Criticality of the lower domination parameters of graphs

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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

Signature: ____________________ Date: ____________________

Abstract

In this thesis we focus on the lower domination parameters of a graph G, denoted $\pi(G)$, for $\pi \in \{i, ir, \gamma\}$. For each of these parameters, we are interested in characterizing the structure of graphs that are critical when faced with small changes such as vertex-removal, edge-addition and edge-removal. While criticality with respect to independence and domination have been well documented in the literature, many open questions still remain with regards to irredundance. In this thesis we answer some of these questions.

First we describe the relationship between transitivity and criticality. This knowledge we then use to determine under which conditions certain classes of graphs are critical. Each of the chosen classes of graphs will provide specific examples of different types of criticality. We also formulate necessary conditions for graphs to be ir-critical and ir-edge-critical.

Opsomming

In hierdie tesis fokus ons op die onderste dominasieparameters van 'n grafiek G, genaamd $\pi(G)$, vir $\pi \in \{i, ir, \gamma\}$. Vir elkeen van hierdie parameters stel ons belang in die karakterisering van die struktuur van grafieke wat krities is met betrekking tot klein veranderinge soos die verwydering van 'n nodus of 'n lyn, of die byvoeging van 'n lyn. Terwyl daar al vele resultate in die literatuur is oor die kritiekheid van onafhanklikheid en dominasie, bestaan daar nog heelwat oop vrae oor onoorbodigheid. In hierdie tesis poog ons om van die vrae te beantwoord.

Ons beskryf eers die verhouding tussen transitiwiteit en kritiekheid. Met hierdie kennis bepaal ons dan die voorwaardes waaronder sekere klasse van grafieke krities is. Elkeen van die gekose klasse verskaf vir ons spesifieke voorbeelde van die verskillende tipes kritiekheid. Ons formuleer ook noodsaaklike voorwaardes waaronder 'n grafiek krities is met betrekking tot onoorbodigheid as 'n nodus verwyder word of 'n lyn bygevoeg word.

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Glossary

Adjacent: Two *vertices* of a *graph* G are said to be *adjacent* if there exists an *edge* of G *joining* the two *vertices.*

Annihilate: For a given *graph* G, if the *private neighbourhood* of $s \in S$ relative to S for $S \subseteq V_G$ is completely contained in the *closed neighbourhood* of $v \in V_G - S$, then we say v *annihilates* s relative to S.

Automorphism: An *automorphism* of a *graph* G is an *isomorphism* of G onto itself.

Circulant: The *circulant* graph $G = C_n \langle a_1, a_2, \cdots, a_l \rangle$ is a graph with $0 < a_1 < a_2 < \cdots < a_l < n$, vertex set $V_G = \{v_1, v_2, \cdots, v_n\}$ and edge set $E_G = \{\{v_i, v_{i+j}\} \ and \ \{v_i, v_{i-j}\} : i = 1, 2, \cdots, n \ \ and \ \ j = a_1, a_2, \cdots, a_l\}$.

Closed Neighbourhood: The *closed neighbourhood* of a *vertex* v in a *graph* G is the set of all *vertices adjacent* to v in G, as well as v itself, and is denoted $N[v]$. The *closed neighbourhood* of a *vertex set* S in G is defined as $N[S] = \{N[v] : \ v \in S\}$.

Complement: The *complement* $\overline{G}$ of a *graph* G is the *graph* for which $V_{\overline{G}} = V_G$ and $e \in E_{\overline{G}}$ if and only if $e \notin E_G$.

Complete Graph: A *complete graph* of *order* n, denoted by K_n, is a *graph* in which every pair of *vertices* are *adjacent.*

Complete Multipartite Graph: The *complete multipartite graph* $K_{n_1, n_2, \cdots, n_m}$ is the complement of the disjoint union of *complete graphs* $K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_m}$.

Component: A *subgraph* H of a *graph* G is called a *component* of G if H is a maximally *connected subgraph* of G.

Connected: For *vertices* u and v of a *graph* G, u is said to be *connected* to v if G contains a $u - v$ *path*. The *graph* G is called a *connected graph* if the *vertices* u and v are connected for any pair $u, v \in V_G$.

Copy: A *copy* of G is a *graph isomorphic* to G.

Degree: The *degree* of a *vertex* v of a *graph* G is the *cardinality* of the *open neighbourhood* of v in G, and is denoted $\deg_G v$.

Disconnected: A *graph* that is not *connected* is said to be *disconnected*.

Disjoint: Two *graphs* G and H are *disjoint* if $V_G \cap V_H = \emptyset$.

Domination Number: The *(lower) domination number*, denoted $\gamma(G)$, of a *graph* G is the minimum *size* over all *minimal dominating sets* of G.

Dominating Set: A *vertex* subset $S \subseteq V_G$ of G is called a *dominating set* if every *vertex* $v \in V_G - S$ is *adjacent* to a *vertex* $u \in S$.

Edge: An *edge* is a 2–element subset of the *vertex set* of a *graph*. *Edges* are indicated by inter–connecting lines between *vertices* in graphical representations of a *graph*.

Edge Set: The set E_G, comprised of all the *edges* of a *graph* G, is called the *edge set* of the *graph*.

Edge-transitive: G is *edge-transitive* if for any $\{u_1, u_2\}, \{v_1, v_2\} \in E_G$ there exists an *automorphism* Φ of G such that $\Phi(\{u_1, u_2\}) = \{v_1, v_2\}$.

External Private Neighbourhood: For a *vertex* subset S of a *graph* G, a *vertex* $w \in V_G - S$ is called an external private neighbour of v relative to S, if $N(w) \cap S = \{v\}$. The set of all epns of v is called the *external private neighbourhood* of v relative to S, and is denoted $\text{epn}(v, S)$.

Graph: A *graph* is a finite, nonempty set of elements, called *vertices*, together with a (possibly empty) set of 2–element subsets of the *vertex set* called *edges*. A *graph* may be represented graphically as a set of nodes with inter–connecting lines.

Independence Number: The maximum *cardinality* over all *maximal independent sets* of a *graph* G is called the *independence number* of G and is denoted $\beta(G)$.

Independent Domination Number: Any *dominating set* of a *graph* G that is also *independent* is called an independent dominating set of G, the minimum *cardinality* of which is called the *independent domination number*, denoted $i(G)$.

Independent Set: A *vertex* subset S of a *graph* G is called *independent* if no two *vertices* in S are *adjacent* in G.

Induced Subgraph: For a non–empty subset $S \subseteq V_G$ of a *graph* G the so–called *induced subgraph* of S in G, denoted $\langle S\rangle_G$, is the *subgraph* of G with *vertex set* $V_{\langle S\rangle_G} = S$ and *edge set* $E_{\langle S\rangle_G} = \{uv \in E_G : u, v \in S\}$.

Irredundance Number: The *irredundance number*, denoted $IR(G)$, is the largest number of vertices in a *maximal irredundant set* of G.

Irredundant: For $S \subseteq V_G$ and $s \in S$, s is a *irredundant* vertex of S if s is not *redundant*.

Irredundant Set: $S \subseteq V_G$ is an *irredundant set* of G if the *private neighbourhood* of each vertex $s \in S$ relative to the set S is empty.

Isolated Vertex: A vertex in *graph* G is *isolated* if it is *adjacent* to no other vertices of G.

Isomorphism: An one–to–one mapping $\phi : V_G \to V_H$ between the vertex sets of two *graphs* G and H such that $uv \in E_G$ if and only if $\phi(u)\phi(v) \in E_H$.

Isomorphic: Two *graphs* G and H are called *isomorphic*, written as $G \cong H$, if there exists an *isomorphism* between their two vertex-sets.

Lower Independence Number: The *lower independence number* $i(G)$ is the smallest number of vertices in a *maximal independent set* of G.

Lower Irredundance Number: The *irredundance number* $IR(G)$ and the *lower irredundance number* $ir(G)$ are the largest and smallest number of vertices in a maximal irredundant set of G, respectively.

Maximal Independent Set: An *independent set* S of *vertices* in a *graph* G is called a *maximal independent* set if S is not a proper subset of any other *independent set* of G.

Maximal Irredundant Set: An *irredundant set* S of G is *maximal irredundant* if and only if $S \cup \{v\}$ is not *irredundant* for every $v \in V_G - S$.

Maximum Degree: The maximum of the degrees of all the vertices in a *graph* G.

Minimal Dominating Set: A *dominating set* S of a *graph* G is called a *minimal dominating* set if no proper subset of S is a *dominating set* of G.

Minimum Degree: The minimum of the degrees of all the vertices in a *graph* G.

Multipartite: An *n–partite graph* is called *multipartite* if $n > 2$.

n–partite: A *graph* G is called n–partite, $n \geq 2$, if the vertex set may be partitioned into n subsets, such that no *edge* of G connects *vertices* from the same subset.

Neighbours: If the unordered pair $\{u, v\} = uv$ is an *edge* of the graph G, it is said that the vertices u and v are *neighbours* in G.

Open Neighbourhood: The *open neighbourhood* of a *vertex* v in a *graph* G is the set of all *vertices adjacent* to v in G, and is denoted $N(v)$. The *open neighbourhood* of a set S is defined as $N[S] = \{N[v] : v \in S\}$.

Order: The *cardinality* of the *vertex set* of a *graph* G is called the *order* of G.

Product of two complete graphs: The product $G = K_m \times K_n$ of two complete graphs K_m and K_n have vertex set

$$V_G = \{v_{ij} | i = 1, 2, \cdots, m \ \ and \ \ j = 1, 2, \cdots, n\}$$

and edge-set

$$E_G = \{\{v_{ij}, v_{kl}\} | i = k \ \ and \ \ j \neq l, \ \ or \ \ j = l \ \ and \ \ i \neq k\}.$$

Private Neighbourhood: If $S \subseteq V_G$ and $s \in S$, then the *private neighbourhood* of s relative to S, denoted by $pn_G(s, S)$, is the set $N_G[s] - N_G[S - \{s\}]$.

Private Neighbours: The vertices of the *private neighbourhood* of S $(pn_G(s, S))$ are called the *private neighbours of s relative to S*.

Redundant: For $S \subseteq V_G$ and $s \in S$, s is a *redundant* vertex of S if the *private neighbourhood* of s relative to S is empty.

Symmetric: A *graph* that is *vertex-transitive* and *edge-transitive*.

Semi-symmetric: A graph that is *edge-transitive* but not *vertex-transitive*.

Singular Isolated Vertex: A vertex $v \in S$ is a *singular isolated vertex* of $S \subseteq V_G$ if $pn_G(v, S) = \{v\}$.

Size: The cardinality of the *edge set* of a *graph* G is called the *size* of G.

Subgraph: A *graph* H is called a *subgraph* of G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

Union: The *union* of two *graphs* H_1 and H_2, written as $H_1 \cup H_2$, is the *graph* H with *vertex set* $V_H = V_{H_1} \cup V_{H_2}$ and *edge set* $E_H = E_{H_1} \cup E_{H_2}$.

Universal Vertex: A *vertex* of G such that it is *adjacent* to all other vertices of G, is defined as an *universal vertex*.

Upper Domination Number: The maximum cardinality over all *minimal dominating sets* of a *graph* G is called the *upper domination number* of G, denoted $\Gamma(G)$.

Vertex: A *vertex* is a combinatorial element in terms of which a *graph* is defined. *Vertices* are indicated by nodes in the graphical representation of a *graph*.

Vertex Set: The set comprised of all *vertices* of a *graph* G, is called the *vertex set* of G.

Vertex-transitive: A *graph* G is *vertex-transitive* if for any $u, v \in V_G$ there exists an *automorphism* Φ of G such that $\Phi(u) = v$.

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Chapter 1

Graph theoretic concepts

This chapter introduces the graph theoretic definitions required in this thesis, including some basic results on independence, domination and irredundance in graphs and their related parameters. For each of these parameters we define six types of criticality and discuss the existence and characterization of these types of criticality.

1.1 Basic definitions

A **graph** $G = (V_G, E_G)$ is a finite, nonempty set of **vertices** V_G, together with a (possibly empty) set of two-element subsets of V_G, the **edges** E_G, which is denoted by $\{u, v\} = uv$. The number of vertices in a graph G is called the **order** of G, while the number of edges in G is called the **size** of G.

The **open neighbourhood** of $v \in V_G$, denoted by $N_G(v)$, is the set $\{u \in V_G | uv \in E_G\}$ and the **closed neighbourhood** $N_G[v]$ is the set $N_G(v) \cup \{v\}$. The **degree** of $v \in V_G$ is the cardinality of the open neighbourhood of v and is denoted by $deg_G(v)$. The **minimum degree** $\delta(G)$ and the **maximum degree** $\Delta(G)$ is, respectively, the minimum and maximum of the degrees taken over all the vertices of G. A vertex of degree 0 is an **isolated vertex** (it is adjacent to no other vertices of G) and a vertex of degree

$|V_G|-1$ is a **universal vertex** of G (it is adjacent to all other vertices of G). A graphical representation of the graph G_1 with order 8 and size 6 is shown in Figure 1.1. From the figure we see that $V_{G_1} = \{v_1, v_2, \cdots, v_8\}$ and $E_{G_1} = \{v_1v_2, v_2v_5, v_5v_7, v_7v_6, v_6v_3, v_3v_4\}$; while $\delta(G_1) = 0$ and $\Delta(G_1) = 2$. Also, v_8 is an isolated vertex of G_1 and there exists no universal vertices in G_1.

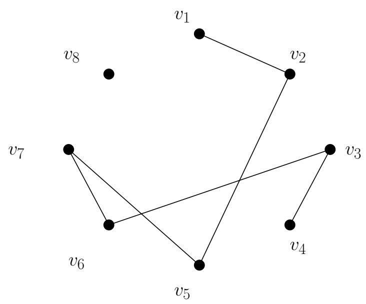

Figure 1.1: *Graphical representation of the graph G_1*

If $uv \in E_G$, it is said that the vertices u and v are **adjacent** in G or that they are **neighbours** in G. With reference to the graph G_1 in Figure 1.1 , we have v_1 adjacent to v_2, while v_4 and v_5 are not adjacent. Also, the neighbours of v_6 include v_7 and v_3.

Two graphs G and H are **isomorphic**, denoted $G \cong H$, if there exists a bijection $\phi : V_G \to V_H$ such that $uv \in E_G$ if and only if $\phi(u)\phi(v) \in E_H$. It is clear that if $G \cong H$, then from the definition of isomorphisms $\overline{G} \cong \overline{H}$. A graph that is isomorphic to a subgraph of G is also called a subgraph of G. The **subgraph induced by a vertex-set** S of G, denoted by $G\langle S\rangle$, has vertex-set S and edge-set $\{uv \in E_G | u, v \in S\}$. A **copy** of G is a graph isomorphic to G. Two graphs G and H are **disjoint** if V_G and V_H are disjoint. The **union** $G \cup H$ of two graphs has $V_{G\cup H} = V_G \cup V_H$ and $E_{G\cup H} = E_G \cup E_H$. Thus the disjoint union of G and H is the union of the disjoint copies of G and H, and the disjoint union of n copies of G will be denoted by nG.

A graph G is **connected** if for any partition $\{V_1, V_2\}$ of V_G there exists a $v_1 \in V_1$ and a $v_2 \in V_2$ such that $v_1v_2 \in E_G$; otherwise G is **disconnected**. A **component** of G is a maximally connected subgraph of G. Suppose G is disconnected and let $\{V_1, V_2\}$ be a partition of V_G such that no vertices of V_1 are adjacent to any vertices of V_2. Then G is the disjoint union of the induced subgraphs $G\langle V_1\rangle$ and $G\langle V_2\rangle$ of G. Each of these subgraphs, if disconnected, can in turn be written as the disjoin union of two induced subgraphs. This procedure terminates in the decomposition of G into its components.

1.2 Domination parameters

The **closed neighbourhood** $N_G[S]$ of a set $S \subseteq V_G$ is the set $\bigcup_{s\in S} N_G[s]$, and the **open neighbourhood** $N_G(S)$ of a set $S \subseteq V_G$ is the set $N_G[S] - S$. If $S, T \subseteq V_G$, then S **dominates** T in G if $T \subseteq N_G[S]$ and if $v \in N_G[S]$, then we say S dominates v. If $S \subseteq V_G$ and $s \in S$, then s is an **isolated vertex of** S **in** G if $N_G(s) \cap S = \emptyset$, *i.e.* s is an isolated vertex of the graph $G\langle S\rangle$.

If $S \subseteq V_G$ and $s \in S$, then the **private neighbourhood** of s relative to S, denoted by $pn_G(s, S)$, is the set $N_G[s] - N_G[S - \{s\}]$. The vertices of $pn_G(s, S)$ are called the **private neighbours of** s **relative to** S. If $pn_G(s, S) = \emptyset$, then s is a **redundant** vertex of S, otherwise it is an **irredundant** vertex of S. We refer to $pn_G(s, S) - S$ as the **external private neighbours** of s relative to S and we denote it by $epn_G(s, S)$. If $pn_G(s, S) \subseteq N[v]$ for $s \in S$ and $v \in V - S$, then we say that v **annihilates** s relative to S.

$S \subseteq V_G$ is an **independent set** of G if $N_G(s) \cap S = \emptyset$ for every $s \in S$, that is, no two vertices in S are adjacent in G. Observe that independence is a **hereditary** property, *i.e.* every subset of an independent set is independent. Consequently, an independent set S of G is maximal independent if and only if $S \cup \{v\}$ is not independent for all $v \in V_G - S$. The **independence number** $\beta(G)$ is the largest number of vertices in a maximal independent set of G, and the **lower independence number** $i(G)$ is the smallest number of vertices in a maximal independent set of G.

$S \subseteq V_G$ is a **dominating set** of G if S dominates V_G, *i.e.* every vertex of $V_G - S$ is adjacent to at least one vertex of S. Domination is a **super-hereditary** property, as clearly every superset of a dominating set is dominating. It follows that a dominating set S of G is minimal dominating if and only if $S - \{s\}$ is not dominating for every $S \in S$. The **domination number** $\gamma(G)$ and the **upper domination number** $\Gamma(G)$ are the smallest and largest number of vertices in a minimal dominating set of G, respectively.

$S \subseteq V_G$ is an **irredundant set** of G if $pn_G(s, S) \neq \emptyset$ for every $s \in S$; thus every vertex of S is either isolated in S, or has an external private neighbour. As in the case of independence, irredundance is also a hereditary property and hence an irredundant set S of G is maximal irredundant if and only if $S \cup \{v\}$ is not irredundant for every $v \in V_G - S$. The **irredundance number** $IR(G)$ and the **lower irredundance number** $ir(G)$ are the largest and smallest number of vertices in a maximal irredundant set of G, respectively.

In this thesis we only consider the **lower domination parameters** ir, γ, i. Also, all the theory related to a graph G assumes that G is a **connected graph**. For a disconnected graph we can just take the decomposition of components as described in Section 1.1, and apply the theory to each of the connected components, since $\pi(G \cup H) = \pi(G) + \pi(H)$ for π a lower domination parameter. Finally, by a π-set of G we mean a subset of V_G realising $\pi(G)$ for $\pi \in \{ir, \gamma, i\}$.

1.3 Some basic results on domination parameters

In this section we briefly state some important results and bounds that will be used in Chapters 2, 3 and 4. The next proposition gives a characterization of maximal independence. Because of this characterization, the lower independence number $i(G)$ of a graph G is also known as the **independent domination number**.

Proposition 1.1 (Berge [3]) *S is a maximal independent set of G if and only if S is an independent dominating set of G.*

Cockayne and Hedetniemi [6] obtained the following characterization of minimal domination.

Proposition 1.2 (Cockayne and Hedetniemi [6]) *S is a minimal dominating set of G if and only if S is an irredundant dominating set of G.*

Using the concept of annihilation, Cockayne, Grobler, Hedetniemi and McRae [7] derived a characterization of maximal irredundance that states the following:

Proposition 1.3 (Cockayne et al [7]) *Suppose S is an irredundant set of G with $U = V_G - N[S]$. Then S is maximal irredundant if and only if for every $v \in N[U]$ there exists an $s_v \in S$ such that v annihilates s_v relative to S.*

The implication of this proposition is that for a given graph G and ir-set S of G, every $v \in U$ annihilates some $s \in S$ and every $u \in N(U)$ must also annihilate some $s \in S$. We use this proposition in our study of graphs that are critical with respect to irredundance, since it helps us to define the structure of these graphs.

Since we will only be examining the lower domination parameters of a graph G, we state the following well-known relationship.

Proposition 1.4 (Cockayne, Hedetniemi and Miller [8]) *For any graph G we have*

$$ir(G) \leq \gamma(G) \leq i(G) \tag{1.1}$$

1.4 Criticality

We are interested in how the lower domination parameters vary when the structure of the graph is slightly changed. For each of the lower domination parameters we define six types of criticality. For $\pi \in \{ir, \gamma, i\}$, the graph G is

- **π-critical** if $\pi(G - v) < \pi(G)$ for all $v \in V_G$.
- **π^+-critical** if $\pi(G - v) > \pi(G)$ for all $v \in V_G$.
- **π-edge-critical** if $\pi(G + uv) < \pi(G)$ for all $uv \in E_{\overline{G}}$.
- **π^+-edge-critical** if $\pi(G + uv) > \pi(G)$ for all $uv \in E_{\overline{G}}$.
- **π-ER-critical** if $\pi(G - uv) > \pi(G)$ for all $uv \in E_G$.
- **π^--ER-critical** if $\pi(G - uv) < \pi(G)$ for all $uv \in E_G$.

Now we need to determine the existence of graphs with these types of criticalities. Firstly, all edgeless graphs with more than one vertex are π-critical and π-edge-critical for $\pi \in \{ir, \gamma, i\}$, while all stars $K_{1,n}$ with $n \geq 1$ are π-ER-critical for $\pi \in \{ir, \gamma, i\}$. In [15], the following was shown: no π^+-critical or π^+-edge-critical graphs for $\pi \in \{ir, \gamma, i\}$ exists, and no γ^--ER-critical graphs exists. It was also shown that there do exist graphs which are i^--ER-critical, but the existence of ir^--ER-critical graphs is still an open question.

Finally, we turn our attention to two useful results. In [15] Grobler proved the following proposition. We use this in proving under which circumstances certain classes of graphs are critical.

Proposition 1.5 (Grobler [15]) *$ir(G) = \gamma(G)$ if and only if there exists an ir-set S of G and an $x \in V_G$ such that $S \cup \{x\}$ is a dominating set of G.*

The following inequality was obtained by Allan and Laskar in 1978 and still proves to be a very useful relationship.

Proposition 1.6 (Allan and Laskar [1]) *For any graph G, $\gamma(G) \leq 2ir(G) - 1$.*

1.5 Vertex-transitivity and edge-transitivity

In this section we focus on vertex- and edge-transitivity. This will be important in our comparison of the different types of criticalities. Let us first define the basic terminology needed for this section, and then we will define the relationship between criticality and transitivity. In this section we will also briefly refer to Group Theory terminology which the reader can find in [14].

An **automorphism** of a graph G is an isomorphism of G onto itself. A graph G is **vertex-transitive** if for any $u, v \in V_G$ there exists an automorphism ϕ of G such that $\phi(u) = v$. This implies that G is vertex-transitive if and only if the group of all automorphisms of G acting on V_G produces only one orbit. Also, from the definition of vertex-transitivity it follows that these automorphisms retain adjacency and non-adjacency between vertices and the between the neighbours of the vertices; hence $|N(u)| = |N(v)| = k$ for all $u, v \in V_G$ and a fixed k. Thus a graph G is vertex-transitive if and only if $deg_G(v) = k$ for all $v \in V_G$ and each $u, v \in V_G$ is contained in the same cycles. A graph G is **edge-transitive** if for any $u_1u_2, v_1v_2 \in E_G$ there exists an automorphism ϕ of G such that $\phi(\{u_1, u_2\}) = \{v_1, v_2\}$. Similarly, this implies that G is edge-transitive if and only if the group of all automorphisms of G acting on E_G produces only one orbit.

A graph that is vertex-transitive and edge-transitive is called **symmetric** (for example K_3), while a graph that is edge-transitive but not vertex-transitive is called a **semi-symmetric** graph. To obtain a semi-symmetric graph, we take any symmetric graph, except a cycle, and replace each edge with a path of two edges through a new vertex of degree 2.

An example of a class of graphs that is vertex-transitive but not edge-transitive is the product of K_2 with any symmetric graph, except K_1 and K_2.

The next proposition shows an important relation between semi-symmetric graphs and bipartite graphs. As mentioned previously, we only consider connected graphs.

Proposition 1.7 *If a graph G is edge-transitive but not vertex-transitive, then it is bipartite.*

Proof: Suppose a graph G is edge-transitive but not vertex-transitive. Let the group Φ of all automorphisms of G act on V_G. Since G is not vertex-transitive, this action produces more than one orbit. We now show that it produces exactly two orbits and that they form two independent sets that partition V_G.

Let A and B be two of the orbits. Consider any $a \in A$ and $b \in B$. Since G is edge-transitive, it follows that for each $x \in N(a)$ and $y \in N(b)$ there exists a $\phi \in \Phi$ such that $\phi(\{a, x\}) = \{b, y\}$. Therefore, for each $x \in N(a)$ and $y \in N(b)$ we have $\phi(a) = y$ and $\phi(x) = b$; hence $N(b) \subseteq A$ and $N(a) \subseteq B$. This holds for any $a \in A$ and $b \in B$; hence A and B are independent sets of G and $A \cup B = V_G$. ■

Thus it follows that a connected non-bipartite edge-transitive graph must be vertex-transitive.

The **complement** $\overline{G}$ of the graph G has $V_{\overline{G}} = V_G$ and $E_{\overline{G}} = \{\{u, v\} | \{u, v\} \notin E_G\}$. We now show that the complement of a vertex-transitive graph is also vertex-transitive.

Proposition 1.8 *If G is vertex-transitive, then $\overline{G}$ is vertex-transitive.*

Proof: Take any $u, v \in V_{\overline{G}}$. Then $u, v \in V_G$. Since G is vertex-transitive, there exists an automorphism ϕ of G such that $\phi(u) = v$. The automorphism group acting on a graph G is the same as the automorphism group acting on $\overline{G}$, thus ϕ is also an automorphism of $\overline{G}$ such that $\phi(u) = v$. Thus $\overline{G}$ is vertex-transitive. ■

Unlike vertex-transitivity, if G is edge-transitive, then $\overline{G}$ is not necessarily edge-transitive. The graph $G = C_6$ is edge-transitive (and vertex-transitive), but the complement $\overline{G} = K_2 \times C_3$ is not, since there exists no automorphism ϕ such that $\phi(\{u, v\}) = \{w, x\}$ for $uv \in C_3$ and $wx \in K_2$. Also, the graph $H = K_{2,5}$ is edge-transitive (but not vertex-transitive), while the complement $\overline{H}$ is not edge-transitive.

1.6 Outline

Since 1979, graphs that are critical with respect to domination have been thoroughly studied by Brigham, Chinn and Dutton [5], Sumner and Blitch [19] and Walikar and Acharya [20]. This was extended by Ao [2] to the study of graphs critical with respect to independence, and the study of irredundance critical graphs was initiated by Grobler [15]. Some open questions still remained, especially involving the characterization of graphs critical with respect to irredundance. The purpose of this thesis is to answer some of these questions.

Chapter 2 deals with vertex-critical graphs. In Section 2.1 we present basic results concerning the lower domination parameters of vertex-critical graphs and the characterization of vertex-critical graphs in terms of singular isolated vertices. We also characterize vertex-critical graphs that are vertex-transitive. This is then implemented in Section 2.2 to determine the vertex-transitivity and vertex-criticality of four classes of graphs. This leads us to the formulation of some conjectures, which we examine further in Section 2.3. In Section 2.3 we also focus on determining necessary conditions for ir-critical graphs.

In Chapter 3 we examine edge-critical graphs. In Section 3.1 we state results and relationships concerning the lower domination parameters of edge-critical graphs and the characterization of edge-critical graphs in terms of singular isolated vertices. We also characterize edge-critical graphs G such that $\overline{G}$ is edge-transitive. This is then implemented in Section 3.2 to determine the edge-criticality of three classes of graphs, leading to some conjectures. In Section 3.3 we focus on the structure of ir-edge-critical graphs, and determine the validity of some of the conjectures in Section 3.2.

Chapter 4 deals with edge-removal-critical graphs. In Section 4.1 we present the results and relationships of the π-ER-critical graphs for $\pi \in \{i, ir, \gamma\}$, and characterize π-ER-critical graphs for $\pi \in \{i, \gamma\}$. In Section 4.2 we determine which of the classes of graphs of Section 2.2 are π-ER-critical for $\pi \in \{i, ir, \gamma\}$. In Section 4.3 we state the necessary conditions for a connected graph to be ir-ER-critical and show that these graphs are neither vertex-transitive nor edge-transitive. We then discuss all the results thus far achieved in the literature with respect to the characterization of ir-ER-critical graphs. In Section 4.4 we examine π^--ER-critical graphs and list some results concerning the lower domination parameters of these graphs. We then finally use this knowledge to determine whether some classes of graphs are π^--ER-critical for $\pi \in \{i, ir\}$.

In Chapter 5 we conclude by briefly listing some remaining open questions and future recommended work. In all the chapters the results and proofs which are given without references are the original work of the author.

Chapter 2

Vertex-criticality of the lower domination parameters

In this chapter we examine π-critical graphs for $\pi \in \{ir, \gamma, i\}$. From Chapter 1 we recall that a graph G is π-critical if and only if $\pi(G - v) < \pi(G)$ for all $v \in V_G$, while G is π^+-critical if and only if $\pi(G - v) > \pi(G)$ for all $v \in V_G$. Also, the edgeless graphs $\overline{K_n}$ with $n \geq 2$ are π-critical, but it was shown by Grobler in [15] that there exist no π^+-critical graphs for $\pi \in \{ir, \gamma, i\}$.

2.1 Basic results on vertex-criticality

We define a k-π-critical graph to be a π-critical graph G such that $\pi(G) = k$ for π a lower domination parameter. As mentioned in Section 1.6, graphs that are γ-critical were initially studied by Brigham, Chinn and Dutton in [5]. Some of their basic results included:

Lemma 2.1 (Brigham, Chinn and Dutton [5]) *The only* 2-γ-*critical graphs are* $\overline{nK_2}$ *with* $n \geq 1$.

Lemma 2.2 (Brigham, Chinn and Dutton [5]) *For any graph G and any $v \in V_G$, $\gamma(G-v) \geq \gamma(G) - 1$.*

This result implies that if G is a γ-critical graph, then $\gamma(G-v) = \gamma(G) - 1$ for all $v \in V_G$.

These three authors also presented some properties for n-γ-critical graphs, concerning bounds on the order of G in terms of $\Delta(G)$, $\gamma(G)$ and the size of G. Clearly G is γ-critical if and only if every component of G is γ-critical. Parallel to this, in [5] it was shown that G is γ-critical if and only if every block of G is γ-critical.

Graphs that are i-critical were then studied by Suqin Ao [2] in her masters thesis, where she obtained results for i-critical graphs analogous to those in [5] for γ-critical graphs.

Lemma 2.3 (Ao [2]) *The only 2-i-critical graphs are $\overline{nK_2}$ with $n \geq 1$.*

Lemma 2.4 (Ao [2]) *For any graph G and any $v \in V_G$, $i(G-v) \geq i(G) - 1$.*

She also obtained bounds on the order of n-i-critical graphs similar to those for n-γ-critical graphs, and showed that G is i-critical if and only if every block of G is i-critical. In her study of γ-critical and i-critical graphs, Lemmas 2.5 and 2.6 played a very important role.

Lemma 2.5 (Ao [2]) *If there exists vertices $u, v \in V_G$ such that $N[v] \subseteq N[u]$, then G is not γ-critical.*

Lemma 2.6 (Ao [2]) *If there exists vertices $u, v \in V_G$ such that $N[v] \subseteq N[u]$, then G is not i-critical.*

Graphs that are ir-critical were first studied by Grobler in [15], where he obtained a result similar to those of Lemmas 2.1 and 2.3. Recall that π a lower domination parameter means that $\pi \in \{ir, \gamma, i\}$.

Proposition 2.1 (Grobler [15]) *For π a lower domination parameter, the only 2-π-critical graphs are $\overline{nK_2}$, $n \geq 1$.* ■

We will see in Section 2.3 that a result similar to Lemmas 2.2 and 2.4 does not hold for ir-critical graphs. The next proposition gives a characterization of γ-critical and i-critical graphs in terms of **singular isolated vertices**. We define $v \in V_G$ as a **singular isolated vertex** of $S \subseteq V_G$ if $pn_G(v, S) = \{v\}$; thus v is an isolated vertex of S that has no external private neighbours relative to S. But first a lemma.

Lemma 2.7 (Grobler [15]) *Let $\pi \in \{\gamma, i\}$. For any graph G with more than one vertex, $\pi(G - v) = \pi(G) - 1$ if and only if v is a singular isolated vertex of some π-set of G.* ■

Proposition 2.2 (Grobler [15]) *Let $\pi \in \{\gamma, i\}$. For any graph G with more than one vertex,*

(a) G is π-critical if and only if $\pi(G - v) = \pi(G) - 1$ for all $v \in V_G$.

(b) G is π-critical if and only if every vertex of G is a singular isolated vertex of some π-set of G. ■

In the case of vertex-transitive graphs, we have

Proposition 2.3 *Let $\pi \in \{\gamma, i\}$. For any vertex-transitive graph G with more than one vertex, G is π-critical if and only if G has a π-set containing a singular isolated vertex.*

Proof: Assume G has a π-set T containing a singular isolated vertex v. Since G is vertex-transitive, there exists for any $u \in V_G$ an automorphism ϕ such that $\phi(v) = u$. Thus $\phi(T)$ is a π-set of G with u a singular isolated vertex. As this is true for any $u \in V_G$, it follows from Proposition 2.2(b) that G is π-critical. ■

Finally, the following result shows an important relationship between i-critical and γ-critical and between γ-critical and ir-critical graphs.

Proposition 2.4 *For any graph G,*

(a) if G is i-critical and $i(G) = \gamma(G)$, then G is γ-critical.

(b) if G is γ-critical and $\gamma(G) = ir(G)$, then G is ir-critical.

Proof:

(a) Suppose G is i-critical and $i(G) = \gamma(G)$. From Proposition 1.4 it follows that $\gamma(G - v) \leq i(G - v) < i(G) = \gamma(G)$ for all $v \in V_G$.

(b) Suppose G is γ-critical and $\gamma(G) = ir(G)$. From Proposition 1.4 it follows that $ir(G - v) \leq \gamma(G - v) < \gamma(G) = ir(G)$ for all $v \in V_G$. ■

This proposition is integral in our proofs of vertex-criticality for the different classes of graphs with respect to their lower domination parameters.

2.2 The vertex-criticality of some classes of graphs

In this section we consider four classes of graphs, namely the complete multipartite graphs, the product of two complete graphs, the complement of the product of two complete graphs and the circulants. In [15] Grobler determined the lower domination parameters for each of these classes. By applying the theory given in Chapter 1 and Section 2.1 and these values for the lower domination parameters, we now determine which of these classes of graphs are vertex-critical.

The **complete multipartite graph** $K_{n_1,n_2,\cdots,n_m}$ is the complement of the disjoint union $K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_m}$. Thus the vertex-set of $K_{n_1,n_2,\cdots,n_m}$ has partition $\{V_1, V_2, \cdots, V_m\}$ with $|V_i| = n_i$ for $1 \leq i \leq m$, and uv is an edge of $K_{n_1,n_2,\cdots,n_m}$ if and only if u and v do not belong to the same partite set. The graph $K_{4,4}$ is shown in Figure 2.1.

Clearly, $K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_m}$ for $m \geq 2$ is vertex-transitive if and only if $n_1 = n_2 = \cdots = n_m$. Thus from Proposition 1.8 it follows that $K_{n_1,n_2,\cdots,n_m}$ for $m \geq 2$ is vertex-transitive if and only if $n_1 = n_2 = \cdots = n_m$.

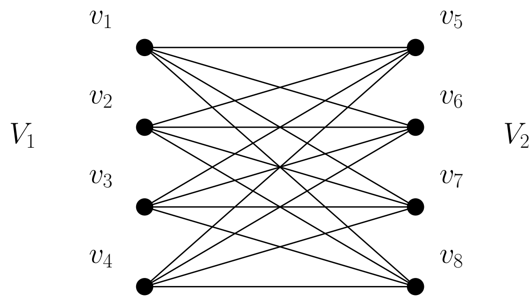

Figure 2.1: $K_{4,4}$

Proposition 2.5 (Grobler [15]) *If $G = K_{n_1,n_2,\cdots,n_m}$ with $m \geq 2$, then*

$$ir(G) = \gamma(G) = \begin{cases} 2 & \text{if} \quad n_i > 1 \quad for \quad all \quad i = 1, 2, \cdots, m \\ 1 & \text{otherwise} \end{cases}$$

$$i(G) = min\{n_i : i = 1, 2, \cdots, m\}$$

■

Now we determine which of the complete multipartite graphs are vertex-critical, using the values of the lower domination parameters as given above.

Proposition 2.6 *Let $G = K_{n_1,n_2,\cdots,n_m}$ with $m \geq 2$.*

1. *For $\pi \in \{ir, \gamma\}$, G is π-critical if and only if $n_1 = n_2 = \cdots = n_m = 2$.*

2. *G is i-critical if and only if $n_1 = n_2 = \cdots = n_m \geq 2$.*

Proof: If $n_i = 1$ for some $i = 1, 2, \cdots, m$, then $\pi(G) = 1$; hence G is not π-critical for $\pi \in \{i, ir, \gamma\}$. Assume therefore that $n_i > 1$ for all $i = 1, 2, \cdots, m$.

1. By Proposition 2.5, if $\pi \in \{ir, \gamma\}$, then $\pi(G) = 2$. Hence by Proposition 2.1, G is π-critical if and only if $G = K_{2,2,\cdots,2}$.

2. If $n_1 = n_2 = \cdots = n_m = n$ (say), then $i(G) = n$ and $i(G - v) = n - 1$ for all $v \in V_G$ by Proposition 2.5. If $n_k > n_j$ for some $k \neq j$, then $i(G) = min\{n_i : i = 1, 2, \cdots, m\} = i(G - v)$ for $v \in V_k$. ■

From this proposition it follows that $K_{n_1,n_2,\cdots,n_m}$ is an example of a graph that is i-critical, but neither γ-critical nor ir-critical for $n_1 = n_2 = \cdots = n_m \geq 3$.

Next we turn our attention to the class of graphs defined as the **product of two complete graphs**. The product $K_m \times K_n$ of two complete graphs K_m and K_n have vertex set

$$V = \{v_{ij} | i = 1, 2, \cdots, m \;\; and \;\; j = 1, 2, \cdots, n\}$$

and edge-set

$$E = \{\{v_{ij}, v_{kl}\} | i = k \;\; and \;\; j \neq l, \;\; or \;\; j = l \;\; and \;\; i \neq k\}$$

Let

$$X_i = \{v_{ik} | k = 1, 2, \cdots, n\} \;\; and \;\; Y_j = \{v_{kj} | k = 1, 2, \cdots, m\}$$

for each $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. Refer to Figure 2.2 for an illustration of $K_m \times K_n$ for $m = n = 3$.

It is easy to see that $G = K_m \times K_n$ with $n \geq m \geq 2$ is vertex-transitive, since there exist an automorphism ϕ_1 of G such that $\phi_1(v_{i,j}) = v_{i,j+1}$ and $\phi_1(v_{i,n}) = v_{i,1}$ for each

$i = 1, 2, \cdots, m$ and an automorphism ϕ_2 of G such that $\phi_2(v_{i,j}) = v_{i+1,j}$ and $\phi_2(v_{m,j}) = v_{1,j}$ for each $j = 1, 2, \cdots, n$.

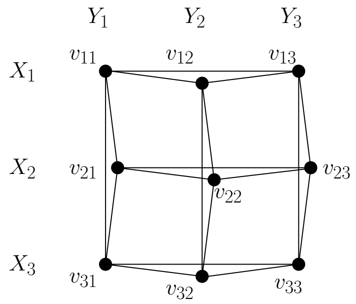

Figure 2.2: $K_3 \times K_3$

Proposition 2.7 (Grobler [15]) *If $G = K_m \times K_n$ with $n \geq m \geq 2$, then*

$$ir(G) = \gamma(G) = i(G) = m$$

■

The following result shows that the product of two complete graphs is vertex-critical for all the lower domination parameters under the assumption that $n = m$.

Proposition 2.8 *Let $G = K_m \times K_n$ with $n \geq m \geq 2$. Then G is π-critical, for $\pi \in \{i, ir, \gamma\}$, if and only if $n = m$.*

Proof: Suppose $m = n$. Then the i-set consisting of the diagonal vertices $\{v_{ii} : 1 \leq i \leq n\}$ contains a singular isolated vertex, namely v_{11}. Since G is vertex-transitive it follows from Proposition 2.3 that G is i-critical, and from Propositions 2.4((a) and (b)) and 2.7 it then follows that G is also ir-critical and γ-critical.

Suppose now that $m < n$. We show that there is no γ-set S that contains a singular isolated vertex. Suppose without loss of generality that v_{11} is a singular isolated vertex of some γ-set S of G. Then $Y_j \cap S \neq \emptyset$ for $j = 2, \cdots, n$. Therefore $|S| \geq n$. This contradicts $|S| = m < n$. It follows that no γ-set of G exists such that v_{11} is a singular isolated vertex of that set. Thus by Proposition 2.2, G is not γ-critical, and from Proposition 2.4 it follows that G is also not i-critical. Thus $\gamma(G - v) = i(G - v) = m$ for any $v \in V_G$.

We now show that $ir(G - v) = m$ for any $v \in V_G$. Since G is vertex-transitive, we can assume without loss of generality that $ir(G - v_{mn}) = p < m$. Let $H = G - v_{mn}$ and consider an ir-set S of H. Since $\gamma(H) = m$, S is not a dominating set of H. Let $u = v_{kl}$ be a vertex of H not dominated by S. Then, since S is maximal irredundant, u annihilates some non-isolated vertex $s = v_{ij}$ of S. Therefore $pn_H(s, S) = \{v_{il}\}$ or $\{v_{kj}\}$. Now if $m = 2$, then $p = 1$ and the graph is not ir-critical. Thus $n > m \geq 3$ for $i \neq k$, $j \neq l$. If $pn_H(s, S) = \{v_{il}\}$, then $S \cap Y_t \neq \emptyset$ for $1 \leq t \leq n-1$, $t \neq l$ and $|S \cap Y_j| \geq 2$. Therefore $|S| \geq n - 1 \geq m$, contradicting $ir(H) < m$; hence $pn_H(s, S) = \{v_{kj}\}$. If $j < n$, then $S \cap X_t \neq \emptyset$ for $1 \leq t \leq m$, $t \neq k$ and $|S \cap X_i| \geq 2$. Therefore $|S| \geq m$, contradicting $ir(H) < m$. Thus $j = n$, *i.e.* $s \in Y_n$. Now $S \cap X_t \neq \emptyset$ for $1 \leq t \leq m-1$, $t \neq k$ and $|S \cap X_i| \geq 2$; hence $|S| \geq m-1$ and therefore $|S| = m-1$. It follows that $S \cap X_m = \emptyset$ and therefore $w = v_{ml}$ is also not dominated by S. Therefore w annihilates some non-isolated vertex t of H. By using the same argument as above, $t \in Y_n$, which is impossible since $pn_H(s, S) = \{v_{kn}\}$. ■

We next consider the complement of the product of two complete graphs (for example Figure 2.3). From Proposition 1.8 it follows that $\overline{K_m \times K_n}$ is vertex-transitive for any $n \geq m \geq 2$.

In [15] the following values for the lower domination parameters of these graphs were obtained:

Proposition 2.9 (Grobler [15]) *If $G = \overline{K_m \times K_n}$ with $n \geq m \geq 2$, then*

$$ir(G) = \gamma(G) = min\{3, m\} \quad \text{and} \quad i(G) = m$$

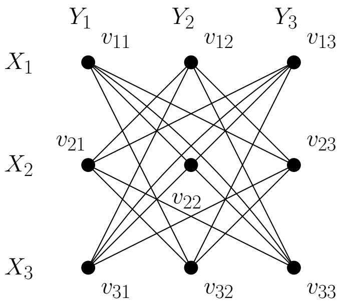

Figure 2.3: $\overline{K_3 \times K_3}$

■

This enables us in the following proposition to determine under which conditions the complement of the product of two complete graphs is vertex-critical.

Proposition 2.10 *Let* $G = \overline{K_m \times K_n}$ *with* $n \geq m \geq 2$.

1. *For* $\pi \in \{ir, \gamma\}$*,* G *is* π*-critical if and only if* $n \geq m = 3$.

2. G *is* i*-critical if and only if* $n \geq m \geq 3$.

Proof: Let $m = 2$. If $\pi \in \{i, ir, \gamma\}$, then from Proposition 2.9 we have $\pi(G) = 2$. Thus it follows from Proposition 2.1 that G is not π-critical.

Now assume $m \geq 3$. The i-set $\{v_{11}, v_{21}, \cdots, v_{m1}\}$ contains m singular isolated vertices. Since G is vertex-transitive, it follows from Proposition 2.2 that G is i-critical. Furthermore, if $m = 3$ it follows from Proposition 2.4 that G is also ir-critical and γ-critical.

Suppose now $m \geq 4$. We want to show that G is neither ir-critical nor γ-critical. From Proposition 2.9 we know that $ir(G) = \gamma(G) = 3$. Without loss of generality remove v_{mn}. Since $H = G - v_{mn}$ contains no universal vertices, $ir(H) > 1$; hence assume $ir(H) = 2$ and let $S = \{s_1, s_2\}$ be an ir-set of H. From Proposition 2.9 we have $i(G) = m$, and since G is i-critical, it follows that $i(H) = m - 1 > 2 = ir(H)$; hence S is not independent; hence

$s_1 = v_{ij}$ and $s_2 = v_{kl}$, with $i \neq k$ and $j \neq l$. Then $S \cup \{v_{xy}\}$ with $v_{xy} \in N(s_1) \cap N(s_2)$ is still an irredundant set in H, since $pn_H(s_1, S \cup \{v_{xy}\}) \neq \emptyset$, $pn_H(s_2, S \cup \{v_{xy}\}) \neq \emptyset$ and $pn_H(v_{xy}, S \cup \{v_{xy}\}) \neq \emptyset$. Thus S is not maximal irredundant as previously assumed; thus G is not ir-critical, and since $ir(G) = \gamma(G)$, it follows from Proposition 2.4 that G is not γ-critical. ■

It follows that if $n \geq m \geq 4$, then $G = \overline{K_m \times K_n}$ is another example of a class of graphs that is i-critical but neither γ-critical nor ir-critical.

Lastly, we define the **circulant** $C_n \langle a_1, a_2, \cdots, a_l \rangle$ with $0 < a_1 < a_2 < \cdots < a_l < n$ by specifying the vertex and edge sets

$$V = \{v_1, v_2, \cdots, v_n\}$$

and

$$E = \{\{v_i, v_{i+j}\} : i = 1, 2, \cdots, n \quad and \quad j = a_1, a_2, \cdots, a_l\}$$

Consider now the circulant $C_n \langle 1, 2, \cdots, r \rangle$ for $n \geq 3$, $1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$. For each $v_i \in V$ let

$$N[v_i] = \{v_{i-r}, \cdots, v_{i-1}, v_i, v_{i+1}, \cdots, v_{i+r}\}.$$

In Figure 2.4 a graphical representation of $C_{11} \langle 1, 2 \rangle$ is given.

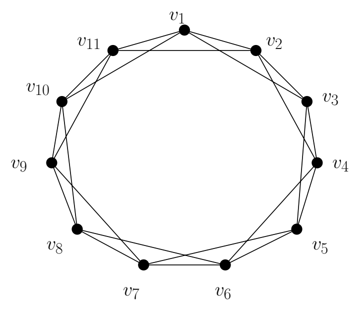

Figure 2.4: $C_{11} \langle 1, 2 \rangle$

Now we determine under which conditions this specific class of circulants is vertex-critical. But first the lower domination parameters of these graphs.

Proposition 2.11 (Grobler [15]) *If* $G = C_n \langle 1, 2, \cdots, r \rangle$ *with* $n \geq 3$, $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$, *then*

$$ir(G) = \gamma(G) = i(G) = \left\lceil \frac{n}{2r+1} \right\rceil$$

■

From this proposition we clearly see that if $r = \lfloor \frac{n}{2} \rfloor$, then $G = K_n$. Then $ir(G) = \gamma(G) = i(G) = 1$; hence G is not π-critical for $\pi \in \{i, ir, \gamma\}$. Therefore we need only examine the vertex-criticality of the circulant $G = C_n \langle 1, 2, \cdots, r \rangle$ for $1 \leq r < \lfloor \frac{n}{2} \rfloor$. Also, $G = C_n \langle 1, 2, \cdots, r \rangle$ with $n \geq 3$ and $1 \leq r < \lfloor \frac{n}{2} \rfloor$ is vertex-transitive, since $deg_G(v) = 2r$ for all $v \in V_G$ and each vertex of G is contained in the same cycles.

Proposition 2.12 *Let* $G = C_n \langle 1, 2, \cdots, r \rangle$ *with* $n \geq 3$ *and* $1 \leq r < \lfloor \frac{n}{2} \rfloor$. *Then* G *is* π*-critical for* $\pi \in \{i, ir, \gamma\}$ *if and only if* $n \equiv 1 \mod (2r+1)$.

Proof: By the division algorithm, there exists unique integers m and q with $0 \leq q \leq 2r$ such that $n = (2r+1)m + q$. First we assume $q = 1$ and show that G is π-critical for $\pi \in \{i, ir, \gamma\}$; then we assume G is π-critical and show that this holds only if $q = 1$.

Assume $q = 1$. Then $n = (2r+1)m + 1$; hence $ir(G) = i(G) = \gamma(G) = \lceil \frac{n}{2r+1} \rceil = m + 1$. If $m = 1$, then $n = 2r + 2$ and $G \cong C_{2r+2} \langle 1, 2, \cdots, r \rangle \cong \overline{(r+1)K_2}$. Therefore from Proposition 2.1 it follows that G is 2-π-critical for $\pi \in \{i, ir, \gamma\}$. Let $m \geq 2$. The set

$$S = \{v_1, v_{1+(2r+1)}, v_{1+2(2r+1)}, \cdots, v_{1+(m-1)(2r+1)}\}$$

of G is independent and dominates all vertices of G except $u = v_{n-r}$. Therefore $S \cup \{u\}$

is an independent dominating set of G with u a singular isolated vertex. Since G is vertex-transitive, it follows from Proposition 2.2 and Proposition 2.4 that G is π-critical.

Let us now assume G is π-critical, and show that $q = 1$. We want to show that $q \neq 0$, so assume to the contrary that $q = 0$. Then $i(G) = \gamma(G) = ir(G) = m$ from Proposition 2.11 and $n = (2r+1)m$. Therefore

$$S = \{v_1, v_{1+(2r+1)}, v_{1+2(2r+1)}, \cdots, v_{1+(m-1)(2r+1)}\}$$

is an i-set of G. Since G is i-critical, $m \geq 2$ and $i(G - v_{2r+1}) = m - 1$. Let S' be an i-set of $G - v_{2r+1}$. Then the $n - 1 = (2r+1)m - 1$ vertices of $G - v_{2r+1}$ are not dominated by the $m - 1$ vertices of S', since $m - 1$ vertices dominates at most $(m-1)(2r+1) = (2r+1)m - 1 - 2r < (2r+1)m - 1$ vertices of $G - v_{2r+1}$. Hence S' is not a dominating set of $G - v_{2r+1}$, which contradicts $q = 0$. Thus $q \neq 0$; hence from Proposition 2.11 it follows that $i(G) = \gamma(G) = ir(G) = m + 1$. If $m = 1$, then $n = 2r + 1 + q$ and since G is 2-π-critical it follows from Proposition 2.1 that $G \cong C_n \langle 1, 2, \cdots, r \rangle \cong \overline{iK_2}$ for $i \geq 2$. Since $i(G - v) = 1$ for all $v \in V_G$, it follows that $deg_v(G) = n - 1 = 2r + 1$ for all $v \in V_G$. Hence $q = 1$. Now let $m \geq 2$. Assume $v_1 \in V_G$ is a singular isolated vertex of some i-set S of G. Then $\{v_{n-r}, v_1, v_{2+r}\} \subset S$ is such that $\{v_{n-r}, v_1, v_{2+r}\}$ dominates $2(2r+1)+1$ vertices of G. Therefore the remaining $(m-2)(2r+1) + q - 1$ vertices of G must be dominated by the remaining $m - 2$ vertices of S. Hence

$$(m-2)(2r+1) + q - 1 - (m-2)(2r+1) = q - 1$$

vertices are not dominated by S, but since S is an dominating set of G, it follows that $q = 1$. ■

From this proposition it follows that $C_{11} \langle 1, 2 \rangle$ (see Figure 2.4) is an example of a π-critical circulant for π a lower domination parameter.

It is interesting to note that in all four classes of graphs, irredundance meets the requirements of Proposition 2.2, namely that a graph G is ir-critical if and only if every vertex of G is a singular isolated vertex of some ir-set of G, and that in all these examples $ir(G) = \gamma(G)$. This motivates the following two conjectures.

Conjecture 2.1 *For any ir-critical graph G, $ir(G) = \gamma(G)$.*

Conjecture 2.2 *For any graph G, G is ir-critical if and only if every vertex of G is a singular isolated vertex of some ir-set of G.*

The open question still remains whether Proposition 2.2 will still be true for $\pi = ir$ if $ir(G) \neq \gamma(G)$. In Section 2.3 we manage to show that given an ir-critical graph G, $ir(G) = \gamma(G)$ for $k \leq 3$.

2.3 Irredundance and Vertex-Criticality

We start this section with a useful result by Favaron. We include a shortened proof. Remember that all graphs referred to are connected; for disconnected graphs, the propositions can just be applied to their components.

Theorem 2.1 (Favaron [12]) *For any graph G with $v \in V_G$ and $ir(G - v) \geq 2$, we have*

$$ir(G) \leq 2ir(G - v) - 1$$

Proof: First note that $ir(G - v) + 1 \leq 2ir(G - v) - 1$ if and only if $ir(G - v) \geq 2$. Let A be an ir-set of the graph $G - v$. Then A is irredundant in G. Thus there exists a maximal irredundant set A' of G such that $A \subseteq A'$. If $|A' - A| \leq 1$, then

$$ir(G) \leq |A'| = |A| + |A' - A| \leq ir(G - v) + 1 \leq 2ir(G - v) - 1$$

So assume $|A' - A| \geq 2$ and let $y \in A' - A$ with $y \neq v$. Then $A \cup \{y\}$ is irredundant in G, but not in $G - v$. Thus there exists an $x \in A \cup \{y\}$ such that $pn_{G-v}(x, A \cup \{y\}) = \emptyset$ but $pn_G(x, A \cup \{y\}) \neq \emptyset$; hence $pn_G(x, A \cup \{y\}) = \{v\}$. This implies that $A \cup \{y, v\}$ is not an irredundant set in G, thus $v \neq A'$. Since $|A' - A| \geq 2$, it then follows that $x \in A$ and $v \in pn_G(x, A)$.

Let U be the set of vertices in $G - v$ not dominated by A and let B be the set of non-isolated vertices of A in $G - v$ which are annihilated by vertices in U.

If $B \neq \{x\}$, let D consist of all vertices in A except for some $t \in B - \{x\}$, together with one private neighbour of each non-isolated vertex of A in the graph $G - v$. Since each vertex of $N_{G-v}[U]$ annihilates some non-isolated vertex of A, D dominates $N_{G-v}[U]$. Furthermore, t is non-isolated and thus dominated by $A - \{t\}$, and v is dominated by $x \in D$. Therefore D is a dominating set of G with $|D| \leq 2|A| - 1$; hence

$$ir(G) \leq \gamma(G) \leq 2ir(G - v) - 1$$

If $B = \{x\}$, let D consist of all vertices of A together with one private neighbour of x in the graph $G - v$. Since each vertex of $N_{G-v}[U]$ annihilates x, and $x \in D$ dominates v, it follows that D is a dominating set of G with $|D| = |A| + 1 \leq 2|A| - 1$; hence

$$ir(G) \leq \gamma(G) \leq 2ir(G - v) - 1$$

■

It is possible to construct a graph G such that $ir(G) = 2ir(G - v) - 1$ holds. We examine any graph G and remove a vertex v. Let $S = \{v_1, v_2, v_3, \cdots, v_k\}$ be an ir-set of the graph $G - v$, $\{u_i\}$ be the external private neighbourhood of v_i for $i = 1, 2, \cdots, k$, and let $|N(v_i) \cap N(v_j)| = m \geq 2k$ for $i, j = 1, 2, \cdots, k$, $j \neq i$. Also each u_i for $i = 1, 2, \cdots, k$ has m private neighbours (refer to Figure 2.5 for $k = 3$ and $m = 6$). Then by taking the ir-set $(S - v_k) \cup \{u_1, u_2, \cdots, u_k\}$, we have $ir(G) = 2k - 1$. Thus we have shown that a result similar to Lemmas 2.2 and 2.4 does not hold for irredundance.

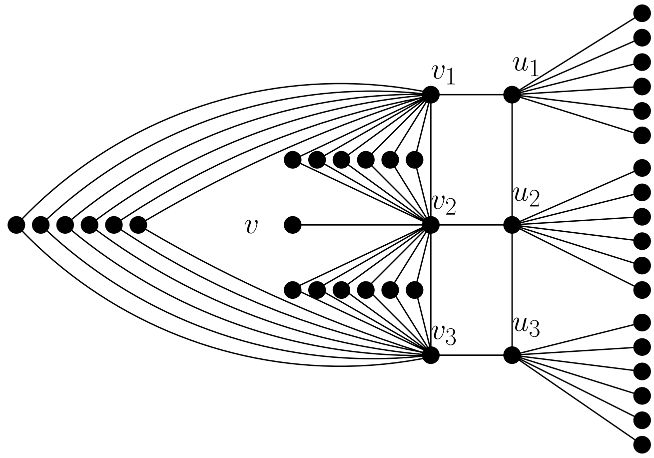

Figure 2.5: *A graph G such that $ir(G) = 2ir(G - v) - 1$*

Now we show that Conjecture 2.2 is true for 2-ir-critical graphs.

Proposition 2.13 *G is 2-ir-critical if and only if every vertex of G is a singular isolated vertex of some ir-set of G.*

Proof: For $\pi \in \{i, ir, \gamma\}$, the only 2-$\pi$-critical graphs are $\overline{nK_2}$ (Proposition 2.1), and every vertex of $\overline{nK_2}$ is clearly a singular isolated vertex of an ir-set of G. ■

From Proposition 1.6 we already know that, if $ir(G) = k$, then $\gamma(G) \leq 2k - 1$. We next show that, if G is k-ir-critical, then this bound can be improved to $\gamma(G) \leq 2k - 2$. Even though Proposition 2.14 is true for $k = 2$, we characterized 2-ir-critical graphs in Proposition 2.13, and now only concentrate on $k \geq 3$.

Proposition 2.14 *If G is k-ir-critical with $k \geq 3$, then $\gamma(G) \leq 2k - 2$. Furthermore, if $\gamma(G) = 2k-2$, then G is also $(2k-2)$-γ-critical with $\gamma(G-v) = 2k-3$ and $ir(G-v) = k-1$ for all $v \in V_G$.*

Proof: By Lemma 2.7 and Proposition 1.6, and the k-ir-criticality of G,

$$\gamma(G) - 1 \leq \gamma(G - v) \leq 2ir(G - v) - 1 \leq 2k - 3 \tag{2.1}$$

for all $v \in V_G$. Therefore $\gamma(G) \leq 2k-2$ and, if $\gamma(G) = 2k-2$, then equation (2.1) reduces to $2k-3 \leq \gamma(G-v) \leq 2ir(G-v) - 1 \leq 2k-3$; hence $\gamma(G-v) = 2k-3$ and $ir(G-v) = k-1$ for all $v \in V_G$. From Proposition 2.2 it then follows that G is $(2k-2)-\gamma$-critical. $\blacksquare$

The next proposition gives necessary conditions for a graph G to be k-ir-critical with $k \geq 3$ and $\gamma(G) = 2k-2$. We will use the following notations in its proof: Take any $v \in V_G$ and suppose $S = \{v_1, v_2, \cdots, v_{k-1}\}$ is an ir-set of $G-v$. Let U, P_i and C denote the (possibly empty) sets of vertices in $V_{G-v} - S$ which are adjacent to no vertices, exactly one vertex v_i, and at least two vertices of S. Thus

$$U = V_{G-v} - N_{G-v}[S]$$

$$P_i = epn_{G-v}(v_i, S) \;\; for \;\; i = 1, 2, \cdots, k-1$$

$$C = N_{G-v}(S) - (\bigcup_{i=1}^{k-1} P_i)$$

Furthermore, let A_i for $i = 1, 2, \cdots, k-1$ denote the (possibly empty) set of vertices in U that annihilates only the vertex v_i and A denote the (possibly empty) set of vertices in U that annihilates two or more vertices of S. We can see that the sets S, U, $\cup P_i$ and C form a disjoint partition of V_{G-v} (since every vertex in V_{G-v} is either in S, or adjacent to no vertices of S, or adjacent to exactly one vertex of S, or adjacent to two or more vertices of S). If we then denote the isolated vertices of S in $G-v$ by I, the non-isolated vertices of S in $G-v$ that are annihilated by some $u \in U$ by B, and $B' = S - (I \cup B)$, it also follows that S is partioned into the disjoint sets I, B and B' with $|S| = |I| + |B| + |B'| = k-1$.

Proposition 2.15 *Given a k-ir-critical graph G with $\gamma(G) = 2k - 2$, $k \geq 3$. Then, for any $v \in V_G$ and an ir-set S of $G - v$,*

1. *$S = B$.*
2. *$P_i = epn_{G-v}(v_i, S) \neq \emptyset$ for all $i = 1, 2, \cdots, k - 1$.*
3. *v is not adjacent to any vertex of $S \cup P_1 \cup P_2 \cup \cdots \cup P_{k-1}$.*
4. *$A_i \neq \emptyset$ for all $i = 1, 2, \cdots, k - 1$.*
5. *v does not dominate A_i for any $i = 1, 2, \cdots, k - 1$.*
6. *$C - N_{G-v}[U] \neq \emptyset$.*
7. *$\{v, p_1, p_2, \cdots, p_{k-1}\}$ does not dominate $C - N_{G-v}[U]$ for any $p_i \in P_i$, $i = 1, 2, \cdots, j$.*

Proof: Let G be a k-ir-critical graph with $\gamma(G) = 2k-2$, $k \geq 3$. For any $v \in V_G$, let S be an ir-set of $G - v$. Take any $v_i \in S$ for $i = 1, 2, \cdots, k - 1$ and let H_i be the set of vertices in $G - v$ consisting of $S - \{v_i\}$ together with one external private neighbour for each vertex in B. It follows from the definition of ir-criticality that every vertex in $N[U]$ must annihilate some vertex of S, thus every $p_i \in P_i$ is adjacent to some P_j, $j = 1, 2, \cdots, k - 1$. Thus by choosing one external private neighbour for each vertex in B, we ensure that every P_i is dominated. Thus the set H_i, with $|H_i| \leq 2k - 3$, is a dominating set of $G - v$, and since $\gamma(G - v) = 2k - 3$ from Proposition 2.14, it follows that $|H_i| = 2k - 3$; hence H_i is a γ-set of $G - v$. Also, $|H_i| = 2k - 3$ implies that $|S| = |B| = k - 1$. Thus $P_i = epn_{G-v}(v_i, S) \neq \emptyset$ for $i = 1, 2, \cdots, k - 1$. Also, since $\gamma(G) = 2k - 2$, it then follows that v is not adjacent to any vertex of $S \cup P_1 \cup P_2 \cup \cdots \cup P_{k-1}$, otherwise the set H_i would be a dominating set of G for some $i = 1, 2, \cdots, k - 1$.

Let H be the set of vertices in $G - v$ consisting of $S - \{v_1\}$ together with one external private neighbour for each vertex in $B - v_2$. Thus $A_2 \neq \emptyset$, otherwise the set H with $|H| = 2k - 4 < \gamma(G - v)$ is a dominating set of $G - v$. Also, v does not dominate A_2, otherwise $H \cup \{v\}$ would be a dominating set of G with $|H \cup \{v\}| = 2k - 3 < \gamma(G)$. Using the same reasoning for all $i = 1, 2, \cdots, k-1$, it follows that $A_i \neq \emptyset$ and v does not dominate any set A_i.

Furthermore, let the set H^* consist of one external private neighbour for each vertex of B in the graph $G-v$. Then H^* is not a dominating set of $G-v$, since $|H^*| = k-1 < \gamma(G-v)$; hence we have that $C - N_{G-v}[U] \neq \emptyset$. Finally, since $\gamma(G) = 2k-2$, the set $H^* \cup \{v\}$ does not dominate $C - N_{G-v}[U]$ in G. ■

This leads us to an important result concerning 3-ir-critical graphs. Again we use the sets as denoted for Proposition 2.15.

Theorem 2.2 *If G is 3-ir-critical, then $\gamma(G) = 3$.*

Proof: Since G is 3-ir-critical, it follows from Proposition 2.14 that $3 \leq \gamma(G) \leq 4$. We want to show that $\gamma(G) = 3$, so suppose to the contrary that $\gamma(G) = 4$. Consider any $v \in V_G$ and let $S = \{v_1, v_2\}$ be an ir-set of $G - v$. From Proposition 2.14 we have $ir(G-v) = 2$ and $\gamma(G-v) = 3$, while from Proposition 2.15 it follows that $S = B$, $C - N_{G-v}[U] \neq \emptyset$, $P_i \neq \emptyset$ and $A_i \neq \emptyset$ for $i = 1, 2$. Also v is not adjacent to any vertex of $S \cup P_1 \cup P_2$, $v \cup p_2$ does not dominate $C - N_{G-v}[U]$ for any $p_2 \in P_2$ and finally, v does not dominate A_i for $i = 1, 2$.

Since G is 4-γ-critical, it now follows from Proposition 2.2 that v_1 is a singular isolated vertex of some 4-γ-set D of G. Thus $G - v_1$ is dominated by the three remaining vertices of $D - v_1$. Let us now choose these 3 vertices such that they dominate $G - v_1$.

To dominate the vertex v_2, with v_1 a singular isolated vertex, v_2 is adjacent to some vertex in $D - v_1$; hence $q \in D$ for some $q \in P_2$. Also, to dominate $C - N_{G-v}[U]$ there then has to exist a $q_2 \in P_2 - q$ such that $C - N_{G-v}[U] \subseteq N(q) \cup N(q_2)$ and $q_2 \in D$ (since we showed in Proposition 2.15.7 that $C - N_{G-v}[U]$ cannot be a subset of q). Then to dominate $A_1 - A_2$ and v (and any vertices of $P_1 \cup P_2 \cup \{C \cap N[U]\}$ not dominated by q and q_2) such that $\gamma(G) = 4$, we need $a \in D$ for some $a \in A_1 \cup A_2$, such that $A_1 - A_2 \subseteq N[a]$

and $av \in E_G$. Thus we have chosen our three remaining vertices of $D - v_1$ in such a way that they dominate $G - v_1$. But then these three vertices $\{a, v_1, q\}$ will be a dominating set of G, which is contrary to $\gamma(G) = 4$. Thus v_1 is not a singular isolated vertex of any γ-set of G; hence $\gamma(G) = 3$. ■

Chapter 3

Edge-criticality of the lower domination parameters

In this chapter we focus on the second main type of criticality, namely edge-criticality, for $\pi \in \{ir, \gamma, i\}$. Recall that a graph G is π-edge-critical if and only if $\pi(G+uv) < \pi(G)$ for all $uv \in E_{\overline{G}}$, while G is π^+-edge-critical if and only if $\pi(G+uv) > \pi(G)$ for all $uv \in E_{\overline{G}}$. The edgeless graphs $\overline{K_n}$ for $n \geq 2$ are clearly π-edge-critical, but in [15] Grobler showed that no π^+-edge-critical graphs exist for $\pi \in \{i, ir, \gamma\}$.

3.1 Basic results on edge-criticality

Initially, γ-edge-critical graphs were studied by Sumner and Blitch in [19], where they proved the following lemma.

Lemma 3.1 (Sumner and Blitch [19]) *The graph G is 2-γ-edge-critical if and only if $\overline{G}$ is the disjoint union of non-trivial stars.*

They also obtained several other useful results, specifically concerning 3-γ-edge-critical graphs. The most interesting of these results include a theorem proving that every 3-γ-

edge-critical graph contains a triangle. They also proved, using Tutte's Theorem, that if the order of the given 3-γ-edge-critical graph is even, then the graph must contain a 1-factor. Finally, the diameter of a connected graph G with $\gamma(G) = 3$ is at most 8, but these authors have shown that for a connected 3-γ-edge-critical graph the diameter is at most 3, and that in the case of the diameter being equal to 3, the graph has $\gamma(G) = i(G)$.

Following this, graphs that are k-γ-edge-critical with $k \geq 4$ were studied by Favaron, Paris, Sumner and Wojcicka in [13] and [18].

In her masters thesis [2], Ao examined graphs that are i-edge-critical where she obtained similar results to those in [19] concerning γ-edge-critical graphs, most notably the following lemma.

Lemma 3.2 (Ao [2]) *The graph G is 2-i-edge-critical if and only if $\overline{G}$ is the disjoint union of non-trivial stars.*

In [15], Grobler examined ir-edge-critical graphs and together with the results from Lemmas 3.1 and 3.2, he obtained the following result.

Proposition 3.1 (Grobler [15]) *For π a lower domination parameter, the graph G is 2-π-edge-critical graphs if and only if G is the complement of the disjoint union of non-trivial stars.* ■

In Corollary 3.1 we give a characterization of γ-edge-critical and i-edge-critical graphs in terms of singular isolated vertices. But first the following proposition.

Proposition 3.2 (Grobler [15]) *Let $\pi \in \{\gamma, i\}$. If G is a non-complete graph, then*

(a) $\pi(G + uv) \geq \pi(G) - 1$ *for all* $uv \in E_{\overline{G}}$,

(b) $\pi(G + uv) = \pi(G) - 1$ *for* $uv \in E_{\overline{G}}$, *if and only if there exists a π-set T of G such that $\{u, v\} \subseteq T$ and one of u and v is a singular isolated vertex of T.* ■

Corollary 3.1 *Let $\pi \in \{\gamma, i\}$. If G is a non-complete graph, then*

(a) G is π-edge-critical if and only if $\pi(G+uv) = \pi(G) - 1$ for all $uv \in E_{\overline{G}}$.

(b) G is π-edge-critical if and only if for every $uv \in E_{\overline{G}}$, there exists a π-set T of G such that $\{u, v\} \subseteq T$ and u or v is a singular isolated vertex of T.

We now explore the relationship between edge-transitivity and edge-criticality.

Proposition 3.3 *Suppose G is a non-complete graph such that $\overline{G}$ is edge-transitive. For $\pi \in \{i, \gamma\}$, if G has a π-set T which contains a singular isolated vertex, then G is π-edge-critical.*

Proof: Suppose G is a non-complete graph such that $\overline{G}$ is edge-transitive. Suppose G has a π-set T containing a singular isolated vertex v. Then $uv \in E_{\overline{G}}$ for some $u \in T$. Therefore, since $\overline{G}$ is edge-transitive, for all edges $xy \in E_{\overline{G}}$ there exist a γ-set T of G such that $\{x, y\} \subseteq T$ and x or y is a singular isolated vertex of T; hence from Corollary 3.1(b) it follows that G is π-edge-critical for $\pi \in \{\gamma, i\}$. ■

It remains an open question whether the same will be true for ir-edge-critical graphs. If the graph G is such that $\overline{G}$ is edge-transitive and vertex-transitive, then from Propositions 2.3 and 3.3 an even stronger characterization follows.

Corollary 3.2 *Suppose G is a non-complete graph such that $\overline{G}$ is a symmetric graph and $\pi \in \{\gamma, i\}$. Then G is π-critical and π-edge-critical if and only if G has a π-set T containing a singular isolated vertex.*

Finally, the following result shows an important relationship between i-edge-critical and γ-edge-critical and between γ-edge-critical and ir-edge-critical graphs.

Proposition 3.4 *For any graph G,* *(a) if G is i-edge-critical and $i(G) = \gamma(G)$, then G is γ-edge-critical.*

(b) if G is γ-edge-critical and $\gamma(G) = ir(G)$, then G is ir-edge-critical.

Proof:

(a) Suppose G is i-edge-critical and $i(G) = \gamma(G)$. Then from Proposition 1.4 it follows that $\gamma(G + uv) \leq i(G + uv) < i(G) = \gamma(G)$ for all $uv \in E_{\overline{G}}$.

(b) Suppose G is γ-edge-critical and $\gamma(G) = ir(G)$. Then from Proposition 1.4 it follows that $ir(G + uv) \leq \gamma(G + uv) < \gamma(G) = ir(G)$ for all $uv \in E_{\overline{G}}$. ■

This proposition is very important in our proofs of edge-criticality for the four classes of graphs with respect to the lower domination parameters.

3.2 The edge-criticality of some classes of graphs

We now examine three classes of graphs, namely the complete multipartite graphs, the product of two complete graphs and the complement of the product of two complete graphs.

Given the complete multipartite graph $G = K_{n_1,n_2,\cdots,n_m}$ with $m \geq 2$, we can clearly see that $\overline{G}$ is edge-transitive if and only if $n_i \in \{1, k\}$ for $i = 1, 2, \cdots, m$ and a fixed k, while $\overline{G}$ is symmetric if and only if $n_1 = n_2 = \cdots = n_m$ for $m \geq 2$.

From the next proposition we can see that for π a lower domination parameter, the complete multipartite graphs are π-critical and π-edge-critical if and only if $n_1 = n_2 = \cdots = n_m = 2$, and i-critical and i-edge-critical if and only if $n_1 = n_2 = \cdots = n_m \geq 2$. Recall that the vertex-set of $K_{n_1,n_2,\cdots,n_m}$ has partition $\{V_1, V_2, \cdots, V_m\}$ with $|V_i| = n_i$ for $1 \leq i \leq m$.

Proposition 3.5 *Let $G = K_{n_1,n_2,\cdots,n_m}$ with $m \geq 2$.*

1. *For $\pi \in \{ir, \gamma\}$, G is π-edge-critical if and only if $n_1 = n_2 = \cdots = n_m = 2$*

2. *G is i-edge-critical if and only if $n_1 = n_2 = \cdots = n_m \geq 2$.*

Proof: If $n_i = 1$ for some $i = 1, 2, \cdots, m$, then $\pi(G) = 1$; hence G is not π-edge-critical for π a lower domination parameter. Assume therefore that $n_i > 1$ for all $i = 1, 2, \cdots, m$.

1. If $\pi \in \{ir, \gamma\}$, then $\pi(G) = 2$ by Proposition 2.5. Thus G must be the complement of the disjoint union of non-trivial stars by Proposition 3.1; which implies that $n_1 = n_2 = \cdots = n_m = 2$.

2. If $n_1 = n_2 = \cdots = n_m = n$ (say), then $\overline{G}$ is symmetric. Since G has an i-set V_1 with each $v \in V_1$ a singular isolated vertex, if follows from Corollary 3.2 that G is i-edge-critical. If $n_k > n_j$ for some $k \neq j$, then it follows from Proposition 2.5 that $i(G) = i(G + uv)$ for any $\{u, v\} \in V_k$ such that $uv \in E_{\overline{G}}$; hence G is not i-edge-critical. ■

We next examine the product of two complete graphs. The complement of the product of two complete graphs, $\overline{K_m \times K_n}$ with $n \geq m \geq 2$, is clearly always symmetric. The next proposition shows that the product of two complete graphs is i-edge-critical, but neither γ-edge-critical nor ir-edge-critical if $n = m$. Recall for the next two propositions that

$$X_i = \{v_{ik} | k = 1, 2, \cdots, n\} \;\; and \;\; Y_j = \{v_{kj} | k = 1, 2, \cdots, m\}$$

for each $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$.

Proposition 3.6 *Let $G = K_m \times K_n$ with $n \geq m \geq 2$. Then G is π-edge-critical, for $\pi \in \{i, ir, \gamma\}$, if and only if $n = m$.*

Proof: Suppose $n = m$. Then the i-set T consisting of diagonal vertices $\{v_{ii} : 1 \leq i \leq n\}$ contains a singular isolated vertex, namely v_{11}. Since $\overline{K_m \times K_n}$ is symmetric, it follows from Proposition 3.3 that G is i-edge-critical. From Proposition 2.7 it follows that $ir(G) = \gamma(G) = i(G)$; hence from Proposition 3.4 G is π-edge-critical.

Suppose now that $m < n$. Assume that the γ-set S of G contains a singular isolated vertex. Thus $Y_j \cap S \neq \emptyset$ for $j = 1, 2, \cdots, n$. Therefore $|S| \geq n$ which contradicts $|S| = m < n$. It follows that no γ-set of G exists such that it contains a singular isolated vertex. Thus by Corollary 3.1(b), G is not γ-edge-critical, and from Proposition 2.7 it follows that $\gamma(G) = i(G)$; hence from Proposition 2.4 G is not i-edge-critical. Thus $\gamma(G + uv) = i(G + uv) = m$ for any $uv \in E_{\overline{G}}$.

We now show that $ir(G + uv) = m$ for any $uv \in E_{\overline{G}}$. Since $\overline{K_m \times K_n}$ is symmetric, we can assume without loss of generality that $ir(G + v_{m-1,n-1}v_{mn}) = p < m$. Let $H = G + v_{m-1,n-1}v_{mn}$ and consider an ir-set S of H. Since $\gamma(H) = m$, S is not a dominating set of H; thus let $u = v_{kl}$ be a vertex of H not dominated by S. Then, since S is maximal irredundant, u annihilates some non-isolated vertex $s = v_{ij}$ of S. Now if $m = 2$, then $p = 1$, and H is not ir-edge-critical. Thus $n > m \geq 3$ for $i \neq k$, $j \neq l$. We now consider 5 cases:

Case 1: Let $k = m - 1$, $l = 1, 2, \cdots, n - 2$, $s = v_{mn}$ and $pn(s, S) = \{v_{m-1,n-1}, v_{m-1,n}\}$. Then $S \cap X_t \neq \emptyset$ for $1 \leq t \leq m$, $t \neq m - 1$, and $S \cap Y_r = \emptyset$ for $r = l, n - 1, n$ and $|S \cap X_m| \geq 2$. Thus $|S| \geq m - 1$, simplifying to $|S| = m - 1$. But then $S \cup v_{1,n-1}$ is irredundant, contradicting S as being maximal irredundant.

Case 2: Let $k = m$, $l = 1, 2, \cdots, n - 2$, $s = v_{m-1,n-1}$ and $pn(s, S) = \{v_{m,n-1}, v_{m,n}\}$. Then $S \cap X_t \neq \emptyset$ for $1 \leq t \leq m - 1$, and $S \cap Y_r = \emptyset$ for $r = l, n - 1, n$ and $|S \cap X_{m-1}| \geq 2$. Thus $|S| \geq m - 1$, simplifying to $|S| = m - 1$. But then $S \cup v_{1,n}$ is irredundant, contradicting S as being maximal irredundant.

Case 3: Let $l = n - 1$, $k = 1, 2 \cdots, m - 2$, $s = v_{mn}$ and $pn(s, S) = \{v_{m-1,n-1}, v_{m,n-1}\}$. Then $S \cap Y_t \neq \emptyset$ for $1 \leq t \leq n - 2$ and $|S \cap Y_n| \geq 2$. Thus $|S| \geq n$, contradicting $ir(H) < m$.

Case 4: Let $l = n$, $k = 1, 2 \cdots, m - 2$, $s = v_{m-1,n-1}$ and $pn(s, S) = \{v_{m-1,n}, v_{m,n}\}$. Then

$S \cap Y_t \neq \emptyset$ for $1 \leq t \leq n-2$ and $|S \cap Y_{n-1}| \geq 2$. Thus $|S| \geq n$, contradicting $ir(H) < m$.

Case 5: In all the other cases we have $pn(s,S) = \{v_{il}\}$ or $\{v_{kj}\}$. If $pn(s,S) = \{v_{il}\}$, then $S \cap Y_t \neq \emptyset$ for $1 \leq t \leq n$, $t \neq l$ and $|S \cap Y_j| \geq 2$. Therefore $|S| \geq n > m$, contradicting $ir(H) < m$. It follows that $pn(s,S) = \{v_{kj}\}$. Then $S \cap X_t \neq \emptyset$ for $1 \leq t \leq m$, $t \neq k$ and $|S \cap X_i| \geq 2$. Thus $|S| \geq m+1$, contradicting $ir(H) < m$.

Thus S is not maximal irredundant as supposed and $ir(H) = m$. ■

Let us now consider the complement of the product of two complete graphs. Similar to the complete multipartite graphs, the product of two complete graphs, $K_m \times K_n$ with $n \geq m \geq 2$, is both vertex-critical and edge-critical under exactly the same assumptions (in this case $n = m$). From the next proposition we can see that this is not true in general. The product of two complete graphs, $K_m \times K_n$ with $n \geq m \geq 2$ is symmetric if and only if $m = n$, since if $m < n$ there exists no automorphism ϕ of $K_m \times K_n$ such that $\phi(\{u,v\}) = \{w,x\}$ for some $u,v \in Y_i$ and $w,x \in X_j$, $i = 1,2,\cdots,n$ and $j = 1,2,\cdots,m$. We now determine under which assumptions the complement of the product of two complete graphs is edge-critical.

Proposition 3.7 *Let $G = \overline{K_m \times K_n}$ with $n \geq m \geq 2$.*

1. *For $\pi \in \{ir, \gamma\}$, G is π-edge-critical if and only if $n = m = 3$.*

2. *G is i-edge-critical if and only if $n = m \geq 3$.*

Proof: Let $m = 2$. If $\pi \in \{i, ir, \gamma\}$, then from Proposition 2.9 we have $\pi(G) = 2$. Thus it follows from Proposition 3.1 that G is not π-edge-critical; hence $m \geq 3$.

Assume $m \geq 3$. If $n = m$, then the i-set $S = \{v_{11}, v_{21}, \cdots, v_{m1}\}$ of G contains m singular isolated vertices. Since $\overline{G}$ is symmetric, it follows from Corollary 3.2 that G is i-edge-critical. Furthermore, if $m = 3$, it follows from Proposition 2.9 that $ir(G) = \gamma(G) = i(G) = 3$; hence from Proposition 3.4 G is also ir-edge-critical and γ-edge-critical. Now let $m \geq 4$. We will show that G is neither γ nor ir-edge-critical. For $v_{11}v_{12} \in E_{\overline{G}}$, there exists no $s \in V_G$ such that either v_{11} or v_{12} is a singular isolated vertex of some γ-set $T = \{v_{11}, v_{12}, s\}$ of G. Thus G is not γ-edge-critical from Corollary 3.1 and thus $\gamma(G+uv) = 3$ for any $uv \in E_{\overline{G}}$. Suppose now that $ir(G+uv) = p < 3$ for some $uv \in E_{\overline{G}}$. Since $H = G + uv$ contains no universal vertices, $ir(H) > 1$; hence assume $ir(H) = 2$ and let $S = \{s_1, s_2\}$ be an ir-set of H. Since $i(H) \geq 3$, S is not a independent set of H, hence $s_1s_2 \in E_H$. Since $\gamma(H) = 3$, S is not a dominating set of H and by Proposition 1.5 for $x \in H$ there exists at least one vertex $u \in H$ such that $S \cup \{x\}$ does not dominate u. Since S is maximal irredundant, u annihilates $s_1 \in S$ (say) with $u \notin N[s_1] \cup N[s_2]$; hence $s_1 \notin N[u]$ and $s_2 \notin N[u]$. But no such $u \in H$ exists; hence S is not maximally irredundant as supposed. Thus G is not ir-edge-critical.

Let us assume $n > m \geq 3$. For $v_{11}v_{12} \in E_{\overline{G}}$, there exists no $s \in V_G$ such that either v_{11} or v_{12} is a singular isolated vertex of some γ-set $T = \{v_{11}, v_{12}, s\}$ of G. Since $\overline{G}$ is symmetric, it then follows from Corollary 3.1 that G is not γ-edge-critical. If $m = 3$, it follows from Proposition 3.4 that G is not i-edge-critical. If $m \geq 4$, then for $v_{11}v_{12} \in E_{\overline{G}}$, either v_{11} or v_{12} is a singular isolated vertex of some i-set S of G with $\{v_{11}, v_{12}\} \in S$. But then $\{v_{13}, v_{14}, \cdots, v_{1n}\} \in S$; hence $|S| > m$, contradicting $i(G) = m$. Thus G is not i-edge-critical. It now remains to prove that G is not ir-edge-critical for $n > m \geq 3$. We have $\gamma(G+uv) = 3 \leq i(G+uv)$ for any $uv \in E_{\overline{G}}$. Suppose now that $ir(G+uv) = p < 3$ for some $uv \in E_{\overline{G}}$. Since $H = G + uv$ contains no universal vertices, $ir(H) > 1$; hence assume $ir(H) = 2$ and let $S = \{s_1, s_2\}$ be an ir-set of H. Since $i(H) \geq 3$, S is not a independent set of H, hence $s_1s_2 \in E_H$. Since $\gamma(H) = 3$, S is not a dominating set of H. Let u be the vertex in H such that S does not dominated it. Since S is maximal irredundant, u annihilates some $s_1 \in S$ with $u \notin N[s_1] \cup N[s_2]$; hence $s_1 \notin N[u]$ and $s_2 \notin N[u]$. But no such $u \in H$ exists; hence S is not maximally irredundant as supposed. Thus G is not ir-edge-critical. ■

It is interesting to note that for each of the above classes of graphs, the given graph is ir-edge-critical if and only for every $uv \in E_{\overline{G}}$, there exists a π-set T of G such that $\{u, v\} \subseteq T$ and u or v is a singular isolated vertex of T. As in the case of vertex-criticality, $ir = \gamma$ for all four classes of graphs. Similar to vertex-criticality, this motivated the following two conjectures.

Conjecture 3.1 *For any ir-edge-critical graph G, $ir(G) = \gamma(G)$.*

Conjecture 3.2 *For any graph G, G is ir-edge-critical if and only for every $uv \in E_{\overline{G}}$, there exists a π-set T of G such that $\{u, v\} \subseteq T$ and u or v is a singular isolated vertex of T.*

The open question still remains for an edge-critical graph G whether Proposition 3.2 will still be true for $\pi = ir$ if $ir(G) \neq \gamma(G)$. In Section 3.3 we manage to show that given a ir-edge-critical graph G, $ir(G) = \gamma(G)$ for $k \leq 3$. It is also interesting to note that for each of the three classes of graphs, the following conjecture holds.

Conjecture 3.3 *For $\pi \in \{i, ir, \gamma\}$, if the graph G is π-edge-critical, then $\overline{G}$ is symmetric.*

3.3 Irredundance and Edge-Criticality

In this section we define some properties of k-ir-edge-critical graph for $k \geq 2$. First, we show that Conjecture 3.2 is true for 2-ir-edge-critical graphs.

Proposition 3.8 *G is 2-ir-edge-critical if and only if for every $uv \in E_{\overline{G}}$, there exists a ir-set T of G such that $\{u, v\} \subseteq T$ and u or v is a singular isolated vertex of T.*

Proof: Assume G is 2-ir-edge-critical. From Proposition 3.1 it follows that G is the complement of the disjoint union of non-trivial stars. Take any $uv \in E_{\overline{G}}$ such that $deg_{\overline{G}}(v) = 1$. Then $T = \{u, v\}$ is an irredundant set of G. Also, since $\{u\} = pn_G(u, T)$

and $\{v\} \subseteq pn_G(v,T)$ with $ir(G)=2$, it follows from the definition of a singular isolated vertex and a maximal irredundant set that T is an ir-set of G such that u is a singular isolated vertex of T. ■

Proposition 1.6 states that, if $ir(G)=k$, then $\gamma(G) \leq 2k-1$. The next proposition shows that, given a k-ir-edge-critical graph G, this bound can be improved to $\gamma(G) \leq 2k-2$ (compare with Proposition 2.14 of Section 2.3). Even though Proposition 3.9 is true for $k=2$, we characterized 2-ir-edge-critical graphs in Proposition 3.8, and now only concentrate on $k \geq 3$.

Proposition 3.9 *If G is k-ir-edge-critical with $k \geq 3$, then $\gamma(G) \leq 2k-2$. Furthermore, if $\gamma(G)=2k-2$, then G is also $(2k-2)$-γ-edge-critical with $\gamma(G+uv)=2k-3$ and $ir(G+uv)=k-1$ for all $uv \in E_{\overline{G}}$.*

Proof: By Propositions 3.2 and 1.6, and the k-ir-edge-criticality of G,

$$\gamma(G)-1 \leq \gamma(G+uv) \leq 2ir(G+uv)-1 \leq 2k-3 \tag{3.1}$$

for all $uv \in E_{\overline{G}}$. Therefore $\gamma(G) \leq 2k-2$ and, if $\gamma(G)=2k-2$, then equation (3.1) reduces to $2k-3 \leq \gamma(G+uv) \leq 2ir(G+uv)-1 \leq 2k-3$; hence $\gamma(G+uv)=2k-3$ and $ir(G+uv)=k-1$ for all $uv \in E_{\overline{G}}$. From Proposition 3.1 it then follows that G is $(2k-2)-\gamma$-edge-critical. ■

The next proposition gives necessary conditions for a graph G that is k-ir-edge-critical with $k \geq 3$ and $\gamma(G)=2k-2$. We will use the following notations in its proof: Take any $uv \in E_{\overline{G}}$ and suppose $S=\{v_1,v_2,\cdots,v_{k-1}\}$ is an ir-set of $G+uv$. Let U, P_i and C denote the (possibly empty) sets of vertices in $V_{G+uv}-S$ which are adjacent to no vertices, exactly one vertex v_i, and at least two vertices of S. Thus

$$U = V_{G+uv} - N_{G+uv}[S]$$

$$P_i = epn_{G+uv}(v_i, S) \quad for \quad i = 1, 2, \cdots, k-1$$

$$C = N_{G+uv}(S) - (\bigcup_{i=1}^{k} P_i)$$

Furthermore, let A_i for $i = 1, 2, \cdots, k-1$ denote the (possibly empty) set of vertices in U that annihilates only the vertex v_i and A denote the (possibly empty) set of vertices in U that annihilates two or more vertices of S. We can see that the sets S, U, $\cup P_i$ and C form a disjoint partition of V_G (since every vertex in V_{G-v} is either in S, or adjacent to no vertices of S, or adjacent to exactly one vertex of S, or adjacent to two or more vertices of S). If we then denote the isolated vertices of S in $G+uv$ by I, the non-isolated vertices of S in $G+uv$ that are annihilated by some $u \in U$ by B, and $B' = S-(I \cup B)$, it also follows that S is partioned into the disjoint sets I, B and B' with $|S| = |I| + |B| + |B'| = k-1$.

Proposition 3.10 *Given a k-ir-edge-critical graph G with $\gamma(G) = 2k-2$, $k \geq 3$. Then, for any $uv \in E_{\overline{G}}$ and an ir-set S of $G+uv$,*

1. $S = B$.
2. $P_i = epn_{G+uv}(v_i, S) \neq \emptyset$ *for all* $i = 1, 2, \cdots, k-1$.
3. $A_i \neq \emptyset$ *for all* $i = 1, 2, \cdots, k-1$.
4. $C - N_{G+uv}[U] \neq \emptyset$.

Proof: Let G be a k-ir-edge-critical graph with $\gamma(G) = 2k-2$, $k \geq 3$. For any $uv \in E_{\overline{G}}$, let S be an ir-set of $G+uv$. Take any $v_i \in S$ for $i = 1, 2, \cdots, k-1$ and let H_i be the set in $G+uv$ consisting of $S - \{v_i\}$ together with one external private neighbour for each vertex in B. It follows from the definition of ir-edge-criticality that every vertex in $N[U]$ must annihilate some vertex of S, thus every $p_i \in P_i$ is adjacent to some P_j, $j = 1, 2, \cdots, k-1$. Thus set H_i, with $|H_i| \leq 2k-3$, is a dominating set of $G+uv$, and since $\gamma(G+uv) = 2k-3$ from Proposition 3.9, it follows that $|H_i| = 2k-3$; hence H_i is a γ-set of $G+uv$. Also, $|H_i| = 2k-3$ implies that $|S| = |B| = k-1$. Thus $P_i = epn_{G+uv}(v_i, S) \neq \emptyset$ for $i = 1, 2, \cdots, k-1$.

Let H be the set in $G+uv$ consisting of $S - \{v_1\}$ together with one external private neighbour for each vertex in $B - \{v_2\}$. Thus $A_2 \neq \emptyset$, otherwise the set H with $|H| = 2k-4 < \gamma(G+uv)$ is a dominating set of $G+uv$. Using the same reasoning, it follows that $A_i \neq \emptyset$ for all $i = 1, 2, \cdots, k-1$.

Furthermore, let the set H^* consist of one external private neighbour for each vertex of B in the graph $G+uv$. Then H^* is not a dominating set of $G+uv$, since $|H^*| = k-1 < \gamma(G+uv)$. Since every vertex of P_i for $i = 1, 2, \cdots, k-1$ annihilates some vertex of S, H^* dominates all vertices in P_i; hence we have $C - N_{G+uv}[U] \neq \emptyset$. ■

Now we show that for any 3-ir-edge-critical graph, we will always have $\gamma = ir$. We use the sets as denoted above.

Theorem 3.1 *If G is 3-ir-edge-critical, then $\gamma(G) = 3$.*

Proof: Let G be a 3-ir-edge-critical. Then from Proposition 3.9 it follows that $\gamma(G) \leq 4$. We want to prove that $\gamma(G) = 3$, so suppose to the contrary that $\gamma(G) = 4$. Consider any $uv \in E_{\overline{G}}$. From Proposition 3.9 it then follows that $\gamma(G+uv) = 3$ and $ir(G+uv) = 2$; hence let $S = \{v_1, v_2\}$ be an ir-set of $H = G+uv$. From Proposition 3.10 it then follows that $S = B$, $C - N_{G+uv}[U] \neq \emptyset$, $P_i \neq \emptyset$ and $A_i \neq \emptyset$ for $i = 1, 2$. If both $|P_1| > 1$ and $|P_2| > 1$, then by removing any $xy \in E_H$ there still exists a γ-set T consisting of either v_1 or v_2, together with one vertex in P_1 and one vertex in P_2. This is contrary to the assumption that $\gamma(G) = 4$; hence assume without loss of generality that $|P_2| = 1$.

We know that G is 4-γ-edge-critical, thus it follows from Corollary 3.1 that for $v_1 q \in E_{\overline{G}}$, $q \in P_2$, there exists a γ-set T of G such that $\{v_1, q\} \in T$ and either v_1 or q is a singular isolated vertex of T in G. Assume v_1 is a singular isolated vertex of $T = \{v_1, q, r, s\}$ in G. Since $\{r, s\} \subseteq V_G - N[v_1]$, we have $s \in U$ (say) with $r \in U \cup N_{G+uv}(v_1)$. Since q does not dominate $C - N_G[U]$ (otherwise $\{v_1, p, q\}$ or $\{v_2, p, q\}$ will be a dominating set of G for some $p \in P_1$), it follows that for $\{r, s\} \subseteq V_G - N[v_1]$, $q \cup r$ dominates $C - N_G[U]$ with $r \in N_{G+uv}(v_1)$ such that $r \notin N_G(v_1)$. Now $r \notin P_1$ (otherwise $\{r, q\}$ is a dominating set of $G + uv$), $r \notin C$ (otherwise $\{r, q, p\}$ is a dominating set of G for some $p \in P_1$) and $r \neq v_2$ (otherwise $\{v_1, p, q\}$ is a dominating set of G for some $p \in P_1$). Hence no such r exists such that $q \cup r$ dominates $C - N_G[U]$ with $r \in N_{G+uv}(v_1)$. Thus v_1 is not a singular isolated vertex; hence assume q is a singular isolated vertex of $T = \{v_1, q, r, s\}$ in G. Thus to dominate $A_1 \cup A_2$, we need $r, s \in V_G - N[q]$. Then v_2, r, s, v_1, p, q or v_2, p, q will be a dominating set of G, which is contrary to $\gamma(G) = 4$; hence from Corollary 3.1 it follows that G is not γ-edge-critical. Thus $\gamma(G) = 3$. ■

Chapter 4

ER-criticality of the lower domination parameters

In this chapter we examine π-ER-critical and π^--ER-critical graphs for $\pi \in \{ir, \gamma, i\}$. Recall from Chapter 1 that a graph G is π-ER-critical if and only if $\pi(G-uv) > \pi(G)$ for all $uv \in E_G$, while G is π^--ER-critical if and only if $\pi(G-uv) < \pi(G)$ for all $uv \in E_G$. All non-trivial stars are π-ER-critical for $\pi \in \{ir, \gamma, i\}$, while Grobler [15] showed that no γ^--ER-critical graphs exists. He also showed that there exist graphs which are i^--ER-critical, but the existence of ir^--ER-critical graphs is still an open question. Please note that in this chapter we do not require the graphs to be connected.

4.1 Basic results on π-ER-criticality

In 1979, Walikar and Acharya [20] published a paper regarding their study of γ-ER-critical graphs. In this paper they proved the following important result.

Lemma 4.1 *A graph G is γ-ER-critical if and only if G is the disjoint union of stars (with at least one of the stars non-trivial).*

In 1994 Ao [2] then extended the characterization of γ-ER-critical graphs to i-ER-critical graphs.

Lemma 4.2 *The graph G is i-ER-critical if and only if G is the disjoint union of stars (with at least one of the stars non-trivial).*

From Lemmas 4.1 and 4.2 the next two corollaries follow.

Corollary 4.1 *For any $\pi = \{i, \gamma\}$, if G is π-ER-critical, then $\pi(G - uv) = \pi(G) + 1$ for all $uv \in E_G$.*

This result implies that the lower independence number and the domination number of any graph can increase at most by one when an edge is removed.

Corollary 4.2 *The graph G is γ-ER-critical if and only if G is i-ER-critical.*

Graphs that are k-ir-ER-critical were studied by Grobler and Mynhardt in [15] and [16]. It is clear that the disjoint union of stars (with at least one of the stars non-trivial) is also ir-ER-critical. This can be stated as follows.

Proposition 4.1 *If $ir(G) = \gamma(G)$, then the graph G is ir-ER-critical if and only if G is γ-ER-critical.*

Proof: Let G be a graph with $ir(G) = \gamma(G)$. Suppose G is ir-ER-critical. Then from Proposition 1.4 it follows that $\gamma(G - uv) \geq ir(G - uv) > ir(G) = \gamma(G)$ for all $uv \in E_G$; hence G is γ-ER-critical. Suppose G is γ-ER-critical. Then from Lemma 4.1 it follows that G is the disjoint union of stars (with at least one star non-trivial), which in turn is ir-ER-critical. ■

Thus we have shown for $ir(G) = \gamma(G)$ that a result similar to Lemmas 4.1 and 4.2 does hold for irredundance, but in Section 4.3 we show that there does exist graphs which are ir-ER-critical but not γ-ER-critical for $ir(G) < \gamma(G)$.

4.2 The π-ER-criticality of some classes of graphs

In this section we examine the complete multipartite graphs, the product of two complete graphs and the complement of the product of two complete graphs and determine which of these three classes of graphs are π-ER-critical for $\pi \in \{i, ir, \gamma\}$.

Each of these classes of graphs we consider will turn out to contain a subclass of graphs that are critical in some respect with regards to the lower domination parameters.

Let us first examine the complete multipartite graphs $K_{n_1,n_2,\cdots,n_m}$ with $m \geq 2$.

Proposition 4.2 *If $G = K_{n_1,n_2,\cdots,n_m}$ with $m \geq 2$, then G is π-ER-critical for $\pi \in \{i, ir, \gamma\}$ if and only if $G = K_{1,n_2}$.*

Proof: Suppose $G = K_{n_1,n_2,\cdots,n_m}$ with $m \geq 2$. From Proposition 2.5 we have

$$ir(G) = \gamma(G) = \begin{cases} 2 & \text{if} \quad n_i > 1 \ \ for \ \ all \ \ i = 1, 2, \cdots, m \\ 1 & \text{otherwise} \end{cases}$$

and $i(G) = min\{n_i : i = 1, 2, \cdots, m\}$. G is isomorphic to a star if and only if $m = 2$ with $n_1 = 1$; hence from Lemmas 4.1 and 4.2 it follows that G is i-ER-critical and γ-ER-critical. Since $\gamma(G) = ir(G) = 1$, it then follows from Proposition 4.1 that G is ir-ER-critical. ■

From the previous proposition we see that the complete multipartite graph $K_{n_1,n_2,\cdots,n_m}$ is π-ER-critical for $\pi = \{i, ir, \gamma\}$ if and only if $K_{n_1,n_2,\cdots,n_m}$ is a non-trivial star.

Next we turn our attention to the product of two complete graphs.

Proposition 4.3 *If $G = K_m \times K_n$ with $n \geq m \geq 2$, then G is not π-ER-critical for $\pi \in \{i, ir, \gamma\}$.*

Proof: Suppose $G = K_m \times K_n$ with $n \geq m \geq 2$. From Proposition 2.7 it follows that $ir(G) = \gamma(G) = i(G) = m$. Then G is not isomorphic to a star; hence from Lemmas 4.1

and 4.2 it follows that G is neither i-ER-critical nor γ-ER-critical, and since $\gamma(G) = ir(G)$, it follows from Proposition 4.1 that G is not ir-ER-critical. ■

Finally we turn our attention to the complement of the product of two complete graphs.

Proposition 4.4 *If $G = \overline{K_m \times K_n}$ with $n \geq m \geq 2$, then G is π-ER-critical if and only if $n = m = 2$ for $\pi \in \{i, ir, \gamma\}$.*

Proof: Suppose $G = \overline{K_m \times K_n}$ with $n \geq m \geq 2$. From Proposition 2.9 we have $ir(G) = \gamma(G) = min\{3, m\}$ and $i(G) = m$. G is isomorphic to the disjoint union of stars if and only if $n = m = 2$, and from Lemmas 4.1 and 4.2 it follows that G is i-ER-critical and γ-ER-critical, and since $\gamma(G) = ir(G)$, it follows from Proposition 4.1 that G is ir-ER-critical. ■

Thus the complement of the product of two complete graphs, $\overline{K_m \times K_n}$ with $n \geq m \geq 2$, is ER-critical with respect to the lower domination parameters if $G = \overline{K_2 \times K_2}$; while $\overline{K_m \times K_n}$ is vertex-critical if $n \geq m = 3$ and edge-critical if $n = m = 3$. Also, we know that $\overline{K_m \times K_n}$ is vertex-transitive for $n \geq m \geq 2$, but $K_m \times K_n$ is edge-transitive only if $n = m$.

4.3 Results on ir-ER-Criticality

The next proposition gives necessary conditions for a connected graph G to be ir-ER-critical, but not γ-ER-critical, as proved by Grobler and Mynhardt in [15] and [16]. We use the following notations: Let S be an irredundant set of the graph G. Let U, P_v and C denote the (possibly empty) sets of vertices in $V_G - S$ which are adjacent to no vertices, exactly one vertex v, and at least two vertices of S. Thus

$$U = V_G - N_G[S]$$

$$P_v = epn_G(v, S) \quad for \quad all \quad v \in S$$

$$C = N_G(S) - (\bigcup_{v \in S} P_v)$$

We can see that the sets S, U, $\cup P_v$ and C form a disjoint partition of V_G (since every vertex in V_G is either in S, or adjacent to no vertices of S, or adjacent to exactly one vertex of S, or adjacent to two or more vertices of S). If we then denote the isolated vertices of S in G by I, the non-isolated vertices of S in G that are annihilated by some $u \in U$ by B, and $B' = S - (I \cup B)$, it also follows that S is partioned into the disjoint sets I, B and B' with $|S| = |I| + |B| + |B'|$. We obtain an even finer partition of S if we let

$$B_1 = \{v \in B : |P_v| = 1\} \quad , \quad B_2 = \{v \in B : |P_v| > 1\}$$

$$B'_1 = \{v \in B' : |P_v| = 1\} \quad , \quad B'_2 = \{v \in B' : |P_v| > 1\}$$

and for a finer partition of P_v for $v \in S$, let

$$E_1 = \bigcup_{v \in B_1} P_v \quad , \quad E_2 = \bigcup_{v \in B_2} P_v$$

such that $E = E_1 \cup E_2$, and

$$F_1 = \bigcup_{v \in B'_1} P_v \quad , \quad F_2 = \bigcup_{v \in B'_2} P_v$$

such that $F = F_1 \cup F_2$.

Proposition 4.5 (Grobler and Mynhardt [15], [16]) *Suppose G is a connected ir-ER-critical graph other than a star. Then the following properties are true for every ir-set*

S of G.

1. *Every $u \in U$ annihilates exactly one vertex $v \in S$ with $N_G(u) = P_v$. In particular, $\langle U \rangle$ is independent and $N_G[U] = U \cup E$.*

2. *Every $p \in E_2$ annihilates exactly one vertex v of S and $v \in B'_2$.*

3. *If p and q are adjacent vertices of $V_G - (S \cup U)$, then, without loss of generality, $p \in E_2$ and $q \in F_2$.*

4. *If $p \in F_2$, then p annihilates no vertices of S.*

5. *Every $c \in C$ has exactly two neighbours and each neighbour is annihilated by a vertex of $U \cup E$. In particular, $\langle C \rangle$ is independent.*

6. *$\langle S \rangle$ is a disjoint union of non-trivial stars. Furthermore, if $v \in S$ has more than one neighbour in S, then each of its neighbours is annihilated by a vertex of $U \cup E$. If v has one neighbour in S, then v or its neighbour is annihilated by a vertex of $U \cup E$.*

7. *$U \neq \emptyset$*

The structure of a graph satisfied by Proposition 4.5 is illustrated in Figure 4.1.

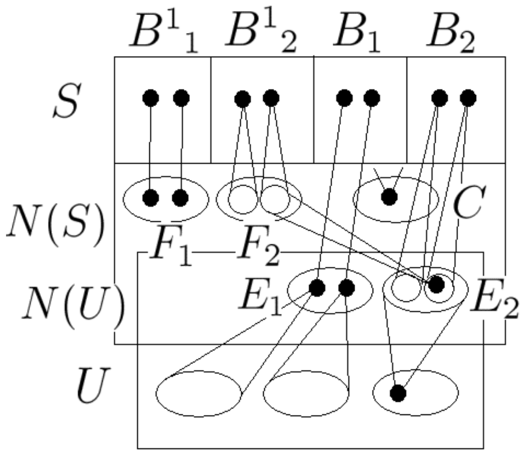

Figure 4.1: *A connected ir-ER-critical graph other than a star*

It follows from Proposition 4.5(1) that $C \cap N_G[U] = \emptyset$ (as we can see from Figure 4.1). From Proposition 4.5(5) it follows that $I = \emptyset$. Also, every vertex in E is annihilated by a vertex (possibly more than one) in U. From Proposition 4.5(5) and Proposition 4.5(3) it follows that no vertex in F_1 can be annihilated; hence $cb \notin E_G$ for any $c \in C$ and $b \in B^1{}_1$.

The following proposition shows that the graph as described in Proposition 4.5 is neither vertex-transitive nor edge-transitive.

Proposition 4.6 *Suppose G is a connected ir-ER-critical graph other than a star. Then G is neither vertex-transitive nor edge-transitive.*

Proof: Let G be a connected ir-ER-critical graph such that G is not γ-ER-critical. Thus from Proposition 1.4 it follows that $ir(G) < \gamma(G)$.

We want to prove that G is not vertex-transitive, thus assume to the contrary that G is vertex-transitive. Proposition 4.5(1) implies that $C \cap N_G[U] = \emptyset$ and $deg_G(c) = 2$ for every $c \in C$. For every $i \in I$, some $c \in C$ and $v \in S - I$ exists such that $\{i, c\}, \{c, v\} \in E_G$. Since I is not annihilated by any vertex in $U \cup E$, it follows from Proposition 4.5(5) that

$I = \emptyset$. Since G is vertex-transitive, $deg_G(v) = 2$ for each $v \in S$; and since each $v \in S$ is adjacent to at least one external private neighbour relative to S and one other vertex of S, it follows that $C = \emptyset$, $E_2 = \emptyset$ and $F_2 = \emptyset$. Take the γ-set T consisting of the external private neighbour of each $v \in S$. Then $\gamma(G) = ir(G)$, which is contrary to our assumption. Thus G is not vertex-transitive.

Assume that G is edge-transitive. Since $deg_G(c) = 2$, $deg_G(v_1) \geq 3$ and $deg_G(v_2) \geq 3$ for $\{v_1, v_2\} \subseteq S$, no automorphism ϕ of G exists such that $\phi(\{c, v_1\}) = \{v_1, v_2\}$ for any $c \in C$ and $\{v_1, v_2\} \subseteq S$. Thus G is not edge-transitive. ■

With the aid of Proposition 4.5, Grobler and Mynhardt ([15], [16]) also characterized the connected 2-ir-ER-critical graphs by defining the graph G as follows. Let G be a connected 2-ir-ER-critical graph. Take $V_G = U_1 \cup U_2 \cup C \cup \{v_1, v_2, p_1, p_2\}$ where U_1, U_2 and C are non-empty, and v_1, v_2, p_1 and p_2 are all distinct. Also E_G is such that for each $c \in C$, $\{cv_1, cv_2\} \subseteq E_G$; for each $i = 1, 2$ and $u \in U_i$, $up_i \in E_G$ and $\{v_1v_2, v_1p_1, v_2p_2\} \subseteq E_G$. The class of all such graphs G will be denoted by $\mathcal{G}$.

Proposition 4.7 (Grobler and Mynhardt [15], [16]) *G is a connected 2-ir-ER-critical graph if and only if $G \in \mathcal{G}$.*

An example of a connected 2-ir-ER-critical graph G is illustrated in Figure 4.2.

It is easy to see that G is neither vertex-transitive nor edge-transitive. From Proposition 4.5(3) it follows that there are no adjacent vertices in $V_G - (S \cup U)$.

In 2001 Cockayne, Favaron and Mynhardt ([10]) characterized the graphs G with $ir(G) = 3$ and $ir(G - uv) > 3$ for all $uv \in E_G$. We begin by describing two classes $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs (which will turn out to be the class of 3-ir-ER-critical graphs).

Let $\mathcal{G}_1$ be the class of all graphs G with $V_G = \{v_1, v_2, v_3, p_2, p_3\} \cup U_2 \cup U_3 \cup C \cup D$, where the sets in the union are non-empty and mutually disjoint. Also, let E_G consists of precisely those edges such that $p_2v_2v_1v_3p_3$ is the vertex sequence of a path in G; $N(u) = \{p_i\}$ for each $u \in U_i$, $i = 2, 3$; $N(d) = \{v_1\}$ for each $d \in D$ and $N(c) = \{v_2, v_3\}$ for each $c \in C$.

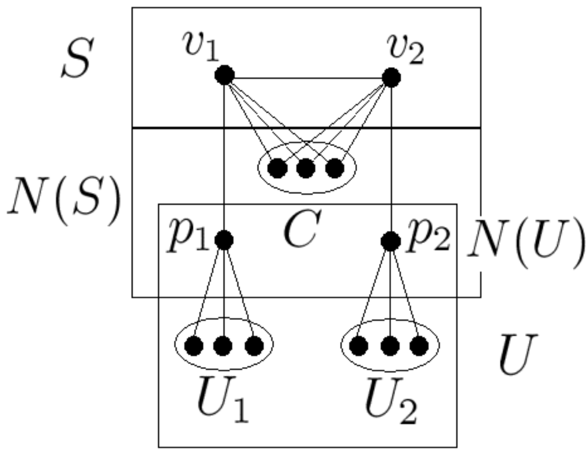

Figure 4.2: *A connected 2-ir-ER-critical graph G*

The structure of such a graph G is illustrated in Figure 4.3. Since $E_2 = \emptyset$, it follows from Proposition 4.5(3) that there exists no adjacent vertices in $V_G - (S \cup U)$. Also, we can see that $D \neq \emptyset$, $C \neq \emptyset$ and $U_i \neq \emptyset$ for $i = 2, 3$. It is clear to see that G is neither vertex-transitive nor edge-transitive.

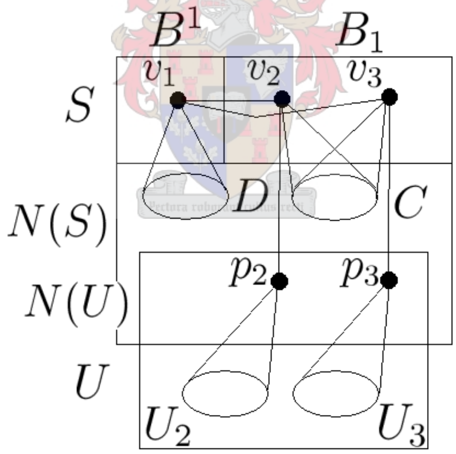

Figure 4.3: *The structure of the graph G*

Let $\mathcal{G}_2$ be the class of all graphs H with $V_H = \{v_1, v_2, v_3, p_1, p_2, p_3\} \cup U_1 \cup U_2 \cup U_3 \cup C$, where the sets in the union are non-empty and mutually disjoint. Also, let E_H consists of precisely those edges such that $v_2v_1v_3$ is the vertex sequence of a path in H; $N(p_i) = U_i \cup \{v_i\}$ for each $i = 1, 2, 3$; $N(c) \subseteq \{v_1, v_2, v_3\}$ and $|N(c)| = 2$ for each $c \in C$, with the requirement that $N(v_i) \cap C \neq \emptyset$ for $i = 1, 2$.

The structure of such a graph H is illustrated in Figure 4.4. H is clearly neither vertex-transitive nor edge-transitive. Each $c \in C$ is adjacent to either $\{v_1, v_2\}$, $\{v_1, v_3\}$ or $\{v_2, v_3\}$. Also, $U_i \neq \emptyset$ for $i = 1, 2, 3$ and $C \cap N_G[U] = \emptyset$.

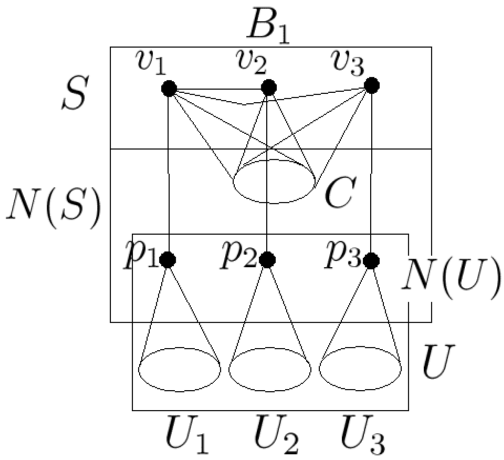

Figure 4.4: *The structure of the graph H*

Proposition 4.8 (Cockayne, Favaron and Mynhardt [10]) *G is a connected 3-ir-ER-critical graph if and only if* $G \in \mathcal{G}_1 \cup \mathcal{G}_2$.

By using the same method as used in [15], [16] and [10], we can construct connected k-*ir*-ER-critical graphs for $k > 3$.

4.4 π^--ER-critical graphs

Graphs that are π^--ER-criticality were briefly studied by Grobler and Mynhardt in [15] and [16] where they exhibited three classes of i^--ER-critical graphs. They used the following simple but useful result in their proofs.

Lemma 4.3 (Grobler and Mynhardt [15], [16]) *Let G be a graph such that for every* $uv \in E_G$, *G has a dominating set T with* $|T| < i(G)$, *and u and v are the only two non-isolated vertices in* $\langle T \rangle$. *Then G is* i^-*-ER-critical.*

Since we know that no γ^--ER-critical graphs exists, the following proposition follows.

Proposition 4.9 *For any graph G, if $i(G) = \gamma(G)$, then G is not i^--ER-critical.*

Proof: Given a graph G such that $i(G) = \gamma(G)$. Suppose G is i^--ER-critical. Then from Proposition 1.4 it follows that $\gamma(G - uv) \leq i(G - uv) < i(G) = \gamma(G)$ for all $uv \in E_G$. Since no γ^--ER-critical graphs exist, it follows that G is not i^--ER-critical. ■

Thus the only way for G to be i^--ER-critical is if $\gamma(G) < i(G)$.

Let us now determine under which assumptions the complete multipartite graph $K_{n_1,n_2,\cdots,n_m}$ is π^--ER-critical for $\pi = \{i, ir\}$.

Proposition 4.10 *If $G = K_{n_1,n_2,\cdots,n_m}$ with $m \geq 2$, then*

1. *G is i^--ER-critical if and only if $n_i \geq 3$ for $i = 1, 2, \cdots, m$.*
2. *G is not ir^--ER-critical.*

Proof:

1. This follows directly from Proposition 2.5 and Proposition 4.9.
2. If $n_i = 1$ for some $i = 1, 2, \cdots, m$, then G is not ir^--ER-critical; hence assume $n_i > 1$ for all $i = 1, 2, \cdots, m$. Thus $ir(G) = 2$ from Proposition 2.5. Since $ir(G - uv) \neq 1$ for any $uv \in E_G$, it follows that G is not ir^--ER-critical. ■

Let us now examine the complement of the product of two complete graphs.

Proposition 4.11 *If $G = \overline{K_m \times K_n}$ with $n \geq m \geq 2$, then G is i^--ER-critical if and only if $n \geq m \geq 4$.*

Proof: If $\gamma(G) = i(G)$, then from Proposition 4.9 G is not i^--ER-critical, thus let $n \geq m \geq 4$. By Proposition 2.9, $i(G) = m > 3 = \gamma(G)$. Consider any $uv \in E_G$. Since G is edge-transitive (as shown in Chapter 3), assume without loss of generality that $u = v_{11}$ and $v = v_{22}$. The set $\{v_{11}, v_{12}, v_{22}\}$ is a γ-set of G with u and v the only non-isolated vertices; hence from Lemma 4.3 it follows that G is i^--ER-critical. ■

Finally, in [15] and [16], Grobler and Mynhardt also showed that some circulants are i^--ER-critical, while we know that $G = K_m \times K_n$ with $n \geq m \geq 2$ is not i^--ER-critical (since $\gamma(G) = i(G)$).

In [11], Cockayne, Favaron and Mynhardt continued the study into i^--ER-critical graphs. They made a few observations, including the following: every i^--ER-critical graph with $\gamma(G) = 2$ is connected. They also formulated the next proposition.

Proposition 4.12 (Cockayne, Favaron and Mynhardt [11]) *A graph G is i^--ER-critical if and only if for every $uv \in E_G$, G has a dominating set D with $|D| < i(G)$ such that u and v are the only non-isolated vertices in $\langle D \rangle$.*

This, together with the observation, they used to prove the following proposition.

Proposition 4.13 (Cockayne, Favaron and Mynhardt [11]) *The graph G is i^--ER-critical with $\gamma(G) = 2$ and $i(G - e) = 2$ for every $e \in E_G$ if and only if G is a complete multipartite graph $K_{n_1,n_2,\cdots,n_m}$ with $m \geq 2$ and $n_i \geq 3$ for $i = 1, 2, \cdots, m$.*

The next corollary follows immediately.

Corollary 4.3 (Cockayne, Favaron and Mynhardt [11]) *The only 3-i^--ER-critical graphs are the complete multipartite graphs $K_{n_1,n_2,\cdots,n_m}$ with $m \geq 2$ and $n_i \geq 3$ for $i = 1, 2, \cdots, m$.*

They constructed two families, $\mathcal{F}_1$ and $\mathcal{F}_2$, of i^--ER-critical graphs such that $\gamma(G) = 2$ and $i(G-e) > 2$ for some $e \in E_G$. They constructed another two families, $\mathcal{F}_3$ and $\mathcal{F}_4$, of i^--ER-critical graphs such that $\gamma(G) = 3$ and $i(G)$ arbitrarily large.

They showed that for the graphs G of $\mathcal{F}_3$, $i(G-e) = 3$ for every $e \in E_G$, while for the graphs G of $\mathcal{F}_4$, $i(G-e) > 3$ for every edge of G. By generalising the family of graphs $\mathcal{F}_4$, they obtained i^--ER-critical graphs with even higher values of $\gamma(G)$. For instance, the graph G obtained from $q \geq 2$ complete bipartite graphs $K_{p,p}$ with $p \geq 3$ by adding an extra vertex w joined to all vertices of one class of each $K_{p,p}$ satisfies $\gamma(G) = q+1$, with $i(G-e) = (q-1)p+2 > \gamma(G)$ and $i(G) = pq > i(G-e)$ for every $e \in E_G$. The graphs in the families of $\mathcal{F}_3$ and $\mathcal{F}_4$ and its generalisation proved the following.

Proposition 4.14 (Cockayne, Favaron and Mynhardt [11]) *Given any two integers $i > \gamma \geq 3$, there exists an i^--ER-critical graph G with $\gamma(G) = \gamma$ and $i(G) = i$.*

They further examined a specific family of the circulant graphs and proved the following two propositions.

Proposition 4.15 (Cockayne, Favaron and Mynhardt [11]) *Let r, m, q be positive integers such that $m \geq 2$ and $1 \leq q \leq 2r-1$, with q odd, and let G be the circulant $C_n\langle 1, 3, \cdots, 2r-1\rangle$ with $n = m(2r+1)+q$. Then $\gamma(G) = m+1$ and $i(G) = m+r+\frac{1}{2}(q-1)$.*

Proposition 4.16 (Cockayne, Favaron and Mynhardt [11]) *Let r, m, q be positive integers such that $m \geq 2$, $r \geq 2$, q odd and $1 \leq q \leq 2r-1$, and let $n = m(2r+1)+q$. Let G be the circulant $C_n\langle 1, 3, \cdots, 2r-1\rangle$ and let e_l, with l odd, $1 \leq l \leq 2r-1$, be any edge of G joining two vertices at distance l on the cycle $C = v_0, v_1, \cdots, v_n$. Then*

$$i(G-e_l) = \begin{cases} m+l+\frac{q-1}{2} & \text{if } \ 1 \leq l \leq q \ \ for \ \ q \ \ odd, \\ m+l+\frac{q+1}{2} & \text{if } \ q < 2r-3 \ \ and \ \ q < l \leq 2r-3 \ \ for \ \ q \ \ odd, \\ m+r+\frac{q-1}{2} = i(G) & \text{if } \ q < 2r-1 \ \ and \ \ l = 2r-1 \end{cases}$$

These results they then used to determine which of the graphs of a special constructed family $\mathcal{F}_5$ are i^--ER-critical.

Corollary 4.4 (Cockayne, Favaron and Mynhardt [11]) *A circulant* $C_n\langle 1,3,\cdots,2r-1\rangle$ *belonging to the family* $\mathcal{F}_5$ *is* i^-*-ER-critical if and only if* $r \geq 2$ *and* $q = 2r-1$.

From this corollary we also have an example of an i^--ER-critical graph satisfying $\gamma(G) = \gamma$ and $i(G) = i$ for two given integers $\gamma \geq 3$ and $i > \gamma$ with $i + \gamma$ even. The existence of ir^--ER-critical graphs is proving to be a very difficult question however, with a lot of work being done on it, but as yet no results or partial results to aid in the discovery of these graphs (or the proving of their non-existence).

Chapter 5

Open Problems

In our examination of the different types of criticality for the lower domination parameters, the following open problems arose:

1. Do any ir-critical (ir-edge-critical) graphs exist which are not γ-critical (γ-edge-critical)?

2. Do any γ-critical (γ-edge-critical) graphs exist which are not ir-critical (ir-edge-critical)?

3. For any graph G, is it true that G is ir-critical if and only if every vertex of G is a singular isolated vertex of some ir-set of G? We showed that this is true for all four classes of graphs investigated in Chapter 2, as well as for any 2-ir-critical graphs.

4. For any graph G, is it true that G is ir-edge-critical if and only for every $uv \in E_{\overline{G}}$, there exists a π-set T of G such that $\{u, v\} \subseteq T$ and u or v is a singular isolated vertex of T? We proved this is true for all three of the classes of graphs investigated in Chapter 3, as well as for 2-ir-edge-critical graphs.

5. Is it true that for any ir-critical (ir-edge-critical) graph G, $ir(G) = \gamma(G)$? We showed in Chapter 2 and 3 that this is true for 2-ir-critical (2-ir-edge-critical) and 3-ir-critical (3-ir-edge-critical) graphs.

6. Will 3-ir-criticality (3-ir-edge-critical) imply 3-γ-criticality (3-γ-edge-critical)?

7. If the graph G is vertex-critical with respect to its lower domination parameters, will its complement always be vertex-transitive? We found it to be true for all four classes of graphs investigated in Chapter 2.

8. If the graph G is edge-critical with respect to its lower domination parameters, will its complement always be symmetric? We found it to be true for the three classes of graphs investigated in Chapter 3.

9. Does any ir^{-}-ER-critical graphs exist?

References

[1] R.B. Allan and R.C. Laskar, *On Domination and some related concepts in Graph Theory*, Congr. Numer. **21**(1978), 43–58.

[2] S. Ao, *Independent Domination Critical Graphs*, Master's Dissertation, University of Victoria, 1994.

[3] C. Berge, *Theory of Graphs and its Applications*, Methuen, London, 1962.

[4] B. Bollobas and E.J. Cockayne, *On the irredundance number and maximum degree of a graph*, Discrete Math. **49**(1984), 197–199.

[5] R.C. Brigham, P.Z. Chinn and R.D. Dutton, *Vertex domination-critical graphs*, Networks **18**(1988), 173–179.

[6] E.J. Cockayne and S.T. Hedetniemi, *Independence Graphs*, Proc. Fifth S.E. Conf. on Combinatorics, Graph Theory and Computing, 471–491, Utilitas Math., Winnipeg, 1974.

[7] E.J. Cockayne, P.J.P. Grobler, S.T. Hedetniemi and A.A. McRae,*What makes an irredundant set maximal*, J. Combin. Math. Combin. Comput., **25** (1997), 213–223.

[8] E.J. Cockayne, S.T. Hedetniemi and D.J. Miller, *Properties of hereditary hypergraphs and middle graphs*, Canad. Math. Bull., **21** (1978), 461–468.

[9] E.J. Cockayne and C.M. Mynhardt, *Irredundance and maximum degree in graphs*, Combin. Prob. Comput. **6**(1997), 153–157.

[10] E.J. Cockayne, O. Favaron and C.M. Mynhardt, *Irredundance-Edge-Removal-Critical Graphs*, Util. Math. **60**(2001), 219–228.

[11] E.J. Cockayne, O. Favaron and C.M. Mynhardt, *On i^--ER-Critical Graphs*, Discrete Math. **276**(2004), 111–125.

[12] O. Favaron, *A Note on the Irredundance Number after vertex-deletion*, Discrete Math. **121**(1993), 51–54.

[13] O. Favaron, D.P. Sumner and E. Wojcicka, *The diameter of domination k-critical graphs*, J. Graph Theory **18**(1994), 723–734.

[14] J.B. Fraleigh, *A First Course in Abstract Algebra*, Addison-Wesley, Canada, 1982.

[15] P.J.P. Grobler, *Critical Concepts in Domination, Independence and Irredundance of Graphs*, Ph.D. Thesis, University of South Africa, 1998.

[16] P.J.P. Grobler and C.M. Mynhardt, *Domination Parameters and Edge-Removal-Critical Graphs*, Discrete Math. **231** (2001), 221–239.

[17] P.J.P. Grobler and C.M. Mynhardt, *Upper Domination Parameters and Edge-Critical Graphs*, J. Combin. Math. Combin. **33**(2000), 239–251.

[18] M. Paris, D.P. Sumner and E. Wojcicka, *Edge-domination critical graphs with cut-vertices*, Congr. Numer. **141**(1999), 111–117.

[19] D.P. Sumner and P. Blitch, *Domination critical graphs*, J. Combin. Theory Ser. B **34**(1983), 65–76.

[20] H.B. Walikar and B.D. Acharya, *Domination critical graphs*, Nat. Acad. Sci. Lett. **2**(1979), 70–72.

[21] H.B. Walikar, E. Sampathkumar and B.D. Acharya, *Two new bounds for the domination number of a graph*, Technical Report **14**, MRI, 1979.

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