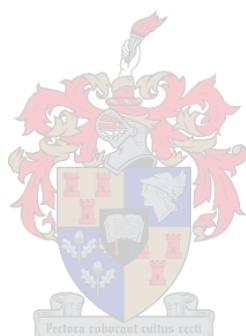


Geometric actions of the absolute Galois group

Paul Joubert



Thesis presented in partial fulfilment of the requirements for the degree of Master of
Science at Stellenbosch University.

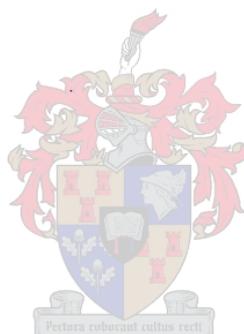
Supervisor: Dr. Florian Breuer
April 2006

Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

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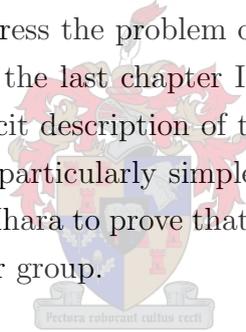
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Abstract

This thesis gives an introduction to some of the ideas originating from A. Grothendieck's 1984 manuscript *Esquisse d'un programme*. Most of these ideas are related to a new geometric approach to studying the absolute Galois group over the rationals by considering its action on certain geometric objects such as *dessins d'enfants* (called *stick figures* in this thesis) and the fundamental groups of certain moduli spaces of curves.

I start by defining stick figures and explaining the connection between these innocent combinatorial objects and the absolute Galois group. I then proceed to give some background on moduli spaces. This involves describing how Teichmüller spaces and mapping class groups can be used to address the problem of counting the possible complex structures on a compact surface. In the last chapter I show how this relates to the absolute Galois group by giving an explicit description of the action of the absolute Galois group on the fundamental group of a particularly simple moduli space. I end by showing how this description was used by Y. Ihara to prove that the absolute Galois group is contained in the Grothendieck-Teichmüller group.



Opsomming

Hierdie tesis gee 'n inleiding tot sommige van die idees beskryf deur A. Grothendieck in 1984 (*Esquisse d'un programme*). Die meeste van hierdie idees hou verband met 'n nuwe meetkundige benadering tot die studie van die absolute Galois groep oor die rasionele getalle. Die benadering maak gebruik van die aksie van hierdie groep op sekere meetkundige voorwerpe soos *dessins d'enfants* en die fundamentele groepe van sekere modulus ruimtes van kurwes.

Ek begin deur *dessins d'enfants* te definieer en die verband tussen hierdie onskuldige kombinatoriese voorwerpe en die absolute Galois groep te verduidelik. Hierna gee ek 'n bietjie agtergrond oor modulus ruimtes. Dit behels 'n beskrywing van hoe Teichmüller ruimtes en afbeeldingsklasgroepe gebruik kan word om die moontlike komplekse strukture op 'n kompakte oppervlak te tel. In die laaste hoofstuk wys ek hoe dit inskakel met die absolute Galois groep deur 'n eksplisiete beskrywing te gee van die aksie van hierdie groep op die fundamentele groep van 'n eenvoudige modulus ruimte. Ek sluit af deur te wys hoe hierdie beskrywing gebruik is deur Y. Ihara om te bewys dat die absolute Galois groep bevat is in die Grothendieck-Teichmüller groep.

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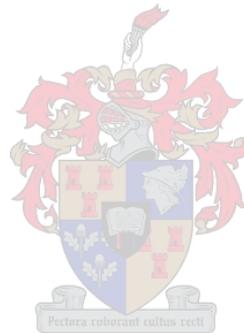
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Chapter 1

Introduction

The material described in this thesis can be divided into two parts, both of which were originally introduced by A. Grothendieck in his (then) unpublished manuscript *Esquisse d'un programme* ([Gro97]) in 1984.

The focus of the first part is on certain objects called *dessins d'enfants* (i.e. children's drawings) by Grothendieck. Although this is the term most widely used in the (mainly French) literature, there is no fixed term in English yet: we will use the phrase *stick figures* in this thesis.

One of the appealing things about stick figures is that they can be described in very different ways. The simplest way is to define them as bipartite graphs embedded into a compact surface such as the sphere. The example in Figure 1.1 makes it clear why they are being called stick figures. We will show that the same information can also be encoded group-theoretically by giving a transitive subgroup of S_n , the symmetric group on the n edges of the stick-figure. Thus the stick figure in Figure 1.1 is in some sense equivalent to the subgroup of S_8 generated by the permutations $(123)(4)(578)(6)$ and $(12)(3456)(7)(8)$.

Stick figures can also be described by giving finite coverings $f : X \rightarrow \mathbb{P}^1(\mathbb{C})$ of the Riemann sphere which are unramified outside the three points $0, 1$ and ∞ . Given such a covering, the pre-image of the unit interval $[0, 1]$ is a stick figure on the compact Riemann

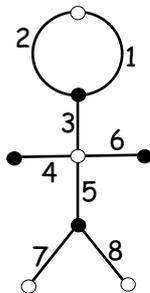


Figure 1.1: *A stick figure drawn on the sphere*

surface X , and conversely every stick figure can be obtained in this way.

Compact Riemann surfaces correspond to algebraic curves defined over \mathbb{C} . By a theorem of Belyi ([Bel80]) it turns out that the compact Riemann surfaces on which one can draw a stick figure are precisely those for which the corresponding algebraic curve is in fact defined over $\overline{\mathbb{Q}} \subset \mathbb{C}$. The natural action of the absolute Galois group $G_{\mathbb{Q}}$ on such curves thus translates into an action of $G_{\mathbb{Q}}$ on the set of stick figures. It turns out that this action is faithful, even when restricted to the genus 0 stick figures (i.e. those drawn on the sphere). Thus an element of $G_{\mathbb{Q}}$ can (in principle) be described by giving its action on every genus 0 stick figure. This gives a first example of viewing $G_{\mathbb{Q}}$ as a subgroup of the automorphism group of an object (the set of stick figures) which is defined without any reference to permutations of roots of equations. Stick figures are the subject of chapter 2.

The rest of the thesis describes the first steps in the direction of another (related) area arising from Grothendieck's *Esquisse*, called Grothendieck-Teichmüller theory. Our goal here is again to show how elements of $G_{\mathbb{Q}}$ can be identified with automorphisms of objects which are defined in a more geometric manner. A starting point is the fact that $G_{\mathbb{Q}}$ can be considered as a subgroup of $\text{Aut}(\hat{F}_2)$ ([Bel80]), where \hat{F}_2 is the profinite completion of the free group on two generators. The connection with the section on stick figures comes from the fact that \hat{F}_2 is the (algebraic) fundamental group of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$.

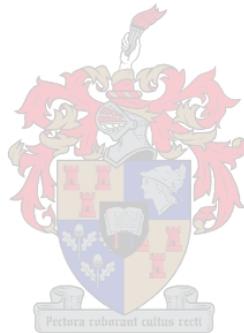
Grothendieck showed that the way to generalize this was to note that $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ is the moduli space (called $\mathcal{M}_{0,4}$) of possible complex structures on the sphere with 4 ordered marked points. So in general one could consider $\mathcal{M}_{g,n}$ for different values of the genus g and n . It turns out that there is a canonical outer action of $G_{\mathbb{Q}}$ on the fundamental groups of these moduli spaces (considered as algebraic stacks which can be shown to be defined over \mathbb{Q}) which generalizes the (outer) action of $G_{\mathbb{Q}}$ on \hat{F}_2 , the algebraic fundamental group of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. Chapter 3 gives some background on moduli spaces. Grothendieck's idea was to join together these moduli spaces using geometric mappings (such as embedding a moduli space into the divisor at infinity of a moduli space of higher dimension) and considering the corresponding *tower* of fundamental groups (or groupoids in his case) with the induced morphisms between them. (Outer) automorphisms of this tower are (outer) automorphisms of the respective groups respecting the morphisms between them. Intuitively, as one adds more moduli spaces and more morphisms between them, the automorphism group of the tower should become smaller. Yet this group always contains $G_{\mathbb{Q}}$ (because $G_{\mathbb{Q}}$ acts faithfully on the fundamental groups of the moduli spaces), and thus the idea is to try and construct a tower such that the automorphism group is precisely $G_{\mathbb{Q}}$.

The elements of the automorphism group of such a tower which come from elements of $G_{\mathbb{Q}}$ can be parametrized by pairs (λ, f) from $\hat{F}'_2 \times \hat{\mathbb{Z}}^{\times}$. In a different context related to quasi-Hopf algebras, Drinfeld ([Dri90]) defined a group called the Grothendieck-

Teichmüller group, \widehat{GT} , whose elements form a subset of $\hat{F}'_2 \times \hat{\mathbb{Z}}^\times$. Ihara ([Iha94]) showed that $G_{\mathbb{Q}}$ is contained in \widehat{GT} , and his proof will be the subject of chapter 4.

The importance of \widehat{GT} can be seen from a result by L. Schneps and P. Lochak ([SL94]), who showed that the automorphism group of the tower of the fundamental groups of genus 0 moduli spaces (linked by subsurface inclusion maps as mentioned before) is exactly \widehat{GT} . In fact, the subtower consisting just of the moduli spaces of dimension 1 and 2 already has \widehat{GT} as its automorphism group.

It is an open question whether $G_{\mathbb{Q}}$ is in fact equal to \widehat{GT} . There are however many variants of \widehat{GT} , defined for example to take into account what happens when moduli spaces of positive genus are added to the tower (see [NS00]). These variants lie between $G_{\mathbb{Q}}$ and \widehat{GT} , but it is not known whether the inclusions are strict.



Chapter 2

Stick figures

2.1 Equivalent definitions of stick figures

Intuitively, stick figures are bipartite graphs drawn on compact surfaces. In this section we will make this definition more precise, and show how the same objects can be described in many different ways.

2.1.1 Topological definition

In this section we will define a stick figure as a purely topological structure. To do so, we start with a 2-dimensional, compact, connected, oriented surface X . Note for future reference that these are precisely the types of surfaces that can be equipped with the analytical structure of a compact Riemann surface.

Now we want to give X a 2-dimensional cell complex structure. This involves partitioning X into three parts, the first (X_V) being a finite discrete set of points (0-cells, or vertices), the second (X_E) homeomorphic to a finite disjoint union of open intervals (1-cells, or edges) and the last (X_F) homeomorphic to a finite disjoint union of open disks (2-cells, or faces).

Then some consideration will show that the closure of an edge must be homeomorphic to either a circle or a closed interval, and contain either one or two vertices corresponding to these two cases. Recall that a graph is defined by a set of vertices and a set of its subsets, each subset having cardinality one or two. Thus there is a natural way to define a graph G_X associated to X with vertices being the elements of X_V , edges the connected components of X_E and the vertices at the end of an edge being the ones contained in its closure.

G_X is required to be a connected bipartite graph. Being bipartite means that we can partition the set of vertices X_V into two sets such that every edge lies between a vertex from the one set and a vertex from the other set. In other words, there exists a function $\mu : X_V \rightarrow \{0, 1\}$ assuming different values on adjacent vertices. This implies for example

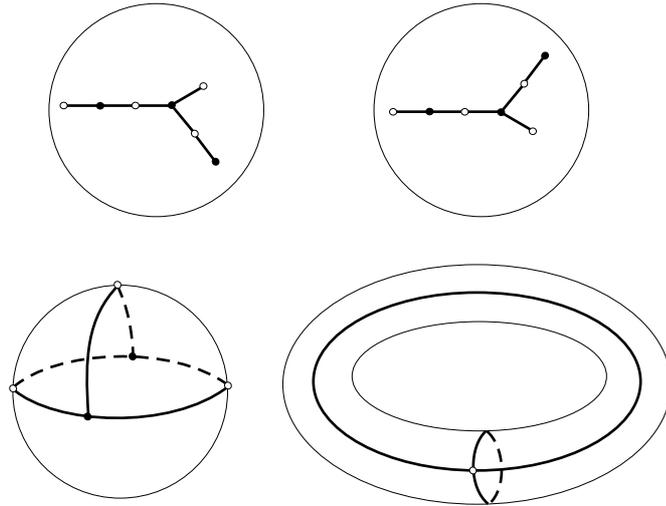


Figure 2.1: *Some examples of stick figures*

that G_X has no loops. We will say a vertex is of *type 0* or *1*.

Definition 2.1.1. A *stick figure* \mathcal{C} is defined by the data $(X = X_V \cup X_E \cup X_F, \mu)$ satisfying the above properties. G_X is referred to as the *associated graph* and X the *underlying surface*. Let \mathcal{C}' be another stick figure with data $(X' = X'_V \cup X'_E \cup X'_F, \mu')$. We say \mathcal{C} is isomorphic to \mathcal{C}' if there exists an orientation preserving homeomorphism $\varphi : X \rightarrow X'$ which induces homeomorphisms $X_V \cong X'_V$, $X_E \cong X'_E$ and $X_F \cong X'_F$, and which respects the bipartite structure by satisfying $\mu = \mu' \circ \varphi$ on X_V .

We will sometimes adopt the convention of speaking of a stick figure as a graph, implying of course the associated graph. It is important to note however that although two isomorphic stick figures necessarily have isomorphic associated graphs, the converse is not true since the way the graph is embedded into the surface is also important. In fact, as we will see in the next section, a stick figure can be completely described by giving its graph and a cyclic ordering of the edges at each vertex corresponding to their ordering on the underlying surface.

Figure 2.1 illustrates some examples of stick figures. Notice that the two stick figures at the top are isomorphic as graphs but not as stick figures.

We will see later that the above description corresponds to taking the pre-image of the unit interval under a function $f : X \rightarrow \mathbb{P}^1(\mathbb{C})$. It will sometimes be more convenient however to take the pre-image of the real line, leading to a slightly different description of a stick figure as a triangulation of a surface.

More precisely, consider a stick figure \mathcal{C} as described above. Enlarge the set of vertices X_V by adding one point in each face. Then extend the set X_E by adding an edge joining each new vertex to each old vertex bordering on the face in which the new vertex was placed. Also extend the function on the vertices to $\mu : X_V \rightarrow \{0, 1, \infty\}$ by letting it take

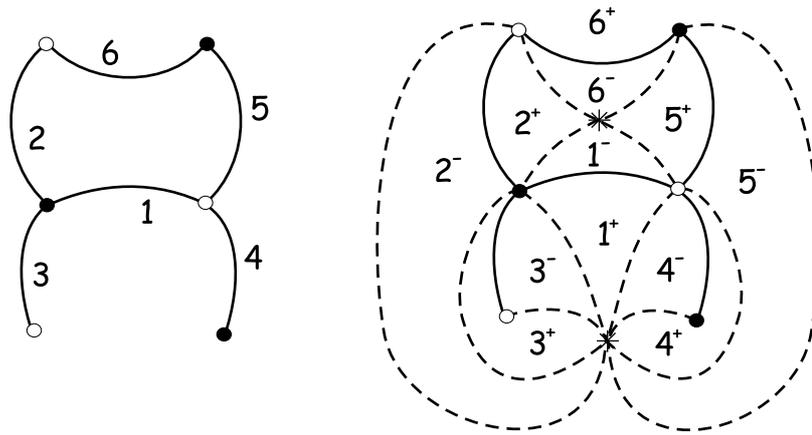


Figure 2.2: A stick figure and its associated triangulation

the value ∞ on the new vertices.

On the right, Figure 2.2 shows the new vertices as stars and the new edges as dotted lines. Vertices of type 0 and 1 are indicated by white and black circles respectively.

If n was the number of edges of the original stick figure, then there will now be $3n$ edges and $2n$ faces. Each face will be bounded by exactly three edges, hence we have a triangulation of the surface X . The vertices have already been partitioned into three types by means of the values assigned to them by μ . Similarly, we can divide the edges into three types according to the types of vertices linked by the edge, namely $\overline{01}$, $\overline{0\infty}$ and $\overline{1\infty}$. Finally, the faces, or triangles, can be divided into two types according to whether the vertices taken in an anticlockwise direction are arranged $(0, 1, \infty)$, or $(0, \infty, 1)$. We call the triangles positive or negative corresponding to these two cases. Two triangles are said to have the same parity if and only if they are both positive or both negative. Clearly two triangles can only share an edge if they are of different parity.

To go in the reverse direction is now easy: given a triangulation as above, we simply disregard the vertices of type ∞ and the edges of type $\overline{0\infty}$ and $\overline{1\infty}$ to get the original stick figure. To summarize, we have shown the following:

Proposition 2.1.2. *A stick figure can also be described by giving a cell complex $X = X_V \cup X_E \cup X_F$ where every face is bounded by exactly three edges, and a function $\mu : X_E \rightarrow \{0, 1, \infty\}$ assuming different values on adjacent vertices. An isomorphism is described exactly as before.*

We conclude this section with a construction which will be useful in the next section. Consider a usual stick figure where the n edges have been labelled from 1 to n . Taking the corresponding triangulation, we have a labelling of the edges of type $\overline{01}$. Since each triangle has exactly one such edge on its boundary, and each such edge is on the boundary of exactly one positive and one negative triangle, we can label the $2n$ triangles as

$\{1^+, 1^-, 2^+, 2^-, \dots, n^+, n^-\}$, where for example triangle k^+ is the positive triangle having edge k on its boundary. This is illustrated in Figure 2.2.

2.1.2 Group-theoretical approach

All the above information can be codified in a visually less appealing, yet more concise form using groups. Let $F_2 := \langle x, y, z \mid xyz = 1 \rangle$ be the free group on two generators x and y . Let F_2 act on the set of edges E of the stick figure as follows: x permutes the edges cyclically around each vertex of type 0 in an anticlockwise direction, while y does the same around the vertices of type 1. For example, the permutations induced on the edges of the stick figure in Figure 2.2 would be $(145)(26)(3)$ and $(123)(4)(56)$ by x and y respectively.

Since F_2 is acting on a set E with n elements, we get a group homomorphism from F_2 to S_n , the symmetric group on n elements. The image of F_2 under this map is a subgroup of S_n generated by the permutations induced by x and y (denoted by g_0 and g_1 respectively). Note that the action of F_2 is transitive because the graph of the stick figure is connected. Then we have the following alternative description of stick figures:

Proposition 2.1.3. *There is a bijection between isomorphism classes of stick figures and equivalence classes of pairs (g_0, g_1) with $g_0, g_1 \in S_n$ for some n , such that $\langle g_0, g_1 \rangle$ is a transitive subgroup of S_n . Two pairs (g_0, g_1) and (g'_0, g'_1) are considered equivalent if they are simultaneously conjugate in S_n , i.e. if there is some h in S_n such that $g'_i = h^{-1}g_i h$ for $i = 0, 1$.*

Proof. For the one direction it was shown above how to associate a pair (g_0, g_1) with each stick figure. In the process it is necessary to label the edges from 1 to n and a different labelling would give another pair which is conjugate to the original one by the permutation h describing the change in labelling.

For the other direction of the proof it is necessary to reconstruct the stick figure from a given pair of permutations (g_0, g_1) where $g_0, g_1 \in S_n$. This will be done by joining together $2n$ triangles along their edges to give a triangulation of a surface which describes a stick figure as explained in the previous section.

Label the triangles as $1^+, 1^-, 2^+, 2^-, \dots, n^+, n^-$ and label the three vertices of each as 0, 1 and ∞ in such a way that the vertices of the positive (resp. negative) triangles are arranged as described in the previous section. Use (k^+, \overline{ij}) to refer to the edge of triangle k^+ lying between vertices i and j . Now join the triangles by identifying edges as follows (while orientating the edges to preserve vertex types):

$$\begin{aligned}(k^+, \overline{0\infty}) &\sim (g_0(k)^-, \overline{0\infty}) \\ (k^+, \overline{01}) &\sim (k^-, \overline{01}) \\ (g_1(k)^+, \overline{1\infty}) &\sim (k^-, \overline{1\infty})\end{aligned}$$

It is clear from this description that each triangle is joined to exactly three other triangles, all with the opposite parity. Thus we can extend the local orientation of each triangle to a global orientation, so we indeed get a compact, oriented surface. Partitioning the surface into vertices, edges and faces, and defining the natural vertex function μ gives a triangulation description of a stick figure as in Proposition 2.1.2.

It only remains to be checked that the permutations of the edges of this stick figure correspond to the g_0 and g_1 we started with. This can be verified using the three rules for joining edges given above. Indeed, if we start with an edge of type $\overline{01}$ labelled k and try to determine the cyclic ordering of the edges around its 0-vertex, we find that we need to apply the first two rules alternately, giving the sequence of edges $k, g_0(k), g_0(g_0(k)), \dots, k$ as desired. The case for g_1 is similar, completing the proof. \square

Given a pair (g_0, g_1) , it is possible to deduce many properties of the stick figure without having to go through the above process to reconstruct it. If we denote the number of cycles of g_i by $|g_i|$, then $|g_0|$ corresponds to the number of vertices of type 0 and the length of each cycle corresponds to the number of edges emanating from that vertex. A similar statement holds for g_1 and if we define g_∞ as $(g_0g_1)^{-1}$, then $|g_\infty|$ corresponds to the number of faces of the stick figure. The length of each cycle of g_∞ is equal to half the number of edges surrounding the corresponding face. Note that some edges can appear twice on the boundary of a face (for example edges 3 and 4 in Figure 2.2). Passing to the corresponding triangulation, $|g_\infty|$ corresponds to the number of vertices of type ∞ . All these properties can be verified for Figure 2.2.

Let $v = |g_0| + |g_1| + |g_\infty|$ denote the total number of vertices of the triangulation, $e = 3n$ the number of edges and $f = 2n$ the number of faces. Then we can calculate the genus g using Euler's formula $2 - 2g = v - e + f$. For example, corresponding to Figure 2.2 we have $g_0 = (145)(26)(3)$, $g_1 = (123)(4)(56)$ and $g_\infty = (1364)(25)$, hence $2 - 2g = (3 + 3 + 2) - 3(6) + 2(6)$, i.e. $g = 0$, which is what it should be since the underlying surface of this specific stick figure is a sphere.

This group-theoretic description will be useful later on, but for our immediate purposes we need the following:

Proposition 2.1.4. *There is a bijection between isomorphism classes of stick figures and conjugacy classes of subgroups of finite index in F_2 .*

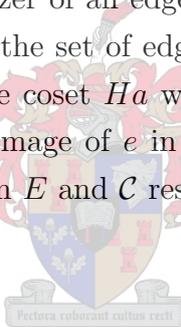
Proof. To associate a conjugacy class of subgroups of F_2 to a given stick figure, consider again the action of F_2 on E , the set of n edges of the stick figure, as described in the

beginning of this section. This action is considered to be a right action. Now fix a specific edge e , and let $H \subset F_2$ be the stabilizer of e , i.e. H is the set of all elements in F_2 which act trivially on e . Since F_2 acts transitively on E , H is a subgroup of index n of F_2 . This is because two elements of F_2 lie in the same coset of H if and only if they both act on e by taking it to the same edge e' . So the number of cosets of H is equal to the number of ways an element of F_2 can act on e , that is, to the number of edges n .

Finally, note that choosing a different edge e gives a conjugate subgroup of F_2 , again using the fact that F_2 acts transitively.

For the converse we are given a subgroup H of index n in F_2 . Let \mathcal{C} be the set of cosets of H . F_2 now acts on \mathcal{C} by taking $a \in F_2$ and $Hb \in \mathcal{C}$ to $Hba \in \mathcal{C}$. This action is clearly transitive, so by denoting the images of the generators x and y of F_2 in \mathcal{C} as g_0 and g_1 , we can appeal to Theorem 2.1.3 to reconstruct a stick figure. It only remains to verify that starting with a subgroup of F_2 corresponding to a specific stick figure, this procedure will yield the same stick figure again, essentially because F_2 acts on the set of cosets in exactly the same way as on the set of edges. More precisely, suppose the H that we started with is the stabilizer of an edge e of some specific stick figure. Then we can establish a bijection between the set of edges E of the stick figure and the cosets \mathcal{C} by associating an edge e' with the coset Ha where $a \in F_2$ takes e to e' and conversely associating a coset Ha with the image of e in E under the action of a . This bijection commutes with the action of F_2 on E and \mathcal{C} respectively, thus completing the proof. \square

2.1.3 Covering spaces



Let $\mathbb{P}^1(\mathbb{C})$ be the Riemann sphere. An object of primary interest throughout this thesis will be the Riemann surface obtained by removing the three points $\{0, 1, \infty\}$, that is $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. Note that for our present purposes, any three distinct points on the sphere would suffice, although later it will be necessary that these three points are actually rational.

Before stating this section's theorem, let us first introduce some notation. Let π_1 denote the fundamental group of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, generated by loops l_0 , l_1 and l_∞ around each of the missing points, with $l_0 l_1 l_\infty = 1$.

Consider also pairs (X, f) , where X is a compact Riemann surface and $f : X \rightarrow \mathbb{P}^1(\mathbb{C})$ is a holomorphic map unramified outside $\{0, 1, \infty\}$. For reasons soon to be explained, such pairs will be referred to as Belyi pairs and the functions f , as Belyi functions. A morphism from one pair (X, f) to another (X', f') is given by a holomorphic map $\rho : X \rightarrow X'$ such that $f' \circ \rho = f$.

Proposition 2.1.5. *There is a bijection between conjugacy classes of subgroups of finite index in π_1 and isomorphism classes of Belyi pairs (X, f) .*

Proof. Firstly, if we consider $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ merely as a topological surface, then we have the standard bijection between conjugacy classes of subgroups of π_1 and classes of unramified coverings of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, where subgroups of finite index correspond to finite coverings. (see for example [Hat01]).

To complete the connection, note that starting with a Belyi pair (X, f) , then considering the restriction of f to $Y = f^{-1}(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\})$ and forgetting about the complex structures, gives the required unramified covering. For the converse, we start with an unramified covering $f : Y \rightarrow \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. The complex structure on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ induces a unique complex structure on Y . There is a unique way up to isomorphism of Y to compactify Y and to extend f to the compactification X to give a branched covering of $\mathbb{P}^1(\mathbb{C})$ unramified outside $\{0, 1, \infty\}$ (see Forster [For81], Thm 8.4). This gives the corresponding Belyi pair (X, f) , completing the proof. \square

The link with the previous section is established by noticing that π_1 is a free group on the two generators l_0 and l_1 . Thus π_1 is isomorphic to F_2 where l_0 and l_1 are identified with x and y respectively.

2.1.4 The Grothendieck correspondence

The results of the previous sections combine to prove the following theorem, known as the Grothendieck correspondence:

Theorem 2.1.6. *There is a bijection between the isomorphism classes of stick figures and the isomorphism classes of Belyi pairs.*

Although we have been working with sets of objects and the concept of an isomorphism between two objects, it would have been possible to define general morphisms between objects and treat the sets of objects as categories. The above correspondence would then have been an equivalence of categories since the bijection respects the morphisms between objects.

As we have seen, the proof of this correspondence takes a detour through group theory. There is however a more explicit way of realizing the correspondence. In the one direction, starting with a Belyi pair (X, f) , we can find the corresponding stick figure on X simply by taking the preimage of the unit interval under f , i.e. $f^{-1}([0, 1])$. The points above 0 will be those of type 0, and the ones above 1 will be those of type 1. This construction makes it clear that the number of elements in the fiber above 0 is precisely the number of vertices of type 0, and order of ramification at each point in the fiber is precisely the number of edges meeting at the corresponding vertex. Furthermore, the action of F_2 on the edges of the stick figure can be seen to correspond directly to the action of the fundamental group π_1 on points of X . More precisely, choose $\frac{1}{2}$ as the base point for the

fundamental group and consider the fiber above this point. From the above description, it is clear that there is exactly one point in the fiber on every edge of the stick figure. Now the fundamental group acts on this fiber via the lifting of loops. The action of the loop l_0 corresponds exactly to the action of $x \in F_2$ on the edges.

To realize the converse direction more concretely, one must make use of the triangulation associated to a stick figure with underlying surface X . Consider the simple triangulation of the sphere by adding the three edges joining the points 0 , 1 and ∞ . There are two triangles, called positive and negative by the original convention. Now, since every triangle is homeomorphic to an open disk, one can define a function f from the triangles of X to the triangles of the sphere, such that f restricted to a single triangle is a homeomorphism between that triangle and the one on the sphere with the same parity. By choosing f appropriately, it is possible to extend it to a continuous function defined on the whole of X , respecting the vertex, edge and face types. This is then the required Belyi function. Using this function we can lift the complex structure on $\mathbb{P}^1(\mathbb{C})$ to a complex structure on X , giving the required Belyi pair.

Note that in the construction just described, one starts with an arbitrary topological surface X , and by drawing a stick figure on it, one can equip X with a complex structure. Two questions arise, namely is this complex structure unique, and which complex structures arise in this way? The positive answer to the first question was first pointed out by Grothendieck [Gro97], but proven essentially before by Jones and Singerman [JS78] and independently by Malgoire and Voisin [VM77] (see also the survey by Wolfart [Wol97]). The second question will be answered by Belyi's Theorem in the next section.

Also note that although the Riemann surface X associated to a given stick figure is unique, the Belyi function f is only unique up to composing with automorphisms of X . Thus in genus 0, f is unique up to $\mathrm{PSL}_2(\mathbb{C})$, in genus 1 up to affine transformation and in genus greater than 1, up to a finite automorphism group. To obtain uniqueness, one needs to restrict f in some way. In genus 0, this can be done by fixing the position of three separate points on the stick figure since the group $\mathrm{PSL}_2(\mathbb{C})$ acts three-transitively on the sphere.

2.2 The arithmetic side of stick figures

As described in the previous section, we start with a topological structure, namely a surface with a stick figure drawn on it. We then construct a uniquely determined complex structure on the surface, taking us to the area of compact Riemann surfaces and algebraic curves. In this section we show how this process can be taken a step further into the realm of number theory, by associating with every stick figure a uniquely determined number field, called its moduli field.

2.2.1 The absolute Galois group

Before returning to stick figures, we briefly recall some ideas from infinite Galois theory pertaining to the absolute Galois group.

The Galois extensions of \mathbb{Q} are precisely the normal algebraic extensions of \mathbb{Q} . To every Galois extension K/\mathbb{Q} we associate its *Galois group*, namely the group of automorphisms of K , called $\text{Gal}(K/\mathbb{Q})$. Denote by $\overline{\mathbb{Q}}$ the field of all algebraic numbers over \mathbb{Q} . The group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is called the *absolute Galois group* and will be denoted by $G_{\mathbb{Q}}$.

Note that we allow infinite Galois extensions. In this setting the Galois correspondence is adjusted to be a bijection between the *closed* subgroups of $G_{\mathbb{Q}}$ and Galois extensions of \mathbb{Q} . For this it is necessary to define a topology on $G_{\mathbb{Q}}$ called the *Krull topology*, which is obtained by viewing $G_{\mathbb{Q}}$ as a profinite group, i.e. as the inverse limit of all its finite quotients.

Let \mathcal{E} be the set of all finite Galois extensions of \mathbb{Q} . The interest in $G_{\mathbb{Q}}$ arises from the fact that these are precisely its finite quotients, thus in some sense all the information about finite Galois extensions is contained in $G_{\mathbb{Q}}$. For every pair $K \subset L$ in \mathcal{E} we can define a group homomorphism $\rho_K^L : \text{Gal}(L/\mathbb{Q}) \rightarrow \text{Gal}(K/\mathbb{Q})$ by restricting an automorphism of L to give an automorphism of K . The groups $\text{Gal}(K/\mathbb{Q})$ and the homomorphisms ρ_K^L form a projective system of which $G_{\mathbb{Q}}$ is the projective limit:

$$\begin{aligned} G_{\mathbb{Q}} &= \varprojlim_{K \in \mathcal{E}} \text{Gal}(K/\mathbb{Q}) \\ &= \left\{ (\sigma_K)_{K \in \mathcal{E}} \in \prod_{K \in \mathcal{E}} \text{Gal}(K/\mathbb{Q}) \mid \rho_K^L(\sigma_L) = \sigma_K \text{ for all } K, L \in \mathcal{E}, K \subset L \right\} \end{aligned}$$

Thus an element of $G_{\mathbb{Q}}$ can be viewed as a consistent choice of automorphisms of every finite Galois extension. To make $G_{\mathbb{Q}}$ into a topological group, start by giving the finite Galois groups the discrete topology. The product of these groups is then given the product topology and $G_{\mathbb{Q}}$ receives the restriction topology as its subgroup. Equivalently, this is the coarsest topology making all the projection maps from $G_{\mathbb{Q}}$ to the finite Galois

groups continuous. Consequently, a subgroup of $G_{\mathbb{Q}}$ is open if and only if it has finite index in $G_{\mathbb{Q}}$.

$G_{\mathbb{Q}}$ is at present not well-understood. One of the ways of studying it is by letting it act faithfully on some well-understood set, thus viewing it as a subgroup of the automorphism group of this set. The rest of this section gives a concrete example of this approach.

2.2.2 Belyi's theorem

One of the much-used dictionaries in mathematics is the correspondence between compact Riemann surfaces and algebraic curves defined over \mathbb{C} . From this perspective we can view a Belyi pair (X, f) as a covering of the projective line $\mathbb{P}^1(\mathbb{C})$ by a projective algebraic curve X defined over \mathbb{C} , where f is unramified outside $\{0, 1, \infty\}$. Then we have the following remarkable result:

Theorem [Belyi] 2.2.1. Let X be a nonsingular projective algebraic curve defined over \mathbb{C} . Then X can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a non-constant morphism $f : X \rightarrow \mathbb{P}^1(\mathbb{C})$ unramified outside $\{0, 1, \infty\}$, i.e. if and only if there exists an f such that (X, f) is a Belyi pair.

Proof. By a classical result following from Weil's criterion [Wei56], we know that X can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a morphism $f : X \rightarrow \mathbb{P}^1(\mathbb{C})$ with all critical values contained in $\overline{\mathbb{Q}}$. Since $\{0, 1, \infty\} \subset \overline{\mathbb{Q}}$, this proves the one direction.

The proof of the other direction is due to Belyi [Bel80], and proceeds along the following lines (following the exposition given in Schneps ([Sch94])): Starting with a curve X defined over $\overline{\mathbb{Q}}$, we can find a morphism $f : X \rightarrow \mathbb{P}^1(\mathbb{C})$ with all critical values in $\overline{\mathbb{Q}}$ as mentioned above or simply by taking a non-constant element of the function field of X . Now the proof is divided into two steps. In the first we compose f on the left with a polynomial defined over \mathbb{Q} to get a function with all its finite critical values lying in \mathbb{Q} instead of just $\overline{\mathbb{Q}}$. This function is then composed on the left with a rational function defined over \mathbb{Q} such that the resulting function has only two finite critical values, both in \mathbb{Q} , which can be taken to be 0 and 1. This function is then still defined on the original X , thus completing the proof.

In more detail: Consider the set \mathcal{C} of all finite critical values of the original function f as well as their conjugates under the action of $G_{\mathbb{Q}}$. Let g_0 be a polynomial with rational coefficients vanishing exactly on \mathcal{C} (we can take g_0 to be the product of the minimal polynomials of the elements of \mathcal{C} , one for each $G_{\mathbb{Q}}$ -orbit). Now the finite critical values of $g_0 \circ f$ will in general consist of the image of the critical values of f under g_0 (in this case just 0) as well as the critical values of g_0 . Then we repeat the process to find a polynomial g_1 vanishing on these critical values and having degree strictly smaller than g_0 (for example by taking g_1 to be the derivative of g_0). Continuing in this manner we

find a g_n with degree 0 for some n . The polynomial $h = g_{n-1} \circ g_{n-2} \circ \cdots \circ g_1 \circ g_0 \circ f$ then has all critical values in $\mathbb{P}^1(\mathbb{Q})$.

Now let \mathcal{C}' be the set of finite critical values of h . If \mathcal{C}' is empty we are done. If $\mathcal{C}' = \{\alpha\}$, then composing h with the linear fractional transformation $z \mapsto z - \alpha$ gives the required function. If $\mathcal{C}' = \{\alpha, \beta\}$, then composing h with the linear fractional transformation $z \mapsto \frac{1}{\beta - \alpha}z + \frac{\alpha}{\alpha - \beta}$ maps the ordered triple (α, β, ∞) to the ordered triple $(0, 1, \infty)$, so again we are done. If $\mathcal{C}' = \{\alpha, \beta, \gamma, \dots\}$, then the previous transformation will take γ to a rational number of the form $\frac{m}{m+n}$. Composing with the rational function

$$z \mapsto \frac{(m+n)^{(m+n)}}{m^m n^n} z^m (1-z)^n$$

takes 0 and 1 to 0, and $\frac{m}{m+n}$ to 1, without introducing new ramification values, thereby giving a function with fewer ramification values than the original h . Repeating this process gives the required function after a finite number of steps. \square

Reinterpreting the Grothendieck correspondence in the light of this theorem, it means that given any stick figure, there is a unique algebraic curve X defined over $\overline{\mathbb{Q}}$ and a unique (up to composing with automorphisms of X) function f in the function field of X (and thus also defined over $\overline{\mathbb{Q}}$) unramified outside $\{0, 1, \infty\}$ such that the stick figure is given by $f^{-1}([0, 1])$ on X seen as a Riemann surface. Note that from now on we will assume that f is defined over $\overline{\mathbb{Q}}$.

This process of recovering the stick figure from the Belyi pair can be formalized as in [Sch94]. But the process of finding the Belyi function in the first place is the challenging part. Algorithms have been found to solve the problem in arbitrary genus (see [CG94]). It is only in the genus 0 case however that they are simple enough to work, and even here it is fast enough only for small stick figures. In the genus 0 case the problem reduces to solving a system of polynomial equations in several variables.

2.2.3 Galois action on stick figures

Since we can view stick figures as Belyi pairs (X, f) with both X and f defined over $\overline{\mathbb{Q}}$, it is natural to consider the action of $G_{\overline{\mathbb{Q}}}$ on the stick figures. An element $\sigma \in G_{\overline{\mathbb{Q}}}$ acts on a pair (X, f) by acting on the coefficients of a defining equation of X and on the coefficients of f . We denote the resulting pair by (X^σ, f^σ) .

The question then arises as to whether the action is faithful, that is, whether for any given $\sigma \in G_{\overline{\mathbb{Q}}}$, $\sigma \neq 1$, one can always find a stick figure on which σ acts non-trivially. That this is indeed the case follows from the following:

Proposition 2.2.2. The action of $G_{\overline{\mathbb{Q}}}$ on the set of genus 1 stick figures is faithful.

Proof. Let σ be an element of $G_{\mathbb{Q}}$. Choose any algebraic number j on which σ acts non-trivially. Let X be an elliptic curve with j -invariant equal to j . Consider any non-constant function on X , and using the process described in the proof of the previous theorem, compose it with a series of functions to produce a Belyi function f on X . Since X^{σ} has j -invariant equal to $\sigma(j)$, σ acts non-trivially on the pair (X, f) . \square

The action on the set of genus 0 stick figures is also faithful. Indeed, consider the set of genus 0 stick figures having only one open cell, that is, only one point above ∞ . These stick figures are called trees, because by choosing this one point to be ∞ , the stick figure can be realised as a planar graph with no cycles. Then a result by Lenstra (proved in [Sch94]) states that the action of $G_{\mathbb{Q}}$ on the set of trees is faithful.

Thus each element of $G_{\mathbb{Q}}$ can be identified with a unique permutation of the set of stick figures, so a better understanding of this action could provide new insight into $G_{\mathbb{Q}}$. A first step would be to characterize the orbit of a stick figure under the group action. By a result of the next section, such an orbit is always finite. However it remains an open problem to decide whether two given stick figures lie in the same Galois orbit or not, and some approaches will be described in the next section.

Another consequence of Belyi's theorem is that we can attach a unique number field to each stick figure, namely the moduli field of the stick figure. This is defined to be the field K of the following lemma (adapted from Wolfart [Wol97]):

Lemma 2.2.3. Let (X, f) be a Belyi pair. Then for a field $K \subset \overline{\mathbb{Q}}$ the following properties are equivalent:

1. K is the minimal field with the property $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/K)$ implies $(X, f) \cong (X^{\sigma}, f^{\sigma})$.
2. For all $\sigma \in G_{\mathbb{Q}}$, $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/K) \iff (X, f) \cong (X^{\sigma}, f^{\sigma})$.
3. K is the fixed field of $\{\sigma \in G_{\mathbb{Q}} \mid (X, f) \cong (X^{\sigma}, f^{\sigma})\}$.

We call L a field of definition of (X, f) if there is some isomorphic Belyi pair (X', f') with both X' and f' defined over L . Clearly any field of definition contains the moduli field, and in fact the moduli field is the intersection of all possible fields of definition.

One can always find a field of definition which is a number field, hence the moduli field is a number field. From the definition of the moduli field, it follows that the number of elements in the Galois orbit of a given stick figure is equal to the degree of its moduli field over \mathbb{Q} .

2.3 Galois invariants

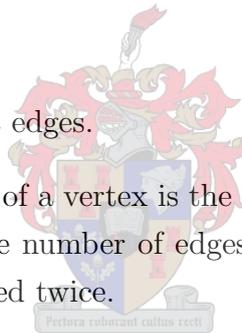
As noted in the previous section, $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of stick figures, dividing it into orbits. Ideally, we would like to have a simple (possibly combinatorial) criterion for telling whether two given stick figures lie in the same orbit or not. Currently, the only sure way of doing this is computing a Belyi function for a given stick figure and using it to find conjugate stick figures. There are however several known necessary (but not sufficient) invariants characterizing the Galois orbit of a stick figure. This section introduces the simpler ones.

Just as the Galois action partitions the set of stick figures into orbits, the Galois invariants also induce partitions which are by definition coarser. We call an invariant *finer in general* than another one if it induces a finer partition. Sometimes we will refer to an invariant as being *finer in some cases* than another simply if it isn't coarser in general. Most invariants seem to be finer than all others in some cases, implying that the best method is a combination of them all.

2.3.1 Valency type

Consider a stick figure \mathcal{S} with n edges.

Definition 2.3.1. The *valency* of a vertex is the number of edges meeting at the vertex. The valency of a face is half the number of edges bordering on the face, where an edge surrounded by the face is counted twice.



Since we are dealing with bipartite graphs, the number of edges bordering on a face will always be even, so this definition makes sense. For now we will refer to the faces as vertices of type ∞ . Then it is clear from the correspondence between valencies of vertices of type i and orbit lengths of g_i where $i \in \{0, 1, \infty\}$ that the sum of the valencies of vertices of type i equals the number of edges, n (see page 8).

To make this precise, let m_i be the number of vertices of type i , $i \in \{0, 1, \infty\}$, and label them $1, 2, \dots, m_i$. Let $k_r^{(i)}$ be the valency of the r 'th vertex of type i . Then for each i there is an unordered partition $n = k_1^{(i)} + k_2^{(i)} + \dots + k_{m_i}^{(i)}$. We say the stick figure has valency type $(k_1^{(0)} + \dots + k_{m_0}^{(0)}; k_1^{(1)} + \dots + k_{m_1}^{(1)}; k_1^{(\infty)} + \dots + k_{m_\infty}^{(\infty)})$. For example the valency type corresponding to Figure 2.2 on page 6 is $(1 + 2 + 3, 1 + 2 + 3, 2 + 4)$.

It can be shown that the valency type remains invariant under the Galois action (see Matzat [Mat87]). Furthermore, there are only a finite number of graphs with a given valency type, hence Galois orbits must be finite. There is no upper bound on their lengths however:

Proposition 2.3.2. Galois orbits of stick figures can be arbitrarily large.

Proof. Let σ be an element of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is not a torsion element. Then for every stick figure s there is a positive number n_s such that $\sigma^{n_s}(s) = s$ (since orbits are finite). Suppose orbits are bounded in length by M , i.e. $n_s \leq M$ for all stick figures s . Now let $n = M!$. Then $\sigma^n(s) = s$ for all s . But since $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully, it follows that σ^n must be the identity, i.e. σ is a torsion element. The contradiction shows that orbits are not bounded in length. \square

The genus of the underlying surface of the stick figure is determined by the valency type (see page 8). Thus from the invariance of the valency type we can also deduce the invariance of the genus.

2.3.2 Monodromy groups and composition

Consider the stick figure given by the pair of permutations (g_0, g_1) from S_n . The subgroup of S_n generated by g_0 and g_1 is called the *monodromy group* of the stick figure. This group is another Galois invariant, in general finer than the valency type.

Variations of the monodromy group invariant can be produced by first composing the given stick figure with another one and then calculating the monodromy group of the resulting stick figure. Given Belyi pairs (X, f) and (Y, g) , we would like to define their composition as the Belyi pair $(X, g \circ f)$. Clearly certain restrictions have to be placed on the pair (Y, g) for this to make sense, leading to the definition:

Definition 2.3.3. A Belyi pair (Y, g) is said to be *composable* if $Y = \mathbb{P}^1(\mathbb{C})$, $g(\{0, 1, \infty\}) \subset \{0, 1, \infty\}$ and g is defined over \mathbb{Q} . Given such a pair and another general pair (X, f) , this is a sufficient condition for $(X, g \circ f)$ to be a Belyi pair, called the *composition* of (X, f) with g .

Since Y is always $\mathbb{P}^1(\mathbb{C})$, we just refer to g as a composable Belyi function. The associated stick figure is called a composable stick figure. The reason for requiring g to be defined over \mathbb{Q} is to ensure that composition commutes with the Galois action:

Proposition 2.3.4. Let g be a composable Belyi function. Then if two stick figures are in the same Galois orbit, their respective compositions with g will also lie in the same Galois orbit.

Proof. Let (X, f) be a Belyi pair, and σ an element of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then the proof of the theorem will follow from the commutativity of the following diagram, where the horizontal arrows indicate composition with g , and the vertical arrows indicate the action of σ .

$$\begin{array}{ccc} (X, f) & \xrightarrow{g} & (X, g \circ f) \\ \downarrow \sigma & & \downarrow \sigma \\ (X, f^\sigma) & \xrightarrow{g} & (X^\sigma, h) \end{array}$$

Going around clockwise we find $h = (g \circ f)^\sigma$, while going anti-clockwise gives $h = g \circ f^\sigma$. Since g is defined over \mathbb{Q} , these functions coincide. \square

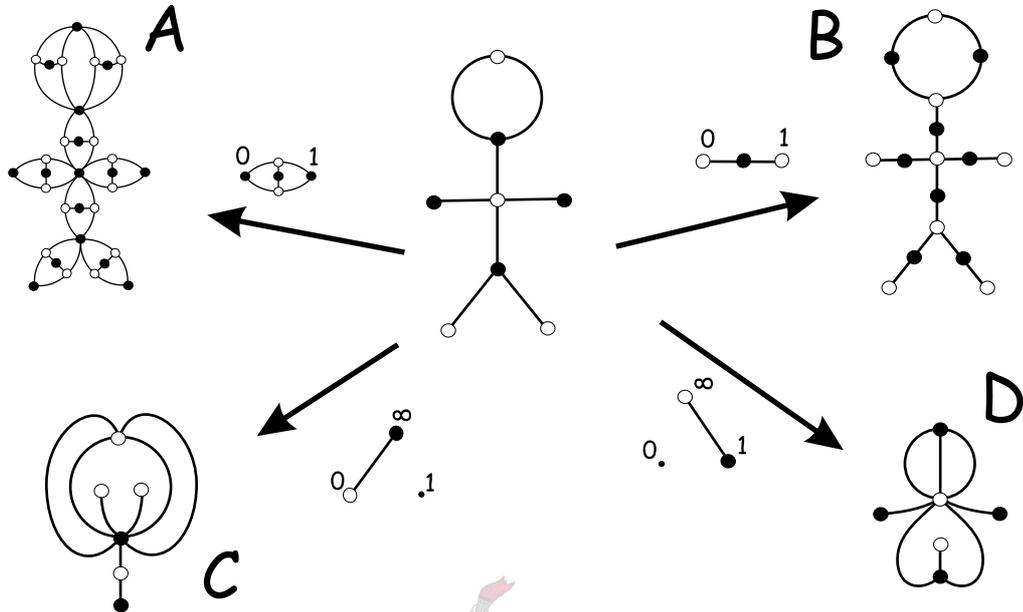


Figure 2.3: Composition of stick figures

Some examples of composing stick figures is shown in Figure 2.3. The stick figure in the middle is composed with the various stick figures drawn next to the arrows, and the resulting compositions are labelled A to D. For the composable stick figures corresponding to various g 's, the condition $g(\{0, 1, \infty\}) \subset \{0, 1, \infty\}$ means that each point in $\{0, 1, \infty\}$ on the underlying surface of g must either be one of the vertices of the stick figure, or lie above ∞ . This is indicated in the figure by labelling the former ones, and simply leaving out the latter ones (or marking them with a point). Thus in the composable stick figure of case A, the point ∞ lies above ∞ , while in case C, the point 1 lies above ∞ .

Before giving further remarks on the stick figures in Figure 2.3, let us first describe the process of using composable stick figures for getting new monodromy group invariants. Let $M(s)$ denote the monodromy group of a stick figure s . Now let g be a composable Belyi function and consider the stick figure found by composing s with g . Denote its monodromy group by $M_g(s)$. As explained above, this is still a Galois invariant. Of course, $M(s)$ corresponds to the case where g is the identity, so $M_g(s)$ could be viewed as a generalized monodromy group invariant.

The most widely used case of this invariant is where g is taken to be $4z(1 - z)$, corresponding to case B in the figure. $M_{4z(1-z)}$ is known as the cartographic group in the literature and is in general a finer invariant than the usual monodromy group (see [JS97]). In some sense it is more general than the monodromy group since it can be defined for any graph embedding, not just bipartite ones. To understand this, note that

in case B the resulting stick figure is found by changing all the vertices of the original stick figure to the same type and adding a new vertex of the other type on every edge. This process can be applied to any graph and will always yield a bipartite one, thus a stick figure. The stick figures that can be created in this way are characterized by the fact that their vertices of type 1 all have valency 2. They are called *clean* stick figures, and are in one-to-one correspondence with general graph embeddings. Thus it is possible to give a more general definition for stick figures (as general graph embeddings with no partitioning of the edges) and then consider the cartographic group as an invariant. This is exactly the same as considering only clean stick figures and their monodromy groups via the bijection just described.

Cases C and D are examples of composing with automorphisms of $\mathbb{P}^1(\mathbb{C})$ leaving $\{0, 1, \infty\}$ fixed. There are six of these, including $\frac{z}{z-1}$ (case C) and $\frac{1}{z}$ (case D). They can easily be visualized by considering the triangulation corresponding to the original stick figure. Then case C corresponds to the $\overline{0\infty}$ edges and case D to the $\overline{1\infty}$ edges. As shown in [Woo03], first composing with such an automorphism and then taking the cartographic group of the resulting stick figure gives an invariant which is finer in some cases than simply taking the cartographic group.

2.3.3 Trees

Trees are genus-0 stick figures with a single face. They are simpler than general stick figures in certain respects, such as always having a polynomial as a Belyi function. Yet as noted before, $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ still acts faithfully on the set of trees. Consequently trees are a natural starting point for studying properties of stick figures. In this section we state (a corollary of) a result of Zapponi [Zap00] which gives an invariant which is strictly finer than valency types for a certain family of trees.

For convenience we change the notation from that given at the beginning of the section. Let T be a specific tree. Let n and m denote the number of vertices of type 1 and 0 respectively. Label the vertices of each type, and let p_i and q_j denote the valencies of the i 'th vertex of type 1 and the j 'th vertex of type 0 respectively. We denote the valency type of the tree by $(p_1, \dots, p_n; q_1, \dots, q_m)$. From Euler's formula then follows that $\sum_i p_i = \sum_j q_j = m + n - 1$.

Actually it is necessary to label the vertices of type 1 (the p_i 's) in a specific way to capture the way the graph is embedded into the sphere. First note that this embedding is described by giving a cyclic permutation of the edges meeting at each vertex corresponding to their order in an anticlockwise direction around the vertex. Now choose a vertex of type 1 to be labelled as number 1 and choose an edge leading away from it as the active edge. We now traverse the graph using the following algorithm:

Walk along the active edge to the next vertex. Replace the current active edge by

the edge following it in the permutation corresponding to this vertex. Now walk along this edge, repeating the process until reaching the original vertex and same active edge. Along the way, label the vertices of type 1 in the order in which they are first reached. Note that every edge will be traversed exactly twice.

Another way to describe it, is to consider the pair of permutations (g_0, g_1) corresponding to the tree. The order in which the edges are traversed can be found by applying g_0 and g_1 alternately.

Having labelled the type 1 vertices, we now make the assumption that they all have distinct valencies, i.e. that the p_i 's are all distinct. There is then a unique permutation σ on n elements such that $p_{\sigma(1)} < p_{\sigma(2)} < \dots < p_{\sigma(n)}$. Let $S(T)$ (the signature) be either 1 or -1 corresponding to whether this is an even or odd permutation.

Note that $S(T)$ depends on the choice of starting vertex, in the sense that starting at a different vertex will give a different permutation which could have a different signature. However, in the case where n as well as all the q_i 's are odd (conditions which are met in the following theorem), it can be verified that the signature is independent of the choice of starting vertex.

Theorem 2.3.5. Let $n, m, p_1, \dots, p_n, q_1, \dots, q_m$ be positive integers such that:

1. $p_1 < \dots < p_n$ and $\sum_i p_i = \sum_j q_j = m + n - 1$.
2. All the q_j 's are odd.
3. $n \equiv 1 \pmod{4}$
4. $p_1 \dots p_n (p_1 + \dots + p_n)$ is a perfect square.

Then $S(T)$ is a Galois invariant for the trees T having valency type $(p_1, \dots, p_n; q_1, \dots, q_m)$. Thus this valency class splits into at least two Galois orbits.

One of the examples which started conjectures in this direction is known as Leila's flowers. They are trees forming a valency class of order 24 which splits into two Galois orbits of equal size. A tree from each orbit is shown in Figure 2.4. The vertices of type 1 have distinct valencies $(2, 3, 4, 5, 6)$ and there are 5 ($\equiv 1 \pmod{4}$) of them. All the vertices of type 0 have valency 1 or 5, hence odd. Finally $p_1 \dots p_n (p_1 + \dots + p_n) = 14400 = (120)^2$ is a perfect square. Thus the theorem applies. Labelling the vertices of type 1 as indicated shows that the tree on the right needs the odd permutation $(12)(3)(4)(5)$ to sort the valencies in increasing order, whereas for the tree on the left they are already sorted. Since the identity permutation is even, this shows that they are in different orbits.

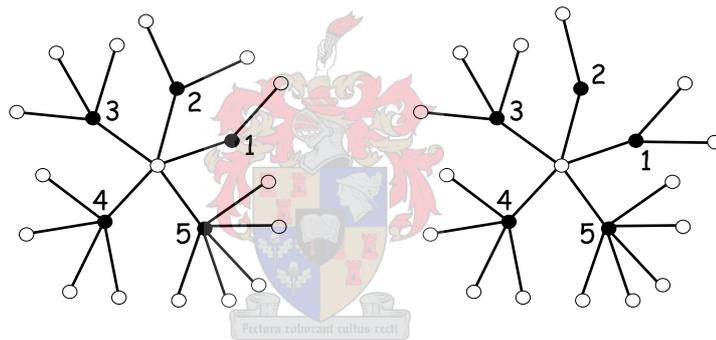


Figure 2.4: *Trees from different Galois orbits*

Chapter 3

Moduli spaces

In the next chapter it will be shown how elements of $G_{\mathbb{Q}}$ can be parametrized by pairs (f, λ) from $\hat{\mathbb{Z}}^{\times} \times \hat{F}'_2$ satisfying certain equations, where $\hat{\mathbb{Z}}^{\times}$ is the group of invertible elements of the profinite completion of \mathbb{Z} and \hat{F}'_2 is the derived subgroup of the profinite completion of the free group on two generators. The equations arise from the action of such elements on (the fundamental groups of) certain geometric objects, namely moduli spaces of Riemann surfaces.

In this chapter we use an analytic rather than algebraic approach to describe these moduli spaces and related objects, such as Teichmüller spaces and mapping class groups.

Most of the material in this chapter is taken from [BFL⁺03], [IT92], [Bir74] and [Sch03].

3.1 Definitions of Teichmüller spaces, moduli spaces and mapping class groups

3.1.1 Counting complex structures

One way to construct a Riemann surface is to start with a two-dimensional orientable manifold and cover it with a set of compatible complex charts, i.e. give it a complex structure. The question arises as to how many distinct ways there are of doing this.

More specifically, we will be counting compact Riemann surfaces of genus g up to isomorphisms respecting a certain number of ordered marked points on each surface. The objects to be counted can be defined as follows:

Definition 3.1.1. A *Riemann surface of type (g, n)* is defined to be a compact Riemann surface of genus g with n ordered marked points. Any morphism between two Riemann surfaces of the same type is required to preserve the ordered marked points as well as the ordering on them.

Then as we will see, there is a $3g - 3 + n$ -dimensional complex manifold $\mathcal{M}_{g,n}$ called

moduli space whose points parametrize the isomorphism classes of Riemann surfaces of type (g, n) .

If the topological manifold that we start with is compact of genus g with n marked points, then giving it a complex structure results in a Riemann surface of type (g, n) , and conversely every Riemann surface of type (g, n) arises in this way. Moreover, two Riemann surfaces of type (g, n) are isomorphic if and only if the complex structures are equivalent in the sense that there is an orientation preserving diffeomorphism of the surface preserving the ordered marked points (and their ordering) making the one complex structure compatible with the other. Consequently, counting Riemann surfaces is the same as counting ways of putting equivalent complex structures on a certain reference surface.

It turns out that it is sometimes more convenient to use a finer notion of equivalence. This is done by requiring the diffeomorphism linking the two complex structures to be isotopic to the identity diffeomorphism. We recall what it means for two diffeomorphisms to be isotopic:

Definition 3.1.2. Let X, Y be topological spaces. Then continuous maps $f, g : X \rightarrow Y$ are *isotopic* if there exists a continuous map $F : X \times [0, 1] \rightarrow Y$ such that $F|_{X \times \{0\}} = f$, $F|_{X \times \{1\}} = g$ and $F|_{X \times \{t\}}$ is a homeomorphism onto its image for every $t \in [0, 1]$.

The resulting space obtained by using this finer notion of equivalence between complex structures is called Teichmüller space and denoted by $\mathcal{T}_{g,n}$. The two spaces are linked by the mapping class group, $\Gamma_{g,n}$, which acts discretely on $\mathcal{T}_{g,n}$ to give $\mathcal{M}_{g,n}$ as a quotient space.

Before making the above more precise, let us cite a familiar example to place things in perspective. The moduli space $\mathcal{M}_{1,1}$ parametrizes isomorphism classes of complex tori. It can be identified with the complex plane through the j -invariant, i.e. through the bijection

$$j : \mathcal{M}_{1,1} \rightarrow \mathbb{C}.$$

The corresponding Teichmüller space $\mathcal{T}_{1,1}$ is the upper-half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$, parametrized by $\tau = \omega_2/\omega_1$ for a uniformizing lattice $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ of a specific complex torus, i.e. via a bijection

$$\tau : \mathcal{T}_{1,1} \rightarrow \mathbb{H}.$$

The mapping class group $\Gamma_{1,1}$ is the Fuchsian group $\mathrm{PSL}_2(\mathbb{Z})$. This group acts discretely on \mathbb{H} with orbits being those values of τ giving isomorphic tori. In other words the familiar isomorphism

$$\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{C}$$

can now be written as

$$\mathcal{T}_{1,1}/\Gamma_{1,1} \cong \mathcal{M}_{1,1}.$$

3.1.2 Analytic approach

There are various ways to define Teichmüller spaces. The analytic approach is to start with a compact, orientable, differentiable surface S of genus g with n marked points labelled as $P = \{x_1, \dots, x_n\}$. S is called the reference surface and is used to *mark* Riemann surfaces of type (g, n) as explained in the following definition:

Definition 3.1.3. A *marked Riemann surface* (of type (g, n)) is a pair (X, f) where X is a Riemann surface of type (g, n) and $f : S \rightarrow X$ is an orientation preserving diffeomorphism. Two marked Riemann surfaces (X, f) and (X', f') are considered to be the same if $f' \circ f^{-1} : X \rightarrow X'$ is a biholomorphism, or equivalently, if f and f' both induce the same complex structure on S . Denote the set of all marked Riemann surfaces by \mathcal{A} .

Note that putting a complex structure on the reference surface S gives rise to a marked Riemann surface if we take $X = S$ and f as the identity on S . Furthermore, every Riemann surface of type (g, n) is the X of some marked Riemann surface (X, f) , because there is always some diffeomorphism from S to X (since there is only one differentiable structure up to diffeomorphism on a topological surface of type (g, n)).

Now define $\text{Diff}^+(S, P)$ as the group of orientation-preserving diffeomorphisms of S fixing every point in P . Those diffeomorphisms isotopic to the identity form a normal subgroup $\text{Diff}_0^+(S, P)$. Since the diffeomorphisms in the isotopy are required to fix P they can also be regarded as diffeomorphisms of $S \setminus P$.

$\text{Diff}^+(S, P)$ acts on \mathcal{A} via composition: A diffeomorphism $\phi \in \text{Diff}^+(S, P)$ takes (X, f) to $(X, f \circ \phi)$. This leads to the following definitions:

Definition 3.1.4. Teichmüller space $\mathcal{T}_{g,n}$ is the quotient space $\mathcal{A}/\text{Diff}_0^+(S, P)$.

Definition 3.1.5. Moduli space $\mathcal{M}_{g,n}$ is the quotient space $\mathcal{A}/\text{Diff}^+(S, P)$.

We show that the last definition is equivalent to the more standard definition of moduli space as isomorphism classes of Riemann surfaces. Let X and X' be isomorphic Riemann surfaces of type (g, n) with $\eta : X' \rightarrow X$ the required biholomorphic mapping. Then as noted before, we can find diffeomorphisms $f : S \rightarrow X$ and $f' : S \rightarrow X'$ to give us pairs (X, f) and (X', f') in \mathcal{A} .

Since a biholomorphic mapping is diffeomorphic, we can define the diffeomorphism $\phi : S \rightarrow S$ as $\phi := f^{-1} \circ \eta \circ f'$, i.e. as the diffeomorphism making the following diagram commute:

$$\begin{array}{ccc}
 S & \xrightarrow{\phi} & S \\
 \downarrow f' & & \downarrow f \\
 X' & \xrightarrow{\eta} & X
 \end{array} \tag{3.1}$$

Then ϕ is in $\text{Diff}^+(S, P)$ and takes (X, f) to (X', f') , hence they represent the same point in moduli space. In general they represent different points in Teichmüller space unless ϕ is in $\text{Diff}_0^+(S, P)$. This is why Teichmüller space can be said to classify Riemann surfaces up to biholomorphism isotopic to the identity.

For the converse, if $\phi \in \text{Diff}^+(S, P)$ takes (X', f') to (X, f) , then by the definition of what it means to be the same in \mathcal{A} , $f' \circ \phi \circ f^{-1}$ must be a biholomorphism.

For example, in the case of tori, if two isomorphic tori (regarded as different complex structures on the same surface) have different τ values, then there is a diffeomorphism of the surface taking the one complex structure to the other (they represent the same point in moduli space), but this diffeomorphism cannot be isotopic to the identity map on the surface (different points in Teichmüller space).

3.1.3 Metric approach

First a note about marked points and punctures. Counting compact Riemann surfaces with marked points up to isomorphisms respecting the marked points is equivalent to counting punctured Riemann surfaces up to isomorphism (taking a neighbourhood of a puncture to a neighbourhood of the same puncture). This is because there is a unique way to replace (compactify) the punctures on a compact Riemann surface. However, when we want to view the Riemann surface as a quotient of its universal covering space (uniformization) then it is more convenient to view the marked points as punctures.

Given a Riemann surface X of type (g, n) , we can calculate its Euler characteristic as $\chi(X) = 2 - 2g - n$. From now on we restrict ourselves to the case where $\chi(X) < 0$. By the uniformization theorem this implies that X has the upper-half plane \mathbb{H} as its universal covering space, and is isomorphic to \mathbb{H}/Γ . Here Γ is a Fuchsian group, i.e. a discrete subgroup of $\text{PSL}_2(\mathbb{R}) = \text{Aut}(\mathbb{H})$.

Recall that there is a standard hyperbolic metric (Riemannian metric with constant curvature -1) on \mathbb{H} having half circles tangent to the real line as geodesics and $\text{PSL}_2(\mathbb{R})$ as its group of isometries. This makes \mathbb{H} into a complete hyperbolic surface, and can be used to define other hyperbolic surfaces which look locally like \mathbb{H} :

Definition 3.1.6. By a *hyperbolic structure* on a surface X we mean a set of pairs (U_i, ϕ_i) where $X = \cup_i(U_i)$ (with the U_i 's being open sets) and $\phi_i : U_i \rightarrow \mathbb{H}$ is an injective continuous map such that the transition functions $\phi_j \circ \phi_i^{-1}$ are isometries on the subsets of \mathbb{H} on which they are defined.

Putting a hyperbolic structure on a surface X gives a hyperbolic surface: The distance between any two points on the surface along any path can be determined by dividing the path up into pieces small enough such that each is contained in some open set $U_i \subset X$, and adding up the lengths of these smaller pieces. The metric on \mathbb{H} then induces a metric

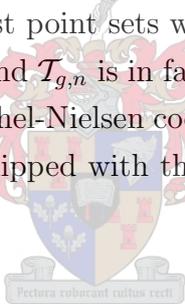
on X by taking the distance between two points as the infimum of the distances along all possible paths between the two points.

Viewing the Riemann surface X as \mathbb{H}/Γ allows us to equip it with a hyperbolic structure. This is done by covering X with open sets U_i such that the projection mapping $\pi : \mathbb{H} \rightarrow \mathbb{H}/\Gamma$ is trivial over each U_i and choosing $\phi_i : X \rightarrow \mathbb{H}$ as a section of π over U_i .

In short, a complex structure on X gives a hyperbolic structure on X . For the converse, start with a surface X of type (g, n) equipped with a hyperbolic structure. Since the transition functions are isometries they are conformal and since X is orientable we can choose them to be orientation-preserving. Thus using the complex structure of \mathbb{H} (as a subset of \mathbb{C}) allows us to view the pairs (U_i, ϕ_i) as defining a complex structure on X .

So there is a correspondence between the complex structures and hyperbolic structures on a differentiable surface. Thus Teichmüller space can also be viewed as a parametrization of the possible hyperbolic structures up to a diffeomorphism isotopic to the identity, and moduli space the possible hyperbolic structures up to isometry on a given differentiable surface. This viewpoint underlies the introduction of Fenchel-Nielsen coordinates to parametrize Teichmüller space, as will be described later on.

Thus far $\mathcal{T}_{g,n}$ and $\mathcal{M}_{g,n}$ are just point sets without a topology. Indeed $\mathcal{T}_{g,n}$ and $\mathcal{M}_{g,n}$ can be given complex structures and $\mathcal{T}_{g,n}$ is in fact simply connected (see [IT92]). We will not prove this, but the use of Fenchel-Nielsen coordinates to parametrize $\mathcal{T}_{g,n}$ later on will show how $\mathcal{T}_{g,n}$ can at least be equipped with the structure of a $6g - 6 + 2n$ dimensional real analytic manifold.



3.1.4 Mapping class group

From now on we will assume that $\mathcal{T}_{g,n}$ and $\mathcal{M}_{g,n}$ are complex manifolds and that $\mathcal{T}_{g,n}$ is simply connected.

Definition 3.1.7. The mapping class group $\Gamma_{g,n}$ is defined as $\text{Diff}^+(S, P)/\text{Diff}_0^+(S, P)$.

By definition it then follows that $\Gamma_{g,n}$ acts on $\mathcal{T}_{g,n}$ with quotient $\mathcal{M}_{g,n}$. Regarding this action there is the following result (see [IT92], chapter 6):

Proposition 3.1.8. The action of the mapping class group $\Gamma_{g,n}$ on $\mathcal{T}_{g,n}$ is properly discontinuous.

By a properly discontinuous action we mean that for any compact subset K of $\mathcal{T}_{g,n}$ there are only a finite number of elements $\lambda \in \Gamma_{g,n}$ such that $\lambda(K) \cap K \neq \emptyset$. In general the action is not free. For example, $\mathcal{T}_{1,1}$ has points with stabilizers of order 2 and 3.

However, in the genus 0 case the action is free. Recalling the fact that $\mathcal{T}_{0,n}$ is simply connected, it follows that $\mathcal{T}_{0,n}$ is a universal covering space of $\mathcal{M}_{0,n}$ and hence $\Gamma_{0,n}$ is isomorphic to the topological fundamental group of $\mathcal{M}_{0,n}$. This provides the connection

to the outer action of $G_{\mathbb{Q}}$ on the algebraic fundamental group of varieties defined over \mathbb{Q} : $\mathcal{M}_{0,n}$ can be viewed as an algebraic variety defined over \mathbb{Q} , hence $G_{\mathbb{Q}}$ acts on the profinite completion of $\Gamma_{0,n}$, denoted by $\hat{\Gamma}_{0,n}$.

Actually, the restriction to genus 0 is not necessary, although the extension is not trivial. The point is that since $\Gamma_{g,n}$ does not act freely in general, it is not (in general) the topological fundamental group of $\mathcal{M}_{g,n}$ regarded as an ordinary manifold. But it is possible to equip $\mathcal{M}_{g,n}$ with some extra structure to remember which points come from points with non-trivial stabilizers in $\mathcal{T}_{g,n}$ and which subgroups of $\Gamma_{g,n}$ are their stabilizers. From the topological side the required generalization of manifolds to what he called *orbifolds* was done by Thurston. The precise definition is not so important to us. We only need to note that $\mathcal{M}_{g,n}$ can in general be regarded as an orbifold, with a related concept of orbifold fundamental group, which is isomorphic to $\Gamma_{g,n}$ in the general case.

The same purpose is served from the algebraic geometric viewpoint by stacks. The important point is that the fundamental group exact sequence given at the beginning of the next chapter can be generalized to fundamental groups of stacks defined over \mathbb{Q} ([Oda97]). Hence we can refer to the outer action of $G_{\mathbb{Q}}$ on the profinite completion of $\Gamma_{g,n}$, denoted by $\hat{\Gamma}_{g,n}$.

3.1.5 Generators of the mapping class groups

We will now proceed to describe generators of $\Gamma_{g,n}$. First, an informal description. By definition, elements of $\Gamma_{g,n}$ are equivalence classes of diffeomorphisms of a reference surface S . Imagine S to be a solid structure covered tightly with some kind of cloth which represents the complex structure. Take $n = 0$ to simplify things. Diffeomorphisms of S can be thought of as twisting and stretching the cloth along the surface. As long as the cloth is not torn, these diffeomorphisms are all isotopic to the identity since the twisting is a continuous motion. To get a diffeomorphism lying outside $\text{Diff}_0^+(S, P)$, we cut the cloth open along a non-trivial simple closed curve on S (such as around one of the handles), twist the cloth on the one side through a full revolution, and then join the two open circles by stitching along the closed loop. The isotopy class of this diffeomorphism is a non-trivial element of the mapping class group, called a Dehn twist.

More precisely, let α be a simple (i.e. with no self-intersections) closed curve on the reference surface $S \setminus P$ (see Figure 3.1). Let U be a tubular neighbourhood of α . Parametrize U by points $(x, \theta) \in (0, 1) \times [0, 2\pi)$ such that points $(\frac{1}{2}, \theta)$ parametrize α . Define a diffeomorphism h_{α} on $S \setminus P$ which is the identity outside U , and acts on U by taking (x, θ) to $(x, \theta + 2\pi x)$.

Definition 3.1.9. A Dehn twist along α is defined to be the isotopy class in $\Gamma_{g,n}$ of h_{α} .

The mapping class group is known to be generated by a finite number of Dehn twists

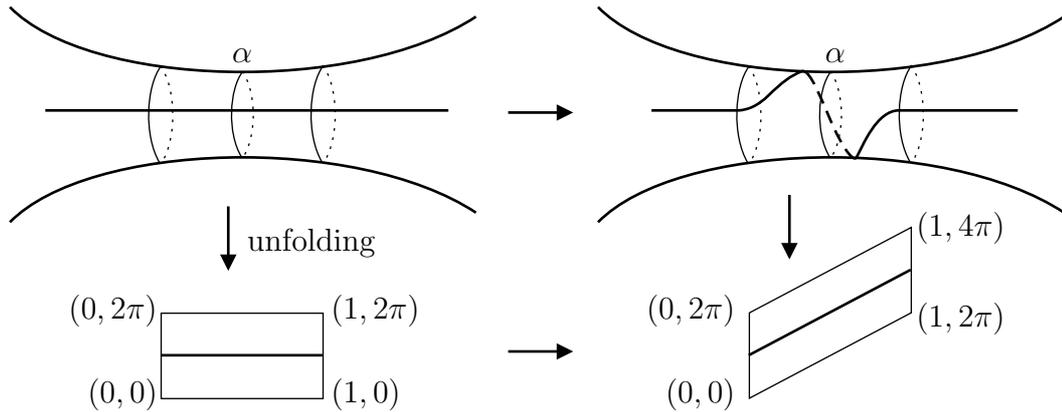


Figure 3.1: *Performing a Dehn twist*

(see Birman, [Bir74]). Jumping ahead of ourselves a bit, we note that the Dehn twists generate certain subgroups of $\Gamma_{g,n}$ known as inertia subgroups. The connection with Galois theory is that the canonical outer action of $G_{\mathbb{Q}}$ on $\hat{\Gamma}_{g,n}$ respects the conjugacy classes of inertia subgroups. In other words, an element of $G_{\mathbb{Q}}$ takes a Dehn twist to a conjugate of a power of the same Dehn twist.

3.2 Fenchel-Nielsen coordinates

The purpose of this section is to show that there exists a homeomorphism between $\mathcal{T}_{g,n}$ and $(\mathbb{R}_{>0})^{3g-3+n} \times \mathbb{R}^{3g-3+n}$. For this it is necessary to take the metric view of Teichmüller space as hyperbolic metrics on a reference surface $S \setminus P$.

The basic idea of this particular way of parametrizing Teichmüller space is to cut up the reference surface into basic building blocks called *pairs of pants*. Counting complex structures on the original surface is then reduced to parametrizing the complex structures on each pair of pants, as well as the possible ways of putting them back together. Both of these types of parameters turn out to be easy to describe.

3.2.1 Pants decomposition

We start by considering the reference surface $S \setminus P$ without any hyperbolic structure on it. Let α be a simple closed loop on $S \setminus P$ which is not freely homotopic to a trivial loop on S , henceforth called a *non-trivial* simple closed loop. (Recall that two closed loops are freely homotopic if there is a homotopy between them which does not necessarily fix any base point.) So for example α can't just be a loop around a point of P . In general we will only be interested in α up to isotopy.

Now suppose we cut the surface $S \setminus P$ open along α . The result is a surface with two

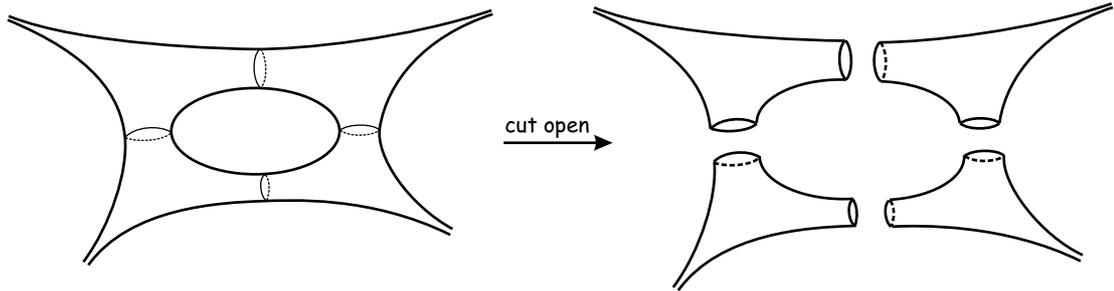


Figure 3.2: *Cutting up a surface of type $(1,4)$ into pairs of pants*

boundary circles, consisting of one or two connected components. Let α' be another non-trivial simple closed loop on this surface, not isotopic to any of the boundary components. Cutting along α' gives a surface with 4 boundary circles and possibly more connected components than before. Continuing in this manner it will be found that after cutting along a finite number of simple closed loops it is not possible to find another simple closed loop satisfying the requirements. In fact, if the original reference surface $S \setminus P$ was of type (g, n) , then it can be shown that the maximal number of loops will always be $3g - 3 + n$. Furthermore, the connected components of the resulting surface will each be homeomorphic to a sphere with three punctures. This prompts the following definition:

Definition 3.2.1. A *maximal multicurve* on $S \setminus P$ is a family of non-trivial and pairwise non-intersecting simple closed loops $\{\alpha_1, \dots, \alpha_{3g-3+n}\}$ on $S \setminus P$ such that no two of them are isotopic to each other. Cutting the surface along all the loops is called performing a *pants decomposition*, and each of the resulting components of $(S \setminus P) \setminus \bigcup_{i=1}^{3g-3+n} \alpha_i$ is called a *pair of pants*.

As seen in Figure 3.2, pairs of pants coming from parts of S with punctures will have boundaries consisting of less than three one-dimensional components (i.e. some of the components are just points instead of circles). These will be referred to as *degenerate* pairs of pants. Also note that the surface in Figure 3.2 is of type $(g, n) = (1, 4)$, hence the number of pairs of pants should be $3(1) - 3 + 4 = 4$, confirming the expression given above for this case.

Now suppose that the reference surface $S \setminus P$ is equipped with some hyperbolic structure. Let \mathcal{C} be some maximal multicurve on $S \setminus P$. Instead of cutting along \mathcal{C} we will prefer to use another maximal multicurve \mathcal{C}' which is isotopic to \mathcal{C} in the sense that each loop in \mathcal{C} is replaced by a loop isotopic to it. \mathcal{C}' is uniquely characterised by the property that all its loops are geodesics for the given hyperbolic structure on $S \setminus P$. This follows from the following result ([IT92], page 54):

Proposition 3.2.2. Let α be a non-trivial simple closed loop on $S \setminus P$. Then there is a unique geodesic on $S \setminus P$ isotopic to α .

One way to visualize it, is to imagine placing an elastic band along α and letting it go. It will then deform into the unique geodesic just mentioned. Henceforth, if we consider a maximal multicurve on a surface equipped with a hyperbolic structure, we will assume that all the loops in the multicurve are geodesics.

For the time being we restrict ourselves to surfaces of type $(g, 0)$, for which all the pairs of pants have three one-dimensional boundary components (i.e. they are non-degenerate). By the above, these boundary components will be geodesics for the hyperbolic structure on each pair of pants.

Thus to each pair of pants we can associate an ordered triple (l_1, l_2, l_3) of the lengths of the respective geodesics. Then there is the following result (see [IT92], page 56):

Proposition 3.2.3. The hyperbolic (thus also complex) structure on a pair of pants is uniquely determined (up to isomorphism) by its ordered triple of geodesic lengths. Furthermore, given any triple of positive real numbers (l_1, l_2, l_3) , there is a unique pair of pants with boundary geodesics of these lengths.

This can be interpreted as parametrizing the moduli space of a genus 0 surface with three boundary components. Thus far we have not considered surfaces with boundaries (only punctures). It is possible however to generalize Teichmüller space to the case of a reference surface S of genus g with n punctures and r boundary components. All isotopies between diffeomorphisms are required to fix boundaries. The corresponding Teichmüller space, moduli space and mapping class group is denoted by $\mathcal{T}_{g,n}^r$, $\mathcal{M}_{g,n}^r$ and $\Gamma_{g,n}^r$ respectively. Then the above theorem can be interpreted as establishing a homeomorphism between $\mathcal{M}_{0,0}^3$ and $(\mathbb{R}_{>0})^3$. Actually, in this case the mapping class group is trivial, hence Teichmüller and moduli space coincide.

If we once again allow surfaces with punctures, then we could get degenerate pants (at least one of the boundary components becomes a point). To extend the parametrization of complex structures to include the degenerate cases simply requires replacing $(\mathbb{R}_{>0})^3$ with $(\mathbb{R}_{\geq 0})^3$. Note that the case where all three boundary components degenerate to points corresponds to the single point $(0, 0, 0)$. This is equivalent to the fact that the automorphism group of the sphere is three-transitive (i.e. can map any three points to any other three points).

3.2.2 Fenchel-Nielsen coordinates

Now we will show how to assign to a hyperbolic structure on given reference surface $S \setminus P$ a set of $3g - 3 + n$ *length* parameters as well as $3g - 3 + n$ *twist* parameters, which will characterise the structure uniquely. To show that these parameters are dependent on a choice of hyperbolic structure, we will consider them as functions of $t \in \mathcal{T}_{g,n}$.

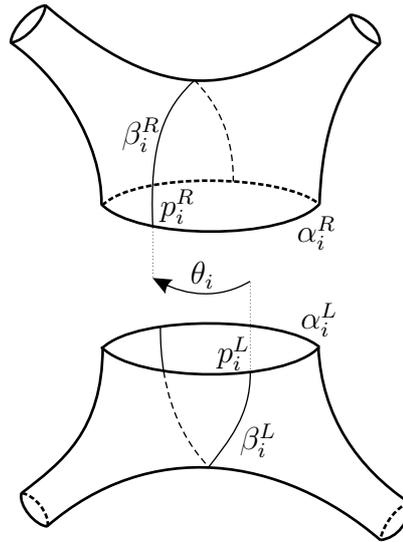


Figure 3.3: *Determining the twist parameter*

The length parameters are obtained by first choosing a maximal multicurve (pants decomposition) $\{\alpha_1, \dots, \alpha_{3g-3+n}\}$ for $S \setminus P$. Let $l_i(t)$ denote the length of the closed geodesic α_i . Then cutting up the surface along the α_i 's into its pair of pants components, and using the results mentioned in the previous section, we can conclude that the l_i 's are uniquely determined by the hyperbolic structure t on $S \setminus P$. Furthermore, the hyperbolic structures on the pairs of pants are uniquely determined by the l_i 's. Hence the only way in which two different hyperbolic structures on $S \setminus P$ can give the same l_i 's, is if we change the way in which the pairs of pants are connected back together again. This is described by the twist parameters.

Informally, these are easy to describe. There are $3g - 3 + n$ twist parameters θ_i , one for each loop α_i in the pants decomposition. Suppose there are two different points in Teichmüller space differing only in the value of one twist parameter θ_i by an amount x . Then given the hyperbolic structure on $S \setminus P$ associated to the one point, we can find the hyperbolic structure of the other point by cutting the surface along α_i , giving the one part a twist of x radians, and attaching them again. In the case where x is a multiple of 2π , this amounts to performing a power of a Dehn twist, taking us to the same point in moduli space but not the same point (unless $x = 0$) in Teichmüller space.

To make this more precise, we have to do without a hyperbolic structure on $S \setminus P$ as a reference point. Start by choosing an orientation for every loop α_i . After cutting up the surface, α_i can be identified with two boundaries of (possibly the same) pairs of pants which can be labelled the left and right one in a canonical way using the orientation on α_i . Call the respective boundaries α_i^L and α_i^R . The idea is now to choose points $p_{L,i}$ and $p_{R,i}$ on the respective boundaries using only the hyperbolic structure of the pants as a

reference, and then determine the distance from $p_{L,i}$ to $p_{R,i}$ along the curve α_i .

To do this we make use of the fact that for every boundary (say α_i^R) on a pair of pants with a given hyperbolic structure, there is a unique (non-closed) geodesic (say β_i^R) joining two points (say c and d) on the boundary which is perpendicular to the boundary (see Figure 3.3). $p_{R,i}$ will be chosen to be one of these two points. To make a consistent choice, start by giving a cyclic ordering of the three boundaries (or punctures) of every pair of pants. Cutting the pair of pants along β_i^R divides it into two parts. Then if the path along α_i^R (according to its orientation) from c to d lies on the same half of the pants as the boundary (or puncture) following α_i^R in the cyclic ordering, we let $p_{R,i}$ be c , otherwise d .

In this way we get two points $p_{L,i}$ and $p_{R,i}$ on every α_i . Let $d_i(t)$ be the distance along α_i from $p_{L,i}$ to $p_{R,i}$. Define $\theta_i(t)$ as $\frac{2\pi d_i(t)}{l_i(t)} \in \mathbb{R}/(2\pi\mathbb{Z})$.

The above can be summarized by saying there is a surjective continuous function (depending on a pants decomposition)

$$\begin{aligned} \Psi : \mathcal{T}_{g,n} &\rightarrow (\mathbb{R}_{>0})^{3g-3+n} \times (\mathbb{R}/(2\pi\mathbb{Z}))^{3g-3+n} \\ t &\mapsto (l_1(t), \dots, l_{3g-3+n}(t), \theta_1(t), \dots, \theta_{3g-3+n}(t)). \end{aligned}$$

Since $\mathcal{T}_{g,n}$ is simply connected, every θ_i has a single-valued branch on $\mathcal{T}_{g,n}$. Choosing such a branch for every θ_i , we can lift Ψ to a single-valued function $\tilde{\Psi}$ on $\mathcal{T}_{g,n}$. Then we have the following theorem ([IT92], p63):

Theorem 3.2.4. The mapping $\tilde{\Psi}$ is a homeomorphism from $\mathcal{T}_{g,n}$ to $(\mathbb{R}_{>0})^{3g-3+n} \times \mathbb{R}^{3g-3+n}$. For every $t \in \mathcal{T}_{g,n}$, $\tilde{\Psi}(t) = (l_1(t), \dots, l_{3g-3+n}(t), \theta_1(t), \dots, \theta_{3g-3+n}(t))$ is called its *Fenchel-Nielsen coordinates*. This gives a global parametrization of Teichmüller space dependent on a choice of pants decomposition.

3.2.3 Compactification of moduli space

$\mathcal{M}_{g,n}$ is in general not compact. In this section we describe the construction of its compactification $\overline{\mathcal{M}}_{g,n}$. The end result is reminiscent of the compactification of affine space. Recall that projective space can be viewed as a union of affine spaces of lower dimension. For example, the projective plane is the union of an affine plane, an affine line and a point. Similarly, $\overline{\mathcal{M}}_{g,n}$ can be seen as the union of $\mathcal{M}_{g,n}$ and several lower dimensional moduli spaces. It is this stratified structure of $\overline{\mathcal{M}}_{g,n}$ that determines the equations that elements of $G_{\mathbb{Q}}$ have to satisfy when viewed as automorphisms of the fundamental group of $\mathcal{M}_{g,n}$.

As a starting point for the compactification, we recall a characterization of compactness in metric spaces by means of sequences. Namely, a space is compact if and only if every sequence has a convergent subsequence. So the goal is to introduce new points as the

limits of certain sequences which do not converge in $\mathcal{M}_{g,n}$. We will start by identifying a certain set of diverging sequences in $\mathcal{M}_{g,n}$ such that any diverging sequence in $\mathcal{M}_{g,n}$ has a sequence from this set as one of its subsequences. Then we will give a description of the points to be added to $\mathcal{M}_{g,n}$ to serve as limits of these diverging sequences. And finally we will describe the topology on the resulting set and indicate why it is compact.

The first problem is to describe sequences in $\mathcal{M}_{g,n}$. We do not have a global parametrization of $\mathcal{M}_{g,n}$. Note that the map Ψ defined in the previous section can be seen as taking the quotient of $\mathcal{T}_{g,n}$ by a subgroup of $\Gamma_{g,n}$, namely the subgroup generated by Dehn twists along the closed loops forming part of the particular maximal multicurve. To obtain $\mathcal{M}_{g,n}$ one would have to take a further quotient by Dehn twists along the other closed loops. But the effect of such Dehn twists on the parameters is too difficult to describe, hence the difficulty of parametrizing $\mathcal{M}_{g,n}$. We thus content ourselves to describing point in $\mathcal{M}_{g,n}$ by choosing whichever maximal multicurve is most useful and using the corresponding parameters of $\mathcal{T}_{g,n}$ for points in the quotient.

Now fix a maximal multicurve \mathcal{C} on $S \setminus P$ and consider some unbounded sequence $(m_i)_{i \in \mathbb{N}}$ in $\mathcal{M}_{g,n}$. Use $\tilde{\mathcal{T}}_{g,n}$ to denote $(\mathbb{R}_{>0})^{3g-3+n} \times (\mathbb{R}/(2\pi\mathbb{Z}))^{3g-3+n}$. As shown before, \mathcal{C} determines a surjective continuous function $\Psi : \mathcal{T}_{g,n} \rightarrow \tilde{\mathcal{T}}_{g,n}$ which lifts the natural projection $\pi : \mathcal{T}_{g,n} \rightarrow \mathcal{M}_{g,n}$. This means that if $\tilde{\pi} : \tilde{\mathcal{T}}_{g,n} \rightarrow \mathcal{M}_{g,n}$ denotes the projection arising from quotienting out by Dehn twists along the closed loops not forming part of \mathcal{C} , then $\pi = \tilde{\pi} \circ \phi$.

Consider any sequence $(n_i)_{i \in \mathbb{N}}$ in $\tilde{\mathcal{T}}_{g,n}$ which maps down to $(m_i)_{i \in \mathbb{N}}$. Since the projection map from $\tilde{\mathcal{T}}_{g,n}$ to $\mathcal{M}_{g,n}$ is continuous, $(n_i)_{i \in \mathbb{N}}$ is also unbounded. While the twist parameters of $\tilde{\mathcal{T}}_{g,n}$ take their values on a circle, which is compact, the length parameters vary along the positive real numbers, which is not compact. Hence unbounded sequences in $\mathcal{T}_{g,n}$ will occur where the lengths of the geodesics in some isotopy class of closed curves tend to infinity or to zero. But it turns out that when the lengths of some geodesics get very large, others must get very small. The following result ([BFL⁺03], p14) is a starting point for showing that we can restrict ourselves to the case where the lengths tend to zero.

Proposition 3.2.5. For every type (g, n) there is a constant $C_{g,n}$, such that for any hyperbolic structure on a surface of type (g, n) , there is a maximal multicurve for which all geodesics have length less than $C_{g,n}$.

Before using this result we need the following lemma which is stated without proof (it will become clearer later on):

Lemma 3.2.6. On a given reference surface $S \setminus P$ there are only a finite number of maximal multicurves up to diffeomorphisms of the surface.

In other words, we can choose a finite set of maximal multicurves (call it \mathcal{D}) such that any maximal multicurve lies in the orbit of one of those under the natural action of $\text{Diff}^+(S, P)$ on the set of all maximal multicurves.

Now we can use Proposition 3.2.5 to prove the following:

Proposition 3.2.7. Given any sequence $(m_i)_{i \in \mathbb{N}}$ in $\mathcal{M}_{g,n}$, there is a subsequence $(m_j)_{j \in J}$, $J \subset \mathbb{N}$ and a sequence $(n_j)_{j \in J}$ in $\mathcal{T}_{g,n}$ mapping down to $(m_j)_{j \in J}$ such that for a certain maximal multicurve the twist parameters of the sequence $(n_j)_{j \in J}$ are bounded and the length parameters are uniformly bounded from above. Hence if the original sequence was unbounded, then we can assume that some of the length parameters of $(n_j)_{j \in J}$ must tend to 0.

Proof. Given the sequence $(m_i)_{i \in \mathbb{N}}$, choose any sequence $(n_i)_{i \in \mathbb{N}}$ in $\mathcal{T}_{g,n}$ mapping down to it. The idea is now to replace elements of the sequence with other elements lying in the same fiber in order to obtain a sequence with a subsequence satisfying the requirements of the proposition.

Since changing the twist parameter of a point in $\mathcal{T}_{g,n}$ by a multiple of 2π gives another point in the same fiber, we can assume that all twist parameters of all elements of the sequence $(n_i)_{i \in \mathbb{N}}$ lie in a compact interval, say $[0, 2\pi]$. Let \mathcal{D} be a finite set of representatives of maximal multicurves up to diffeomorphism as explained above. By Proposition 3.2.5 we can associate to every point n_i in $\mathcal{T}_{g,n}$ a maximal multicurve \mathcal{C} for which the length parameters of n_i are bounded above by a fixed constant $C(g, n)$. By applying the diffeomorphism which takes \mathcal{C} to an element of \mathcal{D} to n_i , we find $n'_i \in \mathcal{T}_{g,n}$ whose length parameters are bounded above by $C(g, n)$ for a maximal multicurve from \mathcal{D} . Replacing n_i by n'_i (which lies in the same fiber), and using the fact that \mathcal{D} is finite, we can find a subsequence $(n_j)_{j \in J}$, $J \subset \mathbb{N}$ whose length parameters are all bounded above by $C(g, n)$ for a fixed maximal multicurve from \mathcal{D} . This subsequence maps down to $(m_j)_{j \in J}$ in $\mathcal{M}_{g,n}$ and satisfies the conditions of the proposition. For the final part: If the original sequence was unbounded, then by taking a further subsequence of $(n_j)_{j \in J}$, we can assume that some of the length parameters tend to 0 while the rest are bounded from below. \square

This suggests that the points which should be added to $\mathcal{M}_{g,n}$ to compactify it, should be the limits of sequences constructed as follows: Choose a maximal multicurve $\{\alpha_1, \dots, \alpha_{3g-3+n}\}$ of the reference surface, giving a parametrization of $\mathcal{T}_{g,n}$. Choose some hyperbolic structure on the surface as a starting point in $\mathcal{T}_{g,n}$. Now choose a subset of curves $\{\alpha_1, \dots, \alpha_r\}$ from the multicurve, and construct a sequence in $\mathcal{T}_{g,n}$ where the length parameters of the curves in this subset all tend to 0. The other length parameters and twist parameters must also converge, for example by staying constant. The image of this sequence in $\mathcal{M}_{g,n}$ will converge to a point in (the compactification) $\overline{\mathcal{M}}_{g,n}$. The process can be described as *pinching* the surface along the selection of curves from the multicurve.

From the hyperbolic structure perspective, pinching along a curve divides the original surface into two disjoint surfaces (or possibly only one), each with a specific hyperbolic structure and a specific puncture corresponding to the place where the two surfaces were divided. Repeating this process for all the curves $\{\alpha_1, \dots, \alpha_r\}$ suggests that a point in $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ (called a *point at infinity*) can be viewed as a disjoint union of hyperbolic surfaces together with a list of r pairs of punctures to indicate where they must be joined to obtain a nearby point in $\mathcal{M}_{g,n}$.

From the complex structure perspective, we can define these extra points as *Riemann surfaces with nodes*.

Definition 3.2.8. A Riemann surface X with nodes is defined to be a connected Hausdorff space with a finite set of r nodes $X_0 = \{x_1, \dots, x_r\}$ such that:

1. $X \setminus X_0$ is a disjoint union of ordinary compact Riemann surfaces with punctures, called the components of X . Thus every point of $X \setminus X_0$ has a neighbourhood homeomorphic to an open subset of the complex plane.
2. The components of $X \setminus X_0$ have negative Euler characteristic.
3. Every x_i has a neighbourhood homeomorphic to $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < 1, |z_2| < 1, z_1 z_2 = 0\}$, i.e. two copies of the open unit disk, joined at the origin, which is the point to which x_i is mapped to.
4. $P \cap X_0 = \emptyset$, where P denotes the set of punctures of X .

We say X is of type (g, n) if both of the following holds:

1. X has n punctures, or equivalently, $X \setminus X_0$ has $n + 2r$ punctures.
2. If $X \setminus X_0$ has p connected components (treating punctures as marked points to make it compact) and g_i is the genus of the i 'th component, then $\sum_i g_i - p + r + 1 = g$.

The last two conditions are to ensure that a Riemann surface with nodes of type (g, n) can be obtained by taking a Riemann surface of type (g, n) and pinching it along a selection of r curves from some maximal multicurve. The requirement that the components have negative Euler characteristic implies that r is bounded above by the maximum number of curves in a multicurve, i.e. $3g - 3 + n$.

A homeomorphism $f : X \rightarrow Y$ of Riemann surfaces with nodes is said to be biholomorphic if it induces a biholomorphic mapping from every component of X to a component of Y . Thus it must take every node of X to some node of Y . If there exists such a biholomorphic mapping then X and Y are said to be isomorphic.

Definition 3.2.9. $\overline{\mathcal{M}}_{g,n}$ is defined as the union of $\mathcal{M}_{g,n}$ with the set of all Riemann surfaces of type (g, n) with at least 1 node.

The topology on $\overline{\mathcal{M}}_{g,n}$

To describe a topology on $\overline{\mathcal{M}}_{g,n}$ it is only necessary to define open neighbourhoods of points in $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ since there is already a topology on $\mathcal{M}_{g,n}$. Consider a Riemann surface X of type (g, n) with r nodes. Let \mathcal{C} be a maximal multicurve on a reference surface $S \setminus P$ of type (g, n) such that X can be obtained by pinching along the first r curves of the multicurve. This means that there is a parametrization of $\mathcal{T}_{g,n}$ (determined by \mathcal{C}) and a sequence in $\mathcal{T}_{g,n}$ which maps to a sequence in $\mathcal{M}_{g,n}$ which converges to X (in $\overline{\mathcal{M}}_{g,n}$).

Denote by $\mathcal{T}_{g,n}^{\mathcal{C}}$ the extension of $\mathcal{T}_{g,n}$ to include points where some of the length parameters are 0, thus $\mathcal{T}_{g,n}^{\mathcal{C}}$ is homeomorphic to $(\mathbb{R}_{\geq 0})^{3g-3+n} \times \mathbb{R}^{3g-3+n}$. The natural projection π of $\mathcal{T}_{g,n}$ onto $\mathcal{M}_{g,n}$ can also be extended to a projection from $\mathcal{T}_{g,n}^{\mathcal{C}}$ to $\overline{\mathcal{M}}_{g,n}$ called $\tilde{\pi}$ (which is no longer onto). Then the sequence in $\mathcal{T}_{g,n}$ converges to a point \tilde{X} in $\mathcal{T}_{g,n}^{\mathcal{C}}$ lying above X , with $l_i(\tilde{X}) = 0$ for $1 \leq i \leq r$. Neighbourhoods of \tilde{X} map down to neighbourhoods of X . Given $\mu > 0$, $\tau > 0$, we define a neighbourhood $U_{\mu,\tau}$ of X :

$$U_{\mu,\tau} = \left\{ \tilde{\pi}(Y) \mid Y \in \mathcal{T}_{g,n}^{\mathcal{C}}, |l_i(Y) - l_i(\tilde{X})| < \mu \text{ for all } i, |\theta_i(Y) - \theta_i(\tilde{X})| < \tau \text{ for } i > r \right\} \quad (3.2)$$

The $U_{\mu,\tau}$'s form a neighbourhood basis of X in $\overline{\mathcal{M}}_{g,n}$. It remains to be shown that the resulting topology is compact. We will only sketch the main idea. Starting with a maximal multicurve on a hyperbolic surface X we can pinch all the curves in the multicurve to obtain a point of maximal degeneration in $\overline{\mathcal{M}}_{g,n}$ (also called a terminal Riemann surface). Pinching along another maximal multicurve which can be obtained via a diffeomorphism of the surface gives the same point. As mentioned in Lemma 3.2.6, it can be shown that there are only a finite number of possible maximal multicurves on a surface up to diffeomorphism, and hence a finite number of points of maximal degeneration. This is proved by labelling each with a trivalent graph (see Figure 3.4). Note that the vertices of degree 3 are in bijection with the pairs of pants, with two vertices sharing an edge only if the corresponding pairs of pants share a boundary. The two sketches on the right of Figure 3.4 show how to obtain two of the three possible points of maximal degeneration of $\mathcal{M}_{0,4}$. The upper one pairs the upper-left and lower-left punctures together while the lower one pairs the upper-left puncture with the upper-right puncture. The third (not shown) would pair the upper-left puncture with the lower-right puncture.

For each point \mathcal{P} of maximal degeneration we can define a neighbourhood consisting of all hyperbolic structures which can degenerate to \mathcal{P} along a maximal multicurve whose lengths are bounded above by $C_{g,n}$. Since every hyperbolic structure admits such a multicurve by Proposition 3.2.5, these open sets cover $\overline{\mathcal{M}}_{g,n}$, showing that it is compact.

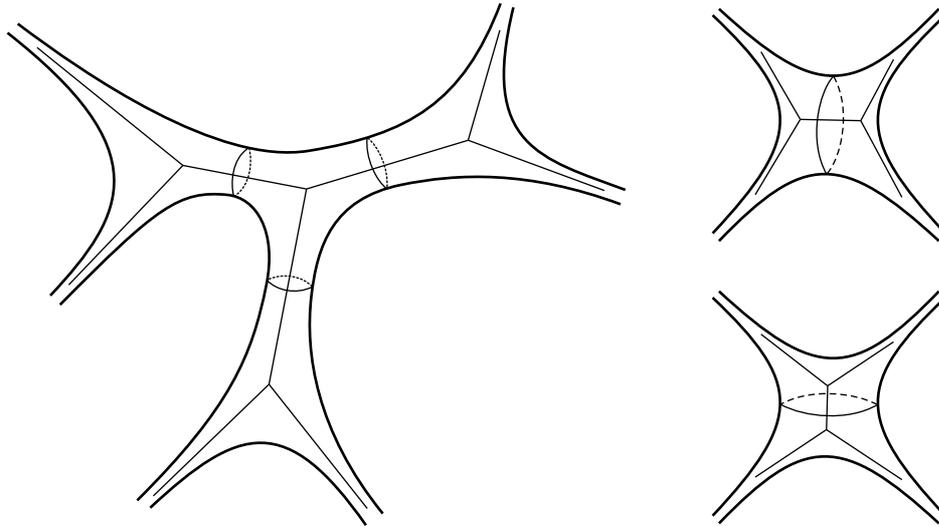


Figure 3.4: *Points of maximal degeneration correspond to trivalent graphs*

The stratified structure of $\overline{\mathcal{M}}_{g,n}$

We now make some general comments on the stratified structure of $\overline{\mathcal{M}}_{g,n}$. The examples later on will make it more concrete.

Recall that the boundary of a subset U of a topological space is defined as $\overline{U} \setminus U$. Thus the boundary of $\mathcal{M}_{g,n}$ is $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$.

Consider a surface of type (g, n) . All points in the boundary of $\mathcal{M}_{g,n}$ can be obtained by pinching the surface along some non-empty multicurve (not necessarily maximal) and giving the resulting surface with nodes the structure of a Riemann surface with nodes. Let $\{\alpha_1, \dots, \alpha_{3g-3+n}\}$ be a maximal multicurve on the surface. Denote by U_1 the set of all points in the boundary of $\mathcal{M}_{g,n}$ which can be obtained by pinching the surface along α_1 only. Then U_1 forms a complex analytic space of (complex) dimension one less than the dimension of $\mathcal{M}_{g,n}$ (which is $3g - 3 + n$). This is because of the disappearance of two real parameters, namely the length and twist parameters corresponding to α_1 . In fact, U_1 is isomorphic either to a single lower dimensional moduli space or to the product of two lower dimensional moduli spaces depending on whether the surface remains connected after cutting along α_1 or whether it is divided into two components. Note that the boundary of U_1 consists of points obtained by pinching the surface along a multicurve containing α_1 and at least one other curve.

Returning to the maximal multicurve $\{\alpha_1, \dots, \alpha_{3g-3+n}\}$, let U_i be the set of all points in the boundary of $\mathcal{M}_{g,n}$ which can be obtained by pinching the surface along the multicurve $\{\alpha_1, \dots, \alpha_i\}$, with $1 \leq i \leq 3g-3+n$. Then U_i has complex co-dimension i relative to $\mathcal{M}_{g,n}$ and contains U_k in its boundary where $k > i$. It follows that U_{3g-3+n} is just a single point, namely a point of maximal degeneration. We can arrange the sets $\mathcal{M}_{g,n}, U_1, \dots, U_{3g-3+n}$

in a lattice under the relation ‘is contained in the boundary of’. So $U_i < U_j$ if U_i is contained in the boundary of U_j . In this case the lattice will simply be a single column with $\mathcal{M}_{g,n}$ at the top and U_{3g-3+n} at the bottom. The vertical position of a set in this lattice determines its complex dimension.

More generally, let \mathcal{C} be any multicurve on the reference surface. Denote by $U_{\mathcal{C}}$ all the points in $\overline{\mathcal{M}}_{g,n}$ which can be obtained by pinching along \mathcal{C} . Define a relation (partial ordering) on the set of all multicurves by $\mathcal{C} < \mathcal{C}'$ if $\mathcal{C} \supset \mathcal{C}'$, or equivalently, if $U_{\mathcal{C}}$ is contained in the boundary of $U_{\mathcal{C}'}$. As before this gives a lattice with $\mathcal{M}_{g,n}$ (corresponding to U_{\emptyset}) at the top and the points of maximal degeneration in the bottom layer. Each set in the lattice is isomorphic to the product of a finite number of moduli spaces, with its dimension determined by its layer in the lattice. Note that the boundary of any set in the lattice is the union of all the sets below it, in particular $\overline{\mathcal{M}}_{g,n}$ is the disjoint union of all the sets in the lattice.

3.3 Examples

In this section we introduce some basic examples which will be useful later on.

3.3.1 $\mathcal{M}_{0,4}$

We have already made reference to the fact that $\mathcal{M}_{0,3}$ is just a point. The next simplest case is $\mathcal{M}_{0,4}$, which plays a very important role in this thesis.

Since there is only one Riemann surface of genus 0, $\mathcal{M}_{0,4}$ can be described as the sphere with 4 (ordered) marked points up to an automorphism of the sphere. The automorphism group of the sphere is three-transitive and an automorphism is fixed by knowing the images of three distinct points. So given a representative of a point in $\mathcal{M}_{0,4}$, there is a unique automorphism taking the first three marked points to 0, 1 and ∞ respectively. The fourth point must lie in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ and uniquely characterizes the point of $\mathcal{M}_{0,4}$. Thus $\mathcal{M}_{0,4}$ is parametrized by $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ and is in fact isomorphic (as a complex manifold) to it by the mapping just described.

$\Gamma_{0,4}$ is just the fundamental group of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, which is generated by the loops x , y and z around the points 0, 1 and ∞ respectively. It can also be viewed as being generated by Dehn twists. There are 3 possible simple closed loops up to diffeomorphism on the reference surface, corresponding to the 3 ways of partitioning the 4 marked points into two sets with two elements each. For instance, if we label the marked points $\{0, 1, \infty, \lambda\}$, then one simple closed loop (say α) will separate $\{0, \lambda\}$ from $\{1, \infty\}$. A Dehn twist along α can be identified with the element x from the fundamental group. Similarly for Dehn twists along the other two loops.

Pinching the sphere along α has the effect of moving λ towards 0. Similarly for the other two loops. Thus there are three points at infinity (three copies of $\mathcal{M}_{0,3}$) and $\overline{\mathcal{M}}_{0,4}$ is just the sphere.

3.3.2 $\mathcal{M}_{0,5}$

Once again we can use an automorphism of the sphere to move the first three marked points to 0, 1 and ∞ . Thus $\mathcal{M}_{0,5}$ can be identified with (and is once again isomorphic to) $(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\})^2 \setminus \Delta$, where Δ is the diagonal of $(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\})^2$ (where the two coordinates coincide). So we can view paths in $\mathcal{M}_{0,5}$ as two distinct points moving around on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. Or as the isomorphism classes of 5 points moving around on the sphere.

This can be generalized to the other genus 0 moduli spaces. We find that $\mathcal{M}_{0,n}$ is isomorphic to $(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta$, where Δ is the thick diagonal where not all coordinates are distinct.

Simple closed loops must separate the marked points into two sets with 2 and 3 points respectively. Thus there are $\binom{5}{2} = 10$ possible simple closed loops up to diffeomorphism. A Dehn twist along such a closed loop can be viewed as rotating 2 adjacent marked points by a full revolution around a point halfway between them. This will become clearer in the section on braid groups.

Pinching along a curve divides the sphere with 5 marked points into a sphere with 3 marked points and a sphere with 4 marked points. This corresponds to a part of $\overline{\mathcal{M}}_{0,5} \setminus \mathcal{M}_{0,5}$ isomorphic to $\mathcal{M}_{0,4}$, which has complex dimension 1 and can be referred to as a line at infinity. Pinching along another curve gives a single point ($\mathcal{M}_{0,3}$) at infinity.

To count these lines and points at infinity, we start by viewing points in $\overline{\mathcal{M}}_{0,5}$ as 5 points moving around on a sphere, call them $\{x_1, \dots, x_5\}$. A line at infinity L_{ij} consists of the cases where $x_i = x_j$ and the other points are distinct. Thus there are 10 lines at infinity, indexed by $1 \leq i < j \leq 5$. A point at infinity $P_{ij,kl}$ is where $x_i = x_j$ and $x_k = x_l$. There are 15 of these, indexed by $1 \leq i, j, k, l \leq 5$, $i < j$, $k < l$ and $i < k$. Thus we have the following expression:

$$\overline{\mathcal{M}}_{0,5} = \mathcal{M}_{0,5} \cup_{i,j} L_{ij} \cup_{i,j,k,l} P_{ij,kl} \quad (3.3)$$

Thus the lattice describing the stratified structure of $\overline{\mathcal{M}}_{0,5}$ consists of $\mathcal{M}_{0,5}$, 10 copies of $\mathcal{M}_{0,4}$ and 15 copies of $\mathcal{M}_{0,3}$. The boundary of each copy of $\mathcal{M}_{0,4}$ consists of 3 copies of $\mathcal{M}_{0,3}$ and each copy of $\mathcal{M}_{0,3}$ is contained in the boundary of 2 different copies of $\mathcal{M}_{0,4}$.

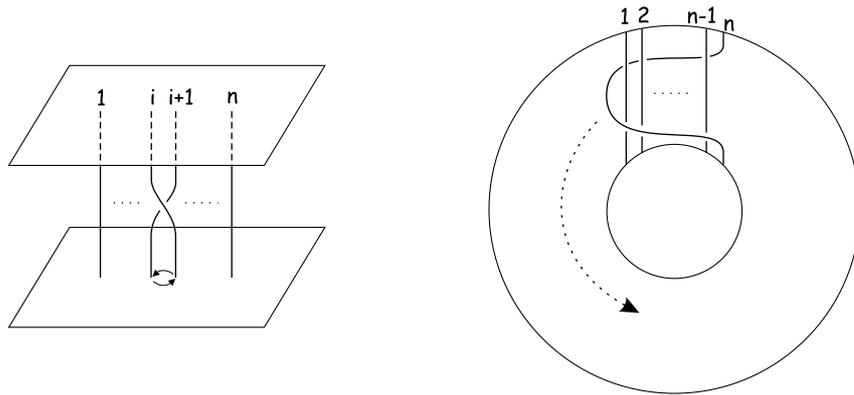


Figure 3.5: Generator σ_i on left and sphere relation $y_n = 1$ on right

3.4 Braid groups

The link between braid groups and moduli spaces is that the genus 0 mapping class group $\Gamma_{0,n}$ is isomorphic to a quotient of the pure Artin braid group on n strings. This allows the possibility of deducing the action of $G_{\mathbb{Q}}$ on the fundamental groups of moduli spaces from its action on braid groups in the genus 0 case.

In this section we will give an introduction to the relevant braid groups, show where the relations come from and how they are connected to the mapping class groups.

Consider two copies of the complex plane lying above each other, each with the same n distinct marked points. The elements of the Artin (or plane) braid group B_n are all the different ways of connecting the marked points on the top plane with those on the bottom plane by means of n strings. Composition is given vertically by placing 2 pairs of complex planes above each other, removing the inner 2 planes and joining the matching strings. The group B_n is generated by the elements $\sigma_1, \dots, \sigma_{n-1}$, where σ_i corresponds to swapping the endpoints of the i 'th and $(i+1)$ 'th strings by letting the i 'th one pass in front of the $(i+1)$ 'th one (see Figure 3.5). This can also be described by viewing the motion of the endpoints on the complex plane: They rotate in an anti-clockwise direction (as seen from above) half a revolution about their midpoint.

Geometric considerations show that these generators have to satisfy the relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for all i , and $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \geq 2$, giving a presentation for B_n :

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } i \text{ and } \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2 \rangle.$$

B_n can also be defined as the fundamental group of a configuration space, namely the space of n distinct *unordered* marked points on the complex plane. The connection with the above braiding definition is that the set of marked points on the plane to which

the strings are attached can be used as a base point for the configuration space and the multipaths traced out by the endpoints of the strings on the lower complex plane are the loops in the configuration space.

Instead of braiding strings between two planes, we can consider strings lying between two concentric spheres, joining points on the outer sphere to those on the inner sphere. The corresponding sphere braid group H_n (also known as the Hurwitz braid group) is obtained from B_n by quotienting out by the sphere relation $y_n = 1$, where $y_i = \sigma_{i-1}\sigma_{i-2}\cdots\sigma_1 \cdot \sigma_1 \cdots \sigma_{i-2}\sigma_{i-1}$. In other words:

$$H_n = B_n / \langle y_n \rangle.$$

Geometrically this corresponds to the fact that braiding the last string around all the others is the same as the trivial braid since the string can be moved around the inner sphere (see Figure 3.5).

H_n may be seen as the fundamental group of the space of n distinct unordered marked points on the sphere. This suggests a connection with $\Gamma_{0,n}$, which however is the fundamental group of the possible complex structures on a sphere with n *ordered* marked points. To establish a connection we introduce a variant of moduli space. Denote by $\mathcal{M}_{0,[n]}$ the (complex analytic) space of complex structures on the sphere up to isomorphism where the isomorphism is allowed to *permute* the marked points. Denote the corresponding Teichmüller space by $\mathcal{T}_{0,[n]}$ and the mapping class group by $\Gamma_{0,[n]}$. (In this case $\Gamma_{0,[n]}$ is once again the *orbifold* fundamental group of $\mathcal{M}_{0,[n]}$ seen as an orbifold.) $\mathcal{M}_{0,[n]}$ is the quotient of $\mathcal{M}_{0,n}$ by the action of the symmetric group S_n permuting the marked points.

$\Gamma_{0,[n]}$ can be obtained from H_n by quotienting out by the centre relation $w_n = 1$, where $w_n = (\sigma_1 \cdots \sigma_{n-1})^n = y_1 \cdots y_n$ is the generator of the centre of H_n (and of B_n) (see Figure 3.6). In other words:

$$\Gamma_{0,[n]} = H_n / \langle w_n \rangle.$$

The picture in the middle of Figure 3.6 shows the movement of the ends of the strings along the surface of the inner sphere (recall that the strings are connected between two concentric spheres). The end point of each string moves along a horizontal path once around the sphere. The picture on the right illustrates this by showing the same movement as seen from above. After moving the end points of the strings in this way, we obtain the braiding pattern shown in the picture on the left of Figure 3.6.

To offer some justification for why we can obtain $\Gamma_{0,[n]}$ in this way, consider the action of H_n on $\mathcal{T}_{0,[n]}$. The multiple path (multipath) traced out by the end points of the strings gives a path in Teichmüller space. Since the multipath of the end points ends in the same set where it started, H_n acts on the fibers above points of the moduli space, just like $\Gamma_{0,[n]}$ does. This gives a mapping from H_n to $\Gamma_{0,[n]}$ which we claim contains w_n in its

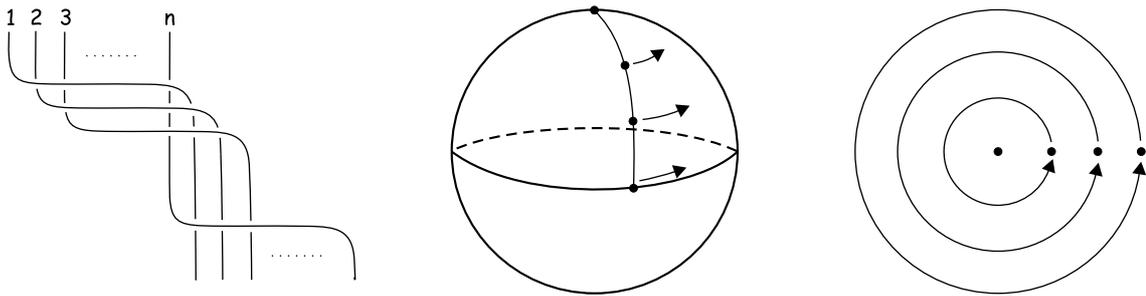


Figure 3.6: *The centre relation $w_n = 1$*

kernel. Note that when viewing a path in $\mathcal{T}_{0,[n]}$ as a multipath traced out by n points on a sphere, we can apply an automorphism of the sphere at every point of the multipath to take the first three points to $\{0, 1, \infty\}$ (or any other 3 fixed points), and then consider the multipath traced out by the other $n - 3$ points. From Figure 3.6 we see that the multipath in $\mathcal{T}_{0,[n]}$ traced out by w_n consist of rotations about an axis of the sphere. Thus every point in the multipath can be returned to the starting point via an automorphism (rotation) of the sphere. Hence w_n induces the trivial path in $\mathcal{T}_{0,[n]}$ and must be quotiented out to get $\Gamma_{0,[n]}$.

The groups defined thus far were all concerned with *unordered* marked points. To go to the ordered case, we start by noting that there is a natural group homomorphism from B_n onto S_n , taking the generator σ_i to the transposition $(i, i + 1)$. The kernel is called the *pure* Artin braid group, denoted by K_n . In other words, K_n is the group fitting into the exact sequence:

$$1 \rightarrow K_n \rightarrow B_n \rightarrow S_n \rightarrow 1.$$

It corresponds to the braids where the end point of each string lies directly beneath its starting point.

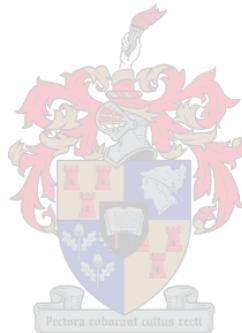
K_n is generated by the elements $x_{ij} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$. Here x_{ij} corresponds to the i 'th point winding counter-clockwise around the j 'th point whilst avoiding the other points. From the braiding perspective, imagine that the strings are attached to the n points on the complex plane corresponding to the n 'th roots of unity, indexed counter-clockwise for example. Then x_{ij} corresponding to twisting the i 'th and j 'th string about each other.

In a similar way we obtain the *pure* sphere braid group P_n and the (pure) mapping class group $\Gamma_{0,n}$ as the kernels of the natural maps from H_n and $\Gamma_{0,[n]}$ to S_n respectively. Equivalently, we could obtain P_n as the image of K_n in the map from B_n to H_n , and $\Gamma_{0,n}$ as the image of P_n in the map from H_n to $\Gamma_{0,[n]}$ (see diagram 3.4 below). The images of the generators x_{ij} of K_n are also generators for P_n and $\Gamma_{0,n}$, and will be denoted by the same symbols.

In fact, the generator x_{ij} of $\Gamma_{0,n}$ corresponds to the Dehn twist along the loop separating the two points i and j from the rest.

The diagram below shows the connections between the various groups considered in this section. Vertical arrows are inclusions, and horizontal arrows are surjections.

$$\begin{array}{ccccc} B_n & \longrightarrow & H_n & \longrightarrow & \Gamma_{0,[n]} \\ \uparrow & & \uparrow & & \uparrow \\ K_n & \longrightarrow & P_n & \longrightarrow & \Gamma_{0,n} \end{array} \tag{3.4}$$



Chapter 4

A different description of $G_{\mathbb{Q}}$

In this chapter we will show how $G_{\mathbb{Q}}$ can be parametrized by certain pairs (λ, f) from $\hat{\mathbb{Z}}^{\times} \times \hat{F}'_2$ where $\hat{\mathbb{Z}}^{\times}$ is the group of invertible elements of the profinite completion of \mathbb{Z} and \hat{F}'_2 is the commutator subgroup of the profinite free group on two generators. We will see how the action of $G_{\mathbb{Q}}$ on the fundamental groups of certain moduli spaces determines the equations that such pairs (λ, f) must satisfy. We will define the Grothendieck-Teichmüller group (\widehat{GT}) as the pairs satisfying these equations. This will imply that the group \widehat{GT} , defined without any reference to algebraic numbers, contains $G_{\mathbb{Q}}$ as a subgroup, with equality being an open question.

4.1 Coverings and fundamental groups

4.1.1 The fundamental group exact sequence

We begin by explaining the connection between the moduli spaces from the previous chapter and $G_{\mathbb{Q}}$.

Let X be a smooth, absolutely irreducible variety over \mathbb{Q} . Let $\overline{X} := X \otimes \overline{\mathbb{Q}}$, and let $\mathbb{Q}(X)$ denote the function field of X . Let M be the maximal extension of $\mathbb{Q}(X)$ unramified at every point of X . Then we have the Galois extension:

$$\mathbb{Q}(X) \subset \mathbb{Q}(\overline{X}) \subset M \tag{4.1}$$

and the corresponding exact sequence:

$$1 \rightarrow \text{Gal}(M/\mathbb{Q}(\overline{X})) \rightarrow \text{Gal}(M/\mathbb{Q}(X)) \rightarrow \text{Gal}(\mathbb{Q}(\overline{X})/\mathbb{Q}(X)) \rightarrow 1. \tag{4.2}$$

Choosing a geometric point x of X as a base point, the following exact sequence is known to follow from the previous one via isomorphisms between the respective groups (due to Grothendieck, [Gro1]):

$$1 \rightarrow \pi_1^{\text{alg}}(\overline{X}, x) \rightarrow \pi_1^{\text{alg}}(X, x) \rightarrow \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow 1. \tag{4.3}$$

Let X^{an} denote the complex manifold obtained from X via analytification. Then it is known that the first group in exact sequence (4.3), i.e. the algebraic fundamental group of \overline{X} , is isomorphic to the profinite completion of the topological fundamental group of X^{an} . In the cases we will study, this will be the well-known object providing the link to the absolute Galois group on the right of exact sequence (4.3). The group in the middle is also known as the arithmetic fundamental group of X .

To introduce some basic group-theoretic notions related to such a short-exact sequence, we replace the above groups by arbitrary groups A , B and C , fitting into the exact sequence:

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1. \quad (4.4)$$

The group of automorphisms of A is denoted by $\text{Aut}(A)$. To any element a of A we can associate an automorphism of A where a acts via conjugation, i.e. $x \in A$ is taken to $a^{-1}xa$. These automorphisms are called *inner* automorphisms and form a normal subgroup of $\text{Aut}(A)$ called $\text{Inn}(A)$. The quotient is denoted by $\text{Out}(A)$ and its elements are called *outer* automorphisms although they do not determine a unique automorphism of A but rather a coset of $\text{Inn}(A)$ in $\text{Aut}(A)$.

Associated to the exact sequence is a canonical group homomorphism from C to $\text{Out}(A)$: Start with an element $c \in C$. Let $b \in B$ be an element mapping to c . Regarding A as a (normal) subgroup of B , we let b act on A via conjugation, i.e. $a \in A$ is taken to $b^{-1}ab$. Note that $b^{-1}ab \in A$ since $A \cong \ker(g) \triangleleft B$. This gives an automorphism of A . Choosing a different element b' in the pre-image of c would give an automorphism of A differing from the first by an inner automorphism given by conjugation by $b^{-1}b'$. Thus to c is associated a well-defined outer automorphism of A .

Sometimes one can choose the b 's in B associated to the different c 's in a consistent way, that is by using a homomorphism h from C to B such that $g \circ h = \text{id}$. Then the exact sequence is said to split, and h is called a section of g . In such a case the outer action is lifted to an action, that is a group homomorphism ψ from C to $\text{Aut}(A)$:

$$\begin{aligned} \psi : C &\rightarrow \text{Aut}(A) \\ c &\mapsto \phi_c \end{aligned}$$

where

$$\begin{aligned} \phi_c : A &\rightarrow A \\ a &\mapsto f^{-1}(g(c)^{-1}f(a)g(c)). \end{aligned}$$

Returning to exact sequence (4.3), we thus have a group homomorphism $G_{\mathbb{Q}} \rightarrow \text{Out}(\pi_1^{\text{alg}}(\overline{X}, x))$. If X is taken to be (the algebraic version of) some moduli space as considered in the previous chapter, then it turns out that this group homomorphism is an injection, and the challenge is to describe its image.

4.1.2 Finite Galois coverings

The main object of interest in this chapter will be the algebraic variety $\mathbb{P}^1(\overline{\mathbb{Q}}) \setminus \{0, 1, \infty\}$. As shown in the previous chapter it is (the algebraic version of) the moduli space $\mathcal{M}_{0,4}$ which has \hat{F}_2 as its algebraic fundamental group. Thus it is the $G_{\mathbb{Q}}$ -action on \hat{F}_2 which will be our focus as explained above. But we will start by considering the more general case of the punctured sphere $X = \mathbb{P}^1(\overline{\mathbb{Q}}) \setminus \{x_1, x_2, \dots, x_n\}$. We require that $\{x_1, x_2, \dots, x_n\}$ all lie in $\mathbb{P}^1(\mathbb{Q})$, thus X is a variety defined over \mathbb{Q} . For simplicity we will often use the same notation X to refer to the analytification X^{an} , relying on the context to make the meaning clear. If X is to be considered as algebraic variety, we could use the notation $X_{\overline{\mathbb{Q}}}$ or $X_{\mathbb{Q}}$. In this subsection we will take a topological perspective, hence for now X is considered as a Riemann surface.

Let $f : Y \rightarrow X$ be a finite (unbranched) Galois covering. There is a unique way via compactification to extend this to a branched covering $\bar{f} : \bar{Y} \rightarrow \bar{X}$, where \bar{X} is the Riemann sphere. The group of fiber-preserving automorphisms of Y is denoted by $\text{Gal}(Y/X)$, which has order equal to the degree of the covering.

Let $x \in X$ be an arbitrary base point and denote the topological fundamental group by $\pi_1(X, x)$. Later on x will be required to be in \mathbb{Q} . There is a natural surjection from $\pi_1(X, x)$ to $\text{Gal}(Y/X)$: Given $y \in Y$ and $\gamma \in \pi_1(X, x)$, let δ be any path from x to $f(y) = x_0$. Then $\gamma(y) \in f^{-1}(x_0)$ is defined to be the endpoint of the path in Y starting at y obtained by lifting $\delta^{-1} \cdot \gamma \cdot \delta$, where paths are composed from the left. This is of course just the standard projection from Galois theory where $\pi_1(X, x)$ is identified with $\text{Gal}(\tilde{X}/X)$ where \tilde{X} is the universal cover of X .

The algebraic fundamental group is just the profinite completion of the topological fundamental group, hence we denote it by $\hat{\pi}_1(X, x)$. From the previous chapter we know that there is an outer action of $G_{\mathbb{Q}}$ on $\hat{\pi}_1(X, x)$, and in this chapter we describe a lifting of this outer action to an action. An automorphism of $\hat{\pi}_1(X, x)$ is determined by the action on a set of topological generators for $\hat{\pi}_1(X, x)$. It turns out that $\hat{\pi}_1(X, x)$ can be topologically generated by elements which we will refer to as *inertia generators*, and that the possible images of such an element under the action of $G_{\mathbb{Q}}$ can be narrowed down rather precisely. These inertia generators generate subgroups known as inertia subgroups.

In general, $G_{\mathbb{Q}}$ takes an element of $\pi_1(X, x) \subset \hat{\pi}_1(X, x)$ to an element outside $\pi_1(X, x)$, thus we postpone describing the $G_{\mathbb{Q}}$ -action until later when $\hat{\pi}_1(X, x)$ has been described. But we can already define the inertia generators contained in $\pi_1(X, x)$ and their images in $\text{Gal}(Y/X)$ from the topological perspective. This has the advantage of helping the intuition and showing the connection with the usual inertia groups from algebraic number theory as well as with the inertia subgroups of the fundamental groups of other moduli spaces.

To each $x_i \in \bar{X}$, $1 \leq i \leq n$ we can associate a conjugacy class of inertia generators in

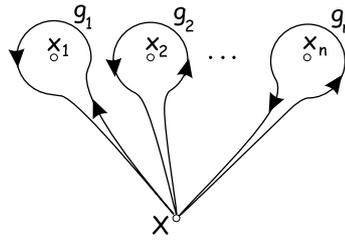


Figure 4.1: *Canonical generators of the fundamental group*

$\pi_1(X, x)$ as follows: Let D_i be a closed disk in \overline{X} centred at x_i , and not containing any of the other x_j 's or the base point x . Let C_i be its boundary. Let δ be a path from x to a point z_i on the boundary of C_i , and let γ be a single loop going anticlockwise around C_i starting and ending at z_i . See Figure 4.1.

Definition 4.1.1. The element $\delta \cdot \gamma \cdot \delta^{-1} \in \pi_1(X, x)$ is called an *inertia generator* associated to x_i . Choosing a different δ , say δ_0 , gives the inertia generator

$$\delta_0 \cdot \gamma \cdot \delta_0^{-1} = (\delta_0 \cdot \delta^{-1}) \cdot (\delta \cdot \gamma \cdot \delta^{-1}) \cdot (\delta_0 \cdot \delta)^{-1}.$$

Thus the collection of such elements gives a well-defined *conjugacy class of inertia generators* associated to x_i . Under the projection to $\text{Gal}(Y/X)$ the images of these elements give a well-defined conjugacy class of inertia generators of $\text{Gal}(Y/X)$ associated to x_i . A cyclic subgroup of $\pi_1(X, x)$ (or $\text{Gal}(Y/X)$) generated by an inertia generator is called an *inertia subgroup*. Thus to every x_i is associated a conjugacy class of inertia subgroups.

The action of $\text{Gal}(Y/X)$ on the fibers of points of X can be extended continuously to an action on the fibers of the x_i 's. The stabilizer in $\text{Gal}(Y/X)$ of $y \in f^{-1}(x_i)$ is one of the inertia subgroups associated to x_i . Thus the number of inertia subgroups in the conjugacy class of x_i equals the cardinality of the fiber above x_i and the order of any inertia generator in $\text{Gal}(Y/X)$ equals the ramification index at y .

This can also be viewed in terms of valuations on the function fields $\mathbb{C}(X) \subset \mathbb{C}(Y)$: The valuation on $\mathbb{C}(X)$ corresponding to x_i can be extended to different valuations on $\mathbb{C}(Y)$ corresponding to points in the fiber $f^{-1}(x_i)$. The Galois group $\text{Gal}(\mathbb{C}(Y)/\mathbb{C}(X))$ (which is isomorphic to $\text{Gal}(Y/X)$) acts on these valuations via composition, giving the same results as above.

This is similar to the situation in algebraic number theory, where one considers a prime ideal \mathfrak{P} lying above a prime ideal \mathfrak{p} in a Galois extension of number fields. The *decomposition group* associated to \mathfrak{P} is the subgroup of the Galois group of the extension taking \mathfrak{P} to itself, and the *inertia group* associated to \mathfrak{P} is the subgroup of the decomposition group which acts trivially on the ring of integers of the extension field modulo \mathfrak{P} . In our

case the inertia group is equal to the decomposition group. Grothendieck's definition of inertia group in [Gro1] is a generalization of both these cases.

Note that it is easy to find n inertia generators for $\pi_1(X, x)$ or $\text{Gal}(Y/X)$ which generate the respective groups. (see figure 4.1) They are often chosen to obey the relation $g_1 g_2 \cdots g_n = 1$ as in the figure. Such a set of generators for $\text{Gal}(Y/X)$ is called a *branch cycle description* of the covering $f : Y \rightarrow X$ and provides a concrete way of parametrizing covers of X as well as being the starting point for a certain approach to attempt to solve the Inverse Galois Problem, which does not concern us here.

To see the connection with Dehn twists, consider the case of $\mathcal{M}_{0,4}$, where $\{x_1, x_2, x_3\}$ are the fixed marked points, and x_4 is the fourth marked point which moves from the base point x around the other x_i 's along the paths of the fundamental group. Then $g_1 \in \pi_1(X, x)$ corresponds to the Dehn twist around a simple closed loop separating x_1 and x_4 from x_2 and x_3 . Similarly for g_2 and g_3 .

In the case of $\mathcal{M}_{0,4}$, the x_i 's are the components of the divisor at infinity. There is a generalization of this idea to higher dimensions, whereby to each component of the divisor at infinity of $\mathcal{M}_{g,n}$ is associated a conjugacy class of inertia subgroups of $\Gamma_{g,n}$.¹

4.1.3 The algebraic fundamental group

Let \mathcal{C} be the category of finite unramified coverings of X . This is equivalent to the category \mathcal{X}_{et} of finite étale covers of $X_{\overline{\mathbb{Q}}}$ ([Gro1]). Choosing a base point x , we can associate to every covering $f : Y \rightarrow X$ in \mathcal{C} the finite set of points in the fiber $f^{-1}(x)$, and to a fiber-preserving map $\phi : Y \rightarrow Y'$ from $f : Y \rightarrow X$ to the covering $f' : Y' \rightarrow X$ (the morphisms of \mathcal{C}), the induced map from the fiber $f^{-1}(x)$ to $f'^{-1}(x)$. This describes a functor F_x :

$$F_x : \mathcal{C} \rightarrow \mathbf{Set}. \quad (4.5)$$

The algebraic fundamental group $\widehat{\pi}_1(X, x)$ can then be defined as (cf. [Gro1]):

$$\widehat{\pi}_1(X, x) := \text{Aut}(F_x). \quad (4.6)$$

To be able to study the $G_{\mathbb{Q}}$ action on $\widehat{\pi}_1(X, x)$ we will start by associating to every point in the fiber of $x \in X$ a specific embedding of the function field $\overline{\mathbb{Q}}(Y)$ into the field of Laurent series at x . Thus the above functor will map an object from \mathcal{C} to a finite set of embeddings instead of just a finite set.

Let $\mathcal{L}_x(\mathbb{C})$ be the set of germs of meromorphic functions at x , and $\mathcal{L}_x(\overline{\mathbb{Q}})$ the subset consisting of those defined over $\overline{\mathbb{Q}}$. Elements of $\mathcal{L}_x(\mathbb{C})$ are given by Laurent series $\sum_{n=N}^{\infty} a_n(z-x)^n$ which converge for values of z in some neighbourhood of x . Let

¹Although this generalization and the inertia-preserving property of the associated $G_{\mathbb{Q}}$ -action is freely used in the literature, I have not been able to find a precise definition or proof. These results are however not needed for our purposes.

$f : Y \rightarrow X$ be a covering from \mathcal{C} and $y \in Y$ a point above x . Then each meromorphic function on Y , say $g \in \mathbb{C}(Y)$, gives a germ of a meromorphic function on Y at y . Since f is a local isomorphism this gives a germ of a meromorphic function on X at x . A meromorphic function on Y is uniquely determined by its germ at any point, thus we have an embedding $\mathbb{C}(Y) \hookrightarrow \mathcal{L}_x(\mathbb{C})$, which restricts to an embedding $\overline{\mathbb{Q}}(Y) \hookrightarrow \mathcal{L}_x(\overline{\mathbb{Q}})$. Note that any such embedding extends the natural embedding of $\overline{\mathbb{Q}}(X) \subset \overline{\mathbb{Q}}(Y)$ into $\mathcal{L}_x(\overline{\mathbb{Q}})$, and in fact every embedding $\overline{\mathbb{Q}}(Y) \hookrightarrow \mathcal{L}_x(\overline{\mathbb{Q}})$ extending this natural embedding is obtained in this way from some y in the fiber of x . Thus there is a 1-1 correspondence between points in the fiber of x and embeddings $\overline{\mathbb{Q}}(Y) \hookrightarrow \mathcal{L}_x(\overline{\mathbb{Q}})$ extending the embedding $\overline{\mathbb{Q}}(X) \hookrightarrow \mathcal{L}_x(\overline{\mathbb{Q}})$, and the functor F_x maps Y in \mathcal{C} to the set of these embeddings.

As already noted, $\widehat{\pi}_1(X, x)$ can be defined as the group of natural transformations of F_x to itself. Given the above description, this can be interpreted more concretely. It means that every $\sigma \in \widehat{\pi}_1(X, x)$ must map an embedding $\epsilon \in F_x(Y)$, $\epsilon : \overline{\mathbb{Q}}(Y) \rightarrow \mathcal{L}_x(\overline{\mathbb{Q}})$ to another embedding $\sigma(\epsilon) \in F_x(Y)$. This is easy to describe for $\sigma \in \pi_1(X, x)$. Firstly, we explain the action of $\pi_1(X, x)$ on $\mathcal{L}_x(\overline{\mathbb{Q}})$. σ acts on $g \in \mathcal{L}_x(\mathbb{C})$ to give $\sigma(g)$, which is the germ of the meromorphic function at x found by continuing g analytically along σ . This gives an action of $\pi_1(X, x)$ on $\mathcal{L}_x(\mathbb{C})$, which restricts to an action on $\mathcal{L}_x(\overline{\mathbb{Q}})$. Now for the action of σ on the set of embeddings. σ acts on the embedding $\epsilon \in F_x(Y)$ to give $\sigma(\epsilon) \in F_x(Y)$, where $\sigma(\epsilon)(g) := \sigma(\epsilon(g))$. This simply means that if ϵ is the embedding corresponding to y in the fiber of x , then $\sigma(\epsilon)$ is the embedding corresponding to the endpoint of the path starting at y found by lifting σ to Y . This shows how every element of $\pi_1(X, x)$ induces an automorphism of F_x .

The above action of $\pi_1(X, x)$ on $\mathcal{L}_x(\overline{\mathbb{Q}})$ can also be interpreted in a slightly different way. Consider a *Galois* cover $f : Y \rightarrow X$ from \mathcal{C} and an embedding ϵ from $F_x(Y)$. Then under the embedding, $\overline{\mathbb{Q}}(Y)$ can be viewed as a finite Galois extension of $\overline{\mathbb{Q}}(X)$. The action of $\sigma \in \pi_1(X, x)$ on $\mathcal{L}_x(\overline{\mathbb{Q}})$ restricts to an action on $\overline{\mathbb{Q}}(Y)$ (since it is Galois) which fixes $\overline{\mathbb{Q}}(X)$. Thus there is a mapping from $\pi_1(X, x)$ to $\text{Gal}(\overline{\mathbb{Q}}(Y)/\overline{\mathbb{Q}}(X))$ dependent on the choice of an embedding ϵ .

To extend this idea we introduce \widetilde{X} , the universal cover of X , with the projection map $\tilde{f} : \widetilde{X} \rightarrow X$. Then $\overline{\mathbb{Q}}(\widetilde{X})$ is the maximal unramified extension of $\overline{\mathbb{Q}}(X)$. Equivalently, it is the injective limit of the function fields $\overline{\mathbb{Q}}(Y)$ as Y varies in \mathcal{C} . And $\text{Gal}(\overline{\mathbb{Q}}(\widetilde{X})/\overline{\mathbb{Q}}(X))$ is the corresponding projective limit of the groups $\text{Gal}(\overline{\mathbb{Q}}(Y)/\overline{\mathbb{Q}}(X))$ as Y varies in \mathcal{C} .

Now just as we did for finite covers, to every point in \widetilde{X} lying above x we can associate an embedding of $\overline{\mathbb{Q}}(\widetilde{X})$ into $\mathcal{L}_x(\overline{\mathbb{Q}})$. Fixing such an embedding, we can restrict the action of $\pi_1(X, x)$ on $\mathcal{L}_x(\overline{\mathbb{Q}})$ to $\overline{\mathbb{Q}}(\widetilde{X})$ as before, giving a mapping from $\pi_1(X, x)$ to $\text{Gal}(\overline{\mathbb{Q}}(\widetilde{X})/\overline{\mathbb{Q}}(X))$. In fact, this can be extended to an isomorphism between $\widehat{\pi}_1(X, x)$ and $\text{Gal}(\overline{\mathbb{Q}}(\widetilde{X})/\overline{\mathbb{Q}}(X))$, which provides a different way of viewing the algebraic fundamental group.

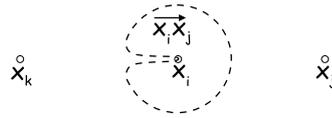


Figure 4.2: *Basepoint at infinity*

4.1.4 Base points at infinity

To explicitly describe the action of $G_{\mathbb{Q}}$ on the inertia generators of the fundamental group, it is necessary to replace the base point x with what is known as a base point at infinity, or a tangential base point. The role of the Laurent series will now be played by Puiseux series.

Let $f : Y \rightarrow X$ be a finite cover and let $y \in f^{-1}(x)$. In the previous subsection we used the fact that a function on Y around y can be viewed as a single-valued function on X around x . Now consider the compactification $\bar{f} : \bar{Y} \rightarrow \bar{X}$, and let $y \in \bar{f}^{-1}(x_i)$ where $x_i \in \bar{X} \setminus X$. The projection \bar{f} locally maps w to z where $w^e = z$ and w and z are local parameters around y and x respectively. Then the single valued function $\sum_{n=N}^{\infty} (a_n w^n)$ around y is mapped to the multivalued function $\sum_{n=N}^{\infty} (a_n (z^{1/e})^n)$ around x_i , where e is the ramification index at y . To select a single-valued branch we must start by removing a radial line of the disk around x_i to get a simply connected region around x_i on X .

We now require that the branch points $\{x_1, x_2, \dots, x_n\}$ lie in $\mathbb{P}^1(\mathbb{Q})$ (in order that $G_{\mathbb{Q}}$ acts trivially on them). The restriction of the natural cyclic ordering of the points of $\mathbb{P}^1(\mathbb{Q})$ gives a cyclic ordering on the branch points, so for every branch point there are two different branch points next to it (assuming $n \geq 3$). One should keep in mind that the case of interest to us is where $n = 3$ and $\{0, 1, \infty\}$ are the branch points. Consider a small punctured disk D_i on X centred at x_i . Suppose x_i lies between branch points x_j and x_k . Then the *base point at infinity* $\overrightarrow{x_i x_j}$ is defined to be the simply connected region found by deleting the line segment between x_i and x_k from D_i (see Figure 4.2). Note that a fundamental group is allowed to have a simply connected region for a base point instead of a point, since any closed loop in a simply connected region is homotopically trivial.

Now analogously to the field of Laurent series \mathcal{L}_x at x we proceed to define the field of Puiseux series at $\overrightarrow{x_i x_j}$. Let z be a local parameter on \bar{X} at x_i which is real positive on the line segment between x_i and x_j . Denote by $z^{1/e}$ the single-valued meromorphic function on $\overrightarrow{x_i x_j}$ which is real positive where z is real positive. Consider the field of all meromorphic functions on $\overrightarrow{x_i x_j}$ which can be expressed as $\sum_{n=N}^{\infty} (a_n (z^{1/e})^n)$ for some positive integer e . We call this the field of convergent Puiseux series at $\overrightarrow{x_i x_j}$ and denote it by $\mathcal{P}_{\overrightarrow{x_i x_j}}$. It clearly contains the field of convergent Laurent series on \bar{X} at x_i and can be shown to be its algebraic closure.

For a given cover $f : Y \rightarrow X$ of degree n we now want to describe the fiber above a base point at infinity $\overrightarrow{x_i x_j}$. For simplicity, consider $X = \mathbb{P}^1(\overline{\mathbb{Q}}) \setminus \{0, 1, \infty\}$ with $\overrightarrow{x_i x_j} = \overrightarrow{01}$. Then the fiber above $\overrightarrow{01}$ is defined to be the set of n preimages of the line segment between 0 and 1. We call each of these a *segment* in the fiber. It will be noted that these are exactly the n edges of the stick figure associated with the cover f .

Now we describe how to map the function field of Y into the field of Puiseux series at $\overrightarrow{01}$ using a choice of a segment in the fiber of $\overrightarrow{01}$. Let y be the unique point on this segment which projects down to 0. Let w be a local parameter at y which is positive real on the segment. Then consider a function $g \in \mathbb{C}(Y)$ expanded around y as $\sum_{n=N}^{\infty} a_n w^n$. g is mapped to a Puiseux series at $\overrightarrow{01}$ by replacing w with $z^{1/e}$. Note that elements of $\overline{\mathbb{Q}}(Y)$ are mapped into $\mathcal{P}_{\overrightarrow{01}}(\overline{\mathbb{Q}})$, i.e. the Puiseux series with coefficients in $\overline{\mathbb{Q}}$.

Similarly to the functor F_x , we can define a functor $F_{\overrightarrow{x_i x_j}}$ from \mathcal{C} to **Set**. A cover Y from \mathcal{C} is taken to the set of embeddings $\overline{\mathbb{Q}}(Y) \rightarrow \mathcal{P}_{\overrightarrow{x_i x_j}}(\overline{\mathbb{Q}})$, one for each segment lying above $\overrightarrow{x_i x_j}$. Denote the topological fundamental group based at the simply connected region $\overrightarrow{x_i x_j}$ by $\pi_1(X, \overrightarrow{x_i x_j})$. Each $\sigma \in \pi_1(X, \overrightarrow{x_i x_j})$ induces an automorphism of $F_{\overrightarrow{x_i x_j}}$ via analytic continuation giving the injection $\pi_1(X, \overrightarrow{x_i x_j}) \hookrightarrow \text{Aut}(F_{\overrightarrow{x_i x_j}})$. The algebraic fundamental group $\widehat{\pi}_1(X, \overrightarrow{x_i x_j})$ is the profinite completion of $\pi_1(X, \overrightarrow{x_i x_j})$ and is once again isomorphic to $\text{Aut}(F_{\overrightarrow{x_i x_j}})$.

We can also consider segments in the fiber above $\overrightarrow{x_i x_j}$ of the universal covering $\tilde{f} : \tilde{X} \rightarrow X$. Each of these gives an embedding of $\overline{\mathbb{Q}}(\tilde{X})$ into $\mathcal{P}_{\overrightarrow{x_i x_j}}(\overline{\mathbb{Q}})$. Restricting the action of $\pi_1(X, \overrightarrow{x_i x_j})$ on $\mathcal{P}_{\overrightarrow{x_i x_j}}(\overline{\mathbb{Q}})$ to the image of this embedding gives a mapping $\pi_1(X, \overrightarrow{x_i x_j}) \rightarrow \text{Gal}(\overline{\mathbb{Q}}(\tilde{X})/\overline{\mathbb{Q}}(X))$ which extends to an isomorphism between $\widehat{\pi}_1(X, \overrightarrow{x_i x_j})$ and $\text{Gal}(\overline{\mathbb{Q}}(\tilde{X})/\overline{\mathbb{Q}}(X))$ as before. Note that this isomorphism is non-canonical since it depends on a choice of a segment in the fiber above $\overrightarrow{x_i x_j}$.

4.1.5 $G_{\mathbb{Q}}$ -action on fundamental groups

The standard action of $G_{\mathbb{Q}}$ on the algebraic numbers can be extended to an action on $\mathcal{P}_{\overrightarrow{x_i x_j}}(\overline{\mathbb{Q}})$ (and thus $\mathcal{L}_x(\overline{\mathbb{Q}})$) by acting on the coefficients of the series. This allows us to define an action of $G_{\mathbb{Q}}$ on the algebraic fundamental group based at or away from infinity.

Let $\sigma \in G_{\mathbb{Q}}$ and $\gamma \in \widehat{\pi}_1(X, x)$. Then γ is determined by an automorphism of each set of embeddings $F_x(Y)$ as Y varies across \mathcal{C} . Such an automorphism takes $\epsilon \in F_x(Y)$ to $\gamma(\epsilon) \in F_x(Y)$ given by analytic continuation of a function in the image of the embedding along γ , as described before. Note that if $\gamma \in \widehat{\pi}_1(X, x) \setminus \pi_1(X, x)$ then we can replace it by some $\gamma' \in \pi_1(X, x)$ giving the same automorphism of $F_x(Y)$ so that the idea of analytic continuation makes sense.

Now given an embedding $\epsilon \in F_x(Y)$, $\epsilon : \overline{\mathbb{Q}}(Y) \hookrightarrow \mathcal{L}_x(\overline{\mathbb{Q}})$, define the embedding

$\sigma(\epsilon) : \overline{\mathbb{Q}}(Y) \hookrightarrow \mathcal{L}_x(\overline{\mathbb{Q}})$ by acting on the coefficients. In other words:

$$\sigma(\epsilon) : \overline{\mathbb{Q}}(Y) \hookrightarrow \mathcal{L}_x(\overline{\mathbb{Q}}) \quad (4.7)$$

$$g \mapsto \epsilon(g) = \left(\sum_{n=N}^{\infty} a_n z^n \in \mathcal{L}_x(\overline{\mathbb{Q}}) \right) \mapsto \sigma(\epsilon(g)) = \left(\sum_{n=N}^{\infty} \sigma(a_n) z^n \in \mathcal{L}_x(\overline{\mathbb{Q}}) \right) \quad (4.8)$$

Note that $\sigma(\epsilon)$ is an embedding over $\mathbb{Q}(X)$, not necessarily over $\overline{\mathbb{Q}}(X)$, hence $\sigma(\epsilon) \in F_x(Y')$ for $Y' \in \mathcal{C}$ possibly different from Y .

σ acts on $\gamma \in \pi_1(X, x)$ to give $\sigma(\gamma)$. We define $\sigma(\gamma)$ by giving its action on $F_x(Y)$:

$$\sigma(\gamma)(\epsilon) := (\sigma \cdot \gamma \cdot \sigma^{-1})(\epsilon). \quad (4.9)$$

Note that the action of σ and γ commute over $\overline{\mathbb{Q}}(X)$, hence $\sigma(\gamma)(\epsilon)$ is an embedding over $\overline{\mathbb{Q}}(X)$, i.e. $\sigma(\gamma)(\epsilon) \in F_x(Y)$. This proves that $\sigma(\gamma)$ gives an automorphism of F_x , i.e. $\sigma(\gamma) \in \widehat{\pi}_1(X, x)$. Described more explicitly, $\sigma(\gamma)$ acts on the image of an embedding by first applying σ^{-1} to the coefficients, then continuing the resulting function analytically along γ (assuming $\gamma \in \pi_1(X, x)$), and finally applying σ to the coefficients.

In a similar way one can define the action of $G_{\mathbb{Q}}$ on the fundamental group at a base point at infinity.

We now indicate how to interpret this $G_{\mathbb{Q}}$ -action as the lifting of the canonical outer action as determined by the fundamental exact sequence described at the beginning of the previous chapter. We again work with $x \in X_{\mathbb{Q}}$ as base point, with the extension to a base point at infinity being clear. As described before, a choice of a point on \widetilde{X} in the fiber of x determines an embedding $\overline{\mathbb{Q}}(X) \rightarrow \mathcal{L}_x(\overline{\mathbb{Q}})$. This gives the following extension of function fields:

$$\mathbb{Q}(X) \subset \overline{\mathbb{Q}}(X) \subset \overline{\mathbb{Q}}(\widetilde{X}) \subset \mathcal{L}_x(\overline{\mathbb{Q}}) \quad (4.10)$$

And a corresponding exact sequence of Galois groups:

$$1 \rightarrow \text{Gal}(\overline{\mathbb{Q}}(\widetilde{X})/\overline{\mathbb{Q}}(X)) \rightarrow \text{Gal}(\overline{\mathbb{Q}}(\widetilde{X})/\mathbb{Q}(X)) \rightarrow G_{\mathbb{Q}} \rightarrow 1 \quad (4.11)$$

The action of $G_{\mathbb{Q}}$ on $\mathcal{L}_x(\overline{\mathbb{Q}})$ restricts to $\overline{\mathbb{Q}}(\widetilde{X})$, giving a section:

$$G_{\mathbb{Q}} \rightarrow \text{Gal}(\overline{\mathbb{Q}}(\widetilde{X})/\mathbb{Q}(X)). \quad (4.12)$$

As described before, $\text{Gal}(\overline{\mathbb{Q}}(\widetilde{X})/\overline{\mathbb{Q}}(X))$ is isomorphic to $\widehat{\pi}_1(X, x)$, so finally we have a representation $G_{\mathbb{Q}} \rightarrow \text{Aut}(\widehat{\pi}_1(X, x))$ as described at the beginning of the chapter. Thus every choice of a point in the fiber of x determines a different lifting of the canonical $G_{\mathbb{Q}}$ outer action.

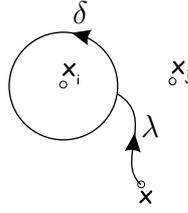


Figure 4.3: *An inertia generator*

4.1.6 $G_{\mathbb{Q}}$ -action on inertia generators

Let $\gamma \in \pi_1(X, x)$ be an inertia generator associated to $x_i \in \overline{X}$, where x is a base point at or away from infinity. Recall that if x is a base point away from infinity, it is required to be rational. As just seen, $\sigma \in G_{\mathbb{Q}}$ acts on γ to give $\sigma(\gamma) \in \widehat{\pi}_1(X, x)$. To get a better description of $\sigma(\gamma)$, it is helpful to divide the path γ into three parts: $\gamma = \lambda \cdot \delta \cdot \lambda^{-1}$, where λ is a path from x to the base point at infinity $\overrightarrow{x_i x_j}$ and δ is a loop going around x_i in an anti-clockwise direction (see Figure 4.3).

To make this more precise, we introduce a generalization of a fundamental group. Let $b_i \in X$, $i = 1, 2$ be base points at or away from infinity. Then the set of paths from b_1 to b_2 modulo homotopy equivalence is denoted by $\pi_1(X; b_1, b_2)$. The set of natural transformations from the functor F_{b_1} to the functor F_{b_2} is denoted by $\widehat{\pi}_1(X; b_1, b_2)$. Since each path from b_1 to b_2 induces such a natural transformation via analytic continuation, we have $\pi_1(X; b_1, b_2) \subset \widehat{\pi}_1(X; b_1, b_2)$. In the case where $b_1 = b_2$, these sets become the usual topological and algebraic fundamental groups.

Given another base point b_3 , there is a natural notion of composition of paths

$$\pi_1(X; b_1, b_2) \times \pi_1(X; b_2, b_3) \rightarrow \pi_1(X; b_1, b_3),$$

and composition of natural transformations

$$\widehat{\pi}_1(X; b_1, b_2) \times \widehat{\pi}_1(X; b_2, b_3) \rightarrow \widehat{\pi}_1(X; b_1, b_3).$$

Just as for the fundamental group, there is a $G_{\mathbb{Q}}$ -action on $\pi_1(X; b_1, b_2)$ and on $\widehat{\pi}_1(X; b_1, b_2)$. From the way the action is defined, it can be deduced that it commutes with the composition of paths.

Now we return to the action of σ on $\gamma = \lambda \cdot \delta \cdot \lambda^{-1}$. From the above we have $\sigma(\gamma) = \sigma(\lambda) \cdot \sigma(\delta) \cdot \sigma(\lambda^{-1}) = \sigma(\lambda) \cdot \sigma(\delta) \cdot (\sigma(\lambda))^{-1}$, where $\lambda \in \pi_1(X; x, x_i)$ and $\delta \in \pi_1(X, \overrightarrow{x_i x_j})$.

To determine $\sigma(\delta)$ more precisely, let $g = \sum_{n=N}^{\infty} a_n z^{n/e} \in \mathcal{P}_{\overrightarrow{x_i x_j}}(\overline{\mathbb{Q}})$ be in the image of the embedding of the function field of Y for some Y in \mathcal{C} . Let $\zeta_e = \exp(\frac{2\pi i}{e})$. Then δ acts on g as follows:

$$g = \sum_n a_n z^{n/e} \xrightarrow{\delta} \sum_n a_n \zeta_e^n z^{n/e} = \delta(g) \tag{4.13}$$

Let $\chi : G_{\mathbb{Q}} \rightarrow \hat{\mathbb{Z}}^{\times}$ denote the cyclotomic character. Then $\sigma(\delta)$ acts on g as follows:

$$\begin{aligned}
 g &= \sum_n a_n z^{n/e} \xrightarrow{\sigma^{-1}} \sum_n \sigma^{-1}(a_n) z^{n/e} \xrightarrow{\delta} \sum_n \sigma^{-1}(a_n) \zeta_e^n z^{n/e} \xrightarrow{\sigma} \\
 &\xrightarrow{\sigma} \sum_n \sigma(\sigma^{-1}(a_n)) (\sigma(\zeta_e))^n z^{n/e} = \sum_n a_n \zeta_e^{n\chi(\sigma)} z^{n/e} = (\sigma(\delta))(g) \quad (4.14)
 \end{aligned}$$

Comparing these two actions, we deduce that $\sigma(\delta) = \delta^{\chi(\sigma)}$. Note that both δ and $\sigma(\delta)$ are inertia generators of the same inertia subgroup of $\hat{\pi}_1(X, \overrightarrow{x_i x_j})$. Now we can prove the following:

Proposition 4.1.2. $\sigma \in G_{\mathbb{Q}}$ acts on the inertia generator $\gamma \in \pi_1(X, x)$ by taking it to a conjugate of $\gamma^{\chi(\sigma)}$ in $\hat{\pi}_1(X, x)$. Hence $G_{\mathbb{Q}}$ takes any inertia subgroup to a conjugate inertia subgroup, equivalently, it preserves conjugacy classes of inertia subgroups.

Proof. First we note that

$$\begin{aligned}
 \gamma^{\chi(\sigma)} &= \lambda \cdot \gamma \cdot \lambda^{-1} \cdot \lambda \cdot \gamma \cdot \lambda^{-1} \cdots \lambda \cdot \gamma \cdot \lambda^{-1} \\
 &= \lambda \cdot \gamma \cdot \lambda^{-1}
 \end{aligned}$$

Using this, and $\sigma(\delta) = \delta^{\chi(\sigma)}$, which was just shown, we obtain:

$$\begin{aligned}
 \sigma(\gamma) &= \sigma(\lambda) \cdot \sigma(\delta) \cdot \sigma(\lambda)^{-1} \\
 &= \sigma(\lambda) \cdot \delta^{\chi(\sigma)} \cdot \sigma(\lambda)^{-1} \\
 &= (\sigma(\lambda) \cdot \lambda^{-1}) \cdot (\lambda \cdot \delta^{\chi(\sigma)} \cdot \lambda^{-1}) \cdot (\sigma(\lambda) \cdot \lambda^{-1})^{-1} \\
 &= k \cdot \gamma^{\chi(\sigma)} \cdot k^{-1} \\
 &= k^{-1} \circ \gamma^{\chi(\sigma)} \circ k
 \end{aligned}$$

where $k = \sigma(\lambda) \cdot \lambda^{-1} \in \hat{\pi}_1(X, x)$, and the inversion in the last line is because composition of paths is from the left and composition in the fundamental group is from the right. If γ generates the inertia subgroup $I_x \subset \hat{\pi}_1(X, x)$ then $\sigma(\gamma)$ generates the inertia subgroup $k^{-1} I_x k$, thus $G_{\mathbb{Q}}$ preserves conjugacy classes of inertia subgroups. \square

4.2 The Grothendieck-Teichmüller group, \widehat{GT}

For this section, let $X = \mathbb{P}^1(\overline{\mathbb{Q}}) \setminus \{0, 1, \infty\}$, which has \hat{F}'_2 as its algebraic fundamental group. Here $\hat{F}'_2 = [\hat{F}_2, \hat{F}_2]$ is the commutator (derived) subgroup of \hat{F}_2 .

4.2.1 Parametrizing $G_{\mathbb{Q}}$

We now show how to lift the canonical outer action of every element of $G_{\mathbb{Q}}$ on \hat{F}'_2 to an action in a way which allows us to assign two unique parameters to every element of $G_{\mathbb{Q}}$.

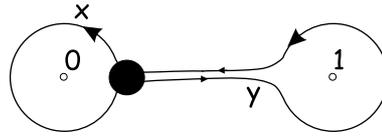


Figure 4.4: *Generators of the fundamental group based at infinity*

The fundamental group can be chosen to have $\overrightarrow{01}$ as its base point with the two generators x and y as indicated in Figure 4.4. Note that by the previous section, such a choice of base point and generators actually already determines a specific lifting of the outer $G_{\mathbb{Q}}$ -action. However, in the following group-theoretic proof, we make no assumptions about the base point and generators. Instead, we start with an arbitrary lifting, which is changed to another lifting by composing with an inner automorphism. This actually amounts to starting with an arbitrary base point and generators, and then moving the base point around and changing the generators. Although we do not show it here, the final lifting is in fact the one associated with the choice of base point and generators given in Figure 4.4.

Proposition 4.2.1. Let $b \in X$ be an arbitrary base point and let x, y denote arbitrary generators of the corresponding fundamental group $\hat{\pi}_1(X, b) \cong \hat{F}_2$. For every $\sigma \in G_{\mathbb{Q}}$ there is a unique pair $\lambda_{\sigma}, f_{\sigma} \in \hat{\mathbb{Z}}^{\times} \times \hat{F}'_2$ such that the canonical outer action of σ on \hat{F}_2 can be uniquely lifted to an action given by:

$$\begin{aligned} \sigma(x) &= x^{\lambda_{\sigma}} \\ \sigma(y) &= (f_{\sigma})^{-1} y^{\lambda_{\sigma}} f_{\sigma} \end{aligned}$$

Proof. Let $\sigma \in G_{\mathbb{Q}}$. From Theorem 4.1.2 we know that there exists $g, h \in \hat{F}_2$ such that the action of σ on \hat{F}_2 is given by:

$$\begin{aligned} \sigma(x) &= g^{-1} x^{\lambda(\sigma)} g \\ \sigma(y) &= h^{-1} y^{\lambda(\sigma)} h \end{aligned}$$

Now consider the abelianization \hat{F}_2^{ab} of \hat{F}_2 :

$$\hat{F}_2^{\text{ab}} = \hat{F}_2 / \hat{F}'_2 \cong \hat{\mathbb{Z}} \times \hat{\mathbb{Z}}.$$

Since the images of x and y still topologically generate \hat{F}_2^{ab} , there exist $a, b \in \hat{\mathbb{Z}}$ such that

$$hg^{-1} \equiv y^b x^a \pmod{\hat{F}'_2}$$

which allows us to define

$$f_{\sigma} := y^{-b} h g^{-1} x^{-a} \in \hat{F}'_2.$$

Define the inner automorphism $\tau \in \text{Inn}(\hat{F}_2)$:

$$\tau(w) = x^a g w g^{-1} x^{-a}, \quad w \in \hat{F}_2$$

Composing with τ gives another lifting of the outer automorphism:

$$\begin{aligned} \tau(\sigma(x)) &= x^a g g^{-1} x^{\chi(\sigma)} g g^{-1} x^{-a} \\ &= x^{\chi(\sigma)} \\ \tau(\sigma(y)) &= x^a g h^{-1} y^{\chi(\sigma)} h g^{-1} x^{-a} \\ &= x^a g h^{-1} y^b y^{\chi(\sigma)} y^{-b} h g^{-1} x^{-a} \\ &= f_{\sigma}^{-1} y^{\chi(\sigma)} f_{\sigma} \end{aligned}$$

This shows that to $\sigma \in G_{\mathbb{Q}}$ we can associate the pair $(\chi(\sigma), f_{\sigma})$. To show uniqueness, suppose conjugation by $k \in \hat{F}_2$ gives an automorphism of the same form. Then from the action on y we can deduce that $k \in \hat{F}'_2$, while the action on x implies that $k = x^c$ for some $c \in \hat{\mathbb{Z}}$. But if $c \neq 0$ then k maps to a non-zero element of \hat{F}_2^{ab} , contradicting $k \in \hat{F}'_2$. Hence k is the identity. \square

Let $A = \{\sigma \in \text{Aut}(\hat{F}_2) \mid \sigma(x) = x^{\lambda}, \sigma(y) = f^{-1} y^{\lambda} f, \lambda \in \hat{\mathbb{Z}}^{\times}, f \in \hat{F}'_2\}$. From the previous section there is a map from $G_{\mathbb{Q}}$ to A . A result of Belyi ([Bel80]) shows that this map is in fact an injective group homomorphism. (Compare this with the *faithful* $G_{\mathbb{Q}}$ -action on the stick figures.) The question now arises as to how the image can be characterized. One would like a list of necessary and sufficient conditions which a pair (λ, f) must satisfy to represent an automorphism in the image of $G_{\mathbb{Q}}$. Thus far, only a list of necessary conditions has been found, which is the subject of the next section.

4.2.2 Defining \widehat{GT}

First a note about notation. Given $f \in \hat{F}_2$, an arbitrary profinite group G and any two elements $a, b \in G$, we denote by $f(a, b) \in G$ the image of f under the unique homomorphism $\hat{F}_2 \rightarrow G$ mapping x to a and y to b . Thus for example $f(x, y) \in \hat{F}_2$ is simply f itself, while $f(y, x)$ is the image of f under the unique automorphism of \hat{F}_2 which switches x and y .

Now we define the following subgroup of $A \subset \text{Aut}(\hat{F}_2)$ (see [Dri90]) with the aim of later showing that it contains the image of $G_{\mathbb{Q}}$:

Definition 4.2.2. The Grothendieck-Teichmüller group, \widehat{GT} , is defined as the group of all automorphisms σ of \hat{F}_2 that can be expressed in the form $\sigma(x) = x^{\lambda}$, $\sigma(y) = f^{-1} y^{\lambda} f$

where $(\lambda, f) \in \widehat{\mathbb{Z}}^{\times} \times \widehat{F}'_2$ is a pair satisfying the following three equations:

- (I) $f(x, y)f(y, x) = 1$
- (II) $f(z, x)z^m f(y, z)y^m f(x, y)x^m = 1$, where $xyz = 1$ and $m = \frac{1}{2}(\lambda - 1)$
- (III) $f(x_{12}, x_{23})f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51}) = 1$

In equations (I) and (II), x and y are the elements of $\widehat{F}'_2 = \widehat{\Gamma}_{0,4}$ described in the previous section, and $z = (xy)^{-1}$. Hence equation (I) and (II) take place in $\widehat{\Gamma}_{0,4}$. Equation (III) takes place in $\widehat{\Gamma}_{0,5}$ however. Here $x_{ij} \in \widehat{\Gamma}_{0,5}$ is the Dehn twist along a loop separating i and j from the other marked points, as described at the end of the previous chapter. The fact that \widehat{GT} is indeed closed under composition was shown by Drinfeld.

4.2.3 The injection $G_{\mathbb{Q}} \rightarrow \widehat{GT}$

To show that $G_{\mathbb{Q}}$ is contained in \widehat{GT} it is necessary to show that the automorphism of \widehat{F}'_2 induced by an element of $G_{\mathbb{Q}}$ satisfies the three defining equations of \widehat{GT} . We will prove only the first two (following Ihara ([Iha94])), which will suffice to show how the geometry of the moduli spaces determines the equations.

We start by collecting all the different base points and natural transformations between the functors associated to them together in one object, called the *fundamental groupoid*. Let \mathcal{B} be the set of 6 tangential base points \vec{i}_j , where $i, j \in \{0, 1, \infty\}, i \neq j$. Recall that from a categorical point of view a group is simply a single object category with the arrows being the group elements (and some notion of inverses). Using this perspective, we define the fundamental groupoid as follows:

Definition 4.2.3. Define $\widehat{\pi}_1(X, \mathcal{B})$ to be the category with \mathcal{B} as its set of objects and $\widehat{\pi}_1(X; a, b)$ the set of arrows from a to b where $a, b \in \mathcal{B}$. Thus the arrows are natural transformations between the functors associated to the respective base points. Composition of arrows is defined by composition of natural transformations where it is possible.

There is again a natural subcategory $\pi_1(X, \mathcal{B})$ consisting of the same objects but with only the arrows which are natural transformations coming from actual paths on X instead of ‘pro-paths’, i.e., with the arrows being $\pi_1(X; a, b)$.

The following summarizes the properties of the $G_{\mathbb{Q}}$ -action which have already been noted before:

Proposition 4.2.4. There is a $G_{\mathbb{Q}}$ -action on $\widehat{\pi}_1(X, \mathcal{B})$.

The automorphism group of X , denoted by $\text{Aut}(X)$, is isomorphic to S_3 , the symmetric group on 3 elements. In fact, $\text{Aut}(X)$ is generated by the automorphisms $\theta : z \rightarrow 1 - z$, and $\omega : z \rightarrow \frac{1}{1-z}$. The important property of any $\lambda \in \text{Aut}(X)$ is that it is defined over \mathbb{Q} . This lies behind the proofs of the following two propositions (see [Iha94] for details).

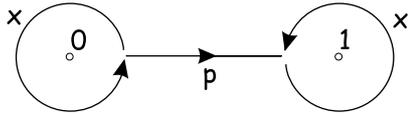


Figure 4.5: *Breaking up the path $y = p^{-1} \circ x' \circ p$. ($y = p \cdot x' \cdot p^{-1}$ as paths.)*

Proposition 4.2.5. There is an $\text{Aut}(X)$ -action on $\widehat{\pi}_1(X, \mathcal{B})$.

The action of $\lambda \in \text{Aut}(X)$ on $\delta \in \pi_1(X, \mathcal{B})$ is easy to describe: Seen as paths, $\lambda(\delta)$ is simply the image of the path δ under the automorphism λ of X . And it is clear that the $\text{Aut}(X)$ -action commutes with the composition of paths.

Proposition 4.2.6. The respective actions of $\text{Aut}(X)$ and $G_{\mathbb{Q}}$ on $\widehat{\pi}_1(X, \mathcal{B})$ commute. In other words, if $\lambda \in \text{Aut}(X)$, $\sigma \in G_{\mathbb{Q}}$ and $\delta \in \widehat{\pi}_1(X, \mathcal{B})$, then $\lambda(\sigma(\delta)) = \sigma(\lambda(\delta))$.

Now we give a more concrete demonstration of how to associate a pair $(\lambda_{\sigma}, f_{\sigma})$ to a given $\sigma \in G_{\mathbb{Q}}$. Consider the fundamental group $\widehat{\pi}_1(X, \overrightarrow{01})$ with the standard generators x and y going anticlockwise around 0 and 1 respectively. Break y up into pieces by writing it as $y = p^{-1} \circ x' \circ p$. Here $p \in \widehat{\pi}_1(X; \overrightarrow{01}, \overrightarrow{10})$ is the straight line between the base points as shown in Figure 4.5, and $x' = \theta(x) \in \widehat{\pi}_1(X, \overrightarrow{10})$.

Let $\sigma \in G_{\mathbb{Q}}$. Define

$$f_{\sigma} = p^{-1} \circ \sigma(p) \in \widehat{\pi}_1(X, \overrightarrow{01})$$

It has been shown before that

$$\sigma(x) = x^{X(\sigma)}$$

So using theorem 4.2.6:

$$\sigma(x') = \sigma(\theta(x)) = \theta(\sigma(x)) = \theta(x^{X(\sigma)}) = p \circ y^{X(\sigma)} \circ p^{-1}$$

Combining these and applying theorem 4.2.4 yields

$$\begin{aligned} \sigma(y) &= \sigma(p)^{-1} \circ \sigma(x') \circ \sigma(p) \\ &= f_{\sigma}^{-1} \circ y^{X(\sigma)} \circ f_{\sigma} \end{aligned}$$

To show that this (i.e. $(\chi(\sigma), f_{\sigma})$) is indeed the same pair $(\lambda_{\sigma}, f_{\sigma})$ that was previously obtained by group-theoretic means, it must just be shown that $f_{\sigma} \in \widehat{F}'_2$:

Proposition 4.2.7. $f_{\sigma} \in \widehat{F}'_2$, where f_{σ} is defined as above.

Proof. Recall that $\hat{F}_2 \cong \hat{\pi}_1(X, \overrightarrow{01}) \cong \text{Gal}(\overline{\mathbb{Q}}(\tilde{X})/\overline{\mathbb{Q}}(X))$. Thus $\hat{F}'_2 \cong \text{Gal}(\overline{\mathbb{Q}}(\tilde{X})/M)$, where M is the maximal abelian extension of $\overline{\mathbb{Q}}(X)$ contained in $\overline{\mathbb{Q}}(\tilde{X})$. So it suffices to show that f_σ fixes M . The field M is generated by the function fields associated to cyclic covers only ramified above $\{0, \infty\}$ and $\{1, \infty\}$, hence M is generated by $z^e, (1-z)^e \in \overline{\mathbb{Q}}(X)$ for all integers e . Consider the action of $f_\sigma = p^{-1} \circ \sigma \circ p = p^{-1} \circ \sigma \circ p \circ \sigma^{-1}$ on z^e :

$$z^e \xrightarrow{\sigma^{-1}} z^e \xrightarrow{p} (1 - (1-z))^e = \left(\sum_n a_n (1-z)^n, a_n \in \mathbb{Q} \right) \xrightarrow{\sigma} \sum_n a_n (1-z)^n \xrightarrow{p^{-1}} z^e$$

Similarly, f_σ acts trivially on $(1-z)^e$, hence $\sigma \in \hat{F}'_2$. □

Now we are in a position to prove the first defining equation of \widehat{GT} :

Theorem 4.2.8. The pair $(\chi(\sigma), f)$ associated to $\sigma \in G_{\mathbb{Q}}$ satisfies equation (I) of $G_{\mathbb{Q}}$.

Proof. Note that the following equations hold on X :

$$\theta(p) \circ p = 1, \quad p^{-1} \circ \theta(x) \circ p = y, \quad p^{-1} \circ \theta(y) \circ p = x$$

Equation (I) is proved by applying σ to the first of these equations:

$$\begin{aligned} 1 &= \sigma(\theta(p) \circ p) \\ &= \sigma(\theta(p)) \circ \sigma(p) \\ &= \theta(\sigma(p)) \circ \sigma(p) \\ &= \theta(p \circ f(x, y)) \circ p \circ f(x, y) \\ &= p^{-1} \circ \theta(f(x, y)) \circ p \circ f(x, y) \\ &= f(p^{-1} \circ \theta(x) \circ p, p^{-1} \circ \theta(y) \circ p) \circ f(x, y) \\ &= f(y, x) \circ f(x, y) \end{aligned}$$

□

To prove the equation (II), we introduce $r \in \hat{\pi}_1(X; \overrightarrow{01}, \overrightarrow{0\infty})$ and $q = \theta(r) \circ p \in \hat{\pi}_1(X; \overrightarrow{01}, \overrightarrow{1\infty})$ as shown in Figure 4.6. Now for the action of σ on r :

Lemma 4.2.9. σ acts on $r \in \hat{\pi}_1(X; \overrightarrow{01}, \overrightarrow{0\infty})$ to give $\sigma(r) = r \circ x^{\frac{1}{2}(\chi(\sigma)-1)}$.

Proof. We determine the action of $r^{-1} \circ \sigma(r) = r^{-1} \circ \sigma \circ r \circ \sigma^{-1} \in \hat{\pi}_1(X, \overrightarrow{01})$ on $a_n z^{n/e} \in \mathcal{P}_{\overrightarrow{01}}$. We write $\log z$ to mean the principal branch of \log defined on the complement of the negative real axis. $\log(-z)$ refers to the branch of \log defined on the complement of the positive real axis and which agrees with $\log z$ on the upper half plane. Note that when the

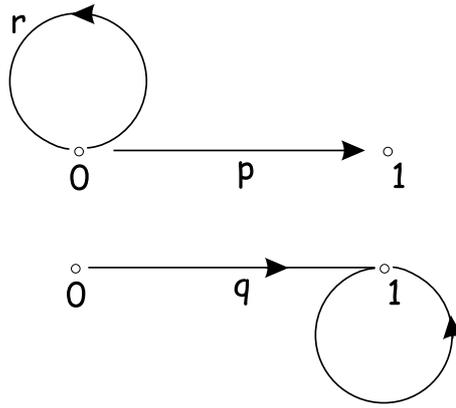


Figure 4.6: Paths used to prove equation (II)

local parameter z at $\overrightarrow{01}$ is changed along r to the local parameter $-z$ at $\overrightarrow{0\infty}$, the function $\log z$ is replaced by $\log(-z) + \pi i$.

$$\begin{aligned}
 a_n z^{n/e} &\xrightarrow{\sigma^{-1}} \sigma^{-1}(a_n) z^{n/e} \\
 &= \sigma^{-1}(a_n) \exp\left(\frac{n}{e}(\log z)\right) \\
 &\xrightarrow{r} \sigma^{-1}(a_n) \exp\left(\frac{n}{e}(\log(-z) + \pi i)\right) \\
 &= \sigma^{-1}(a_n) \exp\left(\frac{n}{e}(\log(-z))\right) \exp\left(\frac{n}{e} \frac{2\pi i}{2}\right) \\
 &= \sigma^{-1}(a_n) \exp\left(\frac{n}{e}(\log(-z))\right) \zeta_e^{\frac{n}{2}} \\
 &\xrightarrow{\sigma} a_n \exp\left(\frac{n}{e} \log(-z)\right) \zeta_e^{\frac{n}{2} \chi(\sigma)} \\
 &\xrightarrow{r^{-1}} a_n \exp\left(\frac{n}{e}(\log z - \pi i)\right) \zeta_e^{\frac{n}{2} \chi(\sigma)} \\
 &= a_n \exp\left(\frac{n}{e} \log z\right) \exp\left(\frac{-n}{2} \frac{2\pi i}{e}\right) \zeta_e^{\frac{n}{2} \chi(\sigma)} \\
 &= a_n z^{n/e} \zeta_e^{-\frac{n}{2}} \zeta_e^{\frac{n}{2} \chi(\sigma)} \\
 &= a_n z^{n/e} \zeta_e^{n(\frac{1}{2}(\chi(\sigma)-1))}
 \end{aligned}$$

Comparing with the action of x on $a_n z^{n/e}$ (i.e. $a_n z^{n/e} \mapsto a_n \zeta_e^n z^{n/e}$), we can deduce:

$$r^{-1} \circ \sigma(r) = x^{\frac{1}{2}(\chi(\sigma)-1)}$$

which is equivalent to the statement in the proposition.

□

Let $m = \frac{1}{2}(\chi(\sigma) - 1)$. The action of σ on $q = \theta(r) \circ p$ can now be determined:

$$\begin{aligned}\sigma(q) &= \sigma(\theta(r)) \circ \sigma(p) \\ &= \theta(r \circ x^m) \circ p \circ f(x, y) \\ &= \theta(r) \circ p \circ p^{-1} \circ \theta(x)^m \circ p \circ f(x, y) \\ &= q \circ (p^{-1} \circ \theta(x) \circ p)^m \circ f(x, y) \\ &= q \circ y^m \circ f(x, y)\end{aligned}$$

Now we can prove the second defining equation of $G_{\mathbb{Q}}$:

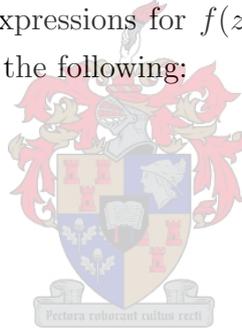
Theorem 4.2.10. The pair $(\chi(\sigma), f)$ associated to $\sigma \in G_{\mathbb{Q}}$ satisfies equation (II) of $G_{\mathbb{Q}}$.

Proof. Similarly to the first equation, the second equation is proved by applying σ to the following equation:

$$\omega^2(q) \circ \omega(q) \circ q = 1$$

This requires knowing the expressions for $f(z, x)$, $f(y, z)$, $\omega(y^m)$ and $\omega^2(y^m)$. They can be derived by starting with the following:

$$\begin{aligned}x &= q^{-1} \circ \omega(z) \circ q \\ y &= q^{-1} \circ \omega(x) \circ q \\ z &= q^{-1} \circ \omega(y) \circ q\end{aligned}$$



Now calculate:

$$\begin{aligned}f(y, z) &= f(q^{-1} \circ \omega(x) \circ q, q^{-1} \circ \omega(y) \circ q) \\ &= q^{-1} \circ \omega(f(x, y)) \circ q \\ f(z, x) &= f(q^{-1} \circ \omega(y) \circ q, q^{-1} \circ \omega(z) \circ q) \\ &= q^{-1} \circ \omega(f(y, z)) \circ q \\ &= q^{-1} \circ \omega(q)^{-1} \circ \omega^2(f(x, y)) \circ \omega(q) \circ q\end{aligned}$$

And for the other two:

$$\begin{aligned}\omega(y) &= q \circ z \circ q^{-1} \\ \omega(z) &= q \circ x \circ q^{-1} \\ \omega^2(y) &= \omega(q \circ z \circ q^{-1}) \\ &= \omega(q) \circ q \circ x \circ q^{-1} \circ \omega(q)^{-1} \\ \omega(y^m) &= q \circ z^m \circ q^{-1} \\ \omega^2(y^m) &= \omega(q) \circ q \circ x^m \circ q^{-1} \circ \omega(q)^{-1}\end{aligned}$$

Now apply σ to the original equation:

$$\begin{aligned}
1 &= \omega^2(\sigma(q)) \circ \omega(\sigma(q)) \circ \sigma(q) \\
&= \omega^2(q \circ y^m \circ f(x, y)) \circ \omega(q \circ y^m \circ f(x, y)) \circ q \circ y^m \circ f(x, y) \\
&= \omega^2(q) \circ \omega^2(y^m) \circ \omega^2(f(x, y)) \circ \omega(q) \circ \omega(y^m) \circ \omega(f(x, y)) \circ q \circ y^m \circ f(x, y) \\
&= (\omega(q) \circ q)^{-1} \circ (\omega(q) \circ q \circ x^m \circ q^{-1} \circ \omega(q)^{-1}) \circ \omega^2(f(x, y)) \\
&\quad \circ \omega(q) \circ (q \circ z^m \circ q^{-1}) \circ \omega(f(x, y)) \circ q \circ y^m \circ f(x, y) \\
&= x^m \circ (q^{-1} \circ \omega(q)^{-1} \circ \omega^2(f(x, y)) \circ \omega(q) \circ q) \circ z^m \circ (q^{-1} \circ \omega(f(x, y)) \circ q) \\
&\quad \circ y^m \circ f(x, y) \\
&= x^m \circ f(z, x) \circ z^m \circ f(y, z) \circ y^m \circ f(x, y)
\end{aligned}$$

The last equation is equivalent to equation (II), hence completing the proof. \square



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