Optimal Asset allocation for South African pension funds under the revised Regualtion 28

by

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Aan die enigste God, Jesus Christus.

Dankie Hemelse Vader vir die lesse wat ek by U kon leer en dankie dat ek kan weet dat U altyd daar is vir my, vir my sorg en my beskerm. Jesus, U is die koning van my lewe en ek is ten volle afhanklik aan U.

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Declaration

I, the undersigned, hereby declare that the work contained in this assignment is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

Signature: ....
F.J. Koegelenberg

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Abstract

On 1 July 2011 the revised version of Regulation 28, which governs the South African pension fund industry with regard to investments, took effect. The new version allows for pension funds to invest up to 25 percent compared to 20 percent, in the previous version, of its total investment in foreign assets. The aim of this study is to determine whether it would be optimal for a South African pension fund to invest the full 25 percent of its portfolio in foreign assets.

Seven different optimization models are evaluated in this study to determine the optimal asset mix. The optimization models were selected through an extensive literature study in order to address key optimization issues, e.g. which risk measure to use, whether parametric or non parametric optimization should be used and if the Mean Variance model for optimization defined by Markowitz, which has been the benchmark with regard to asset allocation, is the best model to determine the long term asset allocation strategies.

The results obtained from the different models were used to recommend the optimal long term asset allocation for a South African pension fund and also compared to determine which optimization model proved to be the most efficient.

The study found that when using only the past ten years of data to construct the portfolios, it would have been optimal to invest in only South African asset classes with statistical differences with regard to returns in some cases. Using the past 20-years of data to construct the optimal portfolios provided mixed results, while the 30-year period were more in favour of an international portfolio with the full 25% invested in foreign asset classes.

A comparison of the different models provided a clear winner with regard to a probability of out performance. The Historical Resampled Mean Variance optimization provided

the highest probability of out performing the benchmark. From the study it also became evident that a 20-year data period is the optimal period when considering the historical data that should be used to construct the optimal portfolio.

Uittreksel

Op 1 Julie 2011 het die hersiene Regulasie 28, wat die investering van Suid-Afrikaanse pensioenfondse reguleer, in werking getree. Hierdie hersiene weergawe stel pensioenfondse in staat om 25% van hulle fondse in buitelandse bateklasse te belê in plaas van 20%, soos in die vorige weergawe. Hierdie studie stel vas of dit werklik voordelig sal wees vir ‘n SA pensioenfonds om die volle 25% in buitelandse bateklasse te belê.

Sewe verskillende optimeringsmodelle is gebruik om die optimale portefeulje te probeer skep. Die optimeringsmodelle is gekies na ’n uitgebreide literatuurstudie sodat van die sleutelkwessies met betrekking tot optimering aangespreek kon word. Die kwessies waarna verwys word sluit in, watter risikomaat behoort gebruik te word in die optimeringsproses, of ‘n parametriese of nie-parametriese model gebruik moet word en of die “Mean-Variance” model wat deur Markowitz in 1952 gedefinieer is en al vir baie jare as maatstaf vir portefeulje optimering dien, nog steeds die beste model is om te gebruik.

Die uiteindelike resultate, verkry van die verskillende optimeringsmodelle, is gevolglik gebruik om die optimale langtermyn bate-allokasie vir ‘n Suid-Afrikaanse pensioenfonds op te stel. Die verskillende optimeringsmodelle is ook met mekaar vergelyk om te bepaal of daar ‘n model is wat beter is as die res.

Vanuit die resultate was dit duidelik dat ’n portfeulje wat slegs uit Suid-Afrikaanse bates bestaan beter sal presteer as slegs die laaste 10-jaar se data gebruik word om die portefeulje op stel. Hierdie resultate is ook in meeste van die gevalle bevestig deur middel van hipotese toetse. Deur gebruik te maak van die afgelope 20-jaar se data om die portefeuljes op te stel, het gemengde resultate gelewer, terwyl die afgelope 30-jaar se data in meeste van die gevalle ’n internasionaal gediversifiseerde portefeulje as die beter portefeulje uitgewys het.

In ’n vergelyking van die verskillende optimeringsmodelle is die “Historical Resampled

Mean Variance" model duidelik as die beter model uitgewys. Hierdie model het die hoogste waarskynlikheid behaal om die vasgstelde maatstafportefeuljes uit te presteer. Die resultate het ook gedui op die 20-jaar periode as die beste data periode om te gebruik as die optimale portfeulje opgestel word.

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Ecclesiastes 11:2 (NIV)

"*Invest in seven ventures, yes, in eight; You do not know what disaster may come upon the land.*"

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Chapter 1

Introduction

1.1 Problem Statement

The pension fund industry in South Africa has more than R1 trillion (WEBMASTER, 2011) in assets and is one of the biggest, if not the biggest, investment force in the South African investment market. That being said, pension funds cannot just invest in any asset class and is highly regulated by the South African government with regard to their investments. Regulation 28 (see Appendix A) regulates pension fund investments in South Africa. In essence Regulation 28 specifies the asset classes and proportions that pension funds are allowed to invest in. It has recently been revised with the new version taking effect as of 1 July 2011.

According to the previous Regulation 28 a South African pension fund could only invest a maximum of 15% in foreign asset classes. Under the revised version, pension funds would be able to invest 25% of the fund in foreign asset classes as well as an additional 5% in African asset classes.

The main focus of this study is to determine if it would be optimal, with regard to the risk return trade off, for a South African pension fund to invest 25% of its funds abroad and if so, to determine the optimal allocation between the foreign and domestic asset classes. In studies done by Cadiz (2004, 2006) they argued that the optimal percentage to invest in foreign assets is greater than 20% and that the inclusion of foreign asset classes, not only moved the efficient frontier (the line created from the risk-reward graph comprised of all the optimal portfolios) to the left, but also upward. Thus including foreign assets in a portfolio not only decreased the total risk of the portfolio, due to diversification, but it also increased the return that was generated by the portfolio. After determining the optimal foreign allocation, the next step would be to determine the optimal portfolio combining the foreign asset classes with the domestic asset classes.

While evaluating the optimal asset allocation question, this study will address another very important question: What optimization model should be used? Should the traditional Mean-Variance model first introduced by Markowitz (Markowitz, 1952) still be used, or should we start moving away from parametric optimization models to non-parametric models? Comparing different asset allocation models will help in answering the question with regard to foreign asset allocation as well as which optimization model performs the best when comparing the returns and allocation of weights across the dfferent asset classes for different portfolios.

Chapter 2

Theoretical Overview: Asset Allocation and Optimization Models

2.1 Introduction

In this chapter the concept of optimal asset allocation and the different optimization models that will be used in this study are discussed. The risk and return measurements of a portfolio, which are essential in the asset allocation process, will also be explained.

2.2 Why Asset Allocation?

Asset allocation is a strategy that aims to balance risk and reward by allocating different weights to the specified assets under consideration. The financial markets are becoming more and more interlinked and dependent on each other especially in times when the market is showing a downward trend, as shown by Asness, Israelov and Liew (2011). It has become a necessity to choose a diversified portfolio (portfolios that is invested in a range of different assets to reduce the underlying risk) to protect against extreme negative price movements, because portfolio diversification still remains one of the most important and fundamental principles of modern finance and portfolio construction (Asness et al., 2011).

Ibbotson and Kaplan (2000) showed that strategic asset allocation explained more than

90% of the variability in returns for the average fund. If asset allocation is detrimental in the variation of returns as shown by Ibbotson and Kaplan (2000), the model used to achieve the asset allocation is very important. Since the future follows a unique path it is becoming increasingly difficult to determine the optimal asset allocation. This is because all financial management approaches are based on past experience and built upon presumptions of outcomes that may never come to pass (Michaud & Michaud, 2008).

Asset allocation models are designed to achieve the optimal weights to be attributed to each asset class such that the portfolio complies with the return expectations of the investor. In the next section six different optimization models will be discussed. The six models were chosen after an extensive sifting of models from articles and text books and each model addresses different short comings of the traditional Mean Variance optimization model.

2.3 Asset Allocation Optimization Models

2.3.1 Traditional Markowitz Mean-Variance (MV) model

The Markowitz mean-variance (MV) optimization model has been the standard theoretical model with regard to asset allocation and portfolio optimization ever since Harry Markowitz published his article, *"Portfolio Selection"*, in 1952. The focus of this article is the so-called second stage of the portfolio selection process, which starts with the relevant beliefs of future performances and ends with the choice of the portfolio (Markowitz, 1952). This new ideology, created by Markowitz, is known as Modern Portfolio Theory (MPT).

According to MPT an investor would prefer portfolio securities or asset classes that provide the maximum return for a given level of risk. In other words, the investor is assumed to be risk averse. The implication is that a rational investor will not invest in a portfolio if a second portfolio exists with a higher level of expected return with the same level of risk, in other words a more favorable risk - expected return profile. The idea behind MPT is then to compile a portfolio from different securities or assets, using the covariance matrix to construct a portfolio with a lower combined variance than the individual assets. Diversification can however not eliminate all the variance from a portfolio. The portfolio with maximum expected return is not necessarily the portfolio with minimum variance and an investor can normally increase his expected return by taking on more variance, or reduce variance by giving up expected return (Markowitz, 1952).

The MV optimization model uses this attribute of MPT to determine a portfolio that maximizes the return for a given level of risk, or minimizes the risk for a given level of return. The MV model uses the expected return and standard deviation to determine the optimal allocation between the assets. From Markowitz (1952) and Brown et al. (2007) follow:

Suppose we have a number of random variables: $R_1, R_2, \ldots, R_n$. In this case these random variables represent the returns of n asset classes. Let R_{ij} be the j'th return for $j = 1, \ldots, m$ for the i'th asset class and let R_{Pj} be a weighted sum of these random variables for $i = 1, \ldots, n$ with x_i the weight assigned to asset i. R_{Pj}, the *j'th* return of the portfolio is then equal to

$$R_{Pj} = \sum_{i=1}^{n}(x_i R_{ij}) \tag{2.3.1}$$

which means R_{Pj} is also a random variable. The expected return of R_{Pj} is also a weighted average of the expected returns on the individual assets. When using the law of expected value equation 2.3.1 reduces to:

$$\bar{R_P} = \sum_{i=1}^{n} x_i \bar{R_i} \tag{2.3.2}$$

with $\bar{R_i}$ being the expected return for asset i. In this study the expected return of asset i will simply be the arithmetic average of the monthly returns contained in the data set (Brown et al., 2007 & Keller, 2010) This method for calculating the expected return for an asset is in line with the method used by Brown et. al. (2007).

With the expected return of the portfolio defined, the next step would be to define the risk measure. Markowitz (1952) uses the standard deviation of return as the risk measure in his MV model. The standard deviation of return for asset *i*, σ_i, is defined as

$$\sigma_i = \sqrt{\frac{\sum_{j=1}^{m}(R_{ij} - \overline{R_i})^2}{m}}, \tag{2.3.3}$$

while the covariance between asset i and k is defined as

$$\sigma_{ik} = \sqrt{\frac{\left(R_{ij} - \bar{R}_i\right)\left(R_{kj} - \bar{R}_k\right)}{m}}. \tag{2.3.4}$$

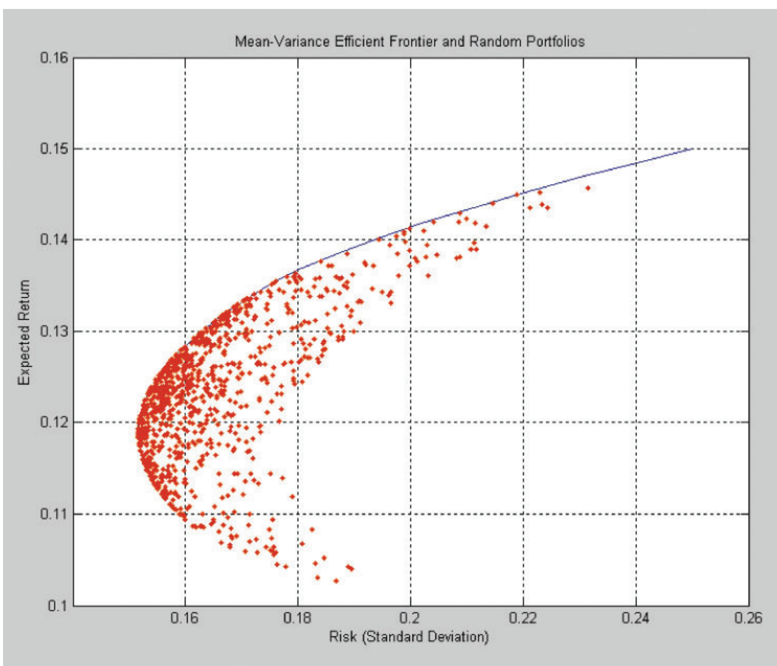
The standard deviation of return for the portfolio σ_P can now be written as

$$\sigma_P = \sqrt{\sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} x_i x_k \sigma_{ik}} = \sqrt{x^{\intercal} \Sigma x} \tag{2.3.5}$$

and is derived from first principles in Appendix C (Brown et al., 2007).

In Markowitz's MPT the efficient frontier is obtained by plotting, for each feasible portfolio constructed from the available asset classes, its expected return against its risk exposure as illustrated in figure 2.3.1 (Keller, 2010).

Figure 2.3.1: Efficient Frontier (Anon, 2012)



The blue line indicates the efficient frontier. It consists of all the portfolios with the highest expected return for a certain amount of risk. The red dots represent all the attainable portfolios from the available asset classes under the specified restrictions, such as the restrictions with regard to the amount to be invested in each asset class. The restrictions for this study is specified in Appendix A, to comply with with Regulation 28.

The Markowitz MV model has various shortcomings which is why practitioners have not fully embraced the model. Probably the most crucial shortcoming is that the MV model assumes asset returns are normally distributed random variables which is not sufficient in modern financial markets (Idzorek, 2006 and Xiong & Idzorek, 2011). The model also assumes that the standard deviation of returns as well as the correlation between asset returns are constant over time.

The optimal portfolios derived using the model tend to be excessively concentrated in a limited subset of the full set of assets, while the MV solution is also overly sensitive to the input parameters (Maillard et al., 2010). A small error in the estimation of expected returns for instance, would be maximized by the model and would give a solution to the optimization process that is not really optimal.

2.3.2 Equally Weighted Portfolio (EW)

The use of the 1/n rule has a long history in asset allocation. In fact, it was recommended in the Talmud, a central text of mainstream Judaism, writing in about the fourth century, a Rabbi Issac bar Aha gave the following asset allocation advice: “A man should always place his money, a third into land, a third into merchandise, and keep a third at hand” (Liang & Weisbenner, 2002). This simple investment strategy of attributing equal weights to each of the assets in the portfolio has in a sense stood the test of time as it is still being used today. This is especially true in the so-called defined contribution saving plans in which investment decisions are made by the plan participants themselves (Liang & Weisbenner, 2002). Due to a lack of knowledge with regard to investment they choose the easiest way out, dividing their cash equally between the available assets.

There are several reasons for studying the 1/n simple asset-allocation rule. One, it does not rely on estimation of moments in other words the calculation of the expected return and standard deviation of asset returns and so it is easy to implement this rule.

Two, despite the sophisticated theoretical models developed in the last fifty years and the advances in methods for estimating the parameters for these models, investors continue to use such simple allocation rules for allocating their wealth across assets. For instance, Benartzi and Thaler (2001) and Liang and Weisbenner (2002) document that investors allocate their wealth across assets using the naive 1/n-rule (Benartzi & Thaler, 2001 and Liang & Weisbenner, 2002).

With the equally weighted portfolio the weight contributed to each of the assets in the portfolio is equal. Let there be n assets in the market, then the weight assigned to each of these assets is equal to 1/n. In other words the weight X_i contributed to asset i will be equal to the weight contributed to asset j, X_j. Substituting these weights for the weights in equations 2.3.2 and 2.3.5, it follows that:

$$\begin{aligned} \bar{R_P} &= \sum_{i=1}^{n} x_i \bar{R}_i \\ \bar{R_P} &= \frac{1}{n} \sum_{i=1}^{n} \bar{R}_i \end{aligned}$$

and

$$\begin{aligned} \sigma_P^2 &= \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{k=1 k\neq i}^{n} x_i x_j \sigma_{ij} \\ \sigma_P^2 &= \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) \sigma_{ij} \end{aligned}$$

which reduces to

$$\sigma_P^2 = \left(\frac{1}{n}\right) \sum_{i=1}^{n} \left[\frac{\sigma_i^2}{n}\right] + \frac{n-1}{n} \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} \left(\frac{\sigma_{ij}}{n(n-1)}\right).$$

2.3.3 Equally-Weighted Risk Contribution Portfolios (ERC)

The problem with regard to the excessively concentrated portfolios achieved under the MV model can be solved by using the equally-weighted risk contribution (ERC) approach. Alternative methods to deal with these issues have been suggested in the literature, such as portfolio re-sampling or robust asset allocation, but these models have their own disadvantages, such as the mathematical complexity and computational burden of these models. In the marketplace investors prefer to use models that they understand, has been used in the market for quite some time, that are computationally simple to implement and are presumed to be robust as they do not depend on the expected returns, but uses historical data (Maillard et al., 2009).

The idea behind the ERC optimization approach is to equalize the risk contributions from the different components of the portfolio. The risk contribution of an asset i is the share of total portfolio risk attributable to that asset. It is calculated by multiplying the allocation to asset i with the marginal risk contribution for that asset. The marginal risk contribution is the change in the total risk of a portfolio for an extremely small increase in the holdings of asset i (Maillard et al., 2009).

Maillard, Roncalli and Teiletche (2009) used standard deviation as the risk measure but other risk measures can also be used. The ERC approach uses risk contributions instead of portfolio weights when compiling a portfolio which has become standard practice for institutional investors over the past years under the label of "risk budgeting". Risk contributions are not solely a mere mathematical decomposition of risk, but it has financial significance since it can be deemed good predictors of the contribution of each position to the losses experienced by a portfolio (Maillard et al., 2009).

The ERC approach to constructing a portfolio mimic the diversification effect of equally weighted portfolios, portfolios where all the assets are given the same weight, while taking into account single and joint risk contributions of the assets. The minimum variance portfolio, the portfolio constructed using the MV method discussed in 2.3.1 with the lowest variance, also equalizes the risk contributions but only on a marginal level (Maillard et al., 2009). In other words, the marginal risk contribution for each of the assets is the same such that the increase in risk contributed by any of the assets would be the same for a small increase in the weight assigned to the asset.

Consider a portfolio of n risky assets with the weight in each asset denoted by $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Let σ_i^2 be the variance of asset i, σ_{ij} be the covariance between assets i and j and $\boldsymbol{\Sigma}$ be the covariance matrix. Let σ_P be the risk of the portfolio as defined in equation 2.3.5. Marginal risk contributions $\partial_{x_i}\sigma_P$, are defined as follows:

$$\partial_{x_i}\sigma_P = \frac{\partial \sigma_P}{\partial x_i} = \frac{x_i\sigma_i^2 + \sum_{j\neq i} x_j\sigma_{ij}}{\sigma_P} \tag{2.3.6}$$

The adjective "marginal" qualifies the fact that those quantities give the change in volatility of the portfolio induced by a small increase in the weight of one asset. Where $\sigma_i(x) = x_i \times \partial_{x_i}\sigma_P$ is defined as the total risk contribution of the i^{th}asset. This can be said because σ, the volatility, satisfies Euler's theorem and is thus a homogeneous function of degree 1.

Euler's Theorem: *A function F(L,K) is homogeneous of degree n if for any values of the parameter* λ, $F(\lambda L, \lambda K) = \lambda n F(L, K)$.

Euler's theorem thus implies that σ can be written as the sum of its arguments multiplied by its first partial derivative (Watkins, 2012). The total risk of the portfolio now decomposes into

$$\sigma_P = \sum_{i=1}^{N} \sigma_i(x). \tag{2.3.7}$$

The risk of the portfolio can thus be written as the sum of the total risk contributions (Maillard et al., 2009).

In vector form the n marginal risk contributions are computed as:

$$\frac{\boldsymbol{\Sigma} x}{\sqrt{x^{\intercal}\boldsymbol{\Sigma} x}} \tag{2.3.8}$$

and the total portfolio risk σ_P can be written as:

$$\sigma_P = x^{\intercal}\frac{\boldsymbol{\Sigma} x}{\sqrt{x^{\intercal}\boldsymbol{\Sigma} x}} = \sqrt{x^{\intercal}\Sigma x}. \tag{2.3.9}$$

2.3.3.1 ERC Strategy

The idea behind the ERC strategy is to find a portfolio where $\sigma_i(x) = \sigma_j(x)$ for all values of i and j, in other words to find a portfolio where the risk contribution, as defined in 2.3.2, from each asset is equal. According to Appendix A short selling will not be allowed and thus $\mathbf{0} < \mathrm{x} < \mathbf{1}$. Mathematically the problem can be written as follows:

$$x^{\star} = \{x \,\epsilon\, [0,1]^n : \sum x_i = 1,\, x_i \times \partial_{x_i}\sigma(x) = x_j \times \partial_{x_j}\sigma(x)\, for\ all\ i,\, j\}. \tag{2.3.10}$$

Where $x^{\star}$is the computed weights such that $\sigma_i(x) = \sigma_j(x)$. Using the vector properties specified in 2.3.8 and noting that $\partial_{x_i}\sigma_P \propto (\Sigma x)$, the problem becomes:

$$x^{\star} = \{x \,\epsilon\, [0,1]^n : \sum x_i = 1,\, x_i \times (\Sigma x)_i = x_j \times (\Sigma x)_j\, for\ all\ i,\, j\}. \tag{2.3.11}$$

where $(\Sigma x)_i$ denotes the i^{th} row of the vector, from the product of Σ with x (Maillard et al., 2009).

2.3.3.2 Theoretical Properties of ERC portfolios

When constructing a portfolio from only 2 asset classes it is relatively easy to compute the optimal weights for the portfolio, such that the risk contributions are equal. Because there is only two assets there will only be one correlation, ρ. Lets assume $x = (\omega, 1-\omega)$, where ω is the weight assigned to asset 1. Then, from equation 2.3.6, the vector of total risk contributions is:

$$\frac{1}{\sigma_P}\begin{pmatrix} \omega^2\sigma_1^2 + \omega(1-\omega)\rho\sigma_1\sigma_2 \\ (1-\omega)^2\sigma_2^2 + \omega(1-\omega)\rho\sigma_1\sigma_2 \end{pmatrix}. \tag{2.3.12}$$

To determine the ERC portfolio we simply need to equate ω such that the two rows are equal. This is easy to do and after some simple algebra the ERC portfolio weights for the two asset case are

$$x^\star = \left(\frac{\sigma_1^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}} ; \frac{\sigma_2^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}} \right). \tag{2.3.13}$$

When using more than 2 assets the solution to the ERC portfolio weight problem becomes more complicated. This is because there is n individual volatilities, one for each asset, as well as *n(n-1)/2* bi-variate correlations to consider. In real life volatilities and bi-variate correlations are rarely equal. For solutions to equal correlations or volatilities the article of Maillard, Roncalli and Teiletche (2009) can be consulted. Only the most general case will be discussed here.

In the most general case the volatilities of the assets and correlations between assets will differ. Contrary to the bi-variate case and the case of constant correlation, for higher order problems, the solution to the ERC problem is endogenous. Being endogenous means that the weight x_i would be a function of itself (Maillard et al., 2009). Let's look at the problem more closely.

The covariance of returns between asset i and the returns of the aggregated portfolio is defined as

$$\begin{aligned} \sigma_{ix} &= cov\left(r_i ; \sum_j x_j r_j \right) \\ &= \sum_j x_j \sigma_{ij} \end{aligned}$$

where r_i is the return from asset i. Equation 2.3.6 can be written as follows

$$\begin{aligned} \partial_{x_i} \sigma_P &= \frac{x_i \sigma_i^2 + \sum_{j \neq i} x_j \sigma_{ij}}{\sigma_P} \\ &= \frac{\sum_j x_j \sigma_{ij}}{\sigma_P} \\ &= \frac{\sigma_{ix}}{\sigma_P}. \end{aligned}$$

From this result follows

$$\sigma_i(x) = x_i \times \partial_{x_i}\sigma_P \tag{2.3.14}$$

$$\sigma_i(x) = x_i \times \frac{\sigma_{ix}}{\sigma_P}. \tag{2.3.15}$$

Now, define β_i as the beta of asset i. β_i *is* the sensitivity of asset *i's* return to a given measure, in this case the portfolio return. From the definition of Beta in Elton et al. (2004) it follows that,

$$\beta_i = \frac{\sigma_{ix}}{\sigma_P^2} \tag{2.3.16}$$

$$\beta_i = \sigma_i(x)\frac{1}{x_i\sigma_P} \tag{2.3.17}$$

The ERC portfolio is determined such that $\sigma_i(x) = \sigma_j(x)$ for all *i,j*. When substituting $\sigma_i(x)$ *with* $\frac{\sigma_P}{n}$ equation (2.3.17) can be simplified to provide the weight that should be allocated to each of the assets that is being considered,

$$x_i = \frac{\beta_i^{-1}}{\sum_{j=1}^{n}\beta_j^{-1}} = \frac{\beta_i^{-1}}{n}. \tag{2.3.18}$$

The weight invested in asset i is inversely proportional to its beta. The higher the asset Beta, the lower the weight attributed to that specific asset which means that assets with high volatility and high correlation with the other assets will be penalized (Maillard et al., 2009). The one problem with this solution is that it is a endogenous solution, which means that the solution of x_i is a function of x_i.

2.3.3.3 Numerical solutions

While equation 2.3.18 allows for the interpretation of the ERC solution in terms of the relative risk of an asset compared to the rest of the portfolio, because of the endogeneity of the program, it does not offer a closed form solution. Finding a solution thus requires the use of a numerical algorithm (Maillard et al., 2009). Quadratic Programming (QP) can be used to solve this numerical algorithm. QP algorithms allow the maximization of the expected return and the minimization of the variance, subject to linear equality and inequality constraints (Michaud & Michaud, 2008).

The optimization problem desired in (2.3.2.2) can be solved using Sequential Quadratic Programming (SQP). The problem can be set up as follows:

$$\begin{aligned} x^{\star} &= argmin\, f(x) \\ & u.c. \quad 1^{\top}x = 1 \, and \, \mathbf{0} \leq x \leq \mathbf{1} \end{aligned}$$

where:

$$f(x) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i\left(\Sigma_x\right)_i - x_j\left(\Sigma_x\right)_j\right)^2. \tag{2.3.19}$$

The existence of the ERC portfolio is ensured only when the condition $f(x^{\star}) = 0$ is verified. In essence this program minimizes the variance of the rescaled risk contributions (Maillard et al., 2009).

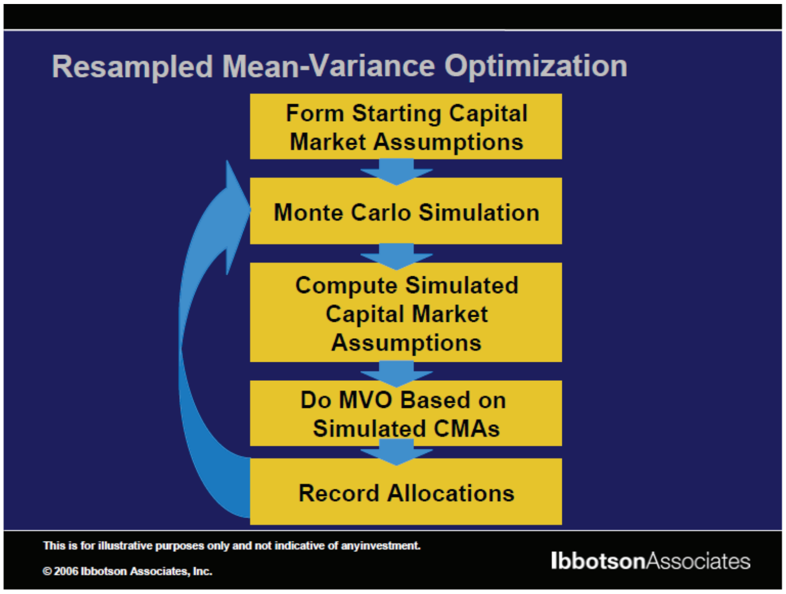
2.3.4 Re-sampled Mean-Variance Optimization (RMV)

The re-sampled Mean-Variance (RMV) optimization method used by Idzorek (2006) is an adaptation of the traditional MV model discussed in 2.3.1. By using Monte Carlo simulation the RMV model speaks to some of the most troubling aspects of the traditional MV model, such as the issues with estimation error, input sensitivity and diversification.

The traditional MV optimization model treats the estimated expected returns, standard deviations and correlations as if they were known with 100% certainty. In the real world these inputs are not known with certainty and the RMV optimization approach takes this uncertainty into account when calculating the optimal portfolio weights (Idzorek 2006).

For this study a parametric approach used by Idzorek (2006) as well as a non-parametric approach or historical approach are used to compute the optimal portfolio weights. The parametric approach involves the use of the capital market assumptions, i.e. the expected returns, standard deviations and correlations as defined in 2.3.2, and Monte Carlo simulation to produce a simulated set of capital market assumptions (Idzorek 2006). The simulated set of capital market assumptions is then fed into the traditional MV optimizer which results in an intermediate frontier called the simulated frontier. The resulting asset allocations from these simulated frontiers are then saved and the process is repeated. After these simulations are done the frontiers are averaged and the optimal asset allocation can be determined. This re-sampling process is depicted in figure 2.3.2.

Figure 2.3.2: Re-sampled Mean-Variance Optimization (Idzorek, 2006)



The non-parametric method involves the use of a bootstrap procedure to extract random data periods from the data population. The MV optimization method is then used on each of these bootstrap samples which results in a simulated frontier for each bootstrap sample. The simulated frontiers are then averaged to construct the RMV efficient frontier.

The averaging is done by first sorting the simulated frontiers according to the annual return generated by the portfolio. This data set is then divided into deciles and the average of the portfolio weights are calculated for each deciles. These decile portfolios are the RMV efficient frontier.

This Parametric re-sampling process works with any method of estimating returns, standard deviations, and correlations, as long as the correlation matrix is well defined (Idzorek 2006).

2.3.4.1 RMV Strategy

Defining the initial capital market assumptions is easy to do. For this model the expected returns, standard deviations and correlations need to be defined for each of the asset classes that is being considered. The expected returns and standard deviations are calculated as specified in 2.3.1. The correlations can be determined from the covariance matrix by using the following formula

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

All of these will be used in the Monte Carlo (MC) simulation. The first step in the MC simulation process is to generate random uniform variables for each of the n asset classes. These independent uniform random variables now need to be transformed into independent standard normal simulations. This can be done using the Inverse Probability Integral Transform, which is defined below.

Inverse Probability Integral Transform: *The probability integral transform (PIT) states that if X is a continuous random variable, with cumulative distribution function F_X and if $Y = F_X(X)$, then Y has a uniform distribution on* [0,1], *i.e.* $Y \sim Unif(0,1)$. *The IPIT is just the inverse of this. Specifically, if $Y \sim Unif(0,1)$ and if X is defined as $X = F_X(Y)$, then X has a cumulative distribution function F_X* (Rice, 2007).

The IPIT therefore implies that to simulate any independent standard normal variable one only need to simulate a random uniformly distributed random variable $U \sim Unif\,(0,1)$ and use this with the inverse of the cumulative distribution function (c.d.f.) for a standard normal distribution. In other words to simulate $r \sim N(0,1)$, an Uniform random variable needs to be generated and used as input with the inverse c.d.f. of the normal distribution which will then give a standard normal random variable as output,$r = F_r^{-1}(U)$. Where $F_r^{-1}(\cdot) = \Phi^{-1}(\cdot)$ is the inverse of the c.d.f. for a standard normal distribution (Van der Merwe, 2010).

This transformation process should be followed for each of the assets that is being considered. The next step in the simulation process is to transform these standard random variables such that they are correlated zero-mean simulations with appropriate variance.

This transformation can be done by using the Cholesky matrix derived from the asset returns covariance matrix. The Cholesky matrix, is the matrix, Q, such that the following condition is satisfied:

$$\Sigma = Q'Q \tag{2.3.20}$$

where Σ is the covariance matrix for the asset returns and Q is a lower triangular $n \times n$ matrix, where n is the total assets being considered (Alexander, 2008 226-227). Now that the zero-mean correlated variables have been computed, the expected return for each asset can be added to give the multivariate normal variables.

From these simulations the new expected returns, standard deviations and correlations can be computed and used as inputs to the traditional MV optimization model. This will give a simulated frontier from which the resulting asset allocation weights are read and is stored. This process is repeated as many times as desired.

Finally the saved portfolio weights from each of the simulated frontiers are averaged. A proprietary "bin approach" in which asset allocations from simulated frontiers are grouped together based on narrowly defined standard deviation ranges that cover the risk spectrum, in other words the total risk environment from the min variance portfolio up to the maximum return portfolio, can be used for this (Idzorek, 2006).

2.3.5 Optimization using Value at Risk as risk measure

All of the optimization models discussed thus far used standard deviation as the risk measure in the optimization process, but to make an asset allocation decision solely on an asset's mean and variance is insufficient in a world where it is known that most asset class returns are not normally distributed (Idzorek 2006 and Xiong & Idozrek, 2011). This section will discuss an alternative risk measure that can be used in the asset allocation process.

The next two models uses Value at Risk (VaR) and Conditional Value at Risk (C-VaR) as risk measures to determine the optimal portfolio. The VaR of a portfolio is defined in Hull (2005) as the loss level over the next T days that we are $\alpha\%$ certain will not be exceeded. In general, when T is the time horizon and $\alpha\%$ is the confidence level, VaR corresponds to the (100-α)th percentile of the distribution of the change in the value of the portfolio over the next T days, i.e. if $T = 5$ and $\alpha = 99$, VaR is the first percentile of the distribution of changes in the value of the portfolio over the next 5 days (Hull, 2005).

The C-VaR of a portfolio is defined as the average amount of loss, given that the VaR of the portfolio is exceeded (Alexander, 2008). In other words, if the losses in the next T days exceeds the α% VaR level for the portfolio, then C-VaR measures by how much this VaR threshold will be exceeded. The C-VaR can be calculated by taking the average of the exceedances over the given VaR level.

2.3.5.1 Optimization by fitting an Extreme Value Distribution (EVD - VaR)

When assuming asset returns are normally distributed, like with the traditional MV model, there is no room for extreme events which is common in financial markets. Financial returns are better represented by extreme value distributions (EVD), which have fatter tails than the normal distribution, in other words, the kurtosis of the distribution is greater than 3. Ignoring this fat-tailed phenomenon, might lead to an optimistic asset allocation result because high risk events are under weighted using the assumption of normality (Bensalah, 2002).

One of the biggest reasons for the assumption of normality of returns is that it provides a closed form solution to the optimization problem and the portfolio that is constructed would then also be normally distributed (Bensalah, 2002), i.e. it is less complicated. Investors are in particular concerned about significant losses meaning the downside risk, which is a function of skewness and kurtosis, which is of significant importance to the portfolio optimization process (Xiong & Idozrek, 2011).

Asset allocation using the $\alpha - quantile$ of the normal distribution, parametric VaR, as risk measure is straightforward. The assumption of normality simplifies the calculus significantly, because a mixture of N marginal normal distributions is a normal distribution. The asset allocation problem is thus reduced to a simple optimization problem, which can be solved for any number of assets because there is an analytic formula for calculating the VaR. Relaxing this assumption to incorporate real world return movements, in this case using an EVD, adds a great deal of complexity to the optimization problem. When the assumption of non-normality is made the closed form solution to the optimization problem disappears (Bensalah, 2002).

In this case the α-quantile of the EVD is used as risk measure in the optimization problem. The inherent problem with the approach is that N EVD distributions is not necessarily an EVD. This problem can be solved by computing the risk-return value of individual portfolios. Then by drawing a surface plot of all the possible portfolios in a risk-return framework, the optimal portfolio can be identified (Bensalah, 2002).

The first step in this process is to generate a random portfolio. Generating a random portfolio is not as straight forward as it may seem because we need different portfolios for the surface plot. The algorithm for generating these portfolios is given in Appendix B. For each of these portfolios the historic returns should be calculated for each of the points in the data set. The number of portfolios to be generated should be enough to give a meaning full risk-return plot.

Next an EVD needs to be fitted to each of the portfolio return sets. To fit an EVD there should be decided on the type of EVD that is to be fitted to the data set. Bensallah (2002) uses the Generalized Extreme Value (GEV) distribution in his study. The GEV distribution uses the maxima of return from non-overlapping time periods to estimate the parameters of the distribution (Bensalah, 2002).

This is however not the best EVD to use. The reason for this is that the highest returns in the one time period could be significantly less than the maximum values in another time period. A scenario could exist where the maximum return for a time period is very low, in fact it would not even be considered an extreme value when compared to the other time periods. The GEV distribution should only be fitted to the extreme values in a data set, if it happens that not all the values are extreme values it can violate the asymptotic support for the model (Coles, 2004).

The GEV distribution will not be used as the EVD in this study. Instead a threshold model known as the Pareto distribution (PD) will be used for this study. The PD uses a threshold to distinguish between so-called extreme values and normal observations. The Generalized Pareto family is defined by Coles (2004) as follows:

Theorem 4.1: *Let X_1, X_2, ... be a sequence of independent random variables with a* *common distribution function F, and let*

$$M_n = max\{X_1, \ldots, X_n\}.$$

Denote an arbitrary term in the X_i sequence by X, and suppose that F satisfies Theorem 3.1.1, so that for large n,

$$Pr\{M_n \leq z\} \approx G(z),$$

where

$$G(z) = exp\{-[1 + \xi(\frac{z-\mu}{\sigma})]^{-\frac{1}{\xi}}\} \tag{2.3.21}$$

for some μ, $\sigma > 0$ and ξ. Then, for a large enough u, the distribution function of $(X - u)$, conditional on $X > u$, is approximately

$$H(y) = 1 - (1 + \frac{\xi y}{\beta})^{-\frac{1}{\xi}} \tag{2.3.22}$$

defined on $\{y : y > 0 \, and \, (1 + \xi y/\beta\}$, where

$$\beta = \sigma + \xi(u - \mu).$$

The conditional distribution of $y = z - u$ given that $z > u$, where z is the portfolio return and u the threshold, can be approximated by the GP distribution, for threshold values that are sufficiently large. These exceedances over a high threshold can be modeled by the GP distribution with distribution function:

$$H(y; \sigma, \xi) = \begin{cases} 1 - (1 + \xi\frac{y}{\beta})^{-\frac{1}{\xi}}, & 1 + \xi\frac{y}{\beta} > 0, \, \xi \neq 0 \\ 1 - exp(-\frac{y}{\beta}), & y > 0, \, \xi = 0 \end{cases} \tag{2.3.23}$$

(Coles, 2004).

The Pareto model that will be used in this study is an adaptation of the Generalized Pareto family of the form:

$$Pa(\alpha) = x^{-\alpha} \tag{2.3.24}$$

where $\alpha = \frac{1}{\gamma}$, $x \geq 1 \, and \, \alpha > 0$.

Before the parameters of the distribution can be estimated, it is necessary to determine an appropriate threshold. The threshold cannot be too high, because a threshold that is too high would reduce the amount of data points which would increase the variance of the estimated parameters. On the other hand if the threshold is too low, there would be too many data points which would violate the asymptotic support for the model (Coles, 2004).

To determine the appropriate threshold the mean residual life plot could be used. In this plot the threshold is plotted against the average of the threshold exceedances. When the appropriate threshold is reached the plot is suppose to become linear in form. For this study the mean residual life plot will not be used, the reason for this is that a mean residual life plot would have to be constructed for each of the generated portfolios and this would cause the threshold to differ across the different portfolios. Instead a constant threshold will be chosen to be used for all the generated portfolios. For the reasons stated above the threshold for this study will be set at the bottom 20% of the return series data.

The only parameter that needs to be estimated for the Pareto model is γ which is estimated using the multiplicative Hill estimator. The Hill estimator is defined in Berning (2010) as follows:

The Hill Estimator

Let $X_1, X_2, ..., X_n$ be a sequence of independent, identically distributed random variables with common Pareto type distribution function F concentrated on $[0, \infty)$. Let k *$(2 \leq k \leq n-1)$ denote the number of excesses, and let $Z_i = \frac{X_{(n-i+1,n)}}{X_{(n-k,n)}}$, i=1,2,...,k, be the multiplicative excesses. The Hill estimator of the (positive) EVI, denoted by $H_{(k,n)}$, is defined as*

$$H_{(k,n)} = \frac{1}{k} \sum_{i=1}^{k} log Z_i.$$

For a comprehensive discussion about the Hill estimator see Berning (2010). It will not be given in this theoretical overview.

The VaR estimate is then simply the α-quantile of the fitted distribution at a specified significance level. For this study α will always be 5%. The quantile is achieved by inverting the Pareto distribution. This inversion is done in Appendix C.3. The inverted Pareto distribution is then given by the following equation:

$$x_\alpha \approx t \left[\frac{n}{k} P\left(X > t\right) \right]^{-\gamma}.$$

This fitting procedure and calculation of the VaR can now be done for each of the generated portfolios. The VaR can then be plotted against the return for each portfolio and the optimal portfolio can then be determined on a risk-return basis (Berning, 2010).

2.3.5.2 Optimization Using C-VaR while incorporating Skewness and Fat Tails

Asset class returns are not normally distributed, but the typical MV optimization proposed by Markowitz and which has dominated the asset allocation process for the past 50 years, relies on only the first two moments of the return distribution. This is a problem, since investors, especially pension funds, are particularly concerned with the negative price movements in the market. This is because pension funds needs to protect the future income of many dependents. Using only the first two moments of a distribution to measure the negative price movements in the market is insufficient, because downside risk is a function of skewness and kurtosis, the third and fourth moments of return distribution (Xiong & Idzorek, 2011).

CVaR, also known as mean shortfall or expected tail loss is closely related to VaR, but it has some characteristics that distinguishes it from normal VaR and is a preferred measure for the downside risk of a portfolio. Artzner, Delbaen, Eber and Heath (1999) showed that a desirable property of a coherent risk measure is subadditivty, which means that the risk of a combination of assets is smaller or equal to the sum of the individual risks. CVaR complies with this sub additivity attribute but not VaR.

Xiong and Idzorek (2011) argues that the higher moments, like skewness and kurtosis are important considerations in the asset allocation process. When using the downside risk as a measure to optimize a portfolio an appropriate risk measure is necessary. In this case the concept of M-CVaR is used by Xiong and Idzorek (2011) to determine the optimal asset allocation. M-CVaR is a form of CVaR that incorporates the skewness and kurtosis of the underlying distribution. This is done by fitting a multivariate truncated Lévy flight (TLF) distribution to the returns data. Monte Carlo simulation is then used to simulate returns from this TLF distribution and from these returns the M-CVaR can be calculated. It is however very hard to acquire a workable version of this TLF distribution and therefore a bootstrapping procedure will be used in this study (Xiong & Idzorek, 2011).

When using CVaR as a risk measure to determine the optimal asset allocation the optimization can be done by maximizing the return for a given level of the CVaR or, equivalently, CVaR can be minimized for a given level of return. In this study CVaR will be minimized for different levels of return, following the procedure described below.

The first step in this optimization process is to take a random sample from our historical returns. This can be done using a simple random sampling. Simple random sampling is defined by Kirchoff (2010) as follows:

Firstly consider the application of the bootstrap in a simple random sample of size n. The sample, $\mathbf{y}$, *is then treated as if it were a population and we re-sample from the sample a large number of times, say B, with replacement. At each re-sample a bootstrap sample is formed,* $\mathbf{y_b^*}=(y_1^*, \ldots, y_n^*)$, *b = 1,..., B. If the sample is similar to the underlying population then the bootstrap samples generated from it should reproduce properties similar to samples drawn directly from the original population* (Kirchoff, 2010).

With regard to the study, $\mathbf{y}$ would be the historical data set of asset returns from which each random sample is drawn with replacement. Each of these samples would then theoretically have similar properties as other samples drawn from the original population. The bootstrapping method simultaneously accounts for input uncertainty and addresses the issues of estimation error, input sensitivity and highly concentrated asset allocations (Xiong & Idzorek, 2011). Bootstrapping is done to simulate the uncertainty that is contained within the financial market. The number data points contained in each of these bootstrap samples should be big enough to accurately calculate the CVaR measure.

The next step would be to determine the optimal portfolio for each of these bootstrap samples, by minimizing CVaR for a given return level. Firstly the historical VaR needs to be determined from each of these bootstrap samples. The historical VaR is defined in Alexander (2008) as the α quantile of an empirical returns distribution. In this case, the empirical return distribution would be each of the bootstrap samples. Doing the optimization in this way can be programmed but would create a lot of problems, because to calculate the VaR we need the portfolio weights, but the VaR is needed to determine the optimal portfolio weights. This study will be using the following route.

After each of the bootstrap samples have been generated, another generation process will begin. A random set of portfolios will be generated for each of the bootstrap samples. This will be done using algorithm B.1 in appendix B. There will however be an extra condition to the generation process, being the fixed return levels. After the set amount of portfolios, say m, have been generated, for each of the bootstrap samples, the VaR can be calculated for each of these portfolios.

After the VaR has been determined the CVaR can be calculated as the average of all the points in the distribution that exceeds the VaR. It is for this reason that their needs to be enough data points in each bootstrap sample to calculate the CVaR. This will result in m portfolios each with a CVaR value. The portfolio with the minimum CVaR will then be selected which would result in a portfolio for each of the bootstrap samples with the minimum CVaR value. The final step in this optimization process is for the portfolio weights to be averaged across the B bootstrap samples.

2.3.6 Optimization using a nonparametric optimization method

All the optimization models discussed thus far had one thing in common, namely an assumption about the underlying distribution of the data. This is known as parametric statistics. Parameters are indices that index individual distributions within a particular family. For example, there are an infinite number of normal distributions, but each normal distribution is uniquely determined by its mean and standard deviation. If you specify all of the parameters, in this case $\mu \, and \, \sigma$ you've specified a unique normal distribution (Dallal, 2009). The optimization models thus far all involved some kind of estimation or calculation of parameters, be it the parameters from the normal distribution or the parameters from the Pareto distribution. Non-parametric statistics can be defined as, *the* *branch of statistics dealing with variables without making assumptions about the form or the parameters of their distribution* (Princeton University, 2012).

By using a nonparametric model for optimization, no assumptions will be made with regard to the distribution of the asset class returns. The standard deviation of returns that was used previously as a risk measure to determine the optimal portfolio is thus absolute, since the standard deviation will not be calculated, neither will the parametric VaR or C-VaR.

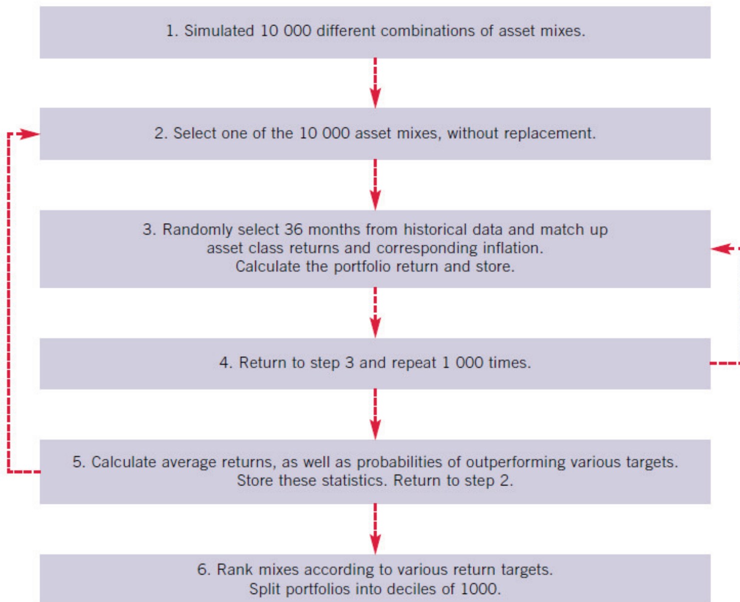
Most portfolio construction methods are based on the belief that an investor will prefer an optimal asset mix that delivers the highest return at the lowest possible level of risk. The models discussed thus far followed the same principle, but what if risk to an investor is not captured by the standard measures of risk, like the volatility of returns or the VaR, but rather by the risk of beating inflation. This is especially important to pension funds, because a pension fund investor invests in a pension fund to protect his money from the effects of inflation. In recent years there has been an increased focus on beating inflation because of the high volatility in the equity markets (Swartz, 2004).

The non-parametric optimization model that will be used in this study is similar to the model that is used by Swartz (2004) . A graphical representation of this model is given by figure 2.3.3. The model that is described below and which will be used in this study is slightly different.

The first step in the optimization process is to randomly generate n portfolios. For this, Algorithm B.1 can be used, or any other method that enables the generation of a random portfolio that complies to the constraints set by Regulation 28. One of these portfolios is then selected and the returns are calculated across the complete historical data set for the selected portfolio. A "bootstrap" procedure is then used to randomly select 36 monthly returns form the historical data set. The "bootstrap" procedure is done with replacement. Because inflation will be used as the risk measure for this optimization model, the inflation index for each of the selected months is also selected and stored.

The 36 monthly portfolio returns are then compared to the corresponding inflation for that given period to determine whether it outperformed the inflation target. Each of the portfolio returns that is higher than the inflation target is then counted and stored, from

Figure 2.3.3: Graphical Description of Methodology followed (Swartz, 2004)



which a probability can be calculated for outperforming the inflation target. This process is repeated m times for the selected portfolio to produce more m different scenarios in which the probability to outperform the inflation target is calculated. All this data is saved and the process is repeated for the next portfolio until it has been done for each of the n generated portfolios (Swartz, 2004).

This process gives a data set of n portfolios, each with its corresponding probabilities to outperform the different inflation-plus targets. These portfolios can hence be ranked in deciles, of probability to outperform the specified inflation-plus targets, and aggregated to determine the corresponding weights for each portfolio. The probability to outperform the inflation-plus target is thus used as the risk measure for the optimization model.

2.4 Conclusion

In this chapter a brief motivation was given for asset allocation and the benefits of diversification for a portfolio. A theoretical overview for each of the models used in this study is also given.

Chapter 3

Literature Review

3.1 Introduction

The articles that are available with regard to the pension fund question, how much to invest abroad, is very limited especially when considering studies on South African pension funds. Most of the articles that evaluate the MVO method is done using countries outside of Africa and only a hand full of articles are available regarding the MVO model in South Africa and even fewer published academic works. The literature review will be split into two parts. Firstly, literature that discusses international studies done with regard to asset allocation and the results that are obtained by the different models under consideration. The second part of the literature review will focus on studies that was conducted in South Africa and the results obtained.

3.2 International Asset Allocation Studies

3.2.1 Asset Allocation

According to Markowitz (1952) there is a rule in the investment world that an investor should diversify and maximize expected return. This rule states that an investor should diversify his funds among those securities which will give the maximum expected return for the portfolio, while taking the risk-return trade-off into account. Diversification can however not eliminate all the variance within a portfolio because of the pairwise correlation between different securities within the market. Markowitz states that the portfolio with the maximum return is not necessarily the one with the minimum variance. Because of this trade-off between risk and return the investor can, theoretically, gain expected return by taking on variance, or reduce variance by giving up expected return (Markowitz, 1952).

Brinson et al. (1986) sought to explain how much the variability of fund returns are explained by the variability of policy returns, in other words the returns from asset allocation. They found that 93.6% of the variation in quarterly returns of a pension fund is due to the asset allocation of that specific fund. Ibbotson and Kaplan (2000) replicated this study and published their results in the article "Does Asset Allocation Policy Explain 40%, 90%, or 100% of Performance?". First, they replicated Brinson's results with two new sets of data, looking at 10 years of monthly returns for 94 balanced (i.e., stock and bond) mutual funds and 10 years of quarterly returns for 58 pension funds (Ibbotson and Kaplan, 2000). Ibbotson and Kaplan (2000) found that 90% of the variability of the monthly returns of a fund can be explained by the variability of the fund's policy benchmark. Based on these results, Ibbotson and Kaplan (2000) stated that more than 90 percent of the variability of the average fund's return across time is explained by that fund's policy mix (Brinson et al., 1986). In other words more or less 90 percent of the variance of a portfolio can be attributed to the strategic asset allocation.

The results of Ibbotson and Kaplan (2000) was confirmed by Andreu, Ferruz and Vicente (2007) looking at Spanish pension funds. They found that strategic asset allocation is a major contributor to Spanish equity personal pension plan performance and found that on average more than 90% of the variability of returns over time and about 70% of the variation of returns among different pension plans, are explained by the strategic asset allocation policy followed by the different pension funds (Adreu et al., 2007).

Why International Asset Allocation?

Odier and Solnik (1993) showed that international diversification is crucial to manage risk in a portfolio and provide attractive risk diversification and profit opportunities to an investor. According to their article the asset allocation decision is the major contributor to the performance and risk of a portfolio and they showed that international assets, both equity and bonds, should be an important component of the asset allocation of any investor. Oldier and Solnik (1993) stated that the risk and return advantages of international diversification are very large for investors in all the major countries (Oldier & Solnik, 1993).

Asness, Israelov and Liew (2011) examined the benefit of international diversification in the long run using a data set of approximately 40 years. They found that the benefit of global diversification can be disappointing in the short term, as world markets tend to crash at the same time, but stated that the critique on global diversification misses the point when viewing only the short term results from market crashes. The long-term returns created from global diversification is far more important to wealth creation. In their study they showed that international markets do crash on the same time but in the long term the country-specific economic conditions tend to dominate, in these cases international diversification protects investors from poor economic performance in certain countries (Asness et al., 2011).

3.2.2 Asset Allocation Models

According to Harlow (1991) downside-risk measures are attractive not only because they are consistent with investors perception of risk, but also because the theoretical assumptions required to justify the use of these measures are very simple. That is, a downside-risk approach can lower risk while maintaining or improving upon the level of expected return offered by mean-variance approaches (Harlow, 1991). VaR is such a downside risk measure.

According to Huisman, Koedjik and Pownall (1999) it is desirable to have an asset allocation model which is well able to asses the risk-return trade-off observed in the financial markets. However not all optimization models succeeds in capturing this risk-return process accurately because of the assumptions made by some of these models i.e. normality of stock returns. Huisman, Koedjik and Pownall propose using VaR as a risk measure to achieve this risk-return trade-off. Their article moves away from VaR as an ex-post assessment of risk in financial markets and highlight the need for taking an ex-ante approach to risk management, such that assets are allocated to maximize expected return and that the risk of a portfolio's value falling below a critical level is known. They incorporated a Student t distribution to calculate the VaR and by doing this they compensated for the fact that US stocks and bonds provide evidence of additional downside risk from skewness and kurtosis (Huisman et al. 1999).

Bensalah (2002) used different VaR estimates as risk measures to determine the optimal asset allocation. In his article he used Normal VaR, extracting the VaR from the normal distribution fitted to the data, Historical VaR, using the quantile of the historical data, and Extreme VaR, where the VaR was calculated by fitting an extreme value distribution, to determine the optimal asset allocation. Bensallah considered portfolios only consisting of bonds and treasury bills and found that it is beneficial to use an extreme value distribution in conjunction with the VaR to determine the optimal asset allocation (Bensalah, 2002). This extreme value distribution could also be applied to other financial data since it since financial data is not normally distributed because of the higher excess kurtosis and skewness (Xiong & Idzorek, 2011).

Boyle and Windcliff (2003) compared the equally weighted portfolio with the traditional MV portfolio. In this study they showed that the equally weighted portfolio is optimal in a very simple market where the assets are indistinguishable and uncorrelated, which is not very realistic. This implies that all the assets have the same expected return and the same volatility and the pairwise correlations are all zero. For this situation the 1/n portfolio would be optimal. Boyle and Windcliff (2003) concluded that when the parameters of the return distribution are known, the 1/n portfolio will lie below the efficient frontier and are thus suboptimal (Boyle & Windcliff, 2003).

Garlappi et al. (2005) compared the MVO model with the 1/n model and other static optimization models. They found that the 1/n portfolio, even though it is the simplest form of asset allocation and ignores the pairwise correlation between the different assets, is not ineffective and it often has a higher Sharpe ratio than portfolios from other static portfolio optimization methods. The turnover is also much lower. According to Garlappi et al. (2005) estimation error with regard to the input parameters can sometimes overwhelm the gains from an optimization model like the MVO model because of its dependance on the input parameters and therefore simple asset allocation strategies, like the 1/n strategy, outperform optimization models that use inputs which may contain estimation error.

Garlappi et al. (2005) concludes that when evaluating the performance of a particular strategy for optimal asset allocation, the simple 1/n naive-diversification rule serves as a useful benchmark.

Idzorek (2006) compared the MVO model with two other optimization models that is used in the market, the Black-Litterman model and a MVO model that uses re-sampling. He wanted to develop an asset allocation model that is more robust than the MVO model. Under most circumstances the MVO model gives a portfolio which is highly concentrated in a few assets and not very diversified even though the portfolio is a minimum variance portfolio. Idzorek found that small changes in the standard deviation of the portfolio correspond to large changes in the asset allocations for the MVO model and that the model excluded nearly half of the assets when constructing the optimal portfolio. According to Idzorek (2006) the MVO model would be optimal in a world where we assume perfect foresight, because then we would want to ignore the losers and bet on the winners. The greatest problem with the MVO model is that it is eleven times more sensitive to estimation error in returns relative to estimation error in risk (variance) and is two times more sensitive to estimation error in risk relative to estimation error in covariances. This results in significant changes in the output from the model due to changes in the input parameters (i.e. the asset returns, variances and covariances) (Idzorek, 2006).

Idzorek (2006) then goes on to compare the MVO model with a re-sampled MVO model, the same model described in section 2.3.4. Idzorek (2006) states that the re-sampled MVO-based asset allocations are more appealing than the corresponding traditional MVO asset allocations because it utilizes more of the available assets and it evolves smoothly across the efficient frontier indicating greater diversification. He found that while re-sampling improved the diversification of the asset allocations, the quality of the asset allocations still depends on the quality of the inputs (Idzorek, 2006).

Maillard et al. (2009) explored the properties of equally-weighted risk contribution (ERC) portfolios, where the total risk contribution for each of the asset classes are set equal instead of the marginal risk contribution as is the case for the traditional MVO model. They compared the ERC, MV and 1/n portfolios with regard to variance, VaR, Sharpe ratio, concentration of weights using the Gini and Herfindahl indices and finally the turnover. From the three back tests they compiled it was found that the ERC approach resulted in better diversification of portfolio risk compared to the MVO model even though it underperformed with regard to return and risk. This is due to the fact that the MVO portfolio is heavily concentrated in a few assets, while the ERC portfolio is better distributed across all the assets. They also found that the ERC portfolio gives a return and risk level that is between the 1/n portfolio and minimum variance portfolio. For their final back test Maillard et al. (2009) constructed a global portfolio from which they found that a ERC portfolio achieved a higher Sharpe ratio as well as higher average returns when compared to the other two models (Maillard et al., 2010).

Idzorek and Xiong (2011) argued that the normal distribution is not a sufficient fit for financial data, because financial data tend to have higher excess kurtosis and skewness. To incorporate the kurtosis and skewness in the asset allocation problem they used bootstrap sampling to create a data set and compute the optimal asset allocation. Idzorek and Xiong (2011) moved away from standard deviation as a risk measure and used C-VaR as the risk measure because it is a tail risk measure. They applied their methodology, with regard to the C-VaR and used a nonparametric bootstrap approach, to a global asset set to incorporate the skewness and kutosis of the data. They found that the C-VaR portfolio provided better results when compared to MVO with regard to the Sharpe ratio and that the optimal portfolio constructed using the C-VaR approach yielded a lower C-VaR than the MVO portfolio (Xiong & Idzorek, 2011).

3.3 South African Asset Allocation Studies

In 2006 Munro and Swartz (Munro & Swartz, 2006) did a study in which they asked whether it is optimal for a South African pension fund to invest 25% of its money in foreign assets. In this study a substantial data set of monthly return data was used and dated over the period from 1925 to 2005. The study used a nonparametric approach to optimization to determine the portfolio allocation with the highest probability to outperform the inflation-plus targets. The portfolios consisted of 5 asset classes namely Local cash, Local equity, Local bonds, Foreign equity and Foreign bonds.

The study found that, when using the nonparametric optimization procedure, the minimum international exposure is 24.9% which is associated with beating a target of inflation + 2%. For higher targets like inflation + 5% the total international exposure was equal to 33.3% for the portfolio in the first decile of simulated portfolios over a 5-year period (Munro & Swartz, 2006).

Bradfield et al. (2010) did a similar study, again evaluating the optimal percentage for a pension fund to invest in foreign assets. The study evaluated three different methods to determine whether it is optimal to invest more than 20% in foreign assets. The first approach they followed was to go as far back in history as possible and determine the optimal asset allocation according to the historical data. Another approach was to rely on theory rather than empirical studies and rigorously estimate the structural relationships between the asset classes to establish how to blend these assets.

For the empirical study a historical data period from 1971 to 2010 was used, i.e. 40 years. The local property index (SAPY) was also included in this study. Working with a 3-year investment horizon, the empirical study found that it is still optimal to invest more than 20% in foreign assets, with the lowest foreign allocation being 29% at inflation +4% and the highest being 46% for inflation + 7%. Their study stress tested the results in environments of local expansion and contraction as well as global expansion and contraction and found similar results. Only in the period of Local expansion did the foreign allocation dip to just below 24% at inflation + 7 and inflation + 8% but it never fell to below 20%. Only when the exchange rate for the rand was held at a stable level for the 40 year period did the foreign asset allocation fall below 20% to 17% at a CPI +3% level (Bradfield & Munro, 2011).

The theoretical model that was used involved dividing the portfolio into 2 portfolios, a local and foreign portfolio. From this study it was also clear that a minimum of 20% would be held in foreign assets except if the excess return over the risk free rate for the foreign portfolio falls to a level below 5.00%, when this happens it would not be beneficial to invest a minimum of 20% in foreign asset classes but less than 20%, how much less is however not stated in the article (Bradfield & Munro, 2011).

A further study by Bradfield, Gopi and Munro (2010) evaluated whether it is beneficial to invest a minimum of 20% of the portfolio value in foreign assets. This study might seem the similar to previous studies, but different asset classes where used for this study. They found that if the foreign assets delivered a return of 0.68% for every 1% return earned by the local part of the portfolio then a minimum of 20% of the total portfolio can be invested in foreign asset classes. If the return achieved by the foreign asset classes would increase for every 1% local return the weight assigned to the foreign asset classes would increase (Bradfield et al., 2010).

In a more recent study, again by Bradfield and Munro (2011), that was done with the revised Regulation 28 in mind, they found that by using the efficient frontier as optimization method as well as the nonparametric approach described by Swartz (2004) it is beneficial for a pension fund to invest 25% of its portfolio in foreign assets. Bradfield and Munro (2011) used the same asset classes as was used by Bradfield et al. (2010) except for the local property index that was replaced by a foreign cash component. When using the efficient frontier as optimization method the complete risk spectrum, from the minimum variance portfolio up to the maximum return portfolio had a foreign asset allocation of more than 25%. The same is true for the nonparametric optimization method (Bradfield & Munro, 2011).

3.4 Conclusion

From the reviewed literature it is clear that there exists a void in South Africa when it comes to asset allocation literature that compares different theoretical models for portfolio optimization especially with regard to published academic work.

All the studies mentioned above used different time periods with regard to the length of historical data used as well as the specified period in time. The differing time periods makes it impossible to compare the different models. This problem of differing time periods will be addressed in the study by varying the historical data periods to be used. The idea behind varying the time periods is to see the effect that different data sets have on the results of the different optimization models. If the same time periods are used for each of the models discussed in Chapter 2, it would be easier to compare the different optimization methods because the same data sets were used.

The length of the data set that is used for optimization is another issue that needs to be addressed. There seems to be no rule of thumb when doing portfolio optimization with regard to the length of the data period that is used. From the reviewed literature the time period used varies from 10 to 80 years.

The other concern is that non of the results discussed in this chapter where tested for statistical significance. This study will attempt to provide statistically significant results, whether it is positive or negative.

Chapter 4

Data and Methodology

4.1 Introduction

This chapter provides a brief overview of the data that is used in this study as well as the methodology used to arrive at the optimum portfolios for each of optimization methods.

4.2 Data

The seven asset classes that was chosen for the study are:

- Local Equity
  - January 1981 to March 1986: Firer and Macload data
  - April 1986 to June 1995: INET, (CI01) total return data
  - July 1995 up to August 2011: INET, (J203T) index return data
- Local Bonds
  - January 1981 to December 1998: Firer and Macload data
  - January 1999 to August 2011: INET, ALBI (JAPI05) index return data
- Local Cash
  - January 1981 to August 2011: INET (TBT3)

- Local Property
  - January 1981 to August 1990: INET, Property Unit Trust Index (cap + DY)
  - September 1990 to January 1993: INET, index comprising of Property Unit Trust Index and Property Loan Stock Index, weighted according to their first market cap (65/35) (Cap + DY)
  - February 1993 to March 2002: INET, Index comprising of Property Unit Trust Index and Property Loan Stock Index, weighted according to their market cap (Total Return)
  - April 2002 to August 2011: INET, Listed Property Index (J253T)
- Foreign Equity
  - January 1981 to August 2011: INET, MSCI Global Equity Index (MSCI.WRLD$)
- Foreign Bonds
  - January 1981 to January 1988: www.globalfinancialdata.com
  - February 1988 to August 2011: INET, JP Morgan World Bond Index (GLOUS)
- Foreign Property
  - January 1981 to August 2011: Reuters, MSCI Global Property Index
- Inflation
  - January 1981 to December 2008: INET, old CPI data
  - January 2009 to August 2011: INET, new CPI series

The monthly log returns for each of the asset classes were collected for a period of 30 years. The Local Equity, Local Bond, Local Cash, Local Property, Foreign Equity, Foreign Bond and Inflation data that were provided by Colin Firer, was already in the log return form and converted into rand returns where applicable. The Foreign Property data were obtained from Reuters and had into be converted to rand log returns.

4.3 Methodology

The main aim of this study is to evaluate whether investment in foreign assets would provide a statistically significant benefit over investment in only South African asset classes, while complying to the rules and restrictions set out by the revised Regulation 28. To test for statistical significance a Paired T-test was used on the return data generated by each of the different portfolios. This will be done for each of the optimization methods for each of the time periods, 10, 20 and 30 years.

The second part is to take each optimal portfolio generated by the models described in Chapter 2 and then compare them on a risk adjusted basis, where possible, or by using a probability of out performing a certain benchmark, to determine which optimization model provides the best portfolio for long term strategic asset allocation. The optimal portfolios were chosen as described in Chapter 2 for each of the different optimization models.

4.3.1 Portfolio Generation

4.3.1.1 Equally Weighted Portfolio (EW)

The optimization process when using the Equally weighted portfolio optimization is quite straight forward because only one portfolio is created. It is thus simply dividing 1 by the total number of assets to be invested in. For the South African only portfolio the weight assigned to each asset classes is simply 1 divided by 4 which equates to 0.25 invested in each asset class.

For the International portfolio it is not that simple, because the restrictions (Regulation 28) need to be taken into account. Only 25% of the total portfolio weight can be invested in foreign asset classes. This implies that the sum of the weights assigned to the three foreign assets should be equal to 0.25. The weights would then be 0.25 divided by 3. In the same way, the sum of the weights assigned to the local asset classes should be equal to 0.75. Then the weights assigned to the local part of the portfolio would be 0.75 divided by 4.

From these two portfolios (local and international) the monthly returns were then calculated for each of the 3 time periods.

4.3.1.2 Equally-Weighted Risk Contribution Portfolio (ERC)

The ERC optimization method is similar to the Equally weighted procedure as it gives one optimal portfolio. The optimization is done using the *fmincon* function in Matlab. *fmincon,* then minimizes the non-linear equation given in equation 2.3.19 on page 32. This function was used specifically due to the fact that it allows for the specification of constraints that is necessary in this optimization process, as is specified in Regulation 28. The two portfolios (local and international) was then used to compute the monthly returns for each of three time periods.

4.3.1.3 Traditional Markowitz Mean-Variance (MV) model

Generating the MV portfolios proved to be a bit more complicated than the ERC and EW portfolios. To generate random portfolios with restrictions as set out by Regulation 28 was done by firstly generating 1 000 000 portfolios using Algorithm B.1, but these are portfolios without any restrictions. To arrive at portfolios with the relevant restrictions all the portfolios that do not comply to the restrictions are eliminated and a random subset of n portfolios is selected form the restricted portfolios.

Next the average monthly returns and standard deviations are calculated for each of the generated portfolios. The return is calculated by multiplying each of the generated portfolios with the average monthly returns of the different asset classes. The standard deviation is simply the square root of the variance, and the variance is calculated by multiplying the covariance matrix by the weight of each portfolio. This results in an average monthly return and average monthly standard deviation.

The Standard deviations are standardized using square root scaling, in other word by multiplying by the square root of 12. The returns are then converted to an effective rate of return and plotted against the annualized standard deviations, to arrive at a risk-return plot. This plot allows the construction of the efficient frontier, which consists of all the portfolios which maximizes the return for a specified level of standard deviation. The optimal portfolio will the be one of these portfolios on the efficient frontier.

To test whether it is optimal to invest in SA only or to invest in International portfolio a paired t-test was done, but in this case it was done on the effective returns of all possible portfolios that could have been generated, instead of the average monthly returns as was done for the ERC and EW portfolios. This is also done on a risk-adjusted return.

4.3.1.4 Re-sampled Mean-Variance Optimization (RMV)

The RMV portfolio generation can be split into two different methods. The greater part of both methods are the same, the only difference is the data that is used. One method uses bootstrap sampling to generate the data, while the other method uses Monte Carlo simulation to generate the data that is used in the optimization process.

The data generation through Monte Carlo simulation is done by using the method described in Algorithm B.2. The Monte Carlo method generates a data set with size equal to that of the original historical data set. For the bootstrap data generation, a bootstrap sample with replacement is done from the historical asset returns. The sample that is selected is the same size as the original historical data set. This process is done 500 times resulting in 500 separate data sets.

The traditional MV procedure is then applied to each of these 500 data sets using 1000 random portfolios and the efficient frontier is calculated and saved. After an efficient frontier has been calculated for each of these data sets it is sorted according to the annual return and divided into 10 bins. Hence the average is calculated for each bin to arrive at the re-sampled efficient frontier.

In a similar fashion to the previous section the returns and standard deviations are calculated and the Paired t-test can be performed on the effective rate of return from all efficient portfolios.

4.3.1.5 Optimization by fitting an Extreme Value Distribution (EVD)

The optimization by fitting an EVD is relatively straight forward. The first step is to generate random portfolios. In this case 10 000 restricted random portfolios were generated using algorithm B.1. These portfolios are multiplied by the monthly asset returns, in other words a weighted sum, resulting in a return series for each of the 10 000 generated portfolios. Using a threshold that includes 20% of the left tail of the return series the hill estimator is used to compute an estimate for γ, the extreme value index, for each of the return series.

The inverted Pareto distribution with $\hat{\gamma}$ as EVI is used to compute the parametric VaR for each of the portfolios at a 5% significance level. The probability to exceed this VaR value will thus be 5%. In this case the optimal portfolio is selected in a similar fashion to the MV approach, by plotting the VaR and effective returns together.

4.3.1.6 Optimization Using C-VaR

Optimization using the C-VaR as the risk measure uses a nonparametric method to establish the optimal portfolio. In this case a bootstrap procedure is used to draw 1 000 random samples from the historical data with replacement. The size of the bootstrap sample is equal to the size of the corresponding historical data set from which the bootstrap were drawn. For each of these bootstrap samples 1 000 portfolios are generated and a return series is calculated for each of these portfolios.

From each of these return series the Historical VaR is selected as the 5% quantile, in other words, 5% of the returns will be less than the selected VaR value. 5% were chosen because of the limited data available. With the 10 year data set a quantile of 5% will result in only about 6 returns less than that value. For a higher significance level more data points is required.

The C-VaR is calculate as the average of all the points exceeding the chosen VaR value for each of the portfolios. The portfolio with the minimum C-VaR value is chosen. This is done for each of the 1 000 bootstrap samples and the portfolios are averaged. If it is required a C-VaR value can be calculated for a specified return level, this will however not be done in this study.

This portfolio can the be used to generate returns and these returns can be compared in the same manner as with the EW and ERC portfolios.

4.3.1.7 Optimization using a nonparametric optimization method

The nonparametric method for optimization moves away from a definite risk measure for optimization and uses the probability of outperforming a benchmark in stead. In this case 10 000 random portfolios are generated. Thirty six months are randomly selected from the historical data sample and the asset returns as well as the inflation are recorded. This is done 1000 times resulting in 1000 data periods of 36 months each. One of the 10 000 generated is selected and the effective returns are calculated for each of the 36 month data periods. This gives 1 000 effective returns. These effective returns are compared to the effective rate of inflation for the specified 36 month period and the probability of outperforming the inflation target is calculated. This process is repeated each of the 10 000 portfolios.

The 10 000 portfolios are sorted into deciles according to the probability of outperforming the inflation target and averaged. The optimal portfolio is thus the portfolio is the first decile, with the highest probability of outperforming the inflation target. This is done for 5 inflation targets, CPI + 0%, +1%, +2%, +4%, +8%.

4.3.2 Portfolio Weight Distribution

One of the focus points of this study is to determine the asset weight distribution for the optimal portfolios. This is done on a risk adjusted return basis where the risk measure that is used to determine the risk adjusted return is simply the risk measure used for specified optimization model. A range of weights to be invested in each asset class can be constructed using the top 250 portfolios with regard to their risk-adjusted returns. This evaluation will be done for each of the optimization models except for the EW, ERC and non-parametric optimization models, because these models only give a single portfolio as solution to the optimization problem.

For the RMV model where an efficient frontier was used to construct the optimal portfolio the weights of the portfolios that was used to construct the efficient frontier is given as these portfolios already provide the best risk adjusted return for the specified return level. The risk adjusted return as well as the return and standard deviation will also be included in this case.

4.3.3 Portfolio Comparison

In Chapter 6 the optimal portfolios from each of the optimization models are compared to see if there is an optimization model that generates the best portfolio. To compare the different models an optimum portfolio have to be chosen for each of the optimization processes. This is done on a risk adjusted return basis where the risk measure is the standard deviation for the models that uses an efficient frontier as an optimization measure.

Where an efficient frontier is not used for the optimization process, the relevant risk measure will be used, in this case the VaR and C-VaR to calculate the risk adjusted return. For the non-parametric optimization model the optimal portfolio will be the portfolio that is obtained by calculating the average of the portfolios with the highest probability of out performance.

The portfolios yielding the highest risk adjusted return for each optimization process will then be used to compare the different optimization methods. The best portfolio will be obtained by using a bootstrap procedure to determine the probability of out performance. The portfolio yielding the highest probability of out performance is then the best portfolio and thus the best optimization method.

Chapter 5

Results: South Africa Only or Restricted International

5.1 Introduction

The main focus of this chapter is to answer the question of whether it is optimal to invest in an portfolio containing only South African (SA) asset classes or to invest in a restricted international portfolio (RI), which is constructed to contain a foreign investment of only 25%. In this chapter the results obtained by the different optimization methods are discussed. Graphical representations of the different optimization models are given where applicable as well as the results from the statistical tests for significance with regard to the foreign asset allocation. This is done for each of the seven optimization procedures obtained using each of the three time periods, 10, 20 and 30 years. The comparison of the different Optimization models is done in Chapter 6.

5.2 Equally Weighted and ERC Optimization models

As discussed in Chapter 4, the EW and ERC methods of optimization only yields one portfolio for a given data period. The EW and ERC portfolios are given in Table 5.1 below.

For each of these portfolios the effective rate of return was calculated as well as the

Table 5.1: EW and ERC Portfolios

(a) Equally Weighted Portfolios

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property
EW SA	25%	25%	25%	25%	-	-	-
EW Restricted	18.75%	18.75%	18.75%	18.75%	8.333%	8.333%	8.333%

(b) ERC Portfolios

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property
ERC SA, 10Y	75%	15.17%	5.7%	4.13%	-	-	-
ERC Restricted, 10Y	75%	10.01%	2.35%	2.93%	3.11%	2.06%	4.54%
ERC SA, 20Y	75%	13.32%	5.41%	6.36%	-	-	-
ERC Restricted, 20Y	73.67%	7.31%	2.51%	4.13%	4.54%	2.48%	5.36%
ERC SA, 30Y	75%	14.26%	4.89%	5.85%	-	-	-
ERC Restricted, 30Y	71.26%	8.85%	2.68%	3.86%	4.40%	2.65%	6.30%

annualised standard deviation to allow for the risk-return scatter plot to be constructed. The risk-return scatter plots for the three time periods are given in Figure 5.2.1:

From the scatter plot in Figure 5.2.1 it is evident that international assets underperformed greatly when compared to the SA assets when looking at the past 10 years of financial data so one would expect a SA portfolio to outperform an International portfolio.

The EW portfolio consisting of only SA assets has a clear advantage with regard to return over the EW International and both ERC portfolios. The EW international portfolio reduces the risk of the portfolio, but it also has a lower return when compared to the EW SA portfolio.

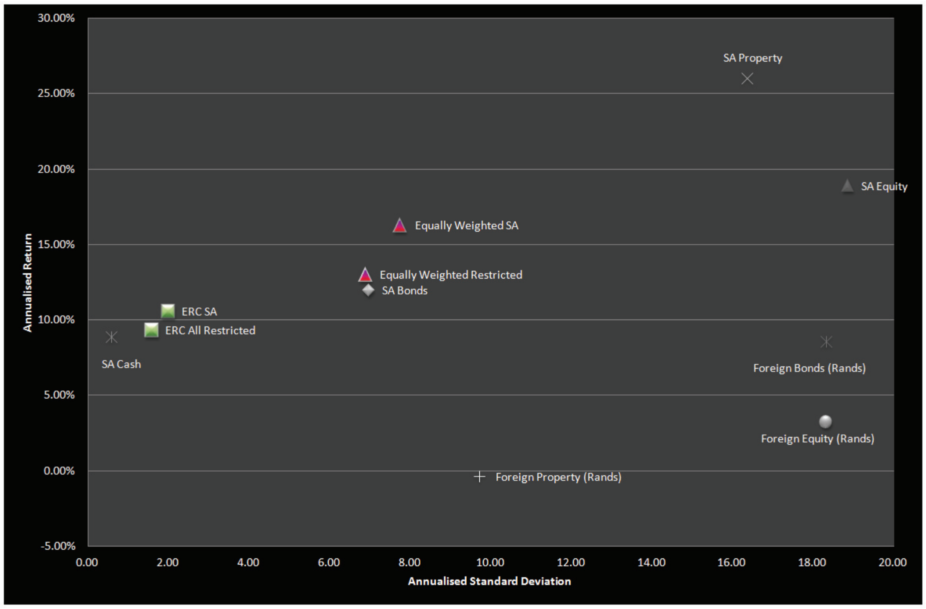
The ERC portfolios outperform the EW portfolio substantially when taking only their standard deviation into account, but with regard to the effective return the ERC portfolios greatly under performs.

The scatter plot of the asset classes as well as the EW and ERC portfolios when using a 20-year data set are showed in Figure 5.2.2. From this plot it is evident that the Foreign asset classes performed much better compared to the 10-year data period.

The EW portfolio invested in only SA assets is still the best portfolio when considering only the return of the portfolio with an effective return of approximately 16%. The EW

Figure 5.2.1: Annualised Risk-Return scatter plot, 10-Year

Scatter plot of the annualised risk and return for the EW and ERC portfolios as well the annualised risk and return for each of the asset classes using a 10-year data period.



portfolio which includes a 25% investment in foreign asset classes has a lower return, but a lower annual standard deviation as well.

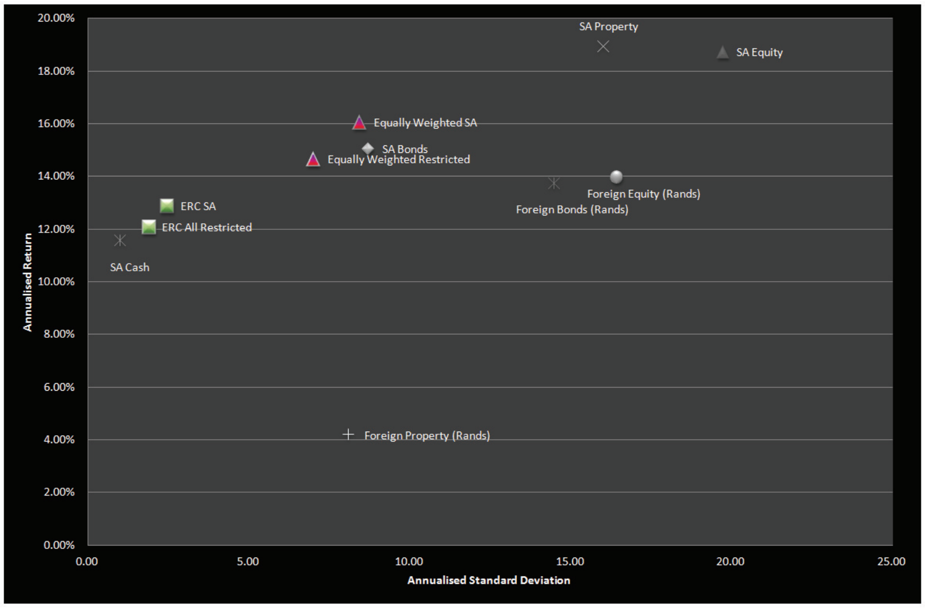
The ERC portfolios again has the lowest return, but also an extremely low standard deviation and one would expect that the ERC portfolio would have a higher risk adjusted return.

The scatter plot using the 30-Year data period is presented in Figure 5.2.3. When looking at the individual asset returns it is interesting to see that SA property and Foreign Equity achieved approximately the same annual return for the 30-year data period. The EW portfolio which includes the foreign asset allocation performs much better when looking at the 30-year data period with almost the same return as the EW SA portfolio but with quite a reduction in standard deviation. The two ERC portfolios again have the lowest return but in this case by only 3 to 5%, while the reduction in standard deviation is quite significant.

A table with the risk adjusted returns would give a better indication of which portfolio to invest in to obtain the best risk adjusted return. The risk adjusted return is calculated

Figure 5.2.2: Annualised Risk-Return scatter plot, 20-Year

Scatter plot of the annualised risk and return for the EW and ERC portfolios as well the annualised risk and return for each of the asset classes using a 20-year data period.



by dividing the annualised return by the standard deviation of the specific portfolio. Table 5.2

Table 5.2: Risk Adjusted Returns

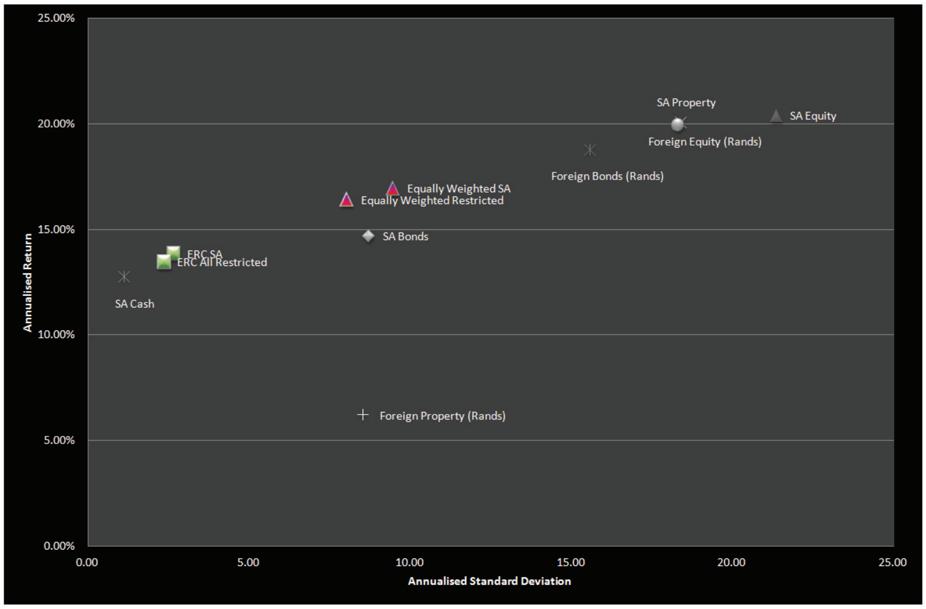
Provides the risk adjusted returns for the EW and ERC portfolios

	10-Year	20-Year	30-Year
EW SA	2.098%	1.900%	1.789%
EW Restricted	1.887%	2.085%	2.042%
ERC SA	5.843%	6.308%	5.682%
ERC Restricted	5.302%	5.202%	5.171%

provides the risk adjusted returns for the EW and ERC portfolios. The percentage value is the average annual return per unit of standard deviation, in other words, the average annual return per unit of risk. On a risk adjusted basis it is clear that the ERC portfolios are now the best performing portfolios especially the ERC portfolio constructed form only SA asset classes. On a risk adjusted basis, the EW RI portfolio outperforms the EW SA portfolio, which implies that foreign asset exposure increases the diversification benefit of the portfolio even for a simple case like the equally weighted portfolio.

Figure 5.2.3: Annualised Risk-Return scatter plot, 30-Year

Scatter plot of the annualised risk and return for the EW and ERC portfolios as well the annualised risk and return for each of the asset classes using a 30-year data period.



5.2.1 Hypothesis testing

To determine whether it is better to invest in a portfolio containing only SA assets or in a portfolio containing SA and Foreign assets a paired t-test was done on the monthly returns generated by the different portfolios. A paired t-test was chosen because the two data sets are dependent. The following hypothesis was tested:

$$\begin{aligned} H_0: \quad \mu_D &= 0 \\ H_{1:} \quad \mu_D &\neq 0, \end{aligned}$$

where $D_i = X_{1,i} - X_{2,i} for i = 1, ..., n$, with n the number of generated monthly returns, $X_{1,i}$ the i'th monthly return for the SA only portfolio and $X_{2,i}$ the i'th monthly return for the restricted international portfolio. μ_D is then the population mean difference between the monthly portfolio returns of the two data sets. If D_i is positive, the SA portfolio outperformed the RI portfolio for that specific month and if it is negative the RI portfolio outperformed the SA portfolio for that specific month. The results for the tests is given in Table 5.3 .

Table 5.3: Paired t-test results

The results in the table contains the p-value for the hypothesis test as well as the confidence interval for the mean difference of the monthly returns.

	$\alpha = 0.05$	**10-Year**	**20-Year**	**30-Year**
Equally Weighted Portfolios	***95% CI***	[0.0478, 0.4269]	[-0.0238, 0.2252]	[-0.0723, 0.1485]
	p-value	**0.0145**	0.1124	0.4975
Equal Risk Contribution Portfolios	***95% CI***	[0.0238, 0.1625]	[-0.0016, 0.1229]	[-0.0239, 0.0813]
	p-value	**0.0218**	0.0560	0.2837

Looking at the results of the paired t-test in table 5.3 it is evident that the only statistically significant result is obtained for the 10-year period. The 95% confidence interval indicate that the SA portfolio return is significantly different from the RI portfolio return. This result is confirmed by a p-value of 0.0145 and 0.0218 for the EW and ERC portfolios respectively implying that the null hypothesis will be rejected at a 1.45% and 2.18% level of significance. There is thus enough evidence to conclude that the mean difference between the two data sets is significantly different from zero. This result seems quite intuitive when looking at the 10-year data scatter plot in Figure5.2.1.

For the EW and ERC portfolios there would not have been any statistically significant benefit from investing 25% of your portfolio in Foreign asset classes for the past 20 or 30 years. When looking only at the past 10 years a portfolio consisting of only SA asset classes would have significantly outperformed a portfolio where 25% were invested abroad. This makes sense given the turmoil in financial markets over the past 10 years with 9/11, the credit crunch of 2008 and with the current debt crises.

5.3 Traditional Mean-Variance Model

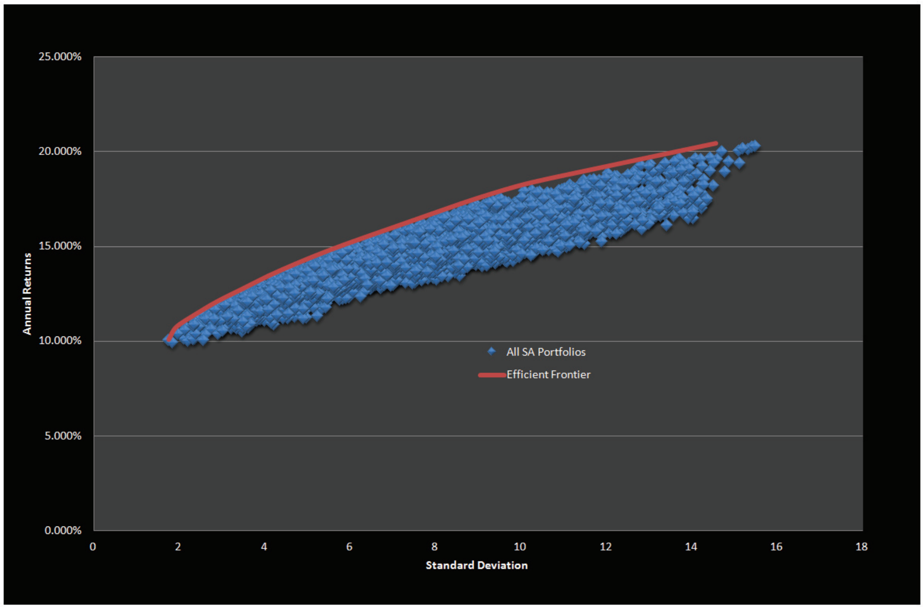
The MV optimization approach differs the the EW and ERC approach as it gives more than one portfolio as output. In fact the number of portfolios that is obtained depend on how comprehensive the optimization process needs to be. In this study 10 000 portfolios were generated for both the SA only and RI portfolio. The results will be given in subsections with regard to the data period used. The hypothesis test results will then be given in the final subsection.

5.3.1 10-Year data period

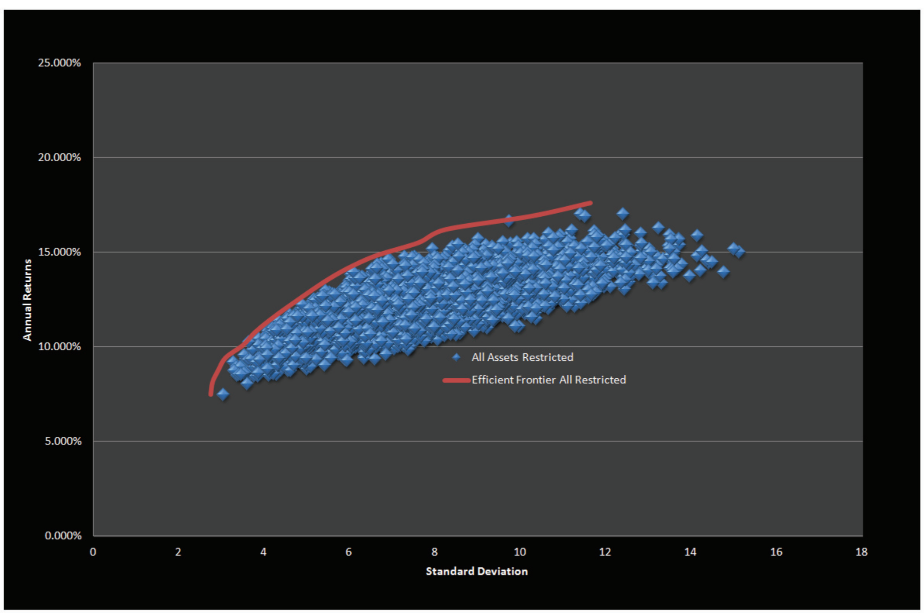
The MV optimization model uses an efficient frontier to determine a subset of optimal portfolios, minimizing the standard deviation for each level of return. The risk-return plots for these generated portfolios together with the efficient frontier are given in 5.3.1 and 5.3.1 respectively.

Figure 5.3.1: Risk-Return Plots, 10-Year

These plots give the risk-return scatter plots for the SA only and RI portfolio, together with the efficient frontier for each of the portfolios.



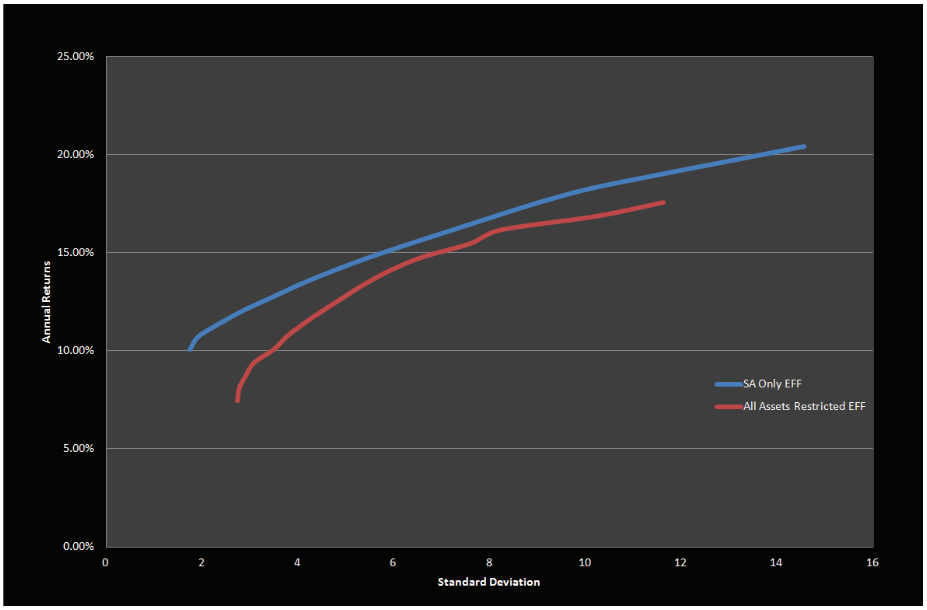
(a) SA Only Portfolios



(b) Restricted International Portfolios

Figure 5.3.2: Efficient Frontiers, 10-Year

A plot comparing the two efficient frontiers.



In Figure 5.3.1 a and b the annual return and standard deviation for each of the 10 000 generated portfolios is plotted. The red line drawn on the edge of each scatter plot is the efficient frontier for that specific portfolio.

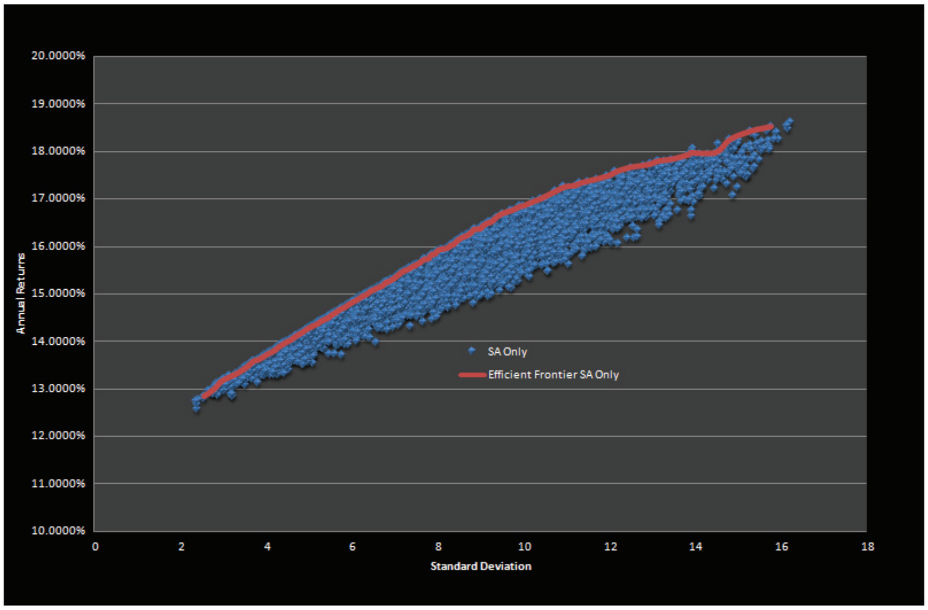
In Figure 5.3.2 the two efficient frontiers is plotted on the same axis form which it is clear that the SA portfolio is substantially better with regard to risk and return as it provides higher returns at the same standard deviation levels than the RI portfolio. In other words the SA portfolio would give a higher return for each level of risk when using the 10-year data period to do the optimization. This is in unison with the results achieved for the EW and ERC portfolios.

5.3.2 20-Year data period

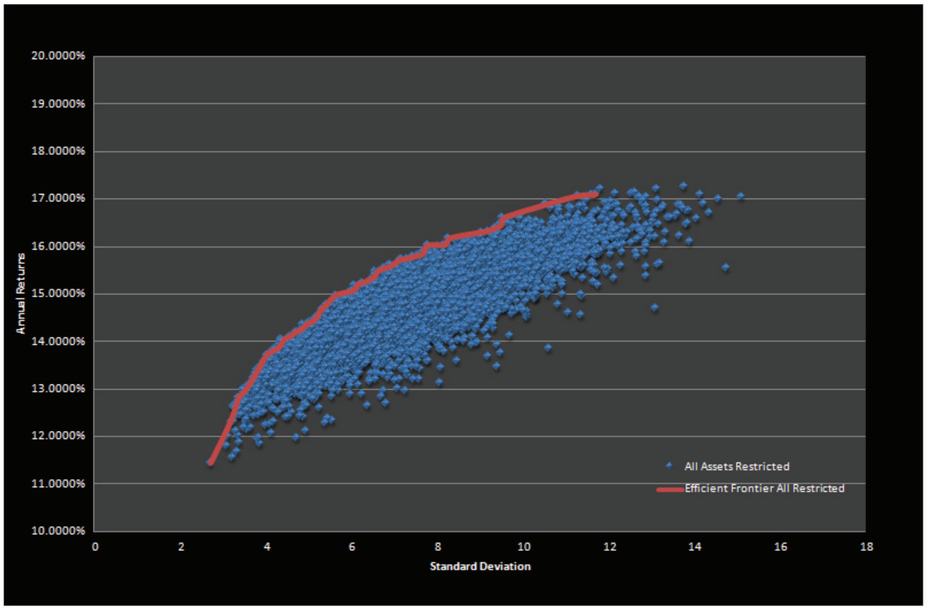
The risk-return scatter plots for the 20-year generated portfolios together with their efficient frontiers are given in 5.3.3 and 5.3.3 respectively.

Figure 5.3.3: Risk-Return Plots, 20-Year

These plots give the risk-return scatter plots for the SA only and RI portfolio, together with the efficient frontier for each of the portfolios



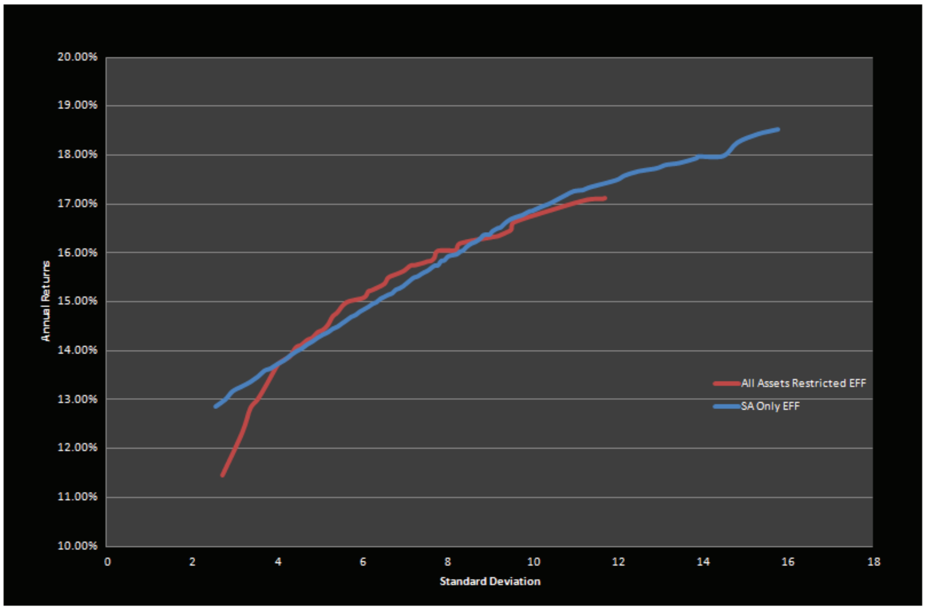
(a) SA Only Portfolios



(b) Restricted International Portfolios

Figure 5.3.4: Efficient Frontiers, 20-Year

A plot comparing the two efficient frontiers.



In Figure 5.3.3 a and b the annual return and standard deviation for each of the 10 000 generated portfolios are plotted. The red line drawn on the upper edge of the scatter plot is the efficient frontier for that specific portfolio.

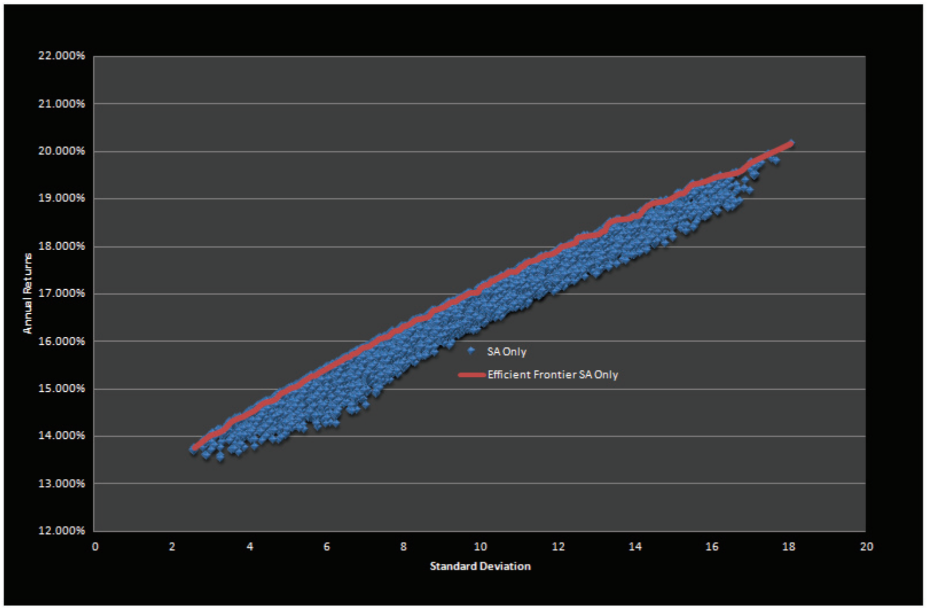
In Figure 5.3.4 the two efficient frontiers are plotted on the same axis. In this case the difference between the SA only and the RI portfolio isn't as obvious. The RI portfolio outperforms the SA portfolio when the standard deviation is between 4 and 8 but under performs for higher and lower standard deviations. The SA portfolio allows for a higher return at higher levels of standard deviation compared to the RI portfolio.

5.3.3 30-Year data period

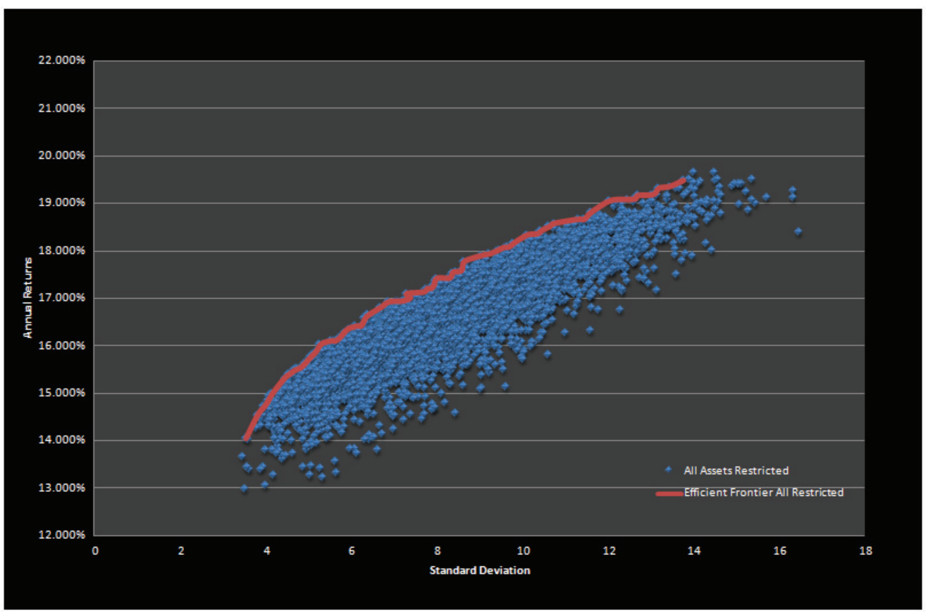
The risk-return scatter plots for the generated portfolios from the 30-year data period, together with the efficient frontiers for each of the portfolios are given in 5.3.5 and 5.3.5 respectively.

Figure 5.3.5: Risk-Return Plots, 30-Year

These plots give the risk-return scatter plots for the SA only and RI portfolio, together with the efficient frontier for each of the portfolios



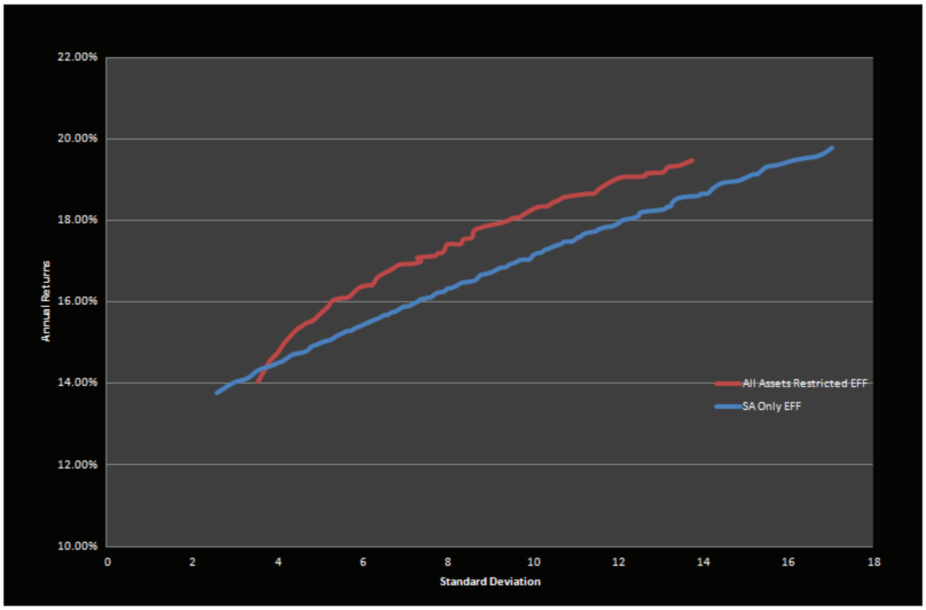
(a) SA Only Portfolios



(b) Restricted International Portfolios

Figure 5.3.6: Efficient Frontiers, 30-Year

A plot comparing the two efficient frontiers.



In Figure 5.3.5 a and b the annual return and standard deviation for each of the 10 000 generated portfolios are plotted. The red line drawn on the edge of each scatter plot is the efficient frontier for that specific portfolio. Figure 5.3.5 a gives all the possible portfolios consisting of only SA asset classes. It is interesting to note that the Efficient frontier takes on an almost linear shape.

In Figure 5.3.6 the two efficient frontiers is plotted on the same axis, form which it is clear that the RI portfolio substantially outperforms the SA only portfolio except at very low levels of standard deviation.

5.3.4 Hypothesis Tests

The paired t-test that is done on the MV portfolios differs slightly from the tests done on the EW and ERC portfolios. The reason for this is that with the MV optimization procedure 10 000 portfolios was generated for which an average annual return and standard deviation can be calculated. The test is thus done on the average annual return for each of the portfolios. In other words the test tests whether there is a difference between the portfolio returns for all the possible portfolios that could be generated for the SA only and RI portfolio. The risk adjusted return is also calculated, which provides another data set on which the paired t-test could be done. The hypothesis to be tested is as follows:

$$H_0: \quad \mu_D \quad = 0$$
$$H_{1:} \quad \mu_D \quad \neq 0,$$

where $D_i = X_{1,i} - X_{2,i} for i = 1, ..., 10000$, where $X_{1,i}$ is the i'th average annual portfolio return for the SA only portfolio and $X_{2,i}$ is the i'th average annual portfolio return for the restricted international portfolio. μ_D is then the population mean difference between the average annual portfolio returns of the two data sets. If $D_i \; for \; i = 1, ..., 10000$ is positive, the SA portfolio outperformed the RI portfolio with regard to it's average annual return and if it is negative the RI portfolio outperformed the SA portfolio. The results for the hypothesis test is given in Table 5.4.

Table 5.4: Paired t-test results

The results in the table contains the p-value for the hypothesis test as well as the confidence interval for the mean difference. A colored p-value implies statistical significance, green if the difference is in favor of the RI portfolio, red if the difference is in favor of the SA portfolio.

	$\alpha = 0.05$	**10-Year**	**20-Year**	**30-Year**
MV Portfolios	***95% CI***	[0.0242,0.0249]	[0.008, 0.0085]	[-0.0025, -0.0019]
	p-value	0	0	0
Risk Adjusted Returns	***95% CI***	[0.0033, 0.0035]	[-0.0017, -0.0014]	[-0.0022, -0.002]
	p-value	0	0	0

Before starting the interpretation of the results it is important to note that the confidence intervals are very narrow, this is because the sample used for the test is 10 000 which is very large and thus a very good representation of the population under consideration. The p-values are all zero which means the null hypothesis will be rejected in favor of the alternative hypothesis in all cases implying that there is a statistically significant difference in the average annual returns of the portfolios.

Lets first examine the results with regard to the average annual return of the portfolios.

The 10-year data period still yields a favorable result for the SA portfolio, which is similar to the result achieved by the EW and ERC portfolios. The level of difference is quite large. According to the confidence interval the SA portfolio outperformed the RI portfolio by a minimum of 2.42% per annum over the past ten years. The 20-year period is also in favor of the SA portfolio, but now the confidence interval is much lower and the minimum exceedance of the SA portfolio over the RI portfolio is only 0.8% for the past 20 years. The 30-year portfolio is in favor of the RI portfolio which is expected when looking at the efficient frontier plot in Figure 5.3.6. According to the 95% confidence interval the RI portfolio outperformed the SA portfolio by a minimum of 0.19% annually over the past 30 year.

The test done on the risk adjusted returns yields some interesting results. Again all the p-values are 0 and thus the null hypothesis is rejected in all cases. In other words their is a statistically significant difference in the risk adjusted returns between the two portfolios. The risk adjusted return is calculated by dividing the average annual return of a portfolio by the annualised standard deviation for that specific portfolio.

The 10-year data still yields a result in favor of the SA portfolio which is to be expected when looking at Figure 5.3.2. For the 20-year period, the result from the hypothesis test favors the RI portfolio instead of the SA only portfolio. In other words the risk associated with the RI portfolio is lower than the risk associated with the SA portfolio, which is a clear indication of the benefit of diversification. For the 30-year data period the result is again in favor of the RI portfolio.

5.4 Re-sampled Mean Variance Optimization

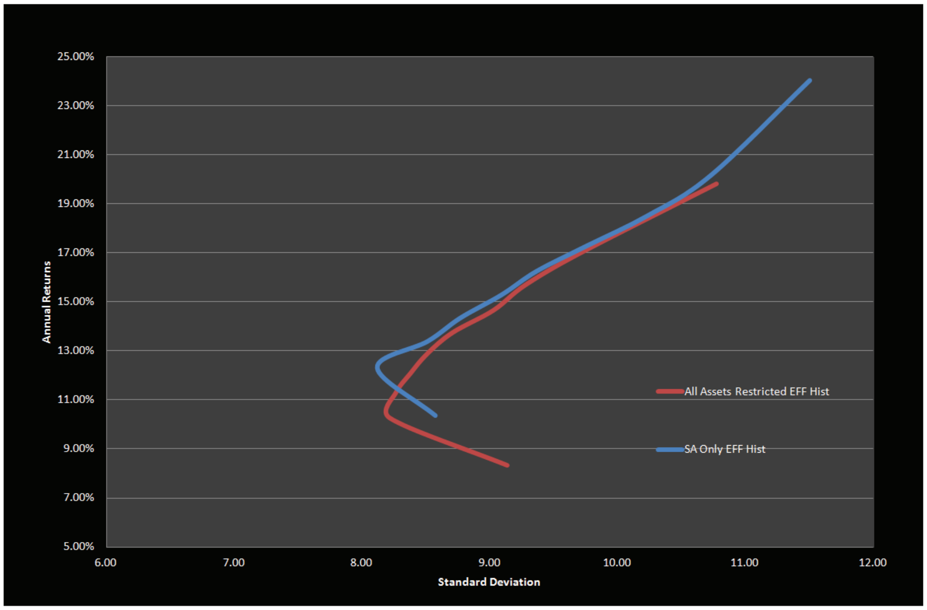
In this section the results for the Re-sampled Mean Variance (RMV) optimization method will be discussed. Two data generation methods were used for the RMV method, a multivariate normal data generation (Normal) and a historical bootstrap (Hist) procedure. Both procedures were used on all three data periods for the RI portfolio as well as the SA only portfolio. Only the averaged efficient frontiers will be discussed in the subsequent subsections as this is the main result and there is too many data points to construct a risk-return scatter plot.

5.4.1 10-Year Data Period

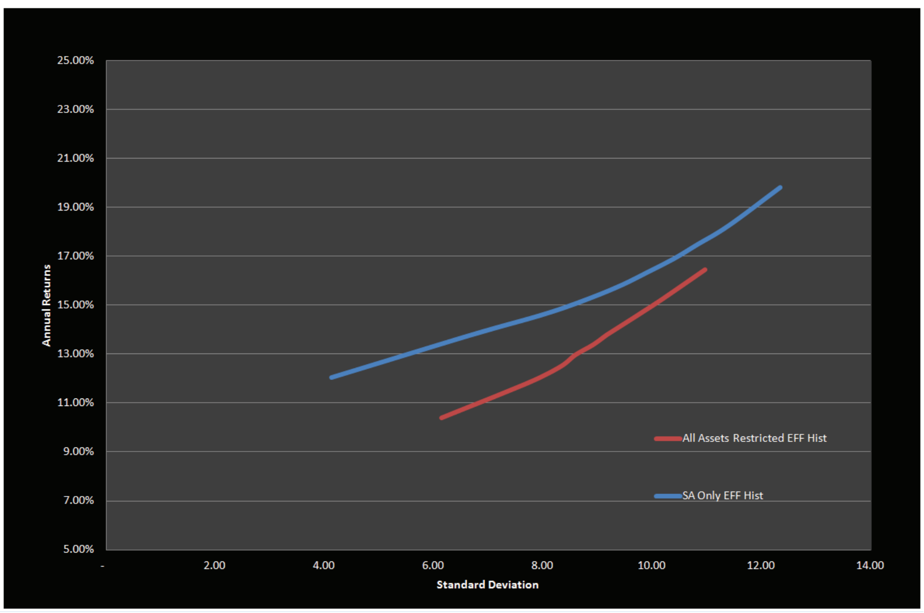
The efficient frontiers for the historical bootstrap and multivariate normal methods of data generation is given in Figure 5.4.1.

Figure 5.4.1: RMV Efficient Frontiers, 10-Year

Contains the averaged Efficient Frontiers for the Historical and Normal data generation methods.



(a) Historical RMV EFF



(b) Normal RMV EFF

Examining the RMV efficient frontier for the historical method, there isn't a real difference between the two frontiers except at the lower levels of standard deviation, for which the SA Only portfolio performs better than the RI portfolio. At a return level of above 19% the SA only portfolio again outperforms the RI portfolio, but for the middle return levels the efficient frontiers are almost identical.

The normal data generation method provides completely different results. If the asset returns were normally distributed, the results for the two procedures would have been more or less the same. Therefore these results could indicate that the returns are not normally distributed. The SA only portfolio is clearly the better method to use according to the normal RMV EFF.

The weight distribution for the portfolios on the efficient frontiers are given below:

Table 5.5: Efficient Frontier Portfolio Weights, 10-Year, Normal RI

Red indicates the maximum allocation, while blue indicates the minimum allocation.

Annual Return	**Annual St. Dev**	**Risk Adjusted Return**	**SA Cash**	**SA Bonds**	**SA Equity**	**SA Property**	**Int Bonds**	**Int Equity**	**Int Property**
10.39%	6.14	1.692%	35.64%	18.11%	11.87%	9.39%	12.75%	6.81%	5.44%
11.84%	7.75	1.529%	21.31%	19.71%	21.89%	12.10%	13.62%	7.65%	3.72%
12.49%	8.32	1.501%	16.03%	20.22%	26.33%	12.42%	13.35%	8.09%	3.56%
12.96%	8.59	1.509%	13.97%	19.86%	28.47%	12.70%	13.18%	8.29%	3.53%
13.39%	8.92	1.501%	12.15%	19.12%	31.08%	12.66%	13.11%	8.43%	3.45%
13.79%	9.17	1.504%	11.09%	18.19%	32.90%	12.82%	12.93%	8.59%	3.48%
14.20%	9.46	1.501%	10.22%	16.93%	34.93%	12.92%	12.84%	8.65%	3.51%
14.67%	9.79	1.499%	9.18%	15.53%	37.21%	13.08%	12.66%	8.95%	3.39%
15.27%	10.20	1.497%	8.20%	13.58%	39.97%	13.25%	12.76%	8.89%	3.35%
16.45%	10.97	1.500%	6.36%	9.85%	44.56%	14.23%	12.62%	9.07%	3.31%

From the portfolio weights it is evident that there is a definite relationship between the return of a portfolio and the portion invested in equity. It is also interesting to note that the risk adjusted return for the portfolios using the normal RMV approach is at a maximum at the lower return scale while the opposite is true for the portfolios using the Historical RMV method.

5.4.2 20 - Year Data

The efficient frontiers for the historical bootstrap and multivariate normal methods of data generation is given in Figure 5.4.2.

Table 5.6: Efficient Frontier Portfolio Weights, 10-Year, Normal SA-Only

Red indicates the maximum allocation, while blue indicates the minimum allocation.

Annual Return	**Annual St. Dev**	**Risk Adjusted Return**	**SA Cash**	**SA Bonds**	**SA Equity**	**SA Property**
12.04%	4.12	2.924%	61.07%	17.76%	9.33%	11.85%
13.77%	6.66	2.067%	41.01%	21.68%	20.05%	17.27%
14.65%	8.03	1.823%	28.50%	26.09%	28.51%	16.90%
15.29%	8.86	1.727%	21.92%	27.42%	34.17%	16.49%
15.85%	9.45	1.676%	17.98%	27.32%	38.60%	16.10%
16.36%	9.91	1.650%	15.84%	26.14%	42.03%	15.99%
16.89%	10.38	1.628%	14.07%	24.52%	45.36%	16.05%
17.49%	10.82	1.616%	12.65%	22.73%	48.30%	16.32%
18.26%	11.39	1.603%	11.62%	19.67%	52.25%	16.45%
19.83%	12.34	1.607%	9.19%	15.34%	58.03%	17.45%

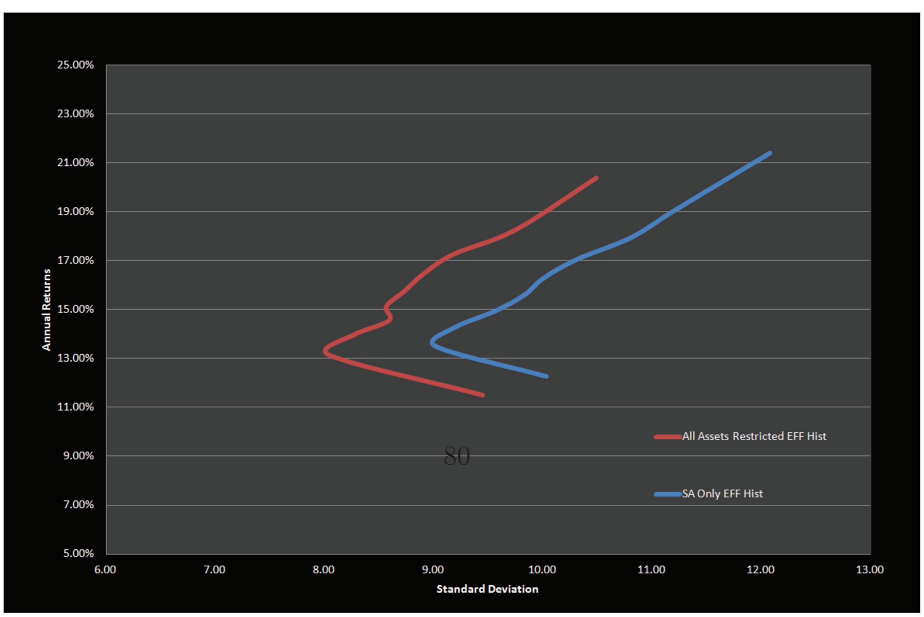
Table 5.7: Efficient Frontier Portfolio Weights, 10-Year, Historical RI

Red indicates the maximum allocation, while blue indicates the minimum allocation.

Annual Return	**Annual St. Dev**	**Risk Adjusted Return**	**SA Cash**	**SA Bonds**	**SA Equity**	**SA Property**	**Int Bonds**	**Int Equity**	**Int Property**
10.26%	8.21	1.249%	20.81%	17.84%	25.37%	10.98%	12.73%	8.10%	4.17%
11.26%	8.26	1.364%	18.39%	18.53%	26.09%	11.99%	13.29%	7.81%	3.90%
12.08%	8.38	1.443%	16.82%	18.73%	26.95%	12.50%	13.33%	8.02%	3.65%
12.89%	8.51	1.515%	15.60%	18.62%	27.96%	12.82%	13.20%	8.25%	3.55%
13.74%	8.70	1.579%	14.30%	18.47%	29.26%	12.96%	13.31%	8.15%	3.54%
14.66%	9.03	1.624%	12.79%	17.99%	31.39%	12.83%	12.84%	8.64%	3.52%
15.69%	9.27	1.692%	11.09%	17.53%	33.42%	12.96%	12.67%	8.86%	3.46%
17.00%	9.71	1.752%	9.80%	15.72%	36.58%	12.90%	12.35%	9.19%	3.46%
19.82%	10.77	1.840%	8.13%	11.67%	42.82%	12.38%	11.59%	9.89%	3.53%

Figure 5.4.2: RMV Efficient Frontiers, 20-Year

Contains the averaged Efficient Frontiers for the Historical and Normal data generation methods.



(a) Historical RMV EFF

Table 5.8: Efficient Frontier Portfolio Weights, 10-Year, Historical SA-Only

Red indicates the maximum allocation, while blue indicates the minimum allocation.

Annual Return	Annual St. Dev	Risk Adjusted Return	SA Cash	SA Bonds	SA Equity	SA Property
12.31%	8.12	1.516%	35.58%	22.03%	28.36%	14.04%
13.39%	8.51	1.573%	31.19%	22.98%	30.56%	15.27%
14.35%	8.77	1.636%	27.63%	24.09%	32.42%	15.87%
15.29%	9.09	1.682%	24.57%	24.32%	34.92%	16.19%
16.26%	9.37	1.736%	21.71%	24.81%	37.25%	16.23%
17.31%	9.77	1.773%	18.97%	24.25%	40.72%	16.07%
18.52%	10.23	1.810%	16.23%	23.57%	44.31%	15.89%
20.18%	10.73	1.881%	14.45%	21.46%	47.95%	16.14%
24.05%	11.50	2.091%	12.15%	18.41%	53.84%	15.59%

Examining the RMV efficient frontier for the historical method of data generation it is clear that the RI portfolio is substantially better than the SA only portfolio. It has almost exactly the same shape as the SA only efficient frontier but it has shifted to the left indicating a significant reduction in standard deviation for the portfolio.

The normal data generation method again provides completely different results. According to the normal efficient frontiers, the SA only portfolio is the better portfolio.

The weight distribution for the portfolios on the efficient frontiers are given below:

Table 5.9: Efficient Frontier Portfolio Weights, 20-Year, Normal RI

Red indicates the maximum allocation, while blue indicates the minimum allocation.

Annual Return	Annual St. Dev	Risk Adjusted Return	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property
13.13%	6.71	1.956%	32.59%	19.05%	14.96%	8.40%	13.59%	7.99%	3.42%
14.05%	7.62	1.843%	19.85%	24.36%	20.51%	10.29%	13.91%	8.78%	2.31%
14.55%	8.07	1.802%	15.61%	25.24%	23.35%	10.80%	13.46%	9.27%	2.26%
14.96%	8.55	1.749%	12.91%	24.67%	26.52%	10.90%	13.05%	9.62%	2.34%
15.33%	8.89	1.724%	10.99%	24.13%	29.01%	10.88%	12.84%	9.81%	2.35%
15.70%	9.15	1.715%	9.61%	23.53%	30.76%	11.10%	12.55%	10.04%	2.41%
16.11%	9.40	1.713%	8.87%	22.41%	32.52%	11.21%	12.24%	10.34%	2.42%
16.57%	9.78	1.695%	8.12%	20.61%	35.46%	10.81%	11.91%	10.55%	2.54%
17.15%	10.24	1.674%	7.09%	18.49%	38.41%	11.01%	11.57%	10.86%	2.57%
18.38%	11.09	1.658%	6.11%	14.31%	43.85%	10.73%	11.01%	11.32%	2.68%

For the 20-Year efficient frontier portfolios similar trends can be observed with regard to the weight distribution. Again there is a positive relationship between the return and

Table 5.10: Efficient Frontier Portfolio Weights, 20-Year, Normal SA-Only

Red indicates the maximum allocation, while blue indicates the minimum allocation.

Annual Return	Annual St. Dev	Risk Adjusted Return	SA Cash	SA Bonds	SA Equity	SA Property
13.41%	6.08	2.208%	50.82%	23.62%	14.24%	11.32%
14.35%	8.03	1.786%	33.86%	29.49%	22.95%	13.70%
14.92%	9.02	1.653%	25.14%	32.17%	28.18%	14.51%
15.43%	9.32	1.656%	20.98%	34.38%	29.96%	14.68%
15.92%	9.87	1.614%	17.80%	33.79%	33.74%	14.68%
16.41%	10.44	1.571%	14.96%	32.92%	37.76%	14.35%
16.91%	10.99	1.539%	12.73%	31.54%	41.61%	14.11%
17.47%	11.46	1.524%	10.93%	30.09%	44.92%	14.06%
18.17%	12.02	1.512%	9.72%	27.18%	49.00%	14.11%
19.63%	12.84	1.529%	8.36%	22.65%	54.95%	14.04%

Table 5.11: Efficient Frontier Portfolio Weights, 20-Year, Historical RI

Red indicates the maximum allocation, while blue indicates the minimum allocation.

Annual Return	Annual St. Dev	Risk Adjusted Return	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property
13.15%	8.04	1.635%	23.27%	20.64%	21.67%	9.42%	13.16%	8.94%	2.90%
13.96%	8.27	1.689%	18.71%	22.41%	23.93%	9.95%	12.88%	9.46%	2.66%
14.55%	8.59	1.694%	15.76%	22.44%	26.35%	10.45%	12.43%	10.00%	2.57%
15.13%	8.57	1.766%	14.69%	23.06%	26.35%	10.90%	12.41%	10.07%	2.52%
15.76%	8.73	1.804%	13.28%	23.04%	27.99%	10.69%	12.24%	10.22%	2.54%
16.45%	8.90	1.847%	11.88%	22.54%	30.00%	10.58%	12.27%	10.26%	2.46%
17.27%	9.18	1.880%	10.23%	22.32%	32.07%	10.39%	11.87%	10.56%	2.56%
18.27%	9.75	1.874%	8.63%	19.30%	36.63%	10.44%	11.57%	10.78%	2.64%
20.40%	10.49	1.944%	7.42%	15.25%	42.12%	10.21%	10.73%	11.48%	2.78%

amount invested in equity and the pattern followed by the risk adjusted return is also similar to that of the 10-Year efficient portfolios. It is however interesting to note that the investment in SA Bonds achieves a maximum and then declines as the return of the portfolio increases with SA property being favored above SA Bonds for higher return levels.

5.4.3 30-Year Data period

The efficient frontiers for the historical bootstrap and multivariate normal methods of data generation is given in Figure 5.4.3.

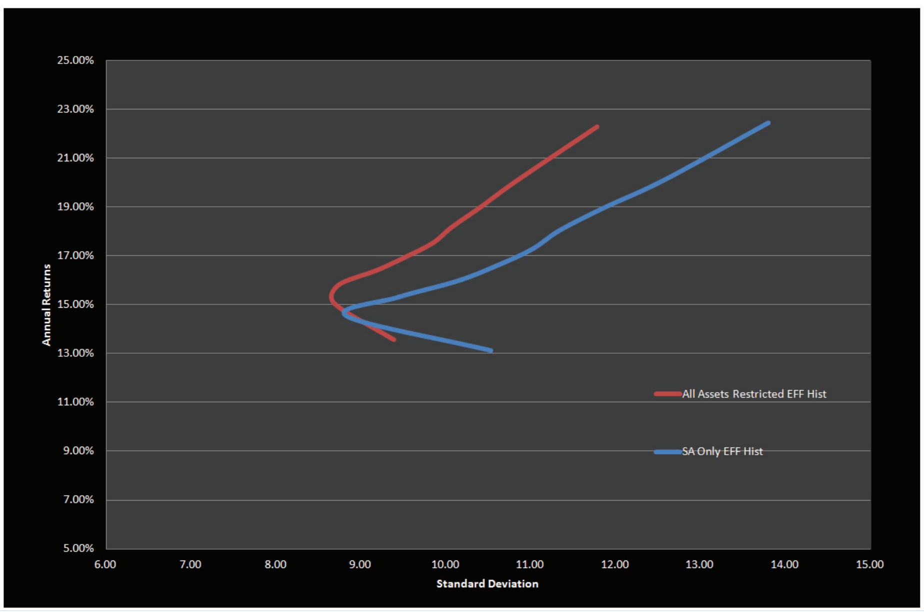
Table 5.12: Efficient Frontier Portfolio Weights, 20-Year, Historical SA-Only

Red indicates the maximum allocation, while blue indicates the minimum allocation.

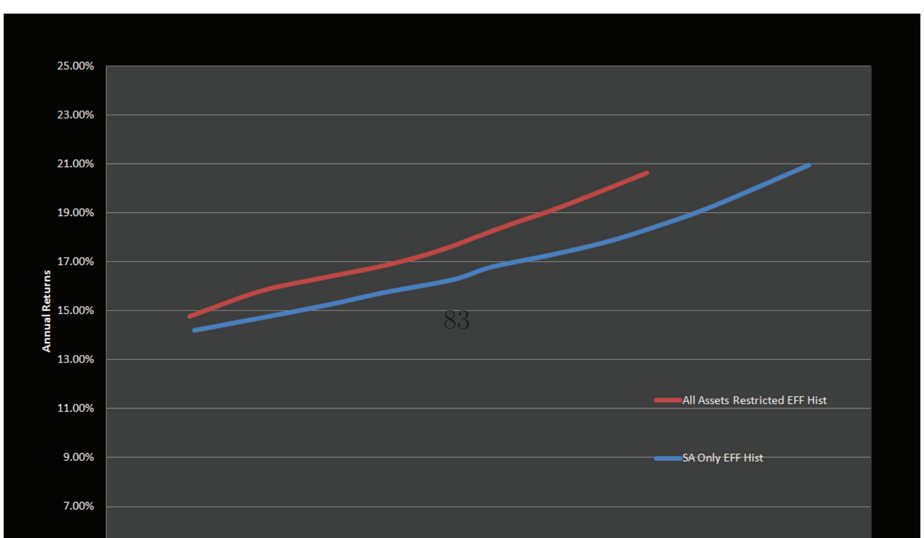
Annual Return	Annual St. Dev	Risk Adjusted Return	SA Cash	SA Bonds	SA Equity	SA Property
13.50%	9.01	1.498%	33.51%	25.95%	28.22%	12.31%
14.26%	9.19	1.552%	30.27%	27.77%	29.57%	12.39%
14.95%	9.57	1.562%	25.99%	29.78%	31.68%	12.55%
15.60%	9.83	1.587%	22.46%	31.45%	33.14%	12.96%
16.28%	10.00	1.628%	19.82%	32.38%	34.53%	13.27%
17.05%	10.31	1.654%	17.64%	32.02%	37.15%	13.19%
17.94%	10.80	1.661%	14.47%	30.89%	41.42%	13.22%
19.09%	11.21	1.702%	12.51%	28.99%	45.14%	13.35%
21.40%	12.08	1.772%	9.68%	23.87%	53.18%	13.27%

Figure 5.4.3: RMV Efficient Frontiers, 30-Year

Contains the averaged Efficient Frontiers for the Historical and Normal data generation methods.



(a) Historical RMV EFF



For the 30-year data period the two types of data generation yields the same results. Both RMV efficient frontiers would suggest that investing in the RI portfolio would provide better returns, because the RI efficient frontiers did not only move upward but also to the left when compared to the SA only RMV efficient frontiers. The hypothesis tests would provide conclusive evidence with regard to the best portfolio and is provided in the next section.

The weight distribution for the portfolios on the efficient frontiers are given below:

Table 5.13: Efficient Frontier Portfolio Weights, 30-Year, Normal RI

Red indicates the maximum allocation, while blue indicates the minimum allocation.

Annual Return	Annual St. Dev	Risk Adjusted Return	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property
14.76%	6.98	2.114%	36.13%	17.72%	13.30%	7.85%	13.17%	8.61%	3.22%
15.79%	7.81	2.023%	27.20%	19.87%	17.77%	10.15%	13.67%	9.32%	2.01%
16.35%	8.56	1.910%	21.77%	20.44%	22.18%	10.62%	13.25%	9.84%	1.91%
16.80%	9.21	1.824%	18.45%	19.62%	26.15%	10.78%	12.78%	10.24%	1.98%
17.21%	9.68	1.778%	16.06%	19.16%	28.96%	10.82%	12.39%	10.56%	2.05%
17.62%	10.06	1.752%	14.60%	18.23%	31.34%	10.83%	12.18%	10.71%	2.11%
18.08%	10.40	1.738%	12.99%	17.80%	33.37%	10.84%	11.92%	10.93%	2.15%
18.60%	10.81	1.720%	11.61%	16.81%	35.83%	10.75%	11.56%	11.19%	2.25%
19.28%	11.37	1.695%	10.00%	15.08%	39.10%	10.81%	11.27%	11.35%	2.38%
20.65%	12.36	1.670%	7.89%	11.64%	44.69%	10.78%	10.69%	11.73%	2.58%

Table 5.14: Efficient Frontier Portfolio Weights, 30-Year, Normal SA-Only

Red indicates the maximum allocation, while blue indicates the minimum allocation.

Annual Return	Annual St. Dev	Risk Adjusted Return	SA Cash	SA Bonds	SA Equity	SA Property
15.14%	8.46	1.789%	39.12%	23.71%	23.49%	13.68%
15.74%	9.27	1.699%	33.19%	24.55%	27.78%	14.48%
16.27%	10.08	1.614%	28.58%	24.60%	32.26%	14.55%
16.80%	10.54	1.593%	24.88%	25.57%	34.97%	14.58%
17.31%	11.27	1.536%	21.18%	25.11%	39.21%	14.50%
17.84%	11.92	1.497%	18.65%	24.02%	42.98%	14.35%
18.47%	12.49	1.478%	16.28%	23.16%	46.24%	14.32%
19.29%	13.15	1.467%	14.41%	21.15%	50.19%	14.25%
20.96%	14.27	1.469%	11.44%	17.91%	56.38%	14.27%

Exactly the same trend is visible for the 30-Year data period as with the 10-Year and 20-Year data periods. The portion invested in Equity again increases as the return of the

Table 5.15: Efficient Frontier Portfolio Weights, 30-Year, Historical RI

Red indicates the maximum allocation, while blue indicates the minimum allocation.

Annual Return	Annual St. Dev	Risk Adjusted Return	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property
15.05%	8.68	1.734%	25.30%	18.99%	21.09%	9.61%	12.69%	9.75%	2.56%
15.80%	8.73	1.810%	23.54%	19.52%	22.13%	9.82%	12.75%	9.95%	2.30%
16.41%	9.18	1.787%	20.15%	19.62%	25.18%	10.04%	12.53%	10.30%	2.17%
16.95%	9.52	1.781%	17.84%	19.37%	27.31%	10.48%	12.39%	10.42%	2.19%
17.53%	9.84	1.781%	16.00%	19.20%	29.24%	10.56%	12.21%	10.61%	2.18%
18.19%	10.07	1.806%	14.55%	18.88%	30.90%	10.68%	11.79%	10.95%	2.26%
19.00%	10.40	1.826%	13.03%	18.02%	33.20%	10.74%	11.65%	11.06%	2.29%
20.09%	10.83	1.855%	11.16%	16.73%	36.45%	10.66%	11.04%	11.49%	2.47%
22.30%	11.78	1.894%	8.98%	12.97%	42.51%	10.54%	10.34%	11.99%	2.67%

Table 5.16: Efficient Frontier Portfolio Weights, 30-Year, Historical SA-Only

Red indicates the maximum allocation, while blue indicates the minimum allocation.

Annual Return	Annual St. Dev	Risk Adjusted Return	SA Cash	SA Bonds	SA Equity	SA Property
14.53%	8.83	1.646%	39.92%	23.49%	24.76%	11.82%
15.31%	9.44	1.623%	34.22%	25.49%	27.86%	12.43%
15.96%	10.13	1.577%	29.36%	26.04%	31.69%	12.91%
16.59%	10.59	1.566%	26.35%	25.72%	34.59%	13.34%
17.27%	11.01	1.568%	24.64%	24.37%	37.52%	13.47%
18.05%	11.34	1.592%	22.23%	24.28%	39.88%	13.62%
18.97%	11.86	1.599%	19.04%	24.14%	43.38%	13.44%
20.13%	12.58	1.600%	15.72%	22.75%	47.92%	13.61%
22.45%	13.80	1.627%	12.45%	18.48%	55.16%	13.91%

portfolio increases and the risk adjusted return for the normal RMV method is exactly the opposite of the historical RMV method.

5.4.4 Hypothesis Tests

The hypothesis tests for the RMV optimization model is done a little differently from the previous hypothesis test. Since only the averaged efficient frontiers are available, the paired t-test is done on the average annual returns of the ten portfolios that make up each efficient frontier. The test is also done on a risk adjusted return basis, which is calculated by dividing the annual return for each portfolio by its corresponding annualised standard deviation. A paired t-test will be used for this test as well. The hypothesis to be tested is as follows:

$$H_0: \quad \mu_D \quad = 0$$
$$H_{1:} \quad \mu_D \quad \neq 0,$$

where $D_i = X_{1,i} - X_{2,i}\, for\, i = 1, ..., 10$, where $X_{1,i}$ is the i'th average annual portfolio return for the SA only portfolio, $X_{2,i}$ is the i'th average annual portfolio return for the restricted international portfolio and μ_D is the population mean difference between the average annual portfolio returns of the two data sets. If $D_i\, for\, i = 1, ..., 10$ is positive, the SA portfolio outperformed the RI portfolio with regard to its average annual return and if it is negative the RI portfolio outperformed the SA portfolio. The results for the test is give in Table 5.17.

Table 5.17: Paired t-test results

The results in the table contains the p-value for the hypothesis test as well as the confidence interval for the mean difference. A colored p-value implies statistical significance, green if the difference is in favor of the RI portfolio, red if the difference is in favor of the SA portfolio.

	$\alpha = 0.05$	**10-Year**	**20-Year**	**30-Year**
Historical RMV	***95% CI***	[0.0214,0.0311]	[0.0043, 0.0075]	[-0.0042, -0.0008]
Efficient Frontier	***p-value***	**0**	**0**	**0.0098**
Historical RMV	***95% CI***	[0.0015, 0.0024]	[-0.002, -0.0011]	[-0.0024,-0.0017]
Risk Adjusted Return	***p-value***	**0**	**0**	**0**
Normal RMV	***95% CI***	[0.0213,0.0287]	[0.0044, 0.0090]	[-0.0053, -0.0009]
Efficient Frontier	***p-value***	**0**	**0.0000109**	**0.011**
Normal RMV	***95% CI***	[0.0006, 0.0056]	[-0.0018,-0.00003]	[-0.0024, -0.0018]
Risk Adjusted Return	***p-value***	**00.02131**	**0.0442**	**0**

The results in Table 5.17 are similar to the results obtained by the traditional MV optimization method in Table 5.4. The null hypothesis is rejected in all cases, historical and normal, returns and risk-adjusted returns. In other words, there is enough evidence to suggest that at a 95% significance level there exists a statistically significant difference in the average annual returns and risk adjusted returns between the SA only and RI portfolio RMV efficient frontiers .

The 10-year data period still have favorable results for the SA only portfolio with regard to the return and risk adjusted returns of the portfolio for the historical and normal data generation process. The SA only portfolio is also the best with regard to returns for the 20-year data period, but not with regard to the risk-adjusted return of the portfolios. Finally, when considering the past 30-years of data, with regard to returns and risk-adjusted returns, the RI portfolio achieved the best results across the board. If $D_i \, for \, i = 1, ..., 10$ is positive, the SA portfolio outperformed the RI portfolio with regard to its average annual return and if it is negative the RI portfolio outperformed the SA portfolio.

It is interesting to note that even though it did not look as if the normal and historical data generation methods yielded the same results when only looking at the plots of the efficient frontiers, the hypothesis tests confirmed that the results from the two methods are similar.

5.5 VaR Optimization, by fitting an EVD

Optimization by fitting an extreme distribution is roughly based on the working paper by Bensalah (2002). According to this paper a EVD is fitted to each of the portfolio returns series generated form 10 000 random portfolios. The optimal portfolio is the portfolio that provides the minimum positive VaR estimate. This VaR estimate is taken as the 5% quantile of the fitted distribution. In the subsections that follows the results for the VaR will always be given as positive values but represent the loss that will not be exceeded in the next month with a certainty of 5%. It is also important to note that the VaR values is a monthly value, because monthly data was used to calculate the VaR.

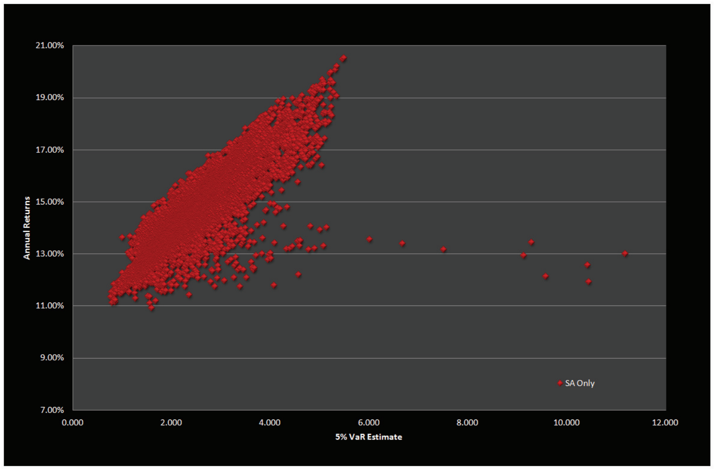
In this section the risk of a the portfolio is represented by the positive VaR value generated for that specific portfolio. The risk-return scatter plots thus consists of the average annual return for the portfolio and the positive monthly VaR value.

5.5.1 10-Year data period

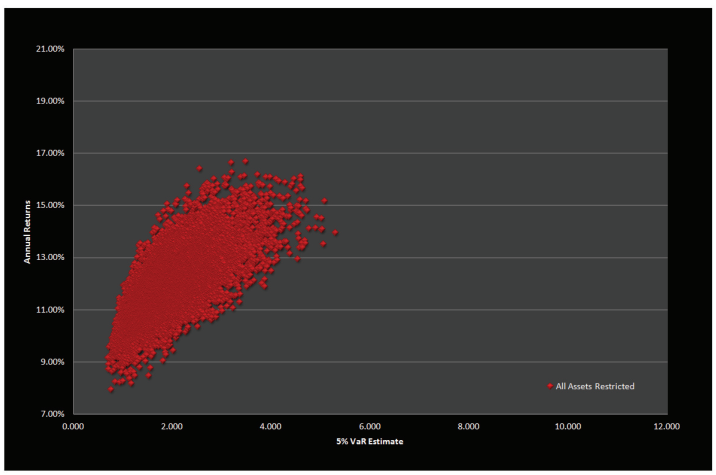
The risk-return scatter plots for the SA only and the Restricted International (RI) portfolios constructed from the 10-year data period are given in Figure 5.5.1.

Figure 5.5.1: Risk-Return Scatter plots, 10-Year

The risk-return scatter plots with the risk component being represented by the VaR of the specific portfolio. The return is the annual return. This is done for all of the 10 000 generated portfolios for the SA only and RI portfolios.



(a) SA Only Portfolio



(b) Restricted International Portfolio

When comparing Figures 5.5.1 (a) and (b) it is evident that the SA portfolios generates returns that is much higher than the returns generated by the RI portfolios. This result makes sense given the results in the previous sections. No obvious difference between the VaR estimates is however evident from the scatter plots.

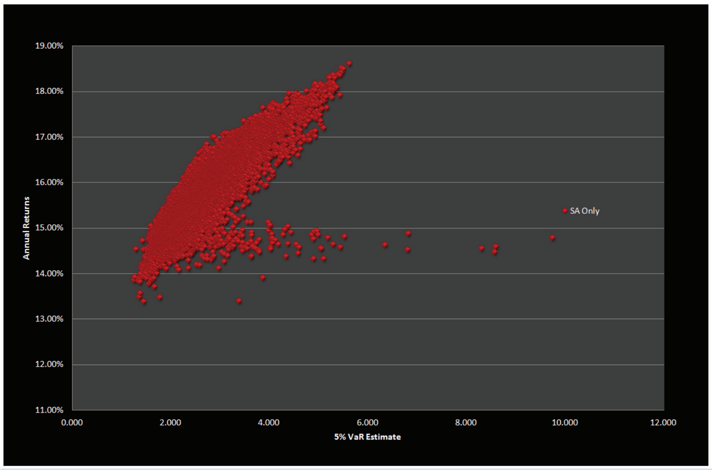
For the SA portfolio, Figure 5.5.1 (a), there is some portfolios that has a VaR estimate that is way out of proportion when compared to the rest of the portfolios. This could be because the fit of the Pareto distribution of the generated return series wasn't that successful.

5.5.2 20-Year data period

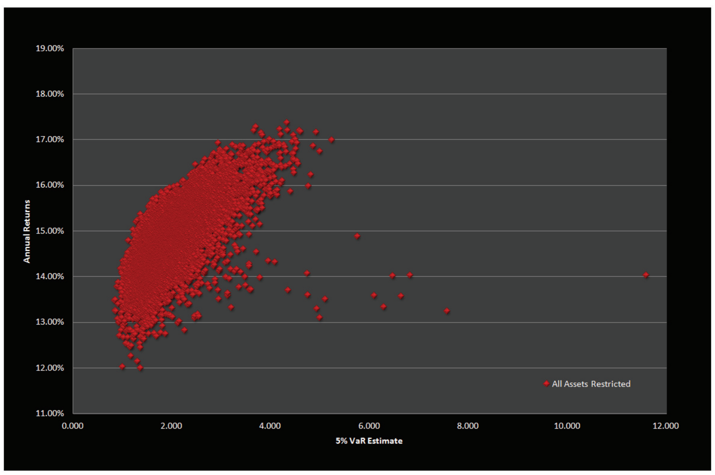
The risk-return scatter plots for the SA only and the Restricted International (RI) portfolios constructed form the 20-year data period are given in Figure 5.5.2.

Figure 5.5.2: Risk-Return Scatter plots, 20-Year

The risk-return scatter plots with the risk component being represented by the VaR of the specific portfolio. The return is the annual return. This is done for all of the 10 000 generated portfolios for the SA only and RI portfolios.



(a) SA Only Portfolio



(b) Restricted International Portfolio

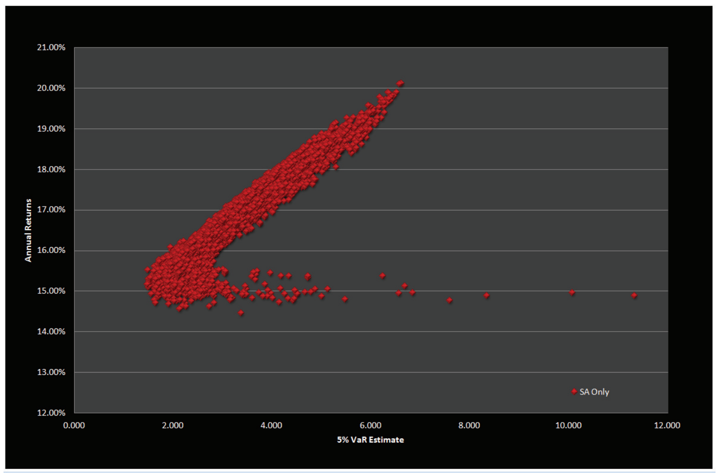
When comparing Figures 5.5.2 (a) and (b) it is evident that the SA only portfolio still provide returns that is higher than the returns generated by the RI portfolios. It does however seem as if the VaR estimates for the RI portfolios is slightly lower than the corresponding estimates for the SA portfolios, which would imply a better risk adjusted return as found in Section 5.3.4. This will be evaluated in Section 5.5.5.

5.5.3 30-Year Data period

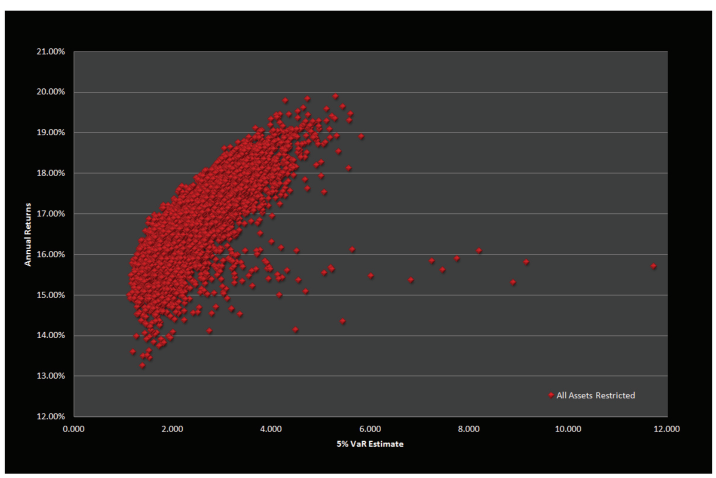
The risk-return scatter plots for the SA only and the Restricted International (RI) portfolios constructed from the 30-year data period is given in Figure 5.5.3.

Figure 5.5.3: Risk-Return Scatter plots, 30-Year

The risk-return scatter plots with the risk component being represented by the VaR of the specific portfolio. The return is the annual return. This is done for all of the 10 000 generated portfolios for the SA only and RI portfolios.



(a) SA Only Portfolio



(b) Restricted International Portfolio

From Figures 5.5.3 (a) and (b) it is evident that there is a more distinct difference between the two subsets of generated portfolios. The RI portfolios looks almost like an efficient frontier scatter plot, while the SA only portfolios has a more linear form. The RI portfolios now have a clear advantage with regard to the estimated VaR levels compared to the VaR levels estimated by the SA only portfolios. One would suspect that the hypothesis tests in the next section would result in a positive result for the RI portfolio.

5.5.4 Portfolio Weight Distribution

The portfolio weight distribution for the EVT optimization model was done on a risk adjusted return basis, with the risk measure being the positive value of the parametric VaR that was achieved for each of the portfolios. The top 100 portfolios where used to derive the range of the weights invested in each asset which is simply the maximum and minimum investments from these 100 portfolios. The results are given in Table 5.18.

Table 5.18: Portfolio Weight Distribution, EVT

Provides the ranges of weights invested in each asset from the top 100 portfolios with regard to risk adjusted return.

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property
10Y SA Only	40.37% - 73.35%	3.86% - 44.24%	2.09% - 24.94%	0.01% - 24.92%	N/A	N/A	N/A
10Y Restricted	29.94% - 63.97%	0.34% - 41.55%	0.08% - 10.23%	0.11% - 19.50%	5.57% - 21.59%	0.12% - 14.41%	0.13% - 15.17%
20Y SA Only	32.20% - 73.88%	0.49% - 40.84%	12.21% - 28.96%	0.01% - 23.55%	N/A	N/A	N/A
20Y Restricted	13.23% - 62.97%	0.39% - 50.19%	0.37% - 14.71%	0.25% - 23.32%	0.05% - 20.32%	0.53% - 22.06%	0.01% - 16.35%
30Y SA Only	34.79% - 65.36%	0.80% - 48.67%	0.05% - 26.37%	0.01% - 24.79%	N/A	N/A	N/A
30Y Restricted	11.93%-58.19%	1.09% - 51.37%	1.20% - 19.62%	0.41% - 22.57%	1.13% - 21.72%	0.36% - 19.50%	0.03% - 10.69%

From Table 5.18 it is evident that the same result as with the RMV were not achieved. Using the parametric VaR as the risk measure shifted the majority of the weight assigned to the assets towards the Cash and Bond part of the portfolio. Equity is no longer the dominant factor in the portfolio, which is to be expected because most of the risk for a downward movement in portfolio value is contained within equity.

5.5.5 Hypothesis Tests

Using the method of Bensalah (2002) the optimal portfolio is chosen as the portfolio with the minimum VaR estimate. These portfolios are given in Table 5.19.

Table 5.19: Optimal Portfolios

Optimal portfolios with VaR estimates for each portfolio as well as the annual return level.

Data Period	SA Cash	SA Bonds	SA Equity	SA Property	Annual Return	5% VaR Estimate
10-Year	64.27%	16.50%	17.93%	1.30%	11.35%	0.776
20-Year	63.75%	7.79%	24.62%	3.84%	13.86%	1.257
30-Year	41.01%	35.32%	9.45%	14.22%	15.18%	1.479

(a) SA Only EVT Portfolios

Data Period	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Annual Return	5% VaR Estimate
10-Year	42.15%	29.98%	2.27%	0.60%	14.32%	3.38%	7.31%	9.17%	0.704
20-Year	44.80%	14.35%	3.27%	12.58%	4.56%	17.30%	3.15%	13.49%	0.840
30-Year	55.09%	2.50%	15.17%	2.24%	10.90%	8.60%	5.50%	15.00%	1.127

(b) Restricted International EVT portfolios

A Paired t-test is used to determine whether there is a significant difference between the returns of the SA portfolios and those of the RI portfolios. The portfolios given in Table 5.19 are used to calculate the monthly returns which are used in the paired t-tests. In other words it is tested if there is a significance difference in the monthly returns of the two portfolios. The risk adjusted return series is also tested for a significant difference. In this case the risk adjusted return is calculated by dividing the monthly return from the portfolio by its corresponding VaR value. In other words the risk adjusted return gives the monthly return per unit of VaR.

The following hypothesis was tested:

$$\begin{aligned} H_0: \quad \mu_D &= 0 \\ H_{1:} \quad \mu_D &\neq 0, \end{aligned}$$

where $D_i = X_{1,i} - X_{2,i} \, for \, i = 1, ..., n$, with n the number of generated monthly returns, $X_{1,i}$ is the i'th monthly return for the SA only portfolio, $X_{2,i}$ is the i'th monthly return for the restricted international portfolio,and μ_D is the population mean difference between the monthly portfolio returns of the two data sets. If $D_i \; for \; i = 1, ..., n$ is positive, the

SA portfolio outperformed the RI portfolio with regard to its monthly return and if it is negative the RI portfolio outperformed the SA portfolio. The results for the test is given in Table 5.17.

Table 5.20: Paired t-test results

The results in the table contains the p-value for the hypothesis test as well as the confidence interval for the mean difference of the monthly returns.

	$\alpha = 0.05$	**10-Year**	**20-Year**	**30-Year**
EVD	***95% CI***	[-0.429, 0.3725]	[-0.1123, 0.1668]	[-0.1483, 0.1739]
Portfolios	***p-value***	0.1190	0.7009	0.8758
Risk Adjusted Returns	***95% CI***	[0.0023,0.0032]	[0.0154,0.0163]	[0.0158,0.0167]
VaR	***p-value***	**0**	**0**	**0**

The results obtained for the Hypothesis test yields no statically significant differences between the two different portfolios. The 10-year data set yields the lowest p-value but this value is not low enough to reject the null hypothesis is only rejected at a 11.9% significance level. Computing the risk adjusted returns, using the VaR for each of the portfolios as the risk measure the same hypothesis was tested and resulted in some surprising results.

The SA portfolio achieved a significantly higher risk adjusted return for all three data periods which is somewhat surprising especially for the generated portfolios of the 30-year data period, given the results in the previous sections. A possible explanation for this could be that the higher returns generated by the SA portfolios, exceeded the benefits of the lower VaR estimates achieved by the RI portfolios.

5.6 C-VaR Optimization

C-VaR or loss given default gives the size of the loss for a portfolio given that the VaR level for a particular time period has been exceeded. In this case the C-VaR is the monthly C-VaR, because monthly return data was used to generate the C-VaR. The optimization process is roughly based on the method used by Xiong and Idzorek (2011). It is a non-parametric method in the sense that no distribution is fitted and no moments are calculated in the procedure. This section will consist of a brief discussion of the risk return scatter plots for each of the portfolios, where the risk measure is the monthly C-VaR estimate. The C-VaR value is given as a positive value and represents a loss. The risk adjusted returns for each portfolio is calculated by dividing the average annual return for the portfolio by the monthly C-VaR value for the corresponding portfolio.

5.6.1 Risk-Return Scatter plots

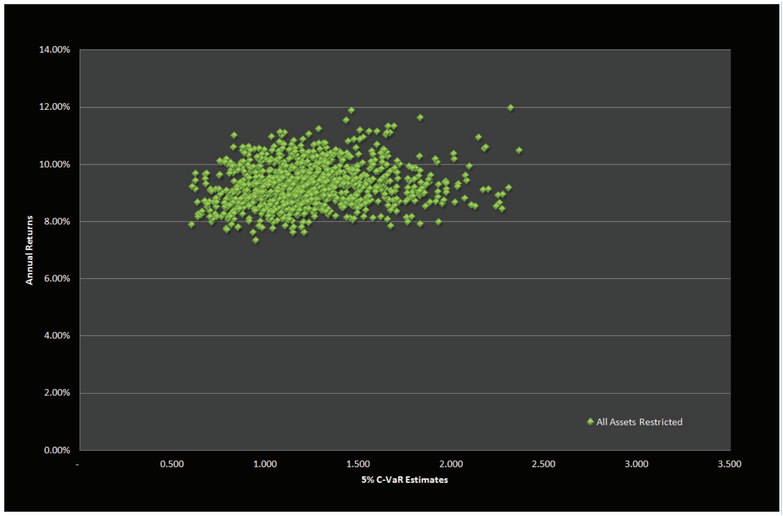
The risk-return scatter plot for the two different portfolio constructions is given in Figures 5.6.1 and 5.6.2.

Figure 5.6.1: SA Only Scatter Plot, 10-Year

The Risk-Return scatter plot for the SA only portfolio with C-VaR as the risk measure.

Figure 5.6.2: Restricted International Scatter Plot, 10-Year

The Risk-Return scatter plot for the RI portfolio with C-VaR as the risk measure.



From Figures 5.6.1 and 5.6.2 it is evident that the SA only portfolio provides a higher return for a lower C-VaR level. The C-VaR is also more centered around one point, which is not the case with the RI portfolio. This could possibly be because the RI portfolio is constructed using seven assets, while the SA portfolio is constructed using only four assets.

Figure 5.6.3: SA Only Scatter Plot, 20-Year

The Risk-Return scatter plot for the SA only portfolio with C-VaR as the risk measure.

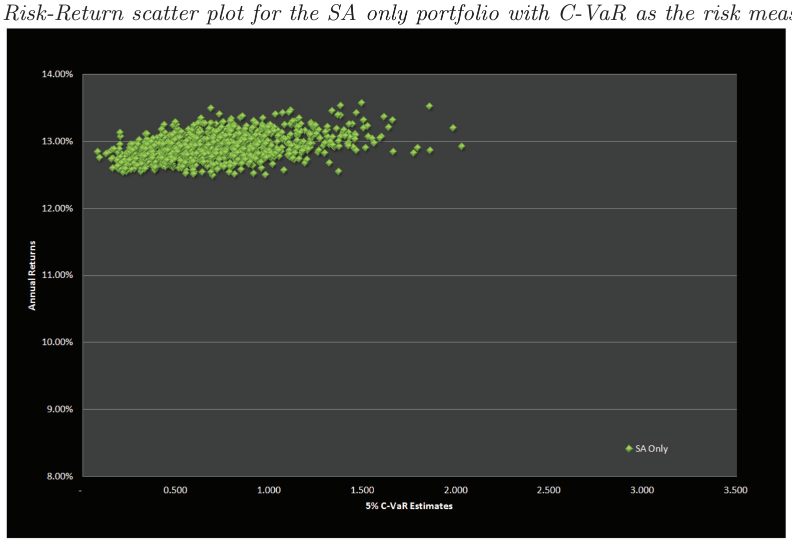
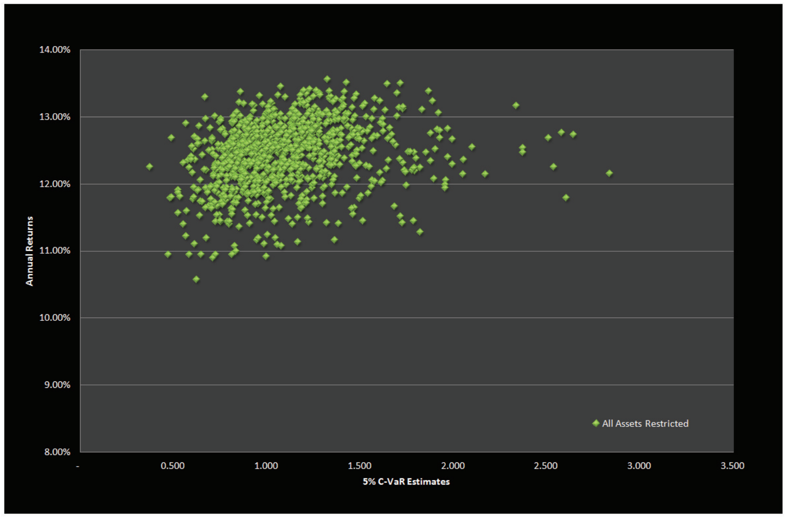


Figure 5.6.4: Restricted International Scatter Plot, 20-Year

The Risk-Return scatter plot for the RI portfolio with C-VaR as the risk measure.



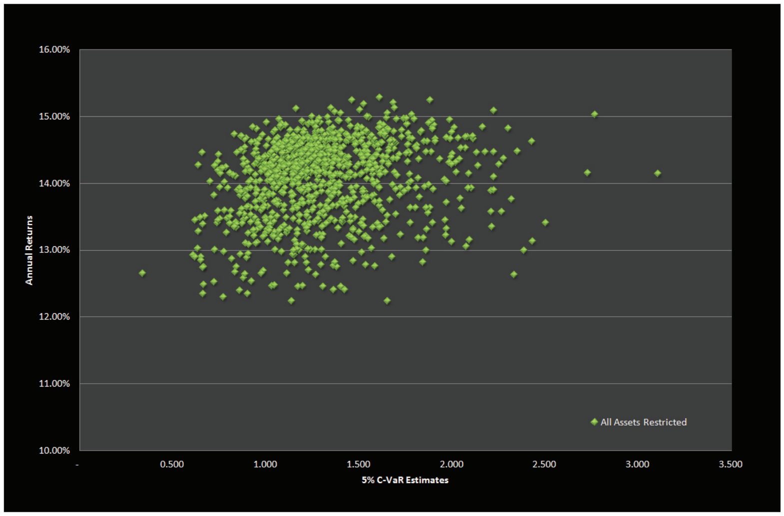
In Figures 5.6.3 and 5.6.4 the same pattern is seen for the 20-year data as with the 10-year data. The points for the SA only portfolio is much closer together compared to the RI portfolio. The RI portfolio also has higher C-VaR values for the corresponding annual return levels.

Figure 5.6.5: SA Only Scatter Plot, 30-Year

The Risk-Return scatter plot for the SA only portfolio with C-VaR as the risk measure.

Figure 5.6.6: Restricted International Scatter Plot, 30-Year

The Risk-Return scatter plot for the RI portfolio with C-VaR as the risk measure.



In Figures 5.6.5 and 5.6.6 it is again evident that the RI portfolio is much more spread out when compared to the SA only portfolio. For the 30-year period as illustrated in Figures 5.6.5 and 5.6.6. It is clear that the RI portfolio has higher return levels for the same C-VaR values when compared to the SA only portfolio.

5.6.2 Portfolio Weight Distribution

The portfolio weight distribution for the C-VaR optimization model was done on a risk adjusted return basis, with the risk measure being the positive C-VaR value that was achieved for each of the portfolios. The top 100 portfolios where used to derive the range of the weights invested in each asset which is simply the maximum and minimum investments from these 100 portfolios. The results are given in Table 5.21.

From Table 5.21 it is evident that the same result as with the RMV were not achieved. The C-VaR method yielded the same results as EVT VaR method, only more conservative. The portion of the portfolio invested in Local Cash is even higher as with the EVT VaR method and the the portion invested in Equity is in some places lower than 10%. It is however not the case with regard to the foreign equity, bonds or property. The only possible reason for this could be the diversification benefits that these assets provide to

Table 5.21: Portfolio Weight Distribution, C-VaR

Provides the ranges of weights invested in each asset from the top 100 portfolios with regard to risk adjusted return.

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property
10Y SA Only	67.72% - 75.00%	10.87% - 27.82%	0.17% - 10.03%	0.02% - 9.94%	N/A	N/A	N/A
10Y Restricted	32.48% - 71.08%	0.09% - 33.57%	0.06% - 12.02%	0.27% - 18.05%	0.72% - 19.97%	0.06% - 16.64%	1.90% - 21.87%
20Y SA Only	67.51% - 74.96%	10.40% - 28.30%	0.07% - 9.95%	0.19% - 10.67%	N/A	N/A	N/A
20Y Restricted	41.62% - 70.26%	0.45% - 25.49%	0.00% - 7.70%	0.25% - 16.06%	1.54% - 18.67%	0.22% - 17.25%	0.03% - 22.72%
30Y SA Only	67.89% - 74.99%	9.91% - 27.56%	0.06% - 8.40%	0.11% - 11.35%	N/A	N/A	N/A
30Y Restricted	41.82% - 72.27%	0.51% - 29.62%	0.03% - 7.70%	0.05% - 11.31%	0.54% - 20.43%	0.31% - 18.30%	0.72% - 21.29%

the portfolio.

5.6.3 Hypothesis Test

For the C-VaR portfolio the annual returns from each of the 1 000 portfolios is used in the paired t-test to determine if there is a significant difference in returns. The risk adjusted return is also tested. The risk adjusted return is calculated by dividing the annual return with its corresponding monthly C-VaR value. The following hypothesis will be tested:

$$\begin{aligned} H_0: \quad \mu_D &= 0 \\ H_{1:} \quad \mu_D &\neq 0, \end{aligned}$$

where $D_i = X_{1,i} - X_{2,i} \, for \, i = 1, ..., 1000$, where $X_{1,i}$ is the i'th average annual portfolio return for the SA only portfolio, $X_{2,i}$ is the i'th average annual portfolio return for the restricted international portfolio, and μ_D is the population mean difference between the average annual portfolio returns of the two data sets. If $D_i \; for \; i = 1, ..., 1000$ is positive, the SA portfolio outperformed the RI portfolio with regard to its average annual return and if it is negative the RI portfolio outperformed the SA portfolio. The results of the test is given in Table 5.22.

The results contained in Table 5.22 is nothing out of the ordinary compared to the results in previous sections. All the null hypotheses are rejected implying a significant difference in the returns of the two portfolios. For the 10 and 20-year data periods the SA

Table 5.22: Paired t-test results

The results in the table contains the p-value for the hypothesis test as well as the 95% confidence interval for the mean difference. A colored p-value implies statistical significance, green if difference is in favor of the RI portfolio, red if the difference is in favor of the SA portfolio.

	$\alpha = 0.05$	**10-Year**	**20-Year**	**30-Year**
C-VaR	***95% CI***	[0.013,0.014]	[0.005, 0.0051]	[-0.0029, -0.0022]
Portfolios	***p-value***	**0**	**0**	**0**
Risk Adjusted	***95% CI***	[0.152,0.173]	[0.102, 0.120]	[-0.0766, -0.0711]
Returns	***p-value***	**0**	**0**	**0**

only portfolio outperformed the RI portfolio with regard to the returns and risk adjusted returns. For the 30-year period the RI portfolio achieves a significantly higher return and risk adjusted return. It is important to note that the risk adjusted return is the average annual return per unit of monthly C-VaR.

5.7 Non-Parametric Optimization

As discussed earlier, the non-parametric optimization method is a direct replication of the method followed by Swartz (2004). For this method three different portfolios is investigated. The SA only portfolio, the Restricted International portfolio and a International portfolio without any restrictions imposed on the portfolio with regard to the percentage invested in foreign assets. Again the statistical tests are done at the end of this section.

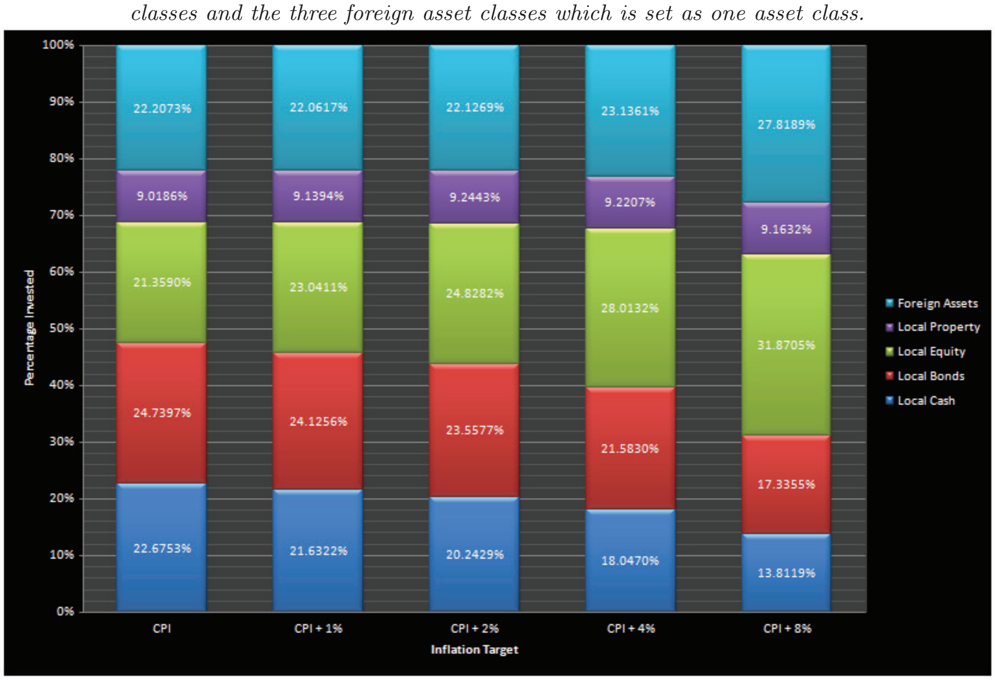
The non-parametric approach involves the use of the probability of outperforming a given benchmark. In this case the benchmark is various CPI + targets over a randomly selected period of 36 months. Not all the results are discussed in this section, only the results regarding the CPI + 2% inflation target. The results for CPI + 0%, 1%, 4% and 8% can be viewed in Appendix D. These results are included because CPI + 2% as done by Swartz (2004) is a fairly conservative benchmark. Again all the results are given in annualized form.

5.7.1 10-Year data period

As was done in the study by Swartz 2004, for each of the data periods a histogram is given were portfolios where constructed without any restriction on the total portion of the portfolio that can be invested in foreign assets. If it happens that the total part of the portfolio invested in foreign assets is above 25% then a portfolio should be constructed containing the full 25% invested in foreign assets. The plot for the portfolio created from the first decile is given in Figure 5.7.1.

Figure 5.7.1: Unrestricted Portfolio Histogram, 10-Year

A Histogram containing the first decile portfolio and the weights assigned to each of the five SA asset classes and the three foreign asset classes which is set as one asset class.



From the histogram in Figure 5.7.1 it is clear that for the 10-year data period only the portfolio that needed to outperform a CPI + 8% target had a foreign component of greater than 25%, this is however a very unrealistic target for a pension fund. A more realistic target is CPI + 2%. The portfolio for the CPI + 2% contains only a 22.1269% investment in foreign assets. The probability of out performance for all 10 deciles can be viewed in Appendix D.1.

This study is however interested in the full 25% of foreign investment against a portfolio only invested in SA assets. Thus far, for the 10-year period, the SA portfolio have dominated with regard to return and risk adjusted return.

Table 5.23: Weights for Out performance of CPI + 2%

These tables contain the weights of portfolios in each decile as well as the probability of outperforming the target and the standard deviation for each of the portfolios for the 10-Year data period.

	SA Cash	**SA Bonds**	**SA Equity**	**SA Property**	**Probability**	**Standard Deviation**
1	43.21%	24.54%	15.87%	16.38%	94.18%	5.33
2	33.04%	31.53%	21.31%	14.12%	92.80%	6.06
3	30.01%	33.00%	24.03%	12.96%	92.04%	6.37
4	29.09%	32.18%	27.08%	11.65%	91.39%	6.65
5	28.10%	32.86%	28.57%	10.47%	90.78%	6.78
6	26.79%	32.15%	31.46%	9.59%	90.14%	7.14
7	26.17%	30.93%	34.46%	8.43%	89.47%	7.49
8	26.89%	30.18%	36.00%	6.92%	88.67%	7.60
9	26.60%	28.80%	38.97%	5.62%	87.62%	7.97
10	29.30%	21.64%	45.42%	3.64%	85.49%	8.86

(a) SA Only Portfolio

	SA Cash	**SA Bonds**	**SA Equity**	**SA Property**	**Int Bonds**	**Int Equity**	**Int Property**	**Probability**	**Standard Deviation**
1	16.18%	22.48%	21.39%	14.95%	15.36%	6.34%	3.31%	86.49%	6.86
2	18.53%	22.35%	22.41%	11.70%	13.17%	8.14%	3.70%	83.80%	6.86
3	19.08%	22.29%	23.60%	10.04%	12.51%	8.77%	3.72%	82.32%	6.97
4	19.65%	22.04%	24.54%	8.78%	11.67%	9.60%	3.73%	81.07%	7.10
5	20.77%	21.57%	24.85%	7.82%	10.91%	10.23%	3.87%	79.89%	7.13
6	21.53%	21.80%	24.87%	6.80%	10.08%	10.84%	4.07%	78.69%	7.13
7	22.27%	22.14%	24.74%	5.85%	9.27%	11.48%	4.25%	77.41%	7.11
8	24.60%	21.63%	23.79%	4.98%	8.35%	12.14%	4.51%	75.96%	6.95
9	27.00%	21.39%	22.53%	4.08%	7.00%	13.45%	4.55%	73.98%	6.81
10	34.48%	22.59%	14.89%	3.04%	6.02%	13.81%	5.17%	69.40%	5.57

(b) RI Portfolio

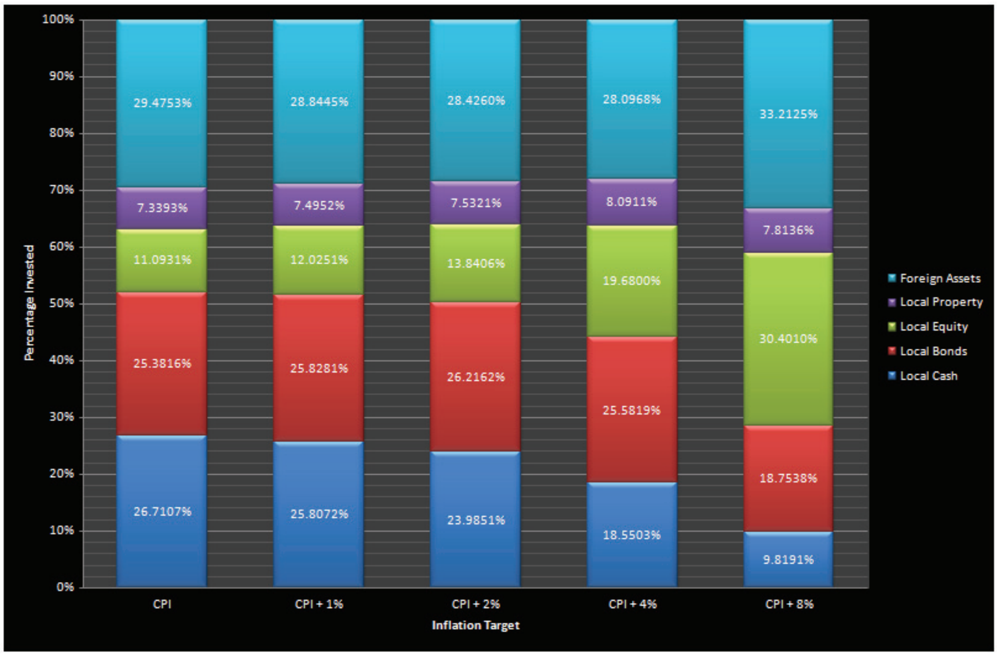
When comparing Tables (5.24a) and (5.24b) it is evident that the SA only portfolio in the first decile has a higher probability of outperforming the CPI + 2% target. When looking at the probability of out performance for the unrestricted portfolio in Appendix D.1, which is at at 83.65%, it is still less than the probability of out performance that is achieved by the SA only portfolio. This results correspond with the results obtained in the earlier sections. It is also interesting to note that the portfolio in the first decile do not have the highest standard deviation as one would expect.

5.7.2 20-Year Data Period

The histogram showing the weights at different inflation targets for the 20-Year data period is given in Figure 5.7.2.

Figure 5.7.2: Unrestricted Portfolio Histogram, 20-Year

A Histogram containing the first decile portfolio and the weights assigned to each of the five SA asset classes and the three foreign asset classes which is set as one asset class.



This time the histogram is drastically different from the histogram of portfolio weights in Figure 5.7.1. The most important difference is that the portion that needs to be invested in the foreign portfolio is now greater than 25% for each of the inflation targets and increases as the inflation target increases.

The decile portfolios are given in Tables (5.25a) and (5.25b).

Table 5.24: Weights for Out performance of CPI + 2%

These tables contain the weights of portfolios in each decile as well as the probability of outperforming the target and the standard deviation for each of the portfolios for the 20-year data period.

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	52.22%	27.10%	9.86%	10.83%	92.70%	4.53
2	40.77%	34.10%	14.52%	10.60%	90.72%	5.65
3	34.62%	35.82%	18.73%	10.84%	89.64%	6.43
4	29.84%	36.85%	22.74%	10.57%	88.75%	7.11
5	27.72%	35.34%	26.89%	10.05%	87.93%	7.67
6	25.98%	33.45%	30.93%	9.65%	87.19%	8.23
7	24.38%	30.33%	35.61%	9.68%	86.41%	8.91
8	23.56%	27.32%	39.93%	9.19%	85.51%	9.52
9	22.85%	22.81%	45.28%	9.06%	84.34%	10.32
10	19.62%	16.43%	55.16%	8.79%	82.30%	11.95

(a) SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	31.92%	25.61%	8.00%	9.47%	14.05%	8.46%	2.49%	91.81%	4.77
2	26.86%	27.43%	11.87%	8.84%	12.58%	9.26%	3.17%	90.27%	5.39
3	24.49%	27.12%	14.84%	8.55%	11.83%	9.55%	3.62%	89.29%	5.82
4	22.16%	26.08%	18.51%	8.24%	11.15%	10.27%	3.58%	88.40%	6.40
5	21.49%	24.37%	21.58%	7.56%	10.51%	10.64%	3.84%	87.60%	6.83
6	20.23%	22.85%	24.22%	7.71%	10.22%	10.65%	4.13%	86.80%	7.24
7	19.87%	21.35%	26.48%	7.30%	9.52%	11.17%	4.31%	85.97%	7.61
8	19.76%	18.92%	29.19%	7.13%	8.91%	11.37%	4.72%	85.01%	8.03
9	18.85%	16.24%	32.82%	7.08%	8.29%	11.80%	4.91%	83.85%	8.65
10	16.90%	12.69%	38.91%	6.49%	6.61%	11.86%	6.54%	81.54%	9.70

(b) RI Portfolio

Comparing the two tables it is again evident that the SA only portfolio has a higher probability of outperforming the inflation target, but in this case the difference between the two portfolios is minimal. It is also interesting that the SA only portfolio still has a higher probability of outperforming the benchmark when compared to the unrestricted International portfolio.

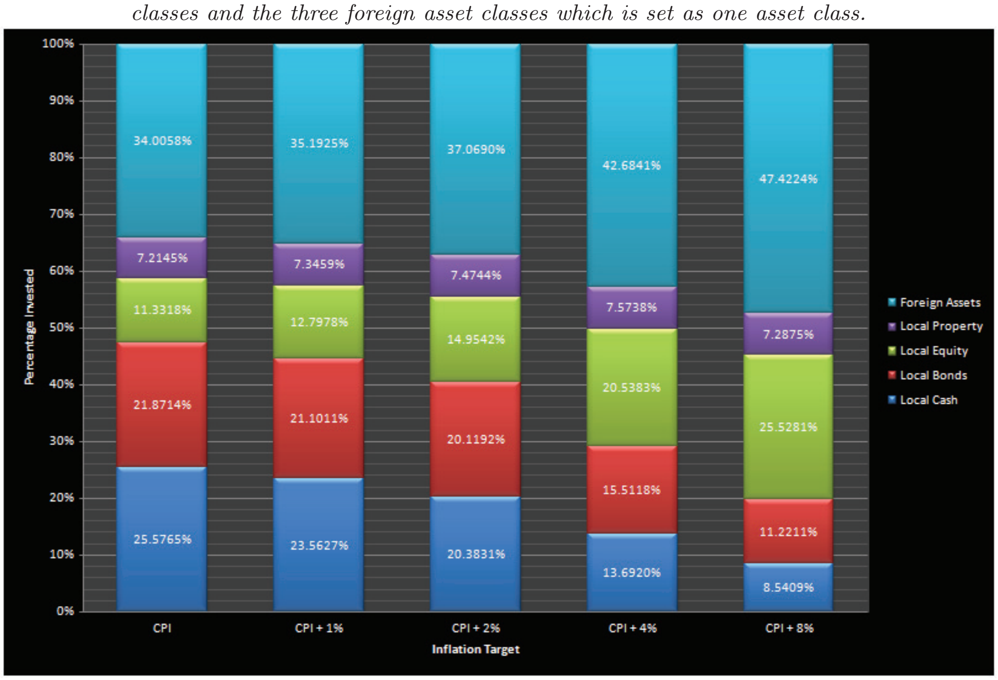
The SA only portfolio also has a lower standard deviation which is interesting because with the MV portfolios the RI portfolio had a higher risk adjusted return and thus a lower standard deviation relative to the SA only portfolio. The result is again approximately the same as the results achieved in the previous sections.

5.7.3 30-Year Data Period

The histogram showing the weights at different inflation targets for the 30-Year data period is given in Figure 5.7.3.

Figure 5.7.3: Unrestricted Portfolio Histogram, 30-Year

A Histogram containing the first decile portfolio and the weights assigned to each of the five SA asset classes and the three foreign asset classes which is set as one asset class.



For the 30-Year data period the international part of each portfolio is a lot more significant even when the inflation target is just equal to CPI. It increases to a point at which approximately 47% of the portfolio should be invested in foreign assets which is a lot irrespective of the inflation target.

The decile portfolios are given in Tables (5.26a) and (5.26b).

Table 5.25: Weights for Out performance of CPI + 2%

These tables contain the weights of portfolios in each decile as well as the probability of outperforming the target and the standard deviation for each of the portfolios for the 30-year data period.

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	51.20%	21.54%	14.83%	12.43%	82.96%	5.65
2	40.54%	27.17%	20.57%	11.73%	81.22%	6.93
3	35.24%	30.32%	23.38%	11.06%	80.37%	7.56
4	31.44%	31.15%	26.85%	10.55%	79.76%	8.21
5	29.20%	31.49%	28.78%	10.53%	79.22%	8.60
6	26.62%	31.73%	31.83%	9.81%	78.68%	9.14
7	24.99%	31.81%	33.83%	9.37%	78.14%	9.49
8	23.68%	31.01%	36.65%	8.66%	77.54%	9.94
9	21.75%	30.29%	39.67%	8.29%	76.81%	10.47
10	19.78%	29.83%	43.18%	7.22%	75.48%	11.06

(a) SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	32.26%	21.91%	10.86%	9.97%	13.77%	9.30%	1.93%	86.41%	5.91
2	27.20%	23.77%	15.28%	8.74%	12.67%	9.85%	2.48%	84.85%	6.60
3	25.05%	24.54%	16.94%	8.47%	11.96%	10.17%	2.87%	83.97%	6.91
4	23.33%	24.23%	19.42%	8.02%	11.28%	10.44%	3.28%	83.27%	7.31
5	21.94%	23.53%	21.62%	7.91%	10.93%	10.54%	3.54%	82.62%	7.67
6	20.86%	23.08%	23.63%	7.43%	10.45%	10.63%	3.93%	81.94%	7.99
7	19.73%	22.35%	25.48%	7.44%	9.84%	10.94%	4.21%	81.25%	8.33
8	18.88%	21.28%	27.65%	7.18%	9.49%	10.72%	4.79%	80.46%	8.67
9	18.10%	19.80%	30.40%	6.70%	8.24%	11.28%	5.49%	79.39%	9.15
10	17.63%	18.87%	32.12%	6.38%	6.74%	9.42%	8.85%	76.96%	9.32

(b) RI Portfolio

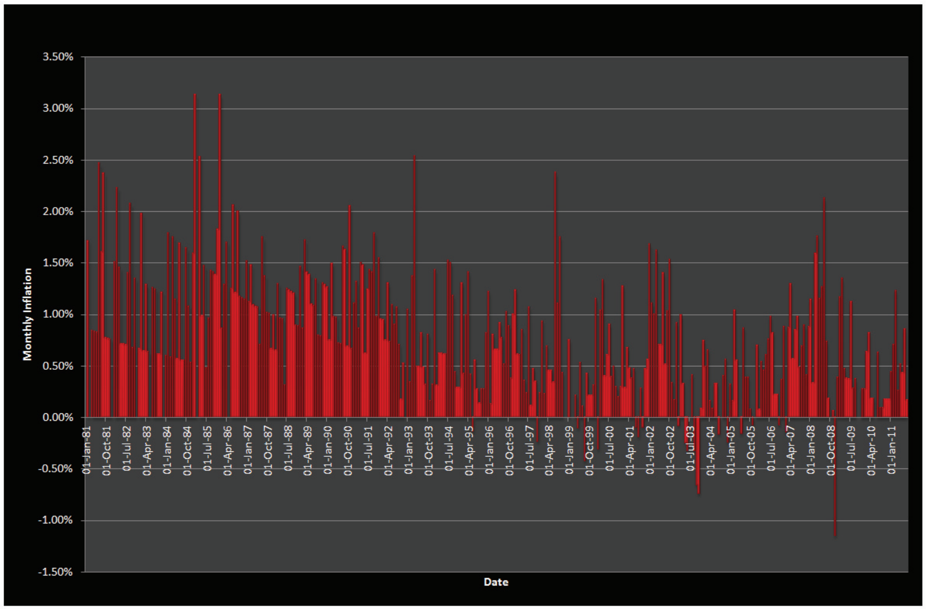
Comparing the two tables it is evident that the International portfolio now have the upper hand with regard to the probability of outperforming the inflation target. The RI portfolio has the highest probability of out performance, even when compared to the Unrestricted portfolio. At 86% it is however significantly less than the probability of outperform ace for the 10 and 20-Year periods.

When comparing the probability of outperforming the CPI for the three different data periods it is evident that the probability of outperform ace decreases as the data period increases. This can be explained by looking at a plot of the monthly log returns for the

CPI index over the past 30 years which is given in Figure 5.7.4.

Figure 5.7.4: CPI Plot

A plot of the monthly CPI value for the past 30 years given as percentage log returns.



The non-parametric method measures the probability that any portfolio will outperform the CPI + target over a random 36 month period. From Figure 5.7.4 it is evident that the inflation was much higher for the period between 1981 and 1991 than for the period between 1991 to 2001 and the inflation is the lowest for the final 10-year time period. It is thus easier for a portfolio to outperform the CPI + target when only the past 10 years are used in the bootstrap procedure to select the random 36 month period. When the time period increases though higher monthly inflation values are included in the data set form which the bootstrap sample is drawn. This results in a higher value for the inflation over the random 36 month period making it harder for the portfolio to outperform the CPI + target.

5.7.4 Hypothesis Tests

In this section the portfolio in the 1st decile is taken and the monthly returns are generated to determine if there is a statistically significant difference between the monthly returns of the SA only portfolio and the RI portfolio. The same test as with the EW and ERC portfolios will be used as there is only one portfolio that is given for the SA and RI portfolios. The hypothesis to be tested is as follows:

$$H_0: \quad \mu_D \quad = 0$$
$$H_{1:} \quad \mu_D \quad \neq 0,$$

where $D_i = X_{1,i} - X_{2,i}\ for\ i = 1, ..., n,$ with n the number of generated monthly returns, $X_{1,i}$ is the i'th monthly return for the SA only portfolio and $X_{2,i}$ is the i'th monthly return for the restricted international portfolio, and μ_D is the population mean difference between the monthly portfolio returns of the two data sets. If D_i is positive, the SA portfolio outperformed the RI portfolio for that specific month and if it is negative the RI portfolio outperformed the SA portfolio for that specific month. The results are given in Table 5.26

Table 5.26: Paired t-test results

The results in the table contains the p-value for the hypothesis test as well as the confidence interval for the mean difference in monthly returns.

	$\alpha = 0.05$	**10-Year**	**20-Year**	**30-Year**
Non Parametric	***95% CI***	[-0.2524, 0.1754]	[-0.1176, 0.1124]	[-0.1702, 0.0432]
Portfolios	***p-value***	0.7224	0.9645	0.2426

Surprisingly non of the portfolios constructed using the probability of outperforming CPI + 2% provides statistically significant results when the monthly returns are compared over the different time periods. The 30-year time period provides the lowest p-value but it is not nearly low enough to reject the null hypothesis. It can thus be concluded that with the non-parametric portfolios at a CPI + 2% level, the null hypothesis cannot be rejected implying no meaningful difference between the SA only and RI portfolios.

5.8 Conclusion

The objective of this chapter was firstly to generate the optimal portfolios for each of the seven different optimization methods. After the optimization has been done for each of the models, hypothesis tests where done to determine whether there is a statistically significant difference in the returns. Where applicable the risk adjusted returns between the SA only and restricted international portfolios were compared.

Most of the hypothesis test arrived at a rejection of the null hypothesis except for the non-parametric optimization method and with the EW and ERC portfolios. In general it seemed that for the 10 and 20-year data periods the SA only portfolio achieved higher returns, while the RI portfolio achieved higher returns when the 30-year data period where used.

With the test regarding the risk-adjusted returns the SA only portfolio again dominated over the past ten years, but for the 20 and 30-year data period the RI portfolios provided better risk-adjusted returns except for the C-VaR optimization model which is an indication of the benefits of international diversification.

Chapter 6

Optimum Optimization method

6.1 Introduction

In this chapter it is seek to determine which of the optimization methods used in Chapter 5 is the best. One optimal portfolio from each optimization method for each time period is chosen. The returns from each portfolio are calculated and compared to different benchmarks to calculate the probability of out performance. If possible, one optimization method which is the best is choosed.

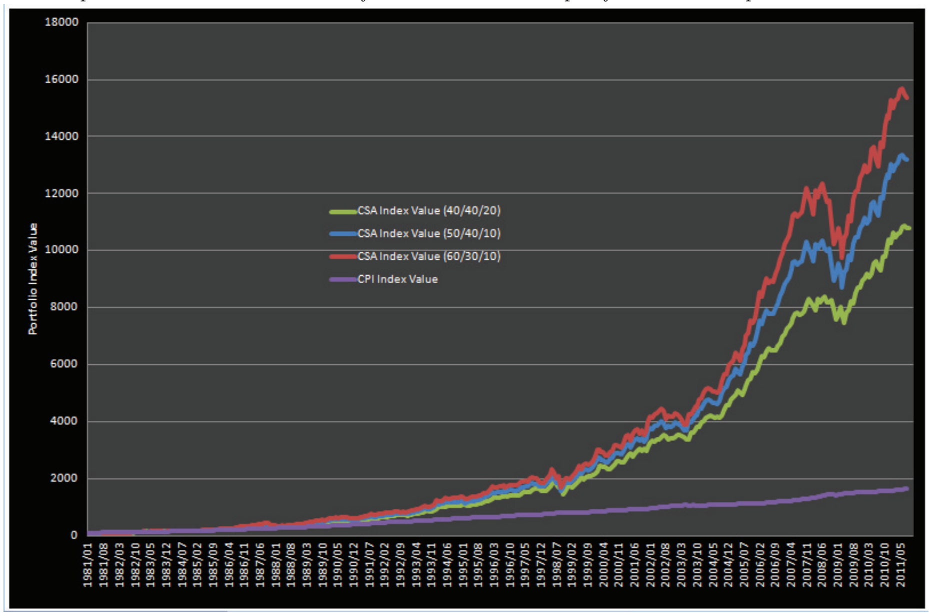
In this chapter the time period was best for constructing the optimal portfolio.

6.2 Benchmarks and Methodology

The benchmarks that are used to compare the different portfolios will be CPI + 4% (CPI) and 3 conservative South African (CSA) portfolios. CPI + 4% was used because it is a more realistic target for a pension fund while not being overly aggresive. Different benchmarks are used because the CPI and CSA portfolios do not necessarily move in the same direction as illustrated in Figure 6.2.1.

Figure 6.2.1: CPI and CSA Portfolio

The plots shows the movement of the CPI and CSA portfolio over the past 30 Years.



The returns of the three benchmarks portfolios were calculated using the weights given in Table 6.1:

Table 6.1: Benchmark portfolio weights

	SA Equity	**SA Bonds**	**SA Cash**
CSA 1	40%	40%	20%
CSA 2	50%	40%	10%
CSA 3	60%	30%	10%

The probability of out performance is calculated using a bootstrap sample of 36 months, randomly selected from the past 10, 20 and 30-Years of data. The average annual return is then calculated for the portfolio as well as the benchmark and compared. This is repeated 100000 times. The number of exceedances is counted and a probability of out performance is calculated by dividing the number of exceedance for each portfolios by 100 000.

6.3 Optimal Portfolios

Choosing the optimal portfolios is not as straight forward as it may seem because some of the Optimization methods generates efficient frontiers like the MV and RMV methods, while the EW and ERC methods only generates one portfolio each. The optimal portfolio is the portfolio that provides the best risk-adjusted portfolio, using the risk measure applicable for the specific portfolio. This method is followed for the MV, RMV, EVD-VaR and C-VaR optimization methods. The EW and ERC methods only provides one portfolio and for the non-parametric optimization method the optimal portfolio is simply the portfolio in the first decile of the CPI + 4% for each data period.

For this evaluation only the RI portfolios are used. This is done due to the fact that there is only four SA asset classes while the RI portfolio is constructed using seven asset classes, which is a better test for the optimization models.

The portfolios that was chosen for the evaluation is given in Table 6.2.

6.4 Probability of out performance

The probability of out performance was calculated for each of the portfolios, against CPI + 4% and the conservative SA portfolios. This was done for all three time periods using 100 000 bootstrap samples of 36 random months each. In other words the probability of a portfolio to outperform the benchmark over the next 3 years was calculated. The results are given in Tables 6.3, 6.4 and 6.5.

The second column of Tables 6.3, 6.4 and 6.5 indicates the time period that was used to construct the given portfolio while the heading indicates the time period that was used in the bootstrap sampling process.

The results contained in Tables 6.3, 6.4 and 6.5 make for some interesting reading. There is a clear winner and a clear loser. The historical re-sampled portfolio optimization method provides the best probability of outperforming CPI+4% and the CSA benchmark portfolios. It is only beaten once, by the non-parametric optimization approach with a higher probability to outperform inflation.

The C-VaR method of optimization is the clear loser in this case. It achieved the lowest

Table 6.2: Optimal Portfolios

This table contains the optimal portfolios for each of the optimization methods, for each of the data periods that was used for the optimization process.

		SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property
Equally	**Weighted**	18.75%	18.75%	18.75%	18.75%	8.333%	8.333%	8.333%
ERC	*10-Year*	75%	10.01%	2.35%	2.93%	3.11%	2.06%	4.54%
	20-Year	73.67%	7.31%	2.51%	4.13%	4.54%	2.48%	5.36%
	30-Year	71.26%	8.85%	2.68%	3.86%	4.40%	2.65%	6.30%
MV	*10-Year*	21.90%	32.49%	11.90%	8.70%	10.48%	9.97%	4.54%
	20-Year	59.61%	10.26%	3.18%	1.96%	8.92%	1.76%	14.32%
	30-Year	59.49%	9.23%	1.06%	5.22%	11.54%	6.24%	7.22%
RMV - Normal	*10-Year*	10.39%	6.14%	35.64%	18.11%	11.87%	9.39%	12.75%
	20-Year	32.59%	19.05%	14.96%	8.40%	13.59%	7.99%	3.42%
	30-Year	36.13%	17.72%	13.30%	7.85%	13.17%	8.61%	3.22%
RMV - Hist	*10-Year*	8.13%	11.67%	42.82%	12.38%	11.59%	9.89%	3.53%
	20-Year	10.49%	7.42%	15.25%	42.12%	10.21%	10.73%	11.48%
	30-Year	8.98%	12.97%	42.51%	10.54%	10.34%	11.99%	2.67%
EVD-VaR	*10-Year*	40.83%	26.97%	4.25%	2.95%	15.55%	0.86%	8.59%
	20-Year	44.80%	14.35%	3.27%	12.58%	4.56%	17.30%	3.15%
	30-Year	22.26%	38.43%	8.19%	6.12%	16.81%	3.49%	4.70%
C-VaR	*10-Year*	49.31%	15.02%	1.31%	9.35%	9.50%	6.26%	9.24%
	20-Year	56.43%	7.97%	0.58%	10.02%	7.74%	7.83%	9.43%
	30-Year	47.61%	23.20%	1.16%	3.03%	5.22%	1.18%	18.59%
Non-Par	*10-Year*	11.49%	18.76%	30.05%	14.70%	14.27%	7.41%	3.32%
	20-Year	17.85%	30.64%	15.57%	10.95%	14.16%	8.87%	1.97%
	30-Year	18.29%	21.15%	24.50%	11.06%	12.85%	10.51%	1.64%

probability of outperforming the given benchmarks for all three data periods.

With regard to the time periods there is no clear pattern that can be seen. It differs across the different optimization methods, but the 10- and 20-year portfolios seem to outperform the 30-year portfolio in most cases even when the complete data period of 30-years was used, indicating that a shorter data period is much more beneficial for use in the optimization process.

It is important to mention that the probability of out performance were caculated using only the return of the respective portfolios, while the portfolios was chosen on a risk adjusted return basis. Tables 6.3, 6.4 and 6.5 is only for the comparison of the different optimization models and nothing else.

Table 6.3: Probability of Out Performing Benchmark using 10 Years of data

The table gives the probability for each portfolio to outperform the given benchmark. Best probability is given in red and the worst probability is given in blue, the maximum probability of out performance for each model is given in green.

		10	Year	Data	Period
		Inf	***CSA 1***	***CSA 2***	***CSA 3***
Equally	**Weighted**	73.35%	33.49%	26.05%	25.01%
ERC	*10-Year*	34.49%	14.77%	**15.25%**	**16.60%**
	20-Year	**35.68%**	**14.88%**	15.13%	16.41%
	30-Year	34.50%	13.98%	14.57%	15.94%
MV	*10-Year*	**63.54%**	**19.22%**	**18.02%**	**19.11%**
	20-Year	22.95%	10.65%	11.31%	12.71%
	30-Year	34.62%	14.27%	14.37%	15.48%
RMV - Normal	*10-Year*	**72.77%**	**49.57%**	**36.63%**	**28.98%**
	20-Year	63.54%	20.76%	18.61%	19.17%
	30-Year	61.14%	19.28%	17.66%	18.37%
RMV - Hist	*10-Year*	74.19%	55.55%	41.32%	30.44%
	20-Year	**84.54%**	**72.32%**	**61.14%**	**53.51%**
	30-Year	72.44%	51.45%	36.62%	26.27%
EVD-VaR	*10-Year*	43.13%	16.64%	16.24%	17.36%
	20-Year	54.07%	15.92%	15.56%	16.75%
	30-Year	**61.10%**	**22.71%**	**20.57%**	**21.24%**
C-VaR	*10-Year*	**44.34%**	**15.64%**	**15.55%**	**16.76%**
	20-Year	40.89%	14.66%	14.61%	15.88%
	30-Year	**22.50%**	**9.61%**	**10.74%**	**12.53%**
Non-Par	*10-Year*	**75.74%**	**47.90%**	**34.07%**	**29.39%**
	20-Year	70.28%	27.98%	23.63%	23.47%
	30-Year	71.49%	32.19%	24.07%	22.75%

6.5 Conclusion

In this chapter the different optimization models were compared using the optimal portfolio from each model which was chosen on a risk-adjusted return basis. A clear winner and loser was found when looking at the different portfolio's probability to outperform a given benchmark. The Historical Re-sampled Mean Variance Optimization method is the best while the C-VaR optimization method was last on all accounts.

With regard to the time period to be used for the optimization process no conclusive results were found, but it seems as if the portfolios constructed form the 10 and 20-year data periods outperformed the portfolios constructed using the 30-year data period.

Table 6.4: Probability of Out Performing Benchmark using 20 Years of data

The table gives the probability for each portfolio to outperform the given benchmark. Best probability is given in red and the worst probability is given in blue, the maximum probability of out perfrmance for each model is given in green.

		20	**Year**	**Data**	**Period**
		Inf	***CSA 1***	***CSA 2***	***CSA 3***
Equally	**Weighted**	76.37%	34.92%	31.40%	31.52%
ERC	*10-Year*	**70.63%**	23.60%	**24.48%**	**25.70%**
	20-Year	68.97%	**24.12%**	24.44%	25.60%
	30-Year	68.25%	23.53%	23.99%	25.22%
MV	*10-Year*	**77.94%**	**32.96%**	**30.69%**	**31.31%**
	20-Year	53.78%	20.84%	21.55%	22.89%
	30-Year	65.84%	25.60%	25.42%	26.32%
RMV - Normal	*10-Year*	74.74%	**48.03%**	**39.73%**	**36.30%**
	20-Year	**77.74%**	32.16%	30.13%	30.51%
	30-Year	77.62%	31.20%	29.76%	30.28%
RMV - Hist	*10-Year*	75.90%	54.61%	44.24%	38.98%
	20-Year	**79.98%**	**55.81%**	**49.37%**	**46.54%**
	30-Year	75.84%	54.21%	44.01%	38.64%
EVD-VaR	*10-Year*	70.70%	26.54%	26.21%	27.32%
	20-Year	75.88%	30.76%	29.45%	29.92%
	30-Year	**78.06%**	**32.72%**	**31.00%**	**31.64%**
C-VaR	*10-Year*	**68.92%**	**25.86%**	**25.71%**	**26.73%**
	20-Year	66.05%	25.06%	25.10%	26.07%
	30-Year	**52.80%**	**19.29%**	**20.40%**	**22.12%**
Non-Par	*10-Year*	78.29%	**46.96%**	**38.51%**	**36.54%**
	20-Year	**80.39%**	38.51%	34.61%	34.47%
	30-Year	78.80%	41.57%	35.51%	34.61%

Table 6.5: Probability of Out Performing Benchmark using 30 Years of data

The table gives the probability for each portfolio to outperform the given benchmark. Best probability is given in red and the worst probability is given in blue, the maximum probability of out perfrmance for each model is given in green.

		30	**Year**	**Data**	**Period**
		Inf	***CSA***	***CSA 2***	***CSA 3***
Equally	**Weighted**	69.97%	47.62%	26.05%	25.01%
ERC	*10-Year*	46.20%	29.02%	**15.25%**	**16.60%**
	20-Year	**48.31%**	**29.82%**	15.13%	16.41%
	30-Year	48.13%	29.10%	14.57%	15.94%
MV	*10-Year*	**70.18%**	**43.13%**	**18.02%**	**19.11%**
	20-Year	41.27%	27.04%	11.31%	12.71%
	30-Year	57.17%	33.51%	14.37%	15.48%
RMV - Normal	*10-Year*	**72.66%**	**64.57%**	**36.63%**	**28.98%**
	20-Year	71.39%	43.68%	18.61%	19.17%
	30-Year	71.03%	42.57%	17.66%	18.37%
RMV - Hist	*10-Year*	72.63%	**70.49%**	41.32%	30.44%
	20-Year	**75.89%**	65.34%	**61.14%**	**53.51%**
	30-Year	72.90%	70.47%	36.62%	26.27%
EVD-VaR	*10-Year*	58.13%	33.01%	16.24%	17.36%
	20-Year	**69.69%**	**41.25%**	15.56%	16.75%
	30-Year	68.40%	39.82%	**20.57%**	**21.24%**
C-VaR	*10-Year*	**58.37%**	**33.40%**	**15.55%**	**16.76%**
	20-Year	56.89%	33.06%	14.61%	15.88%
	30-Year	**36.03%**	**23.94%**	**10.74%**	**12.53%**
Non-Par	*10-Year*	73.48%	**63.18%**	**34.07%**	**29.39%**
	20-Year	73.38%	50.34%	23.63%	23.47%
	30-Year	**73.90%**	57.32%	24.07%	22.75%

Chapter 7

Conclusions and further Research

The main purpose of this study was to determine whether it is optimal to invest in a portfolio consisting of only South African asset classes or to invest in a portfolio consisting of 75% South African Assets and 25% Foreign assets as stipulated in the revised Regulation 28 that took effect on 1 July 2011. The goal was to achieve statistically significant results by generating optimum portfolios using seven different static optimization methods found in the literature. This was done for three time periods, 10, 20 and 30-years and paired t-tests were done to determine if there is a significant difference in returns for each time period.

Next the study sought to determine, if possible, whether one of the optimization methods used are better with regard to out performance of a particular benchmark. The portfolios used for this, was the portfolios that achieved the highest risk adjusted return for each of the optimization methods. From the literature review it became evident that there are no literature comparing optimization models in a South African environment and there is no rule of thumb with regard to the time period that should be used in the optimization process.

7.1 Where to invest, SA only or 25% abroad?

More or less the same results were achieved across all the optimization methods with regard to the SA portfolio question, with a few exceptions. The EW and ERC portfolios provided statistically significant results that favors the SA only portfolio for the 10-year period, but not for the 20 or 30-year periods.

The MV and RMV methods achieved the same results favoring the SA only portfolio for the 10 and 20-year period, but not for the 30-year period, where the RI portfolio gave significantly better returns. For the risk-adjusted returns the SA portfolio were again preferred for the 10-year data period, but not for the 20 and 30-year data periods, which indicates the benefit provided by international diversification.

Optimization by fitting an Extreme Value distribution, did not provide significant results when only the returns were considered, but for the risk-adjusted returns it achieved significant results favoring the SA only portfolio for all three data periods.

The C-VaR optimization method achieved the same results as the MV and RMV optimization methods except for the risk-adjusted return for the 20-year data period which gave significant results favoring the SA only portfolio instead of the RI portfolio.

The non-parametric approach of portfolio optimization did not provide any statistically significant results for any of the data periods.

7.2 Which Optimization method is the best?

Chapter 6 provided probably the most significant results of this study. Using the optimal portfolios on a risk adjusted return basis and computing a probability of outperforming CPI + 4% and a conservative SA only portfolio the best optimization method could be determined. The results gave a resounding yes to the historical RMV method, more specifically the historical RMV portfolio constructed from the 20-year data period. It outperformed the rest of the portfolios in four out of the six cases. With the best performing portfolio over the 30-year data period compared to a conservative SA only portfolio being the 10-year historical RMV portfolio. The C-VaR optimization method achieved the worst results across the board.

With regard to the time period to be used for the optimization process, no conclusive evidence was found, but by considering the results achieved in Chapter 6, it seemed as if the portfolios constructed using the 10 and 20-year data periods had a higher probability of outperforming the benchmarks, compared to the portfolios constructed by using a 30-year data period.

7.3 Areas for further research

This following areas can be considered for further research:

1. The precise percentage to be invested in foreign asset classes would be the next logical step. This study only focused on the portfolios that has invested 25% or 0% in foreign asset classes. It could be that an investment somewhere between 0% and 25% would be optimal.

2. The precise time period to be used for the optimization process. This study briefly touched on this issue and it seems as if a time period of around 20 years is the optimal time period, but a more in depth study is required.

3. The inclusion of dynamic optimization models to compare with the results achieved by these static methods could be explored.

4. The range of different asset classes could be increased to include alternative investment assets like commodities and private equity.

5. The effect of the credit crunch and current debt crises on the portfolio allocation strategy. These two crises had detrimental effects on the world economy and the poor performance of foreign asset classes over the past ten years is probably only due to the time period between 2008 and 2011. It would be interesting to compare an optimized portfolio constructed form a data period that included these two crises and one that did not include these two crises.

Appendix A

Regulation 28 requirements for pension funds

Section 36(1)(bB) of the Pension Funds Act, No 24 of 1956, empowers the Minister of Finance to make regulations limiting the amount and the extent to which a pension fund may invest in particular assets. Of the R5.2 trillion total household savings in South Africa, Regulation 28 currently applies to all private retirement fund assets worth R1.1 trillion.

Regulation 28 prescribes maxima for various types of investments that may be made by a retirement fund. The maxima relate to the fair value of the assets of the fund under the direct control of the trustees. The prevailing maxima applicable to this study are broadly as follows

- Not more than 75 percent may be invested in equities.
- Not more than 25 percent may be invested in property.
- Not more than 90 percent may be invested in a combination of equities and property.
- Not more than 15 percent may be invested in a listed equity with a defined large market capitalization, and not more than 10 percent in any other single equity stock.
- Not more than 20 percent may be invested with any single bank.
- Not more than 15 percent may be invested off-shore, although increased foreign limits

by the South African Reserve Bank are accommodated by the Registrar of Retirement Funds on an application basis.

- Not more than 2,5 percent may be invested in "other assets," which are not specified.

There are no restrictions on investments into bank issued money-market instruments or RSA Government issued bonds. Derivative instruments are not defined, leaving them to fall within the category of "other assets". No guidance is given as to how derivatives may be used.

Regulation 28 has however been revised and the new version has taken effect on 1 July 2011. The new maxima is given below

- Not more than 75% may be invested in corporate and public entity debt.
- Not more than 75% may be invested in equities.
- Not more than 25% in listed property vehicles and not more than 15% in unlisted property vehicles.
- Not more than 10% in listed gold commodities and not more than 5% in other listed commodities.
- Not more than 25% in foreign asset classes as well as an additional 5% in African assets.

A fund should never borrow money for the purpose of investment because of the risks involved (SARB, Febuary 2011 and (Department, March 2011).

Appendix B

Algorithms

B.1 Algorithm (Random portfolio generation for m assets)

Define the number of portfolios as n, as well as the number of asset classes m.

Step *1: Generate m Uniform Random variables.*

Step *2: Compute the sum of the m Random variables and divide each.*

Step *3: Divide each of the m random variables by their sum, to give a portfolio that sums to 1.*

Step *4: Repeat steps 2, 3 and 4 n times to give n random portfolios that sums to 1.*

B.2 Algorithm (Correlated Multivariate normal data generation)

Define:

- *m be the number of asset classes*
- *n be the total number of data points to be generated*
- $x_{i,j}$ $i = 1, ..., n,\ j = 1, ..., m,$ *i.i.d normal variables*
- $y_{i,j}$ $i = 1, ..., n,\ j = 1, ..., m,$ *correlated multivariate random variables*

- μ_j $j = 1, ..., m$, *mean return from sample data*
- $\sigma_{i,j}$ $i = 1, ..., m$, $j = 1, ..., m$, *variance or covariance of asset* i *and* j.
- Q, *upper triangular matrix from Cholesky decomposition.*

Step *1: Calculate the mean return and covariance matrices from existing data.*

Step *2: Do a Cholesky decomposition on the covariance matrix of returns.*

Step *3: Generate n standard normal random variables,* $x_{i,j}$ $i = 1, ..., n$, $j = 1, ..., m$, *resulting in a* n *by* m *matrix.*

Step *4:* $y_{i,j} = x_{i,j}Q + \mu_j$ for $i = 1, ..., n$, $j = 1, ..., m$.

Appendix C

Derivations

C.1 Equation 2.3.5

Define the following variables:

- x_i, be the weight assigned to asset i for portfolio P,
- R_{ij}, is the j*'th* return for asset i and $\bar{R}_i$ is the expected return for asset i,
- $\sigma_i = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(R_i - \overline{R_i})^2}$, is the standard deviation for asset i,
- $\sigma_{ij} = \sqrt{\frac{(R_{ij}-\bar{R}_i)(R_{kj}-\bar{R}_k)}{m}}$, is the covariance between asset i and j for all values of i and j and
- R_P, is the portfolio return and $\bar{R_P} = \sum_{i=1}^{n} x_i\bar{R}_i$ the expected portfolio return.

The portfolio variance, σ_P^2, is then derived from first principles as follows:

$$
\begin{aligned}
\sigma_P^2 &= E\left(R_P - \bar{R_P}\right)^2 \\
\sigma_P^2 &= E\left(\sum_{i=1}^{n} x_i R_{ij} - \sum_{i=1}^{n} x_i \bar{R}_i\right)^2 \\
\sigma_P^2 &= E\left(\sum_{i=1}^{n} x_i \left(R_{ij} - \bar{R}_i\right)\right)^2 \\
\sigma_P^2 &= E\left[\sum_{i=1}^{n} x_i^2 \left(R_{ij} - \bar{R}_i\right)^2 + \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} x_i x_k \left(R_{ij} - \bar{R}_i\right)\left(R_{kj} - \bar{R}_k\right)\right] \\
\sigma_P^2 &= \sum_{i=1}^{n} x_i^2 E\left(R_{ij} - \bar{R}_i\right)^2 + \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} x_i x_k E\left[\left(R_{ij} - \bar{R}_i\right)\left(R_{kj} - \bar{R}_k\right)\right] \\
\sigma_P^2 &= \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k \neq i}}^{n} x_i x_k \sigma_{ik}
\end{aligned}
$$

This result implies equation 2.3.5 since the standard deviation is only the square root of the variance.

C.2 Equation 2.3.13

To arrive at equation 2.12 the following equation needs to be solved for ω.

$$\omega^2 \sigma_1^2 + \omega(1-\omega)\rho\sigma_1\sigma_2 = (1-\omega)^2\sigma_2^2 + \omega(1-\omega)\rho\sigma_1\sigma_2$$

from this equation it is clear that the second term on each side cancels out and this gives

$$
\begin{aligned}
\omega^2\sigma_1^2 &= (1-\omega)^2\sigma_2^2 \\
\frac{\omega^2}{(1-\omega)^2} &= \frac{\sigma_2^2}{\sigma_1^2} \\
\frac{\omega}{1-\omega} &= \frac{\sigma_2}{\sigma_1} \\
\frac{1-\omega}{\omega} &= \frac{\sigma_1}{\sigma_2} \\
\frac{1}{\omega} &= \frac{\sigma_2^{-1}}{\sigma_1^{-1}} + 1 \\
\frac{1}{\omega} &= \frac{\sigma_2^{-1} + \sigma_1^{-1}}{\sigma_1^{-1}} \\
\omega &= \frac{\sigma_1^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}}.
\end{aligned}
$$

C.3 Inverted Pareto Distribution

Let $X_{n,1}, ..., X_{n,n}$ be the ordered data such that $X_{n,1} < ... < X_{n,n}$. Let, t be the sufficiently high threshold, the $(n-k)'^{th}$ order statistic, in other words $t = X_{n,n-k}$.

$$
\begin{aligned}
P\left(\frac{X}{t} > x \,|\, X > t\right) &= x^{-\frac{1}{\gamma}} \\
\frac{P\left(X > xt \cap X > t\right)}{P\left(X > t\right)} &= x^{-\frac{1}{\gamma}} \\
P\left(X > xt\right) &= P\left(X > t\right) x^{-\frac{1}{\gamma}}
\end{aligned}
$$

$$
Let, \quad xt = y \Rightarrow \quad x = \frac{y}{t},
$$

With n observations in the data set $P\left(X>t\right)\approx\frac{k}{n}$, thus implying the following:

$$\begin{aligned}
P\left(X>y\right) &= \frac{k}{n}\left(\frac{y}{t}\right)^{-\frac{1}{\gamma}} \\
P\left(X>x_{\alpha}\right) &\approx \frac{k}{n}\left(\frac{x_{\alpha}}{t}\right)^{-\frac{1}{\gamma}} \\
\frac{x_{\alpha}}{t} &\approx \left[\frac{n}{k}P\left(X>t\right)\right]^{-\gamma} \\
x_{\alpha} &\approx t\left[\frac{n}{k}P\left(X>t\right)\right]^{-\gamma}.
\end{aligned}$$

Hence,$P\left(X>t\right)$ can now be specified to calculate the αquantile.

Appendix D

Results

D.1 Non-Parametric Portfolio Results

D.1.1 10-Year Data Period

D.1.1.1 CPI + 0%

Table D.1: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	51.27%	24.45%	11.66%	12.62%	98.55%	4.29
2	39.05%	32.74%	16.46%	11.74%	97.70%	5.15
3	33.77%	35.01%	19.72%	11.50%	97.15%	5.66
4	29.19%	36.37%	23.37%	11.08%	96.64%	6.18
5	27.58%	35.03%	26.85%	10.54%	96.15%	6.59
6	25.81%	33.25%	31.11%	9.83%	95.64%	7.14
7	23.89%	30.89%	35.61%	9.61%	95.06%	7.80
8	23.52%	28.15%	39.56%	8.76%	94.38%	8.34
9	22.71%	24.27%	44.79%	8.23%	93.41%	9.15
10	22.42%	17.64%	54.06%	5.88%	91.44%	10.57

Table D.2: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	23.84%	23.89%	13.29%	13.98%	15.37%	6.11%	3.53%	94.43%	5.67
2	21.78%	25.34%	16.74%	11.14%	13.48%	7.78%	3.75%	92.82%	6.01
3	21.39%	24.63%	19.35%	9.63%	12.89%	8.20%	3.90%	91.78%	6.28
4	21.54%	23.59%	21.24%	8.63%	11.90%	9.25%	3.85%	90.88%	6.56
5	21.85%	22.58%	22.63%	7.94%	11.06%	9.97%	3.97%	90.00%	6.77
6	21.60%	22.68%	23.63%	7.08%	9.96%	10.93%	4.11%	89.15%	6.96
7	21.67%	21.63%	25.33%	6.36%	9.43%	11.46%	4.11%	88.27%	7.23
8	22.47%	19.99%	26.86%	5.69%	8.39%	12.41%	4.20%	87.22%	7.52
9	22.67%	19.32%	28.37%	4.64%	6.96%	13.44%	4.60%	85.83%	7.82
10	25.27%	16.62%	30.18%	2.93%	4.91%	15.25%	4.84%	83.30%	8.22

Table D.3: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	22.68%	24.74%	21.36%	9.02%	11.99%	5.38%	4.83%	92.47%	6.17
2	19.91%	20.89%	22.42%	7.26%	15.52%	8.48%	5.53%	88.31%	6.90
3	18.50%	19.43%	21.07%	6.88%	17.02%	11.46%	5.64%	85.33%	7.26
4	18.04%	18.59%	19.73%	6.51%	16.80%	14.52%	5.81%	82.77%	7.52
5	17.49%	17.19%	19.21%	6.20%	17.37%	16.99%	5.57%	80.41%	7.87
6	16.39%	16.94%	18.03%	5.89%	17.64%	19.27%	5.85%	78.01%	8.13
7	16.13%	15.82%	17.09%	5.39%	18.43%	20.98%	6.15%	75.49%	8.39
8	15.78%	14.86%	15.53%	5.14%	18.94%	23.21%	6.54%	72.67%	8.65
9	15.88%	14.18%	12.32%	5.01%	20.32%	25.39%	6.89%	69.07%	8.82
10	15.61%	11.65%	8.30%	4.02%	21.75%	30.16%	8.51%	61.73%	9.53

D.1.1.2 CPI + 1%

Table D.4: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	48.64%	23.99%	13.02%	14.35%	96.89%	4.67
2	37.31%	32.08%	17.89%	12.72%	95.73%	5.44
3	31.95%	34.85%	21.32%	11.88%	95.06%	5.92
4	29.79%	34.68%	24.19%	11.35%	94.49%	6.27
5	27.82%	33.92%	27.79%	10.46%	93.90%	6.69
6	25.65%	33.74%	30.91%	9.70%	93.34%	7.11
7	25.17%	31.30%	34.59%	8.94%	92.70%	7.57
8	24.14%	28.73%	38.76%	8.38%	91.98%	8.18
9	24.21%	25.56%	42.99%	7.24%	90.96%	8.76
10	24.55%	18.95%	51.73%	4.77%	88.87%	10.08

Table D.5: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	20.26%	23.34%	16.79%	14.61%	15.37%	6.28%	3.35%	90.91%	6.19
2	20.14%	24.01%	19.37%	11.48%	13.62%	7.64%	3.75%	88.82%	6.38
3	20.43%	23.16%	21.33%	10.08%	12.55%	8.62%	3.83%	87.55%	6.62
4	20.65%	23.10%	22.42%	8.83%	11.91%	9.26%	3.83%	86.45%	6.75
5	20.64%	22.03%	24.52%	7.81%	11.05%	10.11%	3.83%	85.46%	7.07
6	20.99%	22.35%	24.60%	7.07%	10.13%	10.74%	4.13%	84.44%	7.10
7	22.33%	21.45%	25.07%	6.15%	9.30%	11.49%	4.21%	83.40%	7.17
8	23.21%	21.68%	25.00%	5.11%	8.39%	12.13%	4.48%	82.21%	7.16
9	25.22%	19.83%	25.78%	4.17%	7.08%	13.47%	4.45%	80.62%	7.35
10	30.21%	19.34%	22.74%	2.70%	4.93%	15.05%	5.02%	77.45%	6.92

Table D.6: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	21.63%	24.13%	23.04%	9.14%	11.87%	5.53%	4.67%	88.56%	6.43
2	19.21%	20.56%	22.89%	7.41%	15.77%	8.70%	5.45%	83.44%	7.02
3	18.52%	19.29%	21.17%	6.85%	16.62%	11.80%	5.75%	79.95%	7.30
4	17.59%	17.97%	20.45%	6.43%	17.45%	14.45%	5.67%	77.13%	7.66
5	17.74%	17.62%	18.66%	6.15%	17.02%	17.22%	5.58%	74.57%	7.80
6	16.17%	16.60%	18.26%	6.02%	17.36%	19.61%	5.97%	71.99%	8.20
7	16.32%	16.05%	16.55%	5.31%	19.00%	20.63%	6.14%	69.27%	8.32
8	15.97%	14.90%	15.25%	5.07%	19.15%	23.11%	6.55%	66.33%	8.62
9	16.47%	14.80%	11.39%	4.96%	20.29%	25.16%	6.94%	62.58%	8.67
10	16.77%	12.38%	7.39%	3.97%	21.28%	29.62%	8.59%	55.25%	9.28

D.1.1.3 CPI + 2%

Table D.7: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	43.21%	24.54%	15.87%	16.38%	94.18%	5.33%
2	33.04%	31.53%	21.31%	14.12%	92.80%	6.06%
3	30.01%	33.00%	24.03%	12.96%	92.04%	6.37%
4	29.09%	32.18%	27.08%	11.65%	91.39%	6.65%
5	28.10%	32.86%	28.57%	10.47%	90.78%	6.78%
6	26.79%	32.15%	31.46%	9.59%	90.14%	7.14%
7	26.17%	30.93%	34.46%	8.43%	89.47%	7.49%
8	26.89%	30.18%	36.00%	6.92%	88.67%	7.60%
9	26.60%	28.80%	38.97%	5.62%	87.62%	7.97%
10	29.30%	21.64%	45.42%	3.64%	85.49%	8.86%

Table D.8: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	16.18%	22.48%	21.39%	14.95%	15.36%	6.34%	3.31%	86.49%	6.86%
2	18.53%	22.35%	22.41%	11.70%	13.17%	8.14%	3.70%	83.80%	6.86%
3	19.08%	22.29%	23.60%	10.04%	12.51%	8.77%	3.72%	82.32%	6.97%
4	19.65%	22.04%	24.54%	8.78%	11.67%	9.60%	3.73%	81.07%	7.10%
5	20.77%	21.57%	24.85%	7.82%	10.91%	10.23%	3.87%	79.89%	7.13%
6	21.53%	21.80%	24.87%	6.80%	10.08%	10.84%	4.07%	78.69%	7.13%
7	22.27%	22.14%	24.74%	5.85%	9.27%	11.48%	4.25%	77.41%	7.11%
8	24.60%	21.63%	23.79%	4.98%	8.35%	12.14%	4.51%	75.96%	6.95%
9	27.00%	21.39%	22.53%	4.08%	7.00%	13.45%	4.55%	73.98%	6.81%
10	34.48%	22.59%	14.89%	3.04%	6.02%	13.81%	5.17%	69.40%	5.57%

Table D.9: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	16.18%	22.48%	21.39%	14.95%	15.36%	6.34%	3.31%	86.49%	6.86%
2	18.53%	22.35%	22.41%	11.70%	13.17%	8.14%	3.70%	83.80%	6.86%
3	19.08%	22.29%	23.60%	10.04%	12.51%	8.77%	3.72%	82.32%	6.97%
4	19.65%	22.04%	24.54%	8.78%	11.67%	9.60%	3.73%	81.07%	7.10%
5	20.77%	21.57%	24.85%	7.82%	10.91%	10.23%	3.87%	79.89%	7.13%
6	21.53%	21.80%	24.87%	6.80%	10.08%	10.84%	4.07%	78.69%	7.13%
7	22.27%	22.14%	24.74%	5.85%	9.27%	11.48%	4.25%	77.41%	7.11%
8	24.60%	21.63%	23.79%	4.98%	8.35%	12.14%	4.51%	75.96%	6.95%
9	27.00%	21.39%	22.53%	4.08%	7.00%	13.45%	4.55%	73.98%	6.81%
10	34.48%	22.59%	14.89%	3.04%	6.02%	13.81%	5.17%	69.40%	5.57%

D.1.1.4 CPI + 4%

Table D.10: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	24.64%	27.27%	29.24%	18.84%	86.38%	7.67
2	24.56%	28.43%	31.41%	15.59%	84.59%	7.67
3	26.08%	28.59%	31.86%	13.47%	83.56%	7.51
4	27.04%	28.55%	32.35%	12.06%	82.64%	7.43
5	27.26%	29.75%	32.44%	10.55%	81.74%	7.33
6	28.18%	30.10%	33.03%	8.69%	80.78%	7.25
7	30.22%	29.68%	32.73%	7.37%	79.68%	7.06
8	31.58%	30.54%	32.37%	5.52%	78.37%	6.84
9	34.82%	30.21%	30.75%	4.22%	76.76%	6.44
10	44.84%	34.67%	17.02%	3.48%	71.89%	4.31

Table D.11: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	11.49%	18.76%	30.05%	14.70%	14.27%	7.41%	3.32%	75.77%	8.21
2	14.37%	20.45%	29.08%	11.10%	12.21%	9.12%	3.67%	71.83%	7.95
3	16.82%	20.44%	28.53%	9.20%	11.71%	9.76%	3.53%	69.66%	7.78
4	18.62%	21.77%	26.51%	8.11%	11.18%	10.04%	3.79%	67.80%	7.41
5	20.86%	21.67%	25.10%	7.38%	10.51%	10.52%	3.97%	66.08%	7.17
6	23.47%	21.44%	23.40%	6.68%	10.18%	10.71%	4.12%	64.34%	6.86
7	25.28%	22.11%	21.31%	6.30%	9.43%	11.29%	4.29%	62.40%	6.55
8	26.57%	23.22%	19.62%	5.59%	8.37%	12.09%	4.54%	60.00%	6.32
9	29.76%	24.79%	15.45%	5.00%	8.45%	11.98%	4.57%	56.67%	5.65
10	36.83%	25.62%	8.58%	3.97%	8.03%	11.88%	5.08%	47.76%	4.61

Table D.12: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	18.05%	21.58%	28.01%	9.22%	12.17%	6.41%	4.55%	71.13%	7.33
2	17.29%	18.99%	24.40%	7.63%	16.20%	10.18%	5.31%	63.57%	7.50
3	17.29%	18.05%	22.49%	6.90%	16.34%	13.57%	5.36%	59.49%	7.71
4	16.58%	17.33%	21.18%	6.29%	17.25%	15.87%	5.51%	56.27%	7.96
5	16.45%	17.00%	19.52%	6.00%	17.33%	18.02%	5.69%	53.46%	8.08
6	16.70%	16.24%	17.38%	5.83%	18.70%	19.16%	6.00%	50.82%	8.15
7	17.27%	16.40%	15.19%	5.53%	18.93%	20.38%	6.30%	48.13%	8.11
8	17.81%	16.19%	12.43%	5.33%	19.60%	22.09%	6.55%	44.99%	8.15
9	18.18%	16.78%	9.02%	4.99%	20.16%	23.70%	7.17%	41.22%	8.15
10	20.79%	15.74%	5.44%	3.59%	19.11%	26.46%	8.87%	34.88%	8.24

D.1.1.5 CPI + 8%

Table D.13: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	12.65%	22.12%	48.30%	16.93%	67.73%	10.51
2	18.02%	26.30%	42.68%	13.00%	63.11%	9.24
3	21.48%	28.49%	38.60%	11.44%	60.12%	8.43
4	25.44%	28.44%	36.16%	9.96%	57.37%	7.86
5	28.20%	29.53%	33.11%	9.17%	54.66%	7.29
6	31.90%	29.67%	29.44%	9.00%	51.73%	6.66
7	34.17%	31.06%	26.24%	8.52%	48.51%	6.13
8	36.61%	33.05%	22.44%	7.90%	44.21%	5.54
9	41.72%	33.88%	16.18%	8.22%	37.71%	4.71
10	49.01%	35.27%	10.04%	5.67%	23.63%	3.74

Table D.14: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	8.69%	13.90%	40.10%	12.31%	12.06%	9.52%	3.42%	52.02%	9.89
2	12.99%	19.42%	33.09%	9.50%	10.88%	10.37%	3.75%	45.10%	8.62
3	16.41%	20.50%	29.60%	8.49%	10.75%	10.59%	3.66%	41.13%	7.99
4	19.59%	20.57%	27.25%	7.59%	10.45%	10.70%	3.85%	37.71%	7.55
5	21.78%	21.59%	24.20%	7.43%	10.66%	10.36%	3.98%	34.68%	7.01
6	23.42%	22.75%	21.88%	6.95%	9.95%	10.81%	4.24%	31.62%	6.66
7	25.62%	23.67%	18.65%	7.06%	9.95%	10.96%	4.09%	28.44%	6.18
8	27.69%	25.05%	15.37%	6.89%	10.18%	10.40%	4.42%	24.64%	5.65
9	30.69%	26.34%	11.13%	6.84%	10.05%	10.50%	4.45%	19.73%	5.08
10	37.19%	26.50%	6.35%	4.97%	9.40%	10.59%	5.01%	11.52%	4.35

Table D.15: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	13.81%	17.34%	31.87%	9.16%	13.37%	9.70%	4.75%	43.05%	8.47
2	14.06%	16.58%	26.47%	7.25%	16.27%	14.33%	5.04%	35.11%	8.42
3	15.01%	15.62%	24.03%	6.44%	16.94%	16.55%	5.42%	31.51%	8.46
4	15.46%	16.41%	21.12%	6.10%	18.02%	17.41%	5.49%	28.78%	8.27
5	16.57%	16.25%	18.88%	5.92%	18.68%	17.92%	5.79%	26.52%	8.13
6	17.04%	16.27%	15.92%	6.07%	19.36%	19.26%	6.08%	24.32%	8.06
7	17.78%	17.86%	13.28%	5.81%	19.22%	19.88%	6.17%	22.05%	7.82
8	18.77%	18.33%	10.45%	5.53%	20.00%	20.23%	6.69%	19.57%	7.68
9	21.21%	18.77%	7.94%	5.01%	18.67%	21.12%	7.29%	16.81%	7.41
10	26.68%	20.88%	5.10%	4.02%	15.26%	19.46%	8.61%	12.12%	6.48

D.1.2 20-Year Data Period

D.1.2.1 CPI + 0%

Table D.16: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	53.69%	26.46%	9.44%	10.41%	97.89%	4.38
2	41.40%	34.40%	13.94%	10.25%	96.37%	5.55
3	35.81%	34.85%	18.85%	10.50%	95.38%	6.35
4	31.98%	34.82%	23.01%	10.19%	94.52%	6.99
5	28.33%	34.63%	26.96%	10.08%	93.72%	7.64
6	25.90%	33.52%	30.78%	9.80%	92.98%	8.22
7	24.69%	30.51%	35.28%	9.51%	92.19%	8.84
8	22.57%	28.39%	39.65%	9.39%	91.30%	9.54
9	20.67%	24.17%	45.56%	9.60%	90.13%	10.48
10	16.50%	17.79%	56.17%	9.54%	87.90%	12.26

Table D.17: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	37.21%	22.61%	6.61%	8.57%	13.27%	8.50%	3.23%	97.70%	12.65%4.39
2	28.81%	26.79%	10.76%	8.64%	12.22%	9.21%	3.57%	96.56%	5.18
3	24.77%	27.19%	14.59%	8.44%	11.78%	9.59%	3.63%	95.73%	5.78
4	23.02%	26.31%	17.59%	8.07%	11.12%	9.99%	3.89%	94.96%	6.22
5	21.67%	24.59%	21.05%	7.69%	10.44%	10.55%	4.02%	94.24%	6.75
6	20.04%	23.31%	24.05%	7.60%	10.21%	10.67%	4.13%	93.50%	7.22
7	19.29%	21.47%	26.81%	7.43%	9.64%	11.11%	4.25%	92.75%	7.67
8	17.61%	20.53%	29.55%	7.32%	9.19%	11.26%	4.55%	91.91%	8.15
9	16.77%	17.51%	33.50%	7.22%	8.34%	11.92%	4.75%	90.77%	8.83
10	13.35%	12.32%	41.92%	7.42%	7.48%	12.21%	5.30%	88.43%	10.31

Table D.18: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	26.71%	25.38%	11.09%	7.34%	16.76%	8.25%	4.47%	96.17%	5.24
2	21.43%	23.39%	14.42%	6.78%	17.74%	11.20%	5.04%	94.56%	6.04
3	20.09%	21.26%	16.29%	6.28%	17.33%	13.54%	5.20%	93.61%	6.49
4	18.53%	19.84%	17.65%	6.34%	17.41%	14.81%	5.42%	92.82%	6.84
5	17.11%	19.15%	18.46%	6.14%	16.86%	16.32%	5.94%	92.05%	7.13
6	16.56%	17.85%	19.04%	5.81%	16.61%	17.88%	6.26%	91.28%	7.38
7	15.74%	15.63%	19.56%	5.99%	17.10%	19.44%	6.54%	90.48%	7.69
8	14.90%	14.39%	19.56%	5.65%	17.12%	21.62%	6.76%	89.52%	7.97
9	13.29%	11.67%	20.17%	5.60%	17.76%	24.07%	7.43%	88.19%	8.45
10	11.65%	7.95%	18.20%	4.87%	19.78%	28.14%	9.41%	85.30%	9.00

D.1.2.2 CPI + 1%

Table D.19: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	53.17%	26.83%	9.53%	10.47%	95.97%	4.42
2	41.30%	34.12%	14.06%	10.51%	94.07%	5.57
3	35.45%	35.35%	18.71%	10.49%	92.97%	6.36
4	30.97%	35.87%	22.77%	10.39%	92.04%	7.04
5	27.93%	34.90%	27.01%	10.16%	91.18%	7.67
6	25.76%	33.38%	31.01%	9.84%	90.42%	8.26
7	24.53%	30.57%	35.32%	9.58%	89.61%	8.86
8	23.45%	27.37%	39.95%	9.23%	88.67%	9.53
9	21.20%	24.01%	45.42%	9.37%	87.48%	10.42
10	17.80%	17.14%	55.85%	9.21%	85.29%	12.14

Table D.20: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	35.53%	23.62%	6.95%	8.90%	13.67%	8.42%	2.91%	95.42%	4.50
2	28.32%	26.98%	11.06%	8.64%	12.27%	9.32%	3.41%	94.05%	5.24
3	24.49%	27.28%	14.78%	8.45%	11.78%	9.69%	3.53%	93.07%	5.82
4	22.64%	26.56%	17.64%	8.16%	11.07%	10.10%	3.83%	92.21%	6.26
5	21.69%	24.33%	21.35%	7.64%	10.64%	10.31%	4.05%	91.41%	6.77
6	19.87%	23.09%	24.21%	7.83%	10.20%	10.72%	4.08%	90.60%	7.26
7	19.27%	21.59%	26.80%	7.34%	9.59%	11.16%	4.25%	89.80%	7.67
8	18.78%	19.62%	29.32%	7.28%	9.00%	11.42%	4.57%	88.88%	8.09
9	17.47%	17.20%	33.33%	7.00%	8.39%	11.73%	4.88%	87.68%	8.76
10	14.49%	12.39%	40.97%	7.15%	7.07%	12.15%	5.78%	85.31%	10.13

Table D.21: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	25.81%	25.83%	12.03%	7.50%	16.52%	8.28%	4.05%	93.50%	5.37
2	20.62%	23.18%	15.51%	6.91%	17.93%	11.15%	4.70%	91.49%	6.18
3	19.78%	21.28%	17.41%	6.23%	17.07%	13.04%	5.20%	90.37%	6.58
4	18.07%	20.30%	17.68%	6.33%	17.04%	15.17%	5.41%	89.46%	6.88
5	17.63%	19.17%	18.53%	5.90%	16.06%	16.84%	5.87%	88.60%	7.15
6	16.07%	17.72%	19.43%	6.13%	16.61%	17.76%	6.28%	87.72%	7.44
7	15.76%	15.81%	19.55%	5.78%	16.87%	19.67%	6.55%	86.80%	7.70
8	15.21%	13.88%	19.05%	5.74%	17.55%	21.51%	7.07%	85.75%	7.92
9	14.17%	11.27%	19.17%	5.52%	18.18%	24.10%	7.58%	84.30%	8.34
10	12.91%	8.06%	16.08%	4.76%	20.65%	27.77%	9.77%	81.07%	8.76

D.1.2.3 CPI + 2%

Table D.22: SA Only Portfolio

	SA Cash	**SA Bonds**	**SA Equity**	**SA Property**	**Probability**	**Standard Deviation**
1	52.22%	27.10%	9.86%	10.83%	92.70%	4.53
2	40.77%	34.10%	14.52%	10.60%	90.72%	5.65
3	34.62%	35.82%	18.73%	10.84%	89.64%	6.43
4	29.84%	36.85%	22.74%	10.57%	88.75%	7.11
5	27.72%	35.34%	26.89%	10.05%	87.93%	7.67
6	25.98%	33.45%	30.93%	9.65%	87.19%	8.23
7	24.38%	30.33%	35.61%	9.68%	86.41%	8.91
8	23.56%	27.32%	39.93%	9.19%	85.51%	9.52
9	22.85%	22.81%	45.28%	9.06%	84.34%	10.32
10	19.62%	16.43%	55.16%	8.79%	82.30%	11.95

Table D.23: Restricted International Portfolio

	SA Cash	**SA Bonds**	**SA Equity**	**SA Property**	**Int Bonds**	**Int Equity**	**Int Property**	**Probability**	**Standard Deviation**
1	31.92%	25.61%	8.00%	9.47%	14.05%	8.46%	2.49%	91.81%	4.77
2	26.86%	27.43%	11.87%	8.84%	12.58%	9.26%	3.17%	90.27%	5.39
3	24.49%	27.12%	14.84%	8.55%	11.83%	9.55%	3.62%	89.29%	5.82
4	22.16%	26.08%	18.51%	8.24%	11.15%	10.27%	3.58%	88.40%	6.40
5	21.49%	24.37%	21.58%	7.56%	10.51%	10.64%	3.84%	87.60%	6.83
6	20.23%	22.85%	24.22%	7.71%	10.22%	10.65%	4.13%	86.80%	7.24
7	19.87%	21.35%	26.48%	7.30%	9.52%	11.17%	4.31%	85.97%	7.61
8	19.76%	18.92%	29.19%	7.13%	8.91%	11.37%	4.72%	85.01%	8.03
9	18.85%	16.24%	32.82%	7.08%	8.29%	11.80%	4.91%	83.85%	8.65
10	16.90%	12.69%	38.91%	6.49%	6.61%	11.86%	6.54%	81.54%	9.70

Table D.24: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	23.99%	26.22%	13.84%	7.53%	16.76%	8.18%	3.49%	89.61%	5.62
2	20.35%	22.84%	16.89%	7.05%	17.15%	11.31%	4.42%	87.36%	6.35
3	18.76%	21.78%	17.64%	6.47%	16.71%	13.53%	5.11%	86.11%	6.68
4	18.04%	20.15%	18.76%	6.04%	16.51%	15.13%	5.37%	85.11%	6.99
5	17.19%	18.90%	19.12%	6.22%	16.10%	16.57%	5.91%	84.18%	7.22
6	16.39%	17.50%	19.26%	5.91%	16.39%	18.34%	6.20%	83.25%	7.46
7	15.65%	15.66%	18.84%	5.96%	17.21%	20.02%	6.65%	82.27%	7.67
8	15.76%	13.55%	18.92%	5.54%	17.54%	21.39%	7.30%	81.07%	7.88
9	15.14%	11.31%	17.72%	5.49%	18.72%	23.71%	7.91%	79.51%	8.14
10	14.77%	8.60%	13.44%	4.58%	21.38%	27.09%	10.13%	75.97%	8.42

D.1.2.4 CPI + 4%

Table D.25: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	33.63%	34.85%	18.43%	13.09%	81.57%	6.56
2	32.28%	34.93%	20.84%	11.95%	80.45%	6.83
3	30.65%	35.41%	22.52%	11.42%	79.82%	7.07
4	30.04%	34.36%	24.69%	10.91%	79.28%	7.32
5	30.05%	33.03%	26.61%	10.31%	78.77%	7.51
6	28.63%	31.37%	30.67%	9.33%	78.26%	8.04
7	28.92%	30.17%	31.86%	9.06%	77.71%	8.16
8	28.62%	26.45%	36.25%	8.68%	77.09%	8.74
9	29.00%	22.43%	40.56%	8.02%	76.25%	9.31
10	29.73%	16.53%	47.22%	6.51%	74.62%	10.23

Table D.26: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	17.85%	30.64%	15.57%	10.95%	14.16%	8.87%	1.97%	80.63%	6.24
2	19.09%	27.91%	18.63%	9.37%	12.37%	10.06%	2.57%	78.99%	6.56
3	19.58%	26.05%	20.54%	8.83%	11.89%	10.10%	3.01%	78.09%	6.76
4	20.81%	24.57%	21.30%	8.32%	11.26%	10.53%	3.22%	77.36%	6.82
5	21.21%	23.33%	22.67%	7.79%	10.64%	10.82%	3.54%	76.68%	7.00
6	21.62%	21.18%	24.66%	7.55%	10.30%	10.71%	3.98%	75.99%	7.25
7	22.81%	19.73%	25.15%	7.31%	9.48%	11.29%	4.22%	75.24%	7.32
8	23.59%	18.12%	26.66%	6.63%	8.92%	11.35%	4.74%	74.37%	7.50
9	25.87%	16.38%	26.57%	6.17%	8.18%	11.23%	5.59%	73.21%	7.41
10	30.10%	14.74%	24.69%	5.47%	6.46%	10.06%	8.48%	70.28%	6.94

Table D.27: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	18.55%	25.58%	19.68%	8.09%	16.09%	9.35%	2.65%	78.53%	6.60
2	17.47%	22.13%	20.91%	6.88%	16.00%	12.77%	3.83%	76.10%	7.04
3	16.68%	20.73%	20.83%	6.61%	15.85%	14.55%	4.76%	74.69%	7.22
4	16.90%	19.19%	20.23%	6.28%	15.85%	16.32%	5.22%	73.59%	7.32
5	17.15%	18.32%	19.13%	5.92%	16.52%	17.01%	5.93%	72.53%	7.27
6	16.75%	16.88%	18.37%	6.05%	16.99%	18.50%	6.47%	71.48%	7.39
7	17.37%	15.47%	17.36%	5.71%	17.41%	19.76%	6.93%	70.34%	7.43
8	17.15%	14.72%	16.08%	5.28%	18.30%	20.84%	7.63%	68.95%	7.48
9	17.91%	13.14%	13.43%	5.23%	19.26%	22.56%	8.47%	67.13%	7.49
10	20.09%	10.35%	8.41%	4.74%	22.20%	23.62%	10.58%	62.83%	7.45

D.1.2.5 CPI + 8%

Table D.28: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	9.42%	28.06%	49.24%	13.28%	60.83%	11.67
2	14.52%	31.51%	42.62%	11.35%	58.31%	10.42
3	20.17%	30.74%	38.76%	10.33%	56.51%	9.57
4	24.55%	30.44%	35.30%	9.71%	54.84%	8.86
5	29.00%	28.95%	32.79%	9.26%	53.20%	8.28
6	32.58%	28.50%	29.74%	9.18%	51.49%	7.70
7	35.50%	29.56%	26.19%	8.74%	49.52%	7.08
8	38.65%	31.08%	20.82%	9.44%	46.83%	6.32
9	43.51%	31.58%	15.28%	9.63%	42.65%	5.49
10	53.64%	29.13%	8.91%	8.32%	32.90%	4.26

Table D.29: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	7.07%	18.69%	38.96%	10.27%	10.59%	11.74%	2.67%	55.98%	10.03
2	11.20%	22.75%	32.32%	8.74%	10.32%	11.32%	3.36%	52.67%	8.81
3	15.05%	23.35%	28.20%	8.39%	10.42%	11.21%	3.37%	50.51%	8.05
4	18.42%	22.71%	25.88%	7.99%	10.37%	10.93%	3.70%	48.57%	7.57
5	21.11%	22.44%	24.06%	7.39%	10.17%	10.93%	3.90%	46.66%	7.19
6	23.83%	21.84%	22.15%	7.18%	10.25%	10.58%	4.17%	44.72%	6.80
7	25.63%	22.82%	19.32%	7.23%	10.37%	10.18%	4.44%	42.54%	6.33
8	28.48%	23.10%	16.09%	7.33%	10.52%	9.90%	4.58%	39.66%	5.79
9	31.48%	24.19%	12.17%	7.16%	10.35%	9.52%	5.13%	35.69%	5.18
10	40.27%	20.76%	7.26%	6.71%	10.32%	8.69%	5.99%	26.88%	4.25

Table D.30: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	9.82%	18.75%	30.40%	7.81%	13.16%	16.52%	3.53%	51.80%	8.93
2	12.27%	17.71%	25.33%	6.66%	15.79%	17.74%	4.49%	48.10%	8.27
3	13.93%	17.16%	22.73%	6.22%	16.66%	18.22%	5.07%	46.00%	7.95
4	15.47%	16.71%	20.99%	6.03%	17.39%	17.90%	5.51%	44.29%	7.68
5	16.13%	16.84%	18.77%	5.74%	17.73%	18.72%	6.07%	42.63%	7.49
6	17.42%	17.85%	16.06%	5.72%	17.91%	18.87%	6.16%	40.95%	7.16
7	19.09%	17.97%	13.86%	5.66%	18.39%	18.39%	6.65%	39.08%	6.85
8	20.07%	17.49%	11.37%	5.57%	19.62%	18.53%	7.35%	36.92%	6.67
9	22.74%	17.76%	8.74%	5.81%	19.68%	17.37%	7.89%	34.14%	6.25
10	29.08%	18.26%	6.19%	5.56%	18.15%	13.01%	9.76%	27.91%	5.34

D.1.3 30-Year Data Period

D.1.3.1 CPI + 0%

Table D.31: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	54.95%	24.06%	10.23%	10.76%	93.39%	4.78
2	42.98%	31.16%	15.25%	10.60%	90.92%	6.06
3	37.20%	33.16%	19.39%	10.25%	89.63%	6.89
4	33.30%	32.92%	23.59%	10.18%	88.57%	7.65
5	29.77%	32.47%	27.77%	9.99%	87.68%	8.40
6	26.63%	32.67%	31.05%	9.65%	86.88%	9.01
7	23.94%	31.67%	34.65%	9.74%	86.06%	9.69
8	22.01%	29.50%	38.94%	9.54%	85.22%	10.42
9	18.18%	27.87%	44.22%	9.72%	84.16%	11.44
10	15.46%	20.84%	54.48%	9.23%	82.34%	13.21

Table D.32: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	37.28%	21.80%	7.45%	8.47%	13.55%	8.73%	2.72%	94.64%	5.20
2	30.04%	24.85%	11.62%	8.49%	12.10%	9.73%	3.17%	93.17%	6.00
3	26.15%	25.75%	14.70%	8.40%	11.80%	9.83%	3.37%	92.23%	6.54
4	24.20%	25.44%	17.29%	8.07%	10.98%	10.31%	3.71%	91.45%	6.97
5	21.63%	25.02%	20.61%	7.74%	10.83%	10.38%	3.79%	90.71%	7.52
6	20.01%	23.77%	23.48%	7.74%	10.35%	10.71%	3.94%	89.98%	8.02
7	19.36%	21.97%	26.35%	7.31%	10.06%	10.80%	4.14%	89.18%	8.46
8	17.74%	20.81%	29.28%	7.17%	9.42%	11.08%	4.51%	88.34%	8.99
9	15.96%	18.43%	33.00%	7.61%	8.92%	11.02%	5.06%	87.19%	9.68
10	12.60%	15.51%	39.63%	7.26%	7.38%	10.67%	6.95%	84.93%	10.87

Table D.33: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	25.58%	21.87%	11.33%	7.21%	20.23%	10.38%	3.39%	93.02%	6.47
2	21.61%	20.07%	14.33%	6.38%	19.81%	13.54%	4.27%	91.57%	7.24
3	19.93%	19.25%	15.15%	6.26%	19.35%	15.29%	4.78%	90.78%	7.58
4	18.10%	18.62%	16.60%	6.21%	18.85%	16.27%	5.35%	90.14%	7.91
5	17.46%	17.85%	17.09%	6.12%	17.99%	17.66%	5.82%	89.57%	8.12
6	16.43%	16.67%	18.55%	6.04%	17.80%	18.44%	6.08%	89.01%	8.43
7	15.56%	16.47%	19.21%	5.78%	17.34%	19.01%	6.63%	88.41%	8.59
8	15.29%	15.30%	20.15%	5.93%	15.94%	20.36%	7.02%	87.73%	8.84
9	14.84%	14.53%	21.17%	5.88%	14.58%	20.90%	8.10%	86.81%	9.03
10	12.70%	12.19%	23.64%	5.81%	13.17%	22.13%	10.36%	84.77%	9.59

D.1.3.2 CPI + 1%

Table D.34: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	54.52%	22.98%	11.35%	11.15%	88.92%	4.95
2	42.35%	29.74%	16.75%	11.17%	86.61%	6.31
3	37.29%	31.62%	20.61%	10.48%	85.46%	7.04
4	32.47%	33.37%	24.03%	10.13%	84.55%	7.75
5	29.40%	32.43%	27.90%	10.27%	83.81%	8.45
6	26.37%	33.05%	30.92%	9.65%	83.13%	9.01
7	24.63%	30.95%	34.71%	9.71%	82.46%	9.66
8	21.53%	30.63%	38.26%	9.57%	81.70%	10.34
9	18.95%	28.89%	43.04%	9.11%	80.82%	11.18
10	16.91%	22.69%	52.00%	8.40%	79.22%	12.69

Table D.35: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	35.77%	21.71%	8.64%	8.88%	13.75%	8.97%	2.28%	91.13%	5.43
2	29.39%	24.32%	12.56%	8.73%	12.18%	9.89%	2.93%	89.55%	6.17
3	25.66%	25.71%	15.42%	8.20%	12.09%	9.74%	3.17%	88.61%	6.64
4	23.64%	24.79%	18.47%	8.11%	11.30%	10.22%	3.48%	87.81%	7.15
5	22.06%	24.20%	20.54%	8.20%	10.70%	10.58%	3.72%	87.09%	7.53
6	20.10%	24.19%	23.47%	7.24%	10.27%	10.69%	4.04%	86.36%	7.99
7	19.31%	22.47%	25.89%	7.33%	9.95%	10.87%	4.18%	85.62%	8.40
8	18.14%	20.28%	29.21%	7.37%	9.56%	10.90%	4.54%	84.78%	8.96
9	16.56%	18.96%	32.15%	7.33%	8.54%	11.27%	5.19%	83.65%	9.52
10	14.36%	16.73%	37.06%	6.85%	7.02%	10.13%	7.84%	81.37%	10.32

Table D.36: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	23.56%	21.10%	12.80%	7.35%	20.90%	11.49%	2.79%	89.59%	6.85
2	20.33%	19.63%	15.11%	6.51%	20.20%	14.49%	3.73%	88.09%	7.50
3	19.19%	18.96%	16.20%	6.21%	19.20%	15.80%	4.45%	87.27%	7.77
4	17.93%	17.64%	17.06%	6.26%	19.07%	16.86%	5.19%	86.59%	8.06
5	16.95%	17.68%	17.65%	6.04%	18.03%	18.12%	5.52%	85.97%	8.26
6	16.58%	17.06%	18.04%	6.18%	17.50%	18.31%	6.33%	85.41%	8.34
7	16.60%	16.43%	19.00%	5.60%	17.13%	18.56%	6.69%	84.78%	8.46
8	16.15%	15.74%	19.36%	5.93%	15.88%	19.36%	7.59%	84.06%	8.60
9	15.71%	15.68%	20.12%	5.64%	13.91%	20.53%	8.41%	83.10%	8.78
10	14.50%	12.88%	21.89%	5.91%	13.24%	20.46%	11.12%	80.98%	9.12

D.1.3.3 CPI + 2%

Table D.37: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	51.20%	21.54%	14.83%	12.43%	82.96%	5.65
2	40.54%	27.17%	20.57%	11.73%	81.22%	6.93
3	35.24%	30.32%	23.38%	11.06%	80.37%	7.56
4	31.44%	31.15%	26.85%	10.55%	79.76%	8.21
5	29.20%	31.49%	28.78%	10.53%	79.22%	8.60
6	26.62%	31.73%	31.83%	9.81%	78.68%	9.14
7	24.99%	31.81%	33.83%	9.37%	78.14%	9.49
8	23.68%	31.01%	36.65%	8.66%	77.54%	9.94
9	21.75%	30.29%	39.67%	8.29%	76.81%	10.47
10	19.78%	29.83%	43.18%	7.22%	75.48%	11.06

Table D.38: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	32.26%	21.91%	10.86%	9.97%	13.77%	9.30%	1.93%	86.41%	5.91
2	27.20%	23.77%	15.28%	8.74%	12.67%	9.85%	2.48%	84.85%	6.60
3	25.05%	24.54%	16.94%	8.47%	11.96%	10.17%	2.87%	83.97%	6.91
4	23.33%	24.23%	19.42%	8.02%	11.28%	10.44%	3.28%	83.27%	7.31
5	21.94%	23.53%	21.62%	7.91%	10.93%	10.54%	3.54%	82.62%	7.67
6	20.86%	23.08%	23.63%	7.43%	10.45%	10.63%	3.93%	81.94%	7.99
7	19.73%	22.35%	25.48%	7.44%	9.84%	10.94%	4.21%	81.25%	8.33
8	18.88%	21.28%	27.65%	7.18%	9.49%	10.72%	4.79%	80.46%	8.67
9	18.10%	19.80%	30.40%	6.70%	8.24%	11.28%	5.49%	79.39%	9.15
10	17.63%	18.87%	32.12%	6.38%	6.74%	9.42%	8.85%	76.96%	9.32

Table D.39: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	20.38%	20.12%	14.95%	7.47%	22.04%	12.84%	2.19%	85.40%	7.40
2	18.39%	18.59%	16.98%	6.62%	19.98%	16.16%	3.28%	83.85%	7.96
3	18.27%	17.94%	17.05%	6.33%	19.30%	16.98%	4.14%	82.99%	8.05
4	17.30%	17.41%	17.39%	6.24%	19.18%	17.58%	4.91%	82.32%	8.20
5	17.24%	17.08%	17.98%	6.08%	18.10%	18.00%	5.53%	81.70%	8.28
6	16.56%	16.95%	18.17%	5.88%	17.37%	18.73%	6.34%	81.06%	8.39
7	17.11%	16.79%	18.55%	5.75%	16.43%	18.50%	6.87%	80.35%	8.36
8	17.10%	16.40%	19.05%	5.92%	15.53%	18.09%	7.90%	79.57%	8.37
9	17.12%	16.35%	18.42%	5.66%	14.21%	19.26%	8.98%	78.56%	8.39
10	18.03%	15.18%	18.69%	5.66%	12.92%	17.86%	11.66%	76.29%	8.25

D.1.3.4 CPI + 4%

Table D.40: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	19.79%	23.21%	42.46%	14.55%	71.53%	11.40
2	23.01%	25.61%	39.23%	12.15%	70.22%	10.58
3	25.15%	25.97%	37.92%	10.96%	69.45%	10.19
4	26.79%	26.70%	36.05%	10.46%	68.80%	9.79
5	27.89%	28.58%	33.66%	9.87%	68.20%	9.34
6	31.20%	28.36%	31.44%	9.00%	67.57%	8.79
7	31.90%	30.30%	29.18%	8.62%	66.85%	8.40
8	35.22%	32.07%	24.45%	8.26%	65.95%	7.54
9	38.30%	36.00%	17.32%	8.37%	64.50%	6.47
10	45.17%	39.54%	7.87%	7.41%	60.51%	5.08

Table D.41: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	18.29%	21.15%	24.50%	11.06%	12.85%	10.51%	1.64%	75.01%	8.43
2	19.03%	21.07%	25.54%	9.36%	11.89%	10.85%	2.26%	73.50%	8.47
3	20.11%	21.01%	24.95%	8.93%	11.54%	10.81%	2.66%	72.67%	8.31
4	20.56%	21.15%	25.02%	8.27%	11.08%	10.90%	3.02%	71.95%	8.27
5	21.36%	21.92%	23.67%	8.04%	10.56%	10.95%	3.49%	71.28%	8.03
6	21.68%	22.46%	23.62%	7.25%	10.57%	10.54%	3.89%	70.57%	7.94
7	23.60%	22.13%	22.44%	6.83%	10.20%	10.47%	4.33%	69.79%	7.66
8	24.44%	23.13%	20.94%	6.49%	9.74%	10.24%	5.02%	68.84%	7.37
9	26.07%	24.06%	18.79%	6.08%	8.92%	10.13%	5.95%	67.44%	6.97
10	29.87%	25.28%	13.92%	5.93%	8.01%	7.88%	9.11%	63.14%	5.98

Table D.42: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	13.69%	15.51%	20.54%	7.57%	22.43%	18.17%	2.08%	75.51%	8.94
2	14.52%	16.98%	19.70%	6.45%	20.24%	18.94%	3.18%	73.80%	8.76
3	16.13%	16.61%	19.15%	6.31%	18.96%	18.90%	3.94%	72.79%	8.58
4	16.50%	16.75%	19.27%	6.39%	18.15%	18.29%	4.64%	71.92%	8.49
5	16.33%	16.58%	18.58%	6.10%	18.00%	18.62%	5.78%	71.08%	8.47
6	16.69%	17.01%	18.50%	5.95%	16.89%	18.48%	6.48%	70.25%	8.38
7	18.47%	17.28%	17.60%	5.80%	16.73%	17.02%	7.10%	69.32%	8.04
8	18.91%	17.83%	16.51%	5.67%	15.83%	17.17%	8.09%	68.26%	7.88
9	20.98%	18.47%	15.41%	5.58%	14.43%	16.07%	9.06%	66.76%	7.50
10	25.27%	19.77%	11.96%	5.81%	13.39%	12.33%	11.46%	62.98%	6.53

D.1.3.5 CPI + 8%

Table D.43: SA Only Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Probability	Standard Deviation
1	12.60%	19.64%	54.79%	12.97%	54.20%	13.62
2	18.17%	26.06%	44.88%	10.88%	50.43%	11.62
3	22.21%	27.79%	40.23%	9.77%	47.97%	10.62
4	26.53%	27.75%	35.92%	9.80%	45.66%	9.75
5	28.84%	29.70%	32.19%	9.26%	43.33%	9.04
6	31.66%	30.72%	28.08%	9.54%	40.83%	8.31
7	33.55%	33.40%	23.70%	9.36%	37.86%	7.60
8	36.76%	34.81%	18.55%	9.89%	33.94%	6.79
9	42.07%	34.48%	13.47%	9.97%	28.70%	5.90
10	52.04%	31.98%	7.76%	8.22%	18.87%	4.60

Table D.44: Restricted International Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	9.69%	14.55%	40.59%	10.17%	10.33%	11.76%	2.91%	53.91%	11.34
2	14.56%	19.19%	32.72%	8.53%	10.54%	11.43%	3.04%	50.06%	9.75
3	17.20%	20.58%	29.34%	7.87%	10.31%	11.14%	3.55%	47.66%	9.06
4	19.53%	21.18%	26.18%	8.11%	10.39%	10.88%	3.73%	45.64%	8.48
5	21.46%	22.28%	23.86%	7.39%	10.60%	10.51%	3.89%	43.60%	7.99
6	23.02%	23.86%	20.42%	7.71%	10.77%	10.32%	3.91%	41.49%	7.43
7	24.63%	24.75%	18.35%	7.26%	10.55%	10.00%	4.45%	39.07%	7.03
8	27.15%	25.76%	14.60%	7.49%	10.51%	9.80%	4.69%	35.98%	6.42
9	30.69%	26.15%	10.87%	7.29%	10.82%	9.22%	4.96%	31.72%	5.77
10	37.06%	25.06%	6.47%	6.42%	10.55%	8.22%	6.23%	23.79%	4.90

Table D.45: Unrestricted Portfolio

	SA Cash	SA Bonds	SA Equity	SA Property	Int Bonds	Int Equity	Int Property	Probability	Standard Deviation
1	8.54%	11.22%	25.53%	7.29%	20.13%	24.13%	3.17%	54.07%	10.39
2	11.35%	14.41%	22.51%	6.42%	18.79%	21.93%	4.58%	50.63%	9.56
3	13.30%	14.84%	21.46%	6.32%	18.79%	20.00%	5.28%	48.69%	9.14
4	15.21%	16.35%	20.25%	5.95%	17.55%	19.21%	5.48%	47.02%	8.75
5	16.96%	16.56%	18.80%	5.90%	17.94%	18.02%	5.82%	45.53%	8.39
6	18.22%	17.14%	17.52%	5.80%	17.51%	17.40%	6.41%	43.87%	8.10
7	20.05%	18.62%	16.62%	5.85%	16.72%	15.57%	6.58%	41.91%	7.68
8	21.43%	19.67%	13.91%	5.94%	16.83%	14.96%	7.25%	39.59%	7.26
9	23.71%	20.70%	12.02%	6.00%	16.43%	13.33%	7.81%	36.43%	6.77
10	28.73%	23.27%	8.61%	6.16%	14.35%	9.43%	9.44%	28.41%	5.73

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