrage opportunities exist.
7. Securities trade continuously in the market.
8. The option is European and can only be exercised at maturity.

9. All securities are perfectly divisible so that it is possible to borrow any fraction of the price of a security or to hold it at the short-term interest rate.

The Black-Scholes-Merton differential equation is given by

$$rc = \frac{\partial c}{\partial \mathrm{t}} + rS\frac{\partial c}{\partial \mathrm{S}} + \frac{1}{2}\sigma^2 \mathrm{S}^2 \frac{\partial^2 c}{\partial \mathrm{S}^2} \tag{2.4.1}$$

(Hull, 2006).

It can be derived and solved for many different derivatives that can be defined with S as the underlying variable, not only a European call option. It is important to realise that the hedge portfolio used to derive the differential equation is not permanently riskless. It is riskless only for a very short period of time. As S and t change, $\frac{\partial c}{\partial S}$ also changes. To keep the portfolio riskless, it is therefore necessary to change the relative proportions of the derivative and the stock in the portfolio frequently. (Hull, 2006)

The differential equation defines the value of the call option subject to the boundary condition, which specifies the value of the derivative at the boundaries of possible values of S and t. It is known that, at maturity, a call option has the key boundary condition:

$$c = \max\left(0, S - K\right). \tag{2.4.2}$$

Black and Scholes used the heat-exchange equation from physics to solve the differential equation for the call price, c, subject to the boundary condition. A more intuitive solution is suggested in the paper by Cox and Ross (1975). To solve the equation, two observations are made: First, whatever the solution of the differential equation, it is a function only of the variables in (2.4.1) and (2.4.2). Therefore, the solution to the option pricing problem is a function of the five variables:

1) the stock price, S;
2) the instantaneous variance rate on the stock price, σ;
3) the strike price of the option, K;
4) the time to maturity of the option, T, and

5) the risk-free interest rate, r.

The first four of these variables are directly observable; only the variance rate must be estimated.

Secondly, in setting up the hedge portfolio, the only assumption involving the preferences of the individuals in the market is that two assets which are perfect substitutes must earn the same equilibrium rate of return; no assumptions involving risk preference are made. This suggests that if a solution to the problem can be found which assumes one particular preference structure, it must be the solution of the differential equation for any preference structure that permits equilibrium (Smith, 1976). This leads to the principle of risk-neutral valuation. It says that the price of an option or other derivative, when expressed in terms of the price of the underlying stock, is independent of risk preferences. Options therefore have the same value in a risk-neutral world as they have in the real world. It may therefore be assumed that the world is risk-neutral for the purpose of valuing options, to simplify the analysis. In a risk-neutral world all securities have an expected return equal to the risk-free interest rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cashflows is the risk-free interest rate. (Hull, 2006)

The expected value of the option at maturity in the risk-neutral world is

$$\hat{E}\left[\max\left(S_T - K, 0\right)\right],$$

where Ê denotes the expected value in a risk-neutral world. From the risk-neutral argument, the European call option price, *c*, is the expected value discounted at the risk-free interest rate, that is

$$\begin{aligned} c &= e^{-rT}\hat{E}\left(c_T\right) \\ &= e^{-rT}\hat{E}\left[\max\left(S_T - K, 0\right)\right]. \end{aligned} \tag{2.4.3}$$

Start by looking at the value of c_T. If the stock price *S* follows the process

$$dS = \mu S dt + \sigma S dz, \tag{2.4.4}$$

then, using Ito's lemma, it can be found that the process followed by ln S is

$$d(\ln S) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dz. \tag{2.4.5}$$

Because μ and σ are constant, ln S follows a generalised Wiener process with constant drift rate and constant variance. The change in ln S between time zero and some future time, T, is therefore normally distributed with mean and variance given respectively by

$$\left(\mu - \frac{\sigma^2}{2}\right)\mathrm{T} \; and \; \sigma^2 T.$$

This means that

$$\ln S_T - \ln S_0 \sim \phi\left(\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)$$

or

$$\ln S_T \sim \phi\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \ln S_0, \sigma\sqrt{T}\right) \tag{2.4.6}$$

where φ(m,s) denotes the normal distribution with mean m and standard deviation s.

This leads to the well known result for a European call option:

$$c = e^{-rT}\left[SN(d_1)e^{rT} - KN(d_2)\right]. \tag{2.4.7}$$

The proof can be found in the most financial mathematical textbooks (cf. Hull, 2006).

The function $N(x)$ is the cumulative probability function for a standardised normal distribution. In other words, it is the probability that a variable with a standard normal distribution will be less than *x*. The expression $N(d_2)$ is the probability that the option will be exercised in a risk-neutral world, so that $KN(d_2)$ is the strike price times the probability that the strike price will be paid. The expression $SN(d_1)e^{rT}$ is the expected value of a variable that equals S_T if $S_T > K$ and is zero, otherwise, in a risk-neutral world. (Hull, 2006)

Since $c = e^{-rT}\hat{E}(c_T)$, the expected pay-off at maturity, can also be rewritten as

$$\begin{aligned}\hat{E}(c_T) &= SN(d_1)e^{rT} - KN(d_2) \\ &= N(d_2)\left\{Se^{rT}\left[\frac{N(d_1)}{N(d_2)}\right] - K\right\}.\end{aligned}$$

In this expression $N(d_2)$ is the probability that the call finishes in-the-money and is multiplied by the expected in-the-money pay-off (Gemmill, 1993).

To find the European put price, put-call parity can be used:

$$p = c - S + Ke^{-rT}.$$

Substituting from the Black-Scholes equation for *c* gives

$$\begin{aligned}p &= SN(d_1) - Ke^{-rT}N(d_2) - S + Ke^{-rT} \\ &= S\left[N(d_1) - 1\right] - Ke^{-rT}\left[N(d_2) - 1\right] \\ &= Ke^{-rT}N(-d_2) - SN(-d_1).\end{aligned} \tag{2.4.8}$$

The expression in (2.4.8) can also be derived directly from the partial differential equation solved subject to the primary boundary condition for put options given by

$$p = \max(0, K - S).$$

2.4.3 Properties of the Black-Sholes Equation

The properties of the Black-Scholes equation is given by Gemmill (1993), Hull (2006), Smith (1976) and Chappel (1992).

American Options on Non-Dividend Paying Stock

The expressions above were derived for European put and call options on non-dividend-paying stock. Because the European price equals the American price when there are no dividends, (2.4.7) also gives the value of an American call option on non-dividend-paying stock. There is no exact analytic formula for the value of an American put option on non-dividend-paying stock.

Adjusting the Black-Scholes Equation for Dividends

The Black-Scholes model, like the Binomial model, can be used to value dividend-paying stock. It is assumed that the amount and timing of the dividends during the life of the option can be predicted with certainty. When a dividend is paid, the price of a share will fall by an amount reflecting the dividend paid per share. In the absence of any tax effects, the fall in share price is equal to the full dividend amount. A dividend is a pay-out to a shareholder which the holder of a call option does not get, yet the holder suffers the fall in share price. From the other perspective, the holder of a put option will benefit from the fall in share price that follows a dividend. Dividends that will be paid out over the lives of options therefore reduce call prices. (Gemmill, 1993)

<u>European Options</u>
Consider the value of a European option when the stock price is the sum of two components. The one component is that part of the price accounted for by the known dividends during the life of the option and is considered riskless. The riskless component, at any given time, is the present value of all the dividends during the life of the option discounted from the ex-dividend dates to the present. By the time the option matures, the dividends will have been paid and the riskless component will no longer exist. The Black-Scholes formula can be used, provided that the stock price is

reduced by the present value of all dividends during the life of the option, discounted from the ex-dividend dates at the risk-free rate. (Hull, 2006)

American Call Options

If a dividend is sufficiently large, it will be profitable to exercise a call just before the dividend is due. Shares go ex-dividend before the actual payment is made, so the fall in share price occurs on the ex-dividend date. It is optimal to exercise only at a time immediately before the stock goes ex-dividend, because exercising at this time yields an extra dividend, but results in the loss of the time-value on the call. (Gemmill, 1993)

Assume that there are *n* ex-dividend dates expected during the life of the option and that t_1, t_2, …, t_n are the times immediately before the *n* ex-dividend dates where $t_1 < t_2 < ... < t_n$. Let the dividend payments corresponding to these times be $D_1, D_2, ... , D_n$. (Hull, 2006)

Suppose there is no volatility, that is σ = 0. Let S be the share price and K be the strike price. Start by considering early exercise just before the last ex-dividend date, time t_n. It is known that the share price will fall to $S(t_n) - D_n$ on the last ex-dividend date. Assume the call is in-the-money $\left(S(t_n) > K\right)$. Then:

1. Exercising just before the ex-dividend date gives a payoff after the dividend payment equal to

$$\left(S(t_n) - D_n\right) - K + D_n = S(t_n) - K \,;$$

2. not exercising, but waiting until maturity, results in a value today, given that σ = 0, that is equal to the lower bound given by (2.2.4) of

$$S(t_n) - D_n - Ke^{-r(T-t_n)}.$$

With zero volatility, exercise will therefore be worth while if

$$S(t_n) - K > S(t_n) - D_n - Ke^{-r(T-t_n)}.$$

Hence, for optimal exercise at time t_n, it is required that

$$D_n > K - PV(K);$$

that is

$$D_n > K\left(1 - e^{-r(T-t_n)}\right). \qquad (2.4.9)$$

Similarly, it can be shown that (2.4.9) holds for any one of the n ex-dividend dates during the life of the option. This means that early exercise is optimal if

$$D_i > K\left(1 - e^{-r(t_{i+1}-t_i)}\right).$$

This implies that early exercise is more likely if:

1. The dividend (D_i) is large relative to the strike price (K);
2. the time until the next ex-dividend date is fairly close so that $PV(K) \approx K$, and
3. the volatility is low, so that the time value given up to be exercising the option is low.

To value American call options on dividend-paying stock the pseudo-American approach, first outlined by Black (1975), can be used (Gemmill, 1993). This involves calculating the price of European options that mature at time T and t, and setting the American price equal to the greater of the two (Hull, 2006). To demonstrate the procedure, consider the case where there is only one ex-dividend date during the life of the option. The share price will fall at time t, when the share goes ex-dividend, but the option potentially continues until maturity at time T.

The buyer of the American call is now considered to have two separate European call options. The first call option, worth c_{short}, expires at time t, immediately after which the stock pays a dividend of D_t. The call price equation can be written as a functional relationship:

$$c_{short} = f\left(S - De^{-rt}, t, r, \sigma, K - D\right).$$

The stock price is discounted by the present value of the dividend, but this is offset as the strike price is reduced by the dividend payment. The second call, worth c_{long}, expires at T and pays no dividend. It can be written as

$$c_{long} = f\left(S - De^{-rt}, T, r, \sigma, K\right).$$

As before, the stock price is discounted by the present value of the dividend, but this time there is no receipt of dividend to reduce the exercise price. As the holder of the American call effectively has two mutually exclusive European call options, the call will be valued today as the higher of the two,

$$C = \max\left(c_{short}, c_{long}\right).$$

Two Black-Scholes evaluations are made and the larger value is chosen as the correct call value (Gemmill, 1993). As the approach is extended to situations where there are n dividend payments during the life of the option, the number of Black-Scholes evaluations will increase to $(n+1)$. The correct value of the American call option will then be the maximum of these. The pseudo-American adjustment is relatively accurate, but will slightly undervalue the call. The reason is that it assumes that the holder has to decide today when the call will be exercised. In practice, the choice remains open until just before each of the ex-dividend dates.

American Put Options

The Black-Scholes model is inadequate when valuing American put options. The problem is that early exercise may be profitable for a put, especially in the absence of dividends. There is no analytic equivalent of the Black-Scholes equation that allows for this, because in principle exercise could occur at almost any date between today and the maturity of the option. One suggestion is to abandon the Black-Scholes method and use the binomial model instead. The binomial model is accurate because the exercise value is considered at each node of the tree. For the same reason the method is computationally very slow. (Gemmill, 1993)

Following the explanation given by Gemmill (1993), it is found that the Macmillan (1986) method gives a relatively simple equation which is reasonably accurate and quick to calculate. This is an approximation based upon the Black-Scholes equation. Start with the equation

$$P = \mathrm{p} + \mathrm{FA}, \tag{2.4.9}$$

where P is the American put price, p is the European put price and FA is the difference between the European and American puts. This follows from the arbitrage bounds on put options, where it was argued that an American put is at least as valuable as the European contract. To calculate FA, it is regarded as the value of another option, the option to exercise early and it is analysed as follows (Gemmill, 1993).

Early exercise will occur if the stock price, S, falls below some critical level S^{**}. Below this level the put is simply given by its intrinsic value,

$$P = K - S \qquad for\ S \leq S^{**}.$$

Above the critical stock price, the put value is given by

$$P = p + FA \qquad for\ S > S^{**}.$$

The correction factor (FA) depends on how far the stock price, S, is above the critical level, S^{**}. It is

$$FA = A_1\left(\frac{S}{S^{**}}\right)^{q_1},$$

where

$$q_1 = 0.5\left[-(M-1) - \sqrt{(M-1)2 + 4\left(\frac{M}{W}\right)}\right]$$

$$A_1 = -\left(\frac{S^{**}}{q_1}\right)N\left(d_1^{**}\right),$$

in which d_1^{**} is the Black-Scholes d_1 value at $S = S^{**}$

$$M = \frac{2r}{\sigma^2}$$
$$W = 1 - e^{-rt}.$$

The iterative stock price S^{**} is found by an iterative procedure as follows:

1. Calculate q_1 from the equation above, which is a constant.
2. Guess a value for S^{**}, the critical stock price, and calculate A_1 from the equation above.
3. See whether the exercise value, $K - S$, exactly equals the approximated un-exercised value, that is whether

$$K - S = p + A_1\left(\frac{S}{S^{**}}\right)^{q_1}.$$

If this equation holds, then the critical price S^{**} has been found. If the equality does not hold, a new S^{**} is chosen and the algorithm is continued at step (2). This method can be implemented using a software searching algorithm, for example Solver in Microsoft Excel.

2.5 Option Sensitivities

The sensitivity of option prices to its five inputs is measured. This can be done since closed form solutions exist for standard option prices.

2.5.1 Delta

Delta measures the sensitivity of the option price to the share price. It is the ratio of change in the price of the stock option to the change in the price of the underlying stock in the limit. We find delta by taking the first partial derivative of the option price, which also represents the slope of the curve that relates the option price A to the current underlying asset price B.

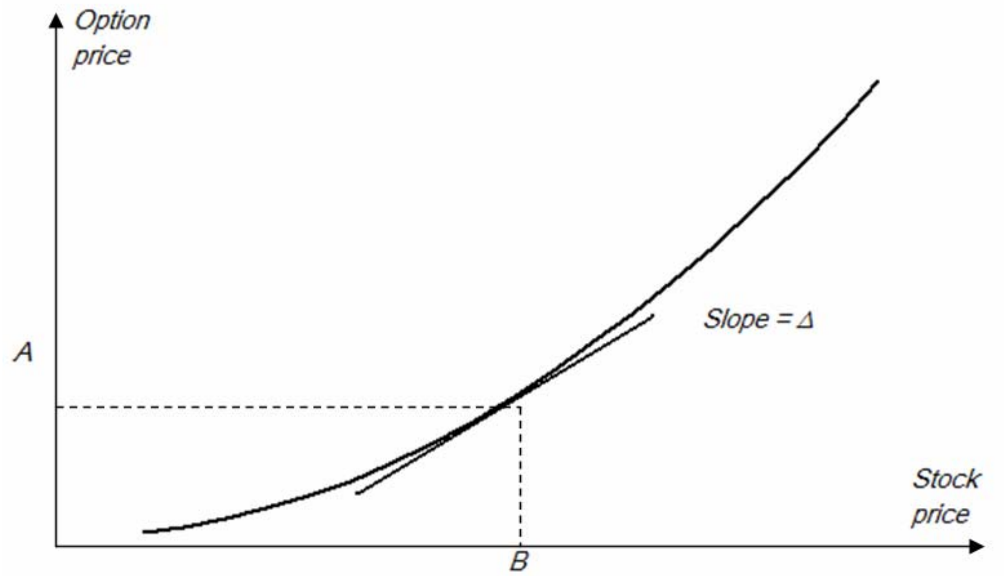

Figure 6: Calculation of delta

The delta of a European call (c) option on non-dividend paying stocks is given by

$$\Delta_c = \frac{\partial c}{\partial S} = N\left(d_1\right),$$

where

$$d_1 = \frac{\ln\left(S/K\right) + \left[r + \left(\sigma^2/2\right)\right]T}{\sigma\sqrt{T}}.$$

Delta of a call option has a positive sign ($N\left(d_1\right)$) for the buyer of a call and a negative sign ($-N\left(d_1\right)$) for the seller of a call.

For a European put on non-dividend paying stocks, delta is

$$\Delta_p = \frac{\partial p}{\partial S} = N\left(d_1\right) - 1.$$

This delta is negative ($N\left(d_1\right) - 1$) for the buyer of a put and positive ($1 - N\left(d_1\right)$) for the seller of a put.

Delta ranges between zero and approximately one and changes as an options goes more into- or out-of-the-money. At-the-money options have a delta of approximately 0.5, while the delta of in-the-money options tends towards one and the delta for out-of-the-money options tends towards 0.

These deltas or hedge ratios can be used to construct a riskless portfolio consisting of a position in an option and a position in the stock. This is known as delta hedging, where hedgers match their exposure to an option position. It is important to remember that because delta changes, the investor's position remains delta hedged only for a relatively short period of time. Therefore the hedge has to be rebalanced periodically. Consider the following:

Long call position

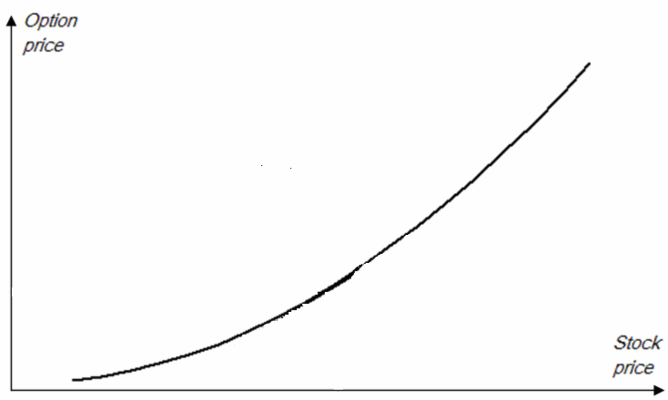

Figure 7: Long call position

A hedger owns a call option. If the stock price increases, the value of the call option also increases. We know that the delta of a long call position is positive, i.e.

$$\Delta_{long\ call} = \frac{\partial c}{\partial S} = N\left(d_1\right) > 0.$$

In order to hedge the position, the hedger sells delta shares of the stock. His portfolio therefore consists of

+1 : option, and

- Δ : shares of the stock.

If the share price increases, the investor makes a profit on the call position equal to the loss he makes on the shares.

The delta for a portfolio of options dependent on a single asset, which price is S, is

$$\Delta = \frac{\partial \Pi}{\partial S},$$

where Π is the value of the portfolio. If the portfolio consists of a quantity w_i of option f_i $(1 \le i \le n)$, that is

$$\Pi = \sum_{i=1}^{n} w_i f_i,$$

then the delta of the portfolio is given by weighted sum of the individual deltas:

$$\Delta = \frac{\partial \Pi}{\partial S} = \sum_{i=1}^{n} w_i \frac{\partial f_i}{\partial S} = \sum_{i=1}^{n} w_i \Delta_i,$$

where Δ_i is the delta of the i^{th} position.

2.5.2 Gamma

The sensitivity of delta to a change in the share price is known as gamma. It is the rate of change of the portfolio's delta with respect to the price of the underlying asset. Gamma is calculated as the second partial derivative of the option price with respect to the share price:

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2} = \frac{\partial \Pi}{\partial S}\left(\frac{\partial \Pi}{\partial S}\right) = \frac{\partial \Pi}{\partial S}(\Delta).$$

The absolute value of the gamma for a European put or call on non-dividend-paying stock is given by

$$\Gamma = \frac{N'(d_1)}{S\sigma\sqrt{t}},$$

where

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2}.$$

The sign is determined by the position taken. Since the delta of a long call is positive, the gamma of a long call will also be positive. The delta of a short call is negative, so it also has a negative gamma. Similarly, a long put has a positive gamma and a short call has a negative gamma.

Gamma can also be seen as the gap between the delta slope and the curve of the option price, relative to the underlying stock price. In Fig. 8 it can be seen that delta is an inaccurate measurement of the relative movement of asset price and option value. When the stock price moves from S to S', using delta assumes the option price moves from C to C', when it really moves from C to C''. Gamma compensates for this error by measuring the curvature of the relationship between the option price and the stock price or the rate at which delta changes.

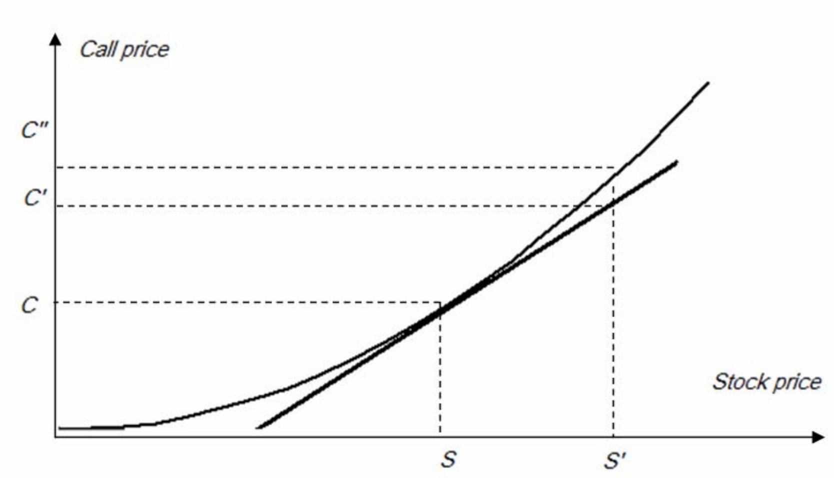

Figure 8: Hedging error introduced by curvature

If the absolute value of gamma is small, the rate of change in delta is small and the delta error is small. If the absolute value of gamma is large, delta changes quickly and the delta error becomes large.

Gamma is greatest for at-the-money options and falls to zero for deeply in-the-money or out-of-the-money options.

2.5.3 Theta

The sensitivity of the option price to the time to expiry, T, is known as theta. It is the rate of change in the value of the portfolio with respect to time. Derived from the Black-Scholes equation for a call option, theta is given as

$$\begin{aligned}\Theta_c &= \frac{\partial c}{\partial T} \\ &= -\left[\frac{S\sigma}{2\sqrt{T}}\right]N'(d_1) - rKe^{-rT}N(d_2),\end{aligned}$$

where

$$d_2 = \frac{\ln(S/K) + \left[r - (\sigma^2/2)\right]T}{\sigma\sqrt{T}},$$

and for a put option it is given by

$$\begin{aligned}\Theta_p &= \frac{\partial p}{\partial T} \\ &= -\left[\frac{S\sigma}{2\sqrt{T}}\right]N'(d_1) + rKe^{-rT}N(-d_2).\end{aligned}$$

The formula calculates the reduction in price of the option for a decrease in time to maturity of one year. In practice, theta is usually quoted as the reduction in price for a decrease in time to maturity of a single day. To calculate theta per day, divide the formula for theta by 365.

Theta has a negative sign for long options and a positive sign for short options. This is because as the time to maturity decreases, with all else remaining the same, the time value of the option decreases. Theta measures this decrease in value and, since the option value decreases at an increasing rate over the lifetime of at-the-money options, theta is lowest just before expiration (Hull, 2006). This is because at-the-money options may become either in-the-money or out-of-the-money on the last day. An out-of-the-money option has some chance of becoming in-the-money before the last few days but, if it is still out-of-the-money in those last few days, has little chance of any pay-off and it has little time value left to decrease (Gemmill, 1993). An in-the-money option follows a similar pattern as the out-of-the-money option, but if it is in-the-money at expiration it will lose a positive amount of time premium.

2.5.4 Vega

The sensitivity of the option price to volatility is called vega. It measures the change in option premium for a 1% change in volatility. For a European call or put option on non-dividend paying stock, vega is given by

$$\begin{aligned} \nu &= \frac{\partial f}{\partial \sigma} \\ &= \mathrm{S}\sqrt{\mathrm{T}}N'(d_1) \\ &> 0. \end{aligned}$$

Vega has a positive sign for long options and a negative sign for short options. This means that if you are an option buyer, vega works for you, but if you are an option seller, vega works against you. If the absolute value of vega is high, the derivative's value is very sensitive to small changes in volatility. Similarly, if the absolute value of vega is low, changes in volatility have little impact on the value of the derivative. Vega is greatest for at-the-money options and decreases to zero for extreme in-the-money or out-of-the-money options.

3.5.5 Rho

The sensitivity of call prices to interest rates is measured by rho. It is the rate of change of the value of a derivative with respect to the interest rate. For a European call option on non-dividend paying stock, it is given by

$$\begin{aligned}\rho &= \frac{\partial f}{\partial r} \\ &= KTe^{-rT} N\left(d_2\right) \\ &> 0,\end{aligned}$$

and for a European put option on non-dividend paying stock

$$\begin{aligned}\rho &= \frac{\partial f}{\partial r} \\ &= -KTe^{-rT} N\left(-d_2\right) \\ &< 0.\end{aligned}$$

Therefore

$$\begin{aligned}\rho_{long\ call} &> 0, \\ \rho_{short\ call} &< 0, \\ \rho_{long\ put} &< 0, \\ \rho_{short\ put} &> 0.\end{aligned}$$

3.

Volatility-dependent derivatives

As Clewlow and Strickland (1997) explain, the value of volatility dependent derivatives depends in an important way on the level of future volatility. Of course, the value of all options is dependent on volatility, but these options are special in that their value is particularly sensitive to volatility over a period which begins not immediately, but in the future. As such they are viewed, in some sense, as forwards or options on future volatility.

These options are particularly useful when there is some event which occurs in the short term which will then potentially affect outcomes further in the future. They are therefore often used as a kind of pre-hedge to lock into the current levels of pricing until more information is known at a later date.

For both compound and chooser options the option is first defined, before an overview of their applicability and use is given and compared to standard options. The option valuations are then derived in detail. A discussion follows on notable aspects of both options. For compound options the arbitrage bounds on valuation of the options are given. These are the limits within which the price of an option should stay, since outside these bounds a risk-free arbitrage would be possible. They allow an investor to constrain an option price to a limited range, and do not require any assumptions about whether the asset price is normally or otherwise distributed. Lastly, the sensitivities or Greeks of the compound options only are given. Simple chooser options decompose exactly into a portfolio of a call option and a put option and their Greeks can be calculated from this portfolio. Each Greek letter measures a different dimension of the risk in an option position, and the aim of a trader is to manage the Greeks so that all risks are acceptable.

3.1 Ordinary Compound Options

3.1.1 Definition

A compound option is a standard European option on an underlying European option. From this definition there are four basic compound options:

- A call on a call,
- a call on a put,
- a put on a call, and
- a put on a put.

If the compound option is exercised, the holder receives a standard European option in exchange for the strike price; otherwise, nothing.

3.1.2 Common Uses

This type of option usually exists for currency or fixed-income markets, where an uncertainty exists regarding the option's risk protection capabilities. Compound options are also used when there is uncertainty about the need for hedging in a certain period. When valuing a compound option there are two possible option premiums. The first premium is paid up front for the compound option. The second premium is paid for the underlying option in the event that the compound option is exercised. Generally, the premium for the compound option is modest. Compound options are also useful in situations where there is a degree of uncertainty over whether the underlying option will be needed at all. The small up-front premium can be viewed as insurance against the underlying option not being required and, since it is a known cost, it can be budgeted for (Clewlow and Strickland, 1997). Therefore, the advantages of compound options are that they allow for large leverage and are cheaper than standard options. However, if the compound option is exercised, the combined premiums will exceed what would have been the premium for purchasing the underlying option outright at the start. (www.investopedia.com; www.riskglossary.com)

Consider the following two examples from www.my.dreamwiz.com:

A major contracting company is tendering for the contract to build two hotels in one months' time. If they win this contract they would need financing for R223.5 million for 3 years. The calculation used in the tender utilizes today's interest rates. The company therefore has exposure to an interest rate rise over the next month. They could buy a 3yr interest rate cap starting in one month but this would prove to be very expensive if they lost the tender. The alternative is to buy a one month call option on a 3yr interest cap. If they win the tender, they can exercise the option and enter into the interest rate cap at the predetermined premium. If they lose the tender they can let the option lapse. The advantage is that the premium will be significantly lower.

Compound Options can also be used to take speculative positions. If an investor is bullish on R/USD exchange rate, they can buy a 6 month call option at say 7.00 for 4.00%. Alternatively, they could purchase a 2 month call on a 4 month R/USD 7.00 call at 2.50%. This will cost say 2.00% upfront. If after 2 months the R/USD is at 7.50, the compound call can be exercised and the investor can pay 2.50% for the 4 month 7.00 call. The total cost has been 4.50%. If the R/USD falls, the option can lapse and the total loss to the investor is only 2.00% instead of 4.00% if they had purchased the straight call.

3.1.3 Valuation

Closed form solutions for compound options in a Black-Scholes framework can be found in the literature (cf. Geske, 1979). For these solutions the following assumptions are made:

1. Security markets are perfect and competitive.
2. Unrestricted short sales of all assets are allowed with full use of proceeds.
3. The risk-free rate of interest is known and constant over time.
4. Trading takes place continuously in time.
5. Changes in the value of the underlying option follow a random walk in continuous time.
6. The variance rate is proportional to the square of the value of the underlying option.

(Geske, 1979)

As with the valuation of standard European options, the principle of risk-neutral valuation is used. The discounting of the expected payoff of the option at expiration by the risk-free interest rate is thus allowed. Also, in a risk-neutral world the underlying asset price has an expected return equal to the risk-free interest rate minus any payouts.

First, K_1 and T_1 are defined as the strike price and maturity of the compound option. The underlying option $c\left(S_{T_1}, K_2, T_2\right)$ has a strike price K_2 and maturity date $T_2 > T_1$. Compound options, therefore, have two strike prices and two exercise dates. S_{T_1} is the value of the underlying asset at time T_1. $PV_{T_1}(.)$ indicates the present value after time T_1 of the quantity in brackets. Two binary variables, φ and Ψ, which are defined below, are used in the derivatives:

$$\phi = \begin{cases} +1 \text{ if the underlying option is a call,} \\ -1 \text{ if the underlying option is a put,} \end{cases}$$

$$\Psi = \begin{cases} +1 \text{ if the compound option is a call,} \\ -1 \text{ if the compound option is a put.} \end{cases}$$

The combined payoff function for a compound option is then given by:

$$\begin{aligned} &\max\left[0, \Psi PV_{T_1}\left[\max\left(0, \phi S_{T_2} - \phi K_2\right)\right] - \Psi K_1\right] \\ &= \max\left[0, \Psi c\left(S_{T_1}, K_2, T_2, \phi\right) - \Psi K_1\right]. \end{aligned} \tag{3.1.1}$$

Consider a call on a call. On the first exercise date, T_1, the holder of the compound option is entitled to pay the strike price, K_1, and receive a call option. The call option gives the holder the right to buy the underlying asset for the second strike price, K_2, on the second exercise date, T_2. The compound option will be exercised on the first exercise date only if the value of the option on that date is greater the first strike price. From (3.1.1) the payoff of a call on a call at time T_1 is:

$$\max\left[0, PV_{T_1}\left[\max\left(0, S_{T_2} - K_2\right)\right] - K_1\right]$$
$$= \max\left[0, c\left(S_{T_1}, K_2, T_2\right) - K_1\right].$$

This is the maximum of the value of the payoff of the underlying option, discounted to the time of expiration of the compound option, T_1, and the strike price of the compound option.

Further define S_t as the value of the underlying option's underlying with a volatility σ. The continuously compounded dividend yield of the underlying asset is q and r is the continuously compounded risk-free interest rate. The payoffs of the four basic European compound options are given in Fig. 9 for S_{T_1} *=100, K_1 = 3, K_2 = 90, r = 5%, q = 3%, T_1 = 1, T_2 = 2* and *σ = 2%.*

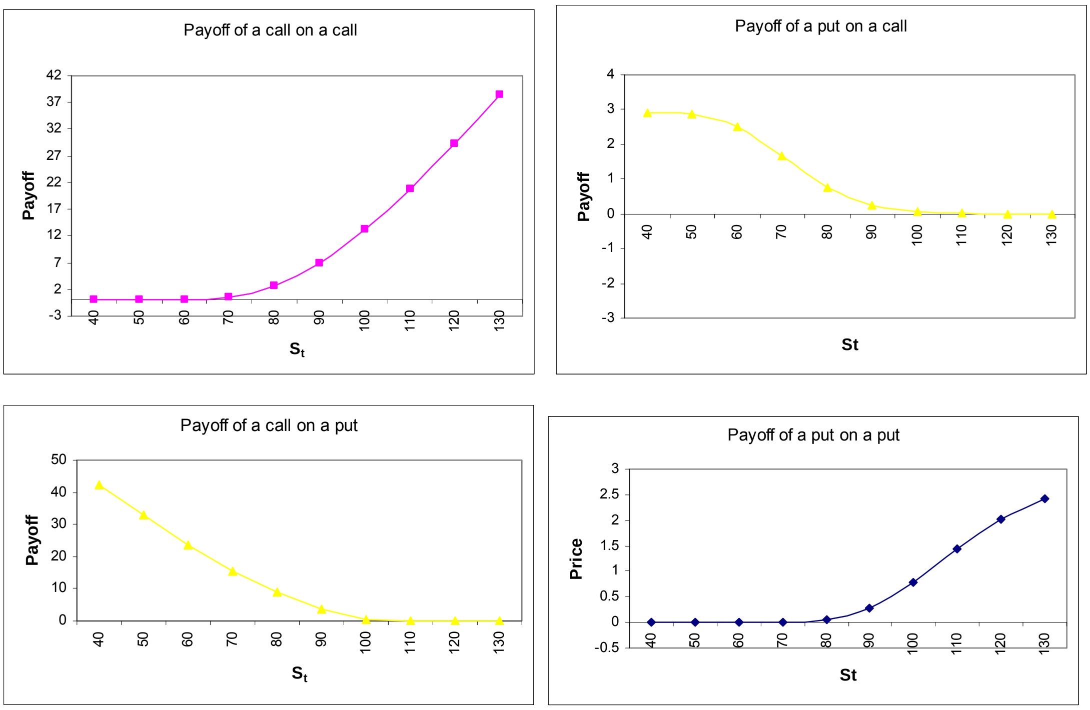

Figure 9: Payoff diagrams for compound options. Parameters: S_{T_1} =100, K_1 = 3, K_2 = 90, r = 5%, q = 3%, T_1 = 1, T_2 = 2, σ = 2%

The results in Lemma 1, considered below, are necessary for the derivation of the value of compound options in Theorem 1.

Lemma 1: (West, 2007)

Define the following for the normal distribution:

$$n(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2},$$

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},$$

$$N(h)=\int_{-\infty}^{h}n(x)\,dx.$$

Then:

$$\int_{-\infty}^{a}n(z)N(A+Bz)\,dz=N_2\left(a,\frac{A}{\sqrt{1+B^2}},\frac{-B}{\sqrt{1+B^2}}\right), \tag{3.1.2}$$

$$\int_{-\infty}^{a}e^{Az}N(C+Bz)n(z)\,dz=e^{\frac{A^2}{2}}N_2\left(a-A,\frac{AB+C}{\sqrt{1+B^2}};\frac{-B}{\sqrt{1+B^2}}\right), \tag{3.1.3}$$

where $N_2(a,b;\rho)$ is the cumulative bivariate standard normal distribution function which is defined by

$$N_2(a,b;\rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{a}\int_{-\infty}^{b}\exp\left(-\frac{u^2-2uv+v^2}{2\left(1-\rho^2\right)}\right)dudv,$$

where ρ is the correlation coefficient between the two bivariate standard normal random variables.

Theorem 1: (Geske, 1979 and Rubinstein, 1991)

Assume that investors are unsatiated, that security markets are perfect and competitive, that unrestricted short sales of all assets are allowed with full use of proceeds, that the risk-free rate of interest is known and constant over time, that trading takes place continuously in time, that changes in the value of the underlying option follow a random walk in continuous time with a variance rate proportional to the square root of the value of the underlying option, and that investors agree on this variance σ^2. *Then the current value at time t of a compound option is given by*

$$C_t\left(S_{T_1},K_1,K_2,T_1,T_2,\sigma,r,q,\phi,\Psi\right)=\phi\Psi S_t e^{-q(T_2-t)}N_2\left(-\phi\Psi\left(X-\sigma\sqrt{\mathrm{T}_1-t}\right),\phi d_+;\Psi\rho\right)$$
$$-\phi\Psi K_2 e^{-r(T_2-t)}N_2\left(-\phi\Psi X,\phi d_-;\Psi\rho\right)$$
$$-\Psi K_1 e^{-r(T_1-t)}N\left(-\phi\Psi X\right)$$

(3.1.4)

where

$$d_+=\frac{\ln\left(\frac{S_t}{K_2}\right)+\left(r-q+\frac{\sigma^2}{2}\right)\left(T_2-t\right)}{\sigma\sqrt{T_2-t}},\qquad(3.1.5)$$

$$d_-=\frac{\ln\left(\frac{S_t}{K_2}\right)+\left(r-q-\frac{\sigma^2}{2}\right)\left(T_2-t\right)}{\sigma\sqrt{T_2-t}}=d_+-\sigma\sqrt{T_2-t},\qquad(3.1.6)$$

and

$$\rho=\sqrt{\frac{\left(T_1-t\right)}{\left(T_2-t\right)}}.\qquad(3.1.7)$$

X is the unique standardised log-return, satisfying

$$H\left(X\right)=c\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)\left(T_1-t\right)+\sigma\times\sqrt{T_1-t}\times X},K_2,T_2,\sigma,r,q,\phi\right)-K_1$$
$$=\phi\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)\left(T_1-t\right)+\sigma\times\sqrt{T_1-t}\times X}e^{-q\left(T_2-T_1\right)}N\left(\phi d_+^{\tau}\right)-K_2e^{-r\left(T_2-T_1\right)}N\left(\phi d_-^{\tau}\right)\right)-K_1$$
$$=0$$

where

$$d_\pm^{\tau}=\frac{\ln\left(\frac{S_{T_1}}{K_2}\right)+\left(r-q\pm\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\qquad(3.1.8)$$

and

$$\tau=T_2-T_1.$$

Proof:

To uniquely determine a price for an option we need a model for the evaluation of the underlying S_t. In the theorem, the Black-Scholes model with constant volatility is assumed, where S_t follows a geometric Brownian motion. The process for S_t is given by

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t, \qquad S_0 > 0, \tag{3.1.9}$$

where $\{W_t\}_{t\geq 0}$ denotes a standard Wiener process, r the continuously compounded risk-free interest rate, q the continuously compounded dividend yield and σ the volatility.

The formula for the value at time T_1 of the underlying option is derived by Wystup (1999) and is given by a generalization of the Black-Scholes formula as:

$$c\left(S_{T_1}, K_2, T_2, \sigma, r, q, \phi\right) = \phi\left(S_{T_1} e^{-q(T_2-T_1)} N\left(\phi d_+^\tau\right) - K_2 e^{-r(T_2-T_1)} N\left(\phi d_-^\tau\right)\right), \tag{3.1.10}$$

$$d_\pm^\tau = \frac{\ln\left(\frac{S_{T_1}}{K_2}\right) + \left(r - q \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}},$$

$$\tau = T_2 - T_1,$$

where φ = 1 for a call and φ = -1 for a put.

The price S_{T_1}, at the current time t is a random variable. Using risk neutral valuation the current value (at time t) of the compound option C is the discounted expectation of the payoff:

$$C_t\left(S_{T_1}, K_1, K_2, T_1, T_2, \sigma, r, q, \phi, \Psi\right) = e^{-r(T_1-t)} \boldsymbol{E}_{T_1}\left[\max\left(0, \Psi c\left(S_{T_1}, K_2, T_2, \sigma, r, q, \phi\right) - \Psi K_1\right)\right].$$

Since S_{T_1}, the price of the underlying asset, is lognormal, the log return $u = \ln\left(\frac{S_{T_1}}{S_t}\right)$ is normally distributed. The probability density function of u follows as

$$f(u)=\frac{1}{\sigma\sqrt{2\pi(T_1-t)}}e^{-\frac{v^2}{2}} \text{ with } v=\frac{u-\mu(T_1-t)}{\sigma\sqrt{T_1-t}} \text{ and } \mu=r-q-\frac{\sigma^2}{2}.$$

Hence C_t can be written as the integral of the payoff over the probability density of S_{T_1} at time t:

$$C_t\left(S_{T_1},K_1,K_2,T_1,T_2,\sigma,r,q,\phi,\Psi\right)=e^{-r(T_1-t)}\int_{-\infty}^{\infty}\max\left(0,\Psi c\left(S_{T_1},K_2,T_2,\sigma,r,q,\phi\right)-\Psi K_1\right)f(u)\,du. \tag{3.1.11}$$

Transforming

$$y=\frac{u-\mu(T_1-t)}{\sigma\sqrt{T_1-t}} \tag{3.1.12}$$

leads to

$$dy=\frac{1}{\sigma\sqrt{T_1-t}}du. \tag{3.1.13}$$

Therefore

$$du=\sigma\sqrt{T_1-t}\,dy. \tag{3.1.14}$$

Note that the transformation in (3.1.12) implies

$$y=\frac{\log\frac{S_{T_1}}{S_t}-\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)}{\sigma\sqrt{T_1-t}},$$

where y is in fact the standardised log-return. Hence

$$\log\frac{S_{T_1}}{S_t}=\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times y,$$

so that the price of the underlying at time T_1 is given by

$$S_{T_1}=S_te^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times y}. \tag{3.1.15}$$

Using (3.1.13), (3.1.14) and (3.1.15), equation (3.1.11) can be written in terms of the standardised log-return:

$$C_t = e^{-r(T_1-t)} \int_{-\infty}^{\infty} \max\left[\Psi\left(c\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times y}, K_2, T_2, \sigma, r, q, \phi \right) - K_1 \right), 0 \right] n(y)\, dy\,. \tag{3.1.16}$$

Notation is simplified by setting

$$H(y) = c\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times y}, K_2, T_2, \sigma, r, q, \phi \right) - K_1\,.$$

Where there is no confusion, the value of a compound option $C_t\left(S_{T_1}, K_1, K_2, T_1, T_2, \sigma, r, q, \phi, \Psi\right)$ will be simplified to C_t. Before valuing C_t it is first written in the form

$$\begin{aligned} C_t &= e^{-r(T_1-t)} \int_{-\infty}^{\infty} \max\left[\Psi H(y), 0\right] n(y)\, dy \\ &= e^{-r(T_1-t)} \int_{\{y:\Psi H(y)\geq 0\}} \left[\Psi H(y)\right] n(y)\, dy. \end{aligned}$$

To evaluate the integral it is noted that the payoff is only positive when $\Psi c\left(S_{T_1}, K_2, T_2, \sigma, r, q, \phi\right) > \Psi K_1$, since the underlying option price is monotonic in the asset price S_{T_1}. The variable S^* is the asset price at time T_1, for which the option price at time T_1 equal K_1. If the actual asset price is more than S^* at time T_1, the first option will be exercised; if it is less than S^*, the option expires worthlessly. To obtain the value S^*, the following equation is solved

$$c\left(S^*, K_2, T_2, \sigma, r, q, \phi\right) - K_1 = \phi\left(S^* e^{-q(T_2-T_1)} N\left(\phi g_+^\tau\right) - K_2 e^{-r(T_2-T_1)} N\left(\phi g_-^\tau\right)\right) - K_1 = 0,$$

where

$$g_{\pm}^{\tau} = \frac{\ln\left(\frac{S^*}{K_2}\right) + \left(r - q \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

Alternatively, the unique standardised log-return X is found which solves the following equation :

$$H(X) = c\left(S_{T_1} e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times X}, K_2, T_2, \sigma, r, q, \phi\right) - K_1 = 0.$$

This can be solved using the Newton-Raphson procedure. Also, because the function *H(y)* is strictly increasing if $\phi = 1$ and strictly decreasing if $\phi = -1$, the value of the compound option can be written as

$$\begin{aligned} C_t &= e^{-r(T_1-t)} \int_{\{y:\Psi H(y)\geq 0\}} \left[\Psi H(y)\right] n(y)\,dy \\ &= e^{-r(T_1-t)} \int_{\{y:\Psi H(y)\geq H(X)\}} \left[\Psi H(y)\right] n(y)\,dy \\ &= e^{-r(T_1-t)} \int_{\{y:-\phi\Psi y\leq -\phi\Psi X\}} \left[\Psi H(y)\right] n(y)\,dy, \end{aligned}$$

simply by checking the four possible cases for the pair (ϕ, Ψ). Now, by substituting $z = -\phi\Psi y,$ the following is obtained for $\phi\Psi = 1$:

$$\begin{aligned} C_t &= -e^{-r(T_1-t)} \int_{-\phi\Psi X}^{-\infty} \left[\Psi H(-\phi\Psi z)\right] n(z)\,dz \\ &= e^{-r(T_1-t)} \int_{-\infty}^{-\phi\Psi X} \left[\Psi H(-\phi\Psi z)\right] n(z)\,dz, \end{aligned}$$

and for $\phi\Psi = -1$,

$$C_t = e^{-r(T_1-t)} \int_{-\infty}^{-\phi\Psi X} \left[\Psi H(-\phi\Psi z)\right] n(z)\,dz.$$

By substituting $z=-\phi\Psi y$ into (3.1.16) and using the results in (3.1.10) and (3.1.11), the value of the compound option follows as:

$$C_t = e^{-r(T_1-t)}\int_{y=-\infty}^{y=+\infty}\left[\Psi\left(c\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times y}, K_2, T_2, \sigma, r, q, \phi\right)-K_1\right)\right]^+ n(y)\,dy$$

$$= e^{-r(T_1-t)}\int_{z=-\infty}^{z=-\phi\Psi X}\left[\Psi\left(c\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)-\sigma\times\sqrt{T_1-t}\times\phi\Psi z}, K_2, T_2, \sigma, r, q, \phi\right)-K_1\right)\right] n(z)\,dz$$

$$= e^{-r(T_1-t)}\times$$

$$\int_{z=-\infty}^{z=-\phi\Psi X}\left[\Psi\left(\begin{array}{l}\phi S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)-\sigma\times\sqrt{T_1-t}\times\phi\Psi z} e^{-q(T_2-T_1)} N\left(\phi\dfrac{\ln\left(\dfrac{S_{T_1}}{K_2}\right)+\left(r-q+\dfrac{\sigma^2}{2}\right)(T_2-T_1)}{\sigma\sqrt{T_2-T_1}}\right)- \\ K_2 e^{-r(T_2-T_1)} N\left(\phi\dfrac{\ln\left(\dfrac{S_{T_1}}{K_2}\right)+\left(r-q-\dfrac{\sigma^2}{2}\right)(T_2-T_1)}{\sigma\sqrt{T_2-T_1}}\right)\end{array}\right)-\Psi K_1\right] n(z)\,dz.$$

To evaluate the integral, it is broken down into three components corresponding to the three payoff variables S_t , K_1 and K_2 inside the square brackets. Then (3.1.15) is substituted in and the familiar forms of (3.1.5), (3.1.6) and (3.1.7) are recognized.

$$[1] \equiv \phi\Psi S_t e^{-r(T_1-t)+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)-q(T_2-T_1)}\times$$

$$\int_{z=-\infty}^{z=-\phi\Psi X} e^{-\sigma\times\sqrt{T_1-t}\times\phi\Psi z} N\left(\phi\frac{\ln\dfrac{S_t}{K_2}-\sqrt{T_1-t}\times\phi\Psi z+\left(r-q-\dfrac{\sigma^2}{2}\right)(T_1-t)+\left(r-q+\dfrac{\sigma^2}{2}\right)(T_2-T_1)}{\sigma\sqrt{T_2-T_1}}\right) n(z)\,dz$$

$$= \phi\Psi S_t e^{-r(T_1-t)+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)-q(T_2-T_1)}\times$$

$$\int_{z=-\infty}^{z=-\phi\Psi X} e^{-\sigma\times\sqrt{T_1-t}\times\phi\Psi z} N\left(-\Psi z\sqrt{\frac{(\mathrm{T}_1-t)}{(T_2-T_1)}}+\phi\frac{\ln\dfrac{S_t}{K_2}+\left(r-q-\dfrac{\sigma^2}{2}\right)(T_1-t)+\left(r-q+\dfrac{\sigma^2}{2}\right)(T_2-T_1)}{\sigma\sqrt{T_2-T_1}}\right) n(z)\,dz$$

$$= \phi\Psi S_t e^{-q(T_2-t)} N_2\left(-\phi\Psi\left(X-\sigma\sqrt{\mathrm{T}_1-t}\right), \phi d_+; \Psi\rho\right)$$

For part [1] the identity in (3.1.3) is applied in the final step with

$$x \text{ in } (3.1.3) = z \text{ in } [1],$$
$$a \text{ in } (3.1.3) = -\phi\Psi X \text{ in } [1],$$
$$A \text{ in } (3.1.3) = -\sigma \times \sqrt{\mathrm{T}_1 - t} \times \phi\Psi \text{ in } [1],$$
$$B \text{ in } (3.1.3) = -\Psi\sqrt{\frac{(\mathrm{T}_1 - t)}{(T_2 - T_1)}} \text{ in } [1], \text{ and}$$
$$C \text{ in } (3.1.3) = \phi\frac{\ln\frac{S_t}{K_2} + \left(r - q - \frac{\sigma^2}{2}\right)(T_1 - t) + \left(r - q + \frac{\sigma^2}{2}\right)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \text{ in } [1].$$

Then

$$\begin{aligned}
a - A &= -\phi\Psi X - \left(-\sigma \times \sqrt{\mathrm{T}_1 - t} \times \phi\Psi\right) \\
&= -\phi\Psi\left(X - \sigma\sqrt{\mathrm{T}_1 - t}\right),
\end{aligned}$$

$$\begin{aligned}
\frac{AB + C}{\sqrt{1 + B^2}} &= \frac{\left(-\sigma \times \sqrt{\mathrm{T}_1 - t} \times \phi\Psi\right)\left(-\Psi\sqrt{\frac{(\mathrm{T}_1 - t)}{(T_2 - T_1)}}\right) + \phi\frac{\ln\frac{S_t}{K_2} + \left(r - q - \frac{\sigma^2}{2}\right)(T_1 - t) + \left(r - q + \frac{\sigma^2}{2}\right)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}}{\sqrt{1 + \left(-\Psi\sqrt{\frac{(\mathrm{T}_1 - t)}{(T_2 - T_1)}}\right)^2}} \\
&= \frac{\sigma^2(\mathrm{T}_1 - t) + \phi\ln\frac{S_t}{K_2} + \phi\left(r - q - \frac{\sigma^2}{2}\right)(T_1 - t) + \phi\left(r - q + \frac{\sigma^2}{2}\right)(T_2 - T_1)}{\sigma\sqrt{T_2 - t}} \\
&= \phi d_+,
\end{aligned}$$

$$\begin{aligned}
\frac{-B}{\sqrt{1 + B^2}} &= \frac{\Psi\sqrt{\frac{(\mathrm{T}_1 - t)}{(T_2 - T_1)}}}{\sqrt{1 + \left(-\Psi\sqrt{\frac{(\mathrm{T}_1 - t)}{(T_2 - T_1)}}\right)^2}} \\
&= \Psi\frac{\sqrt{\frac{(\mathrm{T}_1 - t)}{(T_2 - T_1)}}}{\sqrt{\frac{(\mathrm{T}_2 - t)}{(T_2 - T_1)}}} \\
&= \Psi\rho.
\end{aligned}$$

$$[2] \equiv -\phi\Psi K_2 e^{-r(T_2-t)} \times$$

$$\int_{z=-\infty}^{z=-\phi\Psi X} e^{-\sigma\times\sqrt{T_1-t}\times\phi\Psi z} N\left(\phi\frac{\ln\frac{S_t}{K_2}-\sqrt{T_1-t}\times\phi\Psi z+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\left(r-q+\frac{\sigma^2}{2}\right)(T_2-T_1)}{\sigma\sqrt{T_2-T_1}}\right)n(z)\,dz$$

$$= -\phi\Psi K_2 e^{-r(T_2-t)} \times$$

$$\int_{z=-\infty}^{z=-\phi\Psi X} N\left(-\Psi z\sqrt{\frac{(\mathrm{T}_1-t)}{(T_2-T_1)}}+\phi\frac{\ln\frac{S_t}{K_2}+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\left(r-q+\frac{\sigma^2}{2}\right)(T_2-T_1)}{\sigma\sqrt{T_2-T_1}}\right)n(z)\,dz$$

$$= -\phi\Psi K_2 e^{-r(T_2-t)} N_2\left(-\phi\Psi X, \phi d_-; \Psi\rho\right)$$

For part [2] the identity in (3.1.2) is applied along the same lines as for [1] in the final step with

$$a = -\phi\Psi X,$$

$$A = \phi\frac{\ln\frac{S_t}{K_2}+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\left(r-q+\frac{\sigma^2}{2}\right)(T_2-T_1)}{\sigma\sqrt{T_2-T_1}},$$

$$B = -\Psi\sqrt{\frac{(\mathrm{T}_1-t)}{(T_2-T_1)}}.$$

$$[3] \equiv -\Psi K_1 e^{-r(T_1-t)} N\left(-\phi\Psi X\right),$$

where $N_2(a,b;\rho)$ is the bivariate standard normal distribution function, which is defined in (3.1.3). Standard results for the integral of the product of a normal density and a cumulative normal distribution was used for the first two integrals as given in Lemma 1.

Putting these together, the current value of the compound option is

$$C_t = \phi\Psi S_t e^{-q(T_2-t)} N_2\left(-\phi\Psi\left(X-\sigma\sqrt{\mathrm{T}_1-t}\right), \phi d_+; \Psi\rho\right)$$
$$-\phi\Psi K_2 e^{-r(T_2-t)} N_2\left(-\phi\Psi X, \phi d_-; \Psi\rho\right)$$
$$-\Psi K_1 e^{-r(T_1-t)} N\left(-\phi\Psi X\right),$$

where

$$d_+ = \frac{\ln\left(\frac{S_t}{K_2}\right) + \left(r-q+\frac{\sigma^2}{2}\right)\left(T_2-t\right)}{\sigma\sqrt{T_2-t}},$$

$$d_- = \frac{\ln\left(\frac{S_t}{K_2}\right) + \left(r-q-\frac{\sigma^2}{2}\right)\left(T_2-t\right)}{\sigma\sqrt{T_2-t}} = d_+ - \sigma\sqrt{T_2-t},$$

and

$$\rho = \sqrt{\frac{\left(T_1-t\right)}{\left(T_2-t\right)}}.$$

The formula for the compound option involves the bivariate cumulative normal distribution. This comes from the fact that the option price depends on the joint distribution of the asset price at the maturity dates of the compound and underlying options (Clewlow and Strickland,1997).

Hence, using

$$\phi = \begin{cases} +1 \text{ if the underlying option is a call,} \\ -1 \text{ if the underlying option is a put,} \end{cases}$$

$$\Psi = \begin{cases} +1 \text{ if the compound option is a call,} \\ -1 \text{ if the compound option is a put.} \end{cases}$$

leads to the following:

The value of a European call on a call is

$$C_{call\ on\ call} = S_t e^{-q(T_2-t)} N_2\left(\sigma\sqrt{\mathrm{T}_1-t}-X, d_+; \rho\right) - K_2 e^{-r(T_2-t)} N_2\left(-X, d_-; \rho\right) - K_1 e^{-r(T_1-t)} N\left(-X\right).$$

The value of a European put on a call is

$$C_{put\ on\ call} = -S_t e^{-q(T_2-t)} N_2\left(X-\sigma\sqrt{\mathrm{T}_1-t}, d_+; -\rho\right) + K_2 e^{-r(T_2-t)} N_2\left(X, d_-; -\rho\right) + K_1 e^{-r(T_1-t)} N\left(X\right).$$

The value of a European call on a put is

$$C_{call\ on\ put} = -S_t e^{-q(T_2-t)} N_2\left(X - \sigma\sqrt{T_1 - t}, -d_+; \rho\right) + K_2 e^{-r(T_2-t)} N_2\left(X, -d_-; \rho\right) - K_1 e^{-r(T_1-t)} N\left(X\right).$$

The value of a European put on a put is

$$C_{put\ on\ put} = S_t e^{-q(T_2-t)} N_2\left(\sigma\sqrt{T_1 - t} - X, -d_+; -\rho\right) - K_2 e^{-r(T_2-t)} N_2\left(-X, -d_-; -\rho\right) + K_1 e^{-r(T_1-t)} N\left(-X\right).$$

3.1.4 The Sensitivity of Compound Options to Volatility

Compound option values are extremely sensitive to the volatility of volatility (www.riskglossary.com). This follows from the fact that the price of the underlying option is firstly determined using the implied future rates and volatilities. Then this option value is used as the underlying for the compound option. As with standard options, the volatility of the underlying will be a key factor of the value. However, with compound options, it is more significant as it has a double effect. If volatility rises, this raises the value of the option. With a compound option, an increase in volatility will also increase the value of the underlying asset (another option) (www.my.dreamwiz.com).

The analytic formulas derived above incorporate the Black-Scholes assumption of constant volatility, so they tend to undervalue the options significantly. Research into pricing methodologies in this regard is ongoing (www.riskglossary.com).

In Fig. 10, below S_{T_1} = 100, S_t = 90, K_1 = 90, K_2 = 6, T_1 = 1, T_2 = 2, t = 0.5, r = 5% and q = 3%. As the volatility increases from 0.1 to 0.55, the value of the standard call remains unchanged at R84.43. For a similar change in volatility, the value of the compound call on call option changes with R10.08 or 2584.62%, increasing from R0.39 to R10.47. The change in the compound put on call is R10.08 or 130.91%, increasing from R7.70 to R17.78. The higher the value of the underlying standard call option (deeper in-the-money), the greater the sensitivity of the compound options to changes in the volatility, compared to the standard option. In Fig. 11 where K_1 = 90, K_2 = 100, the underlying option is out-of-the-money and the standard option is more sensitive to changes in volatility than the compound options. The total increase in the value of the standard call option is R 19.23 or 669.15%, from R2.95 to R22.69. The total increase in the value of the compound call on call is R0.57 from R0 to R0.57.

The value of the compound put on call decreases with R17.99 or 20.92%, from R85.98 to R67.99.

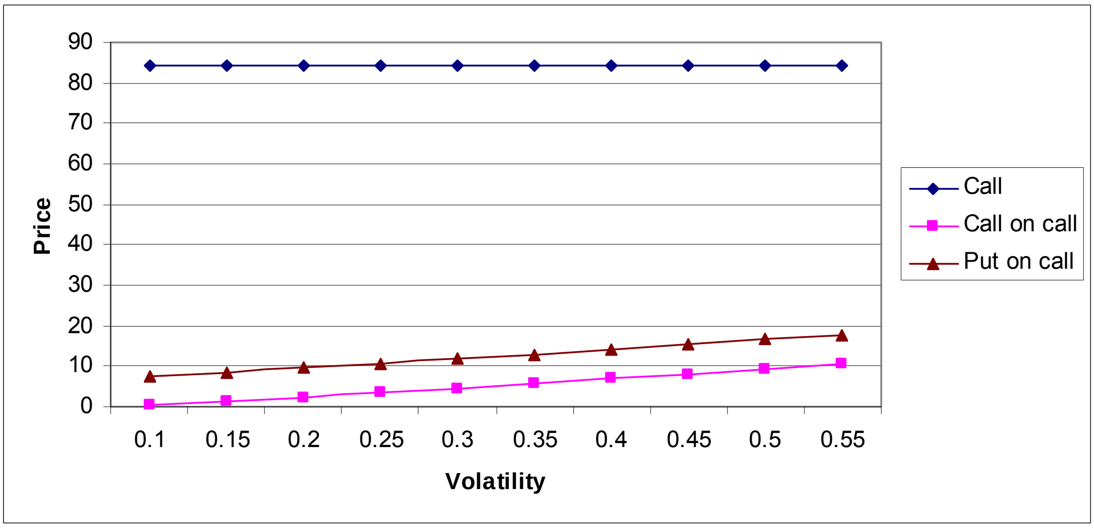

Figure 10: The prices of a standard call and compound options on a call as a function of volatility where if K_1 = 90, K_2 = 6.

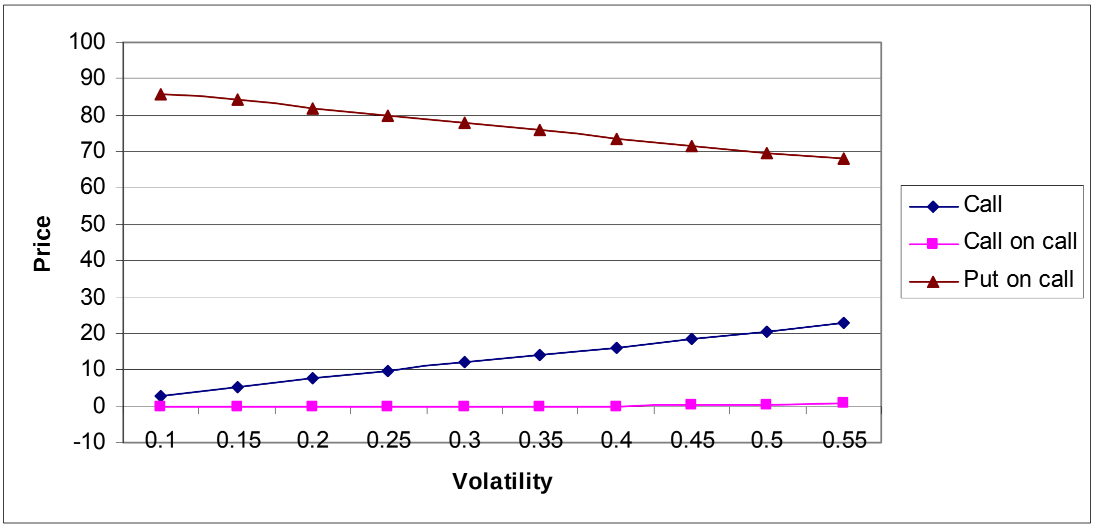

Figure 11: The prices of a standard call and compound options on a call as a function of volatility where K_1 = 6, K_2 = 100.

Fig. 12 shows that the pattern does not hold for underlying standard put options and the compound options on a put.

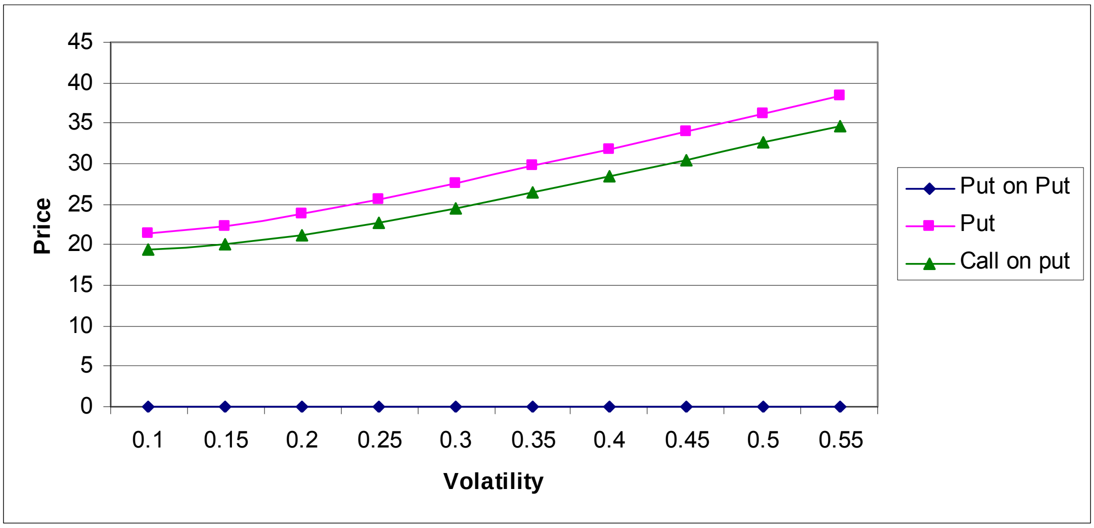

Figure 12: The prices of a standard put and compound options on a put as a function of volatility where $K_1 = 6$, $K_2 = 120$.

We note in Fig. 13 that for compound calls, the higher the second optional payment defined as the strike price of the compound call option $K_{1,}$ the lower the initial compulsory payment. This means that as K_1 increases, the value of a call on a put and a call on a call will decrease. The opposite holds for compound puts: the higher the value of K_1, the higher the price of both a put on a call and a put on a put.

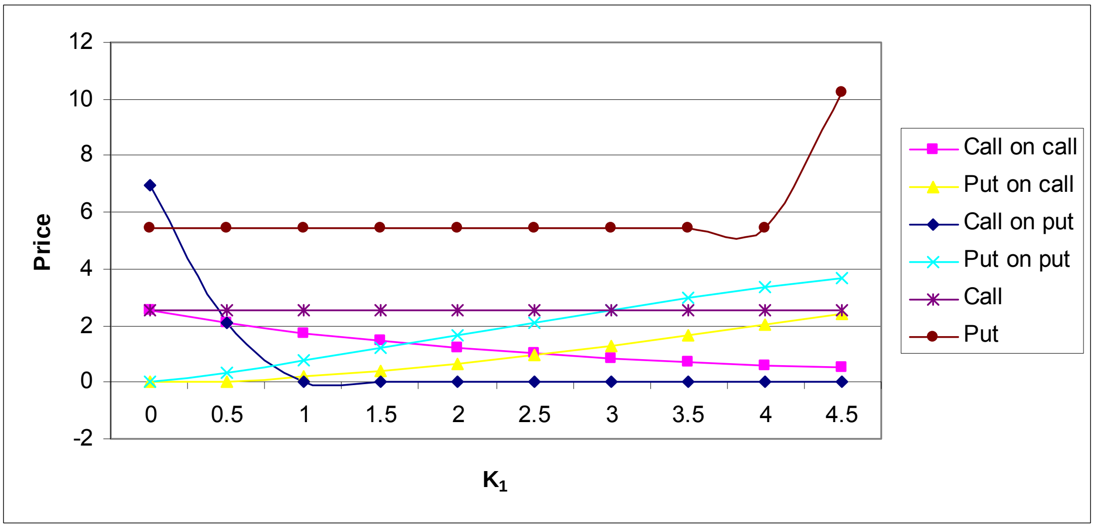

Figure 12: The prices of compound options as a function of the strike price. Parameters: S_t =45, S_{T_1} =100, $K_1 = 3$, $K_2 = 50$, r = 0%, q = 0%, T1 = 1, $T_2 = 2$, σ = 2%.

3.1.5 Arbitrage Bounds on Valuation

Shilling (2001) derives upper and lower bounds for the value of the compound options in terms of the underlying standard options. They are based on the assumption that the market is free of arbitrage opportunities.

Define

- $c_t\left(S_t,t,T_2,K_2,r,q,\phi\right)$: the value of a standard European option,
- $c_t^c\left(S_t,t,T_2,K_2,r,q\right)$: the value of a standard European call option,
- $c_t^p\left(S_t,t,T_2,K_2,r,q\right)$: the value of a standard European put option,
- $C_t^c\left(S_t,t,T_1,T_2,K_1,K_2,r,q,\phi\right)$: the value of a European compound call option, and
- $C_t^p\left(S_t,t,T_1,T_2,K_1,K_2,r,q,\phi\right)$: the value of a European compound put option.

Theorem 2 (Put-call parity for compound options): Shilling (2001)
Given a compound call C_t^c *and a compound put* C_t^p *with the same strike* K_1 *and the same maturity* T_1 *on the same underlying option* $c_t\left(S_t,t,T_2,K_2,r,q,\phi\right)$ *the following relationship holds for* $t\in\left[0,T_1\right]$*:*

$$\begin{aligned}&C_t^c\left(S_t,t,T_1,T_2,K_1,K_2,r,q,\phi\right)+K_1e^{-r(T_1-t)}\\&=C_t^p\left(S_t,t,T_1,T_2,K_1,K_2,r,q,\phi\right)+c_t\left(S_t,t,T_2,K_2,r,q,\phi\right).\end{aligned}\tag{3.1.17}$$

Proof:
We derive this relationship by constructing two portfolios. If they pay the same amount under all conditions at maturity, and cannot be exercised before the expiration date, then they must cost the same today.

Portfolio A: Buy one European compound call option at a price of C_t^c.

At the same time deposit enough money to give the strike price at the time of expiration of the call option. This is the cash amount equal to $K_1e^{-r(T_1-t)}$.

Portfolio B: Buy one European compound put option at a price of C_t^p.

Buy one share of the underlying standard European option at its current price c_t.

The value of the strategies at maturity of the option, time T_1, if $c\left(S_{T_1}, K_2, T_2, \sigma, r, q, \phi\right) > K_1$:

Portfolio A: The compound call option is exercised and the portfolio is worth

$$(c\left(S_{T_1}, K_2, T_2, \sigma, r, q, \phi\right) - K_1) + K_1 = c\left(S_{T_1}, K_2, T_2, \sigma, r, q, \phi\right) = c_{T_1}.$$

Portfolio B: The compound put option expires worthlessly and the portfolio is worth c_{T_1}.

Thus in this case Portfolio A = Portfolio B at time T_1.

The value of the strategies at maturity of the option if $c\left(S_{T_1}, K_2, T_2, \sigma, r, q, \phi\right) < K_1$:

Portfolio A: The compound call option expires worthlessly and the portfolio is worth K_1.

Portfolio B: The compound put option is exercised and the portfolio is worth

$$c\left(S_{T_1}, K_2, T_2, \sigma, r, q, \phi\right) - c\left(S_{T_1}, K_2, T_2, \sigma, r, q, \phi\right) + K_1 = K_1.$$

Thus Portfolio A = Portfolio B at time T_1 in this case also.

Therefore Portfolio A = Portfolio B at the exercise date T_1 in both cases. This means that the result holds independently of whether $c\left(S_{T_1}, K_2, T_2, \sigma, r, q, \phi\right) > K_1$ or $c\left(S_{T_1}, K_2, T_2, \sigma, r, q, \phi\right) < K_1$, hence independently of the value of S_{T_1}. Since the values are the same at time T_1, they must also be equal at time 0. It follows that

$$\begin{aligned} &C_t^c\left(S_t, t, T_1, T_2, K_1, K_2, r, q, \phi\right) + K_1 e^{-r(T_1 - t)} \\ &= C_t^p\left(S_t, t, T_1, T_2, K_1, K_2, r, q, \phi\right) + c_t\left(S_t, t, T_2, K_2, r, q, \phi\right). \end{aligned}$$

■

To derive no-arbitrage bounds on the value of compound options, the following lemma (see Merton ,1973) is useful.

Lemma 2: Shilling (2001)

Suppose there are two standard options $c_t(S_t,t,T,K_1,r,q,\phi)$ *and* $c_t(S_t,t,T,K_2,r,q,\phi)$. *Let* $0<K_1<K_2$. *Then the following inequality holds for* $t\in[0,T_1]$*:*

$$0<\phi\left[c_t(S_t,t,T,K_1,\phi)-c_t(S_t,t,T,K_2,\phi)\right]<e^{-r(T-t)}(K_2-K_1). \qquad (4.1.18)$$

Proof:

The left inequality is obviously true:

If ϕ=1: The lower the strike, the more valuable a call option becomes since the payoff from a call is given by $\max(0,S_t-K)$. Therefore

$$c_t^c(S_t,t,T,K_1)>c_t^c(S_t,t,T,K_2).$$

If ϕ= -1: The higher the strike, the more valuable a put option becomes since the payoff from a put is given by $\max(0,K-S_t)$. Therefore

$$c_t^p(S_t,t,T,K_1)<c_t^p(S_t,t,T,K_2).$$

whereas for put options the contrary is true.

For ϕ = +1, the right inequality can be shown by comparing the payoff profiles of $c_t^c(S_t,t,T,K_2)$ and $e^{-r(T-t)}(K_2-K_1)$ on the one hand and $c_t^c(S_t,t,T,K_1)$ on the other hand. In Fig. 13 the payoff profiles of $c_t^c(S_t,t,T,K_1)-c_t^c(S_t,t,T,K_2)$ and $e^{-r(T-t)}(K_2-K_1)$ is shown.

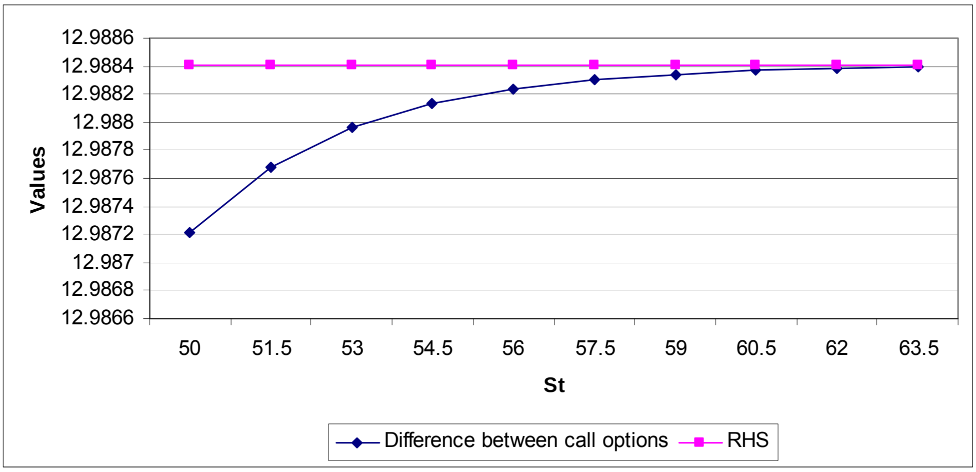

Figure 13: Comparing the payoff profiles of the right inequality for ϕ = +1 Parameters: S_t =50, K_1 = 6, K_2 = 20, r = 0%, q = 0%, T = 2, σ = 2%, t = 0.5.

The proof for ϕ = -1 is analogous: In Fig. 14 the payoff profiles of $c_t^p\left(S_t,t,T,K_1\right)-c_t^p\left(S_t,t,T,K_2\right)$ and $e^{-r(T-t)}\left(K_2-K_1\right)$ are shown.

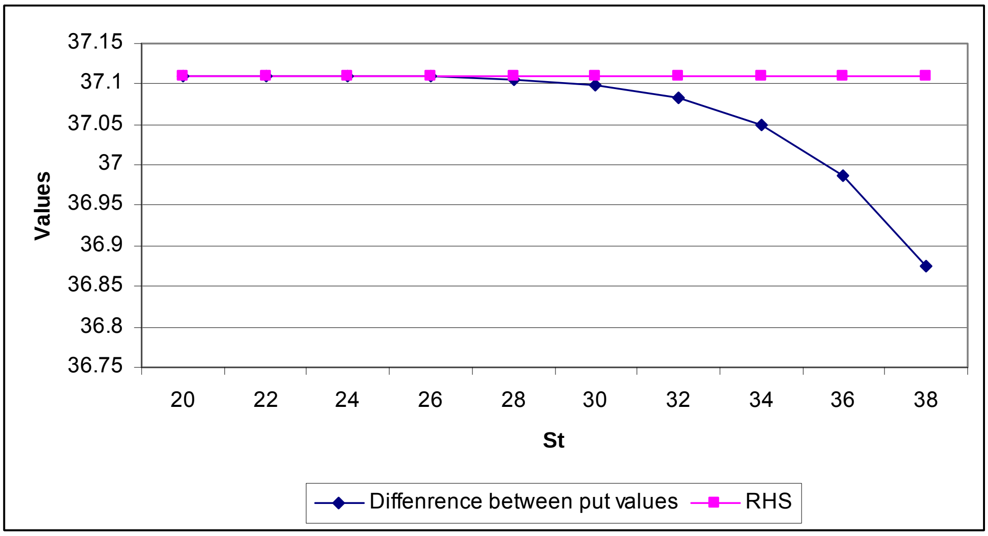

Figure 14: Comparing the payoff profiles of the right inequality for ϕ = -1 Parameters: S_t =50, K_1 = 6, K_2 = 20, r = 0%, q = 0%, T = 2, σ = 2%, t = 0.5.

■

Like standard options, compound options cannot be more valuable than their underlying (in the case of compound calls) or their strike (in the case of compound puts). Hence $c_t\left(S_t,t,T_2,K_2,\phi\right)$ is a upper bound for compound calls and $e^{-r(T_1-t)}K_1$ is a upper bound for a compound put. It is possible to improve these trivial bounds.

Theorem 3 (Upper bound on the value of compound options): Shilling (2001)

Suppose there is a compound option $C_t\left(S_t,t,T_1,T_2,K_1,K_2,r,q,\phi,\Psi\right)$ *with the underlying* $c_t\left(S_t,t,T_2,K_2,r,q,\phi\right)$. *Then the following inequalities hold for* $t\in\left[0,T_1\right]$,

$$C_t^c\left(S_t,t,T_1,T_2,K_1,K_2,\phi\right)<c_t\left(S_t,t,T_2,K_2+\phi K_1e^{r(T_2-T_1)},\phi\right) \tag{3.1.19}$$

$$<c_t\left(S_t,t,T_2,K_2,\phi\right), \tag{3.1.20}$$

$$C_t^p\left(S_t,t,T_1,T_2,K_1,K_2,\phi\right)<c_t\left(S_t,t,T_2,K_2+\phi K_1e^{r(T_2-T_1)},-\phi\right)-c_t\left(S_t,t,T_2,K_2,-\phi\right) \tag{3.1.21}$$

$$<K_1e^{-r(T_1-t)}. \tag{3.1.22}$$

Proof:

Proof of (3.1.19) and (3.1.20).

It is shown that the following relationship holds at T_1:

$$\max\left[0,c_t\left(S_{T_1},T_1,T_2,K_2,\phi\right)-K_1\right]<c_t\left(S_t,t,T_2,K_2+\phi K_1e^{r(T_2-T_1)},\phi\right) \tag{3.1.23}$$

$$<c_t\left(S_t,t,T_2,K_2,\phi\right). \tag{3.1.24}$$

Using Lemma 2 leads to the inequality

$$0<c_t\left(S_{T_1},T_1,T_2,K_2,\phi\right)-c_t\left(S_{T_1},T_1,T_2,K_2+\phi K_1e^{r(T_2-T_1)},\phi\right)<e^{-r(T_2-T_1)}\left(K_2+\phi K_1e^{r(T_2-T_1)}-K_2\right)$$

$$\therefore 0<c_t\left(S_{T_1},T_1,T_2,K_2,\phi\right)-c_t\left(S_{T_1},T_1,T_2,K_2+\phi K_1e^{r(T_2-T_1)},\phi\right)<K_1 \tag{3.1.25}$$

which implies (3.1.24).

On the other hand, inequality (3.1.25) can be transformed to

$$c_t\left(S_{T_1},T_1,T_2,K_2,\phi\right)-K_1<c_t\left(S_{T_1},T,T_2,K_2+\phi K_1e^{r(T_2-T_1)},\phi\right). \tag{3.1.26}$$

As the value of a standard option is always positive, (3.1.26) is equivalent to (3.1.23).

Proof of (3.1.21) and (3.1.22).

These inequalities are derived by using – in this order – the put-call parity for compound options (3.1.17), inequality (3.1.19), the put-call parity for standard options and Lemma 2.

$$
\begin{aligned}
& C_t^p\left(S_t,t,T_1,T_2,K_1,K_2,\phi\right) \\
& = C_t^c\left(S_t,t,T_1,T_2,K_1,K_2,\phi\right) - c_t\left(S_t,t,T_2,K_2,\phi\right) + K_1 e^{-r(T_1-t)} \\
& < c_t\left(S_t,t,T_2,K_2+\phi K_1 e^{r(T_2-T_1)},\phi\right) - c_t\left(S_t,t,T_2,K_2,\phi\right) + K_1 e^{-r(T_1-t)} \\
& = c_t\left(S_t,t,T_2,K_2+\phi K_1 e^{r(T_2-T_1)},\phi\right) - c_t\left(S_t,t,T_2,K_2,-\phi\right) \\
& < K_1 e^{-r(T_1-t)}
\end{aligned}
\qquad (3.1.27)
$$

■

Remark 1. Shilling (2001)

For $T_2 \searrow T_1$ the value of a compound option converges towards the value of its upper no-arbitrage bound.

Proof:

Remark 1 implies that the payoff of a compound option $C_t\left(S_t,t,T_1,T_2,K_1,K_2,\phi,\Psi\right)$ for $T_2 \searrow T_1$ is equal to that of a standard option $c_t\left(S_t,t,T_1,K_2+\phi K_1,\phi\right)$, using the following limit:

$$\lim_{T_2 \searrow T_1}\left(C_t\left(S_{T_1},T_1,T_2,K_1,K_2,\sigma,r,q,\phi,\Psi\right)\right) = \max\left[\Psi\left(\max\left[\phi\left(S_{T_1}-K_2\right),0\right]-K_1\right)\right]. \quad (3.1.28)$$

If the following identities hold (3.1.28) must be true.

For compound calls (Ψ = +1),

$$\max\left[\max\left[\phi\left(S_{T_1}-K_2\right),0\right]-K_1\right] = \max\left[\phi\left(S_{T_1}-K_2\right)-K_1,0\right],$$

and for compound puts (Ψ = -1),

$$\max\left[K_1-\max\left[\phi\left(S_{T_1}-K_2\right),0\right]\right] = \max\left[K_1+\phi\left(K_2-S_{T_1}\right)-K_1,0\right]-\max\left[\phi\left(K_2-S_{T_1}\right),0\right].$$

These identities can be verified by examining all possible cases.

Consider as an example Ψ = +1, ϕ = +1: Firstly,

$$\max\left[\phi\left(S_{T_1}-K_2\right),0\right] = \begin{cases} S_{T_1}-K_2 \ \textit{if}\ S_{T_1} > K_2 \\ 0 \qquad\quad \textit{if}\ S_{T_1} > K_2. \end{cases}$$

Then

$$\max\left[\Psi\left(\max\left[\phi\left(S_{T_1}-K_2\right),0\right]-K_1\right)\right]=\begin{cases}S_{T_1}-K_2-K_1 & \text{if } S_{T_1}>K_2 \text{ and } S_{T_1}-K_2>K_1\\ 0 & \text{otherwise}\end{cases}$$

$$=\max\left[\phi\left(S_{T_1}-K_2\right)-K_1,0\right].$$

The other combinations of Ψ and ϕ follow analogously.

■

As for standard options, the lower arbitrage bound of compound options is equal to its discounted intrinsic value.

Theorem 4 (Lower arbitrage bound on the value of compound options): Shilling (2001)
Given a compound option $C_t\left(S_t,t,T_1,T_2,K_1,K_2,r,q,\phi,\Psi\right)$ *with the underlying option* $c_t\left(S_t,t,T_2,K_2,r,q,\phi\right)$*, the following inequality holds for* $t\in\left[0,T_1\right)$*,*

$$\max\left[\Psi\left(c_t\left(S_t,t,T_2,K_2,\phi\right)-e^{-r(T_1-t)}K_1\right),0\right]<C_t\left(S_t,t,T_1,T_2,K_1,K_2,\phi,\Psi\right). \qquad (3.1.29)$$

Proof: In the case of a compound call (Ψ = +1),

- let portfolio A consist of a compound call $C_t^c\left(S_t,t,T_1,T_2,K_1,K_2,\phi\right)$ and an investment of $e^{-r(T_1-t)}K_1$ in bonds,
- let portfolio B consist of a standard option $c_t\left(S_t,t,T_2,K_2,\phi\right)$.

At time T_1 portfolio A is worth $\max\left[c_t\left(S_{T_1},T_1,T_2,K_2,\phi\right),K_1\right]$, whereas portfolio B is worth $c_t\left(S_{T_1},T_1,T_2,K_2,\phi\right)$. Using the no-arbitrage arguments, portfolio A must be more valuable than portfolio B at time $t<T_1$. This can be transformed to yield

$$c_t\left(S_t,t,T_2,K_2,\phi\right)-e^{-r(T_1-t)}K_1<C_t^c\left(S_t,t,T_1,T_2,K_1,K_2,\phi\right).$$

As the value of the compound option is always positive, the lower bound is given by (3.1.29).

In the case of a compound put (Ψ = -1),

- let portfolio C consist of a compound put $C_t^p\left(S_t,t,T_1,T_2,K_1,K_2,\phi\right)$ and a standard option $c_t\left(S_t,t,T_2,K_2,\phi\right)$;

- let portfolio D consist an investment of $e^{-r(T_1-t)}K_1$ in bonds.

At time T$_1$ portfolio C is worth $\max\left[c_t\left(S_{T_1},T_1,T_2,K_2,\phi\right),K_1\right]$, whereas portfolio D is worth K_1. Using the no-arbitrage arguments, portfolio C must be more valuable than portfolio D at time $t<T_1$. This can be transformed to yield

$$e^{-r(T_1-t)}K_1-c_t\left(S_t,t,T_2,K_2,\phi\right)<C_t^p\left(S_t,t,T_1,T_2,K_1,K_2,\phi\right).$$

As the value of the compound option is always positive, the lower bound is given by (3.1.29).

■

The arbitrage bounds on the four basic compounds options are illustrated in Fig. 15 to Fig. 19 below.

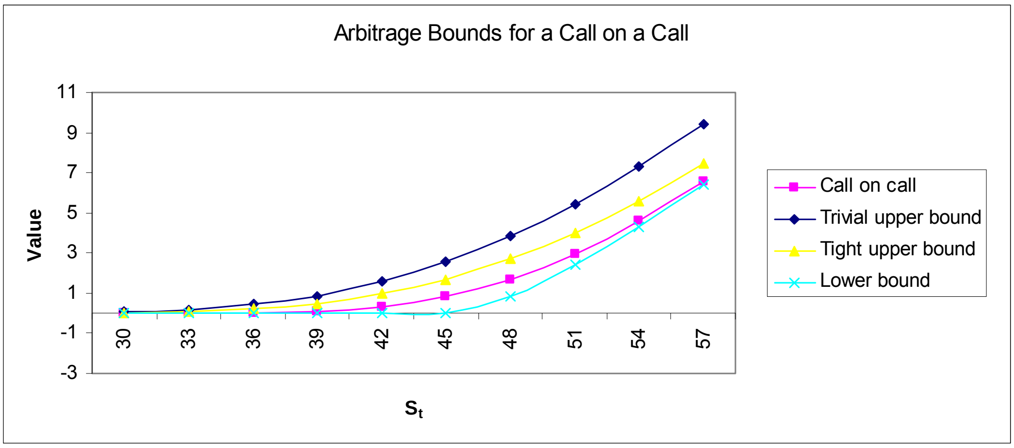

Figure 15: Arbitrage bounds for a Call on a Call. Parameters: S_{T_1} =50, K_1 = 3, K_2 = 50, r = 0%, q = 0%, T_1 = 1, T_2 = 2, t = 0.5, σ = 2%.

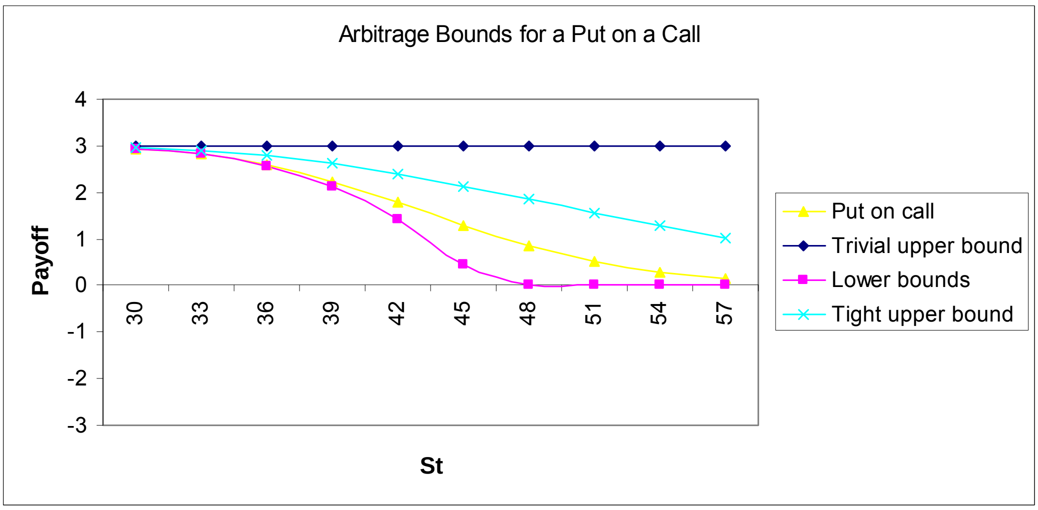

Figure 16: Arbitrage bounds for a Put on a Call. Parameters: S_{T_1} =50, K_1 = 3, K_2 = 50, r = 0%, q = 0%, T_1 = 1, T_2 = 2, t = 0.5, σ = 2%.

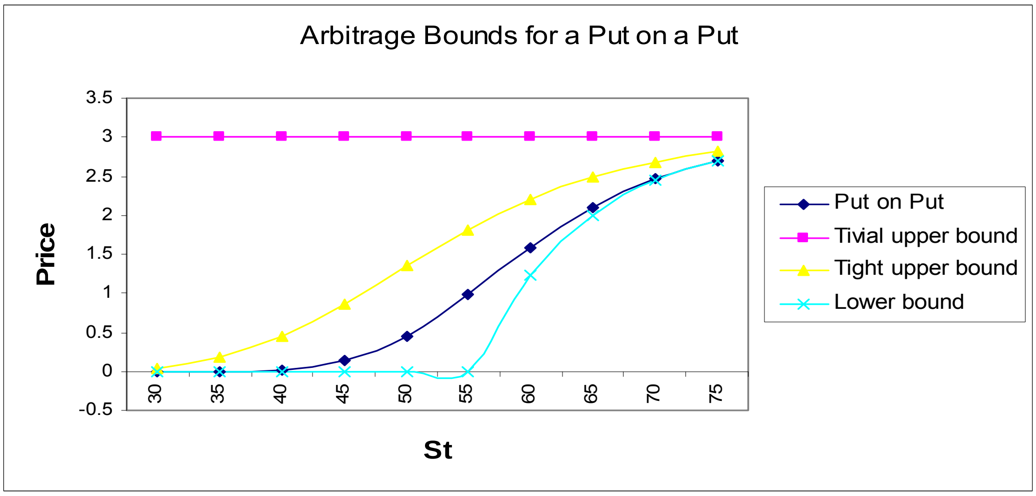

Figure 17: Arbitrage bounds for a Put on a Put. Parameters: S_{T_1} =50, K_1 = 3, K_2 = 50, r = 0%, q = 0%, T_1 = 1, T_2 = 2, t = 0.5, σ = 2%.

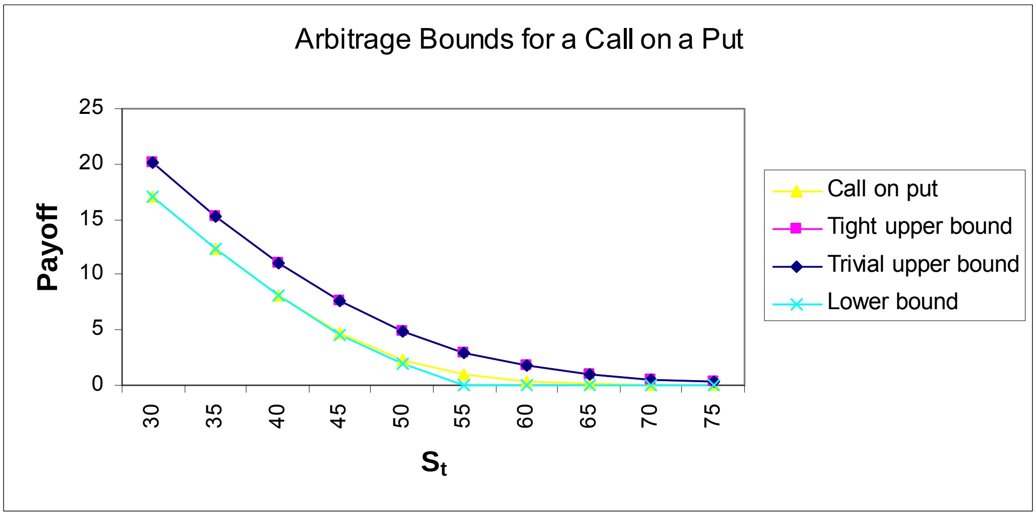

Figure 18: Arbitrage bounds for a Call on a Put. Parameters: S_{T_1} =50, K_1 = 3, K_2 = 50, r = 0%, q = 0%, T_1 = 1, T_2 = 2, t = 0.5, σ = 20%.

With these bounds it is easy to derive the well-known fact that buying a compound call is always cheaper than immediately buying the underlying option, but exercising the compound call (and thus having to pay additional K_1 at maturity) is more expensive than immediately buying the underlying option. The analogous result for compound puts: buying a compound put is always cheaper than the immediate acquisition of a bull or a bear spread, but exercising the compound put leads to higher costs.

3.1.6 Sensitivities

The Greeks in this section were taken directly from Wystup (1999). Define

$$\tau_1 = T_1 - t,\ \ \tau_2 = T_2 - t,\ \ \tau = T_2 - T_1$$

and

$$g = \frac{X\sqrt{\tau_2} + \sqrt{\tau_1}\,d_-}{\sqrt{\tau_{12}}},\quad f = \phi\sqrt{\tau_{12}}\,n\left(d_+^\tau\right).$$

Delta

$$\frac{\partial C_t}{\partial S_t} = \phi\Psi e^{-q\tau_2} N_2\left(-\phi\Psi\left(X - \sigma\sqrt{\tau_1}\right), \phi d_+; \Psi\rho\right)$$

Gamma

$$\frac{\partial C_t}{\partial S_t^{\,2}} = \frac{e^{-q\tau_2}}{\sigma S_t}\left[\frac{1}{\sqrt{\tau_1}} n\left(X - \sigma\sqrt{\tau_1}\right) N\left(\phi d_+^\tau\right) + \frac{\Psi}{\sqrt{\tau_2}} n\left(d_+\right) N\left(\phi\Psi\gamma\right)\right]$$

Theta

$$\begin{aligned}\frac{\partial C_t}{\partial t} &= \phi\Psi r S_t e^{-q\tau_2} N_2\left(-\phi\Psi\left(X - \sigma\sqrt{\tau_1}\right), \phi d_+; \Psi\rho\right)\\ &\quad -\phi\Psi q K_2 e^{-r\tau_2} N_2\left(-\phi\Psi, \phi d_-; \Psi\rho\right)\\ &\quad -\Psi q K_1 e^{-r\tau_1} N\left(-\phi\Psi X\right)\\ &\quad -\frac{1}{2}\sigma S_t e^{-q\tau_2}\left[\frac{1}{\sqrt{\tau_1}} n\left(X - \sigma\sqrt{\tau_1}\right) N\left(\phi d_+^\tau\right) + \frac{\Psi}{\tau_2} n\left(d_+\right) N\left(-\phi\Psi e\right)\right]\end{aligned}$$

Vega

$$\frac{\partial C_t}{\partial \sigma} = S_t e^{-q\tau_2}\left[\sqrt{\tau_1}\,n\left(X - \sigma\sqrt{\tau_1}\right) N\left(\phi d_+^\tau\right) + \Psi\sqrt{\tau_2}\,n\left(d_+\right) N\left(-\phi\Psi e\right)\right]$$

Rho

$$\frac{\partial C_t}{\partial r} = \phi\Psi\tau_2 K_2 e^{-r\tau_2} N_2\left(-\phi\Psi X, \phi d_-; \Psi\rho\right) + \Psi\tau_1 K_1 e^{-r\tau_1} N\left(-\phi\Psi X\right)$$

Analysis of the sensitivities of a call on a call and a call on a put show that they have similar sensitivities to the underlying call and put respectively as shown below in Fig. 19 to Fig. 22.

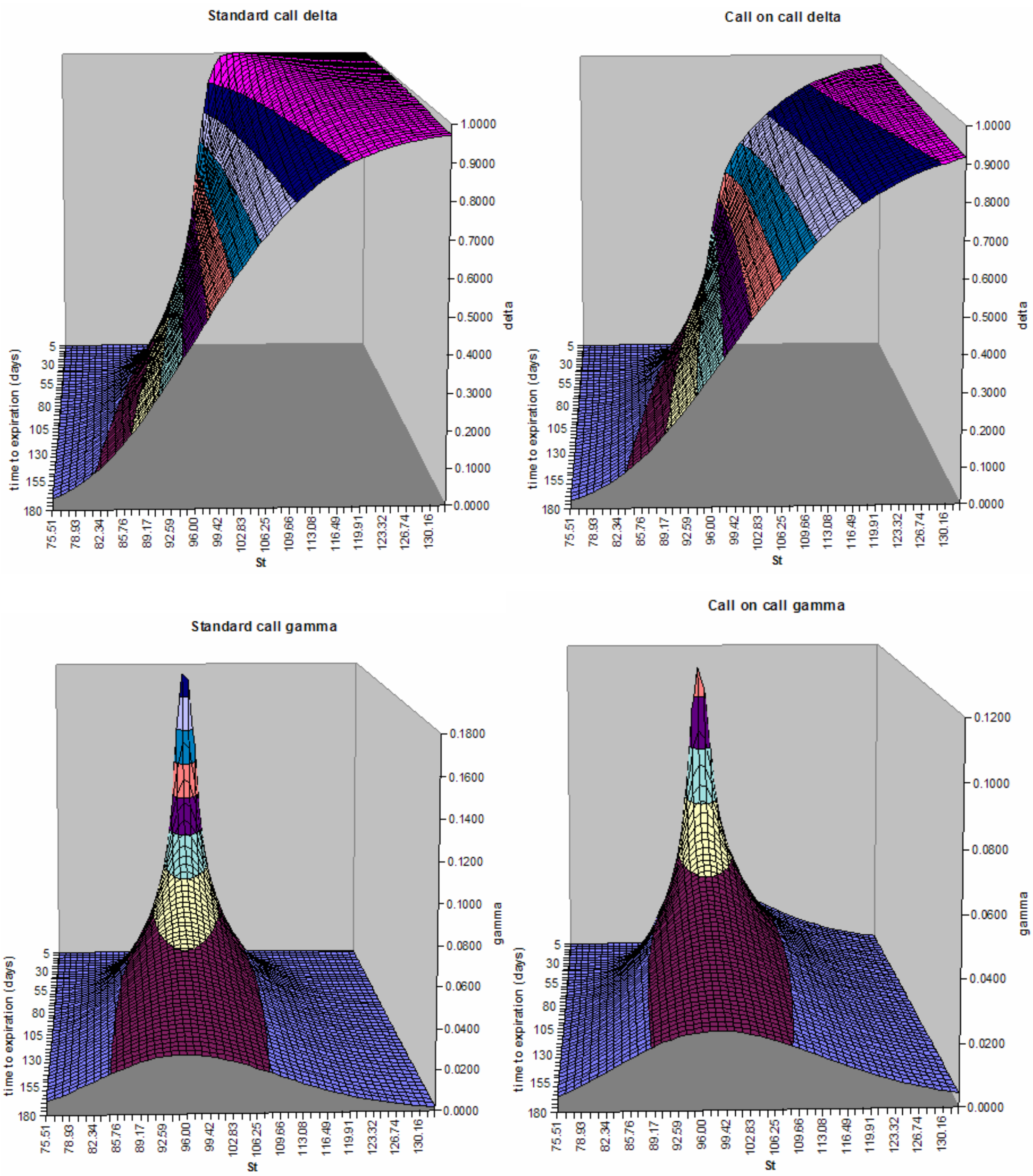

Figure 19: Comparison of the delta and gamma profiles for a call on call and a standard call. Parameters: S_{T_1} =100, K_1 = 6, K_2 = 100, r = 5%, q = 3%, T_1 = 1, T_2 = 2, t = 0.5, σ = 20%.

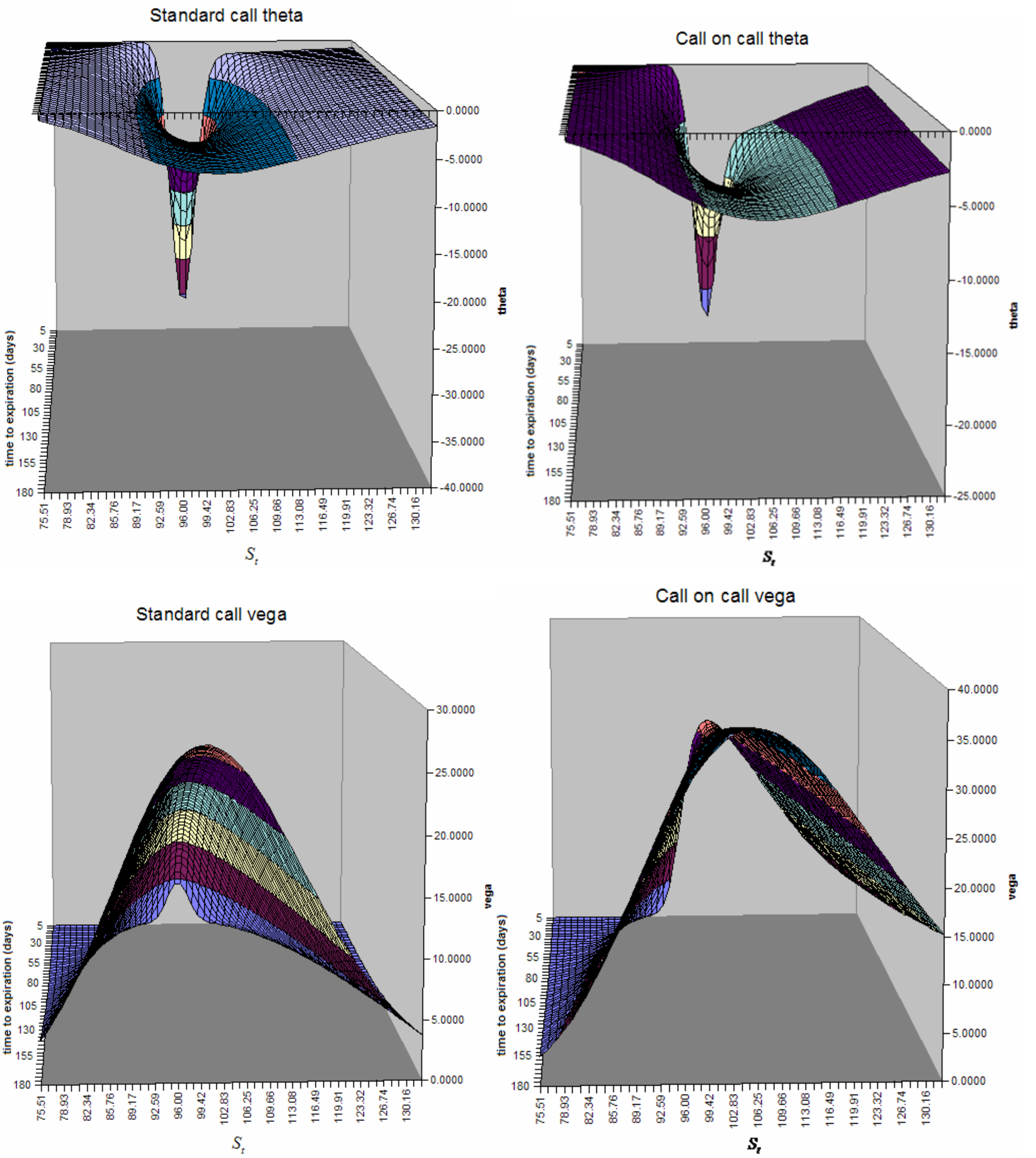

Figure 20: Comparison of the vega and theta profiles for a call on call and a standard call. Parameters: S_{T_1} =100, K_1 = 6, K_2 = 100, r = 5%, q = 3%, T_1 = 1, T_2 = 2, t = 0.5, σ = 20%.

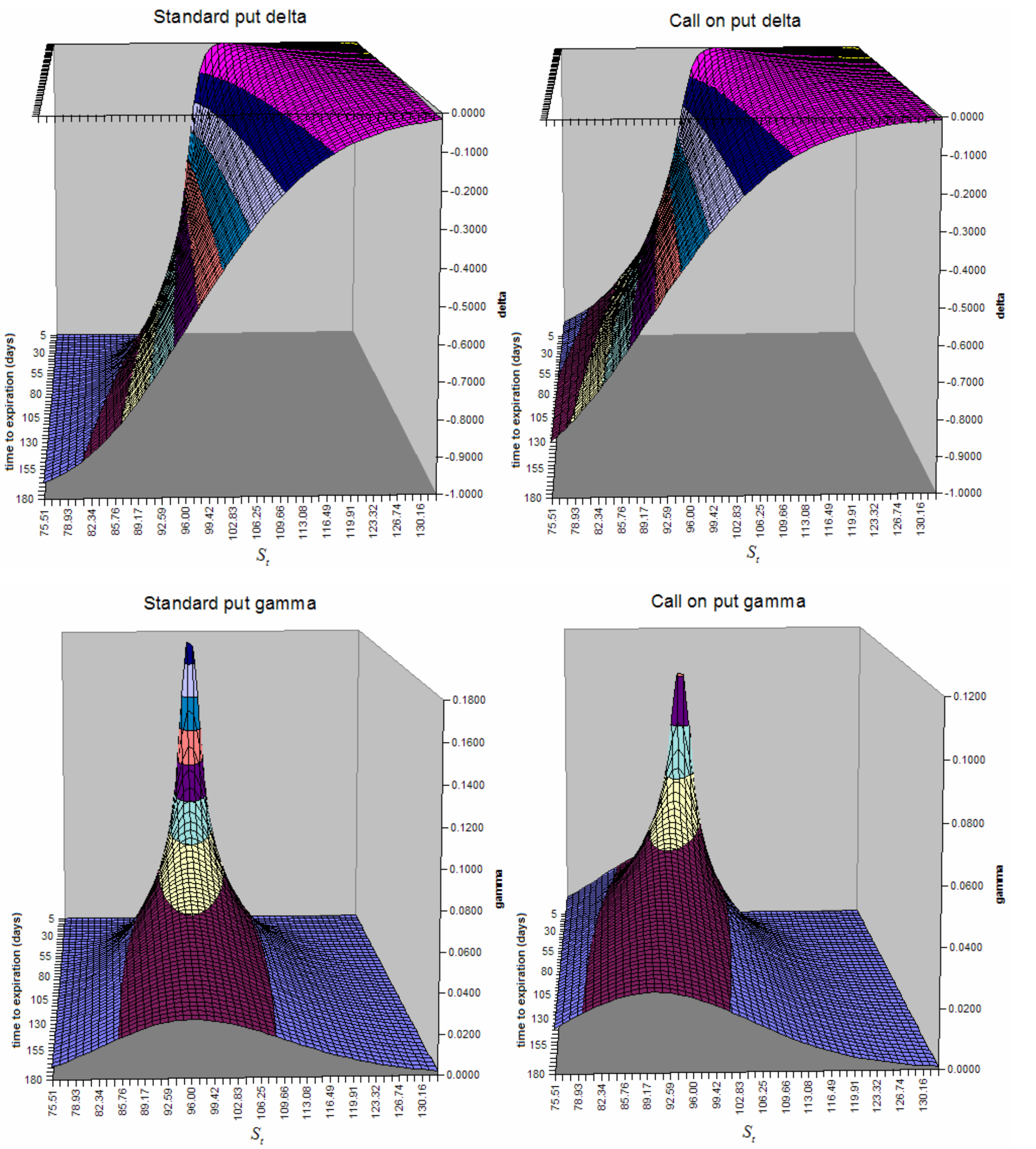

Figure 21: Comparison of the delta and gamma profiles for a call on put and a standard call. Parameters: S_{T_1} =100, K_1 = 6, K_2 = 100, r = 5%, q = 3%, T_1 = 1, T_2 = 2, t = 0.5, σ = 20%.

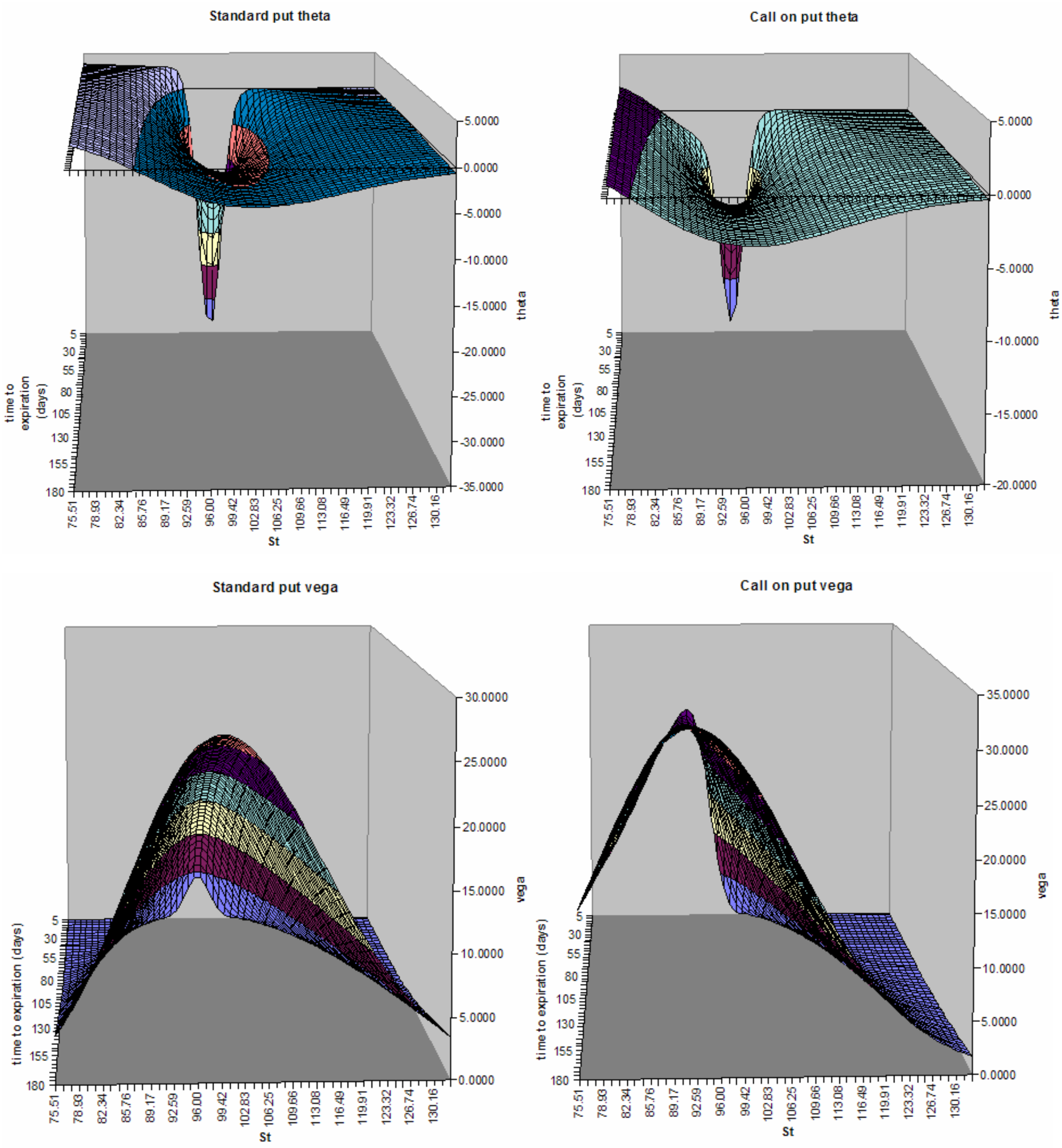

Figure 22: Comparison of the theta and vega profiles for a call on put and a standard call. Parameters: S_{T_1} =100, K_1 = 6, K_2 = 100, r = 5%, q = 3%, T_1 = 1, T_2 = 2, t = 0.5, σ = 20%.

Puts on puts and puts on calls are more complex. Figures 23 to 28 show the price, delta, gamma, theta, vega and rho surfaces with respect to the underlying asset of a put on a call and a put on a put.

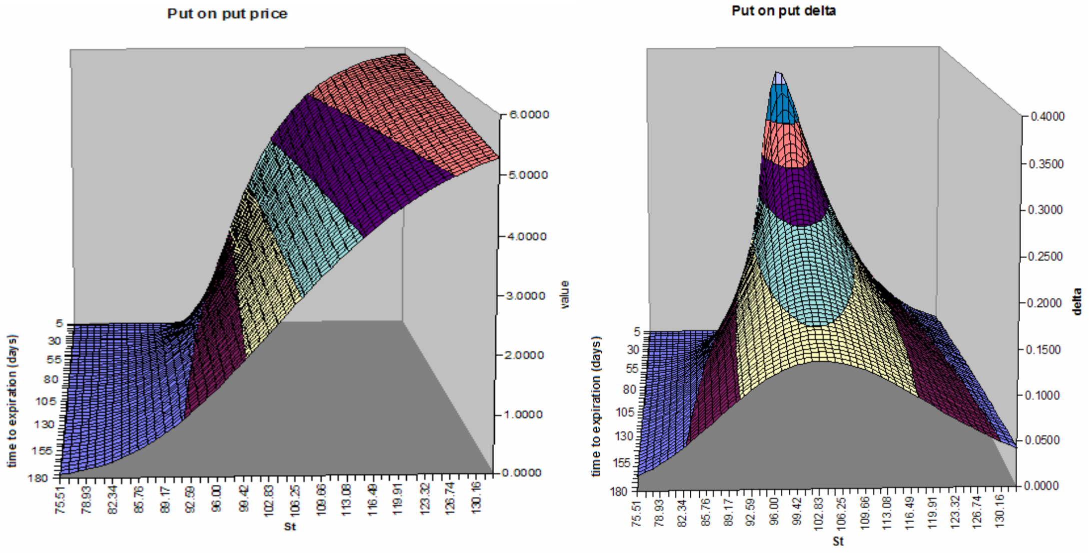

Figure 23: The price and delta profiles of a put on a put. Parameters: S_{T_1} =100, K_1 = 6, K_2 = 100, r = 5%, q = 3%, T_1 = 1, T_2 = 2, t = 0.5, σ = 20%.

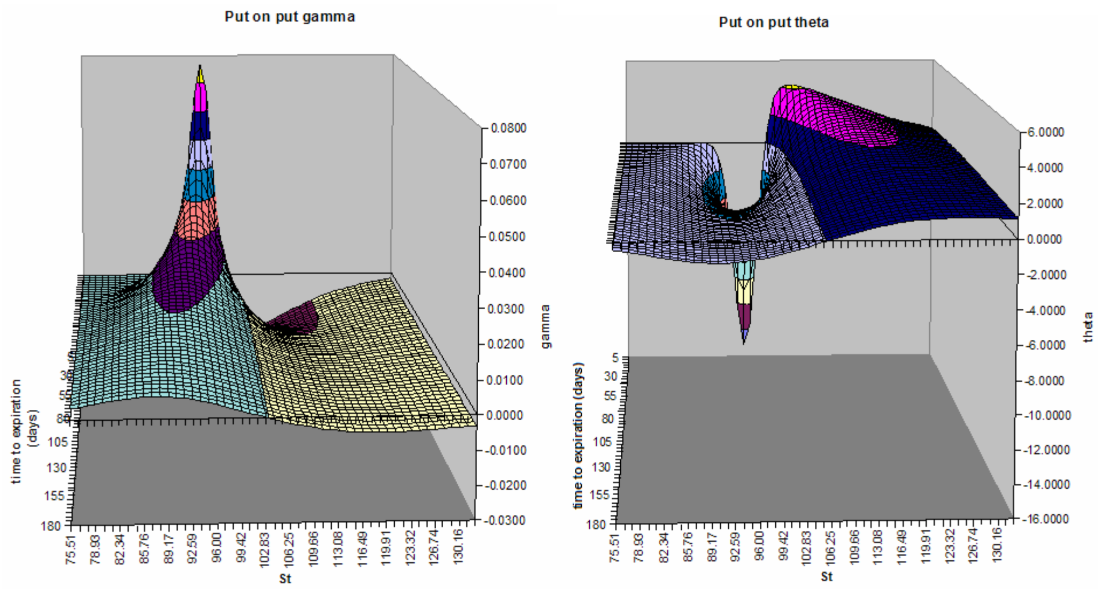

Figure 24: The gamma and theta profiles of a put on a put. Parameters: S_{T_1} =100, K_1 = 6, K_2 = 100, r = 5%, q = 3%, T_1 = 1, T_2 = 2, t = 0.5, σ = 20%.

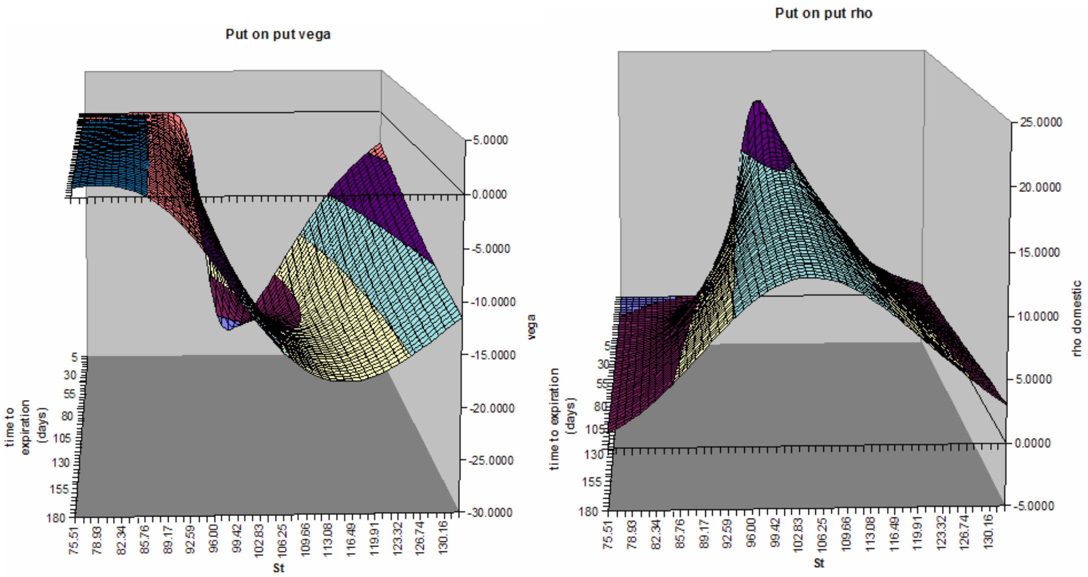

Figure 25: The rho and vega profiles of a put on a put. Parameters: S_{T_1} =100, K_1 = 6, K_2 = 100, r = 5%, q = 3%, T_1 = 1, T_2 = 2, t = 0.5, σ = 20%.

The deltas are quite different to standard call or puts in that they are peaked near the strike price. As a result the gammas of these options can be positive or negative, depending on the level of the underlying asset relative to the strike price. The most critical aspect of these options is the speed at which their vega changes. This confirms their extreme sensitivity to volatility changes (Clewlow and Strickland, 1997).

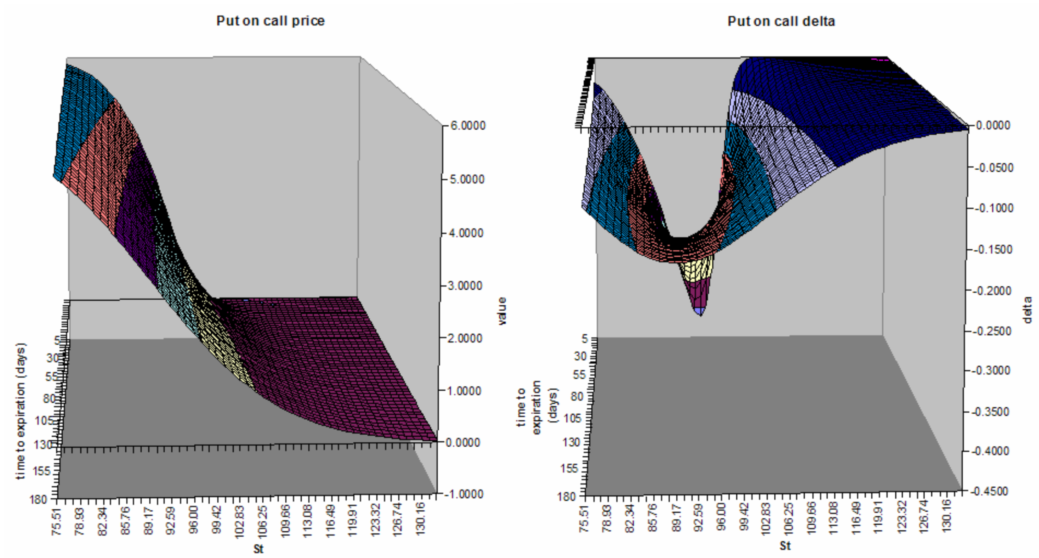

Figure 26: The price and delta profiles of a put on a call. Parameters: S_{T_1} =100, K_1 = 6, K_2 = 100, r = 5%, q = 3%, T_1 = 1, T_2 = 2, t = 0.5, σ = 20%.

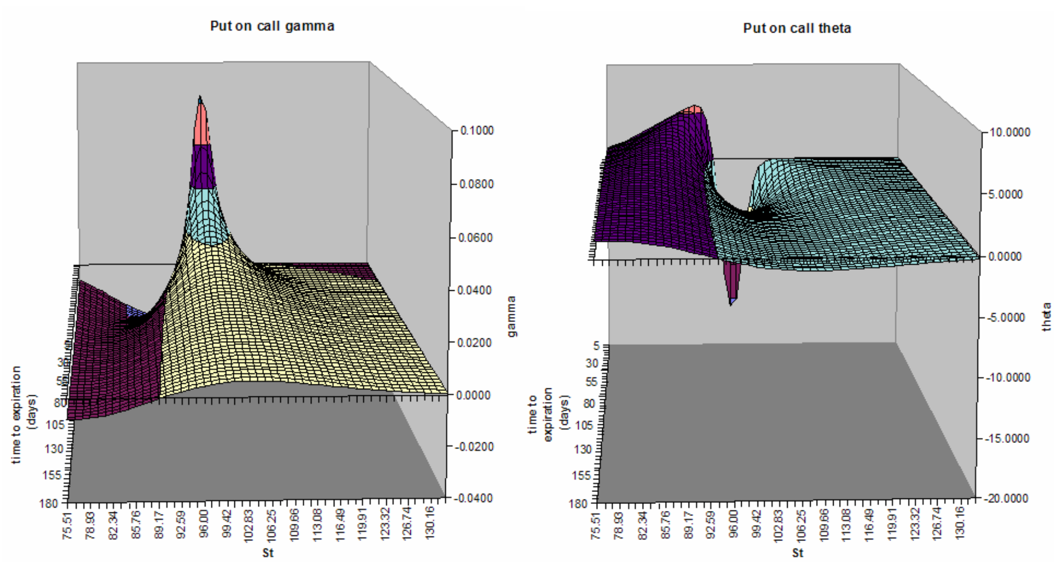

Figure 27: The gamma and theta profiles of a put on a call. Parameters: S_{T_1} =100, K_1 = 6, K_2 = 100, r = 5%, q = 3%, T_1 = 1, T_2 = 2, t = 0.5, σ = 20%.

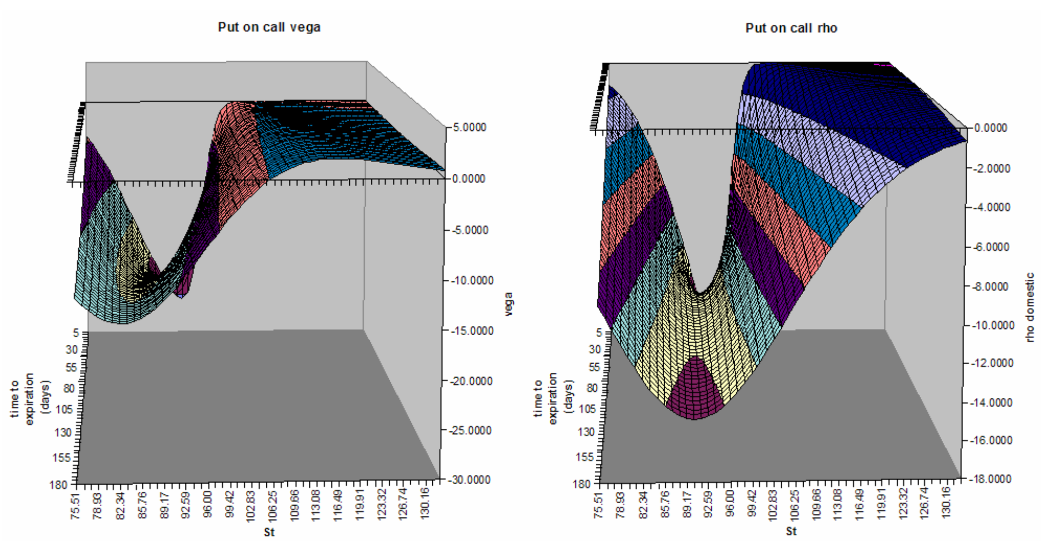

Figure 28: The vega and rho profiles of a put on a call. Parameters: S_{T_1} =100, K_1 = 6, K_2 = 100, r = 5%, q = 3%, T_1 = 1, T_2 = 2, t = 0.5, σ = 20%.

3.2. Chooser Options

3.2.1. Simple Choosers

3.2.1.1 Definition

A standard chooser option, also known as an *as you like it* option, has the feature that, after a specified period of time, the holder can choose whether the option is a call or a put (Hull, 2006). If the structure of a chooser option is considered, one finds that it is identical to that of constructing a *straddle*, or a position in a call and a put simultaneously, with the exception that chooser options are comparatively cheaper (www.global-derivatives.com, 2007).

3.2.1.2 Common Uses

Both a straddle and a chooser can be thought of as a way of speculating on an extreme move in the market. A chooser option is therefore valid for clients who expect strong volatility in the underlying, but who are uncertain about the direction. This makes it an ideal mechanism to take positions on volatility, as seen in Fig. 29 below. A chooser is more appropriate than a straddle when the investor believes information will become available in the future which will indicate the direction of the market move. The advantages of a chooser lie in the flexibility of choosing whether it is a put or a call option and in that the investor does not need to take a directional view. It will therefore always be more expensive than a single standard put or call. A chooser option will be cheaper than a straddle strategy (buying a call and a put at the same strike) since after the chooser date, the buyer has only one option (my.dreamwiz.com, 2007; Clewlow and Strickland,1997)

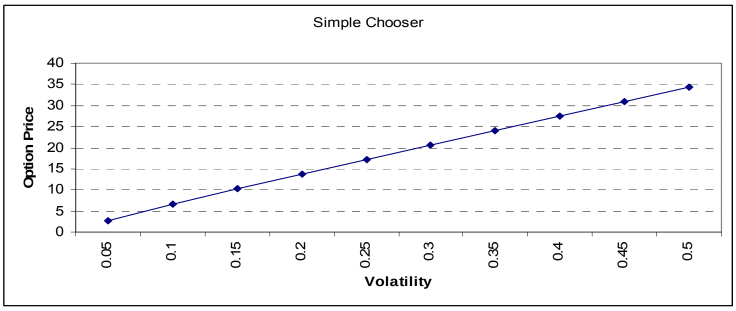

Figure 29: Simple chooser with varying volatility. Parameters: S_{T_1} =100, K = 105,r = 10%, q = 5%, $T_1 = 1$, $T_2 = 2$, t=0.5.

Consider the following example:

A private investor who trades mainly on technical data is convinced that a major movement is about to happen in the FTSE 100 index. On the charts that are available it is clear that the FTSE index is currently trading very close to a major support line at 3000. The investor believes that the support level will not be broken and that the FTSE will move up strongly. On the other hand, a breach of the support level is seen as a major turn in market sentiment and will most likely be followed by a sharp drop in the index. The investor traditionally would enter into a straddle (bought call and bought put). However, a potentially better strategy is to enter into a 1 month chooser option on a 5 month FTSE option with a strike of 3000. At the end of the month, the investor has the choice of a 5 month 3000 put or a 5 month 3000 call (my.dreamwiz.com, 2007).

3.2.1.3 Valuation

A simple chooser option is purchased in the present, but, after a predetermined elapsed time T_1 in the future, it allows the purchaser to choose whether the option is a European standard put or call with a predetermined strike price K and remaining time to maturity T_2-T_1 (Rubinstein, 1991). The payoff from a simple chooser option at the choice date is

$$Chooser_{simple} = \max\left[c, p\right],$$

where c and p denote the respective European call and put values underlying the option.

For a simple chooser, the underlying options are both European with the same strike price and maturity date. Suppose that S_1 is the asset price at time T_1, K is the strike price, T_2 is the maturity of the options, r is the risk-free interest rate and q is the dividend yield. Using the put-call parity relationship, we can re-write the payoff as

$$\begin{aligned} Chooser_{simple}(t) &= \max\left[c, p\right] \\ &= \max\left[c, c + Ke^{-r(T_2-T_1)} - S_t e^{-q(T_2-T_1)}\right] \\ &= c + e^{-q(T_2-T_1)} \max\left[0, Ke^{-r(T_2-T_1)} - S_t e^{-q(T_2-T_1)}\right]. \end{aligned}$$

This shows that the simple chooser option is a package and will have the same payoff today as the payoff from:

1. Buying a call with underlying asset price S_t, strike price K and maturity T_2.
2. Buying a put with underlying asset price $S_t e^{-q(T_2-T_1)}$, strike price $Ke^{-r(T_2-T_1)}$ and maturity T_1.

We can therefore write the formula for a simple chooser as

$$\begin{aligned} Chooser_{simple}(t) = &\left[S_t e^{-q(T_2-t)} N\left(d_+^{\tau_2}\right) - Ke^{-r(T_2-t)} N\left(d_-^{\tau_2}\right)\right] \\ &+\left[Ke^{-r(T_2-t)} N\left(-d_-^{\tau_1}\right) - S_t e^{-q(T_2-t)} N\left(-d_+^{\tau_1}\right)\right], \end{aligned} \tag{3.2.1}$$

where

$$\tau_2 = T_2 - t, \tau_1 = T_1 - t \ ,$$

$$d_+^{\tau_2} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)\left(T_2 - t\right)}{\sigma\sqrt{T_2 - t}}, \qquad d_-^{\tau_2} = d_+^{\tau_2} - \sigma\sqrt{\tau_2} \ ,$$

$$d_+^{\tau_1} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)\left(T_1 - t\right)}{\sigma\sqrt{T_1 - t}}, \qquad d_-^{\tau_1} = d_+^{\tau_1} - \sigma\sqrt{\tau_1} \ .$$

3.2.1.4 The sensitivity of Chooser Options to Varying Time and Strike Price

Define the variables similarly to those defined earlier, with $T_2 - T_1$ being the time to maturity and $T_1 - t$ being the time to choice. Fig. 30 below shows how the time to choice affects the option value. As the time to choice decreases, the value of the option also decreases.

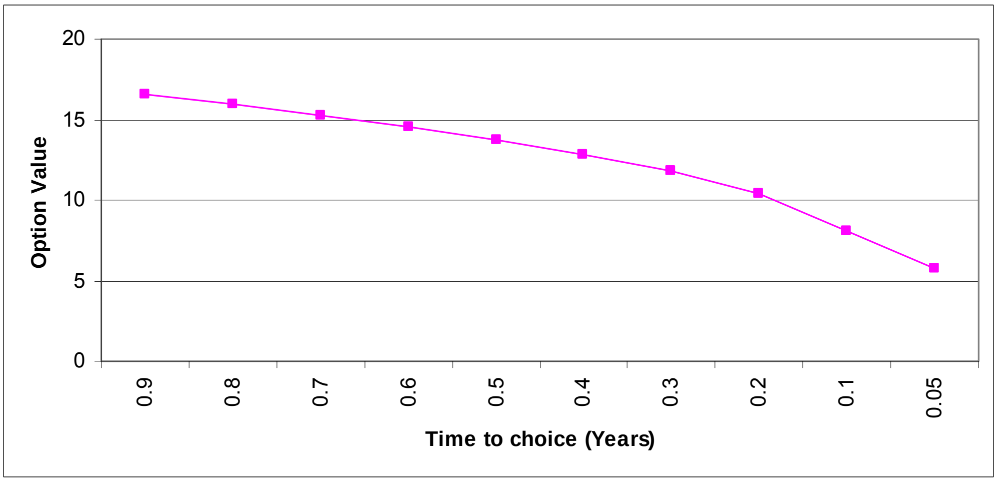

Figure 30: Simple chooser with varying time to choice. Parameters: S_{T_1} =100, K = 105,r = 10%, q = 5%, $T_1 = 1$, $T_2 = 2$, σ = 2%.

Fig. 31 shows that chooser options are generally quite expensive; by varying the strike price, it is shown that even when the asset price is equal to the strike price, the value is still high (www.global-derivatives.com, 2007).

Figure 31: Simple chooser with varying strike price. Parameters: S_{T_1} =100, t = 0.5,r = 10%, q = 5%, $T_1 = 1$, $T_2 = 2$, σ = 20%.

3.2.1.5 Arbitrage Bounds on Valuation

Consider the five simple chooser options valued in Table 1 below, using the Black-Scholes framework. All the options have current underlying asset price $S_t = 100$, strike price $K = 100$, time to expiration $T_2 - t = 1$, $r = 10\%$, $q = 5\%$ and $\sigma = 30\%$. The options only differ by the time to choice.

Table 1: Comparison of Option Prices

Time to choice	Call	Put	Straddle	Chooser
0	12.989	8.349	21.338	12.989
0.1	13.280	8.641	21.921	14.839
0.2	13.369	8.730	22.099	16.317
0.5	13.490	8.851	22.341	19.223
1	13.537	8.898	22.435	22.435

The choice date is obviously the key parameter. If the choice date is today, $T_1 = t$, then the value of the chooser is the same as the value of the call. For a simple chooser, if the choice date is equal to the maturity dates of the call and put, $T_1 = T_2$, then the value of the chooser is the sum of the values of the call and put, since the option has become a straddle (Clewlow and Strickland,1997). These two extreme cases place a minimum and maximum value on the value of the chooser. For all other cases the value of the simple chooser always lies between the value of a single put or call option and the value of a long straddle position.

3.2.2. Complex Choosers

3.2.2.1 Definition

More complex choosers can be defined where the call and the put do not have the same strike price and time to maturity. Because of this property, a complex chooser cannot be broken down in terms of vanilla options. They are not packages and have features that are somewhat similar to compound options (Hull 2006).

3.2.2.2 Valuation

The derivation of the value of complex chooser options are given by Clewlow and Strickland (1997). Define the strike price of the chosen call (put) as K_c (K_p) and maturity dates T_{2c} (T_{2p}). The payoff for a complex chooser on the choice date, T_1, can be written as

$$\max\left[c\left(S_{T_1}, K_c, T_{2c} - T_1\right), p\left(S_{T_1}, K_p, T_{2p} - T_1\right)\right].$$

Using risk-neutral valuation, the current value of a complex chooser option is the discounted expectation of its payoff:

$$Chooser_{complex}\left(t\right)=e^{-r(T_1-t)}\mathrm{E}_t\left[\max\left[c\left(S_{T_1},K_c,T_{2c}\right),p\left(S_{T_1},K_p,T_{2p}\right)\right]\right].$$

Since S_{T_1}, the price of the underlying asset, is lognormal, the log return $u=\ln\left(\frac{S_{T_1}}{S_t}\right)$ is normally distributed. Let $f\left(u\right)$ denote the normal probability density function as given in the proof of Theorem 1 in section 3.1.3. Hence C_t can be written as the integral of the payoff over the probability density of S_{T_1} at time t:

$$Chooser_{complex}\left(t\right)=e^{-r(T_1-t)}\int_{-\infty}^{\infty}\max\left[c\left(Se^u,K_c,T_{2c}\right),p\left(Se^u,K_p,T_{2p}\right)\right]f\left(u\right)du\,.$$

To evaluate the integral we note that since the underlying call and put are monotonic functions of the asset price, the integration space can be divided into two regions. In the lower region we integrate over the put price and in the upper region we integrate over the call price. The regions are divided at that value of the asset price which makes the call and put prices equal. Therefore

$$Chooser_{complex}\left(t\right)=e^{-r(T_1-t)}\left[\int_{-\infty}^{\ln\left(\frac{S^*}{S}\right)}p\left(Se^u,K_p,T_{2p}\right)f\left(u\right)du+\int_{\ln\left(\frac{S^*}{S}\right)}^{\infty}c\left(Se^u,K_c,T_{2c}\right)f\left(u\right)du\right],$$

where S^* is the solution to

$$c\left(S^*,K_c,T_{2c}\right)=p\left(S^*,K_p,T_{2p}\right).$$

The value of S* can be solved iteratively, using the Newton-Raphson search method which satisfies the condition:

$$\begin{gathered}S^*e^{-q\left(T_{2c}-t\right)}N\left(z_{1+}\right)-K_ce^{-r\left(T_{2c}-t\right)}N\left(z_{1-}\right)+\\ S^*e^{-q\left(T_{2p}-t\right)}N\left(-z_{2+}\right)-K_pe^{-r\left(T_{2p}-t\right)}N\left(-z_{2-}\right)=0,\end{gathered}$$

where

$$z_{1+} = \frac{\ln\left(\frac{S^*}{K_c}\right) + \left(r - q + \frac{\sigma^2}{2}\right)\left(T_{2c} - T_1\right)}{\sigma\sqrt{T_{2c} - T_1}}, \quad z_{1-} = z_{1+} - \sigma\sqrt{T_{2c} - T_1}$$

$$z_{2+} = \frac{\ln\left(\frac{S^*}{K_p}\right) + \left(r - q + \frac{\sigma^2}{2}\right)\left(T_{2p} - T_1\right)}{\sigma\sqrt{T_{2p} - T_1}}, \quad z_{2-} = z_{2+} - \sigma\sqrt{T_{2p} - T_1}.$$

The complex chooser can also be valued in terms of the standardised log-return, similarly to the compound option. Define

$$d_{c+} = \frac{\ln\left(\frac{S_t}{K_c}\right) + \left(r - q + \frac{\sigma^2}{2}\right)\left(T_{2c} - t\right)}{\sigma\sqrt{T_{2c} - t}}, \quad d_{c-} = d_{c+} - \sigma\sqrt{T_{2c} - t}, \qquad (3.2.2)$$

$$d_{p+} = \frac{\ln\left(\frac{S_t}{K_p}\right) + \left(r - q + \frac{\sigma^2}{2}\right)\left(T_{2p} - t\right)}{\sigma\sqrt{T_{2p} - t}}, \quad d_{p-} = d_{p+} - \sigma\sqrt{T_{2p} - t}, \qquad (3.2.3)$$

and

$$\rho_c = \sqrt{\frac{\left(T_1 - t\right)}{\left(T_{2c} - t\right)}}, \quad \rho_p = \sqrt{\frac{\left(T_1 - t\right)}{\left(T_{2p} - t\right)}}. \qquad (3.2.4)$$

In this case, start with

$$Chooser_{complex}\left(t\right) = e^{-r\left(T_1 - t\right)} \int_{-\infty}^{\infty} \max\left[\begin{array}{l} c\left(S_t e^{\left(r - q - \frac{\sigma^2}{2}\right)\left(T_1 - t\right) + \sigma \times \sqrt{T_1 - t} \times y}, K_c, T_{2c}, \sigma, r, q\right), \\ p\left(S_t e^{\left(r - q - \frac{\sigma^2}{2}\right)\left(T_1 - t\right) + \sigma \times \sqrt{T_1 - t} \times y}, K_p, T_{2p}, \sigma, r, q\right) \end{array}\right] n\left(y\right) dy.$$

Again, the integral is divided into two regions,

$$Chooser_{complex}(t) = e^{-r(T_1-t)} \int_X^{\infty} c\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times y}, K_c, T_{2c}, \sigma, r, q \right) n(y)\,dy +$$

$$e^{-r(T_1-t)} \int_{-\infty}^{X} p\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times y}, K_p, T_{2p}, \sigma, r, q \right) n(y)\,dy,$$

where X is the unique standardised log-return which solves the following equation:

$$c\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times X}, K_c, T_{2c}, \sigma, r, q \right) = p\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times X}, K_p, T_{2p}, \sigma, r, q \right).$$

This can be solved using the Newton-Raphson procedure.

Before valuing $Chooser_{complex}\left(S_{T_1}, K_c, K_p, T_1, T_{2c}, T_{2p}, \sigma, r, q\right)$ it is first written in the form

$$Chooser_{complex}(t) = e^{-r(T_1-t)} \int_{-\infty}^{-X} c\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times y}, K_c, T_{2c}, \sigma, r, q \right) n(y)\,dy +$$

$$e^{-r(T_1-t)} \int_{-\infty}^{X} p\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times y}, K_p, T_{2p}, \sigma, r, q \right) n(y)\,dy,$$

since the value of the underlying call is strictly increasing.

Now the value of a complex chooser option is determined as follows.

$$Chooser_{complex}(t) = e^{-r(T_1-t)} \int_{-\infty}^{-X} c\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times y}, K_c, T_{2c}, \sigma, r, q \right) n(y)\,dy$$

$$+e^{-r(T_1-t)} \int_{-\infty}^{X} p\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times y}, K_p, T_{2p}, \sigma, r, q \right) n(y)\,dy$$

$$= e^{-r(T_1-t)} \times$$

$$\int_{-\infty}^{-X} \left[\begin{array}{l} S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)-\sigma\times\sqrt{T_1-t}\times y} e^{-q(T_{2c}-T_1)} N\left(\dfrac{\ln\left(\dfrac{S_{T_1}}{K_c}\right)+\left(r-q+\dfrac{\sigma^2}{2}\right)(T_{2c}-T_1)}{\sigma\sqrt{T_{2c}-T_1}} \right) - \\ K_2 e^{-r(T_{2c}-T_1)} N\left(\dfrac{\ln\left(\dfrac{S_{T_1}}{K_c}\right)+\left(r-q-\dfrac{\sigma^2}{2}\right)(T_{2c}-T_1)}{\sigma\sqrt{T_{2c}-T_1}} \right) \end{array} \right] n(y)\,dy +$$

$$e^{-r(T_1-t)} \times$$

$$\int_{-\infty}^{X} \left[\begin{array}{l} K_2 e^{-r(T_{2p}-T_1)} N\left(-\dfrac{\ln\left(\dfrac{S_{T_1}}{K_p}\right)+\left(r-q-\dfrac{\sigma^2}{2}\right)(T_{2p}-T_1)}{\sigma\sqrt{T_{2p}-T_1}} \right) - \\ S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)-\sigma\times\sqrt{T_1-t}\times y} e^{-q(T_{2p}-T_1)} N\left(-\dfrac{\ln\left(\dfrac{S_{T_1}}{K_p}\right)+\left(r-q+\dfrac{\sigma^2}{2}\right)(T_{2p}-T_1)}{\sigma\sqrt{T_{2p}-T_1}} \right) \end{array} \right] n(y)\,dy$$

To evaluate the integral, it is broken down into four components, the first two corresponding to the underlying call and the last two corresponding to the underlying put. Then (3.1.15) is substituted in and the familiar forms of (3.2.2), (3.2.3) and (3.2.3) are recognized.

$$[1] \equiv S_t e^{-r(T_1-t)+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)} e^{-q(T_{2c}-T_1)} \times$$

$$\int_{-\infty}^{-X} e^{-\sigma\times\sqrt{T_1-t}\times y} N\left(\frac{\ln\frac{S_t}{K_c}+\sqrt{T_1-t}\times y+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\left(r-q+\frac{\sigma^2}{2}\right)(T_{2c}-T_1)}{\sigma\sqrt{T_{2c}-T_1}}\right) n(y)\,dy$$

$$= S_t e^{-r(T_1-t)+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)} e^{-q(T_{2c}-T_1)} \times$$

$$\int_{-\infty}^{-X} e^{-\sigma\times\sqrt{T_1-t}\times y} N\left(y\sqrt{\frac{(\mathrm{T}_1-t)}{(T_{2c}-T_1)}} + \frac{\ln\frac{S_t}{K_2}+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\left(r-q+\frac{\sigma^2}{2}\right)(T_{2c}-T_1)}{\sigma\sqrt{T_{2c}-T_1}}\right) n(y)\,dy$$

$$= S_t e^{-q(T_{2c}-t)} N_2\left(\sigma\sqrt{\mathrm{T}_1-t} - X\sigma\sqrt{\mathrm{T}_1-t}, d_{c+}; -\rho_c\right)$$

The identity in (3.1.3) is applied in the final step with

$$\begin{aligned} z &= y, \\ a &= -X, \\ A &= -\sigma\times\sqrt{\mathrm{T}_1-t}, \\ B &= \sqrt{\frac{(\mathrm{T}_1-t)}{(T_{2c}-T_1)}}, \\ C &= \frac{\ln\frac{S_t}{K_2}+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\left(r-q+\frac{\sigma^2}{2}\right)(T_{2c}-T_1)}{\sigma\sqrt{T_{2c}-T_1}}. \end{aligned}$$

$$[2] \equiv -K_c e^{-r(T_{2c}-t)} \times$$

$$\int_{-\infty}^{-X} e^{\sigma\times\sqrt{T_1-t}\times y} N\left(\frac{\ln\frac{S_t}{K_c}+\sqrt{T_1-t}\times y+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\left(r-q-\frac{\sigma^2}{2}\right)(T_{2c}-T_1)}{\sigma\sqrt{T_{2c}-T_1}}\right)n(y)\,dy$$

$$= -K_c e^{-r(T_{2c}-t)} \times$$

$$\int_{-\infty}^{-X} N\left(y\sqrt{\frac{(\mathrm{T}_1-t)}{(T_{2c}-T_1)}}+\frac{\ln\frac{S_t}{K_c}+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\left(r-q-\frac{\sigma^2}{2}\right)(T_{2c}-T_1)}{\sigma\sqrt{T_{2c}-T_1}}\right)n(y)\,dy$$

$$= -K_c e^{-r(T_{2c}-t)} N_2\left(-X, d_{c-}; -\rho_c\right)$$

The identity in (3.1.2) is applied in the final step with

$$z = y,$$
$$a = -X,$$
$$A = \sqrt{\frac{(\mathrm{T}_1-t)}{(T_{2c}-T_1)}},$$
$$B = \frac{\ln\frac{S_t}{K_c}+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\left(r-q+\frac{\sigma^2}{2}\right)(T_{2c}-T_1)}{\sigma\sqrt{T_{2c}-T_1}}.$$

$$[3] \equiv -S_t e^{-r(T_1-t)+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)} e^{-q(T_2-T_1)} \times$$

$$\int_{-\infty}^{X} e^{\sigma\times\sqrt{T_1-t}\times y} N\left(-\frac{\ln\frac{S_t}{K_p}+\sqrt{T_1-t}\times y+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\left(r-q+\frac{\sigma^2}{2}\right)(T_{2p}-T_1)}{\sigma\sqrt{T_{2p}-T_1}}\right)n(y)\,dy$$

$$= S_t e^{-r(T_1-t)+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)} e^{-q(T_2-T_1)} \times$$

$$\int_{-\infty}^{X} e^{\sigma\times\sqrt{T_1-t}\times y} N\left(-y\sqrt{\frac{(\mathrm{T}_1-t)}{(T_{2p}-T_1)}}-\frac{\ln\frac{S_t}{K_2}+\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\left(r-q+\frac{\sigma^2}{2}\right)(T_{2p}-T_1)}{\sigma\sqrt{T_{2p}-T_1}}\right)n(y)\,dy$$

$$= S_t e^{-q(T_{2p}-t)} N_2\left(X-\sigma\sqrt{\mathrm{T}_1-t}, -d_{p+}; \rho_p\right)$$

The identity in (3.1.3) is applied in the final step with

$$
\begin{aligned}
&z = y,\\
&a = X,\\
&A = \sigma \times \sqrt{\mathrm{T}_1 - t},\\
&B = \sqrt{\frac{(\mathrm{T}_1 - t)}{(T_{2p} - T_1)}},\\
&C = -\frac{\ln \frac{S_t}{K_p} + \left(r - q - \frac{\sigma^2}{2}\right)(T_1 - t) + \left(r - q + \frac{\sigma^2}{2}\right)(T_{2p} - T_1)}{\sigma \sqrt{T_{2p} - T_1}}.
\end{aligned}
$$

$$
\begin{aligned}
[4] &\equiv K_p e^{-r(T_{2p}-t)} \times \\
&\int_{-\infty}^{X} e^{\sigma \times \sqrt{T_1 - t} \times y} N\left(-\frac{\ln \frac{S_t}{K_p} + \sqrt{T_1 - t} \times y + \left(r - q - \frac{\sigma^2}{2}\right)(T_1 - t) + \left(r - q + \frac{\sigma^2}{2}\right)(T_{2p} - T_1)}{\sigma \sqrt{T_{2p} - T_1}} \right) n(y)\, dy \\
&= -K_p e^{-r(T_{2p}-t)} \times \\
&\int_{-\infty}^{X} N\left(-y \sqrt{\frac{(\mathrm{T}_1 - t)}{(T_{2p} - T_1)}} - \frac{\ln \frac{S_t}{K_p} + \left(r - q - \frac{\sigma^2}{2}\right)(T_1 - t) + \left(r - q + \frac{\sigma^2}{2}\right)(T_{2p} - T_1)}{\sigma \sqrt{T_{2p} - T_1}} \right) n(y)\, dy \\
&= -K_p e^{-r(T_{2p}-t)} N_2\left(X, -d_{p-}; \rho_p\right)
\end{aligned}
$$

The identity in (3.1.2) is applied in the final step with

$$
\begin{aligned}
&z = y,\\
&a = X,\\
&A = -\frac{\ln \frac{S_t}{K_p} + \left(r - q - \frac{\sigma^2}{2}\right)(T_1 - t) + \left(r - q + \frac{\sigma^2}{2}\right)(T_{2p} - T_1)}{\sigma \sqrt{T_{2p} - T_1}},\\
&B = -\sqrt{\frac{(\mathrm{T}_1 - t)}{(T_{2p} - T_1)}}.
\end{aligned}
$$

The value of the complex chooser option can then be evaluated in a similar way to the compound option to give

$$
\begin{aligned}
&Chooser_{complex}(t) = \\
&S_t e^{-q(T_{2c}-t)} N_2\left(\sigma\sqrt{T_1 - t} - X, d_{c+}; -\rho_c\right) - K_c e^{-r(T_{2c}-t)} N_2\left(-X, d_{c-}; -\rho_c\right) + \\
&K_p e^{-r(T_{2p}-t)} N_2\left(X, -d_{p-}; \rho_p\right) - S_t e^{-q(T_{2p}-t)} N_2\left(X - \sigma\sqrt{T_1 - t}, -d_{p+}; \rho_p\right),
\end{aligned}
$$

where

$$
d_{c+} = \frac{\ln\left(\frac{S_t}{K_c}\right) + \left(r - q + \frac{\sigma^2}{2}\right)\left(T_{2c} - t\right)}{\sigma\sqrt{T_{2c} - t}}, \quad d_{c-} = d_{c+} - \sigma\sqrt{T_{2c} - t},
$$

$$
d_{p+} = \frac{\ln\left(\frac{S_t}{K_p}\right) + \left(r - q + \frac{\sigma^2}{2}\right)\left(T_{2p} - t\right)}{\sigma\sqrt{T_{2p} - t}}, \quad d_{p-} = d_{p+} - \sigma\sqrt{T_{2p} - t},
$$

and

$$
\rho_c = \sqrt{\frac{\left(T_1 - t\right)}{\left(T_{2c} - t\right)}}, \quad \rho_p = \sqrt{\frac{\left(T_1 - t\right)}{\left(T_{2p} - t\right)}}.
$$

3.2.2.3 The Sensitivities of Complex Chooser Options to Some of its Parameters

The formula for the valuation of complex chooser options is quite similar to the formula for valuing compound options. Observe that

$$
\begin{aligned}
&Chooser_{complex}(t) = \\
&S_t e^{-q(T_{2c}-t)} N_2\left(\sigma\sqrt{T_1 - t} - X, d_{c+}; -\rho_c\right) - K_c e^{-r(T_{2c}-t)} N_2\left(-X, d_{c-}; -\rho_c\right) + \\
&K_p e^{-r(T_{2p}-t)} N_2\left(X, -d_{p-}; \rho_p\right) - S_t e^{-q(T_{2p}-t)} N_2\left(X - \sigma\sqrt{T_1 - t}, -d_{p+}; \rho_p\right)
\end{aligned}
$$

- the first two terms of the complex chooser formula are the same as the first two terms of the formula for a call on a call,

$$C_{call\,on\,call} = S_t e^{-q(T_2-t)} N_2\left(\sigma\sqrt{T_1-t}-X, d_+;\rho\right) - K_2 e^{-r(T_2-t)} N_2\left(-X, d_-;\rho\right) - K_1 e^{-r(T_1-t)} N\left(-X\right)$$

- the 3rd and 4th terms of the complex chooser formula are the same as the first two terms of the formula for a call on a put

$$C_{call\,on\,put} = -S_t e^{-q(T_2-t)} N_2\left(X-\sigma\sqrt{T_1-t}, -d_+;\rho\right) + K_2 e^{-r(T_2-t)} N_2\left(X, -d_-;\rho\right) - K_1 e^{-r(T_1-t)} N\left(X\right).$$

The only difference is that the critical underlying asset price S*, or its corresponding unique standardised log-return, is set to the level at which the value of the standard call will equal the value of the standard put after elapsed time T_1: $c\left(S^*, K_c, T_{2c}\right) = p\left(S^*, K_p, T_{2p}\right)$ (Rubinstein, 1991) i.e.

$$c\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times X}, K_c, T_{2c}, \sigma, r, q\right) = p\left(S_t e^{\left(r-q-\frac{\sigma^2}{2}\right)(T_1-t)+\sigma\times\sqrt{T_1-t}\times X}, K_p, T_{2p}, \sigma, r, q\right).$$

With varying strike price in Fig. 32, the complex chooser option has the lowest price when the underlying call and put options have equal values. The price of the option increases as the difference in value between the two underlying options increase.

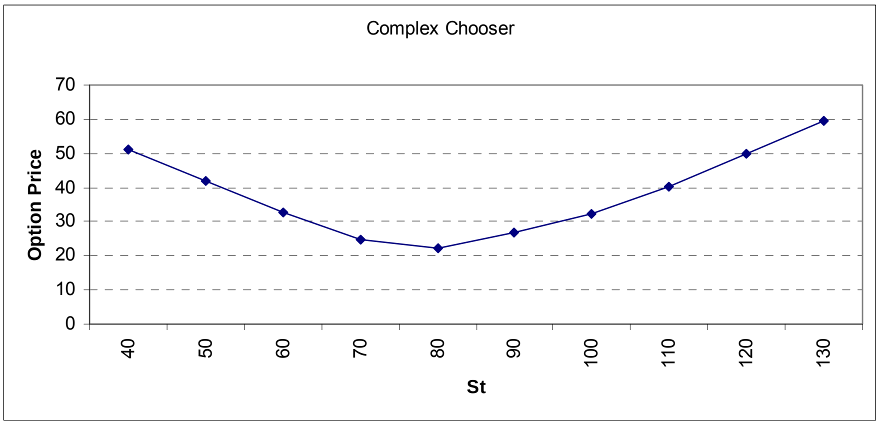

Figure 32: Complex chooser with varying strike price. Parameters: t = 0.5, T_1 =1,T_{2c} = 2,T_{2p} = 2, ,r = 5%, q = 3%, T_1 = 1, T_2 = 2, σ = 20%

The closer the choice date, T_1, is to the maturity dates of the underlying options, T_{2c} and T_{2p}, the higher the option price. This is because the investor has a lot of time to accumulate information on events not very far in the future. As the time to the choice date (T_1-t) decreases, the value of the option also decreases. This is illustrated in Fig. 33 and 34.

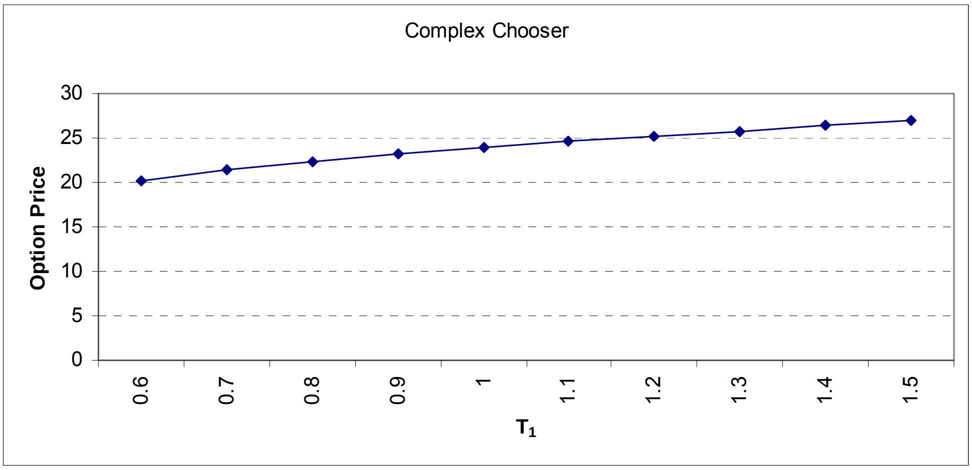

Figure 33: Complex chooser with varying choice date. Parameters: t = 0.5, S_t = 85, T_{2c} = 2,T_{2p} = 2, ,r = 5%, q = 3%, T_1 = 1, T_2 = 2, σ = 20%.

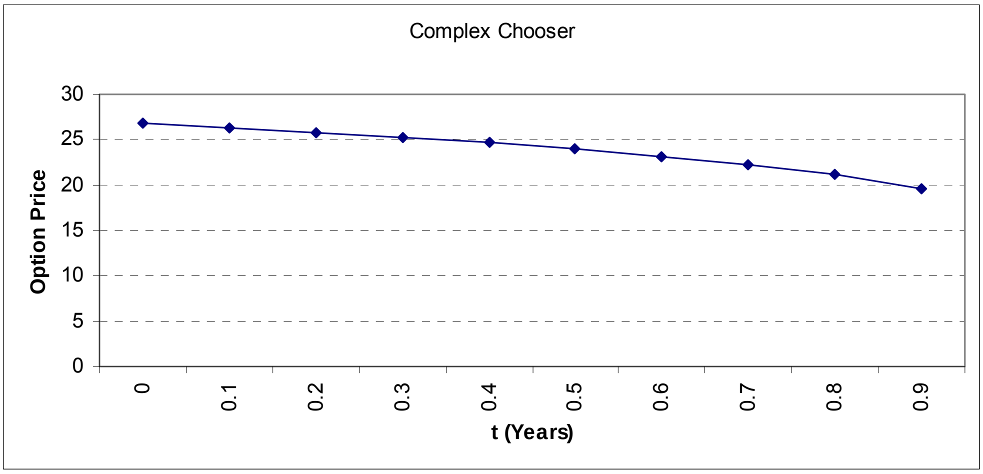

Figure 34: Complex chooser with varying valuation date. Parameters: T_1 =1, S_t = 85, T_{2c} = 2,T_{2p} = 2, ,r = 5%, q = 3%, T_1 = 1, T_2 = 2, σ = 20%.

3.2.3. American Chooser Options

3.2.3.1 Definition

Chooser options can be American, in the sense that the choice of a call and put at the choice date is an American option rather than European in exercise (www.global-derivatives.com, 2007).

3.2.3.2 Valuation

In the Black-Scholes world, allowing the investor to choose at any time, up to some date, does not add any value to the option. This can be seen by recognizing that the value of the chooser is an increasing function of the time to the choice date, so it is optimal for the investor to wait as long as possible. In the real world, however, being able to choose at any time is valuable, in the sense that it allows an immediate profit from a move in the market at any time. (Clewlow and Strickland, 1997.)

An American chooser is priced in a similar fashion to the valuation given for European choosers, but replaces the European payoff function with an American one to find an approximate price (www.global-derivatives.com, 2007).

3.3 Summary

In this chapter two volatility dependent derivatives, compound and chooser options, were discussed. They were first defined, before an overview of their applicability and use was provided and compared to standard options. The option valuations were then derived in detail in the Black-Scholes framework, using properties of the normal distribution. The sensitivity of compound options to volatility was illustrated and the arbitrage bounds on its valuation were given. These are the limits within which the price of an option should stay, since outside these bounds a risk-free arbitrage would be possible. They allow an investor to constrain an option price to a limited range and do not require any assumptions about whether the asset price is normally, or otherwise, distributed. For simple chooser options the sensitivity to varying time and strike price was illustrated. The sensitivities or Greeks of the compound options only were provided and illustrated. They were not provided for simple chooser options, since these decompose exactly into a portfolio of a call and put option and their Greeks can be calculated from this portfolio. Each Greek letter measures a different dimension to the risk in an option position, and the aim of a trader is to manage the Greeks so that all risks are acceptable. In the case of complex chooser options, there is not an exact decomposition, but it is shown that there are similarities between complex chooser options and compound options.

4.

Path-Dependent Derivatives

Barrier options and Asian options are examples of path-dependent options. Path-dependent options are options with payoffs which depend on the complete path taken by the underlying price to reach its expiration value (Clewlow and Strickland,1997).

Barrier options have weak path dependence. This is since the payoff at expiry depends both on whether the underlying hit a prescribed barrier value at some time before expiry, and on the value of the underlying at expiry. Strongly path-dependent contracts have a payoff that depends on some property of the asset path in addition to the value of the underlying asset at the present moment in time. Asian options are strongly path-dependent since their payoff depends on the average value of the underlying asset from inception to expiry (Wilmott, 1998).

4.1 Barrier Options

4.1.1 Definition

Basic barrier options differ in three ways:

1. Kind of option: Call or put.

2. Does the option cancel or come into existence when the underlying price hits or crosses a predetermined barrier?
   A knock-out option ceases to exist immediately when the underlying asset price reaches a certain barrier. A knock-in option comes into existence only when the underlying asset price reaches a barrier. If the underlying price does not hit or cross the barrier, the option does not come into existence and therefore expires worthlessly. In either case, if the option expires inactively, then there may be a cash rebate paid out. This could be nothing, in which case the option ends up worthless, or it could be some fraction of the premium.

3. Does the option knock-in or -out when the underlying price ends up above or below the barrier?

   A down-barrier option knocks in or out when the underlying price ends up below the barrier. An up-barrier option knocks in or out when the underlying price ends up above the barrier.

The four basic types of barrier options are therefore down-and-out, up-and-out, down-and-in and up-and-in options.

4.1.2 Common Uses

Barrier options are attractive to purchasers seeking to pay the lowest possible Rand premium for an option. The weak path-dependency of barrier options makes barrier options less valuable and therefore less expensive than standard options. Their cheapness, relative to standard options, is often why they are used by investors who believe an asset or index will move in a specific manner and who wish to speculate or hedge their portfolios based on their perception of such potential movements. Although there is a greater risk of loss, barrier options are less expensive than standard options, but provide similar potential investment returns. (Braddock, 1997.)

Consider the following examples of scenarios where institutional investors can use barrier options:

- Knock-out calls are used to capture upside stock price movements under the assumption that the underlying asset price will not decline and remain below the barrier level.
- Knock-in puts are used to lock-in profits if upside price moves appear to have peaked.
- Knock-in call options can be used as an inexpensive strategy to participate in the potential for volatile stock price movements.
- Knock-in puts act as "insurance" for bondholders fearing inflation and lower bond prices. (Braddock, 1997.)

4.1.3 Valuation

We follow the derivation of barrier options as given by Reiner and Rubinstein (1991).

Barrier options are valued in a Black-Scholes environment that assumes:

1. The underlying asset follows a jointly lognormal random walk.
2. No arbitrage opportunities exist in the market.

From 2. above, risk-neutral valuation can be applied as follows:

- The riskless interest rate is used as the discount rate.
- The underlying asset price is expected to appreciate at the same riskless rate.
- The expected payoff of the options at expiration are discounted by setting the value of the option at any period equal to the discounted expected value of the option one period later, or its early exercise value, whichever is greater.

To find closed form solutions of barrier options, the density of the natural logarithm of the risk-neutral underlying asset return, u, is needed:

$$f(u) = \frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-\frac{v^2}{2}}$$

with $$v = \frac{u - \mu(T-t)}{\sigma\sqrt{T-t}} \text{ and } \mu = r - q - \frac{\sigma^2}{2}.$$

This is the normal density, where r is the risk-free rate of interest, q is the dividend yield, σ is the volatility of the underlying asset and T is the expiration time of the option. This is the same density used in section 3.1.3 for the valuation of compound options and in section 3.2.2.2 for the valuation of complex chooser options.

If the underlying asset price first starts at S_t **above** the barrier H, the density of the natural logarithm of the underlying asset return, when the underlying asset price breaches the barrier, but ends up above the barrier at expiration, is not equal to the density given above, but is given by

$$g(u) = e^{\frac{2\mu\alpha}{\sigma^2}} \frac{1}{\sigma\sqrt{2\pi(T-t)}} e^{-\frac{v^2}{2}}$$

with $$v = \frac{u - 2\eta\alpha - \eta\mu(T-t)}{\sigma\sqrt{(T-t)}} \text{ and } \alpha = \log\left(\frac{H}{S_t}\right).$$

This is the normal density pre-multiplied by $e^{\frac{2\mu\alpha}{\sigma^2}}$ with $\eta = 1$. Alternatively, given that the underlying asset price first starts **below** the barrier, the density of the natural logarithm of the underlying asset return, when the underlying asset price breaches the barrier, but ends up at expiration below the barrier, is given by the same expression, but with $\eta = -1$.

In order to distinguish between these two situations, define:

$$\eta = \begin{cases} +1 \text{ for the case when the underlying asset price starts above the barrier,} \\ -1 \text{ for the case when the underlying asset price starts below the barrier.} \end{cases}$$

Also define some intermediate values prior to considering the six payoff or probability expressions that cover both in- and out-barrier options:

$$\phi = \begin{cases} +1 \text{ for a barrier call option,} \\ -1 \text{ for a barrier put option.} \end{cases}$$

$$\lambda = \frac{r - q + 0.5\sigma^2}{\sigma^2}$$

$$\mu = r - q - 0.5\sigma^2$$

$$a = \frac{\mu}{\sigma^2},\ b = \frac{\sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}$$

$$w_1 = \frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{(T-t)}$$

$$w_2 = \frac{\ln\left(\frac{S}{H}\right)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{(T-t)}$$

$$w_3 = \frac{\ln\left(\frac{H^2}{SK}\right)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{(T-t)}$$

$$w_4 = \frac{\ln\left(\frac{H}{S}\right)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{(T-t)}$$

$$w_5 = \frac{\ln\left(\frac{H}{S}\right)}{\sigma\sqrt{T-t}} + b\sigma\sqrt{(T-t)}.$$

The prices of the basic barrier options are combinations of the following six expressions denoted by $[i],\ i = 1,...,6.$

$$[1] = \phi S e^{-q(T-t)} N(\phi w_1) - \phi K e^{-r(T-t)} N\left(\phi w_1 - \phi\sigma\sqrt{\mathrm{T}-t}\right)$$

$$[2] = \phi S e^{-q(T-t)} N(\phi w_2) - \phi K e^{-r(T-t)} N\left(\phi w_2 - \phi\sigma\sqrt{\mathrm{T}-t}\right)$$

$$\begin{aligned}
[3] &= \phi S e^{-q(T-t)} \left(\frac{H}{S}\right)^{2\lambda} N(\eta w_4) - \phi K e^{-r(T-t)} \left(\frac{H}{S}\right)^{2\lambda-2} N\left(\eta w_4 - \eta\sigma\sqrt{\mathrm{T}-t}\right) \\
&= \phi \frac{-2\mu}{\sigma^2 S}\left(\frac{H}{S}\right)^{2\lambda-2}\left[S e^{-q(T-t)}\left(\frac{H}{S}\right)^{2} N(\eta w_4) - K e^{-r(T-t)} N\left(\eta w_4 - \eta\sigma\sqrt{\mathrm{T}-t}\right)\right] \\
&\quad -\phi\left(\frac{H}{S}\right)^{2\lambda} e^{-q(T-t)} N(\eta w_4) - \phi\eta e^{-q(T-t)}\left(\frac{H}{S}\right)^{2\lambda} n(w_4)\frac{\left(1-\frac{K}{H}\right)}{\sigma\sqrt{\mathrm{T}-t}}
\end{aligned}$$

$$[4] = \phi S e^{-q(T-t)} \left(\frac{H}{S}\right)^{2\lambda} N(\eta w_3) - K e^{-r(T-t)} \left(\frac{H}{S}\right)^{2\lambda-2} N\left(\eta w_3 - \eta\sigma\sqrt{\mathrm{T}-t}\right)$$

$$[5] = R e^{-r(T-t)}\left[N\left(\eta w_2 - \eta\sigma\sqrt{\mathrm{T}-t}\right) - \left(\frac{H}{S}\right)^{2\lambda-2} N\left(\eta w_4 - \eta\sigma\sqrt{\mathrm{T}-t}\right)\right]$$

$$[6] = R\left[\left(\frac{H}{S}\right)^{a+b} N(\eta w_5) + \left(\frac{H}{S}\right)^{a-b} N\left(\eta w_5 - 2\eta b\sigma\sqrt{T-t}\right)\right]$$

In theorem 5 below the prices of the basic barrier options are informally derived as combinations of expressions [1] to [6].

Theorem 5:

The prices of the basic barrier options are given by the following combinations:

Table 2: Prices of the basic barrier options

Option type	ϕ	η	in/out	Reverse	Combination
Standard up-and-in call	+1	-1	-1	K > H	[1]+[5]
Reverse up-and–in call	+1	-1	-1	K ≤ H	[2]-[4]+[3]+[5]
Standard down-and-in call	+1	+1	-1	K > H	[4]+[5]
Reverse down-and-in call	+1	+1	-1	K ≤ H	[1]-[2]+[3]+[5]
Standard up-and-out call	+1	-1	+1	K > H	[6]
Reverse up-and–out call	+1	-1	+1	K ≤ H	[1]-[2]+[4]-[3]+[6]
Standard down-and-out call	+1	+1	+1	K > H	[1]-[4]+[6]
Reverse down-and-out call	+1	+1	+1	K ≤ H	[2]-[3]+[6]
Standard up-and-in put	-1	-1	-1	K > H	[1]-[2]+[3]+[5]
Reverse up-and–in put	-1	-1	-1	K ≤ H	[4]+[5]
Standard down-and-in put	-1	+1	-1	K > H	[2]-[4]+[3]+[5]
Reverse down-and-in put	-1	+1	-1	K ≤ H	[1]+[5]
Standard up-and-out put	-1	-1	+1	K > H	[2]-[3]+[6]
Reverse up-and–out put	-1	-1	+1	K ≤ H	[1]-[4]+[6]
Standard down-and-out put	-1	+1	+1	K > H	[1]-[2]+[4]-[3]+[6]
Reverse down-and-out put	-1	+1	+1	K ≤ H	[6]

Consider the up-and-in call option given in the first two rows of Table 2. Here $\phi = +1$ indicates a barrier call option, η = -1 indicates that the underlying asset price starts above the barrier, and the knock-in feature is indicated by a binary variable (in/out) set to negative one in the table for distinction from a similar barrier with a knock-out feature. The up-and-in call has two separate price combinations. The price of the first one, the standard up-and-in call, i.e. when K > H, is given by expression [1] plus expression [5]. This is considered to be the standard part. The price of the second one, the reverse up-and-in call, i.e. when K ≤ H, is given by expression [2] minus expression [4] plus expression [3] plus expression [5].

Informal Proof:

European Knock-in Barrier Options

Down-and-in Call

Although the payment for a down-and-in call option is made up front, the call is not received until the underlying asset price reaches a pre-specified barrier level H. If, after elapsed time $t \leq T$, the underlying asset price hits the barrier, the investor receives a standard call with strike K and time to expiration T-t. If, through elapsed time t, the barrier is never hit, the rebate R at that time is received. Let S_t be the price of the underlying after elapsed time t, and S_T the price of the underlying asset at expiration; then the payoff for a down-and-in call option, i.e. where S > H, is given by:

$$\max\left[0, S_T - K\right] \text{ if for some t} \leq \text{T, } \text{S}_\text{t} \leq H;$$
$$R \text{ (at expiry)} \quad \text{ if for all t} \leq \text{T, } \text{S}_\text{t} > H.$$

There are alternative ways that these various payoffs may be earned, particularly when it is possible for the exercise price to be either above or below the barrier price. To price a down-and-in call, five price paths and their associated payoffs are considered as given by Kolb (2003):

1. $K \leq S_T \leq H$; payoff is S_T-K.
2. $K \leq H \leq S_T$, and the barrier was touched; payoff is S_T-K.
3. $K \leq H \leq S_T$, and the barrier was never touched; payoff is R.
4. $H \leq K \leq S_T$, and the barrier was touched; payoff is S_T-K.
5. $H \leq K \leq S_T$, and the barrier was never touched; payoff is R .

For each possible payoff and price path there are associated probabilities and expressions. Consider first the standard case where **K > H,** i.e. 4 and 5 above. The value of the option in this case is the sum of two terms. The first is a *call payoff* corresponding to price path 4 above. The probability that this realises is written as

$$\text{probability (4)} = P[S_T \geq K, S_t \leq H \text{ for some } t \leq T].$$

This means that price path 4 realises if and only if $S_T \geq K$ and $S_t \leq H$ for some $t \leq T$. The second term is a *rebate* and follows accordingly:

$$\text{probability (5)} = P[S_T \geq H] - P[S_T \geq H, S_t \leq H \text{ for some } t \leq T].$$

For price paths 3 and 5 above the probability of being realised is the same, since it is independent of the relationship between S_T and K. This follows since receiving the rebate is independent of the relationship between S_T and K. It only depends on whether the barrier has been breached or not, i.e. if $S_t \leq H$ for some $t \leq T$.

Mathematically, the present value of the pay-off that corresponds to price path 4 follows as

$$E\left[S_T - K \middle| S_T \geq K, S_t \leq H\right] e^{-r(T-t)} = e^{-r(T-t)} \int_{\ln\left(\frac{K}{S}\right)}^{\eta\infty} \phi\left(Se^u - K\right) g\left(u\right) du.$$

The value of this integral is given by Reiner and Rubinstein (1991) as

$$\phi Se^{-q(T-t)} \left(\frac{H}{S}\right)^{2\lambda} N\left(\eta w_3\right) - Ke^{-r(T-t)} \left(\frac{H}{S}\right)^{2\lambda-2} N\left(\eta w_3 - \eta\sigma\sqrt{T-t}\right),$$

which is denoted as equation [4] in Theorem 5. The present value of the pay-off that corresponds to price path 5 follows as

$$E\left[R \middle| S_T \geq K \geq H\right] e^{-r(T-t)} = Re^{-r(T-t)} \int_{\ln\left(\frac{H}{S}\right)}^{\eta\infty} \left[f\left(u\right) - g\left(u\right)\right] du.$$

The value of this integral is given by Reiner and Rubinstein (1991) as

$$Re^{-r(T-t)} \left[N\left(\eta w_2 - \eta\sigma\sqrt{T-t}\right) - \left(\frac{H}{S}\right)^{2\lambda-2} N\left(\eta w_4 - \eta\sigma\sqrt{T-t}\right) \right],$$

which is denoted in Theorem 5 as equation [5].

Hence, the current value of the down-and-in call given that $K > H$ is the present value of the two payoffs $E\left[S_T - K \middle| S_T \geq K, S_t \leq H\right]$ and $E\left[R \middle| S_T \geq K \geq H\right]$ can be expressed as

$$DIC_{(K>H)} = [4] + [5], \left\{\eta = 1, \phi = 1\right\}.$$

Consider the second case where **K < H**. In this case it is necessary to find terms corresponding to the probabilities that the first three price paths realise, i.e.

probability (1) = P[H ≥ S_T > K] = P[S_T > K] - P[S_T > H],

probability (2) = P[S_T > H, S_t ≤ H],

and

probability (3) = probability (5).

Price path 1 is simplified by the fact that since the underlying asset start out above the barrier, if the underlying price then finishes below the barrier, it must have breached the barrier at some time. Therefore it is split up into two terms, P[S_T > K] and P[S_T > H], and considered as two separate price paths. The present values of these two payoffs, together with their solutions, are given by Reiner and Rubinstein (1991) as

$$
\begin{aligned}
E\left[S_T - K \middle| S_T > K\right]e^{-r(T-t)} &= e^{-r(T-t)} \int_{\ln\left(\frac{K}{S}\right)}^{\phi\infty} \phi\left(Se^u - K\right) f\left(u\right) du \\
&= \phi S e^{-q(T-t)} N\left(\phi w_1\right) - \phi K e^{-r(T-t)} N\left(\phi w_1 - \phi\sigma\sqrt{\mathrm{T}-t}\right) \\
&= [1]
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[S_T - K \middle| S_T > H\right]e^{-r(T-t)} &= e^{-r(T-t)} \int_{\ln\left(\frac{H}{S}\right)}^{\phi\infty} \phi\left(Se^u - K\right) f\left(u\right) du \\
&= \phi S e^{-q(T-t)} N\left(\phi w_2\right) - \phi K e^{-r(T-t)} N\left(\phi w_2 - \phi\sigma\sqrt{\mathrm{T}-t}\right) \\
&= [2].
\end{aligned}
$$

The present value of the payoff corresponding to price path 2 above, together with its solutions, is given by Reiner and Rubinstein (1991) as

$$E\left[S_T - K \middle| S_T > H, S_t \le H\right] e^{-r(T-t)} = e^{-r(T-t)} \int_{\ln\left(\frac{H}{S}\right)}^{\eta\infty} \phi\left(Se^u - K\right) g(u)\, du$$

$$= \phi S e^{-q(T-t)} \left(\frac{H}{S}\right)^{2\lambda} N\left(\eta w_4\right) - \phi K e^{-r(T-t)} \left(\frac{H}{S}\right)^{2\lambda-2} N\left(\eta w_4 - \eta\sigma\sqrt{T-t}\right)$$

$$= [3].$$

Using this, the present value of the down-and-in call can be written as

$$DIC_{(K<H)} = [1] - [2] + [3] + [5], \{\eta = 1, \phi = 1\}.$$

Up-and-in Call

This option is identical to a down-and-in call, except that the underlying asset price starts out below, instead of above, the barrier. The payoff from an up-and-in call option, i.e. where S < H, is given by

$$\begin{array}{ll} \max\left[0, S_T - K\right] & \text{if for some t} \le \text{T, S}_\text{t} \ge H; \\ R \text{ (at expiry)} & \text{if for all t} \le \text{T, S}_\text{t} < H. \end{array}$$

An up-and-in call can be priced using the five price paths and their associated payoffs as given for a down-and-in call.

For the **K > H** case there is again a payoff term corresponding to the rebate and a payoff term corresponding to the call payoff.

The rebate payoff is received if

$K \le S_T \le H$ and the barrier was never touched

or $S_T \le H \le K$, and the barrier was never touched,

with the probability given by $P[S_T \le H] - P[S_T \le H, S_t \ge H]$. The density corresponding to the $P[S_T \le H]$ is of course *f(u)*. The density corresponding to $P[S_T \le H, S_t \ge H]$ is identical to *g(u),* but with $\eta = -1$. Therefore, the rebate term is represented by equation [5] with $\eta = -1$ in *g(u)*.

The call payoff is received if $H \le K \le S_T$, with the probability $P[S_T > K]$, which is given by equation [1].

Therefore,

$$UIC_{(K>H)} = [1] + [5], \{\eta = -1, \phi = 1\}.$$

For the **K < H** case, there is a rebate payoff equal to the one derived for the case where H < K. The call payoff is received if

$S_T \ge K \ge H$, payoff is S_T-K;
$H \ge S_T \ge K$, and the barrier was touched; payoff is S_T-K,

with associated probabilities given by $P[S_T \ge H]$ and $P[H > S_T > K, S_t \ge H]$ respectively. The second probability can be restated as

$$P[H > S_T > K, S_t \ge H] = P[S_T < H, S_t \ge H] - P[S_T > K, S_t \ge H].$$

Then immediately it can be written that

$$UIC_{(K<H)} = [2] + [3] - [4] + [5], \{\eta = -1, \phi = 1\}.$$

The remaining knock-in options follow similarly:

<u>Down-and-in Put</u>

A down-and-in put is a put option that ceases to exist when a barrier less than the price is reached (Hull, 2003). The payoff of a down-and-in put, i.e. S > H, is given by

$$\begin{array}{ll} \max\left[0, K - S_T\right] & \text{if for some } t \le T,\ S_t \le H; \\ R \text{ (at expiry)} & \text{if for all } t \le T,\ S_t > H. \end{array}$$

When K > H:

Receive put payoff equal to $\left(K - S_T\right)$ with associated probability

$$P\left[S_T \le H\right] + P\left[H < S_T < K, S_t \le H\right].$$

When H > K:
Receive put payoff equal to $\left(K - S_T\right)$ with associated probability

$$P\left[S_T < K\right].$$

When K > H or H > K:
Receive the rebate payoff equal to R with associated probability

$$P\left[S_T > H\right] - P\left[S_T > H, S_t \leq H\right].$$

Therefore,

$$DIP_{(K>H)} = [2] + [3] - [4] + [5], \left\{\eta = 1, \phi = -1\right\}.$$
$$DIP_{(K<H)} = [1] + [5], \left\{\eta = 1, \phi = -1\right\}.$$

<u>Up-and-in Put</u>

A down-and-in put is a put that comes into existence only if the barrier, H, that is greater than the current asset price is reached (Hull, 2003). The payoff of an up-and-in put, i.e. where S > H, is given by

$$\begin{array}{ll} \max\left[0, K - S_T\right] & \text{if for some } t \leq T,\ S_t \geq H; \\ R \text{ (at expiry)} & \text{if for all } t \leq T,\ S_t < H. \end{array}$$

When K > H:
Receive put payoff equal to $K-S_T$ with associated probability

$$P\left[H \leq S_T < K\right] + P\left[S_T < H, S_t \leq H\right].$$

When H > K:
Receive put payoff equal to $K-S_T$ with associated probability

$$P\left[S_T < K, S_t \geq H\right].$$

When K > H or H > K:
Receive the rebate payoff equal to R with associated probability

$$P\left[S_T < H\right] - P\left[S_T < H, S_t \geq H\right].$$

Therefore,

$$UIP_{(K>H)} = [1] - [2] + [3] + [5], \left\{\eta = -1, \phi = -1\right\}.$$
$$UIP_{(K<H)} = [4] + [5], \left\{\eta = -1, \phi = -1\right\}.$$

For knock-out barrier options there is a sixth price payoff to consider, which will now be presented.

European Knock-out Barrier Options

Down-and –out Call

A down-and-out call is one type of knock-out option. It is a regular call option that ceases to exist if the asset price reaches a certain barrier level. This barrier is below the initial asset price (Hull, 2006). The payoff from a down-and-out call option, i.e. where S > H, is given by

$$\begin{array}{ll} \max\left[0, S_T - K\right] & \text{if for all t} \leq \text{T, } \mathrm{S_t} > H; \\ R \text{ (at hit)} & \text{if for some t} \leq \text{T, } \mathrm{S_t} \leq H. \end{array}$$

If the rebate equals zero the following parity relationship makes it easy to write down the values of European knock-out barrier options:

Payoff from standard option = Payoff from down-and-out option +Payoff from down-and-in option

To show that this identity holds, suppose an investor owns otherwise identical down-and-out and down-and-in options with no rebates. If the common barrier is never hit, he receives the payoff from a standard option. If the common barrier is hit, as the down-and-out option ceases to exist, the down-and-in option delivers him a standard option identical to the one he lost when the down-and-out option was cancelled. Therefore, even in this case, the investor ends up receiving the payoff from a standard option.

If the rebate is not equal to zero, it is necessary to consider that for knock-in options, it is not possible to receive the rebate prior to expiration, since one continues to remain in doubt about whether or not the barrier will never be hit. However, for a knock-out option, it is possible, as well as customary, for the rebate to be paid the moment the barrier is hit. This complicates the risk-neutral valuation problem since the rebate may now be received at a random, rather than pre-specified, time. Therefore, the density of the first passage time (τ) for the underlying asset price to hit the barrier is given by Reiner and Rubinstein (1991) as

$$h(\tau) = \frac{\partial g(u)}{\partial \tau} = -\frac{\eta\alpha}{\sigma\tau\sqrt{2\pi\tau}} e^{-\frac{1}{2}v^2}$$

with $$v = \frac{-\alpha + \mu\tau}{\sigma\sqrt{\tau}}.$$

Here, $\eta = 1$ if the barrier is being approached from above and $\eta = -1$ if the barrier is being approached from below. The present value of the expected rebate follows as the expected rebate discounted by the interest rate, raised to the power of the first passage time:

$$R\int_0^T e^{-rt} h(t)\,dt = R\left[\left(\frac{H}{S}\right)^{a+b} N(\eta w_5) + \left(\frac{H}{S}\right)^{a-b} N\left(\eta w_5 - 2\eta b\sigma\sqrt{T-t}\right)\right]$$
$$= [6].$$

Using this relationship, it is now possible to write down the valuation solutions for the down-and-out call and the three remaining knock-out barrier options:

$$\begin{aligned} DOC_{(K>H)} &= C - DIC_{(K>H)} \\ &= [1] - [4] + [6], \{\eta = 1, \phi = 1\}. \\ DOC_{(K<H)} &= C - DOC_{(K<H)} \\ &= [2] - [3] + [6], \{\eta = 1, \phi = 1\}. \end{aligned}$$

Here C indicates the payoff from a standard call priced in the Black-Scholes framework.

Up-and-out Call

An up-and-out call is a regular call that ceases to exist if the asset price reaches a specified barrier level that is higher than the current asset price (Hull, 2006). The payoff from an up-and-out call option, i.e. where S < H, is given by

$$\begin{aligned} &\max\left[0, S_T - K\right] \text{ if for all t} \le \text{T, } \mathrm{S_t} \le H; \\ &R \text{ (at hit)} \quad \text{if for some t} \le \text{T, } \mathrm{S_t} \ge H. \end{aligned}$$

The valuation solution is

$$
\begin{aligned}
UOC_{(K>H)} &= C - UIC_{(K>H)} \\
&= [6], \{\eta = -1, \phi = 1\}, \\
UOC_{(K<H)} &= C - UIC_{(K<H)} \\
&= [1]-[2]+[4]-[3]+[6], \{\eta = -1, \phi = 1\}.
\end{aligned}
$$

The rebate provides the only contribution to the value of an up-an-out call when the strike price is greater than the barrier. Since S < H < K, in order for the underlying asset price to end up above the strike price it must first breach the barrier, but in this event, the call is extinguished.

Down-and-out Put

A down-and-out put is a put option that ceases to exist when a barrier less than the current asset price is reached (Hull, 2006). The payoff from a down-and-out put option, i.e. where S > H, is given by

$$
\begin{aligned}
&\max\left[0, K - S_T\right] \text{ if for all t} \leq \text{T, } \mathrm{S_t} > H; \\
&R \text{ (at hit)} \quad \text{ if for some t} \leq \text{T, } \mathrm{S_t} \leq H.
\end{aligned}
$$

The valuation solution is

$$
\begin{aligned}
DOP_{(K>H)} &= P - DIP_{(K>H)} \\
&= [1]-[2]+[4]-[3]+[6], \{\eta = 1, \phi = -1\}. \\
DOP_{(K<H)} &= P - DIP_{(K<H)} \\
&= [6], \{\eta = 1, \phi = -1\}.
\end{aligned}
$$

Here P indicates the payoff from a standard put option priced in the Black-Scholes framework. Similarly to an up-and-out call, the rebate provides the only contribution to the value of a down-and-out put when the strike price is less than the barrier.

Up-and-out Put

An up-and-out put is a put option that ceases to exist when a barrier, H, that is greater than the current asset price, is reached (Hull, 2006). The payoff from an up-and-out put option, i.e. where S < H, is given by

$$\begin{array}{ll} \max\left[0, K - S_T\right] & \text{if for all } t \leq T,\ S_t < H; \\ R \text{ (at hit)} & \text{if for some } t \leq T,\ S_t \geq H \end{array}$$

The valuation solution is

$$\begin{aligned} UOP_{(K>H)} &= P - UIP_{(K>H)} \\ &= [2]-[3]+[6], \{\eta = -1, \phi = -1\}. \\ UOP_{(K<H)} &= P - UIP_{(K<H)} \\ &= [1]-[4]+[6], \{\eta = -1, \phi = -1\}. \end{aligned}$$

■

In theorem 5, the prices of the basic barrier options are informally derived as combinations of expressions [1] to [6], where equations [1] to [4] correspond to the payoff of the underlying option and [5] and [6] correspond to the rebate.

Tables 3 through 6 below present the expressions for [1] to [6] for down calls $(\phi = 1, \eta = 1)$, down puts $(\phi = -1, \eta = 1)$, up calls $(\phi = 1, \eta = -1)$ and up puts $(\phi = -1, \eta = -1)$. Table 7 shows the values of each possible barrier option in terms of the expressions in Tables 3 through 6.

Table 3: Valuation Expressions for Down Calls

DC1	$Se^{-q(T-t)}N(w_1)-Ke^{-r(T-t)}N\left(w_1-\sigma\sqrt{T-t}\right)$
DC2	$Se^{-q(T-t)}N(w_2)-Ke^{-r(T-t)}N\left(w_2-\sigma\sqrt{T-t}\right)$
DC3	$Se^{-q(T-t)}\left(\frac{H}{S}\right)^{2\lambda}N(w_4)-Ke^{-r(T-t)}\left(\frac{H}{S}\right)^{2\lambda-2}N\left(w_4-\sigma\sqrt{T-t}\right)$
DC4	$Se^{-q(T-t)}\left(\frac{H}{S}\right)^{2\lambda}N(w_3)-Ke^{-r(T-t)}\left(\frac{H}{S}\right)^{2\lambda-2}N\left(w_3-\sigma\sqrt{T-t}\right)$
DC5	$Re^{-r(T-t)}\left[N\left(w_2-\sigma\sqrt{T-t}\right)-\left(\frac{H}{S}\right)^{2\lambda-2}N\left(w_4-\sigma\sqrt{T-t}\right)\right]$
DC6	$R\left[\left(\frac{H}{S}\right)^{a+b}N(w_5)+\left(\frac{H}{S}\right)^{a-b}N\left(w_5-2b\sigma\sqrt{T-t}\right)\right]$

Table 4: Valuation Expressions for Down Puts

DP1	$Ke^{-r(T-t)}N\left(-w_1-\sigma\sqrt{T-t}\right)-Se^{-q(T-t)}N(-w_1)$
DP2	$Ke^{-r(T-t)}N\left(-w_2-\sigma\sqrt{T-t}\right)-Se^{-q(T-t)}N(-w_2)$
DP3	$Ke^{-r(T-t)}\left(\frac{H}{S}\right)^{2\lambda-2}N\left(w_4-\sigma\sqrt{T-t}\right)-Se^{-q(T-t)}\left(\frac{H}{S}\right)^{2\lambda}N(w_4)$
DP4	$Ke^{-r(T-t)}\left(\frac{H}{S}\right)^{2\lambda-2}N\left(w_3-\sigma\sqrt{T-t}\right)-Se^{-q(T-t)}\left(\frac{H}{S}\right)^{2\lambda}N(w_3)$
DP5	$Re^{-r(T-t)}\left[N\left(w_2-\sigma\sqrt{T-t}\right)-\left(\frac{H}{S}\right)^{2\lambda-2}N\left(w_4-\sigma\sqrt{T-t}\right)\right]$
DP6	$R\left[\left(\frac{H}{S}\right)^{a+b}N(w_5)+\left(\frac{H}{S}\right)^{a-b}N\left(w_5-2b\sigma\sqrt{T-t}\right)\right]$

Table 5: Valuation Expressions for Up Calls

UC1	$Se^{-q(T-t)}N\left(w_1\right)-Ke^{-r(T-t)}N\left(w_1-\sigma\sqrt{\mathrm{T}-t}\right)$
UC2	$Se^{-q(T-t)}N\left(w_2\right)-Ke^{-r(T-t)}N\left(w_2-\sigma\sqrt{\mathrm{T}-t}\right)$
UC3	$Se^{-q(T-t)}\left(\frac{H}{S}\right)^{2\lambda}N\left(-w_4\right)-Ke^{-r(T-t)}\left(\frac{H}{S}\right)^{2\lambda-2}N\left(-w_4+\sigma\sqrt{\mathrm{T}-t}\right)$
UC4	$Se^{-q(T-t)}\left(\frac{H}{S}\right)^{2\lambda}N\left(-w_3\right)-Ke^{-r(T-t)}\left(\frac{H}{S}\right)^{2\lambda-2}N\left(-w_3+\sigma\sqrt{\mathrm{T}-t}\right)$
UC5	$Re^{-r(T-t)}\left[N\left(-w_2+\sigma\sqrt{\mathrm{T}-t}\right)-\left(\frac{H}{S}\right)^{2\lambda-2}N\left(-w_4+\sigma\sqrt{\mathrm{T}-t}\right)\right]$
UC6	$R\left[\left(\frac{H}{S}\right)^{a+b}N\left(-w_5\right)+\left(\frac{H}{S}\right)^{a-b}N\left(-w_5+2b\sigma\sqrt{T-t}\right)\right]$

Table 6: Valuation Expressions for Up Puts

UP1	$Ke^{-r(T-t)}N\left(-w_1-\sigma\sqrt{\mathrm{T}-t}\right)-Se^{-q(T-t)}N\left(-w_1\right)$
UP2	$Ke^{-r(T-t)}N\left(-w_2-\sigma\sqrt{\mathrm{T}-t}\right)-Se^{-q(T-t)}N\left(-w_2\right)$
UP3	$Ke^{-r(T-t)}\left(\frac{H}{S}\right)^{2\lambda-2}N\left(-w_4+\sigma\sqrt{\mathrm{T}-t}\right)-Se^{-q(T-t)}\left(\frac{H}{S}\right)^{2\lambda}N\left(-w_4\right)$
UP4	$Ke^{-r(T-t)}\left(\frac{H}{S}\right)^{2\lambda-2}N\left(-w_3+\sigma\sqrt{\mathrm{T}-t}\right)-Se^{-q(T-t)}\left(\frac{H}{S}\right)^{2\lambda}N\left(-w_3\right)$
UP5	$Re^{-r(T-t)}\left[N\left(-w_2+\sigma\sqrt{\mathrm{T}-t}\right)-\left(\frac{H}{S}\right)^{2\lambda-2}N\left(-w_4+\sigma\sqrt{\mathrm{T}-t}\right)\right]$
UP6	$R\left[\left(\frac{H}{S}\right)^{a+b}N\left(-w_5\right)+\left(\frac{H}{S}\right)^{a-b}N\left(-w_5+2b\sigma\sqrt{T-t}\right)\right]$

Table 7: Valuation of Barrier Options	**Standard K > H**	**Reverse K < H**
Down-and-in Call (DIC)	DC4+DC5	DC1-DC2+DC3+DC5
Up-and-in Call (UIC)	UC1+UC5	UC2-UC4+UC3+UC5
Down-and-in Put (DIP)	DP2+DP3-DP4+DP5	DP1+DP5
Up-and-in Put (UIP)	UP1-UP2+UP3+UP5	UP4+UP5
Down-and-out Call (DOC)	DC1-DC4+DC6	DC2-DC3+DC6
Up-and-out Call (UOC)	UC6	UC1-UC2-UC3+UC4+UC6
Down-and-out Put (DOP)	DP1-DP2-DP3+DP4+DP5	DP6
Up-and-out Put (UOP)	UP2-UP3+UP6	UP1-UP4+UP6

These analytic formulas present a method to price barrier option in continuous time, but in practice the asset price is sampled at discrete times. This means that periodic measurement of the asset price is assumed, rather than a continuous lognormal distribution. Broadie, Glasserman and Kou (1997) arrived at an adjustment to the barrier value to account for discrete sampling as follows:

$$H \to He^{\delta\sigma\sqrt{\frac{T}{n}}},$$

where n is the number of times the asset price is sampled over the period. For "up" options which hit the barrier from underneath, the value of δ is 0.5826. For "down" options where the barrier is hit from the top, the value of δ is -0.5826.

Remark

In Appendix A the mathematical background, used to derive barrier option prices formally, is given.

4.1.4 Remarks on Barrier Options

Here graphs of the shape of the value of barrier options are shown as a function of both the time to expiration and the underlying stock price.

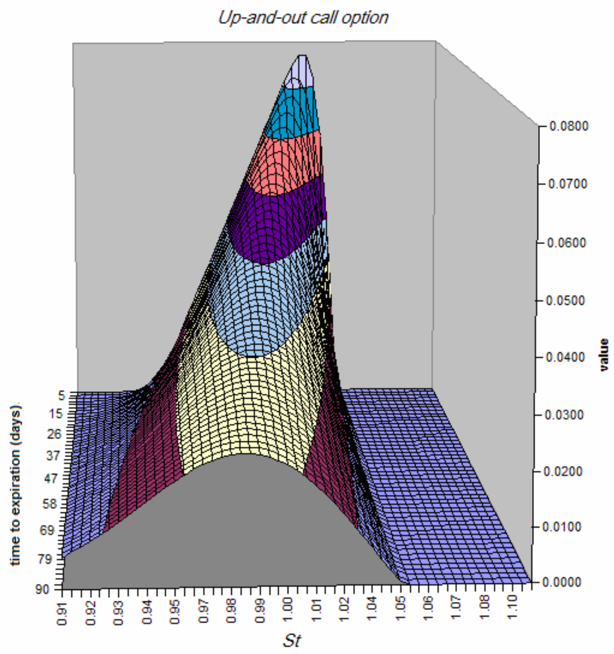

Figure 35: The values of a reverse up-and-out call. Parameters: K = 0.95, H = 1.05, r = 5%, q = 0%, T =90/360, σ = 10%, R = 0. See Table 7.

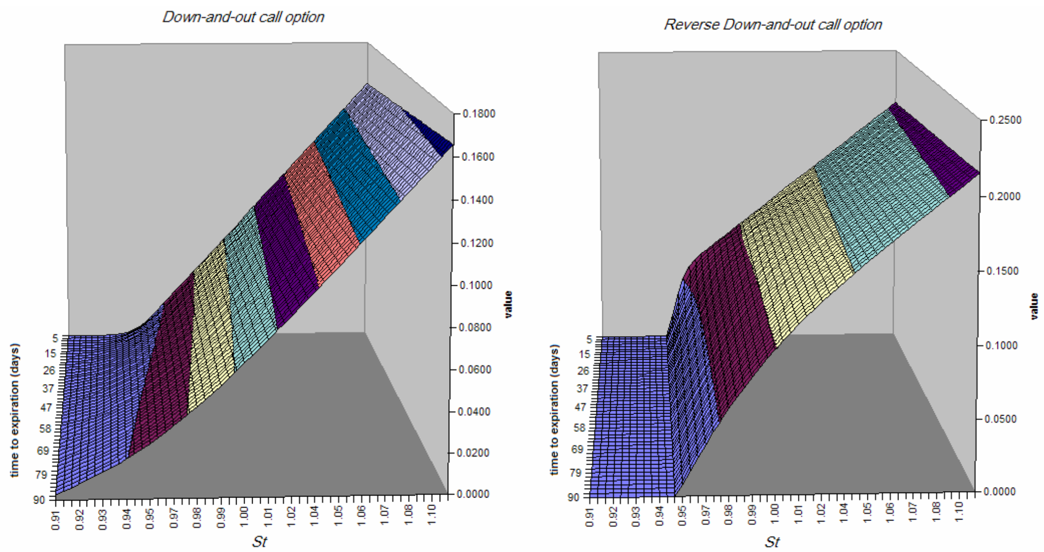

Figure 36: The values of a standard down-and-out call option and a reverse down-and-out cal option. Standard Parameters: K = 0.95, H = 0.90, r = 5%, q = 0%, T =90/360, σ = 10%, R = 0. Reverse Parameters: K = 0.90, H = 0.95, r = 5%, q = 0%, T =90/360, σ = 10%, R = 0. See Table 7.

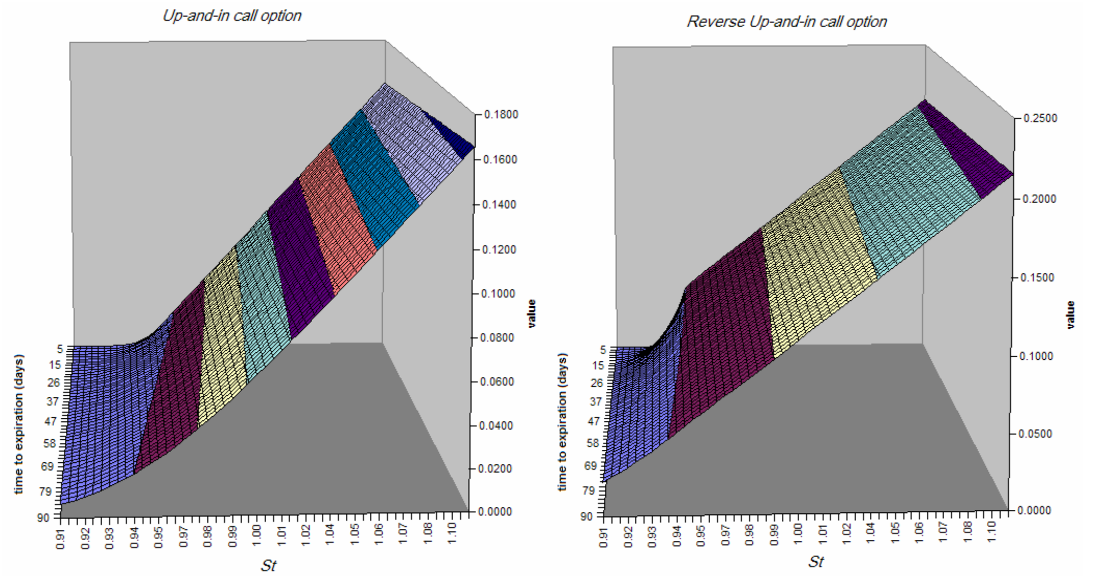

Figure 37: The values of a standard up-and-in call option and a reverse up-and-in call option. Standard Parameters: K = 0.95, H = 0.90, r = 5%, q = 0%, T =90/360, σ = 10%, R = 0. Reverse Parameters: K = 0.90, H = 0.95, r = 5%, q = 0%, T =90/360, σ = 10%, R = 0. See Table 7.

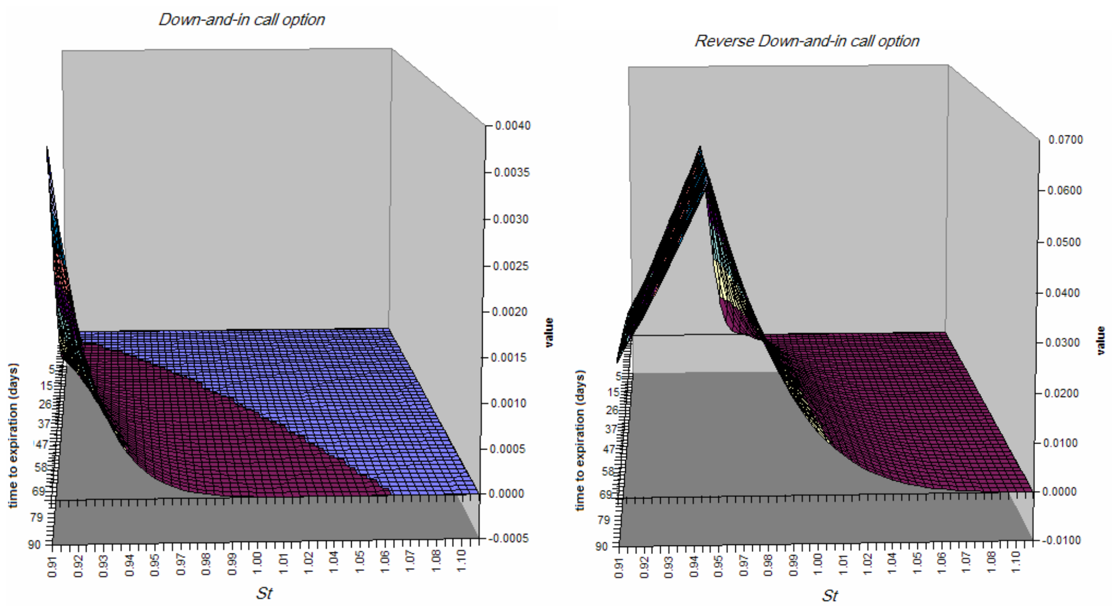

Figure 38: The values of a standard down-and-in call option and a reverse down-and-in call option. Standard Parameters: K = 0.95, H = 0.95, r = 5%, q = 0%, T =90/360, σ = 10%, R = 0. Reverse Parameters: K = 0.90, H = 0.95, r = 5%, q = 0%, T =90/360, σ = 10%, R = 0. See Table 7.

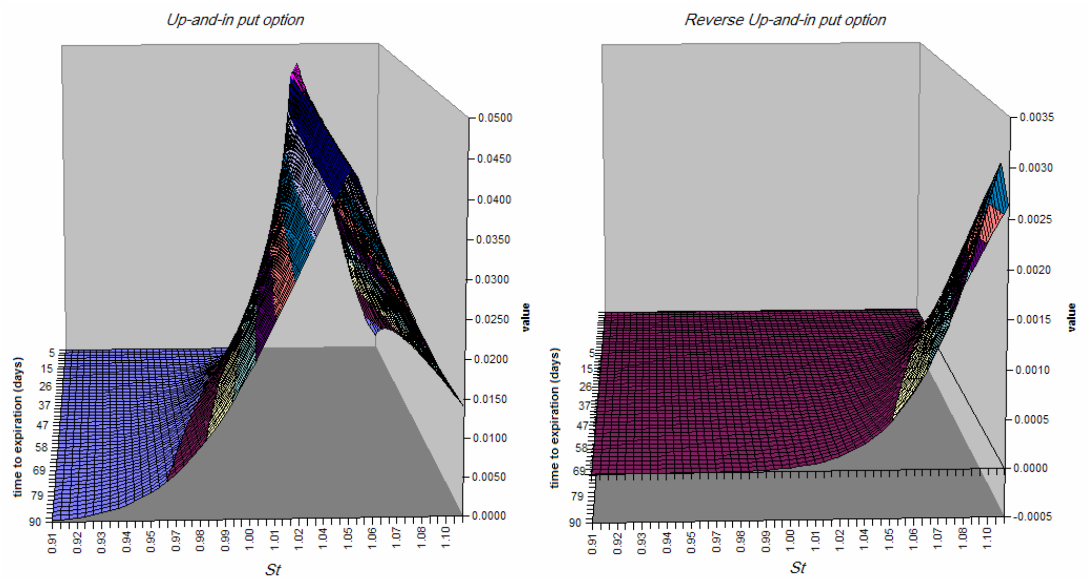

Figure 39: The values of a standard down-and-in call option and a reverse down-and-in call option. Standard Parameters: K = 0.95, H = 0.95, r = 5%, q = 0%, T =90/360, σ = 10%, R = 0. Reverse Parameters: K = 0.90, H = 0.95, r = 5%, q = 0%, T =90/360, σ = 10%, R = 0. See Table 7.

4.1.5 Arbitrage Bounds on Valuation

Plain Vanilla Put-Call Transformation (Haug, 1999)

The American plain vanilla put-call transformation, where S is the asset price, K the strike price, T time to maturity, r the risk free interest rate and b the cost of carry, is given by

$$C(S,K,T,r,b,\sigma) = P(K,S,T,r-b,-b,\sigma).$$

It shows that the value of an American call option is similar to the value of an American put option, with the put asset price equal to the call strike price, the put strike price equal to the call asset price, risk-free rate equal to *r-b* and cost of carry equal to *–b*. This transformation also holds for European options. Rewrite the payoff function from a call option, $\max(S-K,0)$, as $\frac{K}{S}\max\left(\frac{S^2}{K}-S,0\right)$ and combine it with the put-call transformation to get the put-call symmetry:

$$C(S,K,T,r,b,\sigma) = \frac{K}{S} P\left(K,\frac{S^2}{K},T,r-b,-b,\sigma\right).$$

This equation is useful for static hedging and valuation of many exotic options on the basis of plain vanilla options. This is because it is not possible to buy, for instance, a put option with asset price K when the asset price is S (assuming $K \neq S$). However, to buy $\frac{K}{S}$ put options with strike $\frac{S^2}{K}$ and asset price S is a real possibility in the options market.

Barrier Option Put-Call Transformation (Haug, 1999)

The only difference between a plain vanilla put-call transformation and a put-call barrier transformation is the probability of barrier hits. Given the same volatility and drift toward the barrier, the probability of barrier hits only depends on the distance between the asset price and the barrier. In the put-call transformation the drifts are different for the call and the put, *b* versus *–b*. However, given that the asset price of the call is above (below) the barrier and the asset price of the put is below (above) the barrier, this will naturally ensure the same drift towards the barrier. In the case of a put-call transformation between a down-call with asset price S, and an up-put with asset price K, it must be that

$$\ln\left(\frac{S}{H_c}\right) = \ln\left(\frac{H_p}{K}\right),$$

where the call barrier $H_c < S$ and the put barrier $H_p > K$. In the case of a put-call transformation between an up-call and a down-put the barriers and strike must satisfy

$$\ln\left(\frac{H_c}{S}\right) = \ln\left(\frac{K}{H_p}\right),$$

where $H_c > S$ and $H_p < K$. In both cases the put barrier can be rewritten as

$$H_p = \frac{SK}{H_c}.$$

For standard barrier option the put-call transformation and symmetry between “in” option must, from this, be given by

$$
\begin{aligned}
DIC(S,K,H,r,b) &= UIP\left(K,S,\frac{SK}{H},r-b,-b\right) \\
&= \frac{K}{S}UIP\left(S,\frac{S^2}{K},\frac{S^2}{H},r-b,-b\right), \\
UIC(S,K,H,r,b) &= DIP\left(K,S,\frac{SK}{H},r-b,-b\right) \\
&= \frac{K}{S}DIP\left(S,\frac{S^2}{K},\frac{S^2}{H},r-b,-b\right).
\end{aligned}
$$

The put-call transformation and symmetry between "out" option is given by

$$
\begin{aligned}
DOC(S,K,H,r,b) &= UOP\left(K,S,\frac{SK}{H},r-b,-b\right) \\
&= \frac{K}{S}UOP\left(S,\frac{S^2}{K},\frac{S^2}{H},r-b,-b\right), \\
UOC(S,K,H,r,b) &= DOP\left(K,S,\frac{SK}{H},r-b,-b\right) \\
&= \frac{K}{S}DOP\left(S,\frac{S^2}{K},\frac{S^2}{H},r-b,-b\right).
\end{aligned}
$$

If one has a formula for a barrier call, the relationship will give the value for the barrier put and vice versa (Haug, 1999).

4.1.6 Sensitivities

Here are the expressions that correspond to the expressions [1] to [4] of theorem 5 that are relevant for the different sensitivities. These expressions will be used in the same combination used to price each barrier option, to derive its sensitivity to the various variables.

Delta

$$[1]=\phi e^{-q(T-t)}N\left(\phi w_1\right)$$

$$[2]=\phi e^{-q(T-t)}N\left(\phi w_2\right)+e^{-q(T-t)}n\left(\phi w_2\right)\left(1-\frac{K}{H}\right)/\sigma\sqrt{T-t}$$

$$[3]=\phi\left(\frac{\mathrm{H}}{S}\right)^{2\lambda-2}\left[Se^{-q(T-t)}\left(\frac{\mathrm{H}}{S}\right)^{2}N\left(\eta w_4\right)-Ke^{-r(T-t)}N\left(\eta w_4-\eta\sigma\sqrt{T-t}\right)\right]$$

$$[4]=-\phi\left(\frac{\mathrm{H}}{S}\right)^{2\lambda}e^{-q(T-t)}$$

Gamma

$$[1]=e^{-q(T-t)}n\left(w_1\right)/\left(S\sigma\sqrt{(T-t)}\right)$$

$$[2]=e^{-q(T-t)}n\left(w_2\right)/\left(S\sigma\sqrt{(T-t)}\right)\left(1-\left(1-\frac{K}{H}\right)w_2/\sigma\sqrt{(T-t)}\right)$$

$$\left[A_3\right]=-\frac{2\mu}{\sigma^2 S}B_3-\phi\left(\frac{H}{S}\right)^{2\lambda}e^{-q(T-t)}\mathrm{N}\left(\eta w_4\right)-\phi\eta e^{-q(T-t)}\left(\frac{H}{S}\right)^{2\lambda}n\left(w_4\right)\left(1-\frac{K}{H}\right)/\sigma\sqrt{(T-t)}$$

$$[B_3]=\phi\left(\frac{H}{S}\right)^{2\lambda-2}\left[Se^{-q(T-t)}\left(\frac{H}{S}\right)^{2}\mathrm{N}\left(\eta w_4\right)-Ke^{-r(T-t)}\mathrm{N}\left(\eta w_4-\eta\sigma\sqrt{(T-t)}\right)\right]$$

$$[3]=\frac{2\mu}{\sigma^2 S}\left(\frac{B_3}{S}-A_3\right)+\phi e^{-q(T-t)}\frac{H^{2\lambda}}{S^{2\lambda+1}}\left[2\lambda\mathrm{N}\left(\eta w_4\right)+\frac{\eta n\left(w_4\right)}{\sigma\sqrt{(T-t)}}\right]$$
$$+\phi\eta e^{-q(T-t)}\left(1-\frac{K}{H}\right)n\left(w_4\right)/\left(S\sigma\sqrt{(T-t)}\right)\left(2\lambda-\frac{w_4}{\sigma\sqrt{(T-t)}}\right)$$

$$[A_4]=-\frac{2\mu}{\sigma^2 S}B_4-\phi\left(\frac{H}{S}\right)^{2\lambda}e^{-q(T-t)}\mathrm{N}\left(\eta w_3\right)$$

$$[B_4]=\phi\left(\frac{H}{S}\right)^{2\lambda-2}\left[Se^{-q(T-t)}\left(\frac{H}{S}\right)^{2}\mathrm{N}\left(\eta w_3\right)-Ke^{-r(T-t)}\mathrm{N}\left(\eta w_3-\eta\sigma\sqrt{(T-t)}\right)\right]$$

Theta

$$[1]=-\frac{1}{2}\sigma Se^{-q(T-t)}n\left(w_1\right)/\sqrt{T-t}+\phi Se^{-q(T-t)}N\left(\phi w_1\right)q-\phi Ke^{-r(T-t)}N\left(\phi w_1-\phi\sigma\sqrt{T-t}\right)r$$

$$[2]=-\frac{1}{2}\sigma Se^{-q(T-t)}n\left(w_2\right)K/\left(H\sqrt{T-t}\right)+\phi Se^{-q(T-t)}N\left(\phi w_2\right)q-\phi Ke^{-r(T-t)}N\left(\phi w_2-\phi\sigma\sqrt{T-t}\right)r$$
$$-Se^{-q(T-t)}\left(1-\frac{K}{H}\right)w_4/\left(2\left(T-t\right)\right)$$

$$[3]=-\phi\left(\frac{\mathrm{H}}{S}\right)^{2\lambda}Se^{-q(T-t)}\eta n\left(w_4\right)\left[w_2/\left(2\left(T-t\right)\right)\left(1-\frac{K}{H}\right)+\frac{1}{2}\sigma K/\left(H\sqrt{T-t}\right)\right]$$
$$+\phi\left(\frac{\mathrm{H}}{S}\right)^{2\lambda-2}\left[qSe^{-q(T-t)}\left(\frac{\mathrm{H}}{S}\right)^{2}N\left(\eta w_4\right)-rKe^{-r(T-t)}N\left(\eta w_4-\eta\sigma\sqrt{T-t}\right)\right]$$

$$[4]=-\phi\left(\frac{\mathrm{H}}{S}\right)^{2\lambda}Se^{-q(T-t)}\eta n\left(w_3\right)\frac{1}{2}\sigma/\sqrt{T-t}$$
$$+\phi\left(\frac{\mathrm{H}}{S}\right)^{2\lambda-2}\left[qSe^{-q(T-t)}\left(\frac{\mathrm{H}}{S}\right)^{2}N\left(\eta w_3\right)-rKe^{-r(T-t)}N\left(\eta w_3-\eta\sigma\sqrt{T-t}\right)\right]$$

Vega

$$[1]=Se^{-r(T-t)}n\left(w_1\right)\sqrt{T-t}$$

$$[2]=Se^{-r(T-t)}n\left(w_2\right)\left(\sqrt{T-t}-w_2\left(1-\frac{K}{H}\right)/\sigma\right)$$

$$[A_3]=\phi\left(\frac{\mathrm{H}}{S}\right)^{2\lambda-2}\left[Se^{-q(T-t)}\left(\frac{\mathrm{H}}{S}\right)^{2}N\left(\eta w_4\right)-Ke^{-r(T-t)}N\left(\eta w_4-\eta\sigma\sqrt{T-t}\right)\right]$$

$$[3]=\frac{-4}{\sigma^3}\log\left(\frac{H}{S}\right)\left(r-q\right)A_3$$
$$+\phi\left(\frac{\mathrm{H}}{S}\right)^{2\lambda}Se^{-q(T-t)}\eta n\left(w_4\right)\left[\left(\sqrt{T-t}-w_4/\sigma\right)\left(1-\frac{K}{H}\right)+\frac{K}{H}\sqrt{T-t}\right]$$

$$[A_4]=\phi\left(\frac{\mathrm{H}}{S}\right)^{2\lambda-2}\left[Se^{-q(T-t)}\left(\frac{\mathrm{H}}{S}\right)^{2}N\left(\eta w_3\right)-Ke^{-r(T-t)}N\left(\eta w_3-\eta\sigma\sqrt{T-t}\right)\right]$$

$$[4]=\frac{-4}{\sigma^3}\log\left(\frac{H}{S}\right)\left(r-q\right)A_4+\phi\left(\frac{\mathrm{H}}{S}\right)^{2\lambda}Se^{-q(T-t)}\eta n\left(w_3\right)\sqrt{T-t}$$

In figure 40 the sensitivities for a reverse up-and-out call option is illustrated.

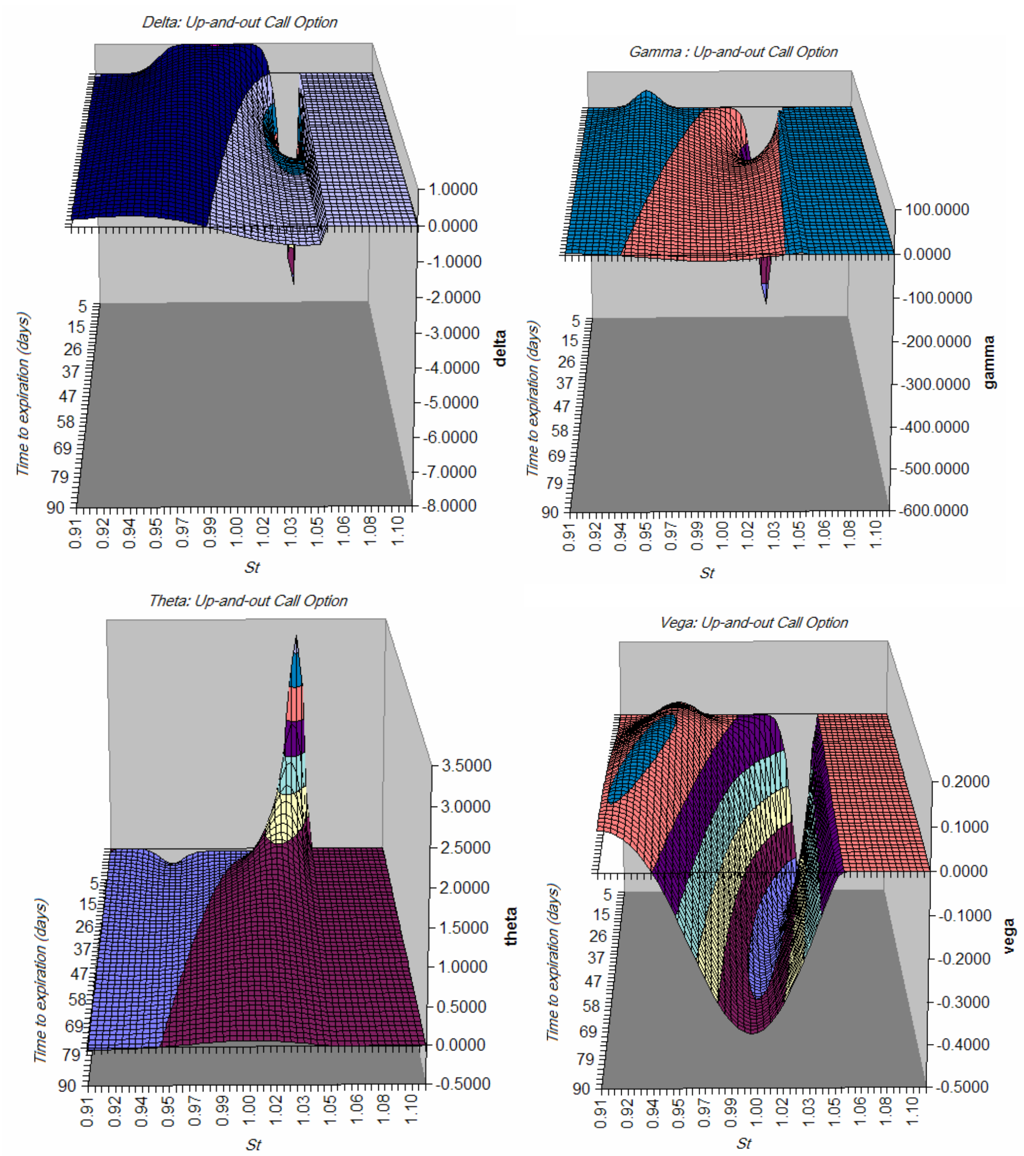

Figure 40: The sensitivities of a reverse up-and-out call. Parameters: K = 0.95, H = 1.05, r = 5%, q = 0%, T =90/360, σ = 10%, R = 0.

4.2 Asian Options

4.2.1 Definition

As discussed in www.riskglossary.com, an **Asian option** is an option of which the payoff is linked to the average value of the underlying asset on a specific set of dates during the life of the option. There are two basic forms:

- An **average rate option** or **average price option** (ARO) is a cash-settled option of which the payoff is based on the difference between the average value of the underlying asset during the life of the option and a fixed strike. Here the expiry date is usually the same date as the last recording date determining the average.
- An **average strike option** (ASO) is a cash settled or physically settled option. It is structured like a vanilla option, except that its strike is set equal to the average value of the asset prices recorded over the life of the option. In this structure it is common for the user to specify an expiry date later than the last recording.

Both types of Asian options can be structured as puts or calls. They are generally exercised as European, but it is possible to specify early exercise provisions based upon an average-to-date.

4.2.2 Common Uses

Asian options were first used in 1987 when Banker's Trust Tokyo office used them for pricing average options on crude oil contracts; hence the name "Asian" option. Asian options are options in which the underlying variable is the average price over a period of time. They are attractive because they tend to be less expensive and sell at lower premiums than comparable vanilla puts or calls. This is because the volatility in the average value of an underlying asset tends to be lower than the volatility of the value of the underlying asset. They are commonly traded on currencies and commodity products which have low trading volumes. In these situations the underlying asset is thinly traded, or there is the potential for its price to be manipulated, and Asian option offers some protection. It is more difficult to manipulate the average value of an underlying asset over an extended period of time than it is to manipulate it just at the expiration of an option (www.global-derivatives.com, www.riskglossary.com).

Consider the following example by Kolb (2003):

A corporate executive is given options on the firm's shares as part of her compensation. If the option payoff were determined by the price of the firm's shares on a particular day, the executive could enrich herself by manipulating the price of her shares for that single day. However, if the payoff of the options depended on the average closing price of the shares over six months, it would be much more difficult for her to profit from manipulation.

4.2.3 Valuation

According to www.derivatives.com, Asian options are broadly segregated into three categories; arithmetic average Asians, geometric average Asians, and both these forms can be averaged on a *weighted average* basis, resulting in the third category, whereby a given weight is applied to each stock being averaged. This can be useful for attaining an average on a sample with a highly skewed sample population.

In other words, averages can be calculated arithmetically:

$$\text{aritmetic average } = \frac{s_1 + s_2 + \ldots + s_m}{m}$$

or geometrically:

$$\text{geometric average} = \sqrt[m]{s_1 s_2 \ldots s_m}\,.$$

They can also be weighted with some weights w_i:

$$\text{weighted aritmetic average } = \frac{w_1 s_1 + w_2 s_2 + \ldots + w_m s_m}{w_1 + w_2 + \ldots + w_m}$$

$$\text{geometric average} = \sqrt[w_1 + w_2 + \ldots + w_m]{{s_1}^{w_1} {s_2}^{w_2} \ldots {s_m}^{w_m}}\;.$$

To this date, there are no known closed form analytical solutions for arithmetic options. The main theoretical reason is that in the standard Black-Scholes environment, security prices are lognormally distributed. Consequently, the geometric Asian option is characterised by the correlated product of lognormal random variables, which is also lognormally distributed. As a result, the state-price density function is lognormal and hence, the analytic no-arbitrage value of the option can be obtained using risk-neutral expectations. In contrast, the arithmetic Asian option depends on the finite sum of correlated lognormal random variables, which is clearly not lognormally distributed and for which there is no recognisable closed-form probability density (Milevsky and Posner, 1998). A further breakdown of these options conclude that Asians are either based on the *average price* of the underlying asset or, alternatively, there is the *average strike* type.

The payoff of geometric Asian options is given as:

$$Payoff_{Asian-Call} = \max\left(0, \left(\prod_{i=1}^{m} S_i\right)^{\frac{1}{m}} - K\right) \text{ and}$$

$$Payoff_{Asian-Put} = \max\left(0, K - \left(\prod_{i=1}^{m} S_i\right)^{\frac{1}{m}}\right).$$

The payoff of arithmetic Asian options is given as:

$$Payoff_{Asian-Call} = \max\left(0, \frac{\sum_{i=1}^{m} S_i}{m} - K\right) \text{ and}$$

$$Payoff_{Asian-Put} = \max\left(0, K - \frac{\sum_{i=1}^{m} S_i}{m}\right).$$

The payoff functions for the Asian options above can also be written in a more general way.

For an average price Asian:

$$V = \max\left(0, \eta\left(S_{ave} - K\right)\right)$$

and average strike Asian:

$$V = \max\left(0, \eta\left(S_T - S_{ave}\right)\right),$$

where η is a binary variable which is set to 1 for a call, and -1 for a put option.

A final consideration is how much data to use in the calculation of the average. If closely-spaced prices over a finite time are used, then the sum that is calculated in the average becomes an integral of the asset over the averaging period. This would give a continuously sampled average. More commonly, only the data at reliable points are taken. Closing prices are used, which is a smaller set of data. This is discrete sampling (Wilmott, 1998).

Formally define

r The continuously compounded risk-free rate of interest, assumed constant over the life of the option.

q The continuous yield on the asset, assumed constant over the life of the option.

S_t The spot price at time t.

A_t The running discrete arithmetic average to date, defined for any time point t, $t_m \le t < t_{m+1}$ given by

$$A_t = \frac{1}{m}\sum_{i=1}^{m} S_{t_i}$$

for a corresponding integer $1 \le m < N$ and $A_t = 0$ for $t < t_1$.

A_{t_N} The arithmetic average of N prices.

G_t The corresponding geometric average given by

$$G_t = \left[S_{t_1}, S_{t_2}, \ldots, S_{t_m}\right]^{\frac{1}{m}}.$$

$ARO_C\left(K,t\right)$ The value at time t of an ARO call option.

$ARO_P\left(K,t\right)$ The value at time t of an ARO put option.

Characterising Valuation Formulae

Option valuation usually assumes instantaneous asset returns to be normally distributed so that asset prices at some future date are log-normally distributed. So in the risk-neutral world, the underlying asset price is assumed to follow the stochastic differential equation

$$dS = \mu S dt + \sigma S dz, \tag{4.2.1}$$

where dz is a Wiener process, μ the drift parameter and σ the volatility parameter. The payoff on Asian options is based on the future path of the spot prices with the process given in (4.2.1). Under (4.2.1) S_{t_i} can be expressed in terms of $S_{t_{i-1}}$ as:

$$S_{t_i} = S_{t_{i-1}} e^{\left(\mu - \frac{1}{2}\sigma^2\right)(t_i - t_{i-1}) + \sqrt{t_i - t_{i-1}} Y_i}, \tag{4.2.2}$$

where $Y_i \sim N(0,1)$. For $t_i > 0$, $\ln S_{t_i} \sim N\left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t_i, \sigma^2 t_i\right)$.

Using the risk-neutral transformation of Cox and Ross (1976), the solution to the Asian option at time $t = 0$ may be characterised as:

ARO call: $$ARO_C(S_0, K) = e^{-rT} E^* \left\{\max\left[A_{t_N} - K, 0\right]\right\} \tag{4.2.3}$$

ASO call: $$ARO_C(S_0) = e^{-rT} E^* \left\{\max\left[S_T - A_{t_N}, 0\right]\right\}, \tag{4.2.4}$$

where E^* is the expectation conditional on S_0 at time $t = 0$ under the risk adjusted density function. This means that in (4.2.1), which gives the process for S_t, μ is replaced by $(r - q)$. Suppose the conditional density function for A_{t_N}, conditional that $A_{t_N} > K$, is denoted by $f^*(w)$, then the expectation term in (4.2.3) can be written as

$$E^* \left\{\max\left[A_{t_N} - K, 0\right]\right\} = \int_K^\infty \left[A_{t_N} - K\right] f^*(w) dw. \tag{4.2.5}$$

For the ASO, the joint density of A_{t_N} and S_T is needed. Denote this by $\Upsilon^*(\xi, w)$, then (4.2.4) can be written as:

$$E^*\left\{\max\left[S_T - A_{t_N}, 0\right]\right\} = \int_0^{\infty}\int_w^{\infty}\left[S_T - A_{t_N}\right]\Upsilon^*(\xi, w)\,d\xi\,dw.$$

Valuing Geometric AROs and ASOs

Two valuation methods are given for geometric AROs and ASOs. The first one uses discrete sampling, while the second one uses continuous sampling.

<u>Discrete Sampling</u>

Because the product of log-normal prices is itself log-normal, the geometric average, G_{t_N}, is also log-normal and the functions $f^*(w)$ and $\Upsilon^*(\xi, w)$ can be determined. Hence the valuation of Asian options, determined by a geometric average of prices, is a relatively simple matter. The derivation given by Clewlow and Strickland (1997) is followed here.

The geometric average is given by $G_{t_N} = \left[S_{t_1}, S_{t_2}, ..., S_{t_N}\right]^{\frac{1}{N}}$. Therefore

$$\ln G_{t_N} = \frac{1}{N}\sum_{i=1}^{N}\ln S_{t_i} \sim N\left(\mu_G, \sigma_G^2\right);$$

since it is a linear combination of normally distributed random variables, it is also normally distributed. It also follows that the distribution $\left[\ln G_{t_N}, \ln S_T\right]$ is bivariate normal with $\rho_G\sigma_G\sigma_T$ the covariance between $\ln G_{t_N}$ and $\ln S_T$. Hence the geometric ARO call and ASO call are, respectively, given by:

$$ARO_C = e^{-rT}E^*\left\{\max\left[G_{t_N} - K, 0\right]\right\} = e^{\mu_G + \frac{1}{2}\sigma_G^2 - rT}\Phi(x_1) - e^{-rT}K\Phi(x_2)$$

where $\Phi(.)$ is the standard normal distribution function ,

$$x_1 = \frac{\left(\mu_G - \ln K + \sigma_G^2\right)}{\sigma_G}$$
$$x_2 = x_1 - \sigma_G;$$

and

$$ASO_C = e^{-rT} E^* \left\{\max\left[S_T - G_{t_N}, 0\right]\right\} = S_t e^{-yT} K\Phi\left(y_1\right) - e^{\mu_G + \frac{1}{2}\sigma_G^2 - rT} \Phi\left(y_2\right)$$

where

$$y_1 = \frac{\left[\ln S_t + \left(r-q\right)T - \mu_G - \frac{1}{2}\sigma_G^2 + \frac{1}{2}\Sigma^2\right]}{\Sigma}$$
$$y_2 = y_1 - \Sigma$$
$$\Sigma^2 = \sigma_G^2 + \sigma^2 T - 2\rho_G \sigma_G \sigma_T .$$

The ARO put and ASO put are, respectively, given by:

$$ARO_P = e^{-rT} E^* \left\{\max\left[K - G_{t_N}, 0\right]\right\} = e^{-rT}\left[K\Phi\left(-x_2\right) - e^{\mu_G + \frac{1}{2}\sigma_G^2}\Phi\left(-x_1\right)\right]$$

and

$$ASO_P = e^{-rT} E^* \left\{\max\left[G_{t_N} - S_T, 0\right]\right\} = e^{\mu_G + \frac{1}{2}\sigma_G^2 - rT}\Phi\left(-y_2\right) - S_t e^{-yT} K\Phi\left(-y_1\right).$$

To calculate the expressions for μ_G, σ_G and $\rho_G \sigma_G \sigma_T$, the mean and variance of the logarithm of the geometric average and its covariance with $\ln S_T$ is first derived for any given time point *t*. At any time *t*, $\ln G_{t_N}$, can be expressed as

$$\ln G_{t_N} = \frac{m}{N}\ln G_t + \frac{1}{N}\sum_{i=m+1}^{N} \ln S_{t_i},$$

where $G_t = \left[S_{t_1}, S_{t_2}, ..., S_{t_m}\right]^{\frac{1}{m}}$, for some $0 \le m < N$. Each $\ln S_{t_i}$ is distributed $N\left(\mu_i, \sigma_i^2\right)$. It follows that the mean of $\ln G_{t_N}$ is immediately

$$\mu_G = \frac{m}{N}\ln G_t + \frac{1}{N}\sum_{i=m+1}^{N}\mu_i \ .$$

For constant *r, q* and σ

$$\mu_i = \ln S_t + \left(r - q - \frac{1}{2}\sigma^2\right)(t_i - t)$$
$$\sigma_i = \sigma^2(t_i - t).$$

Hence,

$$\mu_G = \frac{m}{N}\ln G_t + \frac{N-m}{N}\ln S_t + \frac{1}{N}\left(r - q - \frac{1}{2}\sigma^2\right)\sum_{i=m+1}^{N}(t_i - t).$$

For equidistant fixing intervals from $t_1, (t_{m+i} - t) = (t_{m+1} - t) + (i-1)h,$ where $h = \frac{(t_N - t_1)}{N-1}(i = m+1, ..., N)$, μ_G can be expressed as

$$\mu_G = \frac{m}{N}\ln G_t + \frac{N-m}{N}\left\{\ln S_t + \left(r - q - \frac{1}{2}\sigma^2\right)\left[(t_{m+1} - t) + \frac{h}{2}(N - m - 1)\right]\right\}.$$

The variance of $\ln G_{t_N}$ is given by

$$\sigma_G^2 = \frac{1}{N^2}\left[\sum_{i=m+1}^{N}\sigma_i^2 + 2\sum_{i=m+1}^{N-1}\sum_{j=i+1}^{N}\rho_{ij}\sigma_i\sigma_j\right],$$

where ρ_{ij} is the correlation between $\ln S_{t_i}$ and $\ln S_{t_j}$. For $i \le j$, $\rho_{ij} = \frac{\sigma_i}{\sigma_j}$; hence

$$\sigma_G^2 = \frac{1}{N^2}\left[\sum_{i=m+1}^{N}\sigma_i^2 + 2\sum_{i=m+1}^{N-1}(N-i)\sigma_i^2\right].$$

When $\sigma_i^2 = \sigma^2(t_i - t)$ this becomes

$$\sigma_G^2 = \frac{\sigma^2}{N^2}\left[\sum_{i=m+1}^{N}(t_i - t) + 2\sum_{i=m+1}^{N-1}(N-i)(t_i - t)\right]$$

and for $(t_{m+i} - t) = (t_{m+1} - t) + (i-1)h$

$$\sigma_G^2 = \sigma^2\left(\frac{N-m}{N}\right)^2\left[(t_{m+1} - t) + \frac{h(2N-2m-1)(N-m-1)}{6(N-m)}\right].$$

Finally the covariance term $\rho_G\sigma_G\sigma_T$ is given by

$$\rho_G\sigma_G\sigma_T = Cov\left(\frac{1}{N}\sum_{i=m+1}^{N}\ln S_{t_i}, \ln S_T\right)$$
$$= \frac{1}{N}\sum_{i=m+1}^{N}\sigma_i^2.$$

For $\sigma_i^2 = \sigma^2(t_i - t)$

$$\rho_G\sigma_G\sigma_T = \frac{\sigma^2}{\mathrm{N}}\sum_{i=m+1}^{N}(t_i - t),$$

and for $(t_{m+i} - t) = (t_{m+1} - t) + (i-1)h$

$$\rho_G\sigma_G\sigma_T = \sigma^2\left(\frac{N-m}{\mathrm{N}}\right)\left[(t_{m+1} - t) + \frac{h}{2}(N-m-1)\right].$$

Continuous Sampling

Kemna and Vorst (1991) shows that in a risk neutral world, the probability distribution of the geometric average of an asset price over a certain period is the same as that of the asset price at the end of the period if the asset's expected growth rate is set equal to $\frac{1}{2}\left(r - q - \frac{\sigma^2}{6}\right)$, rather than $(r-q)$, and its volatility is set equal to

$\frac{\sigma}{\sqrt{3}}$, rather than σ. The geometric average option can therefore be treated like a regular option with the volatility set equal to $\frac{\sigma}{\sqrt{3}}$ and the dividend yield equal to

$$r - \frac{1}{2}\left(r - q - \frac{\sigma^2}{6}\right) = \frac{1}{2}\left(r + q + \frac{\sigma^2}{6}\right).$$

The geometric average $G_{t_N} = \left[S_{t_1}, S_{t_2}, \ldots, S_{t_N}\right]^{\frac{1}{N}}$ in the continuous case can be written as

$$G_{t_N} = \exp\left(\frac{1}{t_N - t_1}\int_{t_1}^{t_N} \log S_\tau d\tau\right). \tag{4.2.6}$$

The variable G_{t_N} is lognormally distributed so that its expectation and variance values may be calculated explicitly. Define

$$V_t = \log S_t \text{ and } Z_t = \log G_t. \tag{4.2.7}$$

From Ito's lemma, it follows that (4.2.1) and (4.2.6) give rise to the following system of stochastic differential equations:

$$d\begin{pmatrix} V_t \\ Z_t \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ \frac{1}{t_N - t_1} & 0 \end{pmatrix}\begin{pmatrix} V_t \\ Z_t \end{pmatrix} + \begin{pmatrix} r - \frac{1}{2}\sigma^2 \\ 0 \end{pmatrix}\right] dt + \left[\begin{pmatrix} \sigma \\ 0 \end{pmatrix}\right] dz. \tag{4.2.8}$$

Using Arnold (1974), it follows that since this is a linear stochastic differential equation, $(V_t\ Z_t)'$ must be a Gaussian process. This means that $(V_t\ Z_t)'$ is binormally distributed. Hence, $\log\left(\frac{G_{t_N}}{G_{t_1}}\right)$ is normally distributed. Also from Arnold (1974) follows that

$$d\begin{pmatrix} \boldsymbol{E}V_t \\ \boldsymbol{E}Z_t \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ \dfrac{1}{t_N - t_1} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{E}V_t \\ \boldsymbol{E}Z_t \end{pmatrix} + \begin{pmatrix} r - \dfrac{1}{2}\sigma^2 \\ 0 \end{pmatrix}\right] dt. \tag{4.2.9}$$

The covariance matrix of $(V_t\ Z_t)'$ as defined by

$$K_t = \begin{pmatrix} K_{t(11)} & K_{t(12)} \\ K_{t(21)} & K_{t(22)} \end{pmatrix}, \tag{4.2.10}$$

which is the unique symmetric non-negative definite solution of the following matrix differential equation:

$$d\begin{pmatrix} K_{t(11)} & K_{t(12)} \\ K_{t(21)} & K_{t(22)} \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ \dfrac{1}{t_N - t_1} & 0 \end{pmatrix}\begin{pmatrix} K_{t(11)} & K_{t(12)} \\ K_{t(21)} & K_{t(22)} \end{pmatrix} + \begin{pmatrix} K_{t(11)} & K_{t(12)} \\ K_{t(21)} & K_{t(22)} \end{pmatrix}\begin{pmatrix} 0 & \dfrac{1}{t_N - t_1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \sigma \\ 0 \end{pmatrix}\begin{pmatrix} \sigma & 0 \end{pmatrix}\right] dt. \tag{4.2.11}$$

Solving (4.2.8) and (4.2.10) gives

$$\begin{pmatrix} \boldsymbol{E}\left(V_t - V_{t_1}\right) \\ \boldsymbol{E}\left(Z_t - Z_{t_1}\right) \end{pmatrix} = \begin{pmatrix} \left(r - \dfrac{1}{2}\sigma^2\right)(\varepsilon - t_1) \\ \dfrac{1}{2}\left(\dfrac{1}{t_N - t_1}\right)\left(r - \dfrac{1}{2}\sigma^2\right)(t - t_1)^2 + \left(\dfrac{1}{t_N - t_1}\right)V_{t_1}(t - t_1) \end{pmatrix} \tag{4.2.12}$$

$$\begin{pmatrix} K_{t(11)} & K_{t(12)} \\ K_{t(21)} & K_{t(22)} \end{pmatrix} = \begin{pmatrix} \sigma^2 (t - t_1) & \dfrac{1}{2}\left(\dfrac{1}{t_N - t_1}\right)\sigma^2 (t - t_1)^2 \\ \dfrac{1}{2}\left(\dfrac{1}{t_N - t_1}\right)\sigma^2 (t - t_1)^2 & \dfrac{1}{3}\left(\dfrac{1}{t_N - t_1}\right)^2 \sigma^2 (t - t_1)^3 \end{pmatrix}. \tag{4.2.13}$$

Combining (4.2.7), (4.2.12) and (4.2.13) immediately gives

$$\log G_{t_N} \sim n\left(\frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)(t_N - t_1) + \log S_{t_1}; \frac{1}{3}\sigma^2 (t_N - t_1)\right). \tag{4.2.14}$$

From probability theory it is known that in cases where A is a random variable, such that log A is normally distributed with mean E and variance V, and $K > 0$ is a real number, then:

$$\boldsymbol{E}\max\left(A-K,0\right)=e^{E+\frac{1}{2}V}N\left(\frac{E-\log K+V}{\sqrt{V}}\right)-KN\left(\frac{E-\log K}{\sqrt{V}}\right). \qquad (4.2.15)$$

By combining (4.2.14) and (4.2.15) the geometric average option can be evaluated as follows:

$$\begin{aligned}
\boldsymbol{E}\max\left(G_{t_N}-K,0\right)&=\boldsymbol{E}\left\{G_{t_N}\left|G_{t_N}\geq K\right.\right\}-KP\left(G_{t_N}\geq K\right)\\
&=e^{\frac{1}{2}\left(r-\frac{1}{2}\sigma^2\right)\left(t_N-t_1\right)+\log S_{t_1}+\frac{1}{2}\left[\frac{1}{3}\sigma^2\left(t_N-t_1\right)\right]}\times\\
&\quad N\left(\frac{\frac{1}{2}\left(r-\frac{1}{2}\sigma^2\right)\left(t_N-t_1\right)+\log S_{t_1}-\log K+\left[\frac{1}{3}\sigma^2\left(t_N-t_1\right)\right]}{\sqrt{\frac{1}{3}\sigma^2\left(t_N-t_1\right)}}\right)\\
&\quad -KN\left(\frac{\frac{1}{2}\left(r-\frac{1}{2}\sigma^2\right)\left(t_N-t_1\right)+\log S_{t_1}-\log K}{\sqrt{\frac{1}{3}\sigma^2\left(t_N-t_1\right)}}\right)
\end{aligned}$$

$$\begin{aligned}
&=S_{t_1}e^{\frac{1}{2}\left(r-\frac{1}{6}\sigma^2\right)\left(t_N-t_1\right)}\times N\left(\frac{\log\left(\frac{S_{t_1}}{K}\right)+\frac{1}{2}\left(r-\frac{1}{6}\sigma^2\right)\left(t_N-t_1\right)}{\sigma\sqrt{\frac{1}{3}\left(t_N-t_1\right)}}\right)\\
&\quad -KN\left(\frac{\log\left(\frac{S_{t_1}}{K}\right)+\frac{1}{2}\left(r-\frac{1}{6}\sigma^2\right)\left(t_N-t_1\right)}{\sigma\sqrt{\frac{1}{3}\left(t_N-t_1\right)}}-\sigma\sqrt{\frac{1}{3}\left(t_N-t_1\right)}\right)\\
&=S_{t_1}e^{d^*}N\left(d\right)-KN\left(d-\sigma\sqrt{\frac{1}{3}\left(t_N-t_1\right)}\right),
\end{aligned}$$

where

$$d^* = \frac{1}{2}\left(r - \frac{1}{6}\sigma^2\right)\left(t_N - t_1\right),$$

$$d = \frac{\log\left(\frac{S_{t_1}}{K}\right) + \frac{1}{2}\left(r - \frac{1}{6}\sigma^2\right)\left(t_N - t_1\right)}{\sigma\sqrt{\frac{1}{3}\left(t_N - t_1\right)}}.$$

Valuing Arithmetic AROs and ASOs

When, as is nearly always is the case, Asian options are defined in terms of the arithmetic averages, exact analytic pricing formulas are not available. This is because the distribution of the arithmetic average, which is a sum of log-normal components has no explicit representation or tractable properties. For arithmetic Asian options the functions $f^*(w)$ and $\Upsilon^*(\xi, w)$ in the characterising valuation formulae are non-standard, and to evaluate the necessary integrals, a variety of numeric and approximation methods have been developed.

A variety of techniques have been developed in the literature to analyse arithmetic Asian options. Generally, they can be classified as follows according to Milevsky and Posner (1998):

I. Monte Carlo simulations with variance reduction techniques:
Haykov (1993), Boyle (1977), Corwall et all. (1996) and Kemna and Vorst (1990).

II. Binomial trees and lattices with efficiency enhancements:
Hull and White (1993), and Neave and Turnbull (1993).

III. The PDE approach:
Dewynne and Wilmott (1995), Rogers and Shi (1995), and Alziary, Decamps and Koehl (1993).

IV. General numeric methods:
Carverhill and Clewlow (1990), Curran (1994), and Nielsen and Sandman (1996).

V. Pseudo-analytic characterisations:
German and Yor (1993), Yor (1993), Kramkov and Mordecky (1994), Ju (1997), and Chacko and Das (1997).

VI. Analytic approximations that produce closed-form expressions:

Turnbull and Wakeman (1991), Levy (1992), Vorst (1992), Vorst (1996), and Bouaziz, Briys and Crouhy (1994), Milevsky and Posner (1998).

An Edgeworth Series Expansion

The distribution of the arithmetic average of a set of lognormal distributions is approximately lognormal and this leads to a good analytic approximation for valuing average price options (Hull, 2006). Hull (2006) proposes that if the first two moments of the probability distribution of the arithmetic average in a risk-neutral world is calculated exactly, it can then be assumed that this distribution is the lognormal distribution. This means that the arithmetic average options can be valued similarly to geometric average options where pricing formulas are derived from the fact that the product of log-normal prices is itself log-normal. The moments of an arithmetic average option can be calculated using an edgeworth series expansion.

Turnbull and Wakeman (1991) apply a series expansion for $f^*(w)$ to adjust for higher moments effects. If $f^*(w)$ denotes the true distribution and $a(w)$ an alternative or approximating distribution which, in this case, is a log-normal probability density function, then we can expand $f^*(w)$ as follows:

$$f^*(w) = a(w) - E_1 a^{(1)}(w) + \frac{1}{2!}E_2 a^{(2)}(w) - \frac{1}{3!}E_3 a^{(3)}(w) + \frac{1}{4!}E_4 a^{(4)}(w) - ..., \quad (4.2.16)$$

where $a^{(i)}(w)$ is the ith derivatives of $a(w)$ and $\{E_i\}$ the terms involving the difference between cumulants implied by the log-normal fit and the true cumulants for A_{t_N}.

For a given distribution function F of a random variable X, the first four cumulants are:

$$
\begin{aligned}
\chi_1 &= E(X); \\
\chi_2 &= E\left[X - E(X)\right]^2; \\
\chi_3 &= E\left[X - E(X)\right]^3; \\
\chi_4 &= E\left[X - E(X)\right]^4 - 3\chi_2,
\end{aligned}
$$

where all expectations are with respect to the distribution F. Let $\chi_{i\varepsilon} = \chi_{if} - \chi_{ia}$. Then the first four coefficients $E_i, i = 1,2,3,4,$ are given by

$$
\begin{aligned}
\mathrm{E}_1 &= \chi_{1\varepsilon}, \\
\mathrm{E}_2 &= \chi_{1\varepsilon}^2 + \chi_{2\varepsilon}, \\
\mathrm{E}_3 &= \chi_{1\varepsilon}^3 + 3\chi_{1\varepsilon}\chi_{2\varepsilon} + \chi_{3\varepsilon}, \text{ and} \\
\mathrm{E}_4 &= \chi_{1\varepsilon}^4 + 3\chi_{2\varepsilon}^2 + 4\chi_{1\varepsilon}\chi_{3\varepsilon} + 6\chi_{1\varepsilon}^2\chi_{2\varepsilon} + \chi_{4\varepsilon}.
\end{aligned}
$$

If $a(w)$ is chosen to be the log-normal density with parameters α and ν, then the true moments of A_{t_N} are approximated by $E^*\left[A_{t_N}{}^k\right] = \exp\left(\alpha k + \frac{1}{2}v^2k^2\right)$. Substituting into (4.2.5) and taking the first four terms of (4.2.16), the ARO call is approximated after integrating by

$$
\begin{aligned}
e^{-rt_N}E^*\left\{\max\left[A_{t_N} - K, 0\right]\right\} = {} & e^{\alpha + \frac{1}{2}v^2 - rt_N}\Phi(x_1) - e^{-rt_N}\mathrm{K}\Phi(x_2) \\
& + e^{-rt_N}\left[E_1\Phi(x_2) + \frac{1}{2!}E_2 a(K) + \frac{1}{3!}E_3 a^{(1)}(K) + \frac{1}{4!}E_4 a^{(2)}(K)\right].
\end{aligned}
$$

When α and ν are chosen to equate the first two moments of A_{t_N}, $E_1 = E_2 = 0$ and the approximation becomes

$$
\begin{aligned}
e^{-rt_N}E^*\left\{\max\left[A_{t_N} - K, 0\right]\right\} = {} & e^{\alpha + \frac{1}{2}v^2 - rt_N}\Phi(x_1) - e^{-rt_N}\mathrm{K}\Phi(x_2) \\
& + e^{-rt_N}\left[\frac{1}{3!}E_3 a^{(1)}(K) + \frac{1}{4!}E_4 a^{(2)}(K)\right].
\end{aligned}
$$

To apply the Edgeworth expansion, it is necessary to determine the cumulants of the distribution of the average, A_{t_N} . Although the distribution of A_{t_N} is non-standard, its moments can be found using a recursive relationship. Let A_{t_N} be determined by fixing S_{t_i} for $i = 1,...,N$ and define the price relatives, R_i, by $S_{t_i} = S_{t_{i-1}} R_i$ for $i = 1,...,N$. From (4.2.2) R_i is log-normally distributed and under risk-neutrality is given by

$$R_i = e^{\left(r-q-\frac{1}{2}\sigma^2\right)}\left(t_i - t_{i-1}\right) + \sigma\sqrt{t_i - t_{i-1}}Y_i,$$

where $Y_i \sim N(0,1)$. Thus the moments for R_i are given by

$$E^*\left(R_i^k\right) = \exp\left(\alpha_i k + \frac{1}{2}\theta_i^2 k^2\right),$$

where

$$\alpha_i = \left(r - q - \frac{1}{2}\sigma^2\right)\left(t_i - t_{i-1}\right)$$
$$\theta_i^2 = \sigma\sqrt{t_i - t_{i-1}}Y_i.$$

It follows by definition that A_{t_N} can be written as

$$A_{t_N} = \left(\frac{S_{t_1}}{N}\right)\left(1 + R_2 + R_2R_3 + ... + R_2R_3R_4...R_N\right).$$

Define L_i as follows: $L_N = 1 + R_N$, and $L_i = 1 + R_iL_{i+1}$, $i = N-1,...,2$. Then A_{t_N} can be expressed as

$$A_{t_N} = \left(\frac{S_{t_1}}{N}\right)L_2,$$

and using $S_{t_1} = S_{t_0}R_1$,

$$A_{t_N} = \left(\frac{S_{t_0}}{N}\right)R_1L_2.$$

Since R_1 and L_2 are independent, it follows that

$$E^*\left[A_{t_N}^k\right] = \left(\frac{S_{t_0}}{N}\right)^k E^*\left(R_1^k\right)E^*\left(L_2^k\right).$$

It is known that $E^*\left(R_i^k\right) = \exp\left(\alpha_i k + \frac{1}{2}\theta_i^2 k^2\right)$ and to find $E^*\left(L_2^k\right)$, note that $E^*\left(L_N^k\right) = E^*\left[\left(1+R_N\right)^k\right]$ and for $i < N$, $E^*\left(L_i^k\right) = E^*\left[\left(1+R_i L_{i+1}\right)^k\right]$. Hence, $E^*\left(L_i^k\right)$ can be calculated recursively for $i = N-1,...,2$.

Simple closed-form expressions for the first two moments are now derived. Given that at time $t = 0$, $\ln S_{t_i} \sim N\left(\mu_i, \sigma_i^2\right)$, where

$$\begin{aligned}\mu_i &= \ln S_{t_0} + \left(\mu - \frac{1}{2}\sigma^2\right)t_i,\\ \sigma_i^2 &= \sigma^2 t_i.\end{aligned}$$

It follows that the first moment for A_{t_N} is given by

$$A_{t_N} = \frac{1}{N}\sum_{i=1}^{N} e^{\mu_i + \frac{1}{2}\sigma_i^2} = \frac{1}{N}\sum_{i=1}^{N} F_i,$$

where F_i denotes the forward price of S_{t_i}. For constant interest rates and volatility the following equation holds:

$$E^*\left[A_{t_N}\right] = \frac{S_t}{N}\sum_{i=1}^{N} e^{(r-y)t_i}$$

and for $t_i = t_1 + \frac{(i-1)(t_N - t_1)}{N-1}$,

$$E^*\left[A_{t_N}\right] = \frac{S_t}{N} e^{gt_1} \frac{1-e^{ghN}}{1-e^{gh}}$$

where

$$h = \frac{(t_N - t_1)}{N-1}$$
$$g = (r - y).$$

As the frequency increases, the limit as $N \to \infty$ tends to

$$E^*\left[A_{t_N}\right] = \frac{S_t}{N} e^{gt_1} \frac{e^{gh(t_N - t_1)} - 1}{g(t_N - t_1)}.$$

The second moment for A_{t_N} is given by

$$E^*\left[A_{t_N}^2\right] = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E^*\left[S_{t_i} S_{t_j}\right]$$

or

$$E^*\left[A_{t_N}^2\right] = \frac{1}{N^2} \left\{ \sum_{i=1}^{N} E^*\left[S_{t_i}^2\right] + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} E^*\left[S_{t_i} S_{t_j}\right] \right\}.$$

Noting that $E^*\left[S_{t_i} S_{t_j}\right] = F_i F_j e^{\sigma_i^2}$ for $i \le j$,

$$E^*\left[A_{t_N}^2\right] = \frac{1}{N^2} \left\{ \sum_{i=1}^{N} F_i^2 e^{\sigma_i^2} + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} F_i F_j e^{\sigma_i^2} \right\}.$$

For constant interest rates and volatility

$$E^*\left[A_{t_N}^2\right] = \frac{S_t^2}{N^2} \left[\sum_{i=1}^{N} e^{(2g+\sigma^2)t_i} + 2 \sum_{i=1}^{N-1} e^{(g+\sigma^2)t_i} \sum_{j=i+1}^{N} e^{gt_j} \right],$$

and for $t_i = t_1 + (i-1)h$,

$$E^*\left[A_{t_N}^2\right] = \frac{S_t^2 e^{(2g+\sigma^2)t_1}}{N^2} \left\{ \frac{1 - e^{(2g+\sigma^2)hN}}{1 - e^{(2g+\sigma^2)h}} + \frac{2}{1 - e^{(g+\sigma^2)h}} \left[\frac{1 - e^{ghN}}{1 - e^{gh}} - \frac{1 - e^{(2g+\sigma^2)hN}}{1 - e^{(2g+\sigma^2)h}} \right] \right\}.$$

In the limit as $N \to \infty$ this becomes

$$E^*\left[A_{t_N}^2\right] = \frac{2S_t^2 e^{\left(2g+\sigma^2\right)t_1}}{\left(g+\sigma^2\right)\left(t_N - t_1\right)^2}\left\{\frac{1-e^{g\left(t_N - t_1\right)}}{g} - \frac{1-2e^{\left(2g+\sigma^2\right)\left(t_N - t_1\right)}}{\left(2g+\sigma^2\right)}\right\}.$$

For the limit $N \to \infty$ the average A_{t_N} for continuous sampling is the integral:

$$A_{t_N} = \frac{1}{\left(t_N - t_1\right)}\int_{t_1}^{t_N} S_\tau d\tau .$$

In this instance it is relatively easy to provide closed-form expressions for all the moments for A_{t_N}. The following result is stated by Clewlow and Strickland (1997): for any integer $k \geq 1$, the k*th* moment for A_{t_N} at current time *t* is given by

$$E^*\left[A_{t_N}^k\right] = \frac{k!S_t^k}{\left(t_N - t_1\right)^k} e^{\left(2kg+m(k,k)\sigma^2\right)\left(t_1 - t\right)} M_k ,$$

where M_k is given by

$$M_1 = \left[\frac{e^{g\left(t_N - t_1\right)} - 1}{g}\right]$$

for $k = 1$ and for $k > 1$ by

$$M_k = \frac{1}{g+m(1,k)\sigma^2}\left[\frac{1}{2g+m(2,k)\sigma^2}\left[\ldots \frac{1}{(k-1)g+m(k-1,k)\sigma^2}\left[\frac{e^{\left(kg+m(k,k)\sigma^2\right)\left(t_N - t_1\right)} - 1}{kg+m(k,k)\sigma^2} - M_1\right] - \ldots - M_{k-2}\right] - M_{k-1}\right],$$

with $m(i,k) = i\left[k - \frac{1}{2}(i+1)\right]$ for $i = 1,\ldots,k$. For example, the next three expressions for M_k are:

$$M_2 = \frac{1}{g+\sigma^2}\left[\frac{e^{\left(2g+\sigma^2\right)\left(t_N-t_1\right)}-1}{2g+\sigma^2} - M_1\right],$$

$$M_3 = \frac{1}{g+2\sigma^2}\left[\frac{1}{2g+3\sigma^2}\left[\frac{e^{\left(3g+3\sigma^2\right)\left(t_N-t_1\right)}-1}{3g+3\sigma^2} - M_1\right] - M_2\right], \text{ and}$$

$$M_4 = \frac{1}{g+3\sigma^2}\left[\frac{1}{2g+5\sigma^2}\left[\frac{1}{3g+6\sigma^2}\left[\frac{e^{\left(4g+6\sigma^2\right)\left(t_N-t_1\right)}-1}{4g+6\sigma^2} - M_1\right] - M_2\right] - M_3\right].$$

If it is assumed that the average asset price is lognormal, an option on the average can be regarded as an option on a futures contract. Then the following equations can be used

$$c = e^{-rT}\left[F_0 N\left(d_1\right) - KN\left(d_2\right)\right] \text{ and}$$
$$p = e^{-rT}\left[KN\left(-d_2\right) - F_0 N\left(-d_1\right)\right],$$

where

$$d_1 = \frac{\ln\left(\frac{F_0}{K}\right) + \frac{\sigma^2 T}{2}}{\sigma\sqrt{\mathrm{T}}}$$

$$d_2 = \frac{\ln\left(\frac{F_0}{K}\right) - \frac{\sigma^2 T}{2}}{\sigma\sqrt{\mathrm{T}}}$$

with

$$F_0 = M_1$$

and

$$\sigma^2 = \frac{1}{T}\ln\left(\frac{M_2}{M_1^2}\right).$$

Continuous Sampling

In Zhang (1999) an analytical approximate formula for the pricing of an arithmetic Asian option with continuous sampling is derived by solving a partial differential equation (PDE) numerically. The method is shown to be more accurate than any existing method in the literature, and faster than other PDE methods. The method has a well-controlled error, and therefore the results can be used as a benchmark to justify the error computed by approximation methods, for which the error is unknown.

First AROs are considered. The results can be extended to deal with ASOs. The general explication of Zhang (1999) is followed and expanded on in this section.

The pricing formula is considered only within the averaging period, i.e. $t_0 \le t \le t_N$. The price of the option before the averaging period can be computed by solving the Black-Scholes equation with a particular payoff at t_0, given by the Asian option formula. Take time $t_0 = 0$ for simplicity. Introduce a new variable *I*;

$$I = \int_0^t S_\tau d\tau,$$

which is the sum of the underlying asset price S. Therefore I/t is the Arithmetic average of the underlying over the period of [0,t]. Hence, the payoff of the ARO call with continuous sampling can be written as

$$\max\left(\frac{I}{t_N} - K, 0\right).$$

The following lemma is the well-known Feyman-Kac theorem (cf. Shreve 1997) and will be used to prove Theorem 6.

Lemma 3.

Define

$$v(t,x) = E^{t,x} h\left(X(T)\right), 0 \le t \le T,$$

where

$$dX(t) = a\left(X(t)\right)dt + \sigma\left(X(t)\right)dB_t.$$

Then

$$v_t(t,x) + a(x)v_x(t,x) + \frac{1}{2}\sigma^2(x)v_{xx}(t,x) = 0$$

and

$$v(T,x) = h(x).$$

Theorem 6: (Zhang, 1999)

The price and Greeks of an arithmetic average rate call option with payoff $\max\left(\frac{I}{t_N}-K,0\right)$*, are given by the following analytical approximation formulas:*

$$ARO_{C_0}\left(S,I,t\right)=\frac{S}{t_N}f\left(\zeta,\eta\right)=\frac{S}{t_N}\left[-\zeta\mathrm{N}\left(-\frac{\zeta}{\sqrt{2\eta}}\right)+\sqrt{\frac{\eta}{\pi}}e^{-\frac{\zeta^2}{4\eta}}\right]$$

$$=S\frac{1-e^{-r\tau}}{rt_N}\mathrm{N}\left(-\frac{\zeta}{\sqrt{2\eta}}\right)+\frac{S}{t_N}\sqrt{\frac{\eta}{\pi}}e^{-\frac{\zeta^2}{4\eta}}-e^{-r\tau}\left(K-\frac{I}{t_N}\right)\mathrm{N}\left(-\frac{\zeta}{\sqrt{2\eta}}\right) \tag{4.2.17}$$

$$\Delta_0=\frac{\partial ARO_{C_0}}{\partial S}=\frac{1-e^{-r\tau}}{rt_N}\mathrm{N}\left(-\frac{\zeta}{\sqrt{2\eta}}\right)+\frac{1}{t_N}\sqrt{\frac{\eta}{\pi}}e^{-\frac{\zeta^2}{4\eta}}$$

$$\theta_0=\frac{\partial ARO_{C_0}}{\partial t}=-\frac{S}{t_N}\left(1+r\zeta\right)\frac{1-e^{-r\tau}}{rt_N}\mathrm{N}\left(-\frac{\zeta}{\sqrt{2\eta}}\right)-\frac{S\sigma^2}{4r^2t_N\sqrt{\pi\eta}}\left[1-2e^{-r\tau}+e^{-2r\tau}\right]e^{-\frac{\zeta^2}{4\eta}}$$

$$\Gamma_0=\frac{\partial\Delta_0}{\partial S}=\frac{1}{St_N\sqrt{\pi\eta}}\left[\zeta+\frac{1}{r}\left(1-e^{-r\tau}\right)\right]^2e^{-\frac{\zeta^2}{4\eta}}$$

$$Vega_0=\frac{\partial ARO_{C_0}}{\partial\sigma}=\frac{S}{4t_N\sigma}\sqrt{\frac{\eta}{\pi}}e^{-\frac{\zeta^2}{4\eta}}$$

$$rho_0=\frac{\partial ARO_{C_0}}{\partial r}=\frac{S}{r^2t_N}\left(r^2\tau\zeta+r\tau-1+e^{-r\tau}\right)\mathrm{N}\left(-\frac{\zeta}{\sqrt{2\eta}}\right)$$

$$+\frac{S\sigma^2}{8r^4t_N\sqrt{\pi\eta}}\left[9-4r\tau-\left(12+4r\tau\right)e^{-r\tau}+\left(3+2r\tau\right)e^{-2r\tau}\right]e^{-\frac{\zeta^2}{4\eta}}$$

where

$$\zeta=\frac{t_NK-I}{S}e^{-r\tau}-\frac{1}{r}\left(1-e^{-r\tau}\right),$$

$$\eta=\frac{\sigma^2}{4r^2}\left(-3+2r\tau+4e^{-r\tau}-e^{-2r\tau}\right),$$

$$\tau=t_N-t.$$

Proof:

Suppose the process $\{S_t\}$ is expressed in the usual stochastic manner as used previously. For $t_0\le t\le t_N$ the continuously sampled arithmetic mean is defined as

$$A_t = \frac{1}{t_N - t_0} \int_{t_0}^{t} S_\tau d\tau \qquad (4.2.18)$$

where

t_N is the maturity date and

$[t_0, t_N]$ is the final time over which the average value of the stock is calculated.

Note that A_t is an average only where $t = t_N$. For $t_0 \le t < t_N$, A_t is defined as the part of the final average up to time t, and is a monotonically increasing function of t. When $t_0 \le t \le t_N$, the price of the option ARO_C will depend on (S_t, A_t, t). Where $t < t_N$ the value of A_t will not be relevant. In order to determine the value of the option at $t = 0$, the value of the option in the interval $[t_0, t_N]$ is first calculated and a value found for t_0 used to calculate the value in the interval $[0, t_0]$. Since in the time interval $[0, t_0]$ the value of the option is determined only by *t* and S_t, the standard partial differential equation for the price can be derived using Black and Scholes' (1973) hedging arguments and Merton's (1973) extension:

$$\frac{\partial ARO_C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 ARO_C}{\partial S^2} + rS\frac{\partial ARO_C}{\partial S} - rARO_C = 0.$$

The boundary conditions for a standard call option which expires at t_0 can be expressed as

$$\begin{aligned} ARO_C\left(S_{t_0}, t_0\right) &= \max\left(S_{t_0} - K, 0\right), \\ ARO_C\left(0, t\right) &= 0, \\ ARO_C\left(\infty, t\right) &= 1. \end{aligned}$$

For an ARO, however, the boundary condition at time t_0 implies that the value of the option is equal to $ARO_C\left(S_{t_0}, t_0\right)$. Recall that before this value can be calculated, the ARO_C has to be valued over the time interval $[t_0, t_N]$, and in that case ARO_C depends on (S_t, A_t, t). A partial differential equation is needed for $ARO_C(S_t, A_t, t)$, $t_0 \le t < t_N$. Note that (4.2.18) yields the equation

$$dA_t = \beta S_t dt, \tag{4.2.19}$$

where

$$\beta = \frac{1}{t_N - t_0}.$$

If $ARO_C(S_t, A_t, t)$ is the value of the option at time $t \in [t_0, t_N]$, it is possible to apply Ito's lemma as follows:

$$\partial ARO_{C_t} = \left(\frac{\partial ARO_{C_t}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 ARO_{C_t}}{\partial S^2} + rS\frac{\partial ARO_{C_t}}{\partial S} + \beta S \frac{\partial ARO_{C_t}}{\partial A} \right) dt + \sigma S \frac{\partial ARO_{C_t}}{\partial S} dz.$$

Thus a continuously-adjusted portfolio consisting of $\frac{\partial ARO_{C_t}}{\partial S}$ stocks which is partially financed by a loan $\left(\frac{\partial ARO_{C_t}}{\partial S} S - ARO_{C_t} \right)$, bears an identical risk to the ARO which is given by $\sigma S \frac{\partial ARO_{C_t}}{\partial S} dz$. The portfolio also has costs identical to ARO_{C_t} in payments. Arbitrage arguments imply that the expected instantaneous return on the portfolio and the option must be identical, since the risks and costs are identical. Therefore the following partial differential equation can be derived for the option price

$$\frac{\partial ARO_{C_t}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 ARO_{C_t}}{\partial S^2} + +\beta S \frac{\partial ARO_{C_t}}{\partial A} + rS\frac{\partial ARO_{C_t}}{\partial S} - rARO_{C_t} = 0,$$

which holds in the domain $D = \{(S, A, t) | S \ge 0, A \ge 0, t_0 \le t \le t_N\}$.

Hence within the Black-Scholes (1973) and Merton (1973) framework, the price of an arithmetic average call option $ARO_C(S, I, t)$ satisfies the following partial differential equation, first derived by Kemna and Vorst (1990):

$$\frac{\partial ARO_{C_0}}{\partial t}+S\frac{\partial ARO_{C_0}}{\partial I}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 ARO_{C_0}}{\partial S^2}+rS\frac{\partial ARO_{C_0}}{\partial S}-rARO_{C_0}=0,$$

and final condition

$$ARO_C\left(S,I,t\right)=\max\left(\frac{I}{t_N}-K,0\right).$$

Now define

$$\phi\left(t,x\right)=E\left[\max\left(\int_t^{t_N} S_u\mu\left(du\right)-x,0\right)\middle| S_t=1\right],$$

where the process of S is given in (4.2.1). The random variable $\max\left(\int_t^{t_N} S_u\mu\left(du\right)-x,0\right)$, whose conditional expectation is being computed, does not depend on *t*. Because of this, the tower property implies that $\phi\left(t,x\right),0\le t\le t_N$ is a martingale: For $0\le t\le t_N$,

$$\begin{aligned}
M_t &= E\left[\max\left(\int_0^{t_N} S_u\mu\left(du\right)-K,0\right)\middle|\Im_t\right]\\
&= E\left[\max\left(\int_t^{t_N} S_u\mu\left(du\right)-\left(K-\int_0^t S_u\mu\left(du\right)\right),0\right)\middle|\Im_t\right]\\
&= S_t E\left[\max\left(\int_t^{t_N}\frac{S_u}{S_t}\mu\left(du\right)-\frac{K-\int_0^t S_u\mu\left(du\right)}{S_t},0\right)\middle|\Im_t\right]\\
&= S_t\phi\left(t,\xi_t\right),
\end{aligned}\tag{4.2.20}$$

where for "fixed-strike" Asian options

$$\xi_t=\frac{K-\int_0^t S_u\mu\left(du\right)}{S_t},$$

$$\mu(du) = (t_N)^{-1} I_{[0,t_N]}(u)\, du$$

and it is assumed that the probability measure μ has a density ρ_t in $(0, t_N)$. From its definition, ϕ is jointly continuous and decreasing in *t* and *x*. Now Ito's formula is applied to get the process for ξ_t. Since

$$S_t = S_0 \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t + rt\right),$$

the process for S_t is given by

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dB_t \\ &= S_t\left(rdt + \sigma dB_t\right); \end{aligned}$$

hence

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t.$$

Therefore

$$\xi_t = f\left(S_t, t\right) = \frac{K - c\int_0^t S_u du}{S_t}, \; c = \frac{1}{t_N}$$

$$\begin{aligned} d\xi_t &= \frac{df\left(S_t,t\right)}{dt}dt + \frac{df\left(S_t,t\right)}{dS_t}dS_t + \frac{1}{2}\frac{d^2 f\left(S_t,t\right)}{dt^2}\sigma^2 S_t^2 dt \\ &= \frac{-cS_t}{S_t}dt - \frac{K - c\int_0^t S_u du}{S_t^2}dS_t + \frac{1}{2}2\frac{K - c\int_0^t S_u du}{S_t^3}\sigma^2 S_t^2 dt \\ &= -cdt - \xi_t \frac{dS_t}{S_t} + \sigma^2 \xi_t dt \\ &= -cdt - \xi_t\left(rdt + \sigma dB_t\right) + \sigma^2 \xi_t dt \\ &= -cdt + \xi_t\left(-\sigma dB_t - rdt + \sigma^2 dt\right). \end{aligned}$$

Similarly,

$$
\begin{aligned}
d\phi(\xi_t,t) &= \frac{d\phi(\xi_t,t)}{dt}dt + \frac{d\phi(\xi_t,t)}{d\xi_t}d\xi_t + \frac{1}{2}\frac{d^2\phi(\xi_t,t)}{dt^2}\sigma^2\xi_t^2 dt \\
&= \dot{\phi}dt + \phi'\left[-cdt + \xi_t\left(-\sigma dB_t - rdt + \sigma^2 dt\right)\right] + \frac{1}{2}\phi''\sigma^2\xi_t^2 dt \\
&= \left[\dot{\phi} + \phi'\left(-c + r\xi_t + \sigma^2\xi_t\right) + \frac{1}{2}\phi''\sigma^2\xi_t^2\right]dt - \phi'\xi_t\sigma dB_t,
\end{aligned}
$$

with $\dot{\phi} = \dfrac{d\phi(\xi_t,t)}{dt}$, $\phi' = \dfrac{d\phi(\xi_t,t)}{d\xi_t}$ and $\phi'' = \dfrac{d^2\phi(\xi_t,t)}{dt^2}$.

Assuming that ϕ has enough smoothness to apply Itô's formula to (4.2.20), it gives

$$
\begin{aligned}
dM(S,\phi) &= \phi dS + Sd\phi + dSd\phi \\
&= \phi dS + S\left(\dot{\phi}dt + \phi' d\xi + \frac{1}{2}\phi''\sigma^2\xi^2 dt\right) + dSd\phi \\
&\doteq r\phi Sdt + S\left(\dot{\phi} + \phi'\left(-\rho_t - r\xi + \sigma^2\xi\right) + \frac{1}{2}\sigma^2\xi^2\phi''\right)dt - \sigma S\phi'\sigma\xi dt \\
&= S\left[r\phi + \dot{\phi} - \left(\rho_t + r\xi\right)\phi' + \frac{1}{2}\sigma^2\xi^2\phi''\right]dt,
\end{aligned}
$$

where $\doteq$ signifies that the two sides differ by a local martingale. A local martingale is defined as:

Definition 1: Local Martingale. (Etheridge,2004)

A process $\{X_t\}_{t\geq 0}$ *is a local* $\left(\mathrm{P},\{\Im_t\}_{t\geq 0}\right)-$*martingale if there is a sequence of* $\{\Im_t\}_{t\geq 0}$-*stopping times* $\{T_n\}_{n\geq 1}$ *such that* $\{X_{t\wedge T_n}\}_{t\geq 0}$ *is a* $\left(\mathrm{P},\{\Im_t\}_{t\geq 0}\right)-$*martingale for each n and*

$$
P\left[\lim_{n\to\infty} T_n = \infty\right] = 1.
$$

All martingales are local martingales but the converse is not true.

Because M is a martingale, the sum of the *dt* terms in *dM* must be 0. This implies that

$$0 = \dot{\phi} + r\phi + \frac{1}{2}\sigma^2\xi^2\phi'' - \left(\rho_t + r\xi\right)\phi'. \tag{4.2.21}$$

Set $f\left(t,x\right) = e^{-r\left(t_N - t\right)}\phi\left(t,x\right)$, then it is found by Itô that *f* solves

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} - \left(\rho_t + rx\right)\frac{\partial f}{\partial x} = 0.$$

The boundary condition in the case of the fixed strike Asian option follows from the Feyman-Kac theorem given in lemma 3 as

$$f\left(t_N, x\right) = \max\left(-x, 0\right). \tag{4.2.22}$$

Denote the solutions to the PDE (4.2.21), with the fixed strike boundary condition (4.2.22), by ϕ. Then in the case where μ is uniform on $\left[0, t_N\right]$, the price of the Asian option with maturity t_N, fixed strike *K*, and initial price S_0 is

$$\begin{aligned} e^{-rt_N} E \max\left(\int_0^{t_N} \left(S_u - K\right)\frac{du}{t_N}, 0\right) &= S_0 f\left(0, \frac{K}{S_0}\right) \\ &= e^{-rt_N} S_0 \phi\left(0, \frac{K}{S_0}\right). \end{aligned}$$

Notice that with these parameters, for $x \le 0$

$$\phi\left(t,x\right) = \frac{1}{r}\left(e^{r\tau} - 1\right) - x.$$

Applying a similar transformation to the equations above, to that adopted by Roger and Shi (1995) illustrated above,

$$\begin{aligned} &\zeta = \frac{TK - I}{S} e^{-r\tau} - \frac{1}{r}\left(1 - e^{-r\tau}\right), \\ &\tau = T - t, \end{aligned}$$

which implies that

$$\Rightarrow TK - I = \zeta S e^{r\tau} + S\frac{1}{\tau}\left(e^{r\tau} - 1\right)$$

$$\Rightarrow \frac{I}{T} - K = -\frac{S}{T}\zeta e^{r\tau} - \frac{S}{T}\frac{1}{\tau}\left(e^{r\tau} - 1\right)$$

$$= \frac{S}{T}\left[-\zeta e^{r\tau} - \frac{1}{\tau}\left(e^{r\tau} - 1\right)\right]$$

$$= \frac{S}{T} f^*\left(\zeta, \tau\right)$$

$$\Rightarrow ARO_C\left(S, I, t\right) = \max\left\{\frac{I}{T} - K, 0\right\}$$

$$= \max\left\{\frac{S}{T} f^*\left(\zeta, \tau\right), 0\right\}$$

$$= \frac{S}{T}\max\left\{f^*\left(\zeta, \tau\right), 0\right\}$$

$$= \frac{S}{T} f\left(\zeta, \tau\right). \qquad (4.2.23)$$

To find $f\left(\zeta, \tau\right)$ proceed as follows. Equation (4.2.23) is a linear diffusion equation with variable coefficient and an initial condition

$$\frac{\partial f}{\partial \tau} - \frac{1}{2}\sigma^2\left[\zeta + \frac{1}{r}\left(1 - e^{-r\tau}\right)\right]^2 \frac{\partial^2 f}{\partial \zeta^2} = 0, \qquad -\infty < \zeta < \infty, \qquad (4.2.24)$$

$$f\left(\zeta, 0\right) = \max\left(-\zeta, 0\right). \qquad (4.2.25)$$

Initially

$$\frac{\partial^2 f\left(\zeta, 0\right)}{\partial \zeta^2} = \delta\left(\zeta\right),$$

therefore the effect only exists at $\zeta = 0$ initially, and will be significantly near the region of small ζ. Therefore the ζ is dropped from the (4.2.6). Next solve $f_0\left(\zeta, \tau\right)$, which is an analytical approximation of $f\left(\zeta, \tau\right)$, from the following equations

$$\frac{\partial f_0}{\partial \tau} - \frac{\sigma^2}{2r^2}\left(1-e^{-r\tau}\right)^2 \frac{\partial^2 f_0}{\partial \zeta^2} = 0, \qquad -\infty < \zeta < \infty, \tag{4.2.26}$$
$$f_0\left(\zeta, \tau = 0\right) = \max\left(-\zeta, 0\right).$$

Introducing a new time variable

$$d\eta = \frac{\sigma^2}{2r^2}\left(1-e^{-r\tau}\right)^2 d\tau$$
$$\eta = \int_0^\tau \frac{\sigma^2}{2r^2}\left(1-e^{-r\tau}\right)^2 ds = \frac{\sigma^2}{4r^3}\left(-3+2r\tau+4e^{-r\tau}-e^{-2r\tau}\right),$$

(4.2.26) becomes a standard one dimensional heat equation:

$$\frac{\partial f_0}{\partial \eta} - \frac{\partial^2 f_0}{\partial \zeta^2} = 0, \qquad -\infty < \zeta < \infty,$$
$$f_0\left(\zeta, \eta = 0\right) = \max\left(-\zeta, 0\right).$$

The solution can be obtained by Green's function solution in one dimension given by

$$\begin{cases} u_t = ku_{xx} & -\infty < x < \infty, 0 < t < \infty \\ u\left(x,0\right) = g\left(x\right) \end{cases}$$
$$u\left(x,t\right) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left(-\frac{\left(x-y\right)^2}{4kt}\right) g\left(y\right) dy,$$

where $k = 1, x = \zeta, t = \eta, g\left(x\right) = \max\left(-\zeta, 0\right)$ and $u\left(x,t\right) = f_0\left(\zeta, \eta\right)$, as

$$f_0\left(\zeta,\eta\right) = -\int_{-\infty}^{0} \zeta_0 \frac{1}{\sqrt{4\pi\eta}} e^{\frac{-\left(\zeta_0-\zeta\right)^2}{4\eta}} d\zeta_0 = -\zeta \mathrm{N}\left(-\frac{\zeta}{\sqrt{2\eta}}\right) + \sqrt{\frac{\eta}{\pi}} e^{-\frac{\zeta^2}{4\eta}}. \tag{4.2.27}$$

Substituting (4.2.27) in (4.2.23), for the relevant t, leads to (4.2.17), which proves the theorem. The Greeks are calculated using the following partial derivatives (Zhang, 1999):

$$\frac{\partial f_0}{\partial \zeta} = -N\left(-\frac{\zeta}{\sqrt{2\eta}}\right)$$

$$\frac{\partial^2 f_0}{\partial \zeta^2} = \frac{1}{2\sqrt{\pi\eta}} e^{-\frac{\zeta^2}{\sqrt{2\eta}}} = \frac{\partial f_0}{\partial \zeta}$$

$$\frac{\partial \zeta}{\partial S} = -\frac{1}{S}\left[\zeta + \frac{1}{r}\left(1 - e^{-r\tau}\right)\right]$$

$$\frac{\partial \zeta}{\partial t} = 1 + r\zeta$$

$$\frac{\partial \eta}{\partial t} = -\frac{\sigma^2}{2r^2}\left[1 - 2e^{-r\tau} + e^{-2r\tau}\right]$$

$$\frac{\partial \eta}{\partial \sigma} = \frac{\eta}{2\sigma}$$

$$\frac{\partial \zeta}{\partial r} = -\frac{1}{r^2}\left(r^2\tau\zeta + r\tau - 1 + e^{-r\tau}\right)$$

$$\frac{\partial \eta}{\partial r} = \frac{\sigma^2}{4r^4}\left[9 - 4r\tau - \left(12 + 4r\tau\right)e^{-r\tau} + \left(3 + 2r\tau\right)e^{-2r\tau}\right]$$

This proves the analytic approximation of the option price and its corresponding Greeks and thus completes the proof of the theorem.

■

The analytical approximation formulas in Theorem 6 are corrected using the terms in Theorem 7.

Theorem 7. (Zhang, 1999)

The correction terms of the analytical approximation formulas in Theorem 6 are given by the following:

$$ARO_{C_1}\left(S,I,t\right)=\frac{S}{T}f_1\left(\zeta,\tau;r,\sigma\right) \qquad (4.2.28)$$

$$\Delta_1=\frac{\partial ARO_{C_1}}{\partial S}=\frac{1}{T}f_1-\frac{1}{T}\left[\zeta+\frac{1}{r}\left(1-e^{-r\tau}\right)\right]\frac{\partial f_1}{\partial\zeta}$$

$$\theta_1=\frac{\partial ARO_{C_1}}{\partial t}=-\frac{S}{T}\left(1+r\zeta\right)\frac{\partial f_1}{\partial\zeta}-\frac{S}{T}\frac{\partial f_1}{\partial\zeta}$$

$$\Gamma_1=\frac{\partial\Delta_1}{\partial S}=\frac{1}{ST\sqrt{\pi\eta}}\left[\zeta+\frac{1}{r}\left(1-e^{-r\tau}\right)\right]^2\frac{\partial^2 f_1}{\partial\zeta^2}$$

$$Vega_1=\frac{\partial ARO_{C_1}}{\partial\sigma}=\frac{S}{T}\frac{\partial f_1}{\partial\sigma}$$

$$rho_1=\frac{\partial ARO_{C_1}}{\partial r}=-\frac{S}{r^2T}\left(r^2\tau\zeta+r\tau-1+e^{-r\tau}\right)\frac{\partial f_1}{\partial\zeta}+\frac{S}{T}\frac{\partial f_1}{\partial r}$$

where the function $f_1\left(\zeta,\tau;r,\sigma\right)$ *and its derivatives* $\frac{\partial f_1}{\partial\zeta},\frac{\partial f_1}{\partial\tau},\frac{\partial f_1}{\partial \mathrm{r}},\frac{\partial f_1}{\partial\sigma}$ and $\frac{\partial^2 f_1}{\partial\zeta^2}$ *can be evaluated by solving the following partial differential equation numerically with the finite difference method*

$$\frac{\partial f_1}{\partial\tau}-\frac{1}{2}\sigma^2\left[\zeta+\frac{1}{r}\left(1-e^{-r\tau}\right)\right]^2\frac{\partial^2 f_1}{\partial\zeta^2}=\frac{\sigma^2\zeta}{4\sqrt{\pi\eta}}e^{-\frac{\zeta^2}{4\eta}}\left[\zeta+\frac{2}{r}\left(1-e^{-r\tau}\right)\right],$$
$$f_1\left(\zeta,\tau=0\right)=0.$$

Proof:

The exact value of $f\left(\zeta,\tau\right)$ is equal to the analytic approximation $f_0\left(\zeta,\tau\right)$ plus the correction term $f_1\left(\zeta,\tau\right)$, i.e,

$$f\left(\zeta,\tau\right)=f_0\left(\zeta,\tau\right)+f_1\left(\zeta,\tau\right), \qquad (4.2.29)$$

with $f_0\left(\zeta,\tau\right)$ satisfying (4.2.26), i.e.

$$\frac{\partial f_0}{\partial\tau}-\frac{\sigma^2}{2r^2}\left(1-e^{-r\tau}\right)^2\frac{\partial^2 f_0}{\partial\zeta^2}=0, \qquad -\infty<\zeta<\infty,$$
$$f_0\left(\zeta,\tau=0\right)=\max\left(-\zeta,0\right),$$

is given by (4.2.27) as

$$f_0(\zeta,\eta) = -\int_{-\infty}^{0} \zeta_0 \frac{1}{\sqrt{4\pi\eta}} e^{\frac{-(\zeta_0-\zeta)^2}{4\eta}} d\zeta_0 = -\zeta N\left(-\frac{\zeta}{\sqrt{2\eta}}\right) + \sqrt{\frac{\eta}{\pi}} e^{-\frac{\zeta^2}{4\eta}}.$$

Substitute (4.2.29) into equations (4.2.24) and (4.2.25) then

$$\frac{\partial f}{\partial \tau} - \frac{1}{2}\sigma^2\left[\zeta + \frac{1}{r}\left(1-e^{-r\tau}\right)\right]^2 \frac{\partial^2 f}{\partial \zeta^2} = 0, \qquad -\infty < \zeta < \infty,$$
$$f(\zeta,0) = \max(-\zeta,0).$$

gives

$$\frac{\partial f_1}{\partial \tau} - \frac{1}{2}\sigma^2\left[\zeta + \frac{1}{r}\left(1-e^{-r\tau}\right)\right]^2 \frac{\partial^2 f_1}{\partial \zeta^2} = \frac{1}{2}\sigma^2\zeta\left[\zeta + \frac{2}{r}\left(1-e^{-r\tau}\right)\right]\frac{\partial^2 f_0}{\partial \zeta^2}$$
$$= \frac{\sigma^2\zeta}{4\sqrt{\pi\eta}} e^{-\frac{\zeta^2}{4\eta}}\left[\zeta + \frac{2}{r}\left(1-e^{-r\tau}\right)\right]$$
$$f_1(\zeta,\tau=0) = 0.$$

■

It is very straightforward to implement the present method. The analytical approximation of the average rate call is easy to compute using Theorem 6, since it is a closed-form formula in terms of the cumulative normal distribution function. In order to get the true value of the option, the correction term must also be computed, i.e. it is necessary to solve $f_1(\zeta,\tau;r,\sigma)$ from (4.2.28).

Equation (4.2.28) is an inhomogeneous linear diffusion equation with a variable coefficient. The numerical calculations are done using the Crank-Nicolson scheme. The scheme is popular for solving parabolic partial differential equations.

4.2.4 Arbitrage Bounds on Valuation

An arithmetic Asian Option is Always Worth Less Than a Vanilla Option

For time $t_0 \le t \le t_N$ the continuously sampled arithmetic mean is defined as in (4.2.17). This expression is approximated by the discrete expression

$$A_t = \frac{1}{n+1}\sum_{i=0}^{n} S_{t_i},$$

where $t_i = t_0 + i\left(t - t_0\right)/n$ for large n. Using this approximation the numerical approximation for an arithmetic Asian option at time t_0 follows as:

$$ARO_C\left(S_{t_0}, 0, t_0\right) = e^{-r\left(t_N - t_0\right)} E^* \left\{ \max\left[\sum_{i=0}^{n} \frac{S_{t_i}}{n+1} - K, 0 \right] \right\}. \tag{4.2.30}$$

This formula enables the comparison of the value of an Asian option with that of a standard European option which can be expressed in similar terms:

$$V_C\left(S_{t_0}, t_0\right) = e^{-r\left(t_N - t_0\right)} E^* \left\{ \max\left[\sum_{i=0}^{n} \frac{S_t}{n+1} - K, 0 \right] \right\}, \tag{4.2.31}$$

since $\sum_{i=0}^{n} \frac{S_t}{n+1} = S_t$.

Equations (4.2.30) and (4.2.31) are compared using the following lemmas given in Kemna and Vorst (1991):

Lemma 4.

If U is a random variable with $E\left(U\right) \geq 1$ *then for every* $m \in \mathrm{N}$ *and* $K > 0$

$$\boldsymbol{E}\max\left(\frac{1}{m} + \frac{m-1}{m}U - K, 0\right) \leq \boldsymbol{E}\max\left(U - K, 0\right). \tag{4.2.32}$$

Proof:

Let $p\left(U\right)$ be the density function of $U \geq 0$. It is clear that

$$\frac{1}{m} + \frac{m-1}{m}U - K \geq 0 \;\; iff \;\; U \geq K_0 = \frac{mK - 1}{m-1}.$$

Two cases are distinguished, namely $K_0 \geq 1$ and $K_0 < 1$.

If $K_0 \geq 1$ then

$$\boldsymbol{E}\max\left(U-K,0\right)=\int_K^\infty \left(U-K\right)p\left(U\right)dU$$

$$\geq \int_{K_0}^\infty \left(U-K\right)p\left(U\right)dU$$

$$\geq \int_{K_0}^\infty \left(\frac{1}{m}+\frac{m-1}{m}U-K\right)p\left(U\right)dU$$

$$= \boldsymbol{E}\max\left(\frac{1}{m}+\frac{m-1}{m}U-K,0\right).$$

If $K_0 < 1$ then

$$\boldsymbol{E}\max\left(U-K,0\right)=\int_K^\infty \left(U-K\right)p\left(U\right)dU$$

$$\geq \int_{K_0}^\infty \left(U-K\right)p\left(U\right)dU$$

$$= \boldsymbol{E}\left(U-K\right)-\int_0^{K_0} \left(U-K\right)p\left(U\right)dU$$

$$\geq \boldsymbol{E}\left(\frac{1}{m}+\frac{m-1}{m}U-K\right)-\int_0^{K_0}\left(\frac{1}{m}+\frac{m-1}{m}U-K\right)p\left(U\right)dU$$

$$= \boldsymbol{E}\max\left(\frac{1}{m}+\frac{m-1}{m}U-K,0\right),$$

where the last inequality follows because $U \leq K_0 \leq 1$. It is clear that if $r > 0$ or $\sigma > 0$, at least one of the above inequalities is a strict inequality, which in fact establishes the second part of Theorem 8 below.

■

Theorem 8.

If $r \geq 0$, then

$$E^*\left\{\max\left[\sum_{i=0}^{n}\frac{S_{t_i}}{n+1}-K,0\right]\right\} \leq E^*\left\{\max\left[\sum_{i=0}^{n}\frac{S_t}{n+1}-K,0\right]\right\} \qquad (4.2.33)$$

and strict inequality holds if $r > 0$ or $\sigma > 0$.

Proof:

Define $R_i = \frac{S_{t_i}}{S_{t_{i-1}}}$ and $R_0 = S_{t_0}$ such that $S_{t_i} = R_0 R_1, ..., R_i$. From (4.2.1) it follows that each R_i is lognormally distributed with:

$$\boldsymbol{E}\left(R_0 R_1, ..., R_i\right) = \exp\left\{\frac{r\left(t_N - t_0\right)\left(j - i + 1\right)}{n}\right\} \geq 1. \qquad (4.2.34)$$

Hence, it is necessary to prove

$$\boldsymbol{E} \max\left\{\frac{R_0 + R_0 R_1 + ... + R_0 R_1 ... R_n}{n+1} - K, 0\right\} \leq \boldsymbol{E} \max\left\{R_0 R_1 ... R_n - K, 0\right\} \qquad (4.2.35)$$

using lemma 4 given above. First it is shown how (4.2.35) follows from this lemma. It is enough to show that

$$\boldsymbol{E} \max\left\{\frac{1}{n+1} + \frac{n}{n+1}\frac{R_1 + R_1 R_2 + ... + R_1 R_2 ... R_n}{n} - K', 0\right\} \leq \boldsymbol{E} \max\left\{R_1 R_2 ... R_n - K', 0\right\}$$

for each R_0 with $K' = \frac{K}{R_0}$. Since $E\left(\frac{R_1 + R_1 R_2 + ... + R_1 R_2 ... R_n}{n}\right) \geq 1$ by virtue of (4.2.34), lemma 4 can be applied and hence

$$\begin{aligned} &\boldsymbol{E} \max\left\{\frac{1}{n+1} + \frac{n}{n+1}\frac{R_1 + R_1 R_2 + ... + R_1 R_2 ... R_n}{n} - K', 0\right\} \\ &\leq \boldsymbol{E} \max\left\{\frac{R_1 + R_1 R_2 + ... + R_1 R_2 ... R_n}{n} - K', 0\right\} \\ &\leq \boldsymbol{E} \max\left\{R_1 R_2 ... R_n - K', 0\right\}, \end{aligned}$$

where the last inequality follows from induction on the number of variables.

■

Geometric Asian Options are Worth Less Than or Equal to Arithmetic Asian Options

Geometric Asian options are worth less than or equal to arithmetic Asian options, since for a set of n positive, real numbers $x_1, x_2, ..., x_n$, the following inequality holds

$$\frac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 ... x_n},$$

and that if and only if $x_1 = x_2 = ... = x_n$,

$$\frac{x_1 + x_2 + ... + x_n}{n} = \sqrt[n]{x_1 x_2 ... x_n},$$

which is a well-known result, see for example the proof on www.wikipedia.com.

Arbitrage Conditions for Arithmetic AROs and ASOs

The put-call parity for arithmetic Asian options are derived following the derivation of Clewlow and Strickland (1997). Consider the replication at time t of the payoff of an arithmetic average contract that pays A_{t_N} at time t_N. Suppose that at time t the running average is A_t and $m > 0$ fixings are known. In order to replicate at time t_N a payment of S_{t_i} (for any $i > m$), the following strategy is followed:

- At time t purchase $e^{-q(t_i - t)} e^{-r(t_N - t_i)}$ units of the asset, so that at time t_i the holdings grow to $e^{-r(t_N - t_i)}$ units which can be sold in the market for $S_{t_i} e^{-r(t_N - t_i)}$ units of domestic currency.
- Investing for $(t_N - t_i)$ time units yields S_{t_i} at time t_N.

The cost of this strategy is $S_t e^{-q(t_i - t)} e^{-r(t_N - t_i)}$; hence replicating the payoff A_{t_N} at time t_N would cost at time t:

$$V_{R_t} = \frac{m}{N} A_t e^{-r(t_N - t)} + \frac{1}{N} S_t \sum_{i=m+1}^{N} e^{-q(t_i - t)} e^{-r(t_N - t_i)}.$$

As with conventional European options, a put-call parity condition exists for Asian options. A portfolio of long the call and short the put has the following value at time t_N:

$$ARO_C\left(K,t_N\right)-ARO_P\left(K,t_N\right)=A_{t_N}-K.$$

Hence it follows that, as K is valued at time t as $Ke^{-r(t_N-t)}$ and A_{t_N} can be replicated at cost V_{R_t}, the following put-call parity condition must hold:

$$ARO_P\left(K,t_N\right)=ARO_C\left(K,t_N\right)-V_{R_t}+Ke^{-r(t_N-t)}.$$

Once an ARO call option valuation method is adopted, the above condition is used to value the corresponding ARO put option.

Similarly to deriving the put-call parity condition for arithmetic AROs, it can also be derived for ASOs. A portfolio of long the call and short the put has value at time T:

$$ASO_C\left(T\right)-ASO_P\left(T\right)=S_T-A_{t_N}.$$

The cost at time t of replicating S_T is $S_te^{-q(T-t)}$. Replicating the payoff A_{t_N} at time T would cost at time t:

$$V_{S_t}=\frac{m}{N}A_te^{-r(T-t)}+\frac{1}{N}S_t\sum_{i=m+1}^{N}e^{-q(t_i-t)}e^{-r(T-t_i)},$$

and hence

$$ASO_P\left(t\right)=ASO_C\left(t\right)+V_{S_t}-S_te^{-q(T-t)}.$$

Symmetry Results for Arithmetic Asians Options

Pricing of fixed-strike Asian options has been the subject of much research over the last fifthteen years. The floating-strike Asian option has received far less attention in the literature. It is this fact that means a relationship between the prices of fixed and floating Asian options would be extremely useful. With such a connection, a floating-strike option could be priced using well-known methods for the fixed-strike option.

Define 'forward starting' Asian options, as Asian options where at the current time 0, the averaging has not yet started and where the n variables $S_{T-n+1},...,S_T$ are random. This case states, in contrast with the case that $T-n+1 \le 0$ where only $S_1,...,S_T$ remain random. This Asian option is called "in progress". Henderson and Wojakowski (2002) use the change of numeraire technique to obtain symmetry results between forward starting European-style Asian options with floating and fixed strike in case of continuous averaging. Vanmaele *et al.* (2005) show that those results can be extended to discrete averaging. The symmetry results become very useful for transferring knowledge about one type of option to another. However, there does not exist such a symmetry relation for the option "in progress". Hence the procedures of Henderson and Wojakowski (2002) are given below.

<u>Arithmetic Asian Options with Continuous Averaging</u>

Define the continuous arithmetic average as

$$A_t = \frac{1}{t_N - t_0} \int_{t_0}^{t} S_u du \, .$$

For the fixed strike Asian call option generalised notation is introduced:

$$AC\left(x_1, x_2, x_3, x_4, x_5, x_6\right),$$

where x_1 is the strike price, x_2 the initial value of the process $\left(S_t\right)_{t\ge 0}$, x_3 the risk free interest rate, x_4 the dividend yield, x_5 the starting date of averaging, x_6 the option maturity. Analogous, for a put option set

$$AP\left(x_1, x_2, x_3, x_4, x_5, x_6\right).$$

For floating strike option, introduce a similar generalized notation:

$$ACF\left(y_1, y_2, y_3, y_4, y_5, y_6\right),$$

where y_1 is the initial value of the process $\left(S_t\right)_{t\ge 0}$, y_2 the percentage, y_3 the risk free interest rate, y_4 the dividend yield, y_5 the starting date of averaging, y_6 the option maturity. Analogous, for a floating strike put option set

$$APF\left(y_1, y_2, y_3, y_4, y_5, y_6\right).$$

The percentage y_2 refers to the proportion of S_T that will be received in the floating strike call option or bought in the floating strike put option. The percentage $y_2 = 100$ is the important case in financial option pricing. The floating-strike call is typically interpreted as a call written on *S*, with floating strike A_T. Exercising, the holder receives or buys y_2 units of stock and pays the average of the past prices, A_T.

For the following it is assumed that the averaging period starts at time 0, when the option contract is written, i.e. $t_0 = 0$. Hence it is assumed that the option is vanilla. The results hold, however, for the forward starting case, for prices computed up to and including time t_0.

Theorem 9.

If S follows the exponential Brownian motion process:

$$\frac{dS_t}{S_t} = \left(r - q\right)dt + \sigma dW_t,$$

the following symmetry results hold:

$$ACF\left(S_0, \lambda, r, q, 0, t_N\right) = AP\left(\lambda S_0, S_0, q, r, 0, t_N\right) \qquad (4.2.36)$$

$$AC\left(K, S_0, r, q, 0, t_N\right) = APF\left(S_0, \frac{K}{S_0}, q, r, 0, t_N\right). \qquad (4.2.37)$$

Proof:

Equation (4.2.36) is proved first. The floating-strike Asian call price expressed in units of stock as numeraire is

$$ACF^* = \frac{ACP}{S_0} = \frac{e^{-rt_N}}{S_0} E\left[\max\left(\lambda S_{t_N} - A_{t_N}, 0\right)\right] = E\left[\frac{S_{t_N} e^{-rt_N}}{S_0} - \frac{\max\left(\lambda S_{t_N} - A_{t_N}, 0\right)}{S_{t_N}}\right].$$

By changing the numeraire to S via

$$\frac{S_{t_N} e^{-rt_N}}{S_0 e^{-qt_N}} = e^{-\frac{\sigma^2}{2}t_N + \sigma W_{t_N}} = \frac{dQ^*}{dQ},$$

the measure Q^* is defined. Under Q^*, $W_t^* = W_t - \sigma t$ is a Brownian motion, using the Girsanov theorem. Moreover,

$$\frac{\max\left(\lambda S_{t_N} - A_{t_N}, 0\right)}{S_{t_N}} = \max\left(\lambda - A^*_{t_N}, 0\right)$$

is the terminal payoff in units of stock as numeraire, where $A^*_{t_N} = \frac{A_{t_N}}{S_{t_N}}$. Hence,

$$ACF^* = e^{-qt_N} E^*\left[\max\left(\lambda - A^*_{t_N}, 0\right)\right].$$

It can be seen that the roles of the underlying and exercise price have switched and the new exercise price is λ units of stock. This is a put written on a new asset A^*. To continue, rewrite $A^*_{t_N}$ as

$$A^*_{t_N} = \frac{A_{t_N}}{S_{t_N}} = \frac{1}{t_N}\int_0^{t_N} \frac{S_u}{S_{t_N}} du = \frac{1}{t_N}\int_0^{t_N} S_u^*\left(t_N\right) du .$$

For $u \leq t_N$, a $\Im_t$- measurable random variable is defined as

$$\begin{aligned} S_u^*\left(t_N\right) = \frac{S_u}{S_{t_N}} &= \exp\left\{-\left(r - q - \frac{1}{2}\sigma^2\right)\left(t_N - u\right) - \sigma\left(W_{t_N} - W_u\right)\right\} \\ &= \exp\left\{\left(r - q + \frac{1}{2}\sigma^2\right)\left(u - t_N\right) + \sigma\left(W^*_u - W^*_{t_N}\right)\right\} \end{aligned}$$

using the Q^* Brownian motion W_t^*. Note that if $\forall t\ \hat{W}_t = -W_t^*$ is a reflected Q^*-Brownian motion starting at zero, then from the laws of Brownian motion

$$W_u^* - W_{t_N}^* = \hat{W}_{t_N - u}$$

and

$$A_{t_N}^* = \hat{A}_{t_N} = \frac{1}{t_N} \int_0^{t_N} e^{\sigma \hat{W}_{t_N - u} + \left(r - q - \frac{1}{2}\sigma^2\right)(u - t_N)} du .$$

Reversing time via the variable change $s = t_N - u$, gives

$$\hat{A}_{t_N} = \frac{1}{t_N} \int_0^{t_N} e^{\sigma \hat{W}_s - \left(r - q + \frac{1}{2}\sigma^2\right)s} ds .$$

Thus $S_u^*(t_N)$ are indeed log-normally distributed variables and $A_{t_N}^* = \hat{A}_{t_N}$ is a sum of such log-normally distributed variables. Thus

$$ACF^* = e^{-qt_N} E^*\left[\max\left(\lambda - A_{t_N}^*, 0\right)\right] = e^{-qt_N} E^*\left[\max\left(\lambda - \hat{A}_{t_N}, 0\right)\right],$$

which proves (4.2.36).

To prove (4.2.37), start with a fixed-strike call given by

$$AC(K, S_0, r, q, 0, T) = e^{-rt_N} E\left[\max\left(A_{t_N} - K, 0\right)\right],$$

then (4.2.37) follows form put-call parity results. For the floating strike, it is known that

$$APF(S_0, \lambda, r, q, 0, t_N) - ACF(S_0, \lambda, r, q, 0, t_N) = \frac{1}{(r-q)t_N}\left(e^{-qt_N} - e^{-rt_N}\right)S_0 - \lambda S_0.$$

The analogous result for fixed strike options is

$$AC(K, S_0, r, q, 0, t_N) - AP(K, S_0, r, q, 0, t_N) = \frac{1}{(r-q)t_N}\left(e^{-qt_N} - e^{-rt_N}\right)S_0 - e^{-rt_N}K.$$

Using these put-call parity results the left side of (4.2.37) gives

$$ACF\left(S_0,\lambda,r,q,0,t_N\right)=APF\left(S_0,\lambda,r,q,0,t_N\right)-\frac{1}{\left(r-q\right)t_N}\left(e^{-qt_N}-e^{-rt_N}\right)S_0-\lambda S_0,$$

while the right side of (4.2.37) gives

$$AP\left(\lambda S_0,S_0,q,r,0,t_N\right)=AC\left(\lambda S_0,S_0,q,r,0,t_N\right)-\frac{1}{\left(q-r\right)t_N}\left(e^{-rt_N}-e^{-qt_N}\right)S_0-e^{-qt_N}\lambda S_0.$$

Therefore,

$$\begin{aligned}&APF\left(S_0,\lambda,r,q,0,t_N\right)-\frac{1}{\left(r-q\right)t_N}\left(e^{-qt_N}-e^{-rt_N}\right)S_0-\lambda S_0\\&=AC\left(\lambda S_0,S_0,q,r,0,t_N\right)-\frac{1}{\left(q-r\right)t_N}\left(e^{-rt_N}-e^{-qt_N}\right)S_0-e^{-qt_N}\lambda S_0.\end{aligned}$$

Set $\lambda=\dfrac{K}{S_0}$ and swap the dividend yield and risk-free rate, then

$$\begin{aligned}&APF\left(S_0,\frac{\mathrm{K}}{S_0},q,r,0,t_N\right)-\frac{1}{\left(q-r\right)t_N}\left(e^{-rt_N}-e^{-qt_N}\right)S_0-K\\&=AC\left(K,S_0,r,q,0,t_N\right)-\frac{1}{\left(r-q\right)t_N}\left(e^{-qt_N}-e^{-rt_N}\right)S_0-e^{-rt_N}K,\end{aligned}$$

which gives

$$\begin{aligned}&APF\left(S_0,\frac{\mathrm{K}}{S_0},q,r,0,t_N\right)-K\\&=AC\left(K,S_0,r,q,0,t_N\right)-e^{-rt_N}K,\end{aligned}$$

which proves (4.2.37).

Arithmetic Asian Options with Discrete Averaging

For the fixed strike Asian call option generalized notation is introduced:

$$AC(x_1, x_2, x_3, x_4, x_5, x_6, x_7),$$

where x_1 is the strike price, x_2 the initial value of the process $(S_t)_{t\geq 0}$, x_3 the risk free interest rate, x_4 the dividend yield, x_5 the option maturity, x_6 the number of averaging terms and x_7 the starting date of averaging. Analogous, for a put option set

$$AP(x_1, x_2, x_3, x_4, x_5, x_6, x_7).$$

For floating strike option, introduce a similar generalized notation:

$$ACF(y_1, y_2, y_3, y_4, y_5, y_6, y_7),$$

where y_1 is the initial value of the process $(S_t)_{t\geq 0}$, y_2 the percentage, y_3 the risk free interest rate, y_4 the dividend yield, y_5 the option maturity, y_6 the number of averaging terms in the strike and y_7 the starting date of averaging. Analogous, for a floating strike put option set

$$APF(y_1, y_2, y_3, y_4, y_5, y_6, y_7).$$

Theorem 10.

$$AP(K, S_0, r, q, T, n, T-n+1) = ACF\left(S_0, \frac{K}{S_0}, q, r, T, n, 0\right) \qquad (4.2.38)$$

$$ACF(S_0, \beta, r, q, T, n, T-n+1) = AP(\beta S_0, S_0, q, r, T, n, 0) \qquad (4.2.39)$$

and

$$AC(K, S_0, r, q, T, n, T-n+1) = APF\left(S_0, \frac{K}{S_0}, q, r, T, n, 0\right) \qquad (4.2.40)$$

$$APF(S_0, \beta, r, q, T, n, T-n+1) = AC(\beta S_0, S_0, q, r, T, n, 0). \qquad (4.2.41)$$

Note that the interest rate and dividend yield have switched their roles when going from a floating to a fixed strike Asian option or vice versa.

Proof:

Only the first symmetry result given in (4.2.38) is proved here, since the others follow along similar lines and use put-call parity for Asian options.

$$
\begin{aligned}
&AP\left(K,S_0,r,q,T,n,T-n+1\right)\\
&=e^{-rT}E^{Q}\max\left[K-\frac{1}{n}\sum_{i=0}^{n}S_{T-i},0\right]\\
&=e^{-qT}E^{Q}\max\left[\frac{e^{-(r-q)T}S_T}{S_0}\left(\frac{KS_0}{S_T}-\frac{1}{n}\sum_{i=0}^{n-1}\frac{S_{T-i}S_0}{S_T}\right)-\frac{1}{n}\sum_{i=0}^{n}S_{T-i},0\right]\\
&=e^{-qT}E^{\tilde{Q}}\max\left[\left(\frac{KS_0}{S_T}-\frac{1}{n}\sum_{i=0}^{n-1}S_0\exp\left[-\left(r-q+\frac{\sigma^2}{2}\right)i+\sigma\left(\tilde{B}_{T-i}-\tilde{B}_T\right)\right]\right),0\right],
\end{aligned}
$$

where the probability $\tilde{Q}$ is equivalent to Q by the Radon-Nikodym derivative, where the dividend yield q is stressed:

$$\frac{d\tilde{Q}}{dQ}=\frac{S_T}{S_0e^{(r-q)T}}=\exp\left(-\frac{\sigma^2}{2}T+\sigma B_T\right).$$

Under the probability $\tilde{Q}$, $\tilde{B}_t=B_t-\sigma t$ is a Brownian motion, and therefore, the dynamics of the share under $\tilde{Q}$ are given by

$$\frac{dS_t}{S_t}=\left(\left(r-q\right)+\sigma^2\right)dt+\sigma d\tilde{B}_t.$$

Due to the independent increments, $\tilde{B}_{T-i}-\tilde{B}_T$ has the same distribution as $\tilde{B}_i$ and $-\tilde{B}_i$. Therefore, attention is focussed on the process $\left(S_t^*\right)_t$ defined by

$$S_i^*=S_0\exp\left[-\left(r-q+\sigma^2\right)i+\sigma\tilde{B}_i\right].$$

Indeed, then

$$
\begin{aligned}
AP\left(K,S_0,r,q,T,n,T-n+1\right)&=e^{-qT}E^{\tilde{Q}}\max\left[\left(\frac{KS_T^*}{S_0}-\frac{1}{n}\sum_{i=0}^{n-1}S_i^*\right),0\right]\\
&=e^{-qT}E^{Q}\max\left[\left(\frac{K\tilde{S}_T}{S_0}-\frac{1}{n}\sum_{i=0}^{n-1}\tilde{S}_i\right),0\right]
\end{aligned}
$$

with the process $\left(\tilde{S}_t\right)_t$ defined by

$$\tilde{S}_i = S_0 \exp\left[-\left(r-q+\sigma^2\right)i+\sigma B_i\right]$$

with $\left(B_t\right)_t$ a Brownian motion under Q. As a conclusion,

$$AP\left(K,S_0,r,q,T,n,T-n+1\right)=ACF\left(S_0,\frac{K}{S_0},q,r,T,n,0\right).$$

■

4.2.5 Remarks on Asian Options

Valuing Arithmetic AROs when One or More Fixing is Known

An ARO structure in which the averaging period is a relatively small proportion of the option maturity horizon should be valued close to the price of a conventional European option for the corresponding period. In fact, a European option can be viewed as the limit of an Asian option in which the averaging period is an infinitesimal time period prior to expiry. At another extreme, if just one fixing remains to be determined, the ARO has its terminal value determined by a single asset price. Therefore it can immediately be valued as 1/N times a European option on S_{t_N} with strike $K^*=\left[NK-\left(N-1\right)A_t\right]$.

Extending this idea, AROs of which the recordings have begun can be reformulated as proportional to a new ARO of which the recordings have yet to begin. To see this, consider valuing an arithmetic ARO call when $m>0$ recordings are known, and hence $t>t_m$. A new ARO can be valued by redefining the existing ARO payoff as

$$\max\left[A_{t_N}-K,0\right]=\alpha\max\left[M_t-K^*,0\right],$$

where

$$M_t=\frac{N}{\left(N-m\right)}A_{t_N}-\frac{m}{\left(N-m\right)}A_t=\frac{1}{\left(N-m\right)}\sum_{i=m+1}^{N}S_{t_i}.$$

A_t is again the average of known recordings and the redefined strike K^* is

$$K^* = \frac{N}{(N-m)}K - \frac{m}{(N-m)}A_t,$$

and the proportionalizing factor is

$$\alpha = \frac{N-m}{N}.$$

Thus when $m > 0$, $ARO_C(K,t)$ can be valued as α times a new ARO with *N-m* remaining fixings and strike K^*. Because prices are assumed as always positive, whenever $K^* < 0$, that is $A_t > (N/m)K$, exercise on the call option at time t_N is certain. In this case the call option has the value given by

$$ARO_C(K,t) = V_{R_t} - Ke^{-r(t_N - t)}.$$

4.2.6 Sensitivities

When the start of the averaging period of a forward starting ARO is close to the expiry date, its premium is close to that of a European option with the same maturity. This can be seen in Fig. 41 where the value of the equivalent European call option is R 15.89. It can be explained by the fact that the further away the start of the averaging period is, the higher the variance of the average rate and so the ARO will be more expensive.

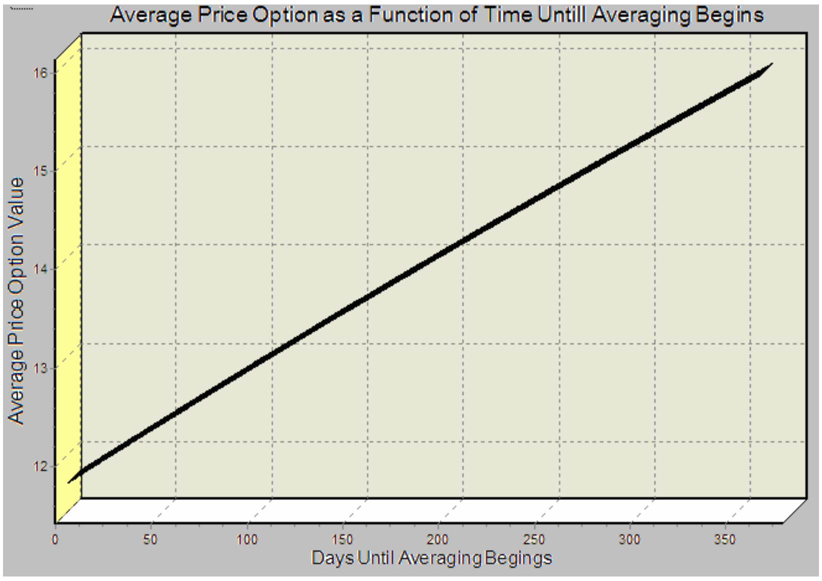

Figure 41: The sensitivity of an Average price option to the time until expiration. Parameters: K = 90, S = 100, r = 10%, q = 5%, t_N =365, σ = 20%.

The effect of altering the number of equidistant fixings over a given averaging period can be seen in Fig. 42. It can be seen that with fewer observations, the higher the variance of the average rate and therefore the higher the price.

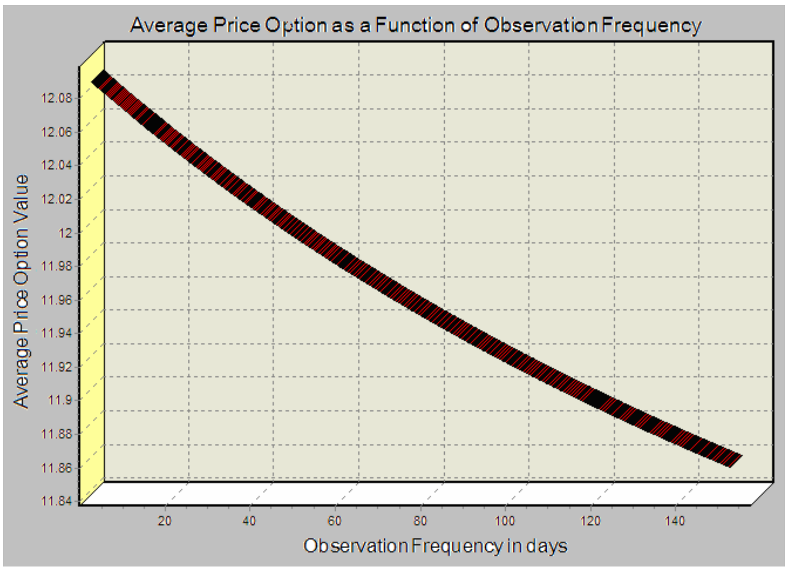

Figure 42: The sensitivity of an Average price option to the observation frequency. Parameters: K = 90, S = 100, r = 10%, q = 5%, t_N =365, σ = 20%.

4.3 Summary

In this lengthy chapter two path-dependent options were discussed. First, weakly path-dependent barrier options were considered. The eight basic barrier options were shown to have prices based on combinations of six evaluation equations. These six evaluation equations were given together with an informal proof of their derivation. The mathematical background and formal derivation on pricing barrier option were referred to in Appendix A. In the discussion the payoff functions of an up-and-out call, down-and-out call, up-and-in call and down-and-in call were illustrated. The put-call parity relationships for barrier options were then derived from the put-call parity relationship for standard options. This led to a relationship between a down-and-in call and up-and-in put, between an up-and-in call and down-and-in put, between a down-and-out call and up-and-out call and a relationship between an up-and-out call and down-and-out put. Lastly the sensitivities of each of the six evaluation equations were given. These will be used in the same combination used to price each barrier option to derive its sensitivity to the various variables.

Secondly, two types of strongly path-dependent Asian options, namely average rate options and average strike options, were defined. To value these options general characterising valuation formulae were derived that showed the dependence of the valuation function of AROs and ASOs on the conditional density functions of the average of N prices for AROs and the joint density of the average of N prices and the final stock price.

It was shown how to value geometric AROs and ASOs using both discrete and continuous sampling. These valuation methods have closed form solutions and are derived from the fact that the average of a set of log-normal prices is itself lognormally distributed. Therefore the joint densities mentioned above are easy to evaluate when the average is defined by the geometric average.

When, as is nearly always the case, Asian options are defined in terms of the arithmetic averages, exact analytic pricing formulas are not available. This is because the distribution of the arithmetic average, which is a sum of log-normal components, has no explicit representation or tractable properties. For arithmetic Asian options the joint densities mentioned above in the characterizing valuation formulae are non-standard, and to evaluate the necessary integrals a variety of numeric and approximation methods have been developed. Two examples of these methods were given, one that approximates the log-normal distribution and one that uses continuous sampling.

Several arbitrage bounds were shown to hold for Asian options: An arithmetic Asian option is always worth less than a vanilla option; geometric Asian options are worth less or equal to arithmetic Asian options, put-call parity for arithmetic Asian options and symmetry results for arithmetic Asian options.

In the remarks on Asian options it was shown how to value arithmetic AROs when one or more fixing is known. Finally, the sensitivity of Asian options to the number of days until averaging begins and the observation frequency in days was illustrated.

5.

Binary Options

5.1 Definition

A binary variable is one which is given a value of either 0 or 1 and nothing else; in the case of derivatives, a binary option is an option which pays either an asset out at expiry, or nothing at all based, on whether or not the option expires in-the-money. The payoff remains the same, no matter how deep in-the-money the option is. These binary options are also known as digital options, a name which reflects the all-or-nothing character of their payoffs. The payoff structure for a binary is discontinuous and these types of exotic options come in one of the following formats:

- A **Cash-or-Nothing** option pays out a prescribed cash amount at expiry if the option expires in the money. The payoffs for a call and put are shown below:

Option Type	*Payout of 0*	*Payout of Cash Amount*
Cash-or-Nothing Call	$S \leq K$	$S > K$
Cash-or-Nothing Put	$S \geq K$	$S < K$

An American Cash-or-Nothing binary is issued out-of-the-money and makes a fixed payment if the underlying asset value ever reaches the strike. The payment can be made immediately, or deferred until the option's expiration date.

- An **Asset-or-Nothing** binary is similar to a cash-or-nothing, with the exception that the positive payoff is the asset itself, given the following payoff criteria which is the same payoff as that of a cash-or-nothing binary:

Option Type	*Payout of 0*	*Payout of Asset*
Asset-or-Nothing Call	$S \leq K$	$S > K$
Asset-or-Nothing Put	$S \geq K$	$S < K$

An asset-or-nothing binary might be structured as an American option with deferred payment, but this structure is not common. (www.global-derivatives.com)

- **American Cash-or-Nothing digital** are often referred to as "one-touch binary/digitals", "binary-at-hit" or the "rebate portion of a knock-out barrier option". This option gives an investor a payout once the price of the underlying asset reaches or surpasses a predetermined barrier. It allows the investor to set the position of the barrier, the time to expiration and the payout to be received once the barrier is broken. Only two outcomes are possible: 1) The barrier is breached and the trader collects the full payout agreed upon at the outset of the contract, or 2) the barrier is not breached and the trader loses the full premium paid to the broker. (www.investopedia.com)

5.2 Common Uses

Binary Options are ideal for short-term trading, offering potentially dramatic short-term returns, but with strictly limited risk. A speculator betting on rising and falling prices can use digital options as cheaper alternatives to regular vanilla options. A hedger uses this cost-effective instrument to draw effectively upon a rebate arrangement that will offer a fixed compensation if the market turned the other direction.

A digital option can be simulated for pricing purposes and replicated for hedging purposes as an aggressive bull spread. A bull spread involves buying an option at a lower strike and selling a similar option at a higher strike; the difference in the strikes is the spread risk. Keep in mind, though, the more aggressive the bull spread, the higher its premium, and therefore the more costly your hedge. On the other hand, the less tight the bull spread, the larger the exposure to spread risk.

Currency markets are event-driven and it is challenging to forecast the direction of market movement prior to important events. Digital options work well in these scenarios. Technical trading does not necessarily bode very well for profit-taking before the scheduled release of key economic and trade reports. However, if you expect increased volatility in light of the announcements, your best choice is to trade options and reduce return-related spikes and whipsaws.

Consider the following Forex example given by www.financial-spread-betting.com:

A digital option lets you wager on whether the exchange rate will trade above or below the strike price at expiration. If exchange rates move unfavourably to the position, the holder exercises his option and trims his losses by a predetermined payout amount, whereas if the market moves favourably, the trader continues to deal in current spot prices and doesn't exercise his option. The reasoning is that, in a volatile market, a digital option presents a cheaper alternative to the traditional vanilla option.

Alternatively, if the trader is expecting a stable or relatively quiet market with low volatility, then the recommended strategy would be to write (sell) options, as doing so will generate profits in an otherwise unprofitable trading environment. Remember, the greater the flexibility and higher the payout for an unfavourable market price movement, the larger the upfront premium associated with purchasing that option.

American cash-or-nothing binary options are useful if a trader believes that the price of an underlying asset will exceed a certain level in the future, but is not sure that the higher price level is sustainable.

5.3 Valuation

Recall the formula of standard European options in the Black-Scholes-Merton environment at time *t* given by

$$\begin{aligned}
c &= S_t e^{-q(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2) \\
&= e^{-r(T-t)} \left[S_t e^{(r-q)(T-t)} N(d_1) - K N(d_2) \right] \\
&= e^{-r(T-t)} \hat{E}\left[\max(S_T - K, 0) \right] \qquad (6.3.1)
\end{aligned}$$

$$\begin{aligned}
p &= K e^{-r(T-t)} N(-d_2) - S_t e^{-q(T-t)} N(-d_1) \\
&= e^{-r(T-t)} \left[K N(-d_2) - S_t e^{(r-q)(T-t)} N(-d_1) \right] \\
&= e^{-r(T-t)} \hat{E}\left[\max(K - S_T, 0) \right], \qquad (6.3.2)
\end{aligned}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

The expressions in (6.3.1) can be decomposed into the difference between two terms and interpreted. Consider the call option: The expression $N(d_2)$ is the probability that the option will be exercised in a risk-neutral world. That means that

$$\begin{aligned} N(d_2) &= P(\text{Call is exercised}) \\ &= P(S_T > K) \text{ with } S_T \sim \log normal \\ &= \int_K^\infty g(x)\,dx, \end{aligned}$$

so that $KN(d_2)$ is the strike price times the probability that the strike price will be paid. The expression $S_t e^{(r-q)(T-t)} N(d_1)$ is the expected value of a variable that equals S_T if $S_T > K$ and zero otherwise in a risk-neutral world. It is therefore the unprotected present value of the underlying asset price conditional upon exercising the options.

Cash-or-Nothing Binary Options

A cash-or-nothing call pays a fixed amount, X, if the stock price, S_T, exceeds the exercise price, K; otherwise, it pays nothing. Similarly, a cash-or-nothing put option pays out a fixed cash amount, X, if the terminal stock price is below the exercise price. These options require no payment of an exercise price. Instead, the exercise price merely determines whether the option owner receives a payoff. If the valuation date is t, then the value of a cash-or-nothing call will be the present value of the fixed cash payoff multiplied by the probability that the terminal stock price will exceed the exercise price. Therefore, the value of a binary cash-or-nothing call is given by

$$v(S_T, K, T, t, \sigma, r, q, \phi = 1) = Call_{cash-or-nothing} = Xe^{-r(T-t)}N(d_2). \qquad (6.3.3)$$

By analogous reasoning the value of a binary cash-or-nothing put that pays of X if the asset price is below the strike price, and nothing, otherwise, is given by

$$v\left(S_T, K, T, t, \sigma, r, q, \phi = -1\right) = Put_{cash-or-nothing} = Xe^{-r(T-t)}N\left(-d_2\right). \quad (6.3.4)$$

Asset-or-Nothing Binary Options

Asset-or-nothing options are similar to cash-or-nothing options, with one major difference. Instead of paying a predetermined cash amount, the payoff of an asset-or-nothing option is the amount equal to the asset price at expiration. To value these, refer to the first term in the Black-Scholes formula, which gives the unprotected present value of the underlying asset price conditional upon exercising the options. Therefore the value of an asset-or-nothing call which pays out nothing if the underling asset price winds up below the strike price and pays of S_T if it ends up above the strike price is given by

$$w\left(S_T, K, T, t, \sigma, r, q, \phi = 1\right) = Call_{asset-or-nothing} = S_T e^{-q(T-t)}N\left(d_1\right). \quad (6.3.5)$$

An asset-or-nothing put pays off nothing if the underlying price ends up above the strike price, and an amount equal to the asset price if it ends up below the strike price. Its value is given by

$$w\left(S_T, K, T, t, \sigma, r, q, \phi = -1\right) = Put_{asset-or-nothing} = S_T e^{-q(T-t)}N\left(-d_1\right). \quad (6.3.6)$$

American-Style Binary Options (ASB)

The derivation of the closed form pricing formula is followed as given by www.mathfinance.de. The ASB pays a cash amount of *X* if a barrier *H* is hit any time before expiry at time *T*. The binary variable is defined as

$$\eta = \begin{cases} 1 & \text{if H is a lower barrier,} \\ -1 & \text{if H is an upper barrier.} \end{cases}$$

The stopping time τ_H is called the first hitting time. Given that the stock price follows the model

$$dS_t = (r-q)S_t dt + \sigma dB_t,$$

the payoff can be written as

$$XI_{\{\tau_H \le T\}},$$
$$\tau_H = \inf\{t \ge 0 : \eta S_t \le \eta \mathrm{H}\}.$$

The modified payoff,

$$XI_{\{\tau_H \ge T\}},$$

describes an ASB which is paid if a knock-in-option has not knocked in by the time it expires and can be valued similarly by exploiting the identity:

$$XI_{\{\tau_H \le T\}} + XI_{\{\tau_H \ge T\}} = X.$$

An ASB option can further be distinguished by whether X is paid at the first hitting time or at the expiry of the option. Denote

$$\omega = \begin{cases} 0 & \text{if X is paid at hit,} \\ 1 & \text{if X is paid at expiry.} \end{cases}$$

The distribution of the first time a stock price hits a barrier is needed to value American binary options. First the hitting time is considered for a Brownian motion without drift before it is expanded to the Brownian motion with drift. Then these results are used when considering the hitting time of a stock price.

Hitting Time for Brownian Motion Without Drift (hit at high)

Now consider the properties of the first hitting time τ_H for Brownian motion. Shreve (1996) gives the mathematics used. Let *B* be a Brownian motion under P without drift and hit level $x > 0$, then define

$$\tau = \inf\{t \ge 0 : B_t = x\}.$$

τ is the first passage time to x. The distribution of τ is computed based on the reflection principle. The crucial observation is that B is bounded from above. Define the maximum level reached by the Brownian motion in the time interval $[0,T]$

$$M_T = \max_{\{0\le t\le T\}} B_t.$$

Then, from the joint distribution of the Brownian motion and its maximum as given in Proposition 7, it follows that

$$\begin{aligned} P\left[M_T \ge m, B_T \le b\right] &= 1 - N\left(\frac{2m-b}{\sqrt{T}}\right) \\ &= \frac{1}{\sqrt{2\pi T}} \int_{2m-b}^{\infty} \exp\left\{-\frac{x^2}{2T}\right\} dx, \end{aligned}$$

where $m > 0, b < m$. Thus, the joint density is

$$\begin{aligned} f_{M_T,B_T}(m,b) &= -\frac{\partial^2}{\partial m \partial b}\left(\frac{1}{\sqrt{2\pi T}} \int_{2m-b}^{\infty} \exp\left\{-\frac{x^2}{2T}\right\} dx\right) dmdb \\ &= -\frac{\partial}{\partial m}\left(\frac{1}{\sqrt{2\pi T}} \exp\left\{-\frac{x^2}{2T}\right\}\right) dmdb \\ &= \frac{2(2m-b)}{T\sqrt{2\pi T}} \exp\left\{-\frac{(2m-b)^2}{2T}\right\} dmdb. \end{aligned}$$

Therefore,

$$\begin{aligned} P\left[M_t \ge x\right] &= \int_x^{\infty}\int_{-\infty}^{m} \frac{2(2m-b)}{t\sqrt{2\pi t}} \exp\left\{-\frac{(2m-b)^2}{2t}\right\} dmdb \\ &= \int_x^{\infty} \frac{2}{\sqrt{2\pi t}} \exp\left\{-\frac{(2m-b)^2}{2t}\right\}\Big|_{b=-\infty}^{b=m} dm \\ &= \int_x^{\infty} \frac{2}{\sqrt{2\pi t}} \exp\left\{-\frac{m^2}{2t}\right\} dm. \end{aligned}$$

Transform $z = \dfrac{m}{\sqrt{t}}$ in the integral to get

$$P\left[M_t \geq x\right] = \int_{\frac{x}{\sqrt{t}}}^{\infty} \frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz.$$

Now the density of the hitting time τ follows from the fact that if

$$F(t) = \int_{a(t)}^{b} g(z)\, dz,$$

then

$$\frac{\partial F}{\partial t} = -\frac{\partial a}{\partial t} g\left(a(t)\right)$$

and (see Appendix A)

$$\tau \leq t \Leftrightarrow M_t \geq x.$$

The density of the hitting time τ is given by

$$\begin{aligned}
f(\tau) &= \frac{\partial}{\partial t} P\{\tau \leq t\}\, dt \\
&= \frac{\partial}{\partial t} P\{\mathrm{M}_\tau \geq x\}\, dt \\
&= \left[\frac{\partial}{\partial t} \int_{\frac{x}{\sqrt{t}}}^{\infty} \frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz\right] dt \\
&= -\frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2t}\right\} \times \frac{\partial}{\partial t}\left(\frac{x}{\sqrt{t}}\right) dt \\
&= \frac{x}{t\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\} dt.
\end{aligned}$$

Therefore, the hitting time density is given by

$$p(t, x) = \frac{x}{t\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}.$$

The Laplace transformation for a Brownian motion without drift is obtained by West (2007). Consider Dólean's exponential of the martingale $\sqrt{2s}B_t^x$, namely $e^{\sqrt{2s}B_t^x - s(t\wedge\tau)}$, which is again a martingale. Since B_t^x is bounded from above, this martingale is bounded from above by $e^{\sqrt{2s}}x$, and below by 0. Thus, the optional sampling theorem can be applied:

$$
\begin{aligned}
E\left[e^{\sqrt{2s}x - s\tau}\right] &= E\left[e^{\sqrt{2s}B_\tau^x - s(\tau\wedge\tau)}\right] \\
&= E\left[e^{\sqrt{2s}B_0^x - s0}\right] \\
&= 1.
\end{aligned}
$$

Let $p(x,t)$ be the hitting time distribution. Let $\boldsymbol{L}$ denote its Laplace transformation. Then

$$
\begin{aligned}
\boldsymbol{L}\left[p(x,t)\right] &= \int_0^\infty e^{-st} p(x,t)\,dt \\
&= E\left[e^{-s\tau}\right] \\
&= E\left[e^{-\sqrt{2s}x + \sqrt{2s}x - s\tau}\right] \\
&= e^{-\sqrt{2s}x} E\left[e^{\sqrt{2s}x - s\tau}\right] \\
&= e^{-\sqrt{2s}x}.
\end{aligned}
$$

Now the hitting time distribution can also be written as

$$
\begin{aligned}
p(x,t) &= \boldsymbol{L}^{-1}\left[e^{-\sqrt{2s}x}\right] \\
&= \frac{x}{t\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\}.
\end{aligned}
\tag{6.3.7}
$$

Hitting Time for Brownian Motion With Drift (hit at high)

Next, consider a Brownian motion with drift θ. For $0 \le t < T$, define

$$\begin{aligned}\tilde{B}_t &= \theta t + B_t \\ Z_t &= \exp\left\{-\theta B_t - \frac{1}{2}\theta^2 t\right\} \\ &= \exp\left\{-\theta \tilde{B}_t + \frac{1}{2}\theta^2 t\right\}.\end{aligned}$$

Define

$$\tilde{\tau} = \inf\left\{t \ge 0 : \tilde{B}_t = x\right\}. \tag{6.3.8}$$

Fix a finite time T and change the probability measure "only up to T". More specifically, with T fixed, define

$$\tilde{P}(A) = \int_A Z_T dP,\ A \in \Im_T.$$

Under $\tilde{P}$, the process $\tilde{B}_t, 0 \le t < T$, is a Brownian motion without drift, so

$$\begin{aligned}\tilde{P}\left[\tilde{\tau} \in dt\right] &= P\left[\tau \in dt\right] \\ &= \frac{x}{t\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\} dt,\ 0 < t \le T.\end{aligned}$$

Then, for $0 \le t < T$,

$$
\begin{aligned}
P\left[\tilde{\tau} \le t\right] &= E\left[1_{\{\tilde{\tau}\le t\}}\right] \\
&= \tilde{E}\left[1_{\{\tilde{\tau}\le t\}}\frac{1}{Z_T}\right] \\
&= \tilde{E}\left[1_{\{\tilde{\tau}\le t\}}\exp\left\{\theta\tilde{B}_T - \frac{1}{2}\theta^2 T\right\}\right] \\
&= \tilde{E}\left[1_{\{\tilde{\tau}\le t\}}\tilde{E}\left[\exp\left\{\theta\tilde{B}_T - \frac{1}{2}\theta^2 T\right\}\middle|\Im_{(\tilde{\tau}\wedge t)}\right]\right] \\
&= \tilde{E}\left[1_{\{\tilde{\tau}\le t\}}\exp\left\{\theta\tilde{B}_{\tilde{\tau}\wedge t} - \frac{1}{2}\theta^2\left(\tilde{\tau}\wedge t\right)\right\}\right] \\
&= \tilde{E}\left[1_{\{\tilde{\tau}\le t\}}\exp\left\{\theta x - \frac{1}{2}\theta^2\tilde{\tau}\right\}\right] \\
&= \int_0^t \exp\left\{\theta x - \frac{1}{2}\theta^2 s\right\}P\left\{\tilde{\tau}\in ds\right\} \\
&= \int_0^t \frac{x}{s\sqrt{2\pi s}}\exp\left\{\theta x - \frac{1}{2}\theta^2 s - \frac{x^2}{2s}\right\}ds \\
&= \int_0^t \frac{x}{s\sqrt{2\pi s}}\exp\left\{-\frac{\left(x-\theta t\right)^2}{2s}\right\}ds.
\end{aligned}
$$

Therefore,

$$
f\left(\tilde{\tau}\right) = \frac{x}{t\sqrt{2\pi t}}\exp\left\{-\frac{\left(x-\theta t\right)^2}{2t}\right\}, 0 < t \le T. \tag{6.3.9}
$$

Since T is arbitrary, this must be the correct formula for all $t > 0$. This result can be obtained more directly using the Laplace transformation as shown by West (2007). From the definition

$$
\tilde{\tau} = \inf\left\{t \ge 0 : \theta t + B_t = x\right\},
$$

it can be seen that B_t^x is the Brownian motion which is stopped when $\theta t + B_t$ first hits x. Again B_t^x is bounded from above by x. Now

$$
\begin{aligned}
\left(\sqrt{2s+\theta^2}-\theta\right)x - s\tau &= \left(\sqrt{2s+\theta^2}-\theta\right)\left[\theta\tau + B_\tau^x\right] - s\left(\tau\wedge\tau\right) \\
&= \left(\sqrt{2s+\theta^2}-\theta\right)B_\tau^x - \left(s - \sqrt{2s+\theta^2}\theta + \theta^2\right)\left(\tau\wedge\tau\right) \\
&= \left(\sqrt{2s+\theta^2}-\theta\right)B_\tau^x - \frac{1}{2}\left(\sqrt{2s+\theta^2}-\theta\right)^2\left(\tau\wedge\tau\right),
\end{aligned}
$$

and so

$$E\left[e^{\left(\sqrt{2s+\theta^2}-\theta\right)x-s\tau}\right]=1,$$

as before. Again, let $p(x,t)$ be the hitting time distribution. Then

$$\begin{aligned}
\boldsymbol{L}\left[p(x,t)\right] &= \int_0^\infty e^{-st}p(x,t)\,dt \\
&= E\left[e^{-s\tau}\right] \\
&= E\left[e^{\left(\theta-\sqrt{2s+\theta^2}\right)x+\left(\sqrt{2s+\theta^2}-\theta\right)x-s\tau}\right] \\
&= e^{\left(\theta-\sqrt{2s+\theta^2}\right)x}E\left[e^{\left(\sqrt{2s+\theta^2}-\theta\right)x-s\tau}\right] \\
&= e^{\left(\theta-\sqrt{2s+\theta^2}\right)x}.
\end{aligned}$$

Thus,

$$\begin{aligned}
p(x,t) &= \boldsymbol{L}^{-1}\left[e^{\left(\theta-\sqrt{2s+\theta^2}\right)x}\right] \\
&= e^{\theta x}\boldsymbol{L}^{-1}\left[e^{-\sqrt{2s+\theta^2}\,x}\right] \\
&= e^{\theta x}\frac{x}{t\sqrt{2\pi t}}\exp\left\{-\frac{x^2}{2t}\right\}e^{-\frac{1}{2}\theta^2 x} \\
&= \frac{x}{t\sqrt{2\pi t}}\exp\left\{-\frac{(x-\theta t)^2}{2t}\right\},
\end{aligned}$$

using the previous Laplace transform result in (6.3.7).

Hitting Time for the Stock Price

Again, consider the Laplace transform. The following theorem by Etheridge (1997) for the hitting time of a sloping line is used.

Theorem 11.

Set $\tau_{x,\theta}=\inf\{t\geq 0: B_t=x+\theta t\}$, *where* $\tau_{x,\theta}$ *is taken to be infinite if no such time exists. Then for* $\alpha>0$, $x>0$ and $\theta\geq 0$

$$E\left[\exp\left(-\alpha\tau_{x,\theta}\right)\right]=\exp\left(-x\left(\theta+\sqrt{2\alpha+\theta^2}\right)\right). \tag{6.3.10}$$

Proof:

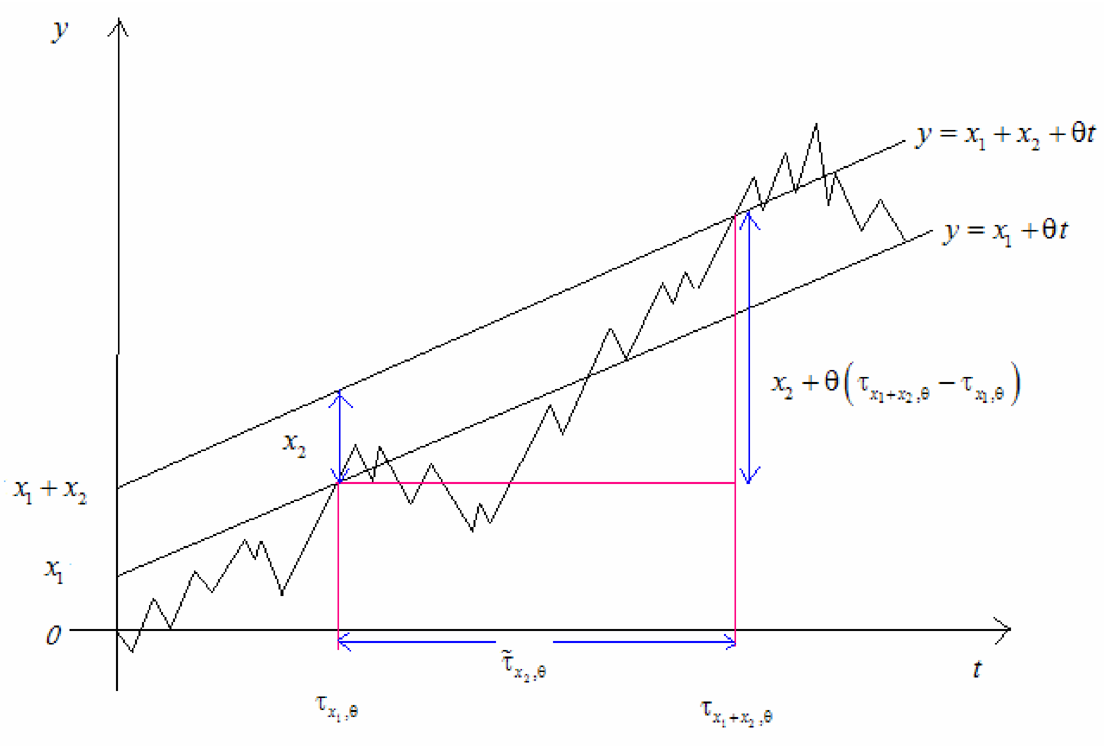

Figure 43: In the notation of theorem 11 $\tau_{x_1+x_2,\theta}=\tau_{x_1,\theta}+\tilde{\tau}_{x_2,\theta}$, where $\tilde{\tau}_{x_2,\theta}$ has the same distribution as $\tau_{x_2,\theta}$.

Fix $\alpha > 0$, and for $x > 0$ and $\theta \ge 0$ set

$$\psi(x,\theta)=E\left[\exp\left(-\alpha\tau_{x,\theta}\right)\right].$$

Now take any two values of x, x_1 and x_2, and notice graphically in Fig. 43 that

$$\tau_{x_1+x_2,\theta}=\tau_{x_1,\theta}+\left(\tau_{x_1+x_2,\theta}-\tau_{x_1,\theta}\right)\underline{\underline{D}}\,\tau_{x_1,\theta}+\tilde{\tau}_{x_2,\theta},$$

where $\tilde{\tau}_{x_2,\theta}$ is independent of $\tau_{x_1,\theta}$ and has the same distribution as $\tau_{x_2,\theta}$. Here $\underline{\underline{D}}$ indicates equality in distribution. In other words,

$$\psi(x_1+x_2,\theta)=\psi(x_1,\theta)\psi(x_2,\theta),$$

which implies that

$$\psi(x,\theta) = e^{-k(\theta)x}, \tag{6.3.11}$$

for some function $k(\theta)$.

Since $\theta \geq 0$, the process must hit the level x before it can hit the line $x + \theta t$. This is used to break $\tau_{x,\theta}$ into two parts. Writing f_{τ_x} for the probability density function of the random variable τ_x and conditioning on τ_x, the following is obtained:

$$\begin{aligned}
\psi(x,\theta) &= \int_0^\infty f_{\tau_x}(t) E\left[\exp\left(-\alpha\tau_{x,\theta}\right) \middle| \tau_x = t\right] dt \\
&= \int_0^\infty f_{\tau_x}(t) e^{-\alpha t} E\left[e^{-\alpha\tau_{\theta t,\theta}}\right] dt \\
&= \int_0^\infty f_{\tau_x}(t) e^{-\alpha t} e^{-k(\theta)\theta t} dt \\
&= E\left[e^{-(\alpha+\theta k(\theta))\tau_x}\right] \\
&= \exp\left(-x\sqrt{2\left(\alpha + \theta k(\theta)\right)}\right).
\end{aligned}$$

Now there are two expressions for $\psi(x,\theta)$. Equating them gives

$$k^2(\theta) = 2\theta + 2\theta k(\theta).$$

Since for $\alpha > 0$, $\psi(x,\theta)$ must be less or equal to 1, choose

$$k(\theta) = \theta + \sqrt{2\alpha + \theta^2}. \tag{6.3.12}$$

Substituting (6.3.12) in (6.3.11) leads to (6.3.10), which proves the theorem.

■

Since $\tilde{B}_t = \theta t + B_t$ is a Brownian motion with a drift θ, $\tau_{x,\theta}$ can be interpreted as the first hitting time of a Brownian motion with drift $-\theta$. Therefore the Laplace-transform of the density of $\tilde{\tau}$ for $\alpha > 0,\ x > 0$ is given by

$$L\left(P\left[\tilde{\tau} \in dt\right]\right) = Ee^{-\alpha\tilde{\tau}} = \int_0^\infty e^{-\alpha t} P\left[\tilde{\tau} \in dt\right] = \exp\left\{x\theta - x\sqrt{2\alpha + \theta^2}\right\}. \quad (6.3.13)$$

If $\tilde{\tau}(\omega) < \infty$, then

$$\lim_{\alpha \downarrow 0} e^{-\alpha\tilde{\tau}(\omega)} = 1;$$

if $\tilde{\tau}(\omega) = \infty$, then $e^{-\alpha\tilde{\tau}(\omega)} = 0$ for every $\alpha > 0$, so

$$\lim_{\alpha \downarrow 0} e^{-\alpha\tilde{\tau}(\omega)} = 0.$$

Therefore,

$$\lim_{\alpha \downarrow 0} e^{-\alpha\tilde{\tau}(\omega)} = 1_{\tilde{\tau} < \infty}.$$

Letting $\alpha \downarrow 0$, and using the Monotone Convergence Theorem in the Laplace transform formula, gives

$$P\left[\tilde{\tau} < \infty\right] = e^{x\theta - x\sqrt{\theta^2}} = e^{x\theta - x|\theta|}.$$

So,

$$P\left[\tilde{\tau} < \infty\right] = \begin{cases} 1 & \text{if } \theta \geq 0, \\ e^{2x\theta} & \text{if } \theta < 0. \end{cases}$$

For upper barriers $H > S_0$ the first passage time τ_H can be rewritten equivalent to (6.3.8) as

$$\tau_H = \inf\{t \geq 0 : S_t = H\}$$
$$= \inf\left\{t \geq 0 : B_t + \left(\frac{r - q - \frac{\sigma^2}{2}}{\sigma}\right)t = \frac{1}{\sigma}\log\frac{H}{S_0}\right\},$$

since $S_t = S_0 \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t + (r-q)t\right)$. Here $x = \frac{1}{\sigma}\log\frac{H}{S_0}$ and define $\theta_- = \left(\frac{r - q - \frac{\sigma^2}{2}}{\sigma}\right)$. The density of τ_H, the first hitting time for the stock price from (6.3.9), is hence

$$P\left[\tilde{\tau}_H \in dt\right] = \frac{\frac{1}{\sigma}\log\frac{H}{S_0}}{t\sqrt{2\pi t}}\exp\left\{-\frac{\left(\frac{1}{\sigma}\log\frac{H}{S_0} - \theta_- t\right)^2}{2t}\right\}dt, t > 0.$$

Using this density function, the valuation function can now be derived as given by Wystup (1999). Consider the value of the paid-at-end $(\omega = 1)$ upper rebate $(\eta = -1)$. It can be written as the following integral:

$$v(T, S_0) = Xe^{-rT}E\left[I_{\{\tau_H \leq T\}}\right]$$
$$= Xe^{-rT}\int_0^T \frac{\frac{1}{\sigma}\log\frac{H}{S_0}}{t\sqrt{2\pi t}}\exp\left\{-\frac{\left(\frac{1}{\sigma}\log\frac{H}{S_0} - \theta_- t\right)^2}{2t}\right\}dt. \qquad (6.3.14)$$

To evaluate this integral, the following notation is introduced:

$$e_{\pm}(t) = \frac{\pm\log\frac{S_0}{H} - \sigma\theta_- t}{\sigma\sqrt{t}}, \qquad (6.4.15)$$

with the properties

$$e_-(t)-e_+(t)=\frac{2}{\sqrt{t}}\frac{1}{\sigma}\log\frac{H}{S_0}, \tag{6.3.16}$$

$$n(e_+(t))=\left(\frac{H}{S_0}\right)^{-\frac{2}{\sigma}\theta_-} n(e_-(t)), \tag{6.3.17}$$

$$\frac{\partial e_\pm(t)}{\partial t}=\frac{e_\mp(t)}{2t}. \tag{6.3.18}$$

The integral in (6.3.14) is evaluated by rewriting the integrand in such a way that the coefficients of the exponentials are the inner derivatives of the exponentials using properties in (6.3.16), (6.3.17) and (6.3.18).

$$\begin{aligned}
v_{up}(T,S_0) &= Xe^{-rT}\int_0^T \frac{\frac{1}{\sigma}\log\frac{H}{S_0}}{t\sqrt{2\pi t}}\exp\left\{-\frac{\left(\frac{1}{\sigma}\log\frac{H}{S_0}-\theta_t\right)^2}{2t}\right\}dt \\
&= Xe^{-rT}\frac{1}{\sigma}\log\frac{H}{S_0}\int_0^T\frac{1}{t^{\frac{3}{2}}}n(e_-(t))\,dt \\
&= Xe^{-rT}\int_0^T\frac{1}{2t}n(e_-(t))\left[e_-(t)-e_+(t)\right]dt \\
&= -Xe^{-rT}\int_0^T n(e_-(t))\frac{e_+(t)}{2t}+\left(\frac{H}{S_0}\right)^{\frac{2\theta_-}{\sigma}} n(e_+(t))\frac{e_-(t)}{2t}\,dt \\
&= Xe^{-rT}\left[\left(\frac{H}{S_0}\right)^{\frac{2\theta_-}{\sigma}} N(e_+(T))+N(-e_-(T))\right]
\end{aligned}$$

Until now a barrier hit from below was considered. Next a barrier hit from above, when the barrier itself is considered to be at a low level, is discussed.

The computation for lower barriers $(\eta=1)$ is similar. As given by West (2007), when $H<S_0$, B_t^x is bounded from below. The martingale that needs to be considered then becomes $-\sqrt{2s}B_t^x$, and it follows that $E\left[e^{-\sqrt{2s}x-s\tau}\right]=1$. It then follows that

$\boldsymbol{L}\left[p(x,t)\right]=e^{\sqrt{2s}x}$. Hence, each calculation is generalised in turn from this stage onward to obtain the value of the paid-at-end $(\omega=1)$ lower rebate $(\eta=-1)$ as

$$v_{down}(T,S_0)=Xe^{-rT}\left[\left(\frac{H}{S_0}\right)^{\frac{2\theta_-}{\sigma}}N\left(-e_+(T)\right)+N\left(e_-(T)\right)\right].$$

Payoff at First Hitting Time

Next consider an American cash-or-nothing that pays off X at the first hitting time $(\omega=0)$, provided it occurs before the maturity date *T*. A similar method to the one just shown for $(\omega=1)$ can be used. If $P\left[\tilde{\tau}_{\mathrm{H}}\in dt\right]$ is the risk-neutral probability distribution of the first time that the stock price hits H when it starts at S_0, at time 0, then the value of the first-touch digital is found by first completing the square; then following the same basic strategy as before. The solution is given by

$$v_\eta(S_0,0;T,H)=X\left[\left(\frac{H}{S_0}\right)^{\frac{\theta_-+\sqrt{\theta_-^2+2r}}{\sigma}}N\left(-\eta g_+(T)\right)+\left(\frac{H}{S_0}\right)^{\frac{\theta_--\sqrt{\theta_-^2+2r}}{\sigma}}N\left(\eta g_-(T)\right)\right],$$

where

$$g_\pm(t)=\frac{\pm\log\frac{S_0}{H}-\sqrt{\theta_-^2\sigma^2+2\sigma^2 r}\times t}{\sigma\sqrt{t}}.$$

Rebates in terms of Binary Options

Ingersoll (2000) notes that an easier solution can be obtained by realising that the first-touch digital is closely related to a digital share with a barrier event. Assume the stock pays no dividend. In this case, the first-touch digital is identical in value to the fraction $\frac{X}{H}$ of a barrier digital share. Suppose $S > H$, so the first touch digital receives a payment of X when the stock price first falls to the level H. Since the stock price then is H, this RX payment can be used to purchase exactly $\frac{X}{H}$ shares of the stock, whenever the stock price reaches H sometime during its life, or nothing if the stock price never falls to H. This is identical to the payoff on $\frac{X}{H}$ digital shares, which pay off if the minimum stock price is less than or equal to H. Similar reasoning applies when $S < H$ relates the first-touch digital to a digital share with a maximum price restriction. Therefore,

$$v\left(S_0, 0; T, Hw\right) = \begin{cases} \frac{X}{H} w\left(S_0, 0; T, S_{\min} < H\right) \text{ for } S_0 > H, \\ \frac{X}{H} w\left(S_0, 0; T, S_{\max} > H\right) \text{ for } S_0 < H, \end{cases}$$

where $w\left(S_0, 0; T, S_{\min} < H\right)$ is the value of an asset-or-nothing call, in the event where $S_{\min} < H$ and $w\left(S_0, 0; T, S_{\max} > H\right)$ is the value of an asset-or-nothing cal,l in the event where $S_{\max} > H$ and w is defined for a call in (6.3.5) and for a put in (6.3.6).

General Pricing Formula

The general value function as given by Wystup (1999) that combine all different forms of ASBs can be written as

$$v\left(t, x\right) = Xe^{-\omega r\tau}\left[\left(\frac{H}{x}\right)^{\frac{\theta_- + \vartheta_-}{\sigma}} N\left(-\eta e_+\left(\tau\right)\right) + \left(\frac{H}{x}\right)^{\frac{\theta_- - \vartheta_-}{\sigma}} N\left(\eta e_-\left(\tau\right)\right)\right], \tag{6.3.19}$$

where

$$\tau = \mathrm{T} - t,$$
$$\theta_\pm = \frac{r-q}{\sigma} \pm \frac{\sigma}{2},$$
$$\vartheta_- = \sqrt{\theta_-^2 + 2(1-\omega)r},$$
$$e_\pm(\tau) = \frac{\pm \log \frac{S_0}{H} - \sigma\vartheta_-\tau}{\sigma\sqrt{\tau}},$$
$$\omega = \begin{cases} 0, & \text{rebate paid at hit} \\ 1, & \text{rebate paid at end.} \end{cases}$$

Note that $\vartheta_- = |\theta_-|$ for X paid at expiry.

For X paid at hit $(\omega = 0)$:

$$\theta_\pm = \frac{r-q}{\sigma} \pm \frac{\sigma}{2},$$
$$\vartheta_- = \sqrt{\theta_-^2 + 2(1)r} = \sqrt{\left(\frac{r-q}{\sigma}\right)^2 + (r+q) + \frac{\sigma^2}{4}},$$
$$e_\pm(\tau) = \frac{\pm \log \frac{S_0}{H} - \sigma\tau\sqrt{\left(\frac{r-q}{\sigma}\right)^2 + (r+q) + \frac{\sigma^2}{4}}}{\sigma\sqrt{\tau}}.$$

5.4 Arbitrage Bounds on Valuation

The following bounds are given by Reib and Wystup (2000).

Put-Call Parity

The put-call parity relationship for cash-or-nothing binary options is given by

$$v(x,K,T,t,\sigma,r,q,\phi=+1) + v(x,K,T,t,\sigma,r,q,\phi=-1) = Xe^{-r(T-t)}.$$

To show this, consider

$$
\begin{aligned}
& v\left(x, K, T, t, \sigma, r, q, \phi=+1\right)+v\left(x, K, T, t, \sigma, r, q, \phi=-1\right) \\
& =X e^{-r(T-t)} N\left(d_2\right)+X e^{-r(T-t)} N\left(-d_2\right) \\
& =X e^{-r(T-t)}\left[N\left(d_2\right)-N\left(-d_2\right)\right] \\
& =X e^{-r(T-t)} .
\end{aligned}
$$

Put-call Delta Parity

The put-call delta parity relationship for cash-or-nothing binary options is given by

$$
\frac{\partial v\left(x, K, T, t, \sigma, r, q, \phi=+1\right)}{\partial x}+\frac{\partial v\left(x, K, T, t, \sigma, r, q, \phi=-1\right)}{\partial x}=0 .
$$

To show this consider

$$
\begin{aligned}
& \frac{\partial v\left(x, K, T, t, \sigma, r, q, \phi=+1\right)}{\partial x}+\frac{\partial v\left(x, K, T, t, \sigma, r, q, \phi=-1\right)}{\partial x} \\
& =X e^{-r(T-t)} \frac{n\left(d_2\right)}{x \sigma \sqrt{\mathrm{T}-t}}-X e^{-r(T-t)} \frac{n\left(d_2\right)}{x \sigma \sqrt{\mathrm{T}-t}} \\
& =0 .
\end{aligned}
$$

Symmetric Strike

Define f as the forward price of the underlying

$$
f=E\left[S_T \mid S_t=x\right]=x e^{(r-q)(T-t)} .
$$

The choice of the strike price $K=f e^{\frac{\sigma^2}{2}(T-t)}=x e^{\left(r-q-\frac{\sigma^2}{2}\right)(T-t)}$ produces identical values and deltas for binary calls and puts, in which case their value is $e^{-\frac{1}{2} r(T-t)}$. This is derived from the identities

$$
N\left(\phi d_2\right)=P\left[\phi S_T \geq \phi \mathrm{K}\right],
$$

where

$$\begin{aligned}
d_2 &= \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\ln\left(\frac{f}{K}\right) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\ln\left(\frac{f}{fe^{\frac{\sigma^2}{2}(T-t)}}\right) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}} \\
&= -\sigma\sqrt{T-t}.
\end{aligned}$$

- Binary call value

$$v\left(x, K = fe^{\frac{\sigma^2}{2}(T-t)}, T, t, \sigma, r, q, \phi = +1\right) = Xe^{-r(T-t)}N\left(-\sigma\sqrt{T-t}\right)$$

- Binary put value

$$v\left(x, K = fe^{\frac{\sigma^2}{2}(T-t)}, T, t, \sigma, r, q, \phi = -1\right) = Xe^{-r(T-t)}N\left(\sigma\sqrt{T-t}\right)$$

- Binary call delta

$$\frac{\partial v\left(x, K = fe^{\frac{\sigma^2}{2}(T-t)}, T, t, \sigma, r, q, \phi = +1\right)}{\partial x} = Xe^{-r(T-t)}\frac{n\left(-\sigma\sqrt{T-t}\right)}{x\sigma\sqrt{\mathrm{T}-t}}$$

- Binary put delta

$$\frac{\partial v\left(x, K = fe^{\frac{\sigma^2}{2}(T-t)}, T, t, \sigma, r, q, \phi = -1\right)}{\partial x} = -Xe^{-r(T-t)}\frac{n\left(-\sigma\sqrt{T-t}\right)}{x\sigma\sqrt{\mathrm{T}-t}}$$

Homogeneity

It may be necessary to measure securities or the underlying in a different unit. Rescaling can have different effects on the value of an option that is dependent on strikes and barrier levels. Let $v(x,k)$ be the value function of an option, where x is the spot price and k is the strike or barrier. Let a be a positive real number.

Definition 2 (Homogeneity classes). *The value function is called k-homogeneous of degree n if for all* $a > 0$

$$v(ax, ak) = a^n v(x, k).$$

An option of which the value is strike homogeneous of degree 1 is called a strike-defined option and similarly an option of which the value function is level-homogeneous of degree 0 a level-defined option.

The overall use of homogeneity equations is to generate double checking benchmarks when computing Greeks.

<u>Space-Homogeneity</u>

When the value of the underlying is measured in a different unit the effect on the option pricing formula, as given by Reiss and Wystup (2000), will be as follows

$$v(x, K, T, t, \sigma, r, q, \phi) = v(ax, aK, T, t, \sigma, r, q, \phi) \text{ for all } a > 0, \qquad (6.3.20)$$

$$aw(x, K, T, t, \sigma, r, q, \phi) = w(ax, aK, T, t, \sigma, r, q, \phi) \text{ for all } a > 0. \qquad (6.3.21)$$

<u>Time-Homogeneity</u>

A similar computation for the time-affected parameters leads to

$$v(x, K, T, t, \sigma, r, q, \phi) = v\left(x, K, \frac{T}{a}, \frac{t}{a}, \sqrt{a}\sigma, ar, aq, \phi\right) \text{ for all } a > 0, \text{ and}$$

$$w(x, K, T, t, \sigma, r, q, \phi) = w\left(x, K, \frac{T}{a}, \frac{t}{a}, \sqrt{a}\sigma, ar, aq, \phi\right) \text{ for all } a > 0.$$

Rates Symmetry

Direct computation shows that the rates symmetry

$$\frac{\partial v}{\partial r} + \frac{\partial v}{\partial q} = -(T - t)v$$

holds for binary options v and w. This relationship holds for a wider class of options, at least for bounded smooth path-dependent payoffs F, because in this case the value function v may be written as

$$v = e^{-r(T-t)} E\left[F\left(xe^{\sigma B_{T-t} + \left(r - q - \frac{\sigma^2}{2} \right)(T-t)} \right) \right];$$

hence

$$\frac{\partial v}{\partial r} = -(T-t)v + (T-t)e^{-r(T-t)} E\left[S_T F'(S_T) \middle| S_t = x \right],$$
$$\frac{\partial v}{\partial q} = -(T-t)e^{-r(T-t)} E\left[S_T F'(S_T) \middle| S_t = x \right].$$

Foreign-Domestic Symmetry

There exists a relationship between the prices of cash-or-nothing digital options $v(x, K, T, t, \sigma, r, q, \phi)$ and asset-or-nothing digital options $w(x, K, T, t, \sigma, r, q, \phi)$. Here $\phi = +1$ for a call option and $\phi = -1$ for a put option. Notice that q can also be regarded as the foreign rate of interest given by r_f and the risk-free rate of interest is now defined as the domestic interest rate r_d. Also, assume the cash-or-nothing option pays the fixed amount $X = 1$ at payoff. Now,

$$\frac{1}{x} v(x, K, T, t, \sigma, r_d, r_f, \phi) = w\left(\frac{1}{x}, \frac{1}{K}, T, t, \sigma, r_f, r_d, -\phi \right). \tag{6.3.22}$$

First consider the left side of (6.3.22):

$$\frac{1}{x} v(x, K, T, t, \sigma, r_d, r_f, \phi) = \frac{1}{x} e^{-r_d(T-t)} N\left(\phi \frac{\ln\left(\frac{x}{K} \right) + \left(r_d - r_f - \frac{\sigma^2}{2} \right)(T-t)}{\sigma\sqrt{T-t}} \right).$$

The right side of (6.8) is given by

$$
\begin{aligned}
w\left(\frac{1}{x},\frac{1}{K},T,t,\sigma,r_f,r_d,-\phi\right) &= \frac{1}{x}e^{-r_d(T-t)}N\left(-\phi d_1\right) \\
&= \frac{1}{x}e^{-r_d(T-t)}N\left(-\phi\frac{\ln\left(\frac{K}{x}\right)+\left(r_f-r_d+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \frac{1}{x}e^{-r_d(T-t)}N\left(\phi\frac{\ln\left(\frac{x}{K}\right)+\left(r_d-r_f-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned}
$$

The reason is that the value of an option can be computed both in a domestic and in a foreign scenario. Wystup (2000) considers the example of S_t, modelling the exchange rate of EUR/ USD. In New York, the cash-or-nothing digital call option costs $v\left(x,K,T,t,\sigma,r_{usd},r_{eur},\phi=+1\right)$ USD and hence $\frac{1}{x}v\left(x,K,T,t,\sigma,r_{eur},r_{usd},\phi=+1\right)$ EUR. If it ends in-the-money, the holder receives 1 USD. For a Frankfurt-based holder of the same option, receiving one USD means receiving asset-or-nothing, where he uses reciprocal values for spot and strike, and for him, domestic currency is the one that is foreign to the New Yorker; and vice versa. Since S_t and $\frac{1}{S_t}$ have the same volatility, the New York value and the Frankfurt value must agree.

5.5 Remarks on Binary Options

The following remark is made by Ingersoll (2000). The pricing of other European derivatives with piecewise linear and path-independent payoffs only requires valuing digital options and shares for events of the type $L < S_T < H$. Any other path-independent event can be described as the union of such events. These digitals can be computed as the difference between two unlimited range binaries given in

$$
\begin{aligned}
v\left(S_T,K,T,t,\sigma,r,q,L<S_T<H\right) &= v\left(S_T,K,T,t,\sigma,r,q,L<S_T\right) \\
&\quad -v\left(S_T,K,T,t,\sigma,r,q,H\le S_T\right).
\end{aligned}
$$

This can be seen by considering the following. A pure European style option is one with a single payoff received on a maturity date known at the contract's inception. The level of the payoff depends on the events that occur between the issuance date and the maturity date. The payoffs on most European contracts are piecewise linear in the underlying stock price on the maturity date. The value of any such contract can be represented as

$$\sum_i a_i v\left(S_T, K, T, t, \sigma, r, q, \xi_i\right) + \sum_j b_j w\left(S_T, K, T, t, \sigma, r, q, \xi_j\right).$$

Here $v\left(S_T, K, T, t, \sigma, r, q, \xi\right)$ is the value of a cash-or-nothing binary at time t of receiving $X = 1$ at time T if, and only if, the event ξ occurs; $w\left(S_T, K, T, t, \sigma, r, q, \xi\right)$ is the value of an asset-or-nothing binary at time t of receiving one share of the stock at time T, if, and only if, ξ occurs. In general, the probability of ξ depends on the stock price being either above or below the strike price K.

This binary portfolio pricing method is illustrated by the following examples.

- A standard European put option can be represented as

$$Xv\left(S_T, K, T, t, \sigma, r, q, \phi = -1\right) - w\left(S_T, K, T, t, \sigma, r, q, \phi = -1\right).$$

  That is, a fixed amount of X is received and one share of stock is given up when the stock price, at maturity, is below the strike price so that the option expires in-the-money.

- A down-and-out call option can be represented as

$$w\left(S_T, K, T, t, \sigma, r, q, S_T > K \cap S_{\min(0,T)} > H\right) - Xv\left(S_T, K, T, t, \sigma, r, q, S_T > K \cap S_{\min(0,T)} > H\right).$$

  For this option, a fixed amount of X is paid to receive a share of the stock if the option is in-the-money at maturity, and the stock price never fell below the knock-out price H.

Relationship Between Cash, Asset and Vanilla

The simple equation of payoffs,

$$\phi\big(w(T) - Kv(T)\big) = \max\left[\phi\left(S_T - K\right), 0\right],$$

leads to the formula

$$vanilla\left(x, K, T, t, \sigma, r, q, \phi\right) = \phi\left[w\left(x, K, T, t, \sigma, r, q, \phi\right) - Kv\left(x, K, T, t, \sigma, r, q, \phi\right)\right].$$

5.6 Sensitivities

First, the sensitivities are given directly from Wystup (1999) for the binary options, where

$$\begin{aligned} v\left(S_T, K, T, t, \sigma, r, q, \phi\right) &= V_{cash-or-nothing} = Xe^{-r(T-t)} N\left(\phi d_2\right) \\ w\left(S_T, K, T, t, \sigma, r, q, \phi\right) &= W_{asset-or-nothing} = S_T e^{-q(T-t)} N\left(\phi d_1\right) \end{aligned}$$

and

$$\phi = \begin{cases} 1 & \text{for a call option,} \\ -1 & \text{for a put option.} \end{cases}$$

Delta

$$\begin{aligned} \frac{\partial v}{\partial S_T} &= \phi X e^{-r(T-t)} \frac{n\left(d_2\right)}{S_T \sigma \sqrt{T-t}} \\ \frac{\partial w}{\partial S_T} &= \phi e^{-q(T-t)} \frac{n\left(d_1\right)}{\sigma \sqrt{T-t}} + e^{-r(T-t)} N\left(\phi d_1\right) \end{aligned}$$

Gamma

$$\begin{aligned} \frac{\partial^2 v}{\partial S_T^2} &= -\phi X e^{-r(T-t)} \frac{n\left(d_2\right) d_1}{S_T^2 \sigma^2 \left(T-t\right)} \\ \frac{\partial^2 w}{\partial S_T^2} &= -\phi X e^{-q(T-t)} \frac{n\left(d_1\right) d_2}{S_T \sigma^2 \left(T-t\right)} \end{aligned}$$

Theta

$$\frac{\partial v}{\partial t} = Xe^{-r(T-t)}\left(rN\left(\phi d_2\right) + \frac{\phi n\left(d_2\right)\tilde{d}_2}{2\left(T-t\right)} \right)$$

$$\frac{\partial w}{\partial t} = S_T e^{-q(T-t)}\left(qN\left(\phi d_1\right) + \frac{\phi n\left(d_1\right)\tilde{d}_1}{2\left(T-t\right)} \right)$$

where

$$\tilde{d}_1 = \frac{\ln\left(\frac{S_t}{K}\right) - \left(r - q + \frac{\sigma^2}{2}\right)\left(T-t\right)}{\sigma\sqrt{T-t}}$$

$$\tilde{d}_2 = d_1 - \sigma\sqrt{T-t}.$$

Vega

$$\frac{\partial v}{\partial \sigma} = -\phi Xe^{-r(T-t)} n\left(d_2\right)\frac{d_1}{\sigma}$$

$$\frac{\partial w}{\partial \sigma} = -\phi S_T e^{-q(T-t)} n\left(d_1\right)\frac{d_2}{\sigma}$$

Rho

$$\frac{\partial v}{\partial r} = Xe^{-r(T-t)}\left(-\left(T-t\right)N\left(\phi d_2\right) + \frac{\phi n\left(d_2\right)\sqrt{T-t}}{\sigma} \right)$$

$$\frac{\partial w}{\partial r} = S_T e^{-q(T-t)}\left(\frac{\phi n\left(d_1\right)\sqrt{T-t}}{\sigma} \right)$$

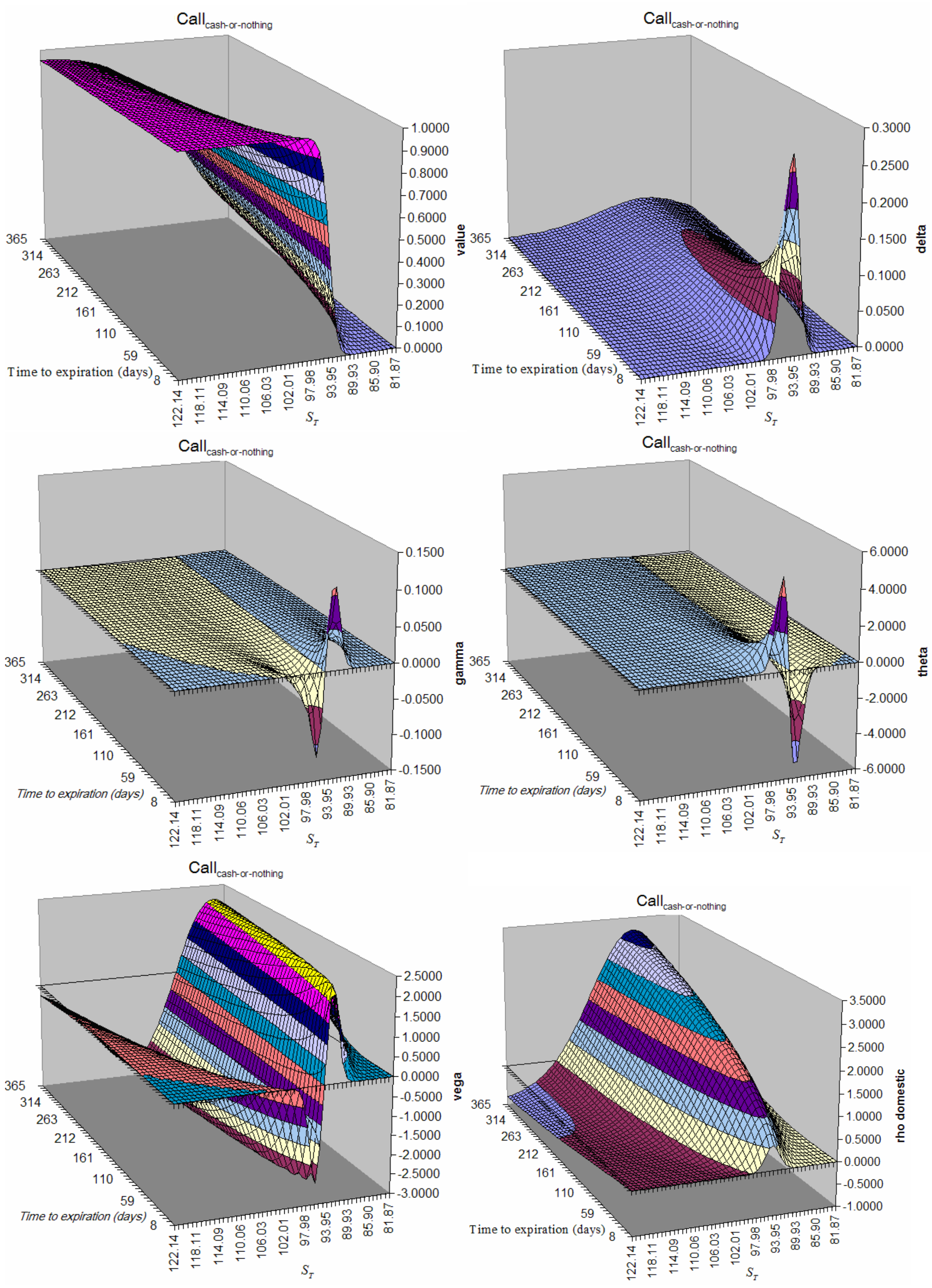

Figure 44: The value, delta, gamma, vega, theta and rho for an cash-or-nothing call option with the parameters: S_T =100, K = 95, r = 5%, q = 3%, T = 365, σ = 10%.

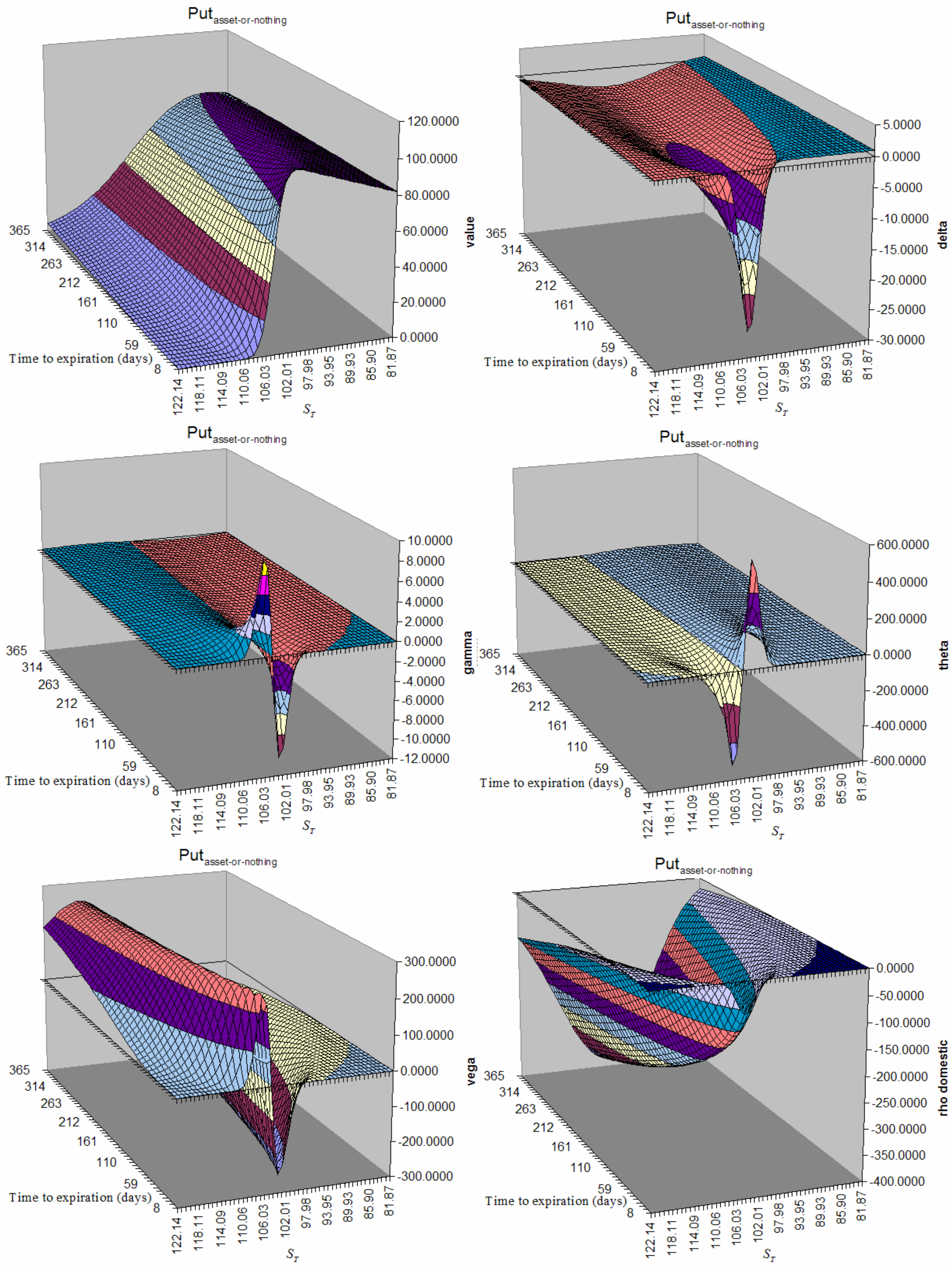

Figure 45: The value, delta, gamma, vega, theta and rho for an asset-or-nothing put option with the parameters: S_T =100, K = 105, r = 5%, q = 3%, T = 365, σ = 10%.

Secondly, the sensitivities of the American cash-or-nothing options are given, using the general formula given in (6.3.19).

Delta

$$\frac{\partial v(t,x)}{\partial x}=-\frac{Xe^{-\omega r\tau}}{\sigma x}\left\{\begin{array}{l}\left[\left(\frac{H}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}\left[(\theta_-+\vartheta_-)\right]N\left(-\eta e_+(\tau)\right)+\frac{\eta}{\sqrt{\tau}}n\left(e_+(\tau)\right)\right]\\+\left(\frac{H}{x}\right)^{\frac{\theta_--\vartheta_-}{\sigma}}\left[(\theta_-+\vartheta_-)N\left(\eta e_-(\tau)\right)+\frac{\eta}{\sqrt{\tau}}n\left(e_-(\tau)\right)\right]\end{array}\right\}$$

Gamma

Gamma can be obtained using

$$\frac{\partial^2 v}{\partial x^2}=\frac{2}{\sigma^2x^2}\left[rv-\frac{\partial v}{\partial t}-(r-q)x\frac{\partial v}{\partial x}\right]$$

and turns out to be

$$\frac{\partial^2 v(t,x)}{\partial x^2}=\frac{2Xe^{-\omega r\tau}}{\sigma^2x^2}\left\{\begin{array}{l}\left(\frac{H}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}N\left(-\eta e_+(\tau)\right)\left[r(1-\omega)+(r-q)\frac{\theta_-+\vartheta_-}{\sigma}\right]\left[+\frac{\eta}{\sqrt{\tau}}n\left(e_+(\tau)\right)\right]\\+\left(\frac{H}{x}\right)^{\frac{\theta_--\vartheta_-}{\sigma}}N\left(-\eta e_-(\tau)\right)\left[r(1-\omega)+(r-q)\frac{\theta_--\vartheta_-}{\sigma}\right]\\+\eta\left(\frac{H}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}n\left(e_+(\tau)\right)\left[-\frac{e_-(\tau)}{\tau}+\frac{r-q}{\sigma\sqrt{\tau}}\right]\\+\eta\left(\frac{H}{x}\right)^{\frac{\theta_--\vartheta_-}{\sigma}}n\left(e_-(\tau)\right)\left[\frac{e_+(\tau)}{\tau}+\frac{r-q}{\sigma\sqrt{\tau}}\right]\end{array}\right\}.$$

Theta

$$\begin{aligned}\frac{\partial v(t,x)}{\partial t}&=\omega rv(t,x)+\frac{\eta Xe^{-\omega r\tau}}{2\tau}\left[\left(\frac{H}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}n\left(e_+(\tau)\right)e_-(\tau)-\left(\frac{H}{x}\right)^{\frac{\theta_--\vartheta_-}{\sigma}}n\left(e_-(\tau)\right)e_+(\tau)\right]\\&=\omega rv(t,x)+\frac{\eta Xe^{-\omega r\tau}}{2\tau^{\frac{3}{2}}}\left(\frac{H}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}n\left(e_+(\tau)\right)\log\left(\frac{H}{x}\right)\end{aligned}$$

The computation exploits the identities given in (6.3.16), (6.3.17) and (6.3.18).

Vega

To compute vega, the following identities are used:

$$\frac{\partial\theta_-}{\partial\sigma} = -\frac{\partial\theta_+}{\partial\sigma}$$

$$\frac{\partial\vartheta_-}{\partial\sigma} = -\frac{\theta_-\theta_+}{\sigma\vartheta_-}$$

$$\frac{\partial e_\pm(\tau)}{\partial\sigma} = \pm\frac{\log\left(\frac{H}{x}\right)}{\sigma^2\sqrt{\tau}} + \frac{\theta_-\theta_+}{\sigma\vartheta_-}\sqrt{\tau}$$

$$A_\pm = \frac{\partial}{\partial\sigma}\frac{\theta_-\pm\vartheta_-}{\sigma} = -\frac{1}{\sigma^2}\left[\theta_+ + \theta_- \pm\left(\frac{\theta_-\theta_+}{\vartheta_-} + \vartheta_-\right)\right].$$

It is calculated as

$$\frac{\partial v(t,x)}{\partial\sigma} = -Xe^{-\omega r\tau}\left\{\begin{array}{l}\left(\frac{H}{x}\right)^{\frac{\theta_-+\vartheta_-}{\sigma}}\left[N\left(-\eta e_+(\tau)\right)A_+\log\left(\frac{H}{x}\right) - \eta n\left(e_+(\tau)\right)\frac{\partial e_+(\tau)}{\partial\sigma}\right]\\ +\left(\frac{H}{x}\right)^{\frac{\theta_--\vartheta_-}{\sigma}}\left[N\left(\eta e_-(\tau)\right)A_-\log\left(\frac{H}{x}\right) + \eta n\left(e_-(\tau)\right)\frac{\partial e_-(\tau)}{\partial\sigma}\right]\end{array}\right\}.$$

5.7 Summary

In this chapter three different types of binary options were given: Cash-or-nothing binary options, asset-or-nothing binary options and American-style binary options. The values of the cash-or-nothing binary options and asset-or-nothing binary options are derived from the Black-Scholes formulae for standard options. In order to value American-style binary options, the distribution of the hitting time of a stock price was derived and applied to four different cases; the value of a digital option with an upper rebate or lower rebate, and for each of these, the case where the rebate is paid at the maturity of the option or at the time of the hit. Several arbitrage bounds were given on the value of binary options. These included the put-call parity relationship, delta put-call parity, and symmetry result for the strike price and interest rates. Homogeneity equations were also given for space and time. These can be used to generate double checking benchmarks when computing the sensitivities of the option. Lastly, the sensitivities of both asset-or-nothing binary options and cash-or-nothing binary options were given and illustrated.

6.

Summary

In this document five different types of exotic options were analysed: compound options, chooser options, barrier options, Asian options and binary options. For each of these options, the payoffs are more complicated than those of standard options. These specialised payoffs can be used to manage risks or to shape a speculative position more exactly.

Many exotic options can be understood in terms of the familiar standard options priced using the Merton model. These exotic options can even sometimes be priced within the Black-Scholes framework. Well-known results were given on standard options as background for the development of the prices and properties of the exotic options discussed, as well as for the sake of completeness. The material given here is expanded where necessary for each exotic option discussed in the preceeding chapters.

Compound and chooser options are volatility-dependent options that depend in an important way on the future level of volatility. Closed form solutions exist for compound options in the Black-Scholes framework. Compound options are second order, because they give an investor the rights over another derivative. Although the Black-Scholes model can, theoretically, cope with second order contracts, it is not clear that the model is completely satisfactory in practice; when the contract is exercised, the investor receives an option at the market price, not at the theoretical price (Willmott, 1998). It is shown that compound options are extremely sensitive to the volatility. As explained in www.riskglossary.com they are also very sensitive to the volatility of the volatility. Since the Black-Scholes framework assumes constant volatility, it undervalues these options.

From a risk management perspective, some of the most flexible options are chooser options. Perhaps surprisingly, given their flexible nature, chooser options have analytical solutions within the Black-Scholes framework. Simple chooser options are closely related to a straddle position. They are not widely used, mainly because of their relatively high cost, even though they are always less expensive than the equivalent straddle. Complex choosers cannot be broken down in terms of standard options, but it is shown that they have features that are somewhat similar to compound options.

Barrier and Asian options are path-dependent options. Barrier options have weak path dependence. This is since the payoff at expiry depends both on whether the underlying hit a prescribed barrier value at some time before expiry, and on the value of the underlying at expiry. Barrier options satisfy the Black-Scholes equation with special boundary conditions.

Strongly path-dependent contracts have a payoff that depends on some property of the asset path in addition to the value of the underlying asset at the present moment in time. Asian options are strongly path-dependent, since their payoff depends on the average value of the underlying asset from inception to expiry (Wilmott, 1998). There exists no continuous time formula for average options where the average is calculated arithmetically, and therefore different methodologies are given that have been proposed to price and risk-manage these options.

Three types of binary options are analysed: Asset-or-Nothing binary options, Cash-or-Nothing binary options and American Asset-or-Nothing binary options. Binary options are similar to barrier options in that their payoff is triggered by whether or not the index trades above or below a given level at or before the maturity of the option. These options are often used in combination with either standard options or barrier options to create other types of options.

The over-the-counter market in exotic options is continuing to expand in both volume and complexity with new and more complicated structures continuing to appear. Many of the simpler exotics which were first to appear in the market, have not become standard. Some, such as Asian options, have even become exchange trade (Clewlow and Strickland, 1997).

This document attempted to provide a comprehensive source of information on some of the above mentioned exotic options with regard to pricing methods, applicability and use. It is hoped that it will contribute to a better understanding of these options when used in derivative courses for the training of financial analysts. It is also believed that practitioners, dealing in exotic options, may benefit from the knowledge that is unlocked in this document.

Appendix A

The mathematical background to derive the prices of barrier options formally is given here. First the joint distribution of Brownian motion and its maximum is derived. It is then shown how this joint distribution is used to derive a general formula for knock-out barrier options. An up-and-out call option is then valued to illustrate how the general knock-out formula can be applied. Finally, the correspondence is shown between this valuation equation for an up-and-out barrier option derived mathematically and the formula given in Table 7. A similar approach is followed by deriving the joint distribution of Brownian motion and its minimum, a general formula for a knock-in barrier option; its application shown with a down-and-in call option and its correspondence to the formula given in Table 7.

The background needed to price barrier options is given in two parts, similar to Roux (2007) and Poulsen (2004). The first recalls Brownian motion and some of its properties as applicable to the pricing of barrier options, and finally two examples are given: the pricing of an up-and-out call and a down-and-in call.

Brownian motion

Suppose that we have a probability space (Ω, F, P). Then Brownian motion can be defined as follows:

Definition A1 (Brownian motion).

A stochastic process $B \equiv (B_t)_{t \geq 0}$ in continuous time, taking real values in a standard Brownian motion under P if

1. $B_0 = 0$ almost surely (P).
2. For each $s \geq 0$ and $t \geq 0$, the random variable $B_{t+s} - B_s$ has the normal distribution with mean zero and variance *t*.
3. For each $n \geq 1$ and any times $0 \leq t_0 < t_1 < \ldots < t_n$, the random variables $\left\{B_{t_k} - B_{t_{k-1}}\right\}_{k=1}^{n}$ are independent.

4. The paths $t \mapsto B_t$ of B are continuous, almost surely (P).

Proposition A1.

If $B \equiv (B_t)_{t\geq 0}$ is a standard Brownian motion, then so are

1. $\{cB_{t/c^2}\}_{t\geq 0}$ for any real c,
2. $\{tB_{1/t}\}_{t\geq 0}$ where $tB_{1/t}$ is taken to be zero when t = 0, and
3. $\{B_{t+s} - B_s\}_{s\geq 0}$ for any fixed s > 0.

Denote the completion of a filtration generated by the Brownian motion B by $F \equiv (F_t)_{t\geq 0.}$

Definition A2 (Stopping time).

A stopping time τ for the Brownian motion B is a random time, such that for each t, the event $\{\tau \leq t\}$ depends only on the history of the process up to and including time t, i.e $(B_s)_{s\in[0,t]}$.

In other words, by observing the Brownian motion up until time t, we can determine whether or not $\tau \leq t$.

Let W be an arithmetic Brownian motion with initial value $W_0 = 0$ i.e.

$$W_t = \mu t + B_t, t \geq 0,$$

where μ and σ = 1 are constants and B is a standard Brownian motion defined on a probability space (Ω, F, P).

Proposition A2 (Girsanov).

Suppose that B is a Brownian motion under P and that the process W is defined by

$$W_t = \mu t + B_t, t \geq 0,$$

for some $\mu \in R$. The measure Q, defined by the means of the Radon-Nikodym derivative

$$\frac{dQ}{dP} = \exp\left\{-\mu B_T - \frac{1}{2}\mu^2 T\right\} = \exp\left\{-\mu W_T + \frac{1}{2}\mu^2 T\right\},$$

is a probability and W is a standard Brownian Motion under Q.

For every stopping time τ, let F_τ be the closure of the σ-algebra generated by the stopped Brownian motion $\left(B_{t\wedge\tau}\right)_{t\geq 0.}$

Proposition A3 (Strong Markov property).

Suppose that B is a Brownian motion. If τ is any stopping time, then the process $\left(B_{t+\tau} - B_\tau\right)_{t\geq 0}$ is also a Brownian motion, and is independent of F_τ.

For some fixed $a \in R$, define the hitting time of the level a as

$$T_a = \inf\left\{t \geq 0 : W_t = a\right\}.$$

We take $T_a = \infty$ if a is never reached. That the random variable τ_a is a stopping time, follows from the continuity of the paths of Brownian motion. Indeed, for $t \geq 0$ we have

$$\left\{\tau_a \leq t\right\} = \left\{B_s = a \text{ for some } s \in [0,t]\right\},$$

which depends only on $\left(B_s\right)_{s\in[0,t]}$. Notice that, again by the continuity of the paths, if $\tau_a < \infty$, we must have $B_{Ta} = a$.

Now calculate the distribution function of τ_a.

Proposition A4.

Let $B \equiv \left(B_t\right)_{t\geq 0}$ be a P-Brownian motion started from $B_0 = 0$ and let α > 0; then

$$P\left[\tau_a < t\right] = 2P\left[B_t > a\right].$$

Proof:

It follows from the strong Markov property that $\left(B_{\tau_\alpha+s}-B_{\tau_\alpha}\right)_{s\geq 0}$ is a standard Brownian motion, and therefore also $\left(B_{\tau_\alpha}-B_{\tau_\alpha+s}\right)_{s\geq 0}$. By symmetry

$$\begin{aligned}1 &= P\left(B_t - B_{\tau_a} > 0 \middle| \tau_\alpha < t\right) + P\left(B_t - B_{\tau_a} < 0 \middle| \tau_\alpha < t\right)\\ &= P\left(B_t - B_{\tau_a} > 0 \middle| \tau_\alpha < t\right) + P\left(B_{\tau_a} - Bt < 0 \middle| \tau_\alpha < t\right)\\ &= 2P\left(B_t - B_{\tau_a} > 0 \middle| \tau_\alpha < t\right),\end{aligned}$$

i.e.

$$P\left(B_{\tau_a} > a \middle| \tau_\alpha < t\right) = \frac{1}{2}.$$

By the continuity of the paths of Brownian motion, now

$$\{B_t > a\} \subseteq \{\tau_a < t\}$$

and therefore

$$\begin{aligned}P\left(B_t\right) &= P\left(B_t > a, \tau_a < t\right)\\ &= P\left(\tau_a < t\right) P\left(B_t > a \middle| \tau_a < t\right)\\ &= \frac{1}{2} P\left(\tau_a < t\right).\end{aligned}$$

■

In a similar fashion, if α < 0, then for $t \geq 0$ it can be shown that

$$P\left(\tau_a < t\right) = 2P\left(B_t < a\right).$$

A more general idea is the following, which shall not be proven here.

Proposition A5 (The reflection principle).

If B is a standard Brownian motion and $a \in R$, then the process $\overline{B} \equiv \left(\overline{B}_t\right)_{t\geq 0}$ defined by

$$\overline{B}_t = \begin{cases} B_t & \text{if } t < \tau_a, \\ 2a - B_t & \text{if } t \geq \tau_a, \end{cases}$$

is also a standard Brownian motion.

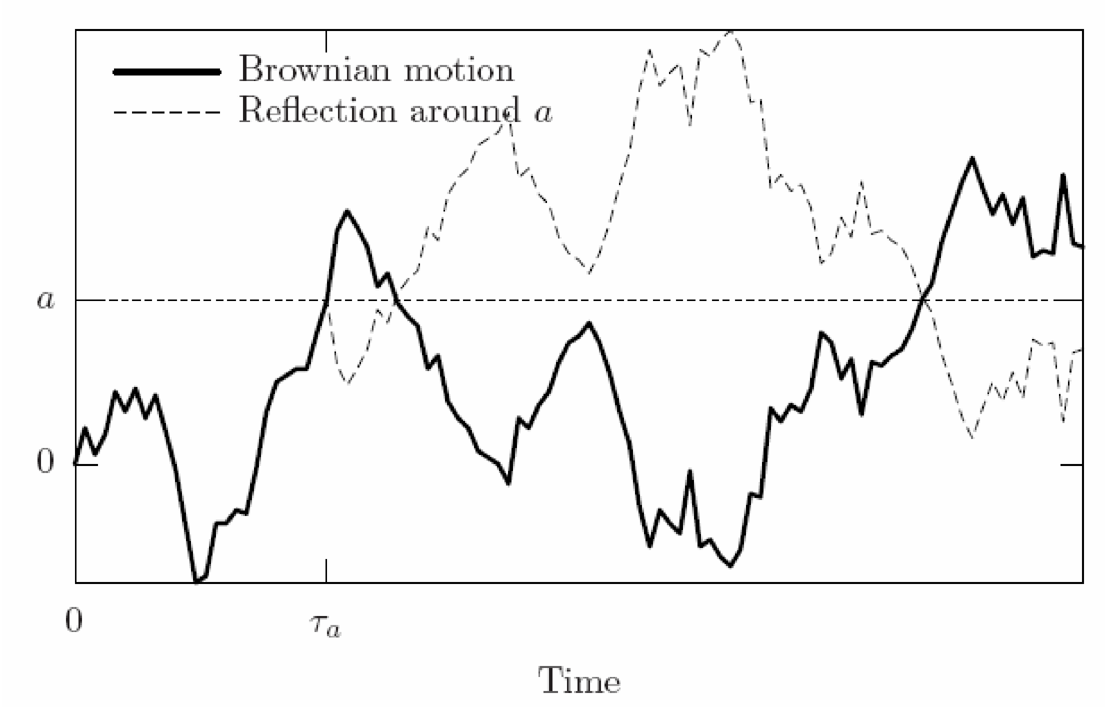

Figure A1: *Reflection of Brownian motion around level* $a > 0$

The reflected process $\bar{B}$ is obtained by observing the Brownian motion B until it first hits the level α, and from that point in time onwards reflects it around the horizontal level α. A sample path of Brownian motion and its reflection appears in figure A1.

Consider the Black-Scholes model with constant short term interest rate and risky asset, the price of which is a Geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dB_t^Q,$$

under the equivalent martingale measure Q. Put

$$p = 1 - \frac{2\mu}{\sigma^2}$$

and consider a claim with a pay-off at time T, specified by a pay-off function g. Its arbitrage free time t price is

$$\pi_t^g = e^{-r(T-t)} E_t^Q \left(g\left(S_T\right) \right) = e^{-r(T-t)} f\left(S_t, t\right),$$

where $f(S_t,t) = E_t^Q(g(S_T))$, and the Markov property of S ensures that this is non-deceptive notation. Let $a > 0$ be a constant and define a new function $\hat{g}$ by

$$\hat{g}(x) = \left(\frac{x}{a}\right)^p g\left(\frac{a^2}{x}\right).$$

We call this the reflection of g through a. The next theorem shows that g- and $\hat{g}$-claims are very closely connected.

Proposition A6 (The reflection theorem). (Poulsen 2004)

Consider a simple claim with pay-off function $\hat{g}$. The arbitrage free price at time t of this $\hat{g}$- claim is

$$\pi_t^{\hat{g}} = e^{-r(T-t)}\left(\frac{S_t}{a}\right)^p f\left(\frac{a^2}{S_t},t\right).$$

Proof:

Using the Ito formula for Geometric Brownian motion on the process of Z defined by

$$Z_t = \left(\frac{S_t}{a}\right)^p,$$

gives

$$dZ_t = p\sigma Z_t dB_t^Q,$$

so Z_t / Z_0 is a positive, mean-1Q-martingale. Here the exact form of p is needed. The result would not hold if σ were time-dependent or stochastic. This means that

$$\frac{dQ^p}{dQ} = \frac{Z_T}{Z_0}$$

defines a probability measure $Q^Z \sim Q$. Now use the abstract Bayes formula for conditional means to write the price of the $\hat{g}$ - claim as

$$\begin{aligned}\pi_t^{\hat{g}} &= e^{-r(T-t)} E_t^Q\left[\left(\frac{S_t}{a}\right)^p g\left(\frac{a^2}{S_T}\right)\right] \\ &= e^{-r(T-t)}\left(\frac{S_t}{a}\right)^p E_t^{Q^Z}\left[g\left(\frac{a^2}{S_T}\right)\right].\end{aligned}$$

Girsanov's theorem tells us that

$$dW_t^{Q^Z} = dW_t^Q - p\sigma dt$$

defines a Q^Z - Brownian motion. Put $Y_t = \frac{a^2}{S_t}$. Then the Ito formula and the definition of W^{Q^Z} gives us that

$$dY_t = \mu Y_t dt + \sigma Y_t\left(-dW_t^{Q^Z}\right),$$

which means that the law of Y under Q^Z is the same as the law of S under Q. Therefore,

$$E_t^{Q^Z}\left(g\left(Y_T\right)\right) = f\left(Y_t, t\right) = f\left(\frac{a^2}{S_t}, t\right).$$

■

The reflection principle is used to determine the joint distribution of Brownian motion B and its maximum, indeed; for $t \geq 0$, define the process M and m by

$$\begin{aligned}&M_t = \max\left\{B_s \mid s \in [0,t]\right\}, \text{ and} \\ &m_t = \min\left\{B_s \mid s \in [0,t]\right\}.\end{aligned}$$

Let M be the maximum of B and m be its minimum. Recalling that –B is also a Brownian motion, and that

$$\max\left\{B_s \mid s \in [0,t]\right\} = -\min\left\{-B_s \mid s \in [0,t]\right\},$$

it follows that the process M has the same law as –m. Consequently, attention is first restricted to the maximum of Brownian motion; corresponding results for the minimum is then derived from the results presented below.

The continuity property of the Brownian paths ensures that its maximum is well-defined. Observe that M is non-negative and non-decreasing, and that

$$\left\{M_t > a\right\} = \left\{\tau_a \le t\right\}$$

for $a > 0$ and $t \ge 0$.

The following result will be key to pricing knock-out barrier options.

Proposition A7 (Joint distribution of Brownian motion and its maximum). Suppose that B is a Brownian motion and M its maximum. For $a > 0$, $x \le a$ and all $t \ge 0$,

$$P\left[M_t \ge a, B_t \le x\right] = 1 - N\left(\frac{2a - x}{\sqrt{t}}\right) = N\left(\frac{x - 2a}{\sqrt{t}}\right),$$

where

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

is the standard normal distribution function.

Proof:

$$\begin{aligned} P\left[M_t \ge a, B_t \le x\right] &= P\left[\tau_a \le t, B_t \le x\right] \\ &= P\left[\tau_a \le t, 2a - x \le \overline{B}_t\right] \\ &= P\left[2a - x \le \overline{B}_t\right] \\ &= 1 - N\left(\frac{2a - x}{\sqrt{t}}\right). \end{aligned}$$

In the third inequality the fact that if $\overline{B}_t \geq 2a-t$, then necessarily $\{\overline{B}_s\}_{s\geq 0}$, and consequently $\{B_s\}_{s\geq 0}$ has hit level α before time t was used. It follows from the reflection principle that $\overline{B}_t$ is a normal random variable with zero mean and variance t. ■

It is known that for any $0 \leq x \leq a$ and $t \geq 0$

$$P(B_t \leq x) = P(M_t < a, B_t \leq x) + P(M_t \geq a, B_t \leq x).$$

It then follows from the proposition that the value of the cumulative distribution function of M_t and B_t at (a,x) is given by

$$\begin{aligned} P(M_t < a, B_t \leq x) &= P(B_t \leq x) - P(M_t \geq a, B_t \leq x) \\ &= N\left(\frac{x}{\sqrt{t}}\right) - N\left(\frac{x-2a}{\sqrt{t}}\right). \end{aligned}$$

Using this result, the joint density function f_{M_t,B_t} of M_t and B_t, with respect to the probability measure P, can be obtained. Indeed, by the definition of a density function it is known that

$$P(M_t < a, B_t \leq x) = \int_{-\infty}^{x}\int_{-\infty}^{a} f_{M_t,B_t}(b,y)\,db\,dy$$

for some function f_{M_t,B_t} on R^2. First of all, whenever it is not true that $x \leq a$, then the density function f_{M_t,B_t} is equal to zero. Moreover, for $x \leq a$, by the fundamental theorem of calculus,

$$\begin{aligned} \int_{-\infty}^{a} f_{M_t,B_t}(b,x)\,db &= \frac{\partial}{\partial x} P(M_t < a, B_t \leq x) \\ &= \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{\sqrt{t}}\phi\left(\frac{x-2a}{\sqrt{t}}\right) \end{aligned} \tag{A.1}$$

and

$$f_{M_t,B_t}(b,x) = \frac{\partial^2}{\partial a \partial x} P(M_t < a, B_t \le x)$$
$$= -2\frac{x-2a}{t\sqrt{t}}\phi\left(\frac{x-2a}{\sqrt{t}}\right). \tag{A.2}$$

The joint distribution and density functions of Brownian motion with drift and its maximum is obtained by means of the Girsanov theorem. Suppose that W is a Brownian motion with drift μ, i.e. there exists a standard Brownian motion B, such that

$$W_t = B_t + \mu t$$

for $t \ge 0$, and suppose that M^W is the process defined by

$$M_t^W = \max\{W_s \mid s \in [0,t]\} = \max\{B_s + \mu s \mid s \in [0,t]\}$$

for $t \ge 0$.

Proposition A8.

Fix $a \ge 0$. For $t > 0$, the joint cumulative distribution function of W_t and M_t^W is given by

$$F_{M_t^W,W_t}(x,a) = N\left(\frac{x-\mu t}{\sqrt{t}}\right) - e^{2a\mu} N\left(\frac{x-2a-\mu t}{\sqrt{t}}\right),$$

whenever $x \le a$, and the joint density function of W_t and M_t^W with respect to P is given by

$$f_{M_t^W,W_t}(x,a) = -2e^{2a\mu}\frac{x-2a}{t\sqrt{t}}\phi\left(\frac{x-2a-\mu t}{\sqrt{t}}\right),$$

whenever $x \le a$, and zero otherwise.

Proof:

The process W is a standard Brownian motion up to time t under the probability measure Q with Radon-Nikodym density:

$$\frac{dQ}{dP} = \exp\left\{-\mu W_t + \frac{1}{2}\mu^2 t\right\}.$$

Also, for $x \le a$,

$$\begin{aligned} F_{M_t^W, W_t}(a, x) &= P\left(M_t^W < a, W_t \le x\right) \\ &= E_P\left(1_{\{M_t^W < a, W_t \le x\}}\right) \\ &= E_Q\left(\frac{1}{\partial Q / \partial P} 1_{\{M_t^W < a, W_t \le x\}}\right) \\ &= E_Q\left(e^{-\mu W_t - \frac{1}{2}\mu^2 t} 1_{\{M_t^W < a, W_t \le x\}}\right). \end{aligned}$$

As the joint distribution of W_t and M_t^W under Q is the same as B_t and its maximum M_t under P, the probability density function f_{M_t, B_t} of (A.1) and (A.2) may be used to obtain

$$\begin{aligned} F_{M_t^W, W_t}(a, x) &= \int_{-\infty}^{x} \int_{-\infty}^{a} e^{\mu W_t - \frac{1}{2}\mu^2 t} f_{M_t, B_t}(b, y)\, db\, dy \\ &= \int_{-\infty}^{x} e^{\mu y - \frac{1}{2}\mu^2 t} \int_{-\infty}^{a} f_{M_t, B_t}(b, y)\, db\, dy \\ &= \int_{-\infty}^{x} e^{\mu y - \frac{1}{2}\mu^2 t} \left[\frac{1}{\sqrt{t}} \phi\left(\frac{y}{\sqrt{t}}\right) - \frac{1}{\sqrt{t}} \phi\left(\frac{y - 2a}{\sqrt{t}}\right)\right] dy. \end{aligned}$$

For the first term in the integrant

$$\frac{1}{\sqrt{t}}e^{\mu y-\frac{1}{2}\mu^2 t}\phi\left(\frac{y}{\sqrt{t}}\right)=\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{1}{2}\left[\frac{y}{\sqrt{t}}\right]^2+\mu y-\frac{1}{2}\mu^2 t\right\}$$

$$=\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{1}{2}\frac{\left(y-\mu t\right)^2}{t}\right\}$$

$$=\frac{1}{\sqrt{t}}\phi\left(\frac{y-\mu t}{\sqrt{t}}\right),$$

and for the second

$$\frac{1}{\sqrt{t}}e^{\mu y-\frac{1}{2}\mu^2 t}\phi\left(\frac{y-2a}{\sqrt{t}}\right)=\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{1}{2}\frac{\left(y-2a\right)^2}{t}+\mu y-\frac{1}{2}\mu^2 t\right\}$$

$$=\frac{e^{2a\mu}}{\sqrt{2\pi t}}\exp\left\{-\frac{1}{2}\frac{\left(y-\mu t\right)^2}{t}+\mu\left(y-2a\right)-\frac{1}{2}\mu^2 t\right\}$$

$$=\frac{e^{2a\mu}}{\sqrt{2\pi t}}\exp\left\{-\frac{1}{2}\frac{\left(y-2a-\mu t\right)^2}{t}\right\}$$

$$=\frac{1}{\sqrt{t}}\phi\left(\frac{y-2a-\mu t}{\sqrt{t}}\right).$$

Consequently,

$$F_{M_t^W,W_t}\left(a,x\right)=\int_{-\infty}^{x}\left[\frac{1}{\sqrt{t}}\phi\left(\frac{y-\mu t}{\sqrt{t}}\right)-\frac{e^{2a\mu}}{\sqrt{t}}\phi\left(\frac{y-2a-\mu t}{\sqrt{t}}\right)\right]dy$$

$$=N\left(\frac{x-\mu t}{\sqrt{t}}\right)-2e^{2a\mu}N\left(\frac{y-2a-\mu t}{\sqrt{t}}\right).$$

Let $f_{M_t^W,W_t}$ now be the joint density function of W_t and M_t^W, so that

$$F_{M_t^W,W_t}\left(x,a\right)=\int_{-\infty}^{x}\int_{-\infty}^{a}f_{M_t^W,W_t}\left(b,y\right)dbdy.$$

Whenever $x\le a$, the fundamental theorem of calculus gives

$$\int_{-\infty}^{a} f_{M_t,B_t}(b,x)\,db = \frac{\partial}{\partial x} F_{M_t^W,W_t}(x,a)$$
$$= \frac{1}{\sqrt{t}}\phi\left(\frac{x-\mu t}{\sqrt{t}}\right) - \frac{e^{2a\mu}}{\sqrt{t}}\phi\left(\frac{x-2a-\mu t}{\sqrt{t}}\right)$$

and

$$f_{M_t,B_t}(b,x) = \frac{\partial^2}{\partial a \partial x} F_{M_t^W,W_t}(x,a)$$
$$= -2e^{2a\mu}\frac{x-2a}{t\sqrt{t}}\phi\left(\frac{x-2a-\mu t}{\sqrt{t}}\right). \qquad (A.3)$$

■

Knock-out barrier options

Here the general equation used to value knock-out barrier options is derived using the joint distribution of Brownian motion and its maximum. Consider the usual Black-Scholes model. Under the risk-neutral measure Q

$$S_t = S_0 e^{r - \frac{\sigma^2}{2} t + \sigma B_t}$$

for $t \in [0,T]$, where B is a Brownian motion under Q, and the interest rate and volatility is given by $r > -1$ and $\sigma > 0$ respectively. Set

$$\mu = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)$$

and

$$W_t = B_t + \mu t$$

for $t \in [0,T]$, then W is a Brownian motion with drift μ under Q, and

$$S_t = S_0 e^{\sigma W_t}$$

for $t \in [0, T]$.

Firstly, consider up-and-out options. The stock price does not cross a fixed barrier $B > S_0$ in the time period $[0, T]$ if, and only if,

$$\max\{S_t \mid t \in [0, T]\} < B,$$

equivalently

$$\max\{S_0 e^{\sigma W_t} \mid t \in [0, T]\} < B,$$

equivalently

$$\max\{W_t \mid t \in [0, T]\} < \frac{1}{\sigma} \ln \frac{B}{S_0}.$$

Consequently, Proposition 8 on the distribution of the maximum of Brownian motion with drift may be used to determine the prices of up-and-out barrier options.

Proposition 9.

The risk-neutral price at time 0 of an option paying $f(S_T)$, if S does not cross the barrier B, and zero if it does, is given by

$$\begin{aligned}
&e^{-rT} E_Q\left(1_{\{\max\{S_t \mid t \in [0,T]\} < B\}} f(S_T)\right) \\
&= e^{-rT} \int_{-\infty}^{b} f\left(S_0 e^{\sigma y}\right) \frac{1}{\sqrt{T}} \phi\left(\frac{y - \mu T}{\sqrt{T}}\right) dy \\
&- e^{-rT} \left(\frac{B}{S_0}\right)^{2\frac{\mu}{\sigma}} \int_{-\infty}^{b} f\left(S_0 e^{\sigma y}\right) \frac{1}{\sqrt{T}} \phi\left(\frac{y - 2b - \mu T}{\sqrt{T}}\right) dy,
\end{aligned}$$

where

$$b = \frac{1}{\sigma} \ln\left(\frac{B}{S_0}\right).$$

Proof:

Set

$$M_T^W = \max\{W_t \mid t \in [0, T]\},$$

then

$$
\begin{aligned}
&e^{-rT}E_Q\left(1_{\{\max\{S_t|t\in[0,T]\}<B\}}f\left(S_T\right)\right)\\
&=e^{-rT}E_Q\left(1_{\{M_T^W<b\}}f\left(S_0e^{\sigma W_T}\right)\right)\\
&=e^{-rT}\int_{-\infty}^{b}f\left(S_0e^{\sigma W_T}\right)\left\{\int_{-\infty}^{b}\left[-2e^{2c\mu}\frac{x-2c}{t\sqrt{T}}\left(\frac{x-2c-\mu T}{\sqrt{T}}\right)\right]dc\right\}dy\\
&=e^{-rT}\int_{-\infty}^{b}f\left(S_0e^{\sigma W_T}\right)\left[\frac{1}{\sqrt{T}}\phi\left(\frac{y-\mu T}{\sqrt{T}}\right)-\frac{e^{2b\mu}}{\sqrt{T}}\phi\left(\frac{y-2b-\mu T}{\sqrt{T}}\right)\right]dy\\
&=e^{-rT}\int_{-\infty}^{b}\int_{-\infty}^{b}f\left(S_0e^{\sigma W_T}\right)\left[-2e^{2c\mu}\frac{x-2c}{t\sqrt{T}}\left(\frac{x-2c-\mu T}{\sqrt{T}}\right)\right]dcdy\\
&=e^{-rT}\int_{-\infty}^{b}f\left(S_0e^{\sigma y}\right)\frac{1}{\sqrt{T}}\phi\left(\frac{y-\mu T}{\sqrt{T}}\right)dy\\
&-e^{-rT}\left(\frac{B}{S_0}\right)^{2\frac{\mu}{\sigma}}\int_{-\infty}^{b}f\left(S_0e^{\sigma y}\right)\frac{1}{\sqrt{T}}\phi\left(\frac{y-2b-\mu T}{\sqrt{T}}\right)dy. \qquad \blacksquare
\end{aligned}
$$

The price of an up-and-in option may be determined by the parity from the price of the corresponding up-and-out option upon realising that

$$1_{\{\max\{S_t|t\in[0,T]\}\geq B\}}f\left(S_T\right)+1_{\{\max\{S_t|t\in[0,T]\}<B\}}f\left(S_T\right)=f\left(S_T\right).$$

The application of the general formula used to value knock-out barrier options is illustrated by valuing an up-and-out call option. The general form derived here is then shown to be equivalent to the formulas given in Table 5 in Chapter 4.

Valuation: Up-and-out call option. An up-and-out call option with strike K and barrier $B>S_0$ gives the same payoff as an ordinary call option with strike K if the barrier B is not breached, and zero if it is. In order to ensure that the payoff of this option is non-trivial, it is customary to assume that $B>K$. At time T, the option is in-the-money but below the barrier if, and only if,

$$K\leq S_T<B,$$

i.e

$$\frac{K}{S_0}\leq e^{\sigma W_T}<\frac{B}{S_0},$$

which is equivalent to

$$\frac{1}{\sigma}\ln\left(\frac{K}{S_0}\right) \le W_T < \frac{1}{\sigma}\ln\left(\frac{B}{S_0}\right).$$

Set

$$k = \frac{1}{\sigma}\ln\left(\frac{K}{S_0}\right),$$

and apply Proposition 9, then the value at time 0 of the up-and-out call option with strike K and barrier B is equal to

$$\begin{aligned}
&e^{-rT}E_Q\left(1_{\{\max\{S_t|t\in[0,T]\}<B\}}\left[S_T-K\right]^+\right)\\
&= e^{-rT}\int_k^b\left(S_0e^{\sigma y}-K\right)\frac{1}{\sqrt{T}}\phi\left(\frac{y-\mu T}{\sqrt{T}}\right)dy\\
&-e^{-rT}\left(\frac{B}{S_0}\right)^{2\frac{\mu}{\sigma}}\int_k^b\left(S_0e^{\sigma y}-K\right)\frac{1}{\sqrt{T}}\phi\left(\frac{y-2b-\mu T}{\sqrt{T}}\right)dy.
\end{aligned}$$

By expanding the brackets, the following is obtained

$$\begin{aligned}
&e^{-rT}E_Q\left(1_{\{\max\{S_t|t\in[0,T]\}<B\}}\left[S_T-K\right]^+\right)\\
&= S_0e^{-rT}\int_k^b e^{\sigma y}\frac{1}{\sqrt{T}}\phi\left(\frac{y-\mu T}{\sqrt{T}}\right)dy - Ke^{-rT}\int_k^b\frac{1}{\sqrt{T}}\phi\left(\frac{y-\mu T}{\sqrt{T}}\right)dy\\
&-S_0e^{2b\mu-rT}\int_k^b e^{\sigma y}\frac{1}{\sqrt{T}}\phi\left(\frac{y-2b-\mu T}{\sqrt{T}}\right)dy + Ke^{2b\mu-rT}\int_k^b\frac{1}{\sqrt{T}}\phi\left(\frac{y-2b-\mu T}{\sqrt{T}}\right)dy. \qquad (\text{A.4})
\end{aligned}$$

Each of the terms on the right hand side of (A.4) is equal to a constant multiple of an integral of the form

$$\frac{1}{\sqrt{2\pi T}}\int_k^b \exp\left\{\beta+\gamma y-\frac{1}{2}\frac{y^2}{T}\right\}dy, \qquad (\text{A.5})$$

for suitable choices of the constants β and γ. This integral may be solved explicitly as follows:

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi T}}\int_k^b \exp\left\{\beta+\gamma y-\frac{1}{2}\frac{y^2}{T}\right\}dy \\
&=\frac{1}{\sqrt{2\pi T}}\int_k^b \exp\left\{\beta+\frac{1}{2}\gamma^2 T-\frac{1}{2}\frac{\left(y-\gamma T\right)^2}{T}\right\}dy \\
&=e^{\beta+\frac{1}{2}\gamma^2 T}\frac{1}{\sqrt{2\pi T}}\int_k^b \exp\left\{-\frac{1}{2}\frac{\left(y-\gamma T\right)^2}{T}\right\}dy \\
&=e^{\beta+\frac{1}{2}\gamma^2 T}\left[N\left(\frac{b-\gamma T}{\sqrt{T}}\right)-N\left(\frac{k-\gamma T}{\sqrt{T}}\right)\right] \\
&=e^{\beta+\frac{1}{2}\gamma^2 T}\left[\left\{1-N\left(\frac{-b+\gamma T}{\sqrt{T}}\right)\right\}-\left\{1-N\left(\frac{-k+\gamma T}{\sqrt{T}}\right)\right\}\right] \\
&=e^{\beta+\frac{1}{2}\gamma^2 T}\left[N\left(\frac{-k+\gamma T}{\sqrt{T}}\right)-N\left(\frac{-b+\gamma T}{\sqrt{T}}\right)\right] \\
&=e^{\beta+\frac{1}{2}\gamma^2 T}\left[N\left(\frac{\ln\left(\frac{K}{S_0}\right)+\gamma\sigma T}{\sigma\sqrt{T}}\right)-N\left(\frac{\ln\left(\frac{B}{S_0}\right)+\gamma\sigma T}{\sigma\sqrt{T}}\right)\right].
\end{aligned}
$$

This result is used to compute each of the four integrals in (A.4). For $x>0$, define

$$
\begin{aligned}
d_+[x]&=\frac{\ln x+\left(r+0.5\sigma^2\right)T}{\sigma\sqrt{\mathrm{T}}}, \\
d_-[x]&=\frac{\ln x+\left(r-0.5\sigma^2\right)T}{\sigma\sqrt{\mathrm{T}}}.
\end{aligned}
$$

For the first term in (A.4), the following is obtained

$$
\begin{aligned}
&S_0 e^{-rT}\int_k^b e^{\sigma y}\frac{1}{\sqrt{T}}\phi\left(\frac{y-\mu T}{\sqrt{T}}\right)dy \\
&=\frac{S_0}{\sqrt{2\pi T}}\int_k^b \exp\left\{-rT-\frac{1}{2}\mu^2 T+\left(\mu+\sigma\right)y-\frac{1}{2}\frac{y^2}{T}\right\}dy \\
&=S_0\left\{N\left(d_+\left[\frac{S_0}{K}\right]\right)-N\left(d_+\left[\frac{S_0}{B}\right]\right)\right\}
\end{aligned}
$$

by substituting $\beta = -rT - \frac{1}{2}\mu^2 T$ and $\gamma = \mu + \sigma$ into (A.5). Similarly by substituting $\beta = -rT - \frac{1}{2}\mu^2 T$ and $\gamma = \mu$ yields for the second term

$$
\begin{aligned}
&-Ke^{-rT}\int_k^b \frac{1}{\sqrt{T}}\phi\left(\frac{y-\mu T}{\sqrt{T}}\right)dy \\
&= \frac{-K}{\sqrt{2\pi T}}\int_k^b \exp\left\{-rT - \frac{1}{2}\mu^2 T + \mu y - \frac{1}{2}\frac{y^2}{T}\right\}dy \\
&= -Ke^{-rT}\left\{N\left(d_-\left[\frac{S_0}{K}\right]\right) - N\left(d_-\left[\frac{S_0}{B}\right]\right)\right\}.
\end{aligned}
$$

For the third term in (A.4), the following is obtained

$$
\begin{aligned}
&-S_0 e^{2b\mu - rT}\int_k^b e^{\sigma y}\frac{1}{\sqrt{T}}\phi\left(\frac{y-2b-\mu T}{\sqrt{T}}\right)dy \\
&= -\frac{S_0}{\sqrt{2\pi T}}\int_k^b \exp\left\{-rT - \frac{2}{T}b^2 - \frac{1}{2}\mu^2 T + \left(\mu + \sigma + \frac{2b}{T}\right)y - \frac{1}{2}\frac{y^2}{T}\right\}dy \\
&= \left(\frac{B}{S_0}\right)^{\frac{2}{\sigma^2}r - \frac{\sigma^2}{2}}\left\{N\left(d_+\left[\frac{B^2}{KS_0}\right]\right) - N\left(d_+\left[\frac{B}{S_0}\right]\right)\right\}
\end{aligned}
$$

upon substituting $\beta = -rT - \frac{2}{T}b^2 - \frac{1}{2}\mu^2 T$ and $\gamma = \mu + \sigma + \frac{2\mathrm{b}}{\mathrm{T}}$ in (A.5). Finally, setting $\beta = -rT - \frac{2}{T}b^2 - \frac{1}{2}\mu^2 T$ and $\gamma = \mu + \sigma + \frac{2\mathrm{b}}{\mathrm{T}}$, the fourth term in (A.4) becomes

$$
\begin{aligned}
&Ke^{2b\mu - rT}\int_k^b \frac{1}{\sqrt{T}}\phi\left(\frac{y-2b-\mu T}{\sqrt{T}}\right)dy \\
&= \frac{K}{\sqrt{2\pi T}}\int_k^b \exp\left\{-rT - \frac{2}{T}b^2 - \frac{1}{2}\mu^2 T + \left(\mu + \frac{2b}{T}\right)y - \frac{1}{2}\frac{y^2}{T}\right\}dy \\
&= Ke^{-rT}\left(\frac{B}{S_0}\right)^{\frac{2}{\sigma^2}r - 1}\left\{N\left(d_-\left[\frac{B^2}{KS_0}\right]\right) - N\left(d_-\left[\frac{B}{S_0}\right]\right)\right\}.
\end{aligned}
$$

In summary, it has been shown that the price of an up-and-out call option with strike K and barrier $B > S_0$ is equal to

$$
\begin{aligned}
& e^{-rT} E_Q\left(1_{\{\max\{S_t | t\in[0,T]\}<B\}}\left[S_T - K\right]^+\right) \\
& = S_0\left\{N\left(d_+\left[\frac{S_0}{K}\right]\right) - N\left(d_+\left[\frac{S_0}{B}\right]\right)\right\} \\
& +B\left(\frac{B}{S_0}\right)^{\frac{2}{\sigma^2}r}\left\{N\left(d_+\left[\frac{B^2}{KS_0}\right]\right) - N\left(d_+\left[\frac{B}{S_0}\right]\right)\right\} \\
& -Ke^{-rT}\left\{N\left(d_-\left[\frac{S_0}{K}\right]\right) - N\left(d_-\left[\frac{S_0}{B}\right]\right)\right\} \\
& -Ke^{-rT}\left(\frac{B}{S_0}\right)^{\frac{2}{\sigma^2}r-1}\left\{N\left(d_-\left[\frac{B^2}{KS_0}\right]\right) - N\left(d_-\left[\frac{B}{S_0}\right]\right)\right\}. \qquad (A.6)
\end{aligned}
$$

■

Equation (A.6) can be shown to lead to the formulas given for the valuation of an up-and-out call option in Table 5 and Table 7 of Chapter 4. Note that in chapter 4 a barrier is denoted by H, instead of the notation B used in this appendix. In (A.6) above the rebate term is not considered, or alternatively, R = 0. Also q= 0 and t = *0*.

First consider the value of the up-an-out call in the case where K > B. When R = 0, the terms UC5 = UC6 = 0. Therefore, the value of the up-and-out call when K > B is zero. Now consider the case where K < B. Using the fact that from the symmetry of the normal distribution $N(-x) = 1 - N(x)$ (A.6) can be written as

$$
\begin{aligned}
& e^{-rT} E_Q\left(1_{\{\max\{S_t|t\in[0,T]\}<B\}}\left[S_T-K\right]^+\right) \\
& = S_0 N\left(d_+\left[\frac{S_0}{K}\right]\right) - Ke^{-rT} N\left(d_-\left[\frac{S_0}{K}\right]\right) \\
& -S_0 N\left(d_+\left[\frac{S_0}{B}\right]\right) + Ke^{-rT} N\left(d_-\left[\frac{S_0}{B}\right]\right) \\
& -B\left(\frac{B}{S_0}\right)^{\frac{2}{\sigma^2}r} N\left(-d_+\left[\frac{B}{S_0}\right]\right) + Ke^{-rT}\left(\frac{B}{S_0}\right)^{\frac{2}{\sigma^2}r-1} N\left(-d_-\left[\frac{B}{S_0}\right]\right) \\
& +B\left(\frac{B}{S_0}\right)^{\frac{2}{\sigma^2}r} N\left(-d_+\left[\frac{B^2}{KS_0}\right]\right) - Ke^{-rT}\left(\frac{B}{S_0}\right)^{\frac{2}{\sigma^2}r-1} N\left(-d_-\left[\frac{B^2}{KS_0}\right]\right).
\end{aligned}
$$

This can be seen to be equal to UC1- UC2 - UC3 + UC4 with q = 0 and t = 0 and is given by

$$
\begin{aligned}
UOC_{\{K<H\}} = {} & SN(w_1) - Ke^{-rT} N\left(w_1 - \sigma\sqrt{\mathrm{T}}\right) \\
& -SN(w_2) + Ke^{-rT} N\left(w_2 - \sigma\sqrt{\mathrm{T}}\right) \\
& -S\left(\frac{H}{S}\right)^{2\lambda} N(-w_4) + Ke^{-rT}\left(\frac{H}{S}\right)^{2\lambda-2} N\left(-w_4 + \sigma\sqrt{\mathrm{T}}\right) \\
& +S\left(\frac{H}{S}\right)^{2\lambda} N(-w_3) - Ke^{-rT}\left(\frac{H}{S}\right)^{2\lambda-2} N\left(-w_3 + \sigma\sqrt{\mathrm{T}}\right)
\end{aligned}
$$

since

$$
\begin{aligned}
& d_+\left[\frac{S_0}{K}\right] = w_1,\ d_-\left[\frac{S_0}{K}\right] = w_1 - \sigma\sqrt{T} \\
& d_+\left[\frac{S_0}{B}\right] = w_2,\ d_-\left[\frac{S_0}{B}\right] = w_2 - \sigma\sqrt{T} \\
& d_+\left[\frac{B^2}{KS_0}\right] = w_3,\ d_-\left[\frac{B^2}{KS_0}\right] = w_3 - \sigma\sqrt{T} \\
& d_+\left[\frac{B}{S_0}\right] = w_4,\ d_-\left[\frac{B}{S_0}\right] = w_4 - \sigma\sqrt{T}
\end{aligned}
\tag{A.7}
$$

and

$$B\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}} = S\left(\frac{B}{S}\right)^{\frac{2r}{\sigma^2}+1} = S\left(\frac{B}{S}\right)^{2\lambda} = S\left(\frac{H}{S}\right)^{2\lambda} \tag{A.8}$$

$$\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}-1} = \left(\frac{H}{S}\right)^{2\lambda-2}. \tag{A.9}$$

This value for $UOC_{\{K<H\}}$ is exactly equal to the corresponding value in Table 7 given the expressions in Table 5.

Using the result obtained in proposition 8 for the joint distribution of Brownian motion and its maximum, the joint distribution of the Brownian motion and its minimum is derived following the derivation given in Etheridge (2004). This joint distribution is then used to derive a general formula for a knock-in barrier option. Its application is shown with a down-and-in call option. Then the correspondence between the formula given in Table 7 and the mathematically derived formula for down-and-in call option is shown.

By symmetry where m is the minimum then for α < 0 and $x \geq a$,

$$P\left[m_t \leq a, B_t \geq x\right] = 1 - N\left(\frac{-2a+x}{\sqrt{t}}\right) = N\left(\frac{2a-x}{\sqrt{t}}\right)$$

or differentiating, if α < 0 and $x \geq a$

$$P\left[m_T \leq a, B_T \in dx\right] = p_T\left(0, -2a+x\right)dx = p_T\left(2a, x\right)dx$$

where

$$p_t\left(x, y\right) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\left|x-y\right|^2 / 2t\right).$$

Similarly to the derivation of the distribution of Brownian motion and its maximum, it is first noted that it is known that for any α < 0 and $x \geq a$

$$P\left(-B_t \le x\right) = P\left(-m_t \le a, -B_t \le x\right) + P\left(-m_t > a, -B_t \le x\right).$$

It then follows from the fact that the value of the cumulative distribution function of m_t and B_t at $\left(a, x\right)$ is given by

$$\begin{aligned}
P\left(-m_t < a, -B_t \le x\right) &= P\left(-B_t \le x\right) - P\left(-m_t \ge a, -B_t \le x\right) \\
&= P\left(B_t > -x\right) - P\left(m_t \le a, B_t \ge x\right) \\
&= 1 - P\left(B_t \le -x\right) - P\left(m_t \le a, B_t \ge x\right) \\
&= 1 - N\left(\frac{-x}{\sqrt{t}}\right) - N\left(\frac{2a - x}{\sqrt{t}}\right) \\
&= N\left(\frac{x}{\sqrt{t}}\right) - N\left(\frac{2a - x}{\sqrt{t}}\right).
\end{aligned}$$

Using this result, the joint density function $f_{-m_t, -B_t}$ of $-m_t$ and $-B_t$, with respect to the probability measure P, can be obtained. Define

$$P\left(-m_t < a, -B_t \le x\right) = \int_{-\infty}^{x} \int_{-\infty}^{a} f_{-m_t, -B_t}\left(b, y\right) db dy = P\left(m_t > a, B_t \ge x\right)$$

for some function $f_{-m_t, -B_t}$ on R^2. First of all, whenever it is not true that $x \ge a$, then the density function $f_{-m_t, -B_t}$ is equal to zero. Moreover, for $x \ge a$, by the fundamental theorem of calculus,

$$\begin{aligned}
\int_{-\infty}^{a} f_{-m_t, -B_t}\left(b, x\right) db &= \frac{\partial}{\partial x} P\left(-m_t < a, -B_t \le x\right) \\
&= \frac{1}{\sqrt{t}} \phi\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{\sqrt{t}} \phi\left(\frac{2a - x}{\sqrt{t}}\right)
\end{aligned} \tag{A.10}$$

and

$$\begin{aligned}
f_{-m_t, -B_t}\left(b, x\right) &= \frac{\partial^2}{\partial a \partial x} P\left(-m_t < a, -B_t \le x\right) \\
&= 2 \frac{2a - x}{t\sqrt{t}} \phi\left(\frac{2a - x}{\sqrt{t}}\right).
\end{aligned} \tag{A.11}$$

Combining these results with two applications of the Girsanov Theorem will allow the calculation of the joint distribution and density functions of Brownian motion with drift and its minimum. Once again suppose that W is a Brownian motion with drift μ, i.e there exists a standard Brownian motion B such that

$$W_t = B_t + \mu t$$

for $t \geq 0$, and suppose that m^W is the process defined by

$$m_t^W = \min\left\{W_s \mid s \in [0,t]\right\} = \min\left\{B_s + \mu s \mid s \in [0,t]\right\}$$

for $t \geq 0$.

Proposition A11. (Etheridge, 2004)
Fix $a < 0$. For $t > 0$, the joint cumulative distribution function of the joint cumulative distribution function of W_t and m_t^W with respect to Q is given by

$$Q\left[m_T^W \leq a, W_T \in dx\right] = \begin{cases} p_T\left(\mu T, x\right) dx & \text{if } x < a, \\ e^{2a\mu} p_T\left(2a + \mu T, x\right) dx & \text{if } x \geq a, \end{cases}$$

where, as above $p_t\left(x, y\right)$ is the Brownian transition density function.

Proof:
By the Girsanov Theorem, there is a measure P, equivalent to Q, under which $\left\{W_t\right\}_{t \geq 0}$ is a P Brownian motion and

$$\frac{dP}{dQ}\Big|_{F_r} = \exp\left(-\mu B_t - \frac{1}{2}\mu^2 t\right).$$

Notice that this depends on $\{B_t\}_{0\le t\le T}$ only through B_t. The Q probability of the event $\{m_t^W \le a\}$ will be the P-probability of that event multiplied by $\frac{dP}{dQ}\Big|_{F_r}$ evaluated at $W_t = x$. Now

$$\frac{dQ}{dP} = \exp\left\{\mu \mathrm{B}_t + \frac{1}{2}\mu^2 t\right\} = \exp\left\{\mu W_t - \frac{1}{2}\mu^2 t\right\}.$$

As the joint distribution of W_t and m_t^W under Q is the same as B_t and its minimum m_t under P, the probability density function f_{m_t,B_t}. Thus for $a<0$ and $x \ge a$

$$\begin{aligned} Q\left[m_T^W \le a, W_T \in dx\right] &= P\left[m_T^W \le a, W_T \in dx\right]\exp\left(\mu x - \frac{1}{2}\mu^2 t\right) \\ &= p_T\left(2a, x\right)\exp\left(\mu x - \frac{1}{2}\mu^2 t\right)dx \\ &= e^{2a\mu} p_T\left(2a + bT, x\right)dx. \end{aligned} \tag{A.12}$$

Evidently for $x \le a$, $\left\{m_T^W \le a, W_T \in dx\right\} = \left\{W_T \in dx\right\}$ and so for $x \le a$

$$\begin{aligned} Q\left[m_T^W \le a, W_T \in dx\right] &= Q\left[W_T \in dx\right] \\ &= Q\left[\mu T + B_T \in dx\right] \\ &= p_T\left(\mu T, x\right)dx. \end{aligned} \tag{A.13}$$

■

Differentiating (A.13) with respect to α, it is seen that in terms of joint densities, for α < 0

$$Q\left[m_T^W \in da, W_T \in dx\right] = \frac{2e^{2a\mu}}{T}\left|x - 2a\right| p_T\left(2a + \mu T, x\right)dxda \quad \text{if } x \ge a.$$

The joint density vanishes if $x < a$ or α > 0. Therefore, the joint distribution function of W_t and m_t^W with respect to P is given by

$$f_{m_t^W, W_t}(x,a) = 2e^{2a\mu}\frac{2a-x}{t\sqrt{t}}\phi\left(\frac{2a+\mu t-x}{\sqrt{t}}\right), \tag{A.14}$$

whenever $x \geq a$, and zero, otherwise. Note how (A.14) corresponds to (A.3), which gives the joint distribution function of W_t and its maximum.

Knock-in barrier options

The same model as used to price knock-out barrier options is used when down-and-in options are considered. The stock price crosses a fixed barrier $B < S_0$ in the time period $[0,T]$, if, and only if,

$$\min\left\{S_t \mid t \in [0,T]\right\} < B,$$

equivalently

$$\min\left\{S_0 e^{\sigma W_t} \mid t \in [0,T]\right\} < B,$$

equivalently

$$\min\left\{W_t \mid t \in [0,T]\right\} < \frac{1}{\sigma}\ln\frac{B}{S_0}.$$

Consequently Proposition 11 on the distribution of the minimum of Brownian motion with drift may be used to determine the prices of down-and-in barrier options. Under the martingale measure Q, $S_t = S_0\exp(\sigma W_t)$ where $W_t = \frac{\left(r-\frac{1}{2}\sigma^2\right)}{\sigma}t + B_t$ and B is a Q Brownian motion. By applying these results with $\mu = \frac{\left(r-\frac{1}{2}\sigma^2\right)}{\sigma}$, the value of any option maturing at time T whose payoff depends on the stock price at time T and its minimum value over the lifetime of the contract can be valued. If the payoff is $C_T = g\left(\min\left\{S_t \middle| t \in [0,T]\right\}, S_T\right)$ and is the riskless borrowing rate, then the value of the option at time zero is

$$V(0,S_0) = e^{-rT} E_Q\left[g\left(\min\left\{S_t \middle| t \in [0,T]\right\}, S_T\right)\right]$$
$$= e^{-rT} \int_{a=-\infty}^{0} \int_{x=a}^{\infty} g\left(S_0 e^{\sigma x}, S_0 e^{\sigma\alpha}\right) Q\left[m_T^W \in da, W_T \in dx\right].$$

The price of a down-and-out option may be determined by the parity from the price of the corresponding down-and-in option upon realising that

$$1_{\{\min\{S_t | t\in[0,T]\}\geq B\}} f(S_T) + 1_{\{\min\{S_t | t\in[0,T]\}<B\}} f(S_T) = f(S_T).$$

Its application of the general procedure for valuing knock-in barrier options is shown with a down-and-in call option. The correspondence between this mathematically derived down-and-in call option and the formula given in Table 7 is shown below.

Valuation: down-and-in call option. A down-and-in call option with strike K and barrier $B < S_0$ gives the same payoff as an ordinary call option with strike K if the barrier B is breached, and zero if it is not. In order to ensure that the payoff of this option is non-trivial, it is customary to assume that $B < K$. The aim is to find the time zero price of a down-and-in call option of which the payoff at time T is

$$C_T = 1_{\{\min\{S_t | t\in[0,T]\}<B\}} \left[S_T - K\right]^+.$$

At time T, the option is in-the-money, but above the barrier, if, and only if,

$$K \leq B < S_T,$$

i.e

$$\frac{K}{S_0} \leq \frac{B}{S_0} < e^{\sigma W_T},$$

which is equivalent to

$$\frac{1}{\sigma}\ln\left(\frac{K}{S_0}\right) \leq \frac{1}{\sigma}\ln\left(\frac{B}{S_0}\right) < W_T.$$

So by using $S_t = S_0 e^{\sigma W_t}$ the payoff is rewritten as

$$C_T = 1_{\left\{m_T^W \le \frac{1}{\sigma}\log\left(\frac{B}{S_0}\right)\right\}} \left[S_0 e^{\sigma W_t} - K\right]^+ .$$

Set $k = \frac{1}{\sigma}\ln\left(\frac{K}{S_0}\right)$, $\mu = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$ and $d = \frac{1}{\sigma}\ln\left(\frac{B}{S_0}\right)$, then the value at time 0 of the down-and-in call option with strike K and barrier B is equal to

$$V\left(0, S_0\right) = e^{-rT}\int_k^\infty \left(S_0 e^{\sigma x} - K\right) Q\left(m_T^W \le a, W_t \in dx\right).$$

Using the expression for the joint distribution of m_T^W and W_t, obtained previously, yields

$$V\left(0, S_0\right) = e^{-rT}\int_k^\infty \left(S_0 e^{\sigma x} - K\right) e^{2a\mu} p_T\left(2a + \mu T, x\right) dx.$$

The fact that since $B < K$, $k \ge a$ was used. First observe that

$$\begin{aligned}
e^{-rT}\int_k^\infty K e^{2a\mu} p_T\left(2a + \mu T, x\right) dx &= K e^{-rT} e^{2a\mu} \int_{(k-2a-\mu T)/\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= K e^{-rT} e^{2a\mu} \int_{-\infty}^{(2a+\mu T-k)/\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= K e^{-rT}\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}-1} \phi\left(\frac{2a + \mu T - k}{\sqrt{T}}\right) \\
&= K e^{-rT}\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}-1} \phi\left(\frac{\log\left(\frac{F}{K}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

where $F = e^{rT}\frac{B^2}{S_0}$.

Similarly,

$$e^{-rT}\int_k^\infty S_0 e^{2ax}e^{2a\mu}p_T\left(2a+\mu T,x\right)dx = S_0e^{-rT}e^{2a\mu}\int_k^\infty \frac{1}{\sqrt{2\pi T}}\exp\left(\frac{\left(x-\left(2a+\mu T\right)\right)^2-2\sigma xT}{2T}\right)dx$$

$$= S_0e^{-rT}e^{2a\mu}\int_{\left(x-\left(2a+\mu T\right)-\sigma T\right)/\sqrt{T}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy\times\exp\left(\frac{1}{2}\sigma^2T+2a\sigma+b\sigma\mathrm{T}\right)$$

$$= e^{-rT}\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}-1}F\phi\left(\frac{\log\left(\frac{F}{K}\right)+\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right).$$

Hence, the value of a down-and-in call option in the case where K > B is given by

$$V\left(0,S_0\right)=e^{-rT}\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}-1}F\phi\left(\frac{\log\left(\frac{F}{K}\right)+\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right)-Ke^{-rT}\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}-1}\phi\left(\frac{\log\left(\frac{F}{K}\right)-\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\right).$$

(A.15)

■

It can now be shown that the expression given in (A.15) is equal to the formulas given in Table 3 and Table 7 of chapter 5 for the valuation of a down-and-in call for the case where K > B. Table 7 gives the value of the down-and-in call as DC4+DC5 where DC 5 corresponds to the rebate term. Therefore, it has to be shown that (A.15) is equal to DC4 with $t = 0$, $q = 0$ and H = B i.e.:

$$S\left(\frac{H}{S}\right)^{2\lambda}N\left(w_3\right)-Ke^{-rT}\left(\frac{H}{S}\right)^{2\lambda-2}N\left(w_3-\sigma\sqrt{\mathrm{T}}\right).$$

Rewrite (A.15) and substitute in $F=e^{rT}\frac{B^2}{S_0}$:

$$V(0,S_0) = B\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}} \phi\left(\frac{\log\left(\frac{B^2}{S_0 K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}-1} \phi\left(\frac{\log\left(\frac{B^2}{S_0 K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

$$= B\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}} N\left(d_+\left[\frac{B^2}{S_0 K}\right]\right) - Ke^{-rT}\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}-1} N\left(d_-\left[\frac{B^2}{S_0 K}\right]\right).$$

The equivalence follows after equations (A.7), (A.8) and (A.9) are applied.

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