Computing the Greeks Using the Integration by Parts Formula for the Skorohod Integral

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Thesis presented in partial fulfillment of the requirements for the degree of **Master of Science** at the **University of Stellenbosch**

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March 2008

Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and I have not previously in its entirety or in part submitted it at any University for a degree.

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3 August 2007

Abstract

The computation of the greeks of an option is an important aspect of financial mathematics. The information gained from knowing the value of a greek of an option can help investors decide whether or not to hold on to or to sell their options to avoid losses or gain a profit.

However, there are technical difficulties that arise from having to do this. Among them is the fact that the mathematical formula for the value some options is complex in nature and evaluating their greeks may be cumbersome. On the other hand the greek might have to be numerically estimated if the option does not posses an explicit evaluation formula. This could be a computationally expensive undertaking.

Malliavin calculus offers us a solution to these problems. We can find formula that can be used in combination with Monte Carlo simulations to give results quickly and which are not computationally expensive to obtain and hence give us an degree of accuracy higher that non Malliavin calculus techniques.

This thesis will develop the Malliavin calculus tools that will enable us to develop the tools which we will then use to compute the greeks of some known options.

Opsomming

Die berekening van die risikos van opsie is belangrike aspek van finansile wiskunde. Die inligting wat verkry word uit kennis omtrent die waarde van risiko van opsie kan beleggers help besluit of hulle hulle opsies moet verkoop of nie ten einde verliese te voorkom of wins te maak.

Daar kan egter tegniese probleme hieruit voortspruit. Onder andere is daar die feit dat die wiskundige formule vir die waarde van party opsies kompleks van aard is en evaluering van hulle opsies kan lastig wees. Aan die ander kant moet die risiko dalk numeries benader word indien die opsie nie oor eksplisiete evalueringsformule beskik nie. Dit kan, wat berekening betref, n duur onderneming wees.

Malliavin-calculus bied aan ons oplossing vir hierdie probleme. Ons kan n formule kry om tesame met Monte Carlo-simulasies te gebruik wat resultate vinnig kan lewer en wat nie wat berekening betref, duur sal wees om te bekom nie. Dit sal ons dus mate van akkuraatheid gee wat groter is as nie-Malliavin-calculustegnieke.

Hierdie tesis sal die Malliavin-calculusinstrumente ontwikkel wat ons in staat sal stel om die instrumente te ontwikkel wat ons dan kan gebruik om die risikos van sommige bekende opsies te bereken.

Contents

Chapter 1

Introduction

1.1 Overview

Options are an integral part of any financial market. They are attractive to investors because they are relatively cheaper to buy and so offer higher net returns than the underlying assets on which they are written. They can be used by speculators to hedge against the risks associated with sharp movements of stock prices. Speculators can use options to generate high returns from options if they bet against the market and their predictions about future market behaviour are accurate.

Due to their great importance, one would need to quantify the risk associated with owning an option. This risk is a derivative (or sensitivity) and it is called a greek. A greek is essentially the derivative of the risk-neutral expected value of the (discounted) option's payoff with respect to a parameter associated with the option.

Greeks can be calculated explicitly or estimated numerically using Monte Carlo simulations. Evaluating the greek may come with some difficulties. In the first place, the function representing the payoff may be complex and carrying out the differentiation may be cumbersome. More than that, if no analytic evaluation existed and the greek were to be estimated using numerical methods such as finite difference methods, the estimation of the greek may be computationally expensive and the result would be inaccurate due to having to make two estimates: estimating the expectation and estimating the derivative of the option's payoff.

This is where Malliavin calculus enters the scene. Due to a result known as the Integration by Parts formula for the Skorohod Integral, we can bypass having to evaluate the derivative of the option's payoff. Instead of computing the derivative of the risk-neutral expected value of the option's payoff, we compute the risk-neutral expected value of the (discounted) option's payoff times a "Malliavin weight". This expectation can either be computed explicitly or estimated by using Monte Carlo simulations. Either way, because the derivative is not estimated, we are assured that our computation is less expensive than if we had to estimate the derivative of the option's payoff.

In this thesis, we will develop the Integration by Parts formula for the Skorohod Integral. We will give a concise explanation of the greeks and we will also give examples of greeks of different types of options. We will then discuss some fundamental concepts in Malliavin calculus and then finally prove the Integration by parts formula for the Skorohod integral. Later, the greeks for a European call option will be computed using Malliavin calculus and then juxtaposed with the analytic results for the greeks of a European call option. We will then repeat this for some exotic options. First, we develop the market model in which we will price the options.

1.2 Black-Scholes Framework

The stock price process S_t, with $S_t : \Omega \to \mathbb{R}$ and $t \in [0,T]$, will be modelled using the Black-Scholes model. Our model will consist of one stock (risky asset) and one bond (risk-free asset). The price process of the risky asset is assumed to have the form

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t.$$

This can be solved to give us *geometric brownian motion*

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right)$$

for $t \in [0,T]$, where W_t for $t \in [0,T]$ is a Brownian motion defined on a complete (filtered) probability space $(\Omega, \mathcal{F}_t, \mathbb{F}, \mathbb{P})$. The filtration is generated by the Brownian motion and completed by the $\mathbb{P}$-null sets and is denoted by $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$, where $\mathcal{F}_t$ is the sigma algebra generated by the Brownian motion at time t. The *mean rate of return* μ and the *volatility* $\sigma \geq 0$, are constant for $t \in [0,T]$.

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t.$$

The bond price process B_t, $t \in [0,T]$ evolves according to the differential equation

$$dB_t = r B_t \, dt, \quad B_0 = 1$$

where the *interest rate* process, $r \geq 0$ is constant for $t \in [0, T]$. In other words, the bond price satisfies

$$B_t = e^{rt}.$$

In the Black-Scholes model, investors are considered to be rational. Such an investor could start off with an initial endowment $x \geq 0$ and he/she could invest it in the assets described above. Let α_t be the number of non-risky assets and β_t be the number of stocks owned by the investor at time t. The pair $\phi_t = (\alpha_t, \beta_t)$ for $t \in [0, T]$, is called a *trading strategy*. The processes α_t and β_t, for $t \in [0, T]$, are measurable with respect to $\mathcal{F}_t$ at time t, and adapted to the filtration $\mathbb{F}$ and are such that

$$\int_0^T |\beta_t \mu| \, dt < \infty, \quad \int_0^T \beta_t^2 \sigma^2 \, dt < \infty, \text{ and } \; r \int_0^T |\alpha_t| \, dt < \infty. \tag{1.1}$$

Then $x = \alpha_0 + \beta_0 S_0$, and the investor's wealth at time t (also called the *value* of the trading strategy) is

$$V_t(\phi) = \alpha_t B_t + \beta_t S_t.$$

The gain of the trading strategy is defined to be

$$G_t(\phi) = \int_0^t \alpha_s \, dB_s + \int_0^t \beta_s \, dS_s.$$

Both integrals are well defined thanks to the conditions in equation (1.1).

Definition 1.1 (Admissible)

A trading strategy ϕ_t for $t \in [0, T]$ is said to be *admissible* for the initial endowment $x \geq 0$ if its value satisfies $V_t(\phi) \geq 0$ for $t \in [0, T]$ almost surely. The investor is then said to be solvent.

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Definition 1.2 (Self-Financing Strategy)

A trading strategy is called self-financing if there are no external injection or withdrawal of funds into the trading strategy. The gains or losses in the strategy are entirely from the movements in the bond and stock prices as well as their positions in the strategy. In this case

$$V_t(\phi) = x + \int_0^t \alpha_s \, dB_s + \int_0^t \beta_s \, dS_s. \tag{1.2}$$

In differential form, we have that

$$dV_t(\phi) = \alpha_t \, dB_t + \beta_s \, dS_t \tag{1.3}$$

for $t \in [0, T]$.

$\blacksquare$

Definition 1.3 (Arbitrage)

An arbitrage opportunity is a self-financing strategy ϕ such that

$$V_0(\phi) = 0, \;\; V_T(\phi) \geq 0$$

and $P(V_T(\phi) > 0) > 0$.

$\blacksquare$

Definition 1.4 (Discounted Stock Price Process)

We say that the stock prices process S_t for $t \in [0, T]$ is discounted if we express these prices as fractions of the bond price process B_t for $t \in [0, T]$. We denote the discounted price process by $\widetilde{S}_t$ for $t \in [0, T]$. In other words

$$\widetilde{S}_t = \frac{S_t}{B_t}$$

for $t \in [0, T]$.

$\blacksquare$

Definition 1.5 (Equivalent Martingale Measure/Risk-Neutral Measure)

When two probability measures have the same null sets, they are said to be equivalent. A probability measure $\mathbb{Q}$ on the σ-algebra $\mathcal{F}_t$, for $t \in [0, T]$ which is equivalent to $\mathbb{P}$, is called an equivalent martingale measure (or a risk-neutral probability measure) if the discounted price process $\widetilde{S}_t$ for $t \in [0, T]$, is a martingale in the probability space $(\Omega, \mathcal{F}_t, \mathbb{Q})$.

$\blacksquare$

In general, market models for the stock price process are assumed to be void of arbitrage. Intuitively, the existence of an arbitrage opportunity is a sign of lack of equilibrium in the market. No real market equilibrium can exist in the long run if there is arbitrage present all the time ([21]). It is therefore imperative to determine if arbitrage is present in the market or not. In the case of discrete time models with *finite* trading horizon, the absence of arbitrage opportunities is equivalent to the existence of equivalent martingale measures. This is known as *the first fundamental theorem of asset pricing* ([7]).

In continuous time, the relation between the absence of arbitrage opportunities and existence of martingale measures is more complicated. Since

$$\sigma > 0$$

for all $t \in [0, T]$ we consider the constant θ, given by

$$\theta = \frac{\mu - r}{\sigma}.$$

We then consider the process Z_t for $t \in [0, T]$, defined by

$$Z_t = \exp\left(-\int_0^t \theta \, dW_s - \frac{1}{2}\int_0^t \theta^2 \, ds\right) \tag{1.4}$$

which is a positive martingale ([20]). By the Girsanov theorem ([21], [15]), the measure $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$$

is a probability measure, which is also equivalent to $\mathbb{P}$, and is such that the process defined by

$$\widetilde{W}_t = W_t + \int_0^t \theta \, ds$$

is a Brownian motion under $\mathbb{Q}$. Notice that in terms of the process $\widetilde{W}_t$ for $t \in [0, T]$, the price process at time t can be expressed as

$$S_t = S_0 \exp\left(\int_0^t \left(r - \frac{\sigma^2}{2}\right) ds + \int_0^t \sigma \, d\widetilde{W}_s\right)$$

for $t \in [0, T]$. The discounted prices are given by

$$\widetilde{S}_t = B_t^{-1} S_t = S_0 \exp\left(\int_0^t \sigma \, d\widetilde{W}_s - \frac{1}{2}\int_0^t \sigma^2 \, ds\right)$$

and the process $\widetilde{S}_t$ forms a martingale under the measure $\mathbb{Q}$ as the differential form of the discounted stock price is $d\widetilde{S}_t = \sigma S_t \, d\widetilde{W}_s$ by the Itô formula.

Definition 1.6 (Derivative Security)

A derivative security or just derivative is a contract on a risky asset that produces a payoff $F_\tau(\omega)$ at a time $\tau \in [0, T]$. The value of the derivative is contingent on the value of an underlying risky asset. An *option* is a type of derivative security that gives the holder some rights at a premium. For European options, $\tau = T$. We will work with this type of option. The payoff is in general an $\mathcal{F}_T$-measurable non-negative random variable F_T, ie $F_T : \Omega \to \mathbb{R}$.

■

A *call option* is an option that gives the holder the right but not the obligation to buy an asset at a predetermined price on or before the expiry date. On the other hand, a *put option* gives the holder the right but not the obligation to sell an asset at a predetermined price on or before the expiry date. For European puts and European calls, the holder decides whether to sell or respectively buy the asset at the expiry date, T.

Definition 1.7 (Attainability/Replicability and Completeness)

A derivative security with nonnegative payoff F_T is said to be *attainable* or *replicable* if there exists a self-financing trading strategy ϕ_t for $t \in [0, T]$ such that $V_t \geq 0$ and $V_T(\phi) = F_T$. A market model in which every derivative security is replicable is called *complete*.

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Theorem 1.1

Let F_T be an $\mathcal{F}_T$-measurable random variable such that $\mathbb{E}(B_T^{-2} Z_T^2 F_T^2) < \infty$. Then F_T is attainable. Here Z_T is as is equation (1.4).

Proof See [20]

□

The Black-Scholes model makes the assumptions that the drift, μ, and the volatility, σ, coefficients in the Itô diffusion that models the stock prices are constant. The Itô diffusion is called the geometric brownian motion: $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$. The solution of geometric Brownian motion is well known to be

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

([21]). Moreover, setting $\theta = \dfrac{\mu - r}{\sigma}$, means that

$$Z_t = \exp\left(-\theta W_t - \frac{\theta^2}{2} t\right),$$

for $t \in [0, T]$.

From the Itô formula, the differential form of Z_t is expressed as

$$dZ_t = -\frac{\theta^2}{2}\, dt - \theta\, dW_t + \frac{(-\theta)^2}{2}\, dt = -\theta\, dW_t.$$

Hence

$$Z_T - Z_0 = -\int_0^T \theta\, dW_t$$

and so

$$Z_T = Z_0 - \int_0^T \theta\, dW_t = 1 - \int_0^T \theta\, dW_t.$$

We see that $\mathbb{E}_{\mathbb{P}}(Z_T) = 1$, since the Itô integral is a martingale and it is zero at time 0. Also, by Girsanov's theorem ([21]), the process $\widetilde{W}_t = W_t + \theta_t$ for $t \in [0, T]$ is a Brownian motion under $\mathbb{Q}$, with $\dfrac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$ on the time interval $[0, T]$.

The Black-Scholes is complete in the sense that any payoff $F_T \geq 0$ satisfying $\mathbb{E}_{\mathbb{Q}}\left(B_T^{-2} Z_T^2 F_T^2\right) < \infty$ is replicable, as by theorem 1.1 ([20]).

1.3 The Greeks

Recall that a derivative security is a contract on a risky asset that produces a payoff on or before a time referred to as the expiry date T. The risky asset is usually a stock with value $S_t(\omega)$ at time t and we will assume this to be the case. Derivative securities can be considered as functions that are dependent on several variables. These are, the value of the underlying stock price process S_t, the time $t \in [0, T]$, the volatility in the market σ for $t \in [0, T]$ the risk-free interest rate r, and the strike price K. The strike price is a predetermined quantity that both the buyer and the seller of the derivative security agree upon before the derivative is written.

We consider a specific type of derivative security called an option, which gives its holder the right without obligation, to carry out certain transactions (such as buying or selling a stock).

In what follows, the strike price K and the expiry date T are fixed once the option is written and the remaining variables S_t, t, r, σ are analysed.

The value of a general option written as a function of these variables is $F(S_t, t, r, \sigma)$, but will be denoted as F for the sake of brevity. The *greeks* describing the sensitivity of an option with respect to the various variables determining the option price are now defined. These sensitivities or derivatives are

- the delta, $\Delta(F) = \dfrac{\partial F}{\partial S_t}$,
- the gamma, $\Gamma(F) = \dfrac{\partial^2 F}{\partial S_t^2}$,
- the theta, $\Theta(F) = \dfrac{\partial F}{\partial t}$,
- the vega, $\vartheta(F) = \dfrac{\partial F}{\partial \sigma}$,
- the rho, $\rho(F) = \dfrac{\partial F}{\partial r}$.

Of the greeks described above, the delta, the gamma and the vega are the most important to an investor. The delta tells the investor by how much the value of the option will change given a change in the value of the underlying stock price $S_t(\omega)$ for $(t, \omega) \in [0, T] \times \Omega$. The delta of an option also coincides with the amount of stock initially required in the replicating strategy of the option. In portfolio theory, the delta is important for another reason: *delta hedging.*

Suppose that one has a portfolio that consists of stocks, bonds and derivatives on the stock. A change in the stock price could adversely affect the value of the portfolio. To immunise the portfolio from changes in the stock price, one could dynamically adjust the delta of the portfolio to zero by holding different amounts (positive or negative) of the different components of the portfolio.

The gamma measures by how much the delta of the option changes given a change in $S_t(\omega)$ for $(t, \omega) \in [0, T] \times \Omega$. In other words, the gamma informs the investor of how frequently he/she should re-hedge when delta-hedging. The vega indicates to the investor by how much the the value of the option will change given the a change in the volatility of the market.

Usually, investors actively buy and sell options based on the greeks of the options at their disposal. Risk averse investors also know as hedgers, will sell options with high greek values fearing a sharp drop in the value of their options should $S_t(\omega)$ or σ, for $(t, \omega) \in [0, T] \times \Omega$, decrease. Speculators on the other hand will find options with high greek values attractive due to the prospect of an increased value in the options should $S_t(\omega)$ or σ, for $(t,\omega) \in [0,T] \times \Omega$, increase. We now look at some examples of options.

1.3.1 European Call Option

A European call option gives the holder the right to buy a stock at the strike price on the expiry date. Using the Black-Scholes framework, it can be shown that if $F = F(T, S_T) = (S_T - K)^+$ is a European call option, then for $t \in [0,T]$

$$F(t, S_t) = S_t N(d_+) - Ke^{-r(T-t)} N(d_-), \tag{1.5}$$

where K is the strike price and

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$$

is the cumulative Gaussian, with σ the volatility,

$$d_+(t) = \frac{\ln \frac{S_0}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{1.6}$$

and

$$d_-(t) = \frac{\ln \frac{S_0}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}. \tag{1.7}$$

Remark 1.1

The parameters $d_+(t)$ and $d_-(t)$ will mainly be used at $t = 0$ and for the sake of brevity will be denoted by d_+ and d_- respectively.

■

This is the much celebrated Black-Scholes evaluation formula for a European Call Option first derived in [4]. We now consider the greeks for this particular option.

The delta of a European call option is the rate of change of its value with respect to the underlying security price. From equation (1.5), it follows that at $t = 0$, we have that $F = F(0, S_0) = S_0 N(d_+) - Ke^{-rT} N(d_-)$, and therefore

$$\Delta(F) = \frac{\partial F}{\partial S_0} = N(d_+) = \int_{-\infty}^{d_+} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy.$$

Differentiating the delta further (by using the Leibniz rule[28]), we see that

$$\Gamma(F) = \frac{\partial^2 F}{\partial S_0^2} = \frac{1}{S_0 \sigma \sqrt{2\pi T}} e^{-\frac{1}{2}d_+^2}.$$

By using the chain rule, we see that the vega of a European Call option is

$$\vartheta(F) = \frac{\partial F}{\partial \sigma} = S_0 \sqrt{T} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_+^2}.$$

The greeks for a European call option are easily derivable. Nonetheless, Malliavin calculus techniques can still be used to obtain the greeks in this case as well and we can use the accuracy of the Malliavin calculus techniques to vindicate their use in obtaining the greeks of options in general.

1.3.2 Knock Out Barrier Options

A barrier option is an option whose payoff is that of another option but with an added constraint. The option becomes invalid if the price process of the underlying asset reaches a certain amount. There are basically four types of barrier options. *Up and in* barriers pay out the payoff at expiry if the underlying price process exceeds a certain amount and nothing otherwise. *Down and in* barriers pay out the payoff at expity if the underlying price process goes below a certain amount. *Up and out* barrier option and *down and out* barrier options are similar but nothing is paid out at expiry if the price process exceeds or goes below a certain level respectively.

In this thesis we will look at an up and out call option and an up and out digital (or knock out digital) barrier option. Computing the greeks for these options is very tedious due to the complicated nature of the price of these options. Using Malliavin calculus proves to be much quicker.

1.3.3 Arithmetic Asian Option

An arithmetic Asian option is an option in which the payoff is determined by the arithmetic average of the price process of the underlying asset. There is no known analytic price for an arithmetic Asian option. The price is usually obtained by using inverse Laplace transforms numerically ([15]). Since a greek is the derivative of the price of the option, there are no formulas for the greeks either. Malliavin calculus techniques can be used to estimate the greeks for arithmetic Asian options. In fact, this is the accepted method for obtaining the greeks for arithmetic Asian options in industry ([1], [18]).

Chapter 2

The Malliavin Derivative and the Skorohod Integral

The derivative is the fountainhead of "ordinary calculus". Likewise, the Malliavin derivative is a fundamental concept in Malliavin calculus. In this chapter, we develop the theory that will enable us to build this derivative and give some of its properties. We shall see that despite being inherently different, the derivative from ordinary calculus and the Malliavin derivative have a very close resemblance. Later we study the Skorohod integral which we will define as the adjoint of the Malliavin derivative.

2.1 Wiener Space

In this section, we describe the framework that we will use to develop Malliavin calculus. Consider the space of real valued continuous functions on $[0, T]$ with value 0 at time 0, i.e.

$$\Omega = C_0([0, T]) = \{\omega_t : [0, T] \to \mathbb{R} | \omega_t \text{ continuous}, \ \omega_0 = 0\}.$$

Consider also another probability space $(\Psi, \mathcal{G}, \nu)$ where $(\beta_t)_{t \in [0,T]}$ is a Brownian motion with respect to the probability measure ν, and let the σ-algebra $\mathcal{G}$ be the σ-algebra generated by this Brownian motion. Since Brownian motion is continuous, it can be regarded as a mapping from Ψ into Ω, namely the mapping of $X \in \Psi$ to the continuous function $X \to \beta_t(X) \in \Omega$. The space Ω is equipped with the σ-algebra $\mathcal{F}$ generated by the finite-dimensional cylinder sets

$$\omega_{t_1} \in A_1, \ldots, \omega_{t_n} \in A_n$$

where $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T$ and $A_1, A_2, \ldots, A_n \in \mathcal{B}$ are Borel sets in $\mathbb{R}$. The Brownian motion $X \mapsto \beta_t(X)$ can be regarded as a measurable mapping from $(\Psi, \mathcal{G}, \nu)$ to $(\Omega, \mathcal{F})$, and therefore induces a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ given by

$$\mathbb{P}(\{\omega | \omega_{t_1} \in A_1, \ldots, \omega_{t_n} \in A_n\}) = \nu(\beta_{t_1} \in A_1, \ldots, \beta_{t_n} \in A_n).$$

This measure is called the *Wiener measure.*

Defining the coordinate mapping process, $W_t : \Omega \to \mathbb{R}$, on the Wiener space by

$$W_t(\omega) = \omega_t,$$

it can be shown that the process $\mathcal{W} = (W_t)_{t\in[0,T]}$ has the same distribution under $\mathbb{P}$ as $(\beta_t)_{t\in[0,T]}$ has under ν ([9],[20]). Hence $\mathcal{W} = (W_t)_{t\in[0,T]}$ is a Brownian motion in the space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover the σ-algebra $\mathcal{F}$ is the σ-algebra generated by the Brownian motion $\mathcal{W} = (W_t)_{t\in[0,T]}$.

2.2 Differentiation on Wiener Space

It is well-known that Brownian motion is nowhere differentiable with respect to time ([15]). However, it is possible to define the concept of differentiation of random variables with respect to perturbations in the underlying Brownian motion. Moreover, there are two ways to establish a derivative on Wiener Space. In this section, the method that uses the notion of a directional derivative is adopted ([32],[9]). The other favoured method employs the Wiener-Itô expansion theorem ([20]).

Let $h \in L^2([0,T])$ be a deterministic function and consider functions in $C_0([0,T])$ (continuous functions on the interval $[0,T]$ with initial value 0) of the form

$$\gamma_t = \int_0^t h_s \, ds \quad \text{for } t \in [0,T]. \tag{2.1}$$

Since the derivative is developed using the notion of a directional derivative, functions of this form are called directions. The set of all such directions in $L^2[0,T]$ is called **Cameron-Martin Space** ([21], [20]). Also, notice that the map $t \mapsto \gamma_t$ is continuous on $[0,T]$ and $\gamma_0 = 0$. Therefore $\gamma \in C_0([0,T])$. For a random variable $F : [0,T] \times \Omega \to \mathbb{R}$, the *directional derivative* of F at the point ω in the direction $\gamma_t = \displaystyle\int_0^t h_s \, ds$ is defined by

$$(D_\gamma F)_t (\omega) = \lim_{\epsilon \to 0} \frac{F_t(\omega + \epsilon\gamma) - F_t(\omega)}{\epsilon}$$

if the limit exists in $L^2([0,T] \times \Omega)$. Further, if there exists a random variable

$$M \in L^2([0,T] \times \Omega),$$

with $M : [0,T] \times \Omega \to \mathbb{R}$, such that

$$\left(D_\gamma F\right)_t(\omega) = \int_0^T M_t(\omega) h_t \, dt \text{ for all } \omega \in \Omega \tag{2.2}$$

and all γ_t of the form in equation (2.1), then F is called a *differentiable* random variable and the *derivative* of F is defined to be the random variable

$$D_t F = M.$$

What this means is that if F is differentiable, then there exists, for each $t \in [0,T]$, a random variable $D_t F : [0,T] \times \Omega \to \mathbb{R}$ such that

$$\left(D_\gamma F\right)_t(\omega) = \int_0^T (D_t F)_t(\omega) h_t \, dt \tag{2.3}$$

for all $\omega \in \Omega$ and $\gamma \in C_0([0,T])$ of the form in equation (2.1). The set of all differentiable random variables is denoted as $\mathcal{D}_{1,2}([0,T] \times \Omega)$.

This definition allows us to define a differentiation operation analogous to the chain rule in ordinary calculus.

Theorem 2.1 (Chain Rule)

Let $F \in \mathcal{D}_{1,2}([0,T] \times \Omega)$, let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise differentiable function and let the direction be of the form $\gamma_t = \displaystyle\int_0^t h_s \, ds$, for some $h \in L^2[0,T]$.

Then $f \circ F \in \mathcal{D}_{1,2}([0,T] \times \Omega)$ and

$$D_t(f \circ F) = (f^{'} \circ F) D_t F.$$

In other words,

$$\left(D_t f(F)\right)_t(\omega) = f^{'}\left(F_t(\omega)\right)\left(D_t F\right)_t(\omega)$$

for all $(t,\omega) \in [0,T] \times \Omega$.

Proof. From the definition of the directional derivative, it can immediately be seen that

$$\left(D_\gamma f(F)\right)_t(\omega) \quad = \quad \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(f(F_t(\omega + \epsilon\gamma)) - f(F_t(\omega))\right).$$

But we can write this as

$$\begin{aligned} &\lim_{\epsilon \to 0} \left[\frac{f(F_t(\omega + \epsilon\gamma)) - f(F_t(\omega))}{F_t(\omega + \epsilon\gamma) - F_t(\omega)} \; \frac{F_t(\omega + \epsilon\gamma) - F(\omega)}{\epsilon}\right] \\ &= f^{'}(F_t(\omega))(D_\gamma F)_t(\omega). \end{aligned}$$

Since $F \in \mathcal{D}_{1,2}([0,T]\times\Omega)$, the random variable D_tF exists in $L^2([0,T]\times\Omega)$ and

$$\begin{aligned}
\left(D_\gamma f(F)\right)_t(\omega) &= f^{'}(F_t(\omega))\left(D_\gamma F\right)_t(\omega) \\
&= f^{'}(F_t(\omega))\int_0^T \left(D_tF\right)_t(\omega)h_t\,dt \\
&= \int_0^T f^{'}(F_t(\omega))\left(D_tF\right)_t(\omega)h_t\,dt
\end{aligned}$$

for all $(t,\omega) \in [0,T]\times\Omega$. Hence $f(F) \in \mathcal{D}_{1,2}([0,T]\times\Omega)$ and

$$\left(D_t f(F)\right)_t(\omega) = f^{'}(F_t(\omega))\left(D_tF\right)_t(\omega)$$

for all $(t,\omega) \in [0,T]\times\Omega$ as by the definition of the directional derivative in equation (2.2).

□

These results can be applied to functions that are Itô integrals of deterministic functions on the interval $[0,T]$.

Theorem 2.2

Let $F_T : \Omega \to \mathbb{R}$ be defined by

$$F_T = \int_0^T f_s\,dW_s = \int_0^T f_s d\omega_s,$$

where $f \in L^2([0,T])$ is a square integrable deterministic function. Then

$$(D_tF_T)_s(\omega) = f_s$$

for all $(s,\omega) \in [0,T]\times\Omega$.

Proof. Let the direction be

$$\gamma_t = \int_0^t h_s\,ds$$

for $t \in [0,T]$ and for some $h \in L^2([0,T])$. From the fact that

$$F_T(\omega+\epsilon\gamma) = \int_0^T f_s\,d(\omega_s+\epsilon\gamma_s),$$

it follows that

$$
\begin{aligned}
(D_\gamma F_T)_s(\omega) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}[F_T(\omega+\epsilon\gamma)-F_T(\omega)] \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_0^T f_s\,d(\omega_s+\epsilon\gamma_s)-\int_0^T f_s\,d\omega_s\right] \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_0^T f_s\,d\omega_s+\epsilon\int_0^T f_s\,d\gamma_s-\int_0^T f_s\,d\omega_s\right] \\
&= \int_0^T f_s\,d\gamma_s \\
&= \int_0^T f_s h_s\,ds.
\end{aligned}
$$

Hence if

$$F_T = \int_0^T f_s\,dW_s,$$

then $(D_t F_T)_s(\omega) = f_s$ for all $s \in [0,T]$ from the definition of the directional derivative, equation (2.3).

$\square$

We now define a norm on the space $\mathcal{D}_{1,2}([0,T]\times\Omega)$. For $F \in \mathcal{D}_{1,2}([0,T]\times\Omega)$, let

$$
\begin{aligned}
||F||_{1,2} &:= ||F||_{L^2(\Omega)} + ||D_t F||_{L^2([0,T]\times\Omega)} \\
&= \sqrt{\int_\Omega (F)^2\ d\mathbb{P}(\omega)} + \sqrt{\int_\Omega\int_0^T (D_t F)^2\ dt\,d\mathbb{P}(\omega)} \\
&= \sqrt{\mathbb{E}_{\mathbb{P}}(F^2)} + \sqrt{\mathbb{E}_{\mathbb{P}}\left(\int_0^T (D_t F)^2\,dt\right)}.
\end{aligned}
$$

However, it is not clear if the given space is closed under the given norm ([9],[21]). To overcome this problem, we now introduce the family $\mathbf{P}$ of *Wiener polynomials*. These are random variables $F : [0,T]\times\Omega \to \mathbb{R}$ of the form

$$F_t(\omega) = f\left(\int_0^T \eta_{1t}\,dW_t(\omega), \int_0^T \eta_{2t}\,dW_t(\omega), \ldots, \int_0^T \eta_{nt}\,dW_t(\omega)\right)$$

for all $\omega \in \Omega$ where f is a polynomial of degree $n \in \mathbb{N}$ and $\eta_1, \ldots, \eta_n \in L^2([0,T])$. The Wiener polynomials are in fact differentiable, that is, $\mathbf{P} \subset \mathcal{D}_{1,2}([0,T]\times\Omega)$ ([32], [9]).

The closure of $\mathbf{P}$ with respect to the norm $||\cdot||_{1,2}$ is the space $\mathbb{D}_{1,2}([0,T]\times\Omega)$ containing all the $F \in L^2([0,T]\times\Omega)$ for which there exists a sequence $(F^{(n)})_{n\in\mathbb{N}}$ in $\mathbf{P}$ such that

$$(F^{(n)})_{n\in\mathbb{N}} \to F \quad \text{in} \;\; L^2([0,T]\times\Omega)$$

and

$$(D_t F^{(n)})_{n\in\mathbb{N}} \quad \text{is convergent in} \quad L^2([0,T]\times\Omega).$$

We have just seen that $\mathbf{P} \subset \mathbb{D}_{1,2}([0,T]\times\Omega)$. Later on, we shall see that $\mathbb{D}_{1,2}([0,T]\times\Omega) \subset L^2([0,T]\times\Omega)$. For now, let us take it for granted that $\mathbf{P} \subset \mathbb{D}_{1,2}([0,T]\times\Omega) \subset L^2([0,T]\times\Omega)$.

Definition 2.1 (Malliavin Derivative)

The *Malliavin derivative* of a random variable $F \in \mathbb{D}_{1,2}([0,T]\times\Omega)$ is defined as the limit of $(D_t F^{(n)})_{n\in\mathbb{N}}$ in $L^2([0,T]\times\Omega)$, where $(F^{(n)})_{n\in\mathbb{N}} \in \mathbf{P}$ is a sequence that converges to F in $\mathbf{P} \subset L^2([0,T]\times\Omega)$.

■

Remark 2.1 *The proof of the fact that the limit is well defined and is unique is a highly technical result and is given in [20]*

We now explore some properties of both the derivative and the Malliavin derivative.

Theorem 2.3 (Product Rule)

If $F \in \mathcal{D}_{1,2}([0,T]\times\Omega)$ and $G \in \mathcal{D}_{1,2}([0,T]\times\Omega)$, then $FG \in \mathcal{D}_{1,2}([0,T]\times\Omega)$ and

$$D_t(FG) = (D_t F)G + F(D_t G).$$

In other words

$$(D_t(FG))_t(\omega) = (D_t F)_t(\omega)G_t(\omega) + F_t(\omega)(D_t G)_t(\omega)$$

for all $(t,\omega) \in [0,T]\times\Omega$.

Proof. For each $\omega \in \Omega$ and each γ_t of the form in equation (2.1) for $t \in [0,T]$, we have that

$$\left(D_\gamma(FG)\right)_t(\omega) = \lim_{\epsilon\to 0} \frac{F_t(\omega+\epsilon\gamma)G_t(\omega+\epsilon\gamma) - F_t(\omega)G_t(\omega)}{\epsilon}.$$

But we can rewrite this as

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[F_t(\omega + \epsilon\gamma) G_t(\omega + \epsilon\gamma) - F_t(\omega) G_t(\omega + \epsilon\gamma) + F_t(\omega) G_t(\omega + \epsilon\gamma) - F_t(\omega) G_t(\omega) \right]$$
$$= \lim_{\epsilon \to 0} \left(\frac{F_t(\omega + \epsilon\gamma) - F_t(\omega)}{\epsilon} G_t(\omega + \epsilon\gamma) + F_t(\omega) \frac{G_t(\omega + \epsilon\gamma) - G_t(\omega)}{\epsilon} \right)$$
$$= (D_\gamma F_t(\omega)) G_t(\omega) + F_t(\omega)(D_\gamma G_t(\omega)).$$

Hence

$$(D_\gamma(FG))_t(\omega) = (D_\gamma F_t(\omega)) G_t(\omega) + F_t(\omega)(D_\gamma G_t(\omega)).$$

But we can write

$$(D_\gamma F_t(\omega)) G_t(\omega) + F_t(\omega)(D_\gamma G_t(\omega))$$

as

$$\left(\int_0^T (D_t F)_t(\omega) h_t \, dt \right) \cdot G_t(\omega) + F_t(\omega) \cdot \left(\int_0^T (D_t G)_t(\omega) h_t \, dt \right).$$

Thus

$$\int_0^T D_t(FG)_t(\omega) h_t \, dt = \int_0^T [(D_t F)_t(\omega) G_t(\omega) + F_t(\omega)(D_t G)_t(\omega)] h_t \, dt.$$

Since the choice of $h \in L^2([0,T])$ was arbitrary, it follows that

$$(D_t FG)_t(\omega)) = (D_t F)_t(\omega) G_t(\omega) + F_t(\omega)(D_t G)_t(\omega)$$

for all $(t, \omega) \in [0,T] \times \Omega$. This means that

$$D_t(FG) = (D_t F)G + F(D_t G).$$

□

We now state a result which we will use later.

Theorem 2.4 (Novikov Condition)

Suppose that $\theta : [0,T] \times \Omega \to \mathbb{R}$ is measurable with respect to the σ-algebra generated by the Brownian motion at time t, $\mathcal{F}_t$, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the following stochastic process:

$$Z_t(\omega) = \exp\left(\int_0^t \theta_t(\omega) \, dW_s - \frac{1}{2} \int_0^t \theta_t^2(\omega) \, dt \right),$$

for $(t, \omega) \in [0,T] \times \Omega$. Then $Z_t(\omega)$ is a martingale if

$$\mathbb{E}_{\mathbb{P}} \left[\exp\left(\frac{1}{2} \int_0^T \theta_t(\omega) \, ds \right) \right] < \infty.$$

Proof See [23]

Proposition 2.1

Let $\mathcal{F}_s$ be the σ-algebra generated by the Brownian motion at time $0 < s < T$ and let $F \in \mathbb{D}_{1,2}([0,T] \times \mathbb{R})$ be measurable with respect to $\mathcal{F}_s$ on the probability space $(\Omega, \mathcal{F}_s, \mathbb{P})$.

Then for $0 < s < t \leq T$, $D_t F$ will be $\mathcal{F}_s$ measurable and the Malliavin derivative evaluates to

$$(D_t F)_s(\omega) = 0$$

for all $\omega \in \Omega$.

Proof. The result is only proved for a special case. A thorough (but highly technical) proof is given in [32]. Consider a random variable of the form

$$F_t = \exp\left(\int_0^t h_u \, dW_u - \frac{1}{2}\int_0^t h_u^2 \, du\right), \tag{2.4}$$

where $h \in L^2([0,T])$ is deterministic. Note that the Novikov condition, theorem 2.4, is satisfied so F_t is an exponential martingale ([24]). Also,

$$D_t F_T = F_T h_t \tag{2.5}$$

for $t \in [0,T]$, by the chain rule and theorem (2.2).

This because the derivative of e^x with respect to x is e^x and the derivative

$$D_t\left(\int_0^T h_u \, dW_u - \frac{1}{2}\int_0^T h_u^2 \, du\right) = h_t + 0,$$

by using theorem 2.2 and the fact the second integral does not depend on the underlying Brownian motion and so its derivative is zero.

For $s \in [0,T]$, we have that

$$\mathbb{E}_{\mathbb{P}}(F_T|\mathcal{F}_s) = F_s = \exp\left(\int_0^s h_u \, dW_u - \frac{1}{2}\int_0^s h_u^2 \, du\right),$$

as F_t is a martingale. It then follows that

$$\begin{aligned} D_t\mathbb{E}_{\mathbb{P}}(F_T|\mathcal{F}_s) &= D_t F_s \\ &= D_t \exp\left(\int_0^s h_u \, dW_u - \frac{1}{2}\int_0^s h_u^2 \, du\right) \\ &= D_t \exp\left(\int_0^T h_u \mathbf{1}_{[0,s]} \, dW_u - \frac{1}{2}\int_0^s h_u^2 \, du\right) \end{aligned}$$

where $\mathbf{1}_{[0,s]}$ is the indicator function on the interval $[0, s]$. Using the result in equation (2.5), we see that

$$\begin{aligned} D_t \mathbb{E}_{\mathbb{P}}(F_T | \mathcal{F}_s) &= \exp\left(\int_0^T \mathbf{1}_{[0,s]} h_u \, dW_u - \frac{1}{2} \int_0^s h_u^2 \, du \right) \mathbf{1}_{[0,s]} \\ &= \exp\left(\int_0^s h_u \, dW_u - \frac{1}{2} \int_0^s h_u^2 \, du \right) \mathbf{1}_{[0,s]} \\ &= F_s \mathbf{1}_{[0,s]}. \end{aligned}$$

But,

$$F_t = \exp\left(\int_0^t h_u \, dW_u - \frac{1}{2} \int_0^t h_u^2 \, du \right)$$

is an exponential martingale and so using the martingale property ([31]),

$$F_s = \mathbb{E}_{\mathbb{P}}(F_T | \mathcal{F}_s).$$

Hence we see that

$$\begin{aligned} D_t \mathbb{E}_{\mathbb{P}}(F_T | \mathcal{F}_s) &= \mathbb{E}_{\mathbb{P}}(F_T | \mathcal{F}_s) h_t \mathbf{1}_{[0,s]} \\ &= \mathbb{E}_{\mathbb{P}}(F_T h_t | \mathcal{F}_s) \mathbf{1}_{[0,s]} \end{aligned}$$

and by equation (2.5) we have that

$$\mathbb{E}_{\mathbb{P}}(F_T | \mathcal{F}_s) h_t \mathbf{1}_{[0,s]} = \mathbb{E}_{\mathbb{P}}(F_T h_t | \mathcal{F}_s) \mathbf{1}_{[0,s]} = \mathbb{E}_{\mathbb{P}}(D_t F_T | \mathcal{F}_s) \mathbf{1}_{[0,s]}. \tag{2.6}$$

Thus

$$D_t \mathbb{E}_{\mathbb{P}}(F_T | \mathcal{F}_s) = \mathbb{E}_{\mathbb{P}}(D_t F_T | \mathcal{F}_s) \mathbf{1}_{[0,s]}.$$

The above computation extends to random variables in the linear span of random variables of the form in equation (2.4). Since the linear span is dense in $L^2([0, T] \times \Omega)$ ([21]), the result also holds for more general random variables. Of course, the result does not hold for all $F_t \in L^2(\Omega)$, since it involves the Malliavin derivative of F_t, which does not exist for all $F_t \in L^2(\Omega)$. (This is seen in theorem 2.7)

In particular, if $F_t \in \mathbb{D}_{1,2}(\Omega)$ is $\mathcal{F}_s$-measurable for $s \in [0, T]$, then

$$D_t F_s = D_t \mathbb{E}_{\mathbb{P}}(F_T | \mathcal{F}_s) = \mathbb{E}_{\mathbb{P}}(D_t F_T | \mathcal{F}_s) \mathbf{1}_{[0,s]}.$$

Hence $D_t F_s$ is $\mathcal{F}_s$-measurable and

$$(D_t F_s)_t(\omega) = 0 \quad \text{if } t > s.$$

□

Theorem 2.5 (Clark-Ocone Formula)

Let $F_T \in \mathbb{D}_{1,2}(\Omega)$ for $\omega \in \Omega$ be $\mathcal{F}_T$-measurable on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F}_t$ is the σ-algebra generated by the Brownian motion at time t, then

$$F_T = \mathbb{E}_{\mathbb{P}}(F_0) + \int_0^T \mathbb{E}_{\mathbb{P}}(D_t F_T | \mathcal{F}_t)\, dW_t.$$

Proof. As in the proof of proposition (2.1) only exponential martingales as in equation (2.4) are used. A more detailed proof can be found in, [20] and [32]. Define a stochastic process F_t by

$$F_t := \exp\left(\int_0^t h_u\, dW_u - \frac{1}{2}\int_0^t h_u^2\, du\right)$$

where $h \in L^2([0,T])$. As explained in the proof of proposition (2.1), on page 19, $F_t = e^{Z_t}$ is an exponential martingale.

Define also the auxiliary process

$$Z_t = \int_0^t h_u\, dW_u - \frac{1}{2}\int_0^t h_u^2\, du.$$

The dynamics of $F_t = e^{Z_t}$ are given by the Itô formula as

$$\begin{aligned} dF_t &= \frac{\partial}{\partial Z} F_t dZ_t + \frac{1}{2}\frac{\partial^2}{\partial Z^2} F_t (dZ_t)^2 \\ &= F_t dZ_t + \frac{1}{2} F_t (dZ_t)^2 \\ &= F_t\left(h_t dW_t - \frac{1}{2}h_t^2\, dt + \frac{1}{2}h_t^2\right) \\ &= F_t h_t\, dW_t. \end{aligned}$$

That is,

$$dF_t = F_t h_t\, dW_t. \tag{2.7}$$

Moreover, using the result in equation (2.5), we have that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(D_t F_T | \mathcal{F}_t) &= \mathbb{E}_{\mathbb{P}}(F_T h_t | \mathcal{F}_t) \\ &= \mathbb{E}_{\mathbb{P}}(F_T | \mathcal{F}_t) h_t \\ &= F_t h_t, \end{aligned}$$

so writing the dynamics of F_t, in equation (2.7), in integral form leads to,

$$\begin{aligned} F_T &= \mathbb{E}_{\mathbb{P}}(F_0) + \int_0^T F_t h_t\, dW_t \\ &= \mathbb{E}_{\mathbb{P}}(F_0) + \int_0^T \mathbb{E}_{\mathbb{P}}(D_t F_T | \mathcal{F}_t)\, dW_t \end{aligned}$$

as $F_0(\omega) = 1 = \mathbb{E}_{\mathbb{P}}(F_0)$. The result is now proved in a special case. Again the result extends to the linear span of all random variables of the exponential form and can be extended to all $F_t \in \mathbb{D}_{1,2}(\Omega)$ ([20]).

$\square$

Remark 2.2 (The Clark-Ocone Formula)

The Clark-Ocone formula is the cornerstone of the martingale approach to determining optimal strategy in an incomplete market ([7],[30],[21]), where the integrand represents the optimal investment strategy. It is also related to the Brownian Martingale Represenation Theorem which states that if $F_T \in L^2(Omega)$ be $\mathcal{F}_T$-measurable on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F}_t$ is the σ-algebra generated by the Brownian motion at time t, then there exists a predictable process ϕ_t, for $0 \leq t \leq T$ such that

$$F_T = \mathbb{E}_{\mathbb{P}}(F_0) + \int_0^T \phi_t \, dW_t.$$

Since the Martingale representation theorem is an existence result and does not give us the explicit form of ϕ, the Clark-Ocone Formula gives us the exact form of ϕ_t whenever $F_T \in \mathbb{D}_{1,2}(\Omega)$ for $\omega \in \Omega$.

In fact, the above result can be generalised to the so called Generalised Clark-Ocone formula with is used to hedge claims, that is, to find a replicating strategy to price options and other derivatives ([20], [32]).

$\blacksquare$

Example 2.1

If W_t for $t \in [0, T]$ is a Brownian motion, then its derivative is

$$D_t W_t = \mathbf{1}_{[0,t]}.$$

To see this, let $F_t(\omega) = W_t(\omega) = \omega_t$ for $\omega \in \Omega$. Then

$$F_t(\omega + \epsilon\gamma) = \omega_t + \epsilon\gamma_t.$$

Using γ_t as in equation (2.1), on page 13, we see that

$$\begin{aligned}
\left(D_\gamma W_t\right)_t(\omega) &= \lim_{\epsilon \to 0} \frac{F_t(\omega + \epsilon\gamma) - F_t(\omega)}{\epsilon} \\
&= \frac{\omega_t + \epsilon\gamma_t - \omega_t}{\epsilon} \\
&= \gamma_t \\
&= \int_0^t h_s \, ds \\
&= \int_0^T \mathbf{1}_{[0,t]} h_s \, ds.
\end{aligned}$$

It can then be concluded from the definition of the directional derivative, equation (2.3) on page 14, that

$$D_t W_t = \mathbf{1}_{[0,t]}. \tag{2.8}$$

From this we can see that

$$D_t W_T = \mathbf{1}_{[0,T]} = 1. \tag{2.9}$$

■

Theorem 2.6

The closure of the Wiener Polynomials is not equal to $L^2([0,T] \times \Omega)$, that is $\mathbb{D}_{1,2}([0,T] \times \Omega) \neq L^2([0,T] \times \Omega)$.

Proof. Let $\mathcal{F}$ be the σ-algebra generated by the random variable $F \in \mathbb{D}_{1,2}([0,T] \times \Omega)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We will prove this result by finding an element in $L^2([0,T] \times \Omega)$ that is not in $\mathbb{D}_{1,2}([0,T] \times \Omega)$. Let $\mathbf{1}_{\mathcal{F}}$ be the indicator function on $\mathcal{F}$. We can show that $\mathbf{1}_{\mathcal{F}} \in \mathbb{D}_{1,2}([0,T] \times \Omega)$ if and only if $\mathbb{P}(t,\omega) = 0$ or $\mathbb{P}(t,\omega) = 1$, i.e. $\mathbb{P}(t,\omega) \in \{0,1\}$ for all $(t,\omega) \in [0,T] \times \Omega$.

First, suppose that $\mathbb{P}(t,\omega) \in \{0,1\}$ for $(t,\omega) \in [0,T] \times \Omega$. Then $\mathbf{1}_{\mathcal{F}}$ is constant almost surely, and hence the Malliavin derivative is zero, i.e.
$\mathbf{1}_{\mathcal{F}} \in \mathbb{D}_{1,2}([0,T] \times \Omega)$ and $(D_t \mathbf{1}_{\mathcal{F}})_t(\omega) = 0$ for $(t,\omega) \in [0,T] \times \Omega$.

Now $\mathbf{1}_{\mathcal{F}} \in \mathbb{D}_{1,2}([0,T] \times \Omega)$ only if $\mathbb{P}(t,\omega) \in \{0,1\}$ for $(t,\omega) \in [0,T] \times \Omega$.

To see this, we begin by noticing that

$$\mathbf{1}_{\mathcal{F}} = (\mathbf{1}_{\mathcal{F}})^2$$

in $[0,T] \times \Omega$. Since the mapping $x \mapsto x^2$ is differentiable, the chain rule yields

$$D_t \mathbf{1}_{\mathcal{F}} = D_t (\mathbf{1}_{\mathcal{F}})^2 = 2\mathbf{1}_{\mathcal{F}} D_t \mathbf{1}_{\mathcal{F}}.$$

For $(t,\omega) \in \mathcal{F}^c$, the indicator function becomes $\mathbf{1}_{\mathcal{F}}(t,\omega) = 0$; therefore,

$$(D_t \mathbf{1}_{\mathcal{F}})_t(\omega) = 2\mathbf{1}_{\mathcal{F}} \cdot (D_t \mathbf{1}_{\mathcal{F}})_t(\omega) = 2 \cdot 0 \cdot (D_t \mathbf{1}_{\mathcal{F}})_t(\omega) = 0,$$

and for $(t,\omega) \in \mathcal{F}$ the Malliavin derivative is

$$D_t \mathbf{1}_{\mathcal{F}} = 2(D_t \mathbf{1}_{\mathcal{F}}).$$

This can only be satisfied if $(D_t \mathbf{1}_{\mathcal{F}})_t(\omega) = 0$ for $(t,\omega) \in [0,T] \times \Omega$. Hence, $(D_t \mathbf{1}_{\mathcal{F}})_t(\omega) = 0$ for all $(t,\omega) \in [0,T] \times \Omega$. Since by assumption

$\mathbf{1}_{\mathcal{F}} \in \mathbb{D}_{1,2}([0,T] \times \Omega)$, the Clark-Ocone formula, theorem 2.5, gives

$$\begin{aligned}
\mathbf{1}_{\mathcal{F}}(t,\omega) &= \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{\mathcal{F}}) + \int_0^T (D_t \mathbf{1}_{\mathcal{F}}|\mathcal{F})(\omega)\, dW_t \\
&= \mathbb{P}((t,\omega)) + \int_0^T \mathbb{E}(0|\mathcal{F})\, dW_t \\
&= \mathbb{P}((t,\omega)).
\end{aligned}$$

which can only be true if $\mathbb{P}(t,\omega)) \in \{0,1\}$ for $(t,\omega) \in [0,T] \times \Omega$ since the indicator function only assume the values 0 and 1. Thus if $\mathbb{P}((t,\omega)) \neq \{0,1\}$, $\mathbf{1}_{\mathcal{F}} \in L^2([0,T] \times \Omega)$ but $\mathbf{1}_{\mathcal{F}} \notin \mathbb{D}_{1,2}([0,T] \times \Omega)$.

We have found an element of $L^2([0,T]\times\Omega)$ that is not in $\mathbb{D}_{1,2}([0,T]\times\Omega)$. The result is established.

□

Theorem 2.7

For all $(t,\omega) \in [0,T] \times \Omega$ we have that $\mathbb{D}_{1,2}([0,T] \times \Omega) \subset L^2([0,T] \times \Omega)$.

Proof. See ([20].

□

2.3 Integration by Parts on Wiener Space

We now derive the integration by parts formula for the **Malliavin derivative** . It strongly resembles the integration by parts formula in Riemann calculus. We first consider the following theorem.

Theorem 2.8

Let $F \in \mathcal{D}_{1,2}([0,T]\times,\Omega)$ for $(t,\omega) \in [0,T] \times \Omega$ on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F}_t$ is the σ-algebra generated by the Brownian motion at time t, and define $\gamma_t = \displaystyle\int_0^t h_s\, ds$ for $h \in L^2([0,T])$. Then

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^T (D_t F) h_t\, dt\right) = \mathbb{E}_{\mathbb{P}}\left(F \int_0^T h_t\, dW_t\right).$$

Proof. From the definition of the directional derivative, we have that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left(D_\gamma F\right) &= \mathbb{E}_{\mathbb{P}}\left(\int_0^T (D_t F) h_t \, dt\right) \\
&= \int_\Omega \left(D_\gamma F\right) \, d\mathbb{P}(\omega) \\
&= \int_\Omega \lim_{\epsilon\to 0} \frac{1}{\epsilon}[F(\omega+\epsilon\gamma) - F(\omega)] \, d\mathbb{P}(\omega) \\
&= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(\int_\Omega F(\omega+\epsilon\gamma \, d\mathbb{P}(\omega) - \int_\Omega F(\omega) \, d\mathbb{P}(\omega)\right), \qquad (2.10)
\end{aligned}$$

where the integrand satisfies the necessary regularity conditions that allow the limit to move out of the integration. Now $h \in L^2([0,T])$, so the process ϵh satisfies the Novikov condition (theorem 2.4) which ensures that

$$M_t := \exp\left(\epsilon \int_0^t h_s \, dW_s - \frac{1}{2}\epsilon^2 \int_0^t h_s^2 \, ds\right)$$

is a $\mathbb{P}$-martingale. By the Girsanov theorem ([24]), the process

$$\widetilde{W_t} = W_t + \epsilon \int_0^t h_s \, ds$$

for $(t,\omega) \in [0,T]\times\Omega$ is a Brownian motion under the measure $\mathbb{Q}$ defined by

$$\begin{aligned}
\frac{d\mathbb{Q}(\omega)}{d\mathbb{P}(\omega)} &= M_T \\
&= \exp\left(-\epsilon \int_0^T h_s \, dWs - \frac{1}{2}\epsilon^2 \int_0^T h_s^2 \, ds\right) \\
&= \exp\left(-\epsilon \int_0^T h_s \, d\widetilde{W}s + \frac{1}{2}\epsilon^2 \int_0^T h_s^2 \, ds\right).
\end{aligned}$$

for all $(t,\omega) \in ([0,T]\times\Omega)$.

Let $\widetilde{\omega} = \omega + \epsilon\gamma$, then the first integral in equation (2.10) can now be written as

$$\begin{aligned}
\int_\Omega F(\omega+\epsilon\gamma) \, d\mathbb{P}(\omega) &= \int_\Omega F(\widetilde{\omega})\exp\left(\epsilon \int_0^T h_s \, d\widetilde{W}_s(\omega) - \frac{1}{2}\epsilon^2 \int_0^T h_s^2 \, ds\right) d\mathbb{Q}(\omega). \\
&= \int_\Omega F(\omega)\exp\left(\epsilon \int_0^T h_s \, dW_s(\omega) - \frac{1}{2}\epsilon^2 \int_0^T h_s^2 \, ds\right) d\mathbb{P}(\omega).
\end{aligned}$$

Since

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^T (D_t F) h_t \, dt\right) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\int_\Omega F(\omega+\epsilon\gamma) \, d\mathbb{P}(\omega) - \int_\Omega F(\omega) \, d\mathbb{P}(\omega)\right]$$

by equation (2.10) and since the right hand side of the equality above this can be rewritten as

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_\Omega F(\omega+\epsilon\gamma)\,d\mathbb{P}(\omega)-\int_\Omega F(\omega)\,d\mathbb{P}(\omega)\right]\\
&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_\Omega F(\omega)\left[\exp\left(\epsilon\int_0^T h_s\,dW_s-\frac{1}{2}\epsilon^2\int_0^T h_s^2\,ds\right)-1\right]\,d\mathbb{P}(\omega)\\
&=\int_\Omega \lim_{\epsilon\to 0}\frac{1}{\epsilon}F(\omega)\left[\exp\left(\epsilon\int_0^T h_s\,dW_s-\frac{1}{2}\epsilon^2\int_0^T h_s^2\,ds\right)-1\right]\,d\mathbb{P}(\omega)\\
&=\int_\Omega F(\omega)\frac{d}{d\epsilon}\left[\exp\left(\epsilon\int_0^T h_s\,dW_s-\frac{1}{2}\epsilon^2\int_0^T h_s^2\,ds\right)\right]_{\epsilon=0}\,d\mathbb{P}(\omega)\\
&=\int_\Omega \left[F(\omega)\int_0^T h_s\,dW_s\right]\,d\mathbb{P}(\omega)\\
&=\mathbb{E}_\mathbb{P}\left(F(\omega)\int_0^T h_s\,dW_s\right)=\mathbb{E}_\mathbb{P}\left(F(\omega)\int_0^T h_t\,dW_t\right),
\end{aligned}
$$

the result follows.

□

Theorem 2.9 (Integration by Parts for Malliavin Derivative)

Let $F\in\mathcal{D}_{1,2}([0,T]\times\Omega)$ and $G\in\mathcal{D}_{1,2}([0,T]\times\Omega)$ for $(t,\omega)\in[0,T]\times\Omega$ on the probability space $(\Omega,\mathcal{F}_t,\mathbb{P})$, where $\mathcal{F}_t$ is the σ-algebra generated by the Brownian motion at time t, and define

$$\gamma_t=\int_0^t h_s\,ds$$

for $h\in L^2([0,T])$.

Then

$$\mathbb{E}_\mathbb{P}\left(G\int_0^T (D_tF)h_t\,dt\right)=\mathbb{E}_\mathbb{P}\left(FG\int_0^T h_t\,dW_t\right)-\mathbb{E}_\mathbb{P}\left(F\int_0^T (D_tG)h_t\,dt\right).$$

Proof. By the product rule, $FG\in\mathcal{D}_{1,2}([0,T]\times\Omega)$ and

$$D_t(FG)=F(D_tG)+G(D_tF).$$

Hence

$$\mathbb{E}_\mathbb{P}\left(\int_0^T D_t(FG)h_t\,dt\right)\;=\;\mathbb{E}_\mathbb{P}\left(\int_0^T (FD_tG+GD_tF)\,h_t\,dt\right),$$

which can be written as

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^T F(D_tG)h_t\,dt\right)+\mathbb{E}_{\mathbb{P}}\left(\int_0^T G(D_tF)h_t\,dt\right). \tag{2.11}$$

Using theorem 2.8, we see that

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^T (D_tFG)h_t\,dt\right) = \mathbb{E}_{\mathbb{P}}\left(FG\int_0^T h_t\,dW_t\right).$$

Thus, by using equation (2.11) we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left(FG\int_0^T h_t\,dW_t\right) &= \mathbb{E}_{\mathbb{P}}\left(\int_0^T (D_tFG)h_t\,dt\right)\\ &= \mathbb{E}_{\mathbb{P}}\left(\int_0^T F(D_tG)h_t\,dt\right)+\mathbb{E}_{\mathbb{P}}\left(\int_0^T G(D_tF)h_t\,dt\right).\end{aligned}$$

We can rearrange this to

$$\mathbb{E}_{\mathbb{P}}\left(G\int_0^T (D_tF)h_t\,dt\right)=\mathbb{E}_{\mathbb{P}}\left(FG\int_0^T h_t\,dW_t\right)-\mathbb{E}_{\mathbb{P}}\left(F\int_0^T (D_tG)h_t\,dt\right).$$

□

2.4 The Skorohod Integral

Definition 2.2 (Skorohod Integral)

Let $H \in L^2([0,T]\times\Omega)$ for $(t,\omega)\in[0,T]\times\Omega$ and let $F\in\mathbb{D}_{1,2}([0,T]\times\Omega)$ for $(t,\omega)\in[0,T]\times\Omega$ on the probability space $(\Omega,\mathcal{F}_t,\mathbb{P})$, where $\mathcal{F}_t$ is the σ-algebra generated by the Brownian motion at time t. Then the Skorohod integral of H, written as

$$\delta(H)=\int_0^T H_t\,\delta W_t,$$

is defined to be *the adjoint operator of the Malliavin derivative* of F ([20], [9], [18]). In other words,

$$\mathbb{E}_{\mathbb{P}}\left(F\delta(H)\right)=\mathbb{E}_{\mathbb{P}}\left(\int_0^T (D_tF)H_t\,dt\right)=\langle D_tF,H\rangle_{[0,T]\times\Omega}. \tag{2.12}$$

More precisely, the Skorohod integral of H is defined as the element $\delta(H)\in L^2(\Omega)$ that satisfies equation (2.12). Equation (2.12) is called the *dual relationship*.

$\blacksquare$

Remark 2.3 (Existence of the Skorohod Integral) *The existence conditions of the Skorohod Integral depend on the some highly technical results that involve Itô-Wiener chaos expansions. We will not discuss this, but the interested reader is advised to consult [20].*

Theorem 2.10

Let $(t, \omega) \in [0, T] \times \Omega$. Also, let $H \in L^2([0, T] \times \Omega)$ be a process on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F}_t$ is the σ-algebra generated by the Brownian motion and which is adapted to the natural filtration generated by the Brownian motion at time t.

Then the Skorohod integral coincides with the Itô integral whenever the latter is defined, that is,

$$\delta(H) = \int_0^T H_t \, \delta W_t = \int_0^T H_t \, dW_t$$

for all $(t, \omega) \in [0, T] \times \Omega$.

Proof. The result is only proved for deterministic functions $H \in L^2([0, T])$. More detailed proofs that apply to general stochastic functions $H \in L^2([0, T] \times \Omega)$ can be found in [20] and [32].

Let $F \in \mathbb{D}_{1,2}([0, T] \times \Omega)$ and $G \in \mathbb{D}_{1,2}([0, T] \times \Omega)$ be two Malliavin differentiable random variables. Then by using the definition of the Skorohod integral, we get that

$$\mathbb{E}_{\mathbb{P}} \left(G\delta(FH) \right) = \mathbb{E}_{\mathbb{P}} \left(\int_0^T (D_t G) F H_t \, dt \right). \tag{2.13}$$

Finally, applying the integration by parts theorem for the Malliavin derivative, theorem 2.9 on page 26, to the right hand side of equation (2.13), it follows that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left(\int_0^T (D_t G) F H_t \, dt \right) &= \mathbb{E}_{\mathbb{P}} \left(GF \int_0^T H_t \, dW_t \right) - \mathbb{E}_{\mathbb{P}} \left(G \int_0^T (D_t F) H_t \, dt \right) \\ &= \mathbb{E}_{\mathbb{P}} \left(G \left(F \int_0^T H_t \, dW_t - \int_0^T (D_t F) H_t \, dt \right) \right). \end{aligned}$$

Thus

$$\mathbb{E}_{\mathbb{P}} \left(G\delta(FH) \right) = \mathbb{E}_{\mathbb{P}} \left(G \left(F \int_0^T H_t \, dW_t - \int_0^T (D_t F) H_t \, dt \right) \right).$$

Since this must hold for all $G \in \mathbb{D}_{1,2}([0,T] \times \Omega)$, and since G, F and H were arbitrary, it follows that

$$\delta(FH) = F \int_0^T H_t \, dW_t - \int_0^T (D_t F) \, H_t \, dt. \tag{2.14}$$

We now look at happens in equation (2.14), for different choices of H and F. With $F = \mathbf{1}_{[0,T]} = 1$, the Itô integral and the Skorohod integral coincide for all deterministic functions in $L^2([0,T])$. This is because

$$D_t F = D_t \mathbf{1}_{[0,T]} = D_t 1 = 0$$

and so the second integral on the right of the equality in equation (2.14) vanishes.

If F is $\mathcal{F}_s$-measurable for $0 \le s \le T$ and $H = \mathbf{1}_{(s,u]}$ for $0 \le s < u \le T$, then equation (2.14) becomes

$$\begin{aligned} \delta\left(F \cdot \mathbf{1}_{(s,u]}\right) &= F \int_0^T \mathbf{1}_{(s,u]} \, dW_t - \int_0^T (D_t F) \mathbf{1}_{(s,u]} \, dW_t && (2.15) \\ &= F \int_s^u dW_t - \int_u^s (D_t F) \, dt && (2.16) \\ &= F(W_u - W_s) - 0 && (2.17) \end{aligned}$$

where the fact that $(D_t F)_s(\omega) = 0$ for $t > s$ is used since F is $\mathcal{F}_s$-measurable, as by theorem 2.1 on page 19.

An elementary process is a process which is defined on subintervals of the time interval $[0,T]$. Now consider the following argument. Let $H = \mathbf{1}_{(s,u]}$ for $0 \le s < u \le T$, thus

$$FH = 0 \cdot \mathbf{1}_{[0,s]} + F \mathbf{1}_{(s,u]} + 0 \cdot \mathbf{1}_{(u,T]},$$

and it can be seen that FH_t is an elementary process and hence

$$\begin{aligned} \int_0^T H_t \, dW_t &= 0 \cdot (W_s - W_0) + F(W_u - W_s) + 0 \cdot (W_T - W_u) && (2.18) \\ &= F(W_u - W_s). && (2.19) \end{aligned}$$

Notice that F is $\mathcal{F}_s$-measurable and so it is "known at" for times greater than s and so it is treated as a constant in the Itô integral defined on the time interval $(s,u]$.

Therefore, the Skorohod integral, equation (2.17), and the Itô integral, equation (2.19), coincide, that is,

$$\delta(FH) = \int_0^T FH_t \, dW_t.$$

The fact that the result holds for any $\mathcal{F}_t$-adapted process $H \in L^2([0,T])$, follows from the fact that any $\mathcal{F}_t$-adapted process defined on a compact set can be approximated by a sequence of elementary processes ([24]).

□

Example 2.2

Let $H = W_t$ for $t \in [0,T]$. The Skorohod integral is

$$\int_0^T W_t \,\delta W_t = \int_0^T W_t \,dW_t = \frac{1}{2}W_T^2 - \frac{1}{2}T$$

since W_t is $\mathcal{F}_t$ adapted and the Skorohod integral coincides with the Itô integral by theorem 2.10.

■

Theorem 2.11

Let $F \in \mathbb{D}_{1,2}([0,T]\times\Omega)$ and $H \in L^2([0,T]\times\Omega)$ on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F}_t$ is the σ-algebra generated by the Brownian motion at time t. Then

$$\int_0^T F H_t \,\delta W_t = F\int_0^T H_t \,\delta W_t - \int_0^T (D_t F) H_t \,dt$$

for all $(t,\omega) \in [0,T]\times\Omega$.

Proof Let $G \in \mathbb{D}_{1,2}([0,T]\times\Omega)$. Using the product rule, we have that

$$D_t(FG) = F(D_tG) + G(D_tF).$$

and so

$$(D_tG)F = D_t(GF) - G(D_tF).$$

Hence

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^T (D_tG)FH_t\,dt\right) = \mathbb{E}_{\mathbb{P}}\left(\int_0^T \left[D_t(GF) - G(D_tF)\right] H_t\,dt\right).$$

By the integration by parts formula, we get that

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^T \left[D_t(GF) - G(D_tF)\right] H_t\,dt\right) = \mathbb{E}_{\mathbb{P}}\left(GF\int_0^T H_t\,\delta W_t - G\int_0^T (D_tF)H_t\,dt\right),$$

which can be rewritten as

$$\mathbb{E}_{\mathbb{P}}\left(G\left[F\int_0^T H_t\,\delta W_t - \int_0^T (D_tF)H_t\,dt\right]\right).$$

But

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left(\int_0^T (D_tG)FH_t\,dt\right) &= \mathbb{E}_{\mathbb{P}}\left(G\delta(FH)\right)\\ &= \mathbb{E}_{\mathbb{P}}\left(G\int_0^T FH_t\,\delta W_t\right),\end{aligned}$$

by the definition of the Skorohod integral. We then see that

$$\mathbb{E}_{\mathbb{P}}\left(G\int_0^T FH_t\,\delta W_t\right) = \mathbb{E}_{\mathbb{P}}\left(G\left[F\int_0^T H_t\,\delta W_t - \int_0^T (D_tF)H_t\,dt\right]\right),$$

and since this should be true for any choice of G, the result follows.

$\square$

Example 2.3

If $H = \mathbf{1}_{[0,T]} = 1$ and $F = W_T$, then by theorem 2.11,

$$\begin{aligned}\int_0^T W_T\,\delta W_t &= W_T\int_0^T 1\,\delta W_t - \int_0^T (D_tW_T)\,dt\\ &= W_T\int_0^T \mathbf{1}_{[0,T]}\,\delta W_t - \int_0^T (D_tW_T)\,dt\end{aligned}$$

Let $\mathbf{1}_{[0,T]}$ be the indicator function on the interval $[0,T]$. Since $\mathbf{1}_{[0,T]} = 1$ is $\mathcal{F}_t$ measurable for $\mathcal{F}_t$ the σ-algebra generated by the Brownian motion for all $t \in [0,T]$, we see that $\mathbf{1}_{[0,T]} = 1$ is adapted to the filtration generated by $\mathcal{F}_t$ for $0 \le t \le T$, and so from theorem 2.10 we have that

$$\delta(1) = \int_0^T 1\,\delta W_t = \int_0^T 1\,dW_t = W_T.$$

Using this and the result from equation (2.9) on page 23, we see that

$$\begin{aligned}W_T\int_0^T \mathbf{1}_{[0,T]}\,\delta W_t - \int_0^T (D_tW_T)\,dt &= W_T(W_T) - \int_0^T \mathbf{1}_{[0,T]}\,dt\\ &= W_T^2 - T.\end{aligned}$$

Thence

$$\int_0^T W_T\,\delta W_t = W_T^2 - T.$$

$\blacksquare$

We now derive the Integration by Parts Formula for the Skorohod Integral

Theorem 2.12 (Integration by Parts Formula for the Skorohod Integral)

Let F and G be two random variables such that $F \in \mathbb{D}_{1,2}([0,T] \times \Omega)$ and $G \in \mathbb{D}_{1,2}([0,T] \times \Omega)$ for $(t,\omega) \in [0,T] \times \Omega$ on the probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F}_t$ is the σ-algebra generated by the Brownian motion at time t. Let $\gamma_t = \int_0^t h_s\,ds$ be such that the directional derivative of F is non zero, for some $h \in L^2[0,T]$, that is,

$$(D_\gamma F)_t\,(\omega) = \int_0^T (D_t F)_t(\omega) h_t\,dt \neq 0 \quad \text{a.s.}$$

for $(t,\omega) \in [0,T] \times \Omega$. Assume also that the

$$\delta\left(G h_t (D_\gamma F)^{-1}\right) = \delta\left(\frac{G h_t}{D_\gamma F}\right)$$

exists for $(t,\omega) \in [0,T] \times \Omega$.

Then for any piecewise continuously differentiable function f with bounded derivative,

$$\mathbb{E}_{\mathbb{P}}(f^{'}(F)G) = \mathbb{E}_{\mathbb{P}}\left(f(F)\mathcal{K}(F,G)\right),$$

where

$$\mathcal{K}(F,G) = \delta\left(G h_t (D_\gamma F)^{-1}\right).$$

Proof. As in the proof of the chain rule, given in theorem 2.1 on page 14, the directional derivative of $f(F)$ is

$$D_\gamma(f(F)) = f^{'}(F) D_\gamma F.$$

Hence

$$f^{'}(F) = \frac{D_\gamma(f(F))}{D_\gamma F} = D_\gamma(f(F))\,(D_\gamma F)^{-1}. \tag{2.20}$$

Using equation (2.20), we see that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(f^{'}(F)G) &= \mathbb{E}_{\mathbb{P}}\left(D_\gamma(f(F)) \cdot \left((D_\gamma F)^{-1} G\right)\right) \\
&= \mathbb{E}_{\mathbb{P}}\left(\int_0^T D_t(f(F)) h_t\,dt \cdot (D_\gamma F)^{-1} G\right) \\
&= \mathbb{E}_{\mathbb{P}}\left(\int_0^T D_t(f(F))\left((D_\gamma F)^{-1} G\right) h_t\,dt\right).
\end{aligned}$$

But by using the dual relationship, given in equation (2.12) on page 27, we see that

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left(\int_0^T D_t(f(F))\left((D_\gamma F)^{-1}G\right) h_t\, dt\right) \\
&= \mathbb{E}_{\mathbb{P}}\left(f(F)\,\delta\left(G h_t (D_\gamma F)^{-1}\right)\right). \\
&= \mathbb{E}_{\mathbb{P}}\left(f(F)\mathcal{K}(F,G)\right)
\end{aligned}
$$

where

$$\mathcal{K}(F,G) = \delta\left(G h_t (D_\gamma F)^{-1}\right)$$

for all $(t,\omega) \in [0,T] \times \Omega$ and the result follows.

$\square$

Chapter 3

The Computation of the Greeks

In the first chapter, we gave the greeks for a European call option. In this chapter, we will derive formulae for the delta, the gamma and the vega for a general European-type option, using Malliavin calculus. The known formulae for the greeks given in chapter 1, derived using the Black-Scholes framework, will then be compared with the formulae derived using Malliavin calculus.

Let $S_T(\omega)$ for $\omega \in \Omega$ be the value of the risky asset at time T. Let $f(S_T)$ be a function that represents the payoff of an option for some function piecewise differentiable function with bounded derivative f. If $\mathbb{Q}$ is the risk-neutral measure, as by definition (1.5), and r the riskless interest rate, then the initial value of the option is given by

$$V_0 = \mathbb{E}_{\mathbb{Q}}(e^{-rT} f(S_T)),$$

([7],[10]). If λ is a parameter of the initial value of the option V_0 , such as the stock price or the risk-free interest rate, then a greek or sensitivity is defined as $\dfrac{\partial V_0}{\partial \lambda}$. Thus

$$\frac{\partial V_0}{\partial \lambda} = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(f^{\prime}(S_T)\frac{dS_T}{d\lambda}\right). \tag{3.1}$$

Using theorem 2.12, on page 32, it follows that

$$\frac{\partial V_0}{\partial \lambda} = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(f(S_T)\mathcal{K}\left(S_T, \frac{dS_T}{d\lambda}\right)\right), \tag{3.2}$$

where

$$\mathcal{K}\left(S_T, \frac{dS_T}{d\lambda}\right) = \delta\left(\frac{dS_T}{d\lambda}h_t(D_\gamma S_T)^{-1}\right)$$

for all $\omega \in \Omega$ and $t \in [0, T]$.

The formula in equation (3.2) provides a better numerical result in combination with Monte Carlo simulations than the formula in equation (3.1). This is because the derivative of the function does not have to be estimated (using techniques such as finite difference methods) if we used the formula in equation (3.2). To this end, the formula in equation (3.2) is less computationally expensive and more accurate than the formula in equation (3.1) ([5]).

3.1 Greeks for European Options Revisited

The most important greek is the delta. It tells the investor by how much the value of the option will change should the stock price increase or decrease by 1. It also coincides with the amount of stock needed in the replicating strategy of the option ([31]). The initial delta for a European-type option with payoff $f(S_T)$ is

$$\Delta = \frac{\partial V_0}{\partial S_0} = \mathbb{E}_{\mathbb{Q}}\left(e^{-rT} f'(S_T) \frac{\partial S_T}{\partial S_0}\right) = \frac{e^{-rT}}{S_0}\mathbb{E}_{\mathbb{Q}}(f'(S_T)S_T)$$

where

$$S_T = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + W_T\sigma\right).$$

The partial derivative of S_T with respect to S_0 is

$$\begin{aligned}
\frac{\partial S_T}{\partial S_0} &= \frac{\partial}{\partial S_0} S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + W_T\sigma\right) \\
&= \exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + W_T\sigma\right) \\
&= \frac{S_T}{S_0}.
\end{aligned}$$

By equation (3.2) on page 34,

$$\Delta = \frac{\partial V_0}{\partial S_0} = \frac{e^{-rT}}{S_0}\mathbb{E}_{\mathbb{Q}}\left(f(S_T)\mathcal{K}\left(S_T, \frac{dS_T}{dS_0}\right)\right) \tag{3.3}$$

where

$$\mathcal{K}\left(S_T, \frac{dS_T}{dS_0}\right) = \delta\left(\frac{dS_T}{dS_0} h_t (D_\gamma S_T)^{-1}\right).$$

The Malliavin derivative of S_T is

$$\begin{aligned} D_t S_T &= D_t S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right) T + W_T \sigma\right) \\ &= \exp\left(\left(r - \frac{1}{2}\sigma^2\right) T\right) D_t \exp\left(W_T \sigma\right) \\ &= \exp\left(\left(r - \frac{1}{2}\sigma^2\right) T\right) \sigma \exp\left(W_T \sigma\right) \end{aligned}$$

by the chain rule, theorem 2.1 on page 14, and the fact that

$$D_t(\sigma W_T) = \sigma D_t W_T = \sigma \cdot \mathbf{1}_{[0,T]} = \sigma,$$

by equation (2.9) on page 23.

Hence

$$\begin{aligned} D_t S_T &= \exp\left(\left(r - \frac{1}{2}\sigma^2\right) T\right) \sigma \exp\left(W_T \sigma\right) \\ &= \sigma S_T. \end{aligned}$$

Since $\mathbf{1}_{[0,T]} = 1$ is $\mathcal{F}_s$ measurable for $\mathcal{F}_s$ the σ-algebra generated by the Brownian motion for all $s \in [0, T]$, we see that $\mathbf{1}_{[0,T]}$ is adapted to the filtration generated by the σ-algebras, and so from theorem 2.10 we have that

$$\delta(1) = \int_0^T 1\, \delta W_t = \int_0^T 1\, dW_t = W_T.$$

Setting $h_t = 1$ for γ_t, for $t \in [0, T]$, in theorem 2.12, and using the fact that the directional derivative of S_T is

$$D_\gamma S_T = \int_0^T D_t S_T\, dt = \int_0^T \sigma S_T\, dt = \sigma S_T T,$$

it follows that

$$\delta\left(S_T \left(\int_0^T \sigma S_T\, dt\right)^{-1}\right) = \delta\left(\frac{S_T}{\sigma T S_T}\right) = \frac{1}{\sigma T}\delta(1) = \frac{W_T}{\sigma T}.$$

As a consequence the Malliavin delta for a European-type option is

$$\Delta = \frac{e^{-rT}}{S_0 \sigma T} \mathbb{E}_{\mathbb{Q}}(f(S_T) W_T).$$

We can see that in this case, the Malliavin delta is simply the discounted risk-neutral expectation of the payoff of the option multiplied by the Wiener process at time T. The term W_T is referred as the Malliavin weight.

The gamma is the second derivative of the option price with respect to S_0. The calculation of the second partial derivative yields that

$$\Gamma = \frac{\partial^2 V_0}{\partial S_0^2} = \mathbb{E}_{\mathbb{Q}}\left(e^{-rT} f''(S_T)\left(\frac{\partial S_T}{\partial S_0}\right)^2\right) = \frac{e^{-rT}}{S_0^2}\mathbb{E}_{\mathbb{Q}}\left(f''(S_T)S_T^2\right).$$

By equation (3.2),

$$\Gamma = \frac{\partial^2 V_0}{\partial S_0^2} = \frac{e^{-rT}}{S_0^2}\mathbb{E}_{\mathbb{Q}}\left(f'(S_T)\mathcal{K}\left(S_T, S_T^2\right)\right) \tag{3.4}$$

where

$$\mathcal{K}\left(S_T, S_T^2\right) = \delta\left(S_T^2 h_t (D_\gamma S_T)^{-1}\right).$$

Taking $h_t = 1$ in γ_t in theorem 2.12 leaves us with

$$\delta\left(S_T^2\left(\int_0^T D_t S_T\, dt\right)^{-1}\right) = \delta\left(\frac{S_T^2}{S_T \sigma T}\right) = \delta\left(\frac{S_T}{\sigma T}\right).$$

To evaluate

$$\delta\left(\frac{S_T}{\sigma T}\right) = \frac{1}{\sigma T}\delta(S_T),$$

since σ and T are constant, we use theorem 2.11. The Skorohod integral of S_T, $\delta(S_T)$, is

$$\delta(S_T) = S_T\int_0^T dW_t - \int_0^T (D_t S_T)\, dt.$$

Since $D_t S_T = \sigma S_T$, as calculated on page 36, the Skorohod integral becomes

$$\delta(S_T) = S_T W_T - \int_0^T \sigma S_T\, dt = S_T W_T - T\sigma S_T.$$

Thus

$$\delta\left(\frac{S_T}{\sigma T}\right) = \frac{1}{\sigma T}\delta(S_T) = \frac{1}{\sigma T}(S_T W_T - T\sigma S_T) = S_T\left(\frac{W_T}{\sigma T} - 1\right).$$

As a result the expectation is

$$\mathbb{E}_Q\left(f''(S_T)S_T^2\right) = \mathbb{E}_{\mathbb{Q}}\left(f'(S_T)S_T\left(\frac{W_T}{\sigma T} - 1\right)\right). \tag{3.5}$$

We can apply theorem 2.12 again to equation (3.5). In that case

$$\begin{aligned}\Gamma = \frac{\partial^2 V_0}{\partial S_0^2} &= \frac{e^{-rT}}{S_0^2}\mathbb{E}_Q\left(f''(S_T)S_T^2\right)\\ &= \frac{e^{-rT}}{S_0^2}\mathbb{E}_{\mathbb{Q}}\left(f'(S_T)S_T\left(\frac{W_T}{\sigma T}-1\right)\right)\\ &= \frac{e^{-rT}}{S_0^2}\mathbb{E}_{\mathbb{Q}}\left(f(S_T)\mathcal{K}\left(S_T, S_T\left(\frac{W_T}{\sigma T}-1\right)\right)\right).\end{aligned}$$

Thus

$$\Gamma = \frac{e^{-rT}}{S_0^2}\mathbb{E}_{\mathbb{Q}}\left(f(S_T)\mathcal{K}\left(S_T, S_T\left(\frac{W_T}{\sigma T}-1\right)\right)\right) \tag{3.6}$$

where

$$\mathcal{K}\left(S_T, S_T\left(\frac{W_T}{\sigma T}-1\right)\right) = \delta\left(S_T\left(\frac{W_T}{\sigma T}-1\right)h_t(D_\gamma S_T)^{-1}\right).$$

Setting $h_t = 1$ yields that

$$\begin{aligned}\delta\left(S_T\left(\frac{W_T}{\sigma T}-1\right)\left(\int_0^T D_t S_T\,dt\right)^{-1}\right) &= \delta\left(S_T\left(\frac{W_T}{\sigma T}-1\right)\frac{1}{S_T\sigma T}\right)\\ &= \delta\left(\frac{W_T}{\sigma^2T^2}-\frac{1}{\sigma T}\right)\\ &= \delta\left(\frac{W_T}{\sigma^2T^2}\right)-\delta\left(\frac{1}{\sigma T}\right)\\ &= \delta\left(\frac{W_T}{\sigma^2T^2}\right)-\frac{1}{\sigma T}\delta\left(1\right).\end{aligned}$$

We use theorem 2.11 to evaluate $\delta\left(\frac{W_T}{\sigma^2T^2}\right)$:

$$\begin{aligned}\delta\left(\frac{W_T}{\sigma^2T^2}\right) = \delta\left(\frac{W_T\cdot 1}{\sigma^2T^2}\right) &= \frac{W_T}{\sigma^2T^2}\int_0^T 1\,dW_t - \int_0^T \frac{D_tW_T}{\sigma^2T^2}\,dt\\ &= \frac{W_T^2}{\sigma^2T^2}-\frac{1}{\sigma^2T}\int_0^T 1\,dt\\ &= \frac{W_T^2}{\sigma^2T^2}-\frac{1}{\sigma^2T}.\end{aligned}$$

The derivative $D_tW_T = \mathbf{1}_{[0,T]} = 1$ as seen in equation (2.9).

Finally, since $\mathbf{1}_{[0,T]}$ is $\mathcal{F}_t$ measurable for $\mathcal{F}_t$ the σ-algebra generated by the Brownian motion at time t for all $t\in[0,T]$, $\mathbf{1}_{[0,T]}$ is adapted to the filtration generated by the σ-algebras, hence we have that

$$\delta(1) = \int_0^T 1\,\delta W_t = \int_0^T 1\,dW_t = W_T, \tag{3.7}$$

as by theorem 2.10.

From equation (3.6), we see that the Malliavin gamma evaluates to

$$\begin{aligned}\Gamma &= \frac{e^{-rT}}{S_0^2}\mathbb{E}_{\mathbb{Q}}\left(f(S_T)\left(\frac{W_T^2}{\sigma^2T^2}-\frac{1}{\sigma^2T}-\frac{W_T}{\sigma T}\right)\right)\\ &= \frac{e^{-rT}}{S_0^2\sigma T}\mathbb{E}_{\mathbb{Q}}\left(f(S_T)\left(\frac{W_T^2}{\sigma T}-\frac{1}{\sigma}-W_T\right)\right).\end{aligned}$$

The Malliavin weight is

$$\left(\frac{W_T^2}{\sigma T}-\frac{1}{\sigma}-W_T\right).$$

The vega is computed in a similar manner to the delta. Indeed,

$$\begin{aligned}\vartheta &= \frac{\partial V_0}{\partial\sigma}=\mathbb{E}_{\mathbb{Q}}\left(e^{-rT}f^{'}(S_T)\frac{\partial S_T}{\partial\sigma}\right)\\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}(f^{'}(S_T)S_T(W_T-\sigma T)).\end{aligned}\tag{3.8}$$

The partial derivative of S_T with respect to σ is

$$\begin{aligned}\frac{\partial S_T}{\partial\sigma} &= \frac{\partial}{\partial\sigma}S_0\exp\left(\left(r-\frac{1}{2}\sigma^2\right)T+W_T\sigma\right)\\ &= S_0\left(W_T-\sigma T\right)\exp\left(\left(r-\frac{1}{2}\sigma^2\right)T+W_T\sigma\right)\\ &= S_T\left(W_T-\sigma T\right).\end{aligned}$$

By equation (3.2), the expression in equation (3.8) becomes

$$\vartheta=\frac{\partial V_0}{\partial\sigma}=e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(f(S_T\mathcal{K}\left(S_T,S_T\left(W_T-\sigma T\right)\right)\right)\tag{3.9}$$

where

$$\mathcal{K}\left(S_T,S_T\left(W_T-\sigma T\right)\right)=\delta\left(S_T\left(W_T-\sigma T\right)h_t(D_\gamma S_T(\omega))^{-1}\right).$$

Now, setting $h_t=1$ in γ_t in theorem (2.12) gives us that

$$\begin{aligned}\delta\left(S_T(W_T-\sigma T)\left(\int_0^T D_tS_T\,dt\right)^{-1}\right) &= \delta\left(\frac{S_T(W_T-\sigma T)}{S_T\sigma T}\right)\\ &= \delta\left(\frac{W_T}{\sigma T}-1\right)\\ &= \delta\left(\frac{W_T}{\sigma T}\right)-\delta\left(1\right).\end{aligned}$$

The directional derivative

$$D_\gamma S_T = \left(\int_0^T D_t S_T \, dt \right)$$

for $h_t = 1$ for $t \in [0, T]$, was worked out on page 36.

We use theorem 2.11 to evaluate first the Skorohod integral:

$$\begin{aligned}
\delta \left(\frac{W_T}{\sigma T} \right) = \delta \left(\frac{W_T \cdot 1}{\sigma T} \right) &= \left(\frac{W_T}{\sigma T} \right) \delta(1) - \left(\frac{1}{\sigma T} \right) \int_0^T D_t W_T \, dt \\
&= \left(\frac{W_T^2}{\sigma T} \right) - \left(\frac{1}{\sigma T} \right) T \\
&= \left(\frac{W_T^2}{\sigma T} \right) - \left(\frac{1}{\sigma} \right).
\end{aligned}$$

The derivative $D_t W_T = \mathbf{1}_{[0,T]} = 1$ as seen in equation (2.9). Finally, as explained on page 38,

$$\delta(1) = \int_0^T 1 \, \delta W_t = \int_0^T 1 \, dW_t = W_T.$$

Hence, the final expression for the Malliavin vega of a European-type option is

$$\vartheta = e^{-rT} \mathbb{E}_{\mathbb{Q}} \left(f(S_T) \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right) \right).$$

In this case, the Malliavin weight is

$$\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right).$$

These results can then be used to develop algorithms to calculate the greeks using Monte Carlo simulations. This will be done in the forthcoming section. We summarise these results in the following table.

greek	$\mathtt{t} = 0$ formula	Malliavin Weight
delta	$\frac{e^{-rT}}{S_0 \sigma T} \mathbb{E}_{\mathbb{Q}}(f(S_T) W_T)$	W_T
gamma	$\frac{e^{-rT}}{S_0^2 \sigma T} \mathbb{E}_{\mathbb{Q}} \left(f(S_T) \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right) \right)$	$\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right)$
vega	$e^{-rT} \mathbb{E}_{\mathbb{Q}} \left(f(S_T) \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right) \right)$	$\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right)$

3.2 Procedure for Computing the Greeks

The stock prices at time T, were modelled with the equation

$$S_T = S_0 \exp \left(\left(r - \frac{1}{2}\sigma^2 \right) T + \sigma W_T \right).$$

Since W_T is a Brownian motion and is Gaussian distributed with mean zero and variance T, it can be expressed as

$$W_T = \sqrt{T}\nu$$

where ν is a random variable that Gaussian distributed with mean 0 and variance 1. So that the stock price is then expressed as

$$S_T = S_0 \exp \left(\left(r - \frac{1}{2}\sigma^2 \right) T + \sigma \sqrt{T}\nu \right).$$

The Malliavin delta was derived to be

$$\Delta = \frac{e^{-rT}}{S_0 \sigma T} \mathbb{E}_{\mathbb{Q}}(f(S_T) W_T).$$

From the above it follows that

$$\Delta = \frac{e^{-rT}}{S_0 \sigma T} \mathbb{E}_{\mathbb{Q}} \left(f \left(S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\nu} \right) \sqrt{T}\nu \right).$$

Similarly, since

$$\Gamma = \frac{e^{-rT}}{S_0^2\sigma T}\mathbb{E}_{\mathbb{Q}}\left(f(S_T)\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right)\right),$$

the Malliavin gamma can be written as

$$\Gamma = \frac{e^{-rT}}{S_0^2\sigma T}\mathbb{E}_{\mathbb{Q}}\left(f\left(S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\nu}\right)\left(\frac{(\sqrt{T}\nu)^2}{\sigma T} - \frac{1}{\sigma} - \sqrt{T}\nu\right)\right).$$

Finally, since the Malliavin vega was found to be

$$\vartheta = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(f(S_T)\left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T\right)\right),$$

it can be expressed as

$$\vartheta = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(f\left(S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\nu}\right)\left(\frac{(\sqrt{T}\nu)^2}{\sigma T} - \frac{1}{\sigma} - \sqrt{T}\nu\right)\right).$$

For European-type options, the Malliavin gamma is actually equal to the Malliavin vega up to a constant; in fact,

$$\Gamma = \frac{\vartheta}{S_0^2\sigma T}.$$

The objective of our Monte Carlo simulation was to approximate this expectation by using the (strong) law of large numbers ([3]), which tells us that if Y_j is a sequence of identically independent distributed random variables then with probability 1, the sequence

$$\frac{1}{N}\sum_{j=1}^{N} Y_j$$

converges to $\mathbb{E}(Y_1)$.

So the algorithm to compute the greeks is clear. Random variables, ν, are drawn from a standard Normal distribution (Gaussian distribution with mean zero and variance 1) and the argument of the expectations is computed. This is done many times (a million times in this case) and the average is taken. The average is then multiplied by the factor $\dfrac{e^{-rT}}{S_0\sigma T}$ in the case of the delta; the discout factor, e^{-rT}, in the case of the vega; and by $\dfrac{e^{-rT}}{S_0^2\sigma T}$ in the case of the Γ.

We now consider the delta, the gamma and the vega of a European call option with strike price, K=100; spot (initial) price, S_0=100; risk-free rate, r=0.1; standard deviation, σ=0.2; and expiry date, T=1.0, i.e. 1 year. We use the explicit formulas for these greeks given in chapter 1 to compute the greeks analytic. The results obtained are given in the following table. Recall that

$$d_+ = \frac{\ln \frac{S_0}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.$$

greek	$\mathtt{T} = 1, \mathtt{r} = 0.1, \mathtt{S}_0 = 100, \mathtt{K} = 100, \sigma = 0.2$	
delta	$N(d_+) = \int_{-\infty}^{d_+} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy$	0.726
gamma	$\frac{1}{S_0\sigma\sqrt{2\pi T}} e^{-\frac{1}{2}d_+^2}$	0.01666
vega	$S_0 \frac{\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{1}{2}d_+}$	33.32

The actual code for computing the greeks was written in C++ and the source code was written to a CD provided along with this thesis.

3.3 Monte Carlo Simulation Results for the Greeks of the European Call Option

The Monte Carlo simulation for the delta of European call option, in figure 3.1, on page 45, shows that after 200 000 simulations the value of the delta is approximately 0.726 and continues to converge towards the true value of the delta. This in agreement with the analytic results.

The Monte Carlo simulation for the gamma of European call option, in figure 3.2, on page 46, shows that after 200 000 simulations the gamma of the option is approximately 0.0166 and continues to converge towards the true value of the gamma. This agrees well with the analytic results.

The Monte Carlo simulation for the vega of European call option, in figure 3.3, on page 47, shows that after 200 000 simulations the vega of the option is approximately 33.3 and continues to converge towards the true value of the gamma, in line with what was obtained analytically.

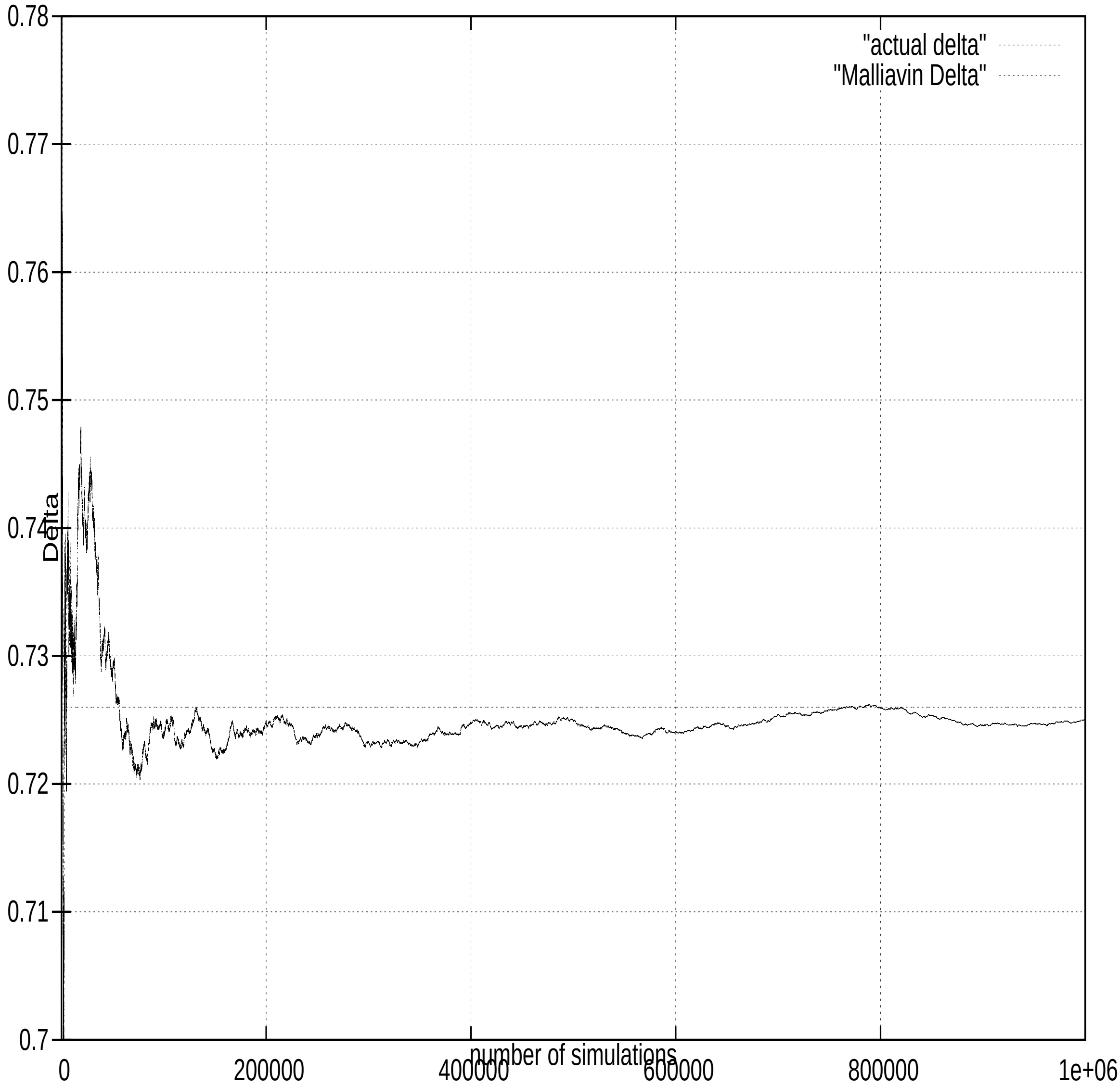

Figure 3.1: The output for the Monte Carlo simulation for the delta of European call option

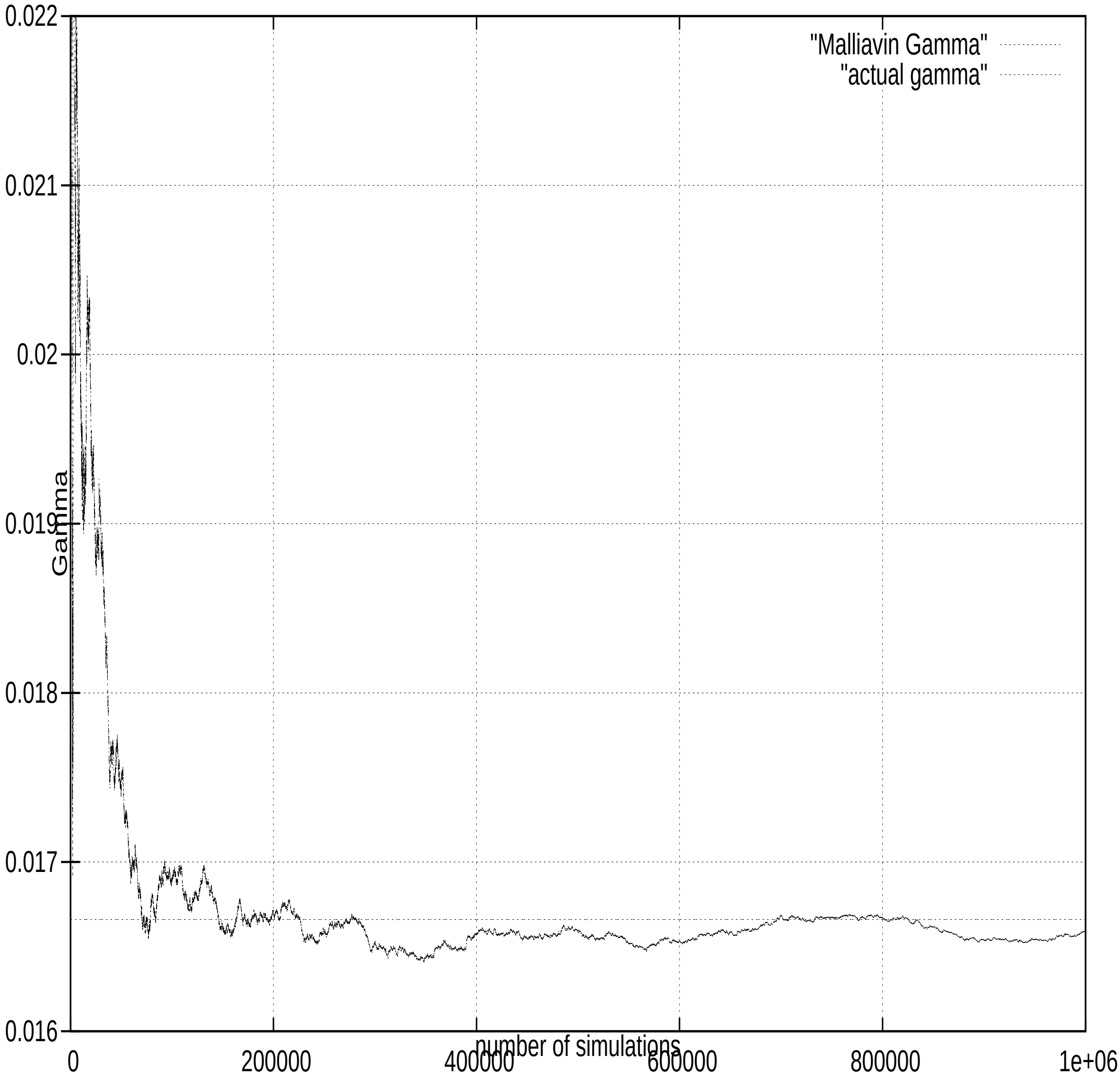

Figure 3.2: The output of the Monte Carlo simulation of the gamma of European call option

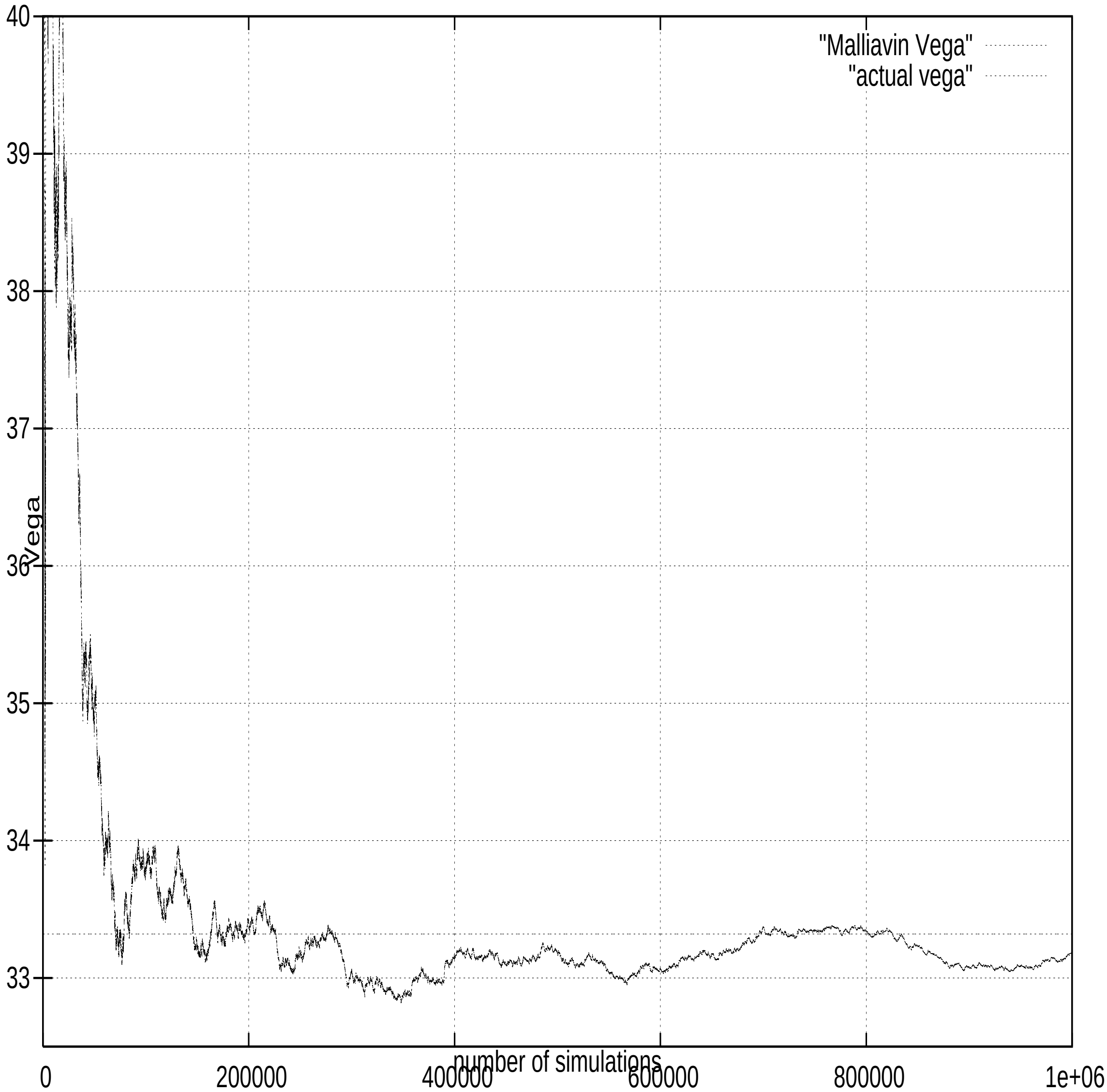

Figure 3.3: The output of the Monte Carlo simulation of the vega of European call option

3.4 Greeks of Exotic Options

We now look at examples of exotic options that have discontinuities in their payoffs. In particular, we study the cases of the the *up and out barrier call option* and *the digital knock out option*. These options have explicit evaluation formulae and we will compare the results that we get from the analytic formulae with the results that we get form the Malliavin formulae for the greeks. We will focus our attention to the greeks at prices which are very near where this discontinuity exists. We will then compare the greeks of these options at this price with the greeks that we obtain from the Malliavin Calculus formulae we derived in section 3.1. Later in this section, we will use Malliavin calculus to compute the delta of an Asian call option which has no know analytic evaluation for its prices and hence no analytic expression for any of its greeks.

3.4.1 Up and Out Call Option

An up and out barrier call option is similar to a call option. However, if the stock price exceeds a certain predetermined barrier level (and hence the name), the option becomes invalid and nothing is paid out regardless of the whether the terminal stock price is greater than the strike price. Let S_0 denote the initial stock price and as usual K, r and σ denote the strike, the risk-free interest rate and the volatility respectively. We designate the barrier level by B. The initial (time 0) value of an up and out barrier call is given as [26]

$$e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(\mathbf{1}_{\{\max S_t | t\in[0,T]\}<B\}}\left(S_T-K\right)^+\right)=$$

$$\begin{aligned}
&S_0\left\{N\left(d_+\left(\frac{S_0}{K}\right)\right)-N\left(d_+\left(\frac{S_0}{B}\right)\right)\right\}\\
&-B\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}}\left\{N\left(d_+\left(\frac{B^2}{KS_0}\right)\right)-N\left(d_+\left(\frac{B}{S_0}\right)\right)\right\}\\
&-Ke^{-rT}\left\{N\left(d_+\left(\frac{S_0}{K}\right)\right)-N\left(d_+\left(\frac{S_0}{B}\right)\right)\right\}\\
&Ke^{-rT}\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}-1}\left\{N\left(d_+\left(\frac{B^2}{KS_0}\right)\right)-N\left(d_+\left(\frac{B}{S_0}\right)\right)\right\},
\end{aligned}$$

where we defined

$$d_+(x)=\frac{\ln(x)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

and
$$d_-(x) = \frac{\ln(x) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

We will consider an up and out call option with strike $K = 100$, barrier $B = 120$, expiry $T = 0.001$ with the risk-free interest rate $r = 0.05$ and volatility $\sigma = 0.25$. The plot of the price is given in figure 3.4.

We see that the price is zero for all initial prices that are less than the strike price. The price then increases and reaches a maximum and then is zero for all initial prices greater than the barrier.

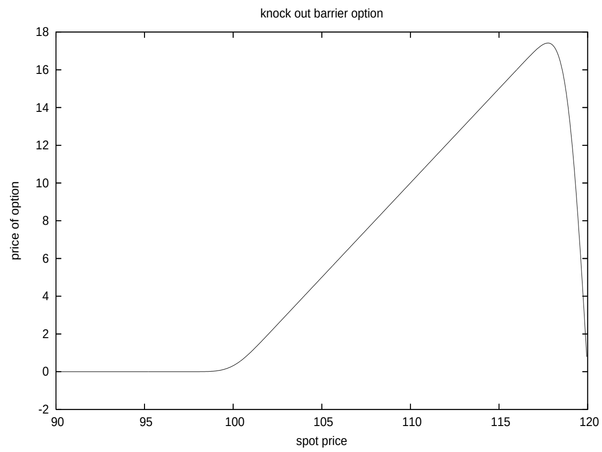

Figure 3.4: Price of Up and Out Call Option

We find that the delta of an up and out barrier call option is given as
$$\Delta = \frac{\partial V_0}{\partial S_0} = \delta_1 - \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6,$$
where
$$\delta_1 = N\left(d_+\left(\frac{S_0}{K}\right)\right),$$
$$\delta_2 = N\left(d_+\left(\frac{S_0}{B}\right)\right),$$
$$\begin{aligned}\delta_3 &= -B(B)^{\frac{2r}{\sigma^2}} S_0^{\frac{-2r}{\sigma^2}}\left(-\frac{2r}{\sigma^2}\right) N\left(d_+\left(\frac{B^2}{KS_0}\right)\right)\\ &\quad -B(B)^{\frac{2r}{\sigma^2}} S_0^{\frac{-2r}{\sigma^2}}\left(\frac{1}{\sqrt{2\pi T}\sigma S_0}\right)\exp\left(-\frac{1}{2}d_+^2\left(\frac{B^2}{KS_0}\right)\right)\\ &\quad -B^{-\frac{2r}{\sigma^2}+1}\left(-\frac{2r}{\sigma^2}\right) S_0^{-\frac{2r}{\sigma^2}-1} N\left(d_+\left(\frac{B^2}{KS_0}\right)\right),\end{aligned}$$

$$
\begin{aligned}
\delta_4 &= -\left(\frac{B}{S_0}\right)^{\frac{-2r}{\sigma^2}+1}\left(\frac{-1}{\sqrt{2\pi T}\sigma}\right)\exp\left(-\frac{1}{2}d_+^2\left(\frac{B^2}{KS_0}\right)\right) \\
&\quad -B^{-\frac{2r}{\sigma^2}+1}\left(-\frac{2r}{\sigma^2}\right)S_0^{-\frac{2r}{\sigma^2}-1}N\left(d_+\left(\frac{B}{S_0}\right)\right) \\
&\quad -\left(\frac{B}{S_0}\right)^{\frac{-2r}{\sigma^2}+1}\left(\frac{-1}{\sqrt{2\pi T}\sigma}\right)\exp\left(-\frac{1}{2}d_+^2\left(\frac{B}{S_0}\right)\right),
\end{aligned}
$$

$$
\delta_5 = K\exp(-rT)B^{\frac{-2r}{\sigma^2}-1}\left(1+\frac{2r}{\sigma^2}\right)S_0^{\frac{-2r}{\sigma^2}}N\left(d_-\left(\frac{B^2}{KS_0}\right)\right),
$$

and

$$
\delta_6 = K\exp(-rT)B^{\frac{-2r}{\sigma^2}-1}S_0^{\frac{-2r}{\sigma^2}}\left(\frac{1}{\sqrt{2\pi T}\sigma}\right)\exp\left(-\frac{1}{2}d_-^2\left(\frac{B^2}{S_0K}\right)\right).
$$

The figure 3.5, on page 50, shows a plot of the delta of an up and out call option. We see that the delta increases slightly and it is then constant for a number of initial prices. After that it decreases to a negative quantity at the barrier.

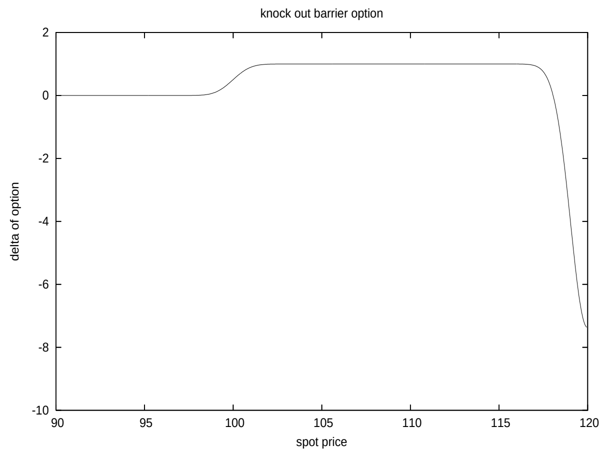

Figure 3.5: delta of up and out call option

When $S_0 = 119.95$, we get that the delta is -7.33. We recall the formula for the Malliavin delta for an option is given as

$$\frac{e^{-rT}}{S_0 \sigma T} \mathbb{E}_{\mathbb{Q}}(f(S_T) W_T). \tag{3.10}$$

In the case of an up and out call option, the payoff is

$$f(S_T) = \mathbf{1}_{\{\max S_t | t \in [0,T]\} < B\}} (S_T - K)^+ .$$

The result of the Monte Carlo simulation of the delta, given as in equation (3.10), of an up and out call option is given in figure 3.6 on page 52. We see that the approximation formula, equation (3.10), is quite good and is very close to the analytical price after 300 000 simulations.

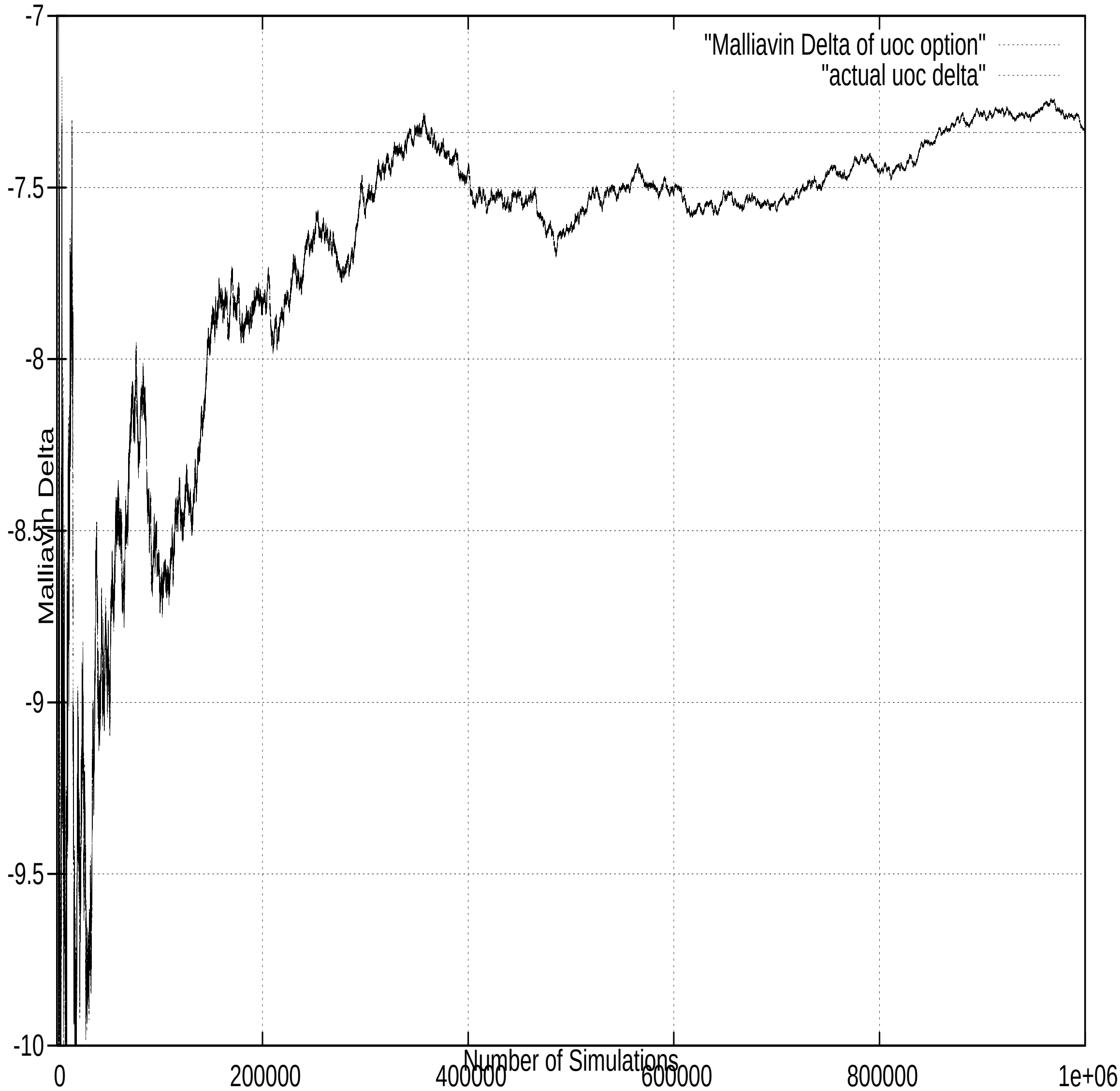

Figure 3.6: Monte Carlo simulation for Malliavin delta for up and out call option

Next, we find that the gamma of an up and out call option is given as

$$\Gamma = \gamma_1 - \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6,$$

where

$$\gamma_1 = \frac{1}{\sqrt{2\pi T}\sigma S_0} \exp\left(-\frac{1}{2} d_+^2 \left(\frac{S_0}{K}\right)\right),$$
$$\gamma_2 = \frac{1}{\sqrt{2\pi T}\sigma S_0} \exp\left(-\frac{1}{2} d_+^2 \left(\frac{S_0}{B}\right)\right),$$

$$\begin{aligned}
\gamma_3 \;=\; & -B^{\frac{2r}{\sigma^2}+1}\left(\frac{-2r}{\sigma^2}\right)\left(-1-\frac{2r}{\sigma^2}\right) S_0^{\frac{-2r}{\sigma^2}-2} N\left(d_+\left(\frac{B^2}{KS_0}\right)\right) \\
& -B^{\frac{2r}{\sigma^2}+1}\left(\frac{2r}{\sigma^2}\right) S_0^{\frac{-2r}{\sigma^2}-1}\left(\frac{-1}{\sqrt{2\pi T}\ \sigma}\right)\exp\left(-\frac{1}{2}d_+^2\left(\frac{B^2}{S_0K}\right)\right) \\
& -B^{\frac{2r}{\sigma^2}+1}\left(-1-\frac{2r}{\sigma^2}\right) S_0^{\frac{-2r}{\sigma^2}-2}\left(\frac{-1}{\sqrt{2\pi T}\ \sigma}\right)\exp\left(-\frac{1}{2}d_+^2\left(\frac{B^2}{S_0K}\right)\right) \\
& -\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}+1}\left(\frac{-1}{\sqrt{2\pi T}\ \sigma}\right)\left(-d_+^2\left(\frac{B^2}{S_0K}\right)\right)\exp\left(-\frac{1}{2}d_+^2\left(\frac{B^2}{S_0K}\right)\right) \\
& \left(\frac{-1}{\sqrt{T}\ \sigma S_0}\right),
\end{aligned}$$

$$\begin{aligned}
\gamma_4 \;=\; & B^{\frac{2r}{\sigma^2}+1}\left(\frac{-2r}{\sigma^2}\right)\left(-1-\frac{2r}{\sigma^2}\right) S_0^{\frac{-2r}{\sigma^2}-2} N\left(d_+\left(\frac{B}{S_0}\right)\right) \\
& +B^{\frac{2r}{\sigma^2}+1}\left(\frac{2r}{\sigma^2}\right) S_0^{\frac{-2r}{\sigma^2}-1}\left(\frac{-1}{\sqrt{2\pi T}\ \sigma}\right)\exp\left(-\frac{1}{2}d_+^2\left(\frac{B}{S_0}\right)\right) \\
& +B^{\frac{2r}{\sigma^2}+1}\left(-1-\frac{2r}{\sigma^2}\right) S_0^{\frac{-2r}{\sigma^2}-2}\left(\frac{-1}{\sqrt{2\pi T}\ \sigma}\right)\exp\left(-\frac{1}{2}d_+^2\left(\frac{B}{S_0}\right)\right) \\
& +\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}+1}\left(\frac{-1}{\sqrt{2\pi T}\ \sigma}\right)\left(-d_+^2\left(\frac{B^2}{S_0K}\right)\right)\exp\left(-\frac{1}{2}d_+^2\left(\frac{B}{S_0}\right)\right) \\
& \left(\frac{-1}{\sqrt{T}\ \sigma S_0}\right),
\end{aligned}$$

$$
\begin{aligned}
\gamma_5 &= Ke^{-rT}B^{\frac{2r}{\sigma^2}-1}\left(1-\frac{2r}{\sigma^2}\right)\left(-\frac{2r}{\sigma^2}\right)S_0^{-\frac{2r}{\sigma^2}-1}N\left(d_-\left(\frac{B^2}{KS_0}\right)\right)\\
&\quad +Ke^{-rT}B^{\frac{2r}{\sigma^2}-1}\left(1-\frac{2r}{\sigma^2}\right)S_0^{-\frac{2r}{\sigma^2}-1}\left(\frac{-1}{\sqrt{2\pi T}\ \sigma S_0}\right)\exp\left(-\frac{1}{2}d_-^2\left(\frac{B^2}{KS_0}\right)\right)\\
&\quad +Ke^{-rT}B^{\frac{2r}{\sigma^2}-1}\left(-\frac{2r}{\sigma^2}\right)S_0^{-\frac{2r}{\sigma^2}-1}\left(\frac{-1}{\sqrt{2\pi T}\ \sigma}\right)\exp\left(-\frac{1}{2}d_-^2\left(\frac{B^2}{KS_0}\right)\right)\\
&\quad +Ke^{-rT}B^{\frac{2r}{\sigma^2}-1}S_0^{-\frac{2r}{\sigma^2}-1}\left(\frac{-1}{\sqrt{2\pi T}\ \sigma}\right)\left(d_-^2\left(\frac{B^2}{KS_0}\right)\right)\exp\left(-\frac{1}{2}d_-^2\left(\frac{B^2}{KS_0}\right)\right)\\
&\quad \left(\frac{-1}{\sqrt{T}\ S_0\sigma}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_6 &= -Ke^{-rT}B^{\frac{2r}{\sigma^2}-1}\left(1-\frac{2r}{\sigma^2}\right)\left(-\frac{2r}{\sigma^2}\right)S_0^{-\frac{2r}{\sigma^2}-1}N\left(d_-\left(\frac{B}{S_0}\right)\right)\\
&\quad -Ke^{-rT}B^{\frac{2r}{\sigma^2}-1}\left(1-\frac{2r}{\sigma^2}\right)S_0^{-\frac{2r}{\sigma^2}-1}\left(\frac{-1}{\sqrt{2\pi T}\ \sigma S_0}\right)\exp\left(-\frac{d_-^2\left(\frac{B}{S_0}\right)}{2}\right)\\
&\quad -Ke^{-rT}B^{\frac{2r}{\sigma^2}-1}\left(-\frac{2r}{\sigma^2}\right)S_0^{-\frac{2r}{\sigma^2}-1}\left(\frac{-1}{\sqrt{2\pi T}\ \sigma}\right)\exp\left(-\frac{1}{2}d_-^2\left(\frac{B}{S_0}\right)\right)\\
&\quad -Ke^{-rT}B^{\frac{2r}{\sigma^2}-1}S_0^{-\frac{2r}{\sigma^2}-1}\left(\frac{-1}{\sqrt{2\pi T}\ \sigma}\right)\left(d_-^2\left(\frac{B^2}{KS_0}\right)\right)\exp\left(-\frac{d_-^2\left(\frac{B}{S_0}\right)}{2}\right)\\
&\quad \left(\frac{-1}{\sqrt{T}\ S_0\sigma}\right).
\end{aligned}
$$

Figure 3.7 shows a plot of the gamma of an up and out call option. We see that the gamma is constant and there is a maximum occurring at the strike price. The gamma then becomes constant again and then it decreases to a negative minimum and then increases to zero at the barrier.

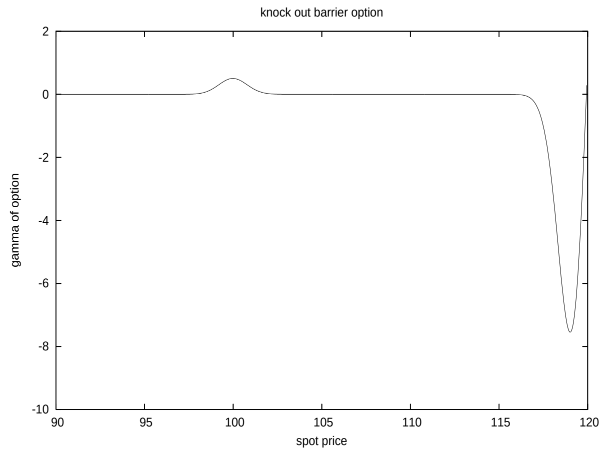

Figure 3.7: gamma of up and out call option

When $S_0 = 119.95$, we get that the gamma is -0.022. The formula for the Malliavin gamma for an option is given as

$$\Gamma = \frac{e^{-rT}}{S_0^2 \sigma T} \mathbb{E}_{\mathbb{Q}} \left(f(S_T) \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right) \right). \tag{3.11}$$

In the case of an up and out call option,

$$f(S_T) = \mathbf{1}_{\{\max S_t | t \in [0,T]\} < B\}} \left(S_T - K \right)^+ .$$

Proceeding with the Monte Carlo simulation of the expression in equation (3.11), we obtain the results in figure 3.8 on page 56. We see that equation (3.11) is a good approximation formula; the gamma converges to -0.0221, which is very close to the analytical value of the gamma.

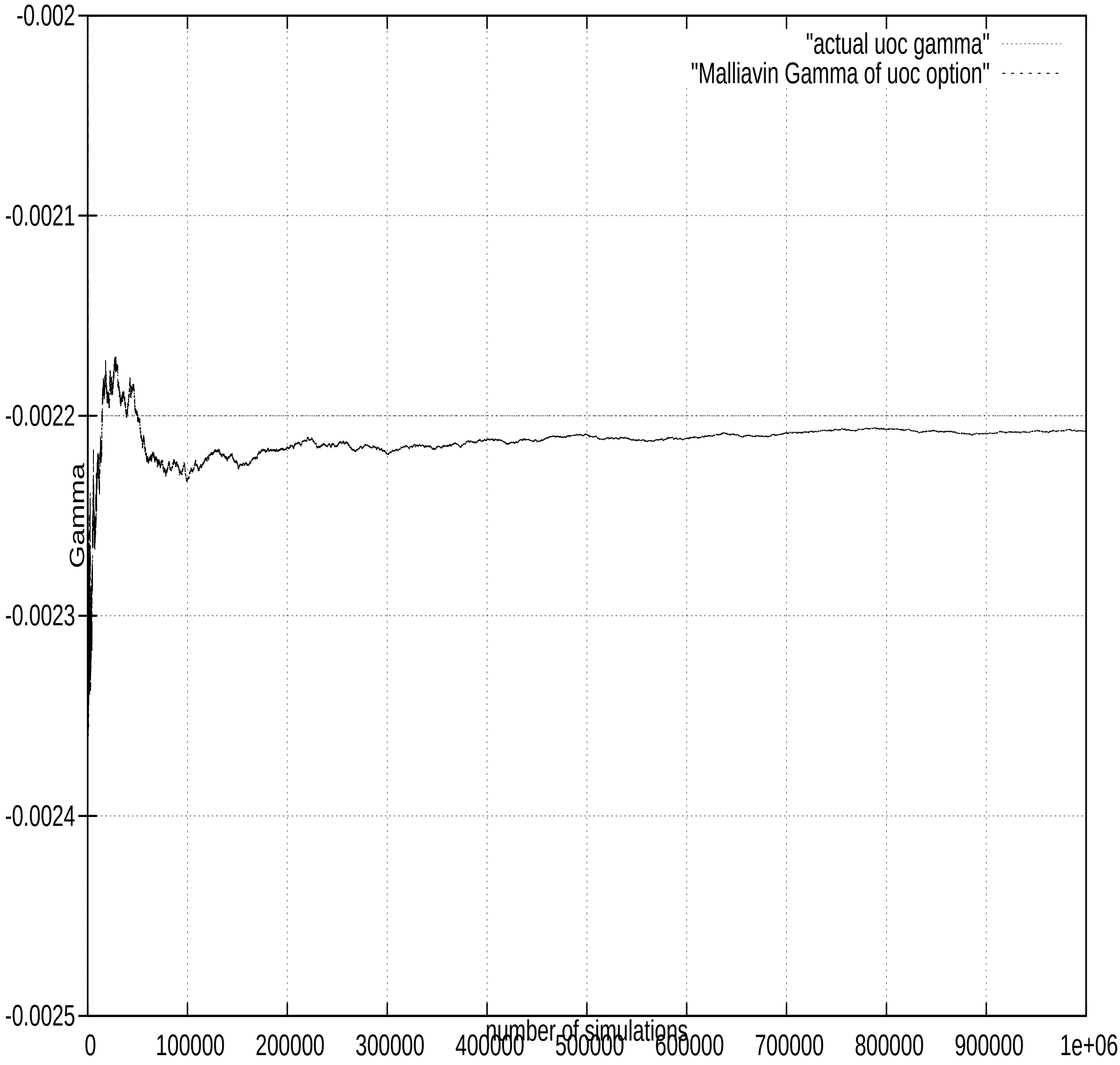

Figure 3.8: Monte Carlo simulation for Malliavin gamma of up and out call

3.4.2 Knock Out Digital Option

A knock out digital option pays one unit of the currency at time T if the price doesn't cross a barrier level $B > S_0$ before or at time T and nothing is paid out if it does. Its price at $t = 0$ is given as [26]

$$
\begin{aligned}
V_0 &= \exp(-rT)N\left(\frac{\log\left(\frac{B}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \\
&- \exp(-rT)\left(\frac{B}{S_0}\right)^{\frac{2r}{\sigma^2}-1} N\left(\frac{-\log\left(\frac{B}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).
\end{aligned}
$$

We will consider a knock out digital option with strike price $K = 100$, expiry date $T = 0.001$, barrier level $B = 120$, interest rate $r = 0.05$ and volatility $\sigma = 0.25$. Figure 3.9 shows the price of a knock out digital option for different spot prices S_0. The price is one (unit of the currency) and it decreases to zero at the barrier.

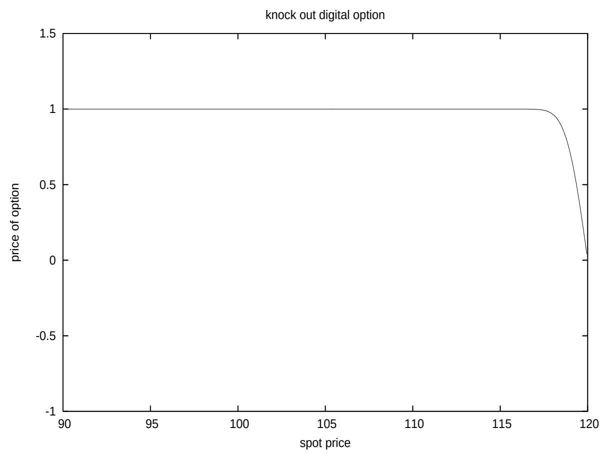

Figure 3.9: price of knock out digital option

We find that the delta of a knock out digital option is given as

$$\Delta = \frac{\partial V_0}{\partial S_0} = \delta_1 - \delta_2 - \delta_3$$

where

$$
\begin{aligned}
\delta_1 &= \exp\left(\frac{-1}{\sqrt{2\pi T}\;\sigma S_0}\right)\exp\left(-\frac{1}{2}\alpha^2\right), \\
\delta_2 &= \exp(-rT)B^{\frac{2r}{\sigma^2}-1}\left(1-\frac{2r}{\sigma^2}\right)S_0^{-\frac{2r}{\sigma^2}}N(\beta) \quad \text{and} \\
\delta_3 &= \exp(-rT)B^{\frac{2r}{\sigma^2}-1}S_0^{-\frac{2r}{\sigma^2}}\left(\frac{1}{\sqrt{2\pi T}\;\sigma}\right)\exp\left(-\frac{1}{2}\beta^2\right),
\end{aligned}
$$

with the substitutions

$$
\begin{aligned}
\alpha &= \frac{\log\left(\frac{B}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad \text{and} \\
\beta &= \frac{-\log\left(\frac{B}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}.
\end{aligned}
$$

Figure 3.10 shows the delta of the digital knock out option for different spot prices. The delta is zero for almost all initial prices and then delta decreases to a negative quantity at the barrier.

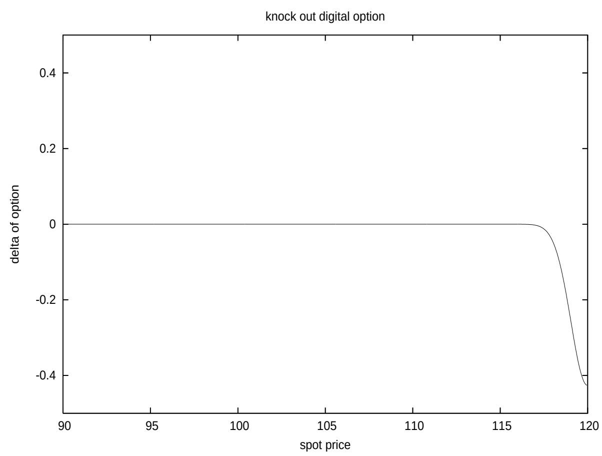

Figure 3.10: delta of knock out digital option

When $S_0 = 119.95$, we get that the delta is -0.426.

The formula for the Malliavin delta for an option is given as

$$\frac{e^{-rT}}{S_0 \sigma T} \mathbb{E}_{\mathbb{Q}} \left(f(S_T) W_T \right). \tag{3.12}$$

In the case of a digital knock out option,

$$f(S_T) = \mathbf{1}_{\{\max S_t | t \in [0,T]\} < B\}}.$$

The Monte Carlo simulation of the formula in equation (3.12) is given in figure 3.11 on page 60. We see that the approximation is very close the analytical prices for about 200 000 to 400 000 simulations and then becomes less than the analytical price; after 900 000 simulations the approximation begins to converge to the analytical price.

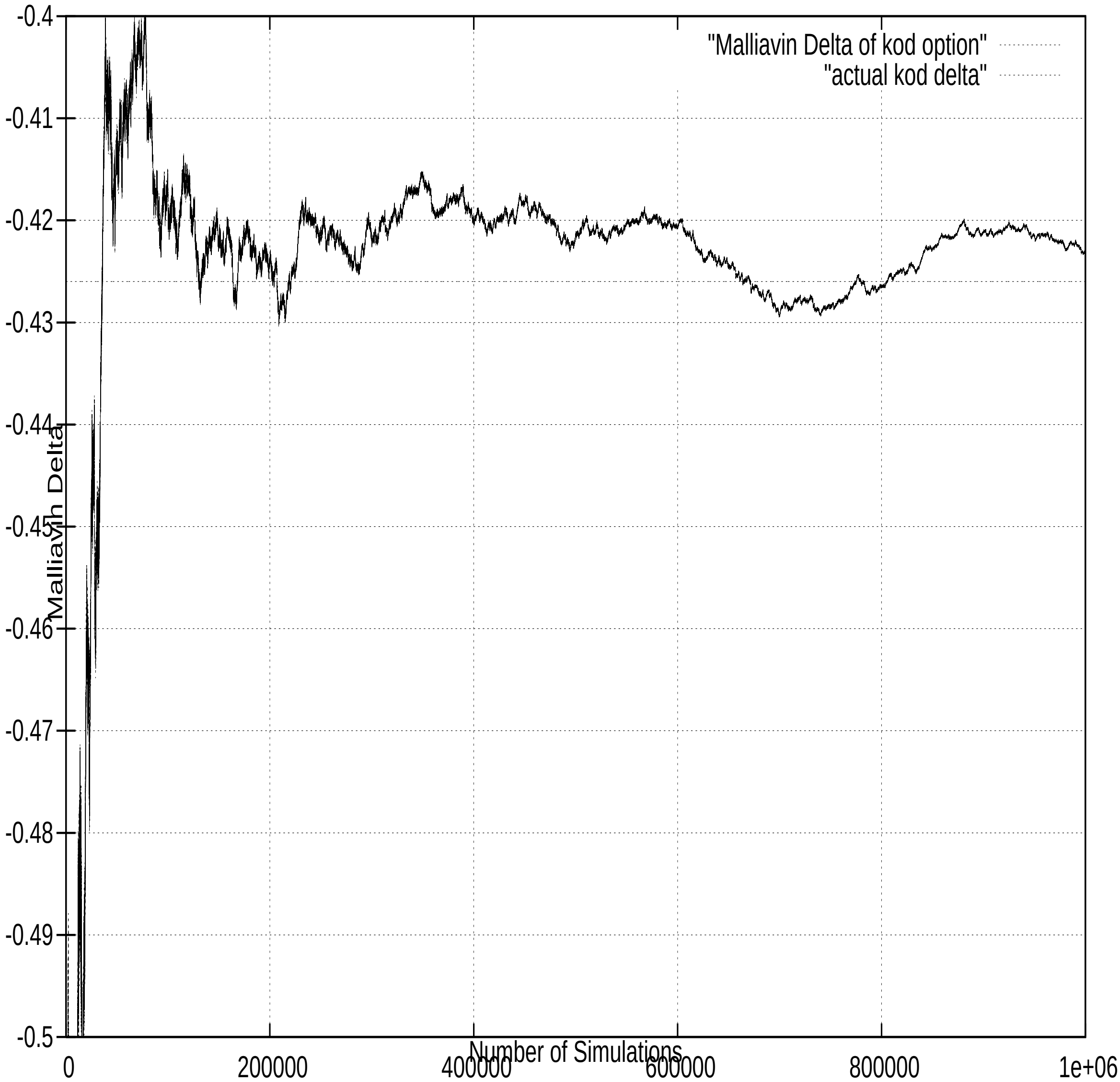

Figure 3.11: Monte Carlo simulation for Malliavin delta of knock out digital option

We find that the gamma of a knock out digital option is given as

$$\Gamma = \frac{\partial^2 V_0}{\partial S_0^2} = \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 + \delta_6$$

where

$$\begin{aligned}
\delta_1 &= \exp(-rT)\left(\frac{1}{\sqrt{2\pi T}\ \sigma S_0^2}\right)\exp\left(-\frac{1}{2}\alpha^2\right), \\
\delta_2 &= +\exp(-rT)\left(\frac{-1}{\sqrt{2\pi T}\ \sigma S_0}\right)\exp\left(-\frac{1}{2}\alpha^2\right)\alpha\left(\frac{-1}{\sqrt{T}\ \sigma S_0}\right), \\
\delta_3 &= -\exp(-rT)B^{\frac{2r}{\sigma^2}-1}\left(1-\frac{2r}{\sigma^2}\right)\left(\frac{-2r}{\sigma^2}\right)S_0^{\frac{-2r}{\sigma^2}-1}N\left(\frac{1}{2}\alpha^2\right), \\
\delta_4 &= -\exp(-rT)B^{\frac{2r}{\sigma^2}-1}\left(1-\frac{2r}{\sigma^2}\right)S_0^{\frac{-2r}{\sigma^2}}N\left(\frac{1}{2}\alpha^2\right), \\
\delta_5 &= -\exp(-rT)B^{\frac{2r}{\sigma^2}-1}\left(\frac{-2r}{\sigma^2}\right)S_0^{\frac{-2r}{\sigma^2}-1}\exp\left(-\frac{1}{2}\beta^2\right)\left(\frac{-1}{\sqrt{2\pi T}\ \sigma S_0}\right), \\
\delta_6 &= -\exp(-rT)B^{\frac{2r}{\sigma^2}-1}S_0^{\frac{-2r}{\sigma^2}}\left(\frac{-1}{\sqrt{2\pi T}\ \sigma S_0}\right)\exp\left(-\frac{1}{2}\beta^2\right)\left(\frac{-1}{\sqrt{T}\ \sigma S_0}\right)\beta,
\end{aligned}$$

with the substitutions

$$\begin{aligned}
\alpha &= \frac{\log\left(\frac{B}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \quad \text{and} \\
\beta &= \frac{-\log\left(\frac{B}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}
\end{aligned}$$

Figure 3.12 on page 62 shows the gamma of the knock out digital option. We see that the zero and then decreases to a minimum near the barrier and then increases to zero at the barrier.

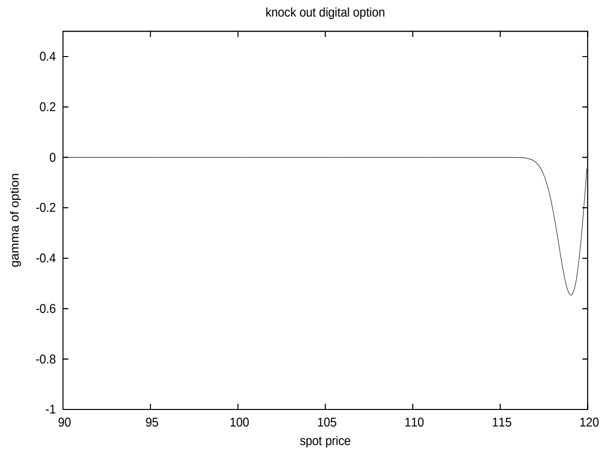

Figure 3.12: gamma of knock out digital option

When $S_0 = 119.95$, we get that the gamma is -0.036. The formula for the Malliavin gamma for an option is given as

$$\frac{e^{-rT}}{S_0^2 \sigma T} \mathbb{E}_{\mathbb{Q}} \left(f(S_T) \left(\frac{W_T^2}{\sigma T} - \frac{1}{\sigma} - W_T \right) \right). \tag{3.13}$$

In the case of a knock out digital option,

$$f(S_T) = \mathbf{1}_{\{\max S_t | t \in [0,T]\} < B\}}.$$

Figure 3.13 on page 63 shows the Monte Carlo simulation for the formula in equation (3.13). We see that the approximation formula in equation (3.13) is good and after 400 000 simulations, the approximation is almost that of the analytical value. After 900 000 simulations, the gamma converges to the analytical value.

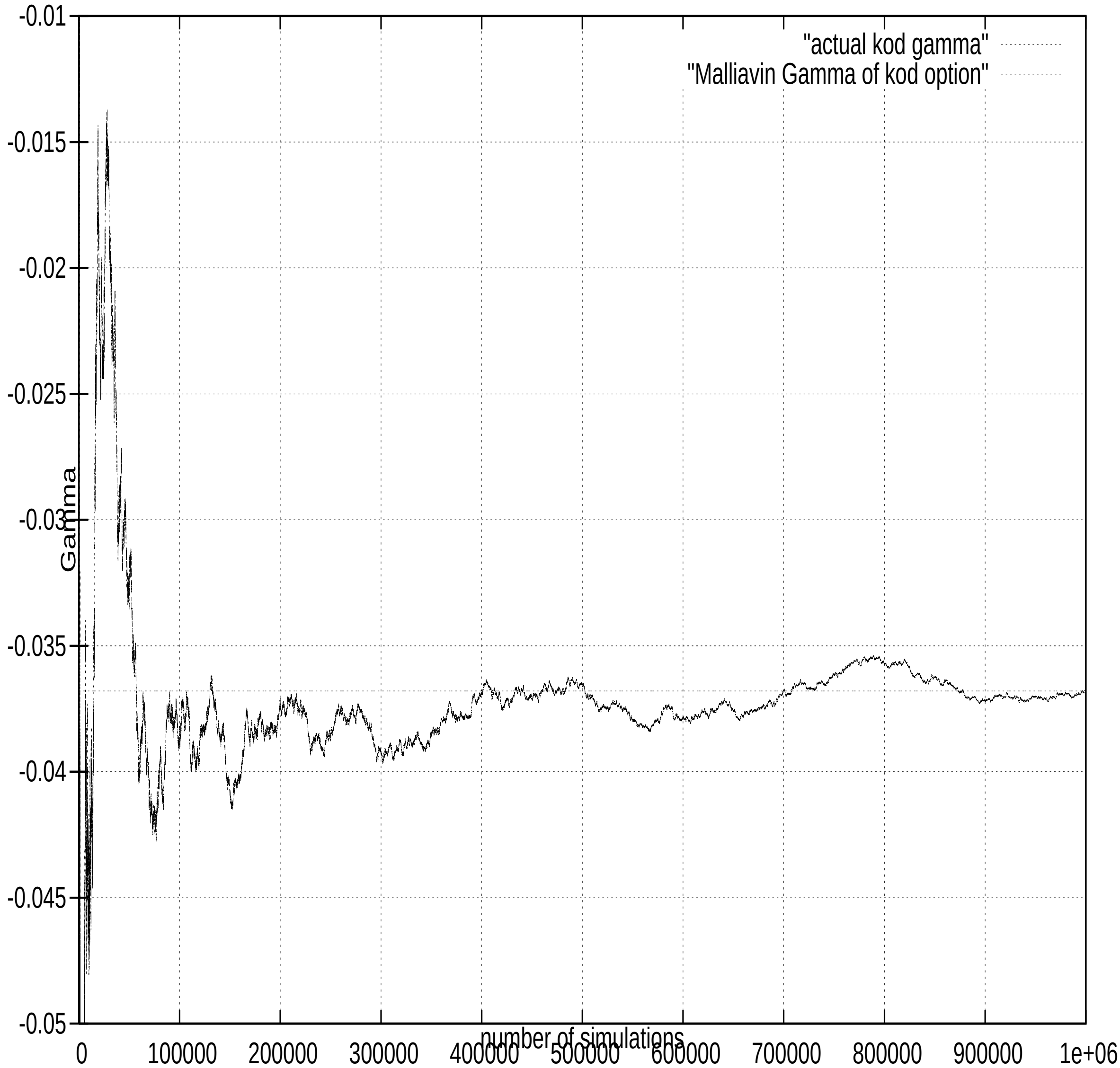

Figure 3.13: Monte Carlo simulation for Malliavin gamma of knock out digital option

3.4.3 Arithmetic Asian Option

An arithmetic Asian option's payoff is a function of the arithmetic average of the stock price

$$\overline{S}_T = \frac{1}{T}\int_0^T S_t \, dt.$$

Hence the payoff can be expressed as

$$f\left(\overline{S}_T\right) = f\left(\frac{1}{T}\int_0^T S_t \, dt\right),$$

where f is the payoff function. The time is expressed as a fraction of 250, as this this the number of business days in a year. That is $T = \frac{i}{250}$ for $0 < i$ and $i \in \mathbb{N}$

For instance, an Asian call option with strike price K, is a derivative security with payoff

$$f(\overline{S}_T) = \left(\frac{1}{T}\int_0^T S_t \, dt - K\right)^+.$$

There is no closed formula for the probability density function of

$$\overline{S}_T = \left(\frac{1}{T}\int_0^T S_t \, dt\right),$$

so there is no explicit formula for the price and thus there is no explicit formula for the greeks since the greeks are derivatives of the price of the option.

As from page 34, we know that the price of the option at time 0 is given by

$$V_0 = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(f\left(\frac{1}{T}\int_0^T S_t \, dt\right)\right) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(f(\overline{S}_T)\right).$$

The delta therefore becomes

$$\Delta = \frac{\partial V_0}{\partial S_0} = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(f'\left(\overline{S}_T\right)\frac{\partial \overline{S}_T}{\partial S_0}\right) = \frac{e^{-rT}}{S_0}\mathbb{E}_{\mathbb{Q}}\left(f'\left(\overline{S}_T\right)\overline{S}_T\right) \tag{3.14}$$

The partial derivative of $\overline{S}_T$ with respect to S_0 is

$$\begin{aligned}\frac{\partial \overline{S}_T}{\partial S_0} &= \frac{\partial}{\partial S_0}\frac{1}{T}\int_0^T S_t\,dt \\ &= \frac{1}{T}\frac{\partial}{\partial S_0}\int_0^T S_t\,dt \\ &= \frac{1}{T}\int_0^T \frac{\partial}{\partial S_0} S_t\,dt\end{aligned}$$

by Leibniz rule ([28]).

And since $S_t = S_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t\right)$,

$$\frac{\partial}{\partial S_0} S_t = \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) = \frac{S_t}{S_0}.$$

Thus

$$\frac{\partial \overline{S}_T}{\partial S_0} = \frac{1}{TS_0}\int_0^T S_t\,dt = \frac{1}{S_0}\overline{S}_T.$$

But by equation (3.2) on page 34,

$$\Delta = \frac{\partial V_0}{\partial S_0} = \mathbb{E}_{\mathbb{Q}}\left(e^{-rT} f'\left(\overline{S}_T\right)\frac{\partial \overline{S}_T}{\partial S_0}\right) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left(f(\overline{S}_T)\mathcal{K}\left(\overline{S}_T, \frac{d\overline{S}_T}{dS_0}\right)\right) \tag{3.15}$$

where

$$\begin{aligned}\mathcal{K}\left(\overline{S}_T, \frac{d\overline{S}_T}{dS_0}\right) &= \mathcal{K}\left(\overline{S}_T, \frac{\overline{S}_T}{S_0}\right) \\ &= \frac{1}{S_0}\delta\left(\overline{S}_T h_t (D_\gamma \overline{S}_T)^{-1}\right) \\ &= \frac{1}{S_0}\mathcal{K}\left(\overline{S}_T, \overline{S}_T\right).\end{aligned}$$

From this, we then have that

$$\Delta = \frac{e^{-rT}}{S_0}\mathbb{E}_{\mathbb{Q}}\left(f(\overline{S}_T)\mathcal{K}\left(\overline{S}_T, \overline{S}_T\right)\right) \tag{3.16}$$

where

$$\mathcal{K}\left(\overline{S}_T, \overline{S}_T\right) = \delta\left(\overline{S}_T h_t (D_\gamma \overline{S}_T)^{-1}\right).$$

We first find the Malliavin derivative of $\overline{S}_T$. This evaluates to

$$D_t\overline{S}_T = D_t\frac{1}{T}\int_0^T S_r\,dr = \frac{1}{T}\int_0^T D_t S_r\,dr = \frac{\sigma}{T}\int_t^T S_r\,dr,$$

using the fact that $D_t S_r = \sigma S_r$ as seen from page 36 and the fact that

$$(D_t F_s)(\omega) = 0$$

for all $\omega \in \Omega$ if F_s, for $s \in [0, T]$, is adapted to the natural filtration generated by the Brownian motion, and $t > s$. Recall that this was shown in proposition 2.1 on page 19. Because of this, the integral is non-zero for $r \in (t, T]$ and this is reflected in the limits of the integral.

For the Skorohod integral

$$\delta\left(\overline{S}_T h_t (D_\gamma \overline{S}_T)^{-1}\right),$$

we use $h_t = S_t$ for $t \in [0, T]$, in

$$D_\gamma \overline{S}_T = \frac{1}{T} \int_0^T (D_t \overline{S}_T) S_t \, dt$$

and we see that

$$\begin{aligned}
\delta\left(\overline{S}_T h_t (D_\gamma \overline{S}_T)^{-1}\right) &= \delta\left(\frac{\overline{S}_T h_t}{D_\gamma \overline{S}_T}\right) \\
&= \delta\left(\frac{\left(\frac{1}{T}\int_0^T S_t \, dt\right) S_t}{\int_0^T (D_t \overline{S}_T) S_t \, dt}\right) \\
&= \delta\left(\frac{\left(\frac{1}{T}\int_0^T S_t \, dt\right) S_t}{\int_0^T \left(\frac{\sigma}{T}\int_t^T S_r \, dr\right) S_t \, dt}\right).
\end{aligned}$$

We need to evaluate the integral

$$\int_0^T \left(\frac{\sigma}{T}\int_t^T S_r \, dr\right) S_t \, dt = \frac{\sigma}{T}\int_0^T \left(\int_t^T S_r \, dr\right) S_t \, dt$$

in the denominator of the Skorohod integral. We see that if

$$\tau = \int_t^T S_r \, dr,$$

then

$$d\tau = -S_t \, dt$$

by the Leibniz rule ([28]). Using these substitutions we get that

$$\begin{aligned}\int \left(\int_t^T S_r\,dr\right) S_t\,dt &= \int -\tau\,d\tau \\ &= -\frac{1}{2}\tau^2 \\ &= -\frac{1}{2}\left(\int_t^T S_r\,dr\right)^2.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^T \left(\int_t^T S_r\,dr\right) S_t\,dt &= -\frac{1}{2}\left(\int_t^T S_r\,dr\right)^2\Bigg|_0^T \\ &= -(0-\frac{1}{2}\left(\int_0^T S_r\,dr\right)^2 \\ &= \frac{1}{2}\left(\int_0^T S_r\,dr\right)^2.\end{aligned}$$

From this we get that

$$\begin{aligned}\delta\left(\frac{\left(\frac{1}{T}\int_0^T S_t\,dt\right) S_t}{\int_0^T \left(\frac{\sigma}{T}\int_t^T S_r\,dr\right) S_t\,dt}\right) &= \delta\left(\frac{\left(\frac{1}{T}\int_0^T S_t\,dt\right) S_t}{\frac{\sigma}{2T}\left(\int_0^T S_t\,dt\right)^2}\right) \\ &= \frac{2}{\sigma}\delta\left(\frac{S_t}{\int_0^T S_t\,dt}\right). \qquad (3.17)\end{aligned}$$

Next we need to evaluate

$$\delta\left(\frac{S_t}{\int_0^T S_t\,dt}\right).$$

For this we use theorem 2.11 on page 30. We recall it for convenience.

Let $F \in \mathbb{D}_{1,2}([0,T]\times\Omega)$ and $H \in L^2([0,T]\times\Omega)$. Then

$$\int_0^T FH\,\delta W_t = F\int_0^T H_t\,\delta W_t - \int_0^T (D_tF)H_t\,dt$$

for all $(t,\omega) \in [0,T]\times\Omega$.

We set $F = \frac{1}{\int_0^T S_t\,dt}$ and $H = S_t$. Now we see that since the stock price process S_t is adapted to the filtration generated by the Brownian motion,

$$\int_0^T S_t\,\delta W_t = \int_0^T S_t\,dW_t$$

as by theorem 2.10 on page 28. Next, to evaluate

$$D_t \frac{1}{\int_0^T S_t \, dt},$$

we use the chain rule with $f(x) = \frac{1}{x}$. Hence

$$D_t \frac{1}{\int_0^T S_t \, dt} = -\frac{1}{\left(\int_0^T S_t \, dt\right)^2} D_t \int_0^T S_t \, dt.$$

After this we evaluate $D_t \int_0^T S_t \, dt$. We find this to be

$$\begin{aligned} D_t \int_0^T S_t \, dt &= \int_0^T D_t S_t \, dt \\ &= \int_t^T \sigma S_t \, dt \\ &= \sigma \int_t^T S_t \, dt \end{aligned}$$

as explained on page 66. Thus applying theorem 2.11, we get that

$$\begin{aligned} \delta\left(\frac{S_t}{\int_0^T S_t \, dt}\right) &= \frac{\int_0^T S_t \, dW_t}{\int_0^T S_t \, dt} + \frac{\int_0^T S_t \left(\int_t^T \sigma S_r\right) \, dt}{\left(\int_0^T S_t \, dt\right)^2} \\ &= \frac{\int_0^T S_t \, dW_t}{\int_0^T S_t \, dt} + \frac{\sigma}{2} \frac{\left(\int_0^T S_t \, dt\right)^2}{\left(\int_0^T S_t \, dt\right)^2} \\ &= \frac{\int_0^T S_t \, dW_t}{\int_0^T S_t \, dt} + \frac{\sigma}{2}. \end{aligned}$$

Recall that

$$\int_0^T S_t \left(\int_t^T \sigma S_r\right) \, dt$$

was worked out on page 67.

Finally from equation (3.17) we get that

$$\frac{2}{\sigma} \delta\left(\frac{S_t}{\int_0^T S_t \, dt}\right) = \frac{2}{\sigma} \frac{\int_0^T S_t \, dW_t}{\int_0^T S_t \, dt} + 1. \tag{3.18}$$

We can evaluate this expression more. The Itô integral

$$\int_0^T S_t \, dW_t$$

can be worked out by applying the Itô formula on $\dfrac{S_t}{\sigma}$. Thus

$$d\left(\frac{S_t}{\sigma}\right) = \frac{1}{\sigma}\left(rS_t \, dt + \sigma S_t dW_t\right).$$

In integral form,

$$\frac{S_T}{\sigma} - \frac{S_0}{\sigma} = \frac{r}{\sigma}\int_0^T S_t \, dt + \int_0^T S_t \, dW_t.$$

Hence

$$\begin{aligned}
\int_0^T S_t \, dW_t &= \frac{S_T}{\sigma} - \frac{S_0}{\sigma} - \frac{r}{\sigma}\int_0^T S_t \, dt \\
&= \frac{1}{\sigma}\left(S_T - S_0 - r\int_0^T S_t \, dt\right).
\end{aligned}$$

And so,

$$\begin{aligned}
\frac{2}{\sigma}\frac{\int_0^T S_t \, dW_t}{\int_0^T S_t \, dt} &= \frac{2}{\sigma}\frac{\frac{1}{\sigma}\left(S_T - S_0 - r\int_0^T S_t \, dt\right)}{\int_0^T S_t \, dt} \\
&= \frac{2(S_T - S_0)}{\sigma^2 \int_0^T S_t \, dt} - \frac{2r}{\sigma^2}.
\end{aligned}$$

Putting all of this together, equation (3.18) becomes

$$\begin{aligned}
\frac{2}{\sigma}\frac{\int_0^T S_t \, dW_t}{\int_0^T S_t \, dt} + 1 &= \frac{2(S_T - S_0)}{\sigma^2 \int_0^T S_t \, dt} - \frac{2r}{\sigma^2} + 1 \\
&= \frac{2}{\sigma^2}\left(\frac{S_T - S_0}{\int_0^T S_t \, dt} - m\right) \\
&= \frac{2}{\sigma^2}\left(\frac{S_T - S_0}{T\left(\frac{1}{T}\int_0^T S_t \, dt\right)} - m\right) \\
&= \frac{2}{\sigma^2}\left(\frac{S_T - S_0}{T\overline{S}_T} - m\right)
\end{aligned}$$

where $m = r - \dfrac{\sigma^2}{2}$.

Finally, from equation 3.16, the delta is

$$\Delta = \frac{2e^{-rT}}{S_0\sigma^2}\mathbb{E}_{\mathbb{Q}}\left(f(\overline{S}_T)\left(\frac{S_T - S_0}{T\overline{S}_T} - m\right)\right).$$

We can then use Monte Carlo simulations to evaluate the delta of any arithmetic Asian option. Using the parameters $K = S_0 = 100$, $\sigma = 0.2$, $r = 0.1$ and 250 trading days and with $f(\overline{S}_T) = (\overline{S}_T - K)^+$, the payoff function for a call option, we find that the delta of an arithmetic Asian call option with the parameters described above is 0.654. Figure 3.14 on page 71 shows the Monte Carlo simulations for this particular option.

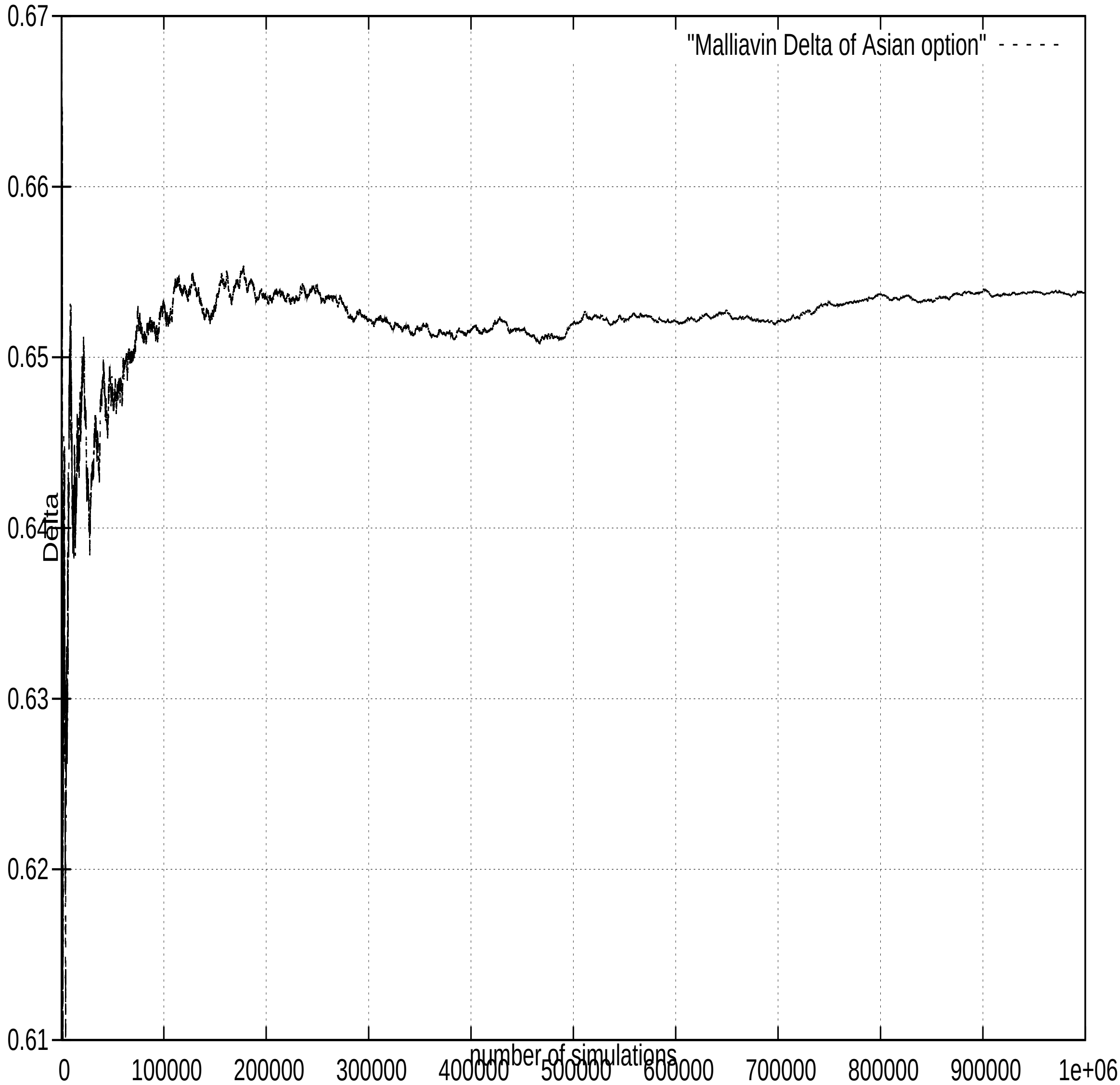

Figure 3.14: Delta of Asian Call option

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