AUTOMORPHISMS OF CURVES AND THE LIFTING CONJECTURE

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DECLARATION

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and has not previously in its entirety or in part been submitted at any university for a degree.

Signature:____ Date:__________________

ABSTRACT

It is an open question whether or not one can always lift Galois extensions of smooth algebraic curves in characteristic p to Galois extensions of smooth relative curves in characteristic 0. In this thesis we study some of the available techniques and partial solutions to this problem.

Our studies include the techniques of Oort, Sekiguchi and Suwa where the lifting problem is approached via a connection with lifting group schemes. We then move to the topic of singular liftings and for this we study the approach of Garuti. Thereafter, we move to the wild *smooth* setting again where we study the crucial $local - global$ principle, and apply it by illustrating how Green and Matignon solved the p^2-lifting problem.

OPSOMMING

Dit is 'n oop vraag of ons altyd Galois uitbreiding van gladde krommes in karakteristiek p na Galois uitbreidings in karakteristiek 0 kan lig. Ons bestudeer hierdie probleem ten opsigte van moderne metodes en kyk na wat bekend is.

Ons tegnieke sluit in die baanbrekers werk van Oort, Sekiguchi en Suwa in 1989 waaruit hulle die probleem via groep teoretiese meetkunde benader het. Ons kyk ook na die tegnieke van Garuti en ook ander soos Green en Matignon. Die hoogtepunt van die werk is die oplossing van die probleem vir p^2-sikliese groepe as ook die uiters belangrike *lokaal-globaal* beginsel.

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Contents

Conventions and Notations

Throughout this work we shall assume that k is an algebraically closed field. We shall use R to denote a complete discrete valuation ring dominating the ring of Witt vectors $W(k)$ of the field k which also has k as residue field. In these cases we shall implicitly assume that k is of characteristic p. Unless otherwise stated, S will denote the scheme $spec(R)$. We shall always let K denote the field of fractions of R and we shall let $| \, |_K$ denote the valuation norm K. When the context is clear we might drop the subscript and simply refer to this as $| \, |$.

Unless otherwise stated C and D will always refer to algebraic curves. However, when we refer to (flat) arithmetic surfaces over R and curves over its residue field k, we shall speak explicitly of C_k or C_K as the fibres and C is then meant to be an arithmetic surface. Again in this case g_C will be meant to be the arithmetic genus on the fibres and we note that the flatness implies that this is well defined. In general we shall let X and Y denote arbitrary schemes and we shall let L and M be field extensions of K. Given a proper fibred surface $C \to S$, we shall let $Pic_{C/S}$ denote the associated Picard Scheme of C and similarly for a curve C/k. We shall also refer to J_C as the Jacobian of a curve C/k.

Given a *Galois* extension of smooth curves $f : C \to D$ over k, given a point $Q \in D$ we shall always let e_Q denote the ramification index of (any) point $P \in f^{-1}(Q)$. We shall also let $ram(C/D)$ denote the set of points $Q \in D$ such that there is ramification above Q in the extension C/D.

When L/K is a finite extension of fields, we shall denote the ring of integers relative to $R = \mathcal{O}_K$ in the field L by $\mathcal{O}_L$. We shall denote by $| \, |_L$ the (extended) norm on the finite extension L/K induced by the norm on K - see [14] p.139 3.2.3 for details.

Given a group G and a ring A, we shall let $A[G]$ denote the group ring. For a scheme X we shall denote by X_E the E site of the scheme X, here E can be the Zariski, étale or flat sites. We shall use the notation $H^i_E(X, G)$ for the cohomology on the E site of the scheme X with values in the E abelian sheaf G. For example $H^i_{Zar}(X, G)$ will mean the usual Zariski cohomology of the scheme X with (usual Zariski) abelian sheaf G on X. When the context is clear we suppress the subscript E.

Furthermore, the groups $\mathbf{Z}/\mathbf{lZ}$ refer to either the cyclic l-group or when the context is clear the associated *étale* group scheme over X. In the latter we might also write $\mathbf{Z/lZ_X}$. Here l is only assumed to be a positive integer and unless otherwise, we shall not make any assumptions on l with respect to characteristics etc.[1]

[1]We should point out at this stage that $\mathbf{Z/pZ}$ is an étale group scheme over a scheme of characteristic p : indeed it is the kernel of the étale map of group schemes $W_{1,\mathbf{F_p}} \stackrel{F-1}{\longrightarrow} W_{1,\mathbf{F_p}}$. Here F is the Frobenius map and $F-1$ is the map $x \mapsto x^p - x$. We note this map is étale since one can check this over the fibres via the Jacobian determinant condition. See also [19] p.67 Exercise 2.19.

Introduction

Let R be a discrete valuation ring with residue field k and assume $X_k \to k$ is some k-scheme with certain properties. It is natural to ask for an extension $X \to R$ which reduces to the given scheme X_k upon specialization. In particular, when $X_k \to k$ is smooth one would be interested in looking for smooth extensions $X \to R$. We say that we can lift the scheme $X_k \to k$ to R if such a scheme $X \to R$ exists.

Alexander Grothendieck has shown that using infinitesimals links the lifting problem to a study of infinitesimals. Using the language of torsors and invoking the powerful algebraization theorem, one can interpret the problem cohomologically, and even more so, certain cohomological groups can even *control* the way liftings are generated.

Perhaps the most striking illustration of Grothendieck's approach is that the cohomology groups involved are those arising from the original given scheme $X_k \to k$. An example of the power of this fact is the case of specializing to when X_k is a smooth curve. In this the higher cohomology of curves vanish and using Grothendieck's theory one can deduce very elegantly

Theorem 1 *Let $X_k \to k$ be a smooth projective k-curve. Then there exists a smooth R-curve $X \to R$ reducing to X_k*

Grothendieck's theory also concerns lifting morphisms and in particular their properties, e.g. given two R-schemes X and Y and a morphism on the special fibre $f_k : Y_k \to X_k$ with property P, can we find a morphism $f : X \to Y$ reducing to f_k which also displays property P.

Most satisfyingly, for projective schemes Grothendieck's theory shows that the property of being étale (or étale Galois) is as above and liftings of this are even unique. In particular one obtains the following useful

Theorem 2 *Let $Y_k \to X_k$ be an étale G-Galois extension of projective k-schemes for some finite group G. Assume X is a projective R-scheme reducing to X_k. Then there exists an étale G-Galois extension $Y \to X$ reducing to $Y_k \to X_k$.*

It is now natural to ask if one can do away with the étale assumption. Simply lifting the scheme Y_k to R without the *étale* assumption is difficult and in view of what we already know one restricts to the case of lifting smooth Galois extensions of curves. Unfortunately ramified coverings in the special fibre need not lift uniquely, and this complicates the problem enormously since there is no obvious lifting candidate. However in a certain case we have some feasible control over the matter

Theorem 3 *Let $X \to R$ be a smooth projective R-curve and assume $Y_k \to X_k$ is a* tame *Galois extension of k-curves. Then there exists a Galois extension of smooth R-curves $Y \to X$ reducing to the given Galois extension in the special fibre.*

The step that remains is that of wild coverings. In particular, lifting wild coverings from characteristic p to characteristic 0. One knows that there may arise obstructions to lifting these extensions and indeed: we know that there are Galois extensions in characteristic p that *cannot* be lifted to characteristic 0.

A closer look at some of these obstructions suggests that one first restrict to cyclic extensions. Here we know that we can always lift p-cyclic Galois extensions to characteristic 0. Later, the list of p^2-cyclic extensions was also added. Using similar ideas, one now has a complete understanding of the liftability of Galois groups of order p^2, even the elementary abelian ones.

However, the higher order cases still elude us. Relaxing the condition of finding smooth liftings, one knows that this problem is solvable, for *any* Galois group G. One can even control the type of lifting homeomorphically. However the method must allow for possible *cusp* singularities.

In a different direction, we know that there is a connection between the theory of coverings of curves and extensions of certain group schemes, namely the *singular Jacobians* associated to curves. Thus instead of formulating the covering-lifting problem, one can ask if we can lift group extensions from characteristic p to characteristic 0.

In yet another direction, we know that the lifting problem is intrinsically tied to constructing certain automorphisms of the rigid disc. Conversely, one can ask for properties of disc automorphisms after reduction. Indeed, this is the backbone of the powerful *local* $-$ *global* lifting principle.

This thesis is concerned with the topics discussed above. In chapter one we shall present certain structural results on the automorphism groups of algebraic curves. The aim is to draw attention to a lifting obstruction related to automorphism group sizes. Our exposition is based on those of Sticthenoth ([11] and [12]) and Nakajima ([5] and [6]). One also finds in this chapter the motivation for first restricting to abelian extensions and needing to possibly extend the ring R.

In chapter two we study the infinitesimal issues we mentioned earlier. We also state the algebraization theorem, which is absolutely crucial in our studies. Indeed, in each of the chapters on lifting one will find that somewhere algebraization has been used.

In chapter three we start out by showing the relation between group extensions and coverings of curves. Our exposition follows that of Serre, but we make some of the (abstract) notions more explicit. This is needed in order to verify a certain smoothness criterion to be used later. We then study the problem of lifting group extensions and afterwards use this information to solve the tame lifting problem. Certain immediate problems arise when one tries to apply this to the wild setting and we leave the matter there temporarily.

In chapter four we examine the singular lifting problem, i.e. lifting extensions up to birationality. We start by stating a theorem of Garuti that étale Galois extensions of the rigid circle can always be lifted to (possibly ramified) extensions of the rigid disc. We then investigate formal fibres of rigid varieties and using this show how to relate Garuti's theorem to the lifting problem. However, here certain existence theorems are used and to actually find extensions one needs a constructive method for doing this. We remark that Sekiguchi-Suwa theory seems to suffice for this.

In chapter five we return to the wild (smooth) lifting problem. We state and comment on the local-global principle, proved by Green and Matignon and later by Henrio. In this context we also mention a special case of Sekiguchi-Suwa *Kummer-Artin-Schreier-Witt* theory. We also mention in the appendices as an aside Henrio's reversal theorem, which concerns automorphisms of the formal R disc. This allows one to construct new liftings of the cyclic-p problem which are slightly different from the one we shall construct in the examples and/or using Sekiguchi-Suwa theory.

After this we give an overview of Green and Matignon's solution to the p^2-lifting problem. Their method relies on the local-global principle and it entails lifting the Artin-Schreier extensions of characteristic p to Kummer extensions in characteristic 0. However, certain singularities might occur and in order to smoothen them one needs to the perturb the Kummer equations in such a way that it reduces *smoothly* to the characteristic p equations.

Finally, we conclude by returning to the cyclic-p problem and its connections to lifting group extensions. Using the *Kummer-Artin-Schreier-Witt* theory, one can give a partial correspondence of extensions of smooth R-curves to the context of group extensions in a style similar to the case of ordinary algebraic curves over k. Unfortunately this correspondence cannot always be reversed, but it does allow one to reduce the wild lifting problem over general curves to that of extensions of the projective line.

The author has tried to give as many of his own proofs of well known theorems as possible. We also try to give instructive examples. We do not prove every theorem, and certain well known theorems or ideas we simply state. An example of this would be the algebraization theorem. Another would be the *local-global* principle, which is a simple, but beautiful idea. We do however include a detailed proof of the latter at the end of chapter 5. To try to give some appreciation for the difficulty of a direct attack on higher order Galois extensions, we also include a calculation on the different of certain p^3-cyclic curve extensions. On the other side of spectrum, we also include a family of *elementary abelian* p^3 extensions which cannot be lifted to characteristic 0.

Chapter 1

Automorphisms of curves

In this chapter we review some of the structure results on curve automorphism groups. This leads in a natural way to a first obstruction for lifting, and a closer look links the problem with non-commutativity in the automorphism groups.

We then move to a structure result of Nakajima regarding abelian automorphism groups and one finds immediately that this obstruction disappears. We conclude with some more structure results of Nakajima which eventually leads to a general obstruction for lifting over the Witt vectors, i.e. without adding roots of unity.

1.1 Automorphism group orders

We start by stating the well known

Theorem 1 ([16] p.348 Ex. 5.2) *Let $C \to k$ be a smooth curve of genus g_C over the algebraically closed field k. Then the group $Aut_k(C)$ is finite if $g_C \geq 2$.*

An immediate corollary of this is the following theorem

Proposition 2 ([16] p.305 Ex. 2.5) *Let C/k be a curve of genus $g_C \geq 2$ and when k is of finite characteristic p, assume that the order of the automorphism group $Aut_k(C)$ is relatively prime to p. Then the order of this group is bounded by $84(g_C - 1)$.*

Proof: For convenience let G denote $Aut_k(C)$ and we set $D = C/G$. Notice that C/D is Galois. By assumption only tame ramification can occur in the extension C/D. We thus have the following simple form of the Hurwitz genus formula

$$\frac{2g_C - 2}{|G|} = 2g_D - 2 + \sum_{Q \in ram(D)} (1 - \frac{1}{e_Q})$$

We set λ equal to the right hand side of the above. Our goal is to show that $\lambda \geq \frac{1}{42}$. By assuming that $g_C \geq 2$ we have that $\lambda > 0$ always. Hence in the case that $g_D \geq 2$ we have that $\lambda \geq 2 \geq \frac{1}{42}$ and we shall be done. The next case is when $g_D = 1$. If there is no ramification present in C/D, then the left hand side will be 0 and hence contradicting the assumption that $g_C \geq 2$. With ramification, the contribution on the right is at least $\frac{1}{2} \geq \frac{1}{42}$ and again we are done.

Finally we consider the case that that $g_D = 0$ and we follow a combinatorial approach by considering the various instances when the ramification indices takes the values 2, 3, 4 and higher. When $|ram(D)| \geq 5$ we shall have that $\lambda \geq \frac{5}{2} - 2 > \frac{1}{42}$. One obtains the same result for when $|ram(D)| = 4$ and at least one of the indices e_Q is 3. When $|ram(D)| = 4$ and all ramification indices are 2, then one obtains a contradiction in that λ would be 0 then. One continues this line of argument until all the possibilities are exhausted, keeping in mind that $\lambda > 0$ and that the e_i are all integers strictly larger than 1. ♣

Remark We remark that the fraction $\frac{1}{42}$ is in fact sharp - see the citation given.

This result was improved considerably by Stichtenoth in his series of papers [11] and [12]

Theorem 3 ([11] Satz 3 p.534) *Let C/k be a smooth curve over the algebraically closed field k and assume its genus $g_C \geq 2$. Let $G = Aut_k(C)$ and we set $D = C/G$. Let $\{Q_1, \ldots, Q_r\}$ denote the subset $ram(D) \subset D$. Then we have that $|G| \leq 84(g_C - 1)$ except for the following (possible) exceptions:*

- *The quotient D is the line and $r = 3$ with one $Q \in D$ wild and the other two tame. In this case $|G| \leq 24g_C^2$.*

- *The quotient D is the line and $r = 2$ with both wild. Then $|G| \leq 16g_C^2$.*

- *The quotient D is the line and $r = 1$ with Q_1 being wild. We then have $|G| \leq 16g_C^3$.*

- *The quotient D is the line and we have one wild point and one tame point. In this case $|G| \leq 16g_C^4$ except in the case when the function field $k(C)$ is of the form $k(x,y)$ with x and y related by $y^{p^n} + y = x^{p^n+1}$ for some positive integer n.*

Remark For examples where these estimates are equalities, one finds general obstructions to lifting the automorphism groups to characteristic 0.

In his second paper [12], Stichtenoth gives a detailed study of the last exceptional case of the above mentioned theorem. One finds there that $|Aut_k(C)|$ certainly is of the order $16g_C^4$ and definitely exceeds the bound $84(g_C - 1)$. Also interesting is

Proposition 4 ([12] Proof of Satz 5 p.623) *Let C/k be the above mentioned exceptional curve over the field k with function field $k(C)$ of the form $k(C) = k(x,y)$ with $y^{p^n} + y = x^{p^n+1}$ for some positive integer n. Let d and e be any solutions of the equations $d^{p^{2n}} - d = 0$ and $e^{p^n} + e - d^{p^n+1} = 0$ and associate to this pair the function $\sigma_{d,e}$ on $k(C) = k(x,y)$ defined by $x \mapsto x + d$ and $y \mapsto y + e + d^{p^n}x$. Then these $\sigma_{d,e}$ are automorphisms and in fact exactly the automorphisms of the inertia group $G(P)$ for some point $P \in C$.*

We choose two solution pairs (d_1, e_1) and (d_2, e_2) (of the equations mentioned in the theorem) such that $d_2^{p^n} d_1 \neq d_1^{p^n} d_2$ and by evaluating the effects of $\sigma_1 \circ \sigma_2$ and $\sigma_2 \circ \sigma_1$ on y one sees non-commutativity of the automorphism group of C. This introduces our next topic.

Abelian subgroups

In his paper [5], Nakajima shows that even though the general automorphism group can become rather large, we still have the following bounds

Theorem 5 ([5] p.23 Theorem 1) *Let C/k be a smooth curve of genus $g_C \geq 2$ and let $G \subset Aut_k(C)$ be an* abelian *subgroup. We then have $|G| \leq 4g_C + 4$ and when $char(k) = 2$ we have that $|G| \leq 4g_C + 2$. These bounds are best possible.*

The proof is rather technical, but may be regarded as a (hard) generalization of the original technique that we employed to bound the automorphism group in the tame case. The fact that G is abelian is used to force certain combinatorial relations between the ramification indices and cuts down the number of cases to be checked in a Hurwitz type argument. See the above citation in the proof of Theorem 2 for details regarding this.

1.2 A numerical obstruction to lifting wild automorphisms

In this section we comment on work done by Nakajima in his paper [6]. Throughout this section C will denote a smooth k-curve. We shall let R denote its ring of Witt Vectors $W(k)$. By definition R/pR is then k. Let G denote a cyclic group of order p acting on C and we let σ be a generator of G.

We set $D = C/G$. We shall denote by $P_1, \ldots, P_r \subset C$ the ramification points of C/D and at each ramification point P_j let π_j be a local parameter. For later use we define the invariants $N_i = ord_{P_i}(\sigma\pi_i - \pi_i)$ attached to the ramification groups - see [22] Chapter 5 for background on these numbers.

$k[G]$-modules

Let V be the k vector space generated by $e_1, \ldots, e_p$ endowed with the G-action $\sigma e_i = e_i + e_{i-1}$. Here we define e_0 to be 0. V is then a $k[G]$-module and if we let V_j denote the $k[G]$-submodule generated by the elements $e_1, \ldots, e_j$, one finds that V_j is exactly the set of elements $\{v \in V | (\sigma - 1)^j v = 0\}$ and that $V_1 \subset V_2 \ldots$.

Nakajima remarks that all finitely generated $k[G]$-modules can be uniquely expressed as direct sums of these modules, i.e. all finitely generated $k[G]$-modules M admit a unique decomposition

$$M = \bigoplus V_j^{m_j}$$

where the m_j are non-negative integers.

Structure result of Nakajima

Let E be a G invariant divisor on C and we assume that E has degree larger than $2g_C - 1$. We can write

$$E = f^*E_D + \sum_{i=1,\dots,r} n_i P_i$$

where E_D is some divisor on the curve D. We let M be the $k[G]$-module of global sections of E_D, i.e. the associated Riemann Roch vector space of E_D on the curve D. We have stated earlier that we can express M uniquely as

$$M = \bigoplus V_j^{m_j}$$

for some integers m_j. An interesting result of Nakajima is the following

Theorem 6 ([6] p.86 Thm.1) *The integers m_j are given as follows :*

$$m_p = \frac{1}{p}\ \ deg(E) - g_D + 1 - \sum_{i=1}^{r} [\frac{1}{p}\ (p-1)N_i + \left\langle \frac{n_i - (p-1)N_i}{p} \right\rangle]$$

$$m_j = \sum [\frac{1}{p}\ N_i + \left\langle \frac{n_i - jN_i}{p} \right\rangle - \left\langle \frac{n_i - (j-1)N_i}{p} \right\rangle]$$

where the second sum ranges over $i = 1, \dots, r$ and the j ranges over $1, \dots, p-1$. We need to mention that the $\langle x \rangle$ here denotes the fractional part of the number of x, i.e. $x = \lfloor x \rfloor + \langle x \rangle$.[1]

We first mention why the expressions for the m_i are integers and we do this for the expression of m_p - the rest is similar. Multiplying the right hand side by p it will be enough to show that p divides

$$deg(E) - \sum_{i=1}^{r} \left[(p-1)N_i + p \left\langle \frac{n_i - (p-1)N_i}{p} \right\rangle \right]$$

which in turn is equivalent to showing that

$$deg(E) - \sum_{i=1}^{r} n_i$$

[1] We mention it here since it will be the only time we use it really.

is divisible by p. By our earlier expression for E we find that

$$deg(E) - \sum_{i=1}^{r} n_i = deg(f^* E_D)$$

where E_D was a divisor on D containing none of the branch points of C/D. Hence the degree of $f^* E_D$ is divisible by p and we are done.

Numerical obstruction

In this section we change notation slightly and denote by C_k a smooth curve over k endowed with a G-action G. We assume that this action lifts to an action on a smooth curve C/R with $\mathcal{O}_C(C) = R$, i.e. without having to extend the domain of definition R.[2] σ will denote both a generator of G over R and in the special fibre.

Our goal is to study the sheaves of powers of the differentials $\Omega^{1\ \otimes\ l}_{C/R}$. It is not too hard to show that the finite R-module $\Omega^{1\ \otimes\ l}_{C/R}(C)$ is flat and hence R-torsion free[3]. Furthermore, since K is flat over R, we have that by flat base change

$$\Omega^{1\ \otimes\ l}_{C/R}(C) \otimes_R K = \Omega^{1\ \otimes\ l}_{C_K/K}(C_K)$$

The latter is a free module over the field K generated by the same set of elements generating $\Omega^{1\ \otimes\ l}_{C/R}(C)$ over R and it is not hard to extract a free R-basis of $\Omega^{1\ \otimes\ l}_{C/R}(C)$ from this. We can thus apply the following result

Theorem 7 (Cohomology of Fibres; [18] p.202 Theorem 3.20) *In the notations above we have that*

$$\Omega^{1\ \otimes\ l}_{C/R}(C) \otimes_R k \to \Omega^{1\ \otimes\ l}_{C_k/k}(C_k)$$

is an isomorphism of k-modules.

[2]We remind the reader that we have *assumed* that R is absolutely unramified or what comes to the same thing, that p is a parameter.

[3]A module is flat over a Dedekind domain iff it is torsion free.

We quote Nakajima's paper which states that because R/pR is a field (and hence we use the fact that R is $W(k)$ and not some ramified extension of it - i.e. no roots of unity adjoined), that we have an explicit form of free R-modules which are also $R[G]$-modules. Indeed, as stated in Nakajima's paper, we have that ${\Omega^1_{C/R}}^{\otimes\, l}(C)$ must be a direct sum of the modules R,

$$I = \frac{R[G]}{\langle 1 + \sigma + \sigma^2 + \ldots + \sigma^{p-1} \rangle}$$

and $R[G]$ itself. Here the

$$\left\langle 1 + \sigma + \sigma^2 + \ldots + \sigma^{p-1} \right\rangle$$

means the $R[G]$-submodule generated by the elements between the $\langle\ \rangle$.

As stated in the paper of Nakajima, the isomorphism in the theorem above is also an isomorphism of $k[G]$-modules and hence we find that the modules $O_{l,k} \stackrel{def}{=} \Omega^{\otimes l}_{C_k/k}(C_k)$ must be direct sums of the modules k, $I_k = V_{p-1}$ and $k[G]$. However, we know that the modules $O_{l,k}$ can be expressed as direct sums of the V_i and we can even make the order to which the sums occur precise; indeed this was exactly the point of Nakajima's Theorem 6. Hence in that notation we must have $m_2 = m_3 = \ldots = m_{p-2} = 0$.

However, by allowing l to become very large and assuming that $g_X \geq 2$ and $p \geq 5$, Nakajima controls these coefficients and he shows that at least one of $m_2, \ldots, m_{p-2}$ must be non-zero when l is large, thus leading to a contradiction. Hence we finally arrive at the following theorem of Nakajima

Theorem 8 ([6] p.92 Theorem 3) *Let $g_C \geq 2$ and $p \geq 5$. Then no lifting of a G-action (on C) to R exists.*

Chapter 2

Infinitesimal lifting techniques

In this chapter we remind the reader of some formal tools to be used in the later exposition. We start with the étale lifting theorem, allowing us to lift étale extensions uniquely to formal schemes. Crucially, one does not need the projectivity assumption here.

Next we move to the topic of lifting morphisms between smooth schemes. This idea allows one to show the projectivity of a formal curve which is projective in the special fibre.

Lastly we conclude with the famous algebraization theorem of Grothendieck.

The étale lifting theorem

Consider a scheme X of finite type over R where R is some complete discrete valuation ring with parameter π. One notes that the closed immersions $spec(R/\pi^j) \hookrightarrow spec(R/\pi^{j+1})$ are topological isomorphisms. Consequently the fibre product closed immersions $X_j \hookrightarrow X_{j+1}$ are also topological isomorphisms. Conversely, given an inverse system of such closed immersions, we can build a formal scheme over the ring R which reduces to these schemes again, i.e. we can construct a formal R-scheme $\mathcal{X} \to R$ such that $\mathcal{X} \otimes_R R/\pi^j = X_j$. In view of the following this technique can be exploited:

Theorem 1 (SGA I.1.5.5; Infinitesimal étale lifting) *Let X, Y be S-schemes and $S_0 \to$*

S a closed immersion of schemes which is a topological isomorphism. Assume X/S is étale. Then $Mor_S(Y, X) \to Mor_{S_0}(Y_0, X_0)$ is a bijection.

Indeed, this technique is used in the following strategy: given a formal scheme $\mathcal{X} \to R$ with special fibre X_1, assume we are given an étale extension Y_1/X_1. The idea is to restrict to a covering of $\mathcal{X}$ such that over each open subset of the covering we can lift Y_1 infinitesimally to an étale extension over X_2 (or rather the open subset of X_2 in the covering). Here X_2 means the scheme obtained from $\mathcal{X}$ after moding out by π^2. This was constructed locally, but the uniqueness of the above theorem imply that one can glue these local constructions on the overlaps, and so one has constructed a *global* covering $Y_2 \to X_2$ lifting $Y_1 \to X_1$. In particular, one obtains

Theorem 2 (Étale lifting theorem) *Let A be an admissible π-adic complete R-algebra with special fibre A_1 and let $spec(B_1) \to spec(A_1)$ be an étale extension of the scheme $spec(A_1)$. Then this lifts to a unique étale extension $Spf(B)$ of $Spf(A)$ and if*

$$spec(B_1) \to spec(A_1)$$

is G-Galois then so is the formal extension.

Lifting morphisms and the connection with differentials

Instead of lifting étale extensions one might be interested in lifting general morphisms. We briefly quote the following in view of using it. As explained in SGA we have

Theorem 3 (SGA 1.3.5.6) *Let $\mathcal{X}$ be a smooth formal scheme over $S = spec(R)$ and assume $g_1 : Y_1 \to X_1$ where $\mathcal{Y} \to S$ is some formal scheme over S and X_1 and Y_1 are the special fibres. We define G_1 as $g_1^*(g_{X/S})$ and assume $H^1_{Zar}(Y_1, G_1) = 0$ (e.g. $\mathcal{Y}$ is affine) where $g_{X/S}$ is the dual of the differentials of the morphisms $X \to S$. Then g_1 prolongs to a formal morphism $\hat{g} : \mathcal{Y} \to \mathcal{X}$.*

As example we can give a quick application with a geometric flavour:

Application of infinitesimal lifting and interpretations

Corollary 4 *Let $\mathcal{X}/R$ be a formal flat R-curve and with map $i_k : \mathcal{X}_k \hookrightarrow \mathbf{P}_k^n$ which is a closed immersion. Then there exists an integer M such that we have a closed immersion of formal schemes $\mathcal{X} \hookrightarrow \hat{\mathbf{P}^M}$.*

Proof*:* Let μ_d be the d-uple embedding

$$\mathbf{P}_k^n \hookrightarrow \mathbf{P}_k^N$$

where N is some integer depending on n and d and we let j_d be the composition $\mu_d \circ i_k$. The idea is to use a map of this kind. Grothendieck's theory suggests we find a d such that

$$H^1_{Zar}(X_k, {j_d}^* g_{\mathbf{P}_k^N}) = 0$$

since we can then lift the map j_d infinitesimally. Consider the exact sequence[1] of coherent sheaves on $\mathbf{P}_k^N$

$$0 \to \mathcal{O}_{\mathbf{P}_k^N} \to \left[\mathcal{O}_{\mathbf{P}_k^N}(1)\right]^{N+1} \to g_{\mathbf{P}_k^N} \to 0.$$

First we pull this back via μ_d to obtain an exact sequence on $\mathbf{P}_k^n$

$$0 \to K_d \to \left[\mathcal{O}_{\mathbf{P}_k^n}(d)\right]^{N+1} \to {\mu_d}^* g_{\mathbf{P}_k^N} \to 0$$

and then via i_k back to X_k

$$0 \to L_{n,d} \to \left[\mathcal{O}_{X_k}(d)\right]^{N+1} \to {j_d}^* g_{\mathbf{P}_k^N} \to 0$$

We consider the induced exact sequence of cohomology

$$\ldots \to \left[H^1(X_k, \mathcal{O}_{X_k}(d))\right]^{N+1} \to H^1(X_k, {j_d}^* g_{\mathbf{P}_k^N}) \to H^2(X_k, L_{n,d}) \ldots$$

All the sheaves involved are coherent and the dimension of X_k is assumed 1, and hence for a large enough d we see that this part of the sequence vanishes[2]. We are not sure if this technique generalizes to higher dimensional schemes, however the theorem does. ♣

[1] The dual of exact sequence II.8.13 in [16].

[2] We recall the theorem of Serre stating that for large enough d we have that $H^p(X, \mathcal{O}_{X_k}(d)) = 0$ for all $p > 0$. This is the special property of being projective. Notice also that the higher cohomology groups $H^2()$ vanish on curves.

Lifting Schemes

Before one can even talk about lifting Galois extensions, we might first want to ask if we can always lift curves, never mind the automorphisms on them. In that regard the following suffices:

Theorem 5 (SGA 1.3.6.10) *We assume $S = spec(R)$ and X_1/k is a smooth scheme over the residue field k of the complete discrete valuation ring R. We define $G_1 = g_{X_1/k}$, the dual of the sheaf of differentials. If $H^2(X_1, G_1)$ vanishes, then this scheme lifts to a formal smooth scheme $\mathcal{X}/\mathcal{S}$. In particular when X_1 is a curve, then this will always be the case.*

Remark We have not proved this here. However an interesting phenomenon occurs in the proof. We lift the scheme X_1/k infinitesimally to X_2/S_2 and then to X_3/S_3 etc. It is shown in SGA that at each step we can control the lifting with the elements of $H^1(X_1, G_1)$. When X_1 is $\mathbf{P}^1$, then $H^1(X_1, G_1) = 0$ and so lifting will be unique. In general $H^1 \neq 0$ and hence lifting Galois coverings is not straight forward from this point of view.

Grothendieck algebraization

Very important in our studies later on is the following theorem of Grothendieck

Theorem 6 ([20] Cor.1.4) *Let X be a proper scheme over R and let GA and $\mathcal{GA}$ denote the categories of G-Galois coherent algebra sheaves on X and $\hat{X}$ respectively. Here $\hat{X}$ is the formal completion of X along the special fibre of R. Then the completion functor*

$$GA \to \mathcal{GA} : F \mapsto \hat{F} = \varprojlim F \otimes_{\mathcal{O}_X} \mathcal{O}_{X \otimes_R R/\pi^i}$$

is an equivalence of categories. In particular, every G-Galois covering of $\hat{X}$ is algebraizable to a G-Galois covering of X and this does not change the special fibre.

The reference above simply states this but does not give a complete proof of this theorem. However see [18] Ex. 10.4.4 for a discussion on algebraization of coherent sheaves.

As a last remark, this theorem assumes that $\hat{X}$ is algebraizable (since we started with an X) and we modelled finite extensions on $\hat{X}$ as coherent algebras on $\hat{X}$. However, if we started with a curve $\hat{X}$ without knowing a priori that it was algebraizable, can we still find an (ordinary) X/R? Our theorem earlier on embedding $\hat{X}$ into some $\mathbf{P}^{\hat{M}}$ deals with exactly that issue and clearly $\mathbf{P}^{\hat{M}}$ is algebraizable to an ordinary R-scheme, namely the ordinary projective M space.

Chapter 3

Jacobian approach to tame lifting

The aim of this chapter is to show how the theory of group extensions can be used to tackle the lifting problem. Essentially the idea is to replace the lifting problem of *curve* extensions by that of lifting group extensions which *models* the given curve extension.

Using a clever (essentially local) smoothness condition we then work our way back to the curve situation. In the current chapter, the method is illustrated for tame extensions. However, in a later chapter (chapter 5), we shall return to this problem in the wild setting, and then we shall be armed with the powerful Sekiguchi-Suwa theory.

We start by recalling some of the technical facts on Jacobians and extensions thereof. We follow essentially the treatment of [21], but we shall use the language of sheaves. The reader should note that all the sheaves and group schemes involved will always be representable *smooth* group schemes.

3.1 Jacobian methods I

We briefly remind the reader of some technical issues on *Ext* cohomology in relation to Jacobians and curve coverings. Throughout let C denote a smooth k-curve defined over the algebraically closed field k. For a modulus δ [1] we denote by C_δ and J_δ the associated singular curves and Jacobians[2]. The subscript δ will be omitted whenever $\delta = 0$, in this case J is then the usual (*projective*) Jacobian. The existence of the rational maps $C \to J_\delta$ is known (by [21]) and we shall refer to them as the canonical maps of $C_\delta \to J_\delta$. One knows that these maps have modulus δ themselves.[3]

Ramified coverings of a curve over a field

An important property we shall use is

Theorem 1 ([21] Proposition 6 p.91) *Let $m \geq m'$ be two moduli on the curve C. One then has a map of k-groups*

$$p^m_{m'} : J_m \to J_{m'}$$

and this map is surjective separable. We shall denote the kernel by $L_{m/m'}$.

Now let

$$(J'_\delta) \ : \ 0 \to N \to J'_\delta \to J_\delta \to 0$$

be an extension (on the flat site k_{fl}) of the algebraic group J_δ by the algebraic group N. We assume N is an abelian étale finite group associated to some abstract finite group, which we also denote by N. In this case we see that the group scheme J'_δ is automatically representable and we remark also that J'_δ/J_δ is an N-Galois étale extension of the variety J_δ.

Using the rational map $C \to J_\delta$, we normalize C inside the *rational* fibre product $C \times_{J_\delta} J_{\delta'}$. Notice this is an N-*Galois* extension of C.[4]

[1] Recall from [21] that a modulus on a non-singular curve is an effective divisor.

[2] See [21] for a discussion.

[3] See [21] for the concept of a rational map from an algebraic curve to an algebraic group to have a *modulus*. It is partially correct to say that it means the domain of definition of the rational map is the complement of the support of the modulus. However, there are also local criteria and for this, see the citation.

[4] Galois on the function fields.

We thus have a map $\phi_\delta^* : Ext^1_{S(k_{fl})}(J_\delta, N) \to Cov(C, N)$ where $Cov(C, N)$ represents the group of isomorphism classes of N-Galois extensions of the scheme C.[5]

Lemma 2 ([21] Proposition 10 p.122) *The map ϕ_δ^* is an injective group homomorphism.*

Lemma 3 *Consider the map $J_m \to J_n$ where $m \geq n$ as modulus objects on C. The map*

$$p_m^{n\,*} : Ext^1_k(J_n, N) \to Ext^1_k(J_m, N)$$

commutes with the maps ϕ_m^* :

$$\phi_m^* \circ p_m^n = \phi_n^*.$$

This means that the covering induced by an extension $(J_n') \in Ext^1_k(J_n, N)$ maps to the (unique) extension in $Ext^1_k(J_m, N)$ inducing the same covering upon rational fibre product.

The above follows directly from the basic properties of tensor product and the important fact that if two projective normal curves over a field agree on some open set, then the two curves are the same.

Theorem 4 ([21] Proposition 11 p.122) *Let C'/C be an abelian N-Galois (ramified) covering of curves. Then there exists a (smallest) modulus δ, named the conductor, such that C'/C is the normal closure of the rational fibre product of the rational map $C \to J_\delta$ and an isogeny $0 \to N \to J_\delta' \to J_\delta \to 0$. The support of this δ in C is exactly the set of ramification points of C'/C in C. The image ϕ_δ^* is the set of all N-coverings of C with conductor not exceeding δ.*

Example ([21] p.124 Example 1) *Let C'/C be a tame extension. Then the conductor is the sum of the ramification points in C, each with coefficient 1.*

We study the effect of taking quotients. Given the $S(k_{fl})$-short exact sequence of finite étale groups $0 \to N \to G \to H \to 0$ one would like to know how to interpret the map

[5] Recall that $Cov(C, N) \cong H^1_{et}(k(C), N)$.

$Ext^1_k(J_\delta, G) \to Ext^1_k(J_\delta, H)$ in terms of coverings of C. Using the explicit description of Ext-groups (Appendix A) we find that it takes a G-covering of C to its quotient by N:

$$\begin{array}{ccccc} N & & & & \\ \downarrow & & & & \\ G & \longrightarrow & J'_\delta & \longrightarrow & J_\delta \\ \downarrow & & \downarrow{\scriptstyle N\ Galois} & & \| \\ H & \longrightarrow & J''_\delta & \longrightarrow & J_\delta \end{array}$$

Let C'/C now be a G-Galois (ramified) covering with conductor δ induced by

$$0 \to G \to J'_\delta \to J_\delta \to 0$$

Consider the maximal unramified subextension E of C'/C, and let this have Galois group H. We have an exact sequence

$$0 \to N \to G \to H \to 0$$

and we have just recalled that E is given by the fibre product

$$E = C \times_J J'$$

for some H-extension of the (usual) Jacobian of C. We see that we can decompose the situation as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & J'_\delta & & J' & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_\delta & \longrightarrow & J_\delta & \longrightarrow & J & \longrightarrow & 0 \end{array}$$

In fact, using the classifications of the Ext-groups and the injectivity of the map

$$Ext^1(J_\delta, \ldots) \to Cov(C, \ldots)$$

we can say slightly more: Let δ and $\delta' < \delta$ be two modulus divisors on the curve C.

Claim 5 *Given any diagram*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & J'_\delta & \longrightarrow & J'_{\delta'} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & J_\delta & \longrightarrow & J'_{\delta'} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

the extension of C induced by the right hand side, is necessarily that of the middle quotient the group N.

We have a converse in the form : Given a G-extension C'/C with conductor δ, let E/C be any subextension with conductor not exceeding δ' with Galois group H the N quotient of G as before, then we can decompose the situation into a diagram as above.

Finally we obtain for a given G-Galois covering C'/C a full decomposition :

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L'_{\delta_k} & \longrightarrow & J'_{\delta_k} & \longrightarrow & J' & \longrightarrow & 0(E_{2,k}) \\
 & & {\scriptstyle f_k}\downarrow & & \downarrow & & {\scriptstyle g_k}\downarrow & & \\
0 & \longrightarrow & L_{\delta_k} & \longrightarrow & J_{\delta_k} & \longrightarrow & J & \longrightarrow & 0\ (E_k) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & \\
 & & (a_k) & & (b_k) & & (c_k) & &
\end{array}$$

Here the column (c_k) represents the unramified part, and we remark similar diagrams hold for the map $J_\delta \to J_{\delta'}$ if $\delta' < \delta$.

Example (Application to $Ext^1(G_m, \mu_n)$) Assume $(n,p) = 1$ and choose a canonical $\zeta \in \mu_n$. This allows an identification of μ_n with the finite étale sheaf $\mathbf{Z/nZ}$. Consider the line $\mathbf{P}^1_k$ and let δ be the modulus $(0) + (\infty)$. Notice that this has generalized Jacobian G_m. By the structure theory just developed we see the group $Ext^1_k(G_m, \mu_n)$ is in direct correspondence with the set of $\mathbf{Z/nZ}$-coverings of $\mathbf{P}^1_k$ with conductor not exceeding δ. These are exactly the classes of $\mathbf{Z/nZ}$-coverings of $\mathbf{P}^1_k$ unramified in $G_m \subset \mathbf{P}^1_k$.

Kummer theory quickly gives the μ_n-Galois étale extensions of G_m to be

$$X_i = spec[\frac{K[T, 1/T][X]}{\langle X^n - T^i \rangle}].$$

In fact, one can identify this set with $\mathbf{Z/nZ}$ via $X_i \to i$ where we understand the Galois group law to act via $X \to \zeta X$ under $1 \in \mathbf{Z/nZ}$ or $\zeta \in \mu_n$.

We can also endow each X_i with a group law $m : X_i \otimes X_i \to X_i$ given in rings as $X \mapsto X \otimes X$ and we note that $\mu_n \hookrightarrow X_i$ by identifying μ_n with the closed subscheme

$$spec[\frac{K[T, 1/T][X]}{\langle X^n - T^i; T - 1 \rangle}] \hookrightarrow X_i.$$

One finds then that this immersion of group schemes is in fact a group homomorphism and that the cokernel is the group G_m. By our theory the set of these X_i is then exactly the group $Ext^1_k(G_m, \mu_n)$. We have $Ext^1_k(G_m, \mu_n) = \{X_i\} = \mathbf{Z/nZ}$ sending $X_i \mapsto i$.

The example extends to the fraction field K of the ring $W(k)$. In this case we note that K need not be algebraically closed, but it does contain all the nth roots. The X_i are still K_{fl}-group schemes and still classify the n *geometric* extensions of $\mathbf{P}^1_K$. Note we are missing the arithmetic extensions of K. More generally one can also prove that

$$Ext^1_S(G_m, \mu_{\mathbf{n}}) = \mathbf{Z/nZ}$$

and that the group schemes X_i are indeed the isomorphism classes.

Singular Jacobians over Arithmetic surfaces

We now let C denote a projective smooth curve over R and we let δ be an effective horizontal R-divisor[6] on C (or a modulus). This induces moduli δ_k and δ_K on the special and generic fibres and the reduction of δ_K is δ_k by the horizontal assumption. Following ([7] Chapter 3) one can define an associated singular curve C_δ inducing the analogous singular curves in the fibres. One can take the degree 0 Picard scheme J_δ and one obtains again the following important theorems

Theorem 6 ([7] p.363 and proof) *There exists an $S = spec(R)$ group homomorphism $p^{\delta}_{\delta'} : J_\delta \to J'_\delta$ for two moduli $\delta > \delta'$ defined over R which induces the analogous maps in the special and generic fibres.*[7] *This map is surjective with smooth S-kernel $L^{\delta}_{\delta'}$.*

When $\delta' = 0$ we shall write L_δ instead of L^{δ}_{0}.
Example ([7] p.365 Eqn 2.8.13) *For $\delta = \sum_{i=1}^{r} s_i$ with $(s_i)_k$ distinct points in the special fibre we have that $L_\delta = G_m^{r-1}$.*

For later use the following is important:

Theorem 7 ([7] p.360 prop 1.5 and proof) *Let C be an arithmetic surface over $S = spec(R)$ and let G be a S-group. Assume we have a S-rational map $h : C \to G$ whose domain of definition intersects the special fibre*[8]*. Assume this map has modulus α_K and α_k on the generic and special fibres respectively. Assume there exists an effective divisor $\delta \subset C$ such that $\delta_K = \alpha_K$ and $\delta_k = \alpha_k$, and furthermore the support of δ consists of S-points. Then the rational map h factors through the generalized Jacobian J_δ as a S-group homomorphism $J_\delta \to G$.*[9]

[6] We take R-divisors to ensure the support are R-points. Recall the situation in curves over non-algebraically closed fields.

[7] With δ defined over R we mean its support consists of R-points. This is always possible after a sufficient extension of the base R.

[8] I.e. is not just defined on the generic fibre, which is also an open subset.

[9] In [7] the authors have stated the theorem slightly differently. They made the modulus condition concrete in terms of principle divisors on the curves $C_\delta \times_R R'$ for all finite R'/R. In view of [21] p.106 Proposition 13 this makes sense; the modulus condition for the curve over the generic fibre is defined only over the algebraic closure, and hence if one is to interpret it in terms of principle divisors as [7] has, one

3.2 Lifting Tame Ramification

Still letting C denote an arithmetic surface over R, we shall now consider G-Galois extensions of its special fibre D_s/C_k. Note we use the symbol D_s: the reason is that we shall later construct a normal arithmetic surface extension D/C and we shall be interested in comparing its special fibre D_k with the given extension on curves D_s/C_k: we shall not know a priori that D_k will actually be exactly D_s.

Throughout this section we shall assume G is tame. Hence the special fibre conductor δ_k has the form $\delta_k = \sum_{i=1}^{r} Q_{k,i}$. Here we have written $Q_{k,i}$ to mean the ramification points on the curve C_k. As we have seen we can decompose the situation into its different stages

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L'_{\delta_k} & \longrightarrow & J'_{\delta_k} & \longrightarrow & J' & \longrightarrow & 0 \ (E_{2,k}) \\
 & & {\scriptstyle f_k}\downarrow & & \downarrow & & {\scriptstyle g_k}\downarrow & & \\
0 & \longrightarrow & L_{\delta_k} & \longrightarrow & J_{\delta_k} & \longrightarrow & J & \longrightarrow & 0 \ (E_k) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & \\
 & & (a_k) & & (b_k) & & (c_k) & &
\end{array}$$

DIAGRAM OSS1

Choose r sections $Q_i \in C(S) \ : S \to C$ lifting the points $Q_{k,i} \in C_k$. We shall denote such a section's generic fibre by $Q_{K,i}$, i.e. $Q_{K,i}$ is a K-point on the generic fibre C_K of C.

The essential aim of this section will be to lift the above diagram to an analogous diagram in characteristic 0. We start with the easy part first - we have indicated earlier (Theorem

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must allow all finite extensions of K.

6) that the extension

$$(E) : 0 \to L_\delta \to J_\delta \to J \to 0$$

lifts the extension (E_k).

The groups G, N and H are finite groups of order prime to the characteristic and we can thus regard each of them as sums of appropriate groups of unity roots. We shall that assume all these groups are cyclic. The argument which follows extends very easily to the general case and we refer the reader to the argument presented in [7]. Assume that $|G| = b$, $|N| = a$ and $|H| = c$.

Since J is an abelian scheme (J here meaning the usual Jacobian), we have that the map

$$Ext^1_S(J, Z/cZ) \to Ext^1_k(J, Z/cZ)$$

is bijective - see (Appendix A Theorem2). Thus we can lift column (c_k) of diagram OSS1 to

$$0 \to H \to J' \to J \to 0 \ .$$

Note here J' is also an abelian scheme.

We can also lift the first column (a_k) :

$$0 \to N \to L'_\delta \xrightarrow{f} L_\delta \to 0 \ .$$

To prove this one uses the fact that $L_\delta = G_m^{r-1}$ and the residue map $Ext^1_S(G_m, \mu_n) \to Ext^1_k(G_m, \mu_n)$ is an isomorphism - this we have seen earlier. We remark that the tame assumption is indirectly used here, which allows us to deduce the simple form of L_δ.

As stated in [7], we have the exact sequence of groups

$$0 \to Ext^1_S(J', N) \to Ext^1_S(J', L'_\delta) \xrightarrow{f_*} Ext^1_S(J', L_\delta) \to 0$$

and similarly for the fibres. This is due to the fact that J' is also an abelian scheme and hence $Ext^2_S(J', N)$ vanishes. Hence we can find an extension $E' \in Ext^1_S(J', L'_\delta)$

$$(E') : 0 \to L'_\delta \to J'_\delta \to J' \to 0$$

such that $f_*(E') = g^*(E)$. In the special fibre the elements $f_{k,*}(E_{2,k})$ and $(f_*(E'))_k$ agree: we have

$$[f_*(E')]_k = [g^*(E)]_k = f_{k,*}(E_{2,k})$$

due to the fact that the residue map commutes with the Ext arrows and we have a diagram in the special fibre

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L'_{\delta_k} & \longrightarrow & J'_{\delta_k} & \longrightarrow & J' & \longrightarrow & 0(E_{2,k}) \\
 & & {\scriptstyle f_k}\big\downarrow & & \big\downarrow & & {\scriptstyle g_k}\big\downarrow & & \\
0 & \longrightarrow & L_{\delta_k} & \longrightarrow & J_{\delta_k} & \longrightarrow & J & \longrightarrow & 0\ (E_k)
\end{array}$$

giving us the latter equality. Hence $E_{2,k} - [E']_k$ is an extension lying in the group $Ext^1_k(J', N_k)$, which is the kernel of the map $f_{k,*}$ in the special fibre. Once again, since J' is an abelian scheme, the reduction map

$$Ext^1_S(J', N) \to Ext^1_k(J'_k, N)$$

is bijective and hence we can lift the extension $E_{2,k} - [E']_k$ to an element E'' in characteristic 0. The extension $E'' + E' \in Ext^1_S(J', L'_\delta)$ satisfies

$$f_*(E'' + E') = g^*(E)$$

and reduces to $E_{2,k}$. We thus have a lifting of diagram OSS1:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & L'_\delta & \longrightarrow & J'_\delta & \longrightarrow & J' & \longrightarrow & 0(E_{2,k}) \\
 & & {\scriptstyle f}\big\downarrow & & \big\downarrow & & {\scriptstyle g}\big\downarrow & & \\
0 & \longrightarrow & L_\delta & \longrightarrow & J_\delta & \longrightarrow & J & \longrightarrow & 0\ (E_k)
\end{array}$$

We now have an extension of J_δ, namely J'_δ. We let D be the normal closure of C inside the generic rational fibre product $C \times_{J_\delta} J'_\delta$ and claim that this D is the correct lifting, a fact which will follow in due course. For reasons becoming clear later, we need to study the ramification indices of the generic fibre D_K in order to estimate its genus. Notice that D_K is the normalization of C_K inside the rational fibre product $C \times_{J_\delta} J'_\delta$.

Lemma 8 *The residue map $Ext^1_S(J_\delta, G) \to Ext^1_k(J_\delta, G)$ is injective.*

Proof*:* One uses the $Ext(-, G)$ exact sequences

$$\begin{tikzcd}
\ldots \arrow[r] & Hom_S(L_\delta, G) \arrow[r] \arrow[d] & Ext^1_S(J, G) \arrow[r] \arrow[d, equal] & Ext^1_S(J_\delta, G) \arrow[r] \arrow[d] & Ext^1_S(L_\delta, G) \arrow[d, equal] \\
\ldots \arrow[r] & Hom_k(L_\delta, G) \arrow[r] & Ext^1_k(J, G) \arrow[r] & Ext^1_k(J_\delta, G) \arrow[r] & Ext^1_k(L_\delta, G)
\end{tikzcd}$$

This sequence commutes with the reduction map and we have that $Hom_k(L_\delta, G) = 0$. This follows from the fact that the kernel of any map $G_m \to G$ would have to be a closed subgroup of G_m and hence either a finite set of points or the entire G_m. But G_m and G are of different dimension and hence the kernel would have to be the entire G_m. The injectivity follows. ♣

Lemma 9 *Let $Q_{i,K}$ be the generic fibre of the S-point $Q_i \in C(S)$. Then its ramification index in the extension D_K/C_K is at most that of $Q_{i,k}$ in the special fibre extension D_s/C_k.*

Proof*:* For the moment we drop all reference to the subscript i. Let $\delta' = \delta - Q$. Let the inertia group of Q_k in D_s/C_k be N_Q [10] and let H_Q be the quotient $H_Q = G/N_Q$. Hence we can define a H_Q-Galois subextension $E_{Q,k}$ of C_k inside the extension D_s/C_k. We note that $E_{Q,k}$ is the largest subextension of D_s admitting a conductor smaller than or equal to δ' since it is unramified at the special point Q_k. The degree of H_Q is the inertia degree $f(Q_k)$ of Q_k inside the extension D_s/C_k. We thus have the following diagrams in the special fibre

$$\begin{tikzcd}
G \arrow[r, "m"] \arrow[d] & H_Q \arrow[r] \arrow[d] & H \arrow[d] \\
J'_{\delta_k} \arrow[r] \arrow[d] & J'_{\delta'_k} \arrow[r] \arrow[d] & J' \arrow[d] \\
J_{\delta_k} \arrow[r, "p^{\delta_k}_{\delta'_k}"] & J_{\delta'_k} \arrow[r, "p_0^{\delta'_k}"] & J_{\delta_k} \\
(b_k) & (b_{Q,k}) & (c_k)
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "m"] \arrow[d] & H_Q \arrow[r, equal] \arrow[d] & H_Q \arrow[d] \\
J'_{\delta_k} \arrow[r] \arrow[d] & m_*(J'_{\delta'}) \arrow[r] \arrow[d] & J'_{\delta'_k} \arrow[d] \\
J_{\delta_k} \arrow[r, equal] & J_{\delta_k} \arrow[r, "p^{\delta_k}_{\delta'_k}"] & J_{\delta'_k} \\
(b_k) & (m_*(b_k)) & (b_{Q,k})
\end{tikzcd}$$

[10] We have assumed the G-extension in abelian. Hence the inertia group of a point $Q_k \in C_k$ inside the Galois extension D_s/C_k is well defined and does not depend on any reference to a point $Q_k \in D_s$ lying over $Q_k \in C_k$.

Note that the right hand diagram implies

$$m_*(b_k) = p^{\delta_k}_{\delta'_k}(b_{Q,k}).$$

Using the technique developed above we can lift these partially to characteristic 0

$$\begin{array}{ccccc}
G & \xrightarrow{m} & H_Q & \longrightarrow & H \\
\downarrow & & \downarrow & & \downarrow \\
J'_\delta & & J'_{\delta'} & \longrightarrow & J' \\
\downarrow & & \downarrow & & \downarrow \\
J_\delta & \xrightarrow{p^\delta_{\delta'}} & J_{\delta'} & \longrightarrow & J \\
(b) & & (b_Q) & & (c)
\end{array}$$

In order to fill in the missing arrow we need to show that inside $Ext^1_S(J_\delta, H_Q)$

$$m_*(b) = p^\delta_{\delta'}(b_Q)$$

This holds in the special fibre and hence

$$[m_*(b) - p^\delta_{\delta'}(b_Q)]_k = 0$$

The injectivity lemma proved previously gives the missing arrow.

Interpreting the lifted diagram above in the generic fibre one sees thus that the extension $D_K/(N_Q)$ with Galois group H_Q is given by an extension of $J_{\delta'}$ and hence must have conductor not exceeding this. In particular the extension with group H_Q is unramified over Q_K. The result follows. ♣

Remark We shall use this information to check that $D_k = D_s$. However we note that if one knows *a priori* that D reduces to D_s then the above claim is easy to prove : one simply notes that a point on the generic fibre cannot be a branch point if its specialization is not one - $\Omega^1_{D/C}$ cannot vanish at a closed point if it does not do so at a generic point.

We proceed to check that $D_k = D_s$. Invoking the Hurwitz genus formula, we find that the generic genus $g(D_K)$ is at most the genus of the given curve $g(D_s)$. Notice that D_K is normal since normalization commutes with (flat) generic fibre. Hence the arithmetic genus and the geometric genus of the generic fibre D_K agree.

We note that the special fibre arithmetic genus $p_a(D_k)$ satisfies $p_a(D_k) \geq p_a(D_s)$.[11] However, since D/S is flat, the arithmetic genera on the generic and special fibres coincide. Furthermore, the arithmetic genus of D_s is exactly its genus[12] and hence we have the inequalities

$$g(D_K) \stackrel{normal}{=} p_a(D_K) \stackrel{flat}{=} p_a(D_k) \geq p_a(D_s) \stackrel{smooth}{=} g(D_s) \geq g(D_K)$$

and hence they are all equalities. We find that the special fibre of D is normal and we have lifted the extension D/C. [13]

Conclusion and wild situation

We now point out some immediate problems with this approach with respect to the wild situation. The fact that the conductor is so simple in the tame case had two advantages:

- Lifting diagram: we could very easily lift the first column of diagram OSS1 due to the simple structure of schemes involved.

- Genus calculation: this was very easy due to the fact that over each ramified point in the special fibre our candidate had unique points in the generic fibre. Furthermore a ramification point in the generic fibre implies that its specialization be a ramification point in the special fibre. Hence in order to find the generic genus one looks at each ramification points of the special fibre individually and one can thus easily bound the generic different from above.

We shall come back to these issues.

[11] This is exactly [18] p.304 Proposition 5.4. and the fact that D_s is the normalization of D_k.

[12] By assumption D_s is smooth.

[13] We note that D/S is thus smooth since its special fibre D_k/k is smooth and D/S itself is flat.

Chapter 4

Singular liftings of Garuti

In this chapter we briefly survey the ideas of Marco Garuti on singular liftings of extensions. His methods answer the following

Question *Let D_s/C_k be an extension of k-curves in characteristic p, assumed Galois with group G. Can one find a Galois lifting of normal R-curves D/C, where C is some given characteristic 0 arithmetic surface reducing to C_k, such that D_k is G-birationally equivalent to the given D_s.*

Using Sekiguchi-Suwa theory one can answer this for p^n-cyclic Galois covers positively, but Garuti's methods are more general. Moreover, he manages to solve the problem in such a way as to show that the reduction D_k is even homeomorphic to the given D_s, and hence admits only cusp singularities. Although this is slightly weaker than the full lifting conjecture, Garuti introduces methods which are interesting in their own right.

In the following, the first section proves a theorem on extending Galois actions on the boundary of a rigid disc to the entire rigid disc. The proof is based on the ideas of Garuti. In the section following this we discuss the structure of formal fibres of rigid varieties. This is somewhat technical, but it allows us to derive a result to be used in a later section. Finally, we discuss how formal models of the rigid varieties can be glued to give an extension of formal curves with the desired behaviour. We conclude with the powerful algebraization theorem to descent back to the category of (ordinary) projective R-curves.

4.1 Lifting étale Galois extensions of boundaries of the rigid disc

Moving away from curve automorphisms for the moment, we briefly discuss Galois extensions of the rigid circle, denoted by C in this section, and prolonging them to the rigid disc, denoted by D in this section. We state and prove the following important theorem of Garuti:

Theorem 1 ([1]) *Let K be a characteristic 0 complete valued field. Let A_C be the Tate Algebra $K\langle X, X^{-1}\rangle$ of the rigid circle C and let $A_{C'}$ be the Tate Algebra of a* connected étale G-Galois *extension of C algebraically. Then this prolongs to an extension D'/D of the rigid disc which is normal and ramified at most a finite number of points in the interior of D.*

We dedicate the rest of this section to a proof of the statement in the case that G is the cyclic group of order p^n. We need the following results

Lemma 2 *The Tate algebras $A_D = K\langle X\rangle$ and A_C are valued rings with their respective Gauss norms $|\ |_{Gauss}$ and these norms are the same as the supremum semi-norm $|\ |_{sup}$ induced on the rigid spaces D and C respectively.*

Proof*:* For convenience let $|\ |$ be the Gauss norm on A_C and note it extends the Gauss Norm of A_D and that of K.

The fact that A_D is a valued ring with its Gauss norm follows from [14] p.103 Proposition 1. Furthermore let M_D be the fraction field of A_D and note that the valuation norm of A_D extends to M_D. In particular it extends to $A_D[X^{-1}]$. Indeed, if $\alpha = \sum_{i=0}^{\infty} a_i X^i$ then $|\alpha| = \max_i |a_i|$ and we find that $|aX| = |a|$. It follows that the norm of $|aX^{-1}| = |a|$ and hence for any integer n we have $|\sum_{i=0}^{n} a_i X^{-i}| = \max_{i=0,\ldots,n} |a_i|$.

It is known that A_C is an integral domain and that it is a complete K-space with this norm. To show that it is a valued ring it suffices to show that $\forall A$ and B we must have $|AB| = |A||B|$. We let $A = \sum_{-\infty}^{\infty} \alpha_i X^i$ and $B = \sum_{-\infty}^{\infty} \beta_i X^i$. Set $A_n = \sum_{i=-n}^{i=n} \alpha_i X^i$ and similarly for B_n. Notice that for $n \gg 0$ we have $|A_n| = |A|$ and $|B_n| = |B|$. Also for n sufficiently large we have $|AB| = |A_n B_n| = |A_n||B_n|$ and where the latter inequality holds due to the valuation property of M_D. Hence A_C is a valued ring with respect to its Gauss norm.

This implies in particular that its Gauss norm is power multiplicative. However we know that the supremum semi-norm on A_C is complete - this is the content of [14] chapter 6 and the fact that A_C is a K-*complete* algebra with its induced residue norm from the complete Tate Algebra $K\langle X, Y\rangle$. Hence by [14] p.178 Lemma 3 we see that the sup norm and the Gauss valuation must coincide. ♣

Remark We have given a direct proof here. However using the connection between the *smooth* formal schemes $Spf(R\langle T\rangle)$ and $Spf(R\langle T, \frac{1}{T}\rangle)$ and the associated rigid varieties, which are exactly the rigid disc and circle, one can deduce that the rings A_D and A_C are valued rings with their respective *sup* norms using [14] (4.2.1 together with Proposition 5 on p.241). Our proof method was chosen to explicitly show that we can replace the sup norm (which is somewhat abstractly defined) with the concrete Gauss norm.

Throughout what follows we shall let A_D, A_C and $A_{C'}$ denote the Tate algebras of the rigid affinoids D, C and C', and we shall always let M_D, M_C and $M_{C'}$ denote their fraction fields. By assumption $M_{C'}/M_C$ is G-Galois. We shall let $|\ |_{M_D}$ denote the valuation norm on the field M_D and similarly for M_C and $M_{C'}$.

Some technical tools

We urge the reader to have patience while we gather a few more technical results:

Lemma 3 (Krasner's Lemma [14] p.149 Proposition 3) *Let M be a complete valued field and let $f(T) \in M[T]$ be a separable polynomial of degree $n > 1$. Assume f is* *irreducible. Then there exists an $\epsilon_f > 0$ such that for all g with $|f - g| < \epsilon_f$ we have that*

$$M[T]/f(T) \cong M[T]/g(T)$$

as M-algebras and hence as fields. For each root α_f of f and for every g satisfying this condition we can find a root α_g of g such that we have equality of fields $M(\alpha_f) = M(\alpha_g)$ inside the algebraic closure of M.

Lemma 4 (Finite Extensions of Affinoid Algebras) *Let N be a proper finite extension of either $M = M_C$ or $M = M_D$ and let B be the normalization of either $A = A_C$ or $A = A_D$ respectively inside N. Then B is a Tate Algebra with sup norm being the valuation induced by the finite extension N/M.*

Proof*:* This theorem is essentially that proved in [14] p.179 Proposition 6 and refined on p.181 Thm 7. ♣

We also have two algebraic properties of the algebras A_C and A_D which we prove using analysis.

Lemma 5 *Let $m_c \in MaxSpec(A_C)$ correspond to the point $c \in C$ and assume that $f \notin m_c$. Then there exists an $\epsilon_f > 0$ such that if $|f - g| < \epsilon_f$ in the Gauss norm, then $g \notin m_c$.*

Proof*:* We have for any $f \in A_C$ that $f \notin m_c$ iff $f(c) \neq 0$. Let $\epsilon_f = \frac{1}{2}|f(c)|_K > 0$ and let $g \in A_C$ be such that $|f - g| < \epsilon_f$ in the Gauss Norm. By definition of the supremum norm we have $|f(c) - g(c)|_K = |(f - g)(c)|_K \leq |f - g|_{sup} = |f - g| < \epsilon_f$. Hence $|g(c)|_K \neq 0$ and we are done. ♣

Most important of all is the following application of Krasner's lemma

Lemma 6 *Let $\gamma \in A_C$ such that $\gamma^{p^n} = \lambda \in M_D$. Assume that K contains all the p^nth roots of unity. Then $\gamma \in M_D$.*

Proof*:* Assume that $\gamma \notin M_D$. Then none of the p^n roots of $f(T) = T^{p^n} - \lambda$ are in M_D -indeed they all differ from each by some factor of a root of unity, which is assumed to be contained in K. Thus $f(T)$ is an irreducible polynomial over $M_D[T]$. Let $\gamma = \sum_{-\infty}^{\infty} \gamma_i X^i$ and we approximate γ

by partial sums $\{A_m\}$ as follows : for each $m > 0$ set $A_m = \sum_{-m\le i\le m} \gamma_i X^i$. We note that $A_m \to \gamma$ and also $|A_m^{p^n} - \lambda| \to 0$ for $m \gg 0$. However each $A_m \in K(X) \subset M_D$ and so $T^{p^n} - A_m^{p^n}$ cannot be irreducible in $M_D[T]$. By Krasner's lemma we have a contradiction. ♣

Conclusion of proof : Locally in C first

We now set out to prove Theorem 1 stated at the start of this section. We shall work locally about a point $x \in C$ first and show that locally, we can generate the extension $A_{C'}/A_C$ using an equation involving rational functions in T. Afterwards we shall extend the idea to the entire C.

Let $x \in C$ be arbitrary. By Kummer theory there exists a $G_x \in A_C$ and a $F_x \in A_C$ s.t.

$$A_{C'}[G_x^{-1}] = \frac{A_C[G_x^{-1}][T]}{T^{p^n} - F_x/G_x}$$

and $x \in spec(A_C[G_x^{-1}])$ where $\frac{F_x}{G_x}$ is a unit of $A_C[G_x^{-1}]$ -see Theorem B.1. These two elements G_x and F_x depend on x and we shall return to this later. We have that $M_{C'} = \frac{M_C[T]}{T^{p^n} - F_x/G_x}$: we note that $M_{C'}$ is the fraction field of $A_{C'}$ and hence also of $A_{C'}[G_x^{-1}]$. We must remark here that we shall often be working with localizations (e.g. $A_{C'}[G_x^{-1}]$) instead of completed localizations such as $A_{C'}\langle G_x^{-1}\rangle$.

We approximate F_x and G_x with partial sums in $K(X)$ as follows : if $F_x = \sum_{-\infty}^{\infty} F_{x,i}X^i$ then we approach it by $f_{x,m} = \sum_{-m<i<m} F_{x,i}X^i \in K(X)$ and similarly we approximate G_x by $g_{x,m} \to G_x$.

We evaluate the M_C norm[1] of

$$\frac{f_{x,m}}{g_{x,m}} - \frac{F_x}{G_x}.$$

We have for m large enough that

$$|\frac{f_{x,m}}{g_{x,m}} - \frac{F_x}{G_x}|_{M_C} = |\frac{G_x(f_{x,m} - F_x) + F_x(G_x - g_{x,m})}{g_{x,m}G_x}|_{M_C}$$

[1]This is the use of Lemma 2 - we can use the explicit form of the sup norm given there.

$$\leq J_x \big[\max_m \{ \ |f_{x,m} - F_x|, \ |g_{x,m} - G_x| \ \} \ \big]$$

where J_x is now some number completely independent of m.

Notice this implies in particular that by choosing m large enough the difference between $\frac{F_x}{G_x}$ and $\frac{f_{x,m}}{g_{x,m}}$ can be made arbitrarily small. Furthermore, by choosing m large enough we can also get that $f_{x,m}(x) \neq 0$ and $g_{x,m}(x) \neq 0$ - see Lemma 5.

In what follows, we set $f_x = f_{x,m}$ and $g_x = g_{x,m}$ for m large enough and such that the conditions above are all satisfied and that (thanks to Krasner's Lemma) we have $M_{C'} = \frac{M_C[T]}{T^{p^n} - \frac{f_x}{g_x}}$. For convenience we set $r_x = x f_x g_x$. We also set $B_x = A_C[f_x^{-1}, g_x^{-1}] = A_C[r_x^{-1}]$ and $B'_x = A_{C'}[f_x^{-1}, g_x^{-1}] = A_{C'}[r_x^{-1}]$. Then:

Claim 7 *We have that* $B'_x = \frac{B_x[T]}{T^{p^n} - f_x/g_x}$.

Proof*:* We start by noting that B'_x is *étale* over B_x, since by assumption $A_{C'}/A_C$ is an *étale* extension. Furthermore, we see that $\frac{f_x}{g_x}$ is a unit in B_x and hence the extension defined by $\frac{B_x[T]}{T^{p^n} - f_x/g_x}$ is a finite Kummer *étale* extension of B_x and hence *normal*. Furthermore, it is contained inside the extension $M_{C'}/M_C$ and hence must be contained in the normalization of B_x inside $M_{C'}$. But this is exactly B'_x and hence the two algebras coincide. ♣

Dependence on $x \in C$

We found that locally about $x \in C$ we can generate the extension $A_{C'}$ as a Kummer equation. However this equation depended on the point x. We study the dependence of the f_x/g_x on the point x in C with the aim of explicitly showing that for two points x_1 and x_2, the corresponding extensions that are generated locally about them are the same.

Consider two points x_1 and x_2 and set $B_{12} = A_C[r_{x_1}^{-1}, r_{x_2}^{-1}]$ and let $B'_{12} = A_{C'}[r_{x_1}^{-1}, r_{x_2}^{-1}]$. These are the algebras corresponding to the open subschemes of $spec(A_C)$ and $spec(A_{C'})$ given by the intersection of the distinguished opens $D_+(r_{x_1})$ and $D_+(r_{x_2})$. We see that

B_{12} is a localization of both B_{x_1} and B_{x_2}. Over the latter two algebras we have that B'_{x_1} and B'_{x_2} are Kummer extensions over the B_{x_i} given by the p^nth roots of the elements $\frac{f_{x_1}}{g_{x_1}}$ and $\frac{f_{x_2}}{g_{x_2}}$ respectively. Thus B'_{12} is a Kummer extension over B_{12} also given by the p^nth roots of *any* of these elements.

However by Kummer theory (B.1) this implies that the two elements f_{x_1}/g_{x_1} and f_{x_2}/g_{x_2} must differ by at most a p^nth power of a unit of B_{12}. Let us call this unit γ. Hence we see that $\gamma^{p^n} \in K(X)$, recalling that $f_{x_i} \in K(X)$ and $g_{x_i} \in K(X)$. Notice that $\gamma = \frac{\gamma'}{P}$, where $\gamma' \in A_C$ and P is some product of appropriate powers of the f_{x_i} and g_{x_i}. Thus this $(\gamma')^{p^n} \in K(X)$ and hence by Lemma 6 it follows that γ' and γ are both in M_D. Hence

$$\frac{M_D[T]}{T^{p^n} - \frac{f_{x_1}}{g_{x_1}}} = \frac{M_D[T]}{T^{p^n} - \frac{f_{x_2}}{g_{x_2}}}$$

is independent of x.

We let $M_{D'}$ be the above extension, let $A_{D'}$ to be the normalization of A_D in $M_{D'}$ and we note that this is a Tate Algebra with supremum norm being the extended valuation of M_D to $M_{D'}$. Also *locally* we have that A'_D is given by Kummer extensions over A_D which we can control up to a certain extent:

Claim 8 *Let x be an arbitrary point in C and we assume the notations as above. Then we have*

$$A'_D[r_x^{-1}] = \frac{A_D[r_x^{-1}][T]}{T^{p^n} - \frac{f_x}{g_x}}$$

inside $M_{D'}/M_D$.

Proof*:* The proof is similar to the one given for the algebras $A_{C'}/A_C$. We note that f_x/g_x is a unit inside $A_D[k_x^{-1}]$ and that the extension given by adjoining a p^nth root of f_x/g_x to this ring is an étale extension of $A_D[k_x^{-1}]$. This latter extension must be normal and contained in $M_{D'}$ and hence must coincide with $A_{D'}[k_x^{-1}]$. Here we recall $r_x = xf_xg_x$ and hence inverting r_x amounts to inverting x, f_x and g_x. ♣

Final step: Desired Algebra

We claim that $A_{D'}$ is the *desired* Tate Algebra as stated in Garuti's theorem. In order to do this we need to check that it gives back the étale Galois extension $A_{C'}/A_C$ that we started with. Our first aim is to show that as A_D-algebras, i.e. forgetting the Tate structure, we have $A_C \otimes_{A_D} A_{D'} \cong A_{C'}$. To this end we state *without proof*:

Claim 9 *It is enough to prove $A_C[X^{-1}] \otimes_{A_D[X^{-1}]} A_{D'}[X^{-1}] \equiv A_{C'}[X^{-1}]$. Here we remember that $A_C[X^{-1}] = A_C$ since X is already invertible in A_C.*

We note that $A_{D'} \hookrightarrow A_{C'}$ and $A_C \hookrightarrow A_{C'}$. Hence we have the unique map

$$E = A_C \otimes_{A_D[X^{-1}]} A_{D'}[X^{-1}] \to A_{C'}$$

which commutes with these maps. Let this map be ϕ. We note that ϕ is an A_C-algebra morphism and that upon localization by $r_x \in A_D[X^{-1}] \subset A_C$ we have a map $\phi_x : E[r_x^{-1}] \to A_{C'}[r_x^{-1}]$, where x here was an arbitrary point of C and r_x the element of A_C associated to x defined earlier.

Claim 10 *For any choice of $x \in C$, we have ϕ_x an isomorphism of A_C-algebras.*

Proof*:* Notice that ϕ_x is the map induced on the tensor product of the embeddings $A_C[r_x^{-1}] \hookrightarrow A_{C'}[r_x^{-1}]$ and $A_{D'}[r_x^{-1}] \hookrightarrow A_{C'}[r_x^{-1}]$. We note however that $A_{D'}[r_x^{-1}] \equiv \frac{A_D[r_x^{-1}][T]}{T^{p^n} - \frac{g_x}{h_x}}$ and hence (by the usual argument) we have $A_C[r_x^{-1}] \otimes_{A_D[r_x^{-1}]} A_{D'}[r_x^{-1}] \xrightarrow{\phi_x} A_{C'}[r_x^{-1}]$ is an isomorphism. ♣

We have just shown that locally about any point $x \in A_C$, the induced restriction of ϕ (namely ϕ_x) was an isomorphism. However $x \in C$ was arbitrary and hence ϕ is a global isomorphism. To conclude we recall that $A_{D'}/A_D$ is finite and hence we have that $A_{C'}$ is indeed the rigid tensor product, so we find that we have found a (possibly) ramified Galois extension of the closed disc, which upon fibre product with the closed disc C gives us back the original Galois extension C' of the closed disc C.

Arbitrary rigids

In the first section we found that given an étale Galois extension of the rigid circle C'/C we can extend this to a Galois extension of the disc D'/D.

Garuti manages to extend his theorem to a more general setting. We state his theorem here, but without proof. Instead we give an intuitive explanation afterwards.

Theorem 11 ([1] Cor 2.14.) *Let $\mathcal{X}_K$ be a rigid normal curve over K and let $\mathcal{U}_K$ be an open subaffinoid space of $\mathcal{X}_K$. Assume there exists an étale morphism $\mathcal{X}_K \to \mathcal{D}_K$ to the rigid closed disc identifying the complement of $\mathcal{U}_K$ with the open disc $\mathcal{D}_K^0$. Let $\mathcal{V}_K/\mathcal{U}_K$ be some finite étale Galois extension of rigid spaces. Then one can prolong the extension $\mathcal{V}_K/\mathcal{U}_K$ to a finite Galois(ramified) extension of normal rigid varieties $\mathcal{Y}_K/\mathcal{X}_K$ which induces $\mathcal{V}_K/\mathcal{U}_K$ on the fibre of $\mathcal{U}_K \subset \mathcal{X}_K$.*

Garuti's method of attack is to enlarge (using Krasner's lemma again) the open affinoid $\mathcal{U}_K$ inside $\mathcal{X}_K$ slightly to an open rigid subvariety $\mathcal{U}_K(r)$ containing $\mathcal{U}_K$. For a suitable choice[2] of this $\mathcal{U}_K(r)$, one can also prolong $\mathcal{V}_K/\mathcal{U}_K$ to an extension $\mathcal{V}_K(r)/\mathcal{U}_K(r)$.

Garuti then identifies a rigid circle $\mathcal{C}_K = \mathcal{C}_K(r)$ inside the open subdomain $\mathcal{U}_K(r)$ and considers the induced étale Galois extension $\mathcal{C}'_K/\mathcal{C}_K$, given by the fibre product $\mathcal{C}'_K = \mathcal{C}'_K(r) = \mathcal{V}_K(r) \times_{\mathcal{U}_K(r)} \mathcal{C}_K(r)$

However, we have already seen that $\mathcal{C}'_K/\mathcal{C}_K$ prolongs to a Galois (ramified) extension of the rigid disc, and one can then glue this to the rest of the extension $\mathcal{V}_K(r)$ to obtain a new rigid variety $\mathcal{Y}_K/\mathcal{X}_K$. Using Kiehl's theorem, one can show that this $\mathcal{Y}_K$ is finite over $\mathcal{X}_K$.

[2]The reference to the r has to do with a *distance of prolonging*, meant intuitively. The author thinks of it as *extending* the boundary of $\mathcal{U}_K$ with its complement a *distance* of r inwards. See [1] for details and a more precise version.

4.2 Structure of formal fibres

Let $\mathcal{X}/R$ be some formal scheme with special fibre X_k. Assume that Y_k/X_k is some Galois extension and let this be étale outside the point $x \in X_k$ i.e. on $U_k = X_k - x$. We know that we can lift the induced extension V_k/U_k to an étale Galois extension of formal schemes $\mathcal{V}/\mathcal{U}$ and more so $\mathcal{U}$ is some open subscheme of $\mathcal{X}$. Taking the generic fibres $(\otimes_R K)$ we find that we have a rigid variety $\mathcal{X}_K$ and some étale Galois extension over an open sub-affinoid of the rigid variety $\mathcal{X}_K$. In Garuti's Theorem 11 we saw that we can *complete* this extension to one over $\mathcal{X}_K$ **if** we knew that the complement of $\mathcal{U}_K$ had the structure of an open disc.

Our aim in this section is to explain why this latter assumption is in fact true. Essentially one studies the *formal fibre* of the point x. We need to clear up a few things first and then we proceed to the explanation. We remark that in [18] Proposition 10.1.4 a similar situation is studied in the context of reduction of projective schemes and so it might seem strange that we can succeed with the same result here when dealing with just affine schemes. However we must note that the algebras we work with are Tate algebras and so they carry much more structure than the algebras of ordinary algebraic varieties.

Reduction map and structure of formal fibres

Let $\mathcal{A}$ be a *normal* formal admissible R-algebra, such that the special fibre $\mathcal{A}/\pi$ is reduced and of dimension 1 over the field k. Associated to it is the Tate algebra $\mathcal{A} \otimes_R K$, which is endowed with its supremum norm. We have:

Lemma 12 *The R-subalgebra $(\mathcal{A}_K)^\circ$ of elements in $\mathcal{A}_K$ with sup-norm ≤ 1 is exactly the algebra $\mathcal{A}$.*

Proof*:* $[\mathcal{A} \subset (\mathcal{A}_K)^\circ]$ We write $\mathcal{A} = \frac{R\langle T_1,\ldots,T_m\rangle}{Q}$ for some ideal Q, which has coefficients in R. Let $f \in \mathcal{A}$ be given by the class of $f' = \sum a_{(v_i)_i} T_1^{v_1} \ldots T_m^{v_m}$ where each $a_{(v_i)_i} \in R$. It is clear that its supremum norm in $K\langle T_1,\ldots,T_m\rangle$ has value ≤ 1. Hence the induced residue norm on $\mathcal{A} \otimes_R K$ is ≤ 1 since the ideal Q is generated by elements of norm ≤ 1 inside $K\langle T_1,\ldots,T_m\rangle$. However, it is known that any residue norm of this type is never less than the supremum norm - see [14] 6.2. Hence we have that $\mathcal{A} \subset (\mathcal{A}_K)^\circ$.

$[\mathcal{A} \supset (\mathcal{A}_K)^\circ]$ To get the other way around, we have to use Noether Normalization. Let $k[t] \to \mathcal{A}_k = \mathcal{A}/\pi$ be a finite monomorphism and let us assume $t \mapsto a_0$. We lift this element to $\mathcal{A}$ as a_i. Define a map $R\langle T\rangle \to \mathcal{A}$ by sending $T \mapsto a_i$. On the special fibre this map is a finite monomorphism, hence it is finite.

Assume it was not a monomorphism and let $\alpha := \sum a_i T^i$ map to 0 in $\mathcal{A}$. We have assumed $\mathcal{A}$ is R-flat and hence R-torsion free. Hence assuming $\alpha \neq 0$, by dividing by a suitable power of the parameter π of R we can assume that at least one of the elements a_i is not divisible by π inside R. The injectivity on the special fibre implies that the reduction $\overline{\alpha} = 0$ or equivalently each reduction $\overline{a_i} = 0$ and this is a contradiction by our assumption. Hence the map defined above is also an injection.

Now this finite injection $R\langle T\rangle \to \mathcal{A}$ sets up a finite injection

$$K\langle T\rangle \to \mathcal{A} \otimes_{\mathcal{R}} \mathcal{K}$$

and it is known that this implies an integral injection[3]

$$(K\langle T\rangle)^\circ \to (\mathcal{A} \otimes_R K)^\circ.$$

Hence $(\mathcal{A} \otimes_R K)^\circ$ is integral over $R\langle T\rangle$ under the injection $K\langle T\rangle \to \mathcal{A} \otimes_{\mathcal{R}} \mathcal{K}$. But since $\mathcal{A}$ is assumed normal and is already finite over $R\langle T\rangle$, we have equality. ♣

Remark It would be interesting to have theory of the above avoiding normality[4].

We let $(\mathcal{A}_K)^{\circ\circ}$ denote the set of elements $f \in \mathcal{A}_K$ such that $|f|_{sup} < 1$. It is clear that $\pi\mathcal{A} \subset (\mathcal{A}_K)^{\circ\circ}$ and we shall prove the opposite inclusion. We use again the finite injection $R\langle T\rangle \to \mathcal{A}$ and let $|f|_{sup} < 1$ be an arbitrary element of $\mathcal{A}$. We invoke the extremely useful

[3] This is the satisfying classification of integral homomorphisms - see Theorem 1 on p.249 [14].

[4] See [14] p.247 Remark (at bottom) for a start.

Theorem 13 ([14] p.239 Proposition 4) *Let $\phi : E \to F$ be an integral map of Tate Algebras. Then for each $f \in F$ there exists a polynomial $q(T) = T^n + a_1T^{n-1} + \ldots + a_n$ over E such that $q(f) = 0$ and*

$$|f|_{sup} = \max_i(|a_i|^{1/i})$$

Letting $q_f(T)$ be the polynomial predicted above for the element f, we see that $|f| < 1$ implies each of the $a_i < 1$ and hence reduces to 0 under reduction $[mod\ \pi]$. [5] But this implies that $f^n \to 0$ under reduction $[mod\ \pi]$ and by the reducedness hypothesis we see that f reduces to 0, i.e. lies in $\pi\mathcal{A}$. Hence $(\mathcal{A}_K)^{\circ\circ} = \pi\mathcal{A}$.

This implies in particular that the reduction of $\mathcal{A}_K$ in the sense of [14] (p.270) is exactly the special fibre of $\mathcal{A}$. We can apply the usual theory and the map

$$red : Sp(\mathcal{A}_K) \to red(\mathcal{A}_K)$$

is the map sending a maximal ideal $\eta \in \mathcal{A}_K$ to the reduction of the prime ideal $\overline{\eta} = \sqrt{\eta \cap \mathcal{A} + \pi}$ to A_k - see [14] p.270 and notably Proposition 1. The reader should take care here - we have written $\overline{\eta}$ for the prime ideal (or point) in $\mathcal{A}$ and **not** its reduction. We remark that the prime ideal $\sqrt{\eta \cap \mathcal{A} + \pi}$ is the unique[6] maximal ideal of $\mathcal{A}$ containing the prime ideal $\eta \cap \mathcal{A}$.

We now choose an element $x \in X_k$ in the special fibre with associated maximal ideal m_x in $\mathcal{A}$. The formal fibre $\mathcal{X}_K(x+)$ is exactly all those maximal primes $\eta \in \mathcal{A}_K$ such that $\overline{\eta} \subseteq m_x$. Consider $L = \mathcal{A}_K/\eta$ for such an $\eta \mapsto x$. We have the maps $\mathcal{A} \hookrightarrow \mathcal{A}_K \to L$. The latter map is necessarily a contraction with respect to the supremum semi-norm - see [14] p.238 Proposition 1. Since $(\mathcal{A}_K)^\circ = \mathcal{A}$, it maps $\mathcal{A}$ into the ring of integers $\mathcal{O}_L$, which is L°, and thus it also maps the entire ideal m_x into the prime ideal $\mathcal{P}_L \subset \mathcal{O}_L$. However, m_x is maximal and hence is exactly the inverse image of $\mathcal{P}_L$. Consequently we obtain a map

[5] It is quick to prove that $\widetilde{(R\langle T\rangle)} = \pi R\langle T\rangle$.

[6] $\overline{\eta}$ is the unique π-adic open prime of $\mathcal{A}$ containing η since $\dim(\mathcal{A})$ is 2 and another π-adic open prime containing it would have to contain $\overline{\eta}$ - again a contradiction by the dimension.

of local algebras $\mathcal{A}_{m_x} \to \mathcal{O}_L$ which induces the map $\mathcal{A}_K \to L$. Furthermore, it induces a map $\widehat{\mathcal{A}_{m_x}} \to \mathcal{O}_L$ since $\mathcal{O}_L$ is already a complete local ring.

Conversely, given a map $\widehat{\mathcal{A}_{m_x}} \to \mathcal{O}_L$ we can find a map $\mathcal{A}_K \to L$ and it is not too hard to see that this gives an L point in the formal fibre of x. Furthermore these two correspondences are inverse to each other. Summing up we have proven

Lemma 14 *With the notations and conventions above, the L points in the formal fibre of x are naturally identified with $Hom_R(\widehat{\mathcal{A}}_{m_x}, \mathcal{O}_L)$ where m_x denotes the maximal ideal of x in $\mathcal{A}$. This correspondence is functorial: if $\mathcal{X} \to \mathcal{Y} = \mathcal{B}$ is a map sending x to y in the special fibre with $\mathcal{B}$ also satisfying the above conditions and if M/L is a finite extension, then we have diagrams*

$$\begin{array}{ccc} Hom_R(\widehat{\mathcal{A}}_{m_x}, \mathcal{O}_L) & \longrightarrow & Hom_R(\widehat{\mathcal{A}}_{m_x}, \mathcal{O}_M) \\ \downarrow & & \downarrow \\ Hom_R(\widehat{\mathcal{B}}_{m_y}, \mathcal{O}_L) & \longrightarrow & Hom_R(\widehat{\mathcal{B}}_{m_y}, \mathcal{O}_M) \end{array}$$

all commuting with the correspondence.

Smooth formal curves and geometric structure of Formal Fibre

Now let X_k/k be a smooth affine k-curve and we *assume* we can find a $finite$ map of degree r $X_k \to \mathbf{A}^1_k$ which is étale at x. Let the inverse image of $0 \in \mathbf{A}^1_k$ under $X_k \to \mathbf{A}^1_k$ be $x = y_1$ as well as the points $y_2, \ \ldots, \ y_m$. For completeness, the given map $X_k \to \mathbf{A}^1_k$ is not assumed Galois at this stage and hence r need not be the same as m. We do not want to confuse the reader with too many variables, but we shall use the variable r in a moment, and the variable m is for convenience later only.

As before this map lifts to a finite map on the formal schemes $\mathcal{X} \to \mathcal{D}$ which is also an injection on the associated R-algebras. Let D^0_K be the formal fibre of 0 in the closed rigid disc $\mathcal{D}_K$. For convenience we let $\mathcal{A} = \mathcal{O}(\mathcal{D}) = R\langle T \rangle$ and $\mathcal{B} = \mathcal{O}(\mathcal{X})$. Very importantly we have

Claim 15 (Flatness of formal map at x) *The map $\mathcal{X} \to \mathcal{D}$ is flat, and hence formally étale at the point x.*

Proof*:* The algebra $B_k = \mathcal{O}(X_k)$ is a finite flat $A_k = \mathcal{O}(\mathbf{A}^1_k)$ algebra since the latter is a Dedekind domain and the map $\mathcal{O}(\mathbf{A}^1_k) \to \mathcal{O}(X_k)$ is an injection of rings[7]. Hence for any point $z \in \mathbf{A}^1_k$ we have that $dim_{k(z)}\big[B_k \otimes_{A_k} k(z)\big] = r$; see Lemma D.1.2. Regarding z as a point of $\mathcal{D}$ or equivalently a prime ideal of $\mathcal{A} = R\langle T\rangle$ we have that $dim_{k(z)}\big[\mathcal{B} \otimes_{\mathcal{A}} k(z)\big] = r$. This includes the generic point of the special fibre, namely the point corresponding to the prime $\langle\pi\rangle$.

Notice also that the map is flat on the generic fibre[8], and hence for all the prime ideals w in the generic fibre $\mathcal{D}_K$ we have that $dim_{K(w)}\big[\mathcal{B} \otimes_{\mathcal{A}} K(w)\big] = s$ for some integer s. In particular this applies to the prime ideal (0). We claim that $s = r$; allowing this for a moment, invoking Lemma D.1.2 to the finite extension of rings $\mathcal{A} \to \mathcal{B}$ will give us the desired flatness result.

We proceed to proving that $r = s$. We localize the extension of rings $\mathcal{A} \hookrightarrow \mathcal{B}$ at the prime ideal $\langle\pi\rangle \subset \mathcal{A}$ and we find that $\mathcal{A}_\pi$ is a Dedekind domain and hence the extension $\mathcal{A} \hookrightarrow \mathcal{B}$ is flat. However, notice the ring $\mathcal{A}_\pi$ contains both a generic point, namely the prime ideal (0), as well as a special point, namely the prime $\langle\pi\rangle$. Hence applying Lemma D.1.2 to this ring, we find the dimensions r and s have to agree. ♣

Claim 16 *Let Z_K be any affinoid subdomain contained in the open formal fibre (the rigid open disc) of 0 in $\mathcal{D}_K$ and let W_K be the inverse image of this in $\mathcal{X}_K$. Then W_K is a finite disjoint union of affinoid subdomains W_i of $\mathcal{X}_K$ - each W_i lying inside the formal fibre of y_i and disconnected from the other W_j. We also have that W_x is étale finite over Z_K.*

Proof*:* Let f_i be a regular function on X_k, which is a parameter at y_i and a unit at y_j for $i \neq j$ and has poles outside X_k. Note we have assumed X_k an *affine* normal curve and hence by the strong approximation theorem for algebraic curves it follows that such f_i exist. Let $F_i \in \mathcal{O}(\mathcal{X}_K)$ be liftings of the elements f_i.

We write $W_i = W_K\big\{|F_j| \geq 1 \ for\ j \neq i\big\}$. This is an affinoid rigid subvariety of W_K; see p.280 of [14]. Note that set theoretically each W_i is the intersection of $\mathcal{X}_K(y_i+)$ with W_K and hence if $j \neq i$

[7] Any torsion free module over a Dedekind domain is flat.

[8] On the generic fibre the map corresponds to an injection of Dedekind domains.

we have (set theoretically) $W_i \cap W_j = \oslash$; see [14] p.270 Proposition 2. We note also that each W_i is by definition affinoid. Hence we find that W_K is the disjoint union of the W_i; see [14] p.280 lemma 8.

Although each W_i is an affinoid subdomain of W_K, the fact that W_K is the disjoint union of these imply that each W_i is finite over W_K; see [14] p.279 lemma 7.[9] Hence each composition

$$W_i \hookrightarrow W_K \rightarrow Z_K$$

is a finite morphism of rigid varieties.

Let $w \in \mathcal{X}_K(x+)$ and $z \in \mathcal{D}_K(0+)$ such that $w \mapsto z$. Notice that w reduces to x which is assumed étale in the *scheme* map $spec(\mathcal{O}(\mathcal{X})) \rightarrow spec(R\langle T \rangle)$. We have that w lies in each open neighbourhood of x and it is well known that being étale is an open property on affine schemes of finite type (and hence finite) morphisms. ♣

Hence for each $Z_K \hookrightarrow D_K^0$ the inverse image in the formal fibre of x is finite étale over Z_K. Finally we can state the crucial

Lemma 17 *The map $\mathcal{X}_K \rightarrow D_K$ restricted to the formal fibre of x is a bijection of the set of L-points of $\mathcal{X}_K$ inside the formal fibre to the of L-points of D^0. This is so for every L/K finite.*

Proof*:* The set of L points of $\mathcal{X}_K(x+)$ is in bijection with the hom set $Hom_R(\widehat{\mathcal{A}_x}, \mathcal{O}_L)$. However by the local characterization of an étale morphism we have that $\widehat{\mathcal{A}_x} = \widehat{R\langle T \rangle_0}$; see [18] Proposition 4.3.26. Hence we have that

$$Hom_R(\widehat{\mathcal{A}_x}, \mathcal{O}_L) = Hom_R(\widehat{\mathcal{R}\langle T \rangle_0}, \mathcal{O}_L).$$

The latter is the set of L points of $D_K^0 = D_K(0+)$, the formal fibre of 0 in the closed disc. ♣

Corollary 18 *The formal fibre of x in $\mathcal{X}_K$ carries the structure of a rigid open disc.*

Proof*:* As in the proof of Claim 16, we select elements F_i in the Tate algebra $\mathcal{O}(\mathcal{X}_K)$ with the properties as indicated there. We recall that the point y_1 was the point x by definition. We restrict

[9] Essentially, each W_i is a closed subvariety of W_K. Furthermore, closed immersions of rigid varieties are finite morphisms.

the rigid variety $\mathcal{X}_K$ to the rigid open affinoid subvariety $\mathcal{X}_{K,x} = \mathcal{X}_K\{|F_j| \geq 1 \; for \; j \neq i\}$. Notice that this restriction does not affect the set $\mathcal{X}_K(x+) = \mathcal{X}_K(y_1+) \subset \mathcal{X}_K$ since each point w in this set satisfies $|F_j(w)| \geq 1$ for $j \neq i$ and hence is contained inside the new rigid variety $\mathcal{X}_{K,x}$. We now consider only the composed morphism

$$\mathcal{X}_{K,x} \hookrightarrow \mathcal{X}_K \to \mathcal{D}_K$$

Notice also that the inverse image of the rigid open disc D^0_K inside $\mathcal{X}_{K,x}$ is now exactly the set $\mathcal{X}_K(x+)$. We now study the induced (or restricted) morphism $\mathcal{X}_K(x+) \to D^0_K$ and we claim that this is an isomorphism.

Let $Z_K \subset D^0_K$ be any rigid affinoid subvariety and let Z'_K be its inverse image inside $\mathcal{X}_K(x+)$ Notice that by definition of $\mathcal{X}_{K,x}$ this Z'_K is exactly the set $W_x = W_1$ of Claim 16 and there it was also shown that W_x is finite étale over Z_K. Hence $Z'_K = W_x$ is finite étale over Z_K and by the Lemma 17 the two rigid varieties agree on their sets of L points, where L/K is now an arbitrary finite field extension. We now invoke Lemma C.10 to conclude that $W_x \to Z_K$ is an isomorphism. The result follows. ♣

Theorem 19 *Let $\mathcal{X}/R$ be a smooth formal affine curve over R and let $x \in X$ be a closed point. Then the formal fibre of x in the associated rigid variety is a formal rigid open disc (the generic fibre of $R[\![T]\!]$).*

Proof*:* We need to justify our assumption at the beginning regarding the existence of a $finite$ map $X_k \to A^1_k$ which is étale at x. Let $X_k \subset \overline{X_k}$ be an embedding of an affine smooth k-curve into a projective smooth k-curve. Let $\{Q_1, \dots, Q_r\} = \overline{X_k} - X_k$ and let f be a local parameter of x such that it is neither a pole nor zero of the points Q_i. This exists by the strong approximation theorem. We have a map on function fields $k(t) \to k(X_k) : t \mapsto f$ and the induced map on projective curves $\phi_f : \overline{X_k} \to \mathbf{P}^1_k$ is étale finite at x. Furthermore ϕ_f maps the complement of X_k in $\overline{X_k}$ away from $\phi_f(x)$.

We can find a finite map $\mathbf{P}^1_k \to \mathbf{P}^1_k$ which is étale at $\phi_f(x)$ and maps the points $\phi_f(Q_i)$ to the infinity point. Composing this with ϕ_f gives us a finite map $\overline{X_k} \to \mathbf{P}^1_k$ which is étale at x and maps the complement of X_k to the infinity point on the projective line. Let $\mathbf{A}^1_k$ be the complement of the latter and we obtain a map $X'_k \to \mathbf{A}^1_k$ finite and étale at x. Here $X'_k \subset X_k$ is the open inverse image of $\mathbf{A}_k^1$ and it contains x. All we need to do now is to shrink the original formal scheme $\mathcal{X}$ to this new X_k' and we note this does not change the formal fibre of x. ♣

Slightly stronger is

Corollary 20 *Given an affine smooth formal R-curve $\mathcal{X}$, by allowing it to shrink, we can find an étale map of formal schemes $\mathcal{X} \to \mathcal{D}$ establishing an isomorphism of the formal fibre of x with the open rigid disc.*

4.3 Garuti's formal method

Let $\mathcal{X}$ be a formal smooth R-curve and let Y_k/X_k be a G-Galois covering in the special fibre. Let the ramification points in X_k be $\{x_i \in X_k\}$ and we cover X_k by affines $X_{k,i}$ such that each $X_{k,i}$ contains at most one of these ramification points. In what follows we shall be working in one chosen $X_{k,i}$ and we shall use the notation $Y_{k,i} = X_{k,i} \times_{X_k} Y_k$ and we assume x_i is the (unique) ramification point contained in $X_{k,i}$. Let $U_{k,i} = X_{k,i} - x_i$ be the complement open affine in $X_{k,i}$ and we let $V_{k,i} \subset Y_{k,i}$ be the inverse image of $U_{k,i}$. By our definitions each $V_{k,i}/U_{k,i}$ is G-Galois *étale*.

To all the open immersions $U_{k,i} \hookrightarrow X_{k,i} \hookrightarrow X_k$ we can associate open immersions $\mathcal{U}_i \hookrightarrow \mathcal{X}_i \hookrightarrow \mathcal{X}$. We know that we can lift each $V_{k,i}/U_{k,i}$ to a unique étale extension of formal affine schemes $\mathcal{V}_i/\mathcal{U}_i$ and this is G-Galois. We assume here that $Y_{k,i}/X_{k,i}$ is totally ramified over x_i although this is by no means necessary: this implies in particular that any projective curve birational to $Y_{k,i}$ is homeomorphic to it.

Consider the corresponding rigid spaces $\mathcal{X}_{iK} \to K$ and we note the open immersion $\mathcal{U}_{iK} \hookrightarrow \mathcal{X}_{iK}$. We note that the formal fibre of $x_i \in X_{k,i}$ inside the rigid space $\mathcal{X}_{iK}$ is exactly the complement of the $\mathcal{U}_{iK}$. However, we know (see cor. 20) that this carries the structure of an open disc. In particular, we can apply Garuti's Theorem 11 on extending the Galois extension $\mathcal{V}_{iK}/\mathcal{U}_{iK}$ to an affinoid normal rigid extension $\mathcal{Y}_{iK}/\mathcal{X}_{iK}$. We let $\mathcal{Y}_i$

be the normalization of $\mathcal{X}_i$ in $\mathcal{Y}_{iK}$.

Since $\mathcal{X}_i$ is a normal formal affine scheme and since the generic fibre extension $\mathcal{Y}_{iK}/\mathcal{X}_{iK}$ is finite, we have the normalization $\mathcal{Y}_i/\mathcal{X}_i$ is also finite[10].

Over $U_{k,i} \hookrightarrow X_{k,i}$ the special fibre of $\mathcal{Y}_i$ still reduces to $V_{k,i}$ and is thus reduced there. We can show that the special fibre is in fact totally reduced; see Theorem C.11 in the appendices. Gluing all the pieces $\mathcal{Y}_i$ along the étale loci to a formal extension of $\mathcal{X}$ and invoking algebraization gives Garuti's theorem on lifting the extension Y_k/X_k birationally and homeomorphically.

Example (Cyclic p-group extension) Let $C/\mathbf{P}^1_k$ now be a p-cyclic extension (with group G) extension of $\mathbf{P}^1_k$ generated by an equation

$$w^p - wt^{p-1} - \bar{\epsilon}t^{p-m_1} = 0$$

- here $0 < m_1 < p$ and $\bar{\epsilon} \in k^*$ and we have chosen a parameter t of the line $\mathbf{P}^1_k$. This defines (after normalization) an integral curve ramified over exactly the point 0 corresponding to the parameter t. The integral closure of $k[t, \frac{1}{t}]$ in this extension is simply given by

$$\frac{k[t, \frac{1}{t}][w]}{\langle w^p - wt^{p-1} - \bar{\epsilon}t^{p-m_1}\rangle}$$

and this lifts to the formal affine

$$B_{k[t,\frac{1}{t}]} = \frac{R\langle T, \frac{1}{T}\rangle\,[W]}{\langle W^p - WT^{p-1} - \epsilon T^{p-m_1}\rangle}$$

where ϵ is some lifting of $\bar{\epsilon}$. This extension is the unique finite *étale* (and hence G-Galois) extension over $Spf(R\langle T, \frac{1}{T}\rangle)$: it suffices to show its flatness; for this one can invoke an argument similar to that of Lemma 15.

We can perturb the coefficients of this equation by multiples of the parameter of R (a special case of Krasner's lemma) and in view of the proof of Theorem 1 earlier we would like to get this equation into a Kummer form.

[10]We consulted [18] p.121 Proposition 4.1.25 for the finiteness statement.

One finds this extension is the same as

$$\frac{R\langle T, \frac{1}{T}\rangle [W]}{\left\langle (W+\frac{T}{\beta})^p - (\frac{T}{\beta})^p - \epsilon T^{p-m_1} \right\rangle}$$

Here β is an element satisfying $p(\frac{T}{\beta})^{p-1}W^1 = -WT^{p-1}$. We note that working over R we must take care in manipulating the defining equation of this equation since $\frac{1}{\beta}$ is not in R. We switch to the generic fibre ($\otimes_R K$) and this extension is then the same as the *Kummer* extension given by $W^p = T^{p-m_1}(T^{m_1} + \beta^p \epsilon)$ where W is now a new indeterminate. It is immediate that this then extends to the rigid disc. For later use, we state that we can calculate the singular number on the special fibre and one finds this to be exactly 0. Hence the above lifting is actually a smooth lifting. Notice that for $R = W(k)[1^{\frac{1}{p}}]$ we can choose $\beta = \lambda = \zeta_p - 1$ where ζ_p is a p^{th} root of unity.

Example (elementary p^3-extension) The following example is inspired by [2]. Let G be the additive group of the finite field $\mathbf{F}_{p^3}$ of p^3 elements and this acts by translation on the line $\mathbf{P}^1_k$. It is known that the quotient $\mathbf{P}^1_k/G = \mathbf{P}^1_k$. To avoid confusion we shall denote by C_s the first projective line and C_t the second. The covering C_s/C_t is totally ramified at $\infty \in C_s$ and this is the only point of ramification. We shall assume t is chosen a parameter of C_t such that $\frac{1}{t}$ corresponds to the image of $\infty \in C_s$. The different of the extension is $d_{C_s/C_t} = 2(p^3-1)$ and is also the local different at the ∞.

The extension C_s/C_t with this action is generated by equations

$$y_i^p - y_i = \overline{\epsilon_i} t^{-1}$$

where $i = 1, 2, 3$ and ϵ_i are three distinct units of k.

Just as above these equations induce cyclic p coverings of C_t whose compositum is C_s. Applying Garuti's argument we find that for each i we can normalize $R\langle T\rangle$ inside a Kummer equation of the form $W_i^p = F_i(T)$ where the $F_i(T) = T^{p-1}(T + \beta^p \epsilon_i)$ are the polynomials found previously. Using Kummer theory we find 2 different ramification points inside the formal fibre of $\infty \in C_t$ after extending with each $W_i^p = F_i(T)$ and notice that $\langle 0\rangle$ is a common ramification point of all three of these extensions. Using this we can bound the generic different of the compositum extension by $D_K \leq 4p^2(p-1)$. One can thus bound the singularity number δ and one finds in general $\delta \leq \frac{1}{2}[2p^3 - 4p^2 + 2]$. Note for $p = 2$ we have $\delta \leq 1$. We can continue:

Proposition 21 *The elementary p^3-extension above cannot be lifted smoothly to characteristic* 0.

Proof*:* Assume such a lift exists. We have indicated in Chapter 2.2 that the lifting C_s/C_t needs to be of the form P^1_R/P^1_R and hence induces over K a G-Galois extension $\mathbf{P}^1_K/\mathbf{P}^1_K$. The trick is to study possible elementary p^3-subgroups of $Aut_K(\mathbf{P}^1_K)$ possibly lifting the given one in characteristic p. Let ζ_p be a p^{th} root of unity in K. As before in order to avoid confusion between the respective projective lines we let Q_s and Q_t be the projective K lines corresponding to the special fibre lines C_s and C_t respectively. Let the Galois group be G. By assumption the generic different of Q_s/Q_t is $D_K = 2(p^3 - 1)$.

The group G has precisely $\frac{p^3-1}{p-1}$ non trivial order $|p|$ subgroups. Each subgroup $G_i \subset G$ has at least[11] two fixed points a_i and b_i. Assume no two non-trivial order $|p|$ subgroups have a common fixed point. In order to meet the requirement of $D_K = 2(p^3 - 1)$, we must have for all i

$$e(a_i) = e(b_i) = p - 1$$

This is a contradiction since this will partition the set of fixed points in Q_s into equal blocks of size p^2 and hence imply that $p^2 | 2\frac{p^3-1}{p-1}$, a contradiction.

Hence we can assume there exists two disjoint (except for the identity of course) order p subgroups, $G_1 = \langle\sigma_1\rangle$ and $G_2 = \langle\sigma_2\rangle$, which have a common fixed point. Using a suitable transformation we can assume this corresponds to the origin $0 \in Q_s$ $(= \mathbf{P}^1_K)$ with parameter z. We assume again that under this transformation G_1 and G_2 also has b_1 and b_2 respectively as other fixed points. By raising each σ_i to a high enough power we can assume that each σ_i is given (in terms of the parameter $z \in K(z) = K(\mathbf{P}^1_K)$) by[12]

$$\sigma_i : \mathbf{P}^1_K \to \mathbf{P}^1_K : z \mapsto \frac{-b_i\zeta_p z}{(1-\zeta_p)z - b_i}$$

By assumption the automorphisms σ_1 and σ_2 commute and hence we must have $\sigma_2 \circ \sigma_1 = \sigma_1 \circ \sigma_2$.

[11]There are no non-trivial étale extensions of the projective line and all tame extensions of the line admit at least two ramification points.

[12]An automorphism of $\mathbf{P}^1_K$ fixing 0 and ∞ and of order p must be of the form $z \to \zeta_p^j z$ for some j with $(j,p) = 1$. One obtains the general case by a suitable invertible transformation.

Substituting this into the above we find that

$$\sigma_1 \circ \sigma_2 : \ \mathbf{P}^1_K \to \mathbf{P}^1_K : \ z \mapsto \frac{-b_1\zeta_p \frac{-b_2\zeta_p z}{(1-\zeta_p)z-b_2}}{(1-\zeta_p)\frac{-b_2\zeta_p z}{(1-\zeta_p)z-b_2} - b_1}$$

or equivalently

$$\sigma_1 \circ \sigma_2 : \ \mathbf{P}^1_K \to \mathbf{P}^1_K : \ z \mapsto \frac{b_1 b_2 \zeta_p^2 z}{\big[-b_2\zeta_p + b_2\zeta_p^2 - b_1 + b_1\zeta_p\big]z + b_1 b_2}$$

and since a similar formula holds for $\sigma_2 \circ \sigma_1$ we must have

$$-b_2\zeta_p + b_2\zeta_p^2 - b_1 + b_1\zeta_p = -b_1\zeta_p + b_1\zeta_p^2 - b_2 + b_2\zeta_p$$

or equivalently

$$\zeta_p(\zeta_p - 1)b_2 + b_1(\zeta_p - 1) = \zeta_p(\zeta_p - 1)b_1 + b_2(\zeta_p - 1)$$

This implies that $b_1 = b_2$ and hence the two order p automorphisms of $\mathbf{P}^1_K$ σ_1 and σ_2 have the same fixed points, and hence are the same up to powers.[13] This is a contradiction since the groups G_1 and G_2 are two disjoint automorphism groups of $Q_s = \mathbf{P}^1_K$. ♣

The following example is inspired by Sekiguchi-Suwa theory ([8] and [10]). This powerful theory gives an explicit way of lifting p^n generating equations in characteristic p to ones of Kummer form in characteristic 0. Together with Garuti's ideas we thus have an explicit way of lifting extensions, albeit with possible singularities. The following is thus only an explicit example, where we have taken the general *Kummer-Artin-Schreier-Witt* theory (see the citation) and specialized for $p = 2$; see the citation for details.

Example (p^2-Extension: $p = 2$) Consider the cyclic 4 extension of $\mathbf{P}^1_k$ generated by

$$x_0^2 + x_0 = t^{-m_1}$$

and

$$x_1^2 - x_1 + x_0^2 + x_0^3 = t^{-m_2}$$

and assume $m_2 \geq m_1$. Using Sekiguchi-Suwa theory we know that we can lift this (uniquely) to the formal scheme $Spf(R\left\langle T, \frac{1}{T}\right\rangle)$ and then to the rigid circle $Sp(K\left\langle T, \frac{1}{T}\right\rangle)$ as a p^2-Kummer extension

[13] Again this is proved by making a suitable transformation to the case where the two fixed points are 0 and ∞.

given by

$$W_2^4 = T^{-(m_1+2m_2)}(4+T^{m_1})(4+T^{m_2}-2iT^{(m_2-m_1)})^2$$

where $i^2 = -1$. This equation extends visibly to the rigid circle $Sp(K\langle T\rangle)$.

One can check that for certain values of m_1 and m_2 the above gives a smooth lifting. However for values $m_1 = 1$ and $m_2 = 2$ it does not work. Later we shall find a more systematic way of deforming p^2-cyclic extensions. However, recently it was noted that there are also other families of curves in characteristic 0 with C_4 actions which reduce correctly in characteristic 2 and of the form just mentioned (i.e. locally of the family with $m_1 = 1$ and $m_2 = 2$). As this thesis is printed, this is a topic of some more research.

Chapter 5

Wild ramification and local approaches

5.1 Local-global principle

We start by stating the local global principle, proved in [2] and later reproved [4]. The proof given in [2] is a geometric proof relying on a rigid form of algebraization. It too is an interesting approach and we recommend the reader to read that paper. Henrio's precise formulation is

Theorem 1 *Let $\mathcal{X}$ be an affine smooth formal R-curve and let G be some finite abstract group. Let Y_k/X_k be a (ramified) G-Galois extension of k-curves and assume that $x \in X_k$ is the only point of ramification. Let $U_k = X_k - x$ be the étale locus of X_k and let V_k be its inverse image in Y_k. Assume we can find a G-Galois extension $\mathcal{B}_x/\widehat{\mathcal{O}_{\mathcal{X},x}}$ fitting the following diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi\mathcal{B}_x & \longrightarrow & \mathcal{B}_x & \longrightarrow & \mathcal{O}(Y_k) \otimes_{\mathcal{O}(X_k)} \widehat{\mathcal{O}(X_k)}_x & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle G} & & \Big\downarrow{\scriptstyle G} & & \Big\downarrow{\scriptstyle G} & & \\ 0 & \longrightarrow & \pi\widehat{\mathcal{O}_{\mathcal{X},x}} & \longrightarrow & \widehat{\mathcal{O}_{\mathcal{X},x}} & \longrightarrow & \widehat{\mathcal{O}(X_k)}_x & \longrightarrow & 0 \end{array}$$

such that all the arrows are G-equivariant: we recall that the latter part of the diagram has an induced G-action coming from the given extension in the special fibre. Then there *exists a G-Galois lifting of smooth formal R-curves $\mathcal{Y}/\mathcal{X}$ such that this extension reduces to the given G-Galois extension in the special fibre. Furthermore this extension $\mathcal{Y}$ can be chosen such that $\mathcal{O}(\mathcal{Y}) \otimes_{\mathcal{O}(\mathcal{X})} \widehat{\mathcal{O}_{\mathcal{X},x}} \cong \mathcal{B}_x$ G-equivariantly.*

We shall postpone discussing Henrio's proof to a later stage, see 5.4. Essentially the idea is this: one notes that for a smooth normal k-curve C_k and for some $x \in C_k$, by replacing C_k by some open neighbourhood of x we can write $\mathcal{O}(C_k) = \widehat{\mathcal{O}_{C_k,x}} \cap \mathcal{O}(C_k - x)$ where the intersection is taken inside a suitable larger structure containing all rings involved, namely the fraction field of $\widehat{\mathcal{O}_{C_k,x}}$. The idea is to lift this construction to characteristic 0. Here one uses some theory of completed algebras and in fact one cannot do with out it. An added bonus is that it arises naturally that this construction is a G-equivariant one and so follows the Galois lifting. See the references for details.

It is worth stating the following theorem, which in view of above gives some structure and places the lifting found earlier (in the example of chapter 4) in context:

Theorem 2 ([8] p.3–4 Assertions 1 & 2; *Kummer-Artin-Schreier-Witt* Sequences) *There exists smooth group schemes $\mathcal{W}_n$ and $\mathcal{V}_n$ over S such that we have the exact sequence*

$$0 \to \mathbf{Z}/p^n\mathbf{Z} \to \mathcal{W}_n \xrightarrow{\Psi_n} \mathcal{V}_n \to 0$$

on the flat or étale site of $S = spec(R)$ where R is assumed to contain enough roots of unity. If B is a flat local R-algebra then

(5.1) $H^1_{fl}(spec(B), \mathcal{W}_n) = H^1_{et}(spec(B), \mathcal{W}_n) = 0$

Immediately one has the following corollary

Corollary 3 (*Kummer-Artin-Schreier-Witt* Theory) *Let $spec(C) \to spec(B)$ be an unramified étale $\mathbf{Z}/p^n\mathbf{Z}$-Galois extension of flat local R-algebras. Then there exists a map $f : spec(B) \to \mathcal{W}_n$ such that the covering $spec(C) \to spec(B)$ is given by the fibre product $spec(B) \times_{\mathcal{W}_n} \mathcal{V}_n$.*

The corollary follows immediately from 5.1 and the explicit interpretation in terms of Galois extensions of the first boundary map of the associated étale long exact cohomology sequence.

Corollary 4 *Let $k((z))/k((t))$ be a $\mathbf{Z}/p^n\mathbf{Z}$ Galois extension of local fields and* **assume** *we have a characteristic* 0 *extension $\mathcal{B}_x/R[\![T]\!]$ lifting the Galois extension $k((z))/k((t))$. Then the defining equations of the field of fractions of $\mathcal{B}_x$ over that of $R[\![T]\!]$ can be given as the* Kummer-Artin-Schreier-Witt *theory predicts.*

For instance, one can show that in Theorem 2 with $n = 1$ we can take

$$\mathcal{W}_1 = spec(R[X, \frac{1}{1+\lambda X}])$$

and

$$\mathcal{V}_1 = spec(R[Y, \frac{1}{1+\lambda^p Y}])$$

with the map Ψ_1 of Theorem 2 given in rings as

$$\Psi_1^{\#} : \ Y \mapsto \frac{(\lambda X + 1)^p - 1}{\lambda^p}$$

Corollary 3 then tells us that if a lifting $\mathcal{B}_x/R[\![T]\!]$ exists it will necessarily be the normalization of an equation $\Psi_1^{\#}(Y) = F(T)$ where $F(T) \in R[\![T]\!]_\pi$. Note one needs to use $R[\![T]\!]_\pi$ since the *Kummer-Artin-Schreier-Witt* theorem applies only to *étale* extensions, which by assumption is the case at the prime ideal $(\pi) \in spec(R[\![T]\!])$. We shall return to *Kummer-Artin-Schreier-Witt* theory in due time. We also want to mention at this stage the appendix section that we included on Hurwitz trees. It was not the theme of our studies, but is helpful in constructing automorphisms of the disc, which we now know is intrically tied to the lifting problem.

5.2 Local global application : p^2-cyclic lifting

We have already seen earlier that the following deformation gives a lifting of all local p-cyclic wild coverings generated by equation B.4 in appendix B

$$Y_1 = \frac{(\lambda^p X_1 + 1)^p - 1}{\lambda^p} = T^{-m_1}$$

Here one chooses $\lambda = \zeta_p - 1$ for some root of unity such that $p\lambda \equiv -\lambda^p \left[mod\ \lambda^p \pi\right]$ where π is any parameter of R, assumed to contain such an element λ.

We shall now focus on an extension $k(\!(t)\!) \subset k(\!(z_1)\!) \subset k(\!(z_2)\!)$ generated by the equations B.4 and B.5 in appendix B and we assume the notations there. We shall let

$$d_{1/0} = (m_1 + 1)(p - 1)$$

denote the different of $k(\!(t)\!) \subset k(\!(z_1)\!)$ and similarly $d_{2/1}$ that of $k(\!(z_1)\!) \subset k(\!(z_2)\!)$. It is not too difficult to calculate $d_{2/1}$ explicitly

Claim 5 *We can find a parameter of $k(\!(z_1)\!)$, which we also denote by z_1 such that $z_1^{-m_1} = x_1$ and the parameter t is related by*

$$t^{-m_1} = z_1^{-pm_1}(1 + z_1^{m_1(p-1)})$$

We set

(5.2) $y_2 = x_2 - \sum_{i,j} (a_{ij}^{(2)})^{\frac{1}{p}} x_1^j t^{-i}$

We also set

$$\mu_2 = max\{m_1 p^2 - m_1 p + m_1,\ ip^2 + m_1(p(j-1)+1)\}$$

and notice this is relatively prime to p due to our assumption on m_1. We can then find a second parameter z_2 of $k(\!(z_2)\!)$ such that $z_2^{-\mu_2} = y_2$. The different of $k(\!(z_1)\!) \subset k(\!(z_2)\!)$ is

$$d_{2/0} = (\mu_2 + 1)(p - 1) + pd_{1/0} = (p - 1)(\mu_2 + 1 + pm_1 + p)$$

We do not give all the details here, but we comment briefly that the claim is proved by noting that $k(\!(z_2)\!)$ is generated over $k(\!(z_1)\!)$ by an equation of the form $y_2^p - y_2 = z_1^{-\mu_2}\Phi(z_1)$ where $\Phi(z_1)$ is some unit in $k[\![z_1]\!]$.

The first fundamental step in lifting the extension to characteristic 0 is the following theorem of Green and Matignon

Theorem 6 ([2] Lemma 5.3 p.260) *Let π_2 be a parameter of R, here assumed to contain the p^2-roots of unity. We shall use the notation $Exp_p(X)$ to mean the degree p truncated Artin-Hasse Exponential*[1]*. For an element $\mu \in R$ let*

$$F(X) = \frac{Exp_p(\mu X_1)^p - (\lambda X_1 + 1)Exp_p(\mu^p Y_1)}{p\mu^p} \in R[\![X_1]\!][\frac{1}{p\mu_2^p}]$$

where Y_1 was defined earlier in X_1.[2] *Then there exists a parameter μ of R such that $F(X_1) \in R[\![X_1]\!]$ and*

$$F(X_1) \equiv \frac{X_1^{p^2} - X_1^p - (X_1^p - X_1)^p}{p} (mod\ \pi_2)$$

Hence we have lifting of the second Artin Schreier component of equation $B.5$ in appendix B:

$$\frac{(\lambda X_2 + Exp_p(\mu_2 X_1))^p - (\lambda X_1 + 1)Exp_p(\mu_2^p Y_1)}{p\mu_2^p}$$
$$\equiv X_2^p - X_2 - \frac{X_1^{p^2} - X_1^p - (X_1^p - X_1)^p}{p} \left[mod\ \pi_2\right]$$

It is not easy to prove the above and details can be found in the citation. The advantage of the lifting is that it immediately implies a Kummer p^2-cyclic lifting in characteristic 0 and hence Galois. We now study the smoothness properties. A first attempt is to consider the equations

$$(5.3)\quad Y = T^{-m_1}$$

and

$$(5.4)\quad \frac{(\lambda X_2+Exp_p(\mu_2 X_1))^p-(\lambda X_1+1)Exp_p(\mu_2^p Y_1)}{p\mu_2^p} - (\lambda X_1 + 1)\Big[\sum_{i,j} A_{ij}^{(2)} Y^j T_{ip} + \sum_k B_k^{(2)} + V_2\Big] = 0$$

where $A_{ij}^{(2)}$, $B_k^{(2)}$ and V_2 are lifts of the analogous elements in characteristic p of equation B.5. We choose them such that if $a_{ij} = 0$ in k then $A_{ij} = 0$ and similarly for B_k. These

[1] See [2] for details or [8] for generalizations.

[2] We consider X_1 as an indeterminate and no relation to T as yet.

equations can be combined into a single p^2-cyclic Kummer type extension in the generic fibre $R[\![T]\!] \otimes_R K$ as

(5.5) $\psi_1^p = T^{-m_1}(\lambda^p + T^{m_1})$

(5.6) $\psi_2^p = \psi_1 \left[\frac{Exp_p(\mu_2^p Y_1)}{p\mu_2^p} + \sum_{i,j} A_{ij}^{(2)} Y^j T_{ip} + \sum_k B_k^{(2)} + V_2 \right] = \psi_1 T^{-N_2} F_2(T)$

Here N_2 is the maximum degree of the fractional terms T^{-1} and $F_2(T)$ is a suitable polynomial. Notice that $F_2(T)$ need not be of degree N_2 due to the T polynomial V_2. It is readily seen that

$$N_2 = \max_{\substack{0 \le i \le m_2 \\ 1 \le j \le p-1}} \{(p-1)m_1,\ ip + m_1 j\}$$

We can rewrite the equations above as a single equation

$$\psi_2^{p^2} = T^{-m_1 - pN_2} F_1(T)[F_2(T)]^p$$

where $F_1(T) = \lambda^p + T^{m_1}$. We first assume that the second equation above was dominated by the T^{-1} terms of $Exp_p(\mu_2^p Y_1)$, i.e. $N_2 = (p-1)m_1$.

Claim 7 *The generic different is given by* $D_{2/0} = (p-1)[(m_1+1)(p+1) + pN_2]$ *which equals the special different* $d_{2/0}$. *Hence in this case the above results in a smooth lifting.*

Proof*:* We consider the polynomial $F_2(T)$ and write it as

$$F_2(T) = \sum_{0 \le k \le N_2} \gamma_k T^k + \sum_{k > N_2} \gamma_k T^k$$

Here the γ_i are suitable elements of R. By assumption it is seen that the element $\gamma_{N_2} \notin R$ and hence carries a negative π_2 valuation[3]. Our assumption also implies that for $k > N_2$ we have $\gamma_k \in R$.

The formal Weierstrass preparation theorem now implies that we can write $F_2(T) = \widetilde{F_2}(T)\epsilon(T)$ where $\epsilon(T)$ is a unit of $R[\![T]\!] \otimes_R K$ and $\widetilde{F_2}$ is a polynomial of degree N_2. Hence the number points of $R[\![T]\!] \otimes_R K$ where $F_2(T)$ is a zero are bounded by N_2. Kummer theory now implies that

$$D_{2/0} \le (p-1)[(m_1+1)(p+1) + pN_2] = (p-1)[(m_1+1)(p-1) + p(p-1)m_1]$$

The latter is equal to $d_{2/0}$, the former is at least $d_{2/0}$ and hence the equality follows from Theorem 5 in appendix B.♣

[3]Remember that $Exp_p(\mu_2^p Y_1)$ has a constant term 1 in its expansion.

Remark In general when $N_2 \neq (p-1)m_1$ then we can still write $F_2(T) = \widetilde{F_2}(T)\epsilon(T)$ with $F_2(T)$ a polynomial of degree N_2, however the upper bound we find for the generic different exceeds the special different. A moment's thought reveals that the problem lies in the fact that to estimate the different of equation B.5 in the above case, we had to substitute x_2 for y_2 and this resulted in eliminating x_1^{-jp} terms (in the notations of equation B.5). In our equations the terms $T^{-m_1 j}$ are causing the problems and we need to get rid of them. Green and Matignon's trick is to move them to the left hand side of the Kummer equations and we explain this next.

For convenience we set $\Delta_{2,i,j} = T^{-i}\mu_2^j \epsilon_{2,i,j}$ where $\epsilon_{2,i,j}$ are elements of R which we shall make explicit later. We ask the reader's patience so that we can state the following rather technical result found in [2]

Claim 8 *Working modulo $\pi_2\mu_2^p p$ we have an identity*

(5.7) $Exp_p(\mu_2^p Y)[1 + \sum_{\substack{0\le i\le m_2 \\ 1\le j\le (p-1)}} \Delta_{2,i,j}]^p \equiv$

$$\big(1 + \sum_{\substack{0\le i\le m_2 \\ 1\le j\le (p-1)}} \Delta_{2,i,j}\big)^p - 1 - \sum_{\substack{0\le i\le m_2 \\ 1\le j\le (p-1)}} \Delta_{2,i,j}^p + Exp_p(\mu_2^p Y) + \sum_{\substack{0\le i\le m_2 \\ 1\le j\le p-1 \\ 0\le k\le p-i}} \mu_2^{p(k+j)} T^{-pi} Y^k \frac{\epsilon_{2,i,j}^p}{k!}$$

Remark One notes that the π_2 valuation of $\pi_2\mu_2^p p$ is that of $\pi_2\mu_2^{p^2}$. The restriction of the indices and the factors μ_2^j in the $\Delta_{2,i,j}$ makes sure that the above terms are thus the only ones which remain after moding out by $\pi_2\mu_2^p p$. Notice how the μ_2^j acts as a *controller* for combinations of Y^k and T^{-pi}.

The above implies

Corollary 9 *Working modulo $\pi_2\mu_2^p p$ we have the following congruences*

$$[Exp_p(\mu_2 X_2)^p - (\lambda X_1 + 1)Exp_p(\mu_2^p Y)]\{1 + \sum_{i, 1\le j\le (p-1)} \Delta_{2,i,j}\}^p \equiv$$

$$p\mu_2^p[\frac{X_1^{p^2} - X_1^p - (X_1^p - X_1)^p}{p}]$$

and thus

$$(5.8)\quad Exp_p(\mu_2 X_2)^p\{1+\sum_{\substack{0\le i\le m_2\\1\le j\le(p-1)}}\Delta_{2,i,j}\}^p-(\lambda X_1+1)\,G_2(Y,T^{-1})\equiv$$

$$p\mu_2^p\Big[\frac{X_1^{p^2}-X_1^p-(X_1^p-X_1)^p}{p}\Big]+\sum_{\substack{0\le i\le m_2\\1\le j\le p-1\\k=p-j}}\mu_2^{p^2}T^{-pi}Y^k\frac{\epsilon_{2,i,j}^p}{k!}$$

Here $G_2(T,Y)$ are the terms of equation 5.7 with coefficients of valuation less than that of $\mu_2^{p^2}$, i.e. all except the terms $\mu_2^{p(k+j)}T^{-pi}Y^k\frac{\epsilon_{2,i,j}^p}{k!}$ with $j+k=p$.

With some thought the trick of Green and Matignon now becomes clear: we choose $\epsilon_{2,i,j}$ such that modulo $\pi^2 p\mu_2^p$ we have

$$\frac{\epsilon_{2,i,j}^p\mu_2^{p^2}}{k!}\equiv p\mu_2^p a_{i,k}^{(2)}$$

where $k=p-j$. We also set $\epsilon_{2,i,j}=0$ if $a_{i,k}=0$. That such elements exist follows from the fact that k is algebraically closed and that the valuation of $p\mu_2^p$ is the same as $\mu_2^{p^2}$.

We have noted earlier on that the problem lies in the dominating terms of N_2. As illustration assume the highest contributor of equation B.5 was the term corresponding to $i=i_0$ and $j=j_0$. This implies in particular that $a_{i_0,j_0}^{(2)}\neq 0$ but $a_{i_0,j_0+1}^{(2)}=0$, where if $j_0=p-1$ we simply set $a_{i_0,j_0+1}^{(2)}=a_{i_0,p}^{(2)}=0$. In particular $\epsilon_{2,i_0,p-j_0}\neq 0$ and by our choice we find that $\epsilon_{2,i_0,p-j_0-1}=0$. Hence the only term of the form $T^{-i_0p}Y^{-j_0}$ in equation 5.8 are those appearing on the right hand side, i.e. not in $G(T^{-1})$. The idea is now to use $G(T^{-1})$ in a well chosen Kummer equation.

We finally arrive at the following theorem of Green and Matignon

Theorem 10 ([2] lemma 5.4) *We have the following identity modulo $\pi_2 p\mu_2^p$*

$$\Bigg[\lambda X_2+Exp_p(\mu_2X_2)\{1+\sum_{\substack{0\le i\le m_2\\1\le j\le(p-1)}}\Delta_{2,i,j}\}\Bigg]^p-(\lambda X_1+1)\Bigg[G_2(Y,T^{-1})+p\mu_2^p\sum_k B_k^{(2)}+p\mu_2^pV_2\Bigg]\equiv$$

$$p\mu_2^p\Bigg[x_2^p-x_2-\frac{x_1^{p^2}-x_1^p-(x_1^p-x_1)^p}{p}-\sum_{\substack{0\le i\le m_2\\0<j<p}}a_{i,j}^{(2)}(x_1^p-x_1)^jt^{-ip}-\sum_{0\le k\le(N_1-1)(p-1)}b_k^{(2)}t^{-k}-v_2(t)\Bigg]$$

Hence we have an equation

$$\frac{\Psi_2^p - (\lambda X_1 + 1)[G_2(Y, T^{-1}) + p\mu_2^p \sum_k B_k^{(2)} + p\mu_2^p V_2]}{p\mu_2^p} = 0, \tag{5.9}$$

which lifts the Artin Schreier equation B.5. Here Ψ_2 refers to

$$\Psi_2 = \lambda X_2 + Exp_p(\mu_2 X_2)\{1 + \sum_{\substack{0 \le i \le m_2 \\ 1 \le j \le (p-1)}} \Delta_{2,i,j}\}$$

In order to estimate the generic different one considers the expression

$$F_2(T) = G_2(Y, T^{-1}) + p\mu_2^p \sum_k B_k^{(2)} + p\mu_2^p V_2$$

Even if the degree of $V_2(T)$ is very large, the Weierstrass preparation theorem allows us to factor the above F_2 again and this time one finds that the special different is actually an upper bound for the generic different. The extension is thus the correct one. For details on the computations and similar matters, see [2].

5.3 Jacobian method II: Wild ramification and *Kummer-Artin-Schreier-Witt*

As indicated earlier on, one knows that the entire class field theory of a curve over a field is given in terms of group extensions of singular Jacobians. However, restricting to cyclic p extensions, one knows that this theory can also be built up from Kummer or Artin Schreier extensions in characteristics 0 and p respectively. The fundamental step is that one has an exact sequence of smooth group schemes

$$0 \to \mathbf{Z/pZ} \to G_1 \to H_1 \to 0$$

such that for a certain class of schemes $\{X \to k \ or \ K\}$ we have the exact sequence

$$\ldots \to H_1(X) \to H^1_{et}(X, \mathbf{Z/pZ}) \to 0 \to \ldots$$

This works for instance on applying Kummer theory[4] to the function field of a curve over K or Artin Schreier theory[5] again with the function field of a curve over k. For convenience let F be either k or K. Throughout what follows G will always be the cyclic p-group. Let C'/C be a Galois cyclic p cover of smooth curves . Notice that this induces an étale Galois extension on the function fields and hence by the surjections

$$H_1(F(C)) = Hom_F(F(C), H_1) \to H^1_{et}(F(C), \mathbf{Z/pZ}) \to 0$$

we find that the extension $F(C')/F(C)$ is necessarily given by an extension of the type

$$y^p = f \in K(C)$$

or

$$y^p - y = f \in k(C)$$

depending on the characteristic of C. Indeed this is just Kummer or Artin-Schreier theory for the class of k or K-schemes over $fields$. The crucial observation now is that the class of normal curves is equivalent to that of function fields and extensions thereof. We can summarize by the following *rational* fibre product diagram

$$\begin{array}{ccc} C' & \xrightarrow{rational} & G_1 \\ \downarrow & & \downarrow \\ C & \xrightarrow{rational} & H_1 \end{array}$$

Here the Sekiguchi-Suwa theory makes its mark. Let C'/C now be a Galois extension of smooth arithmetic surfaces over R, assuming as always that R contains enough roots of unity. Let $x \in C$ be a point such that C'/C is étale over x and we assume the extension C'/C is also Galois in its special fibre[6]. Locally above x we have an étale Galois cyclic p

[4]With

$$G_1 = G_{m,K} \overset{x \mapsto x^p}{\to} H_1 = G_{m,K}$$

and the surjection is then given by Hilbert Theorem 90

[5]We have

$$G_1 = W_{1,k} = G_{a,k} \overset{F-1}{\to} H_1 = W_{1,k} = G_{a,k}$$

with surjection again by the fact that $H^1_{et}(F, W_1) = 0$ for any characteristic p field F.

[6]In other words not inducing a purely inseparable extension modulo π

extension of rings

$$\mathcal{O}_{C,x} \hookrightarrow \mathcal{O}(C') \otimes_{\mathcal{O}(C)} \mathcal{O}_{C,x}$$

and hence by Sekiguchi-Suwa theory (see Theorem 2) this factors as

$$\begin{array}{ccc} & & (\mathbf{Z}/\mathbf{pZ})_R \\ & & \downarrow \\ spec(\mathcal{O}(C') \otimes_{\mathcal{O}(C)} \mathcal{O}_{C,x}) & \longrightarrow & \mathcal{W}_1 \\ \downarrow & & \downarrow \\ spec(\mathcal{O}_{C,x}) & \longrightarrow & \mathcal{V}_1 \end{array}$$

One finds that we can factor the situation again as in the curve case, but the important property - the rational maps are completely defined on the étale loci of C'/C.

$$\begin{array}{ccc} C' & \longrightarrow & \mathcal{W}_1 \\ \downarrow & & \downarrow \\ C & \longrightarrow & \mathcal{V}_1 \end{array}$$

Let us assume the extension of schemes C'/C is such that there exists an effective divisor $\delta \in C$ such that C'_K/C_K has conductor $\delta_K \subset C_K$ on the generic fibre and C'_k/C_k has special conductor δ_k on the special fibre.[7] Since the *Kummer-Artin-Schreier-Witt* sequence is a deformation of the Kummer sequence one knows that generically the map $C_K \to \mathcal{V}_1 = G_{m,K}$ has a modulus δ_K.[8] One can check that the induced rational map $C \to \mathcal{V}_1$ has moduli δ_k and δ_K on the special and generic fibres respectively. This is the point - we can invoke ([7] p.360 Proposition 5; we listed it as Theorem 7 in chapter 3) and its proof to conclude that the rational map $C \to \mathcal{V}_1$ factors through an unique S-group homomorphism $J_\delta \to \mathcal{V}_1$ inducing the analogous maps on the fibres.

[7]See [21] for the concept of moduli and conductors.

[8]The fact that moduli even exists for rational maps from curves over fields to algebraic groups is a deep matter explained in [21]. For the case that the target group is the multiplicative or additive groups, he gives a precise form for the predicted moduli - this is a beautifully refined form of the Kummer and Artin - Schreier theories. See [21] p.33 (First line of proof of Proposition 5) and p.34(Proposition 6 and remark before.).

However we know that C' is given as the normalization of $C \times_{\mathcal{V}_1} \mathcal{W}_1$ - this is the *Kummer-Artin-Schreier-Witt* theorem. Since the rational map $C \to \mathcal{V}_1$ now factors through J_δ, we can take a fibre product $J'_\delta = J_\delta \otimes_{\mathcal{V}_1} \mathcal{W}_1$ and we find finally that

Claim 11 *Let C'/C be a Galois extension of smooth arithmetic surfaces which is also Galois in the special fibre. Assume there exists some effective divisor $\delta \subset C$ such that this extension has conductors δ_K and δ_k in the generic and special fibres respectively. Just as in the case of smooth curves over fields, the extension C'/C is given as an element*

$$(J_\delta)' : 0 \to G \to J'_\delta \to J_\delta \to 0$$

of $Ext^1_S(J_\delta, G)$ and C' is the normalization of C in the rational fibre product $C \otimes_{J_\delta} J'_\delta$. The reader should take care; not all extensions of the Jacobian will give smooth coverings of C.

In view of this method it is tempting to try the following: Let C'_k/C_k be an extension in the special fibre of modulus δ_k and let the induced extension on the Jacobian J_{δ_k} in $Ext^1_k(J_{\delta_k}, G)$ be

$$(b_k) : 0 \to G \to J'_{\delta_k} \to J_{\delta_k} \to 0$$

Let us *assume* we can lift this extension to $Ext^1_S(J_\delta, G)$ where δ_K is of the form $\sum_i Q_i$ for distinct K-points Q_i and reducing to the modulus δ_k. Let C' be the normalization of $C \times_{J_\delta} J'_\delta$. How far is C' then from a possible lift of C'_k/C_k ?

Claim 12 ([7]) *With our assumptions as above we get that the curve C' is in fact smooth and hence a lifting of C'_k/C_k.*

The above is proved using a genus type argument and details can be found in [7]. One has to admit, the fact that one takes the generic modulus δ_K to consist of distinct points makes the genus estimate in the generic fibre easy. Furthermore, the explicit link between the degree and form of the modulus δ_k and the fact that the extension is a G-extension also makes the special genus of C'_k easy to estimate. It is not clear what the situation in a higher order Galois extension will be. To the author it does not yet seem so straight-forward to estimate the generic genus from the generic modulus δ_K alone in the case of general p^n extensions.

In the p-cyclic case we are thus reduced to lifting the Jacobian extension

$$0 \to G \to J'_{\delta_k} \to J_{\delta_k} \to 0$$

Just as in the tame case we can decompose the extension C'/C into the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L'_{\delta_k} \arrow[r] \arrow[d, "f_k"'] & J'_{\delta_k} \arrow[r] \arrow[d] & J' \arrow[r] \arrow[d, "g_k"'] & 0 (E_{2,k}) \\
0 \arrow[r] & L_{\delta_k} \arrow[r] \arrow[d] & J_{\delta_k} \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 0\ (E_k) \\
 & 0 & 0 & 0 & \\
 & (a_k) & (b_k) & (c_k) &
\end{tikzcd}$$

Let $\delta_k = \sum n_i P_i$ and choose points P'_i on the projective line $\mathbf{P}^1_k$. We let $\delta'_k = \sum n_i P'_i$, a modulus now on $\mathbf{P}^1_k$. Things are somewhat simpler in the case of the projective line. For one its Jacobian J vanishes. Hence the singular Jacobian associated to δ'_k on the projective line $\mathbf{P}^1_k$ is simply the group $L_{\delta'_k}(\mathbf{P^1_k})$ - We use the additional $\mathbf{P^1_k}$ here to distinguish between the L-groups associated to extensions of C_k and those of $\mathbf{P}^1_k$.

The two moduli δ_k and δ'_k *look the same* and in fact the explicit classification theory of the L_k-groups schemes given in [21] p.96 Thm.3 (done for curves) tells us that they are even independent of the curves $\mathbf{P}^1_k$ and C_k and only on the structure of the moduli. The point is this : $L_{\delta'_k}(\mathbf{P}^1_k)$ (for $\mathbf{P}^1_k$) and L_{δ_k} (for C_k) are the same.

It implies that column (a_k) induces an N-extension of $\mathbf{P}^1_k$. If one now <u>assumes</u> that this latter can be smoothly lifted to characteristic 0, then by our result Claim 11 one can then find a lifting of column (a_k) thus solving the problem mentioned at the conclusion of chapter 3. The authors of [7] now use a method similar to what we have used in the tame case to obtain the entire lifting of the Jacobians J_{δ_k}.

We conclude with a remark on the assumption. The authors of [7] gives an explicit lifting of extensions of $\mathbf{P}^1_k$. It seems that one cannot escape from working with the explicit equations at one time or another.

5.4 Local global lifting principle: The proof

In this section we return to the proof of Theorem 1 stated earlier.

Let X_k be an integral normal affine algebraic curve, let $x \in X_k$ and consider the natural injections $i_x : \mathcal{O}(X_k - x) \hookrightarrow k((t_x))$ and $i_f : k[[t_x]] \hookrightarrow k((t_x))$. Here we have chosen t_x to be a parameter of X_k at x. We have the commutative diagrams

$$\begin{array}{ccccc} \mathcal{O}(X_k) & \longrightarrow & \mathcal{O}(X_k - x) & \longrightarrow & k(X_k) \\ \downarrow & & & & \downarrow \\ \mathcal{O}_{X_k,x} & \longrightarrow & k[[t_x]] & \longrightarrow & k((t_x)) \end{array}$$

We warn the reader not to be confused with the missing (middle) arrow in the above diagram, we drew it this way to save space; there is no intention of filling it in later.

By identifying the rings as subrings of $k((t_x))$ we have that the intersection $\mathcal{O}(X_k - x) \cap k[[t_x]]$ is the local ring of the open subset $X - (t_x)_\infty$.

Claim 13 (Intersection theorem) *Concerning the affine line spec$(k[t])$ we have the following exact sequence*

$$0 \to k[t] \to k[t, \frac{1}{t}] \oplus k[[t]] \longrightarrow k((t)) \to 0$$

where the latter map is

$$(a, b) \mapsto i_x(a) - i_f(b).$$

In general if X_k is a normal affine algebraic curve and $x \in X_k$, then by restricting the curve X_k to some open neighbourhood of x we have

$$0 \to \mathcal{O}(X_k) \to \mathcal{O}_{X_k}(X_k') \oplus \widehat{\mathcal{O}_{X_k,x}} \to k((t_x)) \to 0$$

where we have defined X_k' to be the open set $X_k - x$. Also t_x is assumed a suitable parameter for the point $x \in X_k$.

The claim is proved by finding a suitable étale map $X_k \to A^1$ sending $x \to 0$ (and only $x \to 0$) and then using the fact that étale maps are flat and preserve complete local rings. One can also deal directly with the valuations. Intuitively the kernel of the given maps should be interpreted as an intersection in the *bigger structure* $k((t))$. From now on we shall drop the i_f and i_x for these embedding whenever the context is clear.

Let Y_k/X_k be a G-Galois k-curve extension and we assume $Y_k' = Y_k \otimes_{X_k} X_k'$ is étale over $X_k' = X_k - x$. Notice that we have the following G-equivariant isomorphism

$$\mathcal{O}(Y_k') \otimes_{\mathcal{O}(X_k')} k((t_x)) \xrightarrow{\sim} (\mathcal{O}(Y_k) \otimes_{\mathcal{O}(X_k)} k[[t_x]]) \otimes_{k[[t_x]]} k((t_x))$$

and sequence

$$(\vartheta_k) : 0 \to \mathcal{O}(Y_k) \to \mathcal{O}(Y_k') \oplus (\mathcal{O}(Y_k) \otimes_{\mathcal{O}(X_k)} k[[t_x]]) \to \mathcal{O}(Y_k) \otimes_{\mathcal{O}(X_k)} k((t_x)) \to 0$$

since Y_k/X_k is flat. By tensoring the above exact sequence with $\mathcal{O}(X_k')$ we also obtain the following exact sequence

$$0 \to \mathcal{O}(Y_k') \to \mathcal{O}(Y_k') \oplus k((t_x)) \to \mathcal{O}(Y_k) \otimes_{\mathcal{O}(X_k)} k((t_x)) \to 0$$

and similarly when tensoring with $k[[t_x]]$. Just as before notice how the exact sequence ϑ_k implies a construction of $\mathcal{O}(Y_k)$ as an *intersection* of two sets of data, namely the etale locus $\mathcal{O}(Y_k')$ and the local ring $O(Y_k) \otimes_{\mathcal{O}(X_k)} k[[t_x]]$.

Now let $\mathcal{X}/R$ be a smooth affine formal curve and $x \in X_k$. We have the diagrams

$$\begin{array}{ccc} \mathcal{O}(\mathcal{X}) & \longrightarrow & \widehat{\mathcal{O}(\mathcal{X})}_\pi \\ \downarrow & & \downarrow \\ \widehat{\mathcal{O}(\mathcal{X})}_x & \longrightarrow & (\widehat{\widehat{\mathcal{O}(\mathcal{X})}_x})_\pi \end{array}$$

lifting those of the special fibre given earlier

$$\begin{array}{ccc} \mathcal{O}(X_k) & \longrightarrow & k(X_k) \\ \downarrow & & \downarrow \\ k[\![t_x]\!] & \longrightarrow & k(\!(t_x)\!) \end{array}$$

Again we obtain

Claim 14 *By restricting $\mathcal{X}$ we obtain the exact sequence in characteristic* 0

$$0 \to \mathcal{O}(\mathcal{X}) \to \mathcal{O}(\mathcal{X} - x) \times \widehat{\mathcal{O}_{\mathcal{X},x}} \longrightarrow (\widehat{\widehat{\mathcal{O}_{\mathcal{X},x}}})_\pi \to 0$$

lifting the special fibre sequences above. Here the latter map is

$$(a, b) \mapsto a - b$$

where one considers $\mathcal{O}(\mathcal{X} - x)$ and $\widehat{\mathcal{O}_{\mathcal{X},x}}$ canonically embedded as subrings in $(\widehat{\widehat{\mathcal{O}_{\mathcal{X},x}}})_\pi$.

For later use we state

Claim 15 *The ring $(\widehat{\widehat{\mathcal{O}_{\mathcal{X},x}}})_\pi$ is a discrete valuation ring with parameter π.*[9]

Local-global assumptions

Assume we can find a G-equivariant lift $\widehat{\mathcal{O}_{\mathcal{X},x}} \hookrightarrow \mathcal{B}_x$ of $k[\![t_x]\!] \hookrightarrow \mathcal{O}(Y_k) \otimes_{\mathcal{O}(X_k)} k[\![t]\!]$. We have seen that the extension $Y' \to X'$ lifts uniquely to a G-Galois étale extension (*of rings*) $\mathcal{O}(\mathcal{X}') \hookrightarrow \mathcal{O}(\mathcal{Y}')$.

Assume we can find a G-equivariant isomorphism

$$\phi : \mathcal{O}(\mathcal{Y}') \otimes_{\mathcal{O}(\mathcal{X}')} (\widehat{\widehat{\mathcal{O}_{\mathcal{X},x}}})_\pi \to \mathcal{B}_x \otimes_{\widehat{\mathcal{O}_{\mathcal{X},x}}} (\widehat{\widehat{\mathcal{O}_{\mathcal{X},x}}})_\pi$$

lifting the equality in the special fibre

$$\mathcal{O}(Y'_k) \otimes_{\mathcal{O}(X'_k)} k(\!(t_x)\!) = \mathcal{B}_{x,k} \otimes_{k[\![t_x]\!]} k(\!(t_x)\!)$$

G-equivariantly and also **assume** that these two modules are separated π-adic modules.

[9] It reduces to the field $k(\!(t_x)\!)$ under reduction modulo π and is also π-adically complete.

Construction via an intersection

Armed with the above assumptions we continue to construct the desired affine formal curve $\mathcal{Y}$ as an *intersection* of the known $\mathcal{Y}'$ and the given local ring $\mathcal{B}_x$ in the same fashion as Y_k can be obtained from Y_k' and $\widehat{\mathcal{O}_{X_k,x}}$. We can define the following map

$$\mathcal{O}(\mathcal{Y}') \times \mathcal{B}_x \overset{(a,b)\mapsto\phi(a)-b}{\longrightarrow} \mathcal{B}_x \otimes_{\widehat{\mathcal{O}_{\mathcal{X},x}}} (\widehat{\widehat{\mathcal{O}_{\mathcal{X},x}}})_\pi$$

This map lifts the corresponding special fibre surjection

$$\mathcal{O}(Y_k') \times (\mathcal{O}(Y_k) \otimes k[\![t_x]\!]) \to \mathcal{O}(Y_k) \otimes k(\!(t_x)\!)$$

of (ϑ_k) and is in fact a surjection.[10] We thus have an exact sequence

$$(F) : 0 \to \mathcal{M} \to \mathcal{O}(\mathcal{Y}') \times \mathcal{B}_x \overset{(a,b)\mapsto\phi(a)-b}{\longrightarrow} \mathcal{B}_x \otimes_{\widehat{\mathcal{O}_{\mathcal{X},x}}} (\widehat{\widehat{\mathcal{O}_{\mathcal{X},x}}})_\pi \to 0$$

lifting the sequence (ϑ_k). Similarly as in the special fibre, we have constructed a candidate object $\mathcal{M}$ as an intersection which lifts the analogous construction of $\mathcal{O}(Y_k)$ in the special fibre and it is this $\mathcal{M}$ which we claim is the desired *coordinate* ring.

We have

$$\mathcal{O}(\mathcal{X}) \subset \mathcal{M}$$

and notice that $\mathcal{M}$ is π-adically complete, R-torsion free and R-flat. By Lemma C.8 and the exactness of sequence (ϑ_k) we have that it also reduces to $\mathcal{O}(Y_k)$ in the special fibre in a G-equivariant way.[11]

We thus also have that $\mathcal{M}$ is a finite $\mathcal{O}(\mathcal{X})$-module since $\mathcal{O}(Y_k)$ is a finite $\mathcal{O}(X_k)$-module. Lastly consider the induced G-equivariant map

$$\mathcal{M} \otimes_{\mathcal{O}(\mathcal{X})} \mathcal{O}(\mathcal{X}') \to \mathcal{O}(\mathcal{Y}')$$

---

[10] We note $\mathcal{B}_x \otimes_{\widehat{\mathcal{O}_{\mathcal{X},x}}} (\widehat{\widehat{\mathcal{O}_{\mathcal{X},x}}})_\pi$ is π-adically separated. We then invoke Lemma C.7 together with the fact that the map is a surjection in the special fibre.

[11] We know that $\mathcal{M}_k$ must be the kernel of the special fibre exact sequence, which is exactly $\mathcal{O}(Y_k)$.

and note that these modules are complete by the fact that $\mathcal{M}/\mathcal{O}(\mathcal{X})$ is finite. On the special fibre this map is an isomorphism (G-equivariant) and hence by Lemma C.9 this is true in characteristic 0 as well. Thus except for checking our assumptions, we have proved the local global principle.

We now look at our assumptions. For illustration we assume again that Y_k/X_k is totally ramified at x. We are given a G lifting $\widehat{\mathcal{O}_{\mathcal{X},x}} \hookrightarrow \mathcal{B}_x$ of the induced special fibre map as above. Let z_y be a parameter for Y_k at the unique point $y \in Y_k$ over x and as before let t_x be a parameter for $x \in X_k$. Let $k((z_y)) = \frac{k((t_x))[w]}{p(w)}$ for some monic polynomial $p(w) \in k((t_x))[w]$. We know already that $(\widehat{\mathcal{O}_{\mathcal{X},x}})_\pi$ is a complete d.v.r reducing to $k((t_x))$ and similarly we have that $\mathcal{B}_x \otimes_{\widehat{\mathcal{O}_{\mathcal{X},x}}} (\widehat{\mathcal{O}_{\mathcal{X},x}})_\pi$ is a d.v.r reducing to $k((z_y))$ such that the following diagram commutes

$$\begin{array}{ccc}
\widehat{\mathcal{O}(\mathcal{X})_{x}}_\pi & \longrightarrow & \widehat{\mathcal{O}(\mathcal{X})_{x}}_\pi \otimes_{\widehat{\mathcal{O}(\mathcal{X})_x}} \mathcal{B}_x \\
\downarrow & & \downarrow \\
k((t_x)) & \longrightarrow & k((z_y))
\end{array}$$

Using the Henselian property we can write $\mathcal{B}_x \otimes_{\widehat{\mathcal{O}_{\mathcal{X},x}}} (\widehat{\mathcal{O}_{\mathcal{X},x}})_\pi = \frac{(\widehat{\mathcal{O}_{\mathcal{X},x}})_\pi[W]}{P(W)}$ where W lifts the root w and $P(W)$ is a monic polynomial lifting $p(w)$ and such that the isomorphism can be chosen G-equivariant. One can do the same thing with $\mathcal{O}(\mathcal{Y}') \otimes_{\mathcal{O}(\mathcal{X}')} (\widehat{\mathcal{O}_{\mathcal{X},x}})_\pi$ and thus we can find our *assumed* G isomorphism of earlier on. The separated condition follows from the fact that all these modules are π-adically complete anyway. We can finally state the important

Theorem 16 ([2], later [4]) *The global lifting problem is reduced to the local lifting problem.*

5.5 p^3-cyclic local differents

In this section, we would like to illustrate one aspect of difficulty when working with higher order Galois extensions. Essentially it is the complexity. Clearly if the Green-Matignon line of attack on the p^3-cyclic lifting problem is to be followed then we shall need estimates for the different of such an extension. We do this calculation for a specific family of p^3-cyclic coverings of the local ring $k[\![t]\!]$. These families admit very high wild ramification and we dub them *extreme* covers, although we apologize for any misconceptions the name might lead to. The methods we follow here are inspired by those employed in [2] and [22] where the differents are studied for p and p^2-cyclic Galois extensions in characteristic p. We gather some technical tools first and we ask the reader's patience for what is a very long deduction.

Some technical tools regarding the valuation ring $k[\![t]\!]$

Again we assume k is characteristic p and algebraically closed and t is some indeterminate. For convenience we write $A = k[\![t]\!]$.

Claim 17 *Let $\epsilon = \epsilon_0 + t^s \epsilon_s$ where $\epsilon_0 \in k^*$ and $s \geq 1$. Then we have $\epsilon^{-1} = \epsilon_0^{-1} + t^s \epsilon_s'$ for some $\epsilon_s' \in A$. Also $\epsilon_s' \in A^*$ iff $\epsilon_s \in A^*$.*

Also useful is

Claim 18 *Let $\epsilon = \epsilon_0 + t^s \epsilon_s$ with $\epsilon_0 \in k^*$ and $s \geq 1$. Let $r = p^j v$ with $(v, p) = 1$ and $j \geq 0$. Then we have $\epsilon = \epsilon_0^r + t^{sp^j} \epsilon'_{sp^j}$ where $\epsilon'_{sp^j} \in A$. The latter is a unit iff ϵ_s is a unit.*

Review of p^2-cyclic different

We have already stated this result earlier, but for convenience we state it here again. Throughout what follows we shall assume given a totally ramified p^3-cyclic Galois extension $k(\!(t)\!) \subset k(\!(z_1)\!) \subset k(\!(z_2)\!) \subset k(\!(z_3)\!)$ where z_i are parameters. We also assume the notation of the Appendix equations B.4 and B.5 for the extensions $k(\!(t)\!) \subset k(\!(z_1)\!) \subset k(\!(z_2)\!)$.

We shall also assume the notation of the following lemma (which was also stated earlier as Claim 5):

Lemma 19 *We can find a parameter of $k((z_1))$, which we also denote by z_1 such that $z_1^{-m_1} = x_1$ and the parameter t is related by*

$$t^{-m_1} = z_1^{-pm_1}(1 + z_1^{m_1(p-1)})$$

We set

$$(5.10)\quad y_2 = x_2 - \sum_{i,j} (a_{ij}^{(2)})^{\frac{1}{p}} x_1^j t^{-i}$$

We also set

$$\mu_2 = max\{m_1p^2 - m_1p + m_1,\ ip^2 + m_1(p(j-1)+1)\}$$

and notice this is relatively prime to p due to our assumption on m_1. We can then find a second parameter z_2 of $k((z_2))$ such that $z_2^{-\mu_2} = y_2$. The different is

$$d_{2/0} = (\mu_2 + 1)(p-1) + pd_{1/0} = (p-1)(\mu_2 + 1 + pm_1 + p)$$

In order to estimate the different of the extension $k((z_2)) \subset k((z_3))$ we shall need to rewrite its defining equation in an Artin-Shreier way such that the right hand side has a z_2 valuation relatively prime to p. We shall do this now.

Assume that $m_1 < p^2$ and consider the alternative equation generating $k((z_2))$ over $k((z_1))$ (which we name the *transformed* second Artin Schreier equation)

$$(5.11)\quad y_2^p - y_2 = x_2^p - x_2 + \sum_{i,j} (a_{ij}^{(2)})^{\frac{1}{p}} x_1^j t^{-i} - \sum_{i,j} (a_{ij}^{(2)}) x_1^{pj} t^{-ip} =$$

$$\frac{x_1^{p^2} - x_1^p - (x_1^p - x_1)^p}{p} + \sum_{\substack{0 \le i \le m_2 \\ 0 < j < p}} a_{i,j}^{(2)} (x_1^p - x_1)^j t^{-ip}$$

$$+ \sum_{1 \le k \le L_1} b_k^{(2)} t^{-k} + v_2(t) + \sum_{i,j} (a_{ij}^{(2)})^{\frac{1}{p}} x_1^j t^{-i} - \sum_{i,j} (a_{ij}^{(2)}) x_1^p j t^{-ip} =$$

$$\frac{x_1^{p^2} - x_1^p - (x_1^p - x_1)^p}{p} + \sum_{\substack{0 \le i \le m_2 \\ 0 < j < p}} a_{i,j}^{(2)} \left[\sum_{1 \le r < p-1} x_1^{p(j-r)+r} \binom{j}{r} \right] t^{-ip}$$

$$+ \sum_{i,j} (a_{ij}^{(2)})^{\frac{1}{p}} x_1^j t^{-i} + \sum_{1 \le k \le L_1} b_k^{(2)} t^{-k} + v_2(t)$$

Assume that $m_2 > pm_1$ and $a_{m_2,p-1} \neq 0$. Then the term in equation (5.11) with the worst[12] z_1-adic valuation is the term corresponding to $r = 1$, $j = p-1$ and $i = m_2$. What is also important (for later use) is that under the assumption $m_1 < p$ the difference between the z_1-adic valuations of the second worst term and that of the worst is at least $(p-1)m_1$: one sees this by minimizing the difference between the expressions

$$m_2 p^2 + \Big[p(p-2)+1\Big] m_1$$

(which is the absolute value of the z_1-adic valuation of the *worst* term, also named μ_2) and the general term[13] has z_1-adic valuation

$$ip^2 - \Big[p(j-1)+1\Big] m_1.$$

From the fact that $i \leq m_2$, $j \leq p-1$ and $m_1 < p^2$ the expression for the lower bound of the difference follows.

We summarize some more facts in the following lemma:

Lemma 20 *We can write $y_2^p - y_2$ in terms of powers of z_1^{-1}:*

$$y_2^p - y_2 = z_1^{-\mu_2}\epsilon_1 + z_1^{-b_1}P(z_1)$$

Here $P(z_1) \in k[\![z_1]\!]$, $\epsilon_1 \in k^*$ and $\mu_2 - b_1 \geq (p-1)m_1$. From our expression of x_2 in terms of y_2, which has z_2 valuation $-\mu_2$, we find that x_2 has z_2 valuation that of $x_1^{p-1}t^{-m_2}$ which is

$$-(p-1)pm_1 - p^2 m_2 = -\mu_2 - (p-1)m_1$$

Generating a family of p^3-cyclic extensions

Lemma 21 *Assume the associated Witt vector obtained as in Lemma B.3 of the extension $k(\!(z_3)\!)/k(\!(t)\!)$ is $[w_1, w_2, w_3]$ where the t-adic valuation of w_2 is negative. Then this extension*

[12] When we talk of the *worst* terms of f under the valuation v we shall mean those that have the least negative valuation, i.e. those terms maximizing $-v(.)$. We apologise for sloppy language.

[13] The other terms we ignore; our assumptions on m_1 and m_2 rule them out.

can be generated by the equations B.4, B.5 and the following

$$x_3^p - x_3 = \frac{x_2^{p^2} - x_2^p - (x_2^p - x_2 - \frac{x_1^{p^2} - x_1^p - (x_1^p - x_1)^p}{p})^p}{p} + \frac{x_1^{p^3} - x_1^{p^2} - (x_1^p - x_1)^p}{p^2}$$
$$+ \sum_{\substack{0 \le i \le m_3 \\ 0 < i < p}} a_{ij}^{(3)} (x_2^p - x_2 - \frac{x_1^{p^2} - x_1^p - (x_1^p - x_1)^p}{p})^j t^{-ip} + \sum_{0 \le k \le (N_2 - 1)(p-1)} b_k^{(3)} t^{-k} + v_3(t) \tag{5.12}$$

Here $-N_2$ is the t-adic valuation of the non-Artin Schreier component on the right hand side of equation B.5.

Proof*:* The proof is as before and we note our assumption on the w_2 implies that

$$x_2^p - x_2 - \frac{x_1^{p^2} - x_1^p - (x_1^p - x_1)^p}{p}$$

has negative t-adic valuation relatively prime to p, namely it has t-adic valuation $-N_2$. The proof goes through exactly as in corollary 4 of Appendix B. ♣

Remark The hypothesis is not necessary; there are many tricks around this. Indeed one notes that $(m_1, p) = 1$ and hence working in a factor of $x_1^p - x_1$ will overcome this problem. However, as the reader will see later, one needs forms which are useful for calculating the different. Our assumption makes this slightly easier.

We ask what the general terms of the right hand side of equation (5.12) is. We shall list them

Claim 22 *The general terms of equation (5.12) are given by*

- *Group 1:* $x_2^{pl+k} \prod_{1 \le u \le p-1} (x_1^{(pu+p-u)n_u})$ *where* $l + k + \sum_{u=1,\dots,p-1} n_u = p$, no $n_u = p$ *and also* $k \neq p$ *and* $l \neq p$
- *Group 2:* $x_1^{p(p^2-r)+r}$ *where* $r = 1, \dots, p^2 - 1$ *and* $r \neq \{p, 2p, \dots, p(p-1)\}$.
- *Group 3:* $t^{-ip} x_2^{pl+k} \prod_{1 \le u \le p-1} x_1^{(pu+p-u)n_u}$ *where* $l + k + \sum_{u=1,\dots,p-1} n_u = j$.
- *Group 4: The t terms of $v_3(t)$ and the* $\sum_k \beta_k^{(3)} t^{-k}$

Our tactic now is to study each group seperably and expand them in z_2 powers. This will allow us to shift out the p-powers of z_2 and determine which non-p-power remains in the Artin-Shreier form, enabling us to read of the different.

Group 3 analysis

The following is a series of crucial but technical statements.

Assume we can write $x_2 = \Omega_2^p + z_2^{-s_2}\Psi_2$ where Ψ_2 is some unit in $k[\![t]\!]$ and $s_2 > 0$. We shall abuse notation and write $|v_2(f)|$ for the absolute value of the z_2-adic valuation of an element $f \in k(\!(z_2)\!)$. We know Ω_2^p has **negative** z_2 valuation and **assume** we know $|v_2(\Omega_2^p)| > s_2$. We rewrite the general term for the *group* 3 elements as

$$t^{-ip}\left[\sum_{r_0=0}^{pl+k} \Omega_2^{pr_0} z_2^{-s_2(pl+k-r_0)} \Psi_2^{pl+k-r_0} \binom{pl+k}{r_0}\right] x_1^{\sum_{1\le u\le p-1}[pu+p-u]n_u}$$

We fix a choice for i and j and ask what the z_2-adic valuation of this expression will be. From the expression above we obtain that the z_2-adic valuation for the term above is (at a choice of $r_0 \in \{0, 1, \ldots, pl+k\}$)

$$-ip^3 - \left[|v_2(\Omega_2)|pr_0 + (pl+k-r_0)s_2\right] - |v_2(x_1)| \left[\sum_{1\le u\le p-1}[pu+p-u]n_u\right]$$

We ask when this will be a minimum or equivalently when

$$\left[|v_2(\Omega_2)|pr_0 + (pl+k-r_0)s_2\right] + |v_2(x_1)|\left[\sum_{1\le u\le p-1}[pu+p-u]n_u\right] \tag{5.13}$$

is a maximum (and we are fixing i and j at first).

Notice that we have $(k+l) + \sum_{1<u<p-1} n_u = j$ and $j \in \{1, \ldots, p\}$. By the assumption $|v_2(\Omega_2)|p > s_2$ one finds that if the value of $k+l$ decreases by one, then the quantity

$$\min_{r_0}\{\ \ |v_2(\Omega_2)|pr_0 + (pl+k-r_0)s_2\ \ \}$$

will decrease by at least s_2. On the other hand, increasing the value of $\sum_{1\le u\le p} n_u$ by one the value of the quantity

$$|v_2(x_1)|\left[\sum_{1\le u\le p-1}[pu+p-u]n_u\right]$$

increases by at most $|v_2(x_1)|\Bigg[p(p-1)+1\Bigg]$.

Hence **if** $s_2 > |v_2(x_1)|\Bigg[p(p-1)+1\Bigg]$ then the largest expressions of equation (5.13) will be in the cases when $k+l=j$ and $\sum_u n_u = 0$.

Earlier we made the assumption that $|v_2(\Omega_2^p)| > s_2$. Thus fixing k and l the two largest terms of equation 5.13 will be when $r_0 = pl+k$ and $r_0 = pl+k-1$ corresponding to the terms

$$a_{i,j}^{(3)} t^{-ip} \Omega_2^{p(lp+k)}$$

and

$$a_{i,j}^{(3)} \binom{lp+k}{1} t^{-ip} \Omega_2^{p(lp+k-1)} z_2^{-s_2} \Psi_2$$

with z_2-adic valuations

(5.14) $-ip^3 - |v_2(\Omega_2)|p(lp+k)$

and

(5.15) $-ip^3 - |v_2(\Omega_2)|p(lp+k-1) - s_2$

respectively. We remark that these terms cannot be cancelled out by other terms; our arguments above show that they have valuations strictly larger than the rest (for a fixed choice of $i,\ j,\ k$ and l).

Notice that for a fixed choice of $i,\ j,\ k$ and l the highest term is a p^{th} power. We can get rid of them with a substitution

$$y_3 = x_3 - \sum_{i,j} (a_{i,j}^{(3)})^{\frac{1}{p}} t^{-i} \Omega_2^{lp+k}$$

but we take care: we must add terms of the form $(a_{i,j}^{(3)})^{\frac{1}{p}} t^{-i} \Omega_2^{lp+k}$ - each of these has valuation $-ip^2 - |v_2(\Omega_2)|(lp+k)$. Hence for a fixed choice of $i,\ j,\ k$ and l, **if** the quantity of expression (5.15) is strictly larger than $-ip^2 - |v_2(\Omega_2)|(lp+k)$ then the *unique* term with the most negative (or *worst*) t-adic valuation of Group 3 (after making the substitution y_3) will be the terms

$$a_{i,j}^{(3)} \binom{lp+k}{1} t^{-ip} \Omega_2^{p(lp+k-1)} z_2^{-s_2} \Psi_2 \tag{5.16}$$

We remind the reader that with *worst* we mean the *least* z_2-adic valuation. In order to use the above as an estimate of the z_2-adic valuation of equation (5.12) we must make sure that it far exceeds the other groups and also that s_2 is relatively prime to p. We remember that in group 3 we have the extra parameter i which we can increase, and this is not present in the other groups. Hence to find examples we can play with this parameter until group 3 far exceeds to the rest in terms of z_2-adic valuations.

Consider for example when $2m_1p^2 < 2m_2p < m_3$. One sees that the worst z_2-adic valuation of group 1 cannot be worse than $2m_2p^4 < m_3p^3$.[14] Similarly the worst z_2-adic valuation of groups 2 and 4 cannot be worse than $m_1p^4 < m_3p^3$. However, the terms found above (for choice $i = m_3$) have z_2 valuation far exceeding this - we remember that $v_2(t) = p^2$ and hence $|v_2(t^{-m_3p})| = p^3m_3$ which are already far worse than all of these terms.

Assuming that $a_{m_3,p-1} \neq 0$ and that $(s_2, p) = 1$, then we can even conclude that the choice of $i,\ j,\ k$ and l that gives the worst z_2-adic valuation is the term

$$a_{m_3,p-1}^{(3)} \binom{(p-1)p+1}{1} t^{-m_3p} \Omega_2^{p((p-1)p+1-1)} z_2^{-s_2}$$

or when $l = p-1$ and $k = 1$ with z_2-adic valuation $-\left[|v_2(\Omega_2^p)|(p-1)p + s_2\right]$ and hence the different of the p-cyclic extension $k((z_3))/k((z_2))$ is given by

$$d_{3/2} = (p-1)\left[|v_2(\Omega_2^p)|(p-1)p + s_2 + 1\right]$$

We obtain the total different by the formula $d_{3/0} = d_{3/2} + pd_{2/0}$ since $k((z_3))/k((z_2))$ is totally ramified with ramification index p.

[14] These estimates are very crude. We recall that $|v_2(x_1)| = pm_1$ and similarly $v_{z_1}(x_2) < 2m_2p^2$.

What remains are the terms $|v_2(\Omega_2^p)|$ and s_2. We need to know what they are. We also need to make sure that $(s,p)=1$ and crucially, that it verifies the assumptions that we made on it earlier: these were that $|v_2(\Omega_2^p)| > s_2$ and that $s_2 > |v_2(x_1)|\Big[p(p-1)+1\Big]$ We deal with this next.

Assumptions on s

We recall from Lemma 20 that we can write a generating equation for $k((z_2))$ by

$$y_2^p - y_2 = z_1^{-\mu_2}\Big[\epsilon_1 + P(z_1)z_1^{\mu_2-b_1}\Big]$$

and that $\mu_2 - b_1 \geq m_1(p-1)$. However, by our choice of z_2 we have $y_2 = z_2^{-\mu_2}$ and hence we have the equality

$$z_2^{-p\mu_2}\Big[1 - z_2^{\mu_2(p-1)}\Big] = z_1^{-\mu_2}[\epsilon_1 + z_1^{\mu_2-b_1}P(z_1)]$$

or equivalently

$$z_2^{-p\mu_2}\frac{\Big[1 - z_2^{\mu_2(p-1)}\Big]}{[\epsilon_1 + z_1^{\mu_2-b_1}P(z_1)]} = z_1^{-\mu_2}$$

However, by Claim 17, we see that we can write

$$\frac{\Big[1 - z_2^{\mu_2(p-1)}\Big]}{[\epsilon_1 + z_1^{\mu_2-b_1}P(z_1)]} = \epsilon_1^{-1} + z_2^c R(z_2)$$

where $c > m_1(p-1)$ and $R(z_2) \in k[\![z_2]\!]$. The reason for this is that by Claim 17 the reciprocal can be expressed as $\epsilon_1^{-1} + z_2^{c'}Q(z_2)$ for some $Q(z_2) \in k[\![z_2]\!]$ and where $c' \geq m_1(p-1)p$ since $\mu_2 - b_1 \geq m_1(p-1)$ and we note that z_1 has z_2 valuation p, and we also note that above the line we have $(p-1)\mu_2 > m_1(p-1)$.

Hence we can write

$$z_2^{-\mu_2 p}\Big[\epsilon_1^{-1} + R(z_2)z_2^c\Big] = z_1^{-\mu_2}$$

Now we realize that $(\mu_2, p) = 1$. Notice that $\Big[\epsilon_1^{-1} + R(z_2)z_2^c\Big]$ is a μ_2-power since μ_2 is relatively prime to p. Hence we can write

$$\Big[\epsilon_1^{-1} + R(z_2)z_2^c\Big] = [\alpha + z_2^d R'(z_2)]^{\mu_2}$$

for some power d, some **unit** α and again some $R'(z_2) \in k[\![z_2]\!]$. However, by Claim 18 we find that

$$d \geq c \geq m_1(p-1)p > m_1(p-1)$$

Raising to m_1 again, which is also relatively prime to p, and again using Claim 18 we find that we can express

$$x_1 = z_2^{-pm_1}\left[\alpha + z_2^e S(z_2)\right]$$

for some $e > m_1(p-1)$.

But this is the point : we know we can write x_2 as a sum of powers of x_1 **and** added to y_2 where $y_2 = z_2^{-\mu_2}$; see equation (5.11). Although x_2 had z_2-adic valuation $-m_2p^2 - (p-1)m_1p = -\mu_2 - (p-1)m_1$, the next worse term in its z_2-adic expansion will necessarily be a term of the type $z_2^{-\mu_2}$ (multiplied by some unit of course) since all the other terms (except the leading terms of course) coming from the x_1 powers are far better (in the z_2-adic sense) than $z_2^{-\mu_2}$. We have found the s_2 we were looking for: it is simply μ_2. Hence for cases when $m_1 < p$, $2pm_1 < m_2$, $2pm_2 < m_3$, $a_{m_2,p-1} \neq 0$ and $a_{m_3,p-1} \neq 0$ we find

Theorem 23 *Under the assumptions above, the relative different of $k(\!(z_2)\!) \subset k(\!(z_3)\!)$ is given by*

$$d_{3/2} = (p-1)\left[p^3 m_3 + (p-1)(\mu_2 + (p-1)m_1) + \mu_2 + 1\right] \tag{5.17}$$

and hence

$$d_{3/0} = d_{3/2} + p(d_{2/0})$$

where $d_{2/0}$ is found in Lemma 19 as

$$d_{2/0} = (\mu_2 + 1)(p-1) + pd_{1/0} = (p-1)(\mu_2 + 1 + pm_1 + p).$$

Combining this with the expression for $d_{2/0}$ we can calculate the different $d_{3/0}$. The reader will note our estimates were extremely crude and one can refine them somewhat, but we leave the matter there for now.

Appendix A

Sheaf cohomology(SC)

The name might seem somewhat fancy, but we mostly use only the notions of Ext^1-groups and the ordinary cohomology groups. We use them on the category of abelian sheaves of a site. Amazingly they are related to the ordinary cohomology groups, a fact which we shall exploit. We shall mostly be concerned with affine group schemes and their torsors and hence we want to point out the theorem on [19] p.121 stating that torsors of affine smooth groups are always representable on the flat (or étale) site of a scheme.

Furthermore we also want to point out the explicit interpretation of the étale cohomology groups in terms of classifying Galois extensions of schemes; see [19] p.123 and perhaps more so Milne's internet notes on Étale Cohomology. Lastly we point out the proper base change theorem, allowing us in certain instances to move between cohomology groups under reduction; see [19] p.224 Cor.2.7.

A.1 Ext-groups : explicit

This appendix section is a general insight into what the Ext^1-group means. We use its explicit interpretation in our studies of the tame lifting problem. It is interesting to remark that it is most specifically used in checking smoothness of the constructed extension, see Lemma 9 of chapter 3 for the use.

Throughout our discussion we shall be working on the flat site of Noetherian schemes. Occasionally we shall switch to the étale site. We shall denote by X_{fl} the flat site and by $S(X_{fl})$ the abelian category of *abelian* group sheaves on X_{fl}. One has the notions of $Ext(-,-)$ cohomology groups. We shall always assume the first argument is an affine smooth S-group scheme and we shall also always assume the second argument is a smooth S-group scheme. In these cases the exact sequences of $Ext(-,-)$ are always representable.

We know that $Ext^1_S(G, A)$ has the interpretation of being the set of all exact sequences in $S(S_{fl})$ of the form

$$(H) : 0 \to A \to H \to G \to 0.$$

One identifies (H) and (H') if there is an isomorphism of sheaves $H \to H'$ making the following sequences commute:

$$\begin{array}{ccccccccc}
(H) : 0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
(H') : 0 & \longrightarrow & A & \longrightarrow & H' & \longrightarrow & G & \longrightarrow & 0
\end{array}$$

Let $A \to A'$ and $B \to B'$. We have the maps

$$Ext^1(A, B) \to Ext^1(A, B')$$

and

$$Ext^1(A', B) \to Ext^1(A, B)$$

We interpret them in terms of exact sequences. The map $Ext^1(A', B) \to Ext^1(A, B)$ takes the extension

$$(H') : 0 \to B \to H' \to A' \to 0$$

to the unique extension

$$(H) : 0 \to B \to H \to A \to 0$$

such that we have a map $H \to H'$ of diagrams

$$\begin{array}{ccccccccc} (H): 0 & \longrightarrow & B & \longrightarrow & H & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ (H'): 0 & \longrightarrow & B & \longrightarrow & H' & \longrightarrow & A' & \longrightarrow & 0 \end{array}$$

A similar situation holds for the map $Ext^1(A, B) \to Ext^1(A, B')$. See [21] chapter 7 for a discussion along the same lines.

A.2 The maps $Ext^i \to H^i$

It is known that we have a canonical map $Ext^i \to H^i$. The use of this map is it allows us to study the group of extensions of a group scheme indirectly by looking at the cohomology group. There are certain useful theorems for the cohomology group related to reduction (e.g. proper base change) and hence if the map is well behaved we can deduce some information on Ext^i. It is amusing to make the map explicit, although we don't need it.[1]

We start with formality: we have functors

$$Hom_{S(S_{fl})}(X, \_) \to H^0_S(X, \_)$$

taking a map $\delta : X \to T$ of sheaves to the element $\delta(id : X \to X) \in T(X)$. We recall that we have assumed X to be a group scheme over S, hence not only is it a sheaf on the flat S-site, we can also regard it as an object on this site and hence we can evaluate another sheaf such as T at X, i.e. we can speak of $T(X)$. Similarly we note that $id : X \to X$ is an element of $X(X)$ and we apply δ to it to obtain an element of $T(X)$. It is not hard to see that this map is a homomorphism of the group functors $Hom_{S(S_{fl})}(X, \_)$ and $H^0_S(X, \_)$.

[1]For interest we mention the cute [16].II.6.1. It is an useful idea to keep in mind when one wants to get a grip on the boundary maps.

This map of functors induces maps on the cohomology $Ext^i(X, \_) \to H^i(X, \_)$.[2]

The following Theorem 2 is very important. We use it in the chapter on tame ramification, and also (without stating) in the chapter on wild ramification. It is powerful and allows us to lift other group extensions by relating them to extensions of projective schemes.

Theorem 1 ([7] Example 2.10 p.355; [19] p.132 Cor. 4.20) *Let S be a strictly Henselian discrete valuation scheme and let X/S be an abelian scheme. Let G be the smooth affine group representing $(\mathbf{Z}/\mathbf{nZ})_S$. Then the map $Ext^1_S(X, G) \to H^1_{fl}(X, G) = H^1_{et}(X, G)$ is an isomorphism.*

As a remark, the above is proved using a very special classification of the image of the Ext-group in H^1. One concludes using rigidity to establish injectivity. For this the fact that X is projective is crucial. We do not need its proof, simply the theorem and hence we ignore the details.

Theorem 2 ([7] Theorem 2.13 p.356) *Let S be a strictly Henselian discrete valuation scheme and let X/S be an abelian scheme over S. We have the following diagram of groups where the vertical arrows indicate the residue maps to the special fibres*

$$\begin{array}{ccc} Ext^1_S(X, \mathbf{Z}/\mathbf{nZ}) & \longrightarrow & H^1_{et}(X, \mathbf{Z}/\mathbf{nZ}) \\ \downarrow & & \downarrow \\ Ext^1_k(X_k, \mathbf{Z}/\mathbf{nZ}) & \longrightarrow & H^1_{et}(X_k, \mathbf{Z}/\mathbf{nZ}) \end{array}$$

Then all the arrows are isomorphisms.

This is a special case of Proper Base Change - see [19] p.224 Cor. 2.7. Notice that using the map of a smooth curve to its (usual) Jacobian we can lift étale extensions of curves in characteristic p to characteristic 0 in a manner similar to that of chapter 3.

[2] One can interpret this explicitly: it takes the extension $0 \to A \to B \to X \to 0$ on S_{fl} to the class of the $A|_X$ torsor $B \to X$ on the flat site X_{fl}.

Appendix B

Cyclic Galois theory

In this chapter we review Galois theory of rings. We have used Kummer theory in the proof of Garuti's theorem (see chapter 4). On the other hand, one knows Artin-Schreier theory is strong enough to generate all cyclic Galois extensions of p-power order in characteristic p. This was used in Green and Matignon's solution of the p^2-cyclic lifting problem. In particular, one can manipulate it as to read off the different in the p^2 case.

We also give a local smoothness criterion related to the different of an extension. This was a crucial component in Green and Matignon's proof since it gives a method to verify whether a certain extension (which is not concretely given, only in terms of an extension and hence too difficult to study directly) admits singularities after reduction. In order to use, one must know what the *generic* different is, for this one needs to know some more Kummer theory.

B.1 Review of cyclic Galois theory over curves

Let A be a K algebra where K is a field of characteristic 0 and containing all p^nth roots of unity. Let G be the p^n-cyclic group. Let $A \hookrightarrow B$ be a G-Galois étale extension[1]. One has the important

[1] I.e. $B^G = A$ where G acts on B in some manner and such that the extension is unramified and flat.

Theorem 1 (Kummer theory - [19] p.125-126) *There exist elements $f_1, \dots, f_n \in A$ such that each extension $B[\frac{1}{f_i}]/A[\frac{1}{f_i}]$ is Kummer of type $y^n = a_i$ where $a_i \in A[\frac{1}{f_i}]^*$. On the intersections $A[\frac{1}{f_i}, \frac{1}{f_j}]$ we have that the elements $\frac{a_i}{a_j}$ are already p^nth powers.*

One also has an analogous statement for the Artin Schreier type extensions

Theorem 2 (Artin Schreier theory - [19] p.127) *With all notations as above, assume instead that A is a characteristic p algebra over the field k and let B/A be G étale Galois as above. Then B is generated over A by the following relations*

$$(F-1)\big[x_1, \dots, x_n\big] = \underline{w} = [w_1, \dots, w_n]$$

where $\underline{w}$ is a suitable length n Witt vector $\underline{w}$ with coefficients in A. Here F is the Frobenius map between Witt vectors. We make this explicit for $n = 3$ to obtain

(B.1) $x_1^p - x_1 = w_1$

(B.2) $x_2^p - x_2 - \frac{x_1^{p^2} - x_1^p - (x_1^p - x_1)^p}{p} = w_2$

(B.3) $x_3^p - x_3 - \frac{x_2^{p^2} - x_2^p - (x_2^p - x_2 - \frac{x_1^{p^2} - x_1^p - (x_1^p - x_1)^p}{p})^p}{p} - \frac{x_1^{p^3} - x_1^{p^2} - (x_1^p - x_1)^p}{p^2} = w_3$

Adding an element of $(F-1)W_3(A)$ to the Witt vector $\underline{w}$ does not change the extension.

We briefly specialize to the case that A is the local field $k((t))$ with parameter t and k is an algebraically closed field of characteristic p. We shall be studying totally ramified extensions of the ring of integers in A, namely the power series ring $k[\![t]\!]$.

Lemma 3 *Let $k[\![z]\!]/k[\![t]\!]$ be some totally ramified p^3-cyclic Artin-Schreier extension and assume this is generated by the Witt vector $\underline{w} = [w_1,\ w_2,\ w_3]$. Then we can choose w_1 to have non-p-multiple t-adic valuation, and similarly w_2 and w_3 to have either a non-p-multiple negative t-adic valuation or a zero t-adic valuation.*[2]

Proof*:* We notice that $(F-1)[0, 0, v] = [0, 0, v^p - v]$ and $[q_1, q_2, q_3] + [0, 0, v^p - v] = [q_1, q_2, q_3 + v^p - v]$. Furthermore, let $p(t) \in tk[\![t]\!]$. Using Hensel's lemma we can find exactly one element $y_{p(t)} \in k[\![t]\!]$

[2] The hypothesis $n = 2$ or $n = 3$ is not necessary; we state it this way because we shall use it this way.

of positive t-adic valuation such that $y_{p(t)}^p - y_{p(t)} = p(t)$. Thus subtracting vectors of the form $(F-1)[y_{p(t)}, 0, 0]$ one can get rid of all the positive t terms occurring in w_1 of the Witt vector $\underline{w}$ given by Theorem 2. Similarly one can get rid of the positive terms occurring in w_2 and w_3 using $(F-1)[0, y_{p(t)}, 0]$ and $(F-1)[0, 0, y_{p(t)}]$. Furthermore, for any term of the form $a_{-ip}t^{-ip}$ where a_{-ip} is some constant and $i > 0$, by subtracting $\left[(a_{-ip})^{\frac{1}{p}} t^{-i}\right]^p - (a_{-ip})^{\frac{1}{p}} t^{-i}$ and repeating this one can get rid of the p-multiples too. Notice it is possible to be left with 0. ♣

We can give a slightly more useful description of p^2-cyclic extensions

Corollary 4 *With the notations as above, all totally ramified p^2-cyclic Galois extensions of $k((t))$ can be given by*

(B.4) $x_1^p - x_1 = t^{-m_1}$

(B.5) $x_2^p - x_2 = \frac{x_1^{p^2} - x_1^p - (x_1^p - x_1)^p}{p} + \sum_{\substack{0 \le i \le m_2 \\ 0 < j < p}} a_{i,j}^{(2)} (x_1^p - x_1)^j t^{-ip} + \sum_{1 \le k \le L_1} b_k^{(2)} t^{-k} + v_2(t)$

Here v_2 is an element of $k[\![t]\!]$, m_1 and m_2 are two non-negative integers, $(m_1, p) = 1$, N_1 is the t valuation of $x_1^p - x_1$ i.e. $N_1 = m_1$ and $L_1 = (N_1 - 1)(p-1)$.

Proof*:* Using the previous claim, we need to check only that the non Artin-Schreier component of the expression on the right hand side of equation B.5 covers all the non-p-multiple negative powers of t. We know $(m_1, p) = 1$ and notice $(x_1^p - x_1)^j t^{-ip} = t^{-(m_1 j + ip)}$. By Lemma D.3 the result follows since all the powers of t that we miss in this expression are stored in the terms $v_2(t)$ (the non-negative t power terms) and $\sum_{1 \le k \le L_1} b_k^{(2)} t^{-k}$ for the rest. This also explains the choice of the number L_1. ♣

B.2 Different considerations

Very important in our studies is the following result found in [2]

Theorem 5 ([2] p.244 Local Criterion for good reduction) *Let $A = R[\![t]\!]$ with fraction field A_0. Let B_0/A_0 be a separable Galois extension of fields and assume this induces a generic different of degree d_K. Assume the normalization of $R[\![t]\!]_\pi$ in B_0 is étale over $R[\![T]\!]_\pi$ and induces the residual extension $k((z))/k((t))$ of special different d_k where z is* *some local parameter in the field $k((z))$. Then $d_K = d_k$ iff the normalization of $R[\![T]\!]$ in B_0 is $R[\![Z]\!]$ for some local parameter Z reducing to z modulo π. In general the difference $d_K - d_k$ is twice the singularity number (see [2]).*

Kummer type differents

Lemma 6 *Consider the normalization of the algebra $R[\![T]\!] \otimes_R K$ in the field extension obtained by adjoining the Kummer roots $Y^n = F(T)\epsilon$ where*

$$F(T) = \prod (T - \alpha_i)^{e_i} \Phi$$

inside $\overline{K}[T]$. Here all the $|\alpha_i|_K < 1$ and $\Phi \in (R[\![T]\!] \otimes_R K)$ is a unit. Then the degree of different D_K is bounded by

$$D_K \leq \sum_i \big(\frac{n}{(n, e_i)} - 1\big)(n, e_i)$$

Appendix C

Rigid and formal tools

C.1 Coherent modules and Kiehl's theorem

On an ordinary scheme the idea of a coherent sheaf is linked to a finite extension of the scheme. When the schemes are Noetherian, one can build these *coherent* objects via modules over affine coverings of the scheme. A signifigant theorem in this regard is the following

Theorem 1 (Local Coherence implies Global Coherence) *Let F be a coherent sheaf on the locally Noetherian scheme X. Let $U \subset X$ be an affine open subscheme of X. Then $F(U)$ is a finite $O(U)$-module. In particular, if X is affine, then F is the sheaf associated to the finite $O(X)$-module $F(X)$.*

See [14] p.377 for the notion of a coherent module on an arbitrary G-ringed space. Again it amounts to saying that locally a coherent sheaf F is associated to finite modules over affinoid subdomains. The remarkable theorem of Kiehl states that we have

Theorem 2 (Kiehl [14] p.378 Cor.4) *Let F be a coherent sheaf on the rigid variety X. Then for any admissible affinoid covering $\mathcal{U}$ of X, it follows that F is $\mathcal{U}$ coherent. In particular, if X is affinoid, then F is the rigid module associated to some finite $O(X)$-module.*

It is not too hard to deduce (similar to the easier ordinary scheme case)

Corollary 3 ([14] p.383 Proposition 3) *Let $f : Y_K \to X_K$ be a morphism of rigid varieties with X_K affinoid. If for some affinoid covering $\{U_{K,i}\}$ of X_K the morphisms $f^{-1}(U_{K,i}) \to U_{K,i}$ are finite in the affinoid sense (see [14] Defn.7.1.4.3) then Y_K is affinoid and associated to a finite algebra over that of X_K.*

This result was used in gluing two finite coverings of two open subaffinoids of a rigid variety in order to make sure tat we obtain a finite covering; see the exposition on Garuti's ideas, chapter 4.

C.2 Finite maps and products of formal schemes

Proposition 4 (Finite Maps) *Let $A \hookrightarrow B$ be an extension of π-adic admissible R-algebras. This extension is finite iff the special fibre $A/\pi \hookrightarrow B/\pi$ is finite.*

Proof*:* We need only show that the special fibre property carries over to B/A. Let $e_1, \ldots, e_n$ be any elements of B such that their images is a set generating the finite module B_1 over A_1. We claim that the e_i generate B over A. Let $N = \sum_i Ae_i$ and notice that it is π-adically complete since it is a finite A-module. Let $f \in B$ and we see that there exists a $g_0 \in B$ and a $f_0 \in N$ such that $f = g_0\pi + f_0$. We can repeat this for g_1 and so obtain a sequence of elements $f_i\pi^i$ such that $f - \sum_{i=0}^n f_i\pi^i \in \pi^{n+1}B$. Furthermore the sequence $\gamma_i = \sum_{i=0}^n f_i\pi^i$ forms a Cauchy Sequence in $N \mathit{subset} B$ and converges in both N and B to the element $\gamma \in N$. Furthermore, since each $f - \gamma_i \in \pi^{i+1}B$ we see that $f - \gamma = 0$ since B too is complete in the π-adic topology. Consequently $f \in N$ and we are done. ♣

Again we restrict to π -adic admissible formal schemes over R. We would like to make the following statement clear

Proposition 5 (EGA 0.7.7.2, 0.7.7.6 and 0.7.7.7) *Let A,B and C be π-adic admissible R-algebras and assume at least one of B or C is* flat *over A. Then under reduction mod (π^i) we have the usual tensor product relations:*

$$(B\hat{\otimes}_A C) \otimes_R R/\pi^i \equiv B_i \otimes_{A_i} C_i$$

and so it does not change the closed fibres.

Remark The flatness condition stated is to ensure the resulting fibre product is R-flat and hence without R torsion - this is to ensure that we stay inside the category of π-adic R admissible algebras. One can define these things more generally.

For completeness we also note the following property of the completed tensor product

Proposition 6 (EGA 0.7.7.8) *Let A,B and C as above and all endowed with π-adic topologies. Then if C is finite over A then the usual tensor product and the completed tensor product agrees.*

Criteria for Surjectivity, Exactness and Isomorphy

The following 3 lemmata show the usefulness of working on completed rings. Many properties of the reduction carry over to the completed ring. These results are used in Henrio's proof of the local-global principle.

Lemma 7 *[Surjectivity Criterion] Let $f : M \to N$ be a map of a complete R-module M into a π-adic separated module N which is a surjection in the special fibre. Then f is a surjection.*

Lemma 8 *[Exactness Criterion] Let T and N be π-adically complete and R-flat (or torsion free) and let Q be a separated flat R-module in the π-adic topology. Then a sequence*

$$0 \to T \to N \to Q \to 0$$

is exact iff

$$0 \to T_k \to N_k \to Q_k \to 0$$

is.

Lemma 9 *[Isomorphism Criterion] Let $M \to N$ be a map of complete π-adic flat R-modules. It is an isomorphism iff $M_k \to N_k$ is.*

C.3 Homeomorphic normal rigid affinoids are isomorphic

The following lemma was used in our exposition on the structure of formal fibres. Essentially the idea was this: we constructed a map $X_K(x+) \to D^0_K$ of rigid varieties and show that it was étale finite over rigid subaffinoids of D^0_K and bijective on L points for all finite L/K. One concluded that $X_K(x+)$ must be isomorphic to D^0_K.

Lemma 10 *Let $W \to Z$ be a map of affinoid K rigid varieties which is étale finite and such that for any finite field extension L/K we have that $W(L) = Z(L)$ under this map. Assume Z is an integral normal rigid (affinoid) curve. Then $W \to Z$ is an isomorphism.*

Proof*:* Let $\mathcal{O}_Z$ and $\mathcal{O}_W$ be the Tate algebras corresponding to the rigid varieties W and Z respectively and consider the ring injection $\mathcal{O}_Z \to \mathcal{O}_W$. We note again the flatness of this extension and hence $\mathcal{O}_W$ is $\mathcal{O}_Z$ torsion free.

For a prime ideal $\langle z \rangle$ let d_z be the dimension of the $K(z)$ vector space $\mathcal{O}_W \otimes_{\mathcal{O}_Z} K(z)$ where $K(z)$ here is the residue field of Z at the point z. The hypothesis on the rigid variety Z implies that its set of prime ideals consists of its maximal ideals (or the points of the rigid space C) and a single *generic* point, namely the prime ideal (0). Let z be some L point of Z for some finite extension L/K and notice that the hypothesis implies that $d_z = 1$.

Hence, using Nakayama's lemma, or using [18] p.174 Ex.5.1.15, we can conclude that the set of prime ideals p of $\mathcal{O}_Z$ for which $d_p \leq 1$, is open in $spec(\mathcal{O}_Z)$; the topological notion here meant scheme theoretically. This implies in particular that the same is true at the generic point (0) of $spec(\mathcal{O}_Z)$. Thus generically, and hence globally, the rings $\mathcal{O}_Z$ and $\mathcal{O}_W$ coincide. ♣

C.4 Reducedness of fibres

Theorem 11 ([18] p.368 Ex.8.3.3) *Let B be a Noetherian normal flat R-algebra where R is a discrete valuation ring. Let K be the ring of fractions of R and L that of B. Let π be a uniformizer for R. Let $X = spec(B)$ and assume that X_k is topologically irreducible* *of dimension 1 and generically reduced (i.e. reduced at its generic points). Then* X_k *is reduced globally.*

Remark One cannot conclude reducedness of a scheme globally from reducedness on some dense open set. Consider for example the curve $spec\left[\frac{k[u,v]}{\langle u^2,uv\rangle}\right]$ which is reduced everywhere (hence *generically*) except the origin.

Appendix D

Miscellaneous tools

There were various ad hoc tools we used which we could not classify and so we put them here.

D.1 Flatness Criteria

One knows that an injection of a Dedekind domain (e.g. curves) into an integral domain renders the latter flat over the former:

Lemma 1 ([18] Cor. 1.2.14) *For any Dedekind domain A, an A-module M is A-flat iff it is A-torsion free. In particular if B is an integral domain containing A, then B is a flat A-module.*

However, when working with relative curves, one would like to have a similar *easy* criterion for deducing flatness. The following are in this direction:

Lemma 2 ([19] Theorem I.2.9, p.11) *Let A be an integral domain and M some* **finite** *A-module. Let $d(w)$ be the dimension of the $k(w)$ vector space $M \otimes_A k(w)$, where $k(w)$ is the residue field of A at a prime ideal w. Then M is flat iff $d(w)$ is independent of the choice of the prime ideal $w \subset A$.*

We used the above in several instances. First of all we used it in constructing an étale map of a smooth formal scheme to the formal disc, see section 2.2 of chapter 4 as well as the examples at the end of that chapter.

D.2 Combinatorial results

We have some purely combinatorial results. The first was used in writing useful expressions for generating p^2- and p^3-cyclic Galois extensions of the ring of formal power series of the field k of characteristic p; see the equations B.5 and 5.12. The second result we used in our negative example at the conclusion of chapter 4. It was a crucial ingredient and we list it here formally.

Lemma 3 *Let a and b be two relatively prime positive integers. Then the set $\{ma + nb \,|m, n > 0\}$ covers the integers larger than $(a-1)(b-1)$.*

Proof*:* To be safe we give the proof here, although it seems to be well known: assume $b < a$ and let j be some integer such that $0 < j \leq b$. We can write $a = bk + r$ with $0 < r < b$ and we study integers of the form

$$M = (a-1)(b-1) + 1 + j = b\big[bk + r - k - 1\big] + (1 + j - r)$$

If $1+j-r \equiv 0$ $[mod\ b]$ then we are done by noting that $1+j-r > -b$ (and hence M will be a *positive* multiple of b). If this is not the case then for some $0 < s < b$ we can write $rs \equiv 1 + j - r[\ mod\ b]$. Hence we can write

$$M = b\big[bk + r - k - 1\big] + (1 + j - r) + sa - sa$$

$$= sa + b\big[bk + r - k - 1 - sk\big] + (1 + j - r - sr)$$

We note that $(j+1-r) = rs - bx$ where x is some integer satisfying $x \leq (r-1)$ since $j+1-r > -r$ and $s < b$. Hence we can write

$$M = sa + b\big[bk + r - k - 1 - sk - x\big]$$

where $s > 0$ and it is clear that

$$bk + r - k - 1 - sk - x = \big[b - 1 - s\big]k + r - 1 - x \geq 0$$

One can do exactly the same for $j > b$ by translation. ♣

Lemma 4 *Let G be an order p^3 elementary abelian group. It has exactly $\frac{p^3-1}{p-1}$ subgroups of order p.*

Proof:There are exactly p^3-1 elements of order p and each generates a subgroup of order p. Two order p subgroups are either identical or intersect in the identity only. Each order p subgroup has exactly $p-1$ non-trivial elements and each generates it. The result follows. ♣

D.3 Birational Construction of the Jacobian Schemes

The following section is an attempt to construct the rational maps between relative curves and their singular Jacobians (see [7] for the notion) used in the paper of [7]. These maps lift those constructed in [21] and to clear it up for ourselves, we wrote it out. The method is that one compares the construction of the Jacobian scheme of a projective (possibly singular integral) R-curve with that of [21]. Serre's exposition gives the theorems we need, however he construct the Jacobians over fields only. We need them (and the rational maps involved) in the relative situation so as to reduce to the correct one in the fibres.

We recall the construction of the relative Jacobian schemes from both the viewpoints of [15] and [21]. Let $X \to S = spec(R)$ be a projective flat curve with integral fibres and arithmetic genus g[1]. Let $X' \subset X$ be the smooth locus. We have the following important

Lemma 5 *For any S-scheme $T \to S$, a section $T \to X'$ induces a relative Cartier divisor on $X' \times_S T$.*[2] *Furthermore, any T point in X' induces a relative Cartier divisor on $X \times_S T$ by embedding $X' \hookrightarrow X$. This induces a map on schemes (or equivalently the representable functors)*

$$(X')^{(g)} \to Div^{(g)}_{X/S}$$

sending the divisor on X' $\sum t_i$ to the divisor $\sum t_i$ on X (by embedding).

As explained in [15] the latter map in the lemma induces a divisor $\mathcal{D}$ on the S-scheme $X \times_S (X')^{(g)}$. It has the following property :

[1]The flatness implies the arithmetic genus is g on all the fibres.

[2]See [15] p.213 Lemma 8.2.6 (ii) for a proof - one does so on the fibres and then lifts it using the lemma.

Claim 6 *Let $w : T \to (X')^{(g)} \times_S T$ be some section of S-schemes. Composing with $(X')^{(g)} \to Div^{(g)}_{X/S}$ one has a section of S-schemes $T \to Div^{(g)}_{X/S} \times_S T$, or equivalently some divisor d_w on $X \times_S T$. This divisor is then exactly the pullback of the divisor $\mathcal{D}$ under the map*

$$X \times_S T \stackrel{(id,w)}{\to} X \times_S (X')^{(g)}$$

From $Div^{(g)}_{X/S}$ one has a map to the degree g Picard functor $Pic^{(g)}_{X/S}$ sending a divisor D to its associated invertible sheaf $\mathcal{O}_X(D)$ in $Pic^{(g)}(X)$.

Example It will become important, so we make it a bit more concrete: Let F be some field extension of either the fraction or the residue fields of R. Then a F point on $Div^{(g)}_{X/S}$ corresponds to some divisor on the (integral) F-curve $X_F = X \otimes_S F$ and the map $Div^{(g)}_{X/S} \to Pic^{(g)}_{X/S}$ sends it to its divisor class in the Picard group of X_F.

We thus have a map

$$(X')^{(g)} \to Div^{(g)}_{X/S} \to Pic^{(g)}_{X/S}$$

which we shall refer to as ϕ.

Example Again let F be a field as in the previous example and assume now that X is the singular curve C_δ associated to a modulus δ and a smooth arithmetic surface C/S.

A section $w_F : F \to (X')^{(g)} \times_S F = (X'_F)^{(g)}$ [3] corresponds to an effective relative divisor on the affine smooth curve X'_F. The map ϕ_F sends w_F to the class of the invertible sheaf $L(w_F)$ on $X_F = C_{\delta_F}$.

Let Q be the singular point on $X_F = C_{\delta_F}$ and let (s) be the divisor on X_F associated to some section of the invertible sheaf $L(w_F)$ on $X_F = C_{\delta_F}$. We assume this s is a section which is a local generator of $L(w_F)$ at the singular point Q.

[3] We know the formation of symmetric products and Hilbert functors agree with base change - see [15] p.253 and p.254.

This (s) is equivalent to the divisor w_F on X_F and this equivalence is necessarily established using a function (g) such that $g \equiv 1[\delta_F]$ in the sense of [21] p.76: if the g did not have this form, it will have poles at the singular point and since both w_F and (s) have support outside this singular point, we have a contradiction. Using the association of [21] p.76 (middle) we find thus that our map sends the divisor w_F to the class of the w_F (now regarded as a divisor in $X'_F = C'_{\delta_F} \subset C_F$) in the ray group $Cl^g_{\delta_F}(C_F)$.

The authors of [15] then constructs a certain open subscheme W of $(X')^{(g)}$ as well as a section $\epsilon : S \to W$ with certain properties. In particular the map

$$W \hookrightarrow (X')^{(g)} \to Div^{(g)}_{X/S} \to Pic^{(g)}_{X/S}$$

is an open immersion of S-schemes. The section $\epsilon : S \to W$ induces a section

$$\widehat{\epsilon} : S \xrightarrow{\epsilon} W \hookrightarrow Pic^{(g)}_{X/S}$$

which corresponds to the $Pic(X)$ class of the degree g divisor $\mathcal{D}_\epsilon$ on X.[4] We consider the translation on the group scheme $Pic_{X/S}$ by the element of the class of $-\mathcal{D}_\epsilon$ and notice this maps $Pic^{(g)}_{X/S}$ to the $Pic^{(0)}_{X/S}$ identity component.

Example Continuing with our examples of the field F earlier, this new map sends a point $w_F : F \to (X'_F)^{(g)}$ to the X_F class of w_F minus the class of the divisor $\epsilon_F : F \to (X'_F)^{(g)}$, or in symbols

$$w_F \mapsto \overline{cl}^0(D_{w_F} - D_{\epsilon_F})$$

where $\overline{cl}^0(E)$ here means the class of the degree 0 divisor E on X_F in $Pic^0(X_F)$. Interpreting the group $Pic^0(X_F)$ as the ray group $Cl_{\delta_F}(C_F)$ of the (smooth) curve C_F we find it maps w_F to the class of $w_F - \epsilon_F$ as divisors on C_F.

Consider now the quotient map

$$(X')^g \to (X')^{(g)}$$

[4] We recall that $\mathcal{D}_\epsilon$ means the divisor on $X \times_S S$ corresponding to the section $S \xrightarrow{\epsilon} W \to Div^{(g)}_{X/S}$.

under the S_g action and let $[e_1, \ldots, e_m]$ be some S-point mapping to the section $\epsilon : S \to W \hookrightarrow (X')^{(g)}$ under the above quotient map.

Consider the closed immersion of schemes

$$X' \hookrightarrow (X')^g$$

induced by sending

$$x \mapsto [x, e_2, \ldots, e_m]$$

for some $x : T \to X' \times_S T$. We compose this with the quotient map

$$(X')^g \to (X')^{(g)}$$

to obtain a map of S-schemes $X' \to (X')^{(g)}$.

Locally about the S-point $e_1 : S \to X$ this maps into the dense open set $W \subset (X')^{(g)}$ Hence for some open subset X'' of X' we have that $X' \to W$ is a well defined rational map and it takes the point $e_1 : S \to X'$ into the section $\epsilon : S \to W$. However, we have seen that W embeds into the Jacobian $Pic^{(0)}_{X/S}$.

Example Continuing once again with the example of the field F, the above rational map $X'_F \to Pic^{(0)}_{X/S}$ sends a *general* point z_F of X'_F to the class of $z - e_1$ in the Jacobian of the singular curve X_F or, in terms of the ray classes the class of the δ_F prime divisor $z - e_1$.

We have seen that W_F is open (and dense) inside $Pic^0_{X_F/F}$. Consider the multiplication

$$m_F : Pic^0 \times_F Pic^0 \to Pic^0$$

which is onto and notice that the inverse image of the open subset W_F intersects the open subset $W_F \times_F W_F$ and let this intersection be V_F :

$$V_F = (W_F \times_F W_F) \cap m_F^{-1}(W_F)$$

$$\subset W_F \times_F W_F$$

$$\subset Pic^0 \times Pic^0$$

Let $w_{F,1}$ and $w_{F,2}$ be two F points of V_F and these correspond to δ_F prime divisors on C_F. They map to the classes of $w_{F,1} - ge_1$ and $w_{F,2} - ge_1$ in $Pic^0_{X_F/F}$ and the inverse image of the sum is that unique element $t \in W_F$ such that t is δ equivalent to $w_{F,1} + w_{F,2} - g_e 1$. But we see that this is exactly the birational group law that Serre puts on his construction of the Jacobian in [21] p.81 (Proposition 6). The situation in [21] is thus completely isomorphic to the one that we have described here and more importantly :

Claim 7 *The map $C'_\delta \to Pic^0_{C_\delta/S}$ induces those of [21] on the fibres.*

This was the point of this section.

D.4 Hurwitz trees

In view of the local global principle a study of automorphisms on the disc $R[\![Z]\!] \otimes_R K$ seems important and it would be a loss to ignore a beautiful theorem of Yannick Henrio that he proved in his PhD thesis ([4]). See the papers [3] and [4] for more on automorphisms on the open disc. Throughout this appendix section we shall let G be the cyclic p-group and this is assumed to act on the open disc $R[\![Z]\!]$. Our aim is not to study the theory, but to at least mention enough so that we can use it to get an interesting example. Whereas the usual liftings of p-cyclic coverings constructed in chapter 4 only require one blow up to split the fixed points, the ones we construct require five blow ups, or as the author sees it two levels. Hopefully the terminology will become clearer as we move through this appendix.

Let the fixed points be $\{x_1, \ldots, x_m\} \subset spec(R[\![Z]\!])$ each with valuation e_i. By systematically blowing up the formal scheme $Spf(R[\![Z]\!])$ at the primes $\langle \pi^{e_i}, Z \rangle$ we obtain a formal scheme with reduction which splits the points x_i to different specializations. We also impose some restriction on the blowups by imposing that we only blow up at such an ideal if there are more than one of the x_i reducing to the same special fibre point. One finds that blowing up causes certain double points to form in the special fibre and to each of these we can associate a thickness and this is related to the various e_i.

Example (Splitting the branch points) We give the easiest example: the action on $R[\![Z]\!]$ for $Z \mapsto \frac{Z}{\zeta+Z}$ where R contains a p^{th} root of unity and has parameter $\lambda = \zeta_p - 1$. This action has fixed points $\langle Z \rangle$ and $\langle Z + \lambda \rangle$ inside $spec(R[\![Z]\!])$. We assume R has parameter $\pi = \lambda$ and take a blow up at $\langle \pi^1, Z \rangle$. This gives the affine schemes $\mathcal{A}_{out,u} = R[\![Z]\!][\frac{\pi}{Z}]$ and $\mathcal{A}_{in,u} = R[\![Z]\!][\frac{\pi}{Z}]$. We can complete these to obtain the affine algebras $\mathcal{A}_{out} = R[\![Z]\!] \left\langle \frac{\pi}{Z} \right\rangle$ and $\mathcal{A}_{in} = R[\![Z]\!] \left\langle \frac{Z}{\pi} \right\rangle = R \langle W \rangle$ where $W = \frac{Z}{\pi}$. One finds that G acts on both these and that the two formal schemes $\mathcal{M}_{in} = Spf(R\langle W \rangle)$ and $\mathcal{M}_{out} = Spf(R[\![Z]\!]\langle \frac{\pi}{Z} \rangle$ glue G-equivariantly along their common boundary the circle $Spf(R\langle W, \frac{1}{W} \rangle)$.

Example (Splitting the branch points in extensions) In the same spirit as the previous example, let R now denote the valuation ring dominating $W(k)[\zeta_p]$ with parameter $p^{\frac{1}{7p^2(p-1)}} = \pi$ and ζ_7 a primitive 7^{th} root of unity. We assume $(p, 7) = 1$. Consider the lifting $R[\![Z]\!]/R[\![T]\!]$ of $k[\![z]\!]/k[\![t]\!]$ given by $x^p - x = t^{-7}$ found earlier: we recall the extension was constructed as the normalization of the equation $W^p = T^{-7}(\lambda^p + T^7)$. The fixed points of $R[\![Z]\!]$ are those lying over the primes $\langle T \rangle$ and $\langle T - \alpha_i \rangle$ of $spec(R[\![T]\!])$ where the α_i are the roots of $\alpha_i^7 = -\lambda^p$.

Notice that for $m \neq 0$ $[mod\ 7]$ we have that $\zeta_7^m \neq 1$ $[\ mod\ \pi]$. The distances between the α_i thus have norms $|\alpha_1| = |\alpha_i| = |\alpha_7|$. Thus taking a blow up of $R[\![T]\!]$ at the circle of norm $|\alpha_i|$ will result in a model which completely splits the points $0, \alpha_1, \ldots, \alpha_7$. The same thing occurs in the normalization $R[\![Z]\!]$ with its 8 fixed points: those lying above the $\langle T \rangle$ and $\langle T - \alpha_i \rangle$.

Our next example concerns the thickness of the (only) double point in the example above. It is easy to work with the blow ups of the ring $R[\![T]\!]$ but working with its extension $R[\![Z]\!]$ is harder. We need to study it indirectly as we did above in order to find the thickness of the double point in the extension.

Example (Outer rim of open disc) With assumptions as above let $\sigma \in G$ and let $\sigma : Z \mapsto$

$ZF_\sigma(Z)$ where $F_\sigma(Z)$ is a unit. Notice that

$$Norm_{R[\![Z]\!]/R[\![T]\!]}(Z) = \prod_{\sigma \in G} \sigma(Z) = TG(T)$$

where $G(T)$ is a unit in $R[\![T]\!]$ and hence we have that $R[\![Z]\!][\frac{\pi^e}{Z}]^G = R[\![T]\!][\frac{\pi^{ep}}{T}]$ and similarly on completions.

In the example above (with special conductor 7) we found that the thickness of the double point in $\mathcal{M}_T$ is p^3; we recall that $R[\![T]\!]$ was blown up at a circle of radius of $|\alpha_i|$ which has a π valuation of p^3. The above relation implies this to be p times that of $\mathcal{M}_Z$ and hence the latter is p^2.

Moving inwards one finds that the action of G on the *inner part* of the disc is trivial after reduction. This implies that for all of the projective lines $C_{Z,i}$ occurring as components in $\mathcal{M}_Z$ are purely inseparable extensions over the corresponding $C_{T,i}$. The only exception is the *root* component which induces the extension $k(\!(t)\!) \hookrightarrow k(\!(z)\!)$ and this can be Galois or purely inseparable - depending on the inertia group of $\langle \pi \rangle$ of $R[\![Z]\!]$ - see [2] for details. In general we shall be interested in the cases when the *root* component induces a Galois extension $k(\!(t)\!) \hookrightarrow k(\!(z)\!)$.

Example (Inner rim blow up:) Again look at $Z \to \frac{Z}{\zeta + Z}$ and the inner rim is $R! \langle W \rangle$ with automorphism $W \mapsto \frac{W}{\zeta + \pi W}$ - here W is identified with $\frac{W}{\pi}$. Reducing the automorphism modulo π gives simply $w \mapsto w$.

Let $C_{Z,i} \to C_{T,i}$ be one such purely inseparable extension and let the generic points be $\zeta_{Z,i}$ and $\zeta_{T,i}$ and the corresponding extension of discrete valuation rings

$$\mathcal{O}_{\mathcal{M}_T, \zeta_{T,i}} \hookrightarrow \mathcal{O}_{\mathcal{M}_Z, \zeta_{Z,i}}$$

These are discrete valuation rings with parameter π and residue fields the functions fields of $C_{Z,i}$ and $C_{T,i}$ respectively. Henrio then associates data in order to classify this purely inseparable extension: he gives a

- $u \in \mathcal{O}_{\mathcal{M}_T, \zeta_{T,i}}$

- a differential w_i on $C_{T,i}$
- The different δ_i

Henrio organizes this data into a *Hurwitz* tree $(V, \sum)$ consisting of:

- a root vertex $r_0 \in V$
- component vertices $v_i \in V_{comp}$ corresponding to $\mathbf{P}^1_k$ components Z_i of the blown up model $\mathcal{M}_T$
- leaf vertices $u_i \in V_{leaf}$ corresponding to reductions of the original branch points
- *double point* edges $e_x \in \sum_{dbl.pt.}$ corresponding to the induced double points with origin and targets the two components on which it lies. We associate two distinct directed edges to a double point with directions opposite.
- *leaf* edges $f_i \in \sum_{leaf}$ connecting the leaf vertices with the component on which they lie
- for each $v \in V_{comp}$ corresponding to $C_{Z,i}$ an element $\overline{u}_v$, a differential w_v on $C_{T,i}$ and the different δ_v classifying the extension $C_{Z,i} \to C_{T,i}$
- for each $e \in \sum_{dbl.pt.}$ corresponding to the double point x in $C_{Z,i}$ lying over the double point $y \in C_{T,i}$: a thickness $\epsilon(e_x)$ of x, two residue numbers $h_{e,t}$ and $h_{e,s}$ corresponding to the residues of the differentials $w_{t(e)}$ and $w_{s(e)}$ at y in the component $C_{T,t(e)}$ and $C_{T,s(e)}$ respectively, as well as two integers $m_{e,t}$ and $m_{e,s}$ related to their orders at y as follows: $ord_{y \in C_{T,t(e)}}(w_{t(e)}) = (m_{e,t} - 1)$ and $ord_{y \in C_{T,s(e)}}(w_{s(e)}) = -(m_{e,s} + 1)$ respectively.[5]

He then proves the following properties

Theorem 8 ([4] Theorem 4.1) *Let G act on the open disc $R[\![Z]\!]$ lifting the Galois extension $k[\![t]\!] \hookrightarrow k[\![z]\!]$. We have*

- $\forall e \in \sum \mid d_{t(e)} = d_{s(e)} + m_{t(e)}\epsilon(e)(p-1)$
- $\forall e \in \sum \mid m_{t,e} = -m_{s,e}$
- *If e_0 is the unique edge with origin r_0 then m_{t,e_0} is the conductor of $k[\![t]\!] \hookrightarrow k[\![z]\!]$*
- *If e is a leaf edge then $m_{t,e} = 0$*
- *If v is a component vertex and $Ar(v)$ the set of edges with origin v, then* $\sum_{e \in Ar(v)} \left[m_{t,e} + 1\right] = 2$

[5]For an edge e we write $s(e)$ and $t(e)$ for the source and target respectively.

- *If v is a leaf vertex, then $d(v) = v_R(p)$.*

Finally a Hurwitz tree whose data satisfies these conditions can be reversed to a G-action on the open disc.

The reversal statement in Henrio's theorem actually permits us to construct new liftings of the characteristic p extension $x^p - x = t^{-7}$, as the following example intend to show:

Example In characteristic $p = 3$ we define differential forms w_i given as

- $w_1 = w_2 = w_3 = w_4 = \frac{d\overline{u}}{\overline{u}}$ where $\overline{u} = \frac{z-1}{z}$.
- $w_v = dG = \frac{dz}{(z^2-1)^2(z^2-\gamma^2)^2}$ where $G(z) = \frac{-1z+\frac{z^5}{5}}{(z^2-1)^3(z^2-\gamma^2)^3}$; here $\gamma^2 \equiv -1[mod\ 3]$

The first differentials have simple poles at 0 and 1 with residues -1 and 1 respectively. One can check that the second has no residues, however it has 4 poles (namely ± 1 and $\pm\gamma$) of order 2 each and a single zero of order 6 at ∞. We associate to it a Hurwitz tree[6] with 5 component vertices, 5 double point edges and 8 leafs. Please see the figure (Hurwitz Tree) included at the end of the thesis for illustration. The figure on the right represents the induced special fibre (of either $\mathcal{M}_Z$ or the quotient $\mathcal{M}_T$: the special fibres are topologically homeomorphic to 5 projective lines Z_0, Z_1, ..., Z_4 intersecting at 4 double points e_1, e_2, e_3 and e_4).

We assume that R has been suitably enlarged such that $v_R(p) = 7(p-1)p^2$. Let the thickness of e_0 be ϵ_0 and that of $e_1, \ldots, e_4$ be ϵ_1. As long as $7\epsilon_0 + \epsilon_1 = 7p^2$ then this set of data verifies the reversal criterion and we thus have new liftings of the conductor 7 problem. One can even control the liftings with the ϵ_0.[7]

[6] We orient the edges positively if the point away from the root v_0. We only indicate them and their corresponding $m_{e,t}$.

[7] We admit that the form of the function $G(z)$ such that $d(G(z))$ has the mentioned poles and zeros was taken from a more general discussion of Stefan Wevers and Irene Bouw in their paper [13]. Their method was so useful that we couldn't resist borrowing it for our example.

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