Multivariate refinable functions with emphasis on box splines

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Declaration

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Date: 2 March 2009

Signed: R. van der Bijl

Summary

The general purpose of this thesis is the analysis of multivariate refinement equations, with focus on the bivariate case. Since box splines are the main prototype of such equations (just like the cardinal B-splines in the univariate case), we make them our primary subject of discussion throughout. The first two chapters are indeed about the origin and definition of box splines, and try to elaborate on them in sufficient detail so as to build on them in all subsequent chapters, while providing many examples and graphical illustrations to make precise every aspect regarding box splines that will be mentioned.

Multivariate refinement equations are ones that take on the form

$$\phi(x) = \sum_{i \in \mathbb{Z}^n} p_i \phi(Mx - i), \tag{1}$$

where ϕ is a real-valued function, called a *refinable function*, on $\mathbb{R}^n$, $p = \{p_i\}_{i \in \mathbb{Z}^n}$ is a sequence of real numbers, called a *refinement mask*, and M is an $n \times n$ matrix with integer entries, called a *dilation matrix*.

It is important to note that any such equation is thus simultaneously determined by all three of ϕ, p and M — and the thesis will try and explain what role each of these plays in a refinement equation.

In Chapter 3 we discuss the definition of refinement equations in more detail and elaborate on box splines as our first examples of refinable functions, also showing that one can actually use them to create even more such functions. Also observing from Chapter

2 that box splines demand yet another parameter from us, namely an *initial direction matrix* $\mathcal{D}$, we focus on the more general instances of these in Chapter 4, while keeping the dilation matrix M fixed. Chapter 5 then in turn deals with the matrix M and tries to generalize some of the results found in Chapter 3 accordingly, keeping the initial direction matrix fixed.

Having dealt with the refinement equation itself, we subsequently focus our attention on the support of a (bivariate) refinable function — that is, the part of the xy-grid on which such a function "lives" — and that of a refinement mask, in Chapter 6, and obtain a few results that are in a sense introductory to our work in the next chapter.

Next, we move on to discuss one area in which refinable functions are especially applicable, namely subdivision, which is analyzed in Chapter 7. After giving the basic definitions of subdivision and subdivision convergence, and investigating the "sum rules" in Section 7.1, we prove our main subdivision convergence result in Section 7.2. The chapter is concluded with some examples in Section 7.3.

The thesis is concluded, in Chapter 8, with a number of remarks on what has been done and issues that are left for future research.

Opsomming

Die algemene doel van hierdie tesis is die analise van meerveranderlike verfyningsvergelykings, met fokus op die bivariate geval. Aangesien bokslatfunksies die hoof-prototipe van sodanige funksies is (net soos die kardinale B-latfunksies in die eendimensionele geval), maak ons hulle regdeur ons primêre onderwerp van bespreking. Die eerste twee hoofstukke handel inderdaad oor die oorsprong en die definisie van bokslatfunksies, en probeer om in genoegsame besonderhede daarop uit te brei, met die doel om verder daarop te bou in alle daaropvolgende hoofstukke, terwyl talle voorbeelde en grafiese illustrasies gegee word om elke aspek aangaande bokslatfunksies vas te lê.

Meerveranderlike verfyningsvergelykings is díe van die vorm

$$\phi(x) = \sum_{i \in \mathbb{Z}^n} p_i \phi(Mx - i), \tag{2}$$

waar ϕ 'n reëlwaardige funksie, genoem 'n *verfynbare funksie*, op $\mathbb{R}^n$, $p = \{p_i\}_{i \in \mathbb{Z}^n}$ 'n ry van reële getalle, genoem 'n *verfyningsmasker*, en M 'n $n \times n$ matriks met heeltallige inskrywings, genoem 'n *dilasiematriks*, is.

Dit is belangrik om daarop te let dat enige sodanige vergelyking dus gelyktydig bepaal word deur al drie van ϕ, p en M — die tesis gaan poog om te verduidelik watter rol elkeen in die verfyningsvergelyking speel.

In Hoofstuk 3 bespreek ons die definisie van verfyningsvergelykings in meer besonderhede en brei ons verder op bokslatfunksies uit as ons eerste voorbeelde van verfynbare funksies, en wys verder dat hulle gebruik kan word om selfs meer sulke funksies te skep. Aangesien dit uit Hoorstuk 2 blyk dat bokslatfunksies nóg 'n parameter van ons verlang, naamlik 'n *aanvanklike rigtingsmatriks* $\mathcal{D}$, fokus ons in Hoofstuk 4 op die meer algemene gevalle hiervan, terwyl ons die dilasiematriks, M, onveranderd hou. In Hoofstuk 5 word dan weer gekyk na die matriks M en word probeer om sommige van die resultate in Hoofstuk 3 sodanig te veralgemeen, terwyl die rigtingsmatriks hier onveranderd gelaat word.

Nadat die verfyningsvergelyking self behandel is, rig ons ons aandag op die steungebied van 'n (bivariate) verfynbare funksie — maw. die deel van die xy-vlak waarop so 'n funksie "lewe" — en díe van 'n verfyningsmasker, in Hoofstuk 6, en verkry 'n aantal resultate wat in 'n sekere sin inleidend is tot ons werk in die daaropvolgende hoofstuk.

Hierna bespreek ons een area waar verfynbare funksies veral toepaslik is, naamlik subdivisie, wat ge-analiseer word in Hoofstuk 7. Nadat ons die basiese definisies van subdivisie en subdivisie-konvergensie gegee het en in Afdeling 7.1 die "som-reëls" bestudeer het, bewys ons ons hoof-subdivisie-konvergensieresultaat in Afdeling 7.2. Die hoofstuk word afgesluit met 'n aantal voorbeelde in Afdeling 7.3.

Die tesis word afgesluit, in Hoofstuk 8, met 'n aantal opmerkings oor die werk wat gedoen is en 'n paar sake wat oorbly vir verdere navorsing.

Acknowledgements

My thanks go in the first place to my supervisor, Prof de Villiers, whose careful proof-reading and imaginative suggestions have been indispensable.

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Contents

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List of Symbols

I^n	the identity matrix of order n, $n \in \mathbb{N}$
1-D	one-dimensional or univariate
$\Pi^{(n)}$	the set of all polynomials of n variables
$\Pi_k^{(n)}$	the set of polynomials in $\Pi^{(n)}$ of total degree $\leq k$
$\mathbb{Z}^{m\times n}$	the set of all matrices of order $m \times n$ with entries in $\mathbb{Z}$
$\mathbb{R}^{m\times n}$	the set of all matrices of order $m \times n$ with entries in $\mathbb{R}$
$M(\mathbb{R}^s)$	the set of functions on $\mathbb{R}^s$
$M_0(\mathbb{R}^s)$	the set of functions that are compactly supported on $\mathbb{R}^s$
$C(\mathbb{R}^s)$	the set of functions that are continuous on $\mathbb{R}^s$
$C_0(\mathbb{R}^s)$	the set of functions that are compactly supported and continuous on $\mathbb{R}^s$
$M(\mathbb{Z}^s)$	the set of sequences $c = \{c_i\}_{i\in\mathbb{Z}^s} \subset \mathbb{R}$
$M_0(\mathbb{Z}^s)$	the set of sequences $c = \{c_i\}_{i\in\mathbb{Z}^s} \subset \mathbb{R}$ that are finitely supported
$D^{(k)}_{(\alpha_1,\ldots,\alpha_s)}$	k'th order directional derivative in the direction $(\alpha_1, \ldots, \alpha_s)$
$C^k(\mathbb{R}^s)$	the set of functions that are k'th order continuous, i.e. those functions that have all k'th order partial derivatives continuous everywhere on $\mathbb{R}^s$
N_m	Cardinal B-spline of order m
$\mathcal{D}$	direction matrix
$\mathcal{D}_k$	direction matrix consisting of $k+1$ direction vectors (except where explicitly stated otherwise)
B	box spline

$B_{\mathcal{D}}$	box spline associated with $\mathcal{D}$
B_k, $B_{\mathcal{D}_k}$	box spline associated with $\mathcal{D}_k$
ϕ	refinable function
p	refinement mask
p_k	2-refinement mask corresponding to the box spline B_k
M	integer dilation matrix
$(p, \phi)_M$	M-refinement pair
(p, ϕ) or $(p, \phi)_2$	2-refinement pair
P	2-refinement mask symbol
P_k	2-refinement mask symbol associated with the pair (p_k, B_k)
$\ell^\infty(\mathbb{Z}^2)$	the set $\left\{c \in M(\mathbb{Z}^2) \subset \mathbb{R} : \sup_{i,j} \lvert c_{i,j}\rvert < \infty\right\}$ of bounded sequences
$\lvert\lvert a\rvert\rvert_\infty$	the infinity-norm of a sequence, $\lvert\lvert a\rvert\rvert_\infty = \sup_{i,j\in\mathbb{Z}} \lvert a_{i,j}\rvert,\ a \in M(\mathbb{Z}^2)$
$\triangle_1$	the operator $\triangle_1 : c_{i,j} \mapsto c_{i-1,j}$
$\triangle_2$	the operator $\triangle_2 : c_{i,j} \mapsto c_{i,j-1}$
$\triangle^\infty(\mathbb{Z}^2)$	the set $\Big\{c \in M(\mathbb{Z}^2) : \triangle_1 c \in \ell^\infty(\mathbb{Z}^2), \triangle_2 c \in \ell^\infty(\mathbb{Z}^2)$, and $\lvert\lvert\triangle c\rvert\rvert_\infty < \infty$, where $\lvert\lvert\triangle c\rvert\rvert_\infty = \max\{\lvert\lvert\triangle_1 c\rvert\rvert_\infty, \lvert\lvert\triangle_2 c\rvert\rvert_\infty\}\Big\}$
$\lceil x\rceil$	the smallest integer greater than or equal to x
$\lfloor x\rfloor$	the largest integer less than or equal to x
$S_{M,p}$	subdivision operator with respect to the integer dilation matrix M and the M-refinement mask p
$S_{2,p}$	subdivision operator with respect to the matrix $M = 2I$ and the 2-refinement mask p
$\mathcal{A}_{k,l}$	the set $\Big\{p \in M_0(\mathbb{Z}^2) : p_{i,j} \geq 0,\ (i,j) \in \mathbb{Z}^2;$ $p_{i,j} = 0,\ i \notin \{0, \ldots, k\},\ j \notin \{0, \ldots, l\};$ $\sum_{i,j} p_{2i,2j} = \sum_{i,j} p_{2i+1,2j} = \sum_{i,j} p_{2i,2j+1} = \sum_{i,j} p_{2i+1,2j+1} = 1\Big\}$

Chapter 1

Introduction

Polynomials are wonderful even after they are cut into pieces, but the cutting must be done with care. One way of doing the cutting leads to the so-called spline functions. —Iso Schoenberg (1964)

Univariate spline functions have been the object of study of many mathematicians since the middle 1900's. They are, in the first place, piecewise polynomials, and they exhibit some special features. Namely, the polynomial fragments are joined at their break points with special care, and they, moreover, possess certain levels of smoothness there, in the sense that their derivatives are continuous up to a certain degree.
A descriptive account on the history and development of spline theory is given in [12]. Carl de Boor was one of the first mathematicians to study splines in earnest and he later started looking at linear combinations of dilated and shifted versions of the B-splines (i.e. splines with minimal support) — the beginning of the theory of refinement equations.

Denoting by $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ the real numbers, integers and natural numbers respectively, a refinement equation is one involving a function ϕ of one (real) variable and a bi-infinite series of real numbers $p_i, i \in \mathbb{Z}$, and reads as follows:

$$\phi(x) = \sum_{i\in\mathbb{Z}} p_i \phi(2x - i), \quad x \in \mathbb{R}. \tag{1.1}$$

If, for a function ϕ, there exists a sequence $p = \{p_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}$ such that (1.1) holds, then (p, ϕ) is called a refinement pair, ϕ is called a refinable function and p the corresponding refinement mask. It has been proved (see [13]) that, if ϕ is a refinable function, then its corresponding refinement mask p is unique, i.e. ϕ can be uniquely expressed as a linear combination of its own dilated shifts.

In [12], it is observed that the work that De Boor had done showed that the B-splines were the first examples of functions that were refinable, and that, moreover, their corresponding refinement masks consisted of weighted binomial coefficients.

Up until the middle 1970's, the study and development of spline functions, including the B-splines, were inherently focussed on the univariate case. Some attempts to generalize the splines to higher dimensions included simplex, cone and tensor product splines, all of which had limited use with respect to practical applications.
Box splines were first mentioned in [7] by De Boor and Ronald de Vore and the central ideas and properties concerning them were established by De Boor, Dahmen, Chui, Micchelli and Höllig in e.g. [4], [6], [7] and [8].

It is the objective of this thesis to study the theory of box splines, and to do so from the viewpoint of bivariate refinement equations.

We define such refinement equations as

$$\phi(x) = \sum_{i\in\mathbb{Z}^2} p_i \phi(Mx - i), \tag{1.2}$$

where ϕ is a real-valued function on $\mathbb{R}^2$ and $\{p_i : i \in \mathbb{Z}^2\}$ is a real-valued sequence such that (1.2) holds. Note that the *dilation factor* 2 of equation (1.1) has been replaced by a dilation *matrix* M in equation (1.2), where M denotes a 2×2 matrix with integer entries, with $\det(M) \neq 0$. In particular, we shall give special attention to the case where

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Of course, one can in general let ϕ be a real-valued function on $\mathbb{R}^n$ for any $n \in \mathbb{N}$ in equation (1.2), with the order of M adjusted accordingly. We restrict our attention to the bivariate case at first, and will conclude the thesis with some remarks and generalizations to the higher-dimensional case.

Chapter 2

Definition

There are three equivalent definitions for box splines, namely geometric, analytic and inductive.

The *geometric definition* is inherited from that of simplical splines and enjoys much attention in the (especially earlier) literature (see e.g. [20], [9], [21]). For simplicity, the univariate linear simplical spline N with knots at points x_0, x_1 and $x_2 \in \mathbb{R}$ is given in Figure 2.1 (b) and can be found as follows: Let P be the orthogonal projection of $\mathbb{R}^2$ onto $\mathbb{R}$ and consider points v_0, v_1 and $v_2 \in \mathbb{R}^2$ such that $P(v_i) = x_i,\ i = 0, 1, 2$, and, lastly, define σ to be the 2-simplex (i.e. the triangle) defined by v_0, v_1 and v_2. The spline $N = N(x),\ x \in \mathbb{R}$, is then given by

$$N(x) = \frac{1}{\mathrm{vol}_2 \sigma} \mathrm{vol}_1 \{v \in \sigma : P(v) = x\},$$

where vol_k denotes k-dimensional volume, $k = 1, 2, ...$, e.g. vol_1 is length, vol_2 is area and vol_3 is volume.

For the definition of box splines, the same idea is used, but with the simplex being replaced by an appropriately chosen cube — meaning that box splines are in fact interpreted as linear projections of higher dimensional cubes (or *boxes* — hence the name *box splines*) (see also [8]).

Formally, if one considers a spline $N = N(x),\ x \in \mathbb{R}^s$, of s variables with knots at points

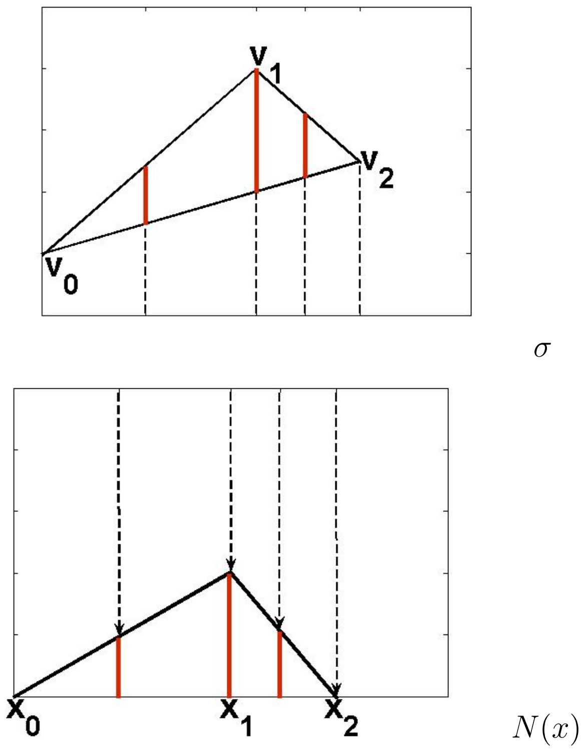


Figure 2.1: Geometrical illustration of a univariate linear simplical spline N

$x_1, x_2, \ldots, x_n,\ x_i \in \mathbb{R}^s, i = 1, 2, \ldots, n$, where $n \in \mathbb{N}$, and if $v_1, v_2, \ldots, v_n,\ v_i \in \mathbb{R}^n, i = 1, 2, \ldots, n$, are linearly independent vectors such that $P(v_i) = x_i,\ i = 1, 2, \ldots, n$, where P is the orthogonal projection from $\mathbb{R}^n$ onto $\mathbb{R}^s$, then

$$N(x) = \frac{1}{\mathrm{vol}_n \mathcal{B}} \mathrm{vol}\left\{ v \in \mathcal{B} : P(v) = x \right\},$$

where, also, $\mathcal{B} = \Big\{ \sum_{i=1}^{n} c_i v_i \ :\ c = \{c_1, \ldots, c_n\} \in [0,1)^n \Big\}$ is an n-dimensional parallelepiped.

The *analytical definition*, as given in, e.g., [9], [21] and [20], characterizes a box spline as the solution, $B \in M(\mathbb{R}^s)$, of the functional equation

$$\int_{\mathbb{R}^s} B(x_1, x_2, \ldots, x_s) f(x_1, x_2, \ldots, x_s)\, dx_1\, dx_2 \ldots\, dx_s$$

$$= \int_{[0,1)^k} f\big(t_1 d_1 \ + \ t_2 d_2 \ + \ \ldots \ + \ t_k d_k\big)\, dt_1\, dt_2 \ldots\, dt_k, \tag{2.1}$$

for all test functions $f \in C(\mathbb{R}^2)$, where, $\{d_1, d_2, \ldots, d_k\} \subset \mathbb{Z}^s$ is a given set of vectors that in fact characterizes the box spline, and where $k \in \mathbb{N},\ s \in \mathbb{N}$.
It is explained in [20] that this characterization follows from the Hermite-Gennochi formulation for the divided difference of a function and by writing this divided difference in turn in terms of cardinal B-splines, for the univariate case, an idea which is then generalized to obtain the above formula for the multivariate case (see also [16]).

Although the geometric definition is helpful to visualize the idea of box splines, neither it nor the analytical one is ideal to actually compute the box splines or derive further their properties. Therefore, an equivalent definition would prove useful. In the next section the inductive definition will be given. In [9] it is shown how the analytical definition is equivalent to the geometrical one, and in [21] the latter is in turn shown to be equivalent to the inductive definition. For the remainder of this thesis we shall therefore interpret box splines as being obtained by the inductive definition, and keep in mind that, had we gone by using any of the other two definitions, our results would not have been altered in any way.

2.1 Preliminary notation

Throughout our work we will make use of the notations $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ that denote the real numbers, integers and natural numbers respectively.
The symbol $\mathbb{Z}_+$ will denote the non-negative integers.
For $k \in \mathbb{N}$, $\mathbb{R}^k$, $\mathbb{Z}^k$, $\mathbb{Z}_+^k$ and $\mathbb{N}^k$ will denote the sets of arrays consisting of k numbers with entries in $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$, respectively.
$\mathbb{Z}^{m\times n}$ (resp. $\mathbb{R}^{m\times n}$) will be the set of all matrices of order $m \times n$ with entries in $\mathbb{Z}$ (resp. $\mathbb{R}$).
We write $\sum_i$ for $\sum_{i\in\mathbb{Z}}$ and $\sum_{i,j}$ for $\sum_{i,j\in\mathbb{Z}}$, unless stated otherwise.
For $n \in \mathbb{N}$, we write $\Pi^{(n)}$ for the set of all Laurent polynomials of n variables and

$\Pi = \cup\{\Pi^{(n)} : n \in \mathbb{N}\}$ for the set of all Laurent polynomials. Also, $\Pi_k^{(n)}$ (resp. Π_k) will be those Laurent polynomials in $\Pi^{(n)}$ (resp. Π) of *total degree* $\leq k$ where $k \in \mathbb{Z}$. Here, we mean by the total degree of

$$f(x_1, \ldots, x_n) = \sum_{\alpha_1, \cdots, \alpha_n \in \mathbb{Z}} a_{(\alpha_1, \ldots, \alpha_n)} x_1^{\alpha_1} \ldots x_n^{\alpha_n},$$

where $a_{(\alpha_1,\ldots,\alpha_n)} \in \mathbb{R}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$, the largest integer m such that $\alpha_1+\alpha_2+\ldots+\alpha_n = m$ and $a_{(\alpha_1,\ldots,\alpha_n)} \neq 0$.

Let s be a natural number.

We denote by $M(\mathbb{R}^s)$ the set of all functions $f : \mathbb{R}^s \to \mathbb{R}$, and by $M(\mathbb{Z}^s)$ the set of all arrays $c = \{c_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathbb{Z}^s} \subset \mathbb{R}$.

We shall say that a sequence $c = \{c_{\boldsymbol{i}}\} \in M(\mathbb{Z}^s)$ is finitely supported if there exist integers $k_{1,1}, k_{1,2}, k_{2,1}, k_{2,2}, \ldots, k_{s,1}, k_{s,2}$ such that $c_{\boldsymbol{i}} = 0,\ \boldsymbol{i} \in \mathbb{Z}^s,\ \boldsymbol{i} \notin [k_{1,1}, k_{1,2}] \times [k_{2,1}, k_{2,2}] \times \ldots \times [k_{s,1}, k_{s,2}]$.

Similarly, a function $f \in M(\mathbb{R}^s)$ is compactly supported if there exist real numbers $x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, \ldots, x_{s,1}, x_{s,2}$ such that $f(\boldsymbol{x}) = 0,\ \boldsymbol{x} \in \mathbb{R}^s,\ \boldsymbol{x} \notin [x_{1,1}, x_{1,2}] \times [x_{2,1}, x_{2,2}] \times \ldots \times [x_{s,1}, x_{s,2}]$.

We write $M_0(\mathbb{Z}^s)$ for the set of sequences that are finitely supported on $\mathbb{Z}^s$.

The subset of $M(\mathbb{R}^s)$ consisting of those functions that are compactly supported will be denoted by $M_0(\mathbb{R}^s)$.

We let $C(\mathbb{R}^s)$ be the subset of all the functions in $M(\mathbb{R}^s)$ that are continuous in $\mathbb{R}^s$, and $C_0(\mathbb{R}^s)$ are those functions that are not only continuous on $\mathbb{R}^s$, but also compactly supported.

We denote by $D_{(\alpha_1,\ldots,\alpha_s)}(f)$ and $D^{(k)}_{(\alpha_1,\ldots,\alpha_s)}(f)$ the directional derivative and k'th order directional derivative of $f \in M(\mathbb{R}^s)$ in the direction $(\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s$, respectively, where $k \in \mathbb{Z}_+$, and with the convention that $D^{(0)}_{(\alpha_1,\ldots,\alpha_s)}(f) = f$, while recalling the definition of the directional derivative of a function $f \in M(\mathbb{R}^s)$, where $f = f(x_1, \ldots, x_s)$, in the direction $(\alpha_1, \ldots, \alpha_s)$, as

$$\begin{aligned} D_{(\alpha_1,\ldots,\alpha_s)}(f) &= \left(\tfrac{\partial f}{\partial x_1}, \ldots, \tfrac{\partial f}{\partial x_s}\right) \cdot (\alpha_1, \ldots, \alpha_s) \\ &= \alpha_1 \tfrac{\partial f}{\partial x_1} + \ldots + \alpha_s \tfrac{\partial f}{\partial x_s}, \end{aligned} \tag{2.2}$$

where $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_s}$ denote the *partial* derivatives of f with respect to $x_1, \ldots, x_s$ respectively.

Moreover, we say that a function $f \in M_0(\mathbb{R}^s)$ has k continuous derivatives (or that f is k'th order continuous), $k \in \mathbb{Z}_+$, if all k'th order partial derivatives of f are continuous everywhere. The set of all such functions f will be denoted by $C^k(\mathbb{R}^s)$, and the subset of these functions that are, moreover, compactly supported, will, naturally, be called $C_0^k(\mathbb{R}^s)$.

2.2 What box splines are

In this section, we introduce the inductive definition of box splines, as given by Prautzch and Boehm in [21], and which is also used in [9] and [4].

It is well known (see e.g. [5]) that one way in which to define the (univariate) cardinal B-spline N_{m+1} of order $m+1$ is by the recurrence relation

$$N_{m+1}(x) = \int_0^1 N_m(x-t)\ dt, \quad x \in \mathbb{R}, \tag{2.3}$$

for $m \in \mathbb{N}$, where N_1 is the *univariate roof function*

$$N_1(x) = \begin{cases} 1 & , \quad x \in [0,1) \\ 0 & , \quad x \in \mathbb{R}\backslash[0,1). \end{cases} \tag{2.4}$$

Note that (2.3) can equivalently be written in the form

$$N_{m+1}(x) = \int_{x-1}^{x} N_m(t)\ dt, \quad x \in \mathbb{R}. \tag{2.5}$$

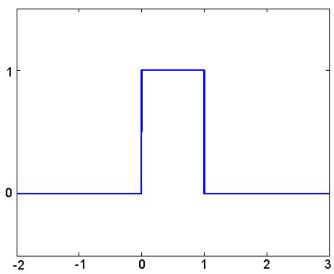
Then (2.4) gives

$$N_2(x) = \int_{x-1}^{x} N_1(t)\ dt,$$

i.e.

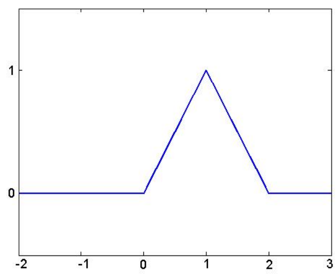
$$
\begin{aligned}
N_2(x) &= \begin{cases} \int_0^x \, dt & , \quad 0 \le x \le 1 \\ \int_{x-1}^1 \, dt & , \quad 1 < x \le 2 \end{cases} \\
&= \begin{cases} x & , \quad 0 \le x \le 1 \\ 2 - x & , \quad 1 < x \le 2. \end{cases}
\end{aligned}
$$

Thus, equation (2.3) graphically implies that the "next" cardinal B-spline is always obtained by integrating (or expanding) the current one, one unit along the x-axis, as is illustrated in Figure 2.2.



N_1

$\longrightarrow$



N_2

Figure 2.2: Graphs of N_1 and N_2

One would expect multivariate splines to exhibit the same feature, i.e. starting with an initial one and each time obtaining a new one by integrating (expanding) the current (or most recent) one. At this stage, however, there would arise a problem, since, while there is only one direction, namely "along the x-axis", to integrate along in 1-D, there are infinitely many possible such directions in the multivariate case. In view of this, it is, when defining a box spline, essential to have it associated with a set of directions (vectors), which should tell us along exactly which direction we must integrate at each step in the induction process. Such a set of directions will be called a *direction matrix* and will typically have the form $\mathcal{D} = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \cdots & \boldsymbol{d}_m \end{bmatrix}$, where $\boldsymbol{d}_i \in \mathbb{Z}^n \backslash \{(0, 0, \ldots, 0)\}$ for all $i = 1, \ldots, m$, and where n is the number of variables we are using and where $m \in \mathbb{N}$. As an example, if $\mathcal{D} = [\, \overbrace{1\ 1\ \ldots\ 1}^{m \text{ times}} \,]$, then its corresponding box spline is obtained by starting with the 1-D roof function (2.4) and repeatedly integrating one unit along the x-axis — hence obtaining, after m steps, the cardinal B-spline of order m, as in (2.3).

For reasons that will become apparent later on, we shall typically refer to the matrix consisting of the first two columns of $\mathcal{D} = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \cdots & \boldsymbol{d}_m \end{bmatrix}$, $\boldsymbol{d}_i \in \mathbb{Z}^n \backslash \{(0,0,\ldots,0)\}, i = 1,\ldots,m$, as the *initial direction matrix*.

As mentioned earlier, we focus on the case $n = 2$.

In view of this, we can now, following [21], give the formal definition of a bivariate box spline B_k that is associated with a direction matrix $\mathcal{D}_k = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \cdots & \boldsymbol{d}_{k+1} \end{bmatrix}$, where $\boldsymbol{d}_i \in \mathbb{Z}^2 \backslash \{(0,0)\}$ for all $i = 1,\ldots,k+1$, as follows:

$$B_k(x,y) := \int_0^1 B_{k-1}((x,y) - t\boldsymbol{d}_{k+1})\, dt, \quad k \geq 2, \ (x,y) \in \mathbb{R}^2, \tag{2.6}$$

where the box spline associated with the initial direction matrix $\mathcal{D}_1 = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the *bivariate roof function* given by

$$B_1(x,y) := \begin{cases} 1 & , \quad (x,y) \in [0,1) \times [0,1), \\ 0 & , \quad (x,y) \in \mathbb{R} \backslash [0,1) \times [0,1), \end{cases} \tag{2.7}$$

as graphically illustrated in Figure 2.3.

In general, it is not necessary to start with the initial direction matrix $\mathcal{D}_1 = I_2$.

For a general initial direction matrix $\mathcal{D}_1 = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$, where $(a_1, b_1) \in \mathbb{Z}^2 \backslash \{(0,0)\}$, $(a_2, b_2) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ and where $a_1 b_2 \neq a_2 b_1$, B_1 is defined as

$$B_1(x,y) := \begin{cases} \frac{1}{a_1 b_2 - a_2 b_1} & , \quad (x,y) \in \mathcal{D}_1[0,1)^2 \\ 0 & , \quad (x,y) \in \mathbb{R}^2 \backslash \mathcal{D}_1[0,1)^2, \end{cases} \tag{2.8}$$

with B_k defined as in (2.6) (see [21]). Here, $\mathcal{D}_1[0,1)^2$ means the (parallelogram-shaped)

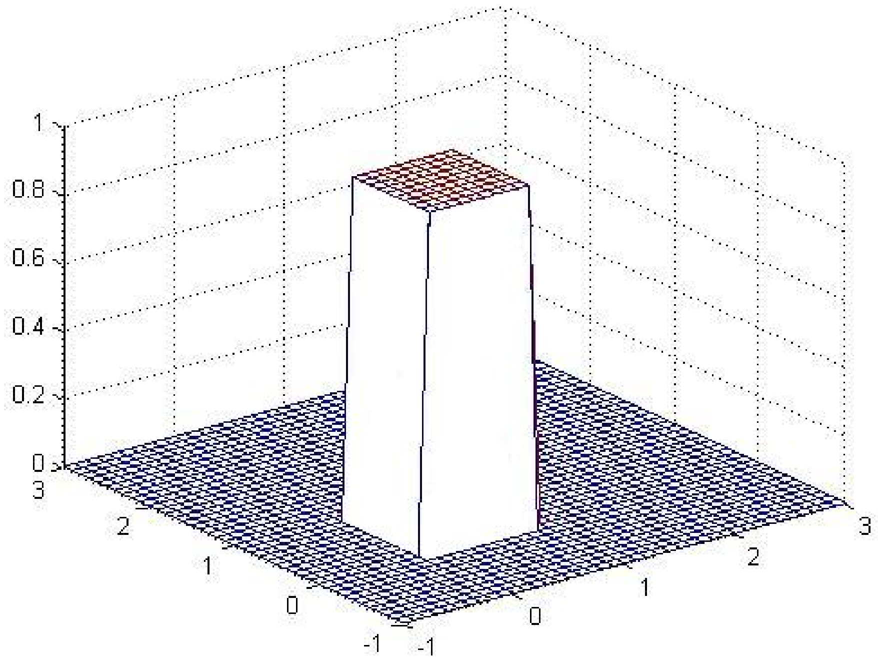


Figure 2.3: Graph of B_1

area formed by a completion of the vectors $\boldsymbol{d}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ and $\boldsymbol{d}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$, namely

$$\mathcal{D}_1[0,1)^2 := \left\{ \sum_{i=1}^{2} \boldsymbol{d}_i t : 0 \leq t \leq 1 \right\},$$

as illustrated in Figure 2.4 for the case $\mathcal{D}_1 = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$.

In Chapters 2 and 3 of this thesis we shall assume that $\mathcal{D}_1 = I^2$, before showing, in Chapter 4, how some of our results can be generalized to arbitrary initial direction matrices.

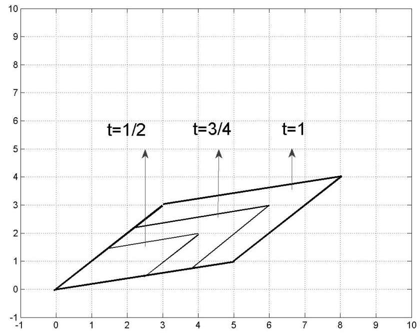


Figure 2.4: Illustration of the area $\mathcal{D}_1[0,1)^2$

2.3 Examples

Let the direction matrix be given by $\mathcal{D}_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

The inductive formula (2.6) is used to calculate $B_{\mathcal{D}_2} =: B_2$, as follows:

- If $x < 0$ or $y < 0$, then $B_2(x,y) = \int_0^1 0\,dt = 0$.
- If $x \in [0,1), y \in [0,1)$, then

$$\begin{aligned} B_2(x,y) &= \int_0^1 B_1(x-t, y-t)\,dt \\ &= \begin{cases} \int_0^x \overbrace{B_1(x-t,y-t)}^{=1}\,dt & = x \quad , \quad \text{if } y \geq x \\ \int_0^y \underbrace{B_1(x-t,y-t)}_{=1}\,dt & = y \quad , \quad \text{if } y < x. \end{cases} \end{aligned}$$

- If $x \in [1,2), y \in [0,1)$, then

$$\begin{aligned} B_2(x,y) &= \int_0^1 B_1(x-t, y-t)\,dt \\ &= \begin{cases} \int_0^1 \overbrace{B_1(x-t,y-t)}^{=0}\,dt & = 0 \quad , \quad \text{if } y \leq x-1 \\ \int_{x-1}^y \underbrace{B_1(x-t,y-t)}_{=1}\,dt & = 1+y-x \quad , \quad \text{if } y > x-1. \end{cases} \end{aligned}$$

Continuing in this fashion, we obtain

$$B_2(x,y) = \begin{cases} y & , \ x \in [0,1), \ y \in [0,1), \ y < x \\ x & , \ x \in [0,1), \ y \in [0,1), \ y \geq x \\ 1+x-y & , \ x \in [0,1), \ y \in [1,2), \ y < x+1 \\ 2-y & , \ x \in [1,2), \ y \in [1,2), \ y \geq x \\ 2-x & , \ x \in [1,2), \ y \in [1,2), \ y < x \\ 1+y-x & , \ x \in [1,2), \ y \in [0,1), \ y \geq x-1 \\ 0 & , \ \text{otherwise.} \end{cases} \tag{2.9}$$

The graph of B_2 is given in Figure 2.5.

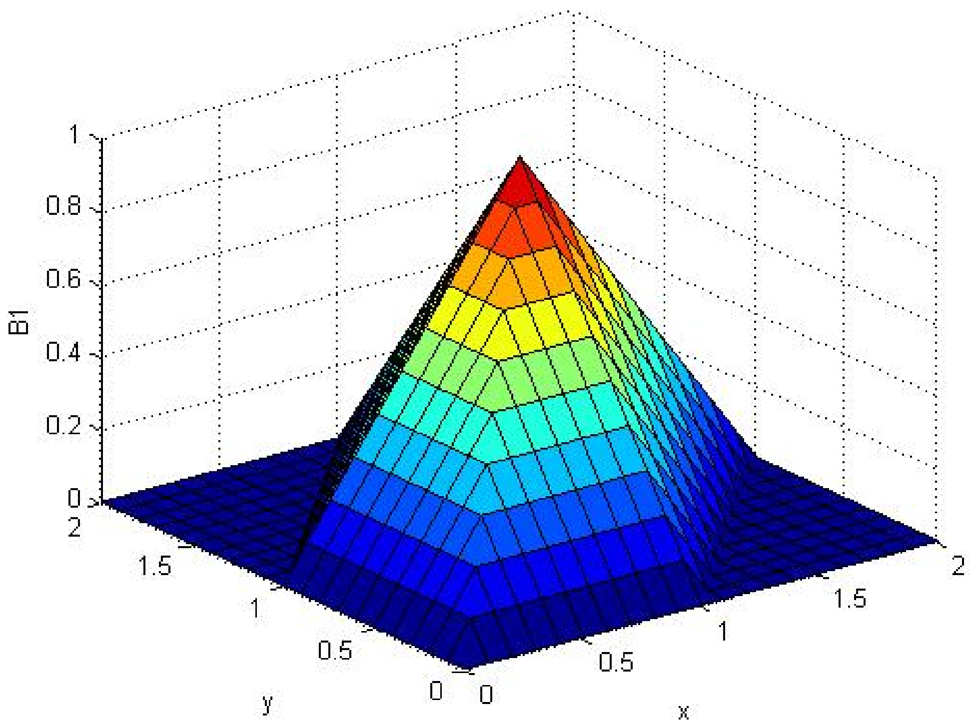


Figure 2.5: Graph of B_2

If $\mathcal{D}_1 = I_2$, then its corresponding box spline B_1 is called the *bivariate roof function*; if $\mathcal{D}_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, then its corresponding box spline B_2 is called the *Courant hat func-*

tion.

Another prominent box spline that appears frequently in the literature ([9], [21]) is the Zwart-Powell function, or ZP-element, which is obtained from the direction matrix

$$\mathcal{D}_{\mathrm{ZP}} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

The graph of the Zwart-Powell function is shown in Figure 2.6. The beauty of the ZP-element lies in the fact that it is on the central part of its support not only piecewise quadratic, but quadratic over the whole part. Namely, $B_{\mathrm{ZP}}(x, y) = \frac{1}{4}(-3 + 2x - 2x^2 + 6y - 2y^2)$ on the whole square $[0, 1] \times [1, 2]$ (see also Figure 6.1 for the support of the ZP-element). From equation (2.9), it is clear that the Courant hat function does not exhibit this property, since it is only linear over each triangular piece of its support.

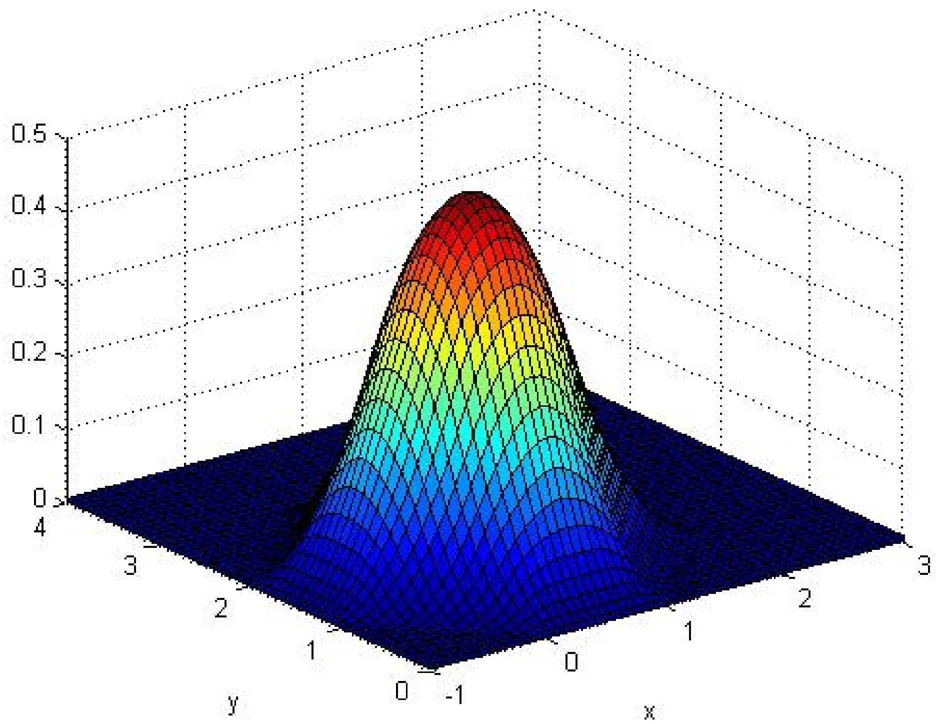


Figure 2.6: Graph of B_{ZP}

NOTE: There are a few variations on the inductive definition of a box spline in the literature. For example, Goodman and Ong (see [15]), as well as Chui (see [4]), use the roof function

$$B_1(x,y) := \begin{cases} 1 & , \quad (x,y) \in [-\frac{1}{2},\frac{1}{2}] \times [-\frac{1}{2},\frac{1}{2}] \\ 0 & , \quad (x,y) \in \mathbb{R}^2 \backslash [-\frac{1}{2},\frac{1}{2}] \times [-\frac{1}{2},\frac{1}{2}], \end{cases}$$

and with the integral boundaries in equation (2.6) as $-\frac{1}{2}$ and $\frac{1}{2}$, instead of 0 and 1, the result of which is exactly the same box spline as (2.6), but shifted so as to be centered around the origin. For computational convenience (and without loss of generality regarding obtained results), we prefer the definition (2.6).

2.4 Preliminary results

It is evident from the examples above that box splines have the following properties:

- they are piecewise polynomials (where by "piecewise" we mean triangular pieces on the xy-grid);
- they are compactly supported;
- they are nonnegative (which follows from the inductive definition, since B_1 is clearly nonnegative and hence so is the integral $\int_0^1 B_1((x,y)-t\boldsymbol{d})\,dt$ for any direction vector $\boldsymbol{d}$);
- the order in which the direction vectors appear in the direction matrix $\mathcal{D}$ has no effect on the associated box spline (this property can easily be verified by looking at the inductive definition, since finite integrals are interchangable) — however, the multiplicity with which any direction appear in $\mathcal{D}$ is very important;
- the box splines are symmetric about the centre of their support.

The last property is perhaps not so clear to see in general. However, recalling the geometrical definition of box splines mentioned in the beginning of the chapter, where the splines are interpreted as the result under linear projections of higher dimensional cubes, it is not so hard to see that, since such cubes are always, naturally, symmetric around *their* centres, this property directly carries over to box splines themselves. This fact (which has been well known to be true for univariate cardinal B-splines), though interesting to note, was mentioned in [21], but not much elaborated on further in the literature.

We now proceed to formalise a few properties that are exhibited in general by box splines. First, it is shown that the shifts of a box spline form a partition of unity, a result that is important in particular in approximation theory (since it implies that continuous functions can be approximated by the space spanned by these shifts (see [9])), and subdivision analysis, as will be seen in Chapter 7.

Lemma 2.1 {*Partition of unity*}

Let $k \in \mathbb{N}$. *If* $\mathcal{D}_k = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \cdots & \boldsymbol{d}_{k+1} \end{bmatrix}$ *is a given direction matrix, with* $\boldsymbol{d}_1 = \begin{bmatrix} a \\ 0 \end{bmatrix}$, $\boldsymbol{d}_2 = \begin{bmatrix} 0 \\ b \end{bmatrix}$, $a, b \in \mathbb{Z}\backslash\{0\}$, *and* $\boldsymbol{d}_i \in \mathbb{Z}^2\backslash(0,0)$, $i = 3, \ldots, k+1$, *and if* B_k *is the associated (bivariate) box spline obtained by the formula (2.6), then*

$$\sum_{(i,j)\in\mathbb{Z}^2} B_k((x,y) - (i,j)) = 1, \quad (x,y) \in \mathbb{R}^2. \tag{2.10}$$

Proof

The proof is by induction on k.

If $k = 1$ and the initial direction matrix is given by $\mathcal{D}_1 = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 \end{bmatrix}$, then it is easy to see that (2.10) holds. Namely, on each of the ab number of unit squares on the xy-grid $[0, a] \times [0, b]$, the value of B_1 is $\frac{1}{ab}$, according to equation (2.8), whereas outside of this grid the value of B_1 is zero.

Now, suppose that (2.10) holds for some $k \in \mathbb{N}$ and let $\mathcal{D}_{k+1} = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \cdots & \boldsymbol{d}_{k+1} & \boldsymbol{d}_{k+2} \end{bmatrix}$.

Then,

$$\begin{aligned}
\sum_{(i,j)\in\mathbb{Z}^2} B_{k+1}((x,y)-(i,j)) &= \sum_{(i,j)\in\mathbb{Z}^2} \left[\int_0^1 B_k((x,y)-(i,j)-t\boldsymbol{d}_{k+2})\,dt\right] \\
&= \int_0^1 \left[\sum_{(i,j)\in\mathbb{Z}^2} B_k\left(((x,y)-t\boldsymbol{d}_{k+2})-(i,j)\right)\right] dt \\
&= \int_0^1 1\,dt \\
&= 1,
\end{aligned}$$

where the second from last step follows from the induction hypothesis and the fact that $(x,y)-t\boldsymbol{d}_{k+2} \in \mathbb{R}^2$.

This completes the proof by induction. □

We will later, in Chapter 7, find that the property of partition of unity is in fact possessed by a whole class of functions, with the box splines as a special case.

Lemma 2.1 was also proved in [9]. Although the result as given in [9] is more general in the sense that it holds for box splines of any number of variables, the proof is quite long and involves results from group theory, and so we do not include it here.

In the following lemma, which is stated in [21], it is shown that, for any direction matrix and its corresponding box spline, the volume under that box spline is always equal to one. Although stated here only for the bivariate case, the lemma holds in general for the multivariate case.

Lemma 2.2 {*Normalization of box splines*}

For an integer $k \in \mathbb{N}$, let $\mathcal{D}_k = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \cdots & \boldsymbol{d}_{k+1} \end{bmatrix}$ be a given direction matrix, with $\boldsymbol{d}_i \in \mathbb{Z}^2\backslash\{(0,0)\}, i = 1,\ldots,k+1$, and B_k the associated bivariate box spline obtained by *the formula (2.6). Then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} B_k(x,y)\,dx\,dy = 1, \qquad k \in \mathbb{N},\ k \geq 2. \tag{2.11}$$

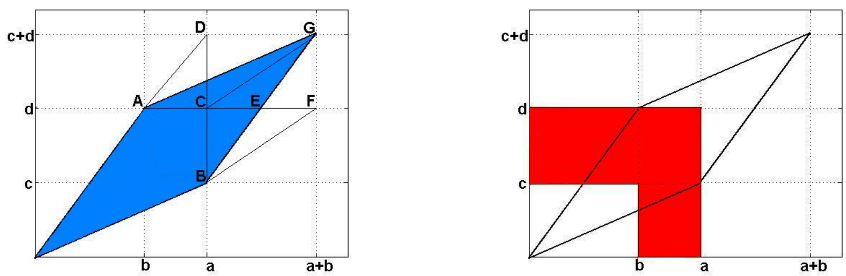
Proof

The proof is by induction on k.

First, let $k = 1$ and suppose the initial direction matrix is given by $\mathcal{D}_1 = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $ad \neq bc$.

Then $B_1(x,y) = \frac{1}{ad-bc}$, $\quad (x,y) \in \mathcal{D}_1[0,1)^2$, according to (2.8), and $\int_{\mathbb{R}}\int_{\mathbb{R}} B_1(x,y)\,dx\,dy = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{ad-bc}\,dx\,dy$ is hence the volume underneath a constant function of which the support forms a parallelogram. If we can prove that the area of this parallelogram is equal to $ad - bc$, then it follows that $\int_{\mathbb{R}}\int_{\mathbb{R}} B_1(x,y)\,dx\,dy = 1$, implying that the lemma holds for $k = 1$. We thus proceed to prove that the area of a parallelogram formed by the vectors (a,c) and (b,d) is indeed equal to $ad - bc$.

To this end, consider the parallelogram (in blue) in Figure 2.7 (a). (For simplicity, we assume here that $ad > bc$. The case $ad < bc$ is similar.) We will show that the area of this parallelogram reduces to the area in Figure 2.7 (b) that is in red.



(a) (b)

Figure 2.7: Area of parallelogram

First, note that Area($\triangle$ACG) $= \frac{1}{2}(a-b)(c) =$ Area($\triangle$ACD). Also, Area($\triangle$CEG) $= \frac{1}{2}(b-\frac{bc}{d})(c) = \frac{1}{2}(\frac{bc}{d})(d-c) =$ Area($\triangle$BFE), which implies that Area($\triangle$BCG) = Area($\triangle$BCF). By a similar rearrangement of the rest of the area of the parallelogram, it follows that the area of the parallelogram is equal to the area in Figure 2.7(b) (red), which is equal to $ad - bc$. This completes the proof for the case $k = 1$.

Next, suppose that $k \geq 2$ and that the lemma holds for $k-1$, and let the direction vector $\boldsymbol{d}_{k+1} \in \mathbb{Z}^2$ be written as $(d_{k+1,1}, d_{k+1,2})$.
Then

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} B_k(x,y)\,dx\,dy &= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\int_0^1 B_{k-1}((x,y) - t(d_{k+1,1}, d_{k+1,2}))\,dt\right) dx\,dy \\
&= \int_0^1 \left(\int_{\mathbb{R}}\int_{\mathbb{R}} B_{k-1}((x,y) - t(d_{k+1,1}, d_{k+1,2}))\,dx\,dy\right) dt \\
&= \int_0^1 1\,dt \\
&= 1,
\end{aligned}$$

where the second from last step again follows by application of the induction hypothesis, so that the lemma also holds for k and hence for every $k \in \mathbb{N}$. $\square$

In the next chapter, we shall discuss the concept of a *refinable function*, and we shall see how the box splines, as introduced in this chapter, are special examples of such functions.

Chapter 3

Refinement pairs

3.1 Introducing refinable functions

As mentioned in Chapter 1, we want to investigate functions for which there exist real coefficients $p_{k,l},\ (k,l) \in \mathbb{Z}^2$, such that the equation

$$\phi(x,y) = \sum_{(k,l)\in\mathbb{Z}^2} p_{k,l}\phi\left(M\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}\right) \tag{3.1}$$

is satisfied for all $(x,y) \in \mathbb{R}^2$, with the exception of at most a finite number of values of (x,y). As remarked in [10], the formula for the function ϕ is usually only known implicitly, as the solution of equation (3.1), while only the values of the coefficients $p_{k,l}$ are known explicitly. Tools like the cascade algorithm (see [10]) can then sometimes be used successfully to generate the function ϕ. This chapter does not attempt to study the generation of ϕ via the cascade algorithm, but instead the definition of refinability is discussed, a couple of examples of refinable functions are given, and it is shown how these can be used to generate more refinable functions. First, some more introductory notation.

If ϕ satisfies equation (3.1) for some sequence $p = \{p_{k,l}\}_{(k,l)\in\mathbb{Z}^2} \subset \mathbb{R}$ and a given matrix $M \in \mathbb{Z}^{2\times 2}$, then ϕ will be called *refinable* with respect to the matrix M, or M*-refinable.* The sequence p will be called an M*-refinement mask* and $(p,\phi)_M$ an M*-refinement pair.*

If the matrix M is of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, with $a \in \mathbb{N}, a \geq 2$, then it will simply be said that ϕ is *a-refinable*; $(p, \phi)_a$ will henceforth be called an *a-refinement pair* and p an *a-refinement mask.*

We restrict ourselves to the case where $M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, in which case (3.1) becomes

$$\phi(x, y) = \sum_{(k,l) \in \mathbb{Z}^2} p_{k,l} \phi(2x - k, 2y - l), \quad (x, y) \in \mathbb{R}^2, \tag{3.2}$$

i.e. ϕ is 2-*refinable.* We will also, for simplicity when working with 2-refinement pairs, sometimes write (p, ϕ) instead of $(p, \phi)_2$.

The functions B_1 and B_2 in (2.7) and (2.9), as obtained from $\mathcal{D}_1 = I_2$ and $\mathcal{D}_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ respectively in Chapter 2, can be shown directly from (2.7) and (2.9) to be 2-refinable, with the only non-zero mask coefficients $p_{k,l}$ in (3.2) given by

$$\begin{aligned} p_{0,0} &= 1 \\ p_{0,1} &= 1 \\ p_{1,0} &= 1 \\ p_{1,1} &= 1 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} p_{0,0} &= 1/2 \\ p_{0,1} &= 1/2 \\ p_{0,2} &= 0 \\ p_{1,0} &= 1/2 \\ p_{1,1} &= 1 \\ p_{1,2} &= 1/2 \\ p_{2,0} &= 0 \\ p_{2,1} &= 1/2 \\ p_{2,2} &= 1/2 \end{aligned} \tag{3.4}$$

respectively.

3.2 Building new ones

We use the coefficients $p_{k,l}$ in (3.2) to form the bivariate Laurent polynomial

$$P(z_1, z_2) := \sum_{(k,l)\in\mathbb{Z}^2} p_{k,l} z_1^k z_2^l, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash (0,0), \tag{3.5}$$

and we call the polynomial $P \in \Pi_{k+l}^{(2)}$ the 2-*refinement mask symbol* corresponding to the 2-refinement pair (p, ϕ). Observe that, if $p_{k,l} = 0$ for $k < 0$ or $l < 0$, then P is a bivariate polynomial, and the origin $(0, 0)$ need not be excluded from the definition (3.5).

It follows from equations (3.3) and (3.4) that the 2-refinement mask symbols of B_1 and B_2 are, respectively,

$$\begin{aligned} P_1(z_1, z_2) &= 1 + z_1 + z_2 + z_1 z_2 \\ &= (1 + z_1)(z + z_2), \qquad (z_1, z_2) \in \mathbb{C}^2, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} P_2(z_1, z_2) &= \tfrac{1}{2} + \tfrac{1}{2} z_1 + \tfrac{1}{2} z_2 + z_1 z_2 + \tfrac{1}{2} z_1^2 z_2 + \tfrac{1}{2} z_1 z_2^2 + \tfrac{1}{2} z_1^2 z_2^2 \\ &= \tfrac{1}{2}(1 + z_1 z_2)(1 + z_1)(1 + z_2), \qquad (z_1, z_2) \in \mathbb{C}^2. \end{aligned} \tag{3.7}$$

It is interesting to note from equations (3.6) and (3.7) that

$$P_2(z_1, z_2) = \frac{1}{2}(1 + z_1 z_2) P_1(z_1, z_2), \quad (z_1, z_2) \in \mathbb{C}^2. \tag{3.8}$$

In other words, while the box spline B_2 was obtained by expanding B_1 along the direction $(1, 1)$ when this vector was included in our direction matrix (according to equation (2.6)), it seems that the 2-refinement mask symbol P_2 was on its turn derived from P_1 by multiplying the latter by the factor $\frac{1}{2}(1 + z_1 z_2)$.

This idea is generalized in Theorem 3.1 below.

The idea is to be able to choose the direction matrix $\mathcal{D}$ in such a way that its corresponding box spline $B_{\mathcal{D}}$ will exhibit certain favourable properties. First of all, we want to ensure that, once a box spline $B_{\tilde{\mathcal{D}}}$ corresponding to a certain direction matrix $\tilde{\mathcal{D}}$ is 2-refinable, then $B_{\mathcal{D}}$ will also be 2-refinable provided that $\mathcal{D}$ is obtained from $\tilde{\mathcal{D}}$ by the addition of specific direction vectors. In other words, 2-refinability must be preserved when including certain combinations of vectors in the direction matrix. Secondly, when moving from one direction matrix to a next one in this fashion, it is desirable to have an efficient inductive formula for computing the successive 2-refinement mask symbols of the corresponding successive box splines, such as in the case P_1 and P_2 above. Finally and most importantly, we would like for this inductive procedure to preserve certain favourable properties. Specifically, we would like the obtained box splines to attain certain levels of smoothness at their break points.

Theorem 3.1 $\{$*Refinement preservation*$\}$
Suppose $(\tilde{p}, \tilde{\phi})$ is a 2-refinement pair, i.e. satisfying

$$\tilde{\phi}(x, y) = \sum_{k,l} \tilde{p}_{k,l} \tilde{\phi}(2x - k, 2y - l), \qquad (x, y) \in \mathbb{R}^2,$$

for some sequence $\tilde{p} = \{\tilde{p}_{k,l}\}_{(k,l)\in\mathbb{Z}^2} \subset \mathbb{R}$, with

$$\tilde{P}(z_1, z_2) = \sum_{k,l} \tilde{p}_{k,l} z_1^k z_2^l, \qquad (z_1, z_2) \in \mathbb{C}^2 \backslash (0, 0)$$

denoting the corresponding 2-refinement mask symbol.
Let P be the Laurent polynomial defined by

$$P(z_1, z_2) = \left(\frac{1 + z_1}{2}\right)\left(\frac{1 + z_2}{2}\right)\tilde{P}(z_1, z_2), \qquad (z_1, z_2) \in \mathbb{C}^2 \backslash (0, 0), \tag{3.9}$$

i.e. $P(z_1, z_2) = \sum_{k,l} p_{k,l} z_1^k z_2^l, \quad (z_1, z_2) \in \mathbb{C}^2 \backslash (0, 0),$ *where*

$$p_{k,l} = \frac{1}{4}(\tilde{p}_{k,l} + \tilde{p}_{k-1,l} + \tilde{p}_{k,l-1} + \tilde{p}_{k-1,l-1}), \quad (k, l) \in \mathbb{Z}^2. \tag{3.10}$$

Also, let $\phi \in M_0(\mathbb{R}^2)$ *be the function defined by*

$$\phi(x,y) = \int_0^1 \int_0^1 \tilde{\phi}(x - t_1, y - t_2)\, dt_1\, dt_2. \tag{3.11}$$

Then (p, ϕ) *is also a 2-refinement pair, with corresponding 2-refinement mask symbol* P. *Moreover, if the function* $\tilde{\phi}$ *satisfies* $\tilde{\phi} \in C^k(\mathbb{R}^2)$ *for some* $k \in \mathbb{Z}_+$, *then* ϕ *will satisfy* $\phi \in C^{k+1}(\mathbb{R}^2)$, *i.e. the recurrence relation (3.11) increases the degree of smoothness of a refinable function by one.*

Proof

For any $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}
\sum_{k,l} p_{k,l}\phi(2(x,y) - (k,l)) &= \sum_{k,l} \frac{1}{4}[\tilde{p}_{k,l} + \tilde{p}_{k-1,l} + \tilde{p}_{k,l-1} + \tilde{p}_{k-1,l-1}] \\
&\quad \left[\int_0^1 \int_0^1 \tilde{\phi}(2(x,y) - (k,l) - (t_1,t_2))\, dt_1\, dt_2\right] \\
&= \frac{1}{4}\int_0^1 \int_0^1 \Bigg[\sum_{k,l} \tilde{p}_{k,l}\tilde{\phi}(2(x,y) - (k,l) - (t_1,t_2)) \\
&\quad + \sum_{k,l} \tilde{p}_{k-1,l}\tilde{\phi}(2(x,y) - (k,l) - (t_1,t_2)) \\
&\quad + \sum_{k,l} \tilde{p}_{k,l-1}\tilde{\phi}(2(x,y) - (k,l) - (t_1,t_2)) \\
&\quad + \sum_{k,l} \tilde{p}_{k-1,l-1}\tilde{\phi}(2(x,y) - (k,l) - (t_1,t_2))\Bigg]\, dt_1\, dt_2 \\
&= \frac{1}{4}\int_0^1 \int_0^1 \Bigg[\sum_{k,l} \tilde{p}_{k,l}\tilde{\phi}(2(x,y) - (k,l) - (t_1,t_2)) \\
&\quad + \sum_{k,l} \tilde{p}_{k,l}\tilde{\phi}(2(x,y) - (k,l) - (1,0) - (t_1,t_2)) \\
&\quad + \sum_{k,l} \tilde{p}_{k,l}\tilde{\phi}(2(x,y) - (k,l) - (0,1) - (t_1,t_2)) \\
&\quad + \sum_{k,l} \tilde{p}_{k,l}\tilde{\phi}(2(x,y) - (k,l) - (1,1) - (t_1,t_2))\Bigg]\, dt_1\, dt_2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\int_0^1\int_0^1 \left[\sum_{k,l}\tilde{p}_{k,l}\tilde{\phi}\left(2\left(x-\frac{t_1}{2}, y-\frac{t_2}{2}\right)-(k,l)\right)\right. \\
&\quad +\sum_{k,l}\tilde{p}_{k,l}\tilde{\phi}\left(2\left(x-\frac{t_1+1}{2}, y-\frac{t_2}{2}\right)-(k,l)\right) \\
&\quad +\sum_{k,l}\tilde{p}_{k,l}\tilde{\phi}\left(2\left(x-\frac{t_1}{2}, y-\frac{t_2+1}{2}\right)-(k,l)\right) \\
&\quad \left.+\sum_{k,l}\tilde{p}_{k,l}\tilde{\phi}\left(2\left(x-\frac{t_1+1}{2}, y-\frac{t_2+1}{2}\right)-(k,l)\right)\right] dt_1\,dt_2 \\
&= \frac{1}{4}\int_0^1\int_0^1 \tilde{\phi}\left(x-\frac{t_1}{2}, y-\frac{t_2}{2}\right) dt_1\,dt_2 \\
&\quad +\frac{1}{4}\int_0^1\int_0^1 \tilde{\phi}\left(x-\frac{t_1+1}{2}, y-\frac{t_2}{2}\right) dt_1\,dt_2 \\
&\quad +\frac{1}{4}\int_0^1\int_0^1 \tilde{\phi}\left(x-\frac{t_1}{2}, y-\frac{t_2+1}{2}\right) dt_1\,dt_2 \\
&\quad +\frac{1}{4}\int_0^1\int_0^1 \tilde{\phi}\left(x-\frac{t_1+1}{2}, y-\frac{t_2+1}{2}\right) dt_1\,dt_2 \\
&= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \tilde{\phi}\left(x-t_1, y-t_2\right) dt_1\,dt_2 + \int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} \tilde{\phi}\left(x-t_1, y-t_2\right) dt_1\,dt_2 \\
&\quad + \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 \tilde{\phi}\left(x-t_1, y-t_2\right) dt_1\,dt_2 + \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 \tilde{\phi}\left(x-t_1, y-t_2\right) dt_1\,dt_2 \\
&= \int_0^1\int_0^{\frac{1}{2}} \tilde{\phi}(x-t_1, y-t_2)\,dt_1\,dt_2 + \int_0^1\int_{\frac{1}{2}}^1 \tilde{\phi}(x-t_1, y-t_2)\,dt_1\,dt_2 \\
&= \int_0^1 \left(\int_0^{\frac{1}{2}} \tilde{\phi}(x-t_1, y-t_2)\,dt_1 + \int_{\frac{1}{2}}^1 \tilde{\phi}(x-t_1, y-t_2)\,dt_1\right) dt_2 \\
&= \int_0^1\int_0^1 \tilde{\phi}(x-t_1, y-t_2)\,dt_1\,dt_2 \\
&= \phi(x,y),
\end{aligned}
$$

i.e. ϕ is indeed 2-refinable, thereby completing the first part of the proof.

Now, suppose that $\tilde{\phi} \in C^k(\mathbb{R}^2)$ for some $k \in \mathbb{Z}_+$. We must show that $\phi \in C^{k+1}(\mathbb{R}^2)$. But, for any $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}
\phi(x,y) &= \int_0^1 \int_0^1 \tilde{\phi}(x - t_1, y - t_2)\, dt_1\, dt_2 \\
&= \int_0^1 \left(\int_0^1 \tilde{\phi}(x - t_1, y - t_2)\, dt_2 \right) dt_1 \\
&= \int_{x-1}^x \left(\int_0^1 \tilde{\phi}(t_1, y - t_2)\, dt_2 \right) dt_1 \\
&= \int_0^x \left(\int_0^1 \tilde{\phi}(t_1, y - t_2)\, dt_2 \right) dt_1 - \int_0^{x-1} \left(\int_0^1 \tilde{\phi}(t_1, y - t_2)\, dt_2 \right) dt_1,
\end{aligned}$$

and it hence follows from the Fundamental Theorem of Calculus that, for a fixed $y \in \mathbb{R}$, $\phi(x) = \phi(x, y)$ is a continuously differentiable function on $\mathbb{R}$, with

$$\begin{aligned}
\frac{\partial \phi}{\partial x}(x,y) &= \int_0^1 \tilde{\phi}(x, y - t_2)\, dt_2 - \int_0^1 \tilde{\phi}(x - 1, y - t_2)\, dt_2 \\
&= \int_0^1 \left(\tilde{\phi}(x, y - t_2) - \tilde{\phi}(x - 1, y - t_2) \right) dt_2,
\end{aligned}$$

i.e. $\frac{\partial \phi}{\partial x} \in C^k(\mathbb{R}^2)$.

Similarly, $\frac{\partial \phi}{\partial y} \in C^k(\mathbb{R}^2)$, and it thus follows from (2.2) that $\phi \in C^{k+1}(\mathbb{R}^2)$, as desired. $\square$

3.3 Looking back at box splines

We know from (3.3) and (3.4) that (p_1, B_1) and (p_2, B_2) are 2-refinement pairs, with

$$P_1(z_1, z_2) = (1 + z_1)(1 + z_2)$$

and

$$P_2(z_1, z_2) = \left(\frac{1 + z_1 z_2}{2} \right) (1 + z_1)(1 + z_2)$$

the corresponding 2-refinement mask symbols, and where B_1 and B_2 are the bivariate roof function and the Courant hat function, respectively. Hence, according to Theorem

3.1, (p_k, B_k) is also a 2-refinement pair for every $k \in \mathbb{N}$, where

$$\begin{aligned} P_k(z_1, z_2) &= \left(\tfrac{1+z_1}{2}\right)\left(\tfrac{1+z_2}{2}\right) P_{k-1}(z_1, z_2) \\ &= \left(\tfrac{1+z_1}{2}\right)^2\left(\tfrac{1+z_2}{2}\right)^2 P_{k-2}(z_1, z_2) \\ &= \cdots \\ &= \left(\tfrac{1+z_1}{2}\right)^{k-2}\left(\tfrac{1+z_2}{2}\right)^{k-2} P_2(z_1, z_2) \\ &= \left(\tfrac{1+z_1}{2}\right)^{k-2}\left(\tfrac{1+z_2}{2}\right)^{k-2} \tfrac{1}{2}(1+z_1z_2)(1+z_1)(1+z_2), \end{aligned}$$

i.e.

$$P_k(z_1, z_2) = 2^{3-2k}(1+z_1)^{k-1}(1+z_2)^{k-1}(1+z_1z_2), \tag{3.12}$$

with

$$B_k(x,y) = \int_0^1\int_0^1 B_{k-1}(x-t_1, y-t_2)\,dt_1\,dt_2, \quad (x,y) \in \mathbb{R}^2,\ k > 2,$$

and where B_1 and B_2 are as in equations (2.7) and (2.9), respectively.
Moreover, since $B_2 \in C(\mathbb{R}^2)$, we have $B_k \in C^{k-1}(\mathbb{R}^2)$ for all $k \in \mathbb{N}, k \geq 2$.

The rationale behind Theorem 3.1 lies in the way in which smoothness can be obtained up to any desired level, provided that we keep integrating our box splines in the appropriate directions, i.e. the correct number of combinations of the directions $(0,1)$ and $(1,0)$ must be included in the direction matrix. If we choose to integrate immediately in the direction $(1,1)$, we would get a function that has continuous derivatives in the direction $(1,1)$ but not necessarily in any other direction, whereas, by integrating in both of the two unit directions *separately*, we are ensured to have the desired regularity. To see why this is true, recall from equation (2.2) that the derivative of a box spline B in any given direction can always be written as a linear combination of the first order partial derivatives of B. For example,

$$B_{(1,1)} \;=\; 1B_{(1,0)} + 1B_{(0,1)} \;=\; \frac{\partial B}{\partial x} + \frac{\partial B}{\partial y},$$

so that, once both $\frac{\partial B}{\partial x}$ and $\frac{\partial B}{\partial y}$ are continuous, then $B_{(1,1)}$ will also be continuous.
The price that we have to pay for an optimum level of smoothness, is therefore the following: first, it is computationally much more difficult and time-consuming to evaluate

box splines by integrating in the two partial directions separately. Secondly, we obtain box splines of which the supports are somewhat bigger (though still compact) than what they would have been, had we integrated in only one direction.

Below are given three examples of box splines to illustrate this.

First, let $\tilde{B}_3$ be the box spline obtained from the direction matrix $\mathcal{D}_{\tilde{B}_3} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

It follows from (2.9) and (2.6) that $\tilde{B}_3$ is given by

$$\tilde{B}_3(x,y) = \begin{cases} \frac{1}{2}y^2 & , \ (x,y) \in \{(x,y) \in [0,1) \times [0,1),\ y \leq x\} & =: \Omega_1 \\ \frac{1}{2}x^2 & , \ (x,y) \in \{(x,y) \in [0,1) \times [0,1),\ y > x\} & =: \Omega_2 \\ \frac{1}{2}x^2 - \frac{1}{2}y^2 + y - \frac{1}{2} & , \ (x,y) \in \{(x,y) \in [0,1) \times [1,2),\ y \leq x+1\} & =: \Omega_3 \\ -\frac{1}{2}x^2 + \frac{1}{2}y^2 + x - \frac{1}{2} & , \ (x,y) \in \{(x,y) \in [1,2) \times [0,1),\ y > x-1\} & =: \Omega_4 \\ -\frac{1}{2}x^2 - \frac{1}{2}y^2 + x + 2y - \frac{3}{2} & , \ (x,y) \in \{(x,y) \in [1,2) \times [1,2),\ y \leq x\} & =: \Omega_5 \\ -\frac{1}{2}x^2 - \frac{1}{2}y^2 + 2x + y - \frac{3}{2} & , \ (x,y) \in \{(x,y) \in [1,2) \times [1,2),\ y > x\} & =: \Omega_6 \\ -\frac{1}{2}x^2 + \frac{1}{2}y^2 + 2x - 3y + \frac{5}{2} & , \ (x,y) \in \{(x,y) \in [1,2) \times [2,3),\ y \leq x+1\} & =: \Omega_7 \\ \frac{1}{2}x^2 - \frac{1}{2}y^2 - 3x + 2y + \frac{5}{2} & , \ (x,y) \in \{(x,y) \in [2,3) \times [1,2),\ y > x-1\} & =: \Omega_8 \\ \frac{1}{2}x^2 - 3x + \frac{9}{2} & , \ (x,y) \in \{(x,y) \in [2,3) \times [2,3),\ y \leq x\} & =: \Omega_9 \\ \frac{1}{2}y^2 - 3y + \frac{9}{2} & , \ (x,y) \in \{(x,y) \in [2,3) \times [2,3),\ y > x\} & =: \Omega_{10} \\ 0 & , \ \text{otherwise,} & \end{cases} \tag{3.13}$$

with the regions $\Omega_1, \Omega_2, \ldots, \Omega_{10}$ as illustrated in Figure 3.1.

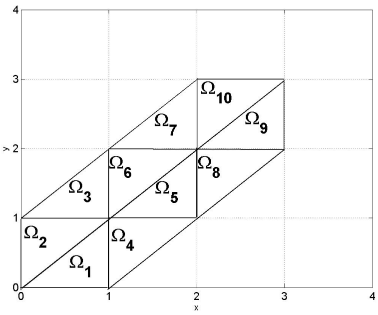


Figure 3.1: Illustration of $\Omega_1, \ldots, \Omega_{10}$

After also making use of (2.2), this yields

$$D_{(1,1)}(\tilde{B}_3)(x,y) = \begin{cases} y & , \ (x,y) \in \Omega_1 \\ x & , \ (x,y) \in \Omega_2 \\ x-y+1 & , \ (x,y) \in \Omega_3 \\ -x+y+1 & , \ (x,y) \in \Omega_4 \\ -x-y+3 & , \ (x,y) \in \Omega_5 \\ -x-y+3 & , \ (x,y) \in \Omega_6 \\ -x+y-1 & , \ (x,y) \in \Omega_7 \\ x-y-1 & , \ (x,y) \in \Omega_8 \\ x-3 & , \ (x,y) \in \Omega_9 \\ y-3 & , \ (x,y) \in \Omega_{10} \\ 0 & , \ \text{otherwise.} \end{cases} \tag{3.14}$$

But it also follows from (2.9) that

$B_2(x,y) - B_2((x,y)-(1,1))$

$$= \begin{cases} (y) & - & (0) & = & y & , \ (x,y) \in \Omega_1 \\ (x) & - & (0) & = & x & , \ (x,y) \in \Omega_2 \\ (1+x-y) & - & (0) & = & 1+x-y & , \ (x,y) \in \Omega_3 \\ (1+y-x) & - & (0) & = & 1+y-x & , \ (x,y) \in \Omega_4 \\ (2-x) & - & (y-1) & = & -x-y+3 & , \ (x,y) \in \Omega_5 \\ (x-y) & - & (x-1) & = & -x-y+3 & , \ (x,y) \in \Omega_6 \\ (0) & - & (1+(x-1)-(y-1)) & = & -x+y-1 & , \ (x,y) \in \Omega_7 \\ (0) & - & (1+(y-1)-(x-1)) & = & x-y-1 & , \ (x,y) \in \Omega_8 \\ (0) & - & (2-(x-1)) & = & x-3 & , \ (x,y) \in \Omega_9 \\ (0) & - & (2-(y-1)) & = & y-3 & , \ (x,y) \in \Omega_{10} \\ (0) & - & (0) & = & 0 & , \ \text{otherwise,} \end{cases}$$

i.e. $D_{(1,1)}\tilde{B}_3(x,y) = B_2(x,y) - B_2((x,y)-(1,1))$ for all $(x,y) \in \mathbb{R}^2$, and, since B_2 is continuous (see Figure 2.5), it follows that $\tilde{B}_3$ is first-order continuous in the $(1,1)$-

direction.

However, $\tilde{B}_3$ does not necessarily have all its other directional derivatives continuous. For example, it follows from equations (3.13) and (2.2) that

$$D_{(1,0)}(\tilde{B}_3)(x,y) = \left(\frac{\partial \tilde{B}_3}{\partial x}\right)(x,y) = 0 \quad \text{on } \Omega_1,$$

while

$$D_{(1,0)}(\tilde{B}_3)(x,y) = \left(\frac{\partial \tilde{B}_3}{\partial x}\right)(x,y) = y \quad \text{on } \Omega_2,$$

i.e. $D_{(1,0)}(\tilde{B}_3)$ is not continuous on $[0,1) \times [0,1)$. As illustrated in Figure 3.2, one can clearly see a non-smooth edge along the intersection of the graph of $\tilde{B}_3$ and the plane $y = x$.

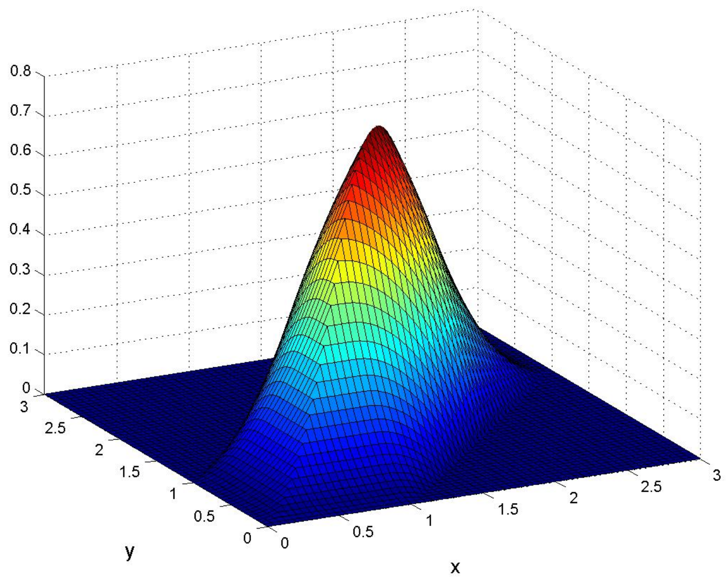
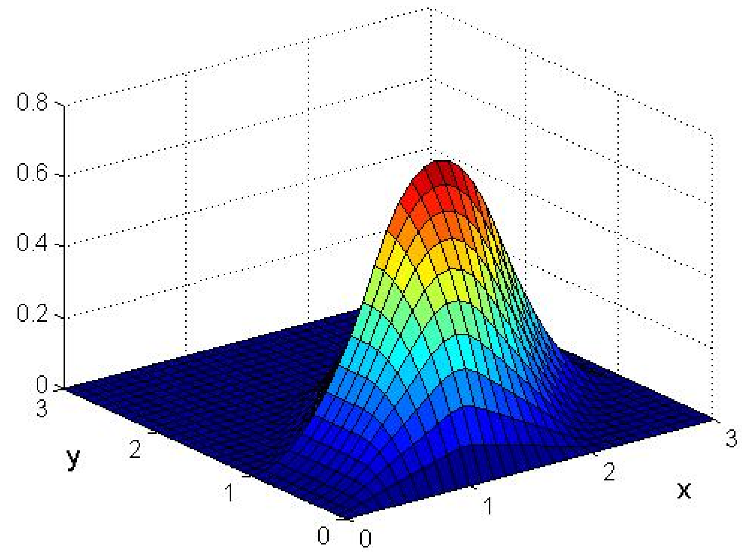


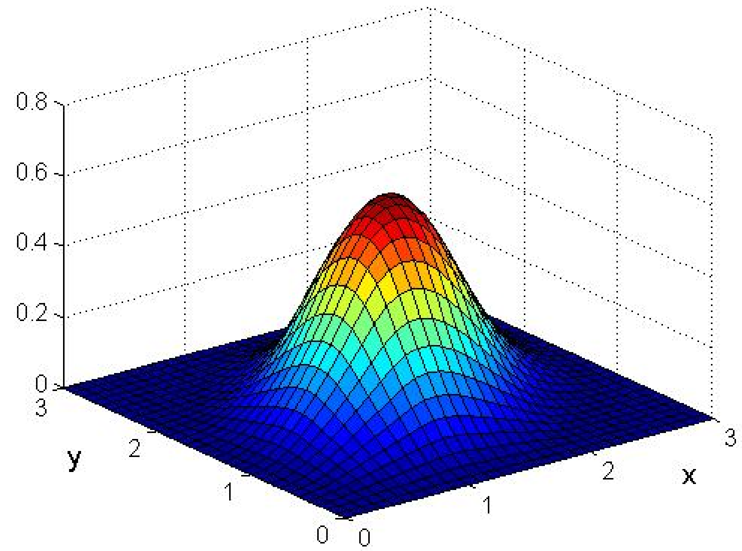
Figure 3.2: Graph of $\tilde{B}_3$

Now, let $\hat{B}_3$ be the box spline corresponding to $\mathcal{D}_{\hat{B}_3} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, (see Figure 3.3(a)), and let B_3 be the box spline corresponding to $\mathcal{D}_{B_3} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$, (see

Figure 3.3(b)), i.e. the matrix $\mathcal{D}_{B_3}$ consists of $\mathcal{D}_{\hat{B}_3}$ with the column $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ added. It follows from Theorem 3.1 and the fact that B_2 as in (2.9) satisfies $B_2 \in C(\mathbb{R})$, that B_3 has (first-order) continuous derivatives in every direction, making it C^1-smooth, and therefore a useful spline function to work with.



(a) $\hat{B}_3$



(b) B_3

Figure 3.3: Graphs of $\hat{B}_3$ and B_3

The formulas for $\hat{B}_3$ and B_3, as obtained by virtue of equations (2.9) and (2.6), are, respectively,

$$\hat{B}_3 = \begin{cases} -\frac{1}{2}y^2 + xy & , \quad (x,y) \in \Lambda_1 \\ \frac{1}{2}x^2 & , \quad (x,y) \in \Lambda_2 \\ \frac{1}{2}(1+x-y)^2 & , \quad (x,y) \in \Lambda_3 \\ \frac{1}{2}y^2 + 2y - xy & , \quad (x,y) \in \Lambda_6 \\ -x^2 - \frac{1}{2}y^2 + 2x + xy - 1 & , \quad (x,y) \in \Lambda_7 \\ -x^2 - \frac{1}{2}y^2 + 2x - y + xy & , \quad (x,y) \in \Lambda_8 \\ \frac{1}{2}y^2 + 2x - y - xy & , \quad (x,y) \in \Lambda_9 \\ \frac{1}{2}(2-x+y)^2 & , \quad (x,y) \in \Lambda_{12} \\ \frac{1}{2}(x-3)^2 & , \quad (x,y) \in \Lambda_{13} \\ -\frac{1}{2}y^2 - 2x - y + xy + 4 & , \quad (x,y) \in \Lambda_{14} \\ 0 & , \quad \text{otherwise;} \end{cases} \tag{3.15}$$

$$B_3 = \begin{cases} \frac{1}{2}xy^2 - \frac{1}{6}y^3 & , \quad (x,y) \in \Lambda_1 \\ -\frac{1}{6}x^3 + \frac{1}{2}x^2y & , \quad (x,y) \in \Lambda_2 \\ -\frac{1}{6}x^3 - x(y-1)^2 + \frac{1}{3}(y-1)^3 + \frac{1}{2}x^2y & , \quad (x,y) \in \Lambda_3 \\ x^2 + \frac{1}{6}x^3 - \frac{1}{2}x^2y & , \quad (x,y) \in \Lambda_4 \\ \frac{1}{6}(2+x-y)^3 & , \quad (x,y) \in \Lambda_5 \\ y^2 - \frac{1}{2}xy^2 + \frac{1}{6}y^3 & , \quad (x,y) \in \Lambda_6 \\ \frac{1}{6}(x-y-1)(7+4x^2-y+y^2-11x-2xy) & , \quad (x,y) \in \Lambda_7 \\ -\frac{5}{3} + \frac{1}{3}x^3 + \frac{3}{2}y - y^2 - \frac{1}{3}y^3 - x^2 - x^2y + \frac{3}{2}x + xy + xy^2 & , \quad (x,y) \in \Lambda_8 \\ -\frac{5}{3} - \frac{1}{3}x^3 - x^2 + x^2y + \frac{3}{2}y - y^2 + \frac{1}{3}y^3 + \frac{3}{2}x + xy - xy^2 & , \quad (x,y) \in \Lambda_9 \\ -\frac{1}{3}x^3 - x^2 + x^2y + \frac{1}{6}(y-2)(1+y)^2 - \frac{1}{2}x(y^2+2y-7) & , \quad (x,y) \in \Lambda_{10} \\ \frac{1}{6}(3x-y)(y-3)^2 & , \quad (x,y) \in \Lambda_{11} \\ -\frac{1}{6}(x-y-2)^3 & , \quad (x,y) \in \Lambda_{12} \\ -\frac{1}{6}(x-3)^2(x-3y) & , \quad (x,y) \in \Lambda_{13} \\ -\frac{1}{3} + \frac{1}{6}x^3 + \frac{21}{6}y - \frac{1}{2}x^2y - y^2 - \frac{1}{3}y^3 - \frac{1}{2}x - xy + xy^2 & , \quad (x,y) \in \Lambda_{14} \\ \frac{1}{6}(x-3)^2(6+x-3y) & , \quad (x,y) \in \Lambda_{15} \\ -\frac{1}{6}(3x-y-6)(y-3)^2 & , \quad (x,y) \in \Lambda_{16} \\ 0 & , \quad \text{otherwise,} \end{cases} \tag{3.16}$$

and where the regions $\Lambda_1, \Lambda_2, \ldots, \Lambda_{16}$ are illustrated in Figure 3.4.

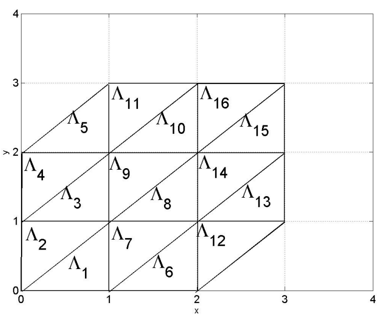


Figure 3.4: Illustration of $\Lambda_1, \ldots, \Lambda_{16}$

In their book [9], De Boor, Höllig and Riemenschneider showed that, for any direction matrix $\mathcal{D}$, the box spline B associated with it satisfies $B \in C^{m-1}(\text{ran}(\mathcal{D}))$, where $\text{ran}(\mathcal{D})$ is the *range* of $\mathcal{D}$, usually $\mathbb{R}^2$, and where the number $m \in \mathbb{Z}_+$ is computed as follows:

Given a direction matrix $\mathcal{D}$, regard $\mathcal{D}$ as a set consisting of the vectors $\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_k$ rather than a matrix, according to which notation we write $\#\mathcal{D}$ for the number of columns in $\mathcal{D}$ and so that, moreover, we say that $\boldsymbol{d}_i \neq \boldsymbol{d}_j,\ i \neq j$, as elements of $\mathcal{D}$, even if $\boldsymbol{d}_i$ and $\boldsymbol{d}_j$ happen to look the same as column vectors of $\mathcal{D}$. In other words, two column vectors in $\mathcal{D}$ are only regarded to be equal if they are obtained by the omission of the same columns from $\mathcal{D}$ — see also [8]. With this understanding, we can say, for any matrix $Z = [\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_l]$, that $Z \subset \mathcal{D}$ iff $\boldsymbol{d}_i \in \mathcal{D}, \quad i = 1, 2, \ldots, l$.

In so doing, one defines a matrix $Z \in \mathcal{D}$ to be *spanning* if $\text{ran}(Z) = \text{ran}(\mathcal{D})$, and defines Z to be a *basis* if Z is minimally spanning, i.e. if no proper submatrix of Z is also spanning. Furthermore, $\mathcal{X}(\mathcal{D})$ is defined to be the set consisting of all such bases.

Finally, the set $\mathcal{A}(\mathcal{D}) := \{Z \in \mathcal{D} : \mathcal{D} \backslash Z \text{ does not span}\}$ is the collection of all $Z \in \mathcal{D}$ which intersect every $X \in \mathcal{X}$.

Then we define $m = m(\mathcal{D}) := \min\{\#Z : Z \in \mathcal{A}(\mathcal{D})\} - 1$.

As an example, let $\mathcal{D} = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} =: \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{d}_4 & \boldsymbol{d}_5 \end{bmatrix}$, in which case $ran(\mathcal{D}) = \mathbb{R}^2$. Then,

$$
\begin{aligned}
\mathcal{X}(\mathcal{D}) \;=\; \Big\{ & \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_4 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_5 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_2 & \boldsymbol{d}_3 \end{bmatrix}, \\
& \begin{bmatrix} \boldsymbol{d}_2 & \boldsymbol{d}_4 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_2 & \boldsymbol{d}_5 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_3 & \boldsymbol{d}_4 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_3 & \boldsymbol{d}_5 \end{bmatrix} \Big\},
\end{aligned}
$$

and

$$\mathcal{A}(\mathcal{D}) \;=\; \Big\{ \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{d}_4 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{d}_4 & \boldsymbol{d}_5 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{d}_4 & \boldsymbol{d}_5 \end{bmatrix} \Big\},$$

so that $m(\mathcal{D}) = \min\{\#Z : Z \in \mathcal{A}(\mathcal{D})\} - 1 = 3 - 1 = 2$,

and therefore the associated box spline B satisfies $B \in C^1(\mathbb{R}^2)$, i.e. B is first-order continuous. See Figure 3.5 for a graphical illustration of the box spline B.

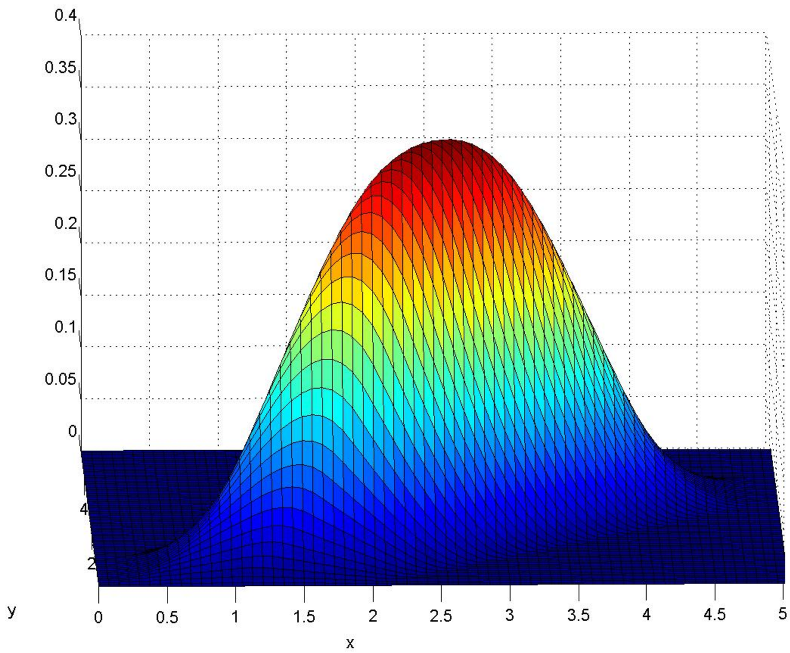


Figure 3.5: Graph of B

As a further example, consider $\mathcal{D}_{\tilde{B}_3} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} =: \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{d}_4 \end{bmatrix}$ as before.

In this case,

$$\mathcal{X}\left(\mathcal{D}_{\tilde{B}_3}\right) \;=\; \Big\{ \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_3 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_4 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_2 & \boldsymbol{d}_3 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_2 & \boldsymbol{d}_4 \end{bmatrix} \Big\},$$

and

$$\mathcal{A}\left(\mathcal{D}_{\tilde{B}_3}\right) = \left\{ \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{d}_4 \end{bmatrix}, \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{d}_4 \end{bmatrix} \right\},$$

so that $m = 2 - 1 = 1$,

i.e. $\tilde{B}_3 \in C(\mathbb{R}^2)$, as expected.

If one uses this method to compute the smoothness class of the box spline B_3 with respect to $\mathcal{D}_3 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$, then it is found that, in fact, $m = 3$, i.e. $B_3 \in C^2(\mathbb{R}^2)$, whereas we have found earlier on that $B_3 \in C^1(\mathbb{R}^2)$. In other words, the result in [9] is a stronger and more general one than ours. However, it is, as far as they describe there, only applicable to box spline theory. As it is our concern to work with refinable functions in general, we leave out the proof for the above method on the regularity of box splines.

In this chapter we have been working with the restricted case where the initial direction matrix was given by $\mathcal{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the dilation matrix by $M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. In the next two chapters we shall consider general initial direction and dilation matrices.

Chapter 4

Other Direction Matrices

We now come to discuss what happens when we allow for the initial direction matrix $\mathcal{D}_1$ to be different from the one we have been using until now, namely $\mathcal{D}_1 = I_2$, while still looking at refinability with dilation matrix $M = 2I_2$, as in the previous chapters.
We first restrict ourselves to the case where $\mathcal{D}_1$ is a diagonal matrix.

Lemma 4.1 $\{$*Diagonal* $\mathcal{D}_1\}$
Suppose $\mathcal{D}_1 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, *with* $a, b \in \mathbb{Z}\backslash\{0\}$, *so that the associated box spline is given, according to equation (2.8), by*

$$B_1(x,y) = \begin{cases} \frac{1}{ab} & , \quad (x,y) \in [0,a) \times [0,b) \\ 0 & , \quad (x,y) \in \mathbb{R}^2 \backslash [0,a) \times [0,b). \end{cases}$$

Then B_1 *is 2-refinable, and*

$$B_1(x,y) = \sum_{k,l} p_{k,l} B_1(2x-k, 2y-l), \tag{4.1}$$

where

$$p_{0,0} = p_{a,0} = p_{0,b} = p_{a,b} = 1,$$

$$\textit{and } p_{k,l} = 0 \textit{ for all other } (k,l) \in \mathbb{Z}^2.$$

Proof

If we define $\tilde{B}_1 \in M(\mathbb{R}^2)$ by

$$\tilde{B}_1(x,y) := B_1(ax, by), \qquad x, y \in \mathbb{R}^2,$$

i.e.

$$B_1(x,y) = \tilde{B}_1\left(\frac{x}{a}, \frac{y}{b}\right), \qquad x, y \in \mathbb{R}^2,$$

then we know from (3.3) that

$$\tilde{B}_1(x,y) = \sum_{k,l} \tilde{p}_{k,l} \tilde{B}_1(2x - k, 2y - l), \tag{4.2}$$

with

$$\tilde{p}_{k,l} = \begin{cases} 1 & , \quad (k,l) \in \{(0,0),(1,0),(0,1),(1,1)\} \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{4.3}$$

Define $\{p_{k,l}\}_{(k,l)\in\mathbb{Z}^2} \subset \mathbb{R}$ by

$$p_{k,l} = \begin{cases} 1 & , \quad (k,l) \in \{(0,0),(a,0),(0,b),(a,b)\} \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{4.4}$$

It then follows from (4.3) and (4.4) that

$$p_{ak,bl} = \begin{cases} \tilde{p}_{k,l} & , \quad (k,l) \in \{(0,0),(1,0),(0,1),(1,1)\} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Hence, we have

$$\begin{aligned}
\sum_{k,l} p_{k,l} B_1(2x-k, 2y-l) &= \sum_{k,l} p_{ak,bl} B_1(2x-ak, 2y-bl) \\
&= \sum_{k,l} \tilde{p}_{k,l} B_1(2x-ak, 2y-bl) \\
&= \sum_{k,l} \tilde{p}_{k,l} \tilde{B}_1\left(2\left(\frac{x}{a}\right)-k, 2\left(\frac{y}{b}\right)-l\right) \\
&= \tilde{B}_1\left(\frac{x}{a}, \frac{y}{b}\right) \\
&= B_1(x,y),
\end{aligned}$$

so that (4.1) holds, as required. $\square$

The phenomenon that presents itself in Lemma 4.1 is that, in the case of diagonal initial direction matrices, the corresponding roof function is always 2-refinable and the corresponding 2-refinement mask coefficients are all zero, except at the *extreme values* (i.e. the corners), $(0,0)$, $(a,0)$, $(0,b)$ and (a,b), of the support of this roof function. Below, we generalize this result, and find that, for *any* non-singular initial direction matrix, the corresponding roof function is still 2-refinable, and the value of the corresponding 2-refinement mask is equal to one at the extreme points of the (parallelogram-shaped) support of the roof function, and zero elsewhere.

Lemma 4.2 $\{$*General* $\mathcal{D}_1\}$
Let $\mathcal{D}_1 = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$, *with* $(a_1,b_1),(a_2,b_2) \in \mathbb{Z}^2 \backslash \{(0,0)\}$, $a_1b_2 \neq a_2b_1$, *so that the corresponding box spline is the roof function given by*

$$B_1(x,y) = \begin{cases} \frac{1}{a_1b_2 - a_2b_1} &, \quad (x,y) \in \mathcal{D}_1[0,1)^2 \\ 0 &, \quad (x,y) \in \mathbb{R}^2 \backslash \mathcal{D}_1[0,1)^2, \end{cases}$$

according to equation (2.8).

Then B_1 is 2-refinable, and

$$B_1(x,y) = \sum_{k,l} p_{k,l} B_1(2x-k, 2y-l), \tag{4.5}$$

where

$$\begin{aligned} &p_{0,0} = p_{a_1,b_1} = p_{a_2,b_2} = p_{a_1+a_2,b_1+b_2} = 1; \\ &p_{k,l} = 0 \textit{ for all other } (k,l) \in \mathbb{Z}^2. \end{aligned} \tag{4.6}$$

Proof

Since $\mathcal{D}_1$ is non-singular, we have

$$\mathcal{D}_1^{-1} = \frac{1}{\triangle} \begin{bmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{bmatrix}, \quad \text{where } \triangle = a_1 b_2 - a_2 b_1.$$

Define

$$J := \{(0,0), (a_1,b_1), (a_2,b_2), (a_1+a_2, b_1+b_2)\}$$

and

$$\hat{J} := \{(0,0), (1,0), (0,1), (1,1)\}.$$

Then $J = \mathcal{D}_1 \hat{J}$ and $\hat{J} = \mathcal{D}_1^{-1} J$.

Define $\tilde{B}_1 \in M_0(\mathbb{R}^2)$ by

$$\tilde{B}_1(x,y) = \tilde{B}_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := B_1\left(\mathcal{D}_1 \begin{bmatrix} x \\ y \end{bmatrix}\right) = B_1(a_1 x + a_2 y, b_1 x + b_2 y), \tag{4.7}$$

i.e.

$$B_1(x,y) = B_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \tilde{B}_1\left(\mathcal{D}_1^{-1} \begin{bmatrix} x \\ y \end{bmatrix}\right) = \tilde{B}_1\left(\frac{b_2 x - a_2 y}{\triangle}, \frac{-b_1 x + a_1 y}{\triangle}\right). \tag{4.8}$$

Then

$$\tilde{B}_1(x,y) = \sum_{k,l} \tilde{p}_{k,l} \tilde{B}_1(2x-k, 2y-l),$$

where $\tilde{p}_{k,l} = \begin{cases} 1 & , \quad (k,l) \in \hat{J} \\ 0 & , \quad \text{otherwise.} \end{cases}$

Define $\{p_{k,l}\}_{(k,l)\in\mathbb{Z}^2} \subset \mathbb{R}$ by

$$p_{k,l} = \begin{cases} 1 & , \quad (k,l) \in J \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then, writing the pair $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ as a column vector $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, it follows that, for $(x,y) \in \mathbb{R}^2$,

$$\begin{aligned} \sum_{k,l} p_{k,l} B_1(2x-k, 2y-l) &= \sum_{(k,l)\in J} p_{\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]} B_1(2x-k, 2y-l) \\ &= \sum_{(k,l)\in\hat{J}} p_{\mathcal{D}_1\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]} B_1\left(\begin{bmatrix} 2x \\ 2y \end{bmatrix} - \mathcal{D}_1 \begin{bmatrix} k \\ l \end{bmatrix}\right). \end{aligned}$$

But $p_{\mathcal{D}_1\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]} = \tilde{p}_{\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]}, \quad (k,l) \in \hat{J}.$

Thus, it follows that

$$\begin{aligned} \sum_{k,l} p_{k,l} B_1(2x-k, 2y-l) &= \sum_{(k,l)\in\hat{J}} \tilde{p}_{\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]} B_1\left(\begin{bmatrix} 2x \\ 2y \end{bmatrix} - \mathcal{D}_1 \begin{bmatrix} k \\ l \end{bmatrix}\right) \\ &= \sum_{(k,l)\in\hat{J}} \tilde{p}_{\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]} \tilde{B}_1\left(2\mathcal{D}_1^{-1} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} k \\ l \end{bmatrix}\right) \\ &= \tilde{B}_1\left(\mathcal{D}_1^{-1} \begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= B_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= B_1(x,y), \end{aligned}$$

hence satisfying (4.5), as required. □

To finish this chapter, we shall make the results obtained thus far even more general, to combine with the work on 2-refinement pairs done in Chapter 3. To this end, we prove the following generalization of Theorem 3.1.

Lemma 4.3 {*Refinement preservation*}

Suppose $(\tilde{p}, \tilde{\phi})$ *is a 2-refinement pair,* $\tilde{p} = \{\tilde{p}_{k,l}\}_{(k,l)\in\mathbb{Z}^2} \subset \mathbb{R}$, $\tilde{\phi} \in M_0(\mathbb{R}^2)$, *and with*

$$\tilde{P}(z_1, z_2) = \sum_{k,l} \tilde{p}_{k,l} z_1^k z_2^l$$

the corresponding 2-refinement mask symbol.

Take any $(\alpha, \beta) \in \mathbb{Z}^2 \backslash \{0, 0\}$, *and define the Laurent polynomial* P *by*

$$P(z_1, z_2) = \left(\frac{1 + z_1^\alpha z_2^\beta}{2} \right) \tilde{P}(z_1, z_2), \tag{4.9}$$

i.e. $P(z_1, z_2) = \sum_{k,l} p_{k,l} z_1^k z_2^l$, *where*

$$p_{k,l} = \frac{1}{2} \left(\tilde{p}_{k,l} + \tilde{p}_{k-\alpha, l-\beta} \right), \quad (k,l) \in \mathbb{Z}^2.$$

Also, let $\phi \in M_0(\mathbb{R}^2)$ *be the function defined by*

$$\phi(x, y) = \int_0^1 \tilde{\phi}(x - \alpha t, y - \beta t)\, dt. \tag{4.10}$$

Then (p, ϕ) *is also a 2-refinement pair, with* $p = \{p_{k,l}\}_{(k,l)\in\mathbb{Z}^2}$.

Proof

For any $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}
\sum_{k,l} p_{k,l} \phi\left(2(x,y) - (k,l)\right) &= \sum_{k,l} \frac{1}{2} (\tilde{p}_{k,l} + \tilde{p}_{k-\alpha,l-\beta}) \int_0^1 \tilde{\phi}(2(x,y) - (k,l) - t(\alpha,\beta))\, dt \\
&= \frac{1}{2} \int_0^1 \left[\sum_{k,l} \tilde{p}_{k,l} \tilde{\phi}(2(x,y) - (k,l) - t(\alpha,\beta)) \right. \\
&\qquad \left. + \sum_{k,l} \tilde{p}_{k,l} \tilde{\phi}(2(x,y) - (k,l) - (\alpha,\beta) - t(\alpha,\beta)) \right] dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^1 \tilde{\phi}\left(x - \frac{t}{2}\alpha, y - \frac{t}{2}\beta\right) dt + \frac{1}{2}\int_0^1 \tilde{\phi}\left(x - \frac{t+1}{2}\alpha, y - \frac{t+1}{2}\beta\right) dt \\
&= \int_0^{1/2} \tilde{\phi}(x - \alpha t, y - \beta t)\, dt + \int_{1/2}^1 \tilde{\phi}(x - \alpha t, y - \beta t)\, dt \\
&= \int_0^1 \tilde{\phi}(x - \alpha t, y - \beta t)\, dt \\
&= \phi(x, y),
\end{aligned}
$$

and thus (p, ϕ) is a 2-refinement pair, as desired. $\square$

Note that the particular reason for working with the special case $\alpha = 1, \beta = 0$ followed by $\alpha = 0, \beta = 1$ in Theorem 3.1, was to not only preserve the 2-refinability, but also to enhance the regularity of ϕ. In view of this, Corollary 4.3 is not as strong a result as Theorem 3.1, but it is more general, as it helps us to come to the following conclusion regarding box splines corresponding to direction matrices of *any* form.

Corollary 4.1 {*General formula for 2-refinement mask symbols*}
Let $k \in \mathbb{N}$, *and let the initial direction matrix be given by* $\mathcal{D}_1 = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$, $(a_1, b_1), (a_2, b_2) \in \mathbb{Z}^2 \backslash \{(0,0)\}$, *and with* $det(\mathcal{D}_1) \neq 0$, *so that its corresponding box spline* B_1 *is 2-refinable, with corresponding 2-refinement mask symbol given, according to (4.6), by*

$$
P_1(z_1, z_2) = 1 + z_1^{a_1} z_2^{b_1} + z_1^{a_2} z_2^{b_2} + z_1^{a_1+a_2} z_2^{b_1+b_2},
$$

and suppose the direction matrix $\mathcal{D} := \mathcal{D}_{k-1}$ *is given, for* $k \geq 3$, *by* $\mathcal{D} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \dots & a_k \\ b_1 & b_2 & b_3 & b_4 & & b_k \end{bmatrix}$, *where* $(a_i, b_i) \in \mathbb{Z}^2 \backslash \{(0,0)\}, \quad i = 3, 4, \dots, k.$
Then the box spline $B = B_{\mathcal{D}}$ *corresponding to* $\mathcal{D}$ *is 2-refinable, and its corresponding*

2-refinement mask symbol is given by

$$P(z_1, z_2) = \left(\frac{1 + z_1^{a_3} z_2^{b_3}}{2}\right)\left(\frac{1 + z_1^{a_4} z_2^{b_4}}{2}\right)\cdots\left(\frac{1 + z_1^{a_k} z_2^{b_k}}{2}\right)\left(1 + z_1^{a_1} z_2^{b_1} + z_1^{a_2} z_2^{b_2} + z_1^{a_1+a_2} z_2^{b_1+b_2}\right). \tag{4.11}$$

Using (4.11), it is possible to write down the 2-refinement mask coefficients corresponding to any box spline, given the box splines's characteristic set of direction vectors. However, if one wants to, moreover, have certain degrees of smoothness of one's box spline, then the correct number of combinations of the vectors (1,0) and (0,1) should be included in the direction matrix, as described in Chapter 3.

Chapter 5

Other Dilation Matrices

When shifting one's attention from the univariate to the multivariate case in the study of refinement equations, one of the principle factors that complicates matters is the dilation factor. In the univariate case, the dilation factor can only be a scalar and has been studied extensively in the past (see e.g. [13]). When more variables are introduced, then the dilation factor becomes a matrix M, the simplest example of which is $M = 2I_2$. Since the ultimate question is whether there are certain choices of M that are more preferable than others, it seems useful to study the effect that different choices of M have on the refinement equation.

In all of the previous chapters, our attention has been focussed on the refinability of bivariate box splines with respect to, specifically, the dilation matrix $M = 2I_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Here, a refinement equation involves a linear combination of shifts of dilated versions of the particular function involved. Particularly, at every point $(x, y) \in \mathbb{R}^2$, the function is evaluated by first dilating (x, y) and then adding all function values of the integer shifts of this dilated point. Since this procedure is the same for all points (x, y), we can say that the *role* that is played by the matrix $M = 2I_2$ in the 2-refinement equation is to *dilate* the domain of the refinable function, and since we will mostly be interested in that part of the domain on which the function is non-zero, we shall say that the role that $M = 2I_2$ plays is to dilate the support region of the refinable function, as is illustrated in Figure 5.1.

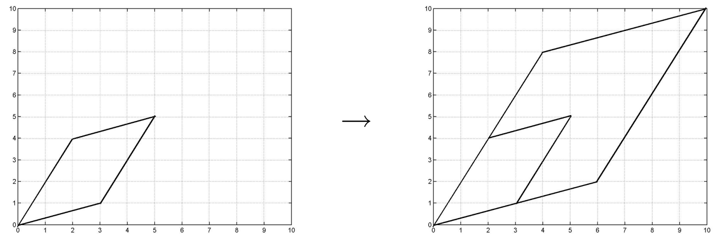


Figure 5.1: The effect of $M = 2I_2$ on the support of a refinable function

When M is not a diagonal matrix, the role it plays is not anymore to just expand the support of ϕ, but also to *rotate* it in the xy-plane. This is demonstrated in Figure 5.2 for the case $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Here, M can also be written as

$$M = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \sqrt{2} \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix},$$

i.e. M rotates the support region of ϕ anti-clickwise by 45 degrees and expands it by factor $\sqrt{2}$.

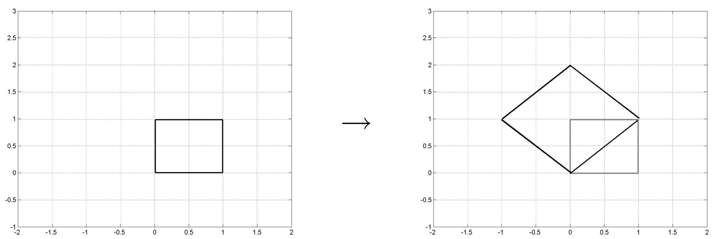


Figure 5.2: The effect of the Quincunx matrix on the area $[0, 1) \times [0, 1)$

The matrix $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ appears frequently in the literature (see e.g. [18]) and is known as the *Quincunx* dilation matrix. A favourable property that this matrix has is that $det(M) = 2$, which is a low value for the determinant of an integer matrix and therefore making it especially useful for application in *wavelet* analysis (see [17] and [18]).

Later in this chapter an example of the refinement equation with respect to the Quincunx matrix will be given.

The objective of this chapter is to find examples of pairs (p, ϕ) such that the M-refinement equation (3.1) is satisfied, where M is a general matrix. It will be assumed that M is invertible and, as usual, that it has integer entries. Since box splines are usually the prototype examples of refinable functions, we shall, specifically, focus on them in Section 5.1, and, for simplicity, we shall restrict our attention to the case where the direction matrix is given by $\mathcal{D}_1 = I_2$, with its corresponding box spline as the bivariate roof function B_1. In Section 5.2, the problem of M-refinement preservation will be discussed for diagonal matrices M. Here we shall give an example of an M-refinable function that is *not* a box spline, and obtain a result regarding M-refinement preservation.

5.1 A formula for the mask of an M-refinable roof function

By setting $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in \mathbb{Z}$, the M-refinement equation (3.1) becomes

$$B_1(x, y) = \sum_{k,l} p_{k,l} B_1(ax + by - k, cx + dy - l). \tag{5.1}$$

Since

$$B_1 = \begin{cases} 1 & , \quad (x, y) \in [0, 1)^2 \\ 0 & , \quad \text{everywhere else,} \end{cases}$$

we proceed to study the action of the matrix M on the region $[0, 1)^2$.

For $x, y \in [0, 1)$, we have the following:

- $\min(ax+by) = \begin{cases} 0 & , \quad a,b \geq 0 \\ a & , \quad a<0, b \geq 0 \\ b & , \quad a \geq 0, b < 0 \\ a+b & , \quad a,b<0; \end{cases}$

- $\max(ax+by) = \begin{cases} a+b & , \quad a,b \geq 0 \\ b & , \quad a<0, b \geq 0 \\ a & , \quad a \geq 0, b < 0 \\ 0 & , \quad a,b<0; \end{cases}$

- $\min(cx+dy) = \begin{cases} 0 & , \quad c,d \geq 0 \\ c & , \quad c<0, d \geq 0 \\ d & , \quad c \geq 0, d < 0 \\ c+d & , \quad c,d<0; \end{cases}$

- $\max(cx+dy) = \begin{cases} c+d & , \quad c,d \geq 0 \\ d & , \quad c<0, d \geq 0 \\ c & , \quad c \geq 0, d < 0 \\ 0 & , \quad c,d<0. \end{cases}$

It therefore follows that, for $x, y \in [0,1)$, we have

- $ax+by \in \begin{cases} (0,a+b) & , \quad a,b \geq 0 \\ (a,b) & , \quad a<0, b \geq 0 \\ (b,a) & , \quad a \geq 0, b < 0 \\ (a+b,0) & , \quad a,b<0; \end{cases}$

- $cx + dy \in \begin{cases} (0, c+d) & , \quad c, d \geq 0 \\ (c, d) & , \quad c < 0, d \geq 0 \\ (d, c) & , \quad c \geq 0, d < 0 \\ (c+d, 0) & , \quad c, d < 0. \end{cases}$

We now consider the following restrictions on the integers a, b, c and d and use the above information on $ax + by$ and $cx + dy$ for equating the left and right hand sides of equation (5.1):

$\underline{\mathbf{a}, \mathbf{b} \geq \mathbf{0}}$:

$\underline{c, d \geq 0}$:

$$\begin{aligned} 1 = B_1(x, y) &= \sum_{k,l} p_{k,l} B_1\left((ax+by) - k, (cx+dy) - l\right) \\ &= p_{0,0} = p_{1,0} = \ldots = p_{a+b-1,0} \\ &= p_{0,1} = p_{1,1} = \ldots = p_{a+b-1,1} \\ &= \ldots \\ &= p_{0,c+d-1} = p_{1,c+d-1} = \ldots = p_{a+b-1,c+d-1}, \end{aligned}$$

implying $p_{k,l} = 1, \quad k \in \{0, 1, \ldots, a+b-1\}, \quad l \in \{0, 1, \ldots, c+d-1\}.$

$\underline{c < 0, d \geq 0}$: Similarly, $p_{k,l} = 1, \quad k \in \{0, 1, \ldots, a+b-1\}, \quad l \in \{c, c+1, \ldots, d-1\}.$

$\underline{c \geq 0, d < 0}$: Similarly, $p_{k,l} = 1, \quad k \in \{0, 1, \ldots, a+b-1\}, \quad l \in \{d, d+1, \ldots, c-1\}.$

$\underline{c, d < 0}$: Similarly, $p_{k,l} = 1, \quad k \in \{0, 1, \ldots, a+b-1\}, \quad l \in \{c+d, c+d+1, \ldots, -1\}.$

$\underline{\mathbf{a} < \mathbf{0}, \mathbf{b} \geq \mathbf{0}}$:

$\underline{c, d \geq 0}$: $p_{k,l} = 1, \quad k \in \{a, a+1, \ldots, b-1\}, \quad l \in \{0, 1, \ldots, c+d-1\}.$

$\underline{c < 0, d \geq 0}$: $p_{k,l} = 1, \quad k \in \{a, a+1, \ldots, b-1\}, \quad l \in \{c, c+1, \ldots, d-1\}.$

$\underline{c \geq 0, d < 0}$: $p_{k,l} = 1, \quad k \in \{a, a+1, \ldots, b-1\}, \quad l \in \{d, d+1, \ldots, c-1\}.$

$\underline{c, d < 0}$: $p_{k,l} = 1, \quad k \in \{a, a+1, \dots, b-1\}, \quad l \in \{c+d, c+d+1, \dots, -1\}$.

$\underline{\mathbf{a \geq 0, b < 0}}$:

$\underline{c, d \geq 0}$: $p_{k,l} = 1, \quad k \in \{b, b+1, \dots, a-1\}, \quad l \in \{0, 1, \dots, c+d-1\}$.

$\underline{c < 0, d \geq 0}$: $p_{k,l} = 1, \quad k \in \{b, b+1, \dots, a-1\}, \quad l \in \{c, c+1, \dots, d-1\}$.

$\underline{c \geq 0, d < 0}$: $p_{k,l} = 1, \quad k \in \{b, b+1, \dots, a-1\}, \quad l \in \{d, d+1, \dots, c-1\}$.

$\underline{c, d < 0}$: $p_{k,l} = 1, \quad k \in \{b, b+1, \dots, a-1\}, \quad l \in \{c+d, c+d+1, \dots, -1\}$.

$\underline{\mathbf{a, b < 0}}$:

$\underline{c, d \geq 0}$: $p_{k,l} = 1, \quad k \in \{a+b, a+b+1, \dots, -1\}, \quad l \in \{0, 1, \dots, c+d-1\}$.

$\underline{c < 0, d \geq 0}$: $p_{k,l} = 1, \quad k \in \{a+b, a+b+1, \dots, -1\}, \quad l \in \{c, c+1, \dots, d-1\}$.

$\underline{c \geq 0, d < 0}$: $p_{k,l} = 1, \quad k \in \{a+b, a+b+1, \dots, -1\}, \quad l \in \{d, d+1, \dots, c-1\}$.

$\underline{c, d < 0}$: $p_{k,l} = 1, \quad k \in \{a+b, a+b+1, \dots, -1\}, \quad l \in \{c+d, c+d+1, \dots, -1\}$.

Examples:

Consider the box spline B_1 associated with the direction matrix $\mathcal{D}_1 = I_2$, and suppose $M = \begin{bmatrix} -2 & 3 \\ 8 & 1 \end{bmatrix}$.

Then B_1 is M-refinable, and

$$p_{k,l} = \begin{cases} 1 & , \quad k \in \{-2, -1, 0, 1, 2\}, \; l \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

We can thus form the M-refinement mask symbol of B_1,

$$P_1(z_1, z_2) = \left(z_1^{-2} + z_1^{-1} + 1 + z_1 + z_1^2\right)\left(1 + z_2 + z_2^2 + z_2^3 + z_2^4 + z_2^5 + z_2^6 + z_2^7 + z_2^8\right).$$

Now, let M be the Quincunx matrix as in the beginning of the chapter. Using the formula above, it follows that B_1 is M-refinable, i.e.

$$B_1(x,y) = \sum_{k,l} p_{k,l} B_1\left(x-y-k, x+y-l\right), \tag{5.2}$$

with, specifically,

$$p_{k,l} = \begin{cases} 1 & , \quad k \in \{-1,0\},\ l \in \{0,1\}, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

The corresponding M-refinement mask symbol is thus given by

$$P_1(z_1,z_2) = \left(\frac{1}{z_1}+1\right)\left(z_2+1\right).$$

Note: If one evaluates the left and right hand sides of equation (5.2) for different values of $(x,y) \in \mathbb{R}^2$, then it is seen that the equation does indeed hold, except for $(x,y) = (0,1)$. Here,

$$B_1(0,1) = 0,$$

while

$$\begin{aligned} \sum_{k,l} p_{k,l} B_1\left(-1-k, 1-l\right) &= B_1(0,1) + B_1(0,0) + B_1(-1,1) + B_1(-1,0) \\ &= 0+1+0+0 \;=\; 1. \end{aligned}$$

However, as stated in the first paragraph of Chapter 3, equation (5.2) need only hold for all but a finite number of values — hence B_1 is still considered M-refinable.

5.2 Preserving M-refinability

In this section, we suppose the dilation matrix M to be diagonal. The example of a function that is refinable with respect to such a matrix, is going to rest on the idea of *tensor products*. Here, a bivariate function is created from two univariate functions by simple multiplication, a process which is described by Loop (see [19]) as "taking a series of parallel curves (in our case, just one curve), and *sweeping* these with another curve in the perpendicular direction to form a surface".

Lemma 5.1 {*Tensor product splines*}
Let $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $a, b \in \mathbb{Z}\backslash\{0\}$, *and suppose* $(p^{(1)}, \phi_1)_a$ *and* $(p^{(2)}, \phi_2)_b$ *are (univariate) a- and b-refinement pairs, respectively, i.e.*

$$\phi_1(x) = \sum_k p_k^{(1)} \phi_1(ax - k), \qquad \phi_2(y) = \sum_l p_l^{(2)} \phi_2(by - l), \qquad x, y \in \mathbb{R}.$$

Let $\phi \in M_0(\mathbb{R}^2)$ *and* $p_{k,l} \in M_0(\mathbb{Z}^2)$ *be defined by*

$$\phi(x, y) := \phi_1(x)\phi_2(y), \quad (x, y) \in \mathbb{R}^2, \tag{5.3}$$

and

$$p_{k,l} := p_k^{(1)} p_l^{(2)}, \quad (k, l) \in \mathbb{Z}^2, \tag{5.4}$$

respectively.
Then (p, ϕ) *is an* M*-refinement pair.*

Proof

For $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned} \sum_{k,l} p_{k,l} \phi(ax - k, by - l) &= \sum_k \sum_l p_k^{(1)} p_l^{(2)} \phi_1(ax - k)\phi_2(by - l) \\ &= \sum_k p_k^{(1)} \phi_1(ax - k) \sum_l p_l^{(2)} \phi_2(by - l) \\ &= \phi_1(x)\phi_2(y) \\ &= \phi(x, y), \end{aligned}$$

as required. □

According to Lemma 5.1, given a diagonal integer dilation matrix M, there always exists a function ϕ that is refinable with respect to M, and with the corresponding mask coefficients obtained explicitly. The rest of this section will build further on this fact and answer the question whether, given such a refinable function, more M-refinable functions can be generated, as was done in Section 3.2.

Theorem 5.1 $\{$*Refinement preservation for diagonal* $M\}$
Suppose $(\tilde{p}, \tilde{\phi})$ *is an* M*-refinement pair, with* $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, *and where* $a, b \in \mathbb{Z}\backslash\{0\}$, *and let*

$$\tilde{P}(z_1, z_2) = \sum_{k,l} \tilde{p}_{k,l} z_1^k z_2^l$$

be the corresponding M*-refinement mask symbol.*
Take any $\alpha \in \mathbb{Z}\backslash\{0\}$, *and define*

$$P(z_1, z_2) := \left(\frac{1 + z_1^{\alpha} + z_1^{2\alpha} + \ldots + z_1^{(a-1)\alpha}}{a} \right) \tilde{P}(z_1, z_2), \tag{5.5}$$

i.e.

$$p_{k,l} = \frac{1}{a} \left(\tilde{p}_{k,l} + \tilde{p}_{k-\alpha,l} + \tilde{p}_{k-2\alpha,l} + \ldots + \tilde{p}_{k-(a-1)\alpha,l} \right). \tag{5.6}$$

Also, let $\phi \in M_0(\mathbb{R}^2)$ *be defined by*

$$\phi(x, y) := \int_0^1 \tilde{\phi}(x - \alpha t, y)\, dt. \tag{5.7}$$

Then (p, ϕ) *is an* M*-refinement pair as well.*

Proof

For any $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}
\sum_{k,l} p_{k,l}\phi(ax-k, by-l) &= \sum_{k,l} \left[\frac{1}{a}\left(\tilde{p}_{k,l} + \tilde{p}_{k-\alpha,l} + \tilde{p}_{k-2\alpha,l} + \ldots + \tilde{p}_{k-(a-1)\alpha,l}\right) \cdot \right. \\
&\qquad \left. \int_0^1 \tilde{\phi}(ax-k-\alpha t, by-l)\, dt \right] \\
&= \frac{1}{a}\int_0^1 \left[\sum_{k,l} \tilde{p}_{k,l}\tilde{\phi}(ax-k-\alpha t, by-l) \right. \\
&\qquad + \sum_{k,l} \tilde{p}_{k,l}\tilde{\phi}(ax-k-\alpha-\alpha t, by-l) \\
&\qquad + \sum_{k,l} \tilde{p}_{k,l}\tilde{\phi}(ax-k-2\alpha-\alpha t, by-l) \\
&\qquad + \ldots \\
&\qquad \left. + \sum_{k,l} \tilde{p}_{k,l}\tilde{\phi}(ax-k-(a-1)\alpha-\alpha t, by-l)\right] dt \\
&= \frac{1}{a}\int_0^1 \left[\sum_{k,l} \tilde{p}_{k,l}\tilde{\phi}\left(a\left(x-\frac{\alpha t}{a}\right)-k, by-l\right) \right. \\
&\qquad + \sum_{k,l} \tilde{p}_{k,l}\tilde{\phi}\left(a\left(x-\frac{\alpha(t+1)}{a}\right)-k, by-l\right) \\
&\qquad + \sum_{k,l} \tilde{p}_{k,l}\tilde{\phi}\left(a\left(x-\frac{\alpha(t+2)}{a}\right)-k, by-l\right) \\
&\qquad + \ldots \\
&\qquad \left. + \sum_{k,l} \tilde{p}_{k,l}\tilde{\phi}\left(a\left(x-\frac{\alpha(t+a-1)}{a}\right)-k, by-l\right)\right] dt \\
&= \frac{1}{a}\int_0^1 \tilde{\phi}\left(x-\frac{\alpha t}{a}, y\right)\, dt + \frac{1}{a}\int_0^1 \tilde{\phi}\left(x-\frac{\alpha(t+1)}{a}, y\right)\, dt \\
&\quad + \frac{1}{a}\int_0^1 \tilde{\phi}\left(x-\frac{\alpha(t+2)}{a}, y\right)\, dt + \ \ldots \\
&\quad + \frac{1}{a}\int_0^1 \tilde{\phi}\left(x-\frac{\alpha(t+a-1)}{a}, y\right)\, dt
\end{aligned}$$

$$\begin{aligned} &= \int_0^{\frac{1}{a}} \tilde{\phi}(x-\alpha t, y)\,dt + \int_{\frac{1}{a}}^{\frac{2}{a}} \tilde{\phi}(x-\alpha t, y)\,dt \\ &\quad + \int_{\frac{2}{a}}^{\frac{3}{a}} \tilde{\phi}(x-\alpha t, y)\,dt + \ \ldots\ + \int_{\frac{a-1}{a}}^{1} \tilde{\phi}(x-\alpha t, y)\,dt \\ &= \int_0^1 \tilde{\phi}(x-\alpha t, y)\,dt \\ &= \phi(x,y). \end{aligned}$$

Hence, (p, ϕ) is an M-refinement pair. □

Corollary 5.1 {*Refinement preservation for diagonal M*} *With all the notation as in Lemma 5.1, one can similarly define, for $\beta \in \mathbb{Z}\backslash\{0\}$,*

$$\hat{P}(z_1, z_2) := \left(\frac{1 + z_2^{\beta} + z_2^{2\beta} + \ldots + z_2^{(b-1)\beta}}{b}\right) \tilde{P}(z_1, z_2), \tag{5.8}$$

i.e.

$$\hat{p}_{k,l} = \frac{1}{b}\left(\tilde{p}_{k,l} + \tilde{p}_{k,l-\beta} + \tilde{p}_{k,l-2\beta} + \ldots + \tilde{p}_{k,l-(b-1)\beta}\right). \tag{5.9}$$

Then, if $\hat{\phi}(x,y) := \int_0^1 \tilde{\phi}(x, y-\beta t)\,dt$, $(\hat{p}, \hat{\phi})$ is also an M-refinement pair.

The proof for this is similar to the proof for Theorem 5.1. □

We now combine the results in Theorem 5.1 and Corollary 5.1 to yield the following:

Corollary 5.2 {*Refinement preservation and smoothness enhancement for diagonal M*}

Suppose $(\tilde{p}, \tilde{\phi})$ is an M refinement pair, with $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and where $a, b \in \mathbb{Z}\backslash\{0\}$, and let

$$\tilde{P}(z_1, z_2) = \sum_{k,l} \tilde{p}_{k,l} z_1^k z_2^l$$

be the corresponding M-refinement mask symbol.

Take any $\alpha, \beta \in \mathbb{Z}\backslash\{0\}$, and define

$$P(z_1, z_2) := \left(\frac{1 + z_1^{\alpha} + z_1^{2\alpha} + \ldots + z_1^{(a-1)\alpha}}{a}\right)\left(\frac{1 + z_2^{\beta} + z_2^{2\beta} + \ldots + z_2^{(b-1)\beta}}{b}\right)\tilde{P}(z_1, z_2), \tag{5.10}$$

i.e.

$$\begin{aligned} p_{k,l} \;=\; & \frac{1}{ab}\big(\tilde{p}_{k,l} + \tilde{p}_{k-\alpha,l} + \tilde{p}_{k-2\alpha,l} + \ldots + \tilde{p}_{k-(a-1)\alpha,l} \\ & +\tilde{p}_{k,l-\beta} + \tilde{p}_{k-\alpha,l-\beta} + \tilde{p}_{k-2\alpha,l-\beta} + \ldots + \tilde{p}_{k-(a-1)\alpha,l-\beta} \\ & +\tilde{p}_{k,l-2\beta} + \tilde{p}_{k-\alpha,l-2\beta} + \tilde{p}_{k-2\alpha,l-2\beta} + \ldots + \tilde{p}_{k-(a-1)\alpha,l-2\beta} \\ & + \;\ldots \\ & +\tilde{p}_{k,l-(b-1)\beta} + \tilde{p}_{k-\alpha,l-(b-1)\beta} + \tilde{p}_{k-2\alpha,l-(b-1)\beta} + \ldots + \tilde{p}_{k-(a-1)\alpha,l-(b-1)\beta}\big). \end{aligned}$$

Also, let $\phi \in M_0(\mathbb{R}^2)$ be defined by

$$\phi(x, y) := \int_0^1 \int_0^1 \tilde{\phi}(x - \alpha t_1, y - \beta t_2)\, dt_1\, dt_2. \tag{5.11}$$

Then (p, ϕ) is an M-refinement pair as well.
Moreover, if $\tilde{\phi} \in C^k(\mathbb{R}^2)$ for some $k \in \mathbb{Z}_+$, then $\phi \in C^{k+1}(\mathbb{R}^2)$.

Proof

The fact that (p, ϕ) forms an M-refinement pair, follows immediately from Theorem 5.1 and Corollary 5.1.

Suppose $\tilde{\phi} \in C^k(\mathbb{R}^2)$ for some $k \in \mathbb{Z}_+$. We must show that $\phi \in C^{k+1}(\mathbb{R}^2)$. Now, for any $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned} \phi(x, y) &= \int_0^1 \int_0^1 \tilde{\phi}\left(x - \alpha t_1, y - \beta t_2\right)\, dt_1\, dt_2 \\ &= \int_0^1 \left[\int_0^1 \tilde{\phi}\left(x - \alpha t_1, y - \beta t_2\right)\, dt_2\right] dt_1 \end{aligned}$$

$$= \frac{1}{\alpha}\int_{x-\alpha}^{x}\left[\int_0^1 \tilde{\phi}\,(t_1, y-\beta t_2)\;dt_2\right]\,dt_1$$
$$= \frac{1}{\alpha}\int_{0}^{x}\left[\int_0^1 \tilde{\phi}\,(t_1, y-\beta t_2)\;dt_2\right]\,dt_1 - \frac{1}{\alpha}\int_{0}^{x-\alpha}\left[\int_0^1 \tilde{\phi}\,(t_1, y-\beta t_2)\;dt_2\right]\,dt_1.$$

As in Theorem 3.1, it thus follows from the Fundamental Theorem of Calculus that, for any fixed $y \in \mathbb{R}$, $\phi(x) = \phi(x, y)$ is a continuously differentiable function on $\mathbb{R}$, with

$$\frac{\partial \phi}{\partial x}(x,y) \;=\; \frac{1}{\alpha}\int_0^1 \tilde{\phi}\,(x, y-\beta t_2)\;dt_2 - \frac{1}{\alpha}\int_0^1 \tilde{\phi}\,(x-\alpha, y-\beta t_2)\;dt_2,$$

i.e. $\frac{\partial \phi}{\partial x} \in C^k(\mathbb{R}^2)$.

Similarly, $\frac{\partial \phi}{\partial y} \in C^k(\mathbb{R}^2)$, and it thus follows from (2.2) that $\phi \in C^{k+1}(\mathbb{R}^2)$, as desired. $\square$

In this section, it has been assumed that the dilation matrix M was diagonal. Further investigation in the future might yield similar results for general dilation matrices of the form $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, i.e. non-diagonal M. The question that remains is, given an M-refinement pair $(\tilde{p}, \tilde{\phi})_M$, how should equations (5.5) and (5.7) be adjusted so that $(p, \phi)_M$ is again an M-refinement pair?

In all our work up until now, we have been concentrating on the refinement equation (3.1), and all the aspects that influence it. In the next chapter, we shall start doing work that will form a basis for the chapter on subdivision, which is an important application area of refinable functions.

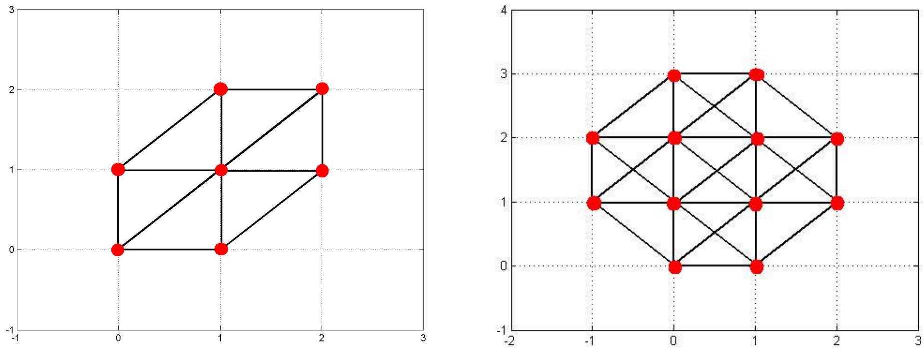
Chapter 6

The support of a refinable function

Many authors have pointed out the particular shape that the support of box splines take on (see e.g. [9], [15], [20], [21], [25]). We observed in Chapter 2 that box splines possess strong symmetric features, and hence so do their regions of support. In [20] an algorithm is even given for fast computer programming to determine the support of any box spline, given its corresponding direction matrix, and a couple of advanced examples is provided there. In [9], a general discussion is given about the triangulation of the support of a box spline, each triangular region on which the box spline is polynomial to a certain degree. The following property is also proved there:

Suppose $\mathcal{D}_1 = I_2$ and that the direction matrix $\mathcal{D}$ is obtained from $\mathcal{D}_1$ *only* by the insertion of either or some of the vectors $\pm e_1,\ \pm e_2,\ \pm e_3,\ \pm e_4$, where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $e_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $e_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. For example, the Courant hat function and the ZP element are box splines corresponding to the direction matrices $[\boldsymbol{e}_1\ \boldsymbol{e}_2\ \boldsymbol{e}_3]$ and $[\boldsymbol{e}_1\ \boldsymbol{e}_2\ -\boldsymbol{e}_4]$, respectively. Their support regions are shown in Figure 6.1.

Note from Figure 6.1 how, with such a direction matrix, the said triangulation of the support seems to always be made up from either (1) a simple square mesh in the plane (no triangulation), (2) the square mesh with the SW-NE diagonal included (forward triangulation), (3) the square mesh with the NW-SE diagonal included (backward triangulation),



(Courant hat function) (Zwart-Powell function)

Figure 6.1: The supports of the Courant hat function and ZP element

or (4) the square mesh and both the SW-NE and SE-NW diagonals included (forward and backward triangulation). It is shown in [9] that, for case (1) to hold, the vectors in $\mathcal{D}$ (except the first two, of course) are only allowed to be picked from $\{\pm e_1, \pm e_2\}$; for case (2), one may use vectors of the form $\pm e_1, \pm e_2$ and $\pm e_3$; for case (3), vectors of the form $\pm e_1, \pm e_2$ and $\pm e_4$ are allowed; finally, for case (3), one may, in addition to $\pm e_1$ and $\pm e_2$, include vectors of both the forms $\pm e_3$ and $\pm e_4$.

In [15], the case is analyzed where $\mathcal{D} = \left[\overbrace{e_1 \ \dots \ e_1}^{\alpha \text{ times}} \ \overbrace{e_2 \ \dots \ e_2}^{\beta \text{ times}} \ \overbrace{e_3 \ \dots \ e_3}^{\gamma \text{ times}} \right]$, for general $\alpha, \beta, \gamma \in \mathbb{Z}_+$ and with e_1, e_2 and e_3 as above, and explicit formulas for the mesh lines (which divide the support region of the corresponding box spline into pieces on which the box spline is polynomial) are given, with, furthermore, results on the continuity of such a box spline across these mesh lines.

It is not the idea to have that sort of discussion of a box spline's support here, or even to restrict ourselves to just box splines. Rather, we study the support of general 2-refinable functions, with the box splines as a special case, as these supports will prove important in the next chapter on subdivision analysis.

Before focussing on 2-refinable functions, however, we first discuss a few results on their corresponding 2-refinement mask sequences, during which time some useful notation will also be introduced.

6.1 The support of a refinement mask

Recall from Theorem 4.2 that the roof function corresponding to an arbitrary 2×2 direction matrix $\mathcal{D}$ is 2-refinable, with non-zero mask coefficients at the extreme values, or corners, of its parallelogram-shaped support region. This phenomenon extends to the case when more columns are introduced to $\mathcal{D}$, in the sense that the support region of its corresponding box spline seems to remain enclosed within the boundaries of its non-zero 2-refinement mask coefficients (see e.g. Figure 6.1, where the support regions of the Courant hat function and the ZP element are shown, with their respective 2-refinement mask coefficients illustrated by red markings). We shall make this notion precise in this chapter, where we deal with 2-refinable functions of which the mask coefficients satisfy certain properties, with box splines as a special case.

First, since a 2-refinement pair involves a mask sequence that is finitely supported, we shall start our discussion by introducing some useful notation for the boundaries of the support of such a sequence. Given a 2-refinement mask sequence $\{p_{i,j}\} \in M_0(\mathbb{Z}^2)$, it is possible to view the entries of $\{p_{i,j}\}$ as values at the coordinate points on the xy-plane, where $p_{i,j}$ denotes the value at the coordinate point $(i,j) \in \mathbb{Z}^2 \subset \mathbb{R}^2$. Then, since such a mask is finitely supported, it is not possible for any $y \in \mathbb{Z}^2$ to have infinitely many non-zero corresponding x-values. In other words, looking at the xy-plane, none of the *horizontal rows* can possibly have infinitely many non-zero values. Hence, looking at any such row, say the row $y = \mu$, it should, if there are any non-zero mask entries $p_{i,\mu}$ to be found at all in that row, be possible to find the first and the last non-zero one and their corresponding x-coordinates. These coordinate values will be denoted, respectively, by $i_{1,\mu}$ and $i_{2,\mu}$, i.e. $p_{i_{1,\mu},\mu}$ and $p_{i_{2,\mu},\mu}$ are the first and last non-zero mask entries of the row $y = \mu$, respectively. Note that it is possible to have $i_{1,\mu} = i_{2,\mu}$, which is the case when the row $y = \mu$ has only one non-zero mask entry. The notation just explained is made precise in the following definition:

Definition: For a 2-refinement pair (p, ϕ), we denote by

$$\{i_{1,\mu_1}, i_{1,\mu_1+1}, \ldots, i_{1,\mu_2}, i_{2,\mu_1}, i_{2,\mu_1+1}, \ldots, i_{2,\mu_2}\} \subset \mathbb{Z},$$

where $\mu_1, \mu_2 \in \mathbb{Z},\ \mu_1 \leq \mu_2$, and where

$$i_{1,\mu_1} \leq i_{2,\mu_1},\ i_{1,\mu_1+1} \leq i_{2,\mu_1+1},\ \ldots,\ i_{1,\mu_2} \leq i_{2,\mu_2},$$

the integers, if they exist, such that

$$\begin{aligned}
&p_{i,\mu_1} = 0, && i \notin \{i_{1,\mu_1}, i_{1,\mu_1}+1, \ldots, i_{2,\mu_1}\}, \quad p_{i_{1,\mu_1},\mu_1} \neq 0 \neq p_{i_{2,\mu_1},\mu_1};\\
&p_{i,\mu_1+1} = 0, && i \notin \{i_{1,\mu_1+1}, i_{1,\mu_1+1}+1, \ldots, i_{2,\mu_1+1}\}, \quad p_{i_{1,\mu_1+1},\mu_1+1} \neq 0 \neq p_{i_{2,\mu_1+1},\mu_1+1};\\
&\quad\vdots\\
&p_{i,\mu_2} = 0, && i \notin \{i_{1,\mu_2}, i_{1,\mu_2}+1, \ldots, i_{2,\mu_2}\}, \quad p_{i_{1,\mu_2},\mu_2} \neq 0 \neq p_{i_{2,\mu_2},\mu_2};\\
&p_{i,j} = 0, && i \in \mathbb{Z},\ j \notin \{\mu_1, \mu_1+1, \ldots, \mu_2\}.
\end{aligned} \tag{6.1}$$

Note: The integers $i_{1,\mu}, i_{2,\mu}$ need not, for a given 2-refinement pair, exist for every $\mu = \mu_1, \mu_1+1, \ldots, \mu_2$. The important point is that, for a given row $y = \mu$ on the xy-plane, if there are any non-zero mask values in that row, then the first and the last one are denoted, respectively, by $p_{i_{1,\mu},\mu}$ and $p_{i_{2,\mu},\mu}$.

Take as an example the box spline B corresponding to $\mathcal{D} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, for which the 2-refinement coefficients are given by (4.6) as

$$p_{0,0} = p_{1,1} = p_{2,3} = p_{3,4} = 1,$$

$$p_{k,l} = 0, \quad (k,l) \in \mathbb{Z}^2 \backslash \{(0,0),(1,1),(2,3),(3,4)\}.$$

Here, $\mu_1 = 0,\ \mu_2 = 4$, and $i_{1,\mu_1} = i_{2,\mu_1} = 0$, $i_{1,\mu_1+1} = i_{2,\mu_1+1} = 1, i_{1,\mu_1+3} = i_{2,\mu_1+3} = 2$, $i_{1,\mu_2} = i_{2,\mu_2} = 3$, while the numbers i_{1,μ_1+2} and i_{2,μ_1+2} do not exist. This example is illustrated in Figure 6.2.

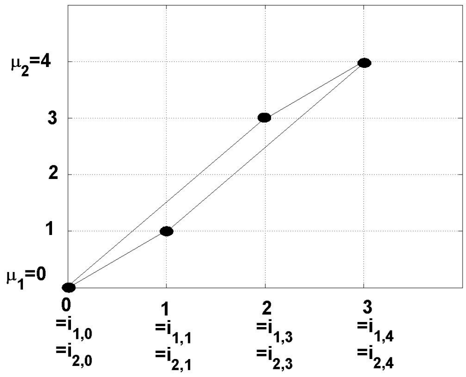


Figure 6.2: Illustration of the integers μ_1, μ_2 and $i_{1,\mu}, i_{2,\mu},\ \mu = \mu_1, \ldots, \mu_2$

Lemma 6.1 $\{$*Mask shifts*$\}$

Suppose (p, ϕ) is a 2-refinement pair, and denote by μ_1, μ_2 and by $i_{1,\mu_1}, i_{1,\mu_1+1}, \ldots, i_{1,\mu_2}$, $i_{2,\mu_1}, i_{2,\mu_1+1}, \ldots, i_{2,\mu_2}$ the integers, if they exist, such that (6.1) holds.
Let $(\rho_1, \rho_2) \in \mathbb{Z}^2$.
The matrix $\tilde{p} \in M_0(\mathbb{Z}^2)$ defined by

$$\tilde{p}_{i,j} = p_{i-\rho_1, j-\rho_2}, \qquad (i,j) \in \mathbb{Z}^2, \tag{6.2}$$

satisfies

$$\begin{aligned}
\tilde{p}_{i,\mu_1+\rho_2} = 0, \qquad & i \notin \{i_{1,\mu_1} + \rho_1, i_{1,\mu_1} + 1 + \rho_1, \ldots, i_{2,\mu_1} + \rho_1\}, \\
& \tilde{p}_{i_{1,\mu_1}+\rho_1, \mu_1+\rho_2} \neq 0 \neq \tilde{p}_{i_{2,\mu_1}+\rho_1, \mu_1+\rho_2}; \\
\tilde{p}_{i,\mu_1+1+\rho_2} = 0, \quad & i \notin \{i_{1,\mu_1+1} + \rho_1, i_{1,\mu_1+1} + 1 + \rho_1, \ldots, i_{2,\mu_1+1} + \rho_1\}, \\
& \tilde{p}_{i_{1,\mu_1+1}+\rho_1, \mu_1+1+\rho_2} \neq 0 \neq \tilde{p}_{i_{2,\mu_1+1}+\rho_1, \mu_1+1+\rho_2}; \\
\vdots \\
\tilde{p}_{i,\mu_2+\rho_2} = 0, \qquad & i \notin \{i_{1,\mu_2} + \rho_1, i_{1,\mu_2} + 1 + \rho_1, \ldots, i_{2,\mu_2} + \rho_1\}, \\
& \tilde{p}_{i_{1,\mu_2}+\rho_1, \mu_2+\rho_2} \neq 0 \neq \tilde{p}_{i_{2,\mu_2}+\rho_1, \mu_2+\rho_2};
\end{aligned} \tag{6.3}$$

$$\tilde{p}_{i,j} = 0, \; i \in \mathbb{Z}, \; j \notin \{\mu_1 + \rho_2, \ldots, \mu_2 + \rho_2\}. \tag{6.4}$$

Moreover, if we define the function $\tilde{\phi} \in M_0(\mathbb{R}^2)$ by

$$\tilde{\phi}(x,y) = \phi(x - \rho_1, y - \rho_2), \qquad (x,y) \in \mathbb{R}^2,$$

then $(\tilde{p}, \tilde{\phi})$ is a 2-refinement pair.

Proof

The fact that equations (6.3) and (6.4) hold, follows directly from equations (6.1) and (6.2).

Further,

$$\begin{aligned}
\sum_{i,j} \tilde{p}_{i,j}\tilde{\phi}(2x - i, 2y - j) &= \sum_{i,j} p_{i-\rho_1, j-\rho_2}\phi(2x - i - \rho_1, 2y - j - \rho_2) \\
&= \sum_{i,j} p_{i,j}\phi(2(x - \rho_1) - i, 2(y - \rho_2) - j) \\
&= \phi(x - \rho_1, y - \rho_2) \\
&= \tilde{\phi}(x,y), \qquad (x,y) \in \mathbb{R}^2.
\end{aligned}$$

Thus, $(\tilde{p}, \tilde{\phi})$ is a 2-refinement pair. □

It follows from Lemma 6.1 that an index shift in a 2-refinement mask produces precisely the same index shift ($= (\rho_1, \rho_2)$) in the corresponding 2-refinable function.

- Choose $\rho_1 = 0$ and $\rho_2 = -\mu_1$, and let $l = \mu_2 - \mu_1$.

  It then follows from (6.1) and Lemma 6.1 that we may, w$l$og, assume that

$$\begin{aligned}
&p_{i,0} = 0 \quad i \notin \{i_{1,0}, i_{1,0}+1, \ldots, i_{2,0}\}, \quad p_{i_{1,0},0} \neq 0 \neq p_{i_{2,0},0}; \\
&p_{i,1} = 0 \quad i \notin \{i_{1,1}, i_{1,1}+1, \ldots, i_{2,1}\}, \quad p_{i_{1,1},1} \neq 0 \neq p_{i_{2,1},1}; \\
&\quad\vdots \\
&p_{i,l} = 0 \quad i \notin \{i_{1,l}, i_{1,l}+1, \ldots, i_{2,l}\}, \quad p_{i_{1,l},0} \neq 0 \neq p_{i_{2,l},l}; \\
&p_{i,j} = 0, \;\; i \in \mathbb{Z}, \; j \notin \{0, 1, \ldots, l\}.
\end{aligned} \tag{6.5}$$

- It also follows that, if we choose $\rho_1 = -\min\{i_{1,0}, i_{1,1}, \ldots, i_{1,l}\}$, and $\rho_2 = -\mu_1$ as before, and if we define $k, l \in \mathbb{Z}$ by $k = \max\{i_{2,0} + \rho_1, i_{2,1} + \rho_1, \ldots, i_{2,l} + \rho_1\}$, $l = \mu_2 - \mu_1$, then

$$p_{i,j} = 0, \quad i \notin \{0, 1, \ldots, k\}, \quad j \notin \{0, 1, \ldots, l\}. \tag{6.6}$$

Moreover,

$$p_{0,j} \neq 0 \text{ for some } j \in \{0, 1, \ldots, l\}, \tag{6.7}$$

$$p_{k,j} \neq 0 \text{ for some } j \in \{0, 1, \ldots, l\}, \tag{6.8}$$

$$p_{i,0} \neq 0 \text{ for some } i \in \{0, 1, \ldots, k\}, \tag{6.9}$$

$$p_{i,l} \neq 0 \text{ for some } i \in \{0, 1, \ldots, k\}. \tag{6.10}$$

6.2 The support of a refinable function

We are now ready to discuss the support of a 2-refinable function in terms of its 2-refinement mask. As mentioned in Section 6.1, it follows from Theorem 4.2 that the roof function corresponding to any direction matrix $\mathcal{D}$ is always 2-refinable and its mask coefficients are zero everywhere except at the corners of the parallelogram-shaped support region of the function. We generalize this phenomenon in this section, and, since we already know that this property holds for roof functions, which are piecewise continuous, we henceforth assume that the refinable functions we are going to work with will be in $C(\mathbb{R}^2)$. We have the following preliminary result, on which we shall also strongly rely in Chapter 7.

Theorem 6.1

For $k \in \mathbb{N},\ l \in \mathbb{Z}_+$, suppose that (p, ϕ), $\phi \in C_0(\mathbb{R}^2)$, is a 2-refinement pair such that (6.6) holds, and with, also, $p_{i,j} \geq 0,\ (i,j) \in \mathbb{Z}^2$, and such that $\phi(x,y) \geq 0,\ (x,y) \in \mathbb{R}^2$. Then, *if we define*

$$\alpha_y := \min\{x \in \mathbb{R} : \phi(x,y) \neq 0\}, \quad y \in \mathbb{R}, \tag{6.11}$$

$$\beta_y := \max\{x \in \mathbb{R} : \phi(x,y) \neq 0\}, \quad y \in \mathbb{R}, \tag{6.12}$$

$$\gamma_x := \min\{y \in \mathbb{R} : \phi(x,y) \neq 0\}, \quad x \in \mathbb{R}, \tag{6.13}$$

$$\delta_x := \max\{y \in \mathbb{R} : \phi(x,y) \neq 0\}, \quad x \in \mathbb{R}, \tag{6.14}$$

it holds that

$$\alpha := \min_{y \in \mathbb{R}}\{\alpha_y\} = 0, \tag{6.15}$$

$$\beta := \max_{y \in \mathbb{R}}\{\beta_y\} = k, \tag{6.16}$$

$$\gamma := \min_{x \in \mathbb{R}}\{\gamma_x\} = 0, \tag{6.17}$$

$$\delta := \max_{x \in \mathbb{R}}\{\delta_x\} = l, \tag{6.18}$$

and thus also

$$\phi(x,y) = 0, \quad (x,y) \notin [0,k] \times [0,l]. \tag{6.19}$$

Proof

Since $\phi \in M_0(\mathbb{R}^2)$, we have $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

It follows from (6.6), and the 2-refinement equation, that

$$\phi(x,y) = \sum_{i=0}^{k}\sum_{j=0}^{l} p_{i,j}\phi(2x-i, 2y-j), \quad (x,y) \in \mathbb{R}^2.$$

Proof of (6.15):

It follows from (6.11) and (6.15) that

$$\min\{x \in \mathbb{R} : \phi(x,y) \neq 0\} = \alpha_y \geq \alpha, \quad y \in \mathbb{R},$$

i.e.

$$\phi(x,y) = 0, \quad (x,y) \in \left\{(x,y) \in \mathbb{R}^2 : \; x < \alpha \text{ and } y \in \mathbb{R}\right\}. \tag{6.20}$$

Let $\tilde{y} \in \mathbb{R}$ be the specific value such that $\alpha = \alpha_{\tilde{y}}$, and suppose first that $\alpha < 0$.

Then there exists a number $x_0 \in [\alpha, \min\{\beta_{\tilde{y}}, \frac{\alpha}{2}\})$ such that $\phi(x_0, \tilde{y}) > 0$, by definition of x_0 and $\alpha_{\tilde{y}}$.

But, for any $y \in \mathbb{R}$, we have

$$\begin{aligned} \phi(x_0, y) &= \sum_{i=0}^{k}\sum_{j=0}^{l} p_{i,j}\phi(2x_0 - i, 2y - j) \\ &= 0, \end{aligned}$$

by virtue of (6.20), since $2x_0 - i < \alpha - i \leq \alpha$ for all $i \geq 0$.

In particular, $\phi(x_0, \tilde{y}) = 0$, a contradiction. Therefore $\alpha \not< 0$.

Suppose next that $\alpha > 0$. Then, with $\tilde{y}$ fixed as before, there exists a number $x_1 \in [\frac{\alpha}{2}, \min\{\frac{\beta_{\tilde{y}}}{2}, \frac{\alpha+1}{2}, \alpha\})$ such that, for any $y \in \mathbb{R}$, $\phi(x_1, y) = 0$, by virtue of (6.20), since $x_1 < \alpha$.

Now, choose any $y_1 \in \mathbb{R}$ such that y_1 can be written as $y_1 = \frac{\tilde{y}+j'}{2}$, for some integer $j' \in \{0, 1, \ldots, l\}$, and such that, also, $p_{0,j'} \neq 0$, as guaranteed possible by (6.7). Then

$$\begin{aligned} \phi(x_1, y_1) &= \sum_{i=0}^{k}\sum_{j=0}^{l} p_{i,j}\phi(2x_1 - i, 2y_1 - j) \\ &= \sum_{j=0}^{l} p_{0,j}\phi(2x_1, 2y_1 - j) \\ &= p_{0,j'}\phi(2x_1, \tilde{y}) \ + \sum_{j \in \{0,\ldots,l\}, j \neq j'} p_{0,j}\phi(2x_1, 2y_1 - j) \\ &> 0, \end{aligned}$$

by virtue of $\alpha = \alpha_{\tilde{y}}$, and since $2x_1 \in [\alpha, \min\{\beta_{\tilde{y}}, \alpha + 1\})$, and after also making use of $p_{i,j} \geq 0, (i,j) \in \mathbb{Z}^2$ and $\phi(x,y) \geq 0, (x,y) \in \mathbb{R}^2$. This is another contradiction, and hence $\alpha \not> 0$. Therefore we have $\alpha = 0$, as desired, so that (6.15) holds.

<u>Proof of (6.16):</u>

It follows from (6.12) and (6.16) that

$$\max\{x \in \mathbb{R} : \phi(x,y) \neq 0\} = \beta_y \leq \beta, \quad y \in \mathbb{R},$$

i.e.

$$\phi(x,y)=0, \quad (x,y)\in\left\{(x,y)\in\mathbb{R}^2:\ x>\beta \text{ and } y\in\mathbb{R}\right\}. \tag{6.21}$$

Let $\hat{y}\in\mathbb{R}$ be the specific value such that $\beta=\beta_{\hat{y}}$, and suppose first that $\beta>k$.

Then there exists a number $\tilde{x}_0\in(\max\{\frac{k+\beta}{2},\alpha_{\hat{y}}\},\beta]$ such that $\phi(\tilde{x}_0,\hat{y})>0$, by definition of $\tilde{x}_0$, $\alpha_{\hat{y}}$ and $\beta_{\hat{y}}$.

Now, for $i\in\{0,1,\ldots,k\}$, we have $2\tilde{x}_0-i\geq 2\tilde{x}_0-k>(k+\beta)-k=\beta$, so that, by virtue of (6.21), we have, for any $y\in\mathbb{R}$,

$$\begin{aligned}\phi(\tilde{x}_0,y) &= \sum_{i=0}^{k}\sum_{j=0}^{l}p_{i,j}\phi(2\tilde{x}_0-i,2y-j)\\ &= 0.\end{aligned}$$

In particular, we have $\phi(\tilde{x}_0,\hat{y})=0$, a contradiction. Hence, $\beta\not>k$.

Suppose next that $\beta<k$. Then, with $\hat{y}$ fixed as before, there exists a number $\tilde{x}_1\in\left(\max\{\frac{k+\alpha_{\hat{y}}}{2},\frac{k+\beta-1}{2},\beta\},\frac{k+\beta}{2}\right]$ such that, for any $y\in\mathbb{R}$, $\phi(\tilde{x}_1,y)=0$, by virtue of (6.21), since $\tilde{x}_1>\beta$.

Now, with $\hat{y}$ fixed as before, choose any $y_2\in\mathbb{R}$ such that y_2 can be written as $y_2=\frac{\hat{y}+j'}{2}$, for some $j'\in\{0,1,\ldots,l\}$, and such that, also, $p_{k,j'}\neq 0$, as guaranteed possible by (6.8).

Then

$$\begin{aligned}\phi(\tilde{x}_1,y_2) &= \sum_{i=0}^{k}\sum_{j=0}^{l}p_{i,j}\phi(2\tilde{x}_1-i,2y_2-j)\\ &= \sum_{j=0}^{l}p_{k,j}\phi(2\tilde{x}_1-k,2y_2-j)\\ &= p_{k,j'}\phi(2\tilde{x}_1-k,\hat{y})+\sum_{j\in\{0,\ldots,l\},j\neq j'}p_{k,j}\phi(2\tilde{x}_1-k,2y_2-j)\\ &> 0,\end{aligned}$$

by virtue of $\beta=\beta_{\hat{y}}$, and since $2\tilde{x}_1-k\in(\max\{\alpha_{\hat{y}},\beta-1\},\beta]$, and after again using the fact that $p_{i,j}\geq 0,(i,j)\in\mathbb{Z}^2$. This is another contradiction, and hence $\beta\not<k$. Therefore $\beta=k$, as desired, i.e. (6.16) holds.

The proofs for equations (6.17) and (6.18) are similar. $\square$

Corollary 6.1 $\{$*Support of* $\phi\}$

For $l \in \mathbb{Z}_+$ *and a 2-refinement pair* $(p, \phi), \phi \in C_0(\mathbb{R}^2)$, *let* $\{i_{1,0}, i_{1,1}, \ldots, i_{1,l}, i_{2,0}, i_{2,1}, \ldots, i_{2,l}\} \subset \mathbb{Z}$ *denote the integers, if they exist, such that (6.5) holds, and with, also,* $p_{i,j} \geq 0, (i,j) \in \mathbb{Z}^2$ *and* $\phi(x,y) \geq 0, (x,y) \in \mathbb{R}^2$. *Then*

$$\alpha := \min_{y \in \mathbb{R}} \{\min\{x \in \mathbb{R} : \phi(x,y) \neq 0\}\} = \min\{i_{1,0}, i_{1,1}, \ldots, i_{1,l}\}, \tag{6.22}$$

$$\beta := \max_{y \in \mathbb{R}} \{\max\{x \in \mathbb{R} : \phi(x,y) \neq 0\}\} = \max\{i_{2,0}, i_{2,1}, \ldots, i_{2,l}\}, \tag{6.23}$$

$$\gamma := \min_{x \in \mathbb{R}} \{\min\{y \in \mathbb{R} : \phi(x,y) \neq 0\}\} = 0, \tag{6.24}$$

$$\delta := \max_{x \in \mathbb{R}} \{\max\{y \in \mathbb{R} : \phi(x,y) \neq 0\}\} = l. \tag{6.25}$$

Proof

The result follows by a direct application of Lemma 6.1 and Theorem 6.1 above, and by definition of l. □

Example: The support as well as the non-zero mask entries corresponding to the Zwart-Powell function B_{ZP} were shown in Figure 6.1. It follows from (3.3) and Lemma 4.3 that $(p := p_{ZP}, B_{ZP})$ is 2-refinable with 2-refinement mask symbol given, according to (4.9) and (3.6), by

$$\begin{aligned} P_{ZP}(z_1, z_2) &= \left(\frac{1 + z_1^1 z_2^1}{2}\right)\left(\frac{1 + z_1^{-1} z_2^1}{2}\right)(1 + z_1 + z_2 + z_1 z_2) \\ &= \frac{1}{4}\Big(1 + z_1 + 2z_2 + 2z_1 z_2 + z_1^{-1} z_2 + z_1^{-1} z_2^2 + 2z_2^2 + z_1^2 z_2 \\ &\qquad +2z_1 z_2^2 + z_1^2 z_2^2 + z_2^3 + z_1 z_2^3\Big), \end{aligned}$$

i.e.

$$p_{0,0} = p_{1,0} = p_{-1,1} = p_{2,1} = p_{-1,2} = p_{2,2} = p_{0,3} = p_{1,3} = \frac{1}{4};$$

$$p_{0,1} = p_{1,1} = p_{0,2} = p_{1,2} = \frac{1}{2};$$

$$p_{i,j} = 0, \text{ all other } (i,j) \in \mathbb{Z}^2.$$

Hence, $\mu_1 = 0$, $\mu_2 = 3$, and

$$i_{1,0} = 0, \quad i_{1,1} = i_{1,2} = -1, \quad i_{1,3} = 0,$$

$$i_{2,0} = 1, \quad i_{2,1} = i_{2,2} = 2, \quad i_{2,3} = 1,$$

i.e. $\min\{i_{1,0}, \ldots, i_{1,3}\} = -1$ and $\max\{i_{2,0}, \ldots, i_{2,3}\} = 2$. Indeed, it can be seen in Figures 2.6 and 6.1 that

$$\alpha = -1, \quad \beta = 2, \quad \gamma = 0, \quad \delta = 3,$$

i.e. (6.22), (6.23), (6.24) and (6.25) do indeed hold.

The next theorem investigates the effect on the support of a 2-refinable function if the corresponding mask has the property of excluding specific points in $\mathbb{Z}^2$. The motivation for this lies in the fact that, at least in the bivariate box spline case, the support of the mask always seems to have some "corner elements" missing, and, furthermore, the corresponding box spline only "lives" exactly inside of the convex hull of the non-zero mask elements. For example, the supports of the Courant hat function and the ZP element were shown in Figure 6.1. The red markings are those points in $\mathbb{Z}^2$ where the respective corresponding mask entries are non-zero.

Theorem 6.2

For $l \in \mathbb{Z}_+$ and a 2-refinement pair $(p, \phi), \phi \in C_0(\mathbb{R}^2)$, let $\{i_{1,0}, i_{1,1}, \ldots, i_{1,l}, i_{2,0}, i_{2,1}, \ldots, i_{2,l}\} \subset \mathbb{Z}$ denote the integers, if they exist, such that (6.5) holds, and with, also, $p_{i,j} \geq 0, (i,j) \in \mathbb{Z}^2$ and $\phi(x,y) \geq 0, (x,y) \in \mathbb{R}^2$, and with, moreover

$$i_{1,0} - 1 = i_{1,1} = i_{1,2} = i_{1,3} = \ldots = i_{1,l-1} = i_{1,l} - 1, \tag{6.26}$$

$$i_{2,0} + 1 = i_{2,1} = i_{2,2} = i_{2,3} = \ldots = i_{2,l-1} = i_{2,l} + 1, \tag{6.27}$$

and

$$\sup\{x : \phi(x,y) \neq 0\} - \inf\{x : \phi(x,y) \neq 0\} \geq 1, \quad y \in \mathbb{R}. \tag{6.28}$$

Then the following are true:

- $\alpha_y := \inf\{x : \phi(x,y) \neq 0\} = m_y, \quad y \in [0,1],$

  where $m_y = (i_{1,1} - i_{1,0})(y-1) + i_{1,1}$;

- $\beta_y := \sup\{x : \phi(x,y) \neq 0\} = n_y, \quad y \in [0,1],$

  where $n_y = (i_{2,1} - i_{2,0})(y-1) + i_{2,1}$;

- $\gamma_y := \inf\{x : \phi(x,y) \neq 0\} = p_y, \quad y \in [0,1],$

  where $p_y = (i_{1,l} - i_{1,l-1})(y-l) + i_{1,l}$;

- $\delta_y := \sup\{x : \phi(x,y) \neq 0\} = q_y, \quad y \in [0,1],$

  where $q_y = (i_{2,l} - i_{2,l-1})(y-l) + i_{2,l}$.

The notation and assertions are illustrated in Figure 6.3 .

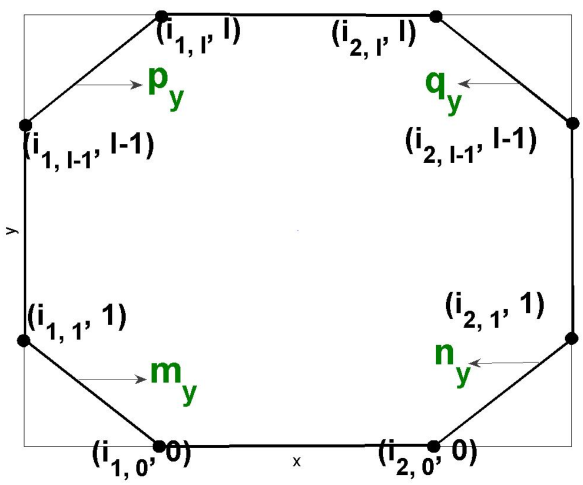


Figure 6.3: Illustration of Theorem 6.2

Proof

We only prove the first assertion, since the proofs for the other three assertions are similar. Note that it follows from Corollary 6.1 that

$$\alpha_y \geq i_{1,1}, \quad \beta_y \leq i_{2,1}, \quad y \in [0,1]$$

and

$$\gamma_y \geq i_{1,l-1}, \quad \delta_y \leq i_{2,l-1}, \quad y \in [l-1,l].$$

Suppose $\alpha_y < m_y$: **(and** $y \in [0,1]$ **is fixed — see Figure 6.4)**

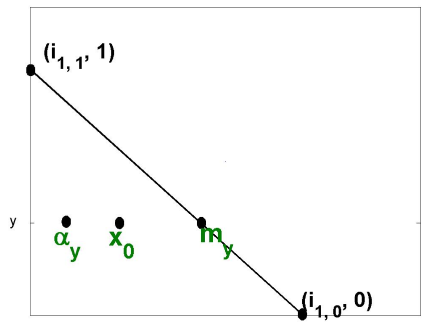


Figure 6.4: $\alpha_y < m_y$

Then there exists a number $x_0 \in \left[\alpha_y, \frac{\alpha_y+m_y}{2}\right)$ such that

$$\phi(x_0, y) \neq 0. \tag{6.29}$$

But, since $y \in [0,1)$, we have $\phi(x, 2y-m) = 0,\ m \geq 2,\ x \in \mathbb{R}$, by virtue of (6.5), (6.13) and (6.17), and it therefore follows from (6.5) and the 2-refinement equation that

$\phi(x_0, y)$

$$= \left[\overbrace{p_{i_{1,0},0}\phi(2x_0 - i_{1,0}, 2y) + p_{i_{1,0}+1,0}\phi(2x_0 - (i_{1,0}+1), 2y) + \ldots + p_{i_{2,0},0}\phi(2x_0 - i_{2,0}, 2y)}^{(A)}\right]$$
$$+ \left[\overbrace{p_{i_{1,1},1}\phi(2x_0 - i_{1,1}, 2y-1) + p_{i_{1,1}+1,1}\phi(2x_0 - (i_{1,1}+1), 2y-1) + \ldots + p_{i_{2,1},1}\phi(2x_0 - i_{2,1}, 2y-1)}^{(B)}\right]. \tag{6.30}$$

It will be shown that, for any fixed $y \in [0, 1]$, both (A) and (B) are equal to zero. This will be done, either directly, or by showing that both

$$2x_0 - i_{1,0} < \alpha_{2y} \tag{6.31}$$

(and hence $2x_0 - i < \alpha_{2y}$ for all $i \geq i_{1,0}$, so that $(A) = 0$, by definition of α_{2y}), and

$$2x_0 - i_{1,1} < \alpha_{2y-1} \tag{6.32}$$

(and hence $2x_0 - i < \alpha_{2y-1}$ for all $i \geq i_{1,1}$, so that $(B) = 0$, by definition of α_{2y-1}).

First, note the following:

(I) If $y \geq \frac{1}{2}$, then:

$$\begin{aligned}
y \geq \frac{1}{2} \quad &\Rightarrow \quad 2y - 1 \geq 0 \\
&\Rightarrow \quad (i_{1,0} - i_{1,1})(2y-1) \geq 0 \qquad (i_{1,0} > i_{1,1}) \\
&\Rightarrow \quad (i_{1,0} - i_{1,1})y + (i_{1,0} - i_{1,1})(y-1) \geq 0 \\
&\Rightarrow \quad (i_{1,0} - i_{1,1})y \geq (i_{1,1} - i_{1,0})(y-1) + (i_{1,1} - i_{1,1}) \\
&\Rightarrow \quad (i_{1,0} - i_{1,1})y \geq m_y - i_{1,1} \\
&\Rightarrow \quad (i_{1,0} - i_{1,1})y > \alpha_y - i_{1,1} \qquad (m_y > \alpha_y) \\
&\Rightarrow \quad (i_{1,1} - i_{1,0})y < i_{1,1} - \alpha_y \\
&\Rightarrow \quad (i_{1,1} - i_{1,0})(y-1) + i_{1,1} - i_{1,0} + \alpha_y < i_{1,1}
\end{aligned}$$

$$\Rightarrow \quad m_y + \alpha_y - i_{1,0} < i_{1,1}$$

$$\Rightarrow \quad 2x_0 - i_{1,0} < i_{1,1}. \qquad (2x_0 < m_y + \alpha_y)$$

Hence, if $y \in [\frac{1}{2}, 1]$, then it follows immediately that (6.31) holds, since $i_{1,1} \leq \alpha_{\tilde{y}}$ for every $\tilde{y} \in [0,1]$, by virtue of equation (6.26) and Corollary 6.1, and so it only remains to show that (6.32) holds.

(II) If $y \leq \frac{1}{2}$, then $2y-1 \leq 0$, so that $\phi(x, 2y-1) = 0,\ x \in \mathbb{R}$, by virtue of Corollary 6.1. In particular, $\phi(2x_0 - i, 2y-1) = 0$ for every $i \geq i_{1,1}$, so that (6.32) automatically holds. In this case, it is therefore only necessary to prove (6.31).

(III) For $y \in [0,1]$,

$$\begin{aligned} 2x_0 - i_{1,0} &< \alpha_y + m_y - i_{1,0} \\ &< 2m_y - i_{1,0} \qquad (\alpha_y < m_y) \\ &= (i_{1,1} - i_{1,0})(2y-2) + 2i_{1,1} - i_{1,0} \\ &= (i_{1,1} - i_{1,0})(2y-1) - i_{1,1} + i_{1,0} + 2i_{1,1} - i_{1,0} \\ &= (i_{1,1} - i_{1,0})((2y)-1) + i_{1,1} \\ &= m_{2y}. \end{aligned}$$

(IV) For $y \in [0,1]$,

$$\begin{aligned} 2x_0 - i_{1,1} &< \alpha_y + m_y - i_{1,1} \\ &< 2m_y - i_{1,1} \qquad (\alpha_y < m_y) \\ &= (i_{1,1} - i_{1,0})(2y-2) + 2i_{1,1} - i_{1,1} \\ &= (i_{1,1} - i_{1,0})((2y-1)-1) + i_{1,1} \\ &= m_{2y-1}. \end{aligned}$$

The fact that (A) and (B) are both equal to zero for every $y \in [0,1]$, can now be proved inductively on every dyadic level, as follows:

First, it is proved for $y = 0$ and $y = 1$,

then for $y = \frac{1}{2}$ (the first level),

then for $y = \frac{1}{4}$ and $y = \frac{3}{4}$ (the second level),

then for $y = \frac{1}{8}$, $y = \frac{3}{8}$, $y = \frac{5}{8}$ and $y = \frac{7}{8}$ (the third level), etc...

Since the dyadic numbers are dense in $\mathbb{R}$, it then follows that (A) and (B) are equal to zero for all $y \in [0, 1]$.

Note that, once it is clear that $(A) = (B) = 0$ for every $y \in [0, 1]$, then $\phi(x_0, y) = 0$ for every $y \in [0, 1]$, by virtue of equation (6.30), thereby contradicting (6.29). Therefore, it will have been proved that $m_y \leq \alpha_y$, as desired.

Induction: basis steps:

$\underline{y = 0}$: $2y = 0 \Rightarrow (A) = 0$ (Corollary 6.1)

$2y - 1 = -1 < 0 \Rightarrow (B) = 0$ (Corollary 6.1)

$(\Rightarrow m_y \leq \alpha_y$ for $y = 0)$

$\underline{y = 1}$: (6.31) holds, by (I).

Also,

$$
\begin{aligned}
2x_0 - i_{1,1} &< m_y + \alpha_y - i_{1,1} \\
&= (i_{1,1} - i_{1,0})(y - 1) + i_{1,1} + \alpha_y - i_{1,1} \\
&= \alpha_y \qquad (y = 1) \\
&= \alpha_{2y-1} \qquad (2y - 1 = 1 = y),
\end{aligned}
$$

so (6.32) holds.

$(\Rightarrow m_y \leq \alpha_y$ for $y = 1)$

1$^{\text{st}}$ level:

$\underline{y = \frac{1}{2}}$: $y \in [\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{2}]$, so (6.31) and (6.32) both hold, by virtue of (I) and (II).

($\Rightarrow m_y \leq \alpha_y$ for $y = \frac{1}{2}$)

Inductive step:

Let $r \in \mathbb{N}$ be fixed.

The "r-th level of dyadic numbers" consists of all the numbers

$$y_{r,j} = \frac{2j-1}{2^r}, \quad j = 1, 2, 3, \ldots, 2^{r-1}.$$

Suppose $\alpha_{y_{r,j}} \geq m_{y_{r,j}}, \quad j = 1, 2, 3, \ldots, 2^{r-1}$ (Induction Hypothesis). We must show that $\alpha_{y_{r+1,j}} \geq m_{y_{r+1,j}}, \quad j = 1, 2, 3, \ldots, 2^r$.

Let $\tilde{y} = y_{r+1,j}$, where $j \in \{1, 2, 3, \ldots, 2^r\}$ is fixed, and suppose first that $j \leq 2^{r-1}$. Then

$$\begin{aligned} & j < 2^{r-1} + \tfrac{1}{2} \\ \Rightarrow \quad & 2j - 1 < 2^r \\ \Rightarrow \quad & \tfrac{2j-1}{2^{r+1}} < \tfrac{1}{2} \\ \Rightarrow \quad & \tilde{y} < \tfrac{1}{2}, \end{aligned}$$

so that (6.32) holds, by (II). Also,

$$2\tilde{y} \;=\; 2y_{r+1,j} \;=\; 2\frac{2j-1}{2^{r+1}} \;=\; \frac{2j-1}{2^r} \;=\; y_{r,j}.$$

It subsequently follows that

$$\begin{aligned} 2x_0 - i_{1,0} &< m_{2\tilde{y}} && \text{(by (III))} \\ &= m_{y_{r,j}} \\ &\leq \alpha_{y_{r,j}} && \text{(Induction Hypothesis, since } j \in \{1, 2, \ldots, 2^{r-1}\}) \\ &= \alpha_{2\tilde{y}}, \end{aligned}$$

so that (6.31) holds. Hence, $\alpha_{y_{r+1,j}} \geq m_{y_{r+1,j}}, \quad j = 1, 2, \ldots, 2^{r-1}$.

Suppose next that $j \geq 2^{r-1} + 1$. Then, similarly as above, we have $\tilde{y} > \frac{1}{2}$, so that (6.31) holds, by virtue of (I). Also,

$$2\tilde{y}-1 \;=\; 2y_{r+1,j}-1 \;=\; 2\frac{2j-1}{2^{r+1}}-1 \;=\; \frac{2j-1}{2^r}-1 \;=\; \frac{2(j-2^{r-1})-1}{2^r} \;=\; y_{r,j'},$$

where $j' = j - 2^{r-1}$. Note that $j \in \{2^{r-1}+1, 2^{r-1}+2, \ldots, 2^r\}$ implies $j - 2^{r-1} \in \{1, 2, \ldots, 2^r - 2^{r-1}\}$, while $2^r - 2^{r-1} = 2^{r-1}(2-1) = 2^{r-1}$, i.e. $j' \in \{1, 2, \ldots, 2^{r-1}\}$. Therefore, it follows that

$$\begin{aligned} 2x_0 - i_{1,1} &< m_{2\tilde{y}-1} && \text{(by (IV))} \\ &= m_{y_{r,j'}} \\ &\leq \alpha_{y_{r,j'}} && \text{(Induction Hypothesis, since } j' \in \{1,2,\ldots,2^{r-1}\}) \\ &= \alpha_{2\tilde{y}-1}, \end{aligned}$$

so that (6.32) holds. Hence, $\alpha_{y_{r+1,j}} \geq m_{y_{r+1,j}}, \quad j = 2^{r-1}+1, 2^{r-1}+2, \ldots, 2^r$.
This completes the induction process, thereby yielding the desired result $m_y \leq \alpha_y$ for all $y \in [0,1]$.

Suppose $\alpha_y > m_y$: **(and $y \in [0,1]$ is fixed — see Figure 6.5)**

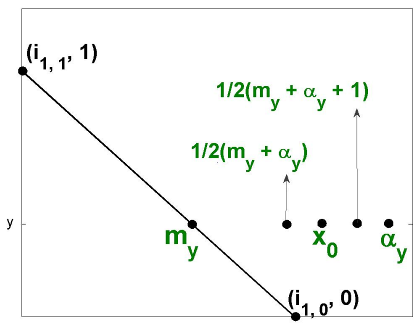


Figure 6.5: $\alpha_y > m_y$

Note that $\alpha_y - m_y < 1$ (see the note at the end of the proof), thus implying $\frac{2m_y+1}{2} > \frac{\alpha_y+m_y}{2}$, so that there exists a number $x_1 \in \left[\frac{\alpha_y+m_y}{2},\ \min\left\{\alpha_y, \frac{2m_y+1}{2}\right\}\right)$ such that

$$\phi(x_1, y) = 0. \tag{6.33}$$

But again, since $y \in [0,1)$, we have $\phi(x, 2y-m) = 0,\ m \geq 2,\ x \in \mathbb{R}$, by virtue of (6.5), (6.13) and (6.17), and it therefore follows from (6.5) and the 2-refinement equation that

$$\begin{aligned} \phi(x_1, y) = \; & \Big[p_{i_{1,0},0}\phi(2x_1 - i_{1,0}, 2y) + p_{i_{1,0}+1,0}\phi(2x_1 - (i_{1,0}+1), 2y) \\ & \quad + \ \ldots \ + \ p_{i_{2,0},0}\phi(2x_1 - i_{2,0}, 2y)\Big] \\ & + \Big[p_{i_{1,1},1}\phi(2x_1 - i_{1,1}, 2y-1) + p_{i_{1,1}+1,1}\phi(2x_1 - (i_{1,1}+1), 2y-1) \\ & \quad + \ \ldots \ + \ p_{i_{2,1},1}\phi(2x_1 - i_{2,1}, 2y-1)\Big]. \end{aligned} \tag{6.34}$$

It will be shown that, for any fixed $y \in [0,1]$, either

$$\alpha_{2y} \leq 2x_1 - i_{1,0} \leq \alpha_{2y} + 1, \tag{6.35}$$

(so that $\phi(2x_1 - i_{1,0}, 2y) \neq 0$, by definition of α_{2y}, and by virtue of (6.28)), or

$$\alpha_{2y-1} \leq 2x_1 - i_{1,1} \leq \alpha_{2y-1} + 1, \tag{6.36}$$

(so that $\phi(2x_1 - i_{1,1}, 2y-1) \neq 0$, by definition of α_{2y-1}, and by virtue of (6.28)).
Since the mask coefficients are assumed to be positive, while ϕ is nonnegative everywhere, so that no cancellation of the terms in (6.34) can take place, this implies that $\phi(x_1, y) \neq 0$, thereby contradicting (6.33), and hence implying that, for any such $y \in [0,1]$, $m_y \not< \alpha_y$, i.e. $m_y = \alpha_y$.
First, note the following:

(V) For $y \in [0,1]$,

$$\begin{aligned} 2x_1 - i_{1,0} &\geq \alpha_y + m_y - i_{1,0} \\ &> 2m_y - i_{1,0} \qquad\qquad (\alpha_y > m_y) \\ &= (i_{1,1} - i_{1,0})(2y-2) + 2i_{1,1} - i_{1,0} \\ &= (i_{1,1} - i_{1,0})(2y-1) - i_{1,1} + i_{1,0} + 2i_{1,1} - i_{1,0} \\ &= (i_{1,1} - i_{1,0})((2y)-1) + i_{1,1} \\ &= m_{2y}. \end{aligned}$$

(VI) For $y \in [0,1]$,

$$\begin{aligned} 2x_1 - i_{1,1} &\geq \alpha_y + m_y - i_{1,1} \\ &> 2m_y - i_{1,1} \qquad\qquad (\alpha_y > m_y) \\ &= (i_{1,1} - i_{1,0})(2y-2) + 2i_{1,1} - i_{1,1} \\ &= (i_{1,1} - i_{1,0})((2y-1)-1) + i_{1,1} \\ &= m_{2y-1}. \end{aligned}$$

(VII) For $y \in [0,1]$,

$$\begin{aligned} 2x_1 - i_{1,0} &< 2m_y + 1 - i_{1,0} \\ &= (i_{1,1} - i_{1,0})(2y-2) + 2i_{1,1} - i_{1,0} + 1 \\ &= (i_{1,1} - i_{1,0})(2y-1) - i_{1,1} + i_{1,0} + 2i_{1,1} - i_{1,0} + 1 \\ &= (i_{1,1} - i_{1,0})\left((2y)-1\right) + i_{1,1} + 1 \\ &= m_{2y} + 1. \end{aligned}$$

(VIII) For $y \in [0,1]$,

$$\begin{aligned} 2x_1 - i_{1,1} &< 2m_y + 1 - i_{1,1} \\ &= (i_{1,1} - i_{1,0})(2y-2) + 2i_{1,1} + 1 - i_{1,1} \\ &= (i_{1,1} - i_{1,0})\left((2y-1)-1\right) + i_{1,1} + 1 \\ &= m_{2y-1} + 1. \end{aligned}$$

The fact that either (6.35) or (6.36) holds for each $y \in [0,1]$, can now be proved inductively on every dyadic level, as before. Again, as the dyadic numbers are dense in $\mathbb{R}$, it will follow that it also holds for every $y \in [0,1]$.

Induction: basis steps:

$\underline{y=0}$:

$$\begin{aligned}2x_1 - i_{1,0} &\geq \alpha_y + m_y - i_{1,0}\\ &= \alpha_y + i_{1,0} - i_{1,0} \qquad (m_y = i_{1,0} \text{ for } y=0)\\ &= \alpha_y\\ &= \alpha_{2y} \qquad (2y = 0 = y);\end{aligned}$$

$$\begin{aligned}2x_1 - i_{1,0} &< 2m_y + 1 - i_{1,0}\\ &= 2m_y + 1 - m_y \qquad (m_y = i_{1,0} \text{ for } y=0)\\ &= m_y + 1\\ &< \alpha_y + 1 \qquad (m_y < \alpha_y)\\ &= \alpha_{2y} + 1 \qquad (2y = 0 = y).\end{aligned}$$

Hence, (6.35) holds.

($\Rightarrow m_y = \alpha_y$ for $y = 0$.)

$\underline{y=1}$:

$$\begin{aligned}2x_1 - i_{1,1} &\geq \alpha_y + m_y - i_{1,1}\\ &= \alpha_y + i_{1,1} - i_{1,1} \qquad (m_y = i_{1,1} \text{ for } y=1)\\ &= \alpha_y\\ &= \alpha_{2y-1} \qquad (2y-1 = 1 = y);\end{aligned}$$

$$\begin{aligned}2x_1 - i_{1,1} &< 2m_y + 1 - i_{1,1}\\ &= 2m_y + 1 - m_y \qquad (m_y = i_{1,1} \text{ for } y=1)\end{aligned}$$

$$\begin{aligned} &= m_y + 1 \\ &< \alpha_y + 1 \qquad (m_y < \alpha_y) \\ &= \alpha_{2y-1} + 1 \qquad (2y - 1 = 1 = y). \end{aligned}$$

Hence, (6.36) holds.

($\Rightarrow m_y = \alpha_y$ for $y = 1$.)

1st level:

$\underline{y = \frac{1}{2}}$:

$$\begin{aligned} 2x_1 - i_{1,0} &> m_{2y} \qquad \text{(by (V))} \\ &= \alpha_{2y} \qquad (2y = 1 \text{ and } m_y = \alpha_y \text{ for } y = 1); \end{aligned}$$

$$\begin{aligned} 2x_1 - i_{1,0} &< m_{2y} + 1 \qquad \text{(by (VII))} \\ &= \alpha_{2y} + 1 \qquad (2y = 1 \text{ and } m_y = \alpha_y \text{ for } y = 1). \end{aligned}$$

Hence, (6.35) holds.

($\Rightarrow m_y = \alpha_y$ for $y = \frac{1}{2}$.)

Inductive step:

Let $r \in \mathbb{N}$ be fixed.

The "r-th level of dyadic numbers" consists of all the numbers

$$y_{r,j} = \frac{2j-1}{2^r}, \quad j = 1, 2, 3, \ldots, 2^{r-1}.$$

Suppose $\alpha_{y_{r,j}} = m_{y_{r,j}}, \quad j = 1, 2, 3, \ldots, 2^{r-1}$. We must show that $\alpha_{y_{r+1,j}} = m_{y_{r+1,j}}, \quad j = 1, 2, 3, \ldots, 2^r$.

Let $\tilde{y} = y_{r+1,j}$, where $j \in \{1, 2, 3, \ldots, 2^r\}$ is fixed, and suppose first that $j \leq 2^{r-1}$ (so that $\tilde{y} < \frac{1}{2}$, as before).

As before,

$$2\tilde{y} = 2\frac{2j-1}{2^{r+1}} = \frac{2j-1}{2^r} = y_{r,j},$$

so that

$$\begin{aligned} 2x_1 - i_{1,0} &> m_{2\tilde{y}} && \text{(by (V))} \\ &= m_{y_{r,j}} \\ &= \alpha_{y_{r,j}} && \text{(Induction Hypothesis)} \\ &= \alpha_{2\tilde{y}} \end{aligned}$$

and

$$\begin{aligned} 2x_1 - i_{1,0} &< m_{2\tilde{y}} + 1 && \text{(by (VII))} \\ &= m_{y_{r,j}} + 1 \\ &= \alpha_{y_{r,j}} + 1 && \text{(Induction Hypothesis)} \\ &= \alpha_{2\tilde{y}} + 1. \end{aligned}$$

Hence, (6.35) holds, implying $\alpha_{y_{r+1,j}} = m_{y_{r+1,j}}, \quad j = 1, 2, 3, \ldots, 2^{r-1}$.

Suppose next that $j > 2^{r-1}$ (so that $\tilde{y} > \frac{1}{2}$, as before).

As before,

$$2\tilde{y} - 1 = 2\frac{2j-1}{2^{r+1}} - 1 = \frac{2j - 2^r - 1}{2^r} = \frac{2(j - 2^{r-1}) - 1}{2^r} = y_{r,j'},$$

where $j' = j - 2^{r-1}$. Note once again that, since $j \in \{2^{r-1}+1, 2^{r-1}+2, \ldots, 2^r\}$, we have $j' \in \{1, 2, 3, \ldots, 2^{r-1}\}$.

It follows that

$$\begin{aligned} 2x_1 - i_{1,1} &> m_{2\tilde{y}-1} && \text{(by (VI))} \\ &= m_{y_{r,j'}} \\ &= \alpha_{y_{r,j'}} && \text{(Induction Hypothesis)} \\ &= \alpha_{2\tilde{y}-1} \end{aligned}$$

and

$$\begin{aligned}2x_1 - i_{1,1} &< m_{2\tilde{y}-1} + 1 \qquad \text{(by (VIII))}\\ &= m_{y_{r,j'}} + 1\\ &= \alpha_{y_{r,j'}} + 1 \qquad \text{(Induction Hypothesis)}\\ &= \alpha_{2\tilde{y}-1} + 1.\end{aligned}$$

Hence, (6.36) holds. It therefore follows that $\alpha_{y_{r+1,j}} = m_{y_{r+1,j}}, \quad j = 2^{r-1}+1, 2^{r-1}+2, \ldots, 2^r$.

This completes the proof by induction.

NOTE: It still remains to prove that $\alpha_y - m_y \leq 1$. Define the number $\alpha \in \mathbb{R}$ by

$$\alpha := \sup\{\alpha_y : y \in [0,1)\},$$

(see Figure 6.6), i.e. $\alpha_y \leq \alpha,\ y \in [0,1)$, or equivalently, by virtue also of (6.11),

$$\phi(x,y) \neq 0, \quad (x,y) \in \left\{(x,y) \in \mathbb{R}^2 : x > \alpha \text{ and } y \in [0,1)\right\}. \tag{6.37}$$

Suppose $\alpha > i_{1,0}$, and let $\tilde{y} \in [0,1)$ by the specific value such that $\alpha = \alpha_{\tilde{y}}$. Then there exists a number $x_0 \in \left[\frac{i_{1,0}+\alpha}{2}, \min\left\{\alpha, \frac{i_{1,0}+\alpha+1}{2}\right\}\right)$ such that

$$\phi(x_0, \tilde{y}) = 0, \tag{6.38}$$

by definition of x_0 and $\alpha_{\tilde{y}}$.

But, for any $y \in [0,1)$, we have $\phi(x, 2y-m) = 0,\ m \geq 2,\ x \in \mathbb{R}$, by virtue of (6.5), (6.11) and (6.15), and so the 2-refinement equation becomes, for any $y \in [0,1)$,

$$\begin{aligned}\phi(x_0, y) &= \sum_{k,l} p_{k,l}\phi(2x_0 - k, 2y - l)\\ &= \sum_k p_{k,0}\phi(2x_0 - k, 2y) + \sum_k p_{k,1}\phi(2x_0 - k, 2y - 1)\end{aligned}$$

$$= \left[p_{i_{1,1},0} \overbrace{\phi(2x_0 - i_{1,1}, 2y)}^{A} + p_{i_{1,1}+1,0} \overbrace{\phi(2x_0 - i_{1,1} - 1, 2y)}^{B} \right] + \left[p_{i_{1,0},1} \overbrace{\phi(2x_0 - i_{1,0}, 2y - 1)}^{C} \right].$$

At this stage, one of the following two cases holds:

$2y \in [0,1)$: Since $i_{1,1} = i_{1,0} - 1$, it follows that

$$\begin{aligned} 2x_0 - i_{1,1} &= 2x_0 - (i_{1,0} - 1) \\ &\in [i_{1,0} + \alpha - (i_{1,0} - 1), i_{1,0} + \alpha + 1 - (i_{1,0} - 1)) \\ &= [\alpha + 1, \alpha + 2), \end{aligned}$$

i.e. $2x_0 - i_{1,1} - 1 \in [\alpha, \alpha + 1)$, so that, by virtue of (6.37) at least one of A or B is not zero.

$2y - 1 \in [0,1)$: Since $2x_0 - i_{1,0}$, it also follows from (6.37) that C is not zero.

In both cases, it thus follows that $\phi(x_0, y) \neq 0$ for any $y \in [0,1)$. In particular, $\phi(x_0, \tilde{y}) \neq 0$, thereby contradicting (6.38). Hence, $\alpha \leq i_{1,0}$.
Since $m_y \in [i_{1,1}, i_{1,0}]$, $y \in [0,1)$, and since $i_{1,0} - i_{1,1} = 1$, it is therefore concluded that $\alpha_y - m_y \leq 1$, as desired. □

If the boundary entries (i.e. the first and last non-zero entries of each row) of a 2-refinement mask sequence satisfy the conditions in Theorem 6.2, then those conditions imply that they also form a convex hull, the area of which, according to Theorem 6.2 has similarities to the support region of the corresponding 2-refinable function. We suspect that the result in Theorem 6.2 can be generalized for more elements to be taken out of the 2-refinement mask, provided that the boundary entries still form a convex hull in the xy-plane. For example, it seems reasonable to expect, when working with a 2-refinement

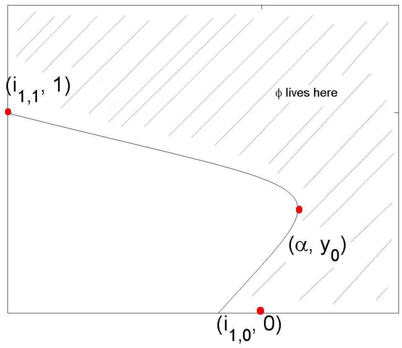


Figure 6.6: Illustration of the proof that $\alpha_y - m_y \leq 1$

mask of, say, $p = \begin{bmatrix} 0 & 0 & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} \\ 0 & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} \\ 0 & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & 0 \\ \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & 0 \\ \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & 0 & 0 \end{bmatrix}$, where $p_{j,i} = \mathrm{x}$ in the matrix denotes a non-zero entry in the sequence $(p_{i,j})$ (see the red markings in Figure 6.7) and all the sequence values that do not appear in the mask are understood to be zero, that the support of the corresponding 2-refinable function will not go outside the convex hull of these markings. We do not attempt to prove this here, but leave it as an issue that can be investigated in the future.

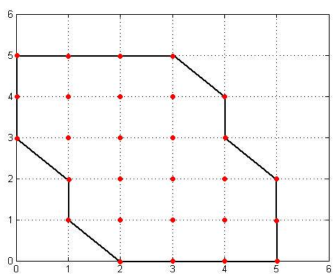


Figure 6.7: Illustration of the support of a 2-refinable function with symmetric mask sequence

In all of the main results in this chapter, it was assumed that $\phi(x,y) \geq 0, (x,y) \in \mathbb{R}^2$. We see no reason why the results would not also hold for functions that would be allowed to take on negative values, but leave this as an issue that can be further investigated in the future.

We end this chapter with another example that illustrates the result in Theorem 6.2.

Example: Consider the box spline B corresponding to the direction matrix $\mathcal{D} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. It follows from (2.7) and (2.6) that B is given by

$$B(x,y) = \begin{cases} x+y & , \quad x \leq 0, y \leq 1, y \geq -x \\ x+1 & , \quad x \geq -1, y \geq 1, y \leq 1-x \\ 2-y & , \quad x \leq 0, y \leq 2, y \geq 1-x \\ y & , \quad x \geq 0, y \geq 0, y \leq 1-x \\ 1-x & , \quad x \leq 1, y \leq 1, y \geq 1-x \\ 2-x-y & , \quad x \geq 0, y \geq 1, y \leq 2-x \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{6.39}$$

It also follows from (3.3), (4.9) and (4.10) that (p, B) is a 2-refinement pair, with corresponding mask symbol given, according to (3.6) and (4.9), by

$$\begin{aligned} P(z_1, z_2) &= \left(\frac{1 + z_1^{-1} z_2^1}{2}\right)(1 + z_1 + z_2 + z_1 z_2) \\ &= \frac{1}{2} + \frac{1}{2} z_1 + z_2 + \frac{1}{2} z_1 z_2 + \frac{1}{2} z_1^{-1} z_2 + \frac{1}{2} z_1^{-1} z_2^2 + \frac{1}{2} z_2^2, \end{aligned}$$

i.e.

$$p_{0,0} = p_{1,0} = p_{1,1} = p_{-1,1} = p_{-1,2} = p_{0,2} = \frac{1}{2};$$

$$p_{0,1} = 1;$$

$$p_{i,j} = 0, \quad (i,j) \in \mathbb{Z}^2 \backslash \{(0,0), (1,0), (1,1), (-1,1), (-1,2), (0,1), (0,1)\}. \tag{6.40}$$

It furthermore follows by a result similar to Lemma 6.1 that $(\tilde{p}, \tilde{B})$ is also a 2-refinement pair, where

$$\tilde{B}(x,y) = B(x-1,y), \quad (x,y) \in \mathbb{R}^2,$$

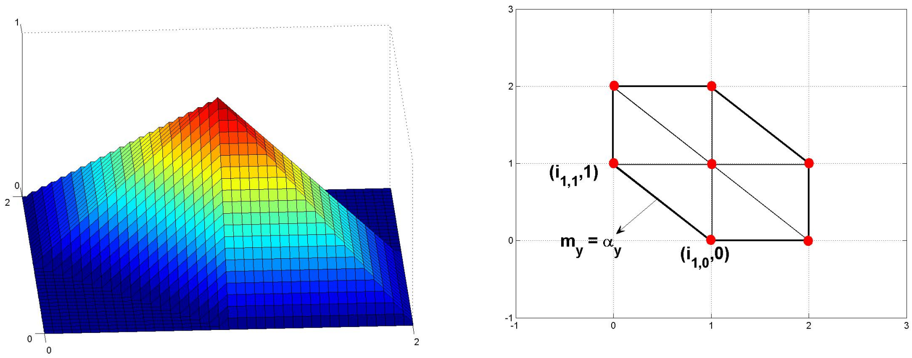
and

$$\tilde{p}_{1,0} = \tilde{p}_{2,0} = \tilde{p}_{2,1} = \tilde{p}_{0,1} = \tilde{p}_{0,2} = \tilde{p}_{1,2} = \frac{1}{2};$$

$$\tilde{p}_{1,1} = 1;$$

$$\tilde{p}_{i,j} = 0, \quad (i,j) \in \mathbb{Z}^2 \backslash \{(1,0),(2,0),(2,1),(0,1),(0,2),(1,1),(2,1)\}. \tag{6.41}$$

Hence, the mask sequence $\{\tilde{p}_{i,j}\}_{(i,j)\in\mathbb{Z}^2}$ satisfies those conditions in Theorem 6.2 that are necessary for the first assertion of the theorem to hold, namely $\alpha_y = m_y, \ y \in [0,1]$. This example furthermore supports the above-mentioned conjecture that the support of the refinable box spline is enclosed within the convex hull of its corresponding mask coefficients. The graph of the box spline $\tilde{B}$ is shown in Figure 6.8(a), with its support and its non-zero mask entries illustrated in Figure 6.8(b).



(a) $\tilde{B}$

(b) Support and non-zero mask entries of $\tilde{B}$

Figure 6.8: Graphs of $\tilde{B}$ and its support

Having thoroughly analysed the support of a 2-refinable function in terms of its corresponding 2-refinement mask, we now move on to the next chapter, where a look into the theory of subdivision is given.

Chapter 7

Subdivision

In [22], one is presented with an overview of the role that mathematics, and in particular subdivision, has played in the technological development of the last two decades. While the 1970's through 1990's have seen the progression of multimedia data types from images and sound to videos, all of which rested mainly on the foundations of Fourier analysis, recent years have witnessed the development of a new data type, namely digital geometry. The latter has found use in a wide variety of applications ranging from laser range scanners to medical MRI and CT scanners, to name a few. P. Schröder describes in [22], from a computer scientist's point of view, the challenge for mathematicians of developing sufficient algorithms for proper use of digital geometry and why algorithms from previous stages are no more satisfactory, and goes on to explain the surfacing of a new approximating tool, called subdivision.

Another example, of a more recreational nature, where subdivision is used on a regular basis is found at the animation film company Pixar (of which Edwin Catmull of the Catmull-Clark subdivision scheme ([1]) was one of the cofounders in 1986 and later also the president). Here, subdivision is used in the producing of movies and short films. In particular, the short film *Geri's game* has won an Academy Award for best animated short film in 1997. The head of the title character, as well as each of his hands, is a single subdivision surface.

Another informative account on the usefulness of subdivision in CAD is given in [19], where it is explained that subdivision yields an algorithm that "allows a designer to create a sculptured surface which is defined and manipulated by a structured set of control points. The designer need only understand the relationship between the control points and the surface, and not the mathematics of the underlying implementation." In [19], Loop also gives a brief overview on existing subdivision schemes, such as Chaikin's algorithm, the Doo/Sabin algorithm (see also [1]) and the Catmull/Clark algorithm (see also [14]). In [2], a comprehensive treatment is given of univariate subdivision schemes.

The purpose of this chapter is to look into (multivariate) subdivision from the point of view of 2-refinable functions.

We define the space

$$\ell^\infty(\mathbb{Z}^2) := \left\{ c \in M(\mathbb{Z}^2) \subset \mathbb{R} : \sup_{i,j\in\mathbb{Z}} |c_{i,j}| < \infty \right\}. \tag{7.1}$$

Also, for $c \in M(\mathbb{Z}^2)$, define the sequences

$$(\triangle_1 c)_{i,j} := c_{i,j} - c_{i-1,j}, \quad (i,j) \in \mathbb{Z}^2, \tag{7.2}$$

$$(\triangle_2 c)_{i,j} := c_{i,j} - c_{i,j-1}, \quad (i,j) \in \mathbb{Z}^2, \tag{7.3}$$

and, with the notation

$$||a||_\infty := \sup_{i,j\in\mathbb{Z}} |a_{i,j}|, \quad a \in M(\mathbb{Z}^2), \tag{7.4}$$

define

$$||\triangle c||_\infty := \max \left\{ ||\triangle_1 c||_\infty, ||\triangle_2 c||_\infty \right\}, \tag{7.5}$$

according to which we then define the space

$$\triangle^\infty(\mathbb{Z}^2) := \left\{ c \in M(\mathbb{Z}^2) : \; ||\triangle c||_\infty < \infty \right\}. \tag{7.6}$$

Definition: Let $(p, \phi)_M$ be an M-refinement pair, with $M \in \mathbb{Z}^{2\times 2}$ and $p \in M_0(\mathbb{Z}^2)$. The *M-subdivision operator* $S_{M,p} : M(\mathbb{Z}^2) \to M(\mathbb{Z}^2)$ is defined by

$$(S_{M,p}c)_{(i,j)} := \sum_{k,l} p_{\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right] - M \left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]} c_{\left[\begin{smallmatrix} k \\ l \end{smallmatrix}\right]}, \qquad (i,j) \in \mathbb{Z}^2, \quad c \in M(\mathbb{Z}^2). \tag{7.7}$$

We call p the *M-subdivision mask* and $(S_{M,p}, c)$ an *M-subdivision scheme*, while c is called the *initial control point sequence*, so that the M-subdivision scheme generates a sequence $\{c^{(r)} : r \in \mathbb{Z}_+\} \subset M(\mathbb{Z}^2)$ recursively as follows:

$$\begin{aligned} c^{(0)} &= c, \\ c^{(r+1)} &= S_{M,p}c^{(r)}, \qquad r \in \mathbb{Z}_+, \end{aligned}$$

or, equivalently,

$$\begin{aligned} c^{(0)} &= c, \\ c^{(r+1)} &= S_{M,p}^{(r)}c, \qquad r \in \mathbb{Z}_+. \end{aligned}$$

We again restrict ourselves to the case $M = 2I_2$, i.e.

$$(S_{2,p}c)_{i,j} := (S_{2I,p}c)_{i,j} = \sum_{k,l} p_{i-2k,j-2l}c_{k,l}, \qquad (i,j) \in \mathbb{Z}^2, \ c \in M(\mathbb{Z}^2), \tag{7.8}$$

and henceforth call p a 2-subdivision mask and $(S_{2,p}, c)$ a 2-subdivision scheme.
For a judiciously chosen subdivision mask $p \in M_0(\mathbb{Z}^2)$ and initial control point sequence $c \in M(\mathbb{Z}^2)$, the generated sequence $\left\{c^{(r)} : r \in \mathbb{Z}_+\right\}$ converges to a smooth surface, in a sense to be made precise below, and which then yields a powerful tool for the generation of smooth surfaces in computer-aided design.

Definition: We say that the subdivision scheme $(S_{2,p}, c)$ converges if there exists a limit function $\Phi \in C(\mathbb{R}^2)$ such that

$$\sup_{(i,j)\in\mathbb{Z}^2} \left| \Phi\left(\frac{i}{2^r}, \frac{j}{2^r}\right) - c^{(r)}_{i,j} \right| \to 0, \quad r \to \infty.$$

Observation: Let $(x,y) \in \mathbb{R}^2$. Since the dyadic set $\left\{\left(\frac{i}{2^r}, \frac{j}{2^r}\right) : (i,j) \in \mathbb{Z}^2, r \in \mathbb{Z}_+\right\}$ is dense in $\mathbb{R}^2$, there exists a sequence $\{(i,j)_r : r \in \mathbb{Z}_+\}$ such that

$$\left|(x,y) - \frac{(i,j)_r}{2^r}\right| \to 0, \quad r \to \infty.$$

Thus, in the context of the above definition, we have

$$\begin{aligned}\left|\Phi(x,y) - c^{(r)}_{(i,j)_r}\right| &\leq \left|\Phi(x,y) - \Phi\left(\frac{(i,j)}{2^r}\right)\right| + \left|\Phi\left(\frac{(i,j)}{2^r}\right) - c^{(r)}_{(i,j)_r}\right| \\ &\to 0 + 0 = 0, \quad r \to \infty\end{aligned}$$

i.e.

$$\sup_{(i,j) \in \mathbb{Z}^2} \left|\Phi(x,y) - c^{(r)}_{(i,j)_r}\right| \to 0, \quad r \to \infty.$$

Hence, the iterates $c^{(r)}_{(i,j)}$, $r = 0, 1, \ldots,$ of a convergent subdivision scheme completely "fills up" its corresponding limit surface described by the function Φ.

7.1 Sum rules

Theorem 7.1 {*Sum rules I*}

For a 2-refinement pair (p, ϕ), $\phi \in C_0(\mathbb{R}^2)$, and such that $p_{i,j} \geq 0, (i,j) \in \mathbb{Z}^2$ and $\phi(x,y) \geq 0, (x,y) \in \mathbb{R}^2$, let $m, n \in \mathbb{N}$ be the integers, according to (6.6), such that

$$p_{i,j} = 0, \quad i \notin \{0, \ldots, m\}, \ j \notin \{0, \ldots, n\}. \tag{7.9}$$

Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y)\, dx\, dy \neq 0 \tag{7.10}$$

if and only if

$$\sum_{i,j} p_{i,j} = 4. \tag{7.11}$$

Proof

Suppose that equation (7.10) holds. Then, from (6.19) and (7.9),

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\,dx\,dy &= \int_0^n\int_0^m \phi(x,y)\,dx\,dy \\
&= \int_0^n\int_0^m \left[\sum_{i=0}^{m}\sum_{j=0}^{n} p_{i,j}\phi(2x-i,2y-j)\right] dx\,dy \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} p_{i,j}\left[\int_0^n\int_0^m \phi(2x-i,2y-j)\,dx\,dy\right] \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} p_{i,j}\left[\frac{1}{2}\frac{1}{2}\int_{-j}^{2n-j}\int_{-i}^{2m-i}\phi(x,y)\,dx\,dy\right] \\
&= \frac{1}{4}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{i,j}\left[\int_0^n\int_0^m \phi(x,y)\,dx\,dy\right] \\
&= \frac{1}{4}\left[\sum_{i=0}^{m}\sum_{j=0}^{n} p_{i,j}\right]\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\,dx\,dy\right],
\end{aligned}$$

from which it follows that (7.11) holds.

Conversely, suppose that equation (7.11) holds. We prove by contradiction that (7.10) then holds. Suppose therefore that (7.10) does not hold, i.e.

$$\int_{\infty}^{\infty}\int_{\infty}^{\infty}\phi(x,y)\,dx\,dy = 0. \tag{7.12}$$

Let $\mu_{i,j} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^i y^j \phi(x,y)\,dx\,dy, \quad (i,j)\in\mathbb{Z}_+^2$,

i.e.

$$\mu_{i,j} = \int_0^n\int_0^m x^i y^j \phi(x,y)\,dx\,dy, \quad (i,j)\in\mathbb{Z}_+^2. \tag{7.13}$$

It follows from (7.13), (7.9), Theorem 6.1 and the Binomial Theorem, that, for $(i,j)\in\mathbb{N}^2$,

$$\begin{aligned}
\mu_{i,j} &= \int_0^n\int_0^m x^i y^j\left[\sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\phi(2x-k,2y-l)\right] dx\,dy \\
&= \sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\left[\int_0^n\int_0^m x^i y^j\phi(2x-k,2y-l)\,dx\,dy\right] \\
&= \sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\left[\frac{1}{2^{i+1}}\frac{1}{2^{j+1}}\int_{-l}^{2n-l}\int_{-k}^{2m-k}(x+k)^i(y+l)^j\phi(x,y)\,dx\,dy\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^{i+1}}\frac{1}{2^{j+1}}\sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\left[\int_0^n\int_0^m (x+k)^i(y+l)^j\phi(x,y)\,dx\,dy\right]\\
&= \frac{1}{2^{i+1}}\frac{1}{2^{j+1}}\sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\left[\sum_{u=0}^{i}\sum_{v=0}^{j}\binom{i}{u}\binom{j}{v}k^{i-u}l^{j-v}\left\{\int_0^n\int_0^m x^u y^v\phi(x,y)\,dx\,dy\right\}\right]\\
&= \frac{1}{2^{i+1}}\frac{1}{2^{j+1}}\sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\sum_{u=0}^{i}\sum_{v=0}^{j}\binom{i}{u}\binom{j}{v}k^{i-u}l^{j-v}\mu_{u,v}.
\end{aligned}
$$

Consequently, the moments $\mu_{i,j}$ can be written as

$$
\mu_{i,j} = \frac{1}{2^{i+1}}\frac{1}{2^{j+1}}\overbrace{\sum_{u=0}^{i}\sum_{v=0}^{j}\left[\sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\binom{i}{u}\binom{j}{v}k^{i-u}l^{j-v}\right]\mu_{u,v}}^{=:A}. \tag{7.14}
$$

Now,

$$
\begin{aligned}
A &= \sum_{u=0}^{i}\sum_{v=0}^{j}\left[\sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\binom{i}{u}\binom{j}{v}k^{i-u}l^{j-v}\right]\mu_{u,v}\\
&= \sum_{u=0}^{i}\Bigg[\left\{\sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\binom{i}{u}k^{i-u}\right\}\mu_{u,j}\\
&\qquad +\sum_{v=0}^{j-1}\left\{\sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\binom{i}{u}\binom{j}{v}k^{i-u}l^{j-v}\right\}\mu_{u,v}\Bigg]\\
&= \left\{\sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\right\}\mu_{i,j}\\
&\qquad +\sum_{v=0}^{j-1}\left[\sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\binom{j}{v}l^{j-v}\right]\mu_{i,v}\\
&\qquad +\sum_{u=0}^{i-1}\left[\sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\binom{i}{u}k^{i-u}\right]\mu_{u,j}\\
&\qquad +\sum_{u=0}^{i-1}\sum_{v=0}^{j-1}\left\{\sum_{k=0}^{m}\sum_{l=0}^{n} p_{k,l}\binom{i}{u}\binom{j}{v}k^{i-u}l^{j-v}\right\}\mu_{u,v},
\end{aligned}
$$

and thus, using also (7.11), equation (7.14) becomes

$$\begin{aligned}
\mu_{i,j} &= \frac{1}{2^{i+1}}\frac{1}{2^{j+1}}\left[4\mu_{i,j}\right] \\
&+\frac{1}{2^{i+1}}\frac{1}{2^{j+1}}\sum_{v=0}^{j-1}\left[\sum_{k=0}^{m}\sum_{l=0}^{n}p_{k,l}\binom{j}{v}l^{j-v}\right]\mu_{i,v} \\
&+\frac{1}{2^{i+1}}\frac{1}{2^{j+1}}\sum_{u=0}^{i-1}\left[\sum_{k=0}^{m}\sum_{l=0}^{n}p_{k,l}\binom{i}{u}k^{i-u}\right]\mu_{u,j} \\
&+\frac{1}{2^{i+1}}\frac{1}{2^{j+1}}\sum_{u=0}^{i-1}\sum_{v=0}^{j-1}\left[\sum_{k=0}^{m}\sum_{l=0}^{n}p_{k,l}\binom{i}{u}\binom{j}{v}k^{i-u}l^{j-v}\right]\mu_{u,v}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu_{i,j} &= \tfrac{1}{4(2^i2^j-1)}\Big\{\textstyle\sum_{v=0}^{j-1}\left[\sum_{k=0}^{m}\sum_{l=0}^{n}p_{k,l}\binom{j}{v}l^{j-v}\right]\mu_{i,v} \\
&\quad+\textstyle\sum_{u=0}^{i-1}\left[\sum_{k=0}^{m}\sum_{l=0}^{n}p_{k,l}\binom{i}{u}k^{i-u}\right]\mu_{u,j} \\
&\quad+\textstyle\sum_{u=0}^{i-1}\sum_{v=0}^{j-1}\left[\sum_{k=0}^{m}\sum_{l=0}^{n}p_{k,l}\binom{i}{u}\binom{j}{v}k^{i-u}l^{j-v}\right]\mu_{u,v}\Big\}.
\end{aligned} \tag{7.15}$$

This means that the $\mu_{i,j}$'s can be computed recursively by means of equation (7.15), for every $(i,j)\in\mathbb{N}^2$.

At this stage it is important to note from equations (7.12) and (7.13) that $\mu_{0,0}=0$. Also, from (7.9) and Theorem 6.1, we have

$$\begin{aligned}
\mu_{1,0} &= \int_0^n\int_0^m x\phi(x,y)\,dx\,dy \\
&= \int_0^n\int_0^m x\left[\sum_{i,j}p_{i,j}\phi(2x-i,2y-j)\right]dx\,dy \\
&= \sum_{i,j}\left[\int_0^n\int_0^m x\phi(2x-i,2y-j)\,dx\,dy\right] \\
&= \frac{1}{2}\sum_{i,j}\left[\int_0^n\int_{-i}^{2m-i}\frac{x+i}{2}\phi(x,2y-j)\,dx\,dy\right] \\
&= \frac{1}{2}\sum_{i,j}\left[\int_0^n\int_0^m\frac{x+i}{2}\phi(x,2y-j)\,dx\,dy\right] \\
&= \frac{1}{4}\sum_{i,j}\left[\int_0^n\int_0^m x\phi(x,2y-j)\,dx\,dy+i\overbrace{\int_0^n\int_0^m\phi(x,2y-j)\,dx\,dy}^{=0}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\sum_{i,j}\left[\int_{-j}^{2n-j}\int_{0}^{m} x\phi(x,y)\,dx\,dy\right] \\
&= \frac{1}{8}\sum_{i,j}\left[\int_{0}^{n}\int_{0}^{m} x\phi(x,y)\,dx\,dy\right] \\
&= \frac{1}{8}\sum_{i,j}\mu_{1,0},
\end{aligned}
$$

thus implying $\mu_{1,0} = 0$.

It similarly follows that $\mu_{0,1} = 0$, and using this together with the recursive formula (7.15) for $\mu_{i,j}$, we deduce that $\mu_{i,j} = 0,\ (i,j) \in \mathbb{Z}_+^2$.

It follows from this that

$$
\int_{-\infty}^{\infty} p(x,y)\phi(x,y)\,dx\,dy = 0, \qquad p \in \Pi^{(2)}. \tag{7.16}
$$

Now, let $\varepsilon > 0$. From the Stone–Weierstrass Theorem, which is a generalization of the Weierstrass Theorem and has the Weierstrass Theorem for more variables as a special case (see [23] and [24]), there exists a polynomial $p_\varepsilon \in \Pi^{(2)}$ such that

$$
\max_{(x,y)\in[a,b]\times[c,d]} |\phi(x,y) - p_\varepsilon(x,y)| < \frac{\varepsilon}{(b-a)(c-d)\max_{(x,y)\in[a,b]\times[c,d]}\left|\phi(x,y)\right|},
$$

i.e.

$$
\begin{aligned}
\int_a^b\int_c^d [\phi(x,y)]^2\,dx\,dy &= \int_a^b\int_c^d \phi(x,y)\big[p_\epsilon(x,y) + \{\phi(x,y) - p_\epsilon(x,y)\}\big]\,dx\,dy \\
&= \int_a^b\int_c^d \phi(x,y)p_\epsilon(x,y)\,dx\,dy \\
&\quad + \int_a^b\int_c^d \phi(x,y)\big[\phi(x,y) - p_\epsilon(x,y)\big]\,dx\,dy \\
&= 0 + \int_a^b\int_c^d \phi(x,y)\big[\phi(x,y) - p_\epsilon(x,y)\big]\,dx\,dy \\
&\leq \left|\int_a^b\int_c^d \phi(x,y)\big[\phi(x,y) - p_\epsilon(x,y)\big]\,dx\,dy\right| \\
&\leq \int_a^b\int_c^d \big|\phi(x,y)\big|\big|\phi(x,y) - p_\epsilon(x,y)\big|\,dx\,dy
\end{aligned}
$$

$$
\begin{aligned}
&< \left(\max_{(x,y)\in[a,b]\times[c,d]} \left|\phi(x,y)\right| \right) (b-a)(c-d) \cdot \\
&\qquad \left(\frac{\varepsilon}{(b-a)(c-d)\max_{(x,y)\in[a,b]\times[c,d]} \left|\phi(x,y)\right|} \right) \\
&= \varepsilon,
\end{aligned}
$$

so that

$$\int_a^b \int_c^d [\phi(x,y)]^2 \, dx\, dy < \varepsilon \text{ for every } \varepsilon > 0,$$

i.e.

$$\int_a^b \int_c^d [\phi(x,y)]^2 \, dx\, dy = 0.$$

Since also $\phi \in C(\mathbb{R})$, it follows that $\phi(x,y) = 0, \quad (x,y) \in \mathbb{R}^2$. Since ϕ is a refinable function and as such not identically zero, this is a contradiction, and it follows that (7.10) holds.

This completes the proof. $\square$

Lemma 7.1 $\{$*Relationship between* $\int\int \phi(x,y)\, dx\, dy$ *and* $\sum \phi(i,j)\}$

For a 2-refinement pair (p,ϕ), $\phi \in C_0(\mathbb{R}^2)$, *and such that* $p_{i,j} \geq 0, (i,j) \in \mathbb{Z}^2$ *and* $\phi(x,y) \geq 0, (x,y) \in \mathbb{R}^2$, *let* $m, n \in \mathbb{N}$ *denote the integers, according to (6.6), such that* $p_{i,j} = 0, \ i \notin \{0,\ldots,m\}, \ j \notin \{0,\ldots,n\}$. *Then*

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y)\, dx\, dy &= \left\{ \lim_{r\to\infty} \left[\frac{1}{4} \sum_{i,j} p_{i,j} \right]^r \right\} \sum_{i,j} \phi(i,j) \\
&= \left\{ \lim_{r\to\infty} \left[\frac{1}{4} \sum_{i,j} p_{i,j} \right]^r \right\} \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \phi(i,j). \qquad (7.17)
\end{aligned}
$$

Proof

Since (6.19) and the assumption that $\phi \in C_0(\mathbb{R}^2)$ imply

$$\phi(0,y) = \phi(m,y) = 0, \ y \in \mathbb{R};$$

$$\phi(x,0) = \phi(x,n) = 0, \ x \in \mathbb{R},$$

we start by writing the integral in (7.17) as a Riemann sum, and find, after again using (6.19), that

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y)\,dx\,dy &= \int_0^m \int_0^n \phi(x,y)\,dx\,dy \\ &= \lim_{r\to\infty}\left[\frac{1}{2^r}\frac{1}{2^r}\sum_{i=0}^{2^r m-1}\sum_{j=0}^{2^r n-1}\phi\left(\frac{i}{2^r},\frac{j}{2^r}\right)\right]\end{aligned}$$

i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y)\,dx\,dy = \lim_{r\to\infty}\left[\frac{1}{4^r}\sum_{i,j}\phi\left(\frac{1}{2^r},\frac{j}{2^r}\right)\right] \tag{7.18}$$

But, for any fixed $r \in \mathbb{N}$, it follows by repeated application of the 2-refinement equation that

$$\begin{aligned}\sum_{i,j}\phi\left(\frac{i}{2^r},\frac{j}{2^r}\right) &= \sum_{i,j}\left[\sum_{k_1,l_1} p_{k_1,l_1}\phi\left(\frac{i}{2^{r-1}}-k_1,\frac{j}{2^{r-1}}-l_1\right)\right] \\ &= \sum_{i,j}\left[\sum_{k_1,l_1} p_{k_1,l_1}\sum_{k_2,l_2} p_{k_2,l_2}\phi\left(\frac{i}{2^{r-2}}-2k_1-k_2,\frac{j}{2^{r-2}}-2l_1-l_2\right)\right] \\ &= \sum_{i,j}\left[\sum_{k_1,l_1} p_{k_1,l_1}\sum_{k_2-2k_1,l_2-2l_1} p_{k_2-2k_1,l_2-2l_1}\phi\left(\frac{i}{2^{r-2}}-k_2,\frac{j}{2^{r-2}}-l_2\right)\right] \\ &= \cdots \\ &= \sum_{k_1,l_1} p_{k_1,l_1}\sum_{k_2,l_2} p_{k_2-2k_1,l_2-2l_1}\cdots\sum_{k_r,l_r} p_{k_r-2k_{r-1},l_r-2l_{r-1}}\left[\sum_{i,j}\phi\left(i-k_r,j-l_r\right)\right] \\ &= \left[\sum_{k_1,l_1} p_{k_1,l_1}\sum_{k_2,l_2} p_{k_2-2k_1,l_2-2l_1}\cdots\sum_{k_r,l_r} p_{k_r-2k_{r-1},l_r-2l_{r-1}}\right]\left[\sum_{i,j}\phi\left(i,j\right)\right] \\ &= \left[\sum_{k_1,l_1} p_{k_1,l_1}\sum_{k_2,l_2} p_{k_2,l_2}\cdots\sum_{k_r,l_r} p_{k_r,l_r}\right]\left[\sum_{i,j}\phi(i,j)\right],\end{aligned}$$

i.e.

$$\sum_{i,j}\phi\left(\frac{i}{2^r},\frac{j}{2^r}\right) = \left[\sum_{k,l} p_{k,l}\right]^r \sum_{i,j}\phi(i,j). \tag{7.19}$$

It now follows from (7.18) and (7.19) that

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\,dx\,dy &= \lim_{r\to\infty}\left[\frac{1}{4^r}\left[\sum_{k,l}p_{k,l}\right]^r\sum_{i,j}\phi(i,j)\right]\\ &= \lim_{r\to\infty}\left[\frac{1}{4}\sum_{k,l}p_{k,l}\right]^r\sum_{i,j}\phi(i,j),\end{aligned}$$

as desired. □

The following is now an immediate consequence of the results in Theorem 7.1 and Lemma 7.1:

Corollary 7.1 {*Relationship between* $\int\int \phi(x,y)\,dx\,dy$ *and* $\sum \phi(i,j)$}
For a 2-refinement pair (p,ϕ), $\phi \in C_0(\mathbb{R}^2)$, *and such that* $p_{i,j} \geq 0, (i,j) \in \mathbb{Z}^2$ *and* $\phi(x,y) \geq 0, (x,y) \in \mathbb{R}^2$, *let* $m,n \in \mathbb{N}$ *be the integers, according to (6.6), such that* $p_{i,j} = 0,\ i \notin \{0,\ldots,m\},\ j \notin \{0,\ldots,n\}$.
Then, if either one of (7.10) or (7.11) holds, then the other one also holds, by virtue of Theorem 7.1, and, from Lemma 7.1,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\,dx\,dy = \sum_{i,j}\phi(i,j) \neq 0. \tag{7.20}$$

We can now prove the following result:

Theorem 7.2 {*Sum rules II*}
Let (p,ϕ) *be a refinement pair with* p *and* ϕ *as in Corollary 7.1, and suppose, moreover, that*

$$\sum_{i,j}p_{2i,2j} = \sum_{i,j}p_{2i+1,2j} = \sum_{i,j}p_{2i,2j+1} = \sum_{i,j}p_{2i+1,2j+1} = 1. \tag{7.21}$$

Then

$$\sum_{i,j}\phi(x-i,y-j) = \sum_{i,j}\phi(i,j) \neq 0, \qquad (x,y)\in\mathbb{R}^2. \tag{7.22}$$

Proof

First, note that (7.21) is equivalent to

$$\sum_{i,j} p_{k-2i,l-2j} = 1, \quad k \in \mathbb{Z}, l \in \mathbb{Z}. \tag{7.23}$$

Also, since (7.21) implies $\sum_{i,j} p_{i,j} = 4$, it follows from Theorem 7.1 and Corollary 7.1 that $\sum_{i,j} \phi(i,j) \neq 0$.

Define the function $F \in M(\mathbb{R}^2)$ by $F(x,y) = \sum_{i,j} \phi(x-i, y-j), \quad (x,y) \in \mathbb{R}^2$.

Then $F \in C(\mathbb{R}^2)$, since $\phi \in C_0(\mathbb{R}^2)$.

Now, since $\left\{\left(\frac{i}{2^r}, \frac{j}{2^r}\right) : (i,j) \in \mathbb{Z}^2, r \in \mathbb{Z}_+\right\}$ is dense in $\mathbb{R}^2$, and $F \in C(\mathbb{R}^2)$, it suffices to show that $F\left(\frac{i}{2^r}, \frac{j}{2^r}\right) = \sum_{k,l} \phi(k,l), \quad (i,j) \in \mathbb{Z}^2, \; r \in \mathbb{Z}_+$.

But, using the 2-refinement equation as well as (7.21),

$$\begin{aligned}
F\left(\frac{i}{2^r}, \frac{j}{2^r}\right) &= \sum_{k,l} \phi\left(\frac{i}{2^r} - k, \frac{j}{2^r} - l\right) \\
&= \sum_{k,l} \sum_{\mu,\nu} p_{\mu,\nu} \phi\left(\frac{i}{2^{r-1}} - 2k - \mu, \frac{j}{2^{r-1}} - 2l - \nu\right) \\
&= \sum_{k,l} \sum_{\mu,\nu} p_{\mu-2k,\nu-2l} \phi\left(\frac{i}{2^{r-1}} - \mu, \frac{j}{2^{r-1}} - \nu\right) \\
&= \sum_{\mu,\nu} \left[\sum_{k,l} p_{\mu-2k,\nu-2l}\right] \phi\left(\frac{i}{2^{r-1}} - \mu, \frac{j}{2^{r-1}} - \nu\right) \\
&= \sum_{\mu,\nu} \phi\left(\frac{i}{2^{r-1}} - \mu, \frac{j}{2^{r-1}} - \nu\right) \\
&= \cdots \\
&= \sum_{\mu,\nu} \phi\left(i - \mu, j - \nu\right) \\
&= \sum_{\mu,\nu} \phi(\mu, \nu),
\end{aligned}$$

as required. $\square$

The conditions in equation (7.21) are generally referred to as the *sum rules.*

Considering the conditions we have placed on the 2-refinement mask sequence p thus far and in order to build on these conditions in our further study of subdivision, we now define, for $(k,l) \in \mathbb{N}^2$, the class of 2-refinement masks

$$\begin{aligned}\mathcal{A}_{k,l} = \quad & \Big\{ p \in M_0(\mathbb{Z}^2) : p_{i,j} \geq 0,\ (i,j) \in \mathbb{Z}^2; \\ & p_{i,j} = 0,\ i \notin \{0,\ldots,k\},\ j \notin \{0,\ldots,l\}; \\ & \textstyle\sum_{i,j} p_{2i,2j} = \sum_{i,j} p_{2i+1,2j} = \sum_{i,j} p_{2i,2j+1} = \sum_{i,j} p_{2i+1,2j+1} = 1 \Big\}.\end{aligned} \tag{7.24}$$

Note from equations (3.3) and (3.4) that $p^{(1)} \in \mathcal{A}_{1,1}$ and $p^{(2)} \in \mathcal{A}_{2,2}$, where $p^{(1)}$ (resp. $p^{(2)}$) is the 2-refinement mask corresponding to B_1, the roof function given by (2.7) (resp. B_2, the Courant hat function given by (2.9)).

It can similarly be seen that $p^{(k)} \in \mathcal{A}_{k,k}$, where $P_k(z_1, z_2) = \sum_{i,j} p^{(k)}_{i,j} z_1^i z_2^j$ is as in equation (3.12).

Further, we define a 2-refinement pair (p, ϕ) to be *normalised* if ϕ satisfies

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x,y)\, dx = \sum_i \sum_j \phi(i,j) = 1, \tag{7.25}$$

according to which we then combine the results thus far to constitute the following theorem, which is a generalization of a result that exists for the univariate case and appears in [11]:

Theorem 7.3 $\{$*Normalised refinement pairs*$\}$

For $(k,l) \in \mathbb{N}^2$, suppose $(p, \tilde{\phi})$ is a 2-refinement pair, with $p \in \mathcal{A}_{k,l}$, and $\tilde{\phi} \in C_0(\mathbb{R}^2)$, such that $\phi(x,y) \geq 0, (x,y) \in \mathbb{R}^2$. Then the definition

$$\phi \;=\; \frac{1}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\phi}(x,y)\, dx\, dy}\tilde{\phi} \;=\; \frac{1}{\sum_i \sum_j \tilde{\phi}(i,j)}\tilde{\phi}$$

yields $\phi \in C_0(\mathbb{R}^2)$, and (p, ϕ) is a normalised 2-refinement pair, with

$$\phi(x,y) = 0, \quad (x,y) \notin (0,k) \times (0,l), \tag{7.26}$$

and

$$\sum_i \sum_j \phi(x-i, y-j) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x,y)\, dx\, dy = 1, \quad (x,y) \in \mathbb{R}^2. \tag{7.27}$$

□

Note that, with the choice $\phi = B_{\mathcal{D}}$, where $B_{\mathcal{D}}$ is the box spline associated with some direction matrix $\mathcal{D}$, we have the box splines as a special case in Theorem 7.3. This is a generalization of the specific case in Lemma 2.1, where we were only working with box splines corresponding to initial direction matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

7.2 Subdivision

We now come to reach the focus point of this chapter, namely subdivision with respect to the dilation matrix $M = 2I$. The result that we have on the convergence of 2-subdivision schemes is an almost direct generalization of the one for the univariate case in [11].

Theorem 7.4 {*Subdivision*}
For $(k,l) \in \mathbb{N}^2,\ k \geq 2,\ l \geq 2,$ let (p, ϕ) be a 2-refinement pair, with $p \in \mathcal{A}_{k,l}$, and such that the conditions in Theorem 7.3 are met.
Take any control point sequence $c \in \triangle^{\infty}(\mathbb{Z}^2)$, and define

$$\Phi(x,y) := \sum_{i,j} c_{i,j} \phi(x-i, y-j), \quad (x,y) \in \mathbb{R}^2. \tag{7.28}$$

Then, if we define the sequence $\left\{c^{(r)} : r \in \mathbb{Z}_+\right\} \subset M(\mathbb{Z}^2)$ recursively by means of

$$c^{(0)} = c, \tag{7.29}$$

$$c_{i,j}^{(r+1)} = \sum_{\alpha,\beta} p_{i-2\alpha, j-2\beta} c_{\alpha,\beta}^{(r)}, \qquad r \in \mathbb{Z}_+, \tag{7.30}$$

then Φ *satisfies*

$$\Phi\left(\frac{\cdot}{2^r}\right) - c^{(r)} \in \ell^\infty(\mathbb{Z}^2), \qquad r \in \mathbb{Z}_+, \tag{7.31}$$

with

$$\sup_{(i,j)\in\mathbb{Z}^2} \left| \Phi\left(\frac{i}{2^r}, \frac{j}{2^r}\right) - c^{(r)}_{(i,j)} \right| \leq K\rho^r ||\triangle c||_\infty, \qquad r \in \mathbb{Z}_+, \tag{7.32}$$

where

$$K = (k-1)(l-1)\sum_{i=1}^{k-1}\sum_{j=1}^{l-1} \left|\phi(i,j)\right|, \tag{7.33}$$

and where

$$\begin{aligned}\rho = \rho(p) = \quad & \tfrac{1}{2}\sup\Big\{ \textstyle\sum_{\mu,\nu} |p_{i-2\mu,j-2\nu} - p_{\alpha-2\mu,\beta-2\nu}| : \\ & (i,j),(\alpha,\beta) \in \mathbb{Z}^2 \text{ and } |i-\alpha| \leq k-1, |j-\beta| \leq l-1 \Big\}.\end{aligned} \tag{7.34}$$

Proof

For $(i,j) \in \mathbb{Z}^2$ and $r \in \mathbb{Z}_+$, it follows from the definition of Φ and the 2-refinement equation that

$$\begin{aligned}
\Phi\left(\frac{i}{2^r}, \frac{j}{2^r}\right) &= \sum_{\alpha,\beta} c^{(0)}_{\alpha,\beta}\phi\left(\frac{i}{2^r} - \alpha, \frac{j}{2^r} - \beta\right) \\
&= \sum_{\alpha,\beta} c^{(0)}_{\alpha,\beta}\left[\sum_{\mu,\nu} p_{\mu,\nu}\phi\left(\frac{i}{2^{r-1}} - 2\alpha - \mu, \frac{j}{2^{r-1}} - 2\beta - \nu\right)\right] \\
&= \sum_{\alpha,\beta} c^{(0)}_{\alpha,\beta}\left[\sum_{\mu,\nu} p_{\mu-2\alpha,\nu-2\beta}\phi\left(\frac{i}{2^{r-1}} - \mu, \frac{j}{2^{r-1}} - \nu\right)\right] \\
&= \sum_{\mu,\nu}\left[\sum_{\alpha,\beta} p_{\mu-2\alpha,\nu-2\beta}c^{(0)}_{\alpha,\beta}\right]\phi\left(\frac{i}{2^{r-1}} - \mu, \frac{j}{2^{r-1}} - \nu\right) \\
&= \sum_{\mu,\nu} c^{(1)}_{\mu,\nu}\phi\left(\frac{i}{2^{r-1}} - \mu, \frac{j}{2^{r-1}} - \nu\right) \\
&= \dots \\
&= \sum_{\alpha,\beta} c^{(r)}_{\alpha,\beta}\phi\left(i-\alpha, j-\beta\right),
\end{aligned}$$

i.e.

$$\Phi\left(\frac{i}{2^r}, \frac{j}{2^r}\right) = \sum_{\alpha=1}^{k-1}\sum_{\beta=1}^{l-1} c^{(r)}_{i-\alpha,j-\beta}\phi(\alpha,\beta), \tag{7.35}$$

where the last equation holds since $\phi(\alpha,\beta) = 0$, $\alpha \notin (0,k)$, $\beta \notin (0,l)$, by virtue of Theorem 6.1.

Also, since all the conditions in Theorem 7.3 hold by assumption, we have

$$\sum_{i,j}\phi(x-i,y-j) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\,dx\,dy = 1, \quad (x,y) \in \mathbb{R}^2. \tag{7.36}$$

Since $(0,0) \in \mathbb{R}^2$, it follows from (7.36) that, for $(i,j) \in \mathbb{Z}^2$,

$$c^{(r)}_{i,j} = c^{(r)}_{i,j}\sum_{\alpha,\beta}\phi(0-\alpha,0-\beta) = \sum_{\alpha,\beta}c^{(r)}_{i,j}\phi(\alpha,\beta). \tag{7.37}$$

Together, equations (7.35) and (7.37) give

$$\phi\left(\frac{i}{2^r}, \frac{j}{2^r}\right) - c^{(r)}_{i,j} = \sum_{\alpha=1}^{k-1}\sum_{\beta=1}^{l-1}\left[c^{(r)}_{i-\alpha,j-\beta} - c^{(r)}_{i,j}\right]\phi(\alpha,\beta), \quad (i,j) \in \mathbb{Z}^2. \tag{7.38}$$

Therefore, we must show that

$$\sup\left\{\left|c^{(r)}_{i,j} - c^{(r)}_{\alpha,\beta}\right| : \ (i,j),(\alpha,\beta) \in \mathbb{Z}^2 \text{ and } |i-\alpha| \le k-1, \ |j-\beta| \le l-1\right\}$$

$$\le (k-1)(l-1)\rho^r||\triangle c||_\infty, \quad r \in \mathbb{Z}_+, \ c \in \triangle^\infty(\mathbb{Z}^2). \tag{7.39}$$

We do so by induction on $r \in \mathbb{Z}_+$. Let $c \in \triangle^\infty(\mathbb{Z}^2)$.

First, consider $(i,j),(\alpha,\beta) \in \mathbb{Z}^2$ such that

$$0 < i-\alpha \le k-1, \quad 0 < j-\beta \le l-1. \tag{7.40}$$

First, let $r = 0$.

It follows from (7.29), (7.6) and (7.40) that

$$\begin{aligned} |c^{(0)}_{i,j} - c^{(0)}_{\alpha,\beta}| &= |c_{i,j} - c_{\alpha,\beta}| \\ &\le \sum_{\mu=\alpha+1}^{i}|(\triangle_1 c)_{\mu,\beta}| + \sum_{\nu=\beta+1}^{j}|(\triangle_2 c)_{\alpha,\nu}| \end{aligned}$$

$$\begin{aligned}
&\leq (i-\alpha)||\triangle_1 c||_\infty + (j-\beta)||\triangle_2 c||_\infty \\
&\leq (i-\alpha)||\triangle c||_\infty + (j-\beta)||\triangle c||_\infty \\
&\leq (k-1)||\triangle c||_\infty + (l-1)||\triangle c||_\infty,
\end{aligned}$$

i.e.

$$|c_{i,j}^{(0)} - c_{\alpha,\beta}^{(0)}| \leq (k-1)(l-1)||\triangle c||_\infty, \tag{7.41}$$

so that (7.39) is satisfied for $r = 0$.

Suppose next that (7.39) holds for a fixed $r \in \mathbb{Z}_+$.

Let $\sigma \in \mathbb{R}$ be an arbitrary number. We then have

$$\begin{aligned}
c_{i,j}^{(r+1)} - c_{\alpha,\beta}^{(r+1)} &= \sum_{\mu,\nu} p_{i-2\mu,j-2\nu} c_{\mu,\nu}^{(r)} - \sum_{\mu,\nu} p_{\alpha-2\mu,\beta-2\nu} c_{\mu,\nu}^{(r)} \\
&= \sum_{\mu,\nu} \left[p_{i-2\mu,j-2\nu} - p_{\alpha-2\mu,\beta-2\nu}\right] c_{\mu,\nu}^{(r)} \\
&= \sum_{\mu,\nu} \left[p_{i-2\mu,j-2\nu} - p_{\alpha-2\mu,\beta-2\nu}\right] \left(c_{\mu,\nu}^{(r)} - \sigma\right),
\end{aligned}$$

where the last equation follows from the assumption that p satisfies the sum rules. Hence, using also the fact that $p \in \mathcal{A}_{k,l}$,

$$c_{i,j}^{(r+1)} - c_{\alpha,\beta}^{(r+1)} = \sum_{\mu=\mu_1}^{\mu_2} \sum_{\nu=\nu_1}^{\nu_2} \left[p_{i-2\mu,j-2\nu} - p_{\alpha-2\mu,\beta-2\nu}\right] \left(c_{\mu,\nu}^{(r)} - \sigma\right), \tag{7.42}$$

where

$$\mu_1 = \left\lceil \frac{\alpha-k}{2} \right\rceil, \quad \mu_2 = \left\lfloor \frac{i}{2} \right\rfloor \tag{7.43}$$

and

$$\nu_1 = \left\lceil \frac{\beta-l}{2} \right\rceil, \quad \nu_2 = \left\lfloor \frac{j}{2} \right\rfloor, \tag{7.44}$$

where the first equation in (7.43) holds by virtue of the following: Since $p \in \mathcal{A}_{k,l}$, we must have $i - 2\mu \leq k$ *and* $\alpha - 2\mu \leq k$, i.e. $\mu \geq \frac{i-k}{2}$ and $\mu \geq \frac{\alpha-k}{2}$, and, since $i > \alpha$ (because $i - \alpha > 0$ by assumption), we must therefore have $\mu \geq \frac{\alpha-k}{2}$. The formula for μ_2 and the formulas in (7.44) follow similarly.

Observe from (7.43) and (7.44) that

$$\mu_2 - \mu_1 \le \frac{i}{2} - \frac{\alpha - k}{2} = \frac{i-\alpha}{2} + \frac{k}{2} \le \frac{k-1}{2} + \frac{k}{2} = k - \frac{1}{2}$$

and

$$\nu_2 - \nu_1 \le \frac{j}{2} - \frac{\beta - l}{2} = \frac{j-\beta}{2} + \frac{l}{2} \le \frac{l-1}{2} + \frac{l}{2} = l - \frac{1}{2},$$

and, since we know $\mu_2 - \mu_1 \in \mathbb{Z}$ and $\nu_2 - \nu_1 \in \mathbb{Z}$, we thus have the inequalities

$$\mu_2 - \mu_1 \le k - 1 \tag{7.45}$$

and

$$\nu_2 - \nu_1 \le l - 1. \tag{7.46}$$

Now, choose the real number σ as

$$\sigma = \frac{1}{2}\left[\min_{\mu_1 \le \mu \le \mu_2,\ \nu_1 \le \nu \le \nu_2} c^{(r)}_{\mu,\nu} + \max_{\mu_1 \le \mu \le \mu_2,\ \nu_1 \le \nu \le \nu_2} c^{(r)}_{\mu,\nu}\right]. \tag{7.47}$$

Then, for $\mu \in \{\mu_1, \ldots, \mu_2\}$ and $\nu \in \{\nu_1, \ldots, \nu_2\}$,

$$\begin{aligned} |c^{(r)}_{i,j}| &\le \tfrac{1}{2}\left[\max_{\mu_1 \le \mu \le \mu_2,\ \nu_1 \le \nu \le \nu_2} c^{(r)}_{\mu,\nu} - \min_{\mu_1 \le \mu \le \mu_2,\ \nu_1 \le \nu \le \nu_2} c^{(r)}_{\mu,\nu}\right] \\ &\le \tfrac{1}{2}\sum\Big\{|c^{(r)}_{i,j} - c^{(r)}_{\alpha,\beta}| : (i,j),(\alpha,\beta) \in \mathbb{Z}^2 \\ &\qquad \text{with } |i-\alpha| \le k-1, |j-\beta| \le l-1\Big\}. \end{aligned} \tag{7.48}$$

It now follows from (7.42), (7.48) and from the Induction Hypothesis, that

$$\begin{aligned} |c^{(r+1)}_{i,j} - c^{(r+1)}_{\alpha,\beta}| &\le \left[\frac{1}{2}\sum_{\mu=\mu_1}^{\mu_1}\sum_{\nu=\nu_1}^{\nu_2}\left|p_{i-2\mu,j-2\nu} - p_{\alpha-2\mu,\beta-2\nu}\right|\right](k-1)(l-1)\rho^r||\triangle c||_\infty \\ &\le (k-1)(l-1)\rho^{r+1}||\triangle c||_\infty, \end{aligned}$$

with ρ as in (7.34).

The proof follows similarly for the cases

$$0 < \alpha - i \le k-1, \qquad 0 < j - \beta \le l-1$$

$$0 < i - \alpha \le k-1, \qquad 0 < \beta - j \le l-1$$

$$0 < \alpha - i \le k-1, \qquad 0 < \beta - j \le l-1.$$

This completes the proof by induction. □

Theorem 7.5 $\{$*Bounds for* $\rho\}$

For $(k,l) \in \mathbb{N}^2$, *suppose that* $p \in \mathcal{A}_{k,l}$, *with* $\mathcal{A}_{k,l}$ *defined as in (7.24), and such that* p *satisfies, moreover, the positivity condition*

$$p_{i,j} > 0, \quad i = 0, 1, \ldots, k, \quad j = 0, 1, \ldots, l. \tag{7.49}$$

Then the positive number $\rho = \rho(p)$ *defined by (7.34) satisfies*

$$0 < \rho \leq 1 - \min\{p_{i,j} : i = 0, 1, \ldots, k,\ j = 0, 1, \ldots, l\} < 1. \tag{7.50}$$

Proof

Take any $(i,j), (\alpha,\beta) \in \mathbb{Z}^2$ such that

$$\begin{aligned} 0 &< i - \alpha \leq k - 1 \\ 0 &< j - \beta \leq l - 1. \end{aligned} \tag{7.51}$$

Define the integers $\mu_0 := \lfloor \frac{1}{2}\alpha \rfloor$, and $\nu_0 := \lfloor \frac{1}{2}\beta \rfloor$, from which it follows that

$$\begin{array}{rcl@{\qquad\qquad}rcl} \alpha - k & \leq & i - k & \beta - l & \leq & j - l \\ & \leq & \alpha - 1 & & \leq & \beta - 1 \\ & \leq & 2\mu_0 & & \leq & 2\nu_0 \\ & \leq & \alpha & & \leq & \beta \\ & < & i & & < & j \end{array}$$

It follows from this and (7.51) that

$$\begin{array}{ccccc@{\qquad\qquad}ccccc} 0 & \leq & i - 2\mu_0 & \leq & k & 0 & \leq & j - 2\nu_0 & \leq & l \\ & & \text{and} & & & & & \text{and} & & \\ 0 & \leq & \alpha - 2\mu_0 & \leq & k & 0 & \leq & \beta - 2\nu_0 & \leq & l. \end{array}$$

Since $p \in \mathcal{A}_{k,l}$, we thus have the inequalities

$$p_{i-2\mu_0,j-2\nu_0} > 0, \qquad p_{\alpha-2\mu_0,\beta-2\nu_0} > 0, \tag{7.52}$$

and if we furthermore define the real number $\delta := \min\{p_{i,j} : i = 0,1,\ldots,k,\ j = 0,1,\ldots,l\}$, then we also have

$$p_{\alpha-2\mu_0,\beta-2\nu_0} \geq \delta. \tag{7.53}$$

Note that the sum rules (7.23) together with (7.49) imply $\delta \in (0,1)$.

Suppose $p_{i-2\mu_0,j-2\nu_0} \geq p_{\alpha-2\mu_0,\beta-2\nu_0}$. Then,

$$\begin{aligned}
\sum_{\mu,\nu} |p_{i-2\mu,j-2\nu} - p_{\alpha-2\mu,\beta-2\nu}| &= \sum_{\mu\neq\mu_0,\nu\neq\nu_0} |p_{i-2\mu,j-2\nu} - p_{\alpha-2\mu,\beta-2\nu}| \\
&\qquad + (p_{i-2\mu_0,j-2\nu_0} - p_{\alpha-2\mu_0,\beta-2\nu_0}) \\
&\leq \sum_{\mu\neq\mu_0,\nu\neq\nu_0} (|p_{i-2\mu,j-2\nu}| + |p_{\alpha-2\mu,\beta-2\nu}|) \\
&\qquad + (p_{i-2\mu_0,j-2\nu_0} - p_{\alpha-2\mu_0,\beta-2\nu_0}) \\
&= \sum_{\mu,\nu} (p_{i-2\mu,j-2\nu} + p_{\alpha-2\mu,\beta-2\nu}) - 2p_{\alpha-2\mu_0,\beta-2\nu_0} \\
&= \sum_{\mu,\nu} p_{i-2\mu,j-2\nu} + \sum_{\mu,\nu} p_{\alpha-2\mu,\beta-2\nu} - 2p_{\alpha-2\mu_0,\beta-2\nu_0} \\
&= 1 + 1 - 2p_{\alpha-2\mu_0,\beta-2\nu_0} \\
&= 2(1 - p_{\alpha-2\mu_0,\beta-2\nu_0}) \\
&\leq 2(1-\delta).
\end{aligned}$$

Similarly, if $p_{i-2\mu_0,j-2\nu_0} \leq p_{\alpha-2\mu_0,\beta-2\nu_0}$, we also have

$$\sum_{\mu,\nu} |p_{i-2\mu,j-2\nu} - p_{\alpha-2\mu,\beta-2\nu}| \leq 2(1-\delta). \tag{7.54}$$

A similar argument shows that (7.54) also holds for the cases

$$\begin{aligned}
&0 < i - \alpha \leq k-1 \quad &&\text{and} \quad 0 < \beta - j \leq l-1; \\
&0 < \alpha - 1 \leq k-1 \quad &&\text{and} \quad 0 < j - \beta \leq l-1; \\
&0 < \alpha - 1 \leq k-1 \quad &&\text{and} \quad 0 < \beta - j \leq l-1.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}&\sup\Big\{\textstyle\sum_{\mu,\nu}|p_{i-2\mu,j-2\nu}-p_{\alpha-2\mu,\beta-2\nu}|:\\&\qquad(i,j),(\alpha,\beta)\in\mathbb{Z}^2 \text{ and } |i-\alpha|\leq k-1,|j-\beta|\leq l-1\Big\}\\&\leq 2(1-\delta),\end{aligned}$$

i.e. $\rho \leq 1-\delta$.

Since, also, $\delta\in(0,1)$ implies $1-\delta\in(0,1)$, this completes the proof. □

It immediately follows from Theorems 7.4 and 7.5 that, for positive masks p that satisfy the sum rules, we have the following subdivision convergence result:

Corollary 7.2 $\{$*Subdivision convergence*$\}$
Suppose, as in Theorem 7.4, that $(k,l)\in\mathbb{N}^2$ *and that* (p,ϕ) *is a 2-refinement pair, with* $p\in\mathcal{A}_{k,l},\phi\in M_0(\mathbb{R}^2)$, *such that the conditions in Theorem 7.3 are satisfied, and suppose furthermore that* $p_{i,j}>0,\quad i=0,1,\ldots,k,\ j=0,1,\ldots,l$. *Then, for any control point sequence* $c\in\triangle^\infty(\mathbb{Z}^2)$, *the subdivision scheme* $(S_{2,p},c)$ *converges to the limit surface* Φ, *as given in (7.28).*
Specifically, if we define the sequence $\{c^{(r)}:r\in\mathbb{Z}_+\}\subset M(\mathbb{Z}^2)$ *recursively by means of (7.29) and (7.30), then*

$$\sup_{(i,j)\in\mathbb{Z}^2}\left|\Phi\left(\frac{i}{2^r},\frac{j}{2^r}\right)-c^{(r)}_{(i,j)}\right| \;\leq\; K\rho^r||\triangle c||_\infty \;\to\; 0,\quad r\to\infty,$$

with K *and* $\rho=\rho(p)$ *defined as in equations (7.33) and (7.34), respectively.*

7.3 Examples

We end this chapter with examples of the 2-subdivision scheme applied to two initial control point sequences. Each time, we show the effective outcome using both the Courant hat function's corresponding mask $\left(p^{(1)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right)$, and that of a quadratic box spline, namely B_3 as at the end of Chapter 3 $\left(p^{(2)} = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{1}{4} & \frac{5}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{2} & \frac{5}{8} & \frac{1}{4} \\ 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix} \right)$.

(Here, we give the 2-refinement mask as a *matrix* $\mathcal{P} = [\mathcal{P}_{i,j}]$, so as to mean that each entry $p_{i,j}$ of the mask sequence corresponds to the entry $\mathcal{P}_{j,i}$ of the matrix. It is understood that, for all points $(i, j) \in \mathbb{Z}^2$ that do not appear in the matrix, the mask sequence values are zero.)

To illustrate that $p^{(1)}$ satisfies the sum rules, its matrix is given below, and each time the values written in the little boxes sum up to one:

$$\begin{bmatrix} \boxed{\mathbf{\frac{1}{2}}} & \frac{1}{2} & \boxed{\mathbf{0}} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \boxed{\mathbf{0}} & \frac{1}{2} & \boxed{\mathbf{\frac{1}{2}}} \end{bmatrix}; \begin{bmatrix} \frac{1}{2} & \boxed{\mathbf{\frac{1}{2}}} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \boxed{\mathbf{\frac{1}{2}}} & \frac{1}{2} \end{bmatrix}; \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \boxed{\mathbf{\frac{1}{2}}} & 1 & \boxed{\mathbf{\frac{1}{2}}} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}; \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \boxed{\mathbf{1}} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The two control point sequences (also shown as matrices, with the same understanding that all values that are not in the matrix equal zero) are $C_0 = \begin{bmatrix} \frac{1}{2} & 2 \\ 3 & 1 \end{bmatrix}$ and $C_1 = \begin{bmatrix} 0 & 5 & 0 \\ 5 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}$ and are shown in Figures 7.1 and 7.2, respectively. Here, the initial control points are plotted each time, and straight lines are inserted to connect the points for improved visualization.

In the graphics below, the iterations of the subdivision schemes are each time plotted by plotting the relevant points, drawing a line between every two points and colouring the surface between the resulting lines. This is done in to order maintain the idea of a "flat surface", so as to emphasize the relationship between the 2-subdivision process at hand and the univariate corner-cutting algorithm of De Rham and Chaikin (see e.g. [3]). Here, the subdivision is carried out by each time splitting an existing surface in four smaller surfaces and cutting some of these smaller surfaces away.

We first show both the initial control point sequences C_0 and C_1, after which we show the 2-subdivision schemes with $p^{(1)}$ and $p^{(2)}$ applied on C_0, and then the 2-subdivision schemes with $p^{(1)}$ and $p^{(2)}$ applied on C_1.

The graphical illustrations in Figures 7.1 through 7.6 below support the fact that, from Corollary 7.2, all four of the 2-subdivision schemes $(S_{2,p^{(1)}}, C_0)$, $(S_{2,p^{(1)}}, C_1)$, $(S_{2,p^{(2)}}, C_0)$, and $(S_{2,p^{(2)}}, C_1)$ converge. In the case where the the subdivision mask is given by $p^{(2)}$, the resulting function is evidently smoother than when the mask $p^{(1)}$ is used, as could be expected by virtue of Corollary 7.2 and equation (7.28).

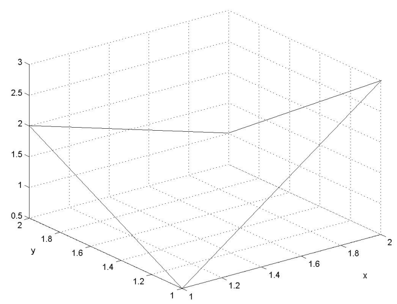


Figure 7.1: Initial plot of C_0

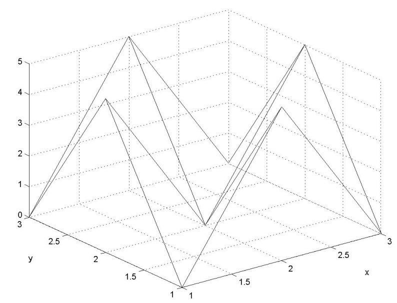
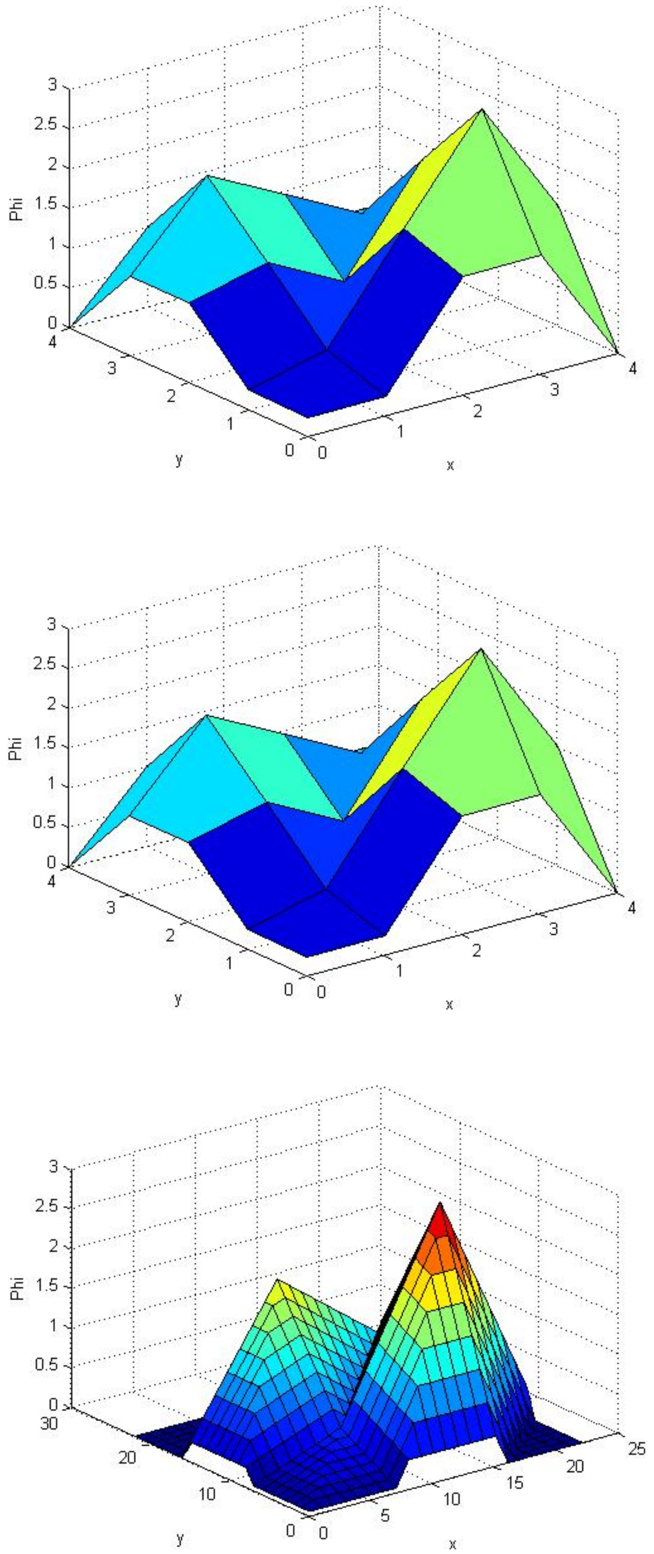


Figure 7.2: Initial plot of C_1

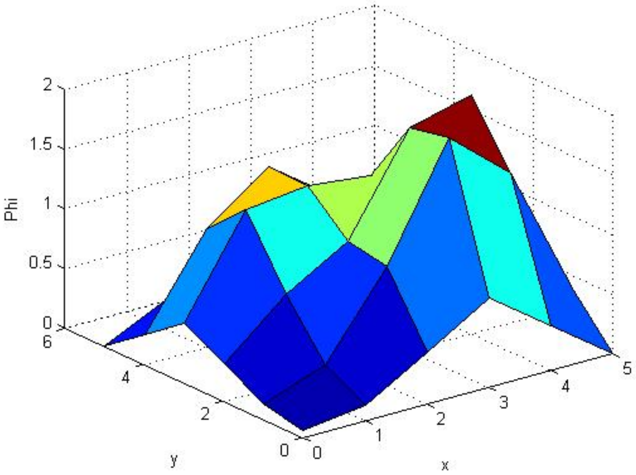


(a) $C_0, p^{(1)}$, Iter1

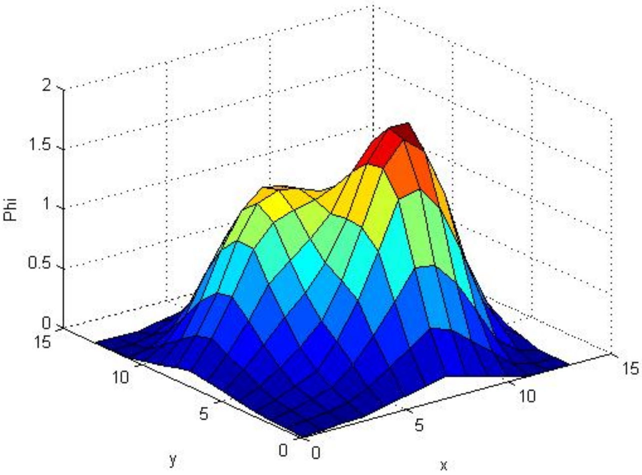
(b) $C_0, p^{(1)}$, Iter2

(c) $C_0, p^{(1)}$, Iter3

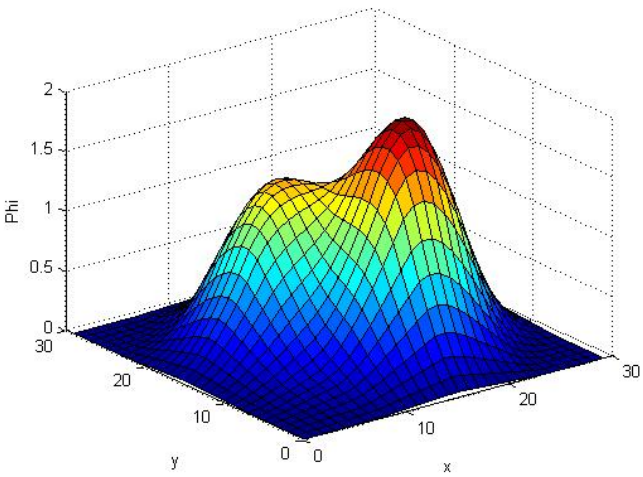
Figure 7.3: Subdivision of C_0 using the Courant hat function



(a) $C_0, p^{(2)}$, Iter1

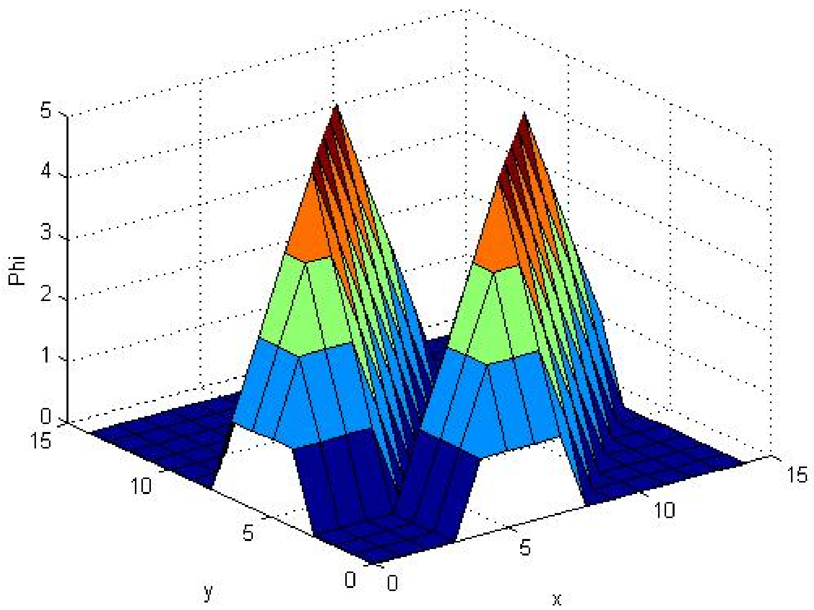


(b) $C_0, p^{(2)}$, Iter2

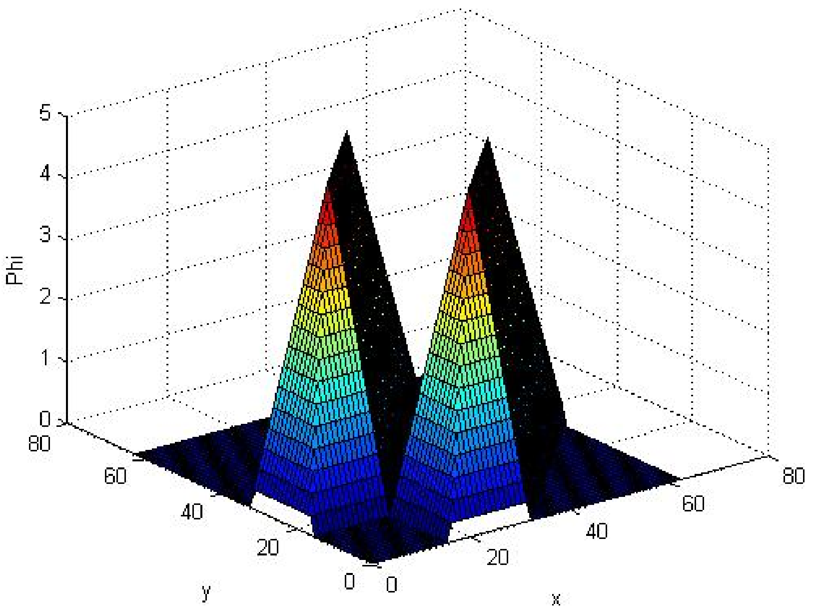


(c) $C_0, p^{(2)}$, Iter3

Figure 7.4: Subdivision of C_0 using a quadratic box spline

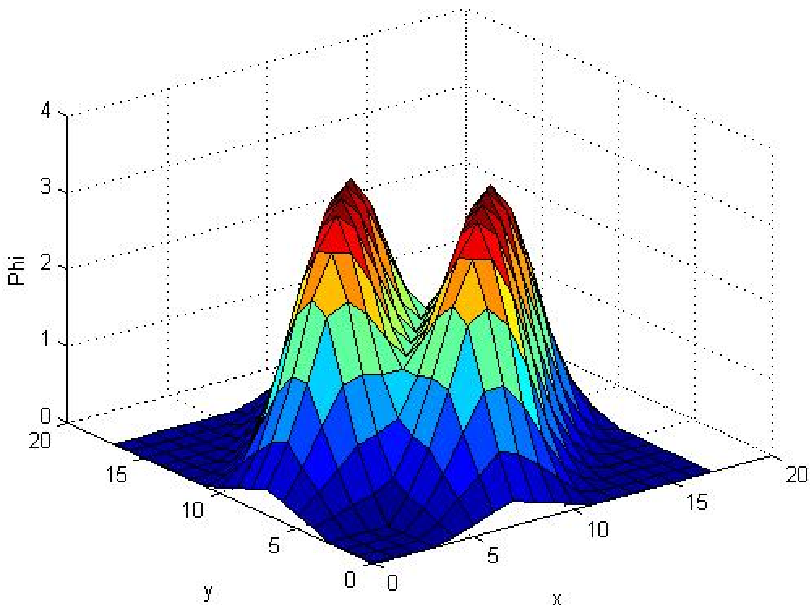


(a) $C_1, p^{(1)}$, Iter2

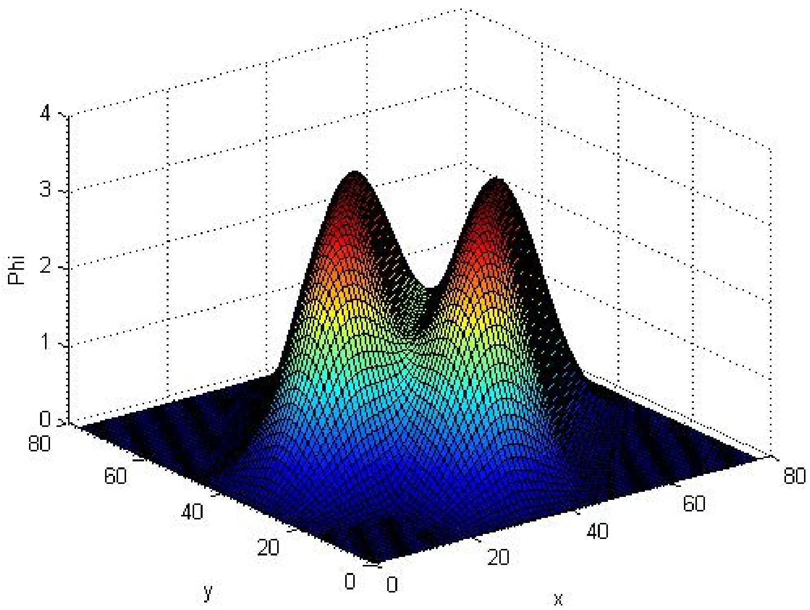


(b) $C_1, p^{(1)}$, Iter4

Figure 7.5: Subdivision of C_1 using the Courant hat function



(a) $C_1, p^{(2)}$, Iter2



(b) $C_1, p^{(2)}$, Iter4

Figure 7.6: Subdivision of C_1 using a quadratic box spline

Chapter 8

More on Refinement Equations

While the main purpose of this thesis has been to obtain a thorough understanding of refinement equations in the bivariate case, it often happened that attention was drawn to issues that were less directly associated with them. For example, as the spline functions (and box splines in particular) are perhaps the simplest examples of refinable functions, it was crucial to get to know their properties first in order to fully use them in refinement equations. In this regard, it was soon noted that a box spline is always connected with a characterizing set of direction vectors. Not only is such a direction matrix directly involved in the definition of the box spline, but properties like degree of smoothness and region of support can easily be derived by studying the direction matrix alone.

With this in mind, it should be interesting to study direction matrices in more depth in future. For instance, the following are, to our best knowledge, still open questions to be investigated: Given certain combinations of certain already familiar direction matrices, what can be said about the corresponding new box splines? What will happen to a box spline if its corresponding direction matrix is multiplied by a non-zero scalar?

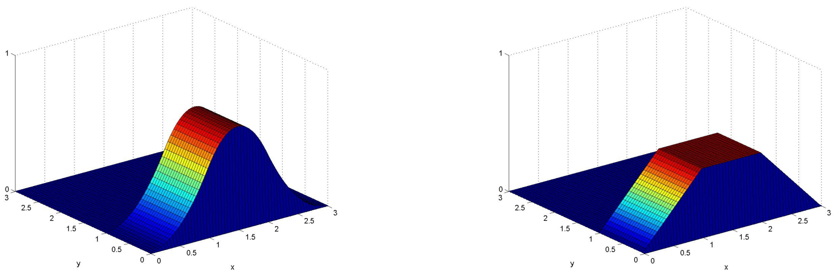
If one considers the roof function B_1 corresponding to the direction matrix $\mathcal{D}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and expands it in the x-direction in two different but seemingly related ways, namely $\hat{\mathcal{D}} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and $\tilde{\mathcal{D}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$, then the corresponding box splines $\hat{B}$ and $\tilde{B}$

(see Figure 8.1) are given, respectively, by

$$\hat{B}(x,y) = \begin{cases} \frac{1}{2}x^2 & (x,y) \in [0,1)^2 \\ -x^2 + 3x - \frac{3}{2} & (x,y) \in [1,2) \times [0,1) \\ \frac{1}{2}x^2 - 3x + \frac{9}{2} & (x,y) \in [2,3) \times [0,1) \\ 0 & (x,y) \in \mathbb{R}^2 \backslash [0,3) \times [0,1) \end{cases}$$

and

$$\tilde{B}(x,y) = \begin{cases} \frac{1}{2}x & (x,y) \in [0,1)^2 \\ \frac{1}{2} & (x,y) \in [1,2) \times [0,1) \\ -\frac{1}{2}x + \frac{3}{2} & (x,y) \in [2,3) \times [0,1) \\ 0 & (x,y) \in \mathbb{R}^2 \backslash [0,3) \times [0,1). \end{cases}$$



$\hat{B}$ $\tilde{B}$

Figure 8.1: Graphs of $\hat{B}$ and $\tilde{B}$

In both cases, the box spline has support region $(0,3) \times (0,1)$, over which it is symmetric. While $\hat{B}$ is a quadratic box spline, it is more difficult to compute. On the other hand, $\tilde{B}$ can be computed in one less step but is only linear over each part of its domain. Moreover, $\hat{B}$ has one more degree of smoothness in the x-direction.

This leads to the issue of when to integrate a box spline in a certain direction all in one step, as opposed to integrating in the same direction in a number of smaller steps.

Focussing on refinement equations again, it was soon found that the issue that complicates matters most (as distinguished from the univariate case) is the dilation matrix that reduces to a scalar in the 1-D case. The simplest dilation matrix is $M = 2I^2$ — Chapter 3 yielded adequate refinement preservation results for $M = 2I^2$, concentrating on improving the degree of smoothness of a box spline, and gave some examples; Chapter 5 considered the consequences of using dilation matrices other that $2I^2$. First, it was found that the roof function is always refinable with respect to M. While refinement preservation was proved for diagonal dilation matrices, it is still an open question whether M-refinability can be preserved for dilation matrices of full rank, and moreover whether this can be done in such a way that the new refinable function possesses improved levels of smoothness. Moreover, the question remains whether certain specific dilation matrices should be preferred in practical applications. As stated in Chapter 5, dilation matrices with low determinant values are preferable in wavelet-related applications. Are there perhaps more such criteria that arise in applications?

The issue of the support of 2-refinable functions was studied in Chapter 6, where Theorem 6.2 yielded specific bounds for the supports of 2-refinable functions that satisfied $\phi(x, y) \geq 0, (x, y) \in \mathbb{R}^2$ and corresponded to 2-refinement masks that satisfied $p_{i,j} \geq 0, (i, j) \in \mathbb{Z}^2$ and of which the support of the non-zero entries formed a convex hull. Future investigation might shed more light to this issue and give bounds for 2-refinable functions corresponding to arbitrary masks, and where negative function or refinement mask values are allowed.

Chapter 7 concluded with a practical application of 2-refinement equations. Subdivision is a very prominent tool in, amongst others, digital design, and much research is still being done with regards to the question of subdivision convergence. For 2-refinement masks that satisfy the strict positivity condition (7.49) as well as $p \in \mathcal{A}_{k,l}$, with $\mathcal{A}_{k,l}$ as in (7.24), subdivision convergence was proved with the aid of Theorems 7.4 and 7.5. It should be interesting to study subdivision schemes for dilation matrices other than $2I^2$ in future. In addition to placing conditions on the refinement mask p, an interesting question is to investigate the properties to be satisfied by the dilation matrix in order to ensure subdivision convergence.

Finally, one might investigate what happens in the refinement equation if more than two variables are introduced.
The definition (2.6) is given in [21] immediately for arbitrarily many variables, and the properties that feature in Chapter 2 are shown in [21] and [9] to still hold in the arbitrary multivariable case. Regarding refinement preservation, it should not be difficult to see that a more-variable extension of Theorem 3.1 holds. However, since functions of more than two variables are not possible to visualize, it is not a simple task to find a 2-refinable function to begin the process with, as was done for the bivariate case in Chapter 3. Similarly, the issues of specific support bounds as raised in Chapter 6 are equally hard to answer. Future research might bring more clarity to these issues. At this point it should be noted, however, that, in practical applications, the bivariate case is the one mostly used in e.g. CAD.

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