

# Growth Optimal Portfolios and Real World Pricing

by

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*Thesis presented in partial fulfilment of the requirements for  
the degree of Master of Science at Stellenbosch University*



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December 2008

# Declaration

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# Abstract

In the Benchmark Approach to Finance, it has been shown that by taking the Growth Optimal Portfolio as numéraire, a candidate for a pricing derivatives formula under the real world probability can be given. This result allows us to price in an incomplete financial market model. The result comes from two different approaches. In the first approach we use the supermartingale property of portfolios in units of the benchmark portfolio which leads to the fact that an equivalent measure is not needed. In the second approach the numéraire property of the Growth Optimal Portfolio is used. The numéraire portfolio defines an equivalent martingale measure and by change of measure using the Radon-Nikodým derivative, a real world pricing formula is derived which is the same as the one given by the first approach stated above.

# Opsomming

In die Maatstaf Benadering tot Finansies kan dit bewys word dat 'n kandidaat vir die prys van 'n finansiële afgeleide bepaal kan word deur gebruik te maak van die Groei-Optimale Portefeulje onder die ware waarskynlikheidsmaat (in plaas van 'n ekwivalente martingaalmaat). Hierdie resultaat maak dit moontlik om pryse te bepaal in sekuriteitsmarkmodelle wat onvolledig is, en spruit uit twee verskillende benaderings: Die eerste benadering maak gebruik van die supermartingaal-eienskap van portefeuljes in eenhede van die maatstaf portefeulje, wat tot gevolg het dat 'n ekwivalente maat onnodig blyk. In die tweede benadering word die numéraire-eienskap van die Groei-Optimale Portefeulje gebruik. Die Numéraire Portefeulje definieer 'n ekwivalente martingaalmaat, en, deur 'n verandering van maat met behulp van 'n Radon-Nikodym afgeleide, kan 'n prys-formule verkry word onder die ware waarskynlikheidsmaat wat gelyk is aan die een bepaal deur die eerste benadering wat hierbo genoem word.

# Acknowledgments

I would like to thank the following people who gave their contributions to the accomplishment of this work. First of all to Stellenbosch University and to AIMS which give the funding to enable me to complete my master degree. To my supervisors: Prof Ekkehard Kopp for his support and for providing all the necessary materials I needed, Dr Maciej Capinski for his patience and time and for providing some valuable ideas and comments which helped me improving the quality of my work. Lastly, many thanks to my mother, to the rest of the family and friends for their kind supports.

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# Introduction

The use of the Growth Optimal Portfolio (GOP) has been recently a point of vast interest in finance. The notion of Growth Optimal Portfolio has been originally introduced by Kelly (1956) where it was shown that there is an optimal gambling strategy which collects more wealth than any other strategy over a long period of time. Since then, the notion of the Growth Optimal Portfolio has been introduced and this portfolio has become known as the one that maximizes its growth rate for any time horizon. It has been shown in Platen and Heath (2007) and some other papers such as Breiman (1961) and Latane (1959) that the GOP outperforms any other strategy as the time horizon increases. This characteristic of the GOP leads to the so-called benchmark approach which consists of taking the GOP as the benchmark or numéraire and using it in asset pricing. In this present work, we present the role of the GOP in pricing derivative securities without the use of equivalent measure in both complete and incomplete market. We reproduce some of the work done by Platen (2006) which is in continuous time setting and by Korn and Schäl (1999) which is in a discrete time case and develop an example which shows the theory in these articles. To do so, we make some modifications in the paper of Korn and Schäl (1999) to be able to adapt it to the continuous time case presented in Platen (2006) and to compare the two approaches.

Chapter 1 gives the background in mathematics that the reader needs throughout the work. This background has been taken mostly from probability theory and stochastic processes.

Chapter 2 which is a reproduction of the work of Platen (2006) studies the GOP in a general continuous time market that involves Stochastic Differential Equations, which give the dynamics of the GOP. The aim of Chapter 2 is to present a pricing formula that uses the real world probability as the measure and the GOP as numéraire. In this chapter, we show that portfolios expressed in units of the GOP are supermartingales and that this supermartingale property does not allow arbitrage opportunities in the market. The existence of equivalent measure is not needed.

Chapter 3 which has been drawn from Korn and Schäl (1999) gives the real world formula in a general discrete time market. We present first the discrete version of the result obtained in Chapter 2. After that we present

another approach which introduces the definition of the numéraire portfolio. The numéraire portfolio defines a martingale measure which guaranties the non existence of arbitrage. We show that the numéraire portfolio is the GOP and vice versa. We finally write the pricing formula by using the measure defined by the numéraire portfolio and by change of measure to the real world probability. We end up with the same real world pricing formula that has been given in Chapter 2.

As the previous chapters were based on already published work , in Chapter 4 we build a new incomplete model example in discrete time which is then used to demonstrate how the prices of an option given by the two approaches in Chapter 1 and Chapter 2 relate to the prices given by the usual risk neutral pricing method in Harrison and Kreps (1979) for a complete model. We use and describe a quadrinomial model composed of two stocks such that each of the two stocks has a behavior of a binomial tree. We study then the Growth Optimal Portfolio and the real world pricing under this quadrinomial model.



# Chapter 1

## Preliminaries

In this chapter, we introduce all the background that the reader needs for the next chapters. We give some results taken from theory of Probability and Stochastic Processes.

### 1.1 Conditional Expectation and Equivalent Probability.

We use the following definitions and propositions throughout the thesis for which Loeve (1978) and Fima C (1998) are good references.

**Definition 1.1.** On a given probability space  $(\Omega, \mathcal{A}, P)$ , an increasing family  $\{\mathcal{A}_t\}_{t \in \mathbb{R}_+}$  of  $\sigma$ -algebras is called a filtration. We call  $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t \in \mathbb{R}_+}, P)$  a filtered probability space.

In finance, we can think of  $\mathcal{A}_t$  as the set of information available to investors at time  $t \in [0, \infty)$ .

Let  $X$  be a random variable  $\mathcal{A}_t$ -measurable. We assume that  $X$  is integrable in the continuous time case. For any  $A \in \mathcal{A}_t$  the expectation of  $X$  given  $A$  is defined to be

$$\frac{E[X, A]}{P(A)}$$

provided  $P(A) \neq 0$  where

$$E[X, A] = E[X\mathbf{1}_A] = \int_A X(\omega)Pd(\omega), \quad (1.1.1)$$

with  $\mathbf{1}_A(\omega) = 1$  if  $\omega \in A$  and 0 otherwise.

**Definition 1.2.** The expectation of  $X$  under any  $\sigma$ -subalgebra  $\mathcal{E}$  of  $\mathcal{A}$ , denoted by  $E[X|\mathcal{E}]$ , is a random variable  $Y$  such that

1.  $Y$  is  $\mathcal{E}$ -measurable,
2.  $E[Y, A] = E[X, A]$  for all  $A \in \mathcal{E}$ .

**Proposition 1.3.** *The random variable  $E[X|\mathcal{E}]$  exists and is unique up to almost sure equivalence.*

The usual properties of the conditional expectation drawn from the book of Loeve (1978) are:

**Proposition 1.4.** *Let  $\mathcal{E}$  be a  $\sigma$ -subalgebra of  $\mathcal{A}$ .*

1. *If  $X$  is  $\mathcal{E}$ -measurable, then  $E[X|\mathcal{E}] = X$ .*
2. *If  $X$  is independent of  $\mathcal{E}$ , then  $E[X|\mathcal{E}] = E[X]$ .*
3. *If  $A_i$  are disjoint sets whose union is  $\Omega$  with  $P(A_i) > 0$  for each  $i$ , and  $\mathcal{E}$  is the  $\sigma$ -algebra generated by the  $A_i$ , then*

$$E[X|\mathcal{E}](\omega) = \sum_i \frac{E[X, A_i]}{P(A_i)} \mathbf{1}_{A_i}(\omega) \quad (1.1.2)$$

4. *If  $Y$  is  $\mathcal{E}$ -measurable (and  $XY$  integrable in the continuous case) then*

$$E[XY|\mathcal{E}] = Y E[X|\mathcal{E}].$$

5. *If  $\mathcal{D} \subseteq \mathcal{E} \subseteq \mathcal{F}$ , then*

$$E[E[X|\mathcal{D}] | \mathcal{E}] = E[X|\mathcal{D}] = E[E[X|\mathcal{E}] | \mathcal{D}]. \quad (1.1.3)$$

**Definition 1.5.** A process  $X = \{X_t, t \in [0, \infty)\}$  is said predictable with respect to the filtration  $\{\mathcal{A}_t\}_{t \in \mathbb{R}_+}$  if for each  $t \geq 0$ ,  $X_t$  is  $\mathcal{A}_{t-1}$ -measurable.

**Definition 1.6.** A process  $X = \{X_t, t \in [0, \infty)\}$  is said adapted to the filtration  $\{\mathcal{A}_t\}_{t \in \mathbb{R}_+}$  if for each  $t \geq 0$ ,  $X_t$  is  $\mathcal{A}_t$ -measurable.

In other words, Definition 1.6 is equivalent to saying that the history of the process  $X$  up to time  $t$  is covered by the information set  $\mathcal{A}_t$ .

**Definition 1.7.** A probability measure  $Q$  on  $(\Omega, \mathcal{A})$  is equivalent to  $P$  if for  $A \subset \Omega$ ,  $P(A) = 0$  if and only if  $Q(A) = 0$ .

For two equivalent probability measures  $Q$  and  $P$  on  $(\Omega, \mathcal{A})$ ,  $E_Q[X] = E_P[LX]$  for some strictly positive density  $L = \frac{dQ}{dP}$ . From now on, we write  $E[.]$  instead of  $E_P[.]$  for expectations under the measure  $P$ .

**Theorem 1.8 (Bayes' Formula (Karatzas and Shreve (1998))).** *If  $Q$  is an equivalent measure to  $P$  with density  $L = \frac{dQ}{dP}$ , then we have for any bounded random variable  $X$ ,*

$$E_Q[X|\mathcal{A}_t] = E[LX|\mathcal{A}_t]/E[L|\mathcal{A}_t].$$

## 1.2 Martingales, Stochastic Integrals and Stochastic Differential Equations.

In this section, we give some results on Wiener processes and Itô integrals. Most of the results are taken from Platen and Heath (2007) and Øksendal (2003) and Gihman and Skorohod (1972). Since later on we work with stochastic differential equations, an existence and uniqueness theorem will be given. We start with some definitions taken from the book of Platen and Heath (2007).

**Notation 1.9.** From now on, if  $u$  and  $v$  are two vectors with components  $u^j$  and  $v^j$ ,  $j = 0, \dots, d$  then  $\langle u, v \rangle$  denotes the inner product of  $u$  and  $v$ . The product  $u * v$  is defined by  $(u * v)^j = u^j v^j$ .

For  $h > 0$  consider the time discretisation

$$t_k = kh, \quad k = 1, 2, \dots \tag{1.2.1}$$

**Definition 1.10.** The quadratic variation of a process  $X$  at time  $t$  denoted by  $[X]_t$  is given by

$$[X]_t = \lim_{h \rightarrow 0} \sum_{k=1}^{i_t} (X_{t_k} - X_{t_{k-1}})^2$$

in probability where  $t_{i_t} = t$ .

**Definition 1.11.** For a left-continuous stochastic process  $e = \{e(t), t \in [0, \infty)\}$ , a stochastic process  $X$  with

$$\int_0^T e(s)^2 d[X]_s < \infty$$

for all  $T \in [0, \infty)$ , we define the Ito integral as

$$\int_0^T e(s) dX_s = \lim_{h \rightarrow 0} \sum_{k=1}^{i_t} e(t_{k-1}) \{X_{t_k} - X_{t_{k-1}}\}$$

in probability.

**Definition 1.12.** A Wiener process  $W = \{W_t, t \in [0, \infty)\}$  is a continuous process that has independent increments and such that for all  $t \in [0, \infty)$  and  $s \in [0, t]$ , the increments  $W_t - W_s$  have a Gaussian distribution with mean 0 and variance  $t - s$  with  $W_0 = 0$ .

The value of the quadratique variation  $[W] = \{[W]_t, t \in [0, \infty)\}$  at time  $t$  for a standard Wiener process  $W$  is given by the relation

$$[W]_t = t$$

for  $t \in [0, \infty)$ .

**Definition 1.13.** An  $m$ -dimensional Wiener process is a vector process such that each of its component  $W^j = \{W_t^j, t \in [0, \infty)\}$ , for  $j \in \{1, 2, \dots, m\}$ , is a scalar  $\mathcal{A}$ -adapted standard Wiener process and  $W^i$  and  $W^j$  are independent for  $i \neq j$ ,  $j, i \in \{1, 2, \dots, m\}$ .

## Itô's Formula

In this subsection, we follow Øksendal (2003) to state the well known Ito's Formula. For the following definition, we still use the discretisation given in (1.2.1). Let  $u : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be a  $d$ -dimensional vector function with components  $u^k$ ,  $k \in \{1, 2, \dots, d\}$  and  $v : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$  be a  $d \times m$  matrix value function. To allow integration on  $u$  and  $v$ , we assume that the components  $u^k$  of  $u$  and  $v^{i,j}$  of  $v$  verify

$$\int_0^\infty |u^k(s, X_s)| ds < \infty \quad a.s \quad \text{for } k \in \{1, 2, \dots, d\},$$

and

$$\int_0^\infty (v^{i,j}(s, X_s))^2 ds < \infty \quad a.s \quad \text{for } i, j \in \{1, \dots, d\}.$$

Let  $X = \{X_t = (X_t^1, \dots, X_t^d)^T, t \in [0, \infty)\}$  be defined by the Itô integral

$$X_t^k - X_0^k = \int_0^t u^k(s, X_s) ds + \sum_{j=1}^m \int_0^t v^{i,k}(s, X_s) dW_s^j.$$

We say that  $X$  has the stochastic differential

$$dX_t = u(t, X_t)dt + v(t, X_t)dW_t$$

and we have the following theorem:

**Theorem 1.14 (The  $d$ -dimensional Ito Formula (Øksendal (2003))).**

Assume that for the function  $f : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ , the partial derivatives  $\frac{\partial f}{\partial t}$ ,  $\frac{\partial f}{\partial x^k}$  and  $\frac{\partial^2 f}{\partial x^k \partial x^i}$  exist for  $k, i \in \{1, \dots, d\}$  and  $x = (x^1, x^2, \dots, x^d)^\top$  and let  $Y = \{Y_t, t \in [0, \infty)\}$  be a process such that  $Y_t = f(t, X_t)$  for  $t \in [0, \infty)$  then:

$$\begin{aligned} dY_t &= df(t, X_t) \\ &= \left\{ \frac{\partial f}{\partial t} + \sum_{k=1}^d u^k(t, X_t) \frac{\partial f}{\partial x^k} + \frac{1}{2} \sum_{j=1}^m \sum_{i,k=1}^d v^{i,j}(t, X_t) v^{k,j}(t, X_t) \frac{\partial^2 f}{\partial x^k \partial x^i} \right\} dt \\ &\quad + \sum_{j=1}^m \sum_{i=1}^d v^{i,j}(t, X_t) \frac{\partial f}{\partial x^i} dW_t^j. \end{aligned}$$

## Existence and Uniqueness of a solution of an SDE.

To be able to write and give the the existence and uniqueness of an SDE that satisfies a random variable  $X$ , we state first the following assumptions:

**Assumption 1.15.** *Let us assume that*

1. *The functions  $a, b : [t_0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  where  $a(t, X_t)$  represents the drift of the process  $X$  and  $b(t, X_t)$  its volatility are square integrable and measurable functions in  $(t, x) \in [t_0, T] \times \mathbb{R}$ ,*
2. *for some  $N$ ,*

$$|a(t, x)|^2 + |b(t, x)|^2 \leq N^2(1 + x^2),$$

3. the coefficients  $a$  and  $b$  satisfies the Lipschitz condition

$$|a(t, x) - a(t, y)| + |b(t, x) - b(t, y)| \leq K|x - y|$$

for some  $K > 0$  and

4. the initial value  $X_{t_0}$  is  $\mathcal{A}_{t_0}$ -measurable with  $E(|X_{t_0}|^2) < \infty$ .

**Theorem 1.16.** (Gihman and Skorohod (1972)). Under the four assumptions in Assumption 1.15, the SDE

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t \quad (1.2.2)$$

has a unique solution  $X = \{X_t, t \in [t_0, T]\}$  with

$$\sup_{t \in [t_0, T]} E(|X_t|^2) < \infty.$$

## Martingales, Local Martingales and Supermartingales.

Definitions and results in this subsection have been taken from Platen and Heath (2007) which can also be obtained from Øksendal (2003).

**Definition 1.17.** A non-negative random variable  $\tau$  is a stopping time with respect to the filtration  $\{\mathcal{A}_t\}_{t \in [0, T]}$  if for each time  $t \in [0, \infty)$ , the event  $\{\tau \leq t\}$  is in  $\mathcal{A}_t$ .

**Definition 1.18.** We call a martingale a continuous-time stochastic process  $X = \{X_t, t \in [0, \infty)\}$  which satisfies  $E(|X_t|) < \infty$  and  $X_s = E(X_t | \mathcal{A}_s)$  a.s. for all  $s \in [0, t]$  and  $t \in [0, \infty)$ .

**Definition 1.19.** An adapted process  $X$  is an  $(\mathcal{A}, P)$ -supermartingale if

$$X_s \geq E(X_t | \mathcal{A}_s) \quad \text{a.s.} \quad (1.2.3)$$

and  $E(|X_t|) \leq \infty$  for  $s \in [0, \infty)$  and  $t \in [0, \infty)$ .

We also use Definitions 1.18 and 1.19 for the discrete time case where  $t = \{0, 1, 2, \dots\}$ .

**Definition 1.20.** The process  $X$  is called an  $(\mathcal{A}, P)$ -local martingale if there exists an increasing sequence of stopping times  $\tau_n \uparrow \infty$  such that the process

$$X_t(\tau_n) = X_{\min\{t, \tau_n\}} \quad (1.2.4)$$

is an  $(\mathcal{A}, P)$ -martingale for every  $n$ .

We only state the following lemmas and theorem as we just need the results but their proofs are in Platen and Heath (2007).

**Lemma 1.21.** *If  $\epsilon : [0, \infty) \rightarrow \mathbb{R}$  is predictable and it holds for this integrand that:*

$$\int_0^T \epsilon(u)^2 du < \infty$$

*a.s. for all  $T \in [0, \infty)$ , then the corresponding Itô integral process*

$$I_{\epsilon, W} = \left\{ I_{\epsilon, W}(t) = \int_0^t \epsilon(s) dW_s, t \in [0, \infty) \right\}$$

*is an  $(\mathcal{A}, P)$ -local martingale.*

**Lemma 1.22.** *A nonnegative  $(\mathcal{A}, P)$ -local martingale  $X = \{X_t, t \in [0, \infty)\}$  such that  $E(X_t | \mathcal{A}_s) < \infty$  for all  $0 \leq s \leq t < \infty$  is an  $(\mathcal{A}, P)$ -supermartingale.*

**Theorem 1.23.** (Doob) *If  $X = \{X_t, t \in [0, \infty)\}$  is a right continuous  $(\mathcal{A}, P)$ -supermartingale, then for two bounded stopping times  $\tau$  and  $\tau'$  with  $\tau \leq \tau'$  we have*

$$E(X_{\tau'} | \mathcal{A}_\tau) \leq X_\tau$$

*almost surely.*

# Chapter 2

## The Continuous Financial Market Model and the Growth Optimal Portfolio.

In this chapter we define and determine the Growth optimal portfolio in a general continuous financial market model that uses self financing strategies. Afterwards, we give the SDE satisfied by its value which gives its dynamics. The aim of this chapter is to show the role of the Growth Optimal Portfolio in option pricing, which leads to the use of the real world probability measure. The whole chapter is based on Platen (2006).

### 2.1 The Continuous Financial Market.

We consider a continuous financial market modelled on the filtered probability space  $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t \in \mathbb{R}_+}, P)$ . The market is composed of  $d+1$  assets. The process  $S^0 = \{S_t^0, t \in [0, \infty)\}$  is the riskless security account process with interest rate  $r = \{r(t), t \in [0, \infty)\}$ . The risky security processes are  $S^j = \{S^j(t), t \in [0, \infty)\}$ ,  $j \in \{1, 2, \dots, d\}$ . Uncertainties are given by a  $d$ -dimensional Wiener process. We consider for the  $j$ -th security the drift process  $a^j = \{a^j(t), t \in [0, \infty)\}$  and the volatility process  $b^{j,k} = \{b^{j,k}(t), t \in [0, \infty)\}$  corresponding to the  $k$ -th Wiener process. The processes  $a^j$  and  $b^{j,k}$  are both predictable processes and satisfy the integrability conditions

$$\int_0^T \sum_{j=0}^d |a^j(s)| ds < \infty \quad (2.1.1)$$



and

$$\int_0^T \sum_{j=1}^d \sum_{k=1}^d (b^{j,k}(t))^2 dt < \infty. \quad (2.1.2)$$

We model the security processes as the following:

$$S^0(t) = \exp \left\{ \int_0^t r(s) ds \right\},$$

and

$$dS^j(t) = S^j(t) \left( a^j(t) dt + \sum_{k=1}^d b^{j,k}(t) dW_t^k \right), \quad (2.1.3)$$

for  $j \in \{1, \dots, d\}$ . To make sense of Equation (2.1.3) we look at the existence of a solution of it.

**Proposition 2.1.** *Under the integrability conditions (2.1.1) and (2.1.2), SDE (2.1.3) has a strong solution up to an initial value  $S^j(0)$  satisfying*

$$E(|S^j(0)|^2) < \infty$$

.

*Proof.* Let  $F$  be a process satisfying

$$dF_t^j = \left( a^j(t) - \frac{1}{2} \sum_{k=1}^d (b^{j,k})^2 \right) dt + \sum_{k=1}^d b^{j,k}(t) dW_t^k \quad (2.1.4)$$

such that  $E(|F_0^j|^2) < \infty$ . Assumption 1.15 is satisfied for the coefficients  $(a^j(t) - \frac{1}{2} \sum_{k=1}^d (b^{j,k})^2)$  and  $b^{j,k}(t)$  in the SDE (2.1.4). It follows then from Theorem 1.16 that the SDE (2.1.4) has a unique solution. Applying Itô's formula to  $e^{F_t^j}$ , we have from (2.1.4) and Theorem 1.14

$$\begin{aligned} d e^{F_t^j} &= \left( a^j(t) - \frac{1}{2} \sum_{k=1}^d (b^{j,k})^2 \right) e^{F_t^j} dt + \frac{1}{2} \sum_{k=1}^d (b^{j,k})^2 e^{F_t^j} dt + \sum_{k=1}^d b^{j,k}(t) e^{F_t^j} dW_t^k \\ &= e^{F_t^j} \left( a^j(t) dt + \sum_{k=1}^d b^{j,k}(t) dW_t^k \right). \end{aligned} \quad (2.1.5)$$

The existence and uniqueness of a solution of the SDE (2.1.5) follows from the existence and uniqueness of a solution of the SDE (2.1.4). Hence, Equation (2.1.3) has a unique and strong solution up to an initial value  $S^j(0)$ .  $\square$

Let us introduce another parameter, the market price of risk, whose importance will be shown later. First, we introduce the vectors  $W(t) = (W_t^1, \dots, W_t^d)^\top$ ,  $S(t) = (S^1(t), \dots, S^d(t))^\top$ , the unit vector  $\mathbf{1} = (1, \dots, 1)^\top$  and the appreciation rate vector  $a(t) = (a^1(t), \dots, a^d(t))^\top$ . We assume that the matrix  $b(t) = [b^{j,k}(t)]_{j,k=1}^d$  is invertible a.s. for  $t \in [0, \infty)$  with inverse  $b^{-1}(t)$ .

**Definition 2.2.** (Platen (2006)). The market price of risk is given by the vector

$$\theta(t) = (\theta^1(t), \dots, \theta^d(t))^\top$$

such that

$$\theta(t) = b^{-1}(t) [a(t) - r(t)\mathbf{1}]. \quad (2.1.6)$$

It is defined as a measure of extra return that investors demand to bear risk.

We can now prove the following proposition which gives the model of the prices in terms of the market price of risk.

**Proposition 2.3.** *Assume that the process  $S^j(t)$  satisfies (2.1.3), with the integrability conditions given to the coefficients  $a^j$  and  $b^{j,k}$ . Moreover, assume that the matrix  $b(t) = [b^{j,k}(t)]_{j,k=1}^d$  is invertible a.s. for  $t \in [0, \infty)$  with inverse  $b^{-1}(t)$ . Then*

$$dS^j(t) = S^j(t) \left( r(t)dt + \sum_{k=1}^d b^{j,k}(t) [\theta^k(t)dt + dW_t^k] \right), \quad (2.1.7)$$

where  $\theta^k(t)$  is the  $k^{\text{th}}$  market price of risk.

*Proof.* Using matrix notation, we have from Equation (2.1.3)

$$dS(t) = S(t) * [a(t)dt + b(t)dW_t].$$

From (2.1.6), we obtain

$$a(t) = b(t)\theta(t) + r(t)\mathbf{1},$$

so

$$dS(t) = S(t) * \left( r(t)dt\mathbf{1} + b(t) [\theta(t)dt + dW_t] \right)$$

and the result follows. □

**Definition 2.4.** (Platen and Heath (2007)) We call a market, in which stock prices satisfy Equation (2.1.7), a continuous financial market.

## 2.2 Portfolios in a Continuous Financial Market.

A portfolio strategy  $\delta$  at time  $t \in [0, \infty)$  is given by the  $\mathbb{R}^{d+1}$  valued random variable  $\delta(t) = (\delta^0(t), \delta^1(t), \dots, \delta^d(t))^\top$ . The value process  $S^\delta(t)$  corresponding to the strategy  $\delta$  is given by

$$S^\delta(t) = \langle \delta(t), S(t) \rangle.$$

Let us recall some definitions in finance from Kopp and Elliott (1998) characterizing the market

**Definition 2.5.** (No arbitrage) We say that the market does not allow arbitrage opportunities if for any strategy  $\delta$ , the following condition is satisfied:

- If  $S^\delta(0) = 0$ , then  $S^\delta(\tau) \geq 0$  a.s. implies  $S^\delta(\tau) = 0$  for a bounded stopping time  $\tau \in (0, \infty)$ .

**Definition 2.6.** A strategy  $\delta$  is called admissible if

- $S^\delta(t) \geq 0$  a.s. for all  $t \in [0, \infty)$ .

**Definition 2.7.** A portfolio corresponding to a strategy  $\delta$  is said to be self-financing if

$$dS^\delta(t) = \langle \delta(t), dS(t) \rangle, \tag{2.2.1}$$

for  $t \in [0, \infty)$ . This means that all the changes in the portfolio value are due to changes in the primary security accounts.

From now on, we assume that all strategies are self-financing and admissible. Assuming that  $S^\delta(t)$  is nonzero, we can introduce the  $j$ -th fraction  $\pi_\delta^j(t)$  of  $S^\delta(t)$  invested in the  $j$ -th security account. It is given by the  $j$ -th component of

$$\pi_\delta(t) = \frac{\delta(t) * S(t)}{S^\delta(t)}. \tag{2.2.2}$$

Let us introduce a new quantity  $b_\delta^k(t)$  which is the  $k$ -th portfolio volatility as the  $k$ -th component of  $b_\delta(t)$  where

$$b_\delta(t) = \pi_\delta(t)b(t). \quad (2.2.3)$$

**Proposition 2.8.** *The value of an admissible self-financing portfolio  $\delta$  is given by the SDE:*

$$dS^\delta(t) = S^\delta(t) \left( [r(t) + \langle b_\delta(t), \theta(t) \rangle] dt + \langle b_\delta(t), dW_t \rangle \right). \quad (2.2.4)$$

*Proof.* Using the fractions given by Equation (2.2.2), Equation (2.2.1) becomes

$$\begin{aligned} dS^\delta(t) &= \langle \delta(t), S(t) * \left( r(t)dt\mathbf{1} + b(t) [\theta(t)dt + dW_t] \right) \rangle \\ &= \langle \delta(t), S(t) \rangle dt + \langle \delta(t), S(t) * b(t) [\theta(t)dt + dW_t] \rangle \\ &= S^\delta(t)r(t)dt + \langle \delta(t) * S(t), b(t) [\theta(t)dt + dW_t] \rangle \\ &= S^\delta(t) \left( r(t)dt + \pi_\delta(t)b(t) [\theta(t)dt + dW_t] \right) \end{aligned} \quad (2.2.5)$$

and the result follows from Equations (2.2.5) and (2.2.3). □

## 2.3 The Growth Optimal Portfolio.

For any strategy  $\delta$ , it comes from Itô's formula that  $\ln(S^\delta(t))$  satisfies

$$d \ln(S^\delta(t)) = g^\delta(t)dt + H_t dW_t \quad (2.3.1)$$

for some process  $H$  and growth rate  $g^\delta(t)$ . A Growth Optimal Portfolio is defined by the following

**Definition 2.9.** (Platen (2006)). In a continuous financial market, a strictly positive portfolio process  $S^{\delta^*}(t)$  is called a Growth Optimal Portfolio if, for any positive portfolio with value  $S^\delta$ , the growth rates satisfy the inequality

$$g^{\delta^*}(t) \geq g^\delta(t)$$

almost surely for all  $t \in [0, \infty)$ .

The dynamic of a GOP is given by the following theorem:

**Theorem 2.10.** (Platen and Heath (2007)). *The growth optimal portfolio process satisfies the SDE*

$$dS^{\delta^*}(t) = S^{\delta^*}(t) \left( [r(t) + \langle \theta(t), \theta(t) \rangle] dt + \langle \theta(t), dW_t \rangle \right) \quad (2.3.2)$$

for all  $t \in [0, \infty)$ .

*Proof.* Looking for the portfolio which maximizes the growth rate  $g_\delta(t)$  is equivalent to looking for the fractions  $\pi_{\delta^*}^j(t)$  solution of

$$\frac{\partial g^\delta(t)}{\partial \pi_{\delta^*}^j(t)} = 0 \quad (2.3.3)$$

Applying Itô's formula to  $\ln(S^\delta(t))$ , we have from Equation (2.2.4) and Theorem 1.14

$$\begin{aligned} d \ln(S^\delta(t)) &= \left( r(t) + \langle b_\delta(t), \theta(t) \rangle - \frac{1}{2} \langle b_\delta(t), b_\delta(t) \rangle \right) dt \\ &\quad + \langle b_\delta(t), dW_t \rangle. \end{aligned}$$

Then

$$g^\delta(t) = r(t) + \langle b_\delta(t), \theta(t) \rangle - \frac{1}{2} \langle b_\delta(t), b_\delta(t) \rangle. \quad (2.3.4)$$

Equations (2.3.4) and (2.3.3) give

$$\sum_{k=1}^d b^{j,k}(t) \left( \theta^k(t) - \sum_{l=1}^d \pi_{\delta^*}^l(t) b^{l,k}(t) \right) = 0$$

for all  $t \in [0, \infty)$  and  $j \in 1, 2, \dots, d$ . Now, let us denote by  $A(t)$  the vector

$$\theta(t) - \pi_{\delta^*}(t)b(t)$$

We have

$$b(t)A(t) = 0.$$

Since  $b(t)$  is invertible,  $A(t) = 0$ , which means

$$\theta(t) = \pi_{\delta^*}(t)b(t)$$

and

$$\pi_{\delta^*}(t) = \theta(t)b^{-1}(t).$$

The vector of optimal fractions is then

$$\pi_{\delta_*}(t) = \theta(t)b^{-1}(t).$$

Using (2.2.3), we have

$$b_{\delta_*}(t) = \pi_{\delta_*}(t)b(t) = \theta(t)b^{-1}(t)b(t) = \theta(t) \quad (2.3.5)$$

and the result follows from substituting (2.3.5) in (2.2.4). □

It is clear from this proof that the value of the GOP as well as its fractions are uniquely determined up to its initial value  $S^{\delta_*}(t)$ .

## 2.4 The Real World Pricing.

In this section, we show that prices can be obtained under the benchmark approach taking the GOP as reference portfolio and without the need of risk neutral measure.

### The GOP as Benchmark Portfolio.

Using the GOP as benchmark plays an important role in the continuous financial market as we will see in this subsection that when using the GOP as reference or benchmark, benchmarked portfolios are supermartingales. We will prove that the supermartingale property of the benchmarked prices does not allow arbitrage opportunities in the market.

**Definition 2.11.** Any security expressed in units of the GOP is called benchmarked security.

**Theorem 2.12.** (*Platen and Heath (2007)*). *In a continuous financial market, the benchmarked portfolio  $\hat{S}^\delta(t) = \frac{S^\delta(t)}{S^{\delta_*}(t)}$  is an  $(\mathcal{A}, P)$ -supermartingale for any nonnegative admissible portfolio.*

*Proof.* Let us apply Ito formula to the process  $Y$  such that  $Y_t = \frac{S^\delta(t)}{S^{\delta^*}(t)}$  for  $t \in [0, \infty)$ . From Theorem 1.14, Equations (2.2.4) and (2.3.2), we have

$$\begin{aligned} dY_t &= \frac{S^\delta(t)}{S^{\delta^*}(t)} ([r(t) + \langle b_\delta(t), \theta(t) \rangle] - [r(t) + \langle \theta(t), \theta(t) \rangle]) dt \\ &\quad + \frac{S^\delta(t)}{S^{\delta^*}(t)} (\langle \theta(t), \theta(t) \rangle - \langle b_\delta(t), \theta(t) \rangle) dt \\ &\quad + \frac{dS^\delta(t)}{S^{\delta^*}(t)} (\langle b_\delta(t), dW_t \rangle - \langle \theta(t), dW_t \rangle) \\ &= Y_t (\langle b_\delta(t) - \theta(t), dW_t \rangle). \end{aligned}$$

Hence,

$$d\hat{S}^\delta(t) = \hat{S}^\delta(t) (\langle b_\delta(t) - \theta(t), dW_t \rangle). \quad (2.4.1)$$

Equation (2.4.1) means that the process  $\hat{S}^\delta$  which are the benchmarked portfolio values is driftless. It follows then from Lemma 1.21 that the process  $\hat{S}^\delta$  is  $(\mathcal{A}, P)$ -local martingale. Since the portfolio  $S^\delta$  is nonnegative, then Lemma 1.22 tells us that the process  $\hat{S}^\delta$  is  $(\mathcal{A}, P)$ -supermartingale.  $\square$

**Theorem 2.13.** *In a continuous financial market, there are no admissible arbitrage opportunities.*

*Proof.* This follows directly from the supermartingale property of the benchmarked portfolio and from Theorem 1.23. In fact, if  $S^\delta(0) = 0$ , we have

$$0 = \hat{S}^\delta(0) \geq E(\hat{S}^\delta(\tau) | \mathcal{A}_0) = E(\hat{S}^\delta(\tau)) \geq 0$$

for any bounded stopping time  $\tau \in [0, \infty)$ . And since the benchmarked value is nonnegative and  $S^{\delta^*}$  is strictly positive, we have

$$P(\hat{S}^\delta(\tau) > 0) = 0.$$

$\square$

## Fair Pricing Formula

In order to be able to write a pricing formula, we state the following definition which has been taken from Platen and Heath (2007).

**Definition 2.14.** A security price process  $V$  is called fair if its benchmarked value  $\hat{V}(t) = \frac{V(t)}{S^{\delta_*}(t)}$  is an  $(\mathcal{A}, P)$ -martingale.

Intuitively, the martingale property of benchmarked prices relates to the notion of fair price. The security price, when expressed in units of the GOP, are the best forecast of its future value. As a result Corollary 2.15 below gives a pricing formula for derivatives securities using the GOP as benchmark. Here, we mean by derivative security a financial instrument whose payoff is an  $\mathcal{A}_t$ -measurable random variable and whose value depends upon the values of the underlying risky securities  $S^j$  for  $j \in \{1, 2, \dots, d\}$ .

**Corollary 2.15.** *The price of a derivative security  $X$  at time  $t$  having a terminal value  $X_T$  is given by*

$$X_t = S^{\delta_*}(t) E \left[ \frac{X_T}{S^{\delta_*}(T)} \middle| \mathcal{A}_t \right] \quad (2.4.2)$$

where  $\pi_{\delta_*}$  denotes a GOP.

Since we have not made any assumption on the completeness of the market, this formula works both in complete and incomplete model. In case of complete model, it is clear that the formula coincide with the risk neutral pricing formula, by change of measure, since in complete model there exists exactly one equivalent martingale measure then a unique and fair price. The example below will demonstrate it by using the Black-Scholes model.

**Example 2.16.** We will now use the real world pricing technique to obtain the price of a European call under the Black-Scholes model. The Black-Scholes model for our market is given by the SDEs

$$\begin{cases} dS(t) &= aS(t)dt + \sigma S(t)dW_t \\ dB_t &= rB_t dt \end{cases} \quad (2.4.3)$$

where  $S = \{S(t), t \in [0, \infty)\}$  denotes the price process of the risky primary security asset and  $B = \{B_t, t \in [0, \infty)\}$  of the riskless bond asset with constant interest rate  $r$ . The variable  $a$  is the appreciation rate and  $\sigma$  the volatility of the risky asset.  $W$  denotes the Wiener process modelling uncertainties. For simplicity we assume  $a$  and  $\sigma$  to be constant. The Black-Scholes model for the risky asset is equivalent to

$$S(t) = S(0) \exp\left\{\left(a - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}$$



The initial price of the European call option maturing at time  $T$  and paying  $H(S(T)) = \max\{S(T) - K, 0\} = (S(T) - K)^+$ , using the real world pricing formula, is

$$U_H(0) = E \left( \frac{S^{(\delta_*)}(0)}{S^{(\delta_*)}(T)} H(S(T)) \middle| \mathcal{A}_0 \right) \quad (2.4.4)$$

And by Equation (2.3.2), the GOP is given by the SDE

$$dS^{(\delta_*)}(t) = S^{(\delta_*)}(t) \left( (r + \theta^2)dt + \theta dW_t \right)$$

which is equivalent to

$$S^{(\delta_*)}(t) = S^{(\delta_*)}(0) \exp\left\{ \left( r + \frac{1}{2}\theta^2 \right)t + \theta W_t \right\}$$

with  $\theta = \frac{a-r}{\sigma}$ . Hence

$$\begin{aligned} U_H(0) &= E \left[ \exp \left( -\left( r + \frac{1}{2}\theta^2 \right)T - \theta W_T \right) H(S(T)) \middle| \mathcal{A}_0 \right] \\ &= \int_{-\infty}^{+\infty} \exp(-rt - \frac{1}{2}\theta^2 T - \theta y) (S(0) \exp\{(a - \frac{1}{2}\sigma^2)T + \sigma y\} - K)^+ \\ &\quad \frac{1}{\sqrt{T}} N' \left( \frac{y}{\sqrt{T}} \right) dy \end{aligned}$$

where  $\frac{1}{\sqrt{T}} N' \left( \frac{y}{\sqrt{T}} \right)$  is the standard normal distribution density function. The integral is zero if  $S(0) \exp\{(a - \frac{1}{2}\sigma^2)T + \sigma y\} \leq K$  that is  $y \leq (\ln \frac{K}{S(0)} - (a - \frac{1}{2}\sigma^2)T)/\sigma$ . Let us denote

$$z = \frac{\ln \frac{K}{S(0)} - (a - \frac{1}{2}\sigma^2)T}{\sigma}$$

then

$$\begin{aligned} U_H(0) &= \int_z^{+\infty} \exp(-rt - \frac{1}{2}\theta^2 T - \theta y) (S(0) \exp\{(a - \frac{1}{2}\sigma^2)T + \sigma y\} - K) \\ &\quad \frac{1}{\sqrt{T}} N' \left( \frac{y}{\sqrt{T}} \right) dy \\ &= \int_z^{+\infty} \exp(-rt - \frac{1}{2}\theta^2 T - \theta y) (S(0) \exp\{(a - \frac{1}{2}\sigma^2)T + \sigma y\}) \\ &\quad \frac{1}{\sqrt{T}} N' \left( \frac{y}{\sqrt{T}} \right) dy \\ &\quad - K \int_z^{+\infty} \exp(-rt - \frac{1}{2}\theta^2 T - \theta y) \frac{1}{\sqrt{T}} N' \left( \frac{y}{\sqrt{T}} \right) dy \end{aligned}$$

Let

$$A = \int_z^{+\infty} \exp\left(-rt - \frac{1}{2}\theta^2 T - \theta y\right) (S(0) \exp\{(a - \frac{1}{2}\sigma^2)T + \sigma y\}) \frac{1}{\sqrt{T}} N'\left(\frac{y}{\sqrt{T}}\right) dy$$

and

$$B = -K \int_z^{+\infty} \exp\left(-rt - \frac{1}{2}\theta^2 T - \theta y\right) \frac{1}{\sqrt{T}} N'\left(\frac{y}{\sqrt{T}}\right) dy.$$

For  $\gamma$  and  $\beta$  positives we have

$$\begin{aligned} & \frac{1}{\sqrt{2\pi T}} \int_z^{+\infty} \exp\left\{\beta + \gamma y - \frac{1}{2} \frac{y^2}{T}\right\} dy \\ &= e^{\beta + \frac{1}{2}\gamma^2 T} \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{-z} \exp\left\{-\frac{1}{2} \frac{(y + \gamma T)^2}{T}\right\} dy \\ &= e^{\beta + \frac{1}{2}\gamma^2 T} N\left(\frac{-z + \gamma T}{\sqrt{T}}\right) \end{aligned}$$

where  $N$  is the standard normal distribution function. Hence, with an appropriate choice of  $\beta$  and  $\gamma$  we have

$$\begin{aligned} A &= S(0) N\left(\frac{-z + (\sigma - \theta)T}{\sqrt{T}}\right) \\ &= S(0) N\left(\frac{\ln\left(\frac{S(0)}{K}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \end{aligned}$$

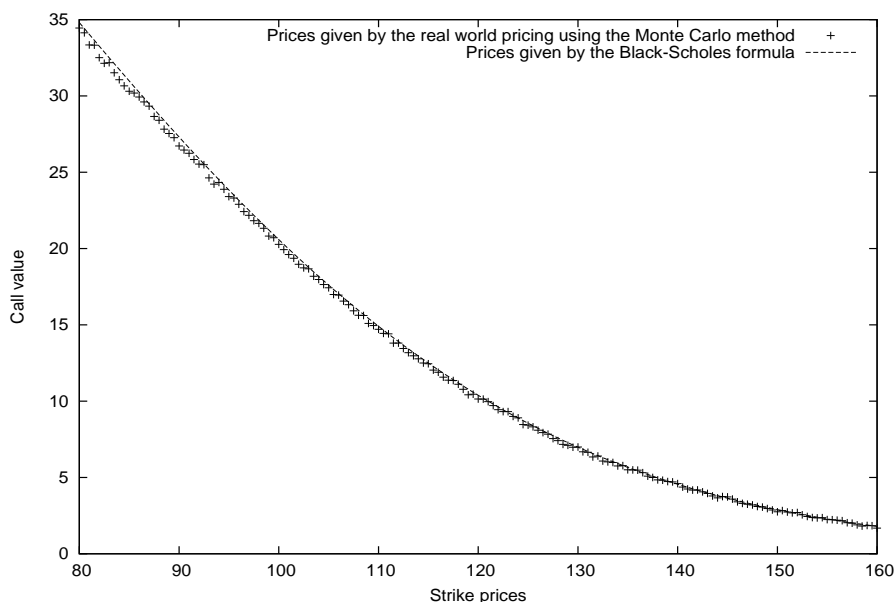
and

$$\begin{aligned} B &= -K e^{-rT} N\left(\frac{-z - \theta T}{\sqrt{T}}\right) \\ &= -K e^{-rT} N\left(\frac{\ln\left(\frac{S(0)}{K}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

Notice that  $A + B$  gives the Black-Scholes formula for European call option which was derived from the risk neutral pricing, that is using the existence of the equivalent martingale measure. This emphasize the fact that the real world pricing formula does not need the equivalent martingale measure to be applied.

As we need to have the value of the GOP to compute the prices using the real world pricing formula, one would figure out how to obtain the GOP without the need of estimation of particular parameters such as the market

price of risk. An alternative way of approximating the GOP is given and developed in Platen (2005) and Platen and Heath (2007). For our numerical example, we use the Black-Scholes model given by (2.4.3) to determine the asset prices and apply directly the Monte Carlo method to compute Equation (2.4.4) where  $H(S(T)) = \max\{S(T) - K, 0\} = (S(T) - K)^+$  is the payoff of the European option. For the inputs, we use  $S(0) = 100$  and  $r = 1/10$  as in the discrete case example given in Chapter 4. We use  $\sigma = 0.17$  for the volatility of the return in the asset and as we did not look closely at the approximation of the GOP, we use  $a = 0.13$  which has been estimated while doing the simulation by comparing and approximating the results with those given by the Black-Scholes formula. The initial value of the GOP is not needed here since it vanishes in the calculation. We compute the option prices and plot versus strike prices which is given by Figure 2.1.



**Figure 2.1:** Call values plotted in function of the strike prices by Black-Scholes formula and the real world pricing.

The prices given by the real world pricing formula have been computed numerically by using Monte Carlo method (see Fishman (1996)). We close the continuous time case with the example above and look at the discrete case in the next chapter where we present another approach of finding the real world pricing by the use of the GOP and the risk neutral measure.

# Chapter 3

## The Growth Optimal Portfolio in Discrete Time.

This chapter, as said in the introduction is due to the article Korn and Schäl (1999). Platen and Bühlmann (2003) is also a good reference.

### 3.1 Analogy of the Continuous Time Market and the GOP.

We use the filtered probability space  $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t \in \{0,1,\dots,T\}}, P)$  where  $T \in [0, \infty)$  is the time horizon. We consider a market of  $d + 1$  assets. The process  $S^0$  is the riskless security account with interest rate  $r = \{r(1), \dots, r(T)\}$  such that  $S^0(0) = 1$  and

$$S^0(t) = (1 + r(1)) \dots (1 + r(t)). \quad (3.1.1)$$

We can assume that the interest rate  $r$  is predictable that is  $\mathcal{A}_{t-1}$ -measurable however such assumption will not be needed for the rest of the work. The risky assets are given by the  $d$ -dimensional stochastic process

$$S = \{S(t) = (S^1(t), \dots, S^d(t)), t \in \{0, \dots, T\}\}.$$

The  $d$ -dimensional predictable process

$$\delta = \{\delta(t) = (\delta^1(t), \dots, \delta^d(t)), t = 1, 2, \dots, T\}$$

represents the strategy that gives the number of shares invested in the risky assets and  $\delta^0(t)$  is the number of shares in bond at time  $t = 0, \dots, T$ . The value  $S^\delta(t)$  of the portfolio at time  $t$  is then given by:

$$S^\delta(t) = \delta^0(t)S^0(t) + \langle \delta(t), S(t) \rangle.$$

**Definition 3.1.** A trading strategy  $\delta = \{\delta(t), t = 1, 2, \dots, T\}$  is self financing if

$$\delta^0(t)S^0(t) + \langle \delta(t), S(t) \rangle = \delta^0(t-1)S^0(t) + \langle \delta(t-1), S(t) \rangle$$

for all  $t = 1, 2, \dots, T$ .

Let  $\pi_\delta$  be the fraction defined by

$$\pi_\delta(t) = \frac{\delta(t) * S(t)}{S^\delta(t)}$$

and

$$\pi_\delta^0 = 1 - (\pi_\delta^1 + \dots + \pi_\delta^d)$$

is the fraction of the bond invested in the portfolio. Now, let us take  $\frac{1}{S^0}$  as the discount factor.

**Definition 3.2.** The discounted price process  $\check{S}(t) = (\check{S}^1(t), \dots, \check{S}^d(t))$  is defined by

$$\check{S}^i(t) = \frac{S^i(t)}{S^0(t)}, \quad (3.1.2)$$

$i = 1, \dots, d$  and  $t = 0, \dots, T$ .

Let  $\Delta X(t) = X(t) - X(t-1)$  define the backward increment of any process  $X$ . We use the notation  $u^i(t)$  for  $\frac{\Delta S^i(t)}{S^i(t-1)}$ .

**Definition 3.3.** (Korn and Schäl (1999)). The relative risk process for the risky assets

$$R = \{R(t) = (R^1(t), \dots, R^d(t))^\top, t \in \{0, 1, \dots, T\}\}$$

is defined by

$$1 + R^i(t) = \frac{\Delta \check{S}^i(t)}{\check{S}^i(t-1)} = \frac{1 + u^i(t)}{1 + r(t)} \quad (3.1.3)$$

for  $i = 1, \dots, d$ .

In a discrete time setting, the value of an admissible self-financing portfolio is given by the following proposition:

**Proposition 3.4.** For an admissible self-financing portfolio, given an initial wealth  $S^\delta(0) = x$ , the portfolio value at time  $t = 1, \dots, T$  is given by

$$S^\delta(t) = xS^0(t) \prod_{m=1}^t (1 + \langle \pi_\delta(m-1), R(m) \rangle). \quad (3.1.4)$$

*Proof.* We have from Equation (3.1.2) that

$$\Delta\check{S}^\delta(t) = \delta(t-1)\Delta\check{S}(t)$$

and from (2.2.2)

$$\pi_\delta^i(t) = \frac{\delta^i(t)\check{S}^i(t)}{\check{S}^{\pi_\delta(t)}},$$

hence

$$\begin{aligned} \check{S}^\delta(t) &= \check{S}^\delta(t-1) + \delta(t-1)\Delta\check{S}(t) \\ &= \check{S}^\delta(t-1) + \frac{\pi_\delta(t-1)\check{S}^\delta(t-1)}{\check{S}(t-1)}\Delta\check{S}(t) \\ &= \check{S}^\delta(t-1)(1 + \langle \pi_\delta(t-1), R(t) \rangle), \end{aligned}$$

and the result follows.  $\square$

By analogy to the continuous time in Chapter 2, the following property still holds in discrete time setting:

**Theorem 3.5.** *All admissible portfolios  $\pi_\delta$  when expressed in unit of the Growth Optimal Portfolio are  $(\mathcal{A}, P)$  – supermartingale that is*

$$E\left(\frac{S^\delta(t+1)}{S^{\delta_*}(t+1)} \middle| \mathcal{A}_t\right) \leq \frac{S^\delta(t)}{S^{\delta_*}(t)}.$$

The proof of Theorem 3.5 will not be needed here but we can refer to Platen and Bühlmann (2003). The supermartingale property of the benchmarked portfolios does not allow arbitrage opportunity in the market and then does not need the existence of an equivalent martingale measure. The approach we have seen in the continuous time of course as an analogy in discrete time and this has been developed in Platen and Bühlmann (2003). Definition 2.14 will again be used for fair securities and it follows the definition of the pricing formula which is

**Definition 3.6.** Let  $\pi_{\delta_*}$  be a Growth Optimal Portfolio. The price  $Pr(X_t)$  of a derivative security  $X$  with terminal value  $X_T$  is given by the relation

$$\frac{1}{S^{\delta_*}(t)}Pr(X_t) = E\left[\frac{X_T}{S^{\delta_*}(T)} \middle| \mathcal{A}_t\right].$$

Using Formula (3.1.4) and Definition 3.6 we have,

$$Pr(X_t) = S^0(t)E \left[ \frac{1}{\prod_{m=t+1}^T (1 + \langle \pi_{\delta_*}(m-1), R(m) \rangle)} \frac{X_T}{S^0(T)} \middle| \mathcal{A}_t \right]. \quad (3.1.5)$$

This approach, that has also been used in the continuous time is not the only one which leads to the real world pricing. For the next section, we present a second approach of the real world pricing theory which uses the existence of a martingale measure in the discrete time financial market. This section gives another property of the Growth Optimal Portfolio which is its numéraire property.

## 3.2 The Numéraire Portfolio.

In this section we give the definition of the numéraire portfolio and show that by change of numéraire the discounted prices are martingales under the given physical probability measure. The notion of numéraire portfolio has been introduced by Korn and Schäl (1999). Lemmas and Theorems in this section are due to Korn and Schäl (1999).

**Definition 3.7.** A probability measure  $Q$  which is equivalent to  $P$  (see Definition 1.7) is an equivalent martingale measure if the discounted price process  $\check{S}(t)$  is a martingale under  $Q$ .

Using the relative risk process given by Equation (3.1.3),  $Q$  is an equivalent martingale measure if and only if

$$E_Q[R^i(t) | \mathcal{A}_{t-1}] = 0 \quad (3.2.1)$$

for  $1 \leq i \leq d$ ,  $1 \leq t \leq T$ .

**Definition 3.8.** (Korn and Schäl (1999)). An admissible portfolio process  $\pi_\delta$  is called a numéraire portfolio if

$$L_t = \frac{1}{1 + \langle \pi_\delta(t-1), R(t) \rangle} = (1 + r(t)) \frac{S^\delta(t-1)}{S^\delta(t)} \quad (3.2.2)$$

defines some equivalent martingale measure  $Q$  by  $\frac{dQ}{dP} = \prod_{t=1}^T L_t$ .

**Lemma 3.9.** (Korn and Schäl (1999)). A probability measure  $Q$  defined by  $\frac{dQ}{dP} = \prod_{t=1}^T L_t$ , for some positive process  $L_t$ , is an equivalent martingale measure if and only if

$$E[L_t | \mathcal{A}_{t-1}] = 1, \quad (3.2.3)$$

and

$$E[L_t R^i(t) | \mathcal{A}_{t-1}] = 0, \quad (3.2.4)$$

$1 \leq i \leq d$ .

*Proof.* Suppose  $Q$  is an equivalent martingale measure. We need to find  $L_t$  such that  $L = \frac{dQ}{dP} = \prod_{t=1}^T L_t$  and Equation (3.2.3) and (3.2.4) are satisfied. Set

$$\begin{aligned} L_1 &= E[L | \mathcal{A}_1] \\ L_1 L_2 &= E[L | \mathcal{A}_2] \\ &\dots = \dots \\ L_1 L_2 \cdots L_T &= E[L | \mathcal{A}_T]. \end{aligned}$$

We have from Bayes formula (1.8):

$$E_Q[R^i(t) | \mathcal{A}_{t-1}] = \frac{E[LR^i(t) | \mathcal{A}_{t-1}]}{E[L | \mathcal{A}_{t-1}]}. \quad (3.2.5)$$

As

$$\begin{aligned} E[LR^i(t) | \mathcal{A}_{t-1}] &= E[R^i(t) E[L | \mathcal{A}_t] | \mathcal{A}_{t-1}] \\ &= E[L_1 \cdots L_t R^i(t) | \mathcal{A}_{t-1}] \\ &= L_1 \cdots L_{t-1} E[L_t R^i(t) | \mathcal{A}_{t-1}] \\ &= E[L | \mathcal{A}_{t-1}] E[L_t R^i(t) | \mathcal{A}_{t-1}], \end{aligned}$$

then

$$\begin{aligned} E[L_t R^i(t) | \mathcal{A}_{t-1}] &= \frac{E[LR^i(t) | \mathcal{A}_{t-1}]}{E[L | \mathcal{A}_{t-1}]} \\ &= 0 \quad (\text{by Equations (3.2.5) and (3.2.1)}). \end{aligned}$$



We have also

$$1 = \frac{E[L|\mathcal{A}_{t-1}]}{E[L|\mathcal{A}_{t-1}]} = \frac{E[E[L|\mathcal{A}_t]|\mathcal{A}_{t-1}]}{E[L|\mathcal{A}_{t-1}]} = \frac{E[L_1 L_2 \cdots L_t | \mathcal{A}_{t-1}]}{L_1 L_2 \cdots L_{t-1}} = E[L_t | \mathcal{A}_{t-1}].$$

Let us now prove the inverse. Suppose there exists  $L_t$  such that  $\frac{dQ}{dP} = \prod_{t=1}^T L_t$  and Equations (3.2.3) and (3.2.4) are satisfied. We want to show that

$$E_Q[R^i(t)|\mathcal{A}_{t-1}] = 0$$

for  $t \in [1, T]$ .

$$\begin{aligned} E_Q[R^i(t)|\mathcal{A}_{t-1}] &= \frac{E[LR^i(t)|\mathcal{A}_{t-1}]}{E[L|\mathcal{A}_{t-1}]} \\ &= \frac{L_1 L_2 \cdots L_{t-2} L_{t-1} E[L_t \cdots L_T R^i(T)|\mathcal{A}_{t-1}]}{L_1 L_2 \cdots L_{t-1} E[L_t \cdots L_T | \mathcal{A}_{t-1}]} \\ &= \frac{E[L_t \cdots L_T R^i(t)|\mathcal{A}_{t-1}]}{E[L_t \cdots L_T | \mathcal{A}_{t-1}]} \\ &= \frac{E[E[L_t | \mathcal{A}_{t-1}] \cdots E[L_T | \mathcal{A}_{T-1}] R^i(t) | \mathcal{A}_{t-1}]}{E[L_{t-1} | \mathcal{A}_{t-1}] \cdots E[L_T | \mathcal{A}_{T-1}]} \\ &= 0 \quad (\text{by (3.2.3) and (3.2.4)}). \end{aligned}$$

□

**Theorem 3.10.** (*Korn and Schäl (1999)*). *The portfolio  $\pi_\delta$  is a numéraire portfolio if and only if the processes  $S^i/S^\delta = \{S^i(t)/S^\delta(t), t = 1, \dots, T\}$ ,  $i = 0, \dots, d$  are martingales under the real world probability  $P$ .*

*Proof.* Let us take  $L_t$  as in Definition 3.8 and such that  $L = \frac{dQ}{dP}$  satisfies

$$E[L|\mathcal{A}_t] = \prod_{m=1}^t L_m. \quad (3.2.6)$$

We want to calculate  $E[S^0(t)/S^\delta(t)|\mathcal{A}_{t-1}]$  and  $E[S^i(t)/S^\delta(t)|\mathcal{A}_{t-1}]$  for  $i = 1, \dots, d$ . From Proposition 3.4, Equations (3.1.1), (3.1.2), (3.1.3) and (3.2.2), we have

$$\begin{aligned} E\left[\frac{S^0(t)}{S^\delta(t)}|\mathcal{A}_{t-1}\right] &= E\left[\frac{S^0(t)}{S^\delta(t-1)(1 + \langle \pi_\delta(t-1), R(t) \rangle)(1 + r(t))}|\mathcal{A}_{t-1}\right] \\ &= \frac{S^0(t-1)}{S^\delta(t-1)} E[L_t | \mathcal{A}_{t-1}] \end{aligned} \quad (3.2.7)$$

and

$$\begin{aligned}
E \left[ \frac{S^i(t)}{S^\delta(t)} \middle| \mathcal{A}_{t-1} \right] &= E \left[ \frac{S^i(t)}{S^\delta(t-1)(1+r(t))(1+\langle \pi_\delta(t-1), R(t) \rangle)} \middle| \mathcal{A}_{t-1} \right] \\
&= \frac{1}{S^\delta(t-1)} E \left[ \frac{S^i(t-1)(1+R^i(t))}{(1+\langle \pi_\delta(t-1), R(t) \rangle)} \middle| \mathcal{A}_{t-1} \right] \\
&= \frac{S^i(t-1)}{S^\delta(t-1)} E[L_t | \mathcal{A}_{t-1}] + \frac{S^i(t-1)}{S^\delta(t-1)} E[L_t R^i(t) | \mathcal{A}_{t-1}].
\end{aligned} \tag{3.2.8}$$

Suppose now that the processes  $S^i/S^\delta$  are martingales under the real world probability. In Equations (3.2.7) and (3.2.8), we have  $E[L_t | \mathcal{A}_{t-1}] = 1$  and  $E[L_t R^i(t) | \mathcal{A}_{t-1}] = 0$ . It follows then from Definition 3.8 and Lemma 3.9 that  $\pi$  is a numéraire portfolio. Conversely if  $\pi$  is a numéraire portfolio, Equations (3.2.3) and (3.2.4) are satisfied and then the martingale property of the processes  $S^i/S^\delta$  is obtained from Equations (3.2.7) and (3.2.8).  $\square$

## The Growth Optimal Portfolio (GOP) as numéraire Portfolio

We define the Growth Optimal Portfolio in this section and show its numéraire property. We will see that using the GOP as numéraire, the discounted prices are martingales under the real world probability. Then as an analogous of the continuous time, we can find a pricing formula. Similarly to Definition 2.9, we have the following definition of the Growth Optimal Portfolio in discrete time:

**Definition 3.11.** (Korn and Schäl (1999)). A portfolio  $\pi_{\delta^*}$  is a Growth Optimal Portfolio if

$$E \left[ \ln \left( \frac{S^{\delta^*}(t)}{S^{\delta^*}(0)} \right) \right] = \sup_{\pi_\delta \in \Pi} E \left[ \ln \left( \frac{S^\delta(t)}{S^\delta(0)} \right) \right]. \tag{3.2.9}$$

We have

$$\begin{aligned}
E \left[ \ln \left( \frac{S^\delta(t)}{S^\delta(0)} \right) \right] &= E \left[ \ln \left( \frac{S^\delta(t)}{S^\delta(t-1)} \right) \right] + E \left[ \ln \left( \frac{S^\delta(t-1)}{S^\delta(t-2)} \right) \right] \\
&\quad + \cdots + E \left[ \ln \left( \frac{S^\delta(1)}{S^\delta(0)} \right) \right]
\end{aligned}$$

and the growth rate  $g^\delta(t)$  in Equation (2.3.1) corresponds to  $E \left[ \ln \frac{S^\delta(t+dt)}{S^\delta(t)} \right]$ .

**Lemma 3.12.** *If  $\Omega$  is finite and  $\pi_\delta$  is a GOP, the following conditions named as **first order conditions** are satisfied:*

$$E \left[ \frac{R^i(t)}{1 + \langle \pi_\delta(t-1), R(t) \rangle} \middle| \mathcal{A}_{t-1} \right] = 0 \quad (3.2.10)$$

and

$$E \left[ \frac{1}{1 + \langle \pi_\delta(t-1), R(t) \rangle} \middle| \mathcal{A}_{t-1} \right] = 1 \quad (3.2.11)$$

for  $1 \leq t \leq T$ ,  $i \in \{0, 1, \dots, d\}$ .

*Proof.* We use the Lagrange multiplier method to solve Equation (3.2.9). We are going to prove first the theorem for a single period  $T = 1$ . We have

$$\frac{S^\delta(1)}{S^\delta(0)} = (1 + r(1))\pi_\delta^0(0) + (1 + u^1(1))\pi_\delta^1(0) + \dots + (1 + u^d(1))\pi_\delta^d(0)$$

Let  $\lambda$  be the Lagrange multiplier and let

$$\begin{aligned} g(\pi_\delta(0)) &= \pi_\delta^0(0) + \pi_\delta^1(0) + \dots + \pi_\delta^d(0) - 1 \\ f(\pi_\delta(0)) &= E \left[ \ln \frac{S^\delta(1)}{S^\delta(0)} \right] - \lambda g(\pi_\delta(0)). \end{aligned}$$

We have to solve the following equations for  $\pi_\delta^0, \dots, \pi_\delta^d$  and  $\lambda$ :

$$\left\{ \begin{array}{l} \frac{\partial f(\pi_\delta(0))}{\partial \lambda} = -g(\pi_\delta(0)) = 0 \\ \frac{\partial f(\pi_\delta(0))}{\partial \pi_\delta^0(0)} = E \left[ \frac{1 + r(0)}{(1 + r(1))\pi_\delta^0(0) + \langle (1 + u(1)), \pi_\delta(0) \rangle} \right] - \lambda = 0 \\ \frac{\partial f(\pi_\delta(0))}{\partial \pi_\delta^1(0)} = E \left[ \frac{1 + u^1(1)}{(1 + r(1))\pi_\delta^0(0) + \langle (1 + u(1)), \pi_\delta(0) \rangle} \right] - \lambda = 0 \\ \vdots \\ \frac{\partial f(\pi_\delta(0))}{\partial \pi_\delta^d(0)} = E \left[ \frac{1 + u^d(1)}{(1 + r(1))\pi_\delta^0(0) + \langle (1 + u(1)), \pi_\delta(0) \rangle} \right] - \lambda = 0 \end{array} \right. \quad (3.2.12)$$

From Equation (3.1.3) and (3.2.12), we have:

$$\begin{aligned} E \left[ \frac{1}{\pi_\delta^0(0) + \frac{1+u^1(1)}{1+r(1)}\pi_\delta^1(0) + \dots + \frac{1+u^d(1)}{1+r(1)}\pi_\delta^d(0)} \right] &= E \left[ \frac{1}{1 + \langle R(1), \pi_\delta(0) \rangle} \right] \\ &= \lambda \end{aligned} \quad (3.2.13)$$

and

$$\begin{aligned}
 \frac{\partial f(\pi_\delta(0))}{\partial \pi_\delta^i(0)} &= E \left[ \frac{1 + R^i(1)}{\pi_\delta^0(0) + \langle (1 + R(1)), \pi_\delta(0) \rangle} \right] - \lambda = 0 \\
 &= E \left[ \frac{1 + R^i(1)}{\pi_\delta^0(0) + \pi_\delta^1(0) + \cdots + \pi_\delta^d(0) + \langle R(1), \pi_\delta(0) \rangle} \right] - \lambda = 0 \\
 &= E \left[ \frac{1 + R^i(1)}{1 + \langle R(1), \pi_\delta(0) \rangle} \right] - \lambda = 0
 \end{aligned} \tag{3.2.14}$$

for  $i \in \{1, \dots, d\}$ . Equations (3.2.13), (3.2.14) give the result which is

$$E \left[ \frac{R^i(1)}{1 + \langle R(1), \pi_\delta(0) \rangle} \right] = 0$$

for  $i \in \{1, 2, \dots, d\}$ . Using the additive property of the logarithm, we can now prove the theorem for a multiperiod model. In fact, we have

$$\begin{aligned}
 E \left[ \ln \left( \frac{S^\delta(t)}{S^\delta(0)} \right) \right] &= E \left[ \ln \left( \frac{S^\delta(t)}{S^\delta(t-1)} \right) \right] + E \left[ \ln \left( \frac{S^\delta(t-1)}{S^\delta(t-2)} \right) \right] \\
 &\quad + \cdots + E \left[ \ln \left( \frac{S^\delta(1)}{S^\delta(0)} \right) \right] \\
 &= E \left[ E \left[ \ln \left( \frac{S^\delta(t)}{S^\delta(t-1)} \right) \mid \mathcal{A}_{t-1} \right] \right] \\
 &\quad + E \left[ E \left[ \ln \left( \frac{S^\delta(t-1)}{S^\delta(t-2)} \right) \mid \mathcal{A}_{t-2} \right] \right] \\
 &\quad + \cdots + E \left[ E \left[ \ln \left( \frac{S^\delta(2)}{S^\delta(1)} \right) \mid \mathcal{A}_1 \right] \right] + E \left[ \ln \left( \frac{S^\delta(1)}{S^\delta(0)} \right) \mid \mathcal{A}_0 \right]
 \end{aligned} \tag{3.2.15}$$

and for each  $t \in [0, \infty)$ ,

$$\sup_{\pi_\delta(t-1)} E \left[ E \left[ \ln \left( \frac{S^\delta(t)}{S^\delta(t-1)} \right) \mid \mathcal{A}_{t-1} \right] \right] = \sup_{\pi_\delta(t-1)} E \left[ \ln \left( \frac{S^\delta(t)}{S^\delta(t-1)} \right) \mid \mathcal{A}_{t-1} \right].$$

For the first expectation, we have

$$\frac{S^\delta(t)}{S^\delta(t-1)} = (1 + r(t))\pi_\delta^0(t-1) + \langle (1 + u(t)), \pi_\delta(t-1) \rangle.$$

Let  $U(t) = (1 + r(t))\pi_\delta^0(t-1) + \langle (1 + u^1(t)), \pi_\delta(t-1) \rangle$ . We have

$$E \left[ \ln(U(t)) \mid \mathcal{A}_{t-1} \right] = \sum_{i=1}^k \frac{E \left[ \ln(U(t)), A_{t-1}^i \right]}{P(A_{t-1}^i)} 1_{A_{t-1}^i}.$$

Besides,

$$\frac{E [\ln (U(t)), A_{t-1}^i]}{P(A_{t-1}^i)} = \frac{\sum_{\omega' \in A_{t-1}^i} P(\omega') \ln (U(t)) (\omega')}{P(A_{t-1}^i)}$$

then

$$E [\ln (U(t)) | \mathcal{A}_{t-1}] (\omega) = \sum_{i=1}^k \frac{\sum_{\omega' \in A_{t-1}^i} P(\omega') \ln (U(t)) (\omega')}{P(A_{t-1}^i)} 1_{A_{t-1}^i} (\omega).$$

Finally, computing the derivative of the Lagrange function, we have:

$$\frac{\partial f(\pi_{t-1})}{\partial \pi_{t-1}^j} = \sum_{i=1}^k \frac{\sum_{\omega' \in A_{t-1}^i} P(\omega') \frac{1 + u^j(t)}{U(t)} (\omega')}{P(A_{t-1}^i)} 1_{A_{t-1}^i} (\omega) - \lambda = 0 \quad (3.2.16)$$

$$= E \left[ \frac{1 + u^j(t)}{U(t)} \middle| \mathcal{A}_{t-1} \right] - \lambda = 0 \quad (3.2.17)$$

Using (3.1.3), applying the same computation as in single period to (3.2.17), we get the result. Similar to what we have just done, we can continue with the rest of the terms in Equation (3.2.15) and get Equation (3.2.10) for  $1 \leq t \leq T$ .

For Equation (3.2.11), we have:

$$\begin{aligned} & E \left[ \frac{1}{1 + \langle \pi_\delta(t-1), R(t) \rangle} - 1 \middle| \mathcal{A}_{t-1} \right] \\ &= E \left[ \frac{1 - 1 - (\langle \pi_\delta(t-1), R(t) \rangle)}{1 + \langle \pi_\delta(t-1), R(t) \rangle} \middle| \mathcal{A}_{t-1} \right] \\ &= -E \left[ \frac{1 + \langle \pi_\delta(t-1), R(t) \rangle}{1 + \langle \pi_\delta(t-1), R(t) \rangle} \middle| \mathcal{A}_{t-1} \right] \\ &= -\pi^i(t-1) E \left[ \frac{R^i(t)}{1 + \langle \pi_\delta(t-1), R(t) \rangle} \middle| \mathcal{A}_{t-1} \right] \\ &= 0. \end{aligned}$$

□

**Theorem 3.13.** (Korn and Schäl (1999)). *If  $\pi_\delta$  is a GOP, then  $\pi_\delta$  is a numéraire Portfolio.*

*Proof.* From Lemma 3.9 and the first order conditions (3.2.10) and (3.2.11), we see that

$$\frac{1}{1 + \langle \pi_\delta(t-1), R(t) \rangle}$$

defines an Equivalent martingale measure. Hence  $\pi_{\delta}$  is a numéraire portfolio by Definition 3.8.  $\square$

The numéraire property of the GOP leads to a pricing formula which is described in the next subsection.

### Pricing Formula

From the numéraire property of the GOP,

$$L_t = \frac{1}{1 + \langle \pi_{\delta_*}(t-1), R(t) \rangle} \quad (3.2.18)$$

defines some equivalent martingale measure  $Q^*$  by  $\frac{dQ^*}{dP} = \prod_{t=1}^T L_t$  and the real world pricing formula is given by:

**Theorem 3.14.** *Let  $\pi_{\delta_*}$  be a Growth Optimal Portfolio, then the price  $Pr(X_t)$  of a derivative security  $X$  at time  $t = 1, \dots, T$  is given by*

$$Pr(X_t) = S^0(t) E \left[ \frac{1}{\prod_{m=t+1}^T (1 + \langle \pi_{\delta_*}(m-1), R(m) \rangle)} \frac{X_T}{S^0(T)} \middle| \mathcal{A}_t \right] \quad (3.2.19)$$

*Proof.* The proof comes directly from the standard pricing formula using the risk neutral measure  $Q^*$  which is

$$Pr(X_t) = S^0(t) E_{Q^*} \left[ \frac{X_T}{S^0(T)} \middle| \mathcal{A}_t \right]. \quad (3.2.20)$$

By change of measure from  $Q^*$  to  $P$ , using the probability density  $\frac{dQ^*}{dP} = \prod_{t=1}^T L_t$ , where  $L_t$  is given by Equation (3.2.18) and using the Bayes Formula given by

Theorem 1.8 we have

$$S^0(t)E_{Q^*} \left[ \frac{X_T}{S^0(T)} \middle| \mathcal{A}_t \right] \quad (3.2.21)$$

$$= S^0(t) \frac{E \left[ \frac{dQ^*}{dP} \frac{X_T}{S^0(T)} \middle| \mathcal{A}_t \right]}{E \left[ \frac{dQ^*}{dP} \middle| \mathcal{A}_t \right]} \quad (3.2.22)$$

$$= S^0(t) \frac{E \left[ \frac{X_T}{S^0(T) \prod_{m=1}^T 1 + \langle \pi_{\delta_*}(m-1), R(m) \rangle} \middle| \mathcal{A}_t \right]}{E \left[ \frac{1}{\prod_{m=1}^T 1 + \langle \pi_{\delta_*}(m-1), R(m) \rangle} \middle| \mathcal{A}_t \right]} \quad (3.2.23)$$

$$= S^0(t) \frac{E \left[ \frac{X_T}{S^0(T) \prod_{m=t+1}^T 1 + \langle \pi_{\delta_*}(m-1), R(m) \rangle} \middle| \mathcal{A}_t \right]}{E \left[ \frac{1}{\prod_{m=t+1}^T 1 + \langle \pi_{\delta_*}(m-1), R(m) \rangle} \middle| \mathcal{A}_t \right]}. \quad (3.2.24)$$

Using Equation (1.1.3) and Lemma 3.9, the denominator in the last line becomes

$$E \left[ \frac{1}{\prod_{m=t+1}^T 1 + \langle \pi_{\delta_*}(m-1), R(m) \rangle} \middle| \mathcal{A}_t \right] \quad (3.2.25)$$

$$= E [L_{t+1} \cdots L_T | \mathcal{A}_t] \quad (3.2.26)$$

$$= E [L_{t+1} \underbrace{E[L_{t+2} | \mathcal{A}_{t+1}]}_{=1} \underbrace{E[L_{t+3} | \mathcal{A}_{t+2}]}_{=1} \cdots \underbrace{E[L_T | \mathcal{A}_{T-1}]}_{=1} | \mathcal{A}_t] \quad (3.2.27)$$

$$= 1 \quad (3.2.28)$$

and hence,

$$Pr(X_t) = S^0(t)E \left[ \frac{X_T}{S^0(T) \prod_{m=t+1}^T 1 + \langle \pi_{\delta_*}(m-1), R(m) \rangle} \middle| \mathcal{A}_t \right]. \quad (3.2.29)$$

□

We notice that this pricing formula is the same as the one given by Definition 3.6. We conclude that the two approaches end up with the same pricing formula. The existence of an EMM is not needed to price derivatives by using the real world probability. Besides, defining an EMM by the numéraire portfolio which is nothing but the GOP, we also obtain the pricing formula which

uses the probability  $P$  by change of measure. From what we have obtained, we conclude that the real world pricing theory does not require the market to be complete. The next chapter consists of an example of an incomplete market where we price options under the real world probability.



# Chapter 4

## Example: A Quadrinomial Model

As the theory of pricing under the real world works both in complete and incomplete model, an example of incomplete model is given in this chapter. We consider a model composed of two stocks  $S^1$  and  $S^2$  observed at time  $t = 1, \dots, T$  with  $T$  the last period and a riskless asset with constant interest rate  $r$  at each period. We model each asset so that at each period they only move up or down with probability depending on whether the other asset goes up or down. This can be seen as a cartesian product of a two binomial models, one on the first stock and one on the second stock. We denote by  $u_i$  the rate of change on the  $i$ -th stock,  $i = 1, 2$ , when its price moves up and  $d_i$  when it moves down. The rates  $u_i$  and  $d_i$  are constants over the time.

### 4.1 The Model in Single Period.

The model is a two stocks single period quadrinomial model where

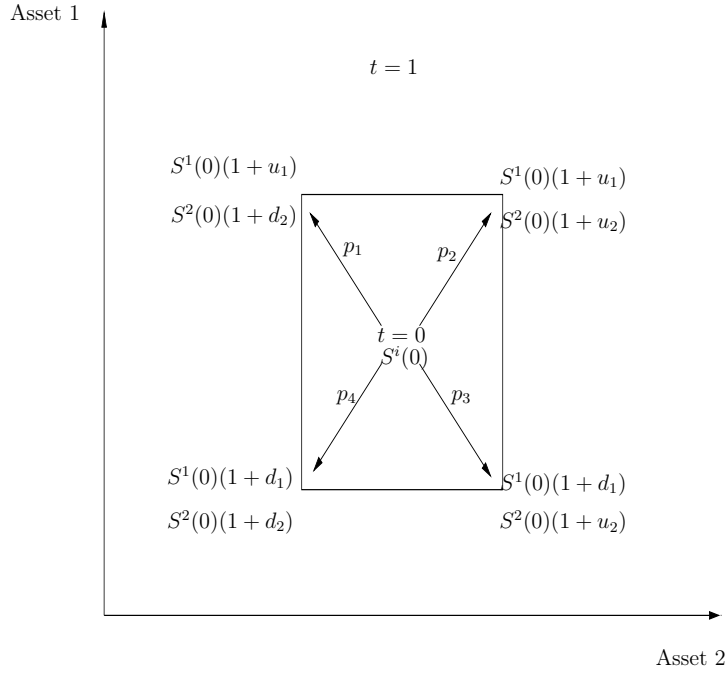
$$\Omega = \{\omega_1 = (u_1 d_2), \omega_2 = (u_1 u_2), \omega_3 = (d_1 u_2), \omega_4 = (d_1 d_2)\} \quad (4.1.1)$$

with respective probabilities

$$\begin{cases} p_1 = P(u_1 d_2) \\ p_2 = P(u_1 u_2) \\ p_3 = P(d_1 u_2) \\ p_4 = P(d_1 d_2) \end{cases} \quad (4.1.2)$$

**Proposition 4.1.** *The model is incomplete.*

*Proof.* The model is complete if and only if for any derivative  $D$ , we can find a strategy that replicates  $D$ . That is there exist  $x$ ,  $y$  and  $z$  such that for any



**Figure 4.1:** The model at time  $t = 1$ .

derivative  $D$  depending on  $\omega \in \Omega$  we have

$$xB_1 + yS^1(1)(\omega) + zS^2(1)(\omega) = D(\omega). \quad (4.1.3)$$

In our case we have the following equations

$$\begin{cases} xB_0(1+r) + yS^1(0)(1+u_1) + zS^2(0)(1+d_2) = D(u_1d_2) \\ xB_0(1+r) + yS^1(0)(1+u_1) + zS^2(0)(1+u_2) = D(u_1u_2) \\ xB_0(1+r) + yS^1(0)(1+d_1) + zS^2(0)(1+u_2) = D(d_1u_2) \\ xB_0(1+r) + yS^1(0)(1+d_1) + zS^2(0)(1+d_2) = D(d_1d_2) \end{cases} \quad (4.1.4)$$

From the first and the second equations, we have

$$z = \frac{D(u_1d_2) - D(u_1u_2)}{S^2(0)(d_2 - u_2)} \quad (4.1.5)$$

and from the third and the last equations we have also

$$z = \frac{D(d_1u_2) - D(d_1d_2)}{S^2(0)(u_2 - d_2)}. \quad (4.1.6)$$

Equations (4.1.5) and (4.1.6) gives

$$D(u_1d_2) - D(u_1u_2) = D(d_1d_2) - D(d_1u_2) \quad (4.1.7)$$

which is of course not always true for any derivative  $D$ .  $\square$

The Growth Optimal Portfolio is determined by the first order condition given in Lemma 3.12

$$E \left[ \frac{R^i(1)}{1 + yR^1(1) + zR^2(1)} \middle| \mathcal{A}_0 \right] \quad (4.1.8)$$

where  $y$  and  $z$  are the fractions of  $S^1$  and  $S^2$  invested in the GOP at time 0. The fraction  $x$  of the bond invested in the GOP is  $x = 1 - (y + z)$ . We solve then the system

$$\left\{ \begin{array}{l} p_1 \frac{R^1(1)(u_1)}{1 + yR^1(1)(u_1) + zR^2(1)(d_2)} + p_2 \frac{R^1(1)(u_1)}{1 + yR^1(1)(u_1) + zR^2(1)(u_2)} \\ + p_3 \frac{R^1(1)(d_1)}{1 + yR^1(1)(d_1) + zR^2(1)(u_2)} + p_2 \frac{R^1(1)(d_1)}{1 + yR^1(1)(d_1) + zR^2(1)(d_2)} = 0 \\ p_1 \frac{R^2(1)(d_2)}{1 + yR^1(1)(u_1) + zR^2(1)(d_2)} + p_2 \frac{R^2(1)(u_2)}{1 + yR^1(1)(u_1) + zR^2(1)(u_2)} \\ + p_3 \frac{R^2(1)(u_2)}{1 + yR^1(1)(d_1) + zR^2(1)(u_2)} + p_2 \frac{R^2(1)(d_2)}{1 + yR^1(1)(d_1) + zR^2(1)(d_2)} = 0. \end{array} \right. \quad (4.1.9)$$

## 4.2 Multiperiod Quadrinomial Model

From now on we use the notation  $u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2}$  to denote the event realized at time  $t$  where  $x_1 + y_1 = t$  and  $x_2 + y_2 = t$ . The stock prices corresponding to these events are

$$S^1(t) = S^1(0)(1 + u_1)^{x_1}(1 + d_1)^{y_1} \quad (4.2.1)$$

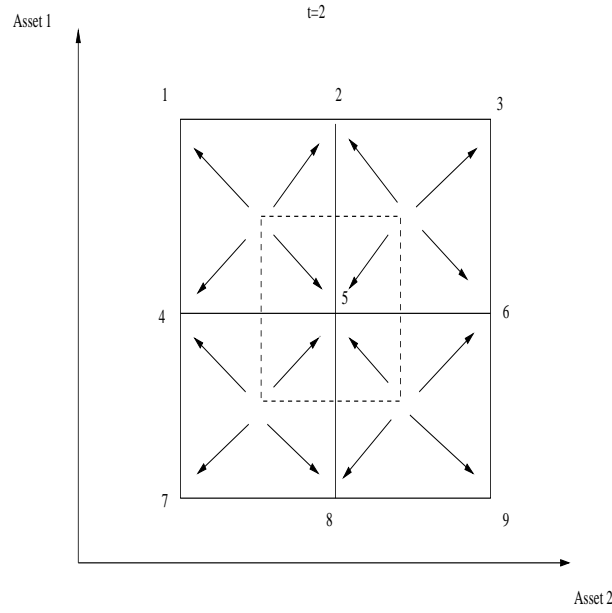
and

$$S^2(t) = S^1(0)(1 + u_2)^{x_2}(1 + d_2)^{y_2} \quad (4.2.2)$$

with probability  $P(u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2})$ . We use  $p_1, p_2, p_3$  and  $p_4$  same as in single period model to determine  $P(u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2})$ . At time terminal time  $T = 2$  for instance, the model is described by Figure 4.2:

We can see from Figure 4.2 that at time  $T = 2$ ,  $|\Omega| = 9$  i.e  $\Omega$  is composed by 9 elements which are:

- Point 1  
 $u_1^2 d_1^0 u_2^0 d_2^2$  with probability  $P(u_1^2 d_1^0 u_2^0 d_2^2) = p_1^2$ .



**Figure 4.2:** The model at time  $t = 2$

- Point 2  
 $u_1^2 d_1^0 u_2^1 d_2^1$  with probability  $P(u_1^2 d_1^0 u_2^1 d_2^1) = 2p_1 p_2$ .
- Point 3  
 $u_1^2 d_1^0 u_2^2 d_2^0$  with probability  $P(u_1^2 d_1^0 u_2^2 d_2^0) = p_2^2$ .
- Point 4  
 $u_1^1 d_1^1 u_2^0 d_2^2$  with probability  $P(u_1^1 d_1^1 u_2^0 d_2^2) = 2p_1 p_4$ .
- Point 5  
 $u_1^1 d_1^1 u_2^1 d_2^1$  with probability  $P(u_1^1 d_1^1 u_2^1 d_2^1) = 2(p_1 p_3 + p_2 p_4)$ .
- Point 6  
 $u_1^1 d_1^1 u_2^2 d_2^0$  with probability  $P(u_1^1 d_1^1 u_2^2 d_2^0) = 2p_2 p_3$ .
- Point 7  
 $u_1^0 d_1^2 u_2^0 d_2^2$  with probability  $P(u_1^0 d_1^2 u_2^0 d_2^2) = p_4^2$ .
- Point 8  
 $u_1^0 d_1^2 u_2^1 d_2^1$  with probability  $P(u_1^0 d_1^2 u_2^1 d_2^1) = 2p_3 p_4$ .
- Point 9  
 $u_1^0 d_1^2 u_2^2 d_2^0$  with probability  $P(u_1^0 d_1^2 u_2^2 d_2^0) = p_3^2$ .

**Proposition 4.2.** Let  $\Omega_t$  be the set of possible states of the world at time  $t$ ,  $t \in [0, T]$ . The cardinal of  $\Omega_t$  is  $|\Omega_t| = (t + 1)^2$

*Proof.* Here, the cardinal of  $\Omega_t$  is just the number of solutions of the system

$$\begin{cases} x_1 + y_1 = t \\ x_2 + y_2 = t. \end{cases}$$

The solutions of the first equation are  $(0, t), (1, t - 1), \dots, (t, 0)$  which are  $t + 1$ . The same as for the second equation. So, the number of solutions is  $(t + 1)^2$  for the two equations together.  $\square$

### Expectation under the quadrinomial model

For a random variable  $X$  we want to compute

$$E[X_T | \mathcal{A}_t] \quad (4.2.3)$$

for a  $\sigma$ -algebra  $\mathcal{A}_t$ ,  $t \in [0, T]$ . For the quadrinomial model, let

$$\begin{aligned} A_{x_1, y_1, x_2, y_2} = \{ \omega = \omega_1 \times \omega_2, \text{ with } \omega_1 = u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2}, \omega_2 = u_1^{m_1} d_1^{n_1} u_2^{m_2} d_2^{n_2} \\ \text{and } m_1 + n_1 = T - t = m_2 + n_2 \}. \end{aligned} \quad (4.2.4)$$

We have the following proposition

**Proposition 4.3.** *Let  $\mathcal{A}_t$  be the  $\sigma$ -algebra generated by the set*

$$g(\mathcal{A}_t) = \{ A_{x_1, y_1, x_2, y_2}, \text{ with } x_1 + y_1 = t = x_2 + y_2 \}.$$

$\mathcal{A}_t$  defines a filtration  $\{\mathcal{A}_t\}_{t \in [0, T]}$  where the  $A_{x_1, y_1, x_2, y_2}$  are given by Equation (4.2.4).

*Proof.* Let  $A_{x_1, y_1, x_2, y_2}$  be an element of  $g(\mathcal{A}_{t-1})$  and let  $\omega_{t-1} \in A_{x_1, y_1, x_2, y_2}$  then there exist  $A_{x'_1, y'_1, x'_2, y'_2}$  an element of  $g(\mathcal{A}_t)$  and a  $\omega_t \in A_{x'_1, y'_1, x'_2, y'_2}$  such that

$$\omega_t = \omega_{t-1} u_1^{n_1} d_1^{m_1} u_2^{n_2} d_2^{m_2}$$

with  $m_1 + n_1 = m_2 + n_2 = 1$ . Is clear that  $x_i \leq x'_i$  and  $y_i \leq y'_i$  for  $i = 1, 2$ . From that let us define a relation  $\mathcal{R}$  between the elements of  $g(\mathcal{A}_t)$  and those of  $g(\mathcal{A}_{t-1})$  by  $A_{x_1, y_1, x_2, y_2} \mathcal{R} A_{x'_1, y'_1, x'_2, y'_2}$  for any  $A_{x_1, y_1, x_2, y_2} \in g(\mathcal{A}_{t-1})$  and  $A_{x'_1, y'_1, x'_2, y'_2} \in g(\mathcal{A}_t)$  if and only if  $x_i \leq x'_i$  and  $y_i \leq y'_i$  for  $i = 1, 2$ .

- $A_{x_1, y_1, x_2, y_2} \mathcal{R} A_{x_1, y_1, x_2, y_2}$
- if  $x_i \leq x'_i$  and  $x'_i \leq x''_i$  then  $x_i \leq x''_i$  and the same for  $y_i, y'_i, y''_i$  hence the relation  $\mathcal{R}$  is transitive.
- if  $x_i \leq x'_i$  and  $x'_i \leq x_i$  then  $x_i = x'_i$  hence the relation  $\mathcal{R}$  is antisymmetric.

We have defined an order relation between the elements of  $g(\mathcal{A}_{t-1})$  and those of  $g(\mathcal{A}_t)$ . We can prove now that  $\mathcal{A}_{t-1} \subset \mathcal{A}_t$ :

Let  $A_{x_1, y_1, x_2, y_2} \in g(\mathcal{A}_{t-1})$ , there exists  $A_{x'_1, y'_1, x'_2, y'_2} \in g(\mathcal{A}_t)$  such that we have  $A_{x_1, y_1, x_2, y_2} \mathcal{R} A_{x'_1, y'_1, x'_2, y'_2}$  and

$$A_{x_1, y_1, x_2, y_2} = \bigcup \{A_{x'_1, y'_1, x'_2, y'_2} \in g(\mathcal{A}_t), A_{x_1, y_1, x_2, y_2} \mathcal{R} A_{x'_1, y'_1, x'_2, y'_2}\}$$

Hence,  $g(\mathcal{A}_{t-1}) \subset \mathcal{A}_t$  and  $\mathcal{A}_{t-1} \subset \mathcal{A}_t$  as  $\mathcal{A}_t$  is a  $\sigma$ -algebra.  $\square$

We can now use Equation (1.1.2) to compute the expectation (4.2.3). We have

$$E[X_T | \mathcal{A}_t](\omega) = \sum_{\substack{x_1+y_1=t \\ x_2+y_2=t}} \frac{E[X_T, A_{x_1, y_1, x_2, y_2}]}{P(A_{x_1, y_1, x_2, y_2})} \mathbf{1}_{A_{x_1, y_1, x_2, y_2}}(\omega).$$

Let

$$E(t, x_1, y_1, x_2, y_2) = \sum_{\substack{m_1+n_1=T-t \\ m_2+n_2=T-t}} P(u_1^{m_1} d_1^{n_1} u_2^{m_2} d_2^{n_2}) X_T(u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2} \times u_1^{m_1} d_1^{n_1} u_2^{m_2} d_2^{n_2}),$$

we have

$$\begin{aligned} E[X_T, A_{x_1, y_1, x_2, y_2}] &= E[X \mathbf{1}_{A_{x_1, y_1, x_2, y_2}}] \\ &= \sum_{\omega \in A_{x_1, y_1, x_2, y_2}} P(\omega) X_T(\omega) \\ &= P(u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2}) E(t, x_1, y_1, x_2, y_2) \end{aligned}$$

then

$$\frac{E[X_T, A_{x_1, y_1, x_2, y_2}]}{P(A_{x_1, y_1, x_2, y_2})} = E(t, x_1, y_1, x_2, y_2)$$

and finally

$$E[X_T | \mathcal{A}_t](\omega) = \sum_{\substack{x_1+y_1=t \\ x_2+y_2=t}} E(t, x_1, y_1, x_2, y_2) \mathbf{1}_{A_{x_1, y_1, x_2, y_2}}(\omega). \quad (4.2.5)$$

Writing  $E(t, x_1, y_1, x_2, y_2)$  in function of the probabilities  $p_1, p_2, p_3$  and  $p_4$ , we have

$$E(t, x_1, y_1, x_2, y_2) = \sum_{\substack{m_1+n_1=T-t=m_2+n_2 \\ \alpha_1+\alpha_2=m_1, \quad \alpha_3+\alpha_4=n_1 \\ \alpha_2+\alpha_3=m_2, \quad \alpha_1+\alpha_4=n_2}} p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4} X_T(u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2} \times u_1^{m_1} d_1^{n_1} u_2^{m_2} d_2^{n_2}). \quad (4.2.6)$$

### The GOP for the quadrinomial model.

For the fractions of the GOP in the model, we have the following proposition:

**Proposition 4.4.** *If  $(x_t, y_t, z_t)$  is the vector that gives the fraction of respectively  $S^0, S^1$  and  $S^2$  invested in the GOP at time  $t + 1$  then*

$$(x_t, y_t, z_t) = (x_{t-1}, y_{t-1}, z_{t-1}) = \cdots = (x_0, y_0, z_0).$$

*That is, the GOP is constant over time for the quadrinomial model.*

*Proof.* Recall that to have the value of the GOP at time  $t$ , we compute the fractions at time  $t - 1$ . Let us prove the proposition for  $T = 2$ , that is, we need to find the fractions of the 3 assets invested in the GOP at time  $t = 0$  and  $t = 1$ . For  $t = 0$ , the GOP is already calculated in single period and of course this does not change by increasing  $T$ . We need to solve

$$E \left[ \frac{R^i(2)}{1 + y_1 R^1(2) + z_1 R^2(2)} \middle| \mathcal{A}_1 \right] \quad (4.2.7)$$

with

$$g(\mathcal{A}_1) = \{A_{x_1, y_1, x_2, y_2} \quad \text{with} \quad x_1 + y_1 = x_2 + y_2 = 1\} \quad (4.2.8)$$

$$= \{A_{1,0,0,1}, A_{1,0,1,0}, A_{0,1,1,0}, A_{0,1,0,1}\}. \quad (4.2.9)$$

Let

$$V^i = \frac{R^i(2)}{1 + yR^1(2) + zR^2(2)}.$$

Using Formula (1.1.2), we have

$$E[V|\mathcal{A}_1] = \sum_{\substack{x_1+y_1=1 \\ x_2+y_2=1}} \frac{E[Z, A_{x_1, y_1, x_2, y_2}]}{P(A_{x_1, y_1, x_2, y_2})} \mathbf{1}_{A_{x_1, y_1, x_2, y_2}} \quad (4.2.10)$$

where

$$P(A_{x_1, x_2, x_3, x_4}) = P(u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2}) p_1 + P(u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2}) p_2 \quad (4.2.11)$$

$$+ P(u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2}) p_3 + P(u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2}) p_4 \quad (4.2.12)$$

$$= P(u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2}) (p_1 + p_2 + p_3 + p_4) \quad (4.2.13)$$

$$= P(u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2}). \quad (4.2.14)$$

for  $x_1 + y_1 = 1 = x_2 + y_2$ . Let

$$E(x_1, y_1, x_2, y_2) = \frac{E[V, A_{x_1, y_1, x_2, y_2}]}{P(A_{x_1, y_1, x_2, y_2})}. \quad (4.2.15)$$

For each  $\omega \in \Omega$ , we have

$$\begin{aligned} E[V | \mathcal{A}_1](\omega) &= E(1, 0, 0, 1) \mathbf{1}_{A_{1,0,0,1}}(\omega) + E(1, 0, 1, 0) \mathbf{1}_{A_{1,0,1,0}}(\omega) \\ &\quad + E(0, 1, 1, 0) \mathbf{1}_{A_{0,1,1,0}}(\omega) + E(0, 1, 0, 1) \mathbf{1}_{A_{0,1,0,1}}(\omega). \end{aligned}$$

The relative risk  $R$  computed at time 2 depends only on the event that happened from time 1 to time 2. These events are always those that occurred at time 1 in single period which are

$u_1 d_2$  with probability  $p_1$ ,

$u_1 u_2$  with probability  $p_2$ ,

$d_1 u_2$  with probability  $p_3$ ,

$d_1 d_2$  with probability  $p_4$ .

Hence by (1.1.1), Equation (4.2.11) and (4.2.15), we have to solve for each  $\omega$  the same following equations for  $y$  and  $z$  in  $V^i$

$$\begin{cases} p_1 V^1(u_1 d_2) + p_2 V^1(u_1 u_2) + p_3 V^1(d_1 u_2) + p_4 V^1(d_1 d_2) = 0 \\ p_1 V^2(u_1 d_2) + p_2 V^2(u_1 u_2) + p_3 V^2(d_1 u_2) + p_4 V^2(d_1 d_2) = 0 \end{cases} \quad (4.2.16)$$

which are exactly the same equation that has been solved in single period while finding the fractions of the assets invested in the GOP at time 0, see



(4.1.8). For  $T \geq 2$ , we must have the same equation to solve for each  $t \in [0, T]$  since at each time  $t$ , the relative risk  $R$  always takes value  $R(u_1d_2)$ ,  $R(u_1u_2)$ ,  $R(d_1u_2)$  or  $R(d_1d_2)$  with each respective probability which means that  $R$  does not depend on the time  $t$  but only on the event that happens between  $t - 1$  and  $t$ .  $\square$

Since now we know the value of the GOP for the model, we are able to price any derivative security using the real world pricing formula.

### Pricing under the quadrinomial model.

Since in our model, the relative risk process is not a function of the time  $t$  but depends only on whether the asset prices go up or down, we denote it by  $R(w_i)$ ,  $i = 1, 2$  and  $w_i \in \{u_i, d_i\}$ . Let

$$f(y, z, w_1, w_2) = 1 + yR(w_1) + zR(w_2), \quad (4.2.17)$$

then we have the following proposition

**Proposition 4.5.** *The price  $Pr(X_t)$  of an option  $X$  at time  $t$  maturing at  $T$  is*

$$Pr(X_t)(\omega) = \sum_{\substack{x_1+y_1=t \\ x_2+y_2=t}} E(t, x_1, y_1, x_2, y_2) \mathbf{1}_{A_{x_1, y_1, x_2, y_2}} \quad (4.2.18)$$

where

$$E(t, x_1, x_2, x_3, x_4) = \frac{1}{(1+r)^{T-t}} \sum_{\substack{m_1+n_1=T-t=m_2+n_2 \\ \alpha_1+\alpha_2=m_1, \quad \alpha_3+\alpha_4=n_1 \\ \alpha_2+\alpha_3=m_2, \quad \alpha_1+\alpha_4=n_2}} p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4} \\ \times \frac{X_T(u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2}) \times u_1^{m_1} d_1^{n_1} u_2^{m_2} d_2^{n_2}}{f^{\alpha_1}(y, z, u_1, d_2) f^{\alpha_2}(y, z, u_1, u_2) f^{\alpha_3}(y, z, d_1, u_2) f^{\alpha_4}(y, z, d_1, d_2)}. \quad (4.2.19)$$

*Proof.* The pricing formula is given by Equation (3.2.29). Applying the formula given by Equation (4.2.5) and Equation (4.2.6) and using (4.2.17), we

have:

$$\begin{aligned}
Pr(X_t) &= \frac{1}{(1+r)^{T-t}} E \left[ \frac{X_T}{\prod_{t+1}^T 1 + \langle \pi_{\delta_*}(m-1), R(m) \rangle} \middle| \mathcal{A}_t \right] \\
&= \frac{1}{(1+r)^{T-t}} \sum_{\substack{m_1+n_1=T-t=m_2+n_2 \\ \alpha_1+\alpha_2=m_1, \quad \alpha_3+\alpha_4=n_1 \\ \alpha_2+\alpha_3=m_2, \quad \alpha_1+\alpha_4=n_2}} p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4} \\
&\quad \times \frac{X_T(u_1^{x_1} d_1^{y_1} u_2^{x_2} d_2^{y_2} \times u_1^{m_1} d_1^{n_1} u_2^{m_2} d_2^{n_2})}{f^{\alpha_1}(y, z, u_1, d_2) f^{\alpha_2}(y, z, u_1, u_2) f^{\alpha_3}(y, z, d_1, u_2) f^{\alpha_4}(y, z, d_1, d_2)}.
\end{aligned}$$

□

### Pricing a European Call Option on one of the assets.

For a numerical example, we price an European call option on one of the two assets under the quadrinomial model. From Proposition 4.5 the value of the European call option on the asset  $S^1$  at time  $t$  maturing at time  $T$  with strike price  $K$  is given by Equation (4.2.18) where

$$\begin{aligned}
E(t, x_1, x_2, x_3, x_4) &= \sum_{\substack{m_1+n_1=T-t=m_2+n_2 \\ \alpha_1+\alpha_2=m_1, \quad \alpha_3+\alpha_4=n_1 \\ \alpha_2+\alpha_3=m_2, \quad \alpha_1+\alpha_4=n_2}} p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4} \\
&\quad \times \frac{\max\{S^1(0)(1+u_1)^{x_1+m_1}(1+d_1)^{y_1+n_1} - K\}}{f^{\alpha_1}(y, z, u_1, d_2) f^{\alpha_2}(y, z, u_1, u_2) f^{\alpha_3}(y, z, d_1, u_2) f^{\alpha_4}(y, z, d_1, d_2)}.
\end{aligned} \tag{4.2.20}$$

We compare the results from the above formula (4.2.20) with the results given by the usual risk neutral pricing formula which

$$Pr(X_t) = B_t E_Q \left[ \frac{X_T}{B_T} \middle| \mathcal{A}_t \right] \tag{4.2.21}$$

for a price of an option  $X$  at time  $t$  where  $Q$  is the risk neutral measure. To be able to apply this risk neutral formula, we use the well known CRR model (see Cox *et al.* (1979)) given by Figure 4.3 For more details about the CRR-model, we also refer to Capinski and Zastawniak (2003). Let us assign values to all the parameters of the model. The initial value of the first asset is  $S^1(0) = 100$ , the strike price is  $K = 110$ . For the rates, we use  $u_1, d_1 = 2/10, -1/10$  for  $S^1$ ,  $u_2, d_2 = 3/10, -5/100$  for  $S^2$  and  $r = 1/10$  is the risk free rate. For the real world probability, we use  $\{p_1, p_2, p_3, p_4\} = \{2/15, 4/15, 4/10, 1/5\}$ . For  $T = 2$  the computation of the prices at time  $t = 0$  and  $t = 1$  gives the following results:

- At  $t = 0$  we obtain the same prices  $Pr(X_0) = 12.48$  from the CRR-model and the quadrinomial model.

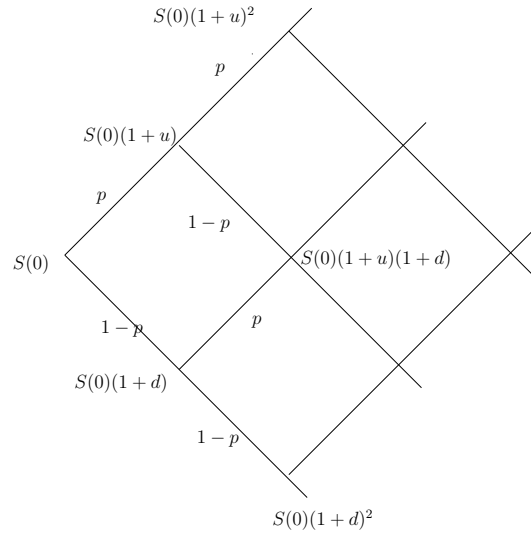


Figure 4.3: CRR model

- For the CRR-model, at  $t = 1$ , we have

$$E_Q \left[ \frac{X_T}{1+r} \middle| \mathcal{A}_1 \right] (\omega) = \begin{cases} 20.60 & \text{if } \omega(t=1) = u_1 \\ 0 & \text{if } \omega(t=1) = d_1 \end{cases} \quad (4.2.22)$$

and at  $t = 2$

$$E_Q \left[ \frac{X_T}{1+r} \middle| \mathcal{A}_2 \right] (\omega) = \begin{cases} 34.0 & \text{if } \omega(t=2) = u_1 \times u_1 \\ 0 & \text{if } \omega(t=2) = d_1 \times u_1 = u_1 \times d_1 \\ 0 & \text{if } \omega(t=2) = d_1 \times d_1 \end{cases} \quad (4.2.23)$$

- The value of the option at time  $t = 1$  obtained by using the quadrinomial model is

$$Pr(X_1)(\omega) = \begin{cases} 20.60 & \text{if } \omega(t=1) = u_1 d_2 \\ 20.60 & \text{if } \omega(t=1) = u_1 u_2 \\ 0 & \text{if } \omega(t=1) = d_1 u_2 \\ 0 & \text{if } \omega(t=1) = d_1 d_2 \end{cases} \quad (4.2.24)$$

and at time  $t = 2$

$$Pr(X_2)(\omega) = \begin{cases} 34.0 & \text{if } \omega(t=2) = u_1d_2 \times u_1d_2 \\ 34.0 & \text{if } \omega(t=2) = u_1d_2 \times u_1u_2 \\ 0 & \text{if } \omega(t=2) = u_1d_2 \times d_1u_2 \\ 0 & \text{if } \omega(t=2) = u_1d_2 \times d_1d_2 \\ 34.0 & \text{if } \omega(t=2) = u_1u_2 \times u_1u_2 \\ 0 & \text{if } \omega(t=2) = u_1u_2 \times d_1u_2 \\ 0 & \text{if } \omega(t=2) = d_1u_2 \times d_1u_2 \\ 0 & \text{if } \omega(t=2) = d_1u_2 \times d_1d_2 \\ 0 & \text{if } \omega(t=2) = d_1d_2 \times d_1d_2 \end{cases} \quad (4.2.25)$$

with  $u_1u_2 \times d_1d_2 = u_1d_2 \times d_1u_2$ .

This is the same as given by the result from the CRR-model when we do not look at the second component of  $\omega(t = 1)$ , that is  $d_2$  and  $u_2$ . From these we can conclude that using the real world pricing formula, the quadrinomial model we have used allows us to price an option using only one of the stock and get the same fair price as given by the complete CRR-model. And since the model gives a unique fair price using the approach of the real world for just one stock, we will also price an option using both the two stocks which we give in the next subsection and see the accuracy of the results. In order to compare the prices with those that will be given in the next subsection which uses simultaneously  $S^1$  and  $S^2$ , we price also an European call option on the stock  $S^2$  with the same strike price  $K$ . Here are the results we obtained:

- The value of the option at time  $t = 1$  given by the quadrinomial model is

$$Pr(X_1)(\omega) = \begin{cases} 5.25 & \text{if } \omega(t=1) = u_1d_2 \\ 29.99 & \text{if } \omega(t=1) = u_1u_2 \\ 29.99 & \text{if } \omega(t=1) = d_1u_2 \\ 5.25 & \text{if } \omega(t=1) = d_1d_2 \end{cases} \quad (4.2.26)$$

and at time  $t = 2$

$$\begin{cases} 0.00 & \text{if } \omega(t=2) = u_1d_2 \times u_1, d_2 \\ 13.50 & \text{if } \omega(t=2) = u_1d_2 \times u_1u_2 \\ 13.50 & \text{if } \omega(t=2) = u_1d_2 \times d_1u_2 \\ 0.00 & \text{if } \omega(t=2) = u_1d_2 \times d_1d_2 \\ 59.00 & \text{if } \omega(t=2) = u_1u_2 \times u_1u_2 \\ 59.00 & \text{if } \omega(t=2) = u_1u_2 \times d_1u_2 \\ 59.00 & \text{if } \omega(t=2) = d_1u_2 \times d_1u_2 \\ 13.50 & \text{if } \omega(t=2) = d_1u_2 \times d_1d_2 \\ 0.00 & \text{if } \omega(t=2) = d_1d_2 \times d_1d_2 \end{cases} \quad (4.2.27)$$

It has been verified that the prices we have obtained from the quadrinomial model coincide with the prices from the CRR-model of the European call option on one of the asset. The next subsection is to price an option using both the 2 stocks, we call such an option a two-stock option.

### Pricing a two-stock option.

Let us consider an option maturing at time  $T$  using both Stock  $S^1$  and Stock  $S^2$  with payoff  $\max\{\max(S^1(T), S^2(T)) - K, 0\}$  (see Hauge (1997)) where  $K$  is again the same strike price that we have used for both the stocks. Below are the results when  $T = 2$ .

- At time  $t = 1$

$$Pr(X_1)(\omega) = \begin{cases} 22.89 & \text{if } \omega(t = 1) = u_1d_2 \\ 37.90 & \text{if } \omega(t = 1) = u_1u_2 \\ 29.99 & \text{if } \omega(t = 1) = d_1u_2 \\ 5.25 & \text{if } \omega(t = 1) = d_1d_2. \end{cases} \quad (4.2.28)$$

- At time  $t = 2$

$$Pr(X_2)(\omega) = \begin{cases} 34.00 & \text{if } \omega(t = 2) = u_1d_2 \times u_1d_2 \\ 34.00 & \text{if } \omega(t = 2) = u_1d_2 \times u_1u_2 \\ 13.50 & \text{if } \omega(t = 2) = u_1d_2 \times d_1u_2 \\ 0.00 & \text{if } \omega(t = 2) = u_1d_2 \times d_1d_2 \\ 59.00 & \text{if } \omega(t = 2) = u_1u_2 \times u_1u_2 \\ 59.00 & \text{if } \omega(t = 2) = u_1u_2 \times d_1u_2 \\ 59.00 & \text{if } \omega(t = 2) = d_1u_2 \times d_1u_2 \\ 13.50 & \text{if } \omega(t = 2) = d_1u_2 \times d_1d_2 \\ 0.00 & \text{if } \omega(t = 2) = d_1d_2 \times d_1d_2. \end{cases} \quad (4.2.29)$$

The reason why we have chosen this option is that it is in fact just a call on the maximum of the two assets which allow to verify easily if the results are correct by comparing it to the results we obtained from previous examples. This is better than the call on just one of the asset in the seller view point since the prices are much more higher. One can also price a call on the minimum of the two assets or a put on the maximum as well as on the minimum of the assets. These kind of options can be seen in Hauge (1997) which were originally priced by Stulz (1982). Some other two stock options could of course also be considered by using this quadrinomial model.

### Numerical results for a higher period

To close the chapter, we give some more numerical results for a higher period. We price two two-stock European options maturing in one month (31 days) which are a call option on the maximum of the two assets and a put option on the minimum of the two assets with strike price  $K = 123$ . The payoff of the put on the minimum of the assets is  $\max\{K - \min(S^1, S^2), 0\}$ . We calculate the stock prices using the quadrinomial model which has here 30 steps. We set then  $T = 30$  and each time  $t \in \{0, \dots, T\}$  corresponds to the  $(t + 1) - th$  day of the contract. The daily rates of change in the stock prices are:  $u_1 = 2/100$ ,  $u_2 = 3/100$ ,  $d_1 = -1/100$ ,  $d_2 = -5/1000$  and  $r = 1/100$  is the daily interest rate of the saving account. We keep the initial values we have used in the previous examples as well as the probabilities. The prices computed at initial time is given by Table 4.1 in which we also include the prices of European put and call options maturing in one month with the same strike prices as the two-stock options. The call on the maximum of the two assets is here cheaper

Call on $S^1$	18.40	Put on $S^1$	0.44
Call on $S^2$	18.42	Put on $S^2$	0.77
European Call on $\max(S^1, S^2)$	14.03	European put on $\min(S^1, S^2)$	1.16

**Table 4.1:** Prices computed at  $t = 0$

than the calls on each one of the two assets and the put on the minimum is slightly expensive than the put on only one for the assets.

To see the prices of the two-stock options computed at each day and to avoid writing all the possible events we compute the mean of the prices  $Pr(X_t)$  which is the quantity:

$$E[Pr(X_t)] \tag{4.2.30}$$

for  $t \in \{0, \dots, 30\}$ . The results of these computations are given by Table 4.2. Table 4.2 is just to demonstrate that we can handle option pricing at any time in discrete setting by using the quadrinomial model in multisteps and the real world pricing formula.

Day	Put on the $\min(S^1, S^2)$	Call on the $\max(S^1, S^2)$
0	1.16	14.03
1	1.17	14.20
2	1.19	14.47
3	1.24	14.83
4	1.32	15.28
5	1.45	15.82
6	1.61	16.45
7	1.81	17.17
8	2.05	17.98
9	2.33	18.86
10	2.65	19.81
11	3.02	20.84
12	3.43	21.93
14	3.88	23.08
15	4.38	24.28

**Table 4.2:** Mean prices at time  $t = 0, 1, \dots, 15$

# Conclusion

The Growth Optimal Portfolio which is known as the portfolio that maximizes growth rate of wealth over some time horizon  $T$  allows us to price derivative securities in incomplete models by discounting prices by it. We have shown that one approach developed in continuous case demonstrated that a risk neutral measure is not needed to price derivatives. This later comes from the fact that benchmarked prices are supermartingales under the real world probability measure. The second approach is in discrete time in which we saw the numéraire property of the GOP which leads to the risk neutral measure defined by the GOP. Both approaches gave us at the end the real world pricing formula.

The quadrinomial model with 2 stocks used as example is a cartesian product of two binomial models and is an example of incomplete market in discrete time financial market. With this quadrinomial model, we are able to price options with two stocks. In further studies we can improve or enlarge the work by finding an appropriate approximation of the quadrinomial model to continuous time.



# List of References

- Breiman, L. (1961). Optimal gambling systems for favorable games. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, vol. I, pp. 65–78.
- Capínski, M. and Zastawniak, T. (2003). *Mathematics for Finance: An Introduction to Financial Engineering*. Springer Verlag.
- Cox, J., Ross, S. and Rubinstein, M. (1979). Option pricing: A simplified approach. *J. Financial Economics*, , no. 7, pp. 229–263.
- Fima C, K. (1998). *Introduction to Stochastic Calculus*. Imperial College Press.
- Fishman, G. (1996). *Monte Carlo: Concepts, Algorithms and Applications*. Springer.
- Gihman, I. and Skorohod, A. (1972). *Stochastic Differential Equations*. Springer.
- Harrison, M. and Kreps, D. (1979). Martingales and arbitrage in multiperiod security markets. *J. Economic Theory*, , no. 20, pp. 381–408.
- Hauge, E.G. (1997). *The Complete Guide to Option Pricing Formulas*. McGraw-Hill Companies, United States of America.
- Karatzas, I. and Shreve, A. (1998). *Brownian Motion and Stochastic Calculus*. Springer, New York.
- Kelly, J. (1956). A New Interpretation of Information Rate. *Bell System Techn.Journal*, , no. 35, pp. 917–926.

- Kopp, E. and Elliott, R. (1998). *Mathematics of Financial Markets*. Springer Finance.
- Korn, R. and Schäl, M. (1999). On value preserving and growth optimal portfolios. *Mathematical Methods of Operations research*, , no. 50, pp. 189–218.
- Latane, H. (1959). Criteria for choice among risky ventures. *J. Political Economy*, , no. 38, pp. 145–155.
- Loeve, M. (1978). *Probability theory II*. Springer-Verlag.
- Øksendal, B. (2003). *Stochastic Differential Equations: An Introduction with Application*. Springer Finance.
- Platen, E. (2005). Diversified portfolios with jumps in a benchmark framework. *Asia-Pacific Financial Markets*, , no. 11(1), pp. 1–22.
- Platen, E. (2006). A Benchmark Approach to Finance. *Mathematical Finance*, , no. 16, pp. 131–151.
- Platen, E. and Bühlmann, H. (2003). A Discrete Time Benchmark Approach for Finance and Insurance. *Astin Bulletin*, , no. 2(33).
- Platen, E. and Heath, D. (2007). *A Benchmark Approach to Quantitative Finance*. Springer Finance.
- Stulz, R, M. (1982). Options on the minimum or the maximum of two risky assets. *Journal of Financial Economics*, , no. 10, pp. 161–185.