A NON-COMMUTATIVE WALECKA MODEL AS AN EFFECTIVE THEORY FOR INTERACTING NUCLEONS OF FINITE SIZE

by

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ABSTRACT

The finite size of nucleons should play an important role in the description of high density nuclear matter as found in astro-physical objects. Yet we see that the Walecka model, which is generally used to describe these systems, treats the nucleons as point particles. Here we argue that a non-commutative version of the Walecka model may be a consistent and appropriate framework to describe finite nucleon size effects. In this framework the length scale introduced through the non-commutative parameter plays the role of the finite nucleon size. To investigate the consequences of this description, the equations of motion and energy-momentum tensor for the non-commutative Walecka model are derived. We also derived an expression for the total energy of the system, as a function of the non-commutative parameter, in a spatially non-uniform matter approximation. The non-commutative parameter, as a variable dependent on the dynamics of the system, remains to be solved self-consistently.

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OPSOMMING

Die eindige grootte van nukleone moet 'n belangrike rol speel in die beskrywing van hoë-digtheid kern materie soos gevind in astro-fisiese voorwerpe. Tog sien ons dat die Walecka model, wat in die algemeen gebruik word om hierdie stelsels te beskryf, die nukleone as punt deeltjies hanteer. Ons redeneer dus dat 'n nie-kommutatiewe weergawe van die Walecka model 'n konsistente en gepaste raamwerk is om die effekte van eindige nukleon grootte te beskryf. In hierdie raamwerk speel die lengte-skaal wat ingevoer word deur die nie-kommutatiewe parameter die rol van eindige grootte vir nukleone. Om die gevolge van hierdie beskrywing te ondersoek, word die vergelykings van beweging en die energie-momentum tensor afgelei vir die nie-kommutatiewe Walecka model. Ons het ook 'n uitdrukking vir die totale energie van die stelsel, as 'n funksie van die nie-kommutatiewe parameter, afgelei in 'n ruimtelik nie-uniforme materie benadering. Die nie-kommutatiewe parameter, as 'n veranderlike afhanklik van die dinamika van die stelsel, bly steeds om self-konsistent opgelos te word.

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This thesis is dedicated to my Lord Christ, the Creator of this wondrous universe I wish to understand and from Whom I receive my strength to do so.

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CHAPTER 1

Introduction

As a physicist one is blessed, or often cursed, with a very inquisitive nature to understand the world around them and comes up with various explanations of how nature orchestrates its existence. Some of these explanations have become known as laws of nature, while others still remain simply as theories which are only able to describe a subset of physical situations or are only able to give approximations thereof. The theories of nuclear interactions are no exception.

As one of my prime interests in physics, nuclear interaction theories always seemed to be divided into two: The ones describing interactions at low energies or densities with no particle structure, and the theories describing interactions at high energies or densities with intrinsic structure. As soon as one of these subsets of theories try to find a crossover to the other, many complications often arise which deems a theory unsolvable, or at least unsolvable with current technology.

As physicists are also ever searching, small improvements in technology and techniques within physics are made. This thesis will also be such an attempt into improving a technique of describing nuclear interactions dependent on the finite nucleon size. We outline the process of this research first by giving the aims of the thesis we want to reach.

1.1 Aims of this Thesis

It is common knowledge within modern physics that particles on the nuclear scale have finite size and plays a significant role in nuclear interactions. These particles on the nuclear scale, specifically baryons (or nucleons) and mesons, have been used in interaction theories, such as the Walecka model [1]. However, the Walecka model did not contain any information of the finite size of particles, but the particles are all described as point-like particles. This approach of point-like particles gives good quantitative results at low nuclear densities where nucleon size plays a minimal role, but this may not be the case at higher nuclear densities, such as astro-physical objects like Neutron Stars, where nucleon size may be an important factor.

We may turn to QCD which is a theory that already describes baryon interactions with finite baryon size and internal structure. However, QCD is a complicated and very involved theory

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which becomes particularly difficult to solve at low energies. We therefore seek to introduce some kind of length scale or size to baryons using the much simpler Walecka model as a basis.

It is well-known that the introduction of finite sized objects in quantum field theory, or for that matter in any relativistic theory is plagued by inconsistencies. For example, if one would consider baryons as rigid bodies, it would be clear that an interaction on nuclear scale containing rigid bodies would violate causality since all points on the rigid body, which are spatially removed, would move at the same time during an interaction. Of course, as already mentioned, QCD provides us with such a consistent framework in which the baryons supposedly appear as the bound states of three point-like quarks. However, the low energy description of QCD is still an open issue, which we want to avoid. Instead we are looking for a simple, effective description in which we can introduce a length scale that can be associated with the finite size of the nucleons. A possible scenario in which this can be done is non-commutative quantum field theory. In the recent past these theories have been investigated quite thoroughly in such work as [2, 3] and their references within, and it emerged that they are consistent non-local theories involving a fundamental length scale, which we shall take here to be the nucleon size.

The rationale behind this identification is as follows: Non-commutative quantum field theory assumes that different position coordinates of a particle do not commute and by doing so introduces a length scale, which measures the extent of non-commutativity. When you have position coordinates that do not commute, you have an uncertainty area in space for the position of a particle. When introducing these commutation relations to particle interaction theory, this uncertainty area of the position acts as if the particles have a finite size and manifests itself through the non-locality of the interactions. It therefore seems quite plausible that effects associated with the finite size of nucleons may be captured in a non-commutative quantum field theory. Knowing that the Walecka model already provides us with a successful description of nuclear matter at low densities, where nucleon size is irrelevant, the natural step to take is then to introduce a non-commutative version of the Walecka model, which can be done in a systematic way as described in Chapter 5 of this thesis.

In the most naïve setting of the non-commutative Walecka model one can identify the non-commutative length scale with the size of the nucleon and simply investigate the consequences thereof on, e.g., the equation of state. However, there is a more sophisticated point of view one can take. We know from QCD that the size of the nucleons is dynamic and will change with baryon density. In a non-commutative setting this length scale should therefore also be

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determined from dynamical considerations. The most obvious way to do this is to minimize the total energy of the system, at fixed baryon density, with respect to the non-commutative length scale. This should yield the non-commutative length scale, and thus the effective nucleon size, as a function of baryon density, which is what interests us. Of course, this computation may be and, indeed, is still very complicated and can only be performed numerically. However, provided that we can derive the necessary equation for the total energy of the non-commutative Walecka model, we have a clear-cut conceptual framework in which such a computation can be carried out.

The aims of this thesis can therefore be summarised as follows:

- Generalise the Walecka model to a non-commutative setting, thereby consistently introducing a length scale that will be identified with the nucleon size.
- Compute the energy momentum tensor of this non-commutative theory.
- Identify the total energy as a function of the non-commutative scale.
- Minimise this energy at fixed baryon density with respect to the non-commutative length scale and find the effective nucleon size as a function of baryon density.

The first three goals have been carried out successfully in this thesis and are reported on in Chapter 5. The last goal turns out to be a rather involved numerical computation which extends beyond the scope of this thesis and will not be reported on here. However, the conceptual and technical frameworks to carry out this computation has been worked out completely.

The aims mentioned above can, of course, not be reached without a thorough study of quantum field theories and their non-commutative counterparts. A large part of this thesis, Chapters 2-4, therefore focuses on a review of these theories in order to provide the necessary background for computations that follow in Chapter 5.

1.2 Significance of this Research

The recent and rather exhaustive study of non-commutative quantum field theories [2, 3] has mainly focused on the high energy aspect, i.e., whether non-commutative quantum field theories can shed light on quantum gravity. With the exception of work done on quantum Hall systems [4] and topological insulators [5], much less has been done on their applications in ordinary low energy physical systems. Therefore, the application of non-commutative quantum field theory

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in this thesis, which is on the nuclear interaction model of Walecka, in fact represents a novel application. If this tool of a non-commutative nuclear interaction theory proves to be successful, it could open doors to new research in the fundamental physics of particle size and interactions.

1.3 Notation and Conventions

Throughout this thesis there are certain mathematical notation and naming conventions that are used. To set the standard jargon which is used, a proper layout of the notation and conventions will be given. First and foremost, we will always assume natural units, i.e. $\hbar = c = 1$, wherever it is applicable.

1.3.1 Vectors, Inner Products and Differentiation

Whether it is to give position coordinates or the momentum in each axis of particles, we will constantly use single symbols to indicate a compact, summarised version of these degrees of freedom, which we will call vectors. However, we will have to distinguish between vectors in 3-dimensional space and 4-dimensional time-space coordinates, or also known as the Minkowski-space.

Vectors in 3-dimensional space gives us the intuitive picture of a particle's spatial position or the momentum of the particle in the three main Cartesian axes. The 3-dimensional position vectors, denoted by a bold symbol $\boldsymbol{x}$, $\boldsymbol{y}$ or $\boldsymbol{z}$, takes the form of

$$\boldsymbol{x} \equiv (x^1, x^2, x^3), \tag{1.1}$$

for Cartesian coordinates, and for polar coordinates we use

$$\boldsymbol{r} \equiv (r, \theta, \phi). \tag{1.2}$$

In the case of Cartesian coordinates, we will refrain from using the notation of $\boldsymbol{x} \equiv (x, y, z)$, since we will use the labels of x, y and z in vector notation to indicate position vectors of different particles and not the individual coordinates of a single particle. The same does not hold for polar coordinates, since we will rarely use polar coordinates for more than one particle. If there is need to distinguish between the coordinates of more than one particle, we may introduce "prime" notation to do this indication, i.e. $\boldsymbol{r}$ and $\boldsymbol{r}'$.

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For the momentum vectors we have that each component gives the particular momentum along each of the Cartesian axes. We will often denote the momentum vectors as a bold symbol $\boldsymbol{p}$, $\boldsymbol{q}$ or also $\boldsymbol{k}$. The form of the 3-dimensional momentum vectors is given by

$$\boldsymbol{p} \equiv (p^1, p^2, p^3). \tag{1.3}$$

If we now consider vectors in the 4-dimensional Minkowski space, we introduce one additional component to the position and momentum vectors according to Einstein's theory for special relativity. For the position vectors, we add the coordinate of the particle in time. The resulting position vector in 4-dimensional Minkowski-space, labelled by x, y or z, takes the form of

$$x \equiv (x^0, x^1, x^2, x^3) = (t, \boldsymbol{x}). \tag{1.4}$$

In the case of the momentum 4-vector, we will add the particle energy as the zeroth component, also according to special relativity. The momentum vector in 4-dimensional Minkowski-space, denoted by p, q or k, then assumes the form

$$p \equiv (p^0, p^1, p^2, p^3) = (E, \boldsymbol{p}). \tag{1.5}$$

To accompany the notation of our vectors, we will also now introduce the inner product of vectors. The short-hand to indicate an inner product between two vectors is denoted by $a \cdot b$, independent of a and b being bold symbols or not. However, the definition thereof is different in 3-dimensional and 4-dimensional vector notation. For 3-vectors, we have that the inner product of two vectors is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{3} a^i b^i = a^i b^i, \tag{1.6}$$

where the last expression implies summation over the same indices. Here we explicitly use Latin indices to indicate that we are using components in 3-dimensional space. The usage of Latin summation indices for 3-vectors may extend to i, j, k, l, m and n and assume the values of 1 to 3. Also, for the 3-dimensional case there is no differentiation between the indices of the components as a superscript or subscript.

In the case of 4-dimensional Minkowski-space, we first have to introduce the metric tensor for

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special relativity. This tensor, denoted by $g^{\mu\nu}$, is defined in this thesis as

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.7}$$

The metric tensor allows us to write our 4-vectors in Einstein's covariant and contravariant notation. Given that the 4-vectors in contravariant notation, with the indices in superscript, is denoted by

$$a^\mu = (a^0, a^1, a^2, a^3), \tag{1.8}$$

we define the covariant notation of a 4-vector as

$$a_\mu \equiv \sum_{\nu=0}^{3} g_{\mu\nu} a^\nu = g_{\mu\nu} a^\nu = (a^0, -a^1, -a^2, -a^3), \tag{1.9}$$

where we also assume summation over similar superscript and subscript Greek indices. We will use Greek indices explicitly for vectors in the 4-dimensional Minkowski-space, where the labels may extend to μ, ν, ρ and σ and assume the values of 0 to 3. It is important to note that summation is only implied over a superscript and a subscript Greek index that is alike. Two vector components that both have subscript or superscript Greek indices, such as $a_\mu b_\mu$ or $a^\mu b^\mu$, do not imply summation over the indices. Therefore, the inner product between two 4-vectors is given by

$$a \cdot b \equiv a^\mu b_\mu = a_\mu b^\mu = \sum_{\mu,\nu=0}^{3} g^{\mu\nu} a_\mu b_\nu = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} a^\mu b^\nu. \tag{1.10}$$

The short-hand notation for differentiation, ∂_μ, will also be used throughout this thesis. This short-hand notation, in 4-dimensional Minkowski-space, is defined as

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \quad \text{and} \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu}, \tag{1.11}$$

where $\mu = 0, 1, 2, 3$. For 3-dimensional space, the short-hand notation will again make use of Latin indices and assumes the ordinary spatial differentiation operators.

It should be noted that it is also common practice to indicate 3-dimensional vectors using "arrow" notation, i.e. $\vec{a}$. However, we will not use this notation for vectors in this thesis. We will reserve

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the use of arrow notation to indicate the direction of differentiation. For example, we may indicate differentiation to the left as $\overleftarrow{\partial}_\mu$. This mathematically implies that $A\overleftarrow{\partial}_\mu B \equiv (\partial_\mu A)B$, while differentiation to the right, $\overrightarrow{\partial}_\mu$, will be implied to be $A\overrightarrow{\partial}_\mu B \equiv A(\partial_\mu B)$.

1.3.2 The trace of operators

We will often refer to the trace of operators during this thesis and it is important to give the form thereof. Since we will only introduce position operators, introduced in Chapter 4, we will assume that our operators live in an infinite dimensional Hilbert-space for which we can construct a continuous orthogonal basis. Therefore the trace of an operator $\hat{A}$ in 1-dimensional space is given as:

$$\mathrm{Tr}(\hat{A}) \equiv \int dx \ \left\langle x \middle| \hat{A} \middle| x \right\rangle . \tag{1.12}$$

The trace may also be extended to 4-dimensional space in which the integral becomes an integral over all four position coordinates.

1.3.3 Naming Conventions

We will also introduce a few naming conventions in order to avoid confusion.

Firstly, as we will work with particle fields as both functions of position operators and numbers, we will refer to the fields as functions of position operators as “operator valued fields” or “field operators”. The fields which are functions of numbers we will simply call “fields”. Further notation and explanations thereof will be covered in Chapter 4.

Also, as we will be working with field theories, we will often refer to the Lagrangian of the system. By “Lagrangian” we will always refer to the Lagrangian density of the system and not the Lagrangian which is a spatial integral of the Lagrangian density, unless explicitly stated otherwise.

Lastly, when working with a nuclear interaction model, we will often refer to the constituents of the atomic nucleus as either “baryons” or “nucleons”. Baryons, in physics, are composite particles constructed from three quarks, where the quarks may have any valid permutations of the 6 quark flavours. Nucleons, on the other hand, only refer to protons and neutrons, which are baryons with two specific permutations of the 6 quark flavours. Since we will not be working with any exotic matter within a nuclear interaction theory, we may assume that both the terms “baryons” and “nucleons” only refer to protons and neutrons.

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CHAPTER 2

Quantum Field Theory

This chapter will serve as a focused revision of all the theory that will be needed for the chapters to follow. We will briefly revise the most important aspects of relativistic quantum mechanics, classical field theory and the second quantisation that are applicable and relevant to research that will be done in this thesis. We will therefore not delve too deeply into the philosophy behind that which brought these theories into existence and any further information or derivations of these theories can be found in most relativistic quantum mechanics and quantum field theory text books. Literature used for this revision is [1, 6, 7, 8].

2.1 Relativistic Quantum Mechanics

2.1.1 Formulation of the Klein-Gordon and Dirac Equations

The basis for any relativistic theory is to construct the theory so that it remains invariant under Poincaré transformations. That is boosts, rotations and translations in space and time. This requires that we introduce four-vector spatial and momentum coordinates. We therefore have our spatial and momentum coordinates respectively given by

$$\begin{aligned} x^\mu &= (x^0, x^1, x^2, x^3) \equiv (t, \boldsymbol{x}) \\ p^\mu &= (p^0, p^1, p^2, p^3) \equiv (E, \boldsymbol{p}), \end{aligned} \tag{2.1}$$

where we assumed natural units, t is the time coordinate and E the energy.

An invariant quantity that came forth from the relativistic mechanics, is that of the mass of the particle squared, given by

$$M^2 = p^\mu p_\mu, \tag{2.2}$$

which results in the well-known relativistic energy relation given by

$$E^2 = \boldsymbol{p}^2 + M^2. \tag{2.3}$$

It was this relation that led Oskar Klein and Walter Gordon to construct the well-known Klein-

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Gordon equation for a spinless particle (ψ) in relativistic quantum mechanics:

$$(\boldsymbol{\nabla}^2 - M^2)\psi(t,\boldsymbol{x}) = \frac{\partial^2}{\partial t^2}\psi(t,\boldsymbol{x}), \tag{2.4}$$

where quantum mechanical descriptions for momentum and energy is assumed and given by $\boldsymbol{p} \to -i\boldsymbol{\nabla}$ and $E \to i\frac{\partial}{\partial t}$. We can also rewrite the Klein-Gordon equation using covariant notation:

$$(\partial^\mu \partial_\mu + M^2)\psi(t,\boldsymbol{x}) = 0. \tag{2.5}$$

Even though the Klein-Gordon equation could only describe spinless particles, one of the main characteristics of relativistic particles already became apparent. The characteristic, of course, is that solutions exist for particles of both positive and negative energies. This followed from the fact that the energy for a relativistic particle could be rewritten from its original form in eq. (2.3) into

$$E = \pm\sqrt{\boldsymbol{p}^2 + M^2}. \tag{2.6}$$

The interpretation given to these positive and negative energies, after much speculation and discussion, would be that of particles and anti-particles. The solutions to these particles, satisfying the Klein-Gordon equation in (2.4), takes the form of a plane wave solution and is given by

$$\psi(t,\boldsymbol{x}) = e^{-iEt}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} = e^{-ip^\mu x_\mu}. \tag{2.7}$$

However, the Klein-Gordon equation was still limited to describe only spinless particles and a new description was needed for particles with non-zero spin. A description for spin was not the only requirement, but mathematical description containing only terms linear in spatial and time derivatives was also required in order to have a positive-definite probability density for particles. Therefore a general form for the Hamiltonian satisfying all these requirements was suggested by Paul Dirac for a multi-component wavefunction $\psi_m(t,\boldsymbol{x})$, with $m = 1, 2, ..., N$:

$$\begin{aligned} H_{mn}\psi_n(t,\boldsymbol{x}) \equiv\ & i\frac{\partial}{\partial t}\psi_m(t,\boldsymbol{x}) \\ =\ & \Big(-i\boldsymbol{\alpha}_{mn}\cdot\boldsymbol{\nabla} + \beta_{mn}M\Big)\psi_n(t,\boldsymbol{x}), \end{aligned} \tag{2.8}$$

where $\boldsymbol{\alpha}_{mn} = (\alpha^1_{mn}, \alpha^2_{mn}, \alpha^3_{mn})$ and β_{mn} are $N \times N$ matrices. By requiring that the Hamiltonian be Hermitian, probability densities are positive-definite and that each of the components ψ_m

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satisfy the Klein-Gordon equation, it was obtained that the smallest number of components for ψ_m is 4 and that the matrix coefficients in the Dirac equation take the most convenient form of:

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{2.9}$$

where $\boldsymbol{\sigma}$ are the three 2×2 Pauli spin-matrices, and

$$\beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}. \tag{2.10}$$

In order for the Dirac equation to be rewritten in covariant notation, we can introduce four matrices based on the matrix coefficients we have just defined. These 4 matrices are the well-known γ-matrices and are defined as

$$\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) = (\beta, \beta\boldsymbol{\alpha}). \tag{2.11}$$

Using the γ-matrices we rewrite the Dirac equation using covariant notation as

$$(i\gamma^\mu \partial_\mu - M)\psi(t, \boldsymbol{x}) = 0. \tag{2.12}$$

This Dirac equation gives a complete description of a free relativistic particle. It is invariant under Lorentz transformations and is also able to describe particles with non-zero spin.

2.1.2 Free-Particle Solutions

As it was mentioned earlier, our particles can have either positive or negative energy eigenvalues and the particle wave functions for the Dirac equations consists of four components. The positive energy solution for Dirac equation can be written as

$$\psi_+(t, \boldsymbol{x}) = e^{-iE(p)t} e^{+i\boldsymbol{p}\cdot\boldsymbol{x}} u(\boldsymbol{p}), \tag{2.13}$$

where $u(\boldsymbol{p})$ provides us with the 4-component Dirac "spinor" for a positive energy particle and contains information on both the spin and helicity of the particle. When one inserts the positive energy solution (2.13) into the Dirac equation, the result is that the positive energy spinor, $u(\boldsymbol{p})$,

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satisfies:

$$(\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta M)\, u(\boldsymbol{p}) = E(p)u(\boldsymbol{p}). \tag{2.14}$$

This leads us to the normalised solution for $u(\boldsymbol{p})$ given by

$$u(\boldsymbol{p}) = \sqrt{\frac{E(p)+M}{2E(p)}} \begin{pmatrix} \chi \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E(p)+M}\chi \end{pmatrix}, \tag{2.15}$$

where χ is a Pauli two-spinor which can have the form of either $\chi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\chi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For the negative energy solutions we have that the negative energy solution for the Dirac equation is given by

$$\psi_-(t, \boldsymbol{x}) = e^{iE(p)t} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} v(\boldsymbol{p}). \tag{2.16}$$

Here, when the negative energy solution is inserted into the Dirac equation, we find that $v(\boldsymbol{p})$ satisfies:

$$(\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} - \beta M)\, v(\boldsymbol{p}) = E(p)v(\boldsymbol{p}). \tag{2.17}$$

Obtaining a solution for $v(\boldsymbol{p})$ in eq. (2.17) results in

$$v(\boldsymbol{p}) = \sqrt{\frac{E(p)+M}{2E(p)}} \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E(p)+M}\chi \\ \chi \end{pmatrix}. \tag{2.18}$$

The Dirac spinors can also be labelled in a more compact form to explicitly state the exact spin of the particle. The notation given to the spinors takes the form of $u(\boldsymbol{p}, s)$, where s takes on value of $s = -1, 1$ and forces us to consider the spinor with a down-spin ($\chi_\downarrow$) or an up-spin ($\chi_\uparrow$), respectively.

With the free-particle solutions for particles of positive and negative energies in place, we will also briefly mention few important and very useful properties of the Dirac spinors. These properties are calculated to be:

$$u^\dagger(\boldsymbol{p}, s)u(\boldsymbol{p}, s') = v^\dagger(\boldsymbol{p}, s)v(\boldsymbol{p}, s') = \delta_{ss'}, \tag{2.19}$$

$$u^\dagger(\boldsymbol{p}, s)v(-\boldsymbol{p}, s') = v^\dagger(-\boldsymbol{p}, s)u(\boldsymbol{p}, s') = 0 \tag{2.20}$$

and

$$\sum_s \left[u(\boldsymbol{p}, s)u^\dagger(\boldsymbol{p}, s') + v(-\boldsymbol{p}, s)v^\dagger(-\boldsymbol{p}, s')\right] = 0. \tag{2.21}$$

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2.2 Classical Field Theory

The fundamental quantity of a field theory is the action, denoted by S. From the action one is able to extract many important properties and quantities of the system, two of them being the equations of motion and the other the energy-momentum tensor, the derivation of which will be discussed in this chapter.

2.2.1 Lagrangian Formalism

To start off the formalism of a classical field theory, it is given that the action is a time integral of the Lagrangian, L, which we get by integrating the Lagrangian density, $\mathcal{L}$, over all space. Therefore, the action, which is a scalar quantity, is given by

$$S = \int d^4x \; \mathcal{L}(\phi_1, \partial_\mu \phi_1, \phi_2, \partial_\mu \phi_2, \ldots), \tag{2.22}$$

where the Lagrangian density is a function of N fields and their derivatives and gives a compact summary of the dynamics of the system. However, as previously mentioned, we will never explicitly use the Lagrangian, L, but we will often make use if the Lagrangian density, $\mathcal{L}$. We will therefore from this point forward also refer to the Lagrangian density simply as the Lagrangian.

Classically, when a system evolves from one state to another, it will do so along a "path" for which the action will be a minimum, i.e. along the "path" for which the variation of the action is zero:

$$\delta S = 0. \tag{2.23}$$

Making an arbitrary infinitesimal variation of the fields given by

$$\phi_n(\boldsymbol{x}, t) \to \phi_n(\boldsymbol{x}, t) + \delta\phi_n(\boldsymbol{x}, t), \tag{2.24}$$

yields the variation of the action, which we expect to vanish, as

$$\begin{aligned} 0 = \delta S &= \int d^4x \sum_n \left[\frac{\partial \mathcal{L}}{\partial \phi_n} \delta\phi_n + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \delta(\partial_\mu \phi_n) \right] \\ &= \int d^4x \sum_n \left[\frac{\partial \mathcal{L}}{\partial \phi_1} \delta\phi_n - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \right) \delta\phi_n + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \delta\phi_n \right) \right], \end{aligned} \tag{2.25}$$

where $n = 1, 2, \ldots, N$. One notices that the last term for ϕ_n is simply evaluated at the boundaries of the path. Using the fact that the variation at the boundaries should vanish since we know the

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initial and final conditions exactly, the final result for the term would be zero. Also, we know that (2.25) has to hold true for any variation $\delta\phi_n$, and therefore we can conclude that

$$\frac{\partial\mathcal{L}}{\partial\phi_n} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\right) = 0, \quad \forall n. \tag{2.26}$$

Eq. (2.26) is also known as the Euler-Lagrange equation and is used to calculate the equations of motion of the Lagrangian with respect to a field, ϕ_n.

2.2.2 Noether's Theorem

A theorem by Emmy Noether states that every symmetry of a system is associated with a conserved quantity. We illustrate this in classical field theory by considering the infinitesimal variation of fields in (2.24) which corresponds to a symmetry of the action. Since the action should stay invariant under this variation of the fields, one can conclude that the Lagrangian stays invariant up to some 4-dimensional divergence and transforms as:

$$\mathcal{L} \to \mathcal{L} + \alpha\partial_\mu\mathcal{J}^\mu, \tag{2.27}$$

with some constant α and 4-divergence $\mathcal{J}^\mu$. However, when applying the variation of fields to the Lagrangian one finds from (2.25) that

$$\alpha\delta\mathcal{L} = \alpha\sum_n\left[\frac{\partial\mathcal{L}}{\partial\phi_1}\delta\phi_n - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\right)\delta\phi_n + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\delta\phi_n\right)\right], \tag{2.28}$$

where the first two terms vanish because of the Euler-Lagrange equation and the remaining term should equal the four-divergence of the Lagrangian under such a transformation. Therefore we can conclude that

$$\partial_\mu j^\mu \equiv \partial_\mu\left[\sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\delta\phi_n - \mathcal{J}^\mu\right] = 0, \tag{2.29}$$

and j^μ is a conserved current. In particular, as we integrate over all space and time in the action, the action must therefore be invariant under space-time translations: An infinitesimal spatial translation given by

$$x^\mu \to x^\mu + a^\mu. \tag{2.30}$$

The fields then transforms under this spatial translation as

$$\phi_n \to \phi_n + a^\mu\partial_\mu\phi_n, \tag{2.31}$$

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and the Lagrangian as

$$\mathcal{L} \to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L}). \tag{2.32}$$

If we would now compare this to our expression in (2.27), we see that for every a^ν, with $\nu = 0, 1, 2, 3$, we have a four-divergence of $\partial_\mu(\delta^\mu_\nu \mathcal{L})$. Therefore, according to (2.29), we have 4 conserved currents given by

$$T^\mu_\nu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\nu \phi_n - \delta^\mu_\nu \mathcal{L}, \tag{2.33}$$

where we define T^μ_ν as the energy-momentum tensor of the system. Therefore, the energy-momentum tensor is conserved: $\partial_\mu T^\mu_\nu = 0$. From this we can extract conserved quantities of the system of which the most important are the energy

$$E = \int d^3x \; T_{00}, \tag{2.34}$$

and the momentum

$$P_i = \int d^3x \; T_{0i}. \tag{2.35}$$

2.2.3 Hamiltonian Formalism

When one wants to quantise a field theory, it is useful to also introduce the concept of the Hamiltonian of the system. Given the definition of the conjugate momentum to a field given by

$$\Pi_n(\boldsymbol{x}, t) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n(\boldsymbol{x}, t)}, \tag{2.36}$$

we can construct the Hamiltonian as a spatial integral of the Hamiltonian density:

$$\begin{aligned} H &\equiv \int d^3x \; \mathcal{H}\left(\Pi_1(\boldsymbol{x}, t), \ldots, \phi_1(\boldsymbol{x}, t), \ldots\right) \\ &= \int d^3x \left[\Pi_1 \frac{\partial \phi_1}{\partial t} + \Pi_2 \frac{\partial \phi_2}{\partial t} + \ldots - \mathcal{L}(\phi_1, \ldots, \partial_\mu \phi_1, \ldots)\right]. \end{aligned} \tag{2.37}$$

Since the Hamiltonian of a system corresponds to the total energy of the system, one can compare (2.37) with (2.34) and find that the Hamiltonian density, corresponding to the energy density of the system, is given by the energy-momentum tensor:

$$\mathcal{H} = \mathcal{E} = T_{00}. \tag{2.38}$$

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In a quantum system where we would view the $\Pi_n(\boldsymbol{x},t)$ and $\phi_n(\boldsymbol{x},t)$ as Hermitian operators, one can write down equal-time commutation relations for these operators:

$$\left[\Pi_n(\boldsymbol{x},t),\phi_n(\boldsymbol{x}',t)\right] = -i\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{2.39}$$

and

$$\left[\Pi_n(\boldsymbol{x},t),\Pi_n(\boldsymbol{x}',t)\right] = \left[\phi_n(\boldsymbol{x},t),\phi_n(\boldsymbol{x}',t)\right] = 0. \tag{2.40}$$

It also follows that we can write the time derivative in the Heisenberg picture for the fields and conjugate momentum as

$$\begin{aligned}\frac{\partial}{\partial t}\Pi_n(\boldsymbol{x},t) &= i\left[H,\Pi_n(\boldsymbol{x},t)\right], \\ \frac{\partial}{\partial t}\phi_n(\boldsymbol{x},t) &= i\left[H,\phi_n(\boldsymbol{x},t)\right]. \end{aligned}\tag{2.41}$$

2.3 The Dirac Field and the Second Quantisation

We now proceed to discuss the quantisation of the Dirac theory (2.12), and how it is used to describe the dynamics of baryons propagating in space and time. One finds that the Dirac Lagrangian, which yields the Dirac equation as an equation of motion, is given by

$$\mathcal{L}_{Dirac} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - M\right)\psi, \tag{2.42}$$

where $\overline{\psi} \equiv \psi^\dagger\gamma^0$. The conjugate momentum to ψ is given by

$$\Pi_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\dagger, \tag{2.43}$$

and therefore the resulting Hamiltonian is given by

$$H = \int d^3x\ \mathcal{H} = \int d^3x\ \psi^\dagger\left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta M\right)\psi. \tag{2.44}$$

For the energy-momentum tensor we calculate that

$$T_{\mu\nu} = i\overline{\psi}\gamma_\mu\partial_\nu\psi, \tag{2.45}$$

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from which we can calculate the conserved momentum of the system:

$$\boldsymbol{P} = \int d^3x \; i\psi^\dagger \boldsymbol{\nabla}\psi. \tag{2.46}$$

Considering now the particle fields as operators, which we will from this point on refer to as baryon field operators, we now impose equal-time anti-commutation relations on the operators:

$$\begin{aligned} \left\{\psi_\mu(\boldsymbol{x},t), \psi_\nu^\dagger(\boldsymbol{x}',t)\right\} &= \delta_{\mu\nu}\delta(\boldsymbol{x}-\boldsymbol{x}'), \\ \left\{\psi_\mu(\boldsymbol{x},t), \psi_\nu(\boldsymbol{x}',t)\right\} &= \left\{\psi_\mu^\dagger(\boldsymbol{x},t), \psi_\nu^\dagger(\boldsymbol{x}',t)\right\} = 0. \end{aligned} \tag{2.47}$$

We impose anti-commutation relations since imposing the normal commutation relations in (2.39) and (2.40) gives a Hamiltonian unbounded from below. The anti-commutation relations also provides our operators with fermionic properties, i.e. it satisfies Pauli's exclusion principle.

We can also now proceed to expand our baryon operators into normal modes. We derive these modes from the free-particle solutions for the Dirac equation, giving baryon field operators containing discrete momentum solutions for both positive and negative energy particles in a system of volume V:

$$\psi(\boldsymbol{x},t) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k},s}\left[a_{\boldsymbol{k},s}u(\boldsymbol{k},s)e^{-ik\cdot x} + b^\dagger_{\boldsymbol{k},s}v(\boldsymbol{k},s)e^{ik\cdot x}\right], \tag{2.48}$$

and

$$\overline{\psi}(\boldsymbol{x},t) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k},s}\left[a^\dagger_{\boldsymbol{k},s}\overline{u}(\boldsymbol{k},s)e^{ik\cdot x} + b_{\boldsymbol{k},s}\overline{v}(\boldsymbol{k},s)e^{-ik\cdot x}\right], \tag{2.49}$$

for which we have the anti-commutation relations for the mode operators, based on the restrictions in eq. (2.47):

$$\begin{aligned} \left\{a_{\boldsymbol{k},s} \; , \; a^\dagger_{\boldsymbol{k}',s'}\right\} &= \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{s,s'}, \\ \left\{b_{\boldsymbol{k},s} \; , \; b^\dagger_{\boldsymbol{k}',s'}\right\} &= \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{s,s'}, \end{aligned} \tag{2.50}$$

where all other anti-commutation relations are zero. We also have that

$$\begin{aligned} \sum_s u(\boldsymbol{k},s)\overline{u}(\boldsymbol{k},s) &= (\gamma^\mu k_\mu + M)/2E(k), \\ \sum_s v(\boldsymbol{k},s)\overline{v}(\boldsymbol{k},s) &= (\gamma^\mu k_\mu - M)/2E(k). \end{aligned} \tag{2.51}$$

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The mode operators $(a^\dagger_{\boldsymbol{k},s}, a_{\boldsymbol{k},s})$ and $(b^\dagger_{\boldsymbol{k},s}, b_{\boldsymbol{k},s})$ are interpreted as creation and annihilation operator pairs for particles and anti-particles, respectively. The ground state for the system, $|\,\Psi_0\rangle$, contains no anti-particles and is obtained by creating particles up to the Fermi momentum k_F, which implies that

$$\begin{aligned} b_{\boldsymbol{k},s}\,|\,\Psi_0\rangle &= 0, \quad \forall \boldsymbol{k} \\ a_{\boldsymbol{k},s}\,|\,\Psi_0\rangle &= 0, \quad \text{for } |\boldsymbol{k}| > k_F \\ a^\dagger_{\boldsymbol{k},s}\,|\,\Psi_0\rangle &= 0, \quad \text{for } |\boldsymbol{k}| < k_F. \end{aligned} \tag{2.52}$$

Considering all the anti-commutation relations for our baryon field operators and restrictions thereon, it enables us to calculate the Hamiltonian in its second-quantised form:

$$H = \sum_{\boldsymbol{k},s} E(k)\left[a^\dagger_{\boldsymbol{k},s}a_{\boldsymbol{k},s} - b_{\boldsymbol{k},s}b^\dagger_{\boldsymbol{k},s}\right] = \sum_{\boldsymbol{k},s} E(k)\left[a^\dagger_{\boldsymbol{k},s}a_{\boldsymbol{k},s} + b^\dagger_{\boldsymbol{k},s}b_{\boldsymbol{k},s} - 1\right]. \tag{2.53}$$

The last constant term of the Hamiltonian may be ignored because we are working with energies relative to the vacuum. We will, however, discuss this in further detail in Chapter 3.

One can also now proceed to determine the baryon propagator for a particle propagating through time and space. For the discrete case we find that the propagator, using the time-ordered product of the baryon field operators, is given by

$$\begin{aligned} iG_{\mu\nu}(x'-x) &= \left\langle \Psi_0 \,\middle|\, T(\psi_\mu(x')\overline{\psi}_\nu(x)) \,\middle|\, \Psi_0 \right\rangle \\ &= \left\langle \Psi_0 \,\middle|\, \psi_\mu(x')\overline{\psi}_\nu(x) \,\middle|\, \Psi_0 \right\rangle \theta(t'-t) - \left\langle \Psi_0 \,\middle|\, \overline{\psi}_\nu(x)\psi_\mu(x') \,\middle|\, \Psi_0 \right\rangle \theta(t-t') \\ &= \frac{1}{V}\sum_{\boldsymbol{k}} \frac{1}{2E(k)} \left\{(\gamma^\mu k_\mu + M)e^{-ik\cdot(x'-x)}\theta(t'-t) - (\gamma^\mu k_\mu - M)e^{-ik\cdot(x-x')}\theta(t-t')\right\}. \end{aligned} \tag{2.54}$$

The propagator for continuous momentum is also given in [1] and [8].

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CHAPTER 3

The Walecka Model

Having laid the quantum field theoretical basis, we are going to apply this fundamental theory to a physical situation. The situation we are going to consider is that of nucleons interacting within the atomic nucleus. The nucleus consists of protons and neutrons bound together by a sensitive balance of the attractive strong nuclear force and repulsive Coulomb force.

We can use quantum field theory to give a model description to the mechanism that binds these nucleons together into stable nuclear systems. One such model is the Walecka model, which gives a simple, yet useful way of understanding the nuclear system of interacting nucleons. In this chapter we will do a literature study on the Walecka model as outlined and discussed in Serot and Walecka's 1986 publication in the journal for Advances in Nuclear Physics [1]. We will investigate the setup of the Walecka model, starting by identifying the particles which will play a role in the model, and building onwards to obtain a quantum field theory for nucleon-nucleon interactions. Thereafter we will also depict how the Walecka model is used to produce mean-field theoretical descriptions of nuclear matter.

3.1 The Model Lagrangian and Energy-Momentum Tensor

The Walecka model, or sometimes referred to as the Quantum Hadrodynamics-I (QHD-I) model [1], aimed to construct a relativistic quantum field theory to describe nuclear matter in a compact and simplified way. Recognising that the scalar- and also the vector mesons gave the largest contributions to the nucleon-nucleon interactions, Dirk Walecka left out other contributors such as the rho- and pi-mesons in attempt to understand nuclear matter on a qualitative scale. The interaction model, therefore, would only contain contributions of 3 particles:

The baryons (ψ) - The baryons are the main interacting particles and physically describes the nucleons within nuclear matter. Generally they are described as two-component iso-spin fields, but when the distinction between protons and neutrons is not necessary, one can simply regard it as a single particle field. The model assumes that both iso-spins have a mass M.

The vector mesons (V_μ) - The vector mesons, often referred to as ω-particles, are exchange

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particles between the baryons which describes the short range repelling strong forces between the nucleons. The vector mesons, with mass m_v, couple to the baryons through $g_v\overline{\psi}\gamma^\mu V_\mu\psi$, with g_v the coupling strength of the vector mesons to the baryons.

The scalar mesons (ϕ) - The scalar mesons, often referred to as σ-particles, are exchange particles between the baryons which gives us the description for the long range attracting strong forces between the nucleons. The mass of the scalar mesons is given by m_s and they couple to the baryons through $g_s\overline{\psi}\phi\psi$, with g_s the coupling strength of the scalar meson to the baryons.

Given these particles, we construct the model Lagrangian density by

$$\begin{aligned}\mathcal{L} = \overline{\psi}(x)\left[\gamma^\mu\left(i\partial_\mu - g_v V_\mu(x)\right) - \left(M - g_s\phi(x)\right)\right]\psi(x) + \frac{1}{2}\left(\partial^\mu\phi(x)\partial_\mu\phi(x) - {m_s}^2\phi^2(x)\right)\\ -\frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x) + \frac{1}{2}{m_v}^2V^\mu(x)V_\mu(x)\ ,\end{aligned} \tag{3.1}$$

where $F^{\mu\nu}(x) = \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x)$. Integrating over space-time gives the action, where we write the fields without their dependences on (x) in order to shorten the writing,

$$\begin{aligned}S = \int d^4x\ \mathcal{L} = \int d^4x\ \Big\{\overline{\psi}\left[\gamma^\mu\left(i\partial_\mu - g_v V_\mu\right) - \left(M - g_s\phi\right)\right]\psi + \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - {m_s}^2\phi^2\right)\\ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}{m_v}^2V^\mu V_\mu\Big\}.\end{aligned} \tag{3.2}$$

Given the Euler-Lagrange equation in (2.26), we can now calculate the equations of motion of our model. Varying the Lagrangian density in (3.1) with respect to $\overline{\psi}$, ψ, V_μ and ϕ, we get for the equations of motion:

For ψ -

$$\left[\gamma^\mu\left(i\partial_\mu - g_v V_\mu\right) - \left(M - g_s\phi\right)\right]\psi = 0. \tag{3.3}$$

For $\overline{\psi}$ -

$$\overline{\psi}\left[\gamma^\mu\left(i\overleftarrow{\partial}_\mu + g_v V_\mu\right) + \left(M - g_s\phi\right)\right] = 0. \tag{3.4}$$

For V_μ -

$$\partial_\mu F^{\mu\nu} + {m_v}^2V^\nu - g_v\overline{\psi}\gamma^\nu\psi = 0. \tag{3.5}$$

and

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For ϕ -

$$\left(\partial^{\mu}\partial_{\mu} + {m_s}^2\right)\phi - g_s\overline{\psi}\psi = 0. \tag{3.6}$$

After the equations of motion, the next process would be to calculate the energy-momentum tensor through Noether's theorem as outlined in Section 2.2.2. This calculation results in

$$\begin{aligned} T_{\mu\nu} &= \left[\frac{\partial}{\partial(\partial q_i/\partial x_\mu)}\frac{\partial q_i}{\partial x^\nu} - g^{\mu\nu}\right]\mathcal{L} \\ &= \left[(i\gamma_\mu\overline{\psi})\partial_\nu\psi + (-F_{\mu\rho})\partial_\nu V^\rho + (\partial_\mu\phi)\partial_\nu\phi\right] - g_{\mu\nu}\mathcal{L}. \end{aligned}$$

Now, using the equation of motion given by eq. (3.3) and the anti-symmetric property of $F^{\mu\nu}$, we get the resulting energy-momentum tensor for the Walecka model as

$$T_{\mu\nu} = i\overline{\psi}\gamma_\mu\partial_\nu\psi + \partial_\mu\phi\partial_\nu\phi + \partial_\nu V^\rho F_{\rho\mu} - \frac{1}{2}g_{\mu\nu}\left[(\partial^\rho\phi\partial_\rho\phi - {m_s}^2\phi^2) - \frac{1}{2}F^{\rho\sigma}F_{\rho\sigma} + {m_v}^2V^\rho V_\rho\right]. \tag{3.7}$$

From the energy-momentum tensor in (3.7) we can calculate important thermodynamical quantities such as the energy-density ($\mathcal{E}$) and pressure (p). The quantity most important to us, the energy density, is given by the expression

$$\begin{aligned} \mathcal{E} = T_{00} &= i\overline{\psi}\gamma_0\partial_0\psi + (\partial_0\phi)^2 + \partial_0 V^\rho F_{\rho 0} \\ &\quad - \frac{1}{2}\left[(\partial^\rho\phi\partial_\rho\phi - {m_s}^2\phi^2) - \frac{1}{2}F^{\rho\sigma}F_{\rho\sigma} + {m_v}^2V^\rho V_\rho\right]. \end{aligned} \tag{3.8}$$

However, again with the help of the equation of motion given by eq. (3.3), we can rewrite our energy density into the more convenient form of

$$\begin{aligned} \mathcal{E} = T_{00} = &-i\psi^\dagger\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi + \overline{\psi}\left[g_v\gamma^\rho V_\rho + (M - g_s\phi)\right]\psi + (\partial_0\phi)^2 + (\partial_0 V^\rho)F_{\rho 0} \\ &- \frac{1}{2}\left[(\partial^\rho\phi\partial_\rho\phi - {m_s}^2\phi^2) - \frac{1}{2}F^{\rho\sigma}F_{\rho\sigma} + {m_v}^2V^\rho V_\rho\right]. \end{aligned} \tag{3.9}$$

We thus conclude that the total energy for the Walecka model, integrated over the total system volume (Ω), is given by

$$\begin{aligned} H = \int_\Omega d^3x\,\mathcal{E} = \int_\Omega d^3x\,\Big\{ &-i\psi^\dagger\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi + \overline{\psi}\left[g_v\gamma^\rho V_\rho + (M - g_s\phi)\right]\psi + (\partial_0\phi)^2 + (\partial_0 V^\rho)F_{\rho 0} \\ &-\frac{1}{2}\left[(\partial^\rho\phi\partial_\rho\phi - {m_s}^2\phi^2) - \frac{1}{2}F^{\rho\sigma}F_{\rho\sigma} + {m_v}^2V^\rho V_\rho\right]\Big\}. \end{aligned} \tag{3.10}$$

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This concludes the basic formulation of the Walecka model and the energy-momentum tensor. To calculate exact solutions for the equations of motion and the energy-momentum tensor would be a very complicated process, if it is possible at all. The fact that we have many non-linear terms drastically increases the difficulty level of finding solutions. Therefore, we will attempt to find a qualitative solution for nuclear matter by introducing some approximations into the Walecka model. We will briefly discuss two of these approximations, stating how they are used to give approximations for the nuclear matter equation of state.

3.2 The Mean-Field Approximation for Uniform Matter

3.2.1 Formulation

In the first mean-field approximation we are going to assume that our system consists of uniform matter. Another assumption we make is that our system is at zero temperature and we also assume that the system containing the matter is much larger than the interaction range of our baryons and meson fields, or even perhaps to be infinite. Therefore, we can assume that our meson fields stay approximately constant over the system and can be replaced by their expectation values:

$$\phi(x) \rightarrow \langle \phi(x) \rangle \equiv \phi_0 \tag{3.11}$$

and

$$V_\mu(x) \rightarrow \langle V_\mu(x) \rangle \equiv \delta_{\mu 0} V_0, \tag{3.12}$$

where ϕ_0 and V_0 are also constants in the Minkowski-space. We also replaced $\langle V_\mu \rangle$ with only the constant time component, since uniform matter implies rotational invariance which causes the $\langle V_i \rangle$ components to vanish for $i = 1, 2, 3$.

Having made these assumptions, we can rewrite the Walecka model in the mean-field approximation. The Lagrangian density is given by

$$\mathcal{L}_{\mathrm{MFT}} = \overline{\psi} \left[i\gamma^\mu \partial_\mu - g_v \gamma^0 V_0 - (M - g_s \phi_0) \right] \psi - \frac{1}{2} {m_s}^2 {\phi_0}^2 + \frac{1}{2} {m_v}^2 {V_0}^2, \tag{3.13}$$

while the equations of motion in (3.3), (3.5) and (3.6) become

$$\left[i\gamma^\mu \partial_\mu - g_v \gamma^0 V_0 - (M - g_s \phi_0) \right] \psi = 0, \tag{3.14}$$

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$$V_0 = \frac{g_v}{{m_v}^2}\psi^\dagger\psi \tag{3.15}$$

and

$$\phi_0 = \frac{g_s}{{m_s}^2}\overline{\psi}\psi. \tag{3.16}$$

Since V_0 and ϕ_0 are themselves expectation values, we should replace $\psi^\dagger\psi$ and $\overline{\psi}\psi$ with expectation values in eqs. (3.15) and (3.16). We define $\langle\psi^\dagger\psi\rangle \equiv \rho_B$ and $\langle\overline{\psi}\psi\rangle \equiv \rho_s$, respectively, as the baryon density and the Lorentz scalar density. Also, since $g_s\phi_0$ lowers the effective mass of our system, we will also from this point onwards introduce the expression for the effective mass defined by

$$M^* \equiv M - g_s\phi_0. \tag{3.17}$$

In the mean-field approximation for uniform matter the energy-momentum tensor becomes

$$(T_{\mu\nu})_{\text{MFT}} = i\overline{\psi}\gamma_\mu\partial_\nu\psi - \frac{1}{2}g_{\mu\nu}\left({m_v}^2{V_0}^2 - {m_s}^2{\phi_0}^2\right), \tag{3.18}$$

and the energy density is therefore given by

$$\mathcal{E} = \psi^\dagger\left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta M^*\right)\psi + g_v V_0\psi^\dagger\psi - \frac{1}{2}{m_v}^2{V_0}^2 + \frac{1}{2}{m_s}^2{\phi_0}^2. \tag{3.19}$$

At this point it is convenient to apply a second quantisation to the energy density in order to facilitate further computations. We follow the second quantisation as described in [1]. The baryon field operator is obtained as an expansion in the terms of the modes solving (3.14). Therefore, we get for the baryon field operator that

$$\psi(\boldsymbol{x},t) = \frac{1}{\sqrt{\Omega}}\sum_{\boldsymbol{k},\lambda}\left[\boldsymbol{A}_{\boldsymbol{k},\lambda}U(\boldsymbol{k},\lambda)e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\varepsilon_+(k)t} + \boldsymbol{B}^\dagger_{\boldsymbol{k},\lambda}V(\boldsymbol{k},\lambda)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}-i\varepsilon_-(k)t}\right], \tag{3.20}$$

where the symbols we have used are defined as:

- Ω is the total volume of the system.
- The index λ refers to the spin indices of the baryons. One can also include iso-spin indices into λ, which is what we will do in our model.
- $\left(\boldsymbol{A}^\dagger_{\boldsymbol{k},\lambda}, \boldsymbol{A}_{\boldsymbol{k},\lambda}\right)$ and $\left(\boldsymbol{B}^\dagger_{\boldsymbol{k},\lambda}, \boldsymbol{B}_{\boldsymbol{k},\lambda}\right)$ are pairs of creation an annihilation operators for particles of positive and negative energies respectively, with momentum $\boldsymbol{k}$ and spin (and iso-spin)

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λ. These operators obey the anti-commutation relations

$$\left\{\boldsymbol{A}^{\dagger}_{\boldsymbol{k},\lambda}, \boldsymbol{A}_{\boldsymbol{k}',\lambda'}\right\} = \left\{\boldsymbol{B}^{\dagger}_{\boldsymbol{k},\lambda}, \boldsymbol{B}_{\boldsymbol{k}',\lambda'}\right\} = \delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\lambda\lambda'},$$
$$\left\{\boldsymbol{A}^{\dagger}_{\boldsymbol{k},\lambda}, \boldsymbol{B}_{\boldsymbol{k}',\lambda'}\right\} = \left\{\boldsymbol{A}^{\dagger}_{\boldsymbol{k},\lambda}, \boldsymbol{B}^{\dagger}_{\boldsymbol{k}',\lambda'}\right\} = \left\{\boldsymbol{A}_{\boldsymbol{k},\lambda}, \boldsymbol{B}_{\boldsymbol{k}',\lambda'}\right\} = \left\{\boldsymbol{A}_{\boldsymbol{k},\lambda}, \boldsymbol{B}^{\dagger}_{\boldsymbol{k}',\lambda'}\right\} = 0. \tag{3.21}$$

- $U(\boldsymbol{k},\lambda)$ and $V(\boldsymbol{k},\lambda)$ are solutions of eq. (3.14) for baryons and anti-baryons respectively. They also obey

$$U^{\dagger}(\boldsymbol{k},\lambda)U(\boldsymbol{k},\lambda') = V^{\dagger}(\boldsymbol{k},\lambda)V(\boldsymbol{k},\lambda') = \delta_{\lambda\lambda'},$$
$$U^{\dagger}(\boldsymbol{k},\lambda)V(-\boldsymbol{k},\lambda') = V^{\dagger}(-\boldsymbol{k},\lambda)U(\boldsymbol{k},\lambda') = 0. \tag{3.22}$$

- The energies $\varepsilon_{\pm}(k)$ are given by $\varepsilon_{\pm}(k) = g_v V_0 \pm \left(\boldsymbol{k}^2 + M^{*2}\right)^{1/2}$. It is convenient to introduce the notation $E^*(k) = \left(\boldsymbol{k}^2 + M^{*2}\right)^{1/2}$.

With the second quantisation for the baryon fields in place, we are now able to effectively write calculation-friendly expressions for various quantities of our system. It should be noted that we have now replaced the baryon fields by baryon field operators even though we do not denote them with a hat symbol. We will reserve the hat symbols for operators in the non-commutative space which we will introduce in Chapter 4.

One important operator that will be used is the total baryon number operator, from which the mean-field baryon density can be easily calculated. With the help of the properties given in

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eqs. (3.21) and (3.22), we define this operator in the normal ordered operator form by

$$
\begin{aligned}
\boldsymbol{B} = \int_\Omega d^3x\ \rho_B &= \int_\Omega d^3x\ \psi^\dagger\psi \\
&= \frac{1}{\Omega}\int_\Omega d^3x \Bigg\{ \sum_{\boldsymbol{k},\lambda} \Big[\boldsymbol{A}^\dagger_{\boldsymbol{k},\lambda} U^\dagger(\boldsymbol{k},\lambda) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}+i\varepsilon_+(k)t} + \boldsymbol{B}_{\boldsymbol{k},\lambda} V^\dagger(\boldsymbol{k},\lambda) e^{i\boldsymbol{k}\cdot\boldsymbol{x}+i\varepsilon_-(k)t} \Big] \\
&\qquad \times \sum_{\boldsymbol{k}',\lambda'} \Big[\boldsymbol{A}_{\boldsymbol{k}',\lambda'} U(\boldsymbol{k}',\lambda') e^{i\boldsymbol{k}'\cdot\boldsymbol{x}-i\varepsilon_+(k')t} + \boldsymbol{B}^\dagger_{\boldsymbol{k}',\lambda'} V(\boldsymbol{k}',\lambda') e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}-i\varepsilon_-(k')t} \Big] \Bigg\} \\
&= \frac{1}{\Omega}\int_\Omega d^3x \sum_{\boldsymbol{k},\boldsymbol{k}',\lambda,\lambda'} \Big\{ \boldsymbol{A}^\dagger_{\boldsymbol{k},\lambda}\boldsymbol{A}_{\boldsymbol{k}',\lambda'} U^\dagger(\boldsymbol{k},\lambda) U(\boldsymbol{k}',\lambda') e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}} e^{i(\varepsilon_+(k)-\varepsilon_+(k'))t} \\
&\qquad + \boldsymbol{A}^\dagger_{\boldsymbol{k},\lambda}\boldsymbol{B}^\dagger_{\boldsymbol{k}',\lambda'} U^\dagger(\boldsymbol{k},\lambda) V(\boldsymbol{k}',\lambda') e^{-i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}} e^{i(\varepsilon_+(k)-\varepsilon_-(k'))t} \\
&\qquad + \boldsymbol{A}_{\boldsymbol{k},\lambda}\boldsymbol{B}_{\boldsymbol{k}',\lambda'} V^\dagger(\boldsymbol{k},\lambda) U(\boldsymbol{k}',\lambda') e^{i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}} e^{-i(\varepsilon_+(k)-\varepsilon_-(k'))t} \\
&\qquad + \boldsymbol{B}_{\boldsymbol{k},\lambda}\boldsymbol{B}^\dagger_{\boldsymbol{k}',\lambda'} V^\dagger(\boldsymbol{k},\lambda) V(\boldsymbol{k}',\lambda') e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}} e^{i(\varepsilon_-(k)-\varepsilon_-(k'))t} \Big\} \\
&= \sum_{\boldsymbol{k},\lambda} \left(\boldsymbol{A}^\dagger_{\boldsymbol{k},\lambda}\boldsymbol{A}_{\boldsymbol{k},\lambda} + \boldsymbol{B}_{\boldsymbol{k},\lambda}\boldsymbol{B}^\dagger_{\boldsymbol{k},\lambda} \right) \\
&= \sum_{\boldsymbol{k},\lambda} \left(\boldsymbol{A}^\dagger_{\boldsymbol{k},\lambda}\boldsymbol{A}_{\boldsymbol{k},\lambda} - \boldsymbol{B}^\dagger_{\boldsymbol{k},\lambda}\boldsymbol{B}_{\boldsymbol{k},\lambda} \right) + \sum_{\boldsymbol{k},\lambda} 1.
\end{aligned}
\tag{3.23}
$$

The final term is the sum over all negative-energy states in the Dirac sea, which can also be interpreted as the vacuum expectation value since all observations are made relative to the vacuum [1]. In order for this quantity to reflect physical observations that we make, we will subtract the vacuum expectation from the baryon number operator and we therefore conclude that the final expression for the baryon number operator is given by

$$
\begin{aligned}
\boldsymbol{B} &= \int_\Omega d^3x\ \left[\psi^\dagger\psi - \left\langle 0 \middle| \psi^\dagger\psi \middle| 0 \right\rangle \right] \\
&= \sum_{\boldsymbol{k},\lambda} \left(\boldsymbol{A}^\dagger_{\boldsymbol{k},\lambda}\boldsymbol{A}_{\boldsymbol{k},\lambda} - \boldsymbol{B}^\dagger_{\boldsymbol{k},\lambda}\boldsymbol{B}_{\boldsymbol{k},\lambda} \right),
\end{aligned}
\tag{3.24}
$$

where we applied the properties of the annihilation operators on the vacuum, which is given by $\boldsymbol{A}_{\boldsymbol{k},\lambda}\left|0\right\rangle = \boldsymbol{B}_{\boldsymbol{k},\lambda}\left|0\right\rangle = 0$. We have to be very careful that we do not confuse the baryon number operator ($\boldsymbol{B}$) with the negative-energy particle annihilator ($\boldsymbol{B}_{\boldsymbol{k},\lambda}$). The baryon number operator will never have any indices attached, while the annihilation operator will always have indices of momentum and spin.

Having calculated the expression for the baryon number operator, we are now able to compute the quantised Hamiltonian operator. Using the expression of the energy density in eq. (3.19),

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and also the definition of the baryon number operator, we get for the Hamiltonian operator:

$$\begin{aligned}
H &= \int_\Omega d^3x \ \left[\psi^\dagger\left(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta M^*\right)\psi + g_v V_0 \psi^\dagger\psi - \frac{1}{2}{m_v}^2{V_0}^2 + \frac{1}{2}{m_s}^2{\phi_0}^2\right] \\
&= g_v V_0 \boldsymbol{B} - \Omega\left(\frac{1}{2}{m_v}^2{V_0}^2 - \frac{1}{2}{m_s}^2{\phi_0}^2\right) \\
&\quad + \int_\Omega d^3x \ \left[\psi_+^\dagger\left(\boldsymbol{\alpha}\cdot\boldsymbol{k}+\beta M^*\right)\psi_+ - \psi_-^\dagger\left(\boldsymbol{\alpha}\cdot\boldsymbol{k}-\beta M^*\right)\psi_-\right] \\
&= g_v V_0 \boldsymbol{B} - \Omega\left(\frac{1}{2}{m_v}^2{V_0}^2 - \frac{1}{2}{m_s}^2{\phi_0}^2\right) + \sum_{\boldsymbol{k},\lambda} E^*(k)\left(\boldsymbol{A}_{\boldsymbol{k},\lambda}^\dagger\boldsymbol{A}_{\boldsymbol{k},\lambda} - \boldsymbol{B}_{\boldsymbol{k},\lambda}\boldsymbol{B}_{\boldsymbol{k},\lambda}^\dagger\right) \\
&= g_v V_0 \boldsymbol{B} - \Omega\left(\frac{1}{2}{m_v}^2{V_0}^2 - \frac{1}{2}{m_s}^2{\phi_0}^2\right) + \sum_{\boldsymbol{k},\lambda} E^*(k)\left(\boldsymbol{A}_{\boldsymbol{k},\lambda}^\dagger\boldsymbol{A}_{\boldsymbol{k},\lambda} + \boldsymbol{B}_{\boldsymbol{k},\lambda}^\dagger\boldsymbol{B}_{\boldsymbol{k},\lambda}\right) + \sum_{\boldsymbol{k},\lambda} E^*(k),
\end{aligned} \tag{3.25}$$

where ψ_+ is the positive energy part of eq. (3.20) and ψ_- the negative energy part. Again we see an extra term at the end which would potentially cause divergences in our calculations. However, we can again apply the same explanation that this term is due to the vacuum expectation value of the Hamiltonian and if we were to only require the physical energy of the system, then we should subtract this term from the Hamiltonian. This ultimately gives us the Hamiltonian operator in the Mean-Field approximation for uniform matter:

$$H = g_v V_0 \boldsymbol{B} - \Omega\left(\frac{1}{2}{m_v}^2{V_0}^2 - \frac{1}{2}{m_s}^2{\phi_0}^2\right) + \sum_{\boldsymbol{k},\lambda} E^*(k)\left(\boldsymbol{A}_{\boldsymbol{k},\lambda}^\dagger\boldsymbol{A}_{\boldsymbol{k},\lambda} + \boldsymbol{B}_{\boldsymbol{k},\lambda}^\dagger\boldsymbol{B}_{\boldsymbol{k},\lambda}\right). \tag{3.26}$$

3.2.2 Solving the Mean-Field approximation for Nuclear Matter

In the mean-field approximation the ground-state of the baryons is the Hartree-Fock vacuum where all single particle states are filled to the Fermi-energy. We must, however, keep in mind that every single particle state is 4 fold degenerate, that is, 2 protons and 2 neutrons each with spin up and spin down occupy each state. This degeneracy will be indicated with $\gamma = 4$. Calculating the expectation value of the baryon number operator (3.24) in the ground-state, we easily derive a relation between the baryon density and the Fermi-momentum (momentum of the highest filled state) [1]

$$\rho_B = \frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k = \frac{\gamma}{6\pi^2}{k_F}^3, \tag{3.27}$$

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Similarly the energy density can be obtained from the expectation value of the Hamiltonian in eq. (3.26) and the expressions for V_0 and ϕ_0 in eqs. (3.15) and (3.17):

$$\begin{aligned}\mathcal{E} &= g_v V_0 \rho_B - \left(\frac{1}{2}{m_v}^2{V_0}^2 - \frac{1}{2}{m_s}^2{\phi_0}^2\right) + \frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k\ E^*(k) \\ &= \frac{{g_v}^2}{{m_v}^2}{\rho_B}^2 - \left(\frac{g_v^2}{2{m_v}^2}{\rho_B}^2 - \frac{{m_s}^2}{2{g_s}^2}(M-M^*)^2\right) + \frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k\ \left(\boldsymbol{k}^2 + M^{*2}\right)^{1/2} \\ &= \frac{{g_v}^2}{2{m_v}^2}{\rho_B}^2 + \frac{{m_s}^2}{2{g_s}^2}(M-M^*)^2 + \frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k\ \left(\boldsymbol{k}^2 + M^{*2}\right)^{1/2}. \end{aligned} \tag{3.28}$$

The only unknown quantity in the energy density is the effective mass M^*. We can find an expression for M^* by finding the value for M^* that minimises the total energy. However, since M^* does not depend on position, we conclude that we can simply minimise the energy density with respect to M^*. Thus

$$\frac{\partial}{\partial M^*}\mathcal{E}(k, M^*) = 0. \tag{3.29}$$

This equation, when applied to eq. (3.28), results in

$$\begin{aligned}\frac{\partial}{\partial M^*}\mathcal{E} &= \frac{\partial}{\partial M^*}\left[\frac{{g_v}^2}{2{m_v}^2}{\rho_B}^2 + \frac{{m_s}^2}{2{g_s}^2}(M-M^*)^2 + \frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k\ \left(\boldsymbol{k}^2 + M^{*2}\right)^{1/2}\right] \\ &= -\frac{{m_s}^2}{{g_s}^2}M + \frac{{m_s}^2}{{g_s}^2}M^* + \frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k\ \frac{M^*}{\left(\boldsymbol{k}^2 + M^{*2}\right)^{1/2}} \\ &= 0. \end{aligned} \tag{3.30}$$

With that we can conclude that

$$\begin{aligned} M^* &= M - \frac{{g_s}^2}{{m_s}^2}\frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k\ \frac{M^*}{\left(\boldsymbol{k}^2 + M^{*2}\right)^{1/2}} \\ &= M - \frac{{g_s}^2}{{m_s}^2}\frac{\gamma M^*}{(2\pi)^2}\left[k_F E_F^* - M^{*2}\ln\left(\frac{k_F + E_F^*}{M^*}\right)\right], \end{aligned} \tag{3.31}$$

with $E_F^* = \left({k_F}^2 + M^{*2}\right)^{1/2}$. We note that the expression for the effective mass given in eq. (3.31) is very similar to the expression of eq. (3.17). This implies that the expression for the scalar density can be rewritten in its final form as

$$\rho_s = \frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k\ \frac{M^*}{\left(\boldsymbol{k}^2 + M^{*2}\right)^{1/2}}. \tag{3.32}$$

We now have a complete, self-consistent solution for the Walecka model in the mean-field ap-

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proximation for uniform matter. Indeed, the final expression for the reduced mass in eq. (3.31) can also be seen as a self-consistency equation: from the expectation value of (3.16) one notes $\phi_0 = \frac{g_s}{{m_s}^2} \left\langle \overline{\psi}\psi \right\rangle$. Computing this expectation value yields exactly (3.32), while expressing ϕ_0 in terms of M^* from (3.17) leads to equation (3.31) for M^*.

The resulting equations can be used to calculate quantities such as the energy per nucleon, the effective mass relative to baryon mass and ultimately the equation of state for nuclear matter as done in [1]. This also requires that one solves for the pressure within the matter. This, however, require a very similar set of calculations that we have done to solve for the energy-density, since the pressure is defined from the energy-momentum tensor as $p \equiv T_{ii}$ with $i = 1, 2, 3$. These equations can also be altered to describe pure neutron matter, by simply changing the degeneracy to 2. One particular use for the solutions for pure neutron matter is to calculate various properties of neutron stars [1, 9].

3.3 The Mean-Field Approximation for Spatially Non-uniform Matter

3.3.1 Formulation

As we try to move away from uniform matter approximations to more realistic finite non-uniform matter solutions, there is little we can do that does not leave us with a set of non-linear equations. There is, however, the non-uniform approximation given by Walecka [1] which attempts to introduce non-uniform matter that can be solved in a similar fashion as the mean-field approximation for uniform matter.

We introduce this spatially non-uniform (SNU) approximation by assuming that our mean-fields introduced in the previous section have spherical symmetry, but shows varying behaviour in the radial direction. That is, the expectation values of our vector and scalar meson fields are, as before, time independent, but now has a radial dependence. Thus, we obtain new expressions for the mean-fields and they are given by

$$\phi(t, \boldsymbol{x}) \to \langle \phi(t, \boldsymbol{x}) \rangle \equiv \phi_0(|\boldsymbol{x}|) = \phi_0(r) \tag{3.33}$$

and

$$V_\mu(t, \boldsymbol{x}) \to \langle V_\mu(t, \boldsymbol{x}) \rangle \equiv \delta_{\mu 0} V_0(|\boldsymbol{x}|) = \delta_{\mu 0} V_0(r). \tag{3.34}$$

The next assumption that we make is the assumption that these meson fields vary slowly enough

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so that they can be treated as constants locally when solving the baryon equation of motion. Therefore, our Lagrangian has a form much similar to the mean-field Lagrangian in eq. (3.13) and is given by

$$\mathcal{L}_{\mathrm{SNU}} = \overline{\psi}\left[i\gamma^{\mu}\partial_{\mu} - g_v\gamma^0 V_0 - (M - g_s\phi_0)\right]\psi + \frac{1}{2}\left((\boldsymbol{\nabla}\phi_0)^2 - {m_s}^2{\phi_0}^2\right) - \frac{1}{2}\left((\boldsymbol{\nabla}V_0)^2 - {m_v}^2{V_0}^2\right) \tag{3.35}$$

The equation of motion for the baryons stay exactly the same as eq. (3.14), except that our mean-fields now have a radial dependence, while our equations of motion for the vector and meson fields are given by

$$\left(\boldsymbol{\nabla}^2 - {m_v}^2\right)V_0(r) = -g_v\psi^{\dagger}\psi = -g_v\rho_B(r) \tag{3.36}$$

and

$$\left(\boldsymbol{\nabla}^2 - {m_v}^2\right)\phi_0(r) = -g_s\overline{\psi}\psi = -g_s\rho_s(r). \tag{3.37}$$

Here the definitions of the previous section for baryon and scalar densities still apply, now however, with a radial dependence. Therefore, following the definition for the baryon density in eq. (3.27), we may now express this as

$$\rho_B(r) = \frac{\gamma}{(2\pi)^3}\int_0^{k_F(r)} d^3k = \frac{\gamma}{6\pi^2}{k_F}^3(r) \tag{3.38}$$

and the total baryon number as

$$\boldsymbol{B} = \int d^3x\ \frac{\gamma}{6\pi^2}{k_F}^3(r). \tag{3.39}$$

Another modification we should make to the mean-field approximation to accompany spatially non-uniform matter, is that of the effective mass. The effective mass will now also have a radial dependence and is expressed as

$$M^*(r) = M - g_s\phi_0(r). \tag{3.40}$$

This enables us to rewrite expression containing terms with $\phi_0(r)$. Using the expression for the scalar density given by eq. (3.32), we can rewrite the scalar density in the spatially non-uniform approximation as

$$\rho_s(r) = \frac{\gamma}{(2\pi)^3}\int_0^{k_F(r)} d^3k\ \frac{M^*(r)}{\left(\boldsymbol{k}^2 + M^{*2}(r)\right)^{1/2}}. \tag{3.41}$$

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With all these modifications in place, we can write down the expression for the total energy, with the energy density having minor modifications to its form from eq. (3.28). It is given by

$$E = \int d^3x \Bigg\{ g_v V_0 \rho_B - \frac{1}{2}\left((\boldsymbol{\nabla} V_0)^2 + {m_v}^2 {V_0}^2\right) + \frac{1}{2}\left((\boldsymbol{\nabla}\phi_0)^2 + {m_s}^2{\phi_0}^2\right) + \frac{\gamma}{(2\pi)^3}\int_0^{k_F(r)} d^3k \ \left(\boldsymbol{k}^2 + M^*(r)\right)^{1/2} \Bigg\}. \tag{3.42}$$

3.3.2 Solving for Spatially Non-uniform Matter

From this point onwards many assumptions are made to simplify the calculation process for the spatially non-uniform matter approximation. We will briefly discuss a few of these assumptions and observations that are all detailed in ref. [1].

The first assumption that is made, is that system is finite and therefore our scalar, vector and baryon densities have to vanish from a certain radius (r_0) onwards. This enables us to find solutions for V_0'/V_0 and ϕ_0'/ϕ_0 at this finite matter radius, where we would also impose relaxed inner boundary conditions on $V_0'(r)$ and $\phi_0'(r)$, implying $V_0'(0) = \phi_0'(0) = 0$.

Other observations made is that the equations of motion for V_0 and ϕ_0 imply that variations to these fields disappear when we make variations to $k_F(r)$. Given that this minimisation holds, we would then obtain the result

$$g_v V_0 + ({k_F}^2 + {M^*}^2)^{1/2} = \mu, \tag{3.43}$$

where μ is a constant and is used as a Lagrange multiplier in restricting the number of baryons to a fixed number B, where given the total energy E, we would find

$$\delta E - \mu \delta B = 0. \tag{3.44}$$

Using these assumptions and also the equations of motion for the meson fields, one would be able to self-consistently solve for the meson fields, the Fermi-momentum and also the effective mass, both inside and outside of the radius (r_0). While these solutions can only give us information on nuclear matter with an equal number of protons and neutrons, ref. [1] continued with the SNU-approximation to include the rho-meson in order to discuss properties of nuclei with $N \neq Z$.

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CHAPTER 4

Non-commutative Quantum Field Theory

In this chapter we will introduce non-commutative coordinates in quantum field theory and construct non-commutative quantum field theories. In essence, non-commutative quantum field theories are theories constructed out of operator valued fields as the fields are themselves now functions of operator valued coordinates.

However, there are, as we will see, two equivalent ways in which non-commutative theories can be treated. The first is a canonical method of mapping functions of non-commuting coordinate operators into functions of commuting numbers. This map is accompanied by an appropriate non-commutative star product that ensures that operator multiplication is homomorphic under this map. In this approach the Lagrangian of a non-commutative field theory will therefore be a function of ordinary fields composed through the star product rather than the ordinary product as in commutative field theories. Although this is a completely valid description of non-commutative systems, it does complicate the systematic derivation of, for example, the energy-momentum tensor of such theories, mainly due to ordering ambiguities.

The second, equivalent, description involves the formulation of the non-commutative field theory directly as a theory constructed from operator valued fields. This approach offers the advantage of a much more systematic derivation of the equations of motion and energy-momentum tensor.

In what follows we briefly review, for completeness, the first description, but our actual implementation of non-commutative quantum field theories follows from the second description. Here we illustrate this description for the well-known example of the ϕ^4 theory before proceeding to apply this approach to the non-commutative Walecka model in the next chapter.

4.1 Canonical formulation of the Moyal-product

The heart of non-commutative quantum field theory revolves around the assumption that the space-time coordinates do not commute. If we make measurements of multiple coordinates on the same function, different orderings of the measurements will yield different results. We have to construct a rule that tells us exactly how these measurements change and therefore we have to define new commutation relations for the space-time coordinates. One simple way to define

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these commutation relations, in terms of position operators, is given by

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{4.1}$$

where $\theta^{\mu\nu}$ is an anti-symmetric tensor. These commutation relations will give us an uncertainty "area" for the coordinates, much like the Heisenberg uncertainty principle, given by

$$\Delta\hat{x}^\mu \Delta\hat{x}^\nu \geq \frac{\theta^{\mu\nu}}{2}. \tag{4.2}$$

This uncertainty "area" can be translated to a fundamental uncertainty in the position of the particle itself, i.e. it becomes impossible to localise any event on a length scale shorter than that set by $\theta^{\mu\nu}$. This can in turn be interpreted as a particle having some kind of structure where we cannot pin down exactly one position of the particle, but rather that the particle occupies a minimum region in space. It is this argument that will drive us to investigate this non-commutative theory, which implies an effective theory for particles with finite size and structure, further in a field theoretical setting.

To set up a field theory, we must consider functions of the non-commutative coordinates defined in (4.1). Of course these functions will also not commute with each other and it is necessary to investigate more closely the product rule of such functions. In order to do this, it is useful to build the following vocabulary:

- Any function of $\hat{x}^\mu$ will be called "operators" of the coordinates and will be denoted as $\phi(\hat{x})$ or $\hat{\phi}$.
- Any function of the x^μ, which are only numbers, will be called "fields" of the operators and will be denoted as $\phi(x)$ or ϕ, often referred to as the "symbols" of the operators.
- Thus, any function with arguments without the hat symbol will commute.

In order to investigate the product between operators, we will make use of the Fourier transforms for the operators and fields which are given by

$$\phi(\hat{x}) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot\hat{x}} \phi(p) \tag{4.3}$$

and

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \phi(p), \tag{4.4}$$

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where $\phi(p)$ is also just a number. It should be noted that the form in (4.3) will be sensitive to the ordering of coordinates for a non-commutative theory. Depending on the ordering one uses, the value for $\phi(p)$ will differ since different orderings will give different phases due to the commutation relations of the coordinates. Here we view (4.3) as an ordering prescription that we use consistently throughout this thesis.

Also useful for the investigation of the product of operators, are the well known Baker-Campbell-Hausdorff (BCH) formulas

$$\begin{aligned} e^A e^B &= e^{A+B} e^{\frac{1}{2}[A,B]} \\ e^{A+B} &= e^A e^B e^{-\frac{1}{2}[A,B]}. \end{aligned} \tag{4.5}$$

Assuming we have two operators, say $\phi_1(\hat{x})$ and $\phi_2(\hat{x})$ each having the same form as (4.3), we compute the product of these two operators using these BCH-formulas together with our commutation relation given in (4.1):

$$\begin{aligned} \phi_1(\hat{x})\phi_2(\hat{x}) &= \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot\hat{x}}\phi_1(p) \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot\hat{x}}\phi_2(q) \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} e^{ip\cdot\hat{x}} e^{iq\cdot\hat{x}}\phi_1(p)\phi_2(q) \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} e^{i(p+q)\cdot\hat{x}} e^{\frac{1}{2}[ip_\mu\hat{x}^\mu, iq_\nu\hat{x}^\nu]}\phi_1(p)\phi_2(q) \\ &= \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} e^{i(p+q)\cdot\hat{x}} e^{-\frac{i}{2}\theta^{\mu\nu}p_\mu q_\nu}\phi_1(p)\phi_2(q). \end{aligned} \tag{4.6}$$

Relating (4.6) to the product of fields, we see that we have to provide for the extra $e^{\frac{i}{2}\theta^{\mu\nu}p_\mu q_\nu}$ term we acquired which would not be present when taking the product of two fields having the form in (4.4). Thus it is required to change the way we define the product of fields in order to generate the extra term. A product that generates this extra term is the Moyal-product and is given by

$$\star \equiv e^{-\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu}. \tag{4.7}$$

Thus we define the mapping between a product of operators and the product of fields as

$$\phi_1(\hat{x})\phi_2(\hat{x}) \to \phi_1(x) \star \phi_2(x). \tag{4.8}$$

There are a few things that require care. For one, our fields have to be elements of the Schwartz

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class in order for the products to be well defined. Also, even though the fields can be treated as commuting fields, it does not mean that $\phi_1 \star \phi_2 = \phi_2 \star \phi_1$. The Moyal-product is non-commutative by nature and we have to treat fields which are subject to the Moyal-product accordingly.

From here on one would start to introduce the Moyal-product into a field theory. One would proceed to substitute all products between all fields in the Lagrangian with the Moyal-product. After a non-commutative Lagrangian is acquired, one would normally proceed to calculate the equations of motion through the Euler-Lagrange formula, and also the energy-momentum tensor through Noether's theorem. However, the formulas used to calculate the equations of motion and energy-momentum tensor for the commutative case will no longer work for the non-commutative case. Both calculations require that we take derivatives of the Lagrangian which contains derivatives of infinite order in the Moyal-product, which becomes quite problematic and ambiguous.

4.2 The Operator Formulation

In order to resolve the problems and ambiguities mentioned above, we now move to a more fundamental description of non-commutative theories based on an operator treatment. Doing this will ultimately enable us to calculate important quantum field theoretical quantities, such as the equations of motion and the energy-momentum tensor, with a clear, unambiguous and hopefully simple set of calculations. In order to start this derivation we reiterate the commutation relations we set for the coordinate operators:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \tag{4.9}$$

with $\theta^{\mu\nu}$ a rank 4 anti-symmetric tensor. Once again we will assume the planewave expansions for operators and fields as depicted in (4.3) and (4.4). Each of these planewave expansions only give a relationship between $\phi(\hat{x})$ and $\phi(p)$, and $\phi(x)$ and $\phi(p)$, but not between $\phi(\hat{x})$ and $\phi(x)$.

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However, substituting (4.3) into (4.4) results in

$$\begin{aligned}
\phi(\hat{x}) &= \int \frac{d^4p}{(2\pi)^4} \int d^4x \; e^{ip\cdot\hat{x} - ip\cdot x} \phi(x) \\
&= \int d^4x \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot\hat{x} - ip\cdot x} \phi(x) \\
&= \int d^4x \int \frac{d^4p}{(2\pi)^4} \hat{T}(p) e^{-ip\cdot x} \phi(x) \\
&= \int d^4x \; \hat{\Delta}(x)\phi(x), \qquad (4.10)
\end{aligned}$$

where

$$\hat{T}(p) = e^{ip\cdot\hat{x}} \qquad (4.11)$$

resembles a translation operator in momentum space, and

$$\hat{\Delta}(x) = \int \frac{d^4p}{(2\pi)^4} \; \hat{T}(p) e^{-ip\cdot x} \qquad (4.12)$$

is a mapping operator that converts fields into operators after integrations. Before we can investigate the relationship between the products of operators and that of fields, we first need to investigate the properties of the product of translation- and mapping operators and therefor compute the matrix elements of these operators. From there it would also be useful to compute the trace of these operators, since it will be shown later that the trace of our operators $\phi(\hat{x})$ relates to an integral of our fields $\phi(x)$. Also, in order to calculate the trace we should first construct a complete set of eigenstates for the operators. The set of eigenstates in 2 dimensions was constructed in [10], but we would like to do so for 4 dimensions. However, we can define a transformation on our coordinate operators such that

$$\hat{\tilde{x}}^\mu = S^\mu_\lambda \hat{x}^\lambda, \qquad (4.13)$$

where S^μ_λ is an arbitrary rank 2 tensor. Applying this transformation to the commutation relation in (4.9), we get

$$\begin{aligned}
\left[\hat{\tilde{x}}^\mu, \hat{\tilde{x}}^\nu\right] &= \left[S^\mu_\lambda \hat{x}^\lambda, S^\nu_\rho \hat{x}^\rho\right] \\
&= i S^\mu_\lambda S^\nu_\rho \theta^{\lambda\rho} \\
&= i\tilde{\theta}^{\mu\nu}, \qquad (4.14)
\end{aligned}$$

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with $\tilde{\theta}^{\mu\nu} \equiv S^{\mu}_{\lambda} S^{\nu}_{\rho} \theta^{\lambda\rho}$. To investigate the effect of this transformation on the Moyal-product defined in (4.7), we first have to transform our coordinate derivatives as well:

$$\begin{aligned} \partial_\mu &= \frac{\partial}{\partial x^\mu} \\ &= \frac{d\tilde{x}^\lambda}{dx^\mu} \frac{\partial}{\partial \tilde{x}^\lambda} \\ &= S^{\lambda}_{\mu} \frac{\partial}{\partial \tilde{x}^\lambda} \\ &= S^{\lambda}_{\mu} \tilde{\partial}_\lambda, \end{aligned} \tag{4.15}$$

with $\tilde{\partial}_\lambda \equiv \frac{\partial}{\partial \tilde{x}^\lambda}$. The implication is that the Moyal-product transforms under coordinate transformations as:

$$\begin{aligned} e^{-\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} &= e^{-\frac{i}{2}\theta^{\mu\nu} S^{\lambda}_{\mu} S^{\rho}_{\nu} \overleftarrow{\tilde{\partial}}_\lambda \overrightarrow{\tilde{\partial}}_\rho} \\ &= e^{-\frac{i}{2}\tilde{\theta}^{\lambda\rho}\overleftarrow{\tilde{\partial}}_\lambda \overrightarrow{\tilde{\partial}}_\rho}. \end{aligned} \tag{4.16}$$

We can use this fact to choose the transformation tensor S^{μ}_{λ} specifically so that we end up with only two non-commuting pairs of coordinates. This implies that we can construct a $\tilde{\theta}^{\mu\nu}$ so that

$$\widetilde{\theta}^{\mu\nu} = \begin{pmatrix} 0 & \theta_1 & 0 & 0 \\ -\theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_2 \\ 0 & 0 & -\theta_2 & 0 \end{pmatrix}, \tag{4.17}$$

where θ_1 and θ_2 are real numbers which are not necessarily equal to each other. They are also not necessarily equal to any of the values in $\theta^{\mu\nu}$ defined in the original commutation relation in (4.9). This would imply that we only have to work in a 2-dimensional eigenset space instead of our original 4-dimensional system. This leads to the commutation relations:

$$\left[\hat{\tilde{x}}_0, \hat{\tilde{x}}_1\right] = i\theta_1, \tag{4.18}$$

$$\left[\hat{\tilde{x}}_2, \hat{\tilde{x}}_3\right] = i\theta_2. \tag{4.19}$$

Now we can proceed as though we only have a two 2-dimensional system and define the set of eigenstates accordingly. Like [10] we can relate the 4 coordinate operators with those of position and momentum operators. We will define two sets of position and momentum operators as:

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- $\hat{\tilde{x}}_0$ and $\hat{\tilde{x}}_2$ will now represent two commuting position operators: $\hat{x}$ and $\hat{y}$ respectively.
- $\hat{\tilde{x}}_1$ and $\hat{\tilde{x}}_3$ will now represent two commuting momentum operators: $\hat{p}$ and $\hat{q}$ respectively.

We will consider the operators without tildes in order to simplify our notation. We will still assume that these operators operate on the transformed coordinate space.

Since we now only find that $\hat{x}$ does not commute with $\hat{p}$ and $\hat{y}$ with $\hat{q}$, we can, without the loss of generality, say that all the definitions regarding the eigenstates of $\hat{x}$ and $\hat{p}$ also applies to $\hat{y}$ and $\hat{q}$. We now define the eigenstates as

$$\begin{aligned}\hat{x}_0 \,|x\rangle &= \hat{x}\,|x\rangle = x\,|x\rangle \\ \hat{x}_1 \,|p\rangle &= \hat{p}\,|p\rangle = \theta p\,|p\rangle\,, \end{aligned} \tag{4.20}$$

where we write $\theta_1 \equiv \theta$ for simplicity, but again not implying that θ_1 has any relation with values of $\theta^{\mu\nu}$ defined in the original commutation relation in (4.9). It also follows that

$$\left\langle x \,\middle|\, x' \right\rangle = \delta\left(x - x'\right) \tag{4.21}$$

and

$$\int dx\,|x\rangle\langle x\,| = 1, \tag{4.22}$$

while the same applies to $|p\rangle$. Also, the overlap between $|x\rangle$ and $|p\rangle$ is given by

$$\langle x\,|\,p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipx}. \tag{4.23}$$

Now that we have defined our set of eigenstates, we can go on to investigate the properties of our translation operator. The first property we can derive for the translation operator, is the product of two translation operators. Using the BCH-formulas in (4.5) and the commutation relation in (4.9) we get

$$\begin{aligned}\hat{T}(k)\hat{T}(k') &= e^{ik\cdot\hat{x}} e^{ik'\cdot\hat{x}} \\ &= e^{\frac{1}{2}\left[ik_\mu \hat{x}^\mu, ik'_\nu \hat{x}^\nu\right]} e^{i(k+k')\cdot\hat{x}} \\ &= e^{-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu}\hat{T}\left(k+k'\right), \end{aligned} \tag{4.24}$$

which agrees with the fact that translation operators should also be non-commutative. Furthermore, we can also derive an expression for the matrix elements of the translation operator, again

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using the BCH-formulas (4.5) and commutation relations in (4.9) and (4.18):

$$\begin{aligned}
\left\langle x \middle| \hat{T}(k) \middle| x' \right\rangle &= \left\langle x \middle| e^{ik\cdot\hat{x}} \middle| x' \right\rangle \\
&= \int dp \left\langle x \middle| p \right\rangle \left\langle p \middle| e^{ik\cdot\hat{x}} \middle| x' \right\rangle \\
&= \int \frac{dp}{\sqrt{2\pi}}\ e^{ipx} \left\langle p \middle| e^{ik_0\cdot\hat{x}_0 + ik_1\cdot\hat{x}_1} \middle| x' \right\rangle \\
&= \int \frac{dp}{\sqrt{2\pi}}\ e^{ipx} \left\langle p \middle| e^{ik_0\cdot\hat{x}_0} e^{ik_1\cdot\hat{x}_1} e^{-\frac{1}{2}[ik_0\hat{x}_0, ik_1\hat{x}_1]} \middle| x' \right\rangle \\
&= \int \frac{dp}{\sqrt{2\pi}}\ e^{ipx} e^{ik_0 x'} e^{ik_1\theta p} e^{\frac{i}{2}k_0 k_1\theta} \left\langle p \middle| x' \right\rangle \\
&= \int \frac{dp}{2\pi}\ e^{ip(k_1\theta + x - x')} e^{\frac{i}{2}k_0 k_1 \theta + ik_0 x'} \\
&= \delta\left(k_1\theta + x - x'\right) e^{\frac{i}{2}k_0(x+x')}.
\end{aligned} \tag{4.25}$$

Using (4.25), we can now compute the trace of the translation operator with a normalization factor given by [10]:

$$\begin{aligned}
\mathrm{Tr}\left(\hat{T}(k)\right) &= 2\pi\theta \int dx\ \left\langle x \middle| \hat{T}(k) \middle| x \right\rangle \\
&= 2\pi\theta \int dx\ \delta\left(k_1\theta + x - x\right) e^{\frac{i}{2}k_0(x+x)} \\
&= 2\pi\theta\delta\left(k_1\theta\right) \int dx\ e^{ik_0 x} \\
&= \left(2\pi\right)^2 \delta\left(k_1\right)\delta\left(k_0\right) \\
&= \left(2\pi\right)^2 \delta\left(k\right).
\end{aligned} \tag{4.26}$$

Therefore, using the anti-symmetric property of $\widetilde{\theta}^{\mu\nu}$ and eqs. (4.24) and (4.26), we get that the trace of the product of translation operators is given by

$$\mathrm{Tr}\left(\hat{T}(k)\hat{T}(k')\right) = \left(2\pi\right)^2 \delta\left(k + k'\right). \tag{4.27}$$

This gives us enough information on the properties of the translation operator for the moment, so we can continue to investigate the properties of $\hat{\Delta}(x)$ defined in (4.12). Like the translation

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operator, we would first like to investigate the matrix elements for $\hat{\Delta}(x)$:

$$\begin{aligned}
\left\langle x' \middle| \hat{\Delta}(x) \middle| x'' \right\rangle &= \int \frac{d^2k}{(2\pi)^2} \left\langle x' \middle| \hat{T}(k) \middle| x'' \right\rangle e^{-ik\cdot x} \\
&= \int \frac{d^2k}{(2\pi)^2} \delta\left(k_1\theta + x' - x''\right) e^{\frac{i}{2}k_0(x'+x'')} e^{-ik\cdot x} \\
&= \frac{1}{2\pi\theta} e^{\frac{i}{\theta}(x'-x'')x_1} \int \frac{dk_0}{2\pi} e^{\frac{i}{2}k_0(x'+x'')-ik_0x_0} \\
&= \frac{1}{2\pi\theta} e^{\frac{i}{\theta}(x'-x'')x_1} \delta\left(x_0 - \frac{x'+x''}{2}\right).
\end{aligned} \tag{4.28}$$

Using this, we now compute for the trace of $\hat{\Delta}(x)$:

$$\begin{aligned}
\mathrm{Tr}\left(\hat{\Delta}(x)\right) &= 2\pi\theta \int dx \left\langle x \middle| \hat{\Delta}(x) \middle| x \right\rangle \\
&= \int dx\, \delta\left(x_0 - \frac{x+x}{2}\right) e^{\frac{i}{\theta}(x-x)x_1} \\
&= \int dx\, \delta\left(x_0 - x\right) \\
&= 1,
\end{aligned} \tag{4.29}$$

which implies that the trace of $\hat{\Delta}(x)$ is independent of x. Another property for $\hat{\Delta}(x)$ we get using the result for a product of translation operators in (4.27) is:

$$\begin{aligned}
\mathrm{Tr}\left(\hat{\Delta}(x)\hat{\Delta}(x')\right) &= \int \frac{d^2k}{(2\pi)^2}\frac{d^2k'}{(2\pi)^2} \mathrm{Tr}\left(\hat{T}(k)\hat{T}(k')\right) e^{-ik\cdot x - ik'\cdot x'} \\
&= \int \frac{d^2k}{(2\pi)^2}\frac{d^2k'}{(2\pi)^2} (2\pi)^2\delta\left(k+k'\right) e^{-ik\cdot x - ik'\cdot x'} \\
&= \int \frac{d^2k}{(2\pi)^2} e^{-ik\cdot(x-x')} \\
&= \delta\left(x - x'\right).
\end{aligned} \tag{4.30}$$

We now have a sufficient number of properties for $\hat{T}(k)$ and $\hat{\Delta}(x)$ for calculations to follow. We can now move forward and construct derivatives on the operator space and further investigate a possible mapping between operators and their fields. First, however, we have to revert back from a 2-dimensional system, used in calculations (4.24) to (4.30), to a 4-dimensional system required for a quantum field theory in Minkowski-space. We mentioned before that we can easily extend this to 4 dimensions through a basic transformation of our coordinate operators. We will now use all derived properties for the trace of the operators in 4 dimensions since our fields in the upcoming chapter live in the 4-dimensional Minkowski-space.

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Additional to the two operators we have just defined, we also have to define a description for the derivative on the operators which we will denote as $\hat{\partial}_\mu$. This would require that

$$\hat{\partial}_\mu \hat{x}^\nu = \delta_\mu^\nu \tag{4.31}$$

and

$$\left[\hat{\partial}_\mu, \hat{\partial}_\nu\right] = 0. \tag{4.32}$$

Assuming that the inverse for $\theta^{\mu\nu}$ exists, which we will define as $(\theta^{-1})_{\mu\nu}$ with $(\theta^{-1})_{\mu\rho}\theta^{\rho\nu} = \delta_\mu^\nu$, we can construct the derivative operator as:

$$\begin{aligned}
\hat{\partial}_\mu \hat{x}^\nu &\equiv -i\left[\hat{x}'_\mu \ , \ \hat{x}^\nu\right] \\
&= -i\left[(\theta^{-1})_{\mu\rho}\hat{x}^\rho \ , \ \hat{x}^\nu\right] \\
&= -i(\theta^{-1})_{\mu\rho}\left[\hat{x}^\rho, \hat{x}^\nu\right] \\
&= (\theta^{-1})_{\mu\rho}\theta^{\rho\nu} = \delta_\mu^\nu,
\end{aligned} \tag{4.33}$$

with $\hat{x}'_\mu \equiv (\theta^{-1})_{\mu\rho}\hat{x}^\rho$. This implies that for any function of the operator $\hat{x}$, we can write:

$$\hat{\partial}_\mu f(\hat{x}) = -i\left[\hat{x}'_\mu, f(\hat{x})\right]. \tag{4.34}$$

From (4.33) and (4.34) the Leibniz rule also follows:

$$\begin{aligned}
\hat{\partial}_\mu \left(f(\hat{x})g(\hat{x})\right) &= -i\left[\hat{x}'_\mu \ , \ f(\hat{x})g(\hat{x})\right] \\
&= -i\left[\hat{x}'_\mu \ , \ f(\hat{x})\right]g(\hat{x}) - if(\hat{x})\left[\hat{x}'_\mu \ , \ g(\hat{x})\right] \\
&= \left(\hat{\partial}_\mu f(\hat{x})\right)g(\hat{x}) + f(\hat{x})\left(\hat{\partial}_\mu g(\hat{x})\right).
\end{aligned} \tag{4.35}$$

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We can also derive an expression relating to partial integration using the cyclic properties of the trace operator:

$$\begin{aligned}
\mathrm{Tr}\left(\phi_1(\hat{x})\hat{\partial}_\mu\phi_2(\hat{x})\right) &= \mathrm{Tr}\left(-i\phi_1(\hat{x})\left[\hat{x}'_\mu,\phi_2(\hat{x})\right]\right)\\
&= \mathrm{Tr}\left(-i\phi_1(\hat{x})\hat{x}'_\mu\phi_2(\hat{x}) + i\phi_1(\hat{x})\phi_2(\hat{x})\hat{x}'_\mu\right)\\
&= \mathrm{Tr}\left(-i\phi_1(\hat{x})\hat{x}'_\mu\phi_2(\hat{x}) + i\hat{x}'_\mu\phi_1(\hat{x})\phi_2(\hat{x})\right)\\
&= \mathrm{Tr}\left(+i\left[\hat{x}'_\mu,\phi_1(\hat{x})\right]\phi_2(\hat{x})\right)\\
&= \mathrm{Tr}\left(-\hat{\partial}_\mu\phi_1(\hat{x})\phi_2(\hat{x})\right),
\end{aligned} \tag{4.36}$$

which we can now use, together with the commutation relation in (4.9), to get an expression for the derivative of $\hat{\Delta}(x)$:

$$\begin{aligned}
\hat{\partial}_\mu\hat{\Delta}(x) &= \int \frac{d^4k}{(2\pi)^4}\ \left(\hat{\partial}_\mu e^{ik\cdot\hat{x}}\right)e^{-ik\cdot x}\\
&= -i(\theta^{-1})_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\ \left[\hat{x}^\nu, e^{ik\cdot\hat{x}}\right]e^{-ik\cdot x}\\
&= i(\theta^{-1})_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\ \left[\hat{x}^\nu, ik_\rho\hat{x}^\rho\right]e^{ik\cdot\hat{x}}e^{-ik\cdot x}\\
&= i(\theta^{-1})_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\ ik_\rho\theta^{\nu\rho}e^{ik\cdot\hat{x}-ik\cdot x}\\
&= -\int \frac{d^4k}{(2\pi)^4}\ k_\mu e^{ik\cdot\hat{x}-ik\cdot x}\\
&= -\partial_\mu\hat{\Delta}(x).
\end{aligned} \tag{4.37}$$

This would now allow us to define the derivative of our field operators:

$$\begin{aligned}
\hat{\partial}_\mu\phi(\hat{x}) &= \int d^4x\ \hat{\partial}_\mu\left(\phi(x)\hat{\Delta}(x)\right)\\
&= \int d^4x\ \phi(x)\left(\hat{\partial}_\mu\hat{\Delta}(x)\right)\\
&= \int d^4x\ \phi(x)\left(-\partial_\mu\hat{\Delta}(x)\right)\\
&= \int d^4x\ \left(\partial_\mu\phi(x)\right)\hat{\Delta}(x).
\end{aligned} \tag{4.38}$$

Eq. (4.37) implies that we can construct translation operators by $e^{v^\mu\hat{\partial}_\mu}$, with $v^\mu \in \mathbb{R}$, which satisfy:

$$e^{v^\mu\hat{\partial}_\mu}\hat{\Delta}(x)e^{-v^\mu\hat{\partial}_\mu} = \hat{\Delta}(x+v), \tag{4.39}$$

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which in turn proves (4.29) in that the trace of the mapping operator should be independent of the position argument [10, 11]. Therefore, we can now compute the trace of our fields as operators using the form for operators in (4.10) and identity in (4.29):

$$\begin{aligned}\mathrm{Tr}\left(\phi(\hat{x})\right) &= \int d^4x\ \mathrm{Tr}\left(\hat{\Delta}(x)\right)\phi(x) \\ &= \int d^4x\ \phi(x). \end{aligned} \tag{4.40}$$

Also, using the result for the product of mapping operators in (4.30) and some clever thinking, we see that we can define an inverse map to (4.10):

$$\begin{aligned}\mathrm{Tr}\left(\phi(\hat{x})\hat{\Delta}(x)\right) &= \int d^4x'\ \phi(x')\mathrm{Tr}\left(\hat{\Delta}(x')\hat{\Delta}(x)\right) \\ &= \int d^4x'\ \phi(x')\delta\left(x'-x\right) \\ &= \phi(x). \end{aligned} \tag{4.41}$$

If one would now consider $\phi_3(\hat{x}) = \phi_1(\hat{x})\phi_2(\hat{x})$, assuming all $\phi_i(\hat{x})$ has the form in (4.10), we can show after a lengthy calculation, outlined in (A.1) and (A.2), that we get

$$\phi_3(x) = \mathrm{Tr}\left(\phi_1(\hat{x})\phi_2(\hat{x})\hat{\Delta}(x)\right) = \phi_1(x) \star \phi_2(x), \tag{4.42}$$

from which follows that

$$\mathrm{Tr}\left(\phi_3(\hat{x})\right) = \mathrm{Tr}\left(\phi_1(\hat{x})\phi_2(\hat{x})\right) = \int d^4x\ \phi_1(x) \star \phi_2(x). \tag{4.43}$$

Therefore we have for any number of field operators, $\phi_i(\hat{x})$, that

$$\mathrm{Tr}\left(\phi_1(\hat{x})\phi_2(\hat{x})\cdots\phi_n(\hat{x})\right) = \int d^4x\ \phi_1(x) \star \phi_2(x) \star \cdots \star \phi_n(x), \tag{4.44}$$

from which we can see that the Moyal-product has to be re-introduced into our mapping. We have now completed our operator formulation for fields. We have also shown that there is a one-to-one correspondence between the fields and their respective operators. A few, but very important, properties of our mapping should be noted. The first important property we get using the cyclic properties of the trace operator is:

$$\int d^4x\ \phi_1(x) \star \phi_2(x) \star \cdots \star \phi_{n-1}(x) \star \phi_n(x) = \int d^4x\ \phi_n(x) \star \phi_1(x) \star \cdots \star \phi_{n-1}(x). \tag{4.45}$$

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Another very important and useful property can be shown using partial integration, the definition of the Moyal-product in (4.7) and the anti-symmetric property of $\theta^{\mu\nu}$:

$$\begin{aligned}\int d^4x\ \phi_1(x) \star \phi_2(x) &= \int d^4x\ \phi_1(x) \left[\sum_n^{\infty} \frac{1}{n!}\left(-\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu\right)^n\right]\phi_2(x) \\ &= \int d^4x\ \phi_1(x) \left[\sum_n^{\infty} \frac{1}{n!}\left(+\frac{i}{2}\theta^{\mu\nu}\overrightarrow{\partial}_\mu\overrightarrow{\partial}_\nu\right)^n\right]\phi_2(x) \\ &= \int d^4x\ \phi_1(x)\phi_2(x). \end{aligned} \tag{4.46}$$

A third property that will play an important role when deriving the energy-momentum tensor is the following:

$$\mathrm{Tr}\left(\hat{\partial}_\mu \phi(\hat{x})\right) = 0. \tag{4.47}$$

This follows directly from (4.34) from which it is seen that the derivative on operator level is a commutator, which vanishes under a trace due to the cyclic properties of the trace. It should, however, be noted that this only holds if the operators $\phi(\hat{x})$ are of trace class and specifically of Hilbert-Schmidt type [14]. On the level of fields this translates into the fields being of Schwartz class.

Furthermore, one should also always remember that the Moyal-product is associative and distributive, but not commutative.

4.3 The Energy-momentum Tensor on the Operator Level

Now that we have a map between fields and operators, we can continue to discuss the derivation of the equations of motion and energy-momentum tensor for non-commutative fields theories. For this purpose we use the example of the scalar ϕ^4-theory extensively studied in [10, 11, 12, 13]. The commutative action for the scalar ϕ^4-theory is given by

$$S[\phi(x)] = \int d^4x\ \left[\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \frac{m^2}{2}\phi^2(x) + \frac{\lambda}{4!}\phi^4(x)\right]. \tag{4.48}$$

However, since we are working in a non-commutative field theory, we should transform our action to the non-commutative counter-part. We make the transformation by replacing all the fields in the action with their respective operators as denoted in (4.10) and replacing the integral with a

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trace. Also we relabel the non-commutative action of operators as $\overline{S}$, thus the transformation is

$$S\left[\phi_1, \phi_2, \ldots\right] \rightarrow \overline{S}\left[\hat{\phi}_1, \hat{\phi}_2, \ldots\right], \tag{4.49}$$

where the non-commutative action for the ϕ^4 field theory is [10]

$$\overline{S}\left[\phi(\hat{x})\right] = \mathrm{Tr}\left(\frac{1}{2}\hat{\partial}_\mu \phi(\hat{x})\hat{\partial}^\mu \phi(\hat{x}) + \frac{m^2}{2}\phi^2(\hat{x}) + \frac{\lambda}{4!}\phi^4(\hat{x})\right). \tag{4.50}$$

Mapping to fields the equivalent action is:

$$S\left[\phi(x)\right] = \int d^4x \left[\frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) + \frac{m^2}{2}\phi^2(x) + \frac{\lambda}{4!}\phi(x)\star\phi(x)\star\phi(x)\star\phi(x)\right]. \tag{4.51}$$

The equations of motion follow when minimizing the action with respect to the operator fields. Thus

$$\overline{S}\left[\hat{\phi}_1 + \delta\hat{\phi}_1, \hat{\phi}_2 + \delta\hat{\phi}_2, \ldots\right] - \overline{S}\left[\hat{\phi}_1, \hat{\phi}_2, \ldots\right] = 0. \tag{4.52}$$

Applying (4.36), (4.52) and the cyclic properties of the trace operator to the scalar theory we get

$$\begin{aligned}\overline{S}\left[\hat{\phi} + \delta\hat{\phi}\right] - \overline{S}\left[\hat{\phi}\right] &= \mathrm{Tr}\left((\hat{\partial}_\mu \delta\hat{\phi})(\hat{\partial}^\mu \hat{\phi}) + m^2(\delta\hat{\phi})\hat{\phi} + \frac{\lambda}{3!}\hat{(}\delta\hat{\phi})\phi^3\right) \\ &= \mathrm{Tr}\left(-(\delta\hat{\phi})(\hat{\partial}_\mu\hat{\partial}^\mu \hat{\phi}) + m^2(\delta\hat{\phi})\hat{\phi} + \frac{\lambda}{3!}(\delta\hat{\phi})\hat{\phi}^3\right) = 0.\end{aligned} \tag{4.53}$$

which should be equal to 0 for any arbitrary operator $\delta\hat{\phi}$. Since we can reinterpret this as an inner product $(\delta\hat{\phi}, \hat{\partial}^\mu\hat{\partial}_\mu\hat{\phi} - m^2\hat{\phi} - \frac{\lambda}{3!}\hat{\phi}^3)$ in the Hilbert-space of Hilbert-Schmidt operators, and since $\delta\hat{\phi}$ is an arbitrary operator, we conclude that

$$\hat{\partial}_\mu\hat{\partial}^\mu\hat{\phi} - m^2\hat{\phi} - \frac{\lambda}{3!}\hat{\phi}^3 = 0, \tag{4.54}$$

while as fields it reads as

$$\partial_\mu\partial^\mu\phi - m^2\phi - \frac{\lambda}{3!}\phi\star\phi\star\phi = 0. \tag{4.55}$$

Now utilising the equations of motion, we can go further and construct a way of calculating the energy-momentum tensor. The first step is to note that the action (4.50) is translation invariant. This follows simply as translations $\phi(\hat{x}^\mu) \rightarrow \phi(\hat{x}^\mu + \epsilon^\mu)$ are implemented through a unitary transformation that can be explicitly constructed from the commutation relations (4.1), but of

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which the explicit form is irrelevant for our purposes. The mere existence of this transformation implies that (4.50) is invariant under translations as the operators appear under a trace. The same conclusion can be reached from (4.51) as the Moyal-product is invariant under translations $x^\mu \to x^\mu + \epsilon^\mu$ and we integrate over all values of x^μ, thus the action is invariant under translations.

From the above we conclude that the change in the Lagrangian under such a transformation must be a total commutator or derivative as in (4.47), i.e.

$$\delta\overline{\mathcal{L}} = \epsilon^\nu \hat{\partial}_\nu \overline{\mathcal{L}}. \tag{4.56}$$

On the other hand, we can compute the same variation in $\overline{\mathcal{L}}$ from the variation in the operator valued fields under a translation, i.e.

$$\delta\overline{\mathcal{L}} = \epsilon^\nu \hat{\partial}_\mu \hat{\mathcal{J}}^\mu_\nu. \tag{4.57}$$

As in the commutative case, $\hat{\mathcal{J}}^\mu_\nu$ needs to be calculated from the variation in the operator valued fields $\delta\hat{\phi}_i = \epsilon^\nu \hat{\partial}_\nu \hat{\phi}_i$, the equations of motion and partial integrations to bring the variation into the form of a total derivative. The commutative analogue of $\hat{\mathcal{J}}^\mu_\nu$ is given by the first term in (2.33).

As the variations of (4.56) and (4.57) are identical and ϵ^ν arbitrary, we conclude:

$$\hat{\partial}_\mu \left[\hat{\mathcal{J}}^\mu_\nu - \delta^\mu_\nu \overline{\mathcal{L}} \right] \equiv \hat{\partial}_\mu \hat{T}^\mu_\nu = 0, \tag{4.58}$$

which expresses the fact that $\hat{T}^\mu_\nu$ is a conserved quantity, which we identify with the non-commutative energy-momentum tensor. Eq. (4.58) can also be written in the more standard form with the energy-momentum tensor having lower indices only:

$$\hat{\partial}^\mu \left[\hat{\mathcal{J}}_{\mu\nu} - g_{\mu\nu} \overline{\mathcal{L}} \right] \equiv \hat{\partial}^\mu \hat{T}_{\mu\nu} = 0. \tag{4.59}$$

From the variation in the field operators and the equation of motion (4.54), we get for the

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variation of the action:

$$\begin{aligned}
\delta\overline{S} &= \mathrm{Tr}\left(\epsilon^\nu\left[\frac{1}{2}\hat{\partial}_\mu(\hat{\partial}_\nu\hat{\phi})\hat{\partial}^\mu\hat{\phi} + \frac{1}{2}\hat{\partial}_\mu\hat{\phi}\hat{\partial}^\mu(\hat{\partial}_\nu\hat{\phi}) + m^2\hat{\phi}\hat{\partial}_\nu\hat{\phi} + \frac{\lambda}{3!}\hat{\phi}^3\hat{\partial}_\nu\hat{\phi}\right]\right) \\
&= \mathrm{Tr}\left(\frac{1}{2}\epsilon^\nu\hat{\partial}^\mu(\hat{\partial}_\nu\hat{\phi}\hat{\partial}_\mu\hat{\phi} + \hat{\partial}_\mu\hat{\phi}\hat{\partial}_\nu\hat{\phi}) - (\hat{\partial}_\nu\hat{\phi})(-\hat{\partial}_\mu\hat{\partial}^\mu\hat{\phi} + m^2\hat{\phi} + \frac{\lambda}{3!}\hat{\phi}^3)\right) \\
&= \mathrm{Tr}\left(\frac{1}{2}\epsilon^\nu\hat{\partial}^\mu(\hat{\partial}_\nu\hat{\phi}\hat{\partial}_\mu\hat{\phi} + \hat{\partial}_\mu\hat{\phi}\hat{\partial}_\nu\hat{\phi})\right). \qquad (4.60)
\end{aligned}$$

From this we identify

$$\hat{\mathcal{J}}_{\mu\nu} = \frac{1}{2}(\hat{\partial}_\nu\hat{\phi}\hat{\partial}_\mu\hat{\phi} + \hat{\partial}_\mu\hat{\phi}\hat{\partial}_\nu\hat{\phi}), \qquad (4.61)$$

and thus the non-commutative energy-momentum tensor for the ϕ^4 theory results in

$$\hat{T}_{\mu\nu} = \frac{1}{2}(\hat{\partial}_\nu\hat{\phi}\hat{\partial}_\mu\hat{\phi} + \hat{\partial}_\mu\hat{\phi}\hat{\partial}_\nu\hat{\phi}) - g_{\mu\nu}(\frac{1}{2}\hat{\partial}_\rho\hat{\phi}\hat{\partial}^\rho\hat{\phi} + \frac{m^2}{2}\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4). \qquad (4.62)$$

In terms of fields the non-commutative energy-momentum tensor is given by

$$T_{\mu\nu} = \frac{1}{2}(\partial_\nu\phi\star\partial_\mu\phi + \partial_\mu\phi\star\partial_\nu\phi) - g_{\mu\nu}(\frac{1}{2}\partial_\rho\phi\star\partial^\rho\phi + \frac{m^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi), \qquad (4.63)$$

giving us a convenient symmetrised tensor.

4.4 Remarks

We have now completed the construction of a non-commutative quantum field theory which is adequate to our needs. It is important to state again that even though we have constructed a non-commutative quantum field theory generally for 4-dimensions, we will require from this point forward that $\theta_{0\mu} = 0$ for $\mu = 0, 1, 2, 3$. This implies that all coordinates commute with the time component in the Minkowski-space.

We have also not introduced twisting in order to repair the apparently broken Lorentz-invariance [15]. However, this is not extremely important to us since we are only constructing an effective theory and will not affect the aims of this thesis. It should be noted that even though a non-commutative quantum field theory breaks Lorentz-invariance, it does, however, still retain translational invariance, which we have used to construct the formalism of the non-commutative energy-momentum tensor.

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CHAPTER 5

The Non-commutative Walecka Model

Now that we have completed the construction of a non-commutative field theory, one wonders if the Walecka model discussed in Chapter 3 could also be converted to a non-commutative Walecka model, and what would be the motivation to do so. Would there be any physical interpretation one could give to the non-commutative nuclear interaction model that is in any way new or different than the commutative model?

5.1 Motivation

To start off the motivation for a non-commutative Walecka model, we will first look at the properties of the commutative Walecka model and what it lacks when describing nuclear interactions.

Firstly, it is well known in the scientific community that the nucleons within the atomic nucleus have finite size, and the description thereof is not present in the Walecka model. The presence of finite nucleon size should affect the nuclear matter equation of state, however, we would expect that the effect will only be clearly visible at very high baryon densities. This may impact the descriptions of astro-physical objects, e.g. neutron stars.

With a non-commutative Walecka model we aim to mimic the finite size of nucleons. We do this by introducing non-commutative coordinates for our nucleons as we did through eq. (4.1). Non-commutative coordinates creates an uncertainty in the position of the nucleons in space and can be seen as a way to mimic the nucleon finite size. Indeed, if we would assume that all coordinates commute with the time-coordinate, i.e. $[x_0, x_\mu] = 0$, the non-commutative parameter θ provides us with a length scale, which we shall identify with the size of a nucleon, i.e. of the order of $\sim 1fm$.

In the simplest setting one can fix this length scale and study how it may affect the nucleon matter equation of state. On the other hand, from our more fundamental QCD based understanding of baryons and their interactions, we expect the length scale to be linked to dynamics. A more sophisticated approach would therefore be to treat the length scale as a variable in the theory, the value of which is determined by dynamical considerations.

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One way to investigate this is by studying the ground-state properties of nuclear matter as a function of the non-commutative scale, i.e., by minimising the energy as a function of θ at a fixed baryon density. Indeed, whenever the nucleons interact through a short ranged repulsive interaction, caused by the vector meson in the Walecka model, one may expect that the model may favour a larger θ as the nucleons will effectively be kept further apart due to their fermionic nature, thus reducing the repulsive interaction energy. On the other hand, the nucleons would not move too far apart as there will then be a penalty in terms of the long range attractive interaction energy, caused by the scalar meson in the Walecka model. Based on this rather naive argument, one therefore expects an interplay between the non-commutative scale and interactions.

This motivates us to introduce a non-commutative version of the Walecka model which should capture some aspects of the nuclear matter equation of state's dependence on the nucleon size, which was not present in the original Walecka model.

It should be noted that we will, at first, not be making any simplifications to our model such as the Uniform Matter or Spatially Non-Uniform Matter approximations discussed in Sections 3.2 and 3.3. We would prefer to keep our model "assumption free" until the theory for the non-commutative Walecka model is complete. However, the results of these simplifications will be discussed at the end of this chapter.

5.2 The Non-commutative Walecka Action and Equations of Motion

Using the Lagrangian of the Walecka model in eq. (3.1), we now wish to alter it into a non-commutative Lagrangian using the process as discussed in Chapter 4. To reiterate, this involves changing position coordinates to coordinate operators in all of our fields, while we also replace all integrals with a trace. The resulting non-commutative Lagrangian of operators for the Walecka model is

$$\begin{aligned}\overline{\mathcal{L}} = \hat{\overline{\psi}} \left[\gamma^\mu \left(i\hat{\partial}_\mu - g_v\hat{V}_\mu\right) - \left(M - g_s\hat{\phi}\right)\right]\hat{\psi} + \frac{1}{2}\left(\hat{\partial}^\mu\hat{\phi}\hat{\partial}_\mu\hat{\phi} - {m_s}^2\hat{\phi}^2\right) \\ - \frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} + \frac{1}{2}{m_v}^2\hat{V}^\mu\hat{V}_\mu,\end{aligned} \tag{5.1}$$

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which, in turn, provides us with the non-commutative action of operators given by

$$\overline{S} = \operatorname{Tr}\left(\hat{\overline{\psi}}\left[\gamma^\mu\left(i\hat{\partial}_\mu - g_v\hat{V}_\mu\right) - \left(M - g_s\hat{\phi}\right)\right]\hat{\psi} + \frac{1}{2}\left(\hat{\partial}^\mu\hat{\phi}\hat{\partial}_\mu\hat{\phi} - {m_s}^2\hat{\phi}^2\right)\right.$$
$$\left. -\frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} + \frac{1}{2}{m_v}^2\hat{V}^\mu\hat{V}_\mu\right). \tag{5.2}$$

Mapping (5.2), which is an action of operators, to an action of fields, gives

$$S = \int d^4x \left\{\overline{\psi}\left(i\gamma^\mu\partial_\mu - M\right)\psi - \overline{\psi}\star\left[g_v\gamma^\mu V_\mu - g_s\phi\right]\star\psi + \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi - {m_s}^2\phi^2\right)\right.$$
$$\left. -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}{m_v}^2 V^\mu V_\mu\right\}, \tag{5.3}$$

where we have used (4.46) in order to simplify the terms quadratic in the fields.

When we compare our non-commutative action of fields in (5.3) to the action of the commutative Walecka model in (3.2), we see that the only difference is in the interaction part. This already hints towards the interactions depending on, or at least knowing of, the particle structure and the finite size of the nucleons.

Following the process of constructing a quantum field theory, we now proceed to vary the action with respect to the individual operators. This will provide us with the non-commutative equations of motion. For the baryon operators $\hat{\psi}$, we get:

$$\overline{S}\left[\hat{\overline{\psi}} + \delta\hat{\overline{\psi}}\right] - \overline{S}\left[\hat{\overline{\psi}}\right] = \operatorname{Tr}\left(\delta\hat{\overline{\psi}}\left[\gamma^\mu\left(i\hat{\partial}_\mu - g_v\hat{V}_\mu\right) - \left(M - g_s\hat{\phi}\right)\right]\hat{\psi}\right) = 0, \tag{5.4}$$

giving

$$\left[\gamma^\mu\left(i\hat{\partial}_\mu - g_v\hat{V}_\mu\right) - \left(M - g_s\hat{\phi}\right)\right]\hat{\psi} = 0. \tag{5.5}$$

For the anti-baryons $\hat{\overline{\psi}}$, we can simply take the Hermitian conjugate of (5.5), giving

$$\hat{\overline{\psi}}\left[\gamma^\mu\left(i\overleftarrow{\hat{\partial}}_\mu + g_v\hat{V}_\mu\right) + \left(M - g_s\hat{\phi}\right)\right] = 0. \tag{5.6}$$

For the scalar operators $\hat{\phi}$, we get using the cyclic properties of the trace and partial integration

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illustrated in eq. (4.36):

$$\begin{aligned}\overline{S}\left[\hat{\phi}+\delta\hat{\phi}\right]-\overline{S}\left[\hat{\phi}\right] &= \mathrm{Tr}\left(g_s\hat{\overline{\psi}}(\delta\hat{\phi})\hat{\psi}+\frac{1}{2}\left[\hat{\partial}^{\mu}(\delta\hat{\phi})\hat{\partial}_{\mu}\hat{\phi}+\hat{\partial}^{\mu}\hat{\phi}\hat{\partial}_{\mu}(\delta\hat{\phi})\right]-{m_s}^2(\delta\hat{\phi})\hat{\phi}\right)\\ &= \mathrm{Tr}\left(g_s(\delta\hat{\phi})\hat{\psi}\hat{\overline{\psi}}-(\delta\hat{\phi})\hat{\partial}^{\mu}\hat{\partial}_{\mu}\hat{\phi}-{m_s}^2(\delta\hat{\phi})\hat{\phi}\right)\\ &= 0, \end{aligned} \tag{5.7}$$

giving

$$\hat{\partial}^{\mu}\hat{\partial}_{\mu}\hat{\phi}+{m_s}^2\hat{\phi}=g_s\hat{\psi}\hat{\overline{\psi}}. \tag{5.8}$$

For the scalar operators $\hat{V}^{\mu}$, using the anti-symmetric property of $\hat{F}^{\mu\nu}$ and again using the cyclic properties of the trace and partial integration illustrated in eq. (4.36), we get:

$$\begin{aligned}\overline{S}\left[\hat{V}^{\mu}+\delta\hat{V}^{\mu}\right]-\overline{S}\left[\hat{V}^{\mu}\right] &= \mathrm{Tr}\Big(-g_v\hat{\overline{\psi}}\gamma^{\nu}(\delta\hat{V}_{\nu})\hat{\psi}-\frac{1}{4}\hat{F}^{\mu\nu}\left[\hat{\partial}_{\mu}(\delta\hat{V}_{\nu})-\hat{\partial}_{\nu}(\delta\hat{V}_{\mu})\right]\\ &\qquad -\frac{1}{4}\left[\hat{\partial}^{\mu}(\delta\hat{V}^{\nu})-\hat{\partial}^{\nu}(\delta\hat{V}^{\mu})\right]\hat{F}_{\mu\nu}+{m_v}^2\hat{V}^{\nu}(\delta\hat{V}_{\nu})\Big)\\ &= \mathrm{Tr}\left(-g_v\hat{\psi}\hat{\overline{\psi}}\gamma^{\nu}(\delta\hat{V}_{\nu})-\hat{F}^{\mu\nu}\hat{\partial}_{\mu}(\delta\hat{V}_{\nu})+{m_v}^2\hat{V}^{\nu}(\delta\hat{V}_{\nu})\right)\\ &= \mathrm{Tr}\left(-g_v\hat{\psi}\hat{\overline{\psi}}\gamma^{\nu}(\delta\hat{V}_{\nu})+(\hat{\partial}_{\mu}\hat{F}^{\mu\nu})\delta\hat{V}_{\nu}+{m_v}^2\hat{V}^{\nu}(\delta\hat{V}_{\nu})\right)\\ &= 0, \end{aligned} \tag{5.9}$$

giving

$$\hat{\partial}_{\mu}\hat{F}^{\mu\nu}+{m_v}^2\hat{V}^{\nu}=g_v\hat{\psi}\hat{\overline{\psi}}\gamma^{\nu}. \tag{5.10}$$

To summarise, the equations of motion for the non-commutative Walecka model are given by

$$\left[\gamma^{\mu}\left(i\hat{\partial}_{\mu}-g_v\hat{V}_{\mu}\right)-\left(M-g_s\hat{\phi}\right)\right]\hat{\psi}=0,$$

$$\hat{\overline{\psi}}\left[\gamma^{\mu}\left(i\overleftarrow{\hat{\partial}}_{\mu}+g_v\hat{V}_{\mu}\right)+\left(M-g_s\hat{\phi}\right)\right]=0,$$

$$\hat{\partial}^{\mu}\hat{\partial}_{\mu}\hat{\phi}+{m_s}^2\hat{\phi}=g_s\hat{\psi}\hat{\overline{\psi}},$$

and

$$\hat{\partial}_{\mu}\hat{F}^{\mu\nu}+{m_v}^2\hat{V}^{\nu}=g_v\hat{\psi}\hat{\overline{\psi}}\gamma^{\nu}. \tag{5.11}$$

When comparing the non-commutative equations of motion to the original commutative equations of motion, we see that we have some strange ordering in the baryon operators for the

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equations of motion of the scalar- and vector operators. It is important at this point to address this technical issue.

The issue is that of the ordering of the baryon and anti-baryon fields. If one would state that the anti-baryon field is strictly a row vector and the baryon field a column vector, then one would see that the equations of motion do not give us scalar quantities. This would, however, be the incorrect way of viewing our baryon fields. For the equations of motion, and also the Lagrangian density to be scalar quantities, we have accepted that whichever of the two fields is first will always be a row vector and the second will be a column vector. Any possible confusion can be removed by explicitly writing the indices, i.e. $\hat{\psi}_\alpha \hat{\overline{\psi}}_\alpha$. The only point of care then is that we can now no longer simply exchange the order of the operators, due to their non-commutative nature. However, in terms of Lorentz transformation properties, this is still a Lorentz scalar.

5.3 The Energy-momentum Tensor

Now that we have our equations of motion, we can proceed to calculate the energy-momentum tensor for the non-commutative Walecka model using the process discussed in Section 4.3. Computing the variation of the action, resulting from the variation of the operators according to the labelling conventions we have introduce in eqs. (4.53) to (4.55), we get by using the equations of motion (5.11):

$$\delta \overline{S} = \mathrm{Tr}\left(\epsilon^\nu \hat{\partial}^\mu \left[i\hat{\overline{\psi}}\gamma_\mu \hat{\partial}_\nu \hat{\psi} + \frac{1}{2}(\hat{\partial}_\mu \hat{\phi}\hat{\partial}_\nu \hat{\phi} + \hat{\partial}_\nu \hat{\phi}\hat{\partial}_\mu \hat{\phi}) + (\hat{\partial}_\nu \hat{V}^\rho)\hat{F}_{\rho\mu} \right]\right), \tag{5.12}$$

where the calculation is shown in Section A.2. From this we identify $\hat{\mathcal{J}}_{\mu\nu}$ from eq. (4.57) to be

$$\hat{\mathcal{J}}_{\mu\nu} = i\hat{\overline{\psi}}\gamma_\mu \hat{\partial}_\nu \hat{\psi} + \frac{1}{2}(\hat{\partial}_\mu \hat{\phi}\hat{\partial}_\nu \hat{\phi} + \hat{\partial}_\nu \hat{\phi}\hat{\partial}_\mu \hat{\phi}) + (\hat{\partial}_\nu \hat{V}^\rho)\hat{F}_{\rho\mu}. \tag{5.13}$$

From the definition of the energy-momentum tensor given in (4.59), we have to subtract from this $g_{\mu\nu}\overline{\mathcal{L}}$. Using again the equations of motion (5.11) this yields:

$$\begin{aligned}\hat{T}_{\mu\nu} = {} & i\hat{\overline{\psi}}\gamma_\mu \hat{\partial}_\nu \hat{\psi} + \frac{1}{2}(\hat{\partial}_\mu \hat{\phi}\hat{\partial}_\nu \hat{\phi} + \hat{\partial}_\nu \hat{\phi}\hat{\partial}_\mu \hat{\phi}) + (\hat{\partial}_\nu \hat{V}^\rho)\hat{F}_{\rho\mu} \\ & - \ g_{\mu\nu}\frac{1}{2}\left[\hat{\partial}^\rho \hat{\phi}\hat{\partial}_\rho \hat{\phi} - {m_s}^2 \hat{\phi}^2 - \frac{1}{2}\hat{F}^{\rho\sigma}\hat{F}_{\rho\sigma} + {m_v}^2 \hat{V}^\rho \hat{V}_\rho\right].\end{aligned} \tag{5.14}$$

In order for us to compare the commutative- with the non-commutative Walecka model, we will need to compare their expressions for the total energy. However, we cannot compare this on

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the operator level, since the commutative Walecka model did not require such a formulation. therefore, we will calculate the total energy for the non-commutative Walecka model on the level of fields. Firstly, we see that the expression for the energy density on operator level is given by:

$$\overline{\mathcal{E}} = \hat{T}_{00} = i\hat{\overline{\psi}}\gamma_0\hat{\partial}_0\hat{\psi} + (\hat{\partial}_0\hat{\phi})^2 + (\hat{\partial}_0\hat{V}^\rho)\hat{F}_{\rho 0} - \frac{1}{2}\left[\hat{\partial}^\rho\hat{\phi}\hat{\partial}_\rho\hat{\phi} - {m_s}^2\hat{\phi}^2 - \frac{1}{2}\hat{F}^{\rho\sigma}\hat{F}_{\rho\sigma} + {m_v}^2\hat{V}^\rho\hat{V}_\rho\right]. \tag{5.15}$$

What interest us is the total energy, rather than the energy density. It is simple to compute the total energy by taking the trace of the energy density:

$$\mathrm{Tr}\left(\overline{\mathcal{E}}\right) = Tr\left(\hat{T}_{00}\right) = \int d^4x\ \mathcal{E}(x), \tag{5.16}$$

where we used (4.40). From this we can write:

$$\mathrm{Tr}\left(\overline{\mathcal{E}}\right) = \int dt \int d^3x\ \mathcal{E}(t, \boldsymbol{x}), \tag{5.17}$$

and we identify the total energy at a given time slice t' as

$$E(t') = \int d^3x\ \mathcal{E}(t', \boldsymbol{x}). \tag{5.18}$$

It is convenient to express this through the partial trace taken at a fixed time slice t':

$$E(t') = \mathrm{Tr}_{\mathbb{R}^3, t'}\left(\overline{\mathcal{E}}\right) = \int d^4x\ \delta(t - t')\mathcal{E}(t, \boldsymbol{x}). \tag{5.19}$$

In fact, most typical cases that interest us will be stationary and the field, energy-momentum tensor and energy density are time independent. In such a generic case we have that $\hat{\partial}_0\hat{\phi} = 0$. From (4.38) this implies

$$\int d^4x\ (\partial_0\phi)\hat{\Delta}(x) = 0, \tag{5.20}$$

which is satisfied if the field is also time independent, i.e. $\partial_0\phi = 0$. In this case we note that the full trace and partial trace are related by a (divergent) constant:

$$\mathrm{Tr}(\hat{\phi}) = \int d^4x\ \phi(x) = \int dt \int d^3x\ \phi(\boldsymbol{x}) = \left(\int dt\right) \mathrm{Tr}_{\mathbb{R}^3}(\hat{\phi}), \tag{5.21}$$

where we have also dropped the t' subscript on $\mathrm{Tr}_{\mathbb{R}^3}$ as it is now independent of the time slice.

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Using this, the expression for energy density given by (5.15), and also using the equations of motion for operators, we derive the expression for the total energy of the non-commutative Walecka model on the operator level:

$$\begin{aligned}\overline{E} = \mathrm{Tr}_{\mathbb{R}^3}\left(\overline{\mathcal{E}}\right) &= \mathrm{Tr}_{\mathbb{R}^3}\Bigg(i\hat{\overline{\psi}}\gamma_0\hat{\partial}_0\hat{\psi} + (\hat{\partial}_0\hat{\phi})^2 + (\hat{\partial}_0\hat{V}^\rho)\hat{F}_{\rho 0} \\ &\qquad - \frac{1}{2}\left[\hat{\partial}^\rho\hat{\phi}\hat{\partial}_\rho\hat{\phi} - {m_s}^2\hat{\phi}^2 - \frac{1}{2}\hat{F}^{\rho\sigma}\hat{F}_{\rho\sigma} + {m_v}^2\hat{V}^\rho\hat{V}_\rho\right]\Bigg) \\ &= \mathrm{Tr}_{\mathbb{R}^3}\Bigg(- i\hat{\psi}^\dagger\boldsymbol{\alpha}\cdot\hat{\boldsymbol{\nabla}}\hat{\psi} + \hat{\overline{\psi}}\left[g_v\gamma^\rho\hat{V}_\rho + (M - g_s\hat{\phi})\right]\hat{\psi} + (\hat{\partial}_0\hat{\phi})^2 + (\hat{\partial}_0\hat{V}^\rho)\hat{F}_{\rho 0} \\ &\qquad - \frac{1}{2}\left[\hat{\partial}^\rho\hat{\phi}\hat{\partial}_\rho\hat{\phi} - {m_s}^2\hat{\phi}^2 - \frac{1}{2}\hat{F}^{\rho\sigma}\hat{F}_{\rho\sigma} + {m_v}^2\hat{V}^\rho\hat{V}_\rho\right]\Bigg). \end{aligned} \tag{5.22}$$

In terms of fields this reads

$$\begin{aligned} E = \int d^3x\ \Bigg\{ &- i\psi^\dagger\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi + \overline{\psi}\star\left[g_v\gamma^\rho V_\rho + (M - g_s\phi)\right]\star\psi + (\partial_0\phi)^2 + (\partial_0 V^\rho)F_{\rho 0} \\ &- \frac{1}{2}\left[\partial^\rho\phi\partial_\rho\phi - {m_s}^2\phi^2 - \frac{1}{2}F^{\rho\sigma}F_{\rho\sigma} + {m_v}^2 V^\rho V_\rho\right]\Bigg\}. \end{aligned} \tag{5.23}$$

We have now reached the point in our research where we can start to compare the commutative- and the non-commutative Walecka models. We do this comparison of the total energy between the two models by subtracting the total energy for the non-commutative model, labelled as E_1, from the total energy for the commutative model, labelled as E_2. The result is

$$\delta E = E_2 - E_1 = \int d^3x\ \left\{\overline{\psi}\left[g_v\gamma^\rho V_\rho + (M - g_s\phi)\right]\psi - \overline{\psi}\star\left[g_v\gamma^\rho V_\rho + (M - g_s\phi)\right]\star\psi\right\}. \tag{5.24}$$

However, to calculate this expression ab-initio would be a complicated task. Even the term for the commutative Walecka energy would be hard to compute without making any approximations.

5.4 Mean-Field Approximations

One has to wonder whether we can simplify our theory in order to make the computational task easier. We have seen during Chapter 3 that there are 2 well known approximations used for the commutative Walecka model, namely the Mean-Field approximation for uniform matter and also for spatially non-uniform matter. We will briefly discuss what effect each approximation will have on the non-commutative Walecka model.

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5.4.1 Uniform Matter Approximation

During the uniform matter approximation we required that the interaction particles, that is the scalar (ϕ) and vector (V^μ) particles, should be approximated by constants of space and time. When we now consider these constant fields in our non-commutative theory, we will observe that this approximation completely nullifies the effect of a Moyal-product. This can easily be seen since a product, subject to the Moyal-product, between any field (ψ) and a constant field (A) will result in:

$$\begin{aligned} A \star \psi &= A \ e^{-\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}} \ \psi \\ &= A \ e^{-\frac{i}{2}\theta^{\mu\nu}(0)\overrightarrow{\partial_\nu}} \ \psi \\ &= A\psi. \end{aligned} \tag{5.25}$$

Consequently we will see no change in our theory and the value for the difference in total energy in (5.24) will be 0. Indeed, we can clearly see that our model will be identical to the commutative case if we consider a uniform mean-field approximation for the non-commutative Walecka model on operator level. Assuming all the assumptions for a uniform mean-field theory discussed in Section 3.2 and applying the constant fields on our operator formalism, we get that

$$\hat{\phi}(x) \to \left\langle \hat{\phi}(x) \right\rangle \equiv \phi_0 \tag{5.26}$$

and

$$\hat{V}_\mu(x) \to \left\langle \hat{V}_\mu(x) \right\rangle \equiv \delta_{\mu 0} V_0. \tag{5.27}$$

Using these approximations, the energy-density on operator level therefore becomes:

$$\overline{\mathcal{E}} = \hat{T}_{00} = \hat{\psi}^\dagger \left(-i\boldsymbol{\alpha} \cdot \hat{\boldsymbol{\nabla}} + \beta M^* \right) \hat{\psi} + g_v V_0 \hat{\psi}^\dagger \hat{\psi} - \frac{1}{2} {m_v}^2 {V_0}^2 + \frac{1}{2} {m_s}^2 {\phi_0}^2, \tag{5.28}$$

assuming the same notation, M^*, introduced in Section 3.2. Also, for fields we get for the energy-density that

$$\mathcal{E} = \psi^\dagger \left(-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta M^* \right) \psi + g_v V_0 \psi^\dagger \psi - \frac{1}{2} {m_v}^2 {V_0}^2 + \frac{1}{2} {m_s}^2 {\phi_0}^2. \tag{5.29}$$

We see that (5.29) is identical to the energy-density in eq. (3.19) and therefore a uniform matter approximation for the non-commutative Walecka model will deliver no new results.

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5.4.2 Spatially Non-uniform Matter Approximation

Unlike the uniform matter approximation where we required constant fields for our interaction particles, during the spatially non-uniform Matter approximations we require variation in at least one of the spatial coordinates. As discussed in Section 3.3, we will introduce the simplest spatially non-uniform matter approximation for the interaction particles in that the mean fields will have a dependence on only the radial component ($|\boldsymbol{x}|$) or (r). Assuming that we are able to transform the coordinates from (x, y, z) to (r, θ, ϕ) and find a matching tensor for our non-commutative parameter $\theta^{\mu\nu}$, if we now take the Moyal-product between any field (ψ) and a field non-uniform in the radial direction ($A(r)$), we see that the product becomes:

$$\begin{aligned} A(r) \star \psi(t, \boldsymbol{r}) &= A(r)\ e^{-\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}}\ \psi(t, \boldsymbol{r}) \\ &= A(r)\ e^{-\frac{i}{2}\theta^{1\nu}\overleftarrow{\partial_1}\overrightarrow{\partial_\nu}}\ \psi(t, \boldsymbol{r}). \end{aligned} \tag{5.30}$$

This approximation keeps the Moyal-product intact, for the most part, and we can proceed to make the spatially non-uniform matter approximations for the non-commutative Walecka model. These approximations for slowly varying meson fields, varying only in the radial direction, on operator level becomes

$$\hat{\phi}(x) \to \left\langle \hat{\phi}(x) \right\rangle \equiv \hat{\phi}_0(r) \tag{5.31}$$

and

$$\hat{V}_\mu(x) \to \left\langle \hat{V}_\mu(x) \right\rangle \equiv \delta_{\mu 0}\hat{V}_0(r). \tag{5.32}$$

We have, therefore, our energy density for a spatially non-uniform matter approximation on operator level:

$$\begin{aligned} \overline{\mathcal{E}} = \hat{T}_{00} = \hat{\psi}^\dagger \left(-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta \hat{M}^*\right) \hat{\psi} + g_v \hat{\psi}^\dagger \hat{V}_0 \hat{\psi} \\ - \frac{1}{2}\left((\hat{\boldsymbol{\nabla}}\hat{V}_0)^2 + {m_v}^2 \hat{V}_0^2\right) + \frac{1}{2}\left((\hat{\boldsymbol{\nabla}}\hat{\phi}_0)^2 + {m_s}^2 \hat{\phi}_0^2\right), \end{aligned} \tag{5.33}$$

where $\hat{M}^*$, $\hat{V}_0$ and $\hat{\phi}_0$ are all implied to have radial dependence. The energy-density for fields is now given by

$$\begin{aligned} \mathcal{E} = \psi^\dagger \star (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta M^*) \star \psi + g_v \psi^\dagger \star V_0 \star \psi \\ - \frac{1}{2}\left((\boldsymbol{\nabla} V_0)^2 + {m_v}^2 {V_0}^2\right) + \frac{1}{2}\left((\boldsymbol{\nabla}\phi_0)^2 + {m_s}^2 {\phi_0}^2\right), \end{aligned} \tag{5.34}$$

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giving us an expression for the total energy for the spatially non-uniform approximation for the non-commutative Walecka model:

$$E = \int d^3x \left\{ \psi^\dagger \star (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta M^*) \star \psi + g_v \psi^\dagger \star V_0 \star \psi \right.$$
$$\left. - \frac{1}{2}\left((\boldsymbol{\nabla} V_0)^2 + {m_v}^2 {V_0}^2\right) + \frac{1}{2}\left((\boldsymbol{\nabla}\phi_0)^2 + {m_s}^2{\phi_0}^2\right) \right\}. \tag{5.35}$$

Eq. (5.35) provides us with an expression for the total energy of the system as a function of the non-commutative parameter θ that is included within the Moyal-product.

For the equations of motion, we have exactly the same equations of motion as eqs. (3.14), (3.36) and (3.37), with the definitions for scalar en baryon densities intact. The only difference is that we have the order for ψ and $\psi^\dagger$ reversed in (3.36) and (3.37).

The only obstacle that hinders us from solving for the total energy in the spatially non-uniform approximation for the non-commutative Walecka model, is the fact that we are unable to write the non-commutative model's total energy in eq. (5.35) in a form similar to (3.42). In the commutative case the fact that the meson fields could easily switch order with baryon fields made it possible to write convenient expressions down for the baryon and scalar densities and the interaction term as an integral over all state energies up to the Fermi-momentum, getting rid of all the baryon fields. The fact we have the Moyal-product within the interaction terms makes it for us impossible to write down the same terms in our total energy. The consequences of this is also that we are unable to completely solve the meson fields and effective mass in a self-consistent manner.

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CHAPTER 6

Conclusions & Outlook

To summarise what aims have been reached of this thesis, we will look at what has been achieved by the work throughout the chapters.

We have completed a literature study of the Walecka model, its description of nuclear interactions using point-like particles and how it was solved in two mean-field approximations: Spatially uniform and spatially non-uniform matter. We also completed a literature study on how a field theory is built on non-commutative position coordinate operators, how this commutation relation interprets as particle size and how to compute the equations of motion and the energy-momentum tensor of the system in a non-commutative quantum field theory.

Using the theories of the Walecka model and non-commutative quantum field theory, we were successful in building, in a consistent manner, a nuclear interaction model containing a description of the finite nucleon size. We have identified that, during a spatially uniform matter approximation, the non-commutative Walecka model will reduce back to the original commutative Walecka model. In contrast, we have successfully applied a spatially non-uniform matter approximation to the non-commutative Walecka model which differs from the commutative case, and have identified the expression for the total energy of a system within this approximation.

We were, however, unable to determine an exact expression for the non-commutative parameter, which is an indication of the length scale of the baryon size, as a function of baryon density. This was due to the fact that the non-commutative Moyal-product prevented us from easily constructing solvable self-consistent expressions for the particle fields of the non-commutative Walecka model. This also prevented us from minimising the total energy as a function of the non-commutative parameter and we could not therefore determine the effect that the non-commutative parameter would have on the dynamics of the system.

In the outlook of this research, it may be possible to find self-consistent expressions for the particle fields using techniques to remove the infinite orders of derivatives in the Moyal-product. One such a technique would be to convert to expression for the total energy to momentum-space where the Moyal-product would simply become a phase term and would simplify the calculation process. However, to find a numerical solution for the commutative case of the spatially non-

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uniform matter approximation in the Walecka model is no easy task, and having an added phase to the expressions would not make the task any easier. The difficulty of the task and the time necessary to do it, however, is out of the scope of this thesis and prevents us from including it in this thesis as an added research topic.

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APPENDIX A

Detailed Calculations

A.1 The trace of a product of two field operators in 4-dim

Consider $\phi_3(\hat{x}) = \phi_1(\hat{x})\phi_2(\hat{x})$, assuming all $\phi_i(\hat{x})$ has the form in (4.10). Using (4.12), (4.24) and the anti-symmetric property of $\theta^{\mu\nu}$ we get:

$$
\begin{aligned}
\phi_3(x) &= \mathrm{Tr}\left(\phi_3(\hat{x})\hat{\Delta}(x)\right) = \mathrm{Tr}\left(\phi_1(\hat{x})\phi_2(\hat{x})\hat{\Delta}(x)\right) \\
&= \int d^4y \int d^4z\ \phi_1(y)\phi_2(z)\mathrm{Tr}\left(\hat{\Delta}(y)\hat{\Delta}(z)\hat{\Delta}(x)\right) \\
&= \int d^4y\ d^4z\ \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\frac{d^4k''}{(2\pi)^4}\ \phi_1(y)\phi_2(z)\mathrm{Tr}\left(\hat{T}(k)\hat{T}(k')\hat{T}(k'')\right)e^{-ik\cdot y-ik'\cdot z-ik''\cdot x} \\
&= \int d^4y\ d^4z\ \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\frac{d^4k''}{(2\pi)^4}\ \phi_1(y)\phi_2(z)e^{-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu}\mathrm{Tr}\left(\hat{T}(k+k')\hat{T}(k'')\right)e^{-ik\cdot y-ik'\cdot z-ik''\cdot x} \\
&= \int d^4y\cdots\frac{d^4k''}{(2\pi)^4}\ \phi_1(y)\phi_2(z)e^{-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu-\frac{i}{2}\theta^{\mu\nu}(k+k')_\mu k''_\nu}\mathrm{Tr}\left(\hat{T}(k+k'+k'')\right)e^{-ik\cdot y-ik'\cdot z-ik''\cdot x} \\
&= \int d^4y\cdots\frac{d^4k''}{(2\pi)^4}\ \phi_1(y)\phi_2(z)e^{-\frac{i}{2}\theta^{\mu\nu}[k_\mu k'_\nu+(k+k')_\mu k''_\nu]}(2\pi)^4\delta\left(k+k'+k''\right)e^{-ik\cdot y-ik'\cdot z-ik''\cdot x} \\
&= \int d^4y\ d^4z\ \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\ \phi_1(y)\phi_2(z)e^{-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu}e^{-ik\cdot(y-x)-ik'\cdot(z-x)} \\
&= \int d^4y\ d^4z\ \frac{d^4k'}{(2\pi)^4}\ \phi_1(y)\phi_2(z)\ \delta\left[(y-x)^\mu+\frac{1}{2}\theta^{\mu\nu}k'_\nu\right]e^{-k'\cdot(z-x)} \\
&= \frac{1}{|\det(\frac{1}{2}\theta)|}\int d^4y\ d^4z\ \frac{d^4k'}{(2\pi)^4}\ \phi_1(y)\phi_2(z)\ \delta\left[-2(\theta^{-1})_{\mu\nu}(y-x)^\mu+k'_\nu\right]e^{-k'\cdot(z-x)} \\
&= \frac{1}{\pi^4|\det\theta|}\int d^4y\ d^4z\ \phi_1(y)\phi_2(z)\ e^{-2i(\theta^{-1})_{\mu\nu}(y-x)^\mu(z-x)^\nu}
\end{aligned}
\tag{A.1}
$$

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Now using (A.1) and (4.4), we can continue:

$$\begin{aligned}
\phi_3(x) &= \frac{1}{\pi^4|\det\theta|}\int d^4y\ d^4z\ \phi_1(y)\phi_2(z)\ e^{-2i(\theta^{-1})_{\mu\nu}(y-x)^\mu(z-x)^\nu} \\
&= \frac{1}{\pi^4|\det\theta|}\int d^4y\ d^4z\ \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\ \widetilde{\phi}_1(p)\widetilde{\phi}_2(q)\ e^{-2i(\theta^{-1})_{\mu\nu}(y-x)^\mu(z-x)^\nu+ip\cdot y+iq\cdot z} \\
&= \frac{1}{\pi^4|\det\theta|}\int d^4y\ d^4z\frac{d^4p\ d^4q}{(2\pi)^8}\ \widetilde{\phi}_1(p)\widetilde{\phi}_2(q)\ e^{-i\left[2(\theta^{-1})_{\mu\nu}(z-x)^\nu-p_\mu\right]y^\mu}e^{-2i(\theta^{-1})_{\mu\nu}x^\mu(z-x)^\nu+iq\cdot z} \\
&= \frac{1}{\pi^4|\det\theta|}\int d^4z\ d^4p\frac{d^4q}{(2\pi)^4}\ \widetilde{\phi}_1(p)\widetilde{\phi}_2(q)\ \delta\left[2(\theta^{-1})_{\mu\nu}(z-x)^\nu+p_\mu\right]e^{-2i(\theta^{-1})_{\mu\nu}x^\mu(z-x)^\nu+iq\cdot z} \\
&= \frac{1}{(2\pi)^4}\int d^4z\ d^4p\frac{d^4q}{(2\pi)^4}\ \widetilde{\phi}_1(p)\widetilde{\phi}_2(q)\ \delta\left[z^\nu-x^\nu-\frac{1}{2}\theta^{\mu\nu}p_\mu\right]e^{-2i(\theta^{-1})_{\mu\nu}x^\mu(z-x)^\nu+iq\cdot z} \\
&= \int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\ \widetilde{\phi}_1(p)\widetilde{\phi}_2(q)\ e^{-\frac{i}{2}\theta^{\mu\nu}p_\mu q_\nu}e^{ip\cdot x+iq\cdot x} \\
&= \phi_1(x)\star\phi_2(x) \qquad\qquad (A.2)
\end{aligned}$$

according to eqs. (4.6) to (4.8).

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A.2 Calculating the variation of the action of the non-commutative Walecka model

By using the cyclic property of the trace, the equations of motion in (5.11) and by doing a spatial translation, given by $x^\nu \to x^\nu + \epsilon^\nu$, on all the operators in (5.2), we get to first order in ϵ^ν :

$$
\begin{aligned}
\delta\overline{S} = \mathrm{Tr}\Big(& (\epsilon^\nu \hat{\partial}_\nu \hat{\overline{\psi}}) \Big[\gamma_\mu(i\hat{\partial}^\mu - g_v \hat{V}^\mu) - (M - g_s\hat{\phi})\Big] \hat{\psi} \\
& + \hat{\overline{\psi}} \Big[\gamma_\mu(i\hat{\partial}^\mu - g_v \hat{V}^\mu) - (M - g_s\hat{\phi})\Big] (\epsilon^\nu \hat{\partial}_\nu \hat{\psi}) \\
& + \hat{\overline{\psi}} \Big[-g_v\gamma_\rho(\epsilon^\nu \hat{\partial}_\nu \hat{V}^\rho) + g_s(\epsilon^\nu \hat{\partial}_\nu \hat{\phi})\Big] \hat{\psi} \\
& + \frac{1}{2}(\hat{\partial}_\mu(\epsilon^\nu \hat{\partial}_\nu \hat{\phi})\hat{\partial}^\mu \hat{\phi} + \hat{\partial}_\mu \hat{\phi}\hat{\partial}^\mu(\epsilon^\nu \hat{\partial}_\nu \hat{\phi})) - {m_s}^2 \hat{\phi}(\epsilon^\nu \hat{\partial}_\nu \hat{\phi}) \\
& - \frac{1}{2}(\hat{\partial}^\mu(\epsilon^\nu \hat{\partial}_\nu \hat{V}^\rho) - \hat{\partial}^\rho(\epsilon^\nu \hat{\partial}_\nu \hat{V}^\mu))\hat{F}_{\mu\rho} + {m_v}^2 \hat{V}_\rho(\epsilon^\nu \hat{\partial}_\nu \hat{V}^\rho)\Big) \\
= \mathrm{Tr}\Big(& -\hat{\overline{\psi}} \Big[\gamma_\mu(i\overleftarrow{\hat{\partial}^\mu} + g_v \hat{V}^\mu) + (M - g_s\hat{\phi})\Big] (\epsilon^\nu \hat{\partial}_\nu \hat{\psi}) + \epsilon^\nu \hat{\partial}^\mu (i\hat{\overline{\psi}}\gamma_\mu \hat{\partial}_\nu \hat{\psi}) \\
& + \epsilon^\nu \hat{\partial}^\mu \frac{1}{2}(\hat{\partial}_\mu \hat{\phi}\hat{\partial}_\nu \hat{\phi} + \hat{\partial}_\nu \hat{\phi}\hat{\partial}_\mu \hat{\phi}) - (\hat{\partial}_\mu \hat{\partial}^\mu \hat{\phi} + {m_s}^2 \hat{\phi} - g_s \hat{\psi}\hat{\overline{\psi}})(\epsilon^\nu \hat{\partial}_\nu \hat{\phi}) \\
& \epsilon^\nu \hat{\partial}^\mu \Big((\hat{\partial}_\nu \hat{V}^\rho)\hat{F}_{\rho\mu}\Big) + \Big(\hat{\partial}^\mu \hat{F}_{\mu\rho} + {m_v}^2 \hat{V}_\rho - g_v \hat{\psi}\hat{\overline{\psi}}\gamma_\rho\Big)(\epsilon^\nu \hat{\partial}_\nu \hat{V}^\rho)\Big) \\
= \mathrm{Tr}\Big(& \epsilon^\nu \hat{\partial}^\mu \Big[i\hat{\overline{\psi}}\gamma_\mu \hat{\partial}_\nu \hat{\psi} + \frac{1}{2}(\hat{\partial}_\mu \hat{\phi}\hat{\partial}_\nu \hat{\phi} + \hat{\partial}_\nu \hat{\phi}\hat{\partial}_\mu \hat{\phi}) + (\hat{\partial}_\nu \hat{V}^\rho)\hat{F}_{\rho\mu}\Big]\Big)
\end{aligned}
\tag{A.3}
$$

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